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The Statistical Description and Interpretation of Geophysical Potential Fields Using Covariance Functions*

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Abstract. This is a review of the statistical description of random potential fields on spherical surfaces as well as on the plane. On spherical surfaces only the radial component of the random field is homogeneous and isotropic. But this is sufficient to estimate the degree variances of the potential and all the other components. Using the general representation by spherical harmonic series the collocation is described in the case of the radial component in much detail. The covariance function of the random potential field in the plane may be represented by a convolution. The covariance function of the kernel in the source equation of the random potential field is folded with the covariance function of the random source field. For two kernels the autocovariance functions are given and some statistical source models and the possibilities for the determination of the depth parameter of the kernel are mentioned.

Key words: Random potential fields – Spherical surfaces – Planes – Autocovariance functions – Degree variances – Collocation – Kernels and their autocovariance functions – Statistical source models – Parameter estimation – Periodicities in the empirical random density profile.

1. Introduction

For a number of years it has been tried to apply mathematical statistics to geophysical potential fields (a) to increase the knowledge about the spatial distribution of these fields by interpolation between the available observations and by combinations with other types of observations, (b) to analyse the structure of these fields and, if possible, to establish relations between different

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** To whom offprint requests should be sent fields, and (c) to obtain informations about the distribution of the sources of these fields.

The two most important geophysical potential fields are the gravity field of the Earth and the permanent part of the geomagnetic field. In geodesy and geophysics methods were developed for the statistical description and interpretation of the gravity field. This was not done with the same intensity for the geomagnetic field, and, still less, for the other geophysical potential fields.

There it is to take into account that the geomagnetic field shows a very pronounced secular variation and spatial irregularities much stronger than the gravity field in the global scale as well as in the regional and local scale.

The treatment of the potential fields by the methods of mathematical statistics is feasible as they may be understood as a linear superposition of a determined normal component by a multitude of irregularly distributed anomalies. The whole of the anomalies may be treated as a realization of a vectorial random field that has also a scalar potential. Beyond it the assumption is made that this field is homogeneous and isotropic on concentric spherical surfaces or on parallel planes in the case of flat Earth approximation. Then it is sufficient to study the covariance function of the anomalies of the potential.

2. The Determination of a Normal or Systematical Part

The splitting of the potential field into a systematic part and the overlaying anomalies is ambiguous. For instance it is possible to represent the potential field by a series with suitable basic functions and to restrict the systematic part to a very small number of terms. If global problems are considered this is always a spherical harmonic series because of the spherical surface of the Earth but also because of the optimal convergence properties for a statistical analysis of data distributed over a spherical surface. To estimate the serial coefficients usually the method of least squares is applied. There it is to take into account that the differences between the observations and the resulting normal values are correlated significantly. As a disadvantage for the interpretation the very irregular distribution of the observations at the Earth's surface appears. Some of the advantages of the spherical harmonics are efficient only for a continuous or a specialized regular distribution of the observations.

To determine the lower terms of the Earth's gravity field satellite observations are very important. Similarly the socalled I.G.R.F. (International Geomagnetic Reference Field) is based on the analysis of satellite observations of the total intensity, but it also contains terrestrial measurements. Because of the existence of large scale anomalies of great intensity, and because of the slow convergence of the spherical harmonic series the I.G.R.F. seems to be not suitable as normal field for the geological interpretation of the anomalies of the permanent magnetic field.

3. Statistical Description of Global Potential Fields

In considering global fields there are some troubles with the spherical surface. One relates to the definition of the random field on this surface, another to the interrelation between the vector components. Lauritzen (1973) has shown that it is impossible to find a stochastic process which is harmonic outside the sphere and both ergodic and normally distributed. In the present case the ergodicity is of great importance because the average taken over a great number of realizations can be replaced by the average of one realization only taken for a great number of observations distributed over the whole spherical surface.

$V(r, \vartheta, \lambda)$ is assumed to be a scalar random function which is harmonic outside the sphere $r = r_0$ and which is also ergodic. If V is also homogeneous and isotropic on the spherical surface, then the expectance yields

$$M[V(r, \vartheta, \lambda)] = 0, \tag{1}$$

and the covariance function

$$K_{VV}(r, \vartheta, \lambda;\ r', \vartheta', \lambda') = M[V(r, \vartheta, \lambda) \cdot V(r', \vartheta', \lambda')] \tag{2}$$

on every spherical surface depends on the spherical distance between the points (r, ϑ, λ) and $(r', \vartheta', \lambda')$. Here M is the average of the expression in rectangular brackets, taken over all realizations. Owing to the ergodicity M can be replaced by the average over all observations distributed over the whole surface. Because of the great number of observations it is allowed, and because of the simpler treatment it is advantageous to approximate the sum over all observations on the surface by the integral over the whole surface

$$M[\,] = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{1}{2\pi} \int_0^{2\pi} [\,]\, d\alpha \sin\vartheta\, d\vartheta\, d\lambda, \tag{3}$$

where α is the azimuth between the two points (r, ϑ, λ) and $(r', \vartheta', \lambda')$.

Outside the sphere the potential V is represented in series of spherical harmonics

$$V(r, \vartheta, \lambda) = r_0 \sum_{n=1}^{\infty} \sum_{m=0}^{n} (G_n^m \cos m\lambda + H_n^m \sin m\lambda)\, P_n^m(\vartheta) \left(\frac{r_0}{r}\right)^{n+1}. \tag{4}$$

Taking into account that the covariance function for V depends only on the distance τ, for the correlation moments of the series coefficients we get

$$\begin{aligned} &K(G_n^m, G_{n'}^{m'}) = K(H_n^m, H_{n'}^{m'}) = \delta_{mm'}\, \delta_{nn'} \frac{2n+1}{2} \int_0^{\pi} K_{VV}(\tau) \sin\tau\, d\tau = K(G_n^0, G_n^0) \\ &K(G_n^m, H_{n'}^{m'}) = 0 \quad \text{for all combinations of } m, m', \text{ and } n, n'. \end{aligned} \tag{5}$$

These may be understood as coefficients in a Legendre polynom series of the covariance function of the potential

$$K_{VV}(r, r', \tau) = r_0^2 \sum_{n=1}^{\infty} K(G_n^0, G_n^0)\, P_n^0(\tau) \left(\frac{r_0^2}{r r'}\right)^{n+1}. \tag{6}$$

They result from the equation

$$K(G_n^0, G_n^0) = \frac{1}{2n+1} \sum_{m=0}^{n} [(G_n^m)^2 + (H_n^m)^2] \tag{7}$$

with the series coefficients of the known realization G_n^m, H_n^m. The parameters (7) are defined as "degree variances".

If the covariance function of the potential V is known, in general it is rather easy to determine the covariance functions for the field quantities which are related to the potential, for instance the gradient **F**. Usually the gradient is split into components related to the directions of the geographical coordinates

$$X=\frac{1}{r}\frac{\partial V}{\partial \vartheta},\qquad Y=-\frac{1}{r}\frac{1}{\sin\vartheta}\frac{\partial V}{\partial \lambda},\qquad Z=\frac{\partial V}{\partial r}. \tag{8}$$

Taking into account the criteria for the potential $V(r,\vartheta,\lambda)$ the expectances for all three components disappear. Instead of the one covariance function of the potential V the random vector field derived from the potential requires three autocovariance functions K_{XX}, K_{YY}, K_{ZZ} and six cross-covariance functions K_{ZX}, K_{ZY}, ... They are obtained using the relations

$$\begin{aligned}K_{ZZ}(r,\vartheta,\lambda;r',\vartheta',\lambda')&=\frac{\partial^2}{\partial r\,\partial r'}K_{VV}(r,r',\tau)|_{r=r'},\\ K_{ZX}(r,\vartheta,\lambda;r',\vartheta',\lambda')&=\frac{1}{r'}\frac{\partial^2}{\partial r\,\partial\vartheta'}K_{VV}(r,r',\tau)|_{r=r'}\quad\text{etc.}\end{aligned} \tag{9}$$

After some steps analytical expressions result which have the following properties: on the spherical surface $r=r'$ merely the autocovariance function of the radial component Z is a function of the distance τ only. All remaining autocovariance functions explicitly depend on the coordinates of the two reference points (r,ϑ,λ) and (r',ϑ',λ'). Hence, under the given conditions for the potential only the radial component is definitely homogeneous and isotropic on the spherical surface $r=r'$ and, therefore, suitable for an estimation of the covariance on the basis of one realization. Since in the expressions for the covariance functions of the potential and of the remaining components only the correlation moments $K(G_n^0, G_n^0)$ occur as empirical parameters, it should be noted that the radial component is sufficient for the determination of all other remaining correlation functions.

The matrix of the auto- and cross-covariance functions of the components of the vector field **F** can substantially be simplified if, instead of a splitting into geographical coordinates a split of components is used with reference to the interrelated position of the two reference points on the spherical surface. The radial component Z remains unchanged; X and Y are substituted by the components L and T, with L being parallel to the great circle which joins the two reference points, and T being perpendicular to it. They can be obtained from the potential function V using the relations

$$L(r,\tau,\alpha)=\frac{1}{r}\frac{\partial V}{\partial\tau},\qquad T(r,\tau,\alpha)=-\frac{1}{r}\frac{1}{\sin\tau}\frac{\partial V}{\partial\alpha}. \tag{10}$$

For points on the spherical surface $r=r_0$ result the auto- and cross-covariance functions

$$K_{LL}(\tau)=\sum_{n=1}^{\infty}K(G_n^0,G_n^0)\sqrt{\tfrac{1}{2}n(n+1)}\,\frac{dP_n^1(\tau)}{d\tau},$$

$$K_{TT}(\tau)=\sum_{n=1}^{\infty} K(G_n^0, G_n^0)\sqrt{\tfrac{1}{2}n(n+1)}\,\frac{P_n^1(\tau)}{\sin\tau},$$

$$K_{ZZ}(\tau)=\sum_{n=1}^{\infty} K(G_n^0, G_n^0)(n+1)^2 P_n^0(\tau), \tag{11}$$

$$K_{ZL}(\tau)=-K_{LZ}(\tau)=\sum_{n=1}^{\infty} K(G_n^0, G_n^0)\sqrt{\tfrac{1}{2}n(n+1)^3}\,P_n^1(\tau),$$

$$K_{TZ}(\tau)=K_{ZT}(\tau)=K_{LT}(\tau)=K_{TL}(\tau)=0.$$

All covariance functions depend only on the spherical distance τ between the two reference points (ϑ, λ) and (ϑ', λ'), and they can be estimated using the observations of one realization. The numerical calculation, however, is rather laboreous since the horizontal components L and T are to be determined separately for each combination of reference points (Kautzleben, 1966, 1967; Harnisch and Kautzleben, 1976).

Figures 1 and 2 contain the auto-covariance functions of the gravity field and of the geomagnetic main field, for which the empirical values have been

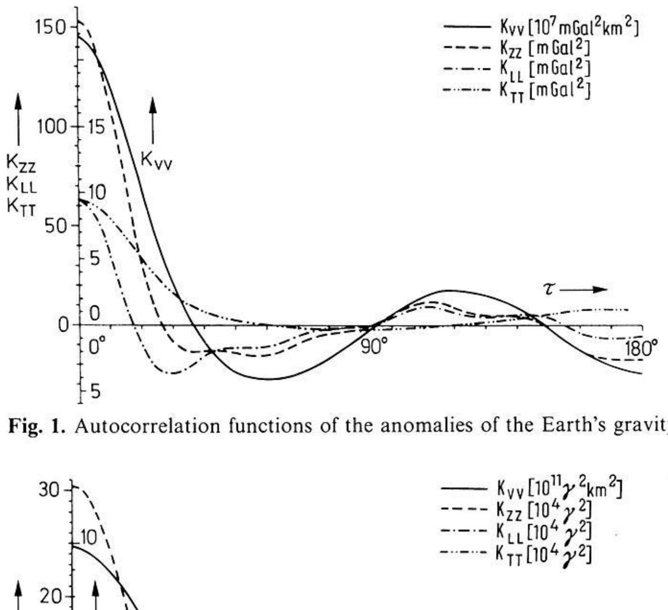

Fig. 1. Autocorrelation functions of the anomalies of the Earth's gravity field

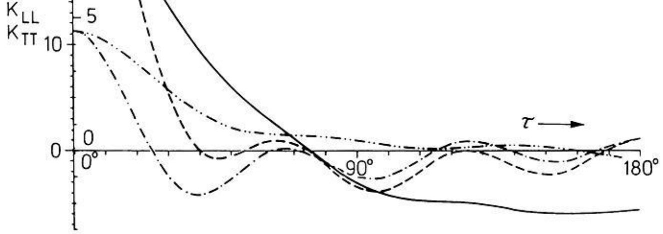

Fig. 2. Autocorrelation functions of the anomalies of the permanent geomagnetic field

drawn from studies by Tscherning and Rapp (1974) and Kautzleben (1969). We can see that the longitudinal covariance function K_{LL} and the covariance function of the radial component oscillate in a more pronounced way, whereas the transversal covariance function K_{TT} over a major part rather smoothly approximates the zero line, however, leaving it again with increasing τ. The largest difference is found between the covariance functions of the potential function of the two fields.

4. Correlation of Different Fields

The correlation analysis of an individual potential field may also be applied to the investigation of the stochastic relations between different fields on a spherical surface. For this end the various fields are taken as components of a multidimensional random function which is defined on the spherical surface. Of particular interest in this case are the cross-covariance functions in the meaning of two-point-correlations. Using here the spherical harmonic series for two functions $F(\vartheta, \lambda)$ with the coefficients A_n^m, B_n^m and $G(\vartheta', \lambda')$ with the coefficients C_n^m, D_n^m for the cross-covariance function the expression

$$K_{FG}(\tau)=\sum_{n=1}^{\infty} K(A_n^0, C_n^0)\, P_n(\tau) \tag{12}$$

is obtained with the correlation moments

$$K(A_n^0, C_n^0)=\frac{1}{2n+1}\sum_{m=0}^{n} (A_n^m C_n^m + B_n^m D_n^m). \tag{13}$$

Only the serial coefficients with equal indices and equal angular functions are different from zero. For equal n the correlation moments are equal to each other. These formulas have been derived by Kaula (1967) and introduced into geophysics. Since then they have been used by several authors.

Let us assume that the functions F and G are field quantities, to each of which a scalar potential can be associated. It is, of course, reasonable to relate every possible cross-covariance function to the relation of the correlation between the two potential fields themselves.

5. Prediction and Collocation

In the determination of the Earth's gravity field statistical methods of interpolation, extrapolation and prediction have been added to the classical least-squares adjustment. Krarup (1969) showed that the prediction formulas can be used to calculate arbitrary parameters of the gravity field if arbitrary elements of this field are measured. Moritz (1973) showed that the prediction methods can be combined with a determination of parameters which represent systematic parts by adjustment. This yields a rather generalized method of least squares, which is called "collocation". It allows the standardized application of any kind of geometrical and physical measurements for an optimal determination of the shape of the Earth and its gravity field. This method can also be applied to any other geophysical potential field, and by the optimal processing of any available observations it provides as much detail as possible about the field variation which are compatible with the observations.

Collocation may always be used if the problem under study contains, apart from the measuring errors **n**, another irregular part **s**, i.e. if the remaining residuals after subtraction of a systematic part may be split into two components which are different in their statistical behaviour.

We start at the observation equation

$$\mathbf{x}=\mathbf{A}\mathbf{X}+\mathbf{s}+\mathbf{n} \tag{14}$$

x designates discrete observations, **A** a given matrix of coefficients, **X** the vector of the wanted parameter, **n** the measuring errors, and **s** the signal as the second random variable. The vectors **n** and **s** are purely random vectors whose expectances vanish. The correlation matrices of the two vectors are designated as **D** and **C**. It is assumed that the measuring errors and the signal are not correlated.

We look for a linear estimation of the signal $\mathbf{s}_P$ at some new points which may be different from the observation points. In analogy to adjustment collocation requires the unknown parameters **X** to be determined in such a way that the total of deviation squares at the observational points and the new points is a minimum. Hence, in the present problem occur additional random parameters which are related to the observations only in an indirect manner via correlation. Due to the requirement of minimum we obtain the formulas

$$\mathbf{X}=(\mathbf{A}^T\bar{\mathbf{C}}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\bar{\mathbf{C}}^{-1}\mathbf{x}, \tag{15}$$

$$\mathbf{s}_P=\mathbf{C}_P^T\bar{\mathbf{C}}^{-1}(\mathbf{x}-\mathbf{A}\mathbf{X}). \tag{16}$$

Equation (15) determines the parameter vector **X**, then follows the signal $\mathbf{s}_P$ at the new points from Equation (16). The matrix $\bar{\mathbf{C}}$ is the correlation matrix of the observation vector **x**. Provided that **n** and **s** are not correlated we have

$$\bar{\mathbf{C}}=\mathbf{C}+\mathbf{D}. \tag{17}$$

The matrix $\mathbf{C}_P^T$ covers the correlations between the signal $\mathbf{s}_P$ and the observation vector **x**.

The correlation values of the signal occurring in the matrices **C** and $\mathbf{C}_P$ are calculated, from the given covariance function of the anomalies of the potential and of further field parameters which are related to it. The observed parameters and those which are to be calculated may be heterogeneous. There is only the requirement that the mathematical relations between them are known, and all correlations must strictly be related to the same covariance function of the potential.

One application of collocation, important for all geophysical potential fields, concerns the combination of spherical harmonic series of the field with direct observation on the Earth's surface for the derivation of an optimum description of the field concerned, by the improvement of the numerical values for the coefficients as well as by prediction of values for the field in areas where direct observations are absent. The method was in much detail tested when combining results of satellite geodesy with terrestrial measurements of the gravity field (Moritz, 1970).

The principle may be demonstrated considering the radial component Z of the anomalies of the field gradient. From direct terrestrial measurements there should be available as observations the M parameters Z_i and from other observations, e.g. via satellites, the $(N+1)^2$ coefficients a_n^m, b_n^m in the finite series

$$Z(\vartheta,\lambda)=\sum_{n=0}^{N}\sum_{m=0}^{n}(a_n^m\cos m\lambda+b_n^m\sin m\lambda)\,P_n^m(\vartheta). \tag{18}$$

We now need the estimates $\bar{\bar{Z}}$ of values at some new points and estimates $\bar{a}$ for a number of serial coefficients, each as a linear combination of all available observations. A systematic part is assumed to be already completely eliminated in the observations. The solution of the problem is furnished by Equation (15), where the signal vector in the defined way consists of the wanted estimate. Following Section 3 the covariance function of Z will be

$$K_{ZZ}(\tau)=\sum_{n=0}^{N}c_n P_n(\tau). \tag{19}$$

For the correlation moments between the serial coefficients a_n^m, b_n^m we obtain using Equations (5) and (19)

$$K(a_n^m,a_n^m)=K(b_n^m,b_n^m)=c_n \tag{20}$$

or zero for all remaining combinations of the indices m, n. The covariance functions between the values of Z and the serial coefficients result by using the series and taking into account the properties of the correlation moments.

$$\begin{aligned}K(Z,a_n^m)&=c_n P_n^m(\vartheta)\cos m\lambda,\\ K(Z,b_n^m)&=c_n P_n^m(\vartheta)\sin m\lambda.\end{aligned} \tag{21}$$

The coordinates are related to the point where Z is considered. Now all parameters are known from which the matrices $\mathbf{C}$ and $\mathbf{C}_P$ in (15) and (16) have been set up. The matrix $\mathbf{D}$ of the observations must be derived from the measuring and the observational errors. The prominent problem of the practical evaluation is the inversion of the matrix $\bar{\mathbf{C}}$ in Equation (15).

6. The Statistical Structure of the Potential Field and That of the Sources

In the previous chapters the considerations were focused on the statistical structure of the potential field only. But from the mathematical and physical point of view the investigation of the relation between the statistical structure of the potential field and the structure of the sources is of much more concern.

To avoid difficulties concerning ergodicity we consider in a first step a model with the following properties: The Earth's surface within the area of investigation is considered as a section of an infinite plane (so-called "flat Earth approximation". For global problems see Tscherning, 1976). Observations are interpreted as a realization of a random function which is defined on this plane. Therefore the observations within the limited area are a sample of the random function. To get in the observation plane a random homogeneous potential field the stochastic source field must also be defined over an infinite plane, i.e. the support (the definition region) concerning the two horizontal coordinates x_1 and x_2 must be infinite.

Naturally the random source field itself must fulfill the condition of homogeneity with respect to the horizontal coordinates. In the vertical direction the source field may be homogeneous or inhomogeneous. If the source field is homogeneous in all three coordinates one gets for the first moment (mean value) the model of a Bouguer plate.

Then the connection between the stochastic source field $\Psi(\mathbf{x})$ and the stochastic potential field $\Phi(\mathbf{x})$ may be written in the form

$$\Phi(\mathbf{x}) = \int_T k(\mathbf{x}-\mathbf{x}')\,\Psi(\mathbf{x}')\,\mathrm{d}\mathbf{x}', \tag{22}$$

where k is the Green function or the kernel of the integral equation, which is different for each discrete expression of a potential, T is the support of the random source field

$$-\infty \leqq x_1 \leqq +\infty, \qquad -\infty \leqq x_2 \leqq +\infty, \qquad t_2 \leqq x_3 \leqq t_1, \qquad t_1 \text{ and } t_2$$

the lower and upper boundaries in the x_3-direction.

For the autocovariance function of the potential field we immediately obtain from (22) the general expression

$$K_{\Phi\Phi}(\mathbf{x}_1, \mathbf{x}_2) = \int_T \int_T k(\mathbf{x}_1 - \mathbf{x}'_1)\, k(\mathbf{x}_2 - \mathbf{x}'_2)\, K_{\Psi\Psi}(\mathbf{x}'_1, \mathbf{x}'_2)\,\mathrm{d}\mathbf{x}'_1\,\mathrm{d}\mathbf{x}'_2 \tag{23}$$

where $K_{\Psi\Psi}$ is the autocovariance function of the random source field. For reasons of simplicity the support of the random source field is suggested to be reduced to a simple layer in the depth ${}_1x'_3 = {}_2x'_3 = t$ parallel to the Earth's surface ${}_1x_3 = {}_2x_3 = a$. Since by the above noted assumptions the source field is a homogeneous random field and the integration is a linear operator, the both autocovariance functions $K_{\Phi\Phi}$ and $K_{\Psi\Psi}$ are dependent only on the horizontal distance

$$\mathbf{s} = (s_1, s_2), \qquad s_1 = {}_2x_1 - {}_1x_1, \qquad s_2 = {}_2x_2 - {}_1x_2$$

between the two points concerned and on the distance

$$h = {}_1x_3 - {}_1x'_3 = {}_2x_3 - {}_2x'_3 = a - t.$$

Commonly the observation plane is the Earth's surface. Then h means the depth of any layer in the Earth's interior.

We can rewrite Equation (23) into

$$K_{\Phi\Phi}(\mathbf{s}, a) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} k(\mathbf{y}, h)\, k(\mathbf{y} + \mathbf{z}, h)\, K_{\Psi\Psi}(\mathbf{s} - \mathbf{z}, t)\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}. \tag{24}$$

Provided that k is absolutely integrable, following the communication theory the expression

$$K_{kk}(\mathbf{z}, h) = \int_{-\infty}^{+\infty} k(\mathbf{y}, h)\, k(\mathbf{y}+\mathbf{z}, h)\, \mathrm{d}\mathbf{y} \tag{25}$$

can be designated as the autocovariance function of the determined Green function k. We obtain the rather unsophisticated relation

$$K_{\Phi\Phi}(\mathbf{s}, a) = \int_{-\infty}^{+\infty} K_{kk}(\mathbf{z}, h)\, K_{\Psi\Psi}(\mathbf{s}-\mathbf{z}, t)\, \mathrm{d}\mathbf{z} \tag{26}$$

between the three covariance functions of the potential field in the measuring plane a, the distribution of sources on the boundary surface t and the Green function k. The first is obtained by a convolution of the latter two functions.

There are not much difficulties in setting up analogous formulas of less simpler models for instance a stack of planes or layers of finite thickness (Schwahn, 1975a). In such cases the autocovariance function $K_{\Psi\Psi}(\mathbf{s}, {}_1x_3', {}_2x_3')$ between the two points $\mathbf{x}_1'$ and $\mathbf{x}_2'$ and the integration concerning ${}_1x_3'$ and ${}_2x_3'$ within the vertical boundaries must be taken into consideration. This shows that the function $K_{\Phi\Phi}$ depends on the properties of the random source field in the vertical direction also. This fact was neglected up to now.

Let us divide the further discussion of Equation (26) into the determination of K_{kk} for distinct Green functions and the consideration of $K_{\Psi\Psi}$ for some simple stochastic source fields.

One yields (Schwahn, 1975b) by a twofold Hankel-transform for the Green function $g(\mathbf{y}, h)$ of the gravity field the autocovariance function (Fig. 3)

$$K_{gg}(\mathbf{s}, h) = 2\pi\, 2h(|\mathbf{s}|^2 + (2h)^2)^{-3/2}. \tag{27}$$

Using the relation between the autocovariance function of a scalar random potential field and his derivatives the autocovariance function K_{zz} of the kernel $z(\mathbf{y}, h)$ of the vertical intensity of the anomalous magnetic field results in the

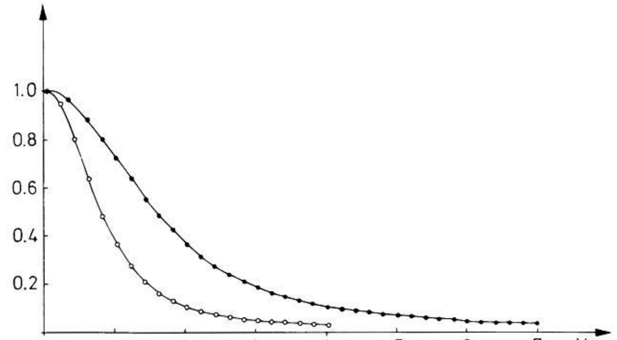

Fig. 3. Autocorrelation function (●—●—●) (normalized autocovariance function) of the kernel of the random gravity field. For comparison the gravity anomaly due to a point mass in the same depth is added (○—○—○). Both functions are given for the relation v = sampling interval · point number · (depth)$^{-1}$

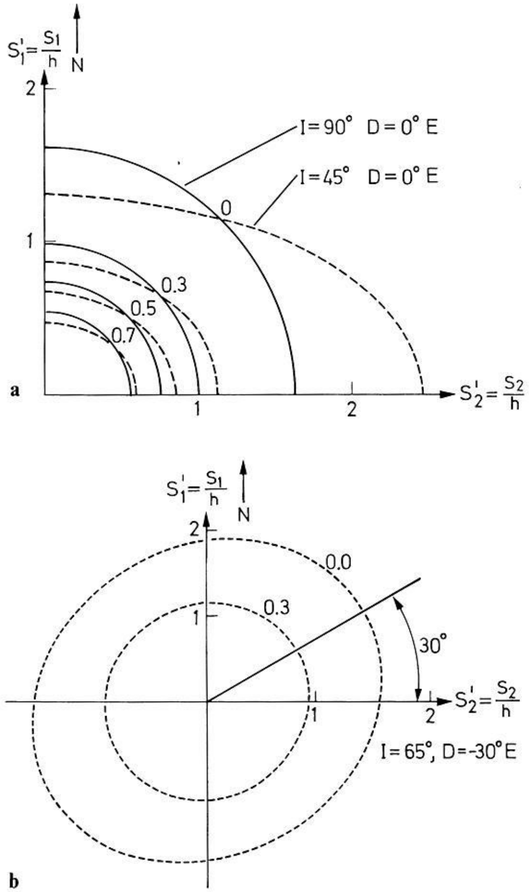

Fig. 4. Twodimensional autocorrelation function of the kernel of the geomagnetic vertical intensity for different inducing geomagnetic main field. **a** Examples for different inclination I (I = 90°, 60°) and declination D = 0°. **b** Example for declination = − 30°, inclination = 65°.
The functions are given for the relation sampling interval · lag number · (depth)$^{-1}$

following form (Schwahn, 1976)

$$K_{zz}(\mathbf{s}, h) = 6\pi\, 2h((2h)^2 + |\mathbf{s}|^2)^{-7/2}(f_1^2((2h)^2 + s_2^2 - 4s_1^2) + f_2^2((2h)^2 + s_1^2 - 4s_2^2) + f_3^2(2(2h)^2 - 3(s_1^2 + s_2^2)) - 10\, f_1 f_2\, s_1\, s_2), \tag{28}$$

whereby **f** stands for the direction-cosinus of magnetization vector (Fig. 4a, b). If the vector has not (as it is assumed here) a constant direction, then we must replace the simple terms of **f** by the corresponding elements of the covariance matrix of the magnetization vector.

If the random field of susceptibility is homogeneous and isotropic an *anisotropic* autocovariance function of the vertical intensity yields for regions apart from the geomagnetic pole. That means, that it is impossible to get an information on the statistical structure of the susceptibility from very long profiles crossing different geomagnetic positions if the data of the vertical intensity are not reduced on the north pole before the further computations.

Let us now consider some random fields of sources. In the rather simple case the sources in the plane t would have to be packed densely, while the related parameters of the involved random material $\Psi(\mathbf{y}, t)$ between two areal elements are – in statistical terms – entirely independent of each other and being $N(0, \sigma^2)$ distributed. From (26) we immediately obtain

$$K_{\Phi\Phi}(\mathbf{s}, a) = \sigma^2 K_{kk}(\mathbf{s}, h). \tag{29}$$

This model, for example, has been successfully applied by a number of authors (e.g. Seron and Hannaford, 1957; Mundt, 1969; Schwahn, 1975c) to the statistical interpretation of anomalies of the gravity or the magnetic field of the Earth.

Except for a constant factor we obtain the same results even for a model with single point sources which are distributed within the plane following a Poisson distribution and the same material parameters as noted above (Serson and Hannaford, 1957; Schwahn, 1975c).

For the cases of infinitely extended strips in the x_1-direction models were considered by Vasiljev (1965) under the assumption of exponential distributed widths of the strips and by Schwahn (1976) under those of equal distribution of the widths.

All the resulting theoretical autocovariance functions have not, with few exceptions, the shape of those autocovariance functions, obtained on the base of empirical data. As an example serves there (Fig. 5) the autocovariance function of gravity $K^e_{\Delta g \Delta g}$ on a profile Schonen – Lappland in Sweden (Schwahn, 1975b). The long periodicities in the gravity data were removed by filters with different length. Periodicities of nearly 25 ... 30 km are remarkable.

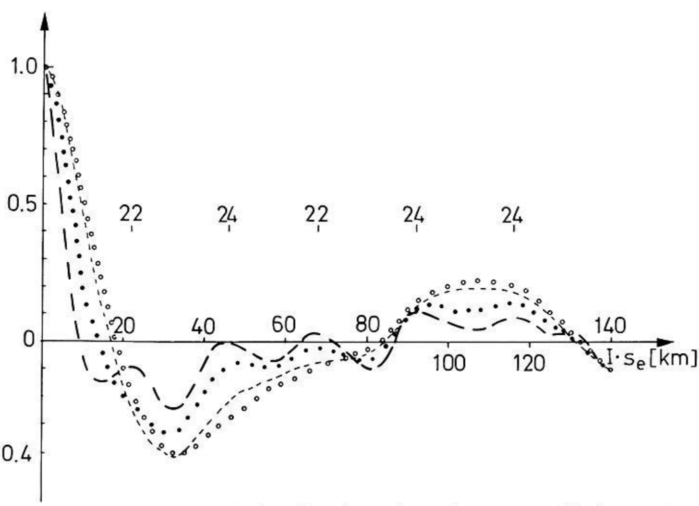

Fig. 5. Empirical autocorrelation function of gravity on a profile in Sweden after high-pass filtering ○○○ 51 coefficients, ----- 41 coefficients, ●●● 31 coefficients, — — — 21 coefficients. The lag distances are smaller than profile length/10. The distances between one maximum to the next are given by numbers above the curve

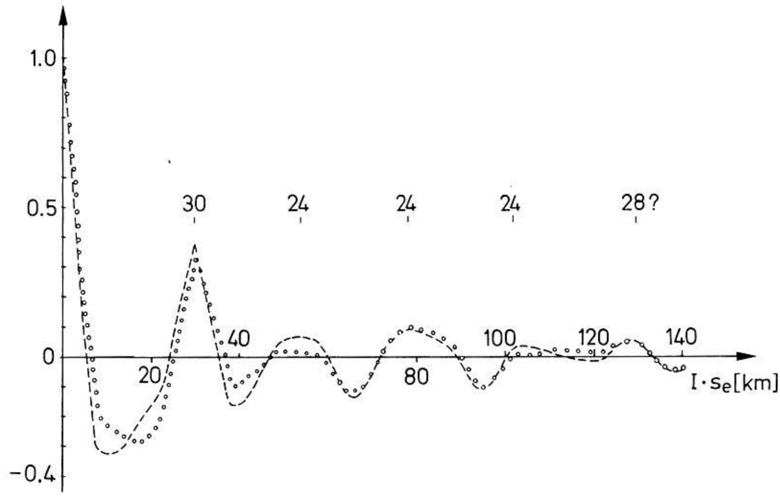

Fig. 6. Empirical autocorrelation function of density on a profile in Sweden after high-pass filtering ○ ○ ○ 51 coefficients, --- 31 coefficients. The lag distances are smaller than profile length/10. The distances between one maximum to the next are given by numbers above the curve

According to Equations (26) and (27) such periodicities can be explained only in terms of the autocovariance function $K_{\Psi\Psi}$ of the density. For a confirmation a density profile for the same location was compiled on the base of the geotectonic situation and rock density measurements (Schwahn, 1975b). After corresponding filtering the autocovariance function of the rock density $K^{e}_{\Psi\Psi}$ on the earth's surface was computed (Fig. 6). Indeed, as it was desired, this function shows very pronounced periodicities in the range of 25 km, i.e. the same range as in the case of gravity.

Within the framework of the statistical analysis of potential fields the consideration of the autocovariance function $K_{\Psi\Psi}$ from the theoretical as well as from the empirical point of view is very important because we have now a close connection to the ideas on the statistical structure of geological features (e.g. Agterberg, 1970; Mundt and Wirth, 1973; Wirth, 1975) and their physical parameters, for instance the viscosity (Schwahn, 1975d).

But not only the connection with geological models are of interest, when the stochastic potential field is considered. Often we are interested in the value of the parameter h, the depth. If we know the random functions both of the stochastic potential field and of the stochastic field of the sources (as in the case of Sweden) and if we make the assumptions that the sources may be concentrated within a plane and that the stochastic fluctuations on the Earth's surface are quite similar in reasonable depth range (the latter supposition is fulfilled presumably within the Fennoscandian shield), then we can get

- the autocovariance function $K_{\Phi\Phi}$ of the stochastic potential field
- the autocovariance function $K_{\Psi\Psi}$ of the stochastic source field
- the cross-covariance function $K_{\Psi\Phi}$.

The kernel $k(\mathbf{y}, h)$ may be obtained using the Wiener-Hopf-equation

$$\int_{-\infty}^{+\infty} k(\mathbf{y}, h)\, K_{\Psi\Psi}(\mathbf{s}-\mathbf{y})\,\mathrm{d}\mathbf{y} - K_{\Psi\Phi}(\mathbf{s}) = 0 \qquad \text{for } s \geq 0. \tag{30}$$

We did not solve this equation. Using the known empirical autocovariance function $K^e_{\Psi\Psi}$ the expression

$$f(h) = E\left\{\left[\int_{-\infty}^{+\infty} K_{gg}(\mathbf{y}, h)\, K^e_{\Psi\Psi}(\mathbf{s}-\mathbf{y})\,\mathrm{d}\mathbf{y} - K^e_{\Delta g \Delta g}(\mathbf{s})\right]^2\right\} \tag{31}$$

was computed for the gravity in Sweden and we looked for the minimum of $f(h)$ by a stepwise choice of the parameter h. One gets the minimum between 18 and 22 km. These values coincide very well with the depth of the seismic Conrad-discontinuity in Sweden.

The detection of periodicities in a sample of the potential field, if there is a stochastic periodicity in the source field depends on several factors. Equation (22) means a smoothing operation (low-pass filter) all the more the greater the distance h is between the two planes. Therefore the detection of any periodicity in the source field on the basis of a sample of the potential field is an estimation problem in dependence on the field under consideration, the wavelength of the periodicity and their amplitude in the stochastic source field, the parameter h and the sampling interval. On the basis of a Rice-distribution of the distance between the two consecutive point-masses we found (Schwahn, 1976), that the relations sampling interval/depth > 0.3 and mean distance/mean square error of the mean distance for the x_2-direction > 2 must be fulfilled for a suitable fixed sampling interval.

7. Conclusions

In this article the problems have been made evident which arise in the application of the general theory of stochastic processes to the potential fields of stochastic nature in global and local scales.

There are troubles which are caused by the finite integration range of the sphere in connection with the ergodic hypothesis and in problems on the plane the assumption of the absolute integrability of the kernels and the demand for homogeneity.

From the statistical treatment of geophysical potential fields it may be concluded that it is necessary to study stochastic source models which are consistent with models of other Earth sciences.

References

Agterberg, F.P.: Autocorrelation function in geology. In: Geostatistics, a colloquium, D.F. Merriam, pp. 113–142. New York-London: Plenum Press 1970

Harnisch, M., Kautzleben, H.: Zur globalen Korrelationsanalyse geophysikalischer Potentialfelder. Gerlands Beitr. Geophys. **85** (5), 415–424, 1976

Kaula, W.M.: Geophysical implications of satellite determinations of the Earth's gravitational field. Space Sci. Rev., Dortrecht (Holland) **7**, 769–794, 1967

Kautzleben, H.: Statistische Analyse des geomagnetischen Hauptfeldes. Habilitationsschrift, Leipzig 1966 (unpublished)

Kautzleben, H.: Zur Interpretation des geomagnetischen Hauptfeldes als vektorielles Zufallsfeld. Z. Geophys. **33**, 415–424, 1967

Kautzleben, H.: K statističeskomu opisaniju glavnogo geomagnitnogo polja, kak slučajnogo vektornogo polja. Geomagnetism i Aeronomija **9** (2), 321–327, 1969

Krarup, T.: A contribution to the mathematical foundation of physical geodesy. Geod. Inst. Meddelelser Nr. 44, Kobenhavn, 1969

Lauritzen, St.: The probabilistic background of some statistical methods in physical geodesy. Geod. Inst. Meddelelser Nr. 48, Kobenhavn, 1973

Moritz, H.: Least squares estimation in physical geodesy. Dtsche Geod. Komm. Bayer. Akad. Wiss., R. A **69**, 34, 1970

Moritz, H.: Least-squares collocation. Dtsche Geod. Komm. Bayer. Akad. Wiss., R. A **75**, 91, 1973

Mundt, W.: Statistische Analyse geophysikalischer Potentialfelder hinsichtlich Aufbau und Struktur der tieferen Erdkruste. Abh. geomagnet. Inst. Potsdam **41**, 57–196, 1969

Mundt, W., Wirth, H.: Objektivierte Potentialfeldanalysen. Z. geolog. Wiss. **1**, 1373–1380, 1973

Schwahn, W.: Eine allgemeine Formulierung der Auto- und Kreuzkorrelationsfunktionen eines beliebigen statistischen Potentialfeldes in einem kartesischen Koordinatensystem. Gerlands Beitr. Geophys. **84** (2), 143-154, 1975

Schwahn, W.: Beziehungen zwischen den Autokovarianzfunktionen der Schwere und der Dichte und deren Anwendung bei der Tiefenbestimmung der Quellen, demonstriert anhand von zwei Profilen über Schweden. Gerlands Beitr. Geophys. **84** (3/4), 281–296, 1975

Schwahn, W.: Zwei statistische Dichtemodelle und die statistischen Momente erster und zweiter Ordnung ihrer Schwerefelder. Gerlands Beitr. Geophys. **84** (6), 478–486, 1975

Schwahn, W.: Das Auftreten von Wiederholungstendenzen in den Dichtestrukturen der Erdkruste und deren Beziehung zu Viskositätsabschätzungen für Substanzen der Erdkruste. Geodät. geophys. Veröff., R. III **36**, 91–93, 1975

Schwahn, W.: Die Beziehungen zwischen den Autokovarianzfunktionen der Potentialfelder und denen der Quellenverteilung unter besonderer Berücksichtigung der realen geologischen Situation. Diss., Zentralinst. Physik d. Erde, Potsdam 1976 (unpublished)

Serson, P.H., Hannaford, W.L.: A statistical analysis of magnetic profiles. J. Geophys. Res. **62**, 1–18, 1957

Tscherning, C.C.: A mass density covariance function consistent with the covariance function of the anomalous potential. Boll. geodes. Sci. aff. **25** (2), 161–172, 1976

Tscherning, C.C., Rapp, R.H.: Closed covariance expressions for gravity anomalies, geoid undulations, and deflections of the vertical implied by anomaly degree variance models. Ohio State Univ., Dep. Geod. Sci. Report 208, 1974

Vasiljev, V.G.: Optimal filtering of the stochastic aggregate of slabs, infinite extended in one horizontal and in the vertical direction (in russian). Geolog. Geofiz. **1**, 146–148, 1965

Wirth, H.: Ansätze zur geodynamischen Interpretation von Potentialfeldanomalien. Gerlands Beitr. Geophys. **84** (2), 137–142, 1975

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