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The Inversion of Surface Wave Dispersion Data with Random Errors*

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Abstract. The ability to use a given set of dispersion data to resolve certain upper mantle model parameters is investigated through the expedient of computing dispersion for a known structure, adding random phases to the result, and then performing an inversion; we use the known structure as the starting model in the inversion. If the phase errors are random and uncorrelated, then the variances in the group velocities are much larger than those for phase velocities. For fundamental mode Love and Rayleigh wave phase and group velocities determined over the range 20–250 s, phase velocity data are considerably more potent resolvents of upper mantle structure than group velocity data. For a continental structure, Love and Rayleigh wave phase velocity data over the same period band have comparable ability to resolve structure, except for low-velocity channel thickness, for which Rayleigh wave data have superior resolution; for an oceanic structure, the two types of dispersion data also give comparable resolution except for lid thickness, for which Rayleigh waves have superior resolution.

Key words: Inverse Problem – Rayleigh waves – Love waves – Phase velocities – Group velocities – Resolution.

Introduction

A belief widely accepted by many practitioners of inversion of surface wave dispersion data is that the inversion of phase velocity dispersion data yields a more tightly knit set of models than does the inversion of group velocity data, for the two sets of data taken over the same span of periods. A "proof" of this statement has been given by Pilant and Knopoff (1970) for the case of noise-free

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data. Because of the usual derivative relationship between group velocities U and phase velocities c,

$$\frac{1}{U}=\frac{d(\omega/c)}{d\omega} \tag{1}$$

where ω is the frequency, it is claimed that different values of the constant of integration arising from the integration of this expression will generate a large family of possible models. The demonstration fails to show with certainty that the function $\{\omega/c(\omega)+\text{constant}\}$ is derivable from a possible real model for the earth, if the function $\omega/c(\omega)$ itself is derived from a real model for the earth. The proposal may be true, but in any event the proof applies only to noise-free data. In the presence of experimental uncertainties among the measurements of phase and group velocities, the span of acceptable models in either case is broadened. In this paper we report on a numerical investigation of the above proposition for the case of dispersion data in the presence of random influences on the measurements.

To study this problem in isolation from other possible influences on the inversion, we have started with a known geophysical structural cross-section for which the dispersion relations $U(T)$ and $c(T)$ are computed with ease by standard methods; T is the period, $T=2\pi/\omega$. To simulate standard inversion procedures, we have digitized the dispersion values into discrete phase and group velocity samples at selected periods, T_i. We assume the velocity samples $U(T_i)$ and $c(T_i)$ have standard deviations $\sigma_U(T_i)$ and $\sigma_c(T_i)$ respectively. We assume the errors in the "data" are random and uncorrelated at each of the sample periods. These "noisy data" are then inverted by a linear inversion procedure using as a starting model, the "exact" solution, namely the starting model.

One aspect of the procedure is inconsistent with customary dispersion analysis. We assume the data errors are uncorrelated from sample to sample. In actual practice, a single seismogram is often used to generate an entire dispersion curve or a major segment of it; the seismogram is usually so heavily processed by the numerical filtering and windowing techniques that are applied to remove extraneous signals, that adjacent sample estimates at nearby periods are likely to be correlated, especially if the periods of the samples are sufficiently close. It is not our purpose to investigate the simulation of a realistic process of data analysis in this paper; as we have indicated we wish to study the problem outlined above, in isolation from other influences. With regard to phase velocities, the problem proposed is therefore similar to the obtaining of each sample estimate perhaps by the processing of an independent earthquake event, and perhaps by some other means not specified. In any case we can imagine that, to the theoretical phase diffeferences for the continuum before digitization, we add a random, uncorrelated phase.

The variances in the group velocities are derived by some process of differentiation of the phases, since the group velocities themselves are the result of a differentiation process. However the method of differentiation deserves some careful attention. Let us assume that the group velocity samples are obtained by taking differences of phase estimates at pairs of relatively closely spaced frequencies with frequency difference $\Delta\omega$. We write (1) as a difference

$$\frac{1}{U}\approx\frac{\Delta(\omega/c)}{\Delta\omega} .$$

If phase is a random variable, then wave number is also a random variable. Hence, by differentiation we find

$$\sigma_U(T)=\frac{U^2}{c^2}\,\frac{\omega}{\Delta\omega}\,2^{1/2}\,\sigma_c(T) \qquad (2)$$

where we have assumed $\sigma_c(T)$ varies slowly with period. The factor $2^{1/2}$ arises from the assumption that the phases at the two sampling periods are independent. The ratio $\sigma_U(T)/\sigma_c(T)$ is evidently of the order of $\omega/\Delta\omega$ since the ratio of group and phase velocities is about unity, for fundamental mode surface waves in the period range we shall consider below. If the values of group velocity are derived from numerical differentiation of the phase velocity curve, then $\Delta\omega$ is evidently the frequency interval between the samples in the differentiation. If group velocities are obtained by reading the times of peak amplitudes from direct inspection of a seismogram, then the differentiation has been performed by the band-pass properties of the signal-group transmitted by the earth and the ability to determine the time of occurence of the peak in the amplitudes depends on the bandwidth, i.e. the ability to resolve independent phases. Numerical bandpass filtering applied to the seismogram merely reduces the effective bandwidth of the signal for analysis and causes the determination of time of the peak amplitude to have greater variance in the presence of noise with random phase properties. From formula (2), if we wish to minimize the effects of noise, we wish to make the frequency sampling interval $\Delta\omega$ as large as possible, but this has the undesirable effect of reducing resolution, i.e. suppressing the details of the group velocity curve itself. In the example given below the ratio $\omega/\Delta\omega$ is about 7 as a rough average over the sample frequencies, which is probably much larger than can be obtained in practice.

Continental Structure

The calculations have been performed for both continental and oceanic models. Our starting reference model for the continental case is one appropriate for a young stable continental region (Knopoff, 1972); this structure is one of those proposed for the South Central United States (Biswas and Knopoff, 1974), and is appropriate for other regions as well. The cross-section is listed in Table 1; it has a 50 km thick crust, a high-velocity lid to a low-velocity channel of moderate contrast to the lid, and a deeper structure which is more-or-less "standard", including rather large step discontinuities in physical properties at 450 and 650 km depth. The starting/standard/reference structure is also shown in Figure 1. We have restricted the problem for the purpose of the analysis to a horizontally layered structure; corrections for sphericity are modifications that are tangential to the main purpose of this study, which is not concerned with a more realistic problem.

The phase and group velocities derived for both Rayleigh and Love Waves in the fundamental mode are shown in Figure 2. All four of these curves have been digitized at the 17 periods

20(5) 40(10) 100(25) 250 s.

The values in parantheses represent the period interval between adjacent "digitized" values. The inversion has been restricted to six parameters in the crust and upper mantle normally deemed capable (or almost so) of being determined from dispersion data over this period range. These six parameters are the lid, channel and sub-channel S-wave velocities β_{LID}, β_{CH}, β_{SUB} and the crustal, lid and channel thicknesses h_{CR}, h_{LID}, h_{CH}. These six adjustable model parameters are indicated in Figure 1 as parameters (P-numbers) $P1$–$P3$ for the S-wave velocities in the order listed above and $P4$–$P6$ for the layer thicknesses. For the case of $P4 = h_{CR}$, it was assumed that the three infrastructural layers of the crust are always found in the ratio of the thicknesses as they appear in the original model 1:2:2.

A description of model variances corresponding to the data variances in six-dimensional parameter space requires the specification of a large number of numbers which is difficult for the casual reader to assess. We have chosen to simplify this task by listing merely the six quantities (Berry and Knopoff, 1967)

$$\left\{\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial c(T_i)}{\partial P_j}\right)^2\right\}^{-1/2}\sigma_c, \qquad N=17 \tag{3}$$

where the T_i are the periods of the digitized dispersion curves. Evidently, if five of the six parameters are held fixed at the starting value and the sixth parameter is allowed to vary by an amount δP_j from its starting value, then the rms difference between the exact result and the model result is

$$\left\{\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial c(T_i)}{\partial P_j}\right)^2\right\}^{1/2}\delta P_j, \qquad N=17$$

Table 1. Crust-upper mantle structure for continent

Depth (km)	Thickness (km)	β(km/s)	α(km/s)	ρ(g/cm^3)
0	10	3.49	6.05	2.75
10	20	3.67	6.35	2.85
30	20	3.85	7.05	3.08
50	65	4.65	8.17	3.45
115	250	4.30	8.35	3.54
365	85	4.75	8.80	3.65
450	200	5.30	9.80	3.98
650	400	6.20	11.15	4.43
1050	240	6.48	11.78	4.63
1290	∞	6.62	12.02	4.71

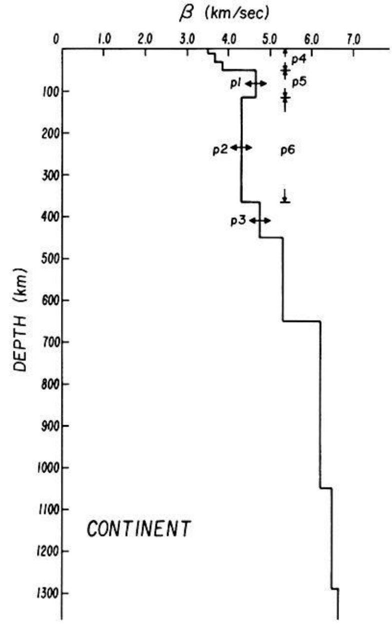

Fig. 1. S-wave velocity cross-section for crust and upper mantle for reference continental structure

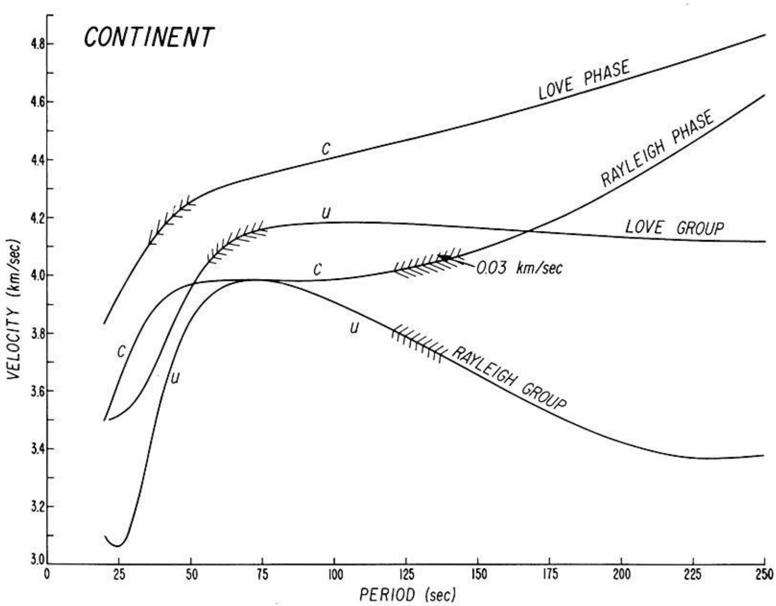

Fig. 2. Dispersion curves for Rayleigh and Love wave phase and group velocities for continental structure

which we set equal to the preassigned value σ_c. The symbol U can be substituted for c as appropriate without change in the reasoning. We have assumed that the assigned "errors" $\sigma_c(T_i)$ are uniform over the entire period range; if they are not uniform, the modification of (3) is not difficult.

Thus according to some criterion for linearized inversion, the quantities in the expression (3) are the intersections of the model variance ellipsoid with the model coordinate axes. These values are therefore the standard deviations in these parameters for the cases in which all the other five parameters are kept fixed at their starting values. What is lost in this description is the information about the inclination of the model ellipsoid to the coordinate axes, i.e. information about the eigenvectors. The tabulation of the items (3) does give some rough information regarding resolution of the parameters P_j by the data set. Whether this is an optimum estimate of the resolution is in part dependent on the criterion used in the inversion. In this paper, we call the quantities (3) the resolution, despite the fact that this definition is inconsistent with other usages in the literature.

In order to preserve the linearity of the inverse process, we have performed the computation (3) with $\sigma_c = \sigma_U = 0.03$ km/s. Although we have indicated at the outset that σ_U is expected to be considerably larger than σ_c, we have taken these two quantities to be equal for the purposes of preserving the linearity of the calculation. A detailed calculation of the quantity σ_U/σ_c given by (2), for the 17 periods, gives an rms value of 7.7 for Rayleigh waves and 8.0 for Love waves, if we assume that the quantity $\Delta\omega$ is the difference between frequencies of nearest neighbors of digitized samples. To use a more realistic value of $\sigma_U = (\sigma_U/\sigma_c) \cdot 0.03$ would take us out of the range of linearity of the inversion. Thus we have preferred, for purposes of the mechanics of the inversion to keep the values of σ_c and σ_U equal and to assess the case for the larger σ_U by a simple multiplication on the results for $\sigma_U = 0.03$ km/s.

The results of the four sets of inversions are given in Table 2. The values listed in the table reveal some valuable insights into the relative resolution of different

Table 2. Results of inversion for continental structure

	Rayleigh waves		Love waves	
	Phase velocities	Group velocities	Phase velocities	Group velocities
rms error in data (km/s)	0.03	0.03[a]	0.03	0.03[a]
$\delta\beta_{LID}$(km/s)	0.144	0.086	0.116	0.101
$\delta\beta_{CH}$(km/s)	0.070	0.062	0.067	0.077
$\delta\beta_{SUB}$(km/s)	0.373	0.435	0.541	1.34
δh_{CR}(km)	4.55	2.46	4.90	3.47
δh_{LID}(km)	30.9	16.5	39.9	27.8
δh_{CH}(km)	49.2	52.6	75.2	142.2

[a] For comparison with results of inversion of phase velocity data, multiply values in this column by the ratio σ_U/σ_c

kinds of dispersion data with the same period range. First, we find that, *at the same level of error in the data*, the inversion of Rayleigh Wave group velocities leads to a better resolution of the lid thickness and S-wave velocity and the crustal thickness, by about a factor of 2, than does the inversion of Rayleigh wave phase velocities. However, since we believe that the intrinsic level of error in group velocity measurements is much more than twice that of phase velocities, for no model parameter does a realistic inversion of group velocity give better resolution than the inversion of phase velocity data; in the worst cases, Rayleigh wave group velocities give no geophysically usable information about channel thickness or sub-channel-S-wave velocities. Similar comments can be made regarding the inversion of Love wave phase and group velocities at the same level of error in the data (crustal and lid thicknesses can be better resolved in the latter case), and similar comments can also be made regarding the relative merits of inversion at more realistic levels of group velocity data errors.

We conclude that, unless methods of obtaining accurate group velocity measurements can be derived, there are no advantages to gathering group velocity data, *if* phase velocity data for the same path are already available. Thus we support the conclusion of Pilant and Knopoff (1970) regarding the relative merits of the inversion of group and phase velocity data; in this case our conclusion rests on our assumption that the errors in the gathering of group velocity data will be intrinsically greater than for phase velocity data because of the error arising from the differentiation of phase velocities having random phase errors. In this paper we have not assessed the effect of reducing the number of degrees of freedom in the group velocity curve by enlarging the frequency interval $\Delta\omega$. Questions of differentiation of phase velocity data with correlated phase errors to yield group velocity information is outside the scope of this paper.

A further result indicated in Table 2 is that Love wave phase velocity data and Rayleigh wave phase velocity data give about the same resolution for crustal thickness, channel velocity and lid thickness. Love wave phase velocity data give a poorer resolution of sub-channel S-wave velocity than do Rayleigh wave phase velocities but both values of $\delta\beta_{\mathrm{SUB}}$ are so large (0.373 km/s and 0.541 km/s) that we conclude that the data set extending over the period range 25–250 s gives very little information regarding sub-channel velocities. Love wave phase velocity data give slightly better resolution regarding β_{LID} and Rayleigh wave phase velocity data give better resolution of channel thickness.

Thus a casual inspection of Table 2 indicates that a Rayleigh wave phase velocity data set extending from 25–250 s will give model information to within roughly

$\pm$ 0.15 km/s	for lid S-wave velocity
$\pm$ 0.07 km/s	channel S-wave velocity
$\pm$ 5 km	crustal thickness
$\pm$ 30 km	lid thickness
$\pm$ 50 km	channel thickness

for an rms error in the data of 0.03 km/s. The inability to resolve lid velocities to better than $\pm$0.15 km/s indicates that we need not strive for too great an accuracy in the specification of the initial model. The most satisfying result is the apparent high resolution of the data for the channel S-wave velocity; this means that it is indeed possible to use these Rayleigh wave data to determine the presence of lateral inhomogeneities in the properties of the channel.

We have not attempted to determine the effect of using both Love and Rayleigh wave data sets in combination, nor have we attempted to determine the effect of changing the span of the periods of the samples. With regard to the latter item, we are well aware that measurement of fundamental mode surface wave dispersion to periods as great as 250 s is quite rare from WWSSN records; it is not appropriate in this preliminary discussion to explore the resolution from data with more practical spans of sample periods; reduced period spans can only reduce the quality of the resolution we have reported here, i.e. increase the sizes of the numbers presented in Table 2.

Oceanic Structure

A similar calculation can be performed for an oceanic structure and the interpretation is similar in most respects. The standard/reference/starting structure we use in the oceanic case is appropriate for the Pacific Basin at 50 my spreading age (Leeds et al., 1974). The structure is listed in Table 3 and illustrated in Figure 3. It has a Moho at 10 km depth below sea level, a high velocity lid to a channel, a channel velocity which has an S-wave velocity that is much lower than in the continental case, and again a "standard" deeper structure with step discontinuities at 450 and 650 km as before. The structure is assumed to be flat, as before, for the purpose of illustration.

The fundamental mode phase and group velocities for Rayleigh and Love waves are shown in Figure 4. The dispersion curves are once again digitized at the same seventeen periods as before. The inversion in this case is restricted to the five upper mantle parameters indicated in Figure 3, namely

$$P1=\beta_{LID}, \quad P2=\beta_{CH}, \quad P3=\beta_{SUB}, \quad P4=h_{LID}, \quad P5=h_{CH}.$$

The rms ratio σ_U/σ_c for the 17 periods in the oceanic case is 8.0 for Rayleigh

Table 3. Crust-upper mantle structure for ocean

Depth (km)	Thickness (km)	β(km/s)	α(km/s)	ρ(g/cm^3)
0				
	4	0.00	1.52	1.03
4				
	1	1.00	2.10	2.10
5				
	5	3.70	6.41	3.07
10				
	50	4.65	8.10	3.40
60				
	150	4.15	7.60	3.40
210				
	240	4.75	8.80	3.65
450				
	200	5.30	9.80	3.98
650				
	400	6.20	11.15	4.43
1050				
	240	6.48	11.78	4.63
1290				
	∞	6.62	12.02	4.71

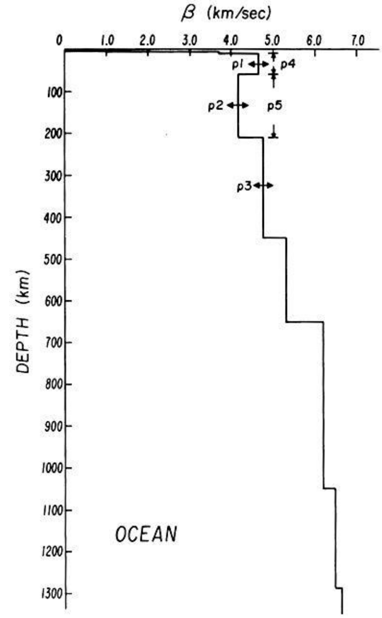

Fig. 3. S-wave velocity cross-section for crust and upper mantle for reference oceanic structure

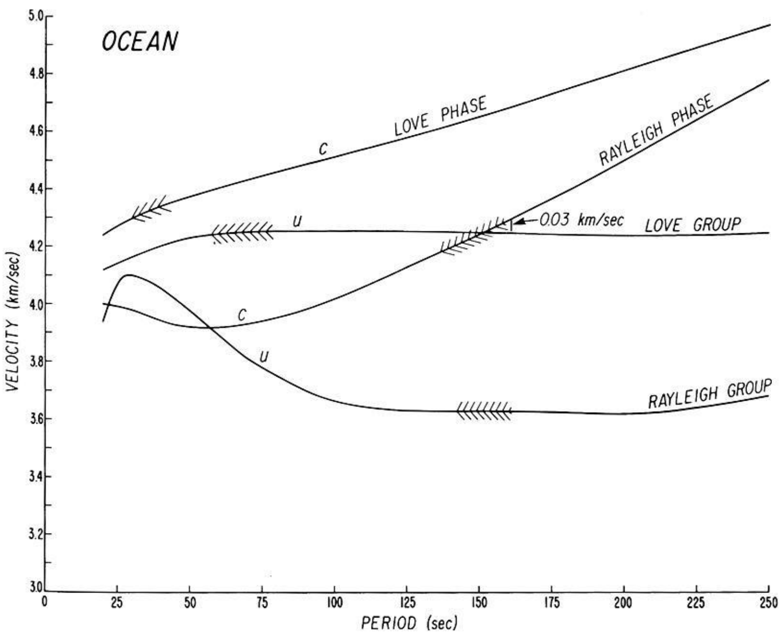

Fig. 4. Dispersion curves for Rayleigh and Love wave phase and group velocities for oceanic structure

Table 4. Results of inversion for oceanic structure

	Rayleigh waves		Love waves	
	Phase velocities	Group velocities	Phase velocities	Group velocities
rms error in data (km/s)	0.03	0.03[a]	0.03	0.03[a]
$\delta\beta_{LID}$(km/s)	0.127	0.063	0.144	0.130
$\delta\beta_{CH}$(km/s)	0.068	0.055	0.046	0.035
$\delta\beta_{SUB}$(km/s)	0.113	0.092	0.116	0.194
δh_{LID}(km)	11.9	6.4	37.0	24.6
δh_{CH}(km)	25.0	17.2	21.3	23.1

[a] For comparison with results of inversion of phase velocity data, multiply values in this column by the ratio σ_U/σ_c

waves and 8.25 for Love waves according to (2), and the assumption that the nearest neighbor periods are those involved in the calculation of $\Delta\omega$.

The results of the four inversions are given in Table 4. In the oceanic case, the group velocity data uniformly provide greater resolution for upper mantle structure than the phase velocity data *at the same level of error* in the data. At an elevated ratio of σ_U/σ_c, which we think is more appropriate (we suggest the value 8), the phase velocities provide the better discrimination. In the case of Love waves, the phase and group velocity data are about equal in resolving power at the same level of error in the data; the phase velocity data provide greater resolution for subchannel velocities. The resolving power of the two data sets are roughly comparable, except for the greater resolving power of the Rayleigh wave data for the lid thickness. In this case the oceanic data provide reasonable estimates of the sub-channel velocity (in comparison with the continental case) and indeed all five parameters can be resolved to reasonable geophysical levels. The Rayleigh wave data set provides model information to within

$\pm$ 0.13 km/s	for	S-wave lid velocity
$\pm$ 0.07 km/s		S-wave channel velocity
$\pm$ 0.11 km/s		S-wave sub-channel velocity
$\pm$12 km		lid thickness
$\pm$25 km		channel thickness

It is our belief that these estimates are applicable as rough resolution criteria to the models that have been constructed for the continents (Knopoff, 1972; Biswas and Knopoff, 1974) and for the oceans (Leeds et al., 1974; Schlue and Knopoff, 1976). These are rough estimates because of the limitations on the calculation reported here:

1. Phase velocity data assumed to be gathered over the period range 25–250 s.
2. The assumptions of uncorrelated phases in the errors.
3. The assumption that resolution can be determined through the variation of one parameter at a time, with the others kept fixed.

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