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Application of Two-Timing Methods in Statistical Geophysics

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Abstract. Two- timing techniques have found wide application in geophysical problems involving resonant interactions between fields or the propagation of particles in random media. It is shown that a rather general class of these applications can be reduced to a simple prototype problem, the diffusion of particles in a statistically homogeneous, stationary turbulence field first considered by Taylor (1921). The similarity of Taylor's diffusion analysis and the more recent resonant-coupling treatment of interacting fields or particles becomes apparent if Taylor's covariance integral expression for the diffusion coefficient is rewritten in terms of the Lagrangian velocity spectrum. Particle diffusion can then be interpreted as a resonant excitation of the particles at their (zero) eigenfrequency. The concepts are illustrated for a number of applications, including climate variability, which is interpreted as a low-frequency random walk problem.

Key words: Two-timing methods – Interacting fields – Climate variability.

1. Basic Equations

In a fundamental paper in 1921 Taylor treated the problem of the diffusion of fluid particles in a statistically homogeneous, stationary turbulent fluid. Taylor showed that for times large compared with the integral correlation scale of the velocity field, the dispersion of the particles is governed by the same laws as molecular diffusion, the effective diffusion coefficient being given by the integral over the covariance function of the random particle velocities. This simple result has found wide application in geophysical turbulence. However, it appears to have been less widely recognised that Taylor's investigation provides the common basis for the analysis of a number of other geophysical problems involving statistical fields, including nonlinear wave-wave scattering, wave-current interactions, particle-field coupling, and, more recently, climate variability. In this note the common mathematical structure of these problems is demonstrated, without going into the algebraic details of particular applications.

Consider generally a system characterised by a state vector $\hat{\mathbf{y}} = (\hat{y}_1, \hat{y}_2, \ldots)$ and governed by a set of first-order equations

$$\frac{d\hat{y}_i}{dt} = \hat{\psi}_i(\hat{\mathbf{y}}, t) \tag{1}$$

The restriction to first-order implies no loss of generality. The dimension of $\hat{\mathbf{y}}$ may be infinite, the system state being represented by a vector rather than an element of an abstract space in order to use more familiar matrix rather than operator notation.

The application of two-timing expansion techniques in our examples assumes that the system has the following property: if the dominant time scale of the problem (to be defined later) is scaled to be of order unity, Equation (1) can be expanded in the form

$$\frac{d\hat{y}_i}{dt} = M_{ij}\hat{y}_j + \varepsilon\hat{\psi}_i^{(1)}(\hat{\mathbf{y}}, t) + \varepsilon^2\hat{\psi}_i^{(2)}(\hat{\mathbf{y}}, t) + \cdots \tag{2}$$

where the matrix M_{ij} is constant (in many applications, zero) $\hat{\psi}_i^{(n)} = O(1)$ and the expansion parameter $\varepsilon \ll 1$.

The linear term can be removed by transforming to "interaction" variables $\mathbf{y} = e^{-Mt}\hat{\mathbf{y}}$ where M denotes the matrix M_{ij} in usual matrix notation, and the matrix e^{-Mt} is defined by the infinite series $e^{-Mt} = 1 - Mt + \frac{M^2 \cdot t^2}{2} - \frac{M^3 \cdot t^3}{3.2} + \cdots$. Equation (2) then becomes

$$\frac{dy_i}{dt} = \varepsilon\psi_i^{(1)}(\mathbf{y}, t) + \varepsilon^2\psi_i^{(2)}(\mathbf{y}, t) + \cdots \tag{3}$$

where

$$\psi_i^{(n)}(\mathbf{y}, t) = e^{-Mt}\hat{\psi}_i^{(n)}(e^{Mt}\mathbf{y}, t). \tag{4}$$

The general solution to Equation (3) under the initial condition

$$y_i(t=0) = y_i^{(0)} \tag{5}$$

can be constructed by a perturbation expansion

$$\mathbf{y} = \mathbf{y}^{(0)} + \varepsilon\mathbf{y}^{(1)} + \varepsilon^2\mathbf{y}^{(2)} + \cdots \tag{6}$$

where

$$|\mathbf{y}^{(0)}| \gg \varepsilon|\mathbf{y}^{(1)}| \gg \varepsilon^2|\mathbf{y}^{(2)}| \gg \cdots \tag{7}$$

The zero'th order term $\mathbf{y}^{(0)} = \text{const}$ is determined by the initial condition (5). The equation for the first-order solution is given by

$$\frac{dy_i^{(1)}}{dt} = \psi_i^{(1)}(\mathbf{y}^{(0)}, t) \tag{8}$$

with the initial condition $y_i^{(1)} = 0$ for $t = 0$.

In general, the solutions $\mathbf{y}^{(n)}$ contain secular contributions which grow indefinitely with time. Thus the condition (7) implies an integration time limit for the validity of the perturbation expansion (6). This can be removed formally by using

a multi-time representation of the solutions in terms of a series of time variables $t^{(n)}=\varepsilon^n t$ (cf. Benney and Saffman, 1966). However, we shall be interested here only in the lowest order secular terms, which can be interpreted rather simply following Taylor's original approach. In this case the method of deriving secular equations from the perturbation solutions can be immediately seen without recourse to the more general multi-time formalism.

2. Statistical Fields

Assume now that the functions $\psi_i^{(n)}$, initial values $y_i^{(0)}$ and solutions $y_i(t)$ represent elements of a statistical ensemble. In particular, let $\psi_i^{(1)}(\mathbf{y}_0, t)$ be a statistically stationary function of t with zero expectation value, $\langle\psi_i^{(1)}\rangle=0$, (this can always be achieved by suitable redefinition of variables) and covariance function

$$R_{ij}(\tau)=\langle\psi_i^{(1)}(t+\tau)\,\psi_j^{(1)}(t)\rangle. \tag{9}$$

It was shown by Taylor (1921) by straightforward integration of (8) that in this case the function $y_i^{(1)}$ is nonstationary, its variance increasing linearly with time t for large t,

$$\langle y_i^{(1)}\,y_j^{(1)}\rangle=2D_{ij}t \quad \text{for } 1\ll t\ll\varepsilon^{-2} \tag{10}$$

where

$$D_{ij}=\tfrac{1}{2}\int_{-\infty}^{\infty} R_{ij}(\tau)\,d\tau. \tag{11}$$

The time scale is normalised here such that the integral correlation time is of order unity, $\max_{i,j}\left|\int_{-\infty}^{\infty} R_{ij}\,d\tau\right|(R_{ii}(0)\,R_{jj}(0))^{-1/2}=O(1)$. The upper limit on t in (10) follows from the inequality (7). The equation is identical to the expression for the increase in the variance of an ensemble of particles with prescribed initial positions $y_i=0$ through molecular diffusion, characterised by a diffusion tensor D_{ij}.

For the following it will be useful to interpret equation (10) also in the frequency domain. In terms of the variance spectrum

$$F_{ij}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ij}(\tau)\,e^{-i\omega\tau}\,d\tau \tag{12}$$

of the forcing function $\psi_i^{(1)}$, the response $y_i^{(1)}$ can be represented in the time interval $1\ll t\ll\varepsilon^{-2}$ by a variance spectrum

$$G_{ij}(\omega)=\begin{cases}\dfrac{F_{ij}(\omega)}{\omega^2} & \text{for } \omega\gg\omega_\varepsilon \qquad (13)\\ 2\pi F_{ij}(0)\,\delta(\omega)\,t & \text{for } \omega\ll\omega_\varepsilon \qquad (14)\end{cases}$$

where the frequency ω_ε satisfies the two-scale inequality

$$\varepsilon^2\ll\omega_\varepsilon\ll 1. \tag{15}$$

Equation (13) follows immediately from the Fourier transform of Equation (8).

The non-stationary δ-function contribution (14) at zero frequency is determined by computing the variance of the response within a small but finite frequency band around zero frequency in the appropriate two-scale limit $1 \ll t \ll \varepsilon^{-2}$. For ω_ε satisfying (15) the spectrum F_{ij} in (14) is constant (white) in the frequency band $\omega \ll \omega_\varepsilon$, so that the response can be expressed in terms of $F_{ij}(0)$. The equation should be read as an integral relation defining the total covariance in the given narrow frequency band around zero frequency.

Equation (14) can be seen to be identical with Equation (10) by writing the diffusion coefficient, using (12), as

$$D_{ij} = \pi F_{ij}(0). \tag{16}$$

Equations (13) and (14) show that the non-stationary response $y_i^{(1)}$ to the stationary forcing $\psi_i^{(1)}$ is concentrated asymptotically entirely in a narrow "resonance" band at zero frequency, the rest of the spectrum representing a stationary process with finite variance. The non-stationary growth of the covariance is accordingly proportional to the spectral density of the forcing at zero frequency (Eqs. (10), (16)).

It should be noted that the location of the dominant forcing and response at zero frequency applies to the "interaction" variables. If the matrix M_{ij} in Equation (2) is non-zero, the transformation back to the original coordinates $\hat{y}_i$ will generally introduce a frequency shift. In many applications the variables $\hat{y}_i$ represent normal-mode variables, i.e. the time-dependent coefficients of an eigenfunction expansion. The matrix then has the diagonal form $M_{jk} = \delta_{jk} i\omega_j$ (no summation convention), where ω_j is the eigenfrequency of the j'th mode. The "interaction" variables $y_j = \hat{y}_j \exp(-i\omega_j t)$ then correspond to the normal-mode amplitudes. In this case, a resonance in the spectrum of the amplitude variables y_j at zero frequency corresponds to a resonance in the spectrum of the normal-mode variables $\hat{y}_j$ at the mode resonance frequency ω_j.

In the following examples, the limitation $t \ll \varepsilon^{-2}$ will be removed by interpreting the linear increase (10) or (14) as the differential rate of change of a slowly varying distribution on a time scale of order ε^{-2}.

3. Applications

3.1. Diffusion of Particles in a Turbulent Fluid

As a generalisation of the dispersion of an ensemble of particles starting from the same initial position, one can consider the evolution of a particle density function $p(\mathbf{y}, t)$ with arbitrary given initial distribution. The characteristic space scale L of p is assumed to satisfy the inequality $L \gg \sqrt{D}$ where $\sqrt{D}(=0(\sqrt{D_{ij}}))$ represents a characteristic "free path" length (assuming again an integral correlation time of R_{ij} of order unity). For particles satisfying the diffusion relation (10), the distribution p then satisfies the heat conduction equation (cf. Wang and Uhlenbek, 1945)

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial y_i}(v_i p) = \frac{\partial}{\partial y_i}\left(D_{ij}\frac{\partial p}{\partial y_j}\right) \tag{17}$$

where the advection velocity (including now a non-zero term $\langle \psi_i^{(1)} \rangle$ in the general case) is given by

$$v_i = \langle \psi_i^{(1)} \rangle - \frac{\partial}{\partial y_j} D_{ij}. \tag{18}$$

Equation (17) may be interpreted as the differential form of (10) with respect to time and the integral form with respect to space. By rewriting equation (10) in time-differential form, the upper time limit $t \ll \varepsilon^{-2}$ is removed, provided the two-scale inequality $L \gg \sqrt{D}$ remains satisfied. The dependence of D_{ij} on $\mathbf{y}^{(0)}$ (through (8), (9) and (11)) is replaced in (17), (18) simply by a dependence on the variable $\mathbf{y}$. (Note that in (17) $\mathbf{y}$ represents a space coordinate in the Eulerian sense, rather than a time-dependent Lagrangian particle position.)

Equations (17) and (18) remain valid if the turbulence field is not strictly homogeneous and stationary, provided D_{ij} and v_i vary slowly in space and time in accordance with the two-scale inequality relations.

The equations have been extensively applied in problems of turbulent diffusion. However, in geophysical turbulence problems the turbulence field may encompass a broad range of natural scales, in which case the two-scale inequality is not always satisfied. In this case a modified form of (17) is often used in which a heuristic dependence of D_{ij} on the scale L is assumed.

3.2. Diffusion of Charged Particles in Random Electro-Magnetic Fields

This problem has received considerable attention in connection with the propagation of particles in the solar wind and the magnetosphere. The problem is basically similar to the diffusion of particles in a turbulent fluid except that the particles experience random accelerations through their interaction with the electromagnetic fields. Thus the perturbation Equations (8) refer here to both momentum and position variables. The heat conduction Equation (17) (or Fokker-Planck equation, as it should now be termed when applied to a generalised phase space) is six dimensional. However, the effective number of degrees of freedom is considerably reduced if the particles are constrained to propagate along a mean magnet field, as in the solar wind and the magnetosphere, and the superimposed weaker field fluctuations are primarily magnetic. In this case the particle distributions are axisymmetric, particle energy is conserved, and the diffusion is limited to the pitch-angles of the helical particle motion and the guiding-center motions perpendicular to the mean field (Jokipii, 1966; Hasselmann and Wibberenz, 1968; Kennel and Petschek, 1966).

Algebraicly, diffusion in the presence of a mean magnetic field is more complex than the classical turbulent diffusion problem or the diffusion of charged particles in a random electromagnetic field with zero mean component, as the lowest order helical particle motion contains both zero eigenfrequencies (for the velocity component parallel to the field) and non-zero eigenfrequencies (for the circular motion perpendicular to the field). The interaction of these motions with the fluctuating field yields an infinite series of resonant perturbations. However, all

resonant contributions to D_{ij} can be determined by application of the basic relations (10) and (16).

3.3. *Climate Variability*

To a good approximation the global climatic system may be described as an interaction between two subsystems with widely separated characteristic time scales, a rapidly varying subsystem $\mathbf{x}=(x_1, x_2, \ldots)$ representing the atmosphere, and a slowly varying subsystem $\mathbf{y}=(y_1, y_2, \ldots)$ containing the remaining components of the climate system, namely the oceans, cryosphere, land vegetation, etc. The governing equations of the complete system are thus of the form

$$\frac{dx_i}{dt}=\varphi_i(\mathbf{x}, \mathbf{y}) \tag{19}$$

$$\frac{dy_i}{dt}=\varepsilon\,\psi_i(\mathbf{x}, \mathbf{y}) \tag{20}$$

with

$$\varepsilon \ll 1,$$

where the time scale and all variables x_i, y_i and functions φ_i, ψ_i are assumed to be normalised to be of order unity.

If the properties of the atmospheric circulation system are known, so that $\mathbf{x}(t)$ may be regarded as a stochastic function with known statistics, Equation (20) has the basic form (3), and the results of Section 2 can immediately be applied (Hasselmann, 1976). In particular, it follows that for times t in the two-timing range $1 \ll t \ll \varepsilon^{-2}$ the covariance tensor of the climate variability increases linearly with time t, and the covariance spectrum has a red distribution in accordance with (13). Observed climate variance spectra agree quite well with these qualitative predictions (Frankignoul and Hasselmann, 1977; Lemke, 1977). If a statistically stationary climate probability distribution is postulated, the results of Section 2 have to be extended to include stabilising feedback terms – for example by inclusion of a non-constant mean excitation $\langle \psi_i^{(1)} \rangle$ which is linearly dependent on $\mathbf{y}^{(1)}$.

The evolution of the probability distribution of climate states is governed by the Fokker-Planck Equation (17). Because of the diffusion terms in the equation, which tend to broaden sharply peaked distributions representing well defined climatic states, climate evolution has only a finite degree of predictability. The predictive skill depends on the relative magnitude of the advective terms, which represent a deterministic evolution of climate states along characteristics in climate phase space (without broadening of the distribution), and the stochastic diffusion terms. In most climate models, both terms are approximately of the same order of magnitude, and the maximal predictive skill is of the order of 50%.

3.4. Wave-Wave Scattering

One of the earliest and most extensive applications of two-timing techniques was in the problem of the energy exchange between nonlinearly interacting random wave fields (e.g. Peierls, 1929; Kadomtsev, 1965; Litvak, 1960; Hasselmann, 1966, 1968). The components y_i represent in this case mode amplitudes, and the functions $\psi_i^{(n)}$ describe the nonlinear coupling between the modes. Normally the perturbation parameter ε represents a scaling parameter proportional to the magnitude of the wave amplitudes, and the functions $\psi_i^{(n)}$ are homogeneous polynomials of degree $(n+1)$ in the wave amplitudes. For non-degenerate modes, the covariance matrix $\langle y_i y_j \rangle$ is diagonal, and the rate of growth of the diagonal term, the wave spectrum, is proportional to the variance spectrum of the nonlinear forcing at zero frequency – which corresponds in the original $\hat{\mathbf{y}}$-representation to resonant forcing of the modes at their resonance frequencies.

The main algebraic difficulty in wave-wave scattering analysis is the evaluation of the variance spectrum of the forcing functions at the resonant frequency. It is a basic result of irreversible statistical thermodynamics (cf. Prigogine, 1962) that in the limit of infinitely weak nonlinear coupling the variance spectrum of the forcing can be determined under the assumption that all mode amplitudes are statistically independent (Gaussian). (In this respect wave-wave interaction theory stands on basically firm ground, as opposed to interaction theories for strongly nonlinear systems, such as turbulence, where no rigorous closure schemes are known.) Applying the Gaussian relations, the general structure of the resonant forcing term (10), (16) can be summarized in a few universal rules in terms of interaction or Feynman diagrams (Hasselmann, 1966, 1968).

The complete analysis of wave-wave scattering actually involves an extension of the first-order perturbation theory presented in Section 2 to second order, since the lowest-order secular contributions of the mode spectra arise both in the term $\langle y_i^{(1)} y_i^{(1)} \rangle$ (no summation convention) given in (10) and the term $\langle y_i^{(0)} y_i^{(2)} \rangle$ involving the second-order response $y_i^{(2)}$. However, if the system is conservative, the second term can be deduced from the first by consideration of the energy and momentum balance.

4. Conclusions

Recognition of the common mathematical structure of various interaction and diffusion problems involving random fields in apparently unrelated areas of geophysics can be very useful. The principal bridge connecting Taylor's classical turbulent diffusion theory to resonant-interaction theories involving wave-wave, wave-field or particle-field interactions is the reformulation of Taylor's expression for the diffusion tensor in terms of the variance spectrum. The diffusion process may then be interpreted as the resonant excitation of the system (the particles) at its eigenfrequency (in this case zero). By transforming to "interaction" variables in which the zero'th order linear response of the system is removed, all weak interaction problems amenable to a statistical two-timing analysis can be cast in the form of the turbulent diffusion problem. Examples include the scattering of

charged particles in random electromagnetic fields, wave-wave scattering problems and the analysis of climate variability.

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