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Scattering of Rayleigh Waves by a Ridge

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Abstract. The scattering of Rayleigh waves by a localised irregularity on the surface of a half-space is studied by the method of matched asymptotic expansions. The method applies only in the limit of long waves but no restriction is placed upon the slope of the irregularity. Solutions can be found for any cross-section that can be mapped conformally by a rational function onto a half-plane, and explicit results are given for a particular one-parameter family of irregularities. Expressions both for the scattered far field and for the ground motion at the irregularity are obtained. The latter confirms that the disturbance produced by an incident Rayleigh wave is amplified at the top of a mountain and reduced at the bottom of a valley.

Key words: Scattering of Rayleigh waves – Topographic irregularities.

1. Introduction

The effect of topography on the propagation of seismic waves is of interest both theoretically and practically, through the need to interpret data from sites near irregularities. Recent experimental studies have been made by Davis and West (1973) and a related model study has been carried out by Rogers et al. (1974). In a previous paper, which will be referred to as *I* in the sequel, Sabina and Willis (1975) studied the effect upon the propagation of SH waves of an irregularity of finite extent on the surface of a half-space. Even this restricted problem is a complex one, for which complete solutions could be obtained only numerically (Boore, 1972, 1973; Bouchon, 1973). Asymptotic solutions were given in *I* which were valid for long waves, by employing the method of matched expansions. The information that is obtained in this way is naturally limited in its range of validity but it does have the virtue of being contained in relatively simple formulae whose physical origins are easy to understand. In contrast to the regular perturbation method of Gilbert and Knopoff (1960), the method of

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matched expansions places no limitation upon the slope of the irregularity and it is possible to obtain expressions for the ground motion at the irregularity, as well as for the far scattered field.

The contribution of the present work is to extend the approach developed in I to the study of the interactions of P and SV motions with a finite irregularity, the scattering of Rayleigh waves being dealt with explicitly. The results have relevance both in seismology and for certain signal-processing devices whose operation is based upon the propagation of Rayleigh waves. In this latter context, Steg and Klemens (1970) studied the scattering of Rayleigh waves by irregularities, considering particularly waves at microwave frequencies propagating over polished surfaces. They modelled their irregularities only approximately, however, as point masses. More recently, with a similar motivation, Tuan and Li (1974) have studied Rayleigh wave reflection from a groove, using a variant of the regular perturbation method of Gilbert and Knopoff (1960). As an example of the use of the present method, simple expressions are derived, for a one-parameter family of irregularities, for the ground motion both near to and far from the irregularity. The expression for the near field confirms the observation of Davis and West (1973), that the disturbance is amplified at the top of a mountain and reduced at the bottom of a valley. The motion of any particle of course is elliptical and the asymptotic expansion of the far field shows that the scattered S-wave always imparts ground motions whose orbits have principal axes parallel to and normal to the free surface. The scattered P-wave, on the other hand, induces ground motions whose orbits are linear and inclined obliquely to the free surface. The angle of inclination is independent of the detail of the topography, however, while amplitudes vary roughly in proportion to the cross-sectional area of the irregularity.

2. Formulation of the Problem

Let the displacement field $\mathbf{u}^I(x, y, t)$ correspond to a wave motion in an isotropic half-space $y>0$, with traction-free surface $y=0$. Thus, $\mathbf{u}^I$ satisfies the equation of motion

$$(\lambda+\mu)\nabla \operatorname{div}\mathbf{u}+\mu\nabla^2\mathbf{u}-\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=0, \tag{2.1}$$

where λ, μ are the Lamé moduli of the half-space and ρ is its density, while on $y=0$, the stress components τ^I_{xy}, τ^I_{yy}, τ^I_{zy} corresponding to $\mathbf{u}^I$ vanish. Suppose, now, that the surface of the half-space is irregular, the free surface having the equation

$$y=f(x), \tag{2.2}$$

with $f(x)$ significantly different from zero only on some finite interval $(-l, l)$. The irregularity will perturb the field $\mathbf{u}^I$, to induce the total displacement

$$\mathbf{u}=\mathbf{u}^I+\mathbf{u}^S, \tag{2.3}$$

where the scattered field $\mathbf{u}^S$ also satisfies (2.1) and is such that the total traction on the free surface (2.2) is zero; thus, on $y=f(x)$,

$$\tau_{ix}^S n_x+\tau_{iy}^S n_y=-\tau_{ix}^I n_x-\tau_{iy}^I n_y \tag{2.4}$$

where $\mathbf{n}=(n_x, n_y)$ is the outward normal to the boundary (2.2) and i denotes x, y or z.

Waves that are harmonic in time will be discussed in the sequel, so that it is convenient to replace $\mathbf{u}(x, y, t)$ by $\mathrm{Re}\{\mathbf{u}(x, y)\, e^{i\omega t}\}$, with superscripts as appropriate, and also to measure lengths in units of l, so that x, y and $\mathbf{u}$ become dimensionless and the irregularity is significant over the interval $(-1, 1)$. For such time-harmonic displacements, Equation (2.1) may then be given in the form

$$(\alpha^2/\beta^2-1)\,\nabla\operatorname{div}\mathbf{u}+\nabla^2\mathbf{u}+\varepsilon^2\mathbf{u}=0, \tag{2.5}$$

where α, β denote respectively the speeds of P and S waves and

$$\varepsilon=\omega l/\beta. \tag{2.6}$$

It is convenient also to measure the stress τ_{ij} in units of the shear modulus μ, so that the stress-strain relations become

$$\tau_{xx}=\frac{\alpha^2}{\beta^2}\frac{\partial u}{\partial x}+\left(\frac{\alpha^2}{\beta^2}-2\right)\frac{\partial v}{\partial y},$$

$$\tau_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},$$

$$\tau_{yy}=\left(\frac{\alpha^2}{\beta^2}-2\right)\frac{\partial u}{\partial x}+\frac{\alpha^2}{\beta^2}\frac{\partial v}{\partial y}$$

$$\tau_{xz}=\frac{\partial w}{\partial x} \tag{2.7}$$

and so on, where $\mathbf{u}$ has components (u, v, w).

The scattering of SH waves (for which $u=v=0$) was considered in *I* and so attention in the present work is directed towards P and SV motion, for which $w=0$. In particular, results will be given for the scattering of a Rayleigh wave, for which

$$u^I=iA\left\{-\exp[-(\nu_\alpha y+ix)\varepsilon\beta/\gamma]+\frac{2\nu_\alpha\nu_\beta}{1+\nu_\beta^2}\exp[-(\nu_\beta y+ix)\varepsilon\beta/\gamma]\right\},$$

$$v^I=A\nu_\alpha\left\{-\exp[-(\nu_\alpha y+ix)\varepsilon\beta/\gamma]+\frac{2}{1+\nu_\beta^2}\exp[-(\nu_\beta y+ix)\varepsilon\beta/\gamma]\right\}, \tag{2.8}$$

where

$$\nu_\alpha^2=1-\gamma^2/\alpha^2, \qquad \nu_\beta^2=1-\gamma^2/\beta^2 \tag{2.9}$$

and γ denotes the speed of Rayleigh waves, so that

$$(1+\nu_\beta^2)^2-4\nu_\alpha\nu_\beta=0. \tag{2.10}$$

As in *I*, a general solution could only be obtained numerically and an asymptotic solution will be sought, when the wavelengths in the incident wave are much greater than the characteristic dimension l of the irregularity; thus, $\varepsilon \ll 1$. The stress components τ_{ij}^I can now be evaluated from (2.7), and a formal expansion in powers of ε yields

$$\begin{aligned}
\tau_{xx}^I &= A[\varepsilon P_{xx} + \varepsilon^2(Q_{xx}y + R_{xx}x)] + O(\varepsilon^3),\\
\tau_{yy}^I &= A\varepsilon^2 Q_{yy}y + O(\varepsilon^3),\\
\tau_{xy}^I &= A\varepsilon^2 Q_{xy}y + O(\varepsilon^3),
\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}
P_{xx} &= -2\beta(v_\alpha^2 - v_\beta^2)/\gamma,\\
Q_{xx} &= -(\beta/\gamma)^2[v_\alpha(v_\beta^2 - 2v_\alpha^2 - 1) + v_\beta(1 + v_\beta^2)],\\
Q_{yy} &= -(\beta/\gamma)^2(1 + v_\beta^2)(v_\alpha - v_\beta),\\
Q_{xy} &= -2i(\beta/\gamma)^2 v_\alpha(v_\alpha - v_\beta),\\
R_{xx} &= 2i(\beta/\gamma)^2(v_\alpha^2 - v_\beta^2).
\end{aligned} \tag{2.12}$$

It is plain that the stresses corresponding to any field $\mathbf{u}^I$ would display the form (2.11), since τ_{xy}^I and τ_{yy}^I must vanish when $y=0$; Equations (2.12), however, apply only to the Rayleigh wave.

The boundary conditions (2.4) now give, on $y=f(x)$,

$$\tau_{ix}^S n_x + \tau_{iy}^S n_y = \varepsilon T_i^{(1)} + \varepsilon^2 T_i^{(2)} + O(\varepsilon^3), \tag{2.13}$$

where

$$\begin{aligned}
T_x^{(1)} &= -AP_{xx}n_x, \qquad T_y^{(1)} = 0,\\
T_x^{(2)} &= -A[(Q_{xx}y + R_{xx}x)n_x + Q_{xy}yn_y],\\
T_y^{(2)} &= -A[Q_{xy}yn_x + Q_{yy}yn_y].
\end{aligned} \tag{2.14}$$

The scattered field $\mathbf{u}^S$ must satisfy the equation of motion (2.5), the boundary conditions (2.13) and a radiation condition to ensure that it consists only of outgoing waves; its construction, correct to order ε^2, will be outlined in the following sections.

3. The Field Near the Irregularity

Equations (2.5) and (2.13) define a singular perturbation problem that can be solved by the method of matched expansions (Van Dyke, 1964; Fraenkel, 1969). A rather detailed study was made of the corresponding problem for SH waves in I, so here the solution will be developed as quickly as possible.

First, when x and y are of order 1, Equations (2.5) and (2.13) can be solved by straightforward perturbation theory, by setting

$$\mathbf{u}^S = \varepsilon\mathbf{u}^{(1)} + \varepsilon^2\mathbf{u}^{(2)} + O(\varepsilon^3). \tag{3.1}$$

Then, $\mathbf{u}^{(1)}$ satisfies the equation

$$(\alpha^2/\beta^2 - 1)\nabla \operatorname{div}\mathbf{u}^{(1)} + \nabla^2\mathbf{u}^{(1)} = 0, \qquad y > f(x), \tag{3.2}$$

with the boundary conditions

$$\tau_{ix}^{(1)} n_x + \tau_{iy}^{(1)} n_y = T_i^{(1)}, \qquad y = f(x). \tag{3.3}$$

The elastostatic problem defined by (3.2) and (3.3) can be solved by the method of Muskhelishvili (1953), so long as the domain $y > f(x)$ can be mapped conformally onto the upper half of the complex ζ-plane by

$$z = g(\zeta) = \zeta + r(\zeta), \tag{3.4}$$

where $r(\zeta)$ is a rational function. In (3.4), we have taken $z = x + iy$; z will retain this meaning throughout the remainder of this work. The function $r(\zeta)$ should tend to zero as ζ tends to infinity and, although a more general development could be given, we shall consider explicitly mappings of the form

$$z = g(\zeta) = \zeta + \sum_{m=1}^{n} r_m (\zeta + i)^{-m}. \tag{3.5}$$

The constants r_m must, of course, be restricted so that $g'(\zeta)$ has no zeros in the upper half-plane.

As shown by Muskhelishvili (1953), the displacements and stresses can be represented in terms of complex potentials $\phi(z)$, $\psi(z)$ as follows

$$2(u + iv) = \left(\frac{\alpha^2/\beta^2 + 1}{\alpha^2/\beta^2 - 1}\right) \phi(z) - z(\phi'(z))^* - (\psi(z))^*, \tag{3.6}$$

$$\tau_{xx} + \tau_{yy} = 2[\phi'(z) + (\phi'(z))^*], \tag{3.7}$$

$$\tau_{yy} - \tau_{xx} + 2i\,\tau_{xy} = 2[z^* \phi''(z) + \psi'(z)], \tag{3.8}$$

where * denotes the complex conjugate. It follows that the total resultant force across a curve starting at A and ending at B is given by

$$F_x + iF_y = -i[\phi(z) + z(\phi'(z))^* + (\psi(z))^*]_A^B \tag{3.9}$$

and hence that, on the boundary $y = f(x)$,

$$\phi^{(1)}(z) + z(\phi^{(1)\prime}(z))^* + (\psi^{(1)}(z))^* = H^{(1)}(x), \tag{3.10}$$

where the superscripts $^{(1)}$ denote the potentials associated with $\mathbf{u}^{(1)}$ and

$$H^{(1)}(x) = i \int_{-\infty}^{x} (T_x^{(1)} + iT_y^{(1)})\, ds, \tag{3.11}$$

the integral being taken along $y = f(x)$.

It must be noted, however, that Muskhelishvili's method assumes that the stresses and displacements are real so that, for example, u really is the real part of $u + iv$. The simplest way to accomodate this is to take the amplitude A to be real, and to redefine the constants P_{xx}, etc. listed in Equations (2.12) as the real parts of $(P_{xx} e^{i\omega t})$, etc. This convention will be adopted without further comment throughout this section. Explicit allowance for it will be made in Section 5. Now from the first of Equations (2.14), we have

$$H^{(1)}(x) = -iAP_{xx} \int_{-\infty}^{x} n_x\, ds = iAP_{xx} f(x) \tag{3.12}$$

since $n_x ds = -dy = -f'(x)dx$ ($\mathbf{n}$ being the outward normal). Equation (3.10) may then be expressed in terms of $\zeta = \xi + i\eta$, to give

$$\phi^{(1)}(\zeta) + \frac{\zeta + r(\zeta)}{(1+(r'(\zeta))^*}(\phi^{(1)\prime}(\zeta))^* + (\psi^{(1)}(\zeta))^* = H^{(1)}(\xi), \tag{3.13}$$

where ζ takes the value $\xi + 0i$, $\phi^{(1)}(\zeta)$, $\psi^{(1)}(\zeta)$ are written in place of $\phi^{(1)}(g(\zeta))$, $\psi^{(1)}(g(\zeta))$ and

$$H^{(1)}(\xi) = \frac{AP_{xx}}{2}[r(\xi) - (r(\xi))^*]. \tag{3.14}$$

The problem defined by (3.13) and (3.14) could be solved at sight but, because a general procedure is needed subsequently, we set

$$\begin{aligned} Q^{(1)}(\zeta) &= \phi^{(1)}(\zeta), \quad \operatorname{Im}(\zeta) = \eta > 0 \\ &= -\frac{\zeta + r(\zeta)}{1 + r'^*(\zeta)}\phi^{(1)*\prime}(\zeta) - \psi^{(1)*}(\zeta), \quad \operatorname{Im}(\zeta) = \eta < 0 \end{aligned} \tag{3.15}$$

where $f^*(\zeta) = (f(\zeta^*))^*$. Equation (3.13) now yields the Hilbert problem

$$Q^{(1)}(\xi + 0i) - Q^{(1)}(\xi - 0i) = H^{(1)}(\xi), \tag{3.16}$$

whose general solution is

$$Q^{(1)}(\zeta) = \left\{\frac{1}{2\pi i}\int \frac{H^{(1)}(\xi)\,d\xi}{\xi - \zeta} + \frac{P(\zeta)}{(\zeta + i)^n}\right\}, \tag{3.17}$$

where $P(\zeta)$ is an entire function since, from its definition (3.15), $Q^{(1)}(\zeta)$ can have a pole of order n at $\zeta = -i$. Thus, evaluating the integral, with $H^{(1)}(\xi)$ given by (3.14),

$$\begin{aligned} Q^{(1)}(\zeta) &= \left\{\frac{A}{2}P_{xx}r(\zeta) + \frac{P(\zeta)}{(\zeta+i)^n}\right\}, \quad \eta > 0 \\ &= \left\{\frac{A}{2}P_{xx}r^*(\zeta) + \frac{P(\zeta)}{(\zeta+i)^n}\right\}, \quad \eta < 0. \end{aligned} \tag{3.18}$$

From (3.15), therefore,

$$\phi^{(1)}(\zeta) = \left\{\frac{A}{2}P_{xx}r(\zeta) + \frac{P(\zeta)}{(\zeta+i)^n}\right\} \tag{3.19}$$

and

$$\begin{aligned} \psi^{(1)}(\zeta) = &-\frac{A}{2}P_{xx}r(\zeta) - \frac{P^*(\zeta)}{(\zeta - i)^n} \\ &-\left(\frac{\zeta + r^*(\zeta)}{1 + r'(\zeta)}\right)\left\{\frac{A}{2}P_{xx}r'(\zeta) + \frac{P'(\zeta)}{(\zeta+i)^n} - \frac{nP(\zeta)}{(\zeta+i)^{n+1}}\right\}. \end{aligned} \tag{3.20}$$

As in the problem for SH waves considered in I, it emerges that matching with the outer expansion (considered later) allows $Q^{(1)}(\zeta) = O(\zeta^{-1})$ as $\zeta \to \infty$ and so $P(\zeta)$ can be taken as a polynomial of degree $(n-1)$, whose coefficients are determined uniquely by the requirement that $\psi^{(1)}(\zeta)$ should be analytic in the upper half-plane.

The solution for $\mathbf{u}^{(2)}$ proceeds similarly. We define

$$H^{(2)}(x)=i\int_{-\infty}^{x}(T_x^{(2)}+iT_y^{(2)})\,ds \tag{3.21}$$

and write $H^{(2)}(\xi)$ for $H^{(2)}(g(\xi))$. An equation exactly like (3.13) is then obtained for complex potentials $\phi^{(2)}(\zeta)$, $\psi^{(2)}(\zeta)$ but its solution is complicated by the fact that $H^{(2)}(\xi)$ tends to a constant value as ξ tends to $+\infty$. This corresponds to the incident field exerting a finite resultant force

$$F=F_x+iF_y=\int_{-\infty}^{\infty}(T_x^{(2)}+iT_y^{(2)})\,ds \tag{3.22}$$

on the boundary $y=f(x)$, and is best handled by subtracting from $\phi^{(2)}(\zeta)$, $\psi^{(2)}(\zeta)$, simple potentials that are associated with the resultant force (3.22). Motivated by knowledge of the potentials associated with a concentrated force F applied to a half-plane, we define

$$\phi_0^{(2)}(\zeta)=-\frac{F}{2\pi}[\ln(\zeta+i)-2\pi i], \tag{3.23}$$

$$\psi_0^{(2)}(\zeta)=\frac{F^*}{2\pi}\ln(\zeta+i)+\frac{F}{2\pi}-\frac{i}{\zeta+i}\left(\frac{F^*}{\pi}+\frac{F}{2\pi}\right), \tag{3.24}$$

the singularity having been moved to $\zeta=-i$. Then, $\phi_0^{(2)}(\zeta)$ and $\psi_0^{(2)}(\zeta)$ satisfy an equation of the form (3.13), but with right side

$$H_0^{(2)}(\xi)=-\frac{F}{2\pi}\left\{\ln\left(\frac{\xi+i}{\xi-i}\right)-2\pi i-\frac{2i}{\xi-i}\right\}+\frac{F^*}{2\pi}\left\{\frac{(\xi-i)r^{*\prime}(\xi)-r(\xi)}{(\xi-i)(1+r^{*\prime}(\xi))}\right\}, \tag{3.25}$$

so that $H_0^{(2)}(\xi)=O(\xi^{-2})$ as $\xi\to-\infty$ and $H_0^{(2)}(\xi)=iF+O(\xi^{-2})$ as $\xi\to+\infty$. We now set

$$\phi_1^{(2)}(\zeta)=\phi^{(2)}(\zeta)-\phi_0^{(2)}(\zeta),\qquad \psi_1^{(2)}(\zeta)=\psi_0^{(2)}(\zeta)-\psi_0^{(2)}(\zeta) \tag{3.26}$$

and

$$H_1^{(2)}(\xi)=H^{(2)}(\xi)-H_0^{(2)}(\xi). \tag{3.27}$$

The potentials $\phi_1^{(2)}(\zeta)$, $\psi_1^{(2)}(\zeta)$ are related to $H_1^{(2)}(\xi)$ by an equation like (3.13) and now, because $H_1^{(2)}(\xi)$ tends to zero as ξ tends to infinity, its solution follows the pattern established for $\phi^{(1)}(\zeta)$, $\psi^{(1)}(\zeta)$. It should be remarked, however, that matching will later require $\phi_1^{(2)}(\zeta)$, $\psi_1^{(2)}(\zeta)=O(1)$ as $\zeta\to\infty$, so that the entire function $P(\zeta)$ should this time be taken as a polynomial of degree n, in which the coefficient of ζ^n is for the moment arbitrary, and the coefficients of the lower order terms related to it by requiring $\psi_1^{(2)}(\zeta)$ to be analytic in the upper half-plane. Fortunately, knowledge of the general structure of $\phi_1^{(2)}(\zeta)$, $\psi_1^{(2)}(\zeta)$ is all that will be required in the sequel.

4. The Far Field

The expansion (3.1) is invalid far from the irregularity because, for example, its terms fail to satisfy the radiation condition. An "outer" expansion may be developed by setting

$$x = x'/\varepsilon, \qquad y = y'/\varepsilon \tag{4.1}$$

and allowing x', y' to remain finite as $\varepsilon \to 0$. In terms of x', y', the equation of motion (2.5) becomes

$$(\alpha^2/\beta^2 - 1)\nabla' \operatorname{div}' \mathbf{u}^S + \nabla'^2 \mathbf{u}^S + \mathbf{u}^S = 0, \tag{4.2}$$

the differentiations now being with respect to x', y' and, since the irregularity is now significant only on an interval $(-\varepsilon, \varepsilon)$, the domain may be approximated by the half-space $y' > 0$, and the boundary conditions (2.13) simulated by a point source. A suitable form is

$$\tau^S_{ix} n_x + \tau^S_{iy} n_y = \varepsilon S_i^{(1)} \delta(x') + \varepsilon^2 S_i^{(2)} \delta'(x') + \cdots, \tag{4.3}$$

on $y' = 0$, where $S_i^{(1)}$, $S_i^{(2)}$ may depend upon ε but are $O(1)$ as $\varepsilon \to 0$. It is convenient here to revert to the convention of writing $\mathbf{u}^S$ as the real part of $\mathbf{u}^S(x, y)\, e^{i\omega t}$, so that Equation (4.3) implies source amplitudes $\varepsilon Re\{S_i^{(1)} e^{i\omega t}\}$, $\varepsilon^2 Re\{S_i^{(2)} e^{i\omega t}\}$. The solution of Equations (4.2), (4.3) follows immediately from that of Lamb's problem. If $U_{ij}(x', y')$ represents the i-component of displacement produced at (x', y') by a unit (time-harmonic) point load applied at the origin in the j-direction, we have

$$u^S(x', y') \sim \left[\varepsilon S_x^{(1)} + \varepsilon^2 S_x^{(2)} \frac{\partial}{\partial x'}\right] U_{xx}(x', y') + \left[\varepsilon S_y^{(1)} + \varepsilon^2 S_y^{(2)} \frac{\partial}{\partial x'}\right] U_{xy}(x', y'), \tag{4.4}$$

$$v^S(x', y') \sim \left[\varepsilon S_x^{(1)} + \varepsilon^2 S_x^{(2)} \frac{\partial}{\partial x'}\right] U_{yx}(x', y') + \left[\varepsilon S_y^{(1)} + \varepsilon^2 S_y^{(2)} \frac{\partial}{\partial x'}\right] U_{yy}(x', y') \tag{4.4}$$

and the radiation condition is satisfied automatically. The fields $U_{ij}(x', y')$ are complicated but can be represented in terms of integrals. We have, from Achenbach (1973),

$$U_{xy} = I_{uL} + I_{uT}, \qquad U_{yy} = I_{vL} + I_{vT}, \tag{4.5}$$

where

$$I_{uL} = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{k(2k^2 - 1)}{F(k)}\, e^{-ikx' - (k^2 - \beta^2/\alpha^2)^{\frac{1}{2}} y'}\, dk, \tag{4.6}$$

$$I_{uT} = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{k(k^2 - 1)^{\frac{1}{2}} (k^2 - \beta^2/\alpha^2)^{\frac{1}{2}}}{F(k)}\, e^{-ikx' - (k^2 - 1)^{\frac{1}{2}} y'}\, dk, \tag{4.7}$$

$$I_{vL}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}(2k^2-1)}{F(k)}\,e^{-ikx'-(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}y'}\,dk, \tag{4.8}$$

$$I_{vT}=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{k^2(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}}{F(k)}\,e^{-ikx'-(k^2-1)^{\frac{1}{2}}y'}\,dk, \tag{4.9}$$

$$F(k)=(2k^2-1)^2-4k^2(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}(k^2-1)^{\frac{1}{2}}. \tag{4.10}$$

The path of integration in the complex k-plane is shown in Figure 1. It differs from the path given by Achenbach but is consistent with that chosen by Ewing, Jardetsky and Press (1957) and is needed to ensure that the waves travel away from the source. The equation $F(k)=0$ defines the (dimensionless) wavenumber for Rayleigh waves; thus, $F(k)=0$ when $k=\pm\beta/\gamma$ and the integrands have poles at these points.

The analysis of Achenbach (1973) is also readily adapted to yield

$$U_{xx}=J_{uL}+J_{uT}, \qquad U_{yx}=J_{vL}+J_{vT}, \tag{4.11}$$

where

$$J_{uL}=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{k^2(k-1)^{\frac{1}{2}}}{F(k)}\,e^{-ikx'-(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}y'}\,dk, \tag{4.12}$$

$$J_{uT}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(k^2-1)^{\frac{1}{2}}(2k^2-1)}{F(k)}\,e^{-ikx'-(k^2-1)^{\frac{1}{2}}y'}\,dk, \tag{4.13}$$

$$J_{vL}=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{k(k^2-1)^{\frac{1}{2}}(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}}{F(k)}\,e^{-ikx'-(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}y'}\,dk, \tag{4.14}$$

$$J_{vT}=\frac{-i}{2\pi}\int_{-\infty}^{\infty}\frac{k(2k^2-1)}{F(k)}\,e^{-ikx'-(k^2-1)^{\frac{1}{2}}y'}\,dk. \tag{4.15}$$

The I- and J-integrals cannot be evaluated explicitly for arbitrary x', y' but can be estimated asymptotically for both small and large values of x', y'. In particular, when x', y' are small, the equations governing U_{ij} reduce asymptotically to those of elastostatics, so that it is plain that U_{ij} should reduce to the corresponding static Green's function. The static Green's function is defined

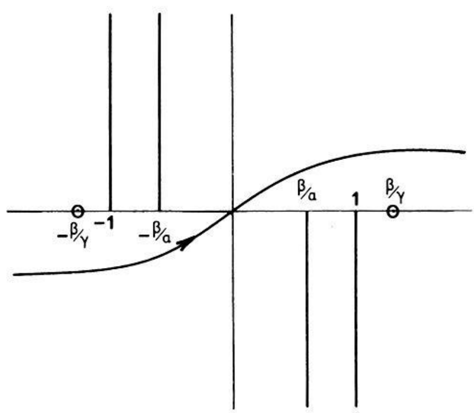

Fig. 1. The complex k-plane and the path of integration employed for the representation of U_{ij}

only up to an arbitrary constant for which the expansion of U_{ij} yields a definite value. To find the constant, it suffices to evaluate U_{ij} asymptotically for $y'=0$, with x' small; this simplifies the exponential factors and allows the two integrals for each of the U_{ij} to be grouped together. For example,

$$U_{xy}(x',0)=\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{k[(2k^2-1)-2(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}(k^2-1)^{\frac{1}{2}}]}{F(k)}e^{-ikx'}\,dk \tag{4.16}$$

in which the integrand has the asymptotic form

$$-\frac{1}{2(\alpha^2/\beta^2-1)}\frac{e^{-ikx}}{k} \tag{4.17}$$

when $|k|$ is large. Now as $x'\to 0$,

$$\int_{k_1}^{\infty}\frac{e^{-ikx'}}{k}\,dk\sim -\ln(k_1|x'|)+\gamma_0-\frac{i\pi}{2}\operatorname{sgn}(x') \tag{4.18}$$

and

$$\int_{-\infty}^{-k_1}\frac{e^{-ikx'}}{k}\,dk\sim \ln(k_1|x'|)-\gamma_0-\frac{i\pi}{2}\operatorname{sgn}(x'), \tag{4.19}$$

where γ_0 is Euler's constant. It is now easy to deduce that, as $x'\to 0$,

$$U_{xy}(x',0)\sim -\frac{1}{4(\alpha^2/\beta^2-1)}\operatorname{sgn}(x')+U_{xy}^0, \tag{4.20}$$

where

$$U_{xy}^0=\frac{i}{2\pi}\int_{-k_1}^{k_1}\frac{k[(2k^2-1)-2(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}(k^2-1)^{\frac{1}{2}}]}{F(k)}\,dk$$
$$+\frac{i}{2\pi}\left\{\left(\int_{-\infty}^{-k_1}+\int_{k_1}^{\infty}\right)\left[\frac{k[(2k^2-1)-2(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}(k^2-1)^{\frac{1}{2}}]}{F(k)}\right.\right.$$
$$\left.\left.+\frac{1}{2(\alpha^2/\beta^2-1)k}\right]dk\right\}. \tag{4.21}$$

The integrands in (4.21) have poles at $k=\pm\beta/\gamma$ and care is required in their evaluation. The paths of integration remain segments of the path shown in Figure 1.

It can be shown similarly that

$$U_{yy}(x',0)\sim\frac{1}{2\pi(1-\beta^2/\alpha^2)}\{-\ln(k_1|x'|)+\gamma_0\}+U_{yy}^0, \tag{4.22}$$

$$U_{xx}(x',0)\sim\frac{1}{2\pi(1-\beta^2/\alpha^2)}\{-\ln(k_1|x'|)+\gamma_0\}+U_{xx}^0, \tag{4.23}$$

the constants U_{yy}^0, U_{xx}^0 being given by expressions of the same type as U_{xy}^0. We also have

$$U_{xy}(x',0)=-U_{yx}(x',0), \tag{4.24}$$

exactly.

Of course, even when x', y' are of order 1, the field point (x, y) is far from the irregularity but still, the asymptotic forms of the displacements for large x', y' are of interest. In particular, at the surface $y'=0$, Achenbach (1973) quotes the results obtained by Lamb (1904)

$$U_{xy} \sim -H e^{-i\beta x'/\gamma} + (2/\pi)^{\frac{1}{2}}(1-\beta^2/\alpha^2)^{\frac{1}{2}} x'^{-\frac{1}{2}} e^{-i(x'+\pi/4)} - (2/\pi)^{\frac{1}{2}} \frac{(1-\beta^2/\alpha^2)^{\frac{1}{2}}}{(1-2\beta^2/\alpha^2)^3} (\beta/\alpha)^{\frac{3}{2}} x'^{-\frac{3}{2}} e^{-i(\beta x'/\alpha - \pi/4)}, \tag{4.25}$$

$$U_{yy} \sim -iK e^{-i\beta x'/\gamma} + 2(2/\pi)^{\frac{1}{2}}(1-\beta^2/\alpha^2) x'^{-\frac{1}{2}} e^{-i(x'-\pi/4)} + \frac{(2\pi)^{-\frac{1}{2}}(\beta/\alpha)^{\frac{1}{2}}}{(1-2\beta^2/\alpha^2)^2} x'^{-\frac{3}{2}} e^{-i(\beta x'/\alpha - \pi/4)}, \tag{4.26}$$

where

$$H = -\frac{(\beta/\gamma)[2\beta^2/\gamma^2 - 1 - 2(\beta^2/\gamma^2 - \beta^2/\alpha^2)^{\frac{1}{2}}(\beta^2/\gamma^2 - 1)^{\frac{1}{2}}]}{F'(\beta/\gamma)} \tag{4.27}$$

and

$$K = -\frac{(\beta^2/\gamma^2 - \beta^2/\alpha^2)^{\frac{1}{2}}}{F'(\beta/\gamma)}. \tag{4.28}$$

Similar analysis yields, for large x',

$$U_{xx} \sim iL e^{-i\beta x'/\gamma} + (2\pi)^{-\frac{1}{2}} x'^{-\frac{3}{2}} e^{-i(x'-\pi/4)} - \frac{2(\beta/\alpha)^{\frac{5}{2}}(1-\beta^2/\alpha^2)}{(1-2\beta^2/\alpha^2)^4} (2/\pi)^{\frac{1}{2}} x'^{-\frac{1}{2}} e^{-i(\beta x'/\alpha - \pi/4)} \tag{4.29}$$

where

$$L = (\beta^2/\gamma^2 - 1)^{\frac{1}{2}}/F'(\beta/\gamma). \tag{4.30}$$

The reciprocal relation (4.24) gives the asymptotic form of U_{yx}, from (4.25).

5. Matching

The near field described in Section 3 contains, so far, an undetermined constant, which we have asserted is of order ε^2 and the far field of Section 4 is expressed in terms of the unknown source amplitudes $S_i^{(1)}$, $S_i^{(2)}$. These constants will be fixed now by employing the asymptotic matching principle of Van Dyke (1964), which implies in the present context that the 2-term outer expansion of the near field should be identical with the two-term inner expansion of the far field. The outer expansion of the near field is obtained by expressing it in terms of the variables x', y' and expanding to order ε^2. Dually, the inner expansion of the far field is obtained by expressing it in terms of x, y and expanding to order ε^2. In performing these expansions, terms like $\varepsilon \ln \varepsilon$ are grouped with terms containing ε alone; the desirability of this has been discussed by Crighton and Leppington (1973).

Proceeding first with the near field, we set $z=z'/\varepsilon$, where $z'=x'+iy'$, so that, from (3.5),

$$\zeta\sim\varepsilon^{-1}(z'-\varepsilon^2 r_1/z'+O(\varepsilon^3)). \tag{5.1}$$

Hence, $\zeta\to\infty$ as $\varepsilon\to 0$ and, from (3.19) and (3.20),

$$\phi^{(1)}(\zeta)\sim\left(\frac{A}{2}P_{xx}r_1+p_{n-1}\right)\Big/\zeta+O(\zeta^{-2}), \tag{5.2}$$

$$\psi^{(1)}(\zeta)\sim-(p_{n-1}+p^*_{n-1})/\zeta+O(\zeta^{-2}), \tag{5.3}$$

where p_{n-1} represents the coefficient of ζ^{n-1} in $P(\zeta)$.

Also, from (3.23) and (3.24),

$$\phi_0^{(2)}(\zeta)\sim-\frac{F}{2\pi}[\ln\zeta-2\pi i]+O(\zeta^{-1}), \tag{5.4}$$

$$\psi_0^{(2)}(\zeta)\sim\frac{F^*}{2\pi}\ln\zeta+\frac{F}{2\pi}+O(\zeta^{-1}), \tag{5.5}$$

while, from the equations analogous to (3.15) and (3.17),

$$\phi_1^{(2)}(\zeta)\sim q_n+O(\zeta^{-1}), \tag{5.6}$$

$$\psi_1^{(2)}(\zeta)\sim-q_n^*+O(\zeta^{-1}) \tag{5.7}$$

where q_n represents the coefficient of ζ^n in the n-th degree polynomial that corresponds to $P(\zeta)$.

Hence, writing $\phi=\varepsilon\phi^{(1)}+\varepsilon^2\phi^{(2)}$ and so on, we have, in terms of z',

$$\phi\sim\varepsilon^2\left[\left(\frac{A}{2}P_{xx}r_1+p_{n-1}\right)\Big/z'-\frac{F}{2\pi}(\ln z'-\ln\varepsilon-2\pi i)+q_n\right]+O(\varepsilon^3), \tag{5.8}$$

$$\psi\sim\varepsilon^2\left[-(p_{n-1}+p^*_{n-1})/z'+\frac{F^*}{2\pi}(\ln z'-\ln\varepsilon)+\frac{F}{2\pi}-q_n^*\right]+O(\varepsilon^3), \tag{5.9}$$

there being no term of order ε. Finally, the outer expansion of the near field approximation to $\mathbf{u}^S$ is obtained by substituting (5.8) and (5.9) into (3.6). Thus, including the superscript N to emphasise that this is an expansion of the near field,

$$\begin{aligned}u^{SN}+iv^{SN}\sim\frac{\varepsilon^2}{2}\Bigg\{&\left(\frac{\alpha^2/\beta^2+1}{\alpha^2/\beta^2-1}\right)\left[\left(\frac{A}{2}P_{xx}r_1+p_{n-1}\right)\Big/z'\right.\\&\left.-\frac{F}{2\pi}(\ln z'-\ln\varepsilon-2\pi i)+q_n\right]\\&+z'\left[\left(\frac{A}{2}P_{xx}r_1+p_{n-1}\right)\Big/z'^2-\frac{F}{2\pi z'}\right]^*\\&+\left[(p_{n-1}+p^*_{n-1})/z'-\frac{F^*}{2\pi}(\ln z'-\ln\varepsilon)+\frac{F}{2\pi}-q_n^*\right]^*\Bigg\}+O(\varepsilon^3)\end{aligned} \tag{5.10}$$

which reduces, as $y' \to 0$, to the form

$$u^{SN} + iv^{SN} \sim \varepsilon^2 \left\{ -\left(\frac{1}{1-\beta^2/\alpha^2}\right) \frac{F}{2\pi} \ln(|x'|/\varepsilon) \right.$$
$$+ \frac{1}{2x'} \left[\left(\frac{A}{2} P_{xx} r_1 + p_{n-1} \right) \left(\frac{\alpha^2/\beta^2 + 1}{\alpha^2/\beta^2 - 1} \right) \right.$$
$$\left. + \frac{A}{2} P_{xx} r_1^* + p_{n-1} + 2p_{n-1}^* \right]$$
$$+ \frac{F}{2\pi} \left(\frac{1}{\alpha^2/\beta^2 - 1} \right) \left(\frac{i\pi}{2} \operatorname{sgn}(x') + q_n \right)$$
$$\left. + \frac{iF}{8} + \frac{3iF}{4} \left(\frac{\alpha^2/\beta^2 + 1}{\alpha^2/\beta^2 - 1} \right) \right\}. \tag{5.11}$$

Correspondingly, the inner expansion of the far field is obtained by setting $x' = \varepsilon x$, $y' = \varepsilon y$ and expanding to order ε^2. As mentioned in Section 4, this expansion is bound to have the same form as (5.10) and therefore, for matching, it suffices to obtain the inner limit of the surface displacements, for comparison with (5.11). We have, from (4.20), with $x' = \varepsilon x$,

$$U_{xy}(x', 0) \sim -\frac{1}{4(\alpha^2/\beta^2 - 1)} \operatorname{sgn}(x) + U_{xy}^0 + O(\varepsilon) \tag{5.12}$$

and, from (4.22),

$$U_{yy}(x', 0) \sim \frac{1}{2\pi(1-\beta^2/\alpha^2)} \{ -\ln(k_1 |x|) - \ln \varepsilon + \gamma_0 \} + U_{yy}^0 + O(\varepsilon), \tag{5.13}$$

with a similar expression for $U_{xx}(x', 0)$. It can be shown, too, that

$$\frac{\partial U_{xy}(x', 0)}{\partial x'} \sim U_{xy}^1 + O(\varepsilon), \tag{5.14}$$

$$\frac{\partial U_{yy}(x', 0)}{\partial x'} \sim -\frac{1}{2\pi(1-\beta^2/\alpha^2)\varepsilon x} + U_{yy}^1 + O(\varepsilon), \tag{5.15}$$

with a similar expression for $\partial U_{xx}/\partial x'$, where

$$U_{xy}^1 = \frac{1}{2\pi} \int_{-k_1}^{k_1} \frac{k^2[(2k^2-1) - 2(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}(k^2-1)^{\frac{1}{2}}]}{F(k)} dk$$
$$+ \frac{1}{2\pi} \left(\int_{-\infty}^{-k_1} + \int_{k_1}^{\infty} \right) \left[\frac{k^2[(2k^2-1) - 2(k^2-\beta^2/\alpha^2)^{\frac{1}{2}}(k^2-1)^{\frac{1}{2}}]}{F(k)} \right.$$
$$\left. + \frac{1}{2(\alpha^2/\beta^2 - 1)} \right] dk, \tag{5.16}$$

with similar expressions for U_{xx}^1, U_{yy}^1.

Therefore, as $\varepsilon \to 0$, using the superscript F to denote the far field,

$$u^{SF}(x',0) \sim \varepsilon S_x^{(1)} \left\{ \frac{1}{2\pi(1-\beta^2/\alpha^2)} [-\ln(k_1|x|) - \ln\varepsilon + \gamma_0] + U_{xx}^0 + \varepsilon U_{xx}^1 x \right\}$$
$$+ \varepsilon^2 S_x^{(2)} \left\{ -\frac{1}{2\pi(1-\beta^2/\alpha^2)\varepsilon x} + U_{xx}^1 \right\}$$
$$+ \varepsilon S_y^{(1)} \left\{ -\frac{1}{4(\alpha^2/\beta^2-1)} \operatorname{sgn}(x) + U_{xy}^0 + \varepsilon U_{xy}^1 x \right\}$$
$$+ \varepsilon^2 S_y^{(2)} U_{xy}^1, \tag{5.17}$$

$$v^{SF}(x',0) \sim \varepsilon S_x^{(1)} \left\{ \frac{1}{4(\alpha^2/\beta^2-1)} \operatorname{sgn}(x) - U_{xy}^0 - \varepsilon U_{xy}^1 x \right\}$$
$$- \varepsilon^2 S_x^{(2)} U_{xy}^1 + \varepsilon S_y^{(1)} \left\{ \frac{1}{2\pi(1-\beta^2/\alpha^2)} [-\ln(k_1|x|) - \ln\varepsilon + \gamma_0] \right.$$
$$\left. + U_{yy}^0 + \varepsilon U_{yy}^1 x \right\} + \varepsilon^2 S_y^{(2)} \left\{ -\frac{1}{2\pi(1-\beta^2/\alpha^2)\varepsilon x} + U_{yy}^1 \right\}. \tag{5.18}$$

With $x' = \varepsilon x$, the asymptotic matching principle asserts that Equation (5.11) should agree with Equations (5.17) and (5.18). It should be noted, however, that (5.11) contains the time t explicitly through the definition of P_{xx} etc. that was adopted in Section 3. Therefore, to be precise, $\operatorname{Re}(u^{SF} e^{i\omega t})$ should agree with u^{SN} and $\operatorname{Re}(v^{SF} e^{i\omega t})$ should agree with v^{SN}. First, by comparing coefficients of $\ln|x|$, we see that $S_x^{(1)}$ and $S_y^{(1)}$ are in fact of order ε:

$$\operatorname{Re}\{S_x^{(1)} e^{i\omega t}\} = \varepsilon F_x,$$
$$\operatorname{Re}\{S_y^{(1)} e^{i\omega t}\} = \varepsilon F_y. \tag{5.19}$$

These relations also balance the coefficients of $\operatorname{sgn}(x)$. The constant terms in u^{SF} and v^{SF} are therefore of order ε^2 while the terms linear in x are of order ε^3 and hence are to be neglected; these observations show that it is indeed possible to match (5.11) with (5.17) and (5.18) and justify the degree of indeterminancy admitted in the construction of $\phi^{(2)}(\zeta)$, $\psi^{(2)}(\zeta)$ in Section 3. Balancing the terms in $1/x$ gives

$$\operatorname{Re}\{S_x^{(2)} e^{i\omega t}\} = -\pi \left[\frac{A}{2} P_{xx}(r_1 + r_1^*) + (p_{n-1} + p_{n-1}^*)(2 - \beta^2/\alpha^2) \right]$$

$$\operatorname{Re}\{S_y^{(2)} e^{i\omega t}\} = i\pi(\beta^2/\alpha^2) \left[\frac{A}{2} P_{xx}(r_1 - r_1^*) + (p_{n-1} - p_{n-1}^*) \right]. \tag{5.20}$$

Finally, balancing the constant terms fixes q_n; the expression for q_n is long and will not be displayed.

Before proceeding to an example, it may be noted that the resultant force components F_x, F_y can be evaluated quite simply. We have

$$F_x = \int_{-\infty}^{\infty} T_x^{(2)}\, ds = -A \int_{-\infty}^{\infty} [(Q_{xx} y + R_{xx} x) n_x + Q_{xy} y n_y]\, ds \tag{5.21}$$

so that, using the relations $n_x ds = -dy$, $n_y ds = -dx$ and evaluating the integrals,

$$F_x = -A(Q_{xy} - R_{xx})S, \tag{5.22}$$

where S denotes the area between the irregularity and the y-axis:

$$S = -\int_{-\infty}^{\infty} y\,dx. \tag{5.23}$$

Similarly,

$$F_y = -AQ_{yy}S. \tag{5.24}$$

Equations (2.12) show that $(Q_{xy} - R_{xx})$ is imaginary and hence, in the sense used in Section 3 and above, contains $\sin \omega t$ as a factor, while Q_{yy} is given as real and hence contains the factor $\cos \omega t$. Thus, from (5.19), $S_x^{(1)}$ is imaginary while $S_y^{(1)}$ is real.

6. Example

In this section, we consider the mapping

$$z = g(\zeta) = \zeta + \frac{b}{\zeta + i} + \frac{ib}{(\zeta + i)^2} \tag{6.1}$$

with b real; this is capable of yielding either a ridge or a valley. The restriction that $g'(\zeta)$ should have no zeros in the upper half-plane implies

$$b > -\tfrac{1}{3}. \tag{6.2}$$

This restricts the depth of the valley that can be modelled by (6.1) but allows a ridge of any height. The boundary $y = f(x)$ is obtained by setting $\zeta = \xi$, real, in (6.1), and equating real and imaginary parts. Plots of this boundary for particular cases of a ridge ($b = 0.4$) and a valley ($b = -0.3$) are shown in Figure 2.

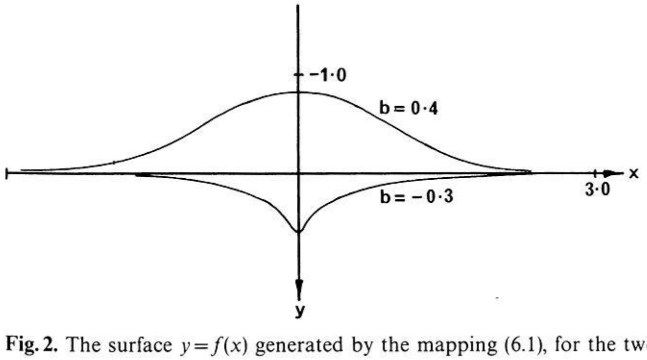

Fig. 2. The surface $y = f(x)$ generated by the mapping (6.1), for the two cases $b = 0.4$, $b = -0.3$

For the mapping (6.1), Equations (3.19) and (3.20) may be given in the form

$$\phi^{(1)}(\zeta)=\frac{A}{2}P_{xx}\left[\frac{b}{\zeta+i}+\frac{ib}{(\zeta+i)^2}\right]+\frac{p_1}{(\zeta+i)}+\frac{ip_0}{(\zeta+i)^2}, \tag{6.3}$$

$$\begin{aligned}\psi^{(1)}(\zeta)= & -\frac{A}{2}P_{xx}\left[\frac{b}{\zeta+i}+\frac{ib}{(\zeta+i)^2}\right]-\frac{p_1^*}{(\zeta-i)}+\frac{ip_0^*}{(\zeta-i)^2}\\ & -\left\{\frac{\zeta+b/(\zeta-i)-ib/(\zeta-i)^2}{1-b/(\zeta+i)^2-2ib/(\zeta+i)^3}\right\}\left\{\frac{A}{2}P_{xx}\left[-\frac{b}{(\zeta+i)^2}-\frac{2ib}{(\zeta+i)^3}\right]\right.\\ & \left.-\frac{p_1}{(\zeta+i)^2}-\frac{2ip_0}{(\zeta+i)^3}\right\},\end{aligned} \tag{6.4}$$

by setting

$$P(\zeta)=p_1(\zeta+i)+ip_0, \tag{6.5}$$

which retains the meaning of p_1 that was used in Section 5. The conditions for $\psi^{(1)}(\zeta)$ to be analytic in the upper half-plane can be satisfied by taking p_1 and p_0 real. They reduce to

$$p_0+\tfrac{1}{4}\left(\frac{b}{1+b/2}\right)[AP_{xx}b+p_1+p_0]=0, \tag{6.6}$$

$$\begin{aligned}&-p_1-\tfrac{1}{16}\left(\frac{b}{1+b/2}\right)[9AP_{xx}b+8p_1+10p_0]\\ &\quad+\tfrac{5}{32}\left(\frac{b}{1+b/2}\right)^2[AP_{xx}b+p_1+p_0]=0.\end{aligned} \tag{6.7}$$

Before substituting into the results of Section 5, it remains to evaluate the constant S, that is, the area under the ridge. The integral is elementary and gives

$$S=b[1+\tfrac{9}{8}b]\pi. \tag{6.8}$$

From (5.19), therefore,

$$S_x^{(1)}=2i\varepsilon A(\beta/\gamma)^2(2v_\alpha+v_\beta)(v_\alpha-v_\beta)S, \tag{6.9}$$

$$S_y^{(1)}=\varepsilon A(\beta/\gamma)^2(1+v_\beta^2)(v_\alpha-v_\beta)S, \tag{6.10}$$

while from (5.20),

$$S_x^{(2)}=-\pi[AP_{xx}b+2p_1(2-\beta^2/\alpha^2)], \tag{6.11}$$

$$S_y^{(2)}=0, \tag{6.12}$$

where p_1 is obtained by solving (6.6) and (6.7), P_{xx} being given by the first of Equations (2.12).

The asymptotic form of the surface displacement very far from the irregularity can now be obtained by substituting the source amplitudes (6.9)–(6.12) into (4.4) and using (4.24), (4.25), (4.26) and (4.29). This show that the perturbation of $\mathbf{u}^I$ produced by $\mathbf{u}^S$ is, asymptotically,

$$u^S(x',0) \sim \varepsilon^2 \{A_1 e^{-i\beta x'/\gamma} + B_1 x'^{-\frac{3}{2}} e^{-ix'} + C_1 x'^{-\frac{3}{2}} e^{-i\beta x'/\alpha}\} \tag{6.13}$$

$$v^S(x',0) \sim \varepsilon^2 \{A_2 e^{-i\beta x'/\gamma} + B_2 x'^{-\frac{3}{2}} e^{-ix'} + C_2 x'^{-\frac{3}{2}} e^{-i\beta x'/\alpha}\}, \tag{6.14}$$

where

$$\varepsilon^2 A_1 = [i\varepsilon S_x^{(1)} + \varepsilon^2 S_x^{(2)}(\beta/\gamma)]\, L - \varepsilon S_y^{(1)} H,$$

$$\varepsilon^2 A_2 = [\varepsilon S_x^{(1)} - i\varepsilon^2 S_x^{(2)}(\beta/\gamma)]\, H - i\varepsilon S_y^{(1)} K, \tag{6.15}$$

$$\varepsilon^2 B_1 = (2\pi)^{-\frac{1}{2}} e^{-i\pi/4} [i\varepsilon S_x^{(1)} + \varepsilon^2 S_x^{(2)} + 2\varepsilon S_y^{(1)}(1-\beta^2/\alpha^2)^{\frac{1}{2}}],$$

$$\varepsilon^2 B_2 = (2/\pi)^{\frac{1}{2}} e^{i\pi/4} (1-\beta^2/\alpha^2)^{\frac{1}{2}} [i\varepsilon S_x^{(1)} + \varepsilon^2 S_x^{(2)} + 2\varepsilon S_y^{(1)}(1-\beta^2/\alpha^2)^{\frac{1}{2}}], \tag{6.16}$$

$$\varepsilon^2 C_1 = -\frac{(2/\pi)^{\frac{1}{2}}(\beta/\alpha)^{\frac{3}{2}}(1-\beta^2/\alpha^2)^{\frac{1}{2}}}{(1-2\beta^2/\alpha^2)^3} e^{i\pi/4}$$

$$\cdot \left[(\varepsilon S_x^{(1)} - i\varepsilon^2 S^{(2)}(\beta/\alpha)) \frac{2(\beta/\alpha)(1-\beta^2/\alpha^2)^{\frac{1}{2}}}{(1-2\beta^2/\alpha^2)} + \varepsilon S_y^{(1)}\right],$$

$$\varepsilon^2 C_2 = \frac{(2\pi)^{-\frac{1}{2}}(\beta/\alpha)^{\frac{1}{2}}}{(1-2\beta^2/\alpha^2)^2} e^{i\pi/4}$$

$$\cdot \left[(\varepsilon S_x^{(1)} - i\varepsilon^2 S_x^{(2)}(\beta/\alpha)) \frac{2(\beta/\alpha)(1-\beta^2/\alpha^2)^{\frac{1}{2}}}{(1-2\beta^2/\alpha^2)} + \varepsilon S_y^{(1)}\right]. \tag{6.17}$$

The perturbation of $\mathbf{u}^I$ is small, being of order ε^2, but $\mathbf{u}^S$ contains P-waves and S-waves that are absent from $\mathbf{u}^I$. Inspection of (6.16) shows that the u and v components of the S-wave are out of phase by $\pi/2$, so that the S-wave induces an elliptical displacement of the point $(x',0)$, the axes of the ellipse being respectively parallel and normal to the surface $y'=0$. With regard to the P-wave, we have

$$\frac{C_2}{C_1} = -\frac{(\alpha/\beta)(1-2\beta^2/\alpha^2)}{2(1-\beta^2/\alpha^2)^{\frac{1}{2}}},$$

which implies that the associated ground motion is linear and inclined at an angle $\tan^{-1}(C_2/C_1)$ to the surface $y'=0$, independently of the detail of the topography. The amplitudes of the motions vary, however, roughly in correspondence with the area S.

Finally, the effect of $\mathbf{u}^S$ at the surface of the irregularity can be found from the results of Section 3. To order ε, it can be shown that, at the point $(0,2b)$, corresponding to the top of the ridge or the bottom of the valley,

$$u^S \sim 0, \tag{6.18}$$

$$v^S \sim \frac{-\varepsilon}{(\alpha^2/\beta^2-1)} [AP_{xx} b + (\alpha^2/\beta^2)(p_0+p_1)], \tag{6.19}$$

which perturbs

$$u^I \sim iA\left\{\left(\frac{2\nu_\alpha \nu_\beta}{1+\nu_\beta^2}-1\right)+2\varepsilon b\,\nu_\alpha(\beta/\gamma)\left(\frac{1-\nu_\beta^2}{1+\nu_\beta^2}\right)\right\}, \tag{6.20}$$

$$v^I \sim A\nu_\alpha\left\{\left(\frac{2}{1+\nu_\beta^2}-1\right)+2\varepsilon b(\beta/\gamma)\left[\nu_\alpha-\frac{2\nu_\beta}{1+\nu_\beta^2}\right]\right\}. \tag{6.21}$$

Thus, to first order in ε, u^I is unperturbed while v^S is in phase with v^I and so produces a first order change in amplitude. This contrasts with the SH case considered in I, in which there was a phase change of order ε but only a second order change in amplitude. When $\alpha^2=3\beta^2$, we have, for the ridge ($b=0.4$),

$$v^I \sim 0.620+0.122\,\varepsilon, \qquad v^S \sim 0.096\,\varepsilon \tag{6.22}$$

so that

$$v=v^I+v^S \sim 0.620+0.218\,\varepsilon. \tag{6.23}$$

For the valley ($b=-0.3$),

$$v^I \sim 0.620-0.092\,\varepsilon, \qquad v^S \sim -0.424\,\varepsilon \tag{6.24}$$

so that

$$v=v^I+v^S \sim 0.620-0.516\,\varepsilon. \tag{6.25}$$

The results (6.23) and (6.25) show that the displacement at the top of the ridge is amplified, while that at the bottom of a valley is reduced, in agreement with the observations of Davis and West (1973). It may be noted, too, that the amplification or reduction of v is enhanced by v^S, beyond the effect that would be found by evaluating v^I, by itself, off the plane $y=0$.

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References

Achenbach, J.D.: Wave propagation in elastic solids. Amsterdam: North-Holland 1973

Boore, D.M.: A note on the effect of simple topography on seismic SH waves. Bull. Seism. Soc. Am. **62**, 275, 1972

Boore, D.M.: The effect of simple topography on seismic waves: implications for the accelerations recorded at Pacoima dam, San Fernando Valley, California. Bull. Seism. Soc. Am. **63**, 1603, 1973

Bouchon, M.: Effect of topography on surface motion. Bull. Seism. Soc. Am. **63**, 615, 1973

Crighton, D.G., Leppington, F.G.: Singular perturbation methods in acoustics: diffraction by a plate of finite thickness. Proc. Roy. Soc., London A**335**, 313, 1973

Davis, L.L., West, L.R.: Observed effects of topography on ground motion. Bull. Seism. Soc. Am. **63**, 281, 1973

Ewing, W.M., Jardetsky, W.S., Press, F.: Elastic waves in layered media. New York: Mc-Graw-Hill 1957

Fraenkel, L.E.: On the method of matched asymptotic expansions. Part I: A matching principle. Proc. Cambridge Phil. Soc. **65**, 209, 1969

Gilbert, F., Knopoff, L.: Seismic scattering from topographic irregularities. J. Geophys. Res. **65**, 3437, 1960

Lamb, H.: On the propagation of tremors over the surface of an elastic solid. Phil. Trans. Roy. Soc. London A**203**, 1, 1904

Muskhelishvili, N.I.: Some basic problems in the mathematical theory of elasticity. Groningen: Noordhoff 1953

Neerhoff, F.L.: Scattering of SH waves by an irregularity at the mass-loaded boundary of a semi-infinite elastic medium. Proc. Roy. Soc. London A**342**, 237, 1975

Rogers, A.M., Katz, L.J., Bennett, T.J.: Topographic effects on ground motion for incident P-waves: A model study. Bull. Seism. Soc. Am. **64**, 437, 1974

Sabina, F.J., Willis, J.R.: Scattering of SH waves by a rough half-space of arbitrary slope. Geophys. **42**, 685, 1975

Steg, R.G., Klemens, P.G.: Scattering of Rayleigh waves by surface defects. J. Appl. Phys. **45**, 23, 1974

Tuan, H.S., Li, R.C.M.: Rayleigh wave reflection from groove and step discontinuities. J. Acoust. Soc. Am. **55**, 1212, 1974

Van Dyke, M.D.: Perturbation methods in fluid mechanics. New York: Academic Press 1964. (Also: Perturbation methods in fluid dynamics, annotated edition. Stanford: The Parabolic Press 1975)

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Mean-Field Electrodynamics and Dynamo Theory of the Earth's Magnetic Field

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Abstract. Mean-field electrodynamics as a branch of magnetohydrodynamics of turbulently moving electrically conducting media has been proved to be useful for investigations of problems of dynamo theory. Considering the conducting media in the Earth's liquid core carrying out convective – i.e. stochastic – motions, we derive Ohms law for the mean electromagnetic fields by the methods of mean-field electrodynamics. In addition to the induction action of the mean velocity field there are essential effects due to the convective motion: (1) The α-effect, i.e. the occurrence of a mean electromotive force (emf) parallel to the mean magnetic field, (2) the diminution of the conductivity with respect to the mean fields, (3) the diamagnetic behavior with respect to the mean fields, and (4) the turbulent emf parallel to the direction of the crossproduct of angular velocity and current density. Spherical models can excite magnetic fields of different symmetry types. In connection with the Earth's magnetic field new numerical results due to Rädler are presented, which are especially of interest with respect to the observed westward drift of the dipole field.

Key words: Mean-field electrodynamics – Dynamo theory – Westward drift.

1. Basic Ideas and Results of the Investigations of the Turbulent Dynamo

The idea of the Earth's magnetic field being excited by a dynamo is old, due to Larmor (1919). However, the construction of proper models meets with enormous mathematical difficulties. The situation was characterized by Cowlings theorem (Cowling, 1934): Dynamo excitation does not exist for axisymmetric configurations. Therefore, any attempt of solving a problem of this kind is confronted with three-dimensional complexity.

Frenkel (1945) and Gurewitch and Lebedinskii (1945) argued that the small scale convective motions as observed at the solar surface and expected in the