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# Mean-Field Electrodynamics and Dynamo Theory of the Earth's Magnetic Field

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**Abstract.** Mean-field electrodynamics as a branch of magnetohydrodynamics of turbulently moving electrically conducting media has been proved to be useful for investigations of problems of dynamo theory. Considering the conducting media in the Earth's liquid core carrying out convective – i.e. stochastic – motions, we derive Ohms law for the mean electromagnetic fields by the methods of mean-field electrodynamics. In addition to the induction action of the mean velocity field there are essential effects due to the convective motion: (1) The $\alpha$-effect, i.e. the occurrence of a mean electromotive force (emf) parallel to the mean magnetic field, (2) the diminution of the conductivity with respect to the mean fields, (3) the diamagnetic behavior with respect to the mean fields, and (4) the turbulent emf parallel to the direction of the crossproduct of angular velocity and current density. Spherical models can excite magnetic fields of different symmetry types. In connection with the Earth's magnetic field new numerical results due to Rädler are presented, which are especially of interest with respect to the observed westward drift of the dipole field.

**Key words:** Mean-field electrodynamics – Dynamo theory – Westward drift.

## 1. Basic Ideas and Results of the Investigations of the Turbulent Dynamo

The idea of the Earth's magnetic field being excited by a dynamo is old, due to Larmor (1919). However, the construction of proper models meets with enormous mathematical difficulties. The situation was characterized by Cowlings theorem (Cowling, 1934): Dynamo excitation does not exist for axisymmetric configurations. Therefore, any attempt of solving a problem of this kind is confronted with three-dimensional complexity.

Frenkel (1945) and Gurewitch and Lebedinskii (1945) argued that the small scale convective motions as observed at the solar surface and expected in the

Earth's liquid core may have the complicated pattern required, and thus the ability of self-excitation. This becomes more obvious with the papers of Parker (1955, 1957). Parker was able to show that convective motions undergoing the influence of Coriolis forces (cyclonic turbulence) provide for a large-scale induction action which can maintain or excite a magnetic field if combined with differential rotation.

### *1.1. Mean-Field Electrodynamics and the α-Effect*

A useful tool for describing phenomena of this kind has proved to be mean-field electrodynamics founded in 1966 by papers of Steenbeck et al. (Steenbeck et al., 1966; Steenbeck and Krause, 1966; Rädler, 1968a, b)[1]. The idea is that we consider an electrically conducting medium carrying out turbulent (i.e. random) motions $\boldsymbol{u}$ and that there is a magnetic field $\boldsymbol{B}$. We split both in the mean part, designated by a bar, and a fluctuating part, designated by a dash:

$$\boldsymbol{u}=\bar{\boldsymbol{u}}+\boldsymbol{u}', \qquad \boldsymbol{B}=\bar{\boldsymbol{B}}+\boldsymbol{B}'. \tag{1}$$

From the interaction of the velocity field and the magnetic field results the Lorentz electrical field strength $\boldsymbol{u}\times\boldsymbol{B}$. Taking the average we find

$$\overline{\boldsymbol{u}\times\boldsymbol{B}}=\bar{\boldsymbol{u}}\times\bar{\boldsymbol{B}}+\mathscr{E}, \tag{2}$$

with

$$\mathscr{E}=\overline{\boldsymbol{u}'\times\boldsymbol{B}'}. \tag{3}$$

Consequently an additional emf, the turbulent emf $\mathscr{E}$, appears in the equations decribing the behaviour of the mean fields. This emf (3) is the counterpart of the Reynolds stresses in hydrodynamic turbulence. Since $\boldsymbol{B}'$ is caused by $\boldsymbol{u}'$ both are correlated, and therefore generally $\mathscr{E}\neq 0$ has to be expected.

The situation shall be illustrated by a simple example which leads us, in spite of its simplicity, directly in the heart of the theory of the turbulent dynamo. We introduce (Fig. 1) a cartesian coordinate system where the $x$-axis and the $z$-axis are lying in the plane of the drawing, and the $y$-direction perpendicular to it. A mean magnetic field $\bar{\boldsymbol{B}}$ may be parallel to the $y$-direction and, in addition, a left-handed helical motion $\boldsymbol{u}'$ shall be given. We represent $\boldsymbol{u}'$ by the sum $\boldsymbol{u}'=\boldsymbol{u}_1+\boldsymbol{u}_2$, where $\boldsymbol{u}_1$ is a rotational motion about the $z$-axis and $\boldsymbol{u}_2$ is the motion parallel to the $z$-direction. The interaction of the rotational motion $\boldsymbol{u}_1$ with the mean magnetic field provides for an electrical field $\boldsymbol{E}'=\boldsymbol{u}_1\times\bar{\boldsymbol{B}}$, which is directed parallel to the $z$-direction, downwards before the plane of the drawing and upwards behind it. $\boldsymbol{E}'$ drives a current $\boldsymbol{j}'$, in the same manner directed downwards before the plane of the drawing and upwards behind it. This current is combined with a magnetic field $\boldsymbol{B}'$ in the positive $x$-direction. The crossproduct of $\boldsymbol{u}_2$ with

[1] A parallel development was elaborated by Braginskij (1964a, b, c) for the field quantities averaged over the azimuth, using an approximation from high magnetic Reynolds numbers (nearly-symmetric dynamos). The resulting equations are identical with those derived on the basis of mean-field magnetohydrodynamics

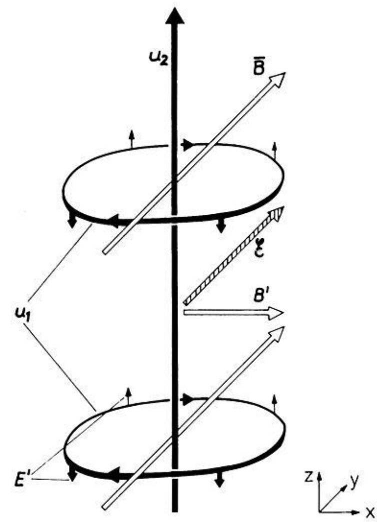



**Fig. 1.** Schematic drawing which shows that a helical motion $u = u_1 + u_2$, in a homogeneous magnetic field $\bar{B}$, provides in second order for an emf, $\mathscr{E}$, parallel to the magnetic field ($\alpha$-effect)

this magnetic field $\boldsymbol{B}'$ is directed in the positive $y$-direction. Consequently, we find an electromotive force $\mathscr{E}$ parallel to the mean magnetic field,

$$\mathscr{E} = \overline{\boldsymbol{u}' \times \boldsymbol{B}'} = \alpha \bar{\boldsymbol{B}}. \tag{4}$$

As can be taken from our example $\alpha$ is positive for left-handed helical motions and negative for right-handed ones. If we now consider a random motion with the property of one kind of helical motions being more probable than the other one we will find in the average an electromotive force parallel, or antiparallel, to the mean magnetic field. This is the $\alpha$-effect wellknown in the theory of the turbulent dynamo.

### *1.2. Experimental Verification of the $\alpha$-Effect*

We now consider (Fig. 2) a box in which we assume an electrically conducting fluid carrying out turbulent motions. The turbulence may be homogeneous and isotropic but has helicity, i.e. one kind of helical motions shall be more probable than the other one. Let us assume we have a higher probability of left-handed helical motions. A magnetic field $\boldsymbol{B}$ may be directed from the left to the right. On this conditions the $\alpha$-effect must be expected and, consequently, an electrical field parallel to the magnetic field. It provides for positive electrical charges at the right wall of the box and for negative at the left. From the outside we can measure a voltage parallel to the applied magnetic field.

Evidence for the existence of the $\alpha$-effect was given by an experiment carried out in the Institute of Physics of the Latvian Academy of Science in Riga (Steenbeck et al., 1967, 1968). Figure 3 shows the "$\alpha$-yashchik" ($\alpha$-box), in which

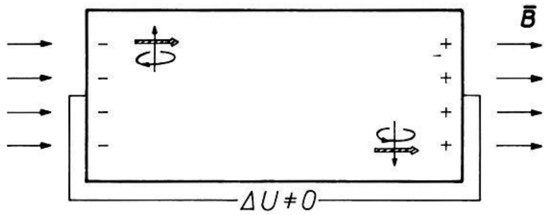



**Fig. 2.** Small-scale left-handed helical motions in a box will provide for the building-up of a voltage along a magnetic field

a motion of liquid sodium according to a right-handed screw was created. A section of the outer wall being removed enables us to see the copper walls inside the box which force the motion to take the helical structure. In Figure 4 the measured voltage parallel to the applied magnetic field is represented in dependence on the Stuart-number. For vanishing Stuart-number the measured voltage was 20% of the theoretically predicted value which was derived from a model of infinite extension in the directions perpendicular to the magnetic field.

### *1.3. Dynamo Excitation Provided by the α-Effect*

The close connection of the α-effect with dynamo excitation is obvious already from simple models (Fig. 5): The two rings shall contain an electrically conducting medium with α-effect. Let us assume a magnetic field $B_0$ being in ring (I). It drives a current $j_1$ because of the α-effect. This current $j_1$ is combined with a magnetic field $B_1$ in ring (II). $B_1$ drives a current $j_2$ *which is combined with a magnetic field* $B_2$ in ring (I). Obviously $B_2$ *supports* $B_0$. Consequently we can expect the fields being maintained inspite of Ohmic losses. The energy of the Joule heat produced by the currents comes from the kinetic energy of the turbulent motion having helicity.

We arrive at a homogeneous model (Fig. 6) by supposing the infinite space filled with an electrically conducting medium with α-effect and by considering a field configuration derived from that of Figure 5 by rotating the fields about the axis designated by the dashed line. $B_{\text{pol}}$ designates the poloidal field part, the field lines of which lie in the planes containing the axis of symmetry; $B_{\text{tor}}$ designates the ring field around this axis. An argumentation analogous to that for Figure 5 clearly shows that such a field configuration is able to maintain itself for sufficient strong α-effect.

### *1.4. Simple Dynamo Models*

The simplest dynamo model which can be treated mathematically is given by a sphere containing an electrically conducting medium with α-effect imbedded in the non-conducting space (Krause and Steenbeck, 1967). There exists a certain

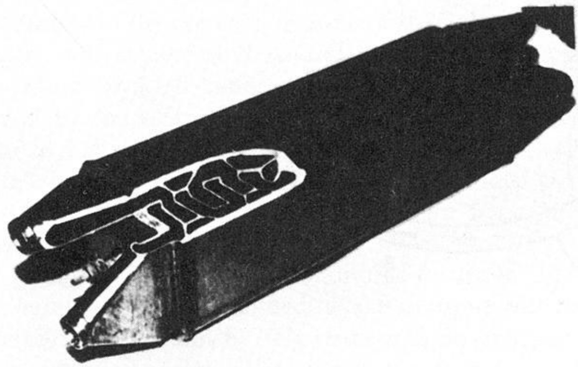


**Fig. 3.** The "α-yashchik" (α-box). Experimental device according to the scheme given in Figure 2. A section of the outer wall being removed enables us to see the copper walls inside the box which force the motion of liquid sodium to take the helical structure

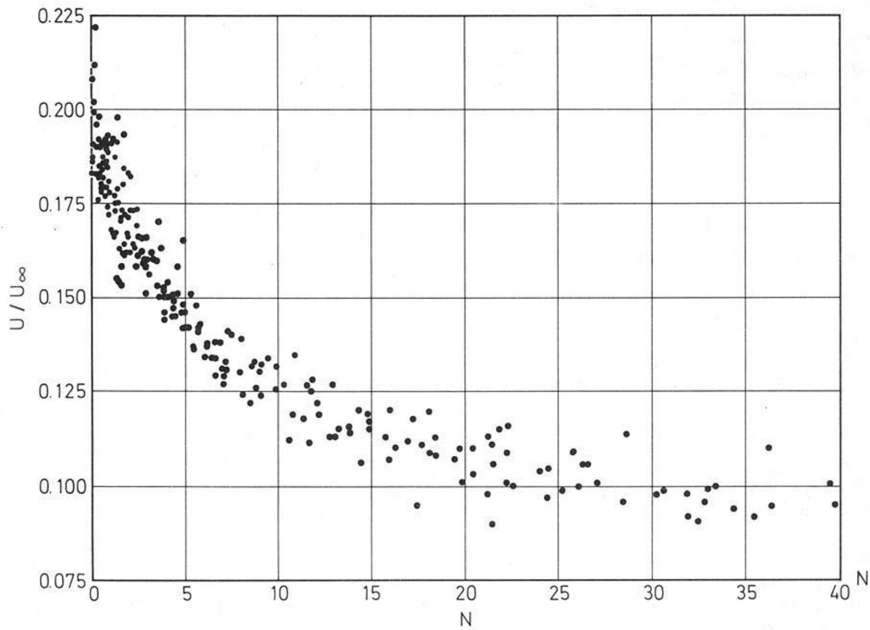
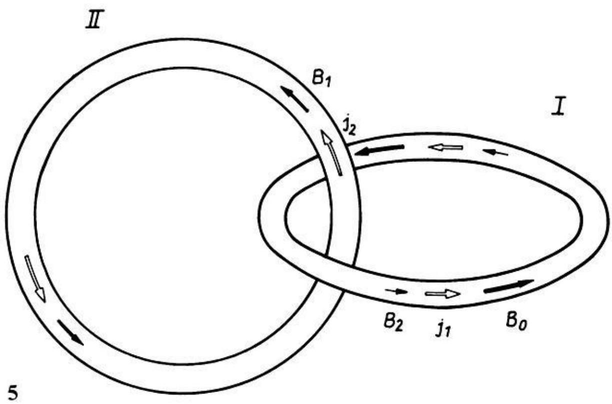



**Fig. 4.** Representation of the voltage measured along the magnetic field in dependence on the Stuart number $N=\frac{B^2 l\sigma}{\rho u}$. ($B$ magnetic induction, $l$ width of the channel, $\rho$ density and $\sigma$ conductivity of liquid sodium, $u$ velocity of the flow)

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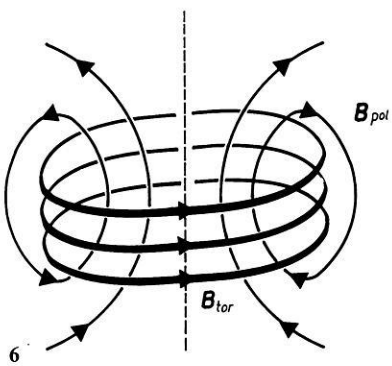


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**Fig. 5.** Model illustrating the self-excited building-up of a magnetic field in a medium with Ohms law $j=\sigma(E+\alpha B)$

**Fig. 6.** A possible self-maintaining field configuration in an infinite extended, conducting medium with $\alpha$-effect. The field is axisymmetric. It is composed of a poloidal part, $B_{\mathrm{pol}}$, with the field lines in the meridional planes with regard to the axis of symmetry, and a toroidal part, $B_{\mathrm{tor}}$, with the field lines encircling the axis of symmetry

value of $\alpha$, upon which self-excitation occurs. This value, $\alpha_{\mathrm{crit}}$, is derived from an eigenvalue problem to be the smallest positive root of the equation

$$J_{3/2}(C)=0, \tag{5}$$

where $C$ is the dimensionless number

$$C=\mu\,\sigma\,\alpha\,R. \tag{6}$$

$\mu$ designates the permeability, $\sigma$ the conductivity and $R$ the radius of the sphere in consideration. Self-excitation of magnetic fields exists if

$$C \geqq C_{\mathrm{crit}}=4.49\ldots. \tag{7}$$

The configuration of the excited field is represented in Figure 7 for the limit $C \to C_{\mathrm{crit}}(C > C_{\mathrm{crit}})$.

### *1.5. Helicity of a Convection on a Rotating Body*

It is quite natural for convective (turbulent, small-scale) motions on a rotating body to have helicity because of the action of Coriolis forces and, consequently, the $\alpha$-effect will be present if the convective medium is electrically conducting. Figure 8 illustrates the situation in a compressible medium; it reflects the conditions in the convection zone of the Sun in the northern hemisphere. Rising matter will expand and rotate because of the action of Coriolis forces, thus providing for a left-handed helical motion. Sinking matter is compressed and is forced by the Coriolis forces to rotate in the opposite direction, again a left-handed helical motion. We realize that the convection in the northern hemisphere shows a higher probality of left-handed helical motions than of right-handed ones. Obviously, in the southern hemisphere the right-handed helical motions will prevail.

Figure 9 illustrates the conditions for an incompressible fluid. A balance of left-handed and right-handed helical motions will exist in the medium layer (the intersection of a and b) only if there is no gradient in the turbulence intensity, since one kind of helical motion comes from below (layer a) and the other from above (layer b). Consequently, a gradient in the turbulence intensity in a turbulent medium on a rotating body will also provide for helicity under the conditions of a flat geometry as considered in Figure 9. For a spherical geometry this balance will occur for a special dependence of the turbulence intensity on the distance from the centre, generally a non-vanishing helicity has to be expected.

At the end of this introductory chapter we present a dynamo model for the earth's magnetic field which is calculated on the basis of the conceptions above (Steenbeck and Krause, 1969). Figure 10 shows the model of the Earth with a certain turbulence profile in the outer core. Figure 11 is a representation of the derived field structure. The eigenvalue $C$ has nearly the same value as that of the $\alpha=$const.-dynamo given in (7).

One should also mention that on the same conceptions a construction of dynamo models for the Sun's alternating magnetic field was possible. In this way basic mechanisms of the solar activity had been clarified (Steenbeck and Krause, 1969a).

## 2. Theoretical Foundation of Mean-Field Electrodynamics

In the first and introductory part of this paper we presented the main ideas leading to a solution of the dynamo problem. We will now direct our attention to the theoretical and mathematical side of this problem.

### *2.1. Averaging Operations*

We are concerned with the theory of the mean quantities in a system where random processes play an important role. The question arises how to define the means. We can take averages over the space coordinates, or the time coordinate, or, and this is most convenient, take statistical averages. In the latter case we assume a great number of systems and take the average over this ensemble of systems. It is in our considerations not important what kind of average we use. The only thing of importance is that the Reynolds rules are fulfilled.

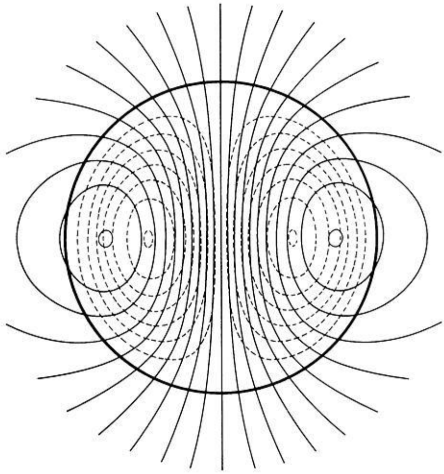

**Fig. 7.** Field lines of the poloidal part (solid lines), and lines of constant field strength of the toroidal part (dashed lines) of the magnetic field excited by the ($\alpha$=constr.)-dynamo in the limit $C \to C_{crit}(C > C_{crit})$

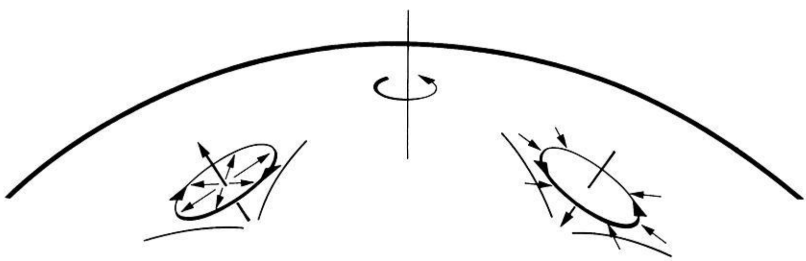

**Fig. 8.** Schematic drawing which explains that left-handed helical motions have a higher probability to appear in the northern hemisphere of the Sun than right-handed ones

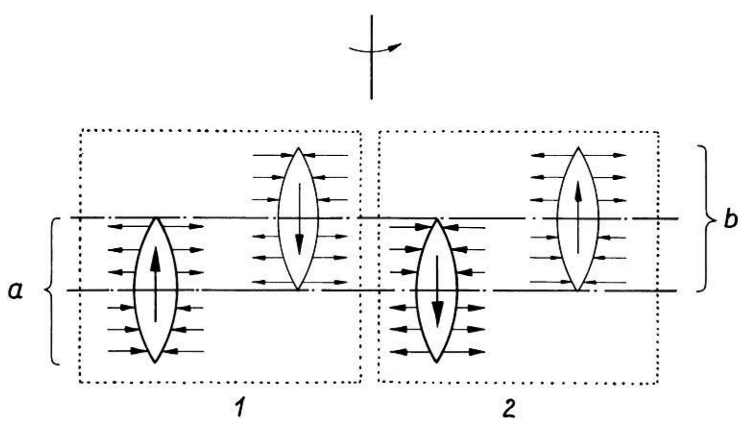


**Fig. 9.** In an incompressible medium the imbalance of right-handed and left-handed helical motions can be due to a vertical gradient of the turbulence intensity $\overline{u'^2}$

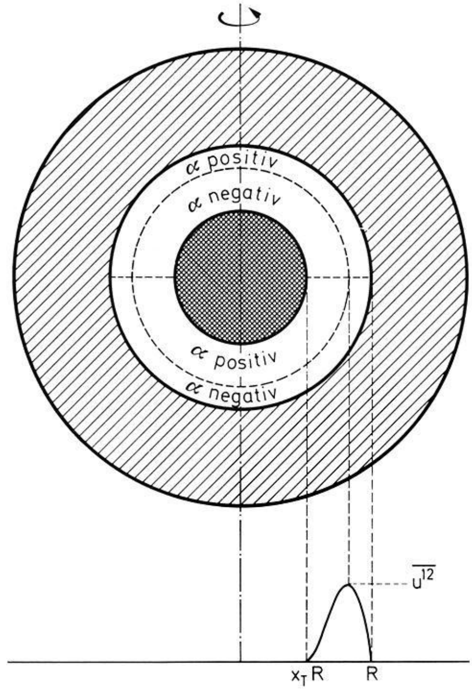


**Fig. 10.** Model of the earth used for the calculations. The hatched outer region represents the mantle with zero conductivity, the crosshatched inner region the rigid inner core assumed being electrically conducting. In the medium region turbulence is assumed to exist as indicated by the curve below

If $F$ and $G$ are random quantities some of these rules are

$$F=\bar{F}+F' \tag{8}$$

$$\bar{F}'=0, \qquad \bar{\bar{F}}=\bar{F}, \tag{9}$$

$$\overline{F+G}=\bar{F}+\bar{G}, \tag{10}$$

$$\overline{F\cdot G}=\bar{F}\cdot\bar{G}+\overline{F'G'}. \tag{11}$$

Exchange with differentiation and integration operation is possible. That is a consequence of (10). For statistical averages the Reynolds rules are valid. For space averages, however, relations (9) will hold only approximately if the characteristic length scale of the random process is small compared with the characteristic length scale of the mean quantity. Similar requirement is made for the characteristic time scales in case we take the average over the time coordinate.

For a turbulent medium we take the correlation length, $\lambda_{\text{cor}}$, and the correlation time, $\tau_{\text{cor}}$, as characteristic scales of the fluctuations. $\bar{\lambda}$ and $\bar{\tau}$ shall be the scales of the mean fields. An application of a theory which is based on the Reynolds rules is indicated if $\lambda_{\text{cor}} \ll \bar{\lambda}$, in case the averages are taken over space coordinates, or if $\tau_{\text{cor}} \ll \bar{\tau}$, in case the averages are taken over the time coordinate. Consequently, it will not be a loss of generality if we make use of a 2-scale property of the turbulence.

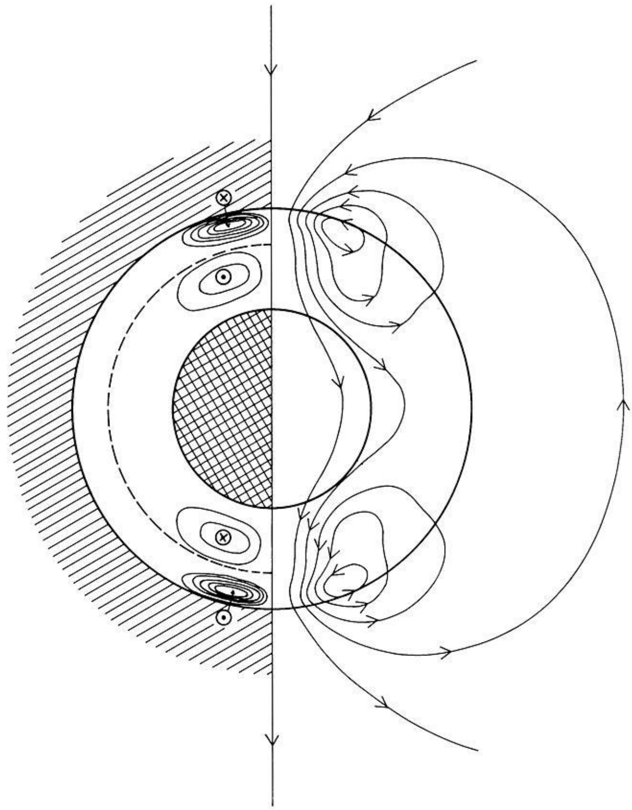

**Fig. 11.** Magnetic field excited by the dynamo model given in Figure 10. At the left the lines of constant field strength of the toroidal field are represented, at the right the field lines of the poloidal field. (After Steenbeck and Krause, 1969b)

### *2.2. Expressions of the Turbulent emf* $\overline{u' \times B'}$

It has become obvious in the foregoing chapter that the determination of the turbulent emf $\mathscr{E}$ defined by (3) is the crucial point of the theory. Here we can only give a short draft of the main lines of ideas, the interested reader may be referred to the literature (Moffatt, 1970a, b; Roberts, 1971; Krause and Rädler, 1971; Roberts and Stix, 1971; Vainshtein and Zeldovich, 1972; Kippenhahn and Möllendorf, 1975; Roberts and Soward, 1975).

We start from the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} - \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{B}) - \eta \Delta \boldsymbol{B} = 0, \tag{12}$$

$u$ denotes the velocity field and $\eta = 1/\mu\sigma$ the magnetic diffusivity. Taking the average we arive at the equation

$$\frac{\partial \bar{\boldsymbol{B}}}{\partial t} - \operatorname{curl}(\bar{\boldsymbol{u}} \times \bar{\boldsymbol{B}}) - \operatorname{curl}\,\overline{(\boldsymbol{u}' \times \boldsymbol{B}')} - \eta \Delta \bar{\boldsymbol{B}} = 0, \tag{13}$$

in the third term one notices the turbulent emf $\mathscr{E}$. Subtracting (13) from (12) we arrive at an equation for the fluctuating magnetic field

$$\frac{\partial \boldsymbol{B}'}{\partial t} - \operatorname{curl}(\bar{\boldsymbol{u}} \times \boldsymbol{B}') - \eta \Delta \boldsymbol{B}' - \operatorname{curl}(\boldsymbol{u}' \times \boldsymbol{B}' - \overline{\boldsymbol{u}' \times \boldsymbol{B}'}) = \operatorname{curl}(\boldsymbol{u}' \times \bar{\boldsymbol{B}}). \tag{14}$$

Obviously, there is no hope to find a sufficiently general solution of Equation (14). It is possible, however, to derive as a general statement that $\boldsymbol{B}'$ is a linear functional of $\bar{\boldsymbol{B}}$. The same is valid for the turbulent emf $\mathscr{E}$. Thus we have

$$\mathscr{E} = \overline{\boldsymbol{u}' \times \boldsymbol{B}'} = \mathscr{L}(\bar{\boldsymbol{B}}), \tag{15}$$

where the functional is denoted by $\mathscr{L}$. In case Equation (14) has decaying solutions only if $\bar{\boldsymbol{B}} = 0$, the functional is homogeneous. Then we can give the alternative representation

$$\mathscr{E}_i = \int\int K_{ik}(\boldsymbol{x}, t, \boldsymbol{\xi}, \tau)\, \bar{B}_k(\boldsymbol{x} + \boldsymbol{\xi}, t + \tau)\, d\boldsymbol{\xi}\, d\tau, \tag{16}$$

where $K_{ik}$ is a certain kernel function (or functional) depending on the mean and the fluctuating velocity field. $K_{ik}$ is, however, a mean quantity.

2.2.1. Further progress is possible if additional simplifications are taken into account. Firstly, there is the two-scale property, which not necessarily means a loss of generality, as pointed out before. We assume the mean magnetic field being so weakly dependent on time and space coordinates over distances of length $\lambda_{\text{cor}}$ and $\tau_{\text{cor}}$ that (16) reduces to the relation

$$\mathscr{E}_i = a_{ik} \bar{B}_k + b_{ikl} \frac{\partial \bar{B}_k}{\partial x_l}. \tag{17}$$

An error of the order $O\left(\frac{\tau_{\text{cor}}}{\bar{\tau}}, \left(\frac{\lambda_{\text{cor}}}{\bar{\lambda}}\right)^2\right)$ must be expected. It is important to mention that the tensors $a_{ik}$, $b_{ikl}$ are skew quantities, since $\mathscr{E}$ is a polar vector and $\bar{\boldsymbol{B}}$ an axial one.

Additional assumptions will provide for further simplifications:

– If the turbulence is steady the tensors $a_{ik}$ and $b_{ikl}$ do not depend on the time coordinate $t$.

– If the turbulence is homogeneous the tensors $a_{ik}$ and $b_{ikl}$ do not depend on the space coordinates.

– If the turbulence is isotropic the tensors $a_{ik}$ and $b_{ikl}$ must be isotropic tensors, i.e.

$$a_{ik} = \alpha \delta_{ik}, \qquad b_{ikl} = \beta \varepsilon_{ikl}, \tag{18}$$

where $\alpha$ is a pseudo-scalar since $a_{ik}$ is skew. $\alpha$ and $\beta$ do not depend on the space coordinates, since an isotropic turbulence is necessarily homogeneous.

– If, in addition, the turbulence is assumed to be mirror symmetric, $\alpha$ as a pseudo scalar must be zero.

2.2.2. As an illustration we will write down Ohms law for the mean fields for the case of a vanishing mean velocity and a steady homogeneous isotropic turbulence. From the usual Ohms law,

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}), \tag{19}$$

we find with (2), (3), (17) and (18)

$$\bar{\boldsymbol{j}}=\sigma_T(\bar{\boldsymbol{E}}+\alpha\bar{\boldsymbol{B}}), \qquad \sigma_T=\frac{\sigma}{1+\mu\sigma\beta}. \tag{20}$$

As an important result we note the $\alpha$-effect derived here by general theoretical arguments. The previous derivation of this effect (p. 423) was based on a consideration of helical motions. Obviously, a turbulence where one kind of helical motions is more probable than the other is a non-mirrorsymmetric turbulence.

In the mirror-symmetric case the effect of the turbulence is a change of the conductivity only. In the cases of main interest $\beta$ is positive (Krause and Roberts, 1973a, b) hence the conductivity with respect to the mean fields is smaller than the molecular conductivity.

### *2.3. Ohms Law for the Mean Electrodynamic Fields in a Rotating Turbulent Fluid*

Homogeneity, isotropy and mirror symmetry are properties of a turbulence which has grown old without any influence. On a rotating gravitating body, however, a turbulence is influenced by the gravity field, $\boldsymbol{g}$, and the rotational motion, the latter may be represented by the vector of angular velocity, $\boldsymbol{\omega}$. Under these conditions the tensors $a_{ik}$, $b_{ikl}$ must be the most general expressions which can be constructed from the quantities $\boldsymbol{g}$, $\boldsymbol{\omega}$ and the isotropic tensors $\delta_{ik}$, $\varepsilon_{ikl}$. Restricting ourselves to linear expressions in $\boldsymbol{g}$ and $\boldsymbol{\omega}$ and taking into account that both tensors are skew we find

$$a_{ik}=\underbrace{\alpha_0(\boldsymbol{g}\cdot\boldsymbol{\omega})}_{\alpha}\delta_{ik}+\alpha_1 g_i\omega_k+\alpha_2 g_k\omega_i+\gamma\varepsilon_{ikl}g l, \tag{21}$$

$$b_{ikl}=\beta\varepsilon_{ikl}+\beta_1\omega_i\delta_{kl}+\beta_2\omega_k\delta_{il}+\beta_3\omega_l\delta_{ik}. \tag{22}$$

$\alpha_0$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\gamma$, $\beta$, $\beta_1$, $\beta_2$, $\beta_3$ are scalars which can be functions of the time and the space coordinates. In a convective layer of a cosmic body we expect a dependence on the distance from the centre only. We again notice the appearance of the $\alpha$-effect given by the first summand in (21). In agreement with our argumentation on page 427 we see in (21) that $\alpha$ changes the sign from one hemisphere to the other, because the scalar product $\boldsymbol{\omega}\cdot\boldsymbol{g}$ does so.

With (2), (3), (17), (21), (22) we easily find Ohms law for the mean electromagnetic fields in a rotating convective layer:

$$\begin{aligned}\bar{\boldsymbol{j}}=\sigma_T\{&\bar{\boldsymbol{E}}+\bar{\boldsymbol{u}}\times\bar{\boldsymbol{B}}+\alpha_0(\boldsymbol{g}\cdot\boldsymbol{\omega})\bar{\boldsymbol{B}}+\alpha_1((\boldsymbol{g}\cdot\bar{\boldsymbol{B}})\boldsymbol{\omega}+(\boldsymbol{\omega}\cdot\bar{\boldsymbol{B}})\boldsymbol{g})\\&+\bar{\boldsymbol{B}}\times(\gamma\boldsymbol{g}+\alpha_2\boldsymbol{\omega}\times\boldsymbol{g})\\&+(\beta_2+\beta_3)\,\mathrm{grad}\,(\boldsymbol{\omega}\cdot\bar{\boldsymbol{B}})-\mu\beta_3\boldsymbol{\omega}\times\mathrm{curl}\,\bar{\boldsymbol{H}}\}.\end{aligned} \tag{23}$$

The second term in the brackets on the right-hand side describes the induction action of the mean velocity field. This field is assumed to be the superposition of a differential rotation and a meridional circulation. The third term describes the $\alpha$-effect. We will speak of the "ideal $\alpha$-effect" if the fourth term is not taken into account, if both are considered we speak of the "real-$\alpha$-effect". There are theoretical arguments that the ratio $\alpha_1/\alpha_0$ is about $-1/4$. The following term, $\bar{\boldsymbol{B}} \times(\gamma \boldsymbol{g}+\alpha_2 \boldsymbol{\omega} \times \bar{\boldsymbol{B}})$, describes a diamagnetic behaviour of the turbulent medium (Rädler, 1968), sometimes also called a "pumping effect" (Drobishefskij and Yuferev, 1975). The term $(\beta_2+\beta_3)\,\mathrm{grad}\,(\boldsymbol{\omega} \cdot \bar{\boldsymbol{B}})$ is of no importance, since it is compensated by space charges. The last term describes the "$\boldsymbol{\omega} \times \boldsymbol{j}$-effect", it can provide for dynamo excitation if combined with differential rotation (Rädler, 1969a, b, 1970).

### 2.4. Theoretical Derivation of the Characteristic Parameters

Inspecting Ohms law for the mean fields as given in (23) we see that by our deductions for a two-scale turbulence the problem is reduced to the determination of the scalar quantities $\alpha_0$, $\alpha_1$, $\alpha_2$, $\gamma$, $\beta_2$, $\beta_3$, $\sigma_T$. For practical purposes like model calculations, estimates are used according to reasonable argumentations. Attempts of theoretical determinations of these quantities have been undertaken. They are, however, in every case confronted with the closure problem of the theory of turbulence, which is not sufficiently solved. Here we want to give only a short survey of this kind of problems:

We have to start from the induction Equation (12), where we take the average thus arriving at (13). There the turbulent emf $\mathscr{E}=\overline{\boldsymbol{u}' \times \boldsymbol{B}'}$ appears as a new quantity which is a statistical moment of second order. We find an equation for this quantity by the following procedure: Substracting Equation (13) from (12) we find Equation (14). Now we substitute in Equation (14) the arguments $\boldsymbol{x}+\boldsymbol{\xi}$ and $t+\tau$, and understand all differentiations as differentiations with respect to the variables $\boldsymbol{\xi}$ and $\tau$. This equation is now multiplied by $\boldsymbol{u}'(\boldsymbol{x}, t)$, and we find by averaging it

$$\frac{\partial \overline{\boldsymbol{u}' \boldsymbol{B}'}}{\partial \tau}-\mathrm{curl}_{\xi}(\bar{\boldsymbol{u}} \times \overline{\boldsymbol{u}' \boldsymbol{B}'})-\eta \Delta_{\xi}\, \overline{\boldsymbol{u}' \boldsymbol{B}'}=\mathrm{curl}_{\xi}(\boldsymbol{Q}^{(2)} \times \bar{\boldsymbol{B}})-\mathrm{curl}_{\xi}(\overline{\boldsymbol{u}' \boldsymbol{u}' \times \boldsymbol{B}'}). \tag{24}$$

Equation (24) is one for the second order statistical moment $\overline{u_i'(\boldsymbol{x}, t)\, B_j'(\boldsymbol{x}+\boldsymbol{\xi}, t+\tau)}$ wherefrom $\mathscr{E}$ can be derived. For brevity a rather unprecise denotation is used, it makes possible however, an explanation of the main features. On the right-hand side we see the second order correlation tensor of the velocity field $\overline{u_i'(\boldsymbol{x}, t)\, u_k'(\boldsymbol{x}+\boldsymbol{\xi}, t+\tau)}$, here denoted by $\boldsymbol{Q}^{(2)}$ which is known according to our assumptions. However, a further unknown quantity appears in this equation, the third order statistical moment $\overline{\boldsymbol{u}' \boldsymbol{u}' \times \boldsymbol{B}'}$. We find an equation for the third order statistical moment by substituting in (14) the arguments $\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\xi}'$. $t+\tau+\tau'$ and understanding all differentiations with respect to $\boldsymbol{\xi}'$ and $\tau'$. If this equation is multiplied with $\boldsymbol{u}'(\boldsymbol{x}, t)\, \boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{\xi}, t+\tau)$ and averaged we arrive at

$$\begin{aligned}\frac{\partial \overline{\boldsymbol{u}' \boldsymbol{u}' \boldsymbol{B}'}}{\partial \tau}&-\mathrm{curl}_{\xi'}(\bar{\boldsymbol{u}} \times \overline{\boldsymbol{u}' \boldsymbol{u}' \boldsymbol{B}'})-\eta \Delta_{\xi'}\, \overline{\boldsymbol{u}' \boldsymbol{u}' \boldsymbol{B}'}=\mathrm{curl}_{\xi'}(\boldsymbol{Q}^{(3)} \times \bar{\boldsymbol{B}}) \\ &-\mathrm{curl}_{\xi'}(\overline{\boldsymbol{u}' \boldsymbol{u}' \boldsymbol{u}' \times \boldsymbol{B}'}).\end{aligned} \tag{25}$$

(25) represents an equation for the third order statistical moment

$$\overline{u'_i(\boldsymbol{x},t)\,u'_j(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)\,B'_k(\boldsymbol{x}+\boldsymbol{\xi}+\boldsymbol{\xi}',t+\tau+\tau')}.$$

On the right we have the third order correlation tensor of the velocity field $\boldsymbol{Q}^{(3)}$, but also the fourth order statistical moment $\overline{\boldsymbol{u}'\boldsymbol{u}'\boldsymbol{u}'\times\boldsymbol{B}'}$, which is an additional unknown quantity. If we proceed in this way we obtain an infinite set of equations. Attempts of solving it meet with the closure problem, i.e. with the problem to derive in a proper way a closed system, e.g. a finite one.

2.4.1. For illustration we present here the expressions for $\alpha$ and $\beta$ derived for a homogeneous isotropic turbulence in case there is no mean motion. The closure is arrived at by ommitting the third order statistical moment in Equation (24). It has been obtained

$$\alpha=-\tfrac{1}{3}\int\int G(\xi,\tau)\overline{\boldsymbol{u}'(\boldsymbol{x},t)\cdot\operatorname{curl}\boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)}\,d\boldsymbol{\xi}\,d\tau, \tag{26}$$

(Krause, 1967, c.f. Roberts and Stix, 1971) and

$$\beta=-\tfrac{1}{3}\int\int \xi\,\frac{\partial G(\xi,\tau)}{\partial\xi}\,f(\xi,\tau)\,d\boldsymbol{\xi}\,d\tau \tag{27}$$

(Rädler, 1966, 1968b). $G(\xi,\tau)$ denotes the Greens function

$$G(\xi,\tau)=\left(\frac{\mu\sigma}{4\pi\tau}\right)^{3/2}\exp\left(-\frac{\mu\sigma\xi^2}{4\tau}\right), \tag{28}$$

and $f$ the longitudinal correlation function defined by

$$f(\xi,\tau)=\frac{1}{\xi^2}\,\overline{(\boldsymbol{\xi}\cdot\boldsymbol{u}'(\boldsymbol{x},t))(\boldsymbol{\xi}\cdot\boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{\xi},t+\tau))}. \tag{29}$$

The expression for $\alpha$ contains the quantity $\overline{\boldsymbol{u}'(\boldsymbol{x},t)\cdot\operatorname{curl}\boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)}$ indicating that one kind of helical motion is more probable than the other one.

## 3. Dynamo Models for the Earth

Investigation of dynamo models for the Earth and the Sun have been carried out on the basis of Ohms law for the mean fields (23). In Figure 11 we represented the field configuration of a dynamo model for the Earth where the ideal $\alpha$-effect is only taken into account. Further investigations have been carried out by Rädler (1969, 1973, 1975), Stix (1971), Roberts (1972), Roberts and Stix (1972), Deinzer and Stix (1971), Krause (1971), Deinzer, Kusserow and Stix (1974), Levy (1972). We take here the opportunity to present some new results concerning the Earth's magnetic field which are due to Rädler (1977).

### *3.1. Spherical Dynamo Models and Symmetry Properties of the Excited Fields*

A conducting sphere embedded in the non-conducting space is considered. A mean velocity field composed of a differential rotation and a meridional motion, and a turbulence are assumed to be given. The motions show axisymmetry with respect to the axis of rotation and symmetry with respect to the equatorial plane, the turbulence shall have these properties on the average. The electromagnetic behaviour of this system is described by the Maxwell equations, by Ohms law for the mean fields (23) and by appropriate boundary conditions.

We call this system a self-excited dynamo, if the equations which are homogeneous in the mean magnetic field possess solutions $\bar{\boldsymbol{B}}$ which do not decay with time. Whether the system is a self-excited dynamo depends on the parameter $C$, defined by (6), and others to be derived from the quantities $\alpha_1, \alpha_2, \gamma, \ldots$. In case the induction effects are too small the solutions decay. If some of the effects are sufficiently strong, e.g. the $\alpha$-effect, dynamo excitation will occur and the magnetic field will grow. Steady or oscillatory solutions exist in the limiting cases which are characterized by certain values to be derived from an eigenvalue problem.

The system under consideration is assumed to be axisymmetric with respect to the axis of rotation, and symmetric with respect to the equatorial plane. In addition, we assume the velocity being steady, the turbulence on the average. Thus all eigen solutions are of the type $e^{i(\Omega t + m\varphi)} F(r, \vartheta)$, where $\Omega$ is a certain frequency, $\varphi$ the azimuth and $\vartheta$ the polar distance of a polar coordinate system with the axis of rotation as the polar axis. In this connection $F$ denotes an arbitrary quantity, e.g. a component of the vector field $\bar{B}$. $m$ is an integer.

We consider the general case where the eigenvalues of our problem are single. As there is symmetry with respect to the equatorial plane of the system the eigen solutions of our problem are either symmetric or antisymmetric with respect to the equatorial plane. Therefore, the function $F$ introduced before is either even with respect to $\vartheta = \frac{\pi}{2}$, i.e.

$$F(r, \vartheta) = F(r, \pi - \vartheta), \tag{30}$$

or odd, i.e.

$$F(r, \vartheta) = -F(r, \pi - \vartheta). \tag{31}$$

For instance, if the eigen solution $\bar{\boldsymbol{B}}$ is symmetric with respect to the equatorial plane (30) is valid for the $r$-component and the $\varphi$-component, but (31) for the $\vartheta$-component. If $\bar{\boldsymbol{B}}$ is antisymmetric, (30) holds for the $\vartheta$-component but (31) for the $r$-component and the $\varphi$-component. We will denote the type of symmetry by $S$ for the symmetric and by $A$ for the antisymmetric case. In addition, we write down the integer $m$ in order to characterize the dependence on the azimuth $\varphi$.

The fields represented in Figures 7 and 11 are of type A0, i.e. antisymmetric with respect to the equatorial plane and axissymmetric with respect to the rotational axis. A field having the appearance of a dipole with its moment in the centre of the sphere and parallel to the equatorial plane is of type S1. A quadrupole of this kind is of type S2 (Fig. 12).

The magnetic field of the Earth is mainly a field of type A0, the inclination of it, however, indicates an additional dipole in the equatorial plane, i.e. an additional field of type S1. The Sun's magnetic field is also of type A0 but alternating with a period of 22 years. According to the observations the magnetic fields of magnetic stars, however, are expected to be of type S1 (Krause, 1972; Krause and Oetken, 1976; Oetken, 1977).

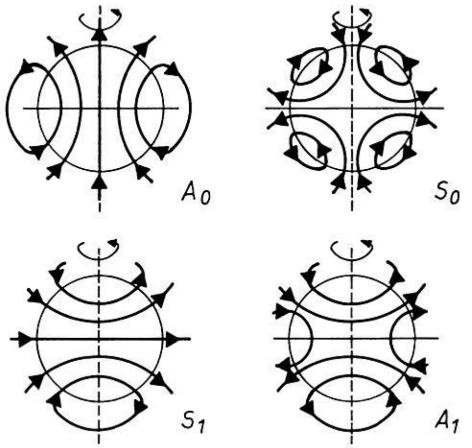


**Fig. 12.** Symmetry types of magnetic fields excited by a spherical dynamo which itself shows axisymmetry with regard to the axis of rotation and mirror symmetry with regard to the equatorial plane. The non-axisymmetric fields (type S1, A1, ...) generally migrate with longitude, i.e. they show an eastward or westward drift

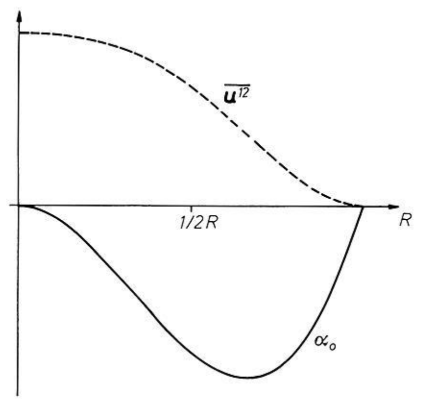


**Fig. 13.** Profiles of the turbulence intensity and the quantity $\alpha_0$ for the first group of models

Finally, it is to be noted that the eigen solutions generally migrate with longitude if $m \neq 0$, due to their special dependence on $t$ and $\varphi$. This remark is important in connection with the observed westward drift of the dipol component of the Earth's magnetic field.

### *3.2. An $\alpha^2$-Dynamo Model for the Earth's Magnetic Field*

For the models presented in the following, we assume the $\alpha$-effect to be due to the gradient of the turbulence intensity. In Figure 13 the turbulence intensity is drawn for a first group of models and the quantity $\alpha_0$ derived. This model shows correspondence to a completely fluid core. In Table 1 the eigenvalues derived by Rädler are listed[2]. $C$ is the parameter defined by (6) where the maximum value of $\alpha_0$ is taken for $\alpha$. $C_\Omega$ is a dimensionless frequency defined by

$$C_\Omega = \mu\sigma\Omega R^2. \tag{32}$$

[2] The results listed for the $A0$ and $S0$ fields confirm those by Steenbeck and Krause (1969b)

**Table 1.** Eigenvalues $C$ and frequency parameters $C_\Omega$ of fields of different types excited by a dynamo model of the first group. The positive sign of $C_\Omega$ indicates an eastward drift of the magnetic field

| | $C$ | $C_\Omega$ |
|---|---|---|
| A0 | 4.58 | 0 |
| S0 | 4.65 | 0 |
| A1 | 4.74 | 0.26 |
| S1 | 4.67 | 2.17 |
| A2 | 6.21 | 0.39 |
| S2 | 6.20 | 1.22 |

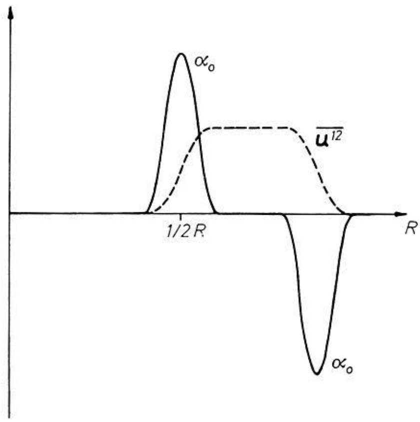


**Fig. 14.** Profiles of the turbulence intensity and the quantity $\alpha_0$ for a second group of models

**Table 2.** Eigenvalues $C$ and frequency parameters $C_\Omega$ of fields of different types excited by a dynamo of the second group. The negative sign of $C_\Omega$ indicates a westward drift of the magnetic field

| | $C$ | $C_\Omega$ |
|---|---|---|
| A0 | 2.68 | 0 |
| S0 | 2.70 | 0 |
| A1 | 2.78 | −1.39 |
| S1 | 2.77 | −.265 |
| A2 | 3.12 | −0.24 |
| S2 | 3.12 | −0.85 |

Positive values of $C_\Omega$ indicate an eastward and negative ones a westward drift of the fields.

In Table 1 we can see that the eigenvalue of the A0 type field is the smallest, which indicates that this field can be excited most easily. The eigenvalues of the other field types are not much larger, especially those of the S0 and the S1 types. The positive sign of $C_\Omega$, however, indicates an eastward-drift. Consequently, an application to the Earth's magnetic field is not possible.

Figure 14 shows the drawing of the qualitative profile of the turbulence intensity assumed for another group of models which have a closer correspondence to the situation in the Earth with its rigid inner core. Table 2 is a list of the eigenvalues.

We find again the smallest eigenvalue for the A0 type field in agreement with the observations. Furthermore, we find a negative sign of $C_\Omega$ which corresponds to a westward drift. This is again in agreement with the observed magnetic field when we consider the S1 type field added to the A0 field.

$C_\Omega$ is the frequency of encircling the Earth once measured in units given by the decay time $\mu\sigma R^2$. From the value of the S1 field (Table 2) we get a migration period of about 10,000 years, since the decay time is generally believed to be of the order of 50,000 years. The former value is in good agreement with the observed westward drift of the dipol component. We should like to underline that we do not consider the weastward drift of the non-dipol component which is faster by a factor 10 or more.

This is not the place to give an exhaustive survey on the variation of the eigenvalues in dependence on the other parameters appearing in Ohms law for the mean fields (23). One point of interest is the influence of the differential rotation, since it leads us in the neighbourhood of the first ever constructed spherical dynamo model by Braginskij (1964). There, the generation of the toroidal from the poloidal field was assumed to be only due to the induction action of the differential rotation ($\alpha\omega$-dynamos). Steady solutions could only be found by adding a meridional motion.

In the meantime, however, Levy (1972) and Deinzer, v. Kusserow and Stix (1974) were able to construct special models of steady $\alpha\omega$-dynamos without meridional motions. Rädler (1977) found steady dynamos by taking into account the full $\alpha$-effect according to Ohms Law (23) (i.e. for the production of the toroidal from the poloidal field also) and differential rotation. His results suggest the Earth being more an $\alpha^2$-dynamo than an $\alpha\omega$-dynamo, if a differential rotation of 0.2° per year is taken into account as derived from the westward drift of the geomagnetic field.

### 3.3. *Estimation for the Characteristic Parameters of the Earth's Core*

We will, finally, derive an estimation of the parameters involved in our problem. Self excitation will appear if a condition of the form

$$C = \mu\sigma_T \alpha_0 R \gtrapprox 5 \tag{33}$$

is fulfilled. The number 5 gives roughly the correct value as can be taken from (7) and Tables 1 and 2. We consider the high conductivity limit, i.e.

$$\tau_{\mathrm{cor}} \ll \mu\sigma\lambda_{\mathrm{cor}}^2 , \tag{34}$$

where $\alpha$ and $\beta$ are given by

$$\alpha = -\tfrac{1}{2}\tau_{\mathrm{cor}}\overline{\boldsymbol{u}'\cdot \mathrm{curl}\,\boldsymbol{u}'}, \qquad \beta = \tfrac{1}{3}\overline{u'^2}\tau_{\mathrm{cor}} \tag{35}$$

(c.f., e.g., Steenbeck and Krause, 1969a) and, generally,

$$\mu\sigma\beta \gg 1. \tag{36}$$

From (20) and (35) we find

$$\mu\sigma_T = \frac{\mu\sigma}{1+\mu\sigma\beta} \approx \frac{1}{\beta} = \frac{3}{\tau_{\mathrm{cor}}\overline{\boldsymbol{u}'^2}}. \tag{37}$$

The helicity in the Earth's core is due to the Coriolis forces and, therefore, a proportionality of $\overline{\boldsymbol{u}' \cdot \operatorname{curl} \boldsymbol{u}'}$ to $(\tau_{\text{cor}}\,\omega)$ has to be expected, if $\omega$ denotes the angular velocity. For a single motion $\boldsymbol{u}' \cdot \operatorname{curl} \boldsymbol{u}'$ is of the order $\overline{u'^2}/\lambda_{\text{cor}}$. In the average, however, the imbalance of right-handed and left-handed helical motions is due to the gradient of the turbulence intensity. Consequently, an additional factor $\lambda_{\text{cor}}/R$ will appear. In this way we find the estimation

$$\overline{\boldsymbol{u}' \cdot \operatorname{curl} \boldsymbol{u}'} \approx (\omega\, \tau_{\text{cor}}) \frac{\overline{u'^2}}{R}, \tag{38}$$

and with (35) and (36)

$$C = \mu \sigma_T \alpha_0 R \approx (\omega\, \tau_{\text{cor}}). \tag{39}$$

According to (33) self excitation will appear for a sufficiently large correlation time. Of importance is to note that the amount of the turbulent velocity does not seem not to be the crucial quantity.

A paper of Pekeris, Accad and Shkoller (1973) is worth to mention in this connection. These authors proved a steady (i.e. $\tau_{\text{cor}} = \infty$) cellular motion in a conducting sphere to show dynamo excitation. The motions are of Beltrami type, i.e. $\operatorname{curl} u = \lambda u$, thus having obviously helicity. Self excitation is found to exist, if a certain parameter exceeds a value of about 30.

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