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Statistical Theory of Electromagnetic Induction in Thin Sheets

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Abstract. The statistical approach to electromagnetic induction in inhomogeneous media, developed in an earlier paper for the case of an infinite medium, is extended to include the induction in very thin plane sheets where a simple boundary condition can be used to match the fields at the sheet. This condition is reformulated for the case of a stochastic conductivity distribution $\sigma=\sigma_0+\sigma_1(x,z)$. The solution of the linearized problem of the stochastic fields (index 1) is given. Then, the surface field correlation tensor $K_{ij}=\overline{B_i B_j}$ is formed. Because of the two-dimensionality of the problem, its components are linearly dependent. Use of one to them, K_{xx}, provides an equation for the mean square (ms) amplitude of the integrated stochastic conductivity $\tau_1(x)$, which can be solved uniquely. The result is a representation of the rms conductivity $(\overline{\tau_1^2})^{1/2}$ through the spatial magnetic power spectrum K_{xx}, the global field $\mathbf{B}_0$, and the mean sheet conductivity τ_0. Since these are assumed to be known, and K_{xx} is a measurable quantity, the rms deviation of the sheet conductivity from τ_0, constant over the corresponding correlation length L, can be determined from the solution. Thus the method provides a solution of the stochastic inverse induction problem in thin sheets. A short discussion is added. In the Appendix a formula for the case of a Gaussian power spectrum K_{xx} is given.

Key words: Geomagnetic deep soundings – Magnetotellurics – Electromagnetic induction – Thin sheets – Inverse problem – Inhomogeneous media

1. Introduction

Changes in the earth's conductivity distribution with depth and in the horizontal space directions often produce very complicated spatial variations of the geomagnetic induction field at the earth's surface (cf. e.g. Schmucker, 1970). In general, the conductivity in the earth's mantle and crust does by no means reflect a smooth and everywhere slowly variable dependence on space but can be considered to be distributed by chance. In the case when this distribution is homogeneous, a mean global conductivity model of the kind used in MTS (magnetotelluric sounding) is appropriate. On the other hand, if the distribution is inhomogeneous, the treatment of the induction problem becomes extremely difficult. To overcome the difficulty, several theoretical methods have been proposed in the past, partly based on calculations of more or less refined conductivity models of the earth (cf. e.g. Hobbs, 1975), partly triggered by the idea of a so-called inverse method. The to our feeling most successful method of the latter kind has been proposed by Weidelt (1972).

All these methods are based on a deterministic view of the whole induction problem and are therefore restricted in two directions: (i) They depend very strongly on the exact knowledge of the field distribution at the earth's surface resp. within the space volume under consideration; this knowledge can be achieved only approximately, for at least the finite spacing of the measuring points along a profile sets a natural limit. (ii) The stochastic distribution of the conductivity is not taken into account; thus only a distinct mean or global conductivity structure can be determined without any hint concerning its real validity or estimate of its accuracy. As for an example we refer to the excellent paper of Weidelt (1972) where his inversion method is applied to an impedance curve previously explained by Fournier. Both interpretations coincide in the mean course of the dependence of the conductivity on depth. There are however significant differences in Fournier's model of a stratified earth and in Weidelt's smooth conductivity curve, and there is no possibility to decide whether the direct inversion method or the stratified model gives the correct conductivity trend. Intuitively one would feel that the correctness depends on the depth itself. One would hence be interested in a measure as e.g. a rms deviation of the conductivity value that has been determined by any conventional method at every space point. Another example is the so-called Northern German conductivity anomaly which in the past has been explained satisfactorily as a deep anomaly and/or a surface anomaly, missing to our knowledge any measure of the accuracy of one or the other model.

We believe that such a measure can be found if the electromagnetic induction problem is reconsidered to include the real stochastic distribution of the conductivity within the earth. Actually, any such conductivity distribution contributes a stochastic part to the electromagnetic induction field measured at the earth's surface via the induction process. Making use of this contribution, a statistical method has been proposed recently (Treumann, 1973; Schäfer and Treumann, 1975; Treumann and Schäfer, 1975) applicable to an infinite medium with the field source embedded in it, showing the possibility of a determination of the ms deviation of the conductivity from its global value at any measuring point by use of a measurable quantity, the magnetic field correlation tensor. Such a structure is, however, far from application to the earth's interior and has been selected primarily because of its mathematical simplicity, and to demonstrate the method (application to ionospheric or magnetospheric conditions seems to be more natural and will be considered in future).

The present paper extends these calculations to another model: that of a thin nonuniformly conducting plane sheet. Since its treatment is relatively simple, electromagnetic induction in thin sheets has experienced wide application in geomagnetic induction theory (cf. e.g. Ashour, 1971; Weidelt, 1971). Here, the model of a thin sheet will be considered for the case, when the stochastic distribution of the conductivity comes into account. In that case, as will be shown below, by use of an appropriate averaging procedure with respect to the scale length L of the conductivity fluctuation, a boundary condition for the secondary stochastic fields is obtained. Subsequently, the stochastic field is calculated up to a first-order approximation, and the second-order field-correlation tensor is constructed. In the fourth section of the present paper, this tensor will be used to obtain a representation of the spatial power spectrum or auto-correlation function of the stochastic part of the sheet conductivity, from which it is possible to determine the mean squared (ms) deviation of the conductivity resp. its rms amplitude from the mean value τ_0. In Section 5, using the representation of the ms conductivity amplitude, the nonlinear contribution of the fluctuating conductivity to the thin sheet boundary condition will be calculated. This leads to the definition of an effective sheet conductivity which is a very complicated function of space and nonlinear in the components of the average field. This suggests that in the presence of a stochastic conductivity distribution in the sheet the surface distribution of the global field becomes extremely complicated, and use of a simple sheet conductivity model is forbidden. Section 6 presents a short discussion of the theoretical results. An application of the derived formulae to a Gaussian power spectrum of the horizontal induction field is given in the Appendix.

2. Formulation of the Problem

Following the program outlined in the Introduction we assume that the conductivity $\sigma(x, z)$ of the sheet is the sum of a *mean* conductivity $\sigma_0(x, z)$ and a *stochastic* conductivity $\sigma_1(x, z)$ (Fig. 1) according to

$$\sigma(x, z)=\sigma_0(x, z)+\sigma_1(x, z),$$

and $\sigma_1(x, z)$ is assumed to vary on a much shorter horizontal scale than $\sigma_0(x, z)$ so that $\sigma_0(x, z)$ can be considered laterally constant when compared with the variation of $\sigma_1(x, z)$. According to the 2 parts of the conductivity any outer inducing primary field leads to the appearance of two parts of the induced secondary field, – mean and stochastic part, respectively –, so that all the field components can be split into mean and stochastic parts distinguished by the indices 0, 1. Further, for simplicity, the sheet is assumed as plane, thin, and two-dimensional ($-\infty \leqq x, y \leqq \infty$).

After integrating with respect to the vertical coordinate z the whole induction process is contained in the equations

$$B_x = -\partial A_y/\partial z = (\mathrm{i}\omega)^{-1}\, \partial E_y/\partial z, \tag{2.1}$$

$$B_z = \partial A_y/\partial x = -(\mathrm{i}\omega)^{-1}\, \partial E_y/\partial x, \tag{2.2}$$

$$j_y = \mu_0^{-1}[B_x(x, 0) - B_x(x, -0)] = \tau(x)\, E_y(x, 0), \tag{2.3}$$

where B_x, B_z and E_y are the magnetic and electric field components, respectively,

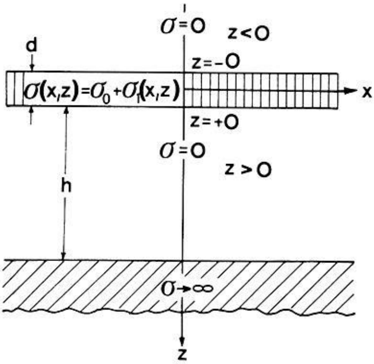

Fig. 1. The model of the thin sheet under consideration

and j_y is the sheet current density. From Equation (2.3) a simple boundary condition for the only component $A_y = -(\mathrm{i}\,\omega)^{-1}\, E_y$ of the magnetic vector potential at $z = \pm 0$ can be derived (e.g. Price, 1949; Weidelt, 1971):

$$\mathrm{i}\,\omega\,\mu_0\,\tau(x)\,A_y(x,0) = (\partial A_y/\partial z)\,|_{z=-0}^{z=+0}. \tag{2.4}$$

Further we have

$$\partial B_x/\partial z = \partial B_z/\partial x \tag{2.5}$$

everywhere outside of the sheet, and $\tau(x) = \int_{-d/2}^{d/2} \sigma(x,z)\,\mathrm{d}z$, the integrated sheet conductivity.

From Equations (2.1), (2.2) and (2.5) the two-dimensional Laplace-equation is derived for A_y outside the sheet:

$$\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} = 0, \qquad z \gtrless 0. \tag{2.6}$$

Together with Equation (2.4) it forms the basis of the calculation. Introducing $A_y(x,z) = A_0(x,z) + A_1(x,z)$, $\tau(x) = \tau_0(x) + \tau_1(x)$, where $\mathbf{B}_{0,1} = \nabla \times \mathbf{A}_{0,1}$, $\tau_{0,1} = \int_{-d/2}^{d/2} \sigma_{0,1}\,\mathrm{d}z$, into Equations (2.4) and (2.6) and averaging with respect to the scale length of the sheet conductivity fluctuation $\tau_1(x)$, L, all terms linear in $\tau_1(x)$ and $A_1(x)$ average out because of the assumption $\bar{\tau}_1 = \bar{A}_1 = 0$, and we get the boundary condition

$$\mathrm{i}\,\omega\,\mu_0\,\tau_0(x)\,A_0(x,0) + \mathrm{i}\,\omega\,\mu_0\,\overline{\tau_1(x)\,A_1(x,0)} = \left.\frac{\partial A_0(x,z)}{\partial z}\right|_{z=-0}^{z=0} \tag{2.7}$$

for the mean vector potential $A_0 \equiv \bar{A}_y$. The bar indicates averaging in the above sense.

Subtracting Equation (2.7) from (2.4) we receive the boundary condition for the vector potential of the stochastic fields, $A_1(x, z)$:

$$\mathrm{i}\omega\mu_0\tau_0(x)A_1(x,0)+\mathrm{i}\omega\mu_0[\tau_1(x)A_1(x,0)-\overline{\tau_1(x)A_1(x,0)}]$$
$$=\frac{\partial A_1(x,z)}{\partial z}\Bigg|_{z=-0}^{z=+0}-\mathrm{i}\omega\mu_0\tau_1(x)A_0(x,0). \tag{2.8}$$

Finally, Equation (2.6) is transformed by the same procedure into

$$\frac{\partial^2 A_{0,1}}{\partial x^2}+\frac{\partial^2 A_{0,1}}{\partial z^2}=0, \qquad (z \neq 0). \tag{2.9}$$

Equation (2.9) has to be solved for both the mean and stochastic fields, and the conditions (2.7), (2.8) have to be imposed subsequently to match the fields above and below the sheet at $z=0$. These conditions are, however, essentially nonlinear in the fluctuating fields. To proceed further, we therefore introduce the linearizing assumption $\max|\tau_1(x)/\tau_0(x)| \ll 1$. Since $A_1(x,z)$ is of first order in this parameter, $\tau_1 A_1$ becomes small of second order and will be neglected in a first-order calculation of $A_1(x,z)$. Moreover, though $\tau_0(x)$, $A_0(x,z)$ are in general (smooth) functions of position, they will be considered constant in the calculation of $A_1(x,z)$ because of their much weaker dependence on space over the characteristic length L of the variation of $\tau_1(x,z)$ when compared with the latter. Hence, τ_0 in our model is any given *constant* mean sheet conductivity, and $A_0(x,z)$ is the vector potential of the mean field which is slowly variable in comparison with $A_1(x,z)$ and which is induced by an arbitrary outer source field, situated in the half-space above the sheet. But $A_1(x,z)$ depends strongly on space, firstly through the space dependence of $\tau_1(x)$, but secondly through that of $A_0(x,z)$ (and in general for other average sheet conductivity models also $\tau_0(x)$). Taking the average of any second order quantity, as for instance $\tau_1(x)A_1(x,z)$ or $A_1(x,z)A_1(x,z)$, $\tau_1(x)\tau_1(x)$, over the "fast" horizontal scale-length L, the short-scale variation of the quantity under consideration is eliminated. What remains is a "slowly" variable quantity (variable on a scale larger than L and comparable with that of $\tau_0(x)$, $A_0(x,z)$), in the expression of which the space dependence of $A_0(x,z)$ (and eventually $\tau_0(x)$) has to be taken into account.

After these introductory remarks we are in the position to select our model (Fig. 1). For simplicity we let $\tau_0=\text{const}$ but include an infinitely conducting half-space at depth $z=h$ below the sheet. Further, in accordance with the above assumptions we drop the term $\overline{\tau_1(x)A_1(x,0)}$ in Equation (2.7). Then the solution for the mean field over a homogeneous sheet becomes the familiar expression

$$A_0(x,z)=\int_{-\infty}^{\infty}\phi(s)\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s\begin{cases}\mathrm{e}^{-|s|z}+\mathrm{e}^{|s|z}\{1-(1+2\mathrm{i}|s|/\omega\mu_0\tau_0)\\ \cdot\exp(-2|s|h)\}/m_0(s) \quad \text{for } z\leqq 0\\ \\ 2\mathrm{e}^{-|s|h}(2\mathrm{i}|s|\omega\mu_0\tau_0)\\ \cdot\sinh\{|s|(h-z)\}/m_0(s) \quad \text{for } h\geqq z\geqq 0\end{cases} \tag{2.10}$$

where $\phi(s)$ is the Fourier-transform of the vector potential of the source field above the sheet, and $m_0(s)=1-2\mathrm{i}|s|/\omega\mu_0\tau_0-\exp(-2|s|h)$. Below, the vector potential A_0 is to be considered constant. The discussion presented above enables us, however, to include the slow spatial variation of A_0, given e.g. by Equation (2.10) for the simple model of constant τ_0, in the final result. The case of a homogeneous inducing primary field is included in Equation (2.10) with $\phi(s) = B_0\,\delta(s)/|s|$, where B_0 is the constant field amplitude.

3. The Stochastic Field

Since there is neither a mean conductivity outside of the sheet, nor are there any fluctuations of the conductivity, the spatially fluctuating first-order part of the induction field possesses no sources above and below the thin sheet. Consequently, the solution of Equation (2.9) for $A_1(x, z)$ outside the sheet can be written as

$$A_1(x,z)=\int_{-\infty}^{+\infty} c(s)\,\mathrm{e}^{\mathrm{i}sx-|s|h}\,\mathrm{d}s\begin{cases}\sinh(|s|\,h)\exp|s|\,z & \text{for } z\leqq 0\\ \sinh|s|(h-z) & \text{for } h\geqq z\geqq 0\end{cases}. \tag{3.1}$$

As has been supposed, this field is of first-order in the parameter $|\tau_1/\tau_0|$, so that Equation (2.8) in a first-order approximation can be linearized, neglecting the second term on the LHS to produce the linear but inhomogeneous boundary condition

$$\mathrm{i}\omega\mu_0\tau_0A_1(x,0)-(\partial A_1(x,z)/\partial z)|_{z=-0}^{z=0}=-\mathrm{i}\omega\mu_0\tau_1(x)\,A_0. \tag{3.2}$$

Here τ_0 and A_0 are constants with respect to the lateral variation of $A_1(x, 0)$ and $\tau_1(x)$. It is thus a simple matter to solve Equation (3.2) with the vector potential field of Equation (3.1) by applying the Fourier transform with respect to x, assuming the existence of

$$\eta(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tau_1(x)\,\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x. \tag{3.3}$$

Now, the solution of Equation (3.2) for the so far unknown function $c(s)$ can be given as

$$c(s)=-\frac{\eta(s)}{m_1(s)}\,\frac{A_0}{\tau_0}, \tag{3.4}$$

where $m_1(s)=1-\exp(-2\,|s|\,h)[1+\mathrm{i}\,|s|/\omega\mu_0\tau_0]$.

The vector potential of the first order stochastic field produced by the spatially fluctuating sheet conductivity reads

$$A_1(x,z)=-A_0\int_{-\infty}^{\infty}\frac{\mathrm{d}s\,\mathrm{e}^{\mathrm{i}sx-|s|h}}{m_1(s)}\,\frac{\eta(s)}{\tau_0}\begin{cases}\sinh(|s|\,h)\exp(|s|\,z) & \text{for } z\leqq 0,\\ \sinh|s|(h-z) & \text{for } h\geqq z\geqq 0,\end{cases} \tag{3.5}$$

and the magnetic field components in the region $z\leqq 0$ resulting from it are

$$B_{1x}(x,z)=A_0\int_{-\infty}^{\infty}\frac{|s|\,\eta(s)}{m_1(s)\,\tau_0}\,\mathrm{e}^{\mathrm{i}sx+|s|(z-h)}\,\mathrm{d}s,\qquad z\leqq 0,$$

$$B_{1z}(x,z)=-A_0\,\mathrm{i}\int_{-\infty}^{\infty}\frac{s\,\eta(s)}{m_1(s)\,\tau_0}\,\mathrm{e}^{\mathrm{i}sx+|s|(z-h)}\,\mathrm{d}s. \tag{3.6}$$

Similarly the fluctuating electric field $E_{1y}(x, z)$ can be found from Equation (3.5), multiplying $A_1(x, z)$ by $-i\omega$. These field components will be used in the next section for the construction of the field correlation tensor. On the other hand, using $E_{1y}(x, z)$ and $B_{1x}(x, z)$, one could construct an appropriate surface impedance fluctuation resulting from the statistical conductivity distribution within the sheet.

We will only use the field components above the sheet in the nonconducting air space for in practice there are no measurements available from the resistosphere below the conducting sheet at the earth's surface but above the conductosphere at $z=h$. On the other hand, if the thin sheet represents the conducting flat ionosphere, and the conductosphere is the conducting earth, both the fields above and below the sheet can be used for probing the ionospheric conductivity by means of satellite and ground based measurements of the magnetic variation field. In that case, however, it would be more appropriate to take into account the anisotropic conductivity tensor of the ionospheric plasma. An extension of the present theory to that case seems to be straightforward.

4. The Correlation Tensor

As in our previous paper (Schäfer and Treumann, 1975) we now come to construct the second-order magnetic field correlation tensor

$$K_{ij}(\xi, z=z_0)=\overline{B_i(x, z=z_0)\,B_j(x+\xi, z=z_0)}=B_{0i}\,B_{0j}+\overline{B_{1i}\,B_{1j}}, \tag{4.1}$$

$i,j=x, z$. Here $\mathbf{B}_0=\nabla\times\mathbf{A}_0$ is the mean magnetic variation field, the components of $\mathbf{B}_1$ are given in Equation (3.6), and the averaging proceeds along any profile $z=z_0$ parallel to the sheet surface. This profile is for instance given by the plane $z=-0$ itself. From the above definition of the correlation tensor it is clear that $K_{ij}(\xi, 0)$ represents a measurable quantity. In practice one has to take the full field components of the magnetic variation field $\mathbf{B}(\omega)$ at fixed frequency ω along any given measuring profile on the earth's surface, where ξ is the (possibly constant) distance between the magnetograph stations, and to perform the cross- and autocorrelation functions between all the field components of neighbouring stations spaced by ξ, to sum up all the contributions along a part of the profile and to divide by the length L of that part. The full set of the correlations calculated in this way provides the value of K_{ij} at ξ. Next this procedure has to be repeated with spacings 2ξ, $3\xi, \ldots$ to get the functional dependence of $K_{ij}(\xi)$. The so far unspecified correlation length L is once restricted by the length of the profile, on the other hand it is required to be comparable with the characteristic variation length of the mean conductivity resp. the mean variation field. Hence, L can be determined from a consideration of the variation field behaviour along the measuring profile by attributing the mean behaviour of the field to the mean conductivity structure.

Despite the practical problem of the determination of the variation field correlation tensor, we are now able to calculate its components theoretically.

First we observe the following symmetries

$$m_1(s)=m_1(-s), \qquad \eta(-s)=\eta^*(s). \tag{4.2}$$

The second symmetry follows from the requirement of reality of the stochastic conductivity.

At the sheet surface $z=-0$ the magnetic field components are given by

$$\left.\begin{matrix} B_{1x}(x,0) \\ B_{1z}(x,0) \end{matrix}\right\} = A_0 \int_{-\infty}^{\infty} \mathrm{d}s \left\{\begin{matrix} |s| \\ -\mathrm{i}\,s \end{matrix}\right\} \frac{\eta(s)}{m_1(s)\,\tau_0} \sinh(|s|\,h)\,\mathrm{e}^{\mathrm{i}sx-|s|h}. \tag{4.3}$$

Inserting these expressions into Equation (4.1) and carrying out the average with respect to x, one finds for instance

$$\overline{B_{1x}B_{1x}}(\xi) = \lim_{L\to\infty} \frac{2\pi A_0^2}{L\tau_0^2} \int_{-L/2}^{L/2} \mathrm{d}s\, \frac{|s|^2\,|\eta(s)|^2}{m_1^2(s)} \sinh^2(|s|\,h)\,\mathrm{e}^{-\mathrm{i}s\xi-2|s|h}. \tag{4.4}$$

Similarly the remaining components $\overline{B_{1z}B_{1z}}(\xi)$, $\overline{B_{1x}B_{1z}}(\xi)$, $\overline{B_{1z}B_{1x}}(\xi)$ are easily calculated. Additionally, taking into account that

$$B_{0i}B_{0j} = \lim_{L\to\infty} \frac{B_{0i}B_{0j}}{L} \int_{-L/2}^{L/2} \mathrm{d}s\,\delta(s)\,\mathrm{e}^{-\mathrm{i}s\xi}, \tag{4.5}$$

one gets for the Fourier-transform of the full correlation tensor

$$K_{ij}(s) = \lim_{L\to\infty} \frac{1}{2\pi L} \int_{-L/2}^{L/2} K_{ij}(\xi)\,\mathrm{e}^{\mathrm{i}s\xi}\,\mathrm{d}\xi \tag{4.6}$$

the following representation

$$K_{ij}(s) = B_{0i}B_{0j}\,\delta(s) + 2\pi \frac{A_0^2|\eta(s)|^2}{m_1^2(s)} |s|^2\,\mathrm{e}^{-2|s|h} \sinh^2(|s|\,h)\,T_{ij}(s), \tag{4.7}$$

where

$$T_{ij}(s) = \begin{pmatrix} 1 & \mathrm{i}\,\mathrm{sgn}\,s \\ -\mathrm{i}\,\mathrm{sgn}\,s & 1 \end{pmatrix}. \tag{4.8}$$

From this expression it is easily observed that det $T_{ij}=0$. Thus T_{ij} has no inverse, and Equation (4.7) cannot be solved for $|\eta(s)|^2$. This is however simply a consequence of the supposed two-dimensionality of the problem. In that case the vanishing determinant of T_{ij} reflects the fact that there exists a linear relationship between the transformed field components Equation (4.3) at the sheet surface. Indeed, from Equation (4.3) we have the relationship

$$B_{1z}(s) = a(s)\,B_{1x}(s), \qquad a(s) = -\mathrm{i}\,\mathrm{sgn}\,s. \tag{4.9}$$

This suggests that $B_{1z}(x,0)$ can be represented as the Faltung integral of the 2 functions $a(x)$ and $B_{1x}(x,0)$ according to

$$B_{1z}(x,0) = \int_{-\infty}^{\infty} a(x-\zeta)\,B_{1x}(\zeta,0)\,\mathrm{d}\zeta, \tag{4.10}$$

where $a(x-\zeta)$ is the inverse Fourier-transform of $a(s)$. Taking advantage of the Dirac identity $\delta_\pm(x)=(1/2)\{\delta(x)\pm(\mathrm{i}/\pi)\,\mathrm{P}(1/x)\}$, where P indicates the principalvalue, $a(x-\zeta)$ can be found to be given by $a(x-\zeta)=(1/\pi)\,\mathrm{P}[1/(x-\zeta)]$. The vertical field component $B_{1z}(x,0)$ is therefore the Hilbert-transform of the horizontal field $B_{1x}(x,0)$, and vice versa. Thus we have recovered the well-known relationships (Weaver, 1964)

$$B_{1z}(x,0)=(-1/\pi)\,\mathrm{P}\int_{-\infty}^{\infty} B_{1x}(\zeta,0)\,\mathrm{d}\zeta/(\zeta-x),$$

$$B_{1x}(x,0)=(1/\pi)\,\mathrm{P}\int_{-\infty}^{\infty} B_{1z}(\zeta,0)\,\mathrm{d}\zeta/(\zeta-x). \tag{4.11}$$

Because of these relations between the field components $B_{1z}(x,0)$, $B_{1x}(x,0)$, it suffices to restrict oneself on one of the four components of $K_{ij}(s)$ for a unique determination of $|\eta(s)|^2$. Without violating generality we choose the (xx)-component

$$\frac{1}{2\pi}\,K_{xx}(s)=\frac{1}{2\pi}\,B_{0x}^2\,\delta(s)+\frac{A_0^2}{\tau_0^2}\,\frac{|\eta(s)|^2}{m_1^2(s)}\,|s|^2. \tag{4.12}$$

Taking the real part of the LHS and rearranging this expression, the spatial power spectrum

$$|\eta(s)|^2=(\tau_0\,B_{0x}/2\pi\,|s|\,A_0)^2\,|\,\mathrm{Re}\{m_1^2(s)[\delta(s)-K_{xx}(s)/(B_{0x}^2)]\}| \tag{4.13}$$

of the fluctuating conductivity $\tau_1(x)$ is obtained.

Equation (4.13) represents the main result of the present paper. It connects the spatial power spectrum of the stochastic conductivity distribution with the auto-correlation function $K_{xx}(s)$ of the horizontal induction field at the surface of the sheet. The latter, however, is a measurable quantity. Thus, $|\eta(s)|^2$ can be determined uniquely from suitable measurements along the sheet surface, if only a mean conductivity of the sheet τ_0 has been determined previously, for which the appropriate global field B_{0x}, needed in Equation (4.13), can be calculated for any primary inducing field from Equation (2.10), and A_0 is given by (2.10) itself. It has been mentioned previously that at this stage the dependence of A_0, B_{0x} (and eventually τ_0) on space becomes appreciable, so that $|\eta(s)|^2$ becomes space-dependent on a scale larger than L both via $K_{xx}(s)$ and A_0, B_{0x} (and eventually τ_0).

The remaining last step is from the spectrum $|\eta(s)|^2$ to the ms conductivity $\overline{\tau_1^2}$. This can easily be done through integrating

$$\tau_1^2(x)=\int_{-\infty}^{\infty} \eta(s)\,\eta(s')\,\mathrm{e}^{\mathrm{i}(s+s')x}\,\mathrm{d}s\,\mathrm{d}s' \tag{4.14}$$

with respect to x over the appropriate scale length L of the conductivity fluctuation:

$$\overline{\tau_1^2}=2\pi\int_{-\infty}^{\infty} |\eta(s)|^2\,\mathrm{d}s. \tag{4.15}$$

Inserting $|\eta(s)|^2$ from Equation (4.13) into this expression, we get finally

$$\overline{\tau_1^2}=\frac{1}{\pi}\,(\tau_0 B_{0x}/A_0)^2\int_0^{\infty}\frac{\mathrm{d}s}{s^2}\,|\mathrm{Re}\{m_1^2(s)[\delta(s)-K_{xx}(s)/(B_{0x}^2)]\}|. \tag{4.16}$$

Equation (4.16) represents a measure of the validity of the mean conductivity model at any measuring point along the profile. It depends firstly on τ_0, but secondly on the auto-correlation function of the horizontal field K_{xx}, and can therefore be calculated if only a mean model of the sheet conductivity can be extracted and the auto-correlation function K_{xx} can be determined from the measured field components.

5. Influence on the Boundary Condition

The presence of a fluctuating part of the conductivity within the sheet changes the boundary condition in second order due to the second term on the LHS of Equation (2.7). Knowing as the solution for $A_1(x, 0)$ (Equation (3.5)) as $|\eta(s)|^2$ (Equation (4.13)) this term can be expressed explicitly through the average fields and the auto-correlation function K_{xx}:

$$\overline{\tau_1(x) A_1(x,0)} = \frac{-2\pi A_0}{\tau_0} \int_{-\infty}^{\infty} \mathrm{d}s \frac{|\eta(s)|^2}{m_1(s)}$$
$$= -\frac{\tau_0 B_{0x}^2}{A_0} \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{|s|^2 m_1(s)} |\mathrm{Re}\{m_1^2(s)[\delta(s) - K_{xx}(s)/(B_{0x}^2]\}|. \tag{5.1}$$

On the other hand, introducing this into Equation (2.7) we get

$$\mathrm{i}\omega\mu_0\tau_0(x)\left\{1 - 2\pi \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{m_1(s)} \frac{|\eta(s)|^2}{\tau_0^2(x)}\right\} A_0(x,0) = \left.\frac{\partial A_0(x,z)}{\partial z}\right|_{z=-0}^{z=0} \tag{5.2}$$

as the new boundary condition for the average field. Here the slow space dependence of $\tau_0(x)$ has been indicated explicitly. Comparison with Equation (2.4) shows that the presence of the stochastic part of the conductivity changes the sheet conductivity $\tau_0(x)$ to the effective onductivity

$$\tau_{\mathrm{eff}}(x) = \tau_0(x)\left\{1 - 2\pi \int_{-\infty}^{\infty} \mathrm{d}s \frac{|\eta(s)|^2}{m_1(s)\tau_0^2(x)}\right\}$$
$$= \tau_0(x)\left\{1 - \frac{B_{0x}^2(x)}{A_0^2(x)} \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{|s|^2 m_1(s,x)} \left|\mathrm{Re}\, m_1^2(s,x)\left[\delta(s) - \frac{K_{xx}(s,x)}{B_{0x}^2(x)}\right]\right|\right\}. \tag{5.3}$$

From this expression it becomes obvious that when $|\tau_1/\tau_0| \ll 1$ the correction term becomes of second order in this parameter, a posteriori justifying our initial assumption of neglecting this term in a first-order calculation of the fluctuating fields. Equation (5.3) suggests a decrease of the effective conductivity when compared with τ_0 due to the stochastic part. Thus, neglecting the latter, an effective conductivity is obtained from a matching procedure of theoretical and measured field values which is lower than the real conductivity of the sheet.

In the case of a non-homogeneous conductivity spectrum $|\eta(s)|^2$, the effective conductivity appearing in Equation (5.2) becomes a very complicated function of space which is moreover nonlinear in $A_0(x)$. Hence, the boundary condition for the average vector potential is in general nonlinear up to second order in $|\tau_1/\tau_0|$. Its solution seems to be impossible though a suitable perturbation technique can be applied. In a first-order calculation the solution is, however, not required since Equation (4.16) opens the possibility to determine a first order rms-correction to the initially assumed conductivity τ_0 of the sheet.

6. Conclusions and Summary

In the present paper we have formulated and solved the statistical induction problem for a thin sheet having a stochastic conductivity distribution $\tau_1(x)$ superposed over an average sheet conductivity τ_0. We have used the boundary condition for a thin sheet initially derived by Price (1949) and reformulated by Weidelt (1971). Our solution of the problem, assuming the mean field distribution to be known, enabled us to construct the magnetic field correlation tensor at the surface of the sheet. We believe the latter to be a measurable quantity appropriate for measuring conditions of the geomagnetic variation field on long profiles at the earth's surface. In our two-dimensional case there exists a unique relationship between the 2 components $B_{1x}(x,0)$, $B_{1z}(x,0)$ at the sheet surface whose representation could be written in the form of a Hilbert-transform. Because of this relationship we were led to the restriction on only one arbitrary component of the correlation tensor, from which the horizontal power spectrum $|\eta(s)|^2$ of the conductivity fluctuation could be derived. Subsequently $|\eta(s)|^2$ has been used to express the ms conductivity $\overline{\tau_1^2}$, averaged over the horizontal scale length L of the fluctuation, through the parameters of the average model of the sheet conductivity and the auto-correlation function of the horizontal magnetic variation field. We believe that our method provides an effective statistical inverse mechanism to determine the rms conductivity deviation of the sheet.

Two points, however, deserve further discussion. The first is concerned with the correlation length L; the second belongs to the informational content of the final result, $\overline{\tau_1^2}$.

The correlation length or characteristic scale length of the conductivity fluctuation, a terminus which has been used throughout the present paper, can be defined as the shortest length, over which the conductivity fluctuation averages out along the profile. It is hence the solution of the equation for $L_{\min}$:

$$\int_{-L_{\min}/2}^{L_{\min}/2} \tau_1(x)\,dx = 0.$$

Since $\tau_1(x)$ is a priori unknown, this equation is of little use for the determination of $L = L_{\min}$. Instead one has to look for practical possibilities of its determination. Two of its practical limitations are the limited length of the measuring profile and the finite distance between the measuring points along the profile. Especially the latter is of crucial importance because it would be of worth to have many near-neighboured stations to receive high precision in the lateral distribution of the fields. L has then to be chosen as small as possible to get some intervals of length L along the profile and hence several values of $\overline{\tau_1^2}$, but simultaneously so large that a mean field can be extracted from the observations that can be brought into connection with the plate of constant conductivity τ_0.

Concerning the informational content of $\overline{\tau_1^2}$ we remark that it gives a measure of the rms conductivity fluctuation, $(\overline{\tau_1^2})^{1/2}$, constant over the length L. Since $(\overline{\tau_1^2})^{1/2}$ has both positive and negative sign, no conclusion can be drawn concerning the direction of the deviation of the conductivity from its mean value τ_0. This, however, would be a deterministic result which has been excluded at the beginning by our stochastic assumption. Instead the rms conductivity fluctuation represents a measure of the scattering width of the real sheet conductivity around τ_0. High values of $(\overline{\tau_1^2})^{1/2}$ within some of the intervals of length L signalize that here the real conductivity does appreciately deviate from τ_0. Such a situation can be taken to stimulate further considerations about the reasons for this deviation and, perhaps, to proceed up to a more realistic model and description of the conductivity distribution. For a more extended application of the statistical method proposed in the present paper it would be of great use to transform the theory to the model of a flat infinite half-space or a spherical Earth. As to the former we refer to our paper presented at Sopron, 1976 (Treumann, 1976).

Finally we conclude that the use of the concept of field correlations in electromagnetic induction theory seems to be not only new but also effective in obtaining an additional information about the conductivity of the Earth from measuring data of the geomagnetic variation field components. In the literature there exist some other quantities called geomagnetic transfer functions or induction tensors. We believe that there exists some close connection between these quantities and the correlation tensor introduced above. We were however unable to work this connection out. This will be left as a problem open for future investigations.

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A. Appendix: The ms Conductivity for a Gaussian Spectrum

Since practical measurements are not available we chose for illustration a theoretical dependence of $K_{xx}(s)$ according to a Gaussian distribution

$$K_{xx}(s)/2\pi B_{0x}^2 = \delta(s) + \beta(s/\Delta s)^2 \exp[-(s-s_0)^2/(\Delta s)^2], \tag{A.1}$$

where $\beta = (b_{1x}/B_{0x})^2$, with b_{1x} the fluctuating field amplitude, $s_0 < \omega\mu_0\tau_0$ a fixed parameter, and Δs a fixed spread of the spectrum. Inserting this into Equation (4.16), using the expression for $m_1(s)$ and carrying out the various integrations with respect to s we obtain for the ms conductivity

$$\overline{\tau_1^2} = \frac{\sqrt{\pi}}{4\Delta s}\left(\frac{\tau_0 b_{1x}}{A_0}\right)^2 \left\{1 + \pi^{-1/2}\left(\frac{\Delta s}{\omega \mu_0 \tau_0}\right)^2 \lim_{z_0' \to z_0} \frac{\partial}{\partial \gamma}\left[\int_{-z_0'}^{\infty} dz\, e^{-z^2} \ln \tanh \gamma(z+z_0) + \int_{z_0'}^{\infty} dz\, e^{-z^2} \ln \tanh \gamma(z-z_0)\right]\right\}. \tag{A.2}$$

Here $\gamma = h\Delta s$, $z_0' > z_0 = s_0/\Delta s$. In every practical problem however no such simple form is available in general, and the calculation must be carried out numerically with $K_{xx}(s)$ a (complex) measuring function which in general cannot be represented in closed form.

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Note Added in Proof. Recently the theory developed in the present paper has been extended to the inclusion of an anisotropic sheet conductivity as it is supposed for a thin ionosphere. It has been found that the ms Cowling conductivity can be determined from the measured field correlation function at the ground or above the ionosphere (R. Treumann, submitted to Gerlands Beiträge zur Geophysik).