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Thermoelastic Deformations of a Half-Space – A Green's Function Approach *

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Abstract. Thermoelastic displacements of a homogeneous half-space are calculated for a point source of harmonic temperature variations at the surface. For distances from the source which are much larger than the skin depth of the temperature wave simple far-field approximations are derived. From these, Green's functions for the strains and tilts are calculated, and the generalization for arbitrary temperature variation at the surface is performed by superposition. The results are given in the form of integrals over the temperature variation in which the Green's functions are influence functions. They strongly decay with distance from the vertical axis through the point of observation, thus showing that thermoelastic strains and tilts normally are due to very localized temperature anomalies. Numerical calculations for a simple example indicate that thermoelastic effects can seriously disturb measurements of earth tides with strainmeters and tiltmeters.

Key words: Thermoelasticity – Green's functions – Earth tides.

Introduction

Strains and tilts in the earth's crust due to earth tides are intensively studied since many years, and the number of extensometers and horizontal or vertical pendulums installed for that purpose in tunnels, mines and boreholes is still increasing. Only in recent years scientists working in this field became fully aware of the many disturbing effects that are seen in the measurements in addition to earth tides, such as strains and tilts due to air pressure variations and rain fall, or modifications of strain and tilt due to the topography of the earth's surface and the form of the underground cavity in which the measurement is made (see, e.g., Harrison (1976)). Thermal and thermoelastic strains and tilts are another disturbing effect. Of major interest are strains and tilts due to

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the diurnal temperature variation because its dominant period is close to the tidal periods in the diurnal period range. Both theoretical and experimental studies are necessary for a better understanding of the magnitude and spatial distribution of these disturbances.

This paper treats theoretically a simple case, namely a homogeneous half-space with a surface temperature distribution $T(x, y)\exp(i\omega t)$, where $T(x, y)$ is the (real) temperature amplitude depending on the horizontal coordinates x and y, ω is the circular frequency of the temperature variation, and t is time. This model is an idealized approximation of reality where the diurnal temperature variation varies over the earth's surface, depending on local soil properties, vegetation cover and ground slopes. The approach used in this paper is to calculate at first Green's functions for the displacements in the half-space due to a point source of temperature variations, i.e., it is assumed that $T(x, y)=T_0\,\delta(x)\,\delta(y)$ where $\delta(x)$ and $\delta(y)$ are delta functions. For distances from the point source which are much larger than the ω-dependent skin depth of the temperature wave, d, simple far-field expressions are derived and from these the strains and tilts. Then, the strains and tilts at depths much larger than d, corresponding to an arbitrary $T(x, y)$, are given as superposition integrals. In these, the Green's functions for strains and tilts represent influence functions which give an instructive picture of how the different parts of the surface contribute to the deformations at depth. Earlier investigations of the same model (Matuzawa, 1942; Jobert, 1960; Popov, 1960; Berger, 1975; Shirokov and Anokhina, 1976) did not include this aspect, but treated mostly the case of temperature variations which are harmonic in one of the horizontal coordinates and independent of the other.

Thermoelastic Green's Functions of a Half-Space

We consider a homogeneous isotropic half-space which occupies the region $z \geqq 0$; the z axis is pointing downwards. The thermoelastic point-source problem is solved in cylindrical coordinates r, φ, z; φ does not appear in the equations because of cylindrical symmetry. The generalization to arbitrary temperature distribution over the surface of the half-space is made in Cartesian coordinates x, y, z.

We first solve the heat-conduction equation

$$\nabla^2 T=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{1}$$

where κ is the thermal diffusivity, and the temperature $T(r, z, t)$ is measured from a constant average temperature. The solution which satisfies the boundary condition at $z=0$,

$$T(r, o, t)=T_0\bar{\delta}(r)\,e^{i\omega t},$$

where $\bar{\delta}(r)$ is the two-dimensional delta function $\delta(x)\,\delta(y)$ and T_0 has the dimension of the product temperature times area, is

$$T(r, z, t)=\frac{T_0 e^{i\omega t}}{2\pi}\int_0^\infty e^{-\gamma z} J_0(kr)\,k\,dk. \tag{2}$$

Here, $J_0(kr)$ is the Bessel function of first kind and order zero, and the radical

$$\gamma = \left(\frac{i\omega}{\kappa} + k^2\right)^{1/2}$$

has a positive real part. Equation (2) is an integral representation of a spherical temperature wave in terms of plane waves, similar to the Sommerfeld integral for a spherical elastic wave.

For later use we introduce at this point as a reference length of the thermoelastic half-space problem the skin depth of a vertically traveling plane temperature wave of circular frequency ω,

$$d = \left(\frac{2\kappa}{\omega}\right)^{1/2}, \tag{3}$$

which is defined as the depth at which the temperature amplitude is reduced to $1/e$ of its surface value.

The thermoelastic stress-strain relations of an isotropic medium are (Love, 1944, p. 108)

$$p_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - (\lambda + \tfrac{2}{3}\mu)\,\alpha\, T\delta_{ij} \tag{4}$$

where λ and μ are Lamé's constants, α is the volume coefficient of thermal expansion, p_{ij} the stress tensor, ε_{ij} the strain tensor and θ its trace, and δ_{ij} the unit tensor. Inserting (4) into the equation of static equilibrium of a deformed homogeneous medium gives the following equation for the displacement vector $\mathbf{u}$:

$$(\lambda + 2\mu)\,\mathrm{grad\,div}\,\mathbf{u} - \mu\,\mathrm{rot\,rot}\,\mathbf{u} = (\lambda + \tfrac{2}{3}\mu)\,\alpha\,\mathrm{grad}\,T. \tag{5}$$

For the special case under study T follows from (2). Equation (5) has to be solved for the boundary conditions of (a) vanishing normal and tangential stresses at the surface $z=0$ of the half-space and (b) vanishing displacements for $z \to \infty$. After writing (5) in components, one obtains two coupled inhomogeneous partial differential equations for the displacement components u in r direction and w in z direction. Inserting into these equations the integral representations

$$u = e^{i\omega t}\int_0^\infty A(z,k)\,J_1(kr)\,k\,dk, \qquad w = e^{i\omega t}\int_0^\infty B(z,k)\,J_0(kr)\,k\,dk,$$

which are similar to (2) (J_1 is the Bessel function of order one), one obtains two coupled inhomogeneous ordinary differential equations for A and B which are solved with the boundary conditions for $z=0$ and $z \to \infty$. The final results for u and w are (Green's functions):

$$u = \frac{(1+\sigma)\alpha\kappa T_0 i e^{i\omega t}}{(1-\sigma)6\pi\omega} \cdot \int_0^\infty \{[(1-2\sigma)k - 2(1-\sigma)\gamma - k(k-\gamma)z]e^{-kz} + ke^{-\gamma z}\}\,J_1(kr)\,k\,dk \tag{6}$$

$$w=\frac{(1+\sigma)\alpha\kappa T_0 i e^{i\omega t}}{(1-\sigma)6\pi\omega} \cdot \int_0^\infty \{[-2(1-\sigma)k+(1-2\sigma)\gamma-k(k-\gamma)z]e^{-kz}+\gamma e^{-\gamma z}\} J_0(kr)k\,dk. \tag{7}$$

Here, σ is Poisson's ratio. In principle these expressions (and likewise the corresponding Green's functions for the strains and tilts) can be calculated numerically. In the following section, however, only an approximate calculation of these quantities is given, corresponding to a far-field approximation. This treatment is sufficient for the calculation of thermoelastic strains and tilts at larger depths.

Far-Field Approximations

For $z>0$ the curly brackets in (6) and (7) are exponentially decaying functions of k, and only small values of k contribute essentially to the integrals for sufficiently large z. In the following we assume at first $z \gg d$ with d according to (3), i.e., the point of observation is at a depth sufficiently large compared with the skin-depth of the temperature wave. Then at $k=0$ $|\gamma z| \gg 1$, and the $e^{-\gamma z}$ terms in (6) and (7) are small compared with the e^{-kz} terms and hence can be dropped. The factors of e^{-kz} are expanded into Taylor series around $k=0$ and only terms up to first order are considered:

$$\begin{aligned}
&(1-2\sigma)k-2(1-\sigma)\gamma-k(k-\gamma)z \\
&\quad=-2(1-\sigma)\left(\frac{i\omega}{\kappa}\right)^{1/2}+\left[1-2\sigma+\left(\frac{i\omega}{\kappa}\right)^{1/2} z\right]k+O(k^2) \\
&-2(1-\sigma)k+(1-2\sigma)\gamma-k(k-\gamma)z \\
&\quad=(1-2\sigma)\left(\frac{i\omega}{\kappa}\right)^{1/2}+\left[-2(1-\sigma)+\left(\frac{i\omega}{\kappa}\right)^{1/2} z\right]k+O(k^2)
\end{aligned} \tag{8}$$

Because of the decay of e^{-kz} essential contributions to the integrals (6) and (7) come only from k values with $kz<5$. In this range the linearization in (8) is permitted if $z>50d$. As another simplification the first term in the square brackets can also be dropped. With these approximations closed-form integration of (6) and (7) is possible:

$$\left.\begin{matrix} u \\ w \end{matrix}\right\} = \frac{(1+\sigma)\alpha T_0 d}{(1-\sigma)6\pi\sqrt{2}} \frac{e^{i\left(\omega t-\frac{\pi}{4}\right)}}{(r^2+z^2)^{5/2}} \left\{\begin{matrix} r[2(1-\sigma)r^2-(1+2\sigma)z^2] \\ z[2\sigma r^2-(3-2\sigma)z^2] \end{matrix}\right. \tag{9}$$

For $z=0$ the integrands in (6) and (7) simplify. Linear approximation of the curly brackets and closed-form integration as before show that (9) applies also for $z=0$. Thus, (9) is valid as soon as the distance $R=(r^2+z^2)^{1/2}$ from the point source is larger than about $50d$, irrespective of z.

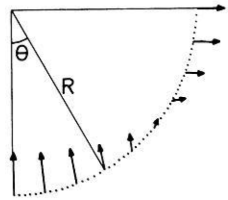

Fig. 1. Thermoelastic far-field displacements in a half-space for a point source at $r=z=0$ ($\sigma=0.25$)

In spherical coordinates R, Θ, φ the displacements are:

$$\left.\begin{matrix} u_R \\ u_\Theta \end{matrix}\right\} = \frac{5\alpha T_0 d}{12\pi\sqrt{2}} \frac{e^{i\left(\omega t-\frac{\pi}{4}\right)}}{R^2} \left\{\begin{matrix} (1-\frac{8}{3}\cos^2\Theta) \\ \frac{1}{3}\sin 2\Theta \end{matrix}\right. \tag{10}$$

$$u_\varphi = 0$$

These formulas apply in the special case $\sigma=0.25$ which we consider in the remainder of this paper. The R dependence is as expected for point sources in elastostatics. The diagram of the displacement vector in Figure 1 shows the dependence on the polar distance Θ. For $\Theta<52°$ the particles of the half-space are pulled towards the point source for half a period, whereas for $\Theta>52°$ they are pushed away. In the following half period the signs are opposite to those in Figure 1. Compared with the temperature variation in the point source the far-field displacements have a constant delay of $\pi/4$, corresponding to one eighth of a period.

For the diurnal temperature variation d is of the order of 10 cm. Hence (9) and (10) are valid approximations of the thermoelastic displacements at depths greater than a few meters.

Strains and Tilts for Arbitrary Temperature Variation

We are interested in the horizontal strain ε_{xx}, the vertical strain ε_{zz} and the tilt β_x in x direction for a point at depth $z>50d$ on the z axis of a Cartesian coordinate system (x, y, z), caused by an arbitrary temperature variation $T(x, y)\,e^{i\omega t}$ at the surface. Decomposing the horizontal radial displacement u in (9) into its x and y components, u_x and u_y, the strains and the tilt for the point source are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \qquad \beta_x = \frac{\partial w}{\partial x}.$$

From these the strains and the tilt for arbitrary temperature variation follow by superposition ($\sigma=0.25$):

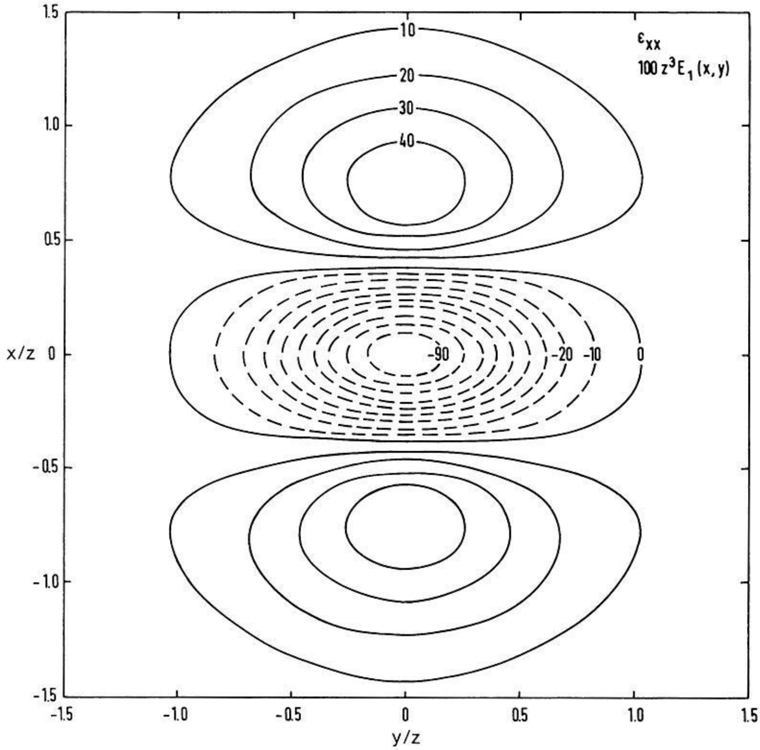

Fig. 2. Influence function $E_1(x, y)$ for the strain ε_{xx} as a function of the normalized distances x/z and y/z (z is the depth of the point of observation; its x and y coordinates are zero). Contour-line interval is 10

$$\left.\begin{matrix}\varepsilon_{xx}\\ \varepsilon_{zz}\\ \beta_x\end{matrix}\right\} = \frac{5\alpha d}{12\pi\sqrt{2}} e^{i\left(\omega t - \frac{\pi}{4}\right)} \iint\limits_{-\infty}^{+\infty} E_{\substack{1\\2\\3}}(x, y)\, T(x, y)\, dx\, dy \tag{11}$$

$$E_1(x, y) = \frac{(-2r^4 + 7r^2 z^2 - z^4)x^2 + (r^4 - z^4)y^2}{r^2(r^2 + z^2)^{7/2}}$$

$$E_2(x, y) = \frac{r^4 - 19r^2 z^2 + 10z^4}{3(r^2 + z^2)^{7/2}}$$

$$E_3(x, y) = \frac{xz(9z^2 - r^2)}{(r^2 + z^2)^{7/2}}$$

$$r^2 = x^2 + y^2.$$

The influence or weight functions E_1, E_2 and E_3 are given in Figures 2–4. They show that thermoelastic strains and tilts depend strongest on variations in temperature amplitude $T(x, y)$ directly above the point of observation in an area with linear dimensions approximately twice the depth. Figures 2–4 can easily be used for estimates of strains and tilts at strainmeters or tiltmeters at a given depth, provided that some knowledge of $T(x, y)$ is available.

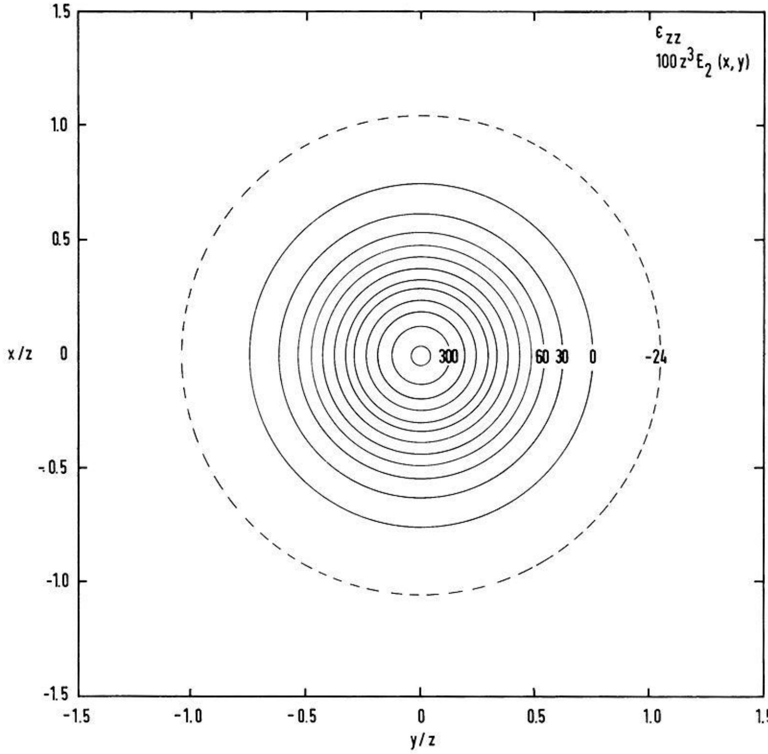

Fig. 3. Influence function $E_2(x, y)$ for the strain e_{zz}. Contour-line interval is 30

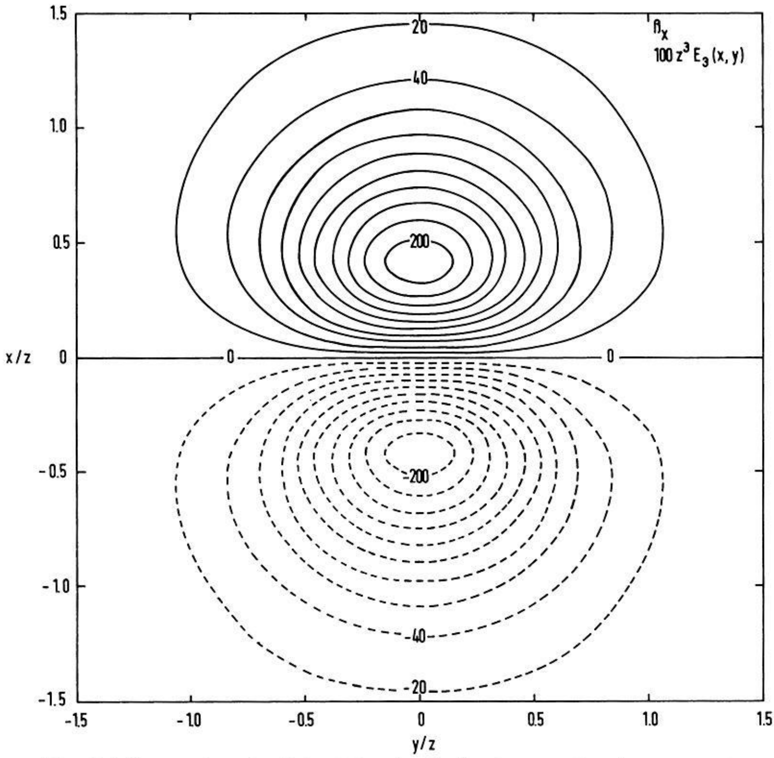

Fig. 4. Influence function $E_3(x, y)$ for the tilt β_x. Contour-line interval is 20

Example

In order to obtain an idea of the magnitude of thermoelastic strains, we investigate the case of a circular temperature anomaly of radius ρ with its center at $x=y=0$:

$$T(x,y)=\begin{cases} T_1=\text{const} & \text{for } (x^2+y^2)^{1/2}\leqq\rho \\ 0 & \text{otherwise.}\end{cases}$$

In this case (11) can be integrated analytically:

$$\left.\begin{matrix}\varepsilon_{xx}\\ \varepsilon_{zz}\end{matrix}\right\}=\left\{\begin{matrix} f\left(\frac{\rho}{z}\right) \\ g\left(\frac{\rho}{z}\right)\end{matrix}\right\}\frac{5\alpha T_1 d}{12\sqrt{2}z}e^{i\left(\omega t-\frac{\pi}{4}\right)}$$

$$f(a)=\frac{a^2(a^2-1)}{(a^2+1)^{5/2}} \qquad g(a)=\frac{2a^2(5-a^2)}{3(a^2+1)^{5/2}}.$$

Because of the symmetry properties of $E_3(x,y)$ (see Fig. 4), the tilt β_x vanishes everywhere on the z-axis.

For a numerical estimate we assume $z=10$ m, $\rho=5$ m, $T_1=1$ °C, $d=10$ cm (diurnal temperature wave), $\alpha=10^{-5}$ per °C. Then the strain amplitudes are $3.15\cdot10^{-9}$ for ε_{xx} and $1.34\cdot10^{-8}$ for ε_{zz}. These values are of the order of tidal strains. At a depth of 50 m the strain amplitudes are reduced by factors of 56 and 70, respectively.

Discussion

The numerical results given show that thermoelastic deformations at depth due to temperature anomalies at the earth's surface can be quite large. However, this can only be taken as a general warning, since the underground normally is much more complicated than the model of this paper. For instance, the wheathered layer has a strongly insulating effect, and at the same time it supports stresses only to a little extent. One can use in this case the above model, but only for the hard rock below the weathered layer. The temperature variation at the surface of the hard rock is considerably reduced compared with that at the earth's surface. As a consequence, thermoelastic strains and tilts at depth will be reduced. Another important difference between model and reality exists if there is topography on the surface of the hard rock. Even in absence of lateral temperature variations thermoelastic deformations develop in this case.

Due to their complicated nature thermoelastic strains and tilts will not only contain a diurnal but probably also a semidiurnal period. Therefore, they may not only disturb the diurnal but also the semidiurnal tides. Thermoelastic strains and tilts may also have a continuous spectrum due to temperature changes by wind, rain, cloud coverage etc., which contributes to the noise level of strainmeters and tiltmeters at periods of interest other than tidal periods.

Thermoelastic deformations will contribute much less to the noise level of gravimeters, as follows from estimates by formula (10).

The influence functions for strains and tilts in Figures 2–4 show that thermoelastic deformations reflect strongest very local temperature anomalies. In the interpretation of observed thermoelastic strains and tilts, one therefore should look at first for quite local causes. Berger (1975) has explained observed anomalous horizontal strains of diurnal period and magnitude $2 \cdot 10^{-9}$ by a spatially harmonic temperature anomaly with a horizontal wavelength of about 50 km. This paper shows that such strains can also be caused by temperature anomalies of much lesser extent. In these cases, installing instruments at greater depths reduces thermoelastic disturbances significantly.

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