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Short Communication

Extreme Models From the Maximum Entropy Formulation of Inverse Problems

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In geophysics an Earth model is usually understood to mean a finite number of functions of the position vector **r** representing the distribution of some physical properties within the Earth. In 1972, Parker proposed that a kind of "extreme model" might be the type of information that should be extracted from data (observations) too limited to allow inversion to a meaningful Earth model by the Backus-Gilbert inversion technique (Backus, Gilbert, 1967, 1968). Such a model is extreme in the sense that it places bounds on the properties of all Earth models complying with a set of given data. In this way Parker showed, for instance, that no Earth model with mean density $\bar{\rho}$ and ratio

$$y = 5I/(2MR^2)$$

(where I, M, and R denote moment of inertia, mass, and radius of the Earth, respectively) can have a maximum density lower than

$$\rho_0 = \bar{\rho}\, y^{-3/2}. \tag{1}$$

It is the purpose of this note to show, via the example of the density distribution within the Earth, that this idea of handling inverse problems can also be implemented by means of the maximum entropy method (Rietsch, 1977).

According to this latter paper (Eq. (57) with $N \to \infty$) the expectation value of the density has the form (with slightly modified notation)

$$\bar{\rho}(x) = \rho_l + 1/h(x) - \Delta\rho/\{\exp[\Delta\rho\, h(x)] - 1\} \tag{2}$$

where $x = r/R$. The abbreviation $h(x)$ stands for

$$h(x) = \lambda_1 + \lambda_2 X^2. \tag{3}$$

The Lagrange multipliers λ_1 and λ_2 are to be determined in such a way that $\bar{\rho}(x)$ satisfies

$$\begin{aligned}\int_0^1 \bar{\rho}(x)\, x^2\, dx &= \bar{\rho}/3\\ \int_0^1 \bar{\rho}(x)\, x^4\, dx &= y\,\bar{\rho}/5.\end{aligned} \tag{4}$$

It is a particular feature of this expectation value of the density that it has been derived with the assumption, that all possible density values within the Earth are bounded by a lower density limit ρ_l and an upper density limit $\rho_u = \rho_l + \Delta\rho$. In general, the range of possible values of the density extends from zero to some maximum density which is usually high enough to make $\Delta\rho\, h(x) \gg 1$ and hence render the last term on the right hand side of Equation (2) negligible.

The density distribution computed from Equation (2) with this assumption and $\rho_l = 0$ and $1\ \mathrm{g/cm^3}$, respectively, turned out to be in good agreement with established density distributions employing additional information.

This is, however, just one way of using Equation (2). An alternative application of this equation is to ask for the lowest possible value of ρ_u for a given ρ_l.

Apparently ρ_u attains this lowest possible value if

$$\bar{\rho}(x) = \rho_u \qquad \text{for some } x, \tag{5}$$

i.e., if the expectation value actually reaches the upper density limit. If in this case $\bar{\rho}(x)$ still allows Equations (4) to be satisfied, it is the density distribution with the lowest upper density limit that complies with the given data.

Explicitly, condition (5) reads

$$\Delta\rho = 1/h(x) - \Delta\rho/\{\exp[\Delta\rho\, h(x)] - 1\}. \tag{6}$$

It is satisfied for

$$1/h(x) \to 0, \qquad \exp[\Delta\rho\, h(x)] \to 0, \tag{7}$$

and, since $\bar{\rho}(x)$ is a monotonic function of $h(x)$, this is the only solution.

Let

$$\lambda_1 = -\lambda \cos\varphi, \qquad \lambda_2 = \lambda \sin\varphi, \qquad \lambda = (\lambda_1^2 + \lambda_2^2)^{1/2}. \tag{8}$$

Then $h(x)$ can be written as

$$h(x) = -\lambda(\cos\varphi - x^2 \sin\varphi),$$

and, for $\lambda \to \infty$, we get

$$\bar{\rho}(x) = \begin{cases}\rho_l & \text{for } \cos\varphi - x^2 \sin\varphi < 0\\ \rho_u & \text{for } \cos\varphi - x^2 \sin\varphi > 0.\end{cases} \tag{9}$$

This means $\bar{\rho}(x)$ is discontinuous at some $x_0 = \sqrt{\cot \varphi}$. Confining our attention to the case

$$\bar{\rho}(x) = \begin{cases} \rho_u & 0 \leqq x < x_0 \\ \rho_l & x_0 < x \leqq 1 \end{cases} \tag{10}$$

we get by substituting (10) for $\bar{\rho}(x)$ in Equations (4)

$$\begin{aligned} x_0^3 \, \rho_u + (1 - x_0^3) \, \rho_l &= \bar{\rho} \\ x_0^5 \, \rho_u + (1 - x_0^5) \, \rho_l &= y \bar{\rho} . \end{aligned} \tag{11}$$

Elimination of x_0 from these equations leads to

$$\rho_u = \rho_l + (\bar{\rho} - \rho_l)^{5/2} / (y \bar{\rho} - \rho_l)^{3/2} \tag{12}$$

which reduces to Equation (1) for $\rho_l = 0$.

Though demonstrated for this particular example only, the method is, of course, applicable to any other inverse problem of this kind. It is perhaps also appropriate to point out that extreme models obtained in this way are independent of the particular prior probability distribution provided it does not vanish at the upper and/or lower limits of the admitted function values.

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