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# Ray Amplitudes of Compressional, Shear, and Converted Seismic Body Waves in 3D Laterally Inhomogeneous Media With Curved Interfaces

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**Abstract.** A compact formula for the leading term of the ray series for the displacement vector of an arbitrary compressional, shear or converted multiply reflected wave in an inhomogeneous medium with curved interfaces is derived. The components of the vector are expressed with respect to a special system of three mutually perpendicular unit vectors moving along the investigated ray. A method of determination of these vectors at any point of the ray is suggested.

**Key words:** Leading term of the ray series – Compressional, shear, and converted multiply reflected waves – Laterally inhomogeneous media with curved interfaces.

## 1. Introduction

It is well-known from the study of propagation of high frequency waves or discontinuities in laterally inhomogeneous isotropic media that there are two wave fronts which propagate independently. One of them corresponds to the compressional ($P$) wave, the other to the shear ($S$) wave.

The complex-valued leading term of the ray series for the displacement vector of the $P$ wave, $\vec{\mathbf{U}}_P$, is tangent to the ray (i.e., both real-valued vectors forming real and imaginary parts of $\vec{\mathbf{U}}_P$ are tangent to the ray) and can be expressed as follows (Červený et al., 1977, p. 23)

$$\vec{\mathbf{U}}_P = U_P F(t-\tau_P)\,\vec{\mathbf{t}}. \tag{1}$$

Complex quantity $U_P$ is usually called the principal component of the $P$ wave in the zero approximation of the ray theory. Complex function $F$ describes in a certain sense the form of the signal of the investigated wave, $t$ is the time, $\vec{\mathbf{t}}$ denotes a unit vector tangent to the ray, $\tau_P$ is the phase function (eikonal) of the $P$ wave. Function $\tau_P$ is a solution of the eikonal equation $(\nabla\tau_P)^2=\alpha^{-2}$, where $\alpha(x_i)$ is the $P$ wave velocity.

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The complex-valued leading term of the ray series for the displacement vector of the $S$ wave, $\vec{\mathbf{U}}_S$, lies in the plane perpendicular to the ray (both the real-valued vectors forming $\vec{\mathbf{U}}_S$ can have different directions in this case). Usually, vector $\vec{\mathbf{U}}_S$ is expressed in terms of the principal components $U_{Sn}$ and $U_{Sb}$ in the zero approximation of the ray theory. The components $U_{Sn}$ and $U_{Sb}$ are taken with respect to the vectors of unit normal $\vec{\mathbf{n}}$ and binormal $\vec{\mathbf{b}}$ to the ray (Červený et al., 1977, p. 23):

$$\vec{\mathbf{U}}_S=(U_{Sn}\vec{\mathbf{n}}+U_{Sb}\vec{\mathbf{b}})\,F(t-\tau_S). \tag{2}$$

Here $\tau_S$ is the eikonal of the $S$ wave, $(\nabla\tau_S)^2=\beta^{-2}$, where $\beta(x_i)$ is the $S$ wave velocity.

The formulae for the determination of the components of the leading term of the ray series for a displacement vector along a ray in a 3D continuous inhomogeneous medium as well as formulae determining the behaviour of these components at an interface in the medium are presented in Sect. (2). The determination of the leading term at any point of an arbitrary multiply reflected wave can then be performed by successively applying the formulae of Sect. (2), following the ray from one interface to another. It would be more desirable, however, to have a compact formula for this purpose. Compact formulae are known for some special types of waves, such as pure $P$ waves in 3D media or $P$-$SV$ waves in 2D media (Červený et al., 1977, pp. 36–39). An alternative compact formula for the determination of the leading term of a general multiply reflected wave is suggested in Sect. (3). In Sect. (2) it is shown that it is useful to express the components of the leading term of the ray series for a displacement vector with respect to a special system of three mutually perpendicular unit vectors moving along the ray. A method of determining these vectors at any point of the ray is suggested in Sect. (4).

## 2. Formulae for a Successive Determination of the Leading Term of the Ray Series

In an inhomogeneous isotropic medium the principal components $U_P$, $U_{Sn}$, $U_{Sb}$ can be given by the following expressions

$$\begin{aligned}
U_P&=(\alpha\rho J)^{-1/2}\,\Psi_P(\gamma_1,\gamma_2),\\
U_{Sn}&=(\beta\rho J)^{-1/2}[\Psi_{Sn}(\gamma_1,\gamma_2)\cos\Theta+\Psi_{Sb}(\gamma_1,\gamma_2)\sin\Theta],\\
U_{Sb}&=(\beta\rho J)^{-1/2}[-\Psi_{Sn}(\gamma_1,\gamma_2)\sin\Theta+\Psi_{Sb}(\gamma_1,\gamma_2)\cos\Theta].
\end{aligned} \tag{3}$$

In (3) symbol $J$ denotes a measure of the cross-sectional area of the ray tube, $\gamma_1$, $\gamma_2$ are ray parameters, $\rho$ denotes density. Functions $\Psi_P$, $\Psi_{Sn}$, $\Psi_{Sb}$ are arbitrary functions of $\gamma_1$ and $\gamma_2$, being constant along the whole ray. Quantity $\Theta$ is given by the formula

$$\Theta=\int_{\tau_0}^{\tau}\beta T\,d\xi, \tag{4}$$

where $T$ denotes the torsion of the ray (for details see Červený et al., 1977, p. 23).

It follows from (3) that the components $U_{Sn}$ and $U_{Sb}$ are mutually coupled unless $T=0$. It also follows from (3) that as the wave progresses, vector $\vec{\mathbf{U}}_S$ rotates in the plane perpendicular to the ray with respect to vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{b}}$. The velocity of the rotation is $d\Theta/d\tau=\beta T$.

To determine the vector $\vec{\mathbf{U}}_S$ at an arbitrary point of a ray, it is necessary to know the unit vectors of the normal and binormal to the ray. They could be determined from Frenet's formulae, which, however, represent an additional system of differential equations to be solved. Moreover, it is necessary to know the torsion and curvature of the ray at any of its points. This can be connected with certain difficulties, e.g. in models containing both homogeneous and inhomogeneous regions (see Červený et al., 1977, pp. 89–90). To avoid these difficulties, the following procedure is suggested.

In the plane perpendicular to the ray, let us introduce two mutually perpendicular unit vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ on the ray (see Fig. 1) by the following relations

$$d\vec{\mathbf{e}}_1/d\tau=-vK\cos\vartheta\,\vec{\mathbf{t}}, \qquad d\vec{\mathbf{e}}_2/d\tau=-vK\sin\vartheta\,\vec{\mathbf{t}}, \tag{5}$$

where $K$ denotes the curvature of the ray and $v$ is the velocity, $v=\alpha$ along the rays of the $P$ wave, $v=\beta$ along the rays of the $S$ wave. The angle $\vartheta$ is given by the formula

$$\vartheta=\int_{\tau_0}^{\tau} v\,T\,d\xi+\vartheta_0, \tag{6}$$

thus

$$\vartheta=\Theta+\vartheta_0. \tag{6'}$$

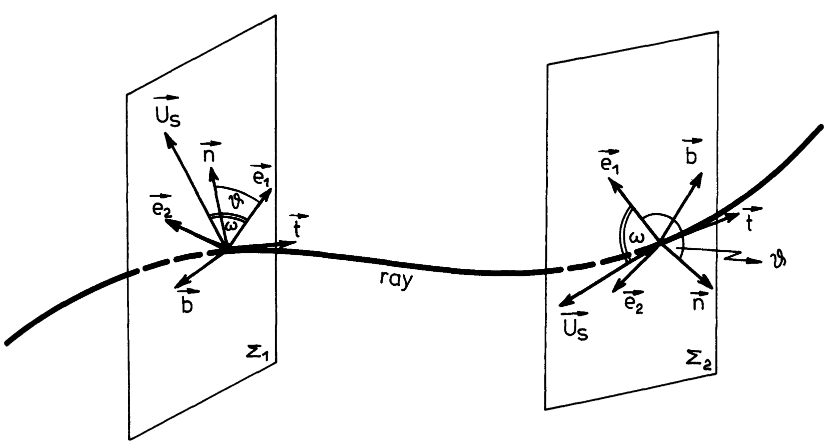


**Fig. 1.** Introduction of vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ in a plane perpendicular to the ray. Unit vector $\vec{\mathbf{t}}$, tangent to the ray, and vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ form a right-handed system of orthonormal vectors. As the wave progresses, the vector $\mathbf{U}_S$ rotates in the plane perpendicular to the ray with respect to normal $\vec{\mathbf{n}}$ and binormal $\vec{\mathbf{b}}$, see $\Sigma_1$ and $\Sigma_2$. The vector $\vec{\mathbf{U}}_S$, however, does not rotate with respect to vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$, the angle $\omega$ is constant along the ray. For details see text

The vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ change according to relations (5) along the ray. The initial orientation of the vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ (for $\tau = \tau_o$) in the plane perpendicular to the ray can be chosen arbitrarily by a choice of the additive constant $\vartheta_0$.

Using Frenet's formulae, we easily obtain

$$\vec{\mathbf{e}}_1 = \vec{\mathbf{n}} \cos\vartheta - \vec{\mathbf{b}} \sin\vartheta, \qquad \vec{\mathbf{e}}_2 = \vec{\mathbf{n}} \sin\vartheta + \vec{\mathbf{b}} \cos\vartheta. \tag{7}$$

It follows from (7) that in the plane perpendicular to the ray, vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ rotate with respect to vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{b}}$ as the wave progresses. The velocity of the rotation is $d\vartheta/d\tau = vT$. Thus, in the case of an $S$ wave, the velocity of the rotation of vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ is the same as the velocity of the rotation of the vector $\vec{\mathbf{U}}_S$. It means that vector $\vec{\mathbf{U}}_S$ does not rotate with respect to vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$, see Fig. 1. Therefore it seems quite natural to express the vector $\vec{\mathbf{U}}_S$ with respect to vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ (instead of $\vec{\mathbf{n}}$ and $\vec{\mathbf{b}}$)

$$\vec{\mathbf{U}}_S = (U_{S1} \vec{\mathbf{e}}_1 + U_{S2} \vec{\mathbf{e}}_2) F(t - \tau_S). \tag{8}$$

Comparing (2) and (8) and taking into account the formulae (7), we obtain

$$U_{S1} = U_{Sn} \cos\vartheta - U_{Sb} \sin\vartheta, \qquad U_{S2} = U_{Sn} \sin\vartheta + U_{Sb} \cos\vartheta. \tag{9}$$

Let us further denote

$$\Psi_{S1}(\gamma_1, \gamma_2) = \Psi_{Sn} \cos\vartheta_0 - \Psi_{Sb} \sin\vartheta_0, \qquad \Psi_{S2}(\gamma_1, \gamma_2) = \Psi_{Sn} \sin\vartheta_0 + \Psi_{Sb} \cos\vartheta_0. \tag{10}$$

After substituting (3) into (9) and taking into account (10), we arrive at final formulae

$$U_{S1} = (\beta \rho J)^{-1/2} \Psi_{S1}(\gamma_1, \gamma_2), \qquad U_{S2} = (\beta \rho J)^{-1/2} \Psi_{S2}(\gamma_1, \gamma_2). \tag{11}$$

Thus, in the medium without interfaces, components $U_{S1}$ and $U_{S2}$ are not coupled to each other and they both change along the ray in the same way. However, this will not hold true after a reflection (transmission) at an interface.

To determine the vector $\vec{\mathbf{U}}_S$ at an arbitrary point of a ray, vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ must be known at this point. If Eqs. (5) or (7) are used for this purpose then the above mentioned difficulties connected with the determination of vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{b}}$ arise again. Fortunately, the vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ can be determined without knowledge of vectors $\vec{\mathbf{n}}$ and $\vec{\mathbf{b}}$, as it is shown in Sect. (4).

If there are curved interfaces of the first order in the medium, the leading term of the ray series changes discontinuously across them. This discontinuity can be expressed by introducing the reflection (transmission) coefficients. In the zero approximation of the ray theory, the process of reflection (transmission) at a curved interface can be investigated locally as a reflection (transmission) of a plane wave at a plane interface. Therefore it is convenient to introduce the $SV$ and the $SH$ components of the leading term of the ray series for the $S$ wave in the vicinity of the point of incidence. We shall denote these components by $U_{SV}$ and $U_{SH}$. At the point of incidence the vector $\vec{\mathbf{U}}_S$ can then be expressed as $\vec{\mathbf{U}}_S = U_{SV} \vec{\mathbf{e}}_{SV} + U_{SH} \vec{\mathbf{e}}_{SH}$. Here $\vec{\mathbf{e}}_{SV}$ and $\vec{\mathbf{e}}_{SH}$ are two mutually perpendicular unit vectors both perpendicular to the ray. Vector

$\vec{e}_{SV}$ lies in the plane of incidence, i.e., in the plane determined by the normal to the interface and the tangent to the ray at the point of incidence. Vector $\vec{e}_{SH}$ is perpendicular to this plane. Vectors $\vec{t}, \vec{e}_{SH}, \vec{e}_{SV}$ form a right-handed system.

When a $P$ or $SV$ wave impinges at an interface, only $P$ and $SV$ reflected and transmitted waves are generated, when an $SH$ wave impinges at an interface, only $SH$ reflected and transmitted waves are generated. Thus, there are five reflection and five transmission coefficients of the types: $R_{PP}$, $R_{PSV}$, $R_{SVP}$, $R_{SVSV}$, $R_{SHSH}$.

If we denote any of the components $U_P$, $U_{SV}$, $U_{SH}$ corresponding to the incident wave at the point of incidence by $U^I$ and any component of a reflected (transmitted) wave at the same point by $U^R$, we can write

$$U^R = U^I \cdot R, \tag{12}$$

where $R$ is the appropriate coefficient of reflection (transmission).

In the case of pure $P$ waves, the component $U_P$ determined from (3) can be immediately substituted in formula (12). Thus, without difficulties, it is possible to get a well-known compact formula for an unconverted multiply reflected wave [see (19)].

The situation becomes more complicated when there is at least one element of the ray along which the wave propagates as an $S$ wave. Then, before applying formula (12) at the point of incidence, it is necessary to transform components $U_{S1}$, $U_{S2}$ (or $U_{Sn}$, $U_{Sb}$) into $U_{SH}$ and $U_{SV}$ components. Then it is possible to apply (12) and to use $U_{SH}$, $U_{SV}$ as $U_{S1}$, $U_{S2}$ on the reflected (transmitted) ray at the point of incidence or return to components $U_{Sn}$, $U_{Sb}$. It is evident that it is not straightforward to get a compact formula for the leading term of the ray series for a displacement vector of a multiply reflected converted wave similar to that for pure $P$ waves. Therefore it has often been proposed to determine the components of this vector following the ray from one interface to another, successively applying relations (3) [or (11)] and (12). An alternative compact formula for the determination of the leading term of the ray series for a displacement vector of a general multiply reflected wave is suggested in the next section.

## 3. Compact Formulae for the Determination of the Leading Term of the Ray Series

Let us introduce a vector $\vec{\mathbf{U}}^T = (U_P, U_{S1}, U_{S2})$, where the superscript $T$ stands for transpose. For a $P$ wave only the first component of the vector $\vec{\mathbf{U}}^T$ is non-zero, for an $S$ wave the second and third components are non-zero, $U_P$ being zero. Let us further introduce the vector $\vec{\boldsymbol{\Psi}}^T = (\Psi_P, \Psi_{S1}, \Psi_{S2})$ which has similar properties to those of the vector $\vec{\mathbf{U}}^T$. Then, the first Eq. in (3) and Eq. (11) can be rewritten as follows

$$\vec{\mathbf{U}} = (v \rho J)^{-1/2} \vec{\boldsymbol{\Psi}}(\gamma_1, \gamma_2), \tag{13}$$

where $v = \alpha$ along the rays of the $P$ wave ($U_{S1} = U_{S2} = \Psi_{S1} = \Psi_{S2} = 0$) or $v = \beta$

along the rays of the $S$ wave ($U_P = \Psi_P = 0$). Relation (12) can be also expressed in terms of the vector $\vec{\mathbf{U}}$. It is

$$\vec{\mathbf{U}}^n = P^{mn} \vec{\mathbf{U}}^m. \tag{14}$$

Index $m$ corresponds to the incident wave: $m=1$ for the incident $P$ wave, $m=-1$ for the incident $S$ wave. Index $n$ corresponds to the generated wave: $n=1$ for the generated $P$ wave, $n=-1$ for the generated $S$ wave. The values $-1$ and 1 were chosen in accordance with the numerical code of waves suggested in Červený et al. (1977), pp. 88–89. The $3\times 3$ matrix $P^{mn}$ has the following form

$$P^{mn} = \begin{pmatrix} R_{PP}\,\delta_{1m}\,\delta_{1n} & R_{SVP}\,\delta_{-1m}\,\delta_{1n}\sin\Omega & R_{SVP}\,d_{-1m}\,\delta_{1n}\cos\Omega \\ 0 & R_{SHSH}\,\delta_{-1m}\,\delta_{-1n}\cos\Omega & -R_{SHSH}\,\delta_{-1m}\delta_{-1n}\sin\Omega \\ R_{PSV}\,\delta_{1m}\,\delta_{-1n} & R_{SVSV}\,\delta_{-1m}\,\delta_{-1n}\sin\Omega & R_{SVSV}\,\delta_{-1m}\,\delta_{-1n}\cos\Omega \end{pmatrix}. \tag{15}$$

Here $R_{PP}$, $R_{PSV}$, $R_{SVP}$, $R_{SVSV}$, $R_{SHSH}$ are standard coefficients of reflection (transmission), see (12), $\delta_{ij}$ is Kronecker's symbol, $\Omega$ is the angle by which it is necessary to rotate vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ at the point of incidence to make them coincide with the vectors $\vec{\mathbf{e}}_{SH}$, $\vec{\mathbf{e}}_{SV}$, respectively, see Sect. (4), formula (26). The matrix $P^{mn}$ includes both the transformation of the components of the leading term $U_P$, $U_{S1}$, $U_{S2}$ into $U_P$, $U_{SH}$, $U_{SV}$ and the coefficients of reflection (transmission) at an interface.

Now it is not difficult to combine relations (13) and (14) and to write a compact formula for a general converted multiply reflected wave at a point $M$ on the ray

$$\vec{\mathbf{U}}(M) = [v(M)\,\rho(M)\,J(M)]^{-1/2} \prod_{j=1}^{N} \left\{ \left[ \frac{\rho'(O_j)\,v'(O_j)\,J'(O_j)}{\rho(O_j)\,v(O_j)\,J(O_j)} \right]^{1/2} P^{mn}(O_j) \right\} \vec{\Psi}. \tag{16}$$

The terms $[(\rho' v' J')/(\rho v J)]^{1/2}$ are introduced to compensate the discontinuities of the function $(\rho v J)^{-1/2}$ at interfaces. $N$ is the number of reflections and transmissions along the ray, $O_j$ is the $j$-th point of incidence of the ray at an interface. The primed (unprimed) quantities are taken on that side of an interface where the generated (incident) wave propagates. For $m=-1$, $v=\beta$, $m=1$ implies $v=\alpha$. The same holds for a generated wave.

For a point source with the directional characteristics $\vec{\mathbf{g}}^T(\varphi_o, \delta_o) = (g_P, g_{S1}, g_{S2})$, formula (16) can be rewritten into the form

$$\vec{\mathbf{U}}(M) = \frac{1}{L(M)} \left[ \frac{\rho_o v_o}{\rho(M)\,v(M)} \right]^{1/2} \prod_{j=1}^{N} \left\{ \left[ \frac{\rho'(O_j)\,v'(O_j)}{\rho(O_j)\,v(O_j)} \right]^{1/2} P^{mn}(O_j) \right\} \vec{\mathbf{g}}(\varphi_o, \delta_o), \tag{17}$$

where $L(M) = \left( \frac{J(M)}{\sin\delta_o} \right)^{1/2} \prod_{j=1}^{N} \left[ \frac{J(O_j)}{J'(O_j)} \right]^{1/2}$, see Červený et al. (1977), p. 38. The symbols $\varphi_o$, $\delta_o$ denote two take off angles of the ray under consideration at the source.

For an unconverted $S$ wave we can again use formulae (16) and (17). The individual quantities can be however, simplified to have the following meaning:

$$\vec{\mathbf{U}}^T=(U_{S1}, U_{S2}), \quad \vec{\mathbf{\Psi}}^T=(\Psi_{S1}, \Psi_{S2}), \quad \vec{\mathbf{g}}^T=(g_{S1}, g_{S2}),$$

$$P=\begin{pmatrix} R_{SHSH}\cos\Omega & -R_{SHSH}\sin\Omega \\ R_{SVSV}\sin\Omega & R_{SVSV}\cos\Omega \end{pmatrix}. \tag{18}$$

For an unconverted $P$ wave formula (16) reduces to the wellknown formula (Červený et al., 1977, p. 38)

$$U_P=(\alpha\rho J)^{-1/2}\prod_{j=1}^{N}\left\{R_j\left[\frac{\rho'(O_j)\,\alpha'(O_j)\,J'(O_j)}{\rho(O_j)\,\alpha(O_j)\,J(O_j)}\right]^{1/2}\right\}\Psi_P(\gamma_1,\gamma_2). \tag{19}$$

In (19) $R_j$ denotes the appropriate reflection (transmission) coefficient at the $j$-th point of incidence $O_j$.

## 4. Determination of the Vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$

To determine the vector $\vec{\mathbf{U}}$ at an arbitrary point of a ray, it is necessary to know the vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ at any point of the ray. Let us note that the knowledge of these vectors is not necessary only for the determination of the vector $\vec{\mathbf{U}}$. The vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ may also play an important role in some methods of computation of geometrical spreading (Červený et al., 1977; Popov and Pšenčík, 1978a; Popov and Pšenčík, 1978b; Červený and Pšenčík, 1979; Hubral, 1979). Thus, once determined, the vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ can serve two purposes: to determine the geometrical spreading and to determine the components of the vector $\vec{\mathbf{U}}$ along the ray.

The vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ can be determined in various ways depending on the form of the ray-tracing system used for the computation of rays. Here, the following ray-tracing system will be considered (Červený et al., 1977, p. 58)

$$dx_i/d\tau=v^2 p_i, \qquad dp_i/d\tau=-v^{-1}v_i. \tag{20}$$

In (20), $p_i$ are the components of the slowness vector $\nabla\tau$, $v_i=\partial v/\partial x_i$, $i=1,2,3$.

The determination of the vector $\vec{\mathbf{t}}$ is straightforward,

$$\vec{\mathbf{t}}=(v p_1, v p_2, v p_3). \tag{21}$$

The vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ can be determined in the following way. At an arbitrary point of the ray, let us define two mutually perpendicular unit vectors

$$\vec{\mathbf{i}}_1=(vD^{-1}p_1\,p_3, vD^{-1}p_2\,p_3, -vD), \qquad \vec{\mathbf{i}}_2=(-D^{-1}p_2, D^{-1}p_1, 0) \tag{22}$$

where $D=(p_1^2+p_2^2)^{1/2}$. Vectors $\vec{\mathbf{i}}_1, \vec{\mathbf{i}}_2$ lie in the plane perpendicular to the ray, see Fig. 2. Vector $\vec{\mathbf{i}}_1$ is always lying in the vertical plane containing the tangent to the ray $\vec{\mathbf{t}}$, vector $\vec{\mathbf{i}}_2$ is always horizontal. The vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{i}}_1$, $\vec{\mathbf{i}}_2$ form a right-handed

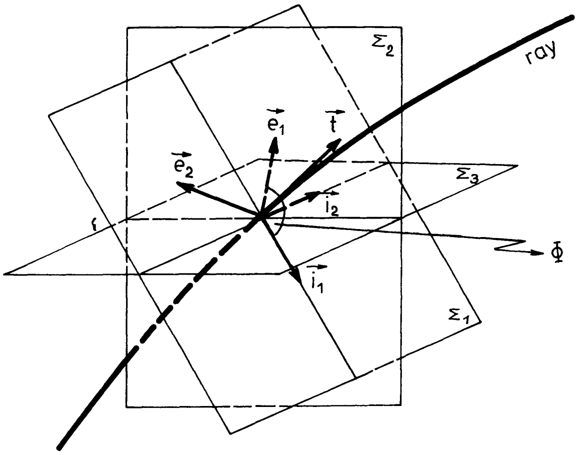


**Fig. 2.** Introduction of auxiliary vectors $\vec{\mathbf{i}}_1$, $\vec{\mathbf{i}}_2$ in the plane $\Sigma_1$ perpendicular to the ray. Vector $\vec{\mathbf{i}}_1$ lies along the intersection of the plane $\Sigma_1$ and vertical plane $\Sigma_2$ containing the tangent to the ray $\vec{\mathbf{t}}$. Vector $\vec{\mathbf{i}}_2$ lies along the intersection of plane $\Sigma_1$ and horizontal $\Sigma_3$. Vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{i}}_1$, $\vec{\mathbf{i}}_2$ form an orthonormal right-handed system. For details see text

system. When the components of the slowness vector $p_i$ are known the vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{i}}_1$, $\vec{\mathbf{i}}_2$ can be simply determined. Let us note that $\vec{\mathbf{i}}_1$, $\vec{\mathbf{i}}_2$ cannot be determined from (22) in the case of the ray parallel with the $z$-axis, i.e., for $p_1 = p_2 = 0$. In such a case, the vectors $\vec{\mathbf{i}}_1$, $\vec{\mathbf{i}}_2$ can be defined in another way, e.g., as follows

$$\vec{\mathbf{i}}_1 = (-v\bar{D}, v\bar{D}^{-1} p_1 p_2, v\bar{D}^{-1} p_1 p_3), \qquad \vec{\mathbf{i}}_2 = (0, -\bar{D}^{-1} p_3, \bar{D}^{-1} p_2), \tag{22'}$$

where $\bar{D} = (p_2^2 + p_3^2)^{1/2}$.

Since the vectors $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2$ also lie in the plane perpendicular to the ray, it is possible to express them as follows (see Fig. 2)

$$\vec{\mathbf{e}}_1 = \vec{\mathbf{i}}_1 \cos\Phi - \vec{\mathbf{i}}_2 \sin\Phi, \qquad \vec{\mathbf{e}}_2 = \vec{\mathbf{i}}_1 \sin\Phi + \vec{\mathbf{i}}_2 \cos\Phi. \tag{23}$$

As it was shown above, the vectors $\vec{\mathbf{i}}_1, \vec{\mathbf{i}}_2$ can be simply determined, thus the problem of the determination of the vectors $\vec{\mathbf{e}}$, $\vec{\mathbf{e}}_2$ reduces to the problem of the determination of the angle $\Phi$.

Taking into account that the relations

$$vK \cos\vartheta = -(\nabla v \cdot \vec{\mathbf{e}}_1), \qquad vK \sin\vartheta = -(\nabla v \cdot \vec{\mathbf{e}}_2)$$

hold identically along the considered ray [Popov and Pšenčík, 1978a, Eq. (3.18)], Eq. (5) can be rewritten as follows

$$d\vec{\mathbf{e}}_i/d\tau = (\nabla v \cdot \vec{\mathbf{e}}_i)\,\vec{\mathbf{t}}, \qquad i = 1, 2.$$

Then, differentiating the first equation in (23) with respect to $\tau$ and taking into account the new form of Eq. (5), we obtain

$$(d\vec{\mathbf{i}}_1/d\tau)\cos\Phi - \vec{\mathbf{i}}_1 \sin\Phi(d\Phi/d\tau)$$
$$-(d\vec{\mathbf{i}}_2/d\tau)\sin\Phi - \vec{\mathbf{i}}_2\cos\Phi(d\Phi/d\tau) = (\nabla v\cdot\vec{\mathbf{e}}_1)\,\vec{\mathbf{t}}.$$

Considering, e.g., the $z$-component of this vectorial equation, after some manipulation using relations (20)–(23), we get

$$d\Phi/d\tau = p_3(p_1^2+p_2^2)^{-1}(v_1 p_2 - v_2 p_1). \tag{24}$$

Let us mention that a similar expression for $\Phi$ was obtained by Popov and Pšenčík (1978b). They, however, used the ray-tracing system in which the polar angles $\varphi$ and $\delta$ were used instead of $p_1, p_2, p_3$.

At each point of incidence $O_j$ of a ray at an interface the vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ must be rotated by an angle $\Omega_j$ to make them coincide with vectors $\vec{\mathbf{e}}_{SH}$, $\vec{\mathbf{e}}_{SV}$, respectively. Let us denote the corresponding value of the angle $\Phi$, which includes $\Omega_j$, by $\Phi(O_j)$. After reflection (transmission) the vectors $\vec{\mathbf{e}}_{SH}$, $\vec{\mathbf{e}}_{SV}$ as well as $\vec{\mathbf{i}}_1$, $\vec{\mathbf{i}}_2$ transform into vectors $\vec{\mathbf{e}}'_{SH}$, $\vec{\mathbf{e}}'_{SV}$, $\vec{\mathbf{i}}'_1$, $\vec{\mathbf{i}}'_2$ corresponding to the reflected (transmitted) ray at the point of incidence $O_j$, and it holds $\vec{\mathbf{e}}'_{SH}=\vec{\mathbf{e}}_{SH}$. Let us choose the vectors $\vec{\mathbf{e}}'_1$, $\vec{\mathbf{e}}'_2$ corresponding to the reflected (transmitted) ray at the point of incidence as follows, $\vec{\mathbf{e}}'_1=\vec{\mathbf{e}}_{SH}$, $\vec{\mathbf{e}}'_2=\vec{\mathbf{e}}'_{SV}$. It corresponds to the choice of $U_{SH}$, $U_{SV}$ components as $U_{S1}$ $U_{S2}$ [Sect. (2)] at the point of reflection (transmission). Then, it follows from (23) that the angle $\Phi$ changes discontinuously across the interface, from the value $\Phi(O_j)$ to a value $\Phi'(O_j)$. Let us denote the difference between these values by $\Omega'_j$, $\Omega'_j=\Phi'(O_j)-\Phi(O_j)$.

If we take into account all the above facts, we can integrate (24) to yield

$$\Phi(\tau)=\int_{\tau_0}^{\tau} p_3(p_1^2+p_2^2)^{-1}(v_1 p_2 - v_2 p_1)\,d\xi + \sum_{j=1}^{N}(\Omega_j+\Omega'_j)+\Phi(\tau_o). \tag{25}$$

The angle $\Omega_j$ is determined by the relations

$$\cos\Omega_j=[\vec{\mathbf{e}}_{SH}(O_j)\cdot\vec{\mathbf{e}}_1(O_j)], \qquad \sin\Omega_j=-[\vec{\mathbf{e}}_{SH}(O_j)\cdot\vec{\mathbf{e}}_2(O_j)]. \tag{26}$$

The angle $\Omega'_j$ is determined by the relations

$$\cos\Omega'_j=(\vec{\mathbf{e}}_{SH}\cdot\vec{\mathbf{i}}'_1)\cos\Phi(O_j)-(\vec{\mathbf{e}}_{SH}\cdot\vec{\mathbf{i}}'_2)\sin\Phi(O_j),$$
$$\sin\Omega'_j=-(\vec{\mathbf{e}}_{SH}\cdot\vec{\mathbf{i}}'_1)\sin\Phi(O_j)-(\vec{\mathbf{e}}_{SH}\cdot\vec{\mathbf{i}}'_2)\cos\Phi(O_j), \tag{27}$$

where, as above, the symbol $\Phi(O_j)$ denotes the value of the angle $\Phi$ corresponding to the vectors $\vec{\mathbf{e}}_{SH}(O_j)$, $\vec{\mathbf{e}}_{SV}(O_j)$ (i.e., the angle which includes only $\Omega_j$, not $\Omega'_j$). The additive constant $\Phi(\tau_0)$ in (25) can be determined from the following relations

$$\cos\Phi(\tau_o)=[\vec{\mathbf{e}}_1(\tau_0)\cdot\vec{\mathbf{i}}_1(\tau_0)], \qquad \sin\Phi(\tau_o)=-[\vec{\mathbf{e}}_1(\tau_o)\cdot\vec{\mathbf{i}}_2(\tau_0)]. \tag{28}$$

Let us emphasize again that the orientation of vectors $\vec{\mathbf{e}}_1(\tau_o)$, $\vec{\mathbf{e}}_2(\tau_o)$ can be chosen arbitrarily, see (6′), (7) and the joined discussion.

Thus, formulae (21)–(28) make possible the unique determination of the vectors $\vec{\mathbf{t}}$, $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ at an arbitrary point of a multiply reflected wave.

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