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Wave Propagation in Stratified Anisotropic Media

An Algorithm for the Computation of the Reflection and Transmission Coefficients as Well as of the Fields*

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Abstract. An algorithm based on the accurate computation of the matrizants of elementary slabs of a stratified medium and also on the properties of the propagation matrix is developed. It leads to the step by step computation of the reflection and transmission matrices of the medium and to the computation of the independent solutions of the differential equation. The multiplication of the matrizants and thus the swamping of the solutions are avoided.

The algorithm is explained in the scope of the special problem of the propagation of E.M. waves in the ionosphere; it works also provided the transmission matrix has the required properties.

Key words: Wave propagation – Anisotropic media – Reflection coefficient – Transmission coefficient.

Introduction

There are numerous studies devoted to the propagation of light and of radio waves through stratified media. Bibliographical references can be found in the textbook of Born and Wolf (1970) and in the review paper of Budden (1969)[1].

The main goal of these studies is either the determination of the coefficients of reflection and transmission of the medium or the determination of the fields with respect to the altitude or both.

The mathematical problem to be solved is the integration of a vectorial differential equation of the form:

$$\frac{d}{dz}\mathbf{f}+ik_0\mathbf{T}\mathbf{f}=0$$

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* Dedicated to Professor Dr. K. Rawer on the occasion of his 65th birthday

[1] Among the more recent papers, it is worth mentioning those of: Altman and Cory (1969a and b); Altman et al. (1970); Altman and Fijalkow (1970); Altman and Postan (1971); Berreman (1972); Honig and den Engelsen (1977); Nagano et al. (1975); Teitler and Henvis (1970)

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where

f is a vector with an even number $(2p)$ of components;
k_0 is a wave number of reference;
T is a square matrix $(2p \times 2p)$ which depends on k_0, on the direction of the wave normal in the vacuum and on the physical parameters of the medium at each height. Moreover, this matrix has the property that one half of its eigenvalues have a negative imaginary part and the other half have a positive imaginary part. Such matrices will be named matrices of class τ_{2p}.

For instance, the matrix **T** is of class τ_2 for isotropic media; it is at least of class τ_4 for anisotropic ones.

The purpose of this study is to develop an algorithm based on the matrizants (or transfer matrices) of elementary layers on which the matrix **T** is approximated by means of a matrix polynomial and on the use of the fact that the matrix **T** is of class τ_{2p} in order to avoid the numerical swamping of the solutions during the integration.

The statement of the method will be made in the frame of the propagation of E.M. waves in an horizontally stratified ionosphere, using the language familiar in the field.[2]

An important part of the algorithm has already been stated in an earlier paper (Bossy and Claes, 1974).

1. Vectorial Differential Equation

The differential equation governing the propagation of E.M. waves in a stratified medium is derived from the Maxwell equations

$$\operatorname{curl} \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} = -\frac{1}{c}\boldsymbol{\mu}\frac{\partial}{\partial t}\mathscr{H}$$

$$\operatorname{curl} \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D} \quad \text{or} \quad \operatorname{curl} \mathscr{H} = \frac{1}{c}\boldsymbol{\varepsilon}\frac{\partial}{\partial t}\mathbf{E} \tag{1.1}$$

with

$$\mathscr{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{H}, \quad \mathbf{B} = \mu_0\boldsymbol{\mu}\mathbf{H} = \frac{1}{c}\boldsymbol{\mu}\mathscr{H}$$

$$\mathbf{D} = \varepsilon_0\boldsymbol{\varepsilon}\mathbf{E}, \quad \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{D} = \frac{1}{c}\boldsymbol{\varepsilon}\mathbf{E} \tag{1.2}$$

where the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ depend only on the height variable z.

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[2] The basic formulation for non horizontal stratifications can be found in Rawer and Suchy (1967, p. 147)

Expressing the fields in the form:

$$\mathbf{E} = \mathbf{e}(z) \exp i[\omega t - k_0(n_1 x + n_2 y)]$$
$$\mathscr{H} = \mathbf{h}(z) \exp i[\omega t - k_0(n_1 x + n_2 y)] \tag{1.3}$$

where

$$k_0 = \omega/c$$
$$\mathbf{n}_0 = n_i \mathbf{e}_i = \sin\theta_0 \cos\Phi_0 \mathbf{e}_1 + \sin\theta_0 \sin\Phi_0 \mathbf{e}_2 + \cos\theta_0 \mathbf{e}_3 \tag{1.4}$$

are defined with respect to the vacuum, one can transform (1.1) in a system of 6 equations; namely in a matricial form:

$$\begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix} + ik_0 \begin{bmatrix} \mathbf{0} & \boldsymbol{\mu} \\ -\boldsymbol{\varepsilon} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix} = 0 \tag{1.5}$$

where the matrix $\mathbf{D}$ represents the operator

$$\mathbf{D} = \begin{bmatrix} 0 & \frac{d}{dz} & -ik_0 n_2 \\ \frac{d}{dz} & 0 & ik_0 n_1 \\ ik_0 n_2 & -ik_0 n_1 & 0 \end{bmatrix}. \tag{1.6}$$

The third and sixth equations are algebraic; they allow the elimination of e_z and h_z, so that finally the fourvector

$$\mathbf{f} = (e_x, e_y, h_x, h_y)^T \tag{1.7}$$

is the solution of the homogeneous differential equation (Clemmow and Heading, 1954)

$$\frac{d}{dz}\mathbf{f} + ik_0 \mathbf{T}\mathbf{f} = \mathbf{0} \tag{1.8}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix} \tag{1.9}$$

with[3]

$$T_{1,jk} = -n_j \varepsilon_{3k}/\varepsilon_{33} + (-1)^{j+k+1} n_{3-k} \mu_{3-j,3}/\mu_{33}$$
$$T_{2,jk} = \varepsilon_{3kl} n_j n_l/\varepsilon_{33} + (-1)^{j+1} (\mu_{3-j,k} \mu_{33} - \mu_{3-j,3} \mu_{3k})/\mu_{33}$$
$$T_{3,jk} = -\varepsilon_{3kl} n_j n_l/\mu_{33} + (-1)^j (\varepsilon_{3-j,k} \varepsilon_{33} - \varepsilon_{3-j,3} \varepsilon_{3k})/\varepsilon_{33}$$
$$T_{4,jk} = -n_j \mu_{3k}/\mu_{33} + (-1)^{j+k} n_{3-k} \varepsilon_{3-j,3}/\varepsilon_{33}.$$

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[3] The indices j, k and l are equal to 1 or 2 and the elements ε_{mnp} are such that $\varepsilon_{mnp} = \varepsilon_{npm} = \varepsilon_{pmn} = -\varepsilon_{mpn} = -\varepsilon_{nmp} = -\varepsilon_{pnm}$ with $\varepsilon_{123} = 1$

As can be seen, the matrix $\mathbf{T}$ depends only on the pulsation ω, on the direction parameters n_1 and n_2 and on the constitutive tensors of the medium. One remarks that trace $\mathbf{T}=0$ for every medium where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are diagonal tensors; this is for instance the case for any isotropic medium.

More precisely, for such isotropic media one has

$$\boldsymbol{\varepsilon}=\varepsilon\mathbf{I}, \qquad \boldsymbol{\mu}=\mu\mathbf{I} \tag{1.10}$$

where $\mathbf{I}$ is the identity matrix, so that

$$\mathbf{T}=\begin{bmatrix} 0 & 0 & \dfrac{n_1 n_2}{\varepsilon} & \mu-\dfrac{n_1^2}{\varepsilon} \\ 0 & 0 & -\mu+\dfrac{n_2^2}{\varepsilon} & -\dfrac{n_1 n_2}{\varepsilon} \\ -\dfrac{n_1 n_2}{\mu} & -\varepsilon+\dfrac{n_1^2}{\mu} & 0 & 0 \\ \varepsilon-\dfrac{n_2^2}{\mu} & \dfrac{n_1 n_2}{\mu} & 0 & 0 \end{bmatrix}. \tag{1.11}$$

On account of the particular structure of $\mathbf{T}$, (1.8) can be put in the form of the differential equation

$$\frac{d}{dz}\begin{bmatrix} e_x \pm i\sqrt{\mu/\varepsilon}\, h_x \\ e_y \pm i\sqrt{\mu/\varepsilon}\, h_y \end{bmatrix} + ik_0 \begin{bmatrix} \mp i\dfrac{n_1 n_2}{n} & \mp i\dfrac{n^2-n_1^2}{n} \\ \pm i\dfrac{n^2-n_2^2}{n} & \pm i\dfrac{n_1 n_2}{n} \end{bmatrix} \begin{bmatrix} e_x \pm i\sqrt{\mu/\varepsilon}\, h_x \\ e_y \pm i\sqrt{\mu/\varepsilon}\, h_y \end{bmatrix} = 0 \tag{1.12}$$

where

$$\varepsilon\mu = n^2 = (M - i\chi)^2. \tag{1.13}$$

The secular equation corresponding to the matrix $\mathbf{T}$ in (1.12) is

$$\lambda^2 = n^2 - (n_1^2 + n_2^2)$$

its eigenvalues have the property required by the matrices of class τ_2.

In the general case, the matrix $\mathbf{T}$ of (1.8) is of class τ_4. As a matter of fact, if one considers the propagation in a homogeneous medium having the physical characteristics of the medium at the level $\bar{z}$, one has to solve the equation with constant coefficients

$$\frac{d}{dz}\mathbf{f} + ik_0\mathbf{T}(\bar{z})\,\mathbf{f} + \mathbf{0} \tag{1.14}$$

the solution of which is (Gantmacher, 1966, p. 119)

$$\mathbf{f}=\mathbf{f}(\bar{z})\,e^{-ik_0\mathbf{T}(\bar{z})(z-\bar{z})}. \tag{1.15}$$

Quite generally, this solution can be expressed in terms of linear combinations of the exponential functions

$$e^{-ik_0 q_\alpha(z-\bar{z})} \qquad (\alpha=1,\ldots,4)$$

where the q_α are the eigenvalues of the matrix $\mathbf{T}$. Booker (1936) has shown that in the present case two eigenvalues have a negative imaginary part and that the two corresponding eigenvectors represent waves the energy-flow of which is directed upwards and that the two other eigenvalues have positive imaginary parts with corresponding waves carrying energy downwards.

This specific property of the matrices $\mathbf{T}$ will be systematically used in the development of the algorithm.

2. Matrizant of an Elementary Layer

It is supposed that the medium considered is situated between the levels z_0 and $z_s(z_0<z_s)$ and that the integration domain is divided into s sub-domains or elementary layers through intermediary levels situated at $z_1, z_2, \ldots, z_{s-1}$.

On the nth elementary layer $z\in[z_{n-1}, z_n]$, one defines the matrizant $\mathbf{M}(z_{n-1}, z)$ through the relation (Volland, 1962b)

$$\mathbf{f}(z_{n-1})=\mathbf{M}(z_{n-1}, z)\,\mathbf{f}(z) \tag{2.1}$$

so that, introducing in (1.8), one gets

$$\frac{d}{dz}\mathbf{M}-ik_0\mathbf{M}\mathbf{T}=\mathbf{0} \qquad \text{with } \mathbf{M}(z_{n-1}, z_{n-1})=\mathbf{I}. \tag{2.2}$$

One knows (Rawer and Suchy, 1967, p. 162) that if a polynomial development of $\mathbf{T}$ on $[z_{n-1}, z_n]$ is used such that

$$\mathbf{T}=\sum_{j=0}\mathbf{T}_j(z-z_{n-1})^j \tag{2.3}$$

then, the corresponding potential development of $\mathbf{M}$ can be written as

$$\mathbf{M}=\sum_{k=0}\mathbf{M}_k(z-z_{n-1})^k; \qquad \mathbf{M}_0=\mathbf{I} \tag{2.4}$$

where the matrices $\mathbf{M}_k$ are obtained through the recurrence relation

$$\mathbf{M}_k=\frac{ik_0}{k}\sum_{m=0}^{k-1}\mathbf{M}_{k-m-1}\mathbf{T}_m; \qquad \mathbf{M}_{k<0}=\mathbf{0}. \tag{2.5}$$

The matrizant ${}_n\mathbf{M}$ related to the nth slab is the matrix defined by

$${}_n\mathbf{M}=\mathbf{M}(z_{n-1}, z_n)=\sum_{k=0}\mathbf{M}_k(z_n-z_{n-1})^k. \tag{2.6}$$

One could verify (Bossy, 1971) that with an approximative representation of **T** by a polynomial of the fifth order it is possible for electronic distributions like those of the **D** region of the ionosphere to work with slabs of 1 km. thickness and to obtain $_n\mathbf{M}$ with a relative precision better than 10^{-7}. One could also ascertain that the number of terms needed for the development of $_n\mathbf{M}$ did practically not depend on the degree of the polynomial representation of **T**.

These remarks are no longer true in the neighbourhood of the levels where **T** diverges; it is then necessary to consider that z varies along a path distorted in the complex z-plane and that, after the circuit around the singularity, the integration proceeds again along the real axis (Budden, 1969). Proceeding in this way, the set of matrizants which can be computed corresponds either to a continuous profile (with some discontinuities in the first derivatives at the subdivision levels) or to a profile with discontinuities at some subdivision levels. Apart the fact that it needs the representation of the matrix **T** by the development (2.3), the volume of computation required exceeds not very much those needed in the method of Inoue and Horowitz (1966) but, for identical slab thicknesses, the approximation of the matrizants is much better.

The multiplicative property of the matrizants such that

$$\mathbf{f}(z_0)=\mathbf{M}(z_0,z_n)\,\mathbf{f}(z_n)={}_1\mathbf{M}\,{}_2\mathbf{M}\,{}_3\mathbf{M}\ldots{}_n\mathbf{M}\,\mathbf{f}(z_n) \tag{2.7}$$

is well known; one knows also (Rawer and Suchy, 1967, p. 158) that the matrizants are nothing else than particular wronskians of the differential equation. Noting the fact that, for the matrices **T** of class τ_{2p}, the particular solutions of the differential equation show extremely important relative variations; one knows that the solutions, the modulus of which grow, tend to mix with the solutions the modulus of which decrease with the effect that the independence of the solutions is destroyed as well as the significance and the usefulness of the wronskian. Therefore, it is fundamental that the integration algorithm avoids the product of matrizants if one needs to be safe from the swamping of the solutions.

3. Propagator and Diffusion Matrix

One considers an elementary slab $z\in[a,b]$, situated between two semi-infinite uniform media. The matrizant **M** related to this slab and the propagation matrices $\mathbf{T}_I$ and $\mathbf{T}_S$ of the uniform media $I(z\leqq a)$ and $S(z\geqq b)$ are known; are also known the matrices $\mathbf{Q}_I$ and $\mathbf{Q}_S$ the columns of which are the eigenvectors respectively of $\mathbf{T}_I$ and $\mathbf{T}_S$.

One knows that the relations

$$\mathbf{f}(a)=\mathbf{Q}_I\mathbf{c}(a);\qquad \mathbf{f}(b)=\mathbf{Q}_S\,\mathbf{c}(b) \tag{3.1}$$

define the decomposition of the vectors **f** with respect to the eigenvectors of **T** as they appear in **Q**. If the media I and S have been selected in such a way that there exist two upgoing and two downgoing waves, then the components of the vectors **c** are the amplitudes of these waves.

The relation between these amplitudes at the levels a and b is

$$\mathbf{c}(a)=\mathbf{Q}_I^{-1}\mathbf{f}(a)=\mathbf{Q}_I^{-1}\mathbf{M}\mathbf{f}(b)=\mathbf{Q}_I^{-1}\mathbf{M}\,\mathbf{Q}_S\,\mathbf{c}(b)=\mathbf{P}\mathbf{c}(b) \tag{3.2}$$

where (Lacoume, 1967) the matrix $\mathbf{P}$ is called the propagator related to the slab $[a,b]$ situated between the two uniform media I and S.

If one classes the eigenvectors in such a way that the two first columns of $\mathbf{Q}$ contain the upgoing waves and the two last the downgoing waves; then the fourvectors $\mathbf{c}$ can be split in two bivectors namely the bivector $\mathbf{u}$ for the upgoing waves and the bivector $\mathbf{d}$ for the downgoing waves. For instance, at the levels a and b, one has

$$\mathbf{c}(a)=\begin{bmatrix}\mathbf{u}_a\\ \mathbf{d}_a\end{bmatrix};\quad \mathbf{c}(b)=\begin{bmatrix}\mathbf{u}_b\\ \mathbf{d}_b\end{bmatrix}. \tag{3.3}$$

The diffusion matrix $\mathbf{S}$ (Volland, 1962a) connects the amplitudes of the waves leaving a slab with the amplitudes of the waves entering this slab in the form

$$\begin{bmatrix}\mathbf{d}_a\\ \mathbf{u}_b\end{bmatrix}=\mathbf{S}\begin{bmatrix}\mathbf{u}_a\\ \mathbf{d}_b\end{bmatrix}=\begin{bmatrix}\mathbf{R}_a^b & \mathbf{D}_b^a\\ \mathbf{D}_a^b & \mathbf{R}_b^a\end{bmatrix}\begin{bmatrix}\mathbf{u}_a\\ \mathbf{d}_b\end{bmatrix}. \tag{3.4}$$

The reflection matrices $\mathbf{R}$ and the transmission matrices $\mathbf{D}$ are 2×2 matrices; they are related to the 2×2 $\mathbf{P}_i$ matrices obtained when partitioning $\mathbf{P}$ according to

$$\mathbf{P}=\begin{bmatrix}\mathbf{P}_1 & \mathbf{P}_2\\ \mathbf{P}_3 & \mathbf{P}_4\end{bmatrix} \tag{3.5}$$

through the relations (Volland, 1968)

$$\mathbf{R}_a^b=\mathbf{P}_3\mathbf{P}_1^{-1},\quad \mathbf{D}_a^b=\mathbf{P}_1^{-1},\quad \mathbf{D}_b^a=\mathbf{P}_4-\mathbf{P}_3\mathbf{P}_1^{-1}\mathbf{P}_2,\quad \mathbf{R}_b^a=-\mathbf{P}_1^{-1}\mathbf{P}_2. \tag{3.6}$$

It appears that as well the propagator as the diffusion matrix depend on the structure of the medium through the matrizant and on the choice of the contiguous uniform media through the eigenvectors of their propagation matrices. It is worth remembering that the choice of these contiguous media is arbitrary as far as their eigenvectors correspond to an equal number of ascending and descending waves.

4. Diffusion Matrix of Two Contiguous Slabs

Starting with the diffusion matrices of two contiguous slabs, one looks for the expression of the diffusion matrix of the sum of the slabs in terms of those of the individual slabs. More precisely, one considers the slab $[a,b]$ situated between the uniform media I and S and the slab $[b,c]$ situated between the media $I^x\equiv S$ and S^x for which one has

$$\begin{bmatrix}\mathbf{d}_a\\ \mathbf{u}_b\end{bmatrix}=\begin{bmatrix}\mathbf{R}_a^b & \mathbf{D}_b^a\\ \mathbf{D}_a^b & \mathbf{R}_b^a\end{bmatrix}\begin{bmatrix}\mathbf{u}_a\\ \mathbf{d}_b\end{bmatrix};\quad \begin{bmatrix}\mathbf{d}_b\\ \mathbf{u}_c\end{bmatrix}=\begin{bmatrix}\mathbf{R}_b^c & \mathbf{D}_c^b\\ \mathbf{D}_b^c & \mathbf{R}_c^b\end{bmatrix}\begin{bmatrix}\mathbf{u}_b\\ \mathbf{d}_c\end{bmatrix} \tag{4.1}$$

and one looks for the expression of

$$\begin{bmatrix}\mathbf{d}_a\\ \mathbf{u}_c\end{bmatrix}=\begin{bmatrix}\mathbf{R}_a^c & \mathbf{D}_c^a\\ \mathbf{D}_a^c & \mathbf{R}_c^a\end{bmatrix}\begin{bmatrix}\mathbf{u}_a\\ \mathbf{d}_c\end{bmatrix} \tag{4.2}$$

in terms of the known diffusion matrices with the condition that the slab $[a,c]$ be situated between the uniform media I and S^x.

One starts from the two equations

$$\begin{aligned}\mathbf{u}_b-\mathbf{R}_b^a\mathbf{d}_b &=\mathbf{D}_a^b\mathbf{u}_a\\ -\mathbf{R}_b^c\mathbf{u}_b+\mathbf{d}_b&=\mathbf{D}_c^b\mathbf{d}_c\end{aligned}$$

eliminating $\mathbf{d}_b$, one obtains

$$(\mathbf{I}-\mathbf{R}_b^a\mathbf{R}_b^c)\,\mathbf{u}_b=\mathbf{D}_a^b\,\mathbf{u}_a+\mathbf{R}_b^a\,\mathbf{D}_c^b\,\mathbf{d}_c$$

or

$$\mathbf{u}_b=(\mathbf{I}-\mathbf{R}_b^a\mathbf{R}_b^c)^{-1}\,(\mathbf{D}_a^b\,\mathbf{u}_a+\mathbf{R}_b^a\,\mathbf{D}_c^b\,\mathbf{d}_c)$$

introducing $\mathbf{u}_b$ in

$$\mathbf{u}_c=\mathbf{D}_b^c\,\mathbf{u}_b+\mathbf{R}_c^b\,\mathbf{d}_c$$

one has

$$\begin{aligned}\mathbf{u}_c&=\mathbf{D}_b^c(\mathbf{I}-\mathbf{R}_b^a\mathbf{R}_b^c)^{-1}\,(\mathbf{D}_a^b\,\mathbf{u}_a+\mathbf{R}_b^a\,\mathbf{D}_c^b\,\mathbf{d}_c)+\mathbf{R}_c^b\,\mathbf{d}_c\\ &=\mathbf{D}_a^b\,\mathbf{u}_a+\mathbf{R}_c^a\,\mathbf{d}_c\end{aligned}$$

so that

$$\begin{aligned}\mathbf{R}_c^a&=\mathbf{R}_c^b+\mathbf{D}_b^c(\mathbf{I}-\mathbf{R}_b^a\mathbf{R}_b^c)^{-1}\,\mathbf{R}_b^a\,\mathbf{D}_c^b\\ &=\mathbf{R}_c^b+\mathbf{D}_b^c\,\mathbf{R}_b^a(\mathbf{I}-\mathbf{R}_b^c\,\mathbf{R}_b^a)^{-1}\,\mathbf{D}_c^b\ {}^{4}\\ \mathbf{D}_a^c&=\mathbf{D}_b^c\,(\mathbf{I}-\mathbf{R}_b^a\mathbf{R}_b^c)^{-1}\,\mathbf{D}_a^b.\end{aligned} \tag{4.3}$$

One gets in a similar way

$$\begin{aligned}\mathbf{R}_a^c&=\mathbf{R}_a^b+\mathbf{D}_b^a(\mathbf{I}-\mathbf{R}_b^c\,\mathbf{R}_b^a)^{-1}\,\mathbf{R}_b^c\,\mathbf{D}_a^b\\ &=\mathbf{R}_a^b+\mathbf{D}_b^a\,\mathbf{R}_b^c\,(\mathbf{I}-\mathbf{R}_b^a\mathbf{R}_b^c)^{-1}\,\mathbf{D}_a^b\\ \mathbf{D}_c^a&=\mathbf{D}_b^a\,(\mathbf{I}-\mathbf{R}_b^c\,\mathbf{R}_b^a)^{-1}\,\mathbf{D}_c^b.\end{aligned} \tag{4.4}$$

With the aid of these relations, it is possible to determine step by step the submatrices of the diffusion matrices using operations which avoid the multiplication of matrizants, and which, except for the inverses, use only matrices the terms of which have modulus less than unity.

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4 Because of the identity

$$(\mathbf{I}-\mathbf{A}\mathbf{B})^{-1}\,\mathbf{A}\equiv\mathbf{A}(\mathbf{I}-\mathbf{B}\mathbf{A})^{-1}.$$

From a formal point of view, these relations are analogous to those obtained by Altman and Cory (1969b) in their application of the thin-film optical method (Born and Wolf, 1970, p. 51–70).

5. Reflection and Transmission Matrices of a Medium

One considers a stratified medium between the levels z_0 and z_s divided into s elementary slabs. With each subslab (say the nth), one associates a lower uniform medium I_n and an upper uniform medium S_n; the set of these media is subject to the conditions $S_n \equiv I_{n+1}$ and $I_1 = S_s$ the latter being related to the vacuum.

If one writes $\mathbf{u}_n = \mathbf{u}(z_n)$ and $\mathbf{d}_n = \mathbf{d}(z_n)$, one knows that

$$\begin{bmatrix} \mathbf{d}_o \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} \mathbf{R}_o^n & \mathbf{D}_n^o \\ \mathbf{D}_o^n & \mathbf{R}_n^o \end{bmatrix} \begin{bmatrix} \mathbf{u}_o \\ \mathbf{d}_n \end{bmatrix} = \mathbf{S}_n \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{d}_n \end{bmatrix} \tag{5.1}$$

where according to (4.3 and 4) the submatrices of S_n are obtained through the relations

$$\begin{aligned}
\mathbf{R}_o^n &= \mathbf{R}_o^{n-1} + \mathbf{D}_{n-1}^o (\mathbf{I} - \mathbf{R}_{n-1}^n \mathbf{R}_{n-1}^o)^{-1} \mathbf{R}_{n-1}^n \mathbf{D}_o^{n-1} \\
&= \mathbf{R}_o^{n-1} + \mathbf{D}_{n-1}^o \mathbf{R}_{n-1}^n (\mathbf{I} - \mathbf{R}_{n-1}^o \mathbf{R}_{n-1}^n)^{-1} \mathbf{D}_o^{n-1} \\
\mathbf{D}_o^n &= \mathbf{D}_{n-1}^n (\mathbf{I} - \mathbf{R}_{n-1}^o \mathbf{R}_{n-1}^n)^{-1} \mathbf{D}_o^{n-1} \\
\mathbf{D}_n^o &= \mathbf{D}_{n-1}^o (\mathbf{I} - \mathbf{R}_{n-1}^n \mathbf{R}_{n-1}^o)^{-1} \mathbf{D}_n^{n-1} \\
\mathbf{R}_n^o &= \mathbf{R}_n^{n-1} + \mathbf{D}_{n-1}^n (\mathbf{I} - \mathbf{R}_{n-1}^o \mathbf{R}_{n-1}^n)^{-1} \mathbf{R}_{n-1}^o \mathbf{D}_n^{n-1} \\
&= \mathbf{R}_n^{n-1} + \mathbf{D}_{n-1}^n \mathbf{R}_{n-1}^o (\mathbf{I} - \mathbf{R}_{n-1}^n \mathbf{R}_{n-1}^o)^{-1} \mathbf{D}_n^{n-1}.
\end{aligned} \tag{5.2}$$

Starting with $\mathbf{R}_o^o = \mathbf{0}$ and $\mathbf{D}_o^o = \mathbf{I}$ and applying s times the relations (5.2), one gets:

$$\begin{bmatrix} \mathbf{d}_o \\ \mathbf{u}_s \end{bmatrix} = \begin{bmatrix} \mathbf{R}_o^s & \mathbf{D}_s^o \\ \mathbf{D}_o^s & \mathbf{R}_s^o \end{bmatrix} \begin{bmatrix} \mathbf{u}_o \\ \mathbf{d}_s \end{bmatrix} \tag{5.3}$$

where $\mathbf{R}_o^s$ and $\mathbf{D}_o^s$ are respectively the reflection and the transmission matrices of the medium when the source is located in the medium I_1 while $\mathbf{R}_s^o$ and $\mathbf{D}_s^o$ are the corresponding matrices when the emission takes place in the medium S_s. Particularly, if the columnvectors of the matrix $\mathbf{Q}$ in the vacuum are put in the following order: upgoing parallel, upgoing perpendicular, downgoing parallel and downgoing perpendicular, one gets with the usual notations

$$\mathbf{R}_o^s = \begin{bmatrix} {}_{\parallel}\mathbf{R}_{\parallel} & {}_{\parallel}\mathbf{R}_{\perp} \\ {}_{\perp}\mathbf{R}_{\parallel} & {}_{\perp}\mathbf{R}_{\perp} \end{bmatrix}; \quad \mathbf{D}_o^s = \begin{bmatrix} {}_{\parallel}\mathbf{D}_{\parallel} & {}_{\parallel}\mathbf{D}_{\perp} \\ {}_{\perp}\mathbf{D}_{\parallel} & {}_{\perp}\mathbf{D}_{\perp} \end{bmatrix}. \tag{5.4}$$

6. Independent Solutions of the Differential Equation

If the application of the algorithm is concerned with the fourvector $\mathbf{f}$ (and of the fields) at the subdivision levels, one has previously (a) to store all the submatrices $\mathbf{R}_n^o$ and $\mathbf{D}_o^n$, (b) to compute the submatrices corresponding to the integration from the top to the bottom of the medium; namely:

$$\begin{aligned}
\mathbf{R}_s^n &= \mathbf{R}_s^{n+1} + \mathbf{D}_{n+1}^s (\mathbf{I} - \mathbf{R}_{n+1}^n \mathbf{R}_{n+1}^s)^{-1} \mathbf{R}_{n+1}^n \mathbf{D}_s^{n+1} \\
&= \mathbf{R}_s^{n+1} + \mathbf{D}_{n+1}^s \mathbf{R}_{n+1}^n (\mathbf{I} - \mathbf{R}_{n+1}^s \mathbf{R}_{n+1}^n)^{-1} \mathbf{D}_s^{n+1} \\
\mathbf{D}_s^n &= \mathbf{D}_{n+1}^n (\mathbf{I} - \mathbf{R}_{n+1}^s \mathbf{R}_{n+1}^n)^{-1} \mathbf{D}_s^{n+1} \\
\mathbf{D}_n^s &= \mathbf{D}_{n+1}^s (\mathbf{I} - \mathbf{R}_{n+1}^n \mathbf{R}_{n+1}^s)^{-1} \mathbf{D}_n^{n+1} \\
\mathbf{R}_n^s &= \mathbf{R}_n^{n+1} + \mathbf{D}_{n+1}^n (\mathbf{I} - \mathbf{R}_{n+1}^s \mathbf{R}_{n+1}^n)^{-1} \mathbf{R}_{n+1}^s \mathbf{D}_n^{n+1} \\
&= \mathbf{R}_n^{n+1} + \mathbf{D}_{n+1}^n \mathbf{R}_{n+1}^s (\mathbf{I} - \mathbf{R}_{n+1}^n \mathbf{R}_{n+1}^s)^{-1} \mathbf{D}_n^{n+1}
\end{aligned} \tag{6.1}$$

starting with $\mathbf{R}_s^s = \mathbf{0}$ and $\mathbf{D}_s^s = \mathbf{I}$ and to store all the submatrices $\mathbf{R}_n^s$ and $\mathbf{D}_s^n$.

Then, the formation of the two pairs of solutions proceeds as follows:

(a) for the first pair, one admits the limiting condition

$$\mathbf{d}_s = \mathbf{0} \tag{6.2}$$

and then, one looks for solutions corresponding to a source situated under the level z_0. The integration then happens, not with initial conditions, but taking this limiting condition into account; in this way, the growing of unwanted solutions is avoided.

Under this condition, one has at the extreme levels:

$$\mathbf{d}_o = \mathbf{R}_o^s \mathbf{u}_o, \qquad \mathbf{u}_s = \mathbf{D}_o^s \mathbf{u}_o \tag{6.3}$$

and at each intermediate level z_n

$$\mathbf{d}_o = \mathbf{R}_o^n \mathbf{u}_o + \mathbf{D}_n^o \mathbf{d}_n, \qquad \mathbf{u}_n = \mathbf{D}_o^n \mathbf{u}_o + \mathbf{R}_n^o \mathbf{d}_n$$

so that

$$\begin{aligned}
&\mathbf{d}_n = (\mathbf{D}_n^o)^{-1} (\mathbf{d}_o - \mathbf{R}_o^n \mathbf{u}_o) = (\mathbf{D}_n^o)^{-1} (\mathbf{R}_o^s - \mathbf{R}_o^n) \mathbf{u}_o = \mathbf{B}_o^n \mathbf{u}_o \\
&\mathbf{u}_n = \mathbf{D}_o^n \mathbf{u}_o + \mathbf{R}_n^o \mathbf{B}_o^n \mathbf{u}_o = \mathbf{A}_o^n \mathbf{u}_o.
\end{aligned} \tag{6.4}$$

If one notes that owing to (4.4)

$$\mathbf{R}_o^s - \mathbf{R}_o^n = \mathbf{D}_n^o \mathbf{R}_n^s (\mathbf{I} - \mathbf{R}_n^o \mathbf{R}_n^s)^{-1} \mathbf{D}_o^n$$

one obtains for $\mathbf{B}_o^n$ the expression

$$\mathbf{B}_o^n = \mathbf{R}_n^s (\mathbf{I} - \mathbf{R}_n^o \mathbf{R}_n^s)^{-1} \mathbf{D}_o^n.$$

Then, one gets

$$\begin{aligned}
\mathbf{A}_o^n &= \mathbf{D}_o^n + \mathbf{R}_n^o \mathbf{B}_o^n = [\mathbf{I} + \mathbf{R}_n^o \mathbf{R}_n^s (\mathbf{I} - \mathbf{R}_n^o \mathbf{R}_n^s)^{-1}] \mathbf{D}_o^n \\
&= (\mathbf{I} - \mathbf{R}_n^o \mathbf{R}_n^s)^{-1} \mathbf{D}_o^n \; {}^{5}
\end{aligned} \tag{6.5}$$

---

[5] According to the identity

$$\mathbf{I} + \mathbf{AB}(\mathbf{I} - \mathbf{AB})^{-1} \equiv (\mathbf{I} - \mathbf{AB})^{-1}$$

and

$$\mathbf{B}_o^n = \mathbf{R}_n^s \mathbf{A}_o^n \tag{6.6}$$

(b) for the second pair, one admits the limiting condition

$$\mathbf{u}_o = \mathbf{0} \tag{6.7}$$

corresponding to a source placed above the medium.

Then one has at the extreme levels:

$$\mathbf{d}_o = \mathbf{D}_s^o \mathbf{d}_s, \qquad \mathbf{u}_s = \mathbf{R}_s^o \mathbf{d}_s \tag{6.8}$$

and at each intermediate level

$$\begin{aligned} \mathbf{d}_n &= \mathbf{A}_s^n \mathbf{d}_s = (\mathbf{I} - \mathbf{R}_n^s \mathbf{R}_n^o)^{-1} \mathbf{D}_s^n \mathbf{d}_s \\ \mathbf{u}_n &= \mathbf{B}_s^n \mathbf{d}_s = \mathbf{R}_n^o \mathbf{A}_s^n \mathbf{d}_s. \end{aligned} \tag{6.9}$$

The linearity of the differential equation allows the grouping of the two pairs in the form of:

$$\mathbf{c}(z_n) = \begin{bmatrix} \mathbf{u}_n \\ \mathbf{d}_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}_o^n & \mathbf{B}_s^n \\ \mathbf{B}_o^n & \mathbf{A}_s^n \end{bmatrix} \begin{bmatrix} \mathbf{u}_o \\ \mathbf{d}_s \end{bmatrix} \tag{6.10}$$

with, at the extreme levels,

$$\mathbf{c}(z_o) = \begin{bmatrix} \mathbf{u}_o \\ \mathbf{d}_o \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{R}_o^s & \mathbf{D}_s^o \end{bmatrix} \begin{bmatrix} \mathbf{u}_o \\ \mathbf{d}_s \end{bmatrix}; \quad \mathbf{c}(z_s) = \begin{bmatrix} \mathbf{u}_s \\ \mathbf{d}_s \end{bmatrix} = \begin{bmatrix} \mathbf{D}_o^s & \mathbf{R}_s^o \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_o \\ \mathbf{d}_s \end{bmatrix}.$$

Finally, the fourvectors $\mathbf{f}(z_n)$ are obtained from

$$\mathbf{f}(z_n) = \mathbf{Q}(z_n)\mathbf{c}(z_n) = \mathbf{Q}(z_n) \begin{bmatrix} \mathbf{A}_o^n & \mathbf{B}_s^n \\ \mathbf{B}_o^n & \mathbf{A}_s^n \end{bmatrix} \begin{bmatrix} \mathbf{u}_o \\ \mathbf{d}_s \end{bmatrix}. \tag{6.11}$$

7. Choice of the Media I_n and S_n. Interpolation

In the present algorithm, the media I_n and S_n are only introduced in order to define four eigenvectors corresponding, from a physical point of view, to a pair of ascending waves and to another pair of descending waves. These eigenvectors constitute a vectorial basis in terms of which the solution is decomposed at each intermediate level; these choice of these vectors (and of the media) is largely a matter of convenience; only physical reasons can restrict this choice. It has evidently no effect on the final results (6.11) of the integration; only the matrices of diffusion bear the mark of the adopted choice. In most algorithms (Inoue and Horowitz, 1966; Altman and Cory, 1969b; Nagano et al., 1975), the media I_n and S_n are defined using the values of the physical parameters at the levels z_{n-1} and z_n. Then, the solution is decomposed in terms of the characteristic waves at the level reached during the integration. These waves are determined starting from the eigenvalues of the matrix $\mathbf{T}(z_n)$ (in ionospheric propagation, they are the roots of the Booker equation) and then forming the corresponding eigenvectors. This type of decomposition bears interest for the physical interpretation of the propagation conditions. However, this interest must not be overrated.

In practice, the numerical treatment gains in simplicity and speed when each of the media I_n and S_n is the same uniform medium. In the special case of electromagnetic waves, it is advisable to use the vacuum for which no problem of normalisation of the eigenvectors arises.

If the fields are decomposed with respect to components linearly polarized in the plane of incidence (∥ component) and perpendicular to it (⊥ component), one uses the matrix:

$$\mathbf{Q}=\begin{bmatrix} \cos\theta_o\cos\Phi_o & \sin\Phi_o & -\cos\theta_o\cos\Phi_o & \sin\Phi_o \\ \cos\theta_o\sin\Phi_o & -\cos\Phi_o & -\cos\theta_o\sin\Phi_o & -\cos\Phi_o \\ -\sin\Phi_o & \cos\theta_o\cos\Phi_o & \sin\Phi_o & -\cos\theta_o\cos\Phi_o \\ \cos\Phi_o & \cos\theta_o\sin\Phi_o & -\cos\Phi_o & -\cos\theta_o\sin\Phi_o \\ \parallel\uparrow & \perp\uparrow & \parallel\downarrow & \perp\downarrow \end{bmatrix} \tag{7.1}$$

and one obtains, with

$$\mathbf{u}_o=\begin{bmatrix}\mathbf{u}_{o\parallel}\\ \mathbf{u}_{o\perp}\end{bmatrix} \quad \text{and} \quad \mathbf{d}_o=\begin{bmatrix}\mathbf{d}_{o\parallel}\\ \mathbf{d}_{o\perp}\end{bmatrix} \tag{7.2}$$

the matrices

$$\mathbf{R}_o^s=\begin{bmatrix}{}_\parallel\mathbf{R}_\parallel & {}_\parallel\mathbf{R}_\perp\\ {}_\perp\mathbf{R}_\parallel & {}_\perp\mathbf{R}_\perp\end{bmatrix} \quad \text{and} \quad \mathbf{D}_o^s=\begin{bmatrix}{}_\parallel\mathbf{D}_\parallel & {}_\parallel\mathbf{D}_\perp\\ {}_\perp\mathbf{D}_\parallel & {}_\perp\mathbf{D}_\perp\end{bmatrix} \tag{7.3}$$

defining respectively the reflection and the transmission of the considered medium limited from both sides by the vacuum. For circular polarizations (r=right, l=left), one gets directly (Budden, 1961)

$$_c\mathbf{R}_o^s=\begin{bmatrix}{}_r\mathbf{R}_r & {}_r\mathbf{R}_l\\ {}_l\mathbf{R}_r & {}_l\mathbf{R}_l\end{bmatrix}=\mathbf{U}^{-1}\mathbf{R}_o^s\mathbf{U}; \qquad {}_c\mathbf{D}_o^s=\begin{bmatrix}{}_r\mathbf{D}_r & {}_l\mathbf{D}_r\\ {}_r\mathbf{D}_l & {}_l\mathbf{D}_l\end{bmatrix}=\mathbf{U}^{-1}\mathbf{D}_o^s\mathbf{U} \tag{7.4}$$

with

$$\mathbf{U}=2^{-1/2}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}. \tag{7.5}$$

Having computed the diffusion matrix related to a medium S_n at the level z_n, it is not difficult to obtain the diffusion matrix related to an other medium S'_n at the same level. If $\mathbf{Q}'$ is the matrix with the eigenvectors of S'_n one forms the matrix and the partition

$$\mathbf{Q}\,\mathbf{Q}'^{-1}=\begin{bmatrix}\mathbf{V}_1 & \mathbf{V}_2\\ \mathbf{V}_3 & \mathbf{V}_4\end{bmatrix} \tag{7.6}$$

and one applies the relations

$$
\begin{aligned}
\mathbf{R}'^{n}_{o} &= \mathbf{R}^{n}_{o} + \mathbf{D}^{o}_{n}\mathbf{V}_3\mathbf{D}'^{n}_{o} \\
\mathbf{D}'^{n}_{o} &= (\mathbf{V}_1 - \mathbf{R}^{o}_{n}\mathbf{V}_3)^{-1}\mathbf{D}^{n}_{o} \\
\mathbf{D}'^{o}_{n} &= \mathbf{D}^{o}_{n}(\mathbf{V}_3\mathbf{R}'^{o}_{n} + \mathbf{V}_4) \\
\mathbf{R}'^{o}_{n} &= (\mathbf{V}_1 - \mathbf{R}^{o}_{n}\mathbf{V}_3)^{-1}(\mathbf{R}^{o}_{n}\mathbf{V}_4 - \mathbf{V}_2).
\end{aligned}
\tag{7.7}
$$

Finally, in regions where the fields vary very fast with height, it can be useful to compute the fields at closer levels in order to describe or to interpret the results of the integration. The easiest process for this interpolation consists in the computation of the matrizants related with the subslabs and to use the relation (2.1). No swamping has to be feared in this case.

Concluding Remarks

All the parts of the algorithm have been programmed in FORTRAN V and tested on the UNIVAC 1100/40 of the Institut Royal Météorologique, including the path distortion around the singularity of **T** which arises at great height where the collision frequency is very low.

There arose no numerical difficulties for the ionization profiles (sum of Chapman functions) used for the *D*- and *E*-regions; the only difficulties were met with the resolution of the Booker equation and the subsequent formation of the eigenvectors, especially (as is well known) as soon as the roots are not well separated.

Therefore, the systematic use of the vacuum as uniform medium in the case of E.M. waves brings advantages as regards speed, usefulness and also accuracy. Further, if one wants to appeal to characteristic waves at any level, it is sufficient to use (7.7); this will have no influence on the results of the present algorithm.

Finally, this algorithm is suitable for the numerical treatment of any differential equation like (1.8) provided **T** is of class τ_{2p}. Such a case arises, after a change of variable, in the theory of gravity waves within the atmosphere (Volland, 1969, p. 500).

References

Altman, C., Cory, H.: The simple thin-film optical method in electromagnetic wave propagation. Radio Sci. **4** (5), 449–457, 1969a

Altman, C., Cory, H.: The generalized thin-film optical method in electromagnetic wave propagation. Radio Sci. **4** (5), 459–470, 1969b

Altman, C., Cory, H., Fijalkow, E.: Coupling processes in the *D*- and *E*-regions at low and very low frequencies – I: Frequencies less than critical: a study of the daytime ionosphere. J. Atmos. Terr. Phys. **32** (8), 1457–1473, 1970

Altman, C., Fijalkow, E.: Coupling processes in the *D*- and *E*-regions at low and very low frequencies – II: Frequencies greater than critical: a study of the night-time ionosphere. J. Atmos. Terr. Phys. **32** (8), 1475–1488, 1970

Altman, C., Postan, A.: Coupling processes in the D- and E-regions at low and medium frequencies – III: The 'coupling echo' and the transition through 'critical coupling'. J. Atmos. Terr. Phys. **33** (2), 329–341, 1971

Berreman, D.W.: Optics in stratified and anisotropic media: 4×4 matrix formulation. J. Opt. Soc. Am. **62** (4), 502–510, 1972

Booker, H.G.: Oblique propagation of electromagnetic waves in a slowly-varying non-isotropic medium. Proc. R. Soc. London, Ser. A: **155**, 235–257, 1936

Born, M., Wolf, E.: Principles of Optics, 4th edn. pp. 51–70. Oxford: Pergamon Press 1970

Bossy, L.: La détermination précise du matrizant pour la résolution des équations de Maxwell dans une ionosphère stratifiée. Ann. Soc. Sci. Bruxelles **85** (1), 29–38, 1971

Bossy, L., Claes, H.: A numerical method for the integration of the equations governing the propagation of E.M. waves in a plane stratified magnetoionic medium, ELF-VLF Radio Wave Propagation, J.A. Holtet, ed.: pp. 111–116. Dordrecht: D. Reidel 1974

Budden, K.G.: Radio waves in the ionosphere. pp. 90–91. Cambridge: University Press 1961

Budden, K.G.: Full wave solutions for radio waves in a stratified magnetoionic medium. Alta Freq., 28 n° speziale, 167–179, 1969

Clemmow, P.C., Heading, J.: Coupled form of the differential equations governing radio propagation in the ionosphere. Proc. Cambridge Philos. **50**, 319–333, 1954

Gantmacher, F.R.: Théorie des matrices, vol. 2. Paris: Dunod 1966

Honig, E.P., den Engelsen, D.: Reflection from stratified anisotropic media: an alternative method. Opt. Acta **24** (1), 89–95, 1977

Inoue, Y., Horowitz, S.: Numerical solution of full-wave equation with mode coupling. Radio Sci. **1** (8), 957–970, 1966

Lacoume, J.L.: Etude de la propagation des ondes électromagnétiques dans un magnétoplasma stratifié. Ann. Géophys. **23** (4), 477–483, 1967

Nagano, I., Mambo, M., Hutatsuishi, G.: Numerical calculation of electromagnetic waves in an anisotropic multilayered medium. Radio Sci. **10** (6), 611–618, 1975

Rawer, K., Suchy, K.: Radio-observations of the ionosphere, Handbuch der Physik, XLIX/2, Geophysik III. Berlin, Heidelberg, New York: Springer 1967

Teitler, S., Henvis, B.W.: Refraction in stratified anisotropic media. J. Opt. Soc. Am. **60** (6), 830–834, 1970

Volland, H.: Die Streumatrix der Ionosphäre. Arch. Elektr. Übertragung **16** (7), 328–334, 1962a

Volland, H.: Kopplung und Reflexion von elektromagnetischen Wellen in der Ionosphäre. Arch. Elektr. Übertragung **16** (10), 515–524, 1962b

Volland, H.: Die Ausbreitung langer Wellen. Braunschweig: Vieweg 1968

Volland, H.: The upper atmosphere as a multiply refractive medium for neutral air motions. J. Atmos. Terr. Phys. **31** (4), 491–514, 1969

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