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Numerical Investigation of the Spectral Resolving Power of Burg's Maximum Entropy Method*

F. Wenzel and W. Zürn

Geophysikalisches Institut, Universität Karlsruhe, Hertzstr. 16, D-7500 Karlsruhe, Federal Republic of Germany

Abstract. The maximum entropy method (MEM) proposed by Burg (1967) for power spectral analysis has often been cited for superior resolving power (Ulrych and Bishop 1975). The resolving power of classical techniques is limited to $\Delta f \geqq T^{-1}$, where T is the length of the time series or the maximum lag, for the periodogram and the power spectrum respectively. The limits of MEM have, to our knowledge, never been explored. Using signals consisting of two and three decaying sinusoids, with random noise superimposed, we tried to find criteria for the limits of resolution with MEM. The best resolution (in the sense that the real frequency separation in the time series is verified by the analysis with relatively small errors) we found was $\Delta f_{\mathrm{SYN}} \cdot T \approx 0.2$; but this is not a well defined limit.

Key words: Maximum entropy method – Spectral resolution.

Introduction

Numerous papers (Ulrych and Bishop 1975 for references) have dealt with the theory and the assumptions of the maximum entropy method (MEM). Numerical experiments have shown, and applications to geophysical time series have indicated, increased spectral resolution of MEM as compared with the classical methods, the periodogram and power spectrum. However, Chen and Stegen (1974) pointed out that MEM has disadvantages also, the most severe being that the spectra may exhibit peaks with no physical meaning (artificial splitting). Another problem they found is the shifting of the locations of spectral peaks (LOSP), an effect which can be rather strong for short signal durations T, and depends on the initial phases φ_i [see Eq. (1)] of the signals. Amplitudes and line shapes are not as readily interpreted as in classical methods; this was recognized in the first publications on MEM (e.g. Lacoss 1971).

Most of the authors who investigated the properties of MEM using synthetic signals confined their interest to the range where the resolution needed is slightly better than classical methods can achieve, i.e., $\Delta f_{\mathrm{SYN}} \cdot T \geqq 0.27$ (Table 1). In all these cases MEM was clearly able to resolve the lines. Here we explored the range $\Delta f_{\mathrm{SYN}} \cdot T$ between $5 \cdot 10^{-7}$ and 10 in the input series, in order to find the limitations of MEM and possibly some criteria for future applications for spectral resolution. In order to suppress the shifting of the LOSP we used only signals with duration T longer than about 20 times the signal period.

* Contribution No. 225, Geophysikalisches Institut, Universität Karlsruhe

Table 1. Frequency separations in synthetic time series used in the literature to show the resolving power of MEM. Δf is the frequency separation, T the length of the series, $\Delta f \cdot T$ indicates the resolving power: if $\Delta f \cdot T$ is less than 1 the resolution is better than $1/T$

Author	Δf	$1/T$	$\Delta f \cdot T$
Lacoss (1971)	0.15	0.1	1.50
Ulrych et al. (1973)	0.04	0.05	0.80
Ulrych and Bishop (1975)	0.03	0.025	1.20
Radoski et al. (1975)	1.0	0.78	1.28
Baggeroer (1977)	0.5	0.42	1.19
Newman (1977)	0.04	0.067	0.60
Bowling (1977)	2.0	4.0	0.50
Frost (1977)	0.016	0.030	0.52
	0.012	0.030	0.41
	0.010	0.030	0.33
	0.0060	0.015	0.39
	0.0042	0.015	0.27

Numerical Experiments

Synthetic Signals

Synthetic signals x_k of the following type

$$x_k = \sum_{i=1}^{L} A_i \cdot \exp(-\delta_i \cdot k \cdot \Delta T) \cdot \cos(2\pi f_i \cdot k \cdot \Delta T - \varphi_i) + n_k$$

$$k = 1, 2, \ldots, M \qquad (1)$$

were analyzed using Burg's MEM algorithm. The frequency separations in the time series are:

$$\Delta f_{\mathrm{SYN}} = |f_i - f_{i+1}|$$

$L = 2, 3$ is the number of decaying harmonic signals in the series; A_i, δ_i, f_i, φ_i are amplitude, attenuation factor, frequency and initial phase of the i-th harmonic; ΔT is the sampling interval and $M \cdot \Delta T$ is the length of the time series. The noise n_k consists of normally distributed random numbers, the signal-to-noise ratio (SNR) is specified in dB and defined as the ratio of the power in the sinusoids to the power in the random numbers.

These signals could simulate a record of free vibrations of the earth after narrow band-pass filtering. The attenuation fac-

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Table 2. List of parameters for the time series used in this paper. For definitions see Eq. (1)

Series	M	ΔT	T	SNR dB	i	A_i	δ_i	f_i	$\varphi_i(°)$	Remarks
B	500	0.001	0.5	5	1	1	5	100	0	Δf varied from 10 to 5×10^{-7}, the corresponding filter length, where resolution is achieved, from 60 to 400
					2	1	5	$100+\Delta f$	180	
F	100	0.01	1.0	5	1	1	1	20	0	Δf varied from 10 to 0.001, the filter length from 20 to 80. In order to see the influence of the SNR and the phase shift, we also operated with SNR=0 and 30 dB and $\varphi_2=45°$. The results remained unchanged
					2	1	1	$20+\Delta f$	90	
H	100	0.01	1.0	30	1	1	0	20	0	Two undamped sinusoids. Δf varied from 10 to 0.01, the filter length from 20 to 95
					2	1	0	$20+\Delta f$	45	
K	100	0.01	1.0	5	1	1	0	20	0	Three undamped sinusoids. Δf varied from 10 to 0.01, the filter length from 20 to 90
					2	1	0	$20+\Delta f$	45	
					3	1	0	$20+2\Delta f$	90	

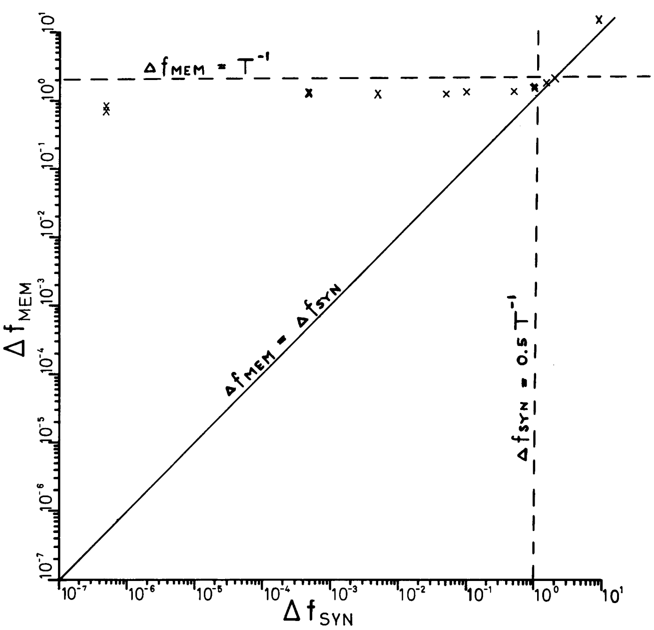

Fig. 1. Plot of $\Delta f_{\rm MEM}$ versus $\Delta f_{\rm SYN}$ for resolved peaks in the MEM spectra of time series B. Δf are measured in units of reciprocal time in the time series (sampling interval $\Delta T=1$ unit)

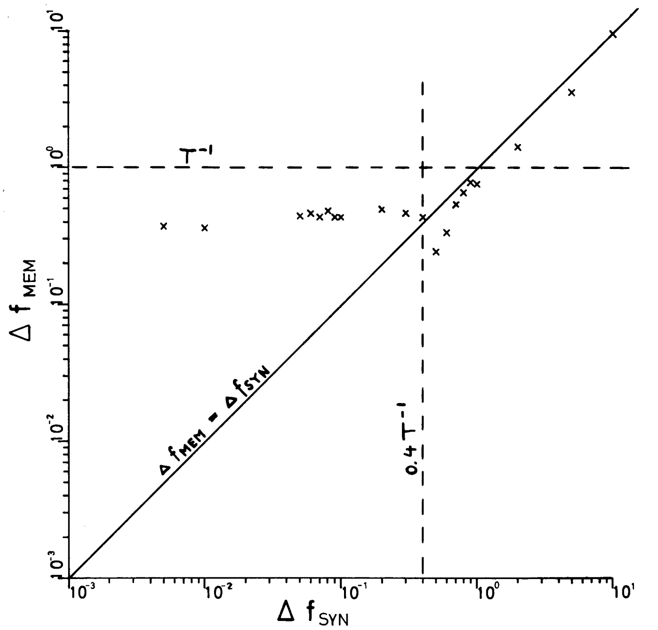

Fig. 2. Plot of $\Delta f_{\rm MEM}$ versus $\Delta f_{\rm SYN}$ for resolved peaks in the MEM spectra of time series F

tors were in fact chosen to be of the order of magnitude observed for free oscillations of the earth.

The frequency differences $\Delta f_{\rm SYN}=|f_i-f_{i+1}|$ were varied over a wide range. The signal parameters and the prediction error filter lengths (NPEF) used are listed in Table 2.

Results

MEM spectra were computed for all these time series and in each case the NPEF was increased until L peaks appeared in the spectrum. The resulting frequency separations $\Delta f_{\rm MEM}$ were plotted in Figs. 1–3 versus the frequency separation in the input time series $\Delta f_{\rm SYN}$. When less or more than L distinct peaks were found, the results were discarded. Ideally the data points should follow a straight line with a slope of 45 degrees, a result which cannot be expected for very small $\Delta f_{\rm SYN}$. For classical techniques no data points would occur for $\Delta f_{\rm SYN}$ smaller than about T^{-1}. It was surprising to find separated peaks with MEM for $\Delta f_{\rm SYN}$ as low as $5\cdot10^{-7}\cdot T^{-1}$ for $L=2$. However, the frequency separation obviously cannot be considered reliable below a certain threshold $f_{\rm SYN}\cdot T=c$. For the three time series B, F, and H we found c to be between 0.4 and 0.5. This threshold does not appear to depend on NPEF, SNR or φ_i.

For the K series, where $L=3$, however, the three constituents were never resolved when $\Delta f_{\rm SYN}\cdot T$ was less than 1. In this case therefore, MEM is only moderately superior to a periodogram as far as resolution is concerned.

It should be mentioned here, that when higher resolution is required than $\Delta f\cdot T\approx1$, for more complicated signals, and when

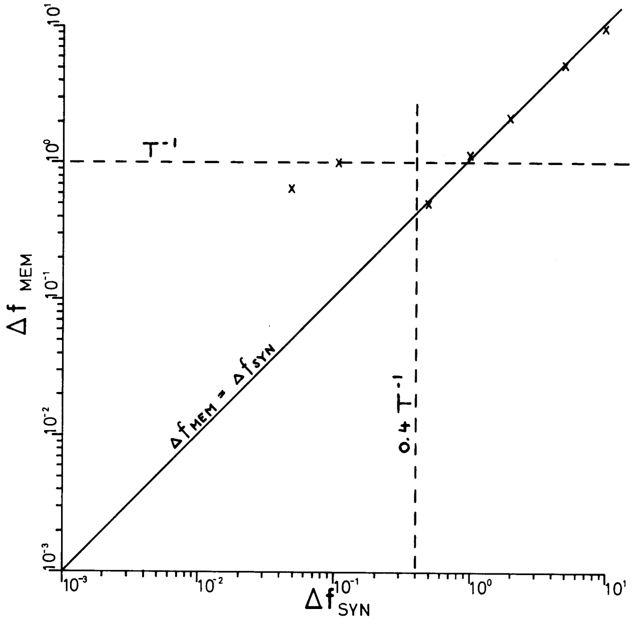

Fig. 3. Plot of Δf_{MEM} versus Δf_{SYN} for resolved peaks in the MEM spectra of time series *H*

NPEF is large, artificial splitting may occur and may complicate, to say the least, the interpretation of real data (e.g. Wenzel 1978).

Discussion

Our experiments demonstrate that for two decaying sinusoids MEM always shows two peaks, but does not give the correct frequency separation if $\Delta f \cdot T$ is less than 0.4. For two undamped sinusoids we found a similar threshold, while for three undamped sinusoids no resolution occurred below $\Delta f \cdot T \approx 1$.

Because of the complexity and nonlinearity of MEM we cannot assess analytically the threshold found, even in the simplest case. Table 1 and our results show that for the simple case of two sinusoids, decaying or not, the resolution of MEM is markedly better than a periodogram's (by a factor of 5 to 10). This property deteriorates as signals become more complicated, as in the free oscillations of the earth, for example (Wenzel 1978).

MEM attenuates the effects of truncation of a time series by prediction filtering and thus virtually increases its length and consequent resolution (Smylie et al. 1973). The quality of the prediction filter found from the data depends on the type of time series under investigation. The best performance of MEM must be expected when autoregressive processes are being analyzed (Ulrych and Bishop 1975).

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