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A Characteristic Method for Numerical Solution of the Inverse Kinematic Seismic Problems*

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Abstract. The problem of determination of a multi-dimensional velocity function, supposed smoothly dependent on coordinates, from the observed travel-times is considered. An accurate mathematical formulation of this problem is obtained by formulating an inverse problem for a Hamilton-Jacobi-type differential equation. A numerical algorithm is constructed for a medium with velocity increasing monotonically with depth and slightly different from a linear function within any small domain of the medium.

In seismic investigations a problem arises in correcting the initial model of the medium, on the basis of comparison of the model with the observed data. An approach to the solution of this problem employing a linearized formulation of the inverse kinematic problem and a numerical method for the solution of some integral geometry problems are considered.

Questions of solvability, stability and practical applicability of the methods developed are discussed.

Key words: Hamilton formalism – Ray method – Inverse problems – Laterally inhomogeneous media – Numerical solution.

Introduction

At present solution of inverse kinematic problems with the assumption of a one-dimensional law of velocity-distribution in the medium is widely used. Such models of real geological media are, from the present-day viewpoint, imperfect. Some progress has been made in the application of mathematical modelling methods to seismic wave-propagation in complicated media. Inverse problems are solved by many of these methods in two steps:

(a) determination of an initial approximation to the medium;

(b) refinement of the model by optimization methods.

The paper deals with formulations and a numerical method for the solution of these problems in the multidimensional case. Although the problems and the method of solution considered here are of interest in themselves, they are presented under the assumption that the procedures proposed can be applied to seismic investigations[1].

An algorithm is described for the determination of a two dimensional velocity-function, supposed smoothly dependent on coordinates, from the observed travel-times of refracted waves. An accurate mathematical formulation of this problem is equivalent to the problem of the determination of unknown functions in a Hamilton-Jacobi-type differential equation, describing the propagation of refracted waves in an inhomogeneous medium, on the basis of information available about this equation.

In the multidimensional case, the mathematical study of the inverse kinematic problem is one of the basic problems in the theory of improperly-posed problems of mathematical physics (Lavrentiev 1967; Lavrentiev et al. 1970; Romanov 1974a). The accurate mathematical formulation of the inverse kinematic problem considered here was first presented by Belonosova and Alekseev (1967). This paper presents a generalized statement of this problem including the case of three-dimensional medium with smooth interfaces.

Inverse Problems for Refracted Waves

Let us consider a two-dimensional medium, where the travel-velocity distribution for seismic waves is described by the function $v(\xi, \eta)$ depending continuously on the horizontal variable ξ and monotonically increasing with the vertical variable η, which characterizes the depth (Fig. 1). Let us assume that, within any small domain of the medium, the velocity function $v(\xi, \eta)$ differs slightly from the linear function $\bar{v}(\xi, \eta) = v_0 + v_1 \xi + v_2 \eta$, where v_0, v_1, v_2 are constants dependent on the size of the domain and on the properties of the medium in this domain. In other words, the assumption of a sufficiently-smooth change of the medium properties is introduced.

1. As is generally well-known, the travel-time of a refracted wave between the points A and B in the medium equals the integral value

$$I = \int_A^B \frac{dS}{v(\xi, \eta)} \tag{1}$$

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[1] Space does not allow presentation of all mathematical calculations and proofs, therefore presentation of some ideas is schematic. But we hope that this will not present difficulties for advanced readers

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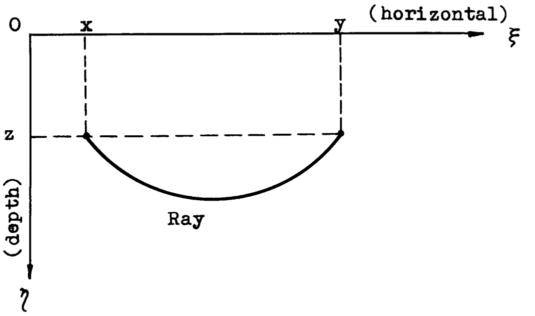

Fig. 1. Orientation of axes and labelling of ray-path in formulation of the inverse problem

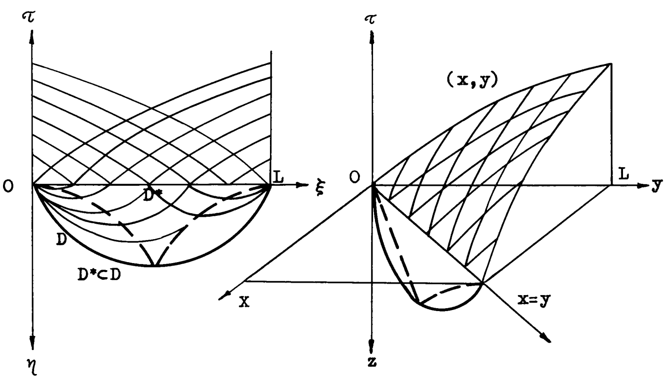

Fig. 2. Travel-time curves and their representation $\varphi(x, y)$, x and y are source and receiver coordinates. D is the lightened domain. D^* is the domain of stability of the inverse-problem solution

taken along the seismic ray-path connecting these points. Let A and B lie on some straight line $\eta = z$ parallel to the axis $O\xi$ and have the coordinates (x, z), (y, z) respectively. Different points A and B on different lines correspond to different values of the travel-time between them, or, in other words, the value of integral (1) is a function of the coordinates of the ends of the ray-path:

$$I = \tau(x, y, z).$$

2. Now let a complete system of travel time curves of refracted waves be given for the profile of observations of length L (or within the interval $[0, L]$ of the medium surface), i.e., the function $\tau(x, y, 0) = \varphi(x, y)$ is known (Fig. 2). The principle of reciprocity of the receiver-point and the shotpoint gives $\varphi(x, y) = \varphi(y, x)$, so one can limit oneself to the case $x \leqq y$. So, in consideration of the travel time-curve system, if $(x, 0)$ are the shotpoint coordinates, then $(y, 0)$ are the receiver coordinates. Assuming monotonic velocity increase with depth for the observational system considered, the seismic rays fill-in the domain D within the medium, limited by the boundary interval $[0, L]$ and the seismic ray joining the boundary points of the profile with the coordinates $(0, 0)$ and $(0, L)$. This domain is often called the domain "lightened" by the given system of travel-time-curves.

The Gaussian curvature of a manifold whose metric is given in the form $ds^2 = v^{-2}(\xi, \eta)(d\xi^2 + d\eta^2)$ equals $K = v^2 \Delta \ln v$. Suppose that everywhere in $D, K \leqq 0$. Then, making use of the Gauss-Bonnet theorem, one can easily show the absence of adjoint points (i.e., intersections of geodesics coming from the same point or caustics of rays). Condition $\partial v/\partial \eta > 0$ provides the return of the ray to the line $\eta = 0$.

3. Let us set the inverse kinematic problem, i.e., determine the velocity function $v(\xi, \eta)$ in the domain D using the observed travel times $\varphi(x, y)$ of refracted waves.

If the point A in (1) has the coordinates (x, z_1), the point B has the coordinates (y, z_2), then $I = T(x, z_1, y, z_2)$ is a function of coordinates of A and B. With A or B fixed, one finds eikonal equations

$$\left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z_2}\right)^2 = v^{-2}(y, z_2), \quad \left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial z_1}\right)^2 = v^{-2}(x, z_1).$$

Consider the functions $z_1(z) = z$, $z_2(z) = z$, $\tau(x, y, z) = T[x, z_1(z), y, z_2(z)]$. Evidently, $\partial\tau/\partial x = \partial T/\partial x$, $\partial\tau/\partial y = \partial T/\partial y$, $\partial\tau/\partial z = \partial T/\partial z_1 + \partial T/\partial z_2$.
Now, expressing the right-hand side of the latter equality from the eikonal equations, having made the necessary substitutions and taken $\partial\tau/\partial z < 0$ into consideration, with $x < y$, we arrive at the equation satisfied by the function $\tau(x, y, z)$:

$$\frac{\partial \tau}{\partial z} + \sqrt{f_1^2(x, z) - \left(\frac{\partial \tau}{\partial x}\right)^2} + \sqrt{f_2^2(y, z) - \left(\frac{\partial \tau}{\partial y}\right)^2} = 0. \tag{2}$$

Here f_1 and f_2 are related to the slowness at the points (x, z), (y, z):

$$f_1(x, z) = -v^{-1}(x, z), \qquad f_2(y, z) = v^{-1}(y, z).$$

An accurate mathematical formulation of the inverse kinematic problem is given as the problem of determining the unknown functions f_1, f_2 in the Hamilton-Jacobi-type Eq. (2), if

$$\tau(x, y, 0) = \varphi(x, y), \qquad 0 \leqq x \leqq y \leqq L < \infty. \tag{3}$$

4. With the above assumptions about the velocity, the characteristic method is applied for the numerical solution of the inverse problem as formulated, whose essence is as follows: with f_1 and f_2 given [that is, given the function $v(\xi, \eta)$], the solution of Cauchy problem (2)–(3) is equivalent to the solution of the characteristic system of ordinary differential equations

$$\frac{dx}{dz} = -\frac{p}{\sqrt{f_1^2 - p^2}}, \qquad \frac{dy}{dz} = -\frac{q}{\sqrt{f_2^2 - q^2}},$$
$$\frac{dp}{dz} = -\frac{f_1 \frac{\partial f_1}{\partial x}}{\sqrt{f_1^2 - p^2}}, \qquad \frac{dq}{dz} = -\frac{f_2 \frac{\partial f_2}{\partial y}}{\sqrt{f_2^2 - q^2}} \tag{4}$$

with the initial conditions

$$x(0) = x^0, \qquad y(0) = y^0, \quad 0 \leqq x^0 \leqq y^0 \leqq L < \infty,$$

$$p(0) = p^0 = \left.\frac{\partial \varphi}{\partial x}\right|_{\substack{x = x^0 \\ y = y^0}}, \qquad q(0) = q^0 = \left.\frac{\partial \varphi}{\partial y}\right|_{\substack{x = x^0 \\ y = y^0}}. \tag{5}$$

A pair of functions $x(z)$, $y(z)$ defines a seismic ray (its ascending and descending branches respectively) joining the points (x, z), (y, z). The travel-time $\tau[x(z), y(z), z]$ is found from the equation

$$\frac{d\tau}{dz} = -\frac{f_1^2}{\sqrt{f_1^2 - p^2}} - \frac{f_2^2}{\sqrt{f_2^2 - q^2}} \tag{6}$$

with the initial condition

$$\tau(0)=\tau^0=\varphi(x^0, y^0). \quad (7)$$

Therefore the inverse problem is to determine the functions $f_1(x, z)$, $f_2(y, z)$, which are part of the right-hand sides of system (4), (6), using the data (5), (7). The domain P_k will be determined in the form of a band, cut out of the domain D by the straight lines $\eta=kh$ and $\eta=(k+1)h$, where h is the value of the numerical integration-step of system (4), (6), $k=0, 1, 2, \ldots$

5. The numerical method for solution of the inverse problem uses successive recalculation of initial data (5), (7) along the characteristics (rays), determined by the system of Eqs. (4), by local determination of the functions f_1, f_2 in the domains P_k using the following algorithm.

(a) Take $k=0$ (i.e., the given band P_0, where $0\leqq z\leqq h$). Let us determine within the domain $0\leqq x\leqq y\leqq L<\infty$ with $\eta=z=0$ a set $M^{(k)}$ of discrete points (x^0, y^0) rather densely distributed. For example, let (x^0, y^0) coincide with the values of coordinates of shotpoints and receivers respectively on the profile; the values p^0, q^0, τ^0 for all the points of the set $M^{(k)}$ are calculated with regard to (5), (7).

(b) Let us specify a certain number δ and take two sets $M_\delta^{(k)}$ and $M_L^{(k)}$ from $M^{(k)}$ with the condition that the point (x^0, y^0) belongs to $M_\delta^{(k)}$ if $y^0-x^0\leqq\delta$ and it belongs to $M_L^{(k)}$ if $y^0-x^0>\delta$ (δ is chosen so that the points of $M_\delta^{(k)}$ are uniformly distributed along the profile and the above assumption, that $|v-\bar{v}|$ is small for all the points of $M_\delta^{(k)}$, is satisfied.

(c) Let us consider the respective Cauchy problem (4) to (7) for the points $M_\delta^{(k)}$. With the given assumptions of locality and the choice of $M_\delta^{(k)}$, in the vicinity of each of (x^0, y^0), $v(\xi, \eta)$ differs little from $\bar{v}(\xi,\eta)$. Therefore, setting $v=\bar{v}$ in (4), (6), relations expressing the parameters v_0, v_1, v_2 in terms of p^0, q^0, τ^0, x^0, y^0 are found (Romanov 1972), i.e., in the vicinity of each point $(x^0, y^0)\in M_\delta^{(k)}$ on the plane $z=kh$, parameters v_0, v_1, v_2 of local approximations $\bar{v}(\xi,\eta)$ to the unknown velocity function $v(\xi,\eta)$ are determined within the band P_k.

(d) Applying a method of smoothing by spline functions we sew the local approximations $\bar{v}(\xi,\eta)$ into the smooth function $v(\xi,\eta)$ within the band P_k.

A stable solution to the inverse problem can be obtained only as follows. Within the domain (Fig. 2) where the solution of the inverse problem is sought, we define the domain D^* as limited by the set of points of maximum depth on each ray on the given observational profile. The function $v(\xi,\eta)$ can be stably determined [according to items (c), (d)] only on the intersection of the band P_k and D^*. Therefore we call D^* the domain of stability of the inverse problem solution.

(e) To determine $v(\xi,\eta)$ outside the domain D^* we use extrapolation or some additional relations for characteristics (see Sect. 3).

(f) Substituting the function obtained, $v(\xi,\eta)$, in the right-hand sides of Eqs. (4) and (6), on the band P_k, the Cauchy problem is solved numerically, with (5) and (7) determined for each point of $M_L^{(k)}$. As a result we have values of the functions $x(z)$, $y(z)$, $p(z)$, $q(z)$, $\tau(z)$ with $z=(k+1)h$. Now a new set $M^{(k+1)}$ is formed by the points $x(z)$, $y(z)$ of the three-dimensional space $\{x, y, z\}$. When solving the Cauchy problem (4) to (7) some characteristics cross the plane given by the equation $x=y$ in three-dimensional space. As a rule these are characteristics originating from points in $M_\delta^{(k)}$. In further calculations those characteristics or rays, whose depth of maximum penetration does not exceed $z=(k+1)h$, do not take part, therefore $M^{(k+1)}$ contains a smaller number of points than $M^{(k)}$.

(g) Changing k into $k+1$, one should come back to item (b) if $M^{(k+1)}$ contains at least one point.

Thus, the algorithm described realizes a recurrent process of successive determination of $v(\xi,\eta)$ within the domain D of uniqueness of the solution of the inverse problem. Here a stable function $v(\xi,\eta)$ is generated in the domain D^*; the accumulation of resultant errors is generally caused by the necessity of using unstable extrapolation procedures beyond the stability domain D^*.

Nevertheless, note that in actual observational systems (especially in seismic prospecting) the range of observational profiles exceeds the maximum distance between the source and receiver in this system: $\max\limits_{(x^0, y^0)}|y^0-x^0|\ll L$. Then the domain of instability $D\backslash D^*$ is significant for determination of the velocity in D^*. Therefore the algorithm with linear extrapolation of the velocity function from D^* to $D\backslash D^*$ is to be applied, for the interpretation of the data from such observational systems.

Method of Refinement for Models of the Medium

The inverse kinematic seismic problem is nonlinear, since seismic ray-paths are to be determined along with the velocity function. In some cases the linear inverse problem can be formulated.

In seismic studies an approximate model of the inhomogeneous medium investigated, or some velocity function $v_0(\xi,\eta)$ can be given on the basis of some a priori data or by approximate methods employing different information in the observed wave field. In particular, the inverse kinematic problem of the determination of the velocity function $v_0(\xi,\eta)$ making use of the observed travel time curves of refracted waves $\varphi(x, y)$ has been considered (see Sect. 1).

Application of the characteristic method results in the function $v_0(\xi,\eta)=v(\xi,\eta)-\Delta v(\xi,\eta)$ where $\Delta v(\xi,\eta)$ is a velocity variation – smaller than $v_0(\xi,\eta)$.

Now consider the problem of determination of $\Delta v(\xi,\eta)$ from the difference $\Delta\varphi(x,y)$ of the observed travel time curves $\varphi(x,y)$ and calculated travel time curves $\varphi_0(x,y)$. This problem was considered in Romanov (1974a) and in Lavrentiev and Romanov (1966) and is reduced to the so-called "problem of integral geometry", if the velocity function $v_0(\xi,\eta)$ is such that the seismic ray $\gamma(x^0, y^0)=\gamma^0$ connecting the points $(x^0, 0)$ and $(y^0, 0)$ is strictly inside that part of the medium that is limited by the ray connecting $(x^1, 0)$ and $(y^1, 0)$ with the condition $0\leqq x^1<x^0<y^0<y^1\leqq L$. This is a linear inverse problem since γ^0 is given. It can be solved numerically by the characteristic method (Romanov 1975).

In fact, let us (similarly to Sect. 1) introduce the function

$$\psi(x, y, z)=\int_{\gamma^0}\Delta n(\xi,\eta)\,ds, \qquad \Delta n=-\frac{\Delta v}{v_0(v_0+\Delta v)}$$

where γ^0 is the seismic ray in the medium with the velocity $v_0(\xi,\eta)$ connecting the points with the coordinates (x, z), (y, z).

Then we arrive at Hamilton-Jacobi equation (see Romanov 1975).

$$\frac{\partial \psi}{\partial z}+\cot \Theta_1 \frac{\partial \psi}{\partial x}+\cot \Theta_2 \frac{\partial \psi}{\partial y}-\frac{f_1}{\sin \Theta_1}-\frac{f_2}{\sin \Theta_2}=0 \qquad (9)$$

satisfied by the function $\psi(x, y, z)$. Here Θ_1, Θ_2 are the angles formed by the axis $O\xi$ and the tangents at the points (x, z), (y, z) of the ray γ^0.

The functions

$$f_1(x, z)=-\Delta n(x, z), \qquad f_2(y, z)=\Delta n(y, z)$$

are to be determined by the given function

$$\psi(x, y, 0)=\Delta \varphi(x, y) \qquad (10)$$

and the rays γ^0 determining the angles Θ_1, Θ_2.

Thus we have an inverse problem similar to that considered in Sect. 1 with the only difference being that the rays in (8) for the model v_0 are given. Hence one can apply the above characteristic method for determination of the velocity model correction $\Delta v(\xi, \eta)$.

Note that initial condition (10) of this problem is approximate if $\psi(x, y, z)$ is introduced in accordance with (8). Condition (10) is determined to within small values of the order of $(\Delta v)^2$ (Romanov 1974a). Therefore, considering $v_1(\xi, \eta)=v_0(\xi, \eta)+\Delta v(\xi, \eta)$ as a new approach to the real velocity function, a new correction can be determined. In other words the method of refinement described for the model of the medium can become the basis of the method of successive approximations for the solution of a multidimensional inverse kinematic problem.

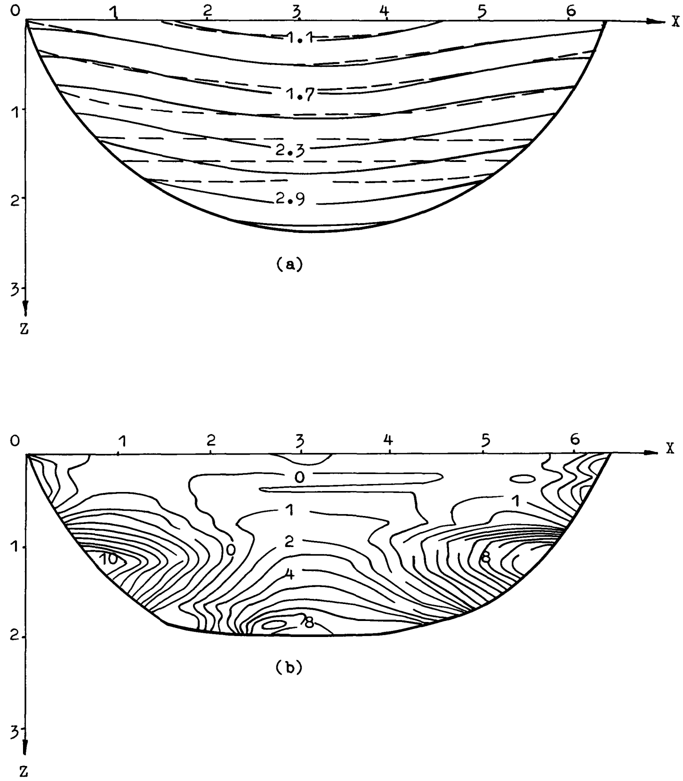

Fig. 3a and b. *Test I.* **a** Isolines of the velocity function $v(x, z)=1.1+0.2\cos(x)+z$ (*solid lines*) and regenerated function (*dashed lines*). **b** Isolines of the error function (in %) show the domain D^*

Discussion of the Results

1. The investigation of the multidimensional inverse kinematic problem is closely connected to that of inverse problems for differential equations and a number of papers of theoretical character have been devoted to the determination of conditions for the uniqueness of solutions. From the point of view of practical applications, the statement of the inverse kinematic problem with initial data on that part of the boundary of the domain where solutions are being sought is most important. It is in this problem that there are some principal difficulties, since the inverse problem here belongs to the class of improperly-posed problems.

In a case where the initial data are given on the whole boundary of the domain of solution, as recent investigations have shown (Mukhometov 1975), the inverse problem appears correct. Geophysical interpretation of this result is that, if the travel-times of a seismic wave between any pairs of points of the boundary of domain, where solution is being sought, are known, the velocity function is determined, stable and unique if it is such that the corresponding family of rays have no caustics, $\Delta \ln v \leqq 0$.

First results on the multidimensional inverse kinematic seismic problem were obtained (Lavrentiev and Romanov 1966; Romanov 1974a) making use of a linearized method. The velocity distribution of seismic waves on the Pamir-Baikal profile was investigated on this basis (Alekseev et al. 1971). Uniqueness of the solution in the class of analytic functions was proved in Anikonov (1969, 1971). In Romanov (1974b) a fairly wide class

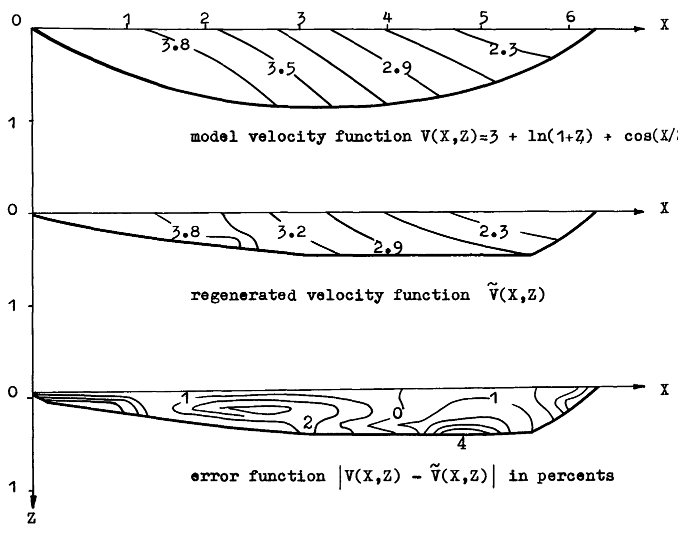

Fig. 4. *Test II:* Isolines of test functions

of functions with unique solutions has been presented, that is the class of functions $n(x, y) \in C^3(D)$, satisfying inequalities $a \leqq n(x, y) \leqq b$, $a \leqq -n'_y(x, y) \leqq b'$ in D and presented in the form

$$f[n(x, y)]=\sum_{k=1}^{N} \varphi_k(x) \psi_k(y)$$

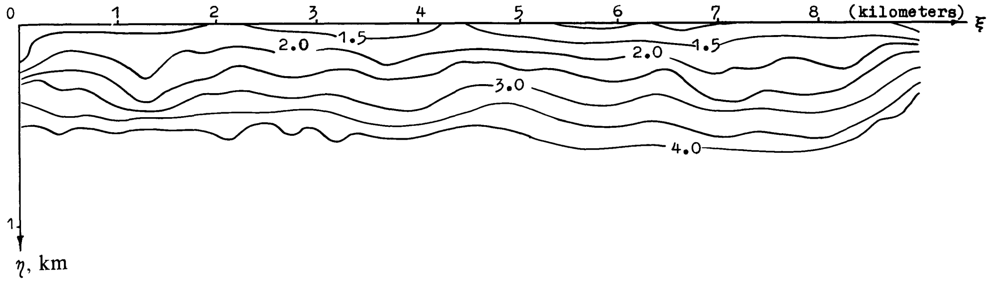

Fig. 5. Example of the velocity cross-section based on seismic prospecting data and regenerated by the characteristic method (values of isolines in km/s)

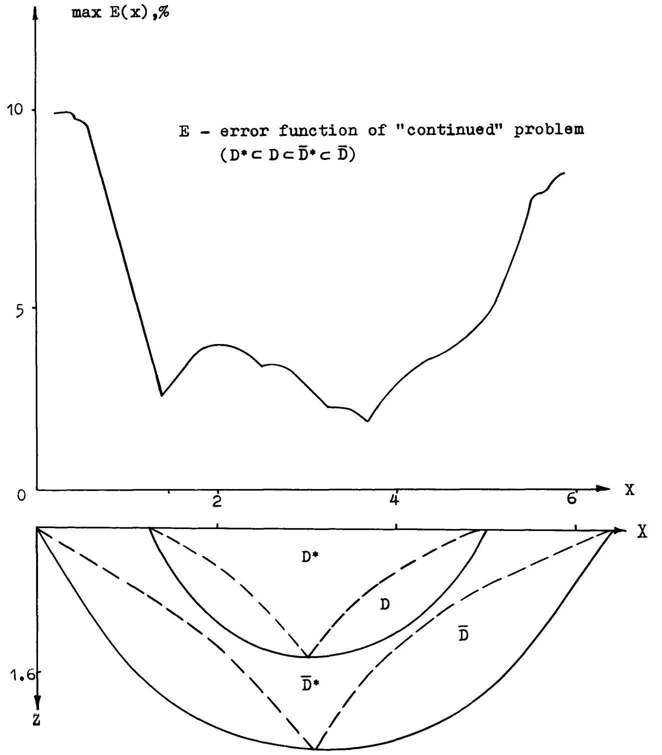

Fig. 6. Results of the regularization experiment for *Test I*. *Dashed lines* correspond to boundary of stability domains, D, D^* are uniqueness and stability domains of the original problem, $\bar{D}$, $\bar{D}^*$ are the same domains of the continued problem. $\bar{D}^*$ contains D, therefore the error-function in D of the continued problem is less than the same function of the original problem (see Fig. 3b)

where

$n(x,y) = v^{-1}(x,y), \quad x=(x_1,\ldots,x_m),$

$D=\{(x,y): |x|<\infty, 0 \leqq y \leqq H\},$

$\varphi_k \in C^3(R^m), \quad \psi_k \in C^3[0,H], \quad a', b'.$

a, b, H are positive constants, $f(z) \in C^3[a,b]$, $|f'(z)|>0$, the variable y being the depth.

Another approach to the investigation of uniqueness of solution of multidimensional inverse problems is described in Anikonov (1971). For the case where the travel time curves satisfy some differential equation, then the velocity function satisfies a corresponding differential equation. On the basis of this approach a number of particular solutions of the multidimensional inverse kinematic problem are obtained (Anikonov and Shasheva 1971; Anikonov 1974). In Jobert (1973) a method for an approximate inversion of the travel-time curves is presented for the three-dimensional case where the surfaces of equal velocity are planes with an infinitesimal dip. In this case our method gives a complete solution for any dips.

2. In our opinion the approach considered here for the solution of the multidimensional inverse kinematic problem has possibilities applicable to the creation of efficient algorithms. A set of programs in ALGOL-60, implementing the above characteristic method for processing real data, obtained from observational systems, was created in Novosibirsk Computing Center, Siberian Branch of the USSR Academy of Sciences. In Figs. 3 and 4 results of test calculations by the characteristic method are shown. An example of isolines of velocity functions generated from the real velocity data are shown in Fig. 5. Data from prospecting profiles, of average length 130 km, have been processed by the characteristic method. The results obtained were used for estimating the velocity parameters of the upper part of the cross-sections of Siberian platform.

3. In the course of numerical experiments it was found that in the uniqueness domain D of the inverse problem the stability domain D^* is distinctly determined. This domain is the set of deeper points of rays in the given finite interval $[0, L]$ of observations $\varphi(x,y)$. Outside the domain D^* the solution of the problem is generally unstable.

If the initial data $\varphi(x,y)$ for problem are given on the whole boundary of the halfspace $\eta \geqq 0$, then $D=\{(\xi,\eta): \eta \geqq 0\}$ and $D^*=D$. In this case the problem is correct (Mukhometov 1975), since the instability domain is absent. But in our problem the data are given on the segment $[0, L]$ thus giving rise to the instability domain $D \backslash D^*$. Therefore our problem belongs to the class of improperly-posed problems. In order to obtain the solution of the improperly-posed problem one should employ a proper regularization method.

However, taking the remark at the end of Sect. 1 into account we can limit ourselves by linear extrapolation of the values f_1 (or f_2) from D^* to $D \backslash D^*$.

The problem can be regularized in three ways. Firstly, if the initial data are continued smoothly beyond the interval of observations, then the stability domain of the continued problem contains the uniqueness domain of the original problem (Fig. 6). In this case an implicit connection between velocity values in D^* with velocity values outside is introduced. Numerical analysis (Romanov 1972) has shown that stability in D can be essentially increased in this way. Secondly, fixing the class of velocity functions where the solution is being sought, one may try to find integrals of Hamilton system (4), (6), thus introducing additional relations on characteristics, determining an explicit connection between the values f_1, f_2 in D. Thirdly, one should consider additionally the variation along the rays of the value

$$r=\frac{\partial \tau}{\partial z}=-\sqrt{f_1^2-p^2}-\sqrt{f_2^2-q^2}.$$

If the value $r(z)$ on the ray is known then the value f_2 is determined from this relation making use of the value f_1 in D^* (or, on the contrary, making use of the value f_2 in D^*, f_1 is determined outside D^*). In this case the Hamilton-Jacobi equation itself is the additional relation which connects the values f_1 and f_2 at the ends of rays.

Stability is also affected by errors arising in numerical implementation of the method: these are errors in obtaining local approximations by the linear functions $\bar{v}(\xi, \eta)$, errors in numerical integration of Eqs. (4), (6) and errors caused by inaccurate initial data (5), (7). However, their estimations require a special discussion and here we note only that an error introduced in local approximation is easily estimated by the method described in Sect. 2. This error is proportional to h^2 and depends on estimations of higher derivatives of the velocity function. To decrease this error in the program for solution of the inverse problem, two blocks are provided: a spline-smoothing block and a block for integration of Eqs. (4), (6) at $v \equiv \bar{v}$ with the step $0.1 \times h$.

The numerical method described for solving the inverse problem is known to have an error of second order in approximation.

4. As compared to other approximate methods employed in seismic practice the methods considered are based on the accurate statement of the inverse problem under conditions providing uniqueness of the inverse-problem solution, i.e., under the a priori assumption of a velocity increase with depth. It is essential that the function of two variables $\varphi(x, y)$ given in the domain $P=\{(x, y)\colon 0 \leqq x \leqq y \leqq L < \infty\}$ is used as the data. The characteristic method considered is based on the discrete set of points $(x^0, y^0) \in P$ where the values $\varphi(x^0, y^0)$, $\partial\varphi(x^0, y^0)/\partial x$, $\partial\varphi(x^0, y^0)/\partial y$ should be given. Therefore, a dense distribution of the points (x^0, y^0) is necessary both for qualitative approximation of $\varphi(x, y)$ and for the determination of the derivatives $\partial\varphi/\partial x$, $\partial\varphi/\partial y$ and the details of the velocity law $v(\xi, \eta)$.

The greater the density of points (x^0, y^0) in the observational system, the better the results.

5. The inverse problem can be also solved when the function $\varphi(x, y)$ is ambiguous (in the presence of loops). In this case, from some point $(x^0, y^0) \in P$, where $\varphi(x, y)$ is ambiguous, rays originate at various angles [different pairs p^0, q^0 in (5)].

6. Sections 1 and 2 describe concrete applications of the theory of Hamilton formalism to the solution of problems in the case of a two-dimensional medium. This formalism can also be applied in the three-dimensional case. In this case the vector-functions

$x(z)=[x_1(z), x_2(z)]$, $y(z)=[y_1(z), y_2(z)]$, $p(z)=[p_1(z), p_2(z)]$,

$q(z)=[q_1(z), q_2(z)]$ should be considered instead of $x(z)$, $y(z)$, $p(z)$, $q(z)$.

Equation (2) also describes the kinematics of reflected waves (see Sect. 4) and the kinematics of waves in a medium with curvilinear interfaces. Therefore the development of this approach to kinematic problems opens possibilities for the de-

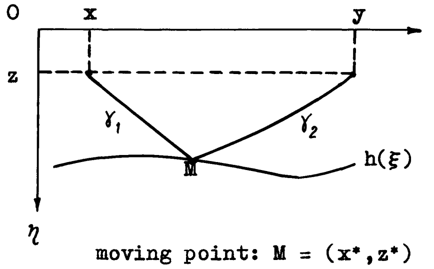

Fig. 7. Orientation of axes and labelling of ray-path and velocity-interface in formulation of the variational problem

velopment of computerized systems of real seismic-data-processing in for example reflection and refraction profiling and seismic deep sounding.

The Characteristic Method – an Application of Hamilton Formalism to Kinematic Problems for Multi-Dimensional Media

Let us consider some general ideas on the application of Hamilton formalism to the kinematic seismic problems. For the sake of simplicity we will restrict ourselves to a two-dimensional medium with one smooth interface, since all the results are valid for three-dimensional and multi-interfaced media.

Let (ξ, η) be a point in the half-space $\eta \geqq 0$ (Fig. 7), $v(\xi, \eta)$—a velocity function with discontinuities on the interface $\eta=h(\xi)$. Consider a curve γ connecting those points in the medium with the coordinates (x, z), (y, z) and consisting of two branches $\gamma_1=\{(\xi, \eta)\colon\ \xi=x(\eta),\ 0 \leqq z \leqq \eta \leqq z^*\}$ and $\gamma_2=\{(\xi, \eta)\colon\ \xi=y(\eta),\ 0 \leqq z \leqq \eta \leqq z^*\}$. For all $\eta \in [z, z^*]$, $x(z^*)=y(z^*)=x^*$, i.e., (x^*, z^*) are the coordinates of the intersection of the branches. Let us define a plane G in the three-dimensional space of the variables $\{x, y, z\}$ by the equation $x=y$ and the interface $\Gamma=\{(x, y, z)\colon x=y=\xi,\ z=h(\xi)\}$. Then all the kinematic seismic problems connected with waves refracted and/or reflected from Γ lead to the consideration of a variational problem of minimizing the functional $J(\gamma)$ with a moving point (x^*, x^*, z^*):

$$J(\gamma)=\int_{z^*}^{z} L(x(\eta), y(\eta), \dot{x}(\eta), \dot{y}(\eta), \eta)\, d\eta \tag{11}$$

where

$$L=f_1[x(\eta), \eta]\sqrt{1+\dot{x}^2(\eta)}-f_2[y(\eta), \eta]\sqrt{1+\dot{y}^2(\eta)},$$
$$\dot{x}=\frac{dx}{d\eta}, \quad \dot{y}=\frac{dy}{d\eta}, \tag{12}$$
$$f_1[x(\eta), \eta]=-\frac{1}{v(x(\eta), \eta)}, \quad f_2[y(\eta), \eta]=\frac{1}{v(y(\eta), \eta)}.$$

For the case of refracted rays the moving point is $(x^*, x^*, z^*) \in G$ and for the case of the rays reflected from Γ, the moving point belongs to the manifold Γ.

If the curve γ is an extremal, i.e., if it realizes the minimum of the functional $J(\gamma)$, the pair of functions $[x(z), y(z)]$ form a seismic ray: either refracted, if $(x^*, x^*, z^*) \in G$, or reflected from Γ, if $(x^*, x^*, z^*) \in \Gamma$. Here $J(\gamma)$ becomes a function of the coordinates of the finite end-points of the extremal γ: $J(\gamma)=\tau(x,y,z)$; the value of this function coincides with the travel-time from the point (x,z) to the point (y,z) and is sometimes called the geodesic distance or eikonal (Courant 1962). The point (x^*, x^*, z^*) along with manifolds G and Γ will be called initial according to $\tau(x^*, x^*, z^*)=0$. On the initial manifolds G and Γ there are transversality conditions, which are obtained from stationary state conditions of the functional $J(\gamma)$ with a moving end. In the case of refracted rays we have

$$\dot{x}(z^*) = -\dot{y}(z^*) = \infty \tag{13}$$

and in the case of reflected rays (Elsgolz 1969) one can easily obtain

$$\frac{h'+\dot{x}}{\sqrt{1+\dot{x}^2}} + \frac{h'+\dot{y}}{\sqrt{1+\dot{y}^2}} = 0, \qquad h' = \left.\frac{dh}{d\xi}\right|_{\xi=x^*} \tag{14}$$

as the condition, describing the law of ray reflection for the moving end.

Considering the function $\tau(x,y,z)$, introduced in the way described above, a theory of direct and inverse kinematic problems in an inhomogeneous medium with interfaces can be developed in the most natural form. Here one may use Hamilton formalism, developed in analytical mechanics, in variational calculus, in the general theory of equations of the first order partial derivatives, as well as in other fields of mathematics, for example, in the theory of optimal control. The application of the results of the latter is especially interesting from the point of view of creating algorithms for the solution of kinematic problems.

It is known (see Sect. 1) that $\tau(x,y,z)$ satisfies the Hamilton-Jacobi equation

$$\frac{\partial\tau}{\partial z} + \sqrt{f_1^2(x,z) - \left(\frac{\partial\tau}{\partial x}\right)^2} + \sqrt{f_2^2(y,z) - \left(\frac{\partial\tau}{\partial y}\right)^2} = 0 \tag{15}$$

where $f_1 = -v^{-1}(x,z)$, $f_2 = v^{-1}(y,z)$. If $z=0$ is the Earth's surface, then in direct seismic problems the problem of determining the function

$$\tau(x,y,0) = \varphi(x,y), \qquad 0 \leqq x \leqq y \leqq L \tag{16}$$

is posed, with a function of velocity distribution $v(\xi,\eta)$ and the interface Γ given (or functions f_1, f_2 in Hamilton-Jacobi equation). Here the function $\varphi(x,y)$ presents a complete set of travel time curves of refracted or reflected waves.

The inverse problem, of the determination of the velocity function $v(\xi,\eta)$ and the interface Γ, can now be posed, using the observed travel time curves, as an inverse problem for the differential Hamilton-Jacobi equation.

In kinematic problems it is natural to pass on to the system of ordinary differential equations equivalent to the Hamilton-Jacobi equation and to determine characteristic curves (from the Hamilton system we have $dz/ds=1$, where s is a parameter, therefore the parameter s is identified with z)

$$\frac{dx}{dz} = \frac{\partial H}{\partial p}, \quad \frac{dy}{dz} = \frac{\partial H}{\partial q}, \quad \frac{dp}{dz} = -\frac{\partial H}{\partial x}, \quad \frac{dq}{dz} = -\frac{\partial H}{\partial y}, \tag{17}$$

$$\frac{d\tau}{dz} = p\frac{\partial H}{\partial p} + q\frac{\partial H}{\partial q} - H \tag{18}$$

$$\frac{dr}{dz} = -\frac{\partial H}{\partial z} \tag{19}$$

where r is a dual variable with respect to z. The system (17) is complete and if the solution of the system is known, then $\tau(z)$ can be found by integration from (18). Equation (19) determines the characteristic function $r(z)$, which can help in solving the inverse problem. In (17)–(19) the Hamilton function is expressed with independent variables

$$H = H(x,y,p,q,z) = \sqrt{f_1^2(x,z) - p^2} + \sqrt{f_2^2(y,z) - q^2}. \tag{20}$$

In direct problems the Cauchy problem is considered for the canonical Hamilton system (17) and Eq. (18), with the initial conditions defined at the moving end. For refracted waves these initial conditions are of the form

$$x(z^*) = x^*, \quad y(z^*) = x^*, \quad p(z^*) = f_1(x^*, z^*),$$
$$q(z^*) = f_2(x^*, z^*), \quad \tau(z^*) = 0. \tag{21}$$

For reflected waves the initial conditions are different only for p and q:

$$p(z^*) = \frac{f_1(x^*, z^*)}{\sqrt{1+\dot{x}^2(z^*)}}\dot{x}(z^*),$$
$$q(z^*) = \frac{-f_2(x^*, z^*)}{\sqrt{1+\dot{y}^2(z^*)}}\dot{y}(z^*), \quad z^* = h(x^*). \tag{22}$$

The values $\dot{x}$, $\dot{y}$ are given in accordance with (14) and determine various values of the reflection angles from the same interface point.

Note that in the case of a three-dimensional medium x, y, p, q are vector-functions $x=(x_1, x_2)$, $y=(y_1, y_2)$, $p=(p_1, p_2)$, $q=(q_1, q_2)$.

In inverse problems unknown functions $f_1(x,z)$, $f_2(y,z)$ [or the function v, see (12)] are included in the right-hand sides of (17) - (19) but the final conditions with $z=0$ are given:

$$x(0) = x^0, \quad y(0) = y^0, \quad p(0) = \left.\frac{\partial\varphi}{\partial x}\right|_{\substack{x=x^0\\ y=y^0}},$$
$$q(0) = \left.\frac{\partial\varphi}{\partial y}\right|_{\substack{x=x^0\\ y=y^0}}, \quad \tau(0) = \varphi(x^0, y^0).$$

Some assumption (see Sect. 1) about the medium and a particular choice of the points (x^0, y^0) on the plane $z=0$ allow one to determine f_1, f_2 approximately, thus reducing the inverse problem to the Cauchy problem similar to (17), (18), (21), the integration of system (17)–(18) being in reverse order (i.e. from the final to the initial point).

Thus the geodesic distance $\tau(x,y,z)$, introduced in the above manner, allows one to consider kinematic seismic problems in inhomogeneous media from a uniform viewpoint.

The approach described is called a characteristic method since its essence lies in the use of the notion of the characteristic curve, i.e., of the ray $[x(z), y(z)]$, along with travel time $\tau(z)$.

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