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Comments on Polarization and Coherence

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Abstract. Measures of the frequency-dependent coherence. and the frequency-dependent degree of polarization have been used for some time in describing the statistical characteristics of multichannel geophysical data. The degree of polarization is rotationally invariant, suggesting that this measure is more useful than the coherence in describing the statistical characteristics of the spatial components of waves. In the present paper generalizations of the bivariate coherence and the degree of polarization for data with an arbitrary number of channels are developed, and the relationship of the coherence to the degree of polarization is described. Some suggestions for applications of the measure of the degree of polarization in the selection and filtering of geophysical data are given.

Key words: Coherence – Polarization – Polarization filters.

Introduction

Measures of the coherence and degree of polarization of vector time-series have been used for some time in evaluating the statistical characteristics of geophysical processes. Jones (1979) has indicated that there does not yet seem to be much appreciation of the close connections between coherence and polarization, at least in the geophysical literature. Jones has limited his discussion and conclusions to two-dimensional processes, but has stated that 'they are equally valid however in three dimensions'. This statement might be somewhat misleading, and consequently I would like to present a more comprehensive evaluation of the measures of coherence and polarization in processes of arbitrary dimension n.

There are numerous reasons for extending these concepts to vector processes of arbitrary dimension, rather than restricting the discussion to three-dimensional spaces. Some examples will serve to illustrate this point. In magnetotellurics, four time series are measured, and the vector process is $\mathbf{x}^T(t)=[e_1(t), e_2(t), b_1(t), b_2(t)]$ where e_j and b_j $(j=1, 2)$ are the electric and magnetic field components respectively. The descriptions of VLF waves in the magnetosphere require a 6-dimensional vector, with 3 magnetic and 3 electric field components (Storey and Lefeuvre 1979). Finally, any array of instruments can be considered to be an $m \times p$ dimensional process, where m is the number of instruments and p is the number of spatial components measured by each instrument.

The Spectral Matrix, Coherence, and Polarization

The information in the vector process $\mathbf{x}^T(t)=[x_1(t), x_2(t) \ldots x_n(t)]$ can be conveniently represented in the frequency domain by using the spectral matrix $\mathbf{S}$ where

$$\mathbf{S}(f, \delta)=\int_{f-\delta}^{f+\delta} \int_{-\infty}^{\infty} \mathbf{C}(\tau)\, e^{-2\pi i g \tau} d\tau\, dg, \tag{1}$$

$$\mathbf{C}(\tau)=\langle \mathbf{x}(t)\, \mathbf{x}^T(t+\tau)\rangle, \tag{2}$$

and $\langle\ \rangle$ denotes the expectation.

The coherence, γ^2, in a coordinate system with bases (j, k) is defined by

$$\gamma_{jk}^2=S_{jk} S_{kj}/(S_{jj} S_{kk}), \tag{3}$$

with $0 \leqq \gamma_{jk}^2 \leqq 1$. For arbitrary directions $\mathbf{r}_1$ and $\mathbf{r}_2$ ($\mathbf{r}_1$ and $\mathbf{r}_2$ are real and orthogonal, i.e., $\mathbf{r}_1^T \mathbf{r}_2=0$), Eq. (3) becomes

$$\gamma^2(\mathbf{r}_1, \mathbf{r}_2)=(\mathbf{r}_1^T \mathbf{S} \mathbf{r}_2\, \mathbf{r}_2^T \mathbf{S} \mathbf{r}_1)/(\mathbf{r}_1^T \mathbf{S} \mathbf{r}_1\, \mathbf{r}_2^T \mathbf{S} \mathbf{r}_2). \tag{4}$$

Equation (4) can be written in a more standard operator format by noting that

$$\mathbf{r}_1^T \mathbf{S} \mathbf{r}_2=\mathrm{Tr}((\mathbf{r}_2 \mathbf{r}_1^T) \mathbf{S}), \tag{5}$$

where Tr denotes the sum of the diagonal elements. Then,

$$\gamma^2(\mathbf{r}_1, \mathbf{r}_2)=\mathrm{Tr}(\mathbf{R}_{12} \mathbf{S})\, \mathrm{Tr}(\mathbf{R}_{21} \mathbf{S})/(\mathrm{Tr}(\mathbf{R}_{11} \mathbf{S})\, \mathrm{Tr}(\mathbf{R}_{22} \mathbf{S})) \tag{6}$$

where $\mathbf{R}_{jk}=\mathbf{r}_j \mathbf{r}_K^T$.

For reasons to be discussed later, it is necessary to further generalize the definition of the coherence by defining this parameter for vector $\mathbf{u}_j$ in a unitary space. A linear vector space will be called unitary if the components of the vector are from the field of complex numbers, and the inner product has the following properties

$$\mathbf{u}_j^\dagger \mathbf{u}_k=(\mathbf{u}_k^\dagger \mathbf{u}_j)^*, \tag{7a}$$

$$(a \mathbf{u}_j)^\dagger \mathbf{u}_k=a \mathbf{u}_j^\dagger \mathbf{u}_k, \tag{7b}$$

$$(\mathbf{u}_j+\mathbf{u}_k)^\dagger \mathbf{u}_l=\mathbf{u}_j^\dagger \mathbf{u}_l+\mathbf{u}_k^\dagger \mathbf{u}_l, \quad \text{and} \tag{7c}$$

$$\mathbf{u}_j^\dagger \mathbf{u}_j=b. \tag{7d}$$

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The symbol † denotes the hermitean adjoint or complex conjugate of the transpose, a and b are real numbers, and b is nonnegative.

For vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in a unitary space ($\mathbf{u}_1^\dagger \mathbf{u}_2 = 0$), the coherence is

$$\gamma^2(\mathbf{u}_1, \mathbf{u}_2) = \mathrm{Tr}(\mathbf{u}_{12}\mathbf{S})\mathrm{Tr}(\mathbf{u}_{21}\mathbf{S})/(\mathrm{Tr}(\mathbf{u}_{11}\mathbf{S})\mathrm{Tr}(\mathbf{u}_{22}\mathbf{S})) \tag{8}$$

where $\mathbf{U}_{jk} = \mathbf{u}_j\mathbf{u}_k^\dagger$. Henceforth, the vectors $\mathbf{r}_j$ will always be in a real space, and the vectors $\mathbf{u}_j$ will be in a unitary space.

In defining the degree of polarization for $n=2$, Jones has used the expansion (Born and Wolf 1964)

$$\mathbf{S} = \mathbf{P} + \mathbf{N} \tag{9}$$

where $\mathbf{P}$ is totally polarized, $\det[\mathbf{P}] = 0$, and $\mathbf{N}$ is unpolarized, $\mathbf{N} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$. The degree of polarization R is given by the ratio of the polarized power to the total power, or

$$R = \mathrm{Tr}\,\mathbf{P}/\mathrm{Tr}\,\mathbf{S}. \tag{10}$$

The expansion in Eq. (8) is not possible for $n>2$ (Samson 1973), and consequently an alternative method must be found to define the degree of polarization.

We first expand $\mathbf{S}$ in the form

$$\mathbf{S} = \sum_{j=1}^{n} \varepsilon_j \mathbf{u}_j \mathbf{u}_j^\dagger \tag{11}$$

where the ε_j and $\mathbf{u}_j$ are respectively the eigenvalues and eigenvectors of $\mathbf{S}$. If $\mathbf{S}$ is purely polarized or is a *pure state*, then $\mathbf{S}$ has only one nonzero eigenvalue and

$$\mathbf{S} = \varepsilon_1 \mathbf{u}_1 \mathbf{u}_1^\dagger. \tag{12}$$

In this case, the vector process can be considered to be 'polarized' in the direction of the unitary vector $\mathbf{u}_1$. Since the information on the polarization is in the eigenvalues of $\mathbf{S}$, we must consider the characteristic equation for the eigenvalues, which is given by

$$\sum_{l=0}^{n} \alpha_l \varepsilon^l = 0. \tag{13}$$

If $\alpha_{n-2} = 0$, then $\mathbf{S}$ is a pure state.

The first three coefficients of the characteristic equation can be written in terms of the invariants $\mathrm{Tr}\,\mathbf{S}$ and $\mathrm{Tr}\,\mathbf{S}^2$ (Samson and Olson 1980) giving

$$\begin{aligned} \alpha_n &= 1.0 \\ \alpha_{n-1} &= -\mathrm{Tr}\,\mathbf{S} \\ \alpha_{n-2} &= \tfrac{1}{2}((\mathrm{Tr}\,\mathbf{S})^2 - \mathrm{Tr}\,\mathbf{S}^2). \end{aligned} \tag{14}$$

Since

$$(\mathrm{Tr}\,\mathbf{S})^2 - (\mathrm{Tr}\,\mathbf{S}^2) \leqq \frac{n-1}{n}(\mathrm{Tr}\,\mathbf{S})^2, \tag{15}$$

(Samson and Olson 1980) we have

$$0 \leqq n(2\alpha_{n-2})/(n-1)(\mathrm{Tr}\,\mathbf{S})^2 \leqq 1, \tag{16}$$

and we can choose our degree of polarization R^2 to be

$$R^2 = (n\,\mathrm{Tr}(\mathbf{S}^2) - (\mathrm{Tr}\,\mathbf{S})^2)/((n-1)(\mathrm{Tr}\,\mathbf{S})^2). \tag{17}$$

If $n=2$, Eq. (16) becomes

$$R^2 = (2(\mathrm{Tr}\,\mathbf{S}^2) - (\mathrm{Tr}\,\mathbf{S})^2)/(\mathrm{Tr}\,\mathbf{S})^2 = 1 - \frac{4\det[\mathbf{S}]}{(\mathrm{Tr}\,\mathbf{S})^2} \tag{18}$$

which is the parameter derived by Born and Wolf (1964).

A Comparison of Coherence and the Degree of Polarization

Having derived the generalizations for the coherence [Eq. (8)], and the degree of polarization [Eq. (17)], we are now in a position to compare the two parameters. Jones (1979) pointed out that in two dimensions, the maximum value of $\gamma^2(\mathbf{r}_1, \mathbf{r}_2)$, under a real rotation, obeys the equality

$$\max(\gamma^2(\mathbf{r}_1, \mathbf{r}_2)) = R^2 \quad (n=2) \tag{19}$$

Relation (19) was derived earlier by Wolf (1959) and by Parrent and Roman (1960) in the study of partially polarized light.

In generalizing the case to $n>2$, I shall show that Eq. (19) is *not true* in general, and that the maximum in the coherence is found by a unitary, not orthonormal (real) transformation. Maximization in a unitary space leads to a maximum value for the coherence which is, for all n, similar to the value for $n=2$. Maximization in a real space leads to quite different results.

The maximization of $\gamma^2(\mathbf{u}_1, \mathbf{u}_2)$ is an extremum problem in a unitary (complex) vector space, and we must determine stationary values of $\gamma^2(\mathbf{u}_1, \mathbf{u}_2)$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal vectors in a unitary space. Thus we must find the stationary points $\mathbf{u}_1$ and $\mathbf{u}_2$ where

$$\begin{aligned} &\gamma^2(\mathbf{u}_1 + \delta\mathbf{u}_1, \mathbf{u}_2) - \gamma^2(\mathbf{u}_1, \mathbf{u}_2) = 0 \quad \text{and} \\ &\gamma^2(\mathbf{u}_1, \mathbf{u}_2 + \delta\mathbf{u}_2) - \gamma^2(\mathbf{u}_1, \mathbf{u}_2) = 0 \end{aligned} \tag{20}$$

with $\delta\mathbf{u}_1$ and $\delta\mathbf{u}_2$ arbitrarily small changes, and subject to the constraint $\mathbf{u}_1^\dagger \mathbf{u}_2 = 0$ (orthogonality).

Using a Lagrangian multiplier formalism, we look for the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ which give stationary values for the function A where

$$\begin{aligned} A = &(\mathbf{u}_1^\dagger \mathbf{S}\mathbf{u}_2)(\mathbf{u}_2^\dagger \mathbf{S}\mathbf{u}_1) + \lambda_0(1 - \mathbf{u}_1^\dagger \mathbf{S}\mathbf{u}_1 \mathbf{u}_2^\dagger \mathbf{S}\mathbf{u}_2) \\ &+ \lambda_1 \mathbf{u}_1^\dagger \mathbf{u}_2 + \lambda_2 \mathbf{u}_2^\dagger \mathbf{u}_1. \end{aligned} \tag{21}$$

The denominator in Eq. (8) has been eliminated by using the constraint with multiplier λ_0. The derivatives $\partial/\partial\lambda_0$, $\partial/\partial\lambda_1$, $\partial/\partial\lambda_2$ give the constraints in finding the stationary values. Then we also have the two equations

$$\begin{aligned} &A(\mathbf{u}_j + \delta\mathbf{u}_j, \mathbf{u}_k) - A(\mathbf{u}_j, \mathbf{u}_k) = 0 \\ &(j=1,\ k=2;\ j=2,\ k=1). \end{aligned} \tag{22}$$

Expanding (22) and neglecting terms second order in $\mathbf{u}_1$ and $\mathbf{u}_2$ we get the two vector-equations

$$\begin{aligned} &\delta\mathbf{u}_j^\dagger(\mathbf{u}_k^\dagger \mathbf{S}\mathbf{u}_j \mathbf{S}\mathbf{u}_k - \lambda_0 \mathbf{u}_k^\dagger \mathbf{S}\mathbf{u}_k \mathbf{S}\mathbf{u}_j + \lambda_j \mathbf{u}_k) \\ &+ (\mathbf{u}_j^\dagger \mathbf{S}\mathbf{u}_k \mathbf{u}_k^\dagger \mathbf{S} - \lambda_0 \mathbf{u}_k^\dagger \mathbf{S}\mathbf{u}_k \mathbf{u}_j^\dagger \mathbf{S} + \lambda_k \mathbf{u}_k^\dagger)\delta\mathbf{u}_j = 0 \\ &(j=1,\ k=2;\ j=2,\ k=1). \end{aligned} \tag{23}$$

Thus we now have two equations of the form

$$\delta\mathbf{u}_1^\dagger \mathbf{a}_1 + \mathbf{a}_2^\dagger \delta\mathbf{u}_1 = 0 \tag{24a}$$

$$\delta\mathbf{u}_2^\dagger \mathbf{b}_1 + \mathbf{b}_2^\dagger \delta\mathbf{u}_2 = 0. \tag{24b}$$

If we consider the variation in the direction $i\delta\mathbf{u}_1$ and $i\delta\mathbf{u}_2$, then Eqs. (24a) and (24b) become

$$\delta\mathbf{u}_1^\dagger\mathbf{a}_1-\mathbf{a}_2^\dagger\delta\mathbf{u}_1=0 \qquad (25\text{a})$$

$$\delta\mathbf{u}_2^\dagger\mathbf{b}_1-\mathbf{b}_2^\dagger\delta\mathbf{u}_2=0. \qquad (25\text{b})$$

It then follows that

$$\mathbf{a}_1=\mathbf{a}_2=\mathbf{b}_1=\mathbf{b}_2=0 \qquad (26)$$

giving four vector-equations.

To determine λ_0 we use the equation

$$\mathbf{u}_1^\dagger\mathbf{a}_1=0 \qquad (27)$$

giving $\lambda_0=\gamma^2(\mathbf{u}_1,\mathbf{u}_2)$. We also note that $\mathbf{b}_1-\mathbf{b}_2=\mathbf{0}$, and consequently $\lambda_1=\lambda_2^*$. Finally we solve for λ_1 using

$$\mathbf{u}_2^\dagger\mathbf{a}_1=0 \quad \text{and} \qquad (28\text{a})$$

$$\mathbf{b}_2^\dagger\mathbf{u}_1=0, \qquad (28\text{b})$$

giving the two equations

$$\lambda_1=\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_1\frac{\mathbf{u}_1^\dagger\mathbf{S}\mathbf{u}}{\mathbf{u}_1^\dagger\mathbf{u}_1}(1-\gamma^2(\mathbf{u}_1,\mathbf{u}_2)), \quad \text{and} \qquad (29\text{a})$$

$$\lambda_1=\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_1\frac{\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_2}{\mathbf{u}_2^\dagger\mathbf{u}_2}(1-\gamma^2(\mathbf{u}_1,\mathbf{u}_2)). \qquad (29\text{b})$$

The multiplier λ_1 is nonzero only if

$$\frac{\mathbf{u}_1^\dagger\mathbf{S}\mathbf{u}_1}{\mathbf{u}_1^\dagger\mathbf{u}_1}=\frac{\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_2}{\mathbf{u}_2^\dagger\mathbf{u}_2}. \qquad (30)$$

Since the value of $\gamma^2(\mathbf{u}_1,\mathbf{u}_2)$ is independent of the magnitudes of $\mathbf{u}_1$ and $\mathbf{u}_2$, we can choose $\mathbf{u}_1^\dagger\mathbf{u}_1=\mathbf{u}_2^\dagger\mathbf{u}_2$, and consequently the condition (30) becomes $\mathbf{u}_1^\dagger\mathbf{S}\mathbf{u}_1=\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_2$.

If either $\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_1=0$ or $\gamma^2(\mathbf{u}_1,\mathbf{u}_2)=1$, then we also have a solution for the stationary values. The former case obviously gives an absolute minimum since $\gamma^2=0$. The latter case gives an absolute maximum, and in this case $\mathbf{u}_1$ and $\mathbf{u}_2$ are not uniquely defined, but need only lie in the plane for which $\gamma^2=1$.

The equations $\mathbf{a}_1=\mathbf{0}$ and $\mathbf{b}_2=\mathbf{0}$ can now be written

$$\mathbf{S}\mathbf{u}_2-\alpha\mathbf{S}\mathbf{u}_1+\beta\mathbf{u}_2=0 \qquad (31\text{a})$$

$$\mathbf{S}\mathbf{u}_1-\alpha^*\mathbf{S}\mathbf{u}_2+\beta\mathbf{u}_1=0 \qquad (31\text{b})$$

where

$$\alpha=\frac{\gamma^2\mathbf{u}_1\mathbf{S}\mathbf{u}_1}{\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_1}=\frac{\gamma^2\mathbf{u}_2\mathbf{S}\mathbf{u}_2}{\mathbf{u}_2^\dagger\mathbf{S}\mathbf{u}_1}, \quad \text{and}$$

$$\beta=\frac{\mathbf{u}_1^\dagger\mathbf{S}\mathbf{u}_1}{\mathbf{u}_1^\dagger\mathbf{u}_1}(1-\gamma^2(\mathbf{u}_1,\mathbf{u}_2)).$$

Solving (31a) and (31b) for $\mathbf{u}_1$ we find

$$\mathbf{u}_1=(\alpha\beta)^{-1}(\alpha^2-1)\mathbf{S}\mathbf{u}_2-\alpha^{-1}\mathbf{u}_2. \qquad (32)$$

Substituting this value for $\mathbf{u}_1$ into Eq. (31a) we obtain the equation

$$\mathbf{A}\mathbf{u}_2+\beta\mathbf{u}_2=0 \qquad (33)$$

where

$$\mathbf{A}=\left(\frac{1-\alpha^2}{\beta}\right)\mathbf{S}^2+2\mathbf{S}.$$

Equation (33) is an eigenvector equation, with eigenvalue β. Since $\mathbf{A}$ is a polynomial of the matrix $\mathbf{S}$, the n eigenvalues of $\mathbf{A}$ are $\left(\frac{1-\alpha^2}{\beta}\right)\varepsilon_j^2+2\varepsilon_j$ $(j=1,n)$, and we have the n equations

$$\beta^2=(1-\alpha^2)\varepsilon_j^2+2\beta\varepsilon_j \quad (j=1,n). \qquad (34)$$

Choosing any two equations and solving for γ^2 gives

$$(\gamma^2(\mathbf{u}_1,\mathbf{u}_2))=\frac{(\varepsilon_j-\varepsilon_k)^2}{(\varepsilon_j+\varepsilon_k)^2}. \qquad (35)$$

An absolute maximum occurs if we choose ε_j to be the maximum eigenvalue of $\mathbf{S}$, and ε_k to be the minimum eigenvalue. Consequently

$$\max(\gamma^2(\mathbf{u}_1,\mathbf{u}_2))=\frac{(\varepsilon_1-\varepsilon_n)^2}{(\varepsilon_1+\varepsilon_n)^2}, \quad \varepsilon_1\geqq\varepsilon_2\ldots\geqq\varepsilon_n. \qquad (36)$$

In general, since the space of real vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is a subspace of the n-dimensional unitary space, we have

$$\max(\gamma^2(\mathbf{r}_1,\mathbf{r}_2))\leqq\max(\gamma^2(\mathbf{u}_1,\mathbf{u}_2)). \qquad (37)$$

The equality applies only when $\mathbf{A}$ in Eq. (33) is a real matrix. For $n=2$ $\mathbf{A}$ can always be written as a real matrix, with a suitable choice for $(1-\alpha^2)/\beta$, and thus the equality is true giving

$$\max(\gamma^2(\mathbf{r}_1,\mathbf{r}_2))=\frac{(\varepsilon_1-\varepsilon_2)^2}{(\varepsilon_1+\varepsilon_2)^2} \quad (n=2). \qquad (38)$$

To complete the discussion, we must consider the relative values of $\max(\gamma^2(\mathbf{u}_1,\mathbf{u}_2))$ and R^2 for arbitrary n. We first rewrite R^2 in the form

$$R^2=\frac{n}{n-1}\frac{\mathrm{Tr}(\mathbf{S}^2)}{(\mathrm{Tr}\,\mathbf{S})^2}-\frac{1}{n-1}. \qquad (39)$$

Now we wish to maximize R^2 by varying the eigenvalues $\varepsilon_j(j=2, n-1)$, and thus we must maximize

$$\mathrm{Tr}(\mathbf{S}^2)/(\mathrm{Tr}\,\mathbf{S})^2=\sum_{j=1}^{n}\varepsilon_j^2\Big/\left(\sum_{j=1}^{n}\varepsilon_j\right)^2 \qquad (40)$$

subject to the constraint $\varepsilon_1\geqq\varepsilon_2\ldots\geqq\varepsilon_n\geqq0$.

Each derivative $\frac{\partial}{\partial\varepsilon_j}(\mathrm{Tr}(\mathbf{S}^2)/(\mathrm{Tr}\,\mathbf{S})^2)$ has at most one stationary point in the region allowed by the constraints, and this point is a local minimum. Thus the maximum must be found at the end points $\varepsilon_j=\varepsilon_1$ $(j=2, n-1)$ or $\varepsilon_j=\varepsilon_n$. Substitution of these two possible sets into (40) shows directly that $\varepsilon_j=\varepsilon_n$, and thus

$$\max(R^2)=\frac{(\varepsilon_1-\varepsilon_n)^2}{(\varepsilon_1+(n-1)\varepsilon_n)^2}. \qquad (41)$$

Comparison of (35) and (41) then shows that

$$\max(\gamma^2(\mathbf{u}_1,\mathbf{u}_2)\geqq R^2. \qquad (42)$$

Discussion

It is clear from the above discussion that the comparison of the coherence values and the degree of polarization is more complicated for $n>2$, and no simple relationship exists to compare

$\max(\gamma^2(\mathbf{u}_1, \mathbf{u}_2))$ or $\max(\gamma^2(\mathbf{r}_1, \mathbf{r}_2))$ with the degree of polarization R^2. For $n=2$, the comparison is far simpler with $\max(\gamma^2(\mathbf{r}_1, \mathbf{r}_2)) = \max(\gamma^2(\mathbf{u}_1, \mathbf{u}_2)) = R^2$ $(n=2)$.

Measures of the degree of polarization of a vector process are in many cases a more objective measure of the statistical characteristics of multichannel data than are the individual coherences. This is particularly true for the analysis of the spatial components of waves. Since the degree of polarization [Eq. (17)] is constructed from the scalar invariants $\mathrm{Tr}\,\mathbf{S}$ and $\mathrm{Tr}\,\mathbf{S}^2$, the value of this measure does not depend on the choice of the coordinate system for orienting the instrument. The value of the coherence depends on the choice of the coordinate-system. Thus the degree of polarization might be considered to be a more 'intrinsic' quantity of the waves.

Measures of the degree of polarization can have many practical applications in the analysis of geophysical data. In most cases, geophysicists are interested in extracting the more polarized waveforms from multidimensional data, and reducing the random noise component. To facilitate these studies, data-adaptive filters which use measures of the degree of polarization can be used to selectively enhance the pure states or polarized waveforms. For example, one can filter the n-dimensional data $\mathbf{x}^T(t) = [x_1(t), x_2(t) \ldots x_n(t)]$ by modulating the Fourier transform of $\mathbf{x}^T(t)$ with the frequency-dependent measure of the degree of polarization. The filtered data $\mathbf{y}(t)$ are then given by

$$\mathbf{y}(t) = \frac{1}{2\pi T} \int_{-\infty}^{\infty} R^2(f)\, \mathbf{s}(f)\, e^{2\pi i f t}\, df, \quad \text{where} \tag{43}$$

$$\mathbf{s}(f) = \int_{-T/2}^{T/2} \mathbf{x}(t)\, e^{-2\pi i f t}\, dt, \tag{44}$$

and T is the temporal length of the data. We have used filters of this type with considerable success in extracting waveforms from multichannel ULF magnetometer data and ULF riometer data.

For a second, and final, example of a practical application of measures of polarization consider the problems inherent in the selection of magnetotelluric data. The magnetotelluric theory assumes that $\mathbf{b} = \mathbf{Z}\mathbf{e}$ where $\mathbf{b}$ is the horizontal magnetic field vector, $\mathbf{e}$ is the two component electric field vector, and $\mathbf{Z}$ is a complex impedance tensor. When the magnetotelluric vector $\mathbf{x}^T(t) = [b_1(t),\ b_2(t),\ e_1(t),\ e_2(t)]$ has a spectral matrix that is completely polarized $(R^2 = 1)$ at some particular frequency, then the data satisfy the impedance-tensor relation. It is possible that selection of magnetotelluric data by using criteria based on measures of the polarization might be more suitable than the coherence-based methods now being used (Goubau et al. 1978, and references therein).

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