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Adiabatic Invariant of the Non-Relativistic Particles in the Field of a Shock Wave*

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Abstract. The asymptotic technique of Krilov and Bogolubov has been used to investigate the motion of charged nonrelativistic particles in the field of the MHD-shock wave. A new adiabatic invariant has been found. For the weakly inhomogeneous field, this coincides with the known one.

In the case of a piston shock wave, after transition through a MHD shock front of limited thickness, the energy of the particle should increase according to the rule $E_\perp/H = \text{const}$.

In the case of an explosive shock wave, when the rarefaction wave follows the compression wave, the energy of the particle does not change.

Key words: Charged particles – MHD shock wave – Adiabatic invariant

This paper deals with the collision of a non-relativistic particle with the front of the MHD shock wave moving across lines of magnetic force $\mathbf{H}=H(x)\,\mathbf{e}_z$. In the coordinate system where the front is immovable, the electric field is constant over the whole area, $\mathbf{E}=E\mathbf{e}_y$ (by choosing the direction of the axis one can always have $E>0$).

The equation of motion of the particle having mass m and charge e acted on by the electric and magnetic fields is:

$$\ddot{\mathbf{r}}=e/m\mathbf{E}+\frac{e}{mc}\,\dot{\mathbf{r}}\times\mathbf{H} \tag{1}$$

where $\mathbf{r}$ is the radius-vector of the particle, c the speed of light.

Let us introduce into Eq. (1) a non-dimensional time equal to the product of dimensional time with the Larmor frequency of the particle in the unperturbed region facing the front $\omega_1=eH_1/mc$, and non-dimensional coordinates equal to the ratio of the dimensional coordinates to the Larmor radius of the particle $R_L=cp_1/eH_1$ in the same region, where p is the particle momentum Projecting (1) on the axis and retaining non-dimensional coordinates denoted by x and y as formely, we have

$$\begin{aligned}\dot{x}&=u\\ \dot{u}&=h(x)\,v\\ \dot{v}&=\varepsilon-h(x)\,u\end{aligned} \tag{2}$$

where $v=\dot{y}$, $\varepsilon=cE/v_1H_1$, $h(x)=H(x)/H_1$. The motion along the axis z is uniform and not connected with the motion along x and y. In the case under consideration $\varepsilon=w/v_1$ is a small value (w being front velocity). In contrast to problems considered by Bogolubov and Mitropolsky (1979), which have been well studied, the function $h(x)$ cannot be assumed to be slowly varying.

Let us have τ as a new variable with the relationship $v=\tau-A(x)$ (see e.g. Khodjayev et al., 1981), where

$$A(x)=\int_0^x h(\xi)\,d\xi.$$

From (2) we have $\dot{\tau}=\varepsilon$, from which we get

$$\tau=\varepsilon(t-t_0)+\tau_0, \qquad \tau_0=v(t_0)+A(x(t_0)).$$

Using the new variables we can write system (2) as

$$\begin{aligned}\dot{x}&=u\\ \dot{u}&=(\tau-A(x))\,h(x)\\ \dot{\tau}&=\varepsilon.\end{aligned} \tag{3}$$

Equation (3) describes the one-dimensional motion of a fictitious particle acted on by a slowly varying field with potential

$$U(x)=\tfrac{1}{2}(\tau-A(x))^2.$$

In the case of the shock wave, the function $h(x)$ does not change its sign. In this case $A(x)$ is a strictly monotonic function and the phase trajectories of system (3) are closed, τ being constant. The motion of the fictitious particle is then oscillation. The centre, amplitude and "period" of these oscillations are slowly varying. The centre of oscillations (guiding centre) x_c, to fairly good accuracy, is the root of the equation

$$\tau-A(x_c)=0$$

the amplitude is x_2-x_1, where x_1, x_2; $x_1<x_2$, are the roots of the equation $2E_\perp=(\tau-A(x))^2$ where

$$E_\perp=\tfrac{1}{2}u^2+\tfrac{1}{2}(\tau-A(x))^2.$$

System (3) has the adiabatic invariant (Arnold, 1979).

$$I(E_\perp,\tau)=\frac{1}{\pi}\int_{x_1}^{x_2}\sqrt{2E_\perp-(\tau-A(\xi))^2}\,d\xi. \tag{4}$$

This means that within the time interval

$$t_0\leqq t\leqq t_0+T/\varepsilon$$

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we have the relationship $|I(t)-I(t_0)| \leqq c\varepsilon$ where c is a number which depends on T but does not depend on ε, i.e. the function is constant to an accuracy of $O(\varepsilon)$ quantities within the time intervals $O(1/\varepsilon)$.

Since

$\partial I/\partial E_\perp = 1/\omega \neq 0$

we can resolve the relationship with respect to $E_\perp = E_\perp(I, \tau)$ and define the energy change to a first order approximation, within time intervals of the order of $1/\varepsilon$ as

$$E_\perp(t_1) - E_\perp(t_0) = E_\perp(I_0, \tau_1) - E_\perp(I_0, \tau_0).$$

It is worth noting that the fluctuation of the magnetic field, having typical scales of the order of the Larmor radius, can easily be shown not to affect the dynamic characteristics of the particle.

The adiabatic invariant $I(E_\perp, \tau)$ is constant within the time intervals $t \sim 1/\varepsilon$ which is equivalent to the displacement along the x axis across the front at a distance of the order of some R_L.

Accordingly, let us consider the energy change of the fast particle transient through the shock front for the three relationships of the front thickness δ and Larmor radius R_L:

1) $\delta \ll R_L$ 2) $\delta \sim R_L$ 3) $\delta \gg R_L$.

In the first two cases we have a transition of the particle from the region of the weakly inhomogeneous field in front of the shock front to that behind the front, moving through the region of strong inhomogeneity. For the weakly inhomogeneous field we have the adiabatic invariant $E_\perp/h(x_c)$ retained, while for the region of strong inhomogeneity the invariant is given by Eq. (4). The adiabatic invariant, according to Eq. (4) for the weakly inhomogeneous field is, with accuracy $O(\varepsilon)$:

$$I = \frac{1}{\pi} \int_{x_1}^{x_2} \sqrt{2E_\perp - h^2(x_c)(\xi - x_c)^2}\, d\xi = E_\perp/h(x_c). \tag{4a}$$

In ofter words adiabatic invariant (4), calculated for the weakly inhomogeneous field conicides with the known one.

Thus, on its transition from the first region to the region of strong inhomogeneity, the particle will have the value of invariant calculated from Eq. (4) which is equal to the value of invariant $E_{\perp 1}/h_1(x_c)$ in the region facing the front. Then, crossing to the region behind the front, the particle will have the value of adiabatic invariant $E_{\perp 2}/h_2(x_c)$ which is equal, by virtue of (4a), to the value of the adiabatic invariant according to Eq. (4) in the region of strong inhomogeneity and the value of invariant $E_{\perp 1}/h_1(x_x)$ in the region before the front. The full transition of the particle from the one region to the other (in spite of strong inhomogeneity) is described by the following relationship

$$\frac{E_{\perp 1}}{h_1} = \frac{E_{\perp 2}}{h_2}. \tag{5}$$

The direct numerical integration of Eq. (2), carried out by the authors, confirms this conclusion.

The numerical study of case $\delta \ll R_L$ was carried out by Pesses [1980] and confirmed our analytical result.

In the third case the field varies slowly over distances of the order of R_L, but the known result discussed by Bogolubov and Mitropolsky [1979] as confirming rule (5) is a trivial consequence of geometry, i.e. $E_\perp$ and h both stay constant at distances of the order of R_L. Obviously further research is needed in order to estimate the energy variation of the particle on its complete transition through the front, i.e. at distances much greater than R_L.

The calculation (using the averaging technique for the equation of motion developed by Krilov, Bogolubov and Mitropolsky) shows that rule (5) is valid within time intervals $t \sim 1/\varepsilon^2$, i.e. within distances much greater than R_L. From the explicit form of the averaged equations it follows that additions to $E_\perp/h$ in the third and higher orders contain grad $h(x)$. Therefore, on the transition of the particle from the region of constant field in front of the shock front to the region of constant field behind the front (both in the case of a compression wave and in the case of a rarefaction wave), rule (5) describing the energy change of the particle holds well.

We should note that generally, in the case of motion of the particle in the slowly varying field, the value $E_\perp/h$ is a non-trivial adiabatic invariant to a first approximation. In this instance, however, the adiabatic invariant occurs as a second approximation and its having a well-known form can be assumed to be a coincidence. It is also important to point out that it is in these time intervals $t \sim 1/\varepsilon^2$ that the adiabatic invariant is conserved.

Thus, in the case of a piston shock wave, while the particle is transient through the MHD shock front at limited thickness, the energy of the particle will increase according to rule (5).

In the case of an explosive shock wave, when the rarefaction wave follows the compression wave, with the field returning to the unperturbed value afterwards, the energy of the particle does not change for any order of the theory of perturbation.

Conclusions

The problem of the transitional charged particle in crossed weak electric and strongly inhomogeneous magnetic fields is reduced to a general solution which is arrived at for the first time. Given the interaction of the non-relativistic charged particle with the field of the MHD shock wave at limited thickness, the adiabatic invariant is found. For any relationship between the radius of gyration of particle R_L and front thickness δ the increase of energy is shown to occur at the transition of the particle through the front according to the following rule: $E_\perp/h = \text{const}$.

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