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A Theory of Forced Magnetohydrodynamic Waves

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Abstract. Geomagnetic micropulsations are believed to be caused by magnetohydrodynamic (MHD) waves in the magnetosphere. The corresponding wave equation is extended by an inhomogeneous term to describe forced vibrations generated by a source. The general solution of this inhomogeneous MHD wave equation is given. It turns out that it is sufficient to solve a simpler equation which is independent of the ambient magnetic field by means of Green's functions and to perform a special transformation characterized by the geometry of the background magnetic field. In an application full space solutions with respect to a homogeneous magnetic field are calculated. In addition, some possible magnetospheric applications are considered.

Key words: MHD waves — Geomagnetic micropulsations — Green's functions — Boundary value problems.

1. Introduction

Since Dungey's (1954) fundamental investigations much effort has been expended in the attempt to find solutions to the magnetohydrodynamic (MHD) wave equation which are in accordance with observed properties of geomagnetic micropulsations. Most of this work deals with the corresponding eigenvalue problem. Of course, any observed pulsation event has to be excited by a source which, roughly speaking, is located in the outer regions of the magnetosphere. Only recently has it been practicable to compare satellite observations accomplished in these source regions with ground based measurements. From these experiments some concepts have been developed relating different kinds of micropulsations to special source mechanisms in the outer magnetosphere (for a recent survey see Southwood, 1981). Therefore, it might be of some interest to study the inhomogeneous MHD wave equation describing MHD vibrations generated by some source distribution. In the present paper a quite general approach will be given to the solution of this problem.

The basic equations are stated the next paragraph. The general procedure and the final integral representation of the solution are presented later, followed by a discussion of the general full space solution in a homogeneous magnetic field and some explicit solutions to some simple source distributions. The final paragraphs summarize the results and consider some possible further applications. In an appendix two simple magnetospheric models are presented to which the methods developed in the paper are applicable in principle.

2. Basic Equations

The theory of geomagnetic micropulsations is usually based on the set of linearized MHD equations. Consider a MHD plasma with density $\rho_0(\mathbf{r})$ in an ambient magnetic field $\mathbf{B}_0(\mathbf{r})$ ($\mathbf{r}$ = position vector). It is assumed that there are no electric currents ($\operatorname{curl}\mathbf{B}_0=0$), no electric fields and volume charges, and no plasma movements. Let this static equilibrium configuration be distorted so that the disturbed magnetic field $\mathbf{B}$ is small compared with $\mathbf{B}_0$, $|\mathbf{B}|\ll|\mathbf{B}_0|$. Neglecting all terms which are small, of second order in the disturbed quantities, the basic equations read (SI units being used throughout the paper)

$$\operatorname{curl}\mathbf{B}(\mathbf{r},t)=\mu_0\mathbf{j}(\mathbf{r},t), \tag{2.1}$$

$$\operatorname{curl}\mathbf{E}(\mathbf{r},t)=-\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}, \tag{2.2}$$

$$\mathbf{E}(\mathbf{r},t)=\mathbf{B}_0(\mathbf{r})\times\mathbf{v}(\mathbf{r},t), \tag{2.3}$$

$$\rho_0(\mathbf{r})\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t}=\mathbf{j}(\mathbf{r},t)\times\mathbf{B}_0(\mathbf{r}). \tag{2.4}$$

Equations (2.1) and (2.2) are Maxwell's equations in the MHD approximation (i.e., neglecting the displacement current) with

μ_0 = magnetic field constant $=4\pi\cdot10^{-7}$ Vs/Am,
$\mathbf{j}$ = electric current density,
$\mathbf{E}$ = electric field,
$\frac{\partial}{\partial t}$ = derivation with respect to time.

Equations (2.3) and (2.4) describe Ohm's law for infinite electrical conductivity and the equation of motion, respectively, where $\mathbf{v}$ is the plasma velocity. The assumption of infinite electrical conductivity as well as neglect-

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ing the gas pressure in Eq. (2.4) are good approximations for the Earth's magnetosphere (Siebert, 1965).

From Eqs. (2.1)–(2.4) one single differential equation can be deduced for any of the vector fields of interest $\mathbf{B}$, $\mathbf{j}$, $\mathbf{E}$, and $\mathbf{v}$. It is customary to choose the electric field $\mathbf{E}$ because it plays the role of the vector potential with respect to the magnetic field disturbance $\mathbf{B}$ for time periodic phenomena (Eq. (2.2)). Supposing a time dependence proportional to $\exp(i\omega t)$ (i = imaginary unit, ω = angular frequency) some manipulations yield ($\text{curl}^2 = \text{curl curl}$)

$$\hat{\mathbf{t}}(\mathbf{r}) \times \hat{\mathbf{t}}(\mathbf{r}) \times \text{curl}^2\, \mathbf{E}(\mathbf{r}) + k^2\, \mathbf{E}(\mathbf{r}) = 0 \tag{2.5}$$

with $\hat{\mathbf{t}} = \mathbf{B}_0/|\mathbf{B}_0|$ the unit vector in the direction of the undisturbed magnetic field and $k = \omega/V_A$. V_A is the Alfvén velocity, $V_A = B_0/\sqrt{\mu_0 \rho_0}$. In the following it is assumed that k = const., i.e., $\mathbf{B}_0(\mathbf{r})$ and $\rho_0(\mathbf{r})$ are related by some function to give a constant Alfvén velocity V_A.

The inhomogeneous MHD wave equation results from (2.5) by adding a "right hand side", say $-\mathbf{Q}(\mathbf{r})$, representing some source function. In terms of an external force density $\mathbf{k}_e(\mathbf{r}, t) = \mathbf{k}_e^0(\mathbf{r}) \exp(i\omega t)$ acting on the configuration, $\mathbf{Q}$ is given by

$$\mathbf{Q} = (i\omega\mu_0/|\mathbf{B}_0|)\, \hat{\mathbf{t}} \times \mathbf{k}_e^0. \tag{2.6}$$

The overall problem of the present paper can then be stated as follows:

Given a scalar constant k and two vector fields $\hat{\mathbf{t}}(\mathbf{r})$ and $\mathbf{Q}(\mathbf{r})$, determine a solution $\mathbf{E}(\mathbf{r})$ of the equation

$$\hat{\mathbf{t}}(\mathbf{r}) \times \hat{\mathbf{t}}(\mathbf{r}) \times \text{curl}^2\, \mathbf{E}(\mathbf{r}) + k^2\, \mathbf{E}(\mathbf{r}) = -\mathbf{Q}(\mathbf{r}) \tag{2.7}$$

in a region V with suitable conditions at the boundaries of V.

3. The Formal Solution of the Inhomogeneous MHD Wave Equation

In this section the formal solution of Eq. (2.7) together with appropriate boundary conditions will be developed using the method of Green's functions. This method for solving inhomogeneous problems is usually based on a knowledge of the eigenfunctions of the corresponding homogeneous equation. Unfortunately, the eigenfunctions of Eq. (2.5) are not generally known. However, a simple transformation of Eq. (2.7) will show a way out of this difficulty.

Making use of the vector identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

for every three vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, Eq. (2.7) can be written as

$$\text{curl}^2\, \mathbf{E} - k^2\, \mathbf{E} = \mathbf{Q} + \mathfrak{P}_t \cdot \text{curl}^2\, \mathbf{E}. \tag{3.1}$$

The second term of the right hand side of Eq. (3.1) represents the scalar product of the dyad (second rank tensor) $\mathfrak{P}_t = \hat{\mathbf{t}}\hat{\mathbf{t}}$ and the vector $\text{curl}^2\, \mathbf{E}$,

$$\mathfrak{P}_t \cdot \text{curl}^2\, \mathbf{E} = \hat{\mathbf{t}}(\hat{\mathbf{t}} \cdot \text{curl}^2\, \mathbf{E}).$$

For the moment, this term involving the influence of the undisturbed magnetic field $\mathbf{B}_0$ will be dropped to shorten the notation. The remaining equation

$$\text{curl}^2\, \mathbf{E} - k^2\, \mathbf{E} = \mathbf{Q} \tag{3.2}$$

is well known from classical electromagnetic theory. It determines the electric field due to a distribution of oscillating electric dipoles with moment $\mathbf{Q}$ (except for dimensional factors). The solution of Eq. (3.2) by means of Green's functions is a classical procedure. Define Green's dyad $\mathfrak{G}(\mathbf{r}|\mathbf{r}')$ by the equation

$$\text{curl}^2\, \mathfrak{G}(\mathbf{r}|\mathbf{r}') - k^2\, \mathfrak{G}(\mathbf{r}|\mathbf{r}') = \mathfrak{J}\,\delta(\mathbf{r} - \mathbf{r}') \tag{3.3}$$

($\mathfrak{J}$ = unit dyad, $\delta(\mathbf{r}-\mathbf{r}')$ = Dirac's delta function, $\mathbf{r}$ = observation point, $\mathbf{r}'$ = source point) satisfying the same boundary conditions as the electric field $\mathbf{E}$. Multiplying Eq. (3.2) by $\mathfrak{G}(\mathbf{r}|\mathbf{r}')$ from the right, Eq. (3.3) by $\mathbf{E}(\mathbf{r})$ from the left, respectively, integrating the difference with respect to $\mathbf{r}$ over the volume V and using Green's vector theorem (Morse and Feshbach, 1953, p. 1768) yields

$$\begin{aligned} \mathbf{E}(\mathbf{r}') = {} & \int_V \mathbf{Q}(\mathbf{r})\mathfrak{G}(\mathbf{r}|\mathbf{r}')\, dV \\ & - \oint_S dS \{\text{curl}\, \mathbf{E}(\mathbf{r}) \cdot [\hat{\boldsymbol{\nu}} \times \mathfrak{G}(\mathbf{r}|\mathbf{r}')] \\ & - [\hat{\boldsymbol{\nu}} \times \mathbf{E}(\mathbf{r})] \cdot \text{curl}\, \mathfrak{G}(\mathbf{r}|\mathbf{r}')\}. \end{aligned} \tag{3.4}$$

In Eq. (3.4) S is the boundary surface of V and $\hat{\boldsymbol{\nu}}$ is the outward normal unit vector.

In the following it is assumed that there are only perfectly conducting boundaries. In this case of zero tangential boundary values the surface integral in Eq. (3.4) vanishes. For exterior problems (i.e., without outer boundaries) a suitable condition at infinity has to be stated. Since Eq. (2.7) belongs to the class of Helmholtz-type differential equations, this condition is the so called Sommerfeld radiation condition. Its appropriate form for the vector Helmholtz equation reads

$$|r\mathbf{E}| < C\,(C = \text{const.}) \quad \text{and} \quad r(\hat{\boldsymbol{\nu}} \times \text{curl}\, \mathbf{E} - ik\mathbf{E}) \to 0 \tag{3.5}$$

uniform with respect to direction as $r \to \infty$ (Jones, 1964, p. 63).

Demanding an equivalent condition for Green's dyad $\mathfrak{G}$, it follows from the boundedness of $r\mathfrak{G}$ that

$$\lim_{r\to\infty} r^2(\hat{\boldsymbol{\nu}} \times \text{curl}\, \mathbf{E} - ik\mathbf{E}) \cdot \mathfrak{G} = 0,$$

and from the boundedness of $r\mathbf{E}$ that

$$\lim_{r\to\infty} r^2 \mathbf{E} \cdot (\hat{\boldsymbol{\nu}} \times \text{curl}\, \mathfrak{G} - ik\mathfrak{G}) = 0.$$

Combining these results yields

$$\lim_{r\to\infty} r^2 [(\hat{\boldsymbol{\nu}} \times \text{curl}\, \mathbf{E}) \cdot \mathfrak{G} - \mathbf{E} \cdot (\hat{\boldsymbol{\nu}} \times \text{curl}\, \mathfrak{G})] = 0.$$

By cyclic permutation of the factors in the triple scalar products and because $dS \propto r^2$ the surface integral in Eq. (3.4) vanishes for all kinds of boundary condition under consideration.

In the remaining terms of Eq. (3.4) interchange $\mathbf{r}$ and $\mathbf{r}'$ and make use of a fundamental property of Green's functions, i.e., their reciprocity,

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}') = \mathfrak{G}^T(\mathbf{r}'|\mathbf{r})$$

($\mathfrak{G}^T$ = transposed dyad). This results in the following integral representation of the solution of Eq. (3.2)

$$\mathbf{E}(\mathbf{r}) = \int_V \mathfrak{G}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{Q}(\mathbf{r}')\, dV' \tag{3.6}$$

where dV' is the volume element with respect to the source point coordinates $(\mathbf{r}')$. Equation (3.6) can be written in a shorter form by introducing an operational notation,

$$\mathbf{E} = G\mathbf{Q}. \tag{3.7}$$

In this form G is an integral operator with dyadic kernel $\mathfrak{G}(\mathbf{r}|\mathbf{r}')$.

Applying this result (Eq. (3.7)) to the complete problem (Eq. (3.1)) the following integral equation, or more exactly, integro-differential equation is obtained

$$\mathbf{E} = G(\mathbf{Q} + \mathfrak{P}_t \cdot \operatorname{curl}^2 \mathbf{E}). \tag{3.8}$$

So far, the anisotropy of the problem caused by the direction of the undisturbed magnetic field and represented by the dyad $\mathfrak{P}_t$ has not been taken into account. A genuine way to overcome this shortcoming is to introduce a special system of reference, i.e., the local triad $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$ with respect to the vector field $\mathbf{B}_0$, as it was brought into discussion by Siebert (1965). The three vectors $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{b}}$ form a right-handed orthogonal set of unit vectors specifying the tangential, the principal normal, and the binormal direction, respectively, at any point of space with respect to the field $\mathbf{B}_0$.

Now, all vector fields are to be decomposed in their components according to that triad (e.g., $\mathbf{F} = F_t\hat{\mathbf{t}} + F_n\hat{\mathbf{n}} + F_b\hat{\mathbf{b}}$ for any vector $\mathbf{F}$). In this sense the following matrix notation is obvious for Green's dyad $\mathfrak{G}$,

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}') = \begin{pmatrix} G_{tt}(\mathbf{r}|\mathbf{r}') & G_{tn}(\mathbf{r}|\mathbf{r}') & G_{tb}(\mathbf{r}|\mathbf{r}') \\ G_{nt}(\mathbf{r}|\mathbf{r}') & G_{nn}(\mathbf{r}|\mathbf{r}') & G_{nb}(\mathbf{r}|\mathbf{r}') \\ G_{bt}(\mathbf{r}|\mathbf{r}') & G_{bn}(\mathbf{r}|\mathbf{r}') & G_{bb}(\mathbf{r}|\mathbf{r}') \end{pmatrix} \tag{3.9}$$

where, e.g., $G_{tt}(\mathbf{r}|\mathbf{r}')$ is given by

$$G_{tt}(\mathbf{r}|\mathbf{r}') = \hat{\mathbf{t}}(\mathbf{r}) \cdot \mathfrak{G}(\mathbf{r}|\mathbf{r}') \cdot \hat{\mathbf{t}}(\mathbf{r}').$$

In a similar way one can introduce a matrix notation for the integral operator G,

$$G = \begin{pmatrix} G_{tt} & G_{tn} & G_{tb} \\ G_{nt} & G_{nn} & G_{nb} \\ G_{bt} & G_{bn} & G_{bb} \end{pmatrix}. \tag{3.10}$$

The components of this operator valued matrix (Eq. (3.10)) are scalar integral operators with kernels given by the corresponding elements of the matrix defined in Eq. (3.9).

This system of reference is of great advantage in describing the fields involved. Special features which frequently are closely connected to the geometry of $\mathbf{B}_0$ can be seen most directly. For example, the condition that the electric field has to be perpendicular to the magnetic field $\mathbf{B}_0$ (Eq. (2.3)) simply reads $E_t = 0$, similarly, $Q_t = 0$ (Eq. (2.6)). Furthermore, rather than specifying the magnetic field $\mathbf{B}_0$ in this stage of analysis, it is possible to continue the calculations quite generally.

Now, after these preliminary remarks concerning the system of reference, Eq. (3.8) can be treated further. It is quite unsatisfactory that the condition $E_t = 0$ cannot be seen directly in this representation. However, it is possible to transform the right hand side of Eq. (3.8) so that it depends only on the given source function $\mathbf{Q}$ (Krummheuer, 1981, p. 60)

$$\mathbf{E} = (G - G\Lambda G)\,\mathbf{Q} \tag{3.11}$$

where

$$\Lambda = \begin{pmatrix} G_{tt}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.12}$$

with G_{tt}^{-1} as the inverse operator to G_{tt}, $G_{tt}G_{tt}^{-1} = I$, I = identity operator. Now it can be shown that the electric field given by Eq. (3.11) possesses all the desired properties. Obviously, the dyad $\mathfrak{P}_t$ can be represented by the operator valued matrix

$$P_t = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then firstly,

$$\mathbf{E}_t = P_t\mathbf{E} = (P_tG - P_tG\Lambda G)\,\mathbf{Q} = (P_tG - P_tG)\,\mathbf{Q} = 0 \tag{3.13}$$

because $P_tG\Lambda = P_t$. Secondly, $\mathbf{E}$ is a solution of the differential equation (3.1) because

$$\operatorname{curl}^2 \mathbf{E} - k^2\mathbf{E} = \mathbf{Q} - \Lambda G\mathbf{Q} \tag{3.14}$$

from Eq. (3.3). Applying the operator P_t to the last equation one obtains

$$P_t \operatorname{curl}^2 \mathbf{E} = -P_t\Lambda G\mathbf{Q} \tag{3.15}$$

using Eq. (3.13), $\mathbf{E}_t = 0$, and $\mathbf{Q}_t = 0$ (Eq. (2.6)). Now, $P_t\Lambda = \Lambda$, and comparing Eqs. (3.15) and (3.14) it follows that

$$\operatorname{curl}^2 \mathbf{E} - k^2\mathbf{E} = \mathbf{Q} + \mathfrak{P}_t \cdot \operatorname{curl}^2 \mathbf{E}$$

which is identical with Eq. (3.1). Thirdly, the representation of the electric field $\mathbf{E}$ by Eq. (3.11) guarantees that the boundary conditions are satisfied, since the kernel $\mathfrak{G}(\mathbf{r}|\mathbf{r}')$ of the integral operator G obeys the same boundary conditions as $\mathbf{E}$.

The remaining problem now is to determine G_{tt}^{-1}. Unfortunately, the components of the matrix defined by Eq. (3.9) do not have the properties of scalar Green's functions. This can be seen by multiplying Eq. (3.3) by $\hat{\mathbf{t}}$, e.g., from the left and the right. The result

$$\hat{\mathbf{t}}(\mathbf{r}) \cdot \operatorname{curl}^2 \mathfrak{G}(\mathbf{r}|\mathbf{r}') \cdot \hat{\mathbf{t}}(\mathbf{r}') - k^2 G_{tt}(\mathbf{r}|\mathbf{r}') = \hat{\mathbf{t}}(\mathbf{r}) \cdot \hat{\mathbf{t}}(\mathbf{r}')\,\delta(\mathbf{r} - \mathbf{r}')$$

shows that the first term cannot be expressed as a scalar differential operator applied to G_{tt}, in general. Therefore, G_{tt}^{-1} is not representable as a scalar differential operator. Only for very simple configurations does it appear possible to find G_{tt}^{-1} analytically. One example will be given in the next Section.

For the general solution (Eq. (3.11)) the following Fredholm integral equation of the first kind has to be solved,

$$G_{tt} f = -(G_{tn} Q_n + G_{tb} Q_b), \tag{3.16}$$

or, more explicitly,

$$\int_V G_{tt}(\mathbf{r}|\mathbf{r}')\, f(\mathbf{r}')\, dV' = -\int_V [G_{tn}(\mathbf{r}|\mathbf{r}')\, Q_n(\mathbf{r}') + G_{tb}(\mathbf{r}|\mathbf{r}')\, Q_b(\mathbf{r}')]\, dV'.$$

Defining a function $\mathbf{Q}'$ by

$$\mathbf{Q}' = f\hat{\mathbf{t}} + \mathbf{Q}$$

with f the solution of Eq. (3.16), the general result can be written as

$$\mathbf{E} = -\hat{\mathbf{t}} \times \hat{\mathbf{t}} \times G\mathbf{Q}'. \tag{3.17}$$

4. Applications to Forced Vibrations in a Homogeneous Magnetic Field

In the preceding Section a general solution of the inhomogeneous MHD wave equation has been found without specifying the ambient magnetic field $\mathbf{B}_0$ (except for $\operatorname{curl} \mathbf{B}_0 = 0$). It is instructive now to look for solutions in simple configurations. Consider a homogeneous magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{x}}$ in a rectangular cartesian coordinate system with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, and require a full space (i.e., an outward radiating) solution $\mathbf{E}(\mathbf{r})$ of Eq. (2.7) generated by an arbitrary source function $\mathbf{Q}(\mathbf{r})$. The corresponding Green's function for this problem is (using dyadic notation)

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}') = \left(\mathfrak{I} + \frac{1}{k^2} \operatorname{grad} \operatorname{grad}\right) \frac{e^{-ikR}}{4\pi R}, \qquad R = |\mathbf{r} - \mathbf{r}'| \tag{4.1}$$

(e.g., Morse and Feshbach, 1953, p. 1780). To apply Eq. (3.11) it is necessary to invert the integral operator $G_{tt} = G_{xx}$ with kernel

$$G_{xx}(\mathbf{r}|\mathbf{r}') = \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial x^2}\right) \frac{e^{-ikR}}{4\pi R}. \tag{4.2}$$

This can be accomplished by looking for a solution $f(\mathbf{r})$ of the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) f(\mathbf{r}) = -q(\mathbf{r}) \tag{4.3}$$

(q some source function) satisfying a radiation condition in *three* dimensional space. Direct substitution into Eq. (4.3) shows that

$$f(\mathbf{r}) = -\frac{1}{2ik} \int_V \frac{e^{-ikR}}{4\pi R} \cdot \left[\int_{-\infty}^{+\infty} e^{-ik|x'-x''|} (\Delta'' + k^2)\, q(\mathbf{r}'')\, dx''\right] dV' \tag{4.4}$$

is the desired solution ($R = |\mathbf{r} - \mathbf{r}'|$, $\mathbf{r} = (x, y, z)$, $\mathbf{r}' = (x', y', z')$, $\mathbf{r}'' = (x'', y', z')$, Δ'' = Laplacian with respect to $\mathbf{r}''$).

To obtain this result one has to proceed quite carefully. For example, the function

$$f(\mathbf{r}) = \frac{1}{2ik} \int_{-\infty}^{+\infty} e^{-ik|x-x'|}\, q(x', y, z)\, dx'$$

is also a solution of Eq. (4.3), but it fulfills 1D radiation condition rather than a 3D one. The radiation condition in 3D space is given by the behavior of the function $\exp(-ikR)/R$ as $R \to \infty$, and it is this kernel in Eq. (4.4) that guarantees the correct behavior at infinity. This dimensional dependence of the radiation condition is the reason for "embedding" the 1D kernel $\exp(-ik|x-x'|)$ in a manner given by Eq. (4.4).

Now, combining Eqs. (4.2) and (4.4),

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) f(\mathbf{r}) = -\frac{k}{2i} \int_V G_{xx}(\mathbf{r}|\mathbf{r}') \cdot \left[\int_{-\infty}^{+\infty} e^{-ik|x'-x''|} (\Delta'' + k^2)\, q(\mathbf{r}'')\, dx''\right] dV' \tag{4.5}$$

and rewritting Eq. (4.3) as

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) f(\mathbf{r}) = -G_{xx} G_{xx}^{-1} q$$

the bracket on the right hand side of Eq. (4.5) together with the factor $k/2i$ provides the representation of G_{xx}^{-1}. Further, the matrix Λ (Eq. (3.12)) can now be established and the complete solution of the equation under consideration reads (after some lengthy calculations)

$$\mathbf{E}(\mathbf{r}) = \int_V \left(\frac{e^{-ikR}}{4\pi R} \mathfrak{I}\right) \cdot \mathbf{Q}(\mathbf{r}')\, dV' + \hat{\mathbf{x}} \times \hat{\mathbf{x}} \times \int_V \frac{1}{k^2} \operatorname{grad} \operatorname{grad} \left(\frac{e^{-ikR}}{4\pi R}\right) \cdot \tilde{\mathbf{Q}}(\mathbf{r}')\, dV'. \tag{4.6}$$

In the second term the influence of the magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{x}}$ is obvious both in the double vector product with $\hat{\mathbf{x}}$ and in the appearance of $\tilde{\mathbf{Q}}$, where

$$\tilde{\mathbf{Q}}(\mathbf{r}') = \frac{k}{2i} \int_{-\infty}^{+\infty} e^{-ik|x'-x''|}\, \mathbf{Q}(x'', y', z')\, dx''. \tag{4.6a}$$

It can be ascertained by direct substitution into Eq. (2.7) (with $\hat{\mathbf{t}}(\mathbf{r})$ replaced by $\hat{\mathbf{x}}$) that Eq. (4.6) together with Eq. (4.6a) is indeed the solution of the inhomogeneous MHD wave equation for a homogeneous magnetic field $\mathbf{B}_0$ and a radiation condition.

The representations Eqs. (4.6) and (4.6a) allow determination of exact solutions of the inhomogeneous MHD wave equation for simple source distributions $\mathbf{Q}(\mathbf{r})$.

As a first example consider a homogeneous excitation in the plane $x = 0$, i.e.,

$$\mathbf{Q}(\mathbf{r}) = \mathbf{Q}\,\delta(x), \mathbf{Q} = \text{const.}, \qquad Q_x = 0.$$

Straightforward integration shows that the second term in Eq. (4.6) makes no contribution whereas the first term simply leads to

$$\mathbf{E}=\frac{1}{2ik}\mathbf{Q}e^{-ik|x|}. \tag{4.7}$$

As expected, this result represents a plane wave travelling in the positive x direction for $x>0$ and in the negative x direction for $x<0$, respectively.

As a second example consider a point source excitation by a Hertzian dipole at the origin, i.e.,

$$\mathbf{Q}(\mathbf{r})=\mathbf{Q}\,\delta(\mathbf{r}),\ \mathbf{Q}=\text{const.},\qquad Q_x=0.$$

Even in this more complicated case all integrals can be evaluated explicitly to give the electric field

$$\mathbf{E}=\mathbf{Q}\frac{e^{-ikr}}{4\pi r}+\frac{i}{4\pi k}\hat{\mathbf{x}}\times\hat{\mathbf{x}}\times\operatorname{grad}\left[\frac{e^{-ikr}}{\rho^2}(\mathbf{r}\cdot\mathbf{Q})\right]. \tag{4.8}$$

($\rho^2=(\hat{\mathbf{x}}\times\hat{\mathbf{x}}\times\mathbf{r})^2=r^2-x^2$). Its components turn out to be quite simple when we introduce cylindrical polar coordinates (x,ρ,ϕ). In addition, without loss of generality, let

$$\mathbf{Q}=Q\hat{\mathbf{y}}=Q(\hat{\boldsymbol{\rho}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi).$$

With these assumptions one obtains

$$E_\rho=\frac{i}{4\pi k}\frac{e^{-ikr}}{\rho^2}Q\cos\phi, \tag{4.9a}$$

$$E_\phi=\frac{i}{4\pi k}\frac{e^{-ikr}}{\rho^2}Q\sin\phi\left(1+\frac{ik\rho^2}{r}\right). \tag{4.9b}$$

A striking feature of this solution (Eqs. (4.8) or (4.9a, b)) is the existence of a singularity at $\rho=0$, i.e., a singularity along the x axis. In particular, it is impossible to satisfy the radiation condition (Eq. (3.5)) with respect to the direction of the ambient magnetic field $\mathbf{B}_0$. This can be seen in more detail by writing Eq. (3.5) explicitly,

$$r(\hat{\mathbf{v}}\times\operatorname{curl}\mathbf{E}-ik\mathbf{E})=\frac{e^{-ikr}}{4\pi r}Q\cos\phi\frac{1}{\sin\theta}\hat{\mathbf{r}} \tag{4.10}$$

where the angle θ is between the radius vector and the x axis, $\cos\theta=\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}$. From Eq. (4.10) it is clear that the radiation condition is satisfied in every direction as $r\to\infty$ except for $\theta=0$ and $\theta=\pi$, i.e., except for the direction of the magnetic field $\mathbf{B}_0$. This property is not yet completely understood, but it seems to be a quite general feature of wave motion in anisotropic media (Lighthill, 1960). Nevertheless, Eq. (4.8) represents an exact solution of the inhomogeneous MHD wave equation for a point source excitation in a homogeneous magnetic field $\mathbf{B}_0$.

The components of the corresponding magnetic wave field $\mathbf{B}$ can easily be obtained from Eqs. (4.9a, b) using the induction law (Eq. (2.2)),

$$B_x=\frac{i}{\omega}\frac{e^{-ikr}}{4\pi r^3}\rho Q\sin\phi(1+ikr),$$

$$B_\rho=-\frac{i}{\omega}\frac{e^{-ikr}}{4\pi r\rho^2}xQ\sin\phi\left[1+\frac{\rho^2}{r^2}(1+ikr)\right],$$

$$B_\phi=\frac{i}{\omega}\frac{e^{-ikr}}{4\pi r\rho^2}xQ\cos\phi.$$

Obviously, the behavior of the component B_x is regular outside the origin whereas the components B_ρ and B_ϕ show the same singularity at $\rho=0$ as the electric field components (Eqs. (4.9a, b)).

In summary we note that the undisturbed magnetic field $\mathbf{B}_0$ does not enforce one special direction of wave propagation. It is the direction of $\mathbf{B}_0$ that influences the amplitude of the wave field in a striking manner.

As a third example consider a homogeneous excitation within a circular disc of radius a in the plane $x=0$, i.e.,

$$\mathbf{Q}(\mathbf{r})=\begin{cases}\mathbf{Q}\,\delta(x), & \text{if } \rho\leqq a,\ \mathbf{Q}=\text{const.}\\ & \qquad =Q(\hat{\boldsymbol{\rho}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi)\ \text{(say)},\ Q_x=0\\ 0, & \text{if } \rho>a\end{cases}$$

again using cylindrical polar coordinates. In this case the integrals can be calculated only asymptotically for the two extremes $\rho\ll a$ and $\rho\gg a$, respectively, yielding

$$\mathbf{E}=\frac{\mathbf{Q}}{2ik}\left(e^{-ik|x|}-\tfrac{1}{2}e^{-ik\sqrt{x^2+a^2}}\right),\qquad \rho\ll a \tag{4.11a}$$

and

$$\mathbf{E}=\pi a^2 Q\frac{i}{4\pi k}\frac{e^{-ikr}}{\rho^2}\cdot\left[\cos\phi\,\hat{\boldsymbol{\rho}}+\left(1+\frac{ik\rho^2}{r}\right)\sin\phi\,\hat{\boldsymbol{\phi}}\right],\qquad \rho\gg a. \tag{4.11b}$$

Of course, the results of the preceding two examples are included in Eqs. (4.11a, b). From Eq. (4.11a), as $a\to\infty$ (assuming that k possesses a small negative imaginary component), we derive Eq. (4.7). Similarly, from Eq. (4.11b), as $a\to 0$ (and simultaneously $Q\to\infty$ so that $\pi a^2 Q=\text{const.}$), we derive the result for a point source excitation (Eqs. (4.9a, b)).

The first example shows that a plane wave is generated only by a very special excitation. The results of the second and the third example "re-emphasize how unrealistic is a treatment of anisotropic wave motions in terms of plane waves alone" (Lighthill, 1960, p. 415).

5. Conclusions

In the preceding section the feasibility of the method described in Sect. 3 has been tested by applying Eq. (3.11) to simple source distributions in a homogeneous field $\mathbf{B}_0$. The general solution (Eq. (3.11)) shows that it is sufficient to know Green's dyad for a problem which is independent of the field $\mathbf{B}_0$ (Eqs. (3.2) and (3.3)). The anisotropy caused by the geometry of the magnetic field $\mathbf{B}_0$ has then to be taken into account by a transformation of the integral operator G. This transformation can be achieved most simply by introducing a special system of reference, i.e., the local triad with respect to the vector field $\mathbf{B}_0$. After inversion of the component G_{tt} of the integral operator G the (operator valued) matrix Λ can be established, and the influence of the magnetic field $\mathbf{B}_0$ is then given by the additional term $G\Lambda G\mathbf{Q}$ in Eq. (3.11).

This procedure divides the problem into two parts. The first is concerned with the region under consideration together with the boundary conditions, the second with the magnetic field $\mathbf{B}_0$. In principle therefore, it is possible to solve the inhomogeneous MHD wave equation in those regions where Green's dyad can be found, according to Eq. (3.3). The prerequisites for applying this formalism to two configurations which might serve as simple magnetospheric models are given in the appendix.

It is the aim of future work to complete these models (or similar ones) by special source mechanisms to describe the excitation of the various types of micropulsation. This can be accomplished only by detailed numerical calculations. Any heuristic approach to the problem will lead to quite speculative results, since even the conclusions of the preceding section, referring to a far simpler geometry, show some unexpected features.

Finally, it should be noticed that the application of the general result of Sect. 3 is not restricted to the interpretation of geomagnetic micropulsation. It is applicable to all kinds of forced MHD wave phenomena for which the basic equations (Sect. 2) hold.

Appendix. As a first example consider the region outside a sphere of radius a. Let this region be filled with a magnetic dipole field and ask for a solution $\mathbf{E}$ of Eq. (2.7) generated by an arbitrary source distribution $\mathbf{Q}$ with $\hat{\mathbf{r}}\times\mathbf{E}=0$ at $r=a$ and $\mathbf{E}$ satisfying a radiation condition (Eq. (3.5)). This is an exterior problem outside a perfectly conducting sphere of radius a. Green's dyad according to Eq. (3.3) is well known from classical electromagnetic theory. Using dyadic notation and referring to spherical polar coordinates (r,θ,ϕ) it reads (Jones, 1964, p. 497)

$$\mathfrak{G}(\mathbf{r}|\mathbf{r}')=-\frac{i}{4\pi k}\left\{\left(k^2\hat{\mathbf{r}}+\operatorname{grad}\frac{\partial}{\partial r}\right)\left(k^2\hat{\mathbf{r}}'+\operatorname{grad}'\frac{\partial}{\partial r'}\right)rr'\,V_1(\mathbf{r},\mathbf{r}')+k^2(\hat{\mathbf{r}}\times\operatorname{grad})(\hat{\mathbf{r}}'\times\operatorname{grad}')\,V_2(\mathbf{r},\mathbf{r}')\right\}. \tag{A.1}$$

The functions V_1 and V_2 are given by

$$V_1(\mathbf{r},\mathbf{r}')=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(j_n(kr)+A_n h_n^{(2)}(kr))\,h_n^{(2)}(kr')\,P_n(\cos\gamma)$$

$$V_2(\mathbf{r},\mathbf{r}')=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(j_n(kr)+B_n h_n^{(2)}(kr))\,h_n^{(2)}(kr')\,P_n(\cos\gamma) \tag{A.2a}$$

for $r<r'$ and interchanging r and r' for $r>r'$. In Eq. (A.2a) j_n and $h_n^{(2)}$ are spherical Bessel functions of the first and third kind, respectively, P_n are Legendre polynomials, γ is the angle between the observation point $\mathbf{r}$ and the source point $\mathbf{r}'$,

$$\cos\gamma=\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi').$$

A_n and B_n are constants,

$$A_n=-\left[\frac{d}{dr}(rj_n)\right]_{r=a}\Big/\left[\frac{d}{dr}(rh_n^{(2)})\right]_{r=a} \tag{A.2b}$$

$$B_n=-j_n(ka)/h_n^{(2)}(ka).$$

The local triad with respect to a dipole field

$$\mathbf{B}_0=-\frac{\mu_0}{4\pi}\frac{M}{r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}) \tag{A.3}$$

(M = dipole moment) is given by

$$\hat{\mathbf{t}}(\mathbf{r})=-(1+3\cos^2\theta)^{-\frac{1}{2}}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}),$$

$$\hat{\mathbf{n}}(\mathbf{r})=-(1+3\cos^2\theta)^{-\frac{1}{2}}(\sin\theta\,\hat{\mathbf{r}}-2\cos\theta\,\hat{\boldsymbol{\theta}}),$$

$$\hat{\mathbf{b}}(\mathbf{r})=-\hat{\boldsymbol{\phi}}.$$

The second example refers to a closed magnetospheric model. Indeed, the magnetosphere is not extended to infinity, but it is bounded by the magnetopause. The actual shape of the magnetopause is rather awkward, and it might be very difficult to construct Green's dyad for configurations close to reality. Certainly, the following model is oversimplified, but it should serve as a first attempt to take into account an outer boundary.

Consider the region between two concentric, perfectly conducting spheres with radii a and R, respectively, $a<R$, and a magnetic field $\mathbf{B}_0$ which will be referred to later. Green's dyad for this region and the corresponding boundary conditions at $r=a$ and $r=R$ can be represented in a similar way to the case above. The functions V_1 and V_2 in Eq. (A.1) have now to be replaced by

$$V_1(\mathbf{r},\mathbf{r}')=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}[(A_n j_n(kr)+B_n h_n^{(2)}(kr))\,j_n(kr')+((B_n+1)\,j_n(kr)+C_n h_n^{(2)}(kr))\,h_n^{(2)}(kr')]\,P_n(\cos\gamma) \tag{A.4a}$$

$$V_2(\mathbf{r},\mathbf{r}')=\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}[(D_n j_n(kr)+E_n h_n^{(2)}(kr))\,j_n(kr')+((E_n+1)\,j_n(kr)+F_n h_n^{(2)}(kr))\,h_n^{(2)}(kr')]\,P_n(\cos\gamma)$$

for $r<r'$ and interchanging r and r' for $r>r'$. The constants $A_n, B_n, \ldots, F_n$ are given by

$$\begin{aligned}
A_n&=\left[\frac{d}{dr}(rh_n^{(2)})\right]_{r=a}\left[\frac{d}{dr}(rh_n^{(2)})\right]_{r=R}\Big/D_1\\
B_n&=-\left[\frac{d}{dr}(rj_n)\right]_{r=a}\left[\frac{d}{dr}(rh_n^{(2)})\right]_{r=R}\Big/D_1\\
C_n&=\left[\frac{d}{dr}(rj_n)\right]_{r=a}\left[\frac{d}{dr}(rj_n)\right]_{r=R}\Big/D_1\\
D_n&=h_n^{(2)}(ka)\,h_n^{(2)}(kR)/D_2\\
E_n&=-j_n(ka)\,h_n^{(2)}(kR)/D_2\\
F_n&=j_n(ka)\,j_n(kR)/D_2
\end{aligned} \tag{A.4b}$$

with

$$D_1=\left[\frac{d}{dr}(rj_n)\right]_{r=a}\left[\frac{d}{dr}(rh_n^{(2)})\right]_{r=R}-\left[\frac{d}{dr}(rj_n)\right]_{r=R}\left[\frac{d}{dr}(rh_n^{(2)})\right]_{r=a}$$

and

$$D_2 = j_n(ka)\, h_n^{(2)}(kR) - j_n(kR)\, h_n^{(2)}(ka).$$

The ambient magnetic field in this region can be chosen as a superposition of a dipole field (cf. Eq. (A.3)) and the homogeneous field

$$\mathbf{K}_0 = \frac{\mu_0}{4\pi} \frac{2M}{R^3} (\cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}).$$

The local triad with respect to the resulting field is given by

$$\hat{\mathbf{t}}(\mathbf{r}) = -c(r,\theta) \left[2\cos\theta \left(\left(\frac{R}{r}\right)^3 - 1 \right) \hat{\mathbf{r}} + \sin\theta \left(\left(\frac{R}{r}\right)^3 + 2 \right) \hat{\boldsymbol{\theta}} \right],$$

$$\hat{\mathbf{n}}(\mathbf{r}) = -c(r,\theta) \left[\sin\theta \left(\left(\frac{R}{r}\right)^3 + 2 \right) \hat{\mathbf{r}} - 2\cos\theta \left(\left(\frac{R}{r}\right)^3 - 1 \right) \hat{\boldsymbol{\theta}} \right],$$

$$\hat{\mathbf{b}}(\mathbf{r}) = -\hat{\boldsymbol{\phi}}$$

with

$$c(r,\theta) = \left[4 + 4 \left(\frac{R}{r}\right)^3 (1 - 3\cos^2\theta) + \left(\frac{R}{r}\right)^6 (1 + 3\cos^2\theta) \right]^{-\frac{1}{2}}.$$

For $a \leqq r \ll R$ this field is very similar to a dipole field whereas for $r \to R$ the magnetic field lines fit the outer Boundary $r = R$. By construction, it has the same vector field properties as a dipole field (i.e., zero curl and zero divergence).

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