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Comparison of Errors in Local and Reference Estimates of the Magnetotelluric Impedance Tensor

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Abstract. The estimation of the magnetotelluric impedance tensor by a regression analysis from locally measured electromagnetic surface fields includes a bias-error, if both the electric and the magnetic field are degraded by additive noise. The remote reference method developed by Gamble et al. (1979a) avoids this bias-error. The errors of the traditional local estimation and the remaining error in the reference estimation are compared. It is shown that for some different types of noise the standard deviation of the reference estimate may be as large as the bias-error of the local estimate. However in order to get a consistent reference estimate, and in order to take full advantage of this method, generally much more data have to be recorded and analysed if using the reference estimation instead of local estimation.

Key words: Coherent noise – Noise reduction – Error analysis – Bias-error – Variance – Magnetotellurics

Introduction

The magnetotelluric impedance tensor $[Z]$ relates the horizontal electric field $\mathbf{E}$ to the horizontal magnetic field $\mathbf{H}$ at the Earth's surface in the frequency domain

$$\mathbf{E}(\omega) = Z(\omega) \cdot \mathbf{H}(\omega), \tag{1}$$

with angular frequency $\omega = 2\pi/T$. T denotes the period of interest. Written in components Eq. (1) gives

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} \cdot \begin{bmatrix} H_x \\ H_y \end{bmatrix}. \tag{1a}$$

All values are dependent on the angular frequency ω, and on the resistivity distribution of the Earth in the vicinity of the recording location.

In order to find the unknown resistivity distribution below the Earth's surface, first the elements of the impedance tensor are calculated from the measured values of the electromagnetic field. Then data interpretation can be undertaken in several ways, for example, by inversion procedures or model fitting.

Generally the magnetotelluric method including the interpretation of the impedance tensor requires the measured fields $\mathbf{E}$ and $\mathbf{H}$ to be homogeneously induced, because otherwise the elements of the impedance tensor are not only dependent on the resistivity distribution but on the primary source field distribution as well. A homogeneously induced field is the total field at the Earth's surface, if the inducing primary field is homogeneous. This condition is valid for natural fields in middle geomagnetic lattitudes up to periods of about 1,000 s (Wait, 1954; Rikitake, 1966; Madden and Nelson, 1964).

Magnetotelluric measurements are degraded by equipment noise, but often much more by artificial and man made noise. This noise in general, results in an erroreous impedance tensor. In order to reduce this error influence, the impedance tensor is estimated from a lot of measurements of the electromagnetic field. Normally a regression analysis is used (Swift, 1967; Rankin and Reddy, 1969). Starting point of this method is the postulated regression model, for example

$$\mathbf{E}_i = [Z]\,\mathbf{H}_i + \delta\mathbf{H}_i, \qquad (i = 1 \ldots M) \tag{2}$$

with a noise term $\delta\mathbf{E}_i$ in the measured electric field data $\mathbf{E}_i$. Correlating the *output* data $\mathbf{E}_i$ of the regression model with the *input* data $\mathbf{H}_i$ gives the estimate.

$$[\hat{Z}]_L = \begin{bmatrix} \hat{Z}_{xxL} & \hat{Z}_{xyL} \\ \hat{Z}_{yxL} & \hat{Z}_{yyL} \end{bmatrix} = [C_{EH}] \cdot [P_{HH}]^{-1} \tag{3a}$$

where $[C_{EH}]$ is the estimated crosspower matrix and $[P_{HH}]$ is the estimated autopower matrix of the measured data $\mathbf{E}_i$ and $\mathbf{H}_i$. Exchanging the input and output data in the regression model leads to another estimate of the impedance tensor

$$[Z]'_L = [P_{EE}] \cdot [C_{EH}]^{-1}. \tag{3b}$$

In the following $[\hat{Z}]_L$ and $[\hat{Z}]'_L$ are called the *local estimates of the impedance tensor* because they are calculated from measured data of only *one* location.

Inherent in the local regression models is the assumption that the input data is free of noise. Only in this case the local estimates are bias-free estimates of the impedance tensor, and with a large number of samples $[\hat{Z}]_L$ and $[\hat{Z}]'_L$ tend towards the "true" impedance tensor $[Z]$. In practice this assumption often is violated. Noise in the input data causes at least an error in the estimated autopower matrices $[P_{HH}]$ and $[P_{EE}]$. Different authors have shown that in

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this case the estimates are biased downward and upward respectively, if Eq. (3b) is used, compared with the true value (e.g. Sims et al., 1971). Moreover, if the noise in the input data is coherent with the noise in the output data, an error also occurs in the estimated crosspower matrix $[C_{EH}]$. In this situation the values of the bias-errors in the local estimates of the impedance tensor cannot be seen immediately. One of the aims of this paper is to describe this type of error (see below).

Several methods have been presented to reduce the bias in the local estimates of the impedance tensor (Sims et al., 1971; Kao and Rankin, 1977; Gundel, 1977). All these estimates are based on local measurements of the electromagnetic field. To obtain a reduction of the bias-error, these methods require the components of noise in the estimate to be incoherent. Otherwise a separation of homogeneously induced data and noise by a local measurement is impossible (Goubau et al., 1978).

A definite improvement in the reduction of the bias-error, even in the case of coherent noise, was obtained by the so-called reference magnetotellurics, developed by Goubou et al. (1978). A detailed description of this method can be found in Gamble et al. (1979a). In this method the impedance tensor at a measuring station is estimated by using reference data $\mathbf{R}_i$, which are recorded synchronously at a remote station, the reference station. Normally a magnetic reference is used. The *reference estimate* $[\hat{Z}]_R$ of the impedance tensor is obtained by the correlation of data from the measuring station and the reference station.

$$[\hat{Z}]_R = \begin{bmatrix} \hat{Z}_{xxR} & \hat{Z}_{xyR} \\ \hat{Z}_{yxR} & \hat{Z}_{yyR} \end{bmatrix} = [C_{ER}] \cdot [C_{HR}]^{-1}. \tag{4}$$

This estimate only contains crosspowers between data from the measuring station and the reference station. Provided the noise at the measuring station is incoherent with the noise at the reference station the reference estimate Eq. (4) is bias-free. In practice this condition is satisfied by a sufficiently large distance between the measuring and reference station, usually some tens of km. This condition should not be confused with coherency between local noise in the electric and magnetic channels at the measuring station, which is permitted in the remote reference method. Indeed the reference method theoretically leads to a bias-free estimate, but there is some practical expense in comparison with the local method, which should be mentioned. First, more measurement equipment is needed. Also measuring and reference data have to be recorded synchronously within certain bounds. This either requires very stable time bases in both sets of equipment, or a telemetry connection for synchronising. In each case the sampling at both stations has to be done automatically. Foremost, in the last few years automatically recording MT-equipment has been developed which enables synchronous recording in the period range below 1,000 s.

Besides the differences in cost of equipment there is a second main difference between the local and the reference method. The reference method requires measuring of two more data channels, which may also suffer interference. Due to this additional noise it is to be expected that the reference estimate has a larger associated variance than that of the local estimates, provided the same quantity of data is used.

The increase of measurement equipment, the demand for automatic recording and the possible increased amount of data required seem to be the reasons why the reference method has not, up to now, been used as the standard method in MT surveys, although its advantage in avoiding bias errors is obvious. In particular, the quantity of data necessary to obtain a consistent estimate is as yet an unsolved quantity. Also it is not yet known how large the bias-errors in the local estimates for different kinds of noise are. In the literature there are derivations of the bias-errors only for the case of incoherent noise at the measuring station (e.g. Sims et al., 1971). Furthermore, the question arises of how large the bias-error is in practice. This paper will illustrate one example, which shows that for locations with coherent noise only the reference method is suitable for increasing the estimation accuracy in MT soundings.

By comparing the errors in both the local and the reference estimates of the impedance tensor this paper will help to answer the above mentioned questions. First, the value of the bias-error in one of the local estimates in the case of different kinds of noise shall be evaluated theoretically. The variance of the reference estimate in these cases will be derived. Secondly the analysis of real MT data will show which kind of noise in practice has to be expected and how large the bias-error and the variances are.

Definition of Signal and Noise

The MT-method requires homogeneously induced fields, which in the following are called the signal. Therefore noise is that part in the measured data $\mathbf{E}_i$ and $\mathbf{H}_i$ which either is not induced or not homogeneously induced.

The first group, the non-induced part, is noise measurable in the magnetic and electric channels. The noise measurements in these channels are independent from each other, i.e. *incoherent*. Examples are the noise of the measurement equipment, activity caused by moving vehicles, or mechanical vibration of the sensors.

The second group comprises noise caused by inhomogeneously induced fields. Physically these are the man made electromagnetic fields of electrical power lines or industrial areas. A simple model of such a field is given by Kröger (1981). In contrast to the first group, the noise in the electrical field is related to that in the magnetic channels, i.e. the noise is *coherent*.

Signal and noise are always generated by different sources and therefore the signal is not coherent with the noise.

Definition of Different Types of Noise

The measured data $\mathbf{E}_i$ and $\mathbf{H}_i$ are composed of the signals $\mathbf{E}_{si}$, $\mathbf{H}_{si}$ and the noise $\mathbf{E}_{ni}$, $\mathbf{H}_{ni}$.

$$\begin{aligned} \mathbf{E}_i &= \mathbf{E}_{si} + \mathbf{E}_{ni} \\ \mathbf{H}_i &= \mathbf{H}_{si} + \mathbf{H}_{ni}. \end{aligned} \tag{5}$$

Table 1. Definition of the types of noise distinguished by their coherency characteristics

Type	Definition	COH	$[\hat{Z}]_L$ biased by:
1	incoherent	$\mathrm{COH}(\mathbf{E}_n;\mathbf{H}_n)=0$ $\mathrm{COH}(H_{xn},H_{yn})=0$	autopowers of $\mathbf{H}_n$
2	input-coherent	$\mathrm{COH}(\mathbf{E}_n,\mathbf{H}_n)=0$ $\mathrm{COH}(H_{xn},H_{yn})\neq 0$	autopowers of $\mathbf{H}_n$ crosspowers of H_{xn} and H_{yn}
3	multiple-coherent	$\mathrm{COH}(\mathbf{E}_n,\mathbf{H}_n)\neq 0$ $\mathrm{COH}(H_{xn},H_{yn})\neq 0$	autopowers of $\mathbf{H}_n$ crosspowers of H_{xn} and H_{yn} crosspowers of $\mathbf{E}_n$ and $\mathbf{H}_n$

The "true" tensor $[Z]$ relates the signals

$$\mathbf{E}_{si}=[\mathbf{Z}]\cdot\mathbf{H}_{si}. \tag{6}$$

Due to this relation in the following $[Z]$ is called the *signal impedance.*

Concerning the different influence on the bias-error, the noise has to be distinguished by its coherency characteristic. Possible types of noise are summarized in Table 1.

1. The noise components $\mathbf{E}_n$ and $\mathbf{H}_n$ are incoherent and furthermore the input components H_{xn}, H_{yn} are incoherent. This type shall be referred to as *incoherent noise.*

2. The noise components $\mathbf{E}_n$ and $\mathbf{H}_n$ are incoherent but the input components of the noise H_{xn}, H_{yn} are coherent. This type is called an *input-coherent noise.*

3. The noise $\mathbf{E}_n$ is coherent with the noise $\mathbf{H}_n$ and furthermore H_{xn} and H_{yn} are coherent. This type shall be referred to as *multiple-coherent noise.*

Bias Error in the Local Estimate of the Impedance Tensor

A general two dimensional error analysis for each case of different types of noise in Table 1 is given by Kröger (1981). For simplification and demonstration of the main effects on bias and variance only, the absolute values of the diagonal elements Z_{xx}, Z_{yy} in the tensor $[Z]$ are assumed to be negligible small in comparison with the off-diagonal elements Z_{xy}, Z_{yx}. In practice, this is a good approximation for many existing conductivity distributions. In this case the local estimates may be reduced to one-dimensional estimates (Scheelke, 1972). Also for simplification in the following we want to restrict ourselves to the demonstration of the bias (and variance) for one of the local estimates, i.e. $[\hat{Z}]_L$, Eq. (3a). The reader may easily transfer the results if $[\hat{Z}]'_L$, Eq. (3b) instead of $[\hat{Z}]_L$ is used. For example, the one-dimensional local estimate for the element Z_{xy} is given by

$$\hat{Z}_{xyL}=\frac{\sum\limits_i E_{xi}H^*_{yi}}{\sum\limits_i |H_{yi}|^2}\qquad \left(\sum_i=\sum_{i=1}^{M}\right), \tag{7}$$

where H^*_{yi} is the complex conjugate of H_{yi}.

In this estimate noise of type 2 has no bias influence and that of type 3 simply reduces to coherent noise. This is not very restrictive, because, as shown by Kröger (1981), the multiple coherent noise usually will have much more error influence than the input coherent noise.

The estimate of the bias error ΔZ_{xy} in the local estimate $\hat{Z}_{xyL}$ is defined by

$$\Delta\hat{Z}_{xy}=\langle\hat{Z}_{xyL}\rangle-Z_{xy} \tag{8}$$

where $\langle\;\rangle$ denotes the ensemble average and Z_{xy} is the signal impedance (true but unknown value). In the following error analysis stationarity of signals and noise is always assumed. For a sufficiently large number M of samples in Eq. (7) the variance of $\hat{Z}_{xyL}$ becomes small compared with the bias of $\hat{Z}_{xyL}$ (see next section) and $\langle\hat{Z}_{xyL}\rangle$ may be approximated by

$$\langle\hat{Z}_{xyL}\rangle\simeq\hat{Z}_{xyL}. \tag{9}$$

Seperating the measured data into signals and noise and considering incoherency between them, Eq. (7) leads to

$$\hat{Z}_{xyL}=\frac{\sum\limits_i E_{xsi}H^*_{ysi}+\sum\limits_i E_{xni}H^*_{yni}}{\sum\limits_i|H_{ysi}|^2+\sum\limits_i|H_{yni}|^2}. \tag{10}$$

The signals are related by $E_{xsi}=Z_{xy}\cdot H_{ysi}$. For the noise a regression model may be set up and described by

$$E_{xni}=Z_{xyN}H_{yni}+\delta\delta E_{xi}. \tag{11}$$

Here, in the case of coherent noise Z_{xyN} is a well defined function and shall be called the *interference impedance.* $\delta\delta E_{xi}$ is the incoherent (with H_{yni}) part of the noise E_{xni}. Assuming known noise, Z_{xyN} can be estimated in accordance with Eq. (7)

$$\text{as}\quad \hat{Z}_{xyN}=\frac{\sum\limits_i E_{xni}H^*_{yni}}{\sum\limits_i|H_{yni}|^2}. \tag{12}$$

Inserting the signal impedance Z_{xy} and the interference impedance $\hat{Z}_{xyN}$, given by Eq. (12), into Eq. (10) yields

$$\hat{Z}_{xyL}=\frac{Z_{xy}\hat{S}_{Hy}+\hat{Z}_{xyN}\hat{N}_{Hy}}{\hat{S}_{Hy}+\hat{N}_{Hy}}, \tag{13}$$

where the following abbreviations were used:

$$\hat{S}_{Hy}=\sum_i|H_{ysi}|^2:\ \text{estimated signal power in } H_y$$

$$\hat{N}_{Hy}=\sum_i|H_{yni}|^2:\ \text{estimated noise power in } H_y. \tag{13a}$$

With Eqs. (8), (9) and (13) the bias-error is

$$\Delta\hat{Z}_{xy}\simeq\hat{Z}_{xyL}-Z_{xy}=\frac{1}{1+\left(\frac{S}{N}\right)_{Hy}}\cdot(\hat{Z}_{xyN}-Z_{xy}). \tag{14}$$

$\left(\frac{S}{N}\right)_{Hy}$ is the estimated signal-to-noise-ratio $\hat{S}_{Hy}/\hat{N}_{Hy}$. A similar result may be derived for the estimated bias-error $\Delta\hat{Z}_{yx}$ in the element $\hat{Z}_{yxL}$.

The bias-error depends on the noise parameters: signal-to-noise ratio and interference impedance. The difference between the interference impedance and the signal impedance is determined by the inhomogeneity of the source field of the noise. For example, in the near region of such a noise source the interference impedance due to the high inhomogeneity is very different from the signal impedance (absolute and phase value) as shown by Kröger (1981). With a low signal-to-noise ratio the bias-error may be much larger than the signal impedance, that means the local estimate $\hat{Z}_{xyL}$ is "unusable" for modelling. If the distance between the location of measurement and the source increases, the inhomogeneity of the inducing field decreases. This means the interference impedance tends towards the signal impedance and the resulting bias-error becomes smaller. For a sufficient large distance from a coherent source of noise you will get the far field solution which is equivalent to the homogenous inducing (natural) field and $\hat{Z}_{xyN}$ becomes equal to Z_{xy}. If there is no other type of noise, then the bias-error is zero.

If the noise is incoherent (type 1), according to Eq. (12), $\hat{Z}_{xyN}$ is zero and Eq. (14) leads to the well known expression for the bias-error (Sims et al., 1971)

$$\Delta\hat{Z}_{xy} \simeq -\frac{1}{1+\left(\frac{S}{N}\right)_{Hy}} \cdot Z_{xy}. \tag{15}$$

For incoherent noise the phase of the relative bias-error $\Delta\hat{Z}_{xy}/Z_{xy}$ is 180° and therefore the local estimate $\hat{Z}_{xyL}$ is downward biased compared with the signal impedance Z_{xy}.

For coherent noise the phase in Eq. (14) depends on the interference and signal impedance and, in this case, $\hat{Z}_{xyL}$ may be upward biased (see also practical results below).

In contrast to the local estimate $[\hat{Z}]_L$, the reference estimate $[\hat{Z}]_R$ does not depend on the autopowers and the crosspowers of the locally measured data and therefore it is not biased, provided the noise at the measuring station and that at the reference station is incoherent.

Variance of the Estimates of the Impedance Tensor

For the different types of noise the variances of the local and reference estimate shall be compared. Again, for simplification, it is assumed $|Z_{xx}|, |Z_{yy}| \ll |Z_{xy}|, |Z_{yx}|$.

The estimated variance of the local estimate, for example, is given by Bendat and Piersol (1971). For the element $\hat{Z}_{xyL}$ it is approximately

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\} \simeq \frac{1}{M} \cdot \frac{\sum_i |\delta E_{xi}|^2}{\sum_i |H_{yi}|^2}, \tag{16}$$

where δE_{xi} is the residuum

$$\delta E_{xi} = E_{xi} - \hat{Z}_{xyL} \cdot H_{yi} \tag{17}$$

and M is the number of samples in the regression analysis.

With Eqs. (17) and (7), Eq. (16) transformes into

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\} \simeq \frac{1}{M} \cdot \left[\frac{\sum_i |E_{xi}|^2}{\sum_i |H_{yi}|^2} - |\hat{Z}_{xyL}|^2\right]. \tag{18}$$

The total power in the electric channel, $\sum_i |E_{xi}|^2$, can be decomposed into the signal power $\hat{S}_{Ex}$, the noise $\hat{N}_{ExC}$, (which is totally coherent to the noise power $\hat{N}_{Hy}$) and the noise power $\hat{N}_{ExU}$ (which is totally incoherent to $\hat{N}_{Hy}$)

$$\sum_i |E_{xi}|^2 = \hat{S}_{ExC} + \hat{N}_{ExU} + \hat{N}_{ExC}. \tag{19}$$

In Appendix A1 it is shown how the expression for the variance in Eq. (18) by use of this decomposition becomes

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\} \simeq \frac{1}{M} \cdot \frac{1}{1+\left(\frac{N}{S}\right)_{Hy}} \cdot \left[\left(\frac{N_U}{S}\right)_{Ex} \cdot |Z_{xy}|^2 + \frac{1}{1+\left(\frac{S}{N}\right)_{Hy}} \cdot |\hat{Z}_{xyN} - Z_{xy}|^2\right]. \tag{20}$$

A similar expression may be derived for the element $\hat{Z}_{yxL}$. The variance of the local estimate depends on the already defined noise parameters $\left(\frac{S}{N}\right)_{Hy}$ and $\hat{Z}_{xyN}$ and on the signal-to-noise ratio $\left(\frac{S}{N_U}\right)_{Ex}$ in the electric channel. N_U is the incoherent part of the noise power in E_x.

A general expression for the variance of the reference estimate was given by Gamble et al. (1979b). The expressed variance is dependent on the crosspowers of the measured data. For the aims of this paper, namely comparing bias and variance, the variance shall be given by its dependency of the above noise parameters. This is done by Kröger (1981). With $|Z_{xx}|, |Z_{yy}| \ll |Z_{xy}|, |Z_{yx}|$, the variance for $\hat{Z}_{xyR}$ is

$$\hat{\mathrm{VAR}}\{Z_{xyR}\} \simeq \frac{1}{M} \cdot \frac{1}{|\hat{K}_{yy}|^2 \cdot \sum_i |R_{yi}|^2} [\hat{N}_{ExU} + |\hat{Z}_{xyN} - Z_{xy}|^2 \cdot \hat{N}_{Hy}] \tag{21}$$

$$\text{with } |\hat{K}_{yy}|^2 = \frac{\hat{S}_{Hy}/\hat{S}_{Ry}}{\left[1+\left(\frac{N}{S}\right)_{Ry}\right]^2}. \tag{21a}$$

$\hat{S}_{Ry}$ is the estimated signal and $\left(\frac{S}{N}\right)_{Ry}$ the estimated signal-to-noise ratio at the reference station. In Appendix A2 it is shown that Eq. (21) can be transformed into

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyR}\} \simeq \frac{1}{M} \cdot \left[1+\left(\frac{N}{S}\right)_{Ry}\right] \cdot \left[\left(\frac{N_U}{S}\right)_{Ex} \cdot |Z_{xy}|^2 + \left(\frac{N}{S}\right)_{Hy} \cdot |\hat{Z}_{xyN} - Z_{xy}|^2\right]. \tag{22}$$

Besides the noise parameters at the measuring station the signal-to-noise ratio at the reference station appears in Eq. (22) and, as will be shown in the following, the variance of the reference estimate increases compared with the variance of the local estimate.

In Table 2 the ratio of the two variance expressions of Eq. (20) and Eq. (22) is given for the limits of the signal-to-noise ratios $\left(\frac{S}{N}\right)_{Hy}$ and $\left(\frac{S}{N_U}\right)_{Ex}$ and for the interference impedance $\hat{Z}_{xyN}$. This table points out that, due to the finite signal-to-noise ratio at the reference station, the reference estimate always has a larger variance than the local estimate. If the reference method is used, the reference station should be chosen as free of noise as possible because then the variance ratio becomes smallest for fixed noise parameters at the measuring station. Only in the case of large signal-to-noise ratios at the measuring and at the reference station do both methods give nearly the same variance.

The reference method becomes very important in the case of low signal-to-noise ratios at the measuring station $\left(\frac{S}{N}\right)_{Hy} \ll 1$. In this case the local estimation leads to unusable results because of its large bias-error. But, as shown in Table 2, in just this case is $\hat{\mathrm{VAR}}\{\hat{Z}_{xyR}\} \gg \hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\}$. With totally coherent noise (case a) a very consistent local estimate $\hat{Z}_{xyL}$ is obtained, which of course is wrong. The reference method in this case needs much more data to get an estimate with as low a variance. The large bias-error or $\hat{Z}_{xyL}$ is exchanged for a large variance of $\hat{Z}_{xyR}$.

Also in the case of totally incoherent noise and a low $\left(\frac{S}{N}\right)_{Hy}$ (case b) is $\hat{\mathrm{VAR}}\{\hat{Z}_{xyR}\} \gg \{\hat{Z}_{xyL}\}$ and therefore even for this type of noise the reference method needs much more data.

Separation of Signal and Noise

An analysis of magnetotelluric data will show how large the noise parameters can be. In order to determine these parameters the signal and the noise at the measuring station have to be estimated. Gamble et al. (1979b) describe a method for estimating signal and noise power, based on two assumptions:

1. The noise at the measuring station is incoherent with the noise at the reference station.
2. The noise at the measuring station is incoherent.

Condition 1 can be satisfied by a sufficiently large distance between the two stations. However, it is easy to understand that the second assumption can be violated especially for measurements in industrialized areas, for example in central Europe. Our own experiences verify this (see below). Therefore a different method for the separation of signal and noise will be given. This method also uses the magnetic reference $\mathbf{R}_i$; condition 1 has to be valid, but it is not restricted to incoherent noise at the measuring station. Instead of assumption 2 this method requires the reference station to be much less contaminated by noise than the measuring station. The necessity of this restriction will be explained in the following. Methods for choosing such a reference and checking this condition by use of a second reference station are described by Kröger (1981).

Table 2. Comparison of variances for limits of noise parameters

$\left(\frac{S}{N}\right)_{Hy}$	$\left(\frac{N_U}{S}\right)_{Ex}$, $\hat{Z}_{xyN}$	$\hat{\mathrm{VAR}}\{\hat{Z}_{xyR}\}/\hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\}$
$\left(\frac{S}{N}\right)_{Hy} \gg 1$		$1+\left(\frac{N}{S}\right)_{Ry} > 1$
$\left(\frac{S}{N}\right)_{Hy} \ll 1$	a) totally coherent noise $\left(\frac{N_U}{S}\right)_{Ex} = 0$	$\left[1+\left(\frac{N}{S}\right)_{Ry}\right] \cdot \left(\frac{N}{S}\right)^2_{Hy} \gg 1$
	b) totally incoherent noise $\left(\frac{N_U}{S}\right)_{Ex} = \left(\frac{N}{S}\right)_{Ex}$, $\hat{Z}_{xyN} = 0$	$\left[1+\left(\frac{N}{S}\right)_{Ry}\right] \cdot \left(\frac{N}{S}\right)_{Hy} \cdot \frac{\left(\frac{N}{S}\right)_{Hy} + \left(\frac{N}{S}\right)_{Ex}}{1+\left(\frac{N}{S}\right)_{Ex}} \gg 1$

For separating the signal and the noise in the magnetic field a regression model is set up between magnetic data measured at the measuring and the reference station.

$$\mathbf{H}_i = [K] \cdot \mathbf{R}_i + \delta \mathbf{H}_i. \tag{23}$$

The transfer-function $[K]$ can be estimated similarly to Eq. (3)

$$[\hat{K}] = \begin{bmatrix} \hat{K}_{xx} & \hat{K}_{xy} \\ \hat{K}_{xy} & \hat{K}_{yy} \end{bmatrix} = [C_{HR}] \cdot [P_{PR}]^{-1}. \tag{24}$$

Once $[\hat{K}]$ has been estimated the magnetic field at the measuring station can be predicted as

$$\mathbf{H}_{pi} = [\hat{K}] \cdot \mathbf{R}_i. \tag{25}$$

Under the conditions that the reference $\mathbf{R}_i$ is nearly free of noise and the noise at the measuring station is incoherent with the noise at the reference station, $[\hat{K}]$ will be estimated without bias-error and, with a sufficiently large number M of samples, has a low variance. $[\hat{K}]$ then describes the linear relationship between the (coherent) signals at the measuring and the reference stations and therefore the predicted $\mathbf{H}_{pi}$ are nearly the signals at the measuring station

$$\mathbf{H}_{pi} \simeq \mathbf{H}_{si}. \tag{26}$$

The noise in the magnetic data at the measuring station is estimated by the residuals

$$\delta \mathbf{H}_i = \mathbf{H}_i - \mathbf{H}_{pi} \simeq \mathbf{H}_{ni}. \tag{27}$$

Only under the condition of a low interference reference station the separation is exact.

In a second regression model the predicted (signals)

$\mathbf{H}_{pi}$ are linearily related to the electric data $\mathbf{E}_i$ at the measuring station

$$\mathbf{E}_i = [Z] \cdot \mathbf{H}_{pi} + \delta \mathbf{E}_i. \tag{28}$$

The estimate of this impedance tensor is

$$[\hat{Z}]_R = [C_{EP}] \cdot [P_{pp}]^{-1}. \tag{29}$$

The reference estimate of the impedance tensor is used to predict signals in the electric data

$$\mathbf{E}_{pi} = [\hat{Z}]_R \cdot \mathbf{H}_{pi}, \tag{30}$$

which are estimates of the signal at the measuring station and with the above assumption of a reference station nearly free of noise, is

$$\mathbf{E}_{pi} \simeq \mathbf{E}_{si}. \tag{31}$$

The estimated noise in the electric data are the residuals

$$\delta \mathbf{E}_i = \mathbf{E}_i - \mathbf{E}_{pi} \simeq \mathbf{E}_{ni}. \tag{32}$$

The reference method described here gives the signals and the noise at the measuring station and moreover the bias-free reference estimate $[\hat{Z}]_R$. $[\hat{Z}]_R$ tends towards the signal impedance $[Z]$ if a sufficiently large number of samples is used in the regression. From the isolated signals and noise the noise parameters are estimated. Results are shown in the next section.

It is easy to verify, as shown in Appendix A3, that the elements of $[\hat{Z}]_R$ given by Eq. (29) in any case of noise are exactly the same as given by Gamble et al. (1979a). This is because Eq. (4) can be decomposed into a double linear regression analysis given by Eqs. (24) and (29). Therefore the condition of a noise free reference in this method only has to be valid for the decomposition of signals and noise, but not for the estimation of the impedance $[\hat{Z}]_R$.

Results of Measurements

The magnetotelluric measurements for the recognition of noise and errors have been made in northern Germany between Braunschweig and Uelzen. Figure 1 shows the measurement area. The period range of registrations and data analysis extended from 2.5 s up to 128 s. The measuring station was chosen at ADB5 north of Braunschweig and the reference station at UMM1, approximately 20 km remote from the measuring station. Previous recordings by the authors at more than one reference station have shown that UMM1 has much less noise than ADB5 and therefore it is a suitable reference station for separation of signals and noise at ADB5 (see Kröger, 1981).

For the data analysis we used $M = 521$ synchronously recorded samples with sufficiently large "signal activity" which were transformed into the frequency domain by the usual methods.

The results for the local estimate $[\hat{Z}]_L$ and the reference estimate $[\hat{Z}]_R$ at ADB5 are given in Fig. 2. It shows for example the absolute value of the element $\hat{Z}_{xy}$ versus period T. The bars at each estimated point

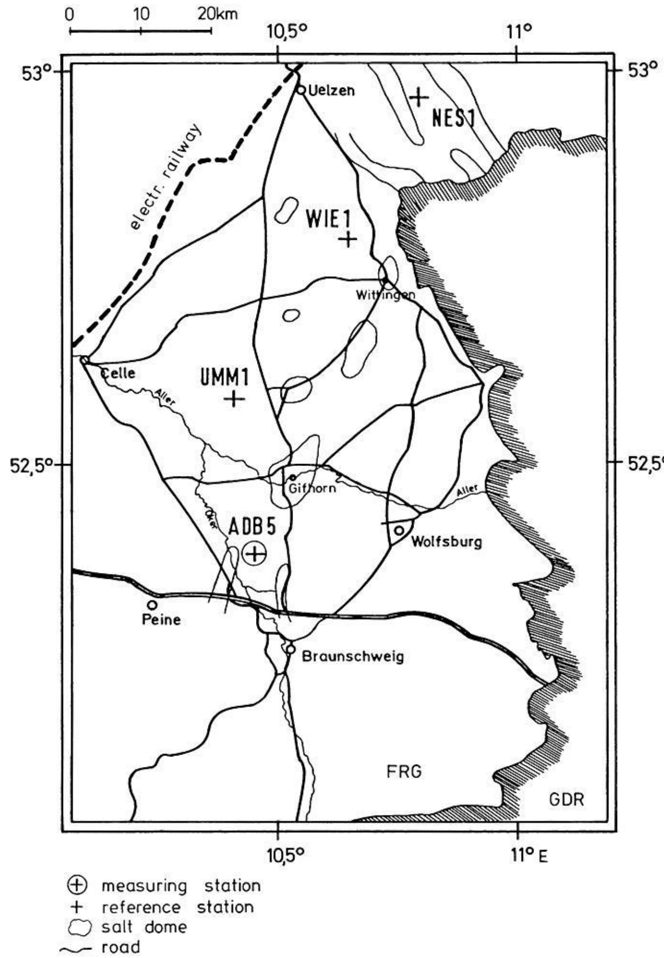

Fig. 1. Map of stations

denote the standard deviation $\sqrt{\mathrm{VAR}}$, determined by standard methods (see e.g. Kröger, 1981). For comparison $|\hat{Z}_{xyL}|$ has been indicated in the drawing of $|\hat{Z}_{xyR}|$.

At periods greater than 80 s both estimates are almost equal in amplitude and phase, and both have a very low variance. By comparison with $\hat{Z}_{xyR}$, which is assumed to be the signal impedance, there is a large bias-error in the local estimate $\hat{Z}_{xyL}$ between about 15 s and 50 s. This indicates a large noise contribution at the measuring station. A more exact analysis, which include the phases, shows that this bias-error is nearly as large as the signal impedance (absolute value). Therefore in this period range for this location the local estimate $\hat{Z}_{xyL}$ is unusable for MT-modelling. $|\hat{Z}_{xyL}|$ is larger than $|\hat{Z}_{xyR}|$ which implies that the noise must be coherent.

As expected, in the whole period range the variance of $\hat{Z}_{xyR}$ is larger than that of $\hat{Z}_{xyL}$. This difference can be recognized particularly in the range between 15 s and 50 s where the bias of $\hat{Z}_{xyL}$ is large (exchange of bias and variance). While $\hat{Z}_{xyL}$ seems to be estimated "very well", but wrongly, $\hat{Z}_{xyR}$ needs much more data to get a similary consistent estimate. Although there are a large number of samples in the regression the element $\hat{Z}_{xyR}$ is badly estimated below about 8 s.

These errors of both estimates can be explained by the magnitude of the noise parameters. Figure 3 shows

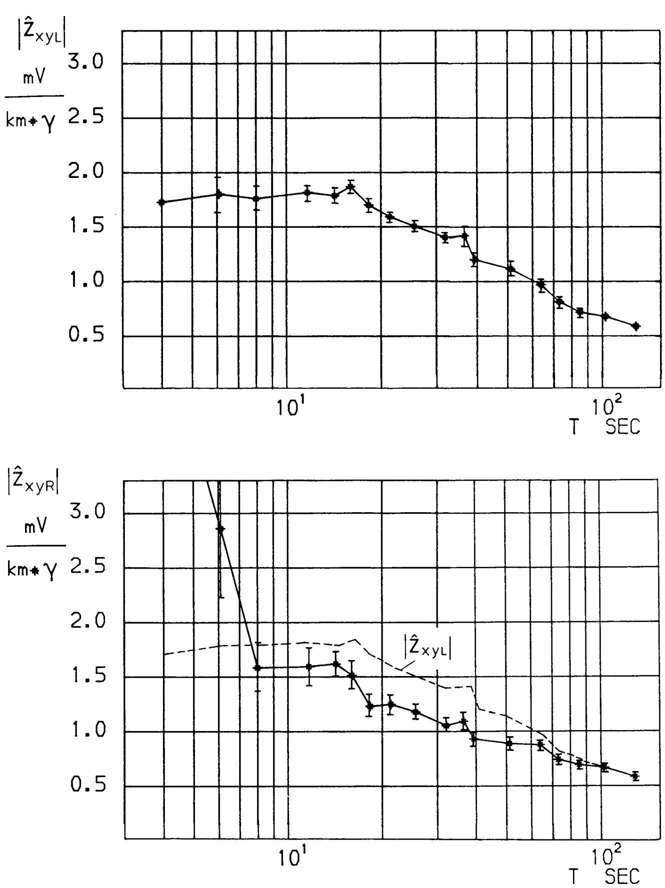

Fig. 2. Absolute value of the local estimate $\hat{Z}_{xyL}$ and the reference estimate $\hat{Z}_{xyR}$

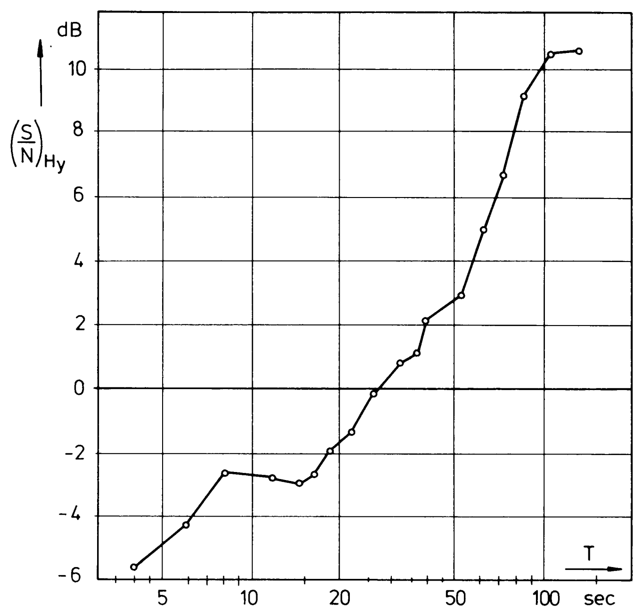

Fig. 3. Estimated signal-to-noise ratio in H_y at the measuring station ADB5

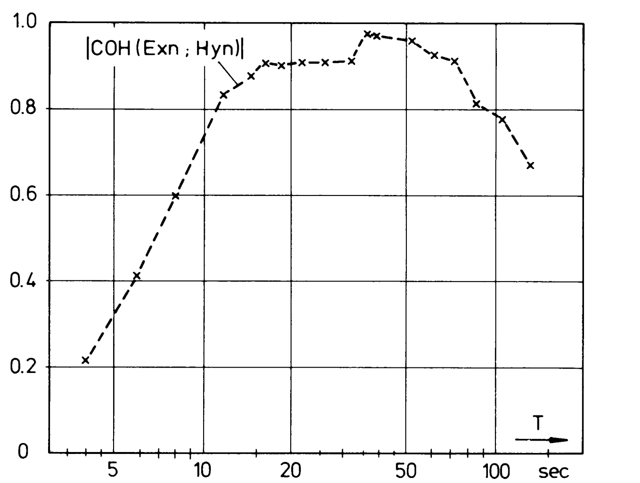

Fig. 4. Coherency function of noise at ADB5

the estimated signal-to-noise ratio in the component H_y versus period. This ratio was calculated from Eq. (26) and Eq. (27) as

$$\left(\frac{S}{N}\right)_{Hy} = \frac{\hat{S}_{Hy}}{\hat{N}_{Hy}} = \frac{\sum_i |H_{ypi}|^2}{\sum_i |\delta H_{yi}|^2}. \tag{33}$$

In the analysed data the estimate of the signal-to-noise ratio is much larger than one above 80 s. Therefore in this range the bias-error in $\hat{Z}_{xyL}$ is very small. Both methods of estimating the impedance tensor lead to nearly the same results. Below 50 s the signal-to-noise ratio decreases rapidly, and is even much smaller than one below 20 s. This explains a large bias-error in $\hat{Z}_{xyL}$ and the large variance of $\hat{Z}_{xyR}$, below 50 s. Below about 8 s there is hardly any signal power compared to noise power, hence $\hat{Z}_{xyR}$ is inconsistent (see Fig. 2 and Table 2 with $\left(\frac{S}{N}\right)_{Hy} \ll 1$).

Figure 4 demonstrates that the noise at the measuring station ADB5 is highly coherent especially in the range from 15 s up to 70 s. The figure shows the absolute value of the coherency

$$|\mathrm{COH}(E_{xn}; H_{yn})| \simeq \frac{|\sum_i \delta E_{xi} \cdot \delta H_{yi}^*|}{\sqrt{\sum_i |\delta E_{xi}|^2 \cdot \sum_i |\delta H_{yi}|^2}}, \tag{34}$$

calculated from Eqs. (27) and (32). The high coherency of the noise does not influence the estimates $\hat{Z}_{xyL}$ and $\hat{Z}_{xyR}$ very much at periods above 80 s, because the signal-to-noise ratio is sufficiently large here. Most importantly, at the low signal-to-noise ratios below 50 s, there is an influence. Due to this high coherency there is a well defined interference impedance, which actually differs greatly from the signal impedance. This leads to an upward biased local estimate $\hat{Z}_{xyL}$ and, furthermore, from Table 2, to a large variance of $\hat{Z}_{xyR}$ below 20 s. On the other hand, due to the high coherency, $\hat{Z}_{xyL}$ appears to be well defined down to 4 s.

Summary

Magnetotelluric recordings and data analysis over the last few years have shown that the homogeneously induced signals are very often degraded by artificial noise content, which sometimes causes large errors in the estimates of the impedance tensor.

In this work two methods of estimation, the local and the reference method, are compared with regard to the errors resulting from different types of noise. The main types are: incoherent and coherent noise.

The advantage of the local estimate is based on the fact that only one set of measurement equipment is needed, and that data analysis is relatively simple. But there can be a very large bias-error in the local estimate $[\hat{Z}]_L$, if the signal-to-noise ratio of the measured data is very low. Moreover, if the noise is coherent, the bias may be too large for MT-modelling. Measurement results prove that this type of noise exists in practice.

The reference method introduced by Goubau et al., (1978) and Gamble et al. (1979a) yields a bias-free estimate of the impedance tensor $[\hat{Z}]_R$ if the noise at the measuring station is incoherent with the the noise at the reference station. However, due to additionally introduced measurement data $\mathbf{R}_i$ this method generally requires more data to obtain an estimate as consistent as $[\hat{Z}]_L$. Coherent noise between electric and magnetic field at the measuring station, in particular, leads to a large variance of $[\hat{Z}]_R$ and, therefore, requires a large number of samples for a consistent estimation. The large bias-error of $[\hat{Z}]_L$ at this type of noise exchanges with a large variance of $[\hat{Z}]_R$.

From the results presented the following conclusions may be drawn: Generally the locations of magnetotelluric measurements are primarily fixed by geological aspects and usually there are only a few degrees of freedom in fixing stations. In the industrialized central Europe especially it is unavoidable that measuring stations are used which are very "noisy". The noise cannot be recognized if measurements are done without a reference and if the noise is coherent. Any local estimate of the impedance tensor, in these cases, can be unusable as shown in this paper. To avoid large errors, future magnetotelluric measurements should be made with an additional reference station and the impedance tensor should be estimated using the reference method as advised by Gamble et al. (1979a). The reference station should be choosen as free of noise as possible. This method, in fact, needs more measurement equipment and, depending on the type of noise, possibly much more data but it yields, in any case, more exact estimates of the impedance tensor.

Appendix A1: Estimated variance of $\hat{Z}_{xyL}$

With Eqs. (13), (18), and (19) follows

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\} \simeq \frac{1}{M} \cdot \left[\frac{\hat{S}_{Ex}+\hat{N}_{ExC}+\hat{N}_{ExU}}{\hat{S}_{Hy}+\hat{N}_{Hy}} - \frac{|Z_{xy}\hat{S}_{Hy}+\hat{Z}_{xyN}\hat{N}_{Hy}|^2}{(\hat{S}_{Hy}+\hat{N}_{Hy})^2}\right]. \tag{a1}$$

For the signal power and the coherent noise power it is

$$\hat{S}_{Ex}=|Z_{xy}|^2\hat{S}_{Hy} \tag{a2}$$

$$\hat{N}_{ExC}=|\hat{Z}_{xyN}|^2\hat{N}_{Hy}. \tag{a3}$$

Equations (a2) and (a3) introduced into Eq. (a1) leads to

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\} \simeq \frac{1}{M} \cdot \frac{1}{1+\left(\frac{N}{S}\right)_{Hy}} \cdot \left[\frac{\hat{N}_{ExU}}{\hat{S}_{Hy}} + \frac{1}{1+\left(\frac{S}{N}\right)_{Hy}} \cdot \left(|Z_{xy}|^2+|\hat{Z}_{xyN}|^2-2\,\mathrm{Re}\{Z_{xy}\cdot\hat{Z}^*_{xyN}\}\right)\right]. \tag{a4}$$

Equation (a4) with Eq. (a2) gives

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyL}\} \simeq \frac{1}{M}\,\frac{1}{1+\left(\frac{N}{S}\right)_{Hy}} \cdot \left[\left(\frac{N_u}{S}\right)_{Ex} \cdot |Z_{xy}|^2 + \frac{1}{1+\left(\frac{S}{N}\right)_{Hy}} \cdot |Z_{xy}-\hat{Z}_{xyN}|^2\right]. \tag{a5}$$

Appendix A2: Estimated variance of $\hat{Z}_{xyR}$

In Eq. (21) the total power at the reference station $\sum_i |R_{yi}|^2$ is separated into signal and noise power (estimated values)

$$\sum_i |R_{yi}|^2 = \hat{S}_{Ry} + \hat{N}_{Ry}. \tag{a6}$$

Then with Eq. (21a)

$$|\hat{K}_{yy}|^2 \cdot \sum_i |R_{yi}|^2 = \hat{S}_{Hy} \cdot \frac{1}{1+\left(\frac{N}{S}\right)_{Ry}}. \tag{a7}$$

Equation (a7) introduced into Eq. (21) leads to

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyR}\} \simeq \frac{1}{M} \cdot \left[1+\left(\frac{N}{S}\right)_{Ry}\right] \cdot \left[\frac{\hat{N}_{ExU}}{\hat{S}_{Hy}} + |\hat{Z}_{xyN}-Z_{xy}|^2 \cdot \frac{\hat{N}_{Hy}}{\hat{S}_{Hy}}\right]$$

or with $\hat{S}_{Ex}=|Z_{xy}|^2\cdot\hat{S}_{Hy}$

$$\hat{\mathrm{VAR}}\{\hat{Z}_{xyR}\} \simeq \frac{1}{M} \cdot \left[1+\left(\frac{N}{S}\right)_{Ry}\right] \cdot \left[\left(\frac{N_u}{S}\right)_{Ex} \cdot |Z_{xy}|^2 + \left(\frac{N}{S}\right)_{Hy} \cdot |\hat{Z}_{xyN}-Z_{xy}|^2\right] \tag{a8}$$

Appendix A3

With Eq. (24) and Eq. (25) it is

$$[C_{EP}]=[C_{ER}]\cdot[\hat{K}]^{*T} \tag{a9}$$

and

$$[P_{PP}]=[\hat{K}]\cdot[P_{RR}]\cdot[\hat{K}]^{*T} = [C_{HR}]\cdot[\hat{K}]^{*T}, \tag{a10}$$

where $[\hat{K}]^{*T}$ is the Hermitian adjoint matrix of $[\hat{K}]$.

Equation (a9) and Eq. (a10) introduced into Eq. (29) gives

$$[\hat{Z}]_R = [C_{ER}] \cdot [\hat{K}]^{*T} \cdot [[C_{HR}] \cdot [\hat{K}]^{*T}]^{-1} = [C_{ER}] \cdot [C_{HR}]^{-1}. \qquad (a11)$$

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