J Geophys (1983) 53: 59–67

Journal of Geophysics

A Study of the Mechanism of Whistler-Triggered VLF Emissions

Toshio Yoshida[1], Jinsuke Ohtsu, and Masashi Hayakawa

Research Institute of Atmospherics, Nagoya University, Toyokawa, Aichi, 442, Japan

Abstract. The conditions for triggering VLF emission by a natural whistler are studied, in terms of the length of interaction between whistler and electron determined by the unperturbed phase variation resulting from the spatial inhomogeneity and wave frequency variation. The computations are performed for a realistic magnetospheric model at $L=4.0$. It is found that we expect the highest probability of VLF emission triggering at an initial frequency slightly above 3 kHz in the whistler spectrum, at a geomagnetic latitude $\sim -15°$ on the upstream side of the electron for the following reasons. (1) The interaction time has a sharp maximum ($\gtrsim 50$ ms) at ~ 3.25 kHz, (2) the interaction time at 3 kHz is very large compared with the phase bunching time for reasonable wave amplitudes over a wide range of resonant electron pitch angles ($\alpha_{eq}=5°$–$60°$), and (3) resonance electron energy at 3 kHz (~ 10–20 keV for $\alpha_{eq}=5°$–$55°$) is less than that at 1,2 and 4 kHz and there is, therefore, a greater flux of electrons available for wave amplification. For the initial frequency $\gtrsim 3$ kHz the variation of the Doppler-shifted wave frequency seen by the electron matches very well with the variation of the resonant electron's gyrofrequency in the inhomogeneous medium, giving rise to the longest interaction time. Our method of analysis would be applicable to the triggering of VLF emissions by other variable frequency waves, in which the ratio of the interaction time to the phase bunching time would be an essential quantity in the wave triggering.

Key words: Whistler-triggered VLF emissions — Whistlers — VLF emissions — Whistler-electron cyclotron interaction — Magnetosphere — Phase bunching

Introduction

Discrete VLF emissions are thought to be generated *around the equator* by cyclotron resonance between whistler mode waves and energetic electrons (Helliwell, 1967; Rycroft, 1972; Gendrin, 1975). Evidence for their equatorial generation includes the observations of banded chorus of Burtis and Helliwell (1969, 1976), who have found that the frequency of chorus is associated with the equatorial gyro-frequency. As further evidence, Maeda (1976) has shown that the missing frequency of two-banded chorus is equal to one half of the equatorial gyro-frequency. In parallel with the studies of naturally generated waves the Siple active experiments using the injected waves with constant frequency (Helliwell, 1977) have yielded a wealth of interesting results on nonlinear whistler instability around the equator, which would also be of great use when interpreting natural discrete emissions.

However, wave-particle interactions in the *off-equatorial* plane have not been thoroughly investigated. Helliwell (1970) has considered a case of linearly changing frequency and has shown simply that the resonance point is situated on the upstream side of the electron when the frequency decreases. A wave with this linearly changing frequency has been added as a triggering signal in the Siple experiment (Miller, 1979), but there has been, as yet, no detailed study on the region and mechanism of wave-particle interactions based on the ramp signal transmissions. Recently Brinca (1981) has proposed a method of enhancing the whistler instability by the use of waves whose frequency is suitably varied so as to compensate the plasma inhomogeneity. In the off-equatorial region the plasma inhomogeneity is greatly enhanced compared with that at the equator so that the region is of great importance in studying the essential points lying in the general wave-particle interactions. Needless to say, the off-equatorial region can be attacked only by the use of variable frequency waves.

In this paper we adopt natural whistlers originating from lightning as a variable frequency wave and elucidate the possible conditions of triggering new emissions by natural whistlers in order to study the mechanism of whistler-triggered emissions. The L value we are concerned with is assumed to be 4.0 where we expect the most frequent occurrence of whistler-triggered emissions (Carpenter, 1968). We study the wave-particle interaction by using the variations of unperturbed phase angle determined by both the spatial inhomogeneity of the medium and wave frequency variation, enabling us to define the "interaction time", with a special reference to its comparison with the bunching time. Then, in order to examine the importance of the ratio of the interaction to bunching time in whistler-triggered

1 *Present address:* Nippon Electric Co. Ltd., Microwave Communications Systems Division, Yokohama, 226, Japan

Offprint requests to: M. Hayakawa

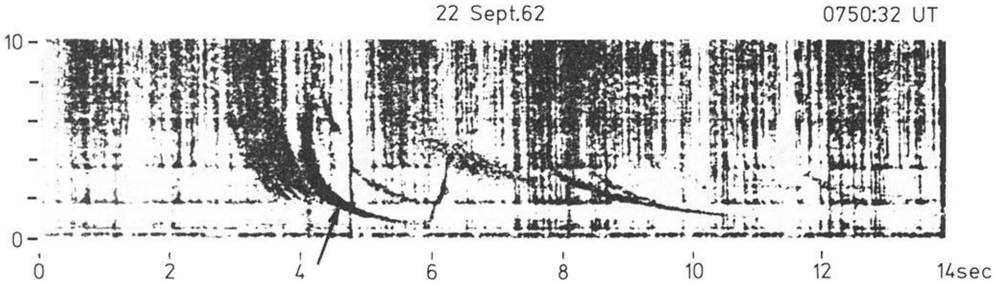

Fig. 1. An example of whistler-triggered emissions (After Helliwell, 1965). The emission shown by an arrow is the one for which the plasma parameters are very similar to those treated in the present paper

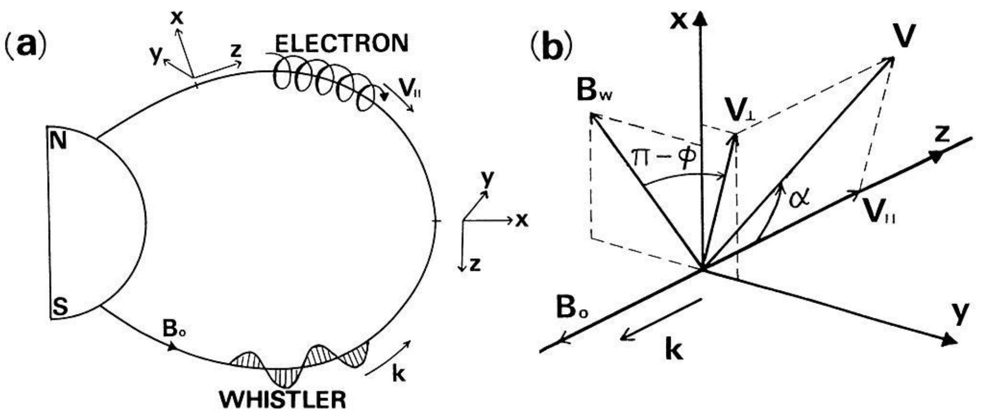

Fig. 2. a Model of interaction between a whistler and an electron. **b** The phase angle ϕ between the electron's perpendicular velocity ($v_\perp$) and the opposite direction of wave magnetic field ($-B_w$). α is the electron's pitch angle

emissions, we calculate the trajectories of electrons in phase space in the presence of a whistler wave with its amplitude being varied over a wide range.

Unperturbed Phase Angle and Bunching Time

Figure 1 shows an example of whistler-triggered hook emission at $L \sim 4.0$ (after Helliwell, 1965). The generation of triggered emissions can be roughly divided into two steps. The first step is that the wave amplitude grows usually at the frequency of the triggering or stimulating signal; we expect phase bunching of resonant electrons by the triggering wave. The second step is that emissions are triggered above the frequency of the triggering wave, and we have drastically varying frequency changes. The present paper is concerned only with the first step for the variable frequency wave.

Figure 2a shows the model in which a constant amplitude whistler wave propagating along the magnetic field interacts with a counter-streaming electron on the same field line. The equations of motion of an electron (charge, -*e* and mass *m*) travelling in the inhomogeneous medium, in the presence of a whistler wave are given by (Dysthe, 1971; Inan et al., 1978; Matsumoto, 1979),

$$\dot{V}_{\parallel} = \Omega_w V_\perp \sin\phi - \frac{V_\perp^2}{2B_0}\frac{\partial B_0}{\partial z}, \tag{1a}$$

$$\dot{V}_\perp = -\Omega_w\left(V_\parallel + \frac{\omega}{k}\right)\sin\phi + \frac{V_\perp V_\parallel}{2B_0}\frac{\partial B_0}{\partial z}, \tag{1b}$$

$$\dot{\phi} = (\Omega_H - \omega - kV_\parallel) - \Omega_w\left(V_\parallel + \frac{\omega}{k}\right)\frac{\cos\phi}{V_\perp} \tag{1c}$$

where $\omega(=2\pi f)$ = angular wave frequency, k = wave number, B_w = wave amplitude, $\Omega_w = eB_w/m$, $\Omega_H(=2\pi f_H) = eB_0/m$, $v_\parallel$ and $v_\perp$ = the electron's parallel and perpendicular velocities, and the phase angle ϕ is defined as an angle between $v_\perp$ and $-B_w$, as shown in Fig. 2b. Equation (1c) implies that as each electron moves through the wave train, ϕ changes because of (1) wave induced longitudinal drift and (2) spatial inhomogeneity of the medium and wave frequency variation. For the conditions we usually meet (not too large wave amplitude and not too small pitch angle; $B_w \lesssim 30\,\mathrm{m}\gamma$ and $\alpha > 5°$ at $L=4$), it is appropriate to neglect the wave term in Eq. (1c) and then it reduces to,

$$\dot{\phi}_u = \Omega_H - \omega - kv_\parallel. \tag{2}$$

The ϕ_u indicates the phase angle under the conditions of extremely small amplitudes ($B_w \sim 0$) and is called the "unperturbed phase angle". In this case we have only the last terms in Eqs. (1a) and (1b), which are the differential form of the first adiabatic invariant. We now define a terminology of "unperturbed resonance" as follows. Suppose that an electron resonates with the whistler initially at a specific point on the field line. As the electron travels adiabatically (for $B_w \sim 0$), the local electron's gyro-frequency changes in the inhomogeneous medium and the Doppler-shifted wave frequency seen by the electron also changes. If the rates of change of the electron's gyro-frequency and of the wave frequency balance each other, there exists a situation for which ϕ_u is maintained nearly constant with time, this being named the unperturbed resonance. Under the unperturbed resonance condition, we cannot ignore the wave amplitude effect any longer. That is, the resonant electrons may be phase bunched by the $ev_\perp \times B_w$ force acting along the field line so as to radiate coherently with consequent wave amplification. By taking into account the finite B_w, we have the motion of a simple pendulum for ϕ around $-B_w$. This oscillation can be considered to be the trapping of electrons by the wave, with its period being given by,

$$T=\left(2\pi\frac{m}{e}\frac{v_p}{v_\perp B_w f}\right)^{1/2} \quad (3)$$

where $v_p(=\omega/k)$=wave phase velocity. The electrons whose absolute initial phase is less than 90° will be bunched at $\phi=0°$ after the period of $T/4$. So we will call this $T/4$ the "bunching time", T_B. The smaller the amplitude B_w becomes, the longer duration we require for the unperturbed resonance because of the large T_B. For natural whistlers, it is generally difficult to maintain the unperturbed resonance condition for an absolutely long time, by making a complete match of the variations between the wave frequency and electron's gyro-frequency. If ϕ_u changes very slowly, or we satisfy the unperturbed resonance for as long as possible, we are able to expect a strong wave-particle interaction. Only in this case, is it possible to expect phase bunching of resonant electrons, when we take into account the wave effect. So, the unperturbed phase variation would be of primary interest and will be studied in the following section.

Calculation of Interaction Time Based on the Unperturbed Phase Angle

In the analysis the background plasma is considered to be cold, and the effects of ions and collisions are neglected so the wave number k is given by (Helliwell, 1965),

$$k=\frac{1}{c}\frac{f_p f^{1/2}}{(f_H-f)^{1/2}} \quad (4)$$

where f_p=plasma frequency. We have employed a centered dipole model for the magnetic field and a diffusive equilibrium model (Angerami, 1966) for the cold plasma density. The parameters of the diffusive equilibrium model are chosen so as to yield a realistic model by making the calculated dispersion agree with the experimental one. The parameters for our computations are summarized in Table 1. In the computations $L=4.0$ is selected because whistlers are most numerous at these latitudes (Helliwell, 1965), and correspondingly we frequently observe the whistler-triggered VLF emissions there (Carpenter, 1968).

The "interaction time", T_I is defined as follows.

$$|\Delta\phi_u|=\left|\int_0^{T_I}\dot{\phi}_u\,dt\right|=\pi/2. \quad (5)$$

This implies that T_I is the time interval for which ϕ_u stays within $\pm\pi/2$ of the exact resonance condition. The value of T_I varies considerably, depending on the position of initiation of resonance, initial wave frequency and electron pitch angle. Given the equatorial pitch angle α_{eq} and initial resonance frequency f_i, we can calculate the variation of ϕ_u for various initial geomagnetic latitudes λ_i. The pitch angle at the initial position is obtained by the first adiabatic invariant,

Table 1. Parameters used for calculations

Field line	$L=4$
Equatorial gyrofrequency	$f_{Heq}=13.65$ kHz
Equatorial cold plasma density	$N_{eq}=420\,\mathrm{cm}^{-3}$ ($f_p=184$ kHz)
Nose frequency	$f_n=5.2$ kHz
Whistler dispersion	$D=110\,\mathrm{s}^{1/2}$
Initial frequency	$f_i=1, 2, 3, 4$ kHz ($\Delta f_i=1$ kHz)
Equatorial pitch angle	$\alpha_{eq}=5, 15, 30, 45, 60°$

$$\sin\alpha_i=\sin\alpha_{eq}(f_{Hi}/f_{Heq})^{1/2} \quad (6)$$

where f_{Heq}=equatorial gyro-frequency and f_{Hi}=gyrofrequency at the initial position. The initial value $v_{\parallel i}$ for the electron's parallel velocity is deduced from the resonance condition (or $\omega-kv_\parallel=\Omega_H$). Then we can determine the electron's total velocity v and energy W, which remain constant in our unperturbed study. As we are interested in triggering around whistler tails as shown in Fig. 1, the initial frequency f_i is chosen to be below the nose frequency. For further details of the calculation procedure, see Appendix.

Now the results of computations are presented. Figure 3 illustrates the variation of ϕ_u for four different initial frequencies for a specific pitch angle of $\alpha_{eq}=30°$. The ϕ_u variation given by a thick line in each panel yields the largest $T_I(T_{Imax})$. In each panel a shift of the initial latitude from that giving the T_{Imax}, even by a small amount, is found to result in remarkably different ϕ_u curves, a typical example of this being seen in the third panel of Fig. 3. This suggests that there is a very restricted range of geomagnetic latitude where we might expect the unperturbed resonance, and its interaction region is far from the equator on the upstream side of the electrons, being significantly different from the case for constant frequency waves. Some of the ϕ_u variations with larger T_I values show conspicuous oscillations as a consequence of a joint effect of the variations of the spatial inhomogeneity and wave frequency, as if they were phase-trapped by the wave.

A summary deduced from Fig. 3 is given in Fig. 4 as the relationship between the interaction time and initial geomagnetic latitude, with initial frequency as a parameter. The variation of frequency of the whistler wave is smaller at lower f_i so that the resonance region limited in latitude should be located in a region of smaller spatial inhomogeneity, or nearer to the equator. Conversely the resonance region would move farther away from the equator for higher f_i. The most important point of Fig. 4 is that the interaction time T_I is a maximum (~50 ms) for $f_i=3$ kHz, among various frequencies.

Figures 5 and 6 are a summary of the results of similar computations when the pitch angle is changed. Figure 5 shows how the initial geomagnetic latitude for which we have the T_{Imax} for each f_i varies with pitch angle. Generally speaking, the initial position where the wave-particle interaction is maximized, is located at higher latitudes for smaller pitch angles, and it shifts towards lower latitudes relatively abruptly at pitch an-

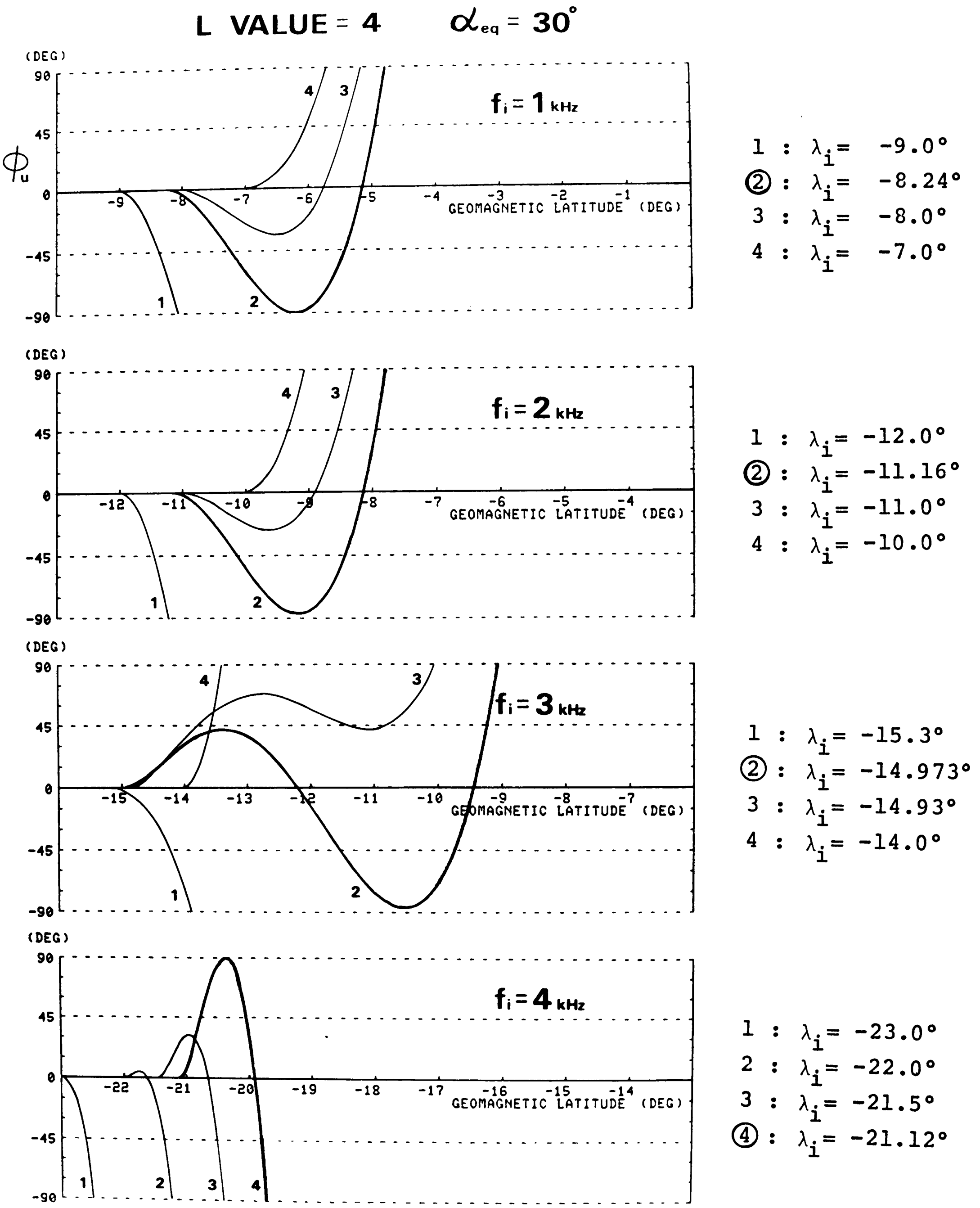

Fig. 3. Variations of unperturbed phase angle ϕ_u starting at various geomagnetic latitudes. Four initial frequencies are studied. A thick curve gives the largest T_I in each panel, and the corresponding initial latitude is described on the right of each panel

gles larger than about 30°. Figure 6 illustrates the resonance energy, as calculated from $\dot{\phi}_u=0$ in Eq. (2), versus pitch angle for different f_i. This shows that the interaction for $f_i=3$ kHz requires smaller resonant electron energy (15–25 keV) than for any other f_i.

In Fig. 7 we compare the calculated T_I with the bunching time T_B discussed above to examine the possibility of phase bunching and subsequent triggering emissions. We take the parameters $f_i=3$ kHz and $\alpha_{eq}=30°$, these conditions being found to be extremely favorable for triggering emissions because of the maximum T_I as seen in Figure 4 and of the minimum resonant electron energy as in Fig. 6. The broken lines indicate the variation of T_B for different wave amplitudes. In the calculations of T_B we have used the initial values of $v_\perp, f$ and v_p at λ_i. The T_B for an appreciable amplitude such as $B_w=10$ mγ is rather small so that T_{Imax} is much larger than T_B, and the region where T_I exceeds T_B for $B_w=10$ mγ is relatively wide (about 5°) in latitude. With a decrease of wave amplitude down to $B_w=1$ mγ, T_B increases up to ~ 14 ms at the geomagnetic latitude of $\sim -15°$, but the maximum $T_I(\sim 50$ ms) is still much larger than the T_B within an extremely restricted region ($\sim 1°$) in latitude. This may suggest that the phase bunching of electrons might be possible with weak whistler amplitude even below 1 mγ. This 1 mγ

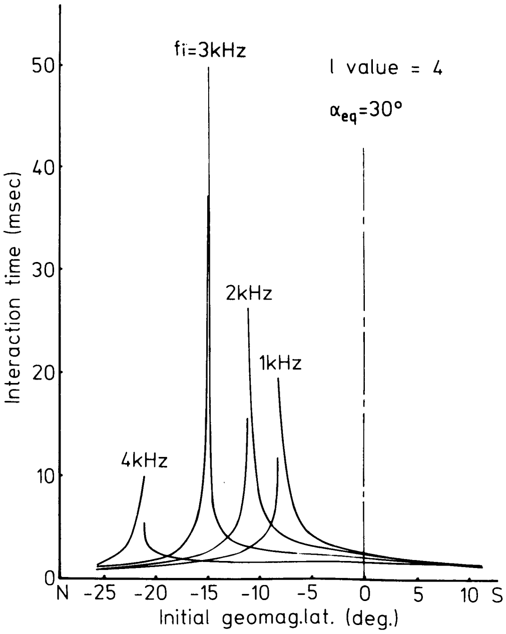

Fig. 4. Relationship between the interaction time and initial latitude with an initial frequency as a parameter. $\alpha_{eq}=30°$

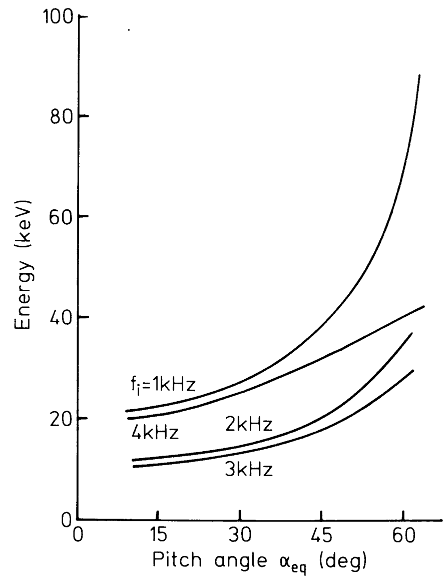

Fig. 6. Resonant electron energy versus pitch angle for different f_i

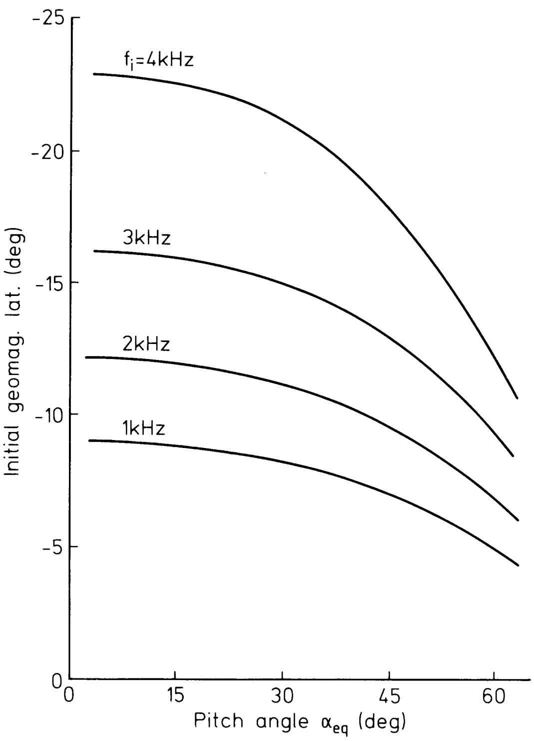

Fig. 5. Initial geomagnetic latitude for which T_I is maximized, versus pitch angle. The initial frequency f_i is a parameter

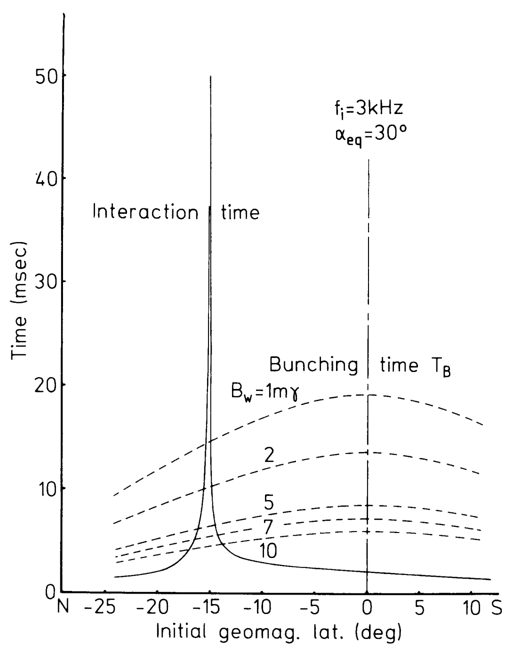

Fig. 7. Comparison between the interaction time and bunching time for various wave amplitudes. $f_i=3$ kHz and $\alpha_{eq}=30°$

(~ 3 mV/m) is a typical value for the intensity of ordinary whistlers at $L\sim 4$ (Helliwell, 1965).

Figure 8 illustrates the relationship of T_{Imax} for each f_i with α_{eq} (solid lines). The broken lines indicate the variation of T_B with pitch angle for a typical $B_w=1$ mγ. Except for the case of $f_i=4$ kHz, the interaction time (T_{Imax}) decreases with an increase of pitch angle. In particular T_{Imax} shows a decrease when the pitch angle is larger than 40°. It is found that T_{Imax} for $f_i=3$ kHz has an extremely large value compared with that for other f_i. The part indicated by thick solid lines corresponds to the region where the condition $T_I > T_B$ is satisfied. A comparison of the maximum interaction times at $f_i=1, 2, 3$ and 4 kHz in Fig. 8 results in Fig. 9, which is a different form of presentation. This figure indicates that T_I is maximised at $f_i\sim 3.25$ kHz. The amplitude of

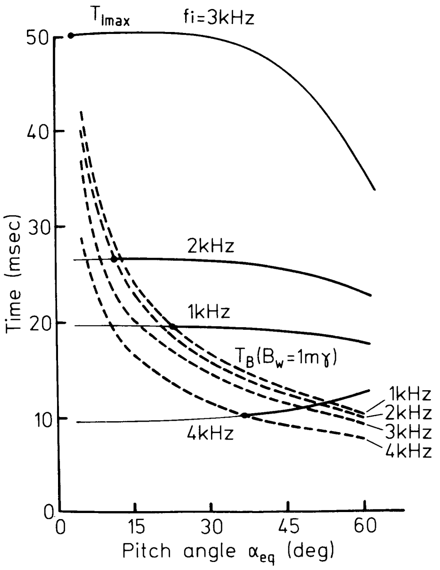

Fig. 8. Maximum T_I (T_{Imax}) versus pitch angle for different f_i and the corresponding variations of T_B for $B_w = 1$ m

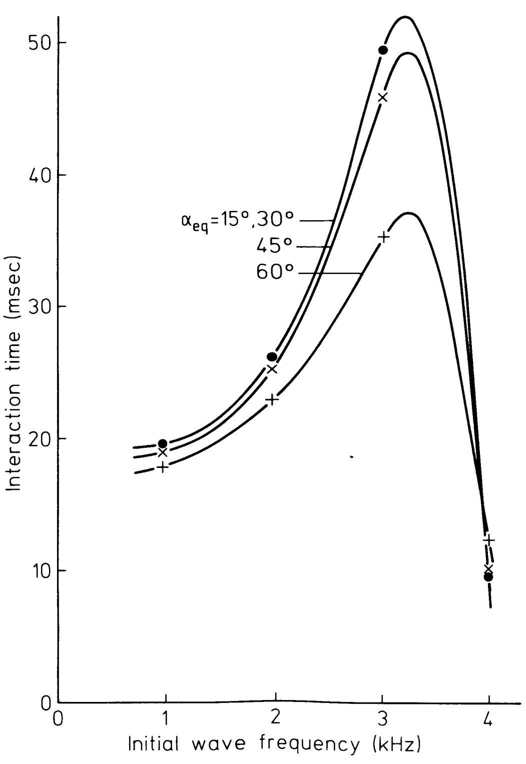

Fig. 9. Relationship of the interaction time with the initial frequency

whistlers at $L \sim 4$ is, normally, about 1 mγ as stated above, and so 1 mγ is taken as a reasonable value in Fig. 8. As the condition favorable for triggering VLF emissions by a natural whistler, we require a large difference between T_{Imax} and T_B. In this case it is highly probable that emissions will be triggered by a natural whistler with $f_i = 3$ kHz even with small amplitude below 1 mγ. The initial f_i of 3 kHz yields the final whistler frequency of 3.0–1.8 kHz after the interaction, depending on the length of the interaction time; the new emissions will appear at that frequency. This gives the ratio of $f/f_{Heq} = 0.22$–0.13. Figure 8 imposes a lower limit on the pitch angle for producing phase bunching, and correspondingly, as shown in Fig. 6, the resonant electron energy increases with increasing pitch angle. So the range of pitch angle appropriate for phase bunching would be in an intermediate range of 30°–45°, as suggested by Helliwell (1967).

Wave Effect and Phase Bunching

The importance of the ratio T_I/T_B discussed in the previous section is quantitatively examined by calculating the trajectories of electrons in phase space based on the complete equations, Eq. (1). Wave amplitude is assumed to be constant over the whole frequency range. Provided that perpendicular velocities of nearly resonant electrons are oriented randomly before they come into resonance with the wave, the actual population can be represented by a sheet of electrons spaced uniformly in phase (Helliwell, 1970; Helliwell and Crystal, 1973). So the test particles are assumed to have initial phases equally spaced at 30° intervals and to be in exact resonance with the wave at λ_i. Since we adopt a test particle approach, our calculations do not include the effects of the electromagnetic fields generated by perturbed electrons. We present the results only for the most promising situation for wave generation, i.e. $f_i = 3$ kHz and $\alpha_{eq} = 30°$. Figure 10 shows the computed trajectories for 12 electrons which are distributed in initial phase angle ϕ_0 and which start the interaction at $\lambda_i = -15$. For this situation we obtained a larger T_I (25.19 ms). Figure 10a refers to the case of $B_w = 1$ mγ and Fig. 10b $B_w = 0.3$ mγ. Figure 11 shows similar results for an initial latitude slightly displaced by 0.5° ($\lambda_i = -14.5°$), for which we have obtained a smaller T_I (7.17 ms). Figure 11a is for $B_w = 1$ mγ, while Fig. 11b for $B_w = 5$ mγ. From Fig. 10a for $B_w = 1$ mγ such that T_I is larger than T_B(~ 12 ms), it is found that the electrons whose initial phase angle is approximately less than 90° are phase bunched at $\phi \cong 0°$ and at $\lambda_i = -13°$. Even if we decrease the wave amplitude down to $B_w = 0.3$ mγ (in Fig. 10b), we have the condition $T_I \simeq T_B$, and we can confirm the presence of the phase bunching phenomenon at $\lambda_i = -11°$ at $\phi \simeq -160°$, but not at $\phi \simeq 0°$. Figure 11a shows that the variations of ϕ starting at various ϕ_0 indicate nearly the same structures, indicating a negligibly small effect of the wave amplitude. This is because T_I is much smaller than T_B for this amplitude of $B_w = 1$ mγ. If we increase the amplitude up to $B_w = 5$ mγ, as given in Fig. 11b, we expect that T_I is approximately equal to T_B, and hence the phase bunching is observable at $\lambda_i = -13.5°$ at $\phi \sim 80°$, but not at $\phi \simeq 0°$. Further increase of wave amplitude so as to satisfy the condition of $T_I > T_B$ is found to result in the phase bunching of resonant electrons at $\phi \simeq 0°$. From these results, we may conclude that the electrons are possibly phase bunched only by the wave whose amplitude satisfies the condition of $T_I > T_B$. Hence, the value of T_I deduced from the variation of unperturbed phase angle seems to be of great use in determining the threshold for phase bunching of resonant electrons by a variable frequency wave.

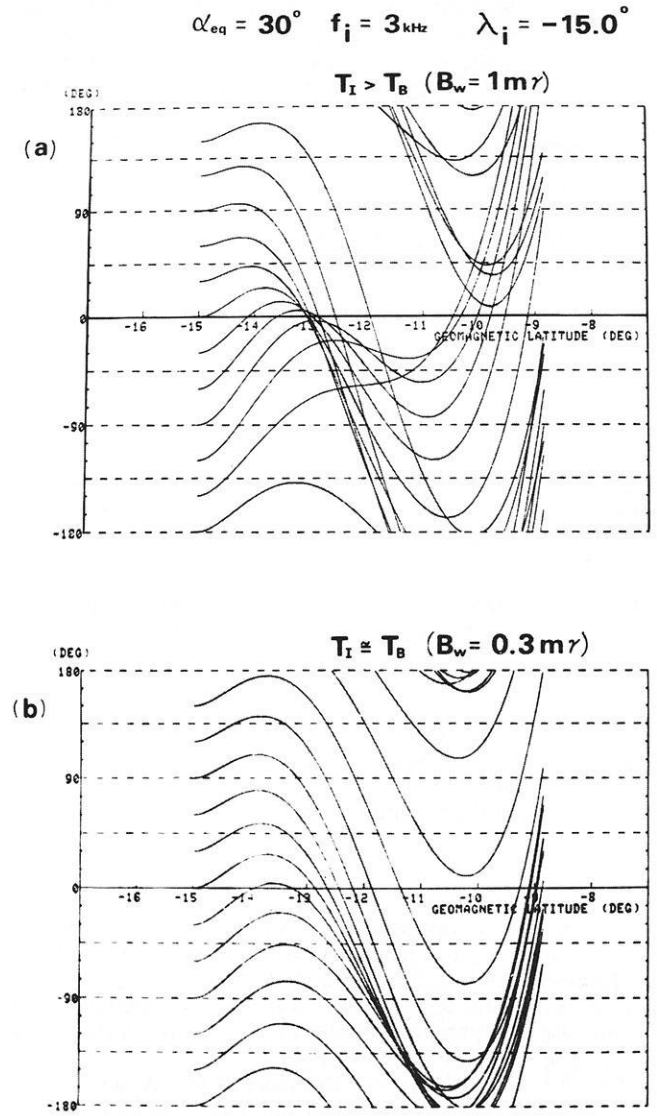

Fig. 10a and b. Variation of phase angle of electrons starting at $\lambda_i = -15$ (large T_I) for **a** $B_w = 1$ mγ ($T_I > T_B$), and **b** for a smaller amplitude, $B_w = 0.3$ mγ ($T_I \simeq T_B$). $f_i = 3$ kHz and $\alpha_{eq} = 30°$. A random distribution in phase is assumed before the interaction

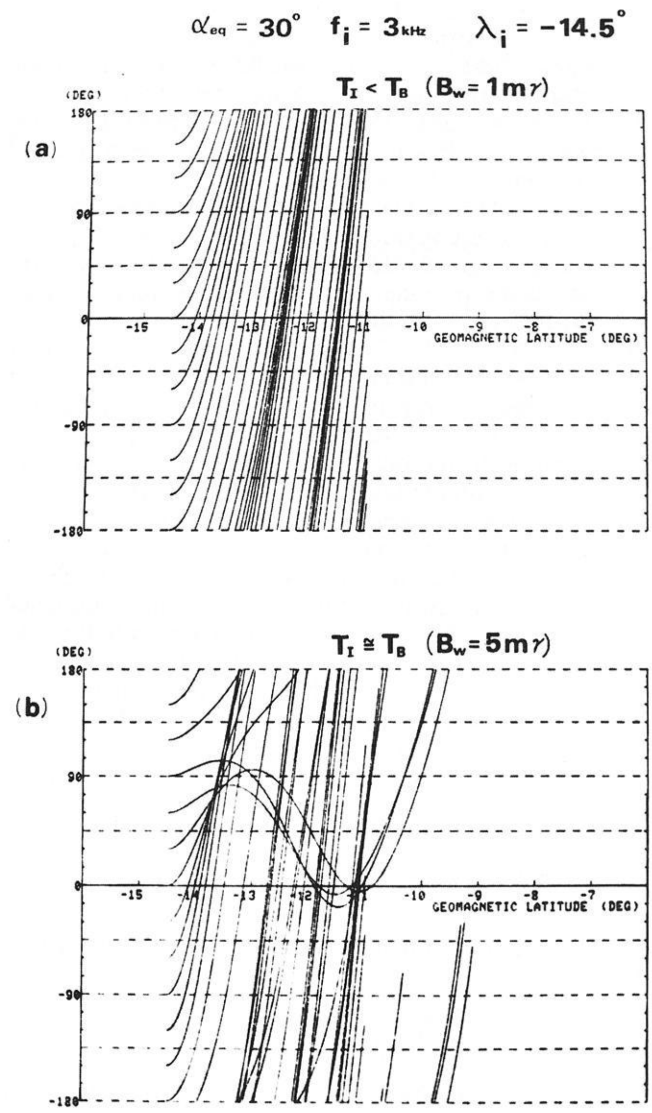

Fig. 11a and b. Variation of phase angle of electrons starting at a slightly displaced initial latitude ($\lambda_i = -14.5°$) (smaller T_I than that in Fig. 10) **a** for a smaller amplitude $B_w = 1$ mγ ($T_I > T_B$) and **b** for a considerable amplitude $B_w = 5$ mγ ($T_I \simeq T_B$). $f_i = 3$ kHz and $\alpha_{eq} = 30°$

Conclusions and Discussion

In conclusion we first summarize the essential points elucidated in the present paper.

(1) The interaction time defined by the variation of unperturbed phase angle determined by the spatial inhomogeneity of the medium and wave frequency variation is found to be very useful in the study of triggering emissions by a variable frequency wave such as a natural whistler, with special reference to its comparison with the bunching time. This point was quantitatively examined by calculating the trajectories of resonant electrons in phase space in the presence of a wave with variable amplitude. Our method of analysis would be applicable to other variable frequency waves such as linearly changing frequency (ramp) and sinusoidally varying frequency waves in the Siple transmissions.

(2) The computations of interaction times for a realistic model at $L = 4$ have shown that the region where electrons stay in resonance with a natural whistler is extremely restricted in latitude, far from the equator on the upstream side of the electrons. A resonance with initial frequency of 3 kHz would be extremely favorable for triggering VLF emissions compared with other initial frequencies. The interaction region for $f_i = 3$ kHz is located at $-15.89°$ for $\alpha_{eq} = 15°$ and at $-9.34°$ for $\alpha_{eq} = 60°$. The resonant electron energy ranges from 10.77 keV ($\alpha_{eq} = 15°$) to 28.35 keV ($\alpha_{eq} = 60°$). Pitch angles appropriate for phase bunching and subsequent coherent radiation would probably be in the intermediate range, 30°–45°. In the present paper we used initial frequencies 1 kHz apart, but Fig. 9 shows that the maximum T_I is achieved for $f_i \sim 3.25$ kHz, yielding the starting frequency of new emissions at 3.25–3.5 kHz (or

$f/f_{Heq}=0.24$–0.17). The lowest frequencies would be more likely starting frequencies for whistler-triggered emissions. These results seem to be in good agreement with the statistics on the starting frequency by Matsumoto and Kimura (1971), who found the highest occurrence probability at about one sixth the equatorial gyro-frequency ($f/f_{Heq}=0.17$).

Because there have been as yet no theoretical reports on the interaction region and starting frequency of whistler-triggered VLF emissions, we believe that the present results provide new information on these subjects. Although the calculations were performed for a specific L value of 4.0, it is generally accepted that there exists a restricted initial geomagnetic latitude and initial frequency appropriate for triggering emissions by a natural whistler even for other L values. These facts invite attention to the use of whistler spectra in studies of wave-particle interactions. We intend to extend the analyses to other L values, to be compared with experimental results. Furthermore, assuming a realistic distribution of energetic electrons, we will be able to calculate the resonant currents formed by the phase bunched electrons and subsequent coherent wave radiation. In the case of emission triggering at whistler tails, as treated in the present paper, we have a prominent property of a rapid frequency change as in Fig. 1. The mechanism of this drastic frequency change seems to be closely related with the second-order resonance condition (Helliwell, 1967; Nunn, 1974) in which the spatial inhomogeneity plays a key role. As is inferred from the present paper, the interaction region is far from the equator where there is a large spatial inhomogeneity, and this off-equatorial interaction region might result in such a rapid frequency change. This requires further study.

Appendix

In Fig. (A1), t_1 indicates the group delay time traversed by a wave with frequency f_1 during the transition from the starting position z_s (the height of 300 km in the ionosphere) to the position z_1 in the magnetosphere, and is given by

$$t_1=\int_{z_s}^{z_1}\frac{dz}{v_g(f_1)} \tag{A1}$$

where $v_g(f)$ is the group velocity at frequency f. Similarly t_2 is given by

$$t_2=\int_{z_s}^{z_2}\frac{dz}{v_g(f_2)}. \tag{A2}$$

Then, $\Delta t(=t_2-t_1)$ is the time interval the electron travels from z_1 to z_2, which is expressed by

$$\Delta t=\int_{z_1}^{z_2}\frac{dz}{v_\parallel}. \tag{A3}$$

Using Eqs. (A1), (A2) and (A3), we obtain the following requation.

$$\int_{z_s}^{z_2}\frac{dz}{v_g(f_2)}-\int_{z_s}^{z_1}\frac{dz}{v_g(f_1)}=\int_{z_1}^{z_2}\frac{dz}{v_\parallel}. \tag{A4}$$

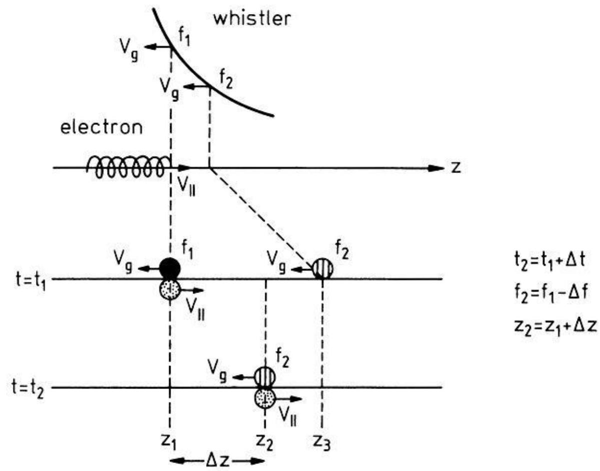

Fig. (A1). Relation between whistler wave packets and resonant electrons between two closely spaced points within the interaction region

Therefore, when we fix the position z_1 and frequency f_1 and we give Δz, we can determine Δf by using a numerical computation. So we can calculate ϕ_u with a step of Δz.

Acknowledgements. The authors wish to express their sincere thanks to Dr. Y. Tanaka of their institute for useful discussions. Thanks are also due to Professors A. Iwai and A. Kimpara for their continual encouragement.

References

Angerami, J.J.: A whistler study of the distribution of thermal electrons in the magnetosphere. Tech. Rept. 3412-7, Radioscience Lab., Standord Univ. 1966

Brinca, A.L.: Enhancing whistler wave-electron interaction by the use of specially modulated VLF wave injection. J. Geophys. Res. **86**, 792–800, 1981

Burtis, W.L., Helliwell, R.A.: Banded chorus—A new type of VLF radiation observation in the magnetosphere by Ogo 1 and Ogo 3. J. Geophys. Res. **74**, 3002–3010, 1969

Burtis, W.L., Helliwell, R.A.: Magnetospheric chorus:occurrence patterns and normalised frequency. Planet. Space Sci. **24**, 1007–1024, 1976

Carpenter, D.L.: Ducted whistler mode propagation in the magnetosphere: A half-gyrofrequency upper intensity cutoff and some associated wave growth phenomena. J. Geophys. Res. **73**, 2919–2928, 1968

Dysthe, K.B.: Some studies of triggered whistler emissions. J. Geophys. Res. **76**, 6915–6931, 1971

Gendrin, R.: Waves and wave-particle interactions in the magnetosphere—A review. Space Sci. Rev. **18**, 145–200, 1975

Helliwell, R.A.: Whistlers and related ionospheric phenomena. Stanford Univ. Press 1965

Helliwell, R.A.: A theory of discrete VLF emissions from the magnetosphere. J. Geophys. Res. **72**, 4773–4790, 1967

Helliwell, R.A.: Intensity of discrete VLF emissions. In: Particles and fields in the magnetosphere, B.M. McCormac, ed. Dodrecht: D. Reidel 1970

Helliwell, R.A.: Active very low frequency experiments on the magnetosphere from Siple Station, Antarctica. Philos. Trans. R. Soc. London Ser. B**279**, 213–224, 1977

Helliwell, R.A., Crystal, T.L.: A feedback model of cyclotron interaction between whistler-mode waves and energetic electrons in the magnetosphere. J. Geophys. Res. **78**, 7357–7371, 1973

Inan, U.S., Bell, T.S., Helliwell, R.A.: Nonlinear pitch angle scattering of energetic electrons by coherent VLF waves in the magnetosphere. J. Geophys. Res. **83**, 3235–3253, 1978

Maeda, K.: Cyclotron side-band emissions from ring current electrons. Planet. Space Sci. **24**, 341–347, 1976

Matsumoto, H.: Nonlinear whistler-mode interaction and triggered emissions in the magnetosphere. In: Wave Instabilities in Space Plasma, P.J. Palmadesso and K. Papadoulos, eds. Dodrecht: D. Reidel, 1979

Matsumoto, H., I. Kimura: Linear and nonlinear Cyclotron instability and VLF emissions in the magnetosphere. Planet. Space Sci. **19**, 567–608, 1971

Nunn, D.: A self-consistent theory of VLF emissions. Planet. Space Sci. **22**, 349–378, 1974

Miller, T.R.: Siple VLF transmitter data 1978: Selected records from Roberval, Canada and ISEE. Tech. Rept. VLF 79-1, Radioscience Lab., Stanford Univ. 1979

Rycroft, M.J.: VLF emissions in the magnetosphere. Radio Sci. **7**, 811–830, 1972

Received November 11, 1982; Revised version January 5, 1983

Accepted January 19, 1983