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Short Communication

On the Generation of Almost Uniform Magnetic Fields Inside Solenoids

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Introduction

Since the first investigations of the magnetic fields of stationary current distributions, the problem of providing almost uniform artificial magnetic fields has touched many branches of physics. For different applications different solutions were found (see, for example, the proposals of Garrett, 1967 and the review of Serson, 1974). Here, the following problem is stated: How can one generate a magnetic field which is as uniform as possible in a long (as compared with its diameter) cylindrical volume, while the length of the field generating device cannot be much longer than that volume but might be much broader for practical reasons?

The weak fields used in some applications, e.g. calibration coils for geomagnetic micropulsation sensors, can best be controlled if a moderate excitation current (about one milliAmp) is allowed. For this, single-layered solenoids with equally spaced turns, of the order of one hundred, are in use. As the axial magnetic field of a uniformly wound solenoid decreases from the center to the ends, the idea of increasing the winding density towards the ends or - more generally - of choosing the winding density as a function of distance from the center is suggested.

In the next section, a numerical method for calculating this winding density function will be presented, and in the last part, the result will be applied to an arbitrarily chosen example.

The Calculation

A single-layered solenoid can be modelled by a series of concentric circular filamentary currents which for practical reasons (easy but precise construction) should have equal radii and carry equal currents. Keeping in mind the symmetry of the solenoid, we choose a cylindrical coordinate system with its centre at the centre of the coil and its z-axis along the central axis.

Let a be the radius of the solenoid and $2z_N$ its length, and let $2N+1$ current filaments be in the (generally not equally spaced) planes $z_n = \text{const.}$ $(n = -N, \ldots, N)$. Let us consider now the (longitudinal) field along the z-axis between the limits $-z_{\max}$ and $z_{\max}$ (remarks on the off-axis field will follow later), for which scalar notation can be used. Following Biot-Savart's law, a current strength I in each turn will produce a field at the point z on the axis (SI units used)

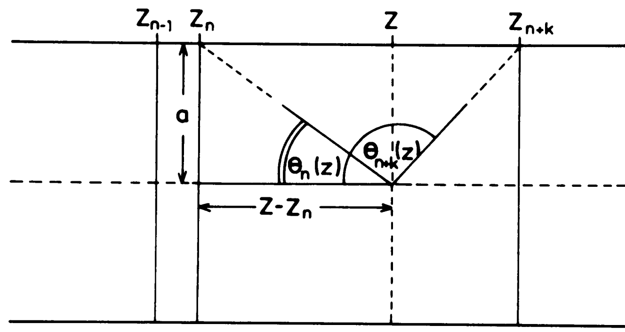

Fig. 1. On the calculation of the magnetic field at the point z on the z-axis caused by circular currents in the planes $z = z_n$ and $z = z_{n+k}$

$$H(z) = \frac{I a^2}{2} \sum_{n=-N}^{N} [a^2 + (z - z_n)^2]^{-3/2}. \tag{1}$$

Let all spatial dimensions be normalized with respect to a and the magnetic field with respect to $I/2a$, and let $\Theta_n(z)$ be the angle between the z-axis and a straight line from z to the filament z_n (see Fig. 1), then we obtain Eq. (1a) instead of (1):

$$H(z) = \sum_{n=-N}^{N} \sin^3 \Theta_n(z) = \sum_{n=-N}^{N} H_n(z). \tag{1a}$$

If $\bar{H}$ denotes the arithmetic mean of the field between the limits $-z_{\max}$ and $z_{\max}$ and

$$D(z) = H(z) - \bar{H} \tag{2}$$

the error at the point z, then the requirement for an almost uniform field between $-z_{\max}$ and $z_{\max}$ means finding the minimum of

$$J_2 = \int_{-z_{\max}}^{z_{\max}} D^2(z)\, dz = z_{\max} \int_{-1}^{1} D^2(z_{\max} \cdot \zeta)\, d\zeta \tag{3}$$

under the condition

$$J_1 = \int_{-z_{\max}}^{z_{\max}} D(z)\,dz = z_{\max} \int_{-1}^{1} D(z_{\max} \cdot \zeta)\,d\zeta = 0 \tag{4}$$

which is identical with taking $\bar{H}$ as the arithmetic mean between the limits $-z_{\max}$ and $z_{\max}$.

(We should use the relative errors instead of the – easier to handle – absolute errors, but we will see later, that $\bar{H}$, once chosen, will hardly be changed by the minimization procedure.) As neither J_2 nor $\partial J_2/\partial z_n$ can be evaluated in analytical form, a numerical integration has to be carried out. $2M+1$ nodes will be chosen in accordance with the integration formula. Let

$$Q = z_{\max} \sum_{m=-M}^{M} g_m D_m^2 \tag{5}$$

be the sum of the absolute squared errors and

$$P = z_{\max} \sum_{m=-M}^{M} g_m D_m \tag{6}$$

be the sum of errors, with $D_m = D(z_{\max} \cdot \zeta_m)$, $\zeta_m(m = -M, \ldots, M)$ being the nodes in the interval $[-1, 1]$ and g_m the respective weights. Writing $\Theta_n^m = \Theta_n(z_{\max} \cdot \zeta_m)$, we find

$$\bar{H} = \tfrac{1}{2} \sum_{m=-M}^{M} g_m \sum_{n=-N}^{N} \sin^3 \Theta_n^m. \tag{7}$$

Minimization of Q and taking into account the symmetries of the problem

$$\begin{aligned}
z_{-n} &= -z_n \\
g_{-m} &= g_m \\
\sin \Theta_n^{-m} &= \sin \Theta_{-n}^m \\
\cos \Theta_n^{-m} &= -\cos \Theta_{-n}^m \\
\partial Q/\partial z_{-n} &= -\partial Q/\partial z_n
\end{aligned}$$

will lead to a $N+1$-dimensional system of equations $(\bar{n} = 0, \ldots, N)$

$$\begin{aligned}
0 &= \frac{1}{6 z_{\max}} \frac{\partial Q}{\partial z_{\bar{n}}} \\
&= \sum_{\bar{m}} g_{\bar{m}} \Big[\sum_n \sin^3 \Theta_n^{\bar{m}} - \sum_m g_m \sum_n \sin^3 \Theta_n^m - \frac{g_0}{2} \sum_n \sin^3 \Theta_n^0\Big] \\
&\cdot \Big[\sin^4 \Theta_{\bar{n}}^{\bar{m}} \cos \Theta_{\bar{n}}^{\bar{m}} - \sin^4 \Theta_{-\bar{n}}^{\bar{m}} \cos \Theta_{-\bar{n}}^{\bar{m}} - g_0 \sin^4 \Theta_{\bar{n}}^0 \cos \Theta_{\bar{n}}^0 \\
&- \sum_m g_m (\sin^4 \Theta_{\bar{n}}^m \cos \Theta_{\bar{n}}^m - \sin^4 \Theta_{-\bar{n}}^m \cos \Theta_{-\bar{n}}^m)\Big] \\
&+ \frac{g_0}{2} \Big[(1 - g_0/2) \sum_n \sin^3 \Theta_n^0 - \sum_m g_m \sum_n \sin^3 \Theta_n^m\Big] \\
&\cdot \Big[(2 - g_0) \sin^4 \Theta_{\bar{n}}^0 \cos \Theta_{\bar{n}}^0 \\
&- \sum_m g_m (\sin^4 \Theta_{\bar{n}}^m \cos \Theta_{\bar{n}}^m - \sin^4 \Theta_{-\bar{n}}^m \cos \Theta_{-\bar{n}}^m)\Big].
\end{aligned} \tag{8}$$

(Summation index m or $\bar{m}$ has to be taken from 1 to M, and n from $-N$ to N.)

As the non-linear system cannot be solved directly, we try to find a minimum by a gradient iteration procedure, i.e. we start with equally spaced loops, for example, calculate $\partial Q/\partial z_{\bar{n}}$, and find after k steps improved values

$$z_{\bar{n}}^{(k+1)} = z_{\bar{n}}^{(k)} - \kappa \cdot \partial Q/\partial z_{\bar{n}}^{(k)} \\ \cdot (\bar{n} = 0, \ldots, N;\ k = 1, \ldots) \tag{9}$$

with a suitable κ. After each step, the new values will be stretched or compressed so that $\pm z_N$ keeps constant to preserve the length of the solenoid and (nearly) the mean field $\bar{H}$. Furthermore, the gradient iteration procedure will guarantee to approach a minimum of Q.

A Numerical Example

Let us consider a solenoid with a length-to-diameter ratio of 5 with 101 turns. We require an almost uniform field along the central 90% of its length. Taking 1001 nodes and using the trapezoidal integration formula, we find the residual sum of squares Q to be decreased to 0.1% of that of equally spaced turn after 78 iterations while the mean field $\bar{H}$ changed less than 5%. Figure 2 compares the relative deviation of the axial field from the field in the center for equally spaced starting values of z_n with that after 78 iterations and after 250 iterations for one half of the cylinder (upper box). The dashed-dotted line marks the 90% region. The $(+)/(-)$ signs indicate positive/negative deviation. Errors of less then 0.1% are supressed for technical reasons: Let the construction errors be normally distributed with a variance of $(\Delta a)^2$ for the radial and $(\Delta z)^2$ for the longitudinal errors of the current filaments. From Eqs. (1) and (1a) we get for the n-th turn

$$\partial H_n/\partial a = (2 - 3 \sin^2 \Theta_n) \cdot \sin^3 \Theta_n.$$

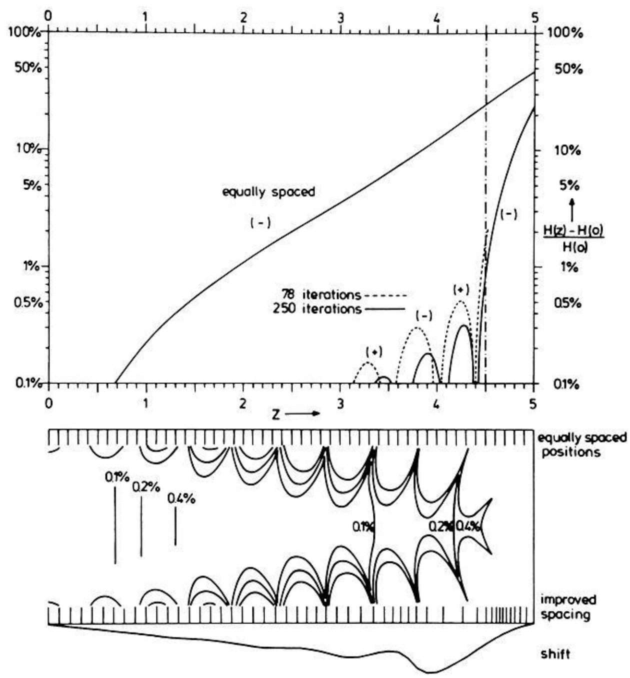

Fig. 2. Deviation of the axial magnetic field component from the field in the coil's center with equally spaced turns and with improved spacing after 78, and 250 iteration steps (upper box). For the last case, envelope of the 0.1%, 0.2%, and 0.4% error regions (lower box) as well as the shift magnitude of each turn (bottom)

Differentiation shows that the error reaches its maximal absolute value at $z=z_n$, decreasing to zero at $|z-z_n| = \sqrt{2}/2$ and increasing again, but only to one fifth of its maximum value before decreasing to zero when $|z-z_n|$ reaches infinity. Therefore, in reasonable cases, the largest error will be found at $z_0=0$:

$$\frac{\Delta H(0)}{\bar{H}} \approx \frac{\Delta a}{\bar{H}} \cdot \left[2 \sum_{n=1}^{N} [(2-3\sin^2\Theta_n^0)\cdot\sin^3\Theta_n^0]^2+1\right]^{1/2}. \quad (10)$$

Similarly, we find

$$\partial H_n/\partial z = 3\sin^4\Theta_n \cos\Theta_n$$

which is zero for $z=z_n$, but increases quickly to its maximum at $|z-z_n|=0.5$ and decreases again to zero. Therefore, the largest error will be found at $z_0=0$ (in all reasonable cases):

$$\frac{\Delta H(0)}{\bar{H}} \approx \frac{3\Delta z}{\bar{H}} \cdot \left[2 \sum_{n=1}^{N} (\sin^4\Theta_n^0 \cos\Theta_n^0)^2\right]^{1/2}. \quad (11)$$

Choosing $a=10$ cm and $\Delta a=\Delta z=0.05$ cm gives a total construction error of $\Delta H(0)/\bar{H} \approx 0.1\%$.

The lower box of Fig. 2 shows one half of the solenoid scale-modelled, with equally spaced turns on the the upper boundary, improved spacing on the lower, and the amount of turn shift below. The undulation of the latter is in accordance with the results of Montgomery (1969, p. 260) for compound coils. The vertical lines indicate the 0.1%-, 0.2%-, and 0.4%-error limits for an equally spaced winding, the contours mark the error limits for the improved device, as calculated from the tables of Hart (1967) for the z-component of the magnetic field.

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