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Rheological properties and velocity dispersion of a medium with power-law dependence of Q on frequency

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Abstract. The Kramers-Krönig relations for magnitude and phase of a linear causal filter are used to derive an exact general expression for the viscoelastic modulus M, corresponding to power laws for the quality factor, $Q \sim \omega^{\gamma}$. The exponent γ varies from -1 to $+1$, such that the spectrum of rheologies extends from a Kelvin-Voigt to a Maxwell body. High- and low-frequency approximations for $M(\omega)$ are derived, and in the special cases $\gamma = \pm 1$, $\pm 1/2$, $\pm 1/3$, $\pm 1/4, \ldots$ closed-form solutions are given which apply for arbitrary frequencies. With $M(\omega)$ at hand, both high-frequency phenomena such as velocity dispersion and low-frequency phenomena such as creep and stress relaxation can be investigated. Results for phase-velocity dispersion are given as well as short- and long-time-scale approximations of the creep and relaxation functions. Simple dissipation operators are derived which can be convolved with theoretical seismograms in order to correct these for the influence of absorption. Some results on relaxation spectra for the case $0 \leqq \gamma \leqq 1$ are summarized in an appendix. Taken together, the results of this paper suggest that media with $0 < \gamma < 1$ should be considered as generalized Maxwell bodies and media with $-1 < \gamma < 0$ as generalized Kelvin-Voigt bodies. Application of the Kramers-Krönig relations to the viscoelastic modulus is better than the use of those relations in conjunction with the wavenumber of a plane wave, which is the procedure that has been employed so far.

Key words: Viscoelasticity – Quality factor – Dispersion – Stress relaxation and creep – Dissipation operators

Introduction

The study of the attenuation-dispersion properties of earth materials is a subject of past and current interest, both in rock physics and seismology. After a period during which the assumption of frequency independence of the quality factor Q in the seismic frequency band was considered sufficient, it is now apparent, from both seismological and laboratory studies, that there may be significant departures from the constant-Q model (Sipkin and Jordan, 1979; Anderson and Minster, 1979; Lundquist and Cormier, 1980; Berckhemer et al., 1982; Minster, 1980; see also Stacey et al., 1981). In this context a power-law dependence of Q on frequency, $Q \sim \omega^{\gamma}$ with $0 < \gamma < 0.5$, has become popular. Approximate results for the corresponding frequency dependence of phase velocity c have been given, e.g., by Brennan (1980); this author also discussed a method for the determination of γ from pulse-form measurements in ultrasonic studies. Brennan (see also Brennan and Smylie, 1981 or Chin, 1980) obtained his results for phase-velocity dispersion, following earlier developments by Strick (1967) and many others, from the Kramers-Krönig relations for the quantity $k/\omega = 1/c - i\beta/\omega$ (k = wavenumber, β = attenuation coefficient of a plane wave):

$$\frac{1}{c(\omega)} = \frac{1}{c(\infty)} + \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\beta(\omega')/\omega'}{\omega' - \omega} d\omega'$$
$$= \frac{1}{c(\infty)} + \frac{1}{\pi\omega} P \int_{-\infty}^{+\infty} \frac{\beta(\omega')}{\omega' - \omega} d\omega', \tag{1}$$

$$\beta(\omega) = -\frac{\omega}{\pi} P \int_{-\infty}^{+\infty} \frac{c^{-1}(\omega')}{\omega' - \omega} d\omega'. \tag{2}$$

The attenuation coefficient is related to Q by

$$\beta(\omega) = \frac{\omega}{2c(\omega)\,Q(\omega)}. \tag{3}$$

This relation is consistent with the definition of Q as the ratio of the real and imaginary part of the viscoelastic modulus $M = M_1 + iM_2$, $Q = M_1/M_2$, only for $Q \gg 1$. Another approximation, after inserting $Q \sim \omega^{\gamma}$ into (3), is the assumption that β is strictly proportional to $\omega^{1-\gamma}$. This implies that in (3) dispersion is neglected and $c(\omega)$ is effectively identified with $c(\infty)$. Then, β is inserted into (1) and the principal-value integral solved, giving $c(\omega)$. Both approximations are perfectly valid for seismological applications where Q, even if frequency-dependent, is usually much larger than 1 and dispersion is only slight.

The purpose of this paper is to study the attenuation-dispersion properties of materials with $Q \sim \omega^{\gamma}$ from a different starting point, namely the Kramers-Krönig relations for magnitude and phase of the viscoelastic modulus $M = Ae^{i\varphi}$ (which follow from the familiar Kramers-Krönig relations for the real part and the imaginary part of the function $\ln M$):

$$\ln A(\omega) = B - \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\varphi(\omega')}{\omega' - \omega} d\omega', \tag{4}$$

$$\varphi(\omega) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\ln A(\omega')}{\omega' - \omega} d\omega' \text{ (minimum phase).} \tag{5}$$

The background of these equations is the assumption that the modulus M provides a *linear* relation between stress σ and strain ε, $\sigma(\omega) = M(\omega)\varepsilon(\omega)$, irrespective of frequency or stress and strain amplitudes. This relation must, of course, be causal; this imposes the conditions (4) and (5) on the filter $M(\omega)$.

The so far undetermined constant B in (4) expresses the fact that the amplitude characteristic of a linear filter is determined by the phase only to within a multiplicative constant. B will be fixed suitably in the following. The quality factor Q is introduced into (4) by

$$\varphi(\omega) = \arctan Q^{-1}(\omega). \tag{6}$$

Then, if the principal-value integral can be calculated, the viscoelastic modulus $M(\omega)$ is known.

There are two advantages in using $M(\omega)$ instead of the wave-number $k(\omega)$ in an investigation of the attenuation-dispersion properties of earth materials. First, the modulus describes both the rheological (long-time-scale) and seismological (short-time-scale) properties: besides the phase velocity the relaxation and creep functions can be determined, and the variation with frequency of the modulus itself is of interest. Second, no restriction has to be imposed on the size of Q, because the phase in (4) is finite for all Q, according to (6). Indeed, as shown below, for $Q \sim \omega^\gamma$ and $\gamma = \pm 1, \pm 1/2, \pm 1/3, \pm 1/4, \ldots$ closed-form solutions can be obtained for $M(\omega)$, valid for arbitrary values of Q and hence frequency. What this means for practical purposes is a different question, but it is at least of general interest to know *completely* the elastic-anelastic behaviour of a material with power-law dependence of Q on frequency. In this regard our study is an extension of a paper by Kjartansson (1979) where the case of frequency-independent Q ($\gamma = 0$) was treated without approximations and restrictions.

In the following, we present first the case $0 < \gamma \leqq 1$, i.e., the case of materials with elastic behaviour at high and viscous behaviour at low frequencies; $\gamma = 1$ corresponds to a Maxwell body. Then the results for media with complementary behaviour and $-1 \leqq \gamma < 0$ are summarized; $\gamma = -1$ is the case of a Kelvin-Voigt body. Finally, simple dissipation operators are presented in the frequency and the time domain by which theoretical seismograms, calculated without absorption, can be approximately corrected.

Positive exponents ($0 < \gamma \leqq 1$)

General solution for the viscoelastic modulus

The power law for Q will be used in the following in two different forms. The form best suited for applications is

$$Q(\omega) = Q(\omega_r)\left(\frac{\omega}{\omega_r}\right)^\gamma \tag{7}$$

with the reference (angular) frequency ω_r for which Q is known; ω_r normally is chosen in or close to the seismic frequency band. An equivalent form is

$$Q(\omega) = \left(\frac{\omega}{\omega_0}\right)^\gamma. \tag{8}$$

Here, the frequency ω_0 is defined by $Q(\omega_0) = 1$; usually it is located far below the seismic frequency band. This frequency is useful in the determination of long- and short-time approximations of the creep and relaxation functions. The relation between ω_0 and ω_r is

$$\omega_0 = \omega_r Q(\omega_r)^{-1/\gamma}.$$

Formulas (7) and (8) apply for $\omega \geqq 0$. In (4) negative frequencies also occur. The values of the phase φ of the viscoelastic modulus at these frequencies follow from the fact that Q and hence φ (via (6)) are odd functions of frequency. For Q according to (7) or (8) there is a jump in the phase from $-\pi/2$ to $\pi/2$ at $\omega = 0$, whereas for $\omega \to \pm\infty$ the phase tends to zero. Applying partial integration in (4) separately for the frequency intervals from $-\infty$ to 0 and from 0 to $+\infty$ and observing that the derivative of the phase, $d\varphi/d\omega'$, is an even function, one obtains

$$\ln A(\omega) = B + \ln|\omega| + \frac{1}{\pi}\int_0^\infty \ln|\omega'^2 - \omega^2| \frac{d\varphi}{d\omega'} d\omega'.$$

Inserting now the analytical form of $d\varphi/d\omega'$,

$$\frac{d\varphi}{d\omega'} = -\frac{\gamma(\omega'/\omega_0)^\gamma}{\omega'[1 + (\omega'/\omega_0)^{2\gamma}]},$$

and splitting the logarithm,

$$\ln|\omega'^2 - \omega^2| = 2\ln|\omega| + \ln\left|1 - \frac{\omega'^2}{\omega^2}\right|,$$

we have

$$\ln A(\omega) = B - \frac{\gamma}{\pi}\int_0^\infty \frac{Q(\omega')}{\omega'[1 + Q^2(\omega')]} \ln\left|1 - \frac{\omega'^2}{\omega^2}\right| d\omega'.$$

From this we find $B = \ln A(\infty)$. The final step is the variable change from ω' to $z = (\omega'/\omega)^{-\gamma}$ which yields

$$A(\omega) = A(\infty)\, e^{I(\omega,\gamma)},$$

$$I(\omega, \gamma) = -\frac{Q(\omega)}{\pi}\int_0^\infty \frac{1}{z^2 + Q^2(\omega)} \ln|1 - z^{-2/\gamma}|\, dz, \tag{9}$$

with $Q(\omega)$ from (7) or (8). The complete viscoelastic modulus is

$$M(\omega) = A(\infty)\exp\{I(\omega,\gamma) + i\arctan Q^{-1}(\omega)\}. \tag{10}$$

Its limiting value $M(\infty) = A(\infty)$, the unrelaxed modulus, is real.

In the following, (9) and (10) are evaluated approximately and exactly, and further results for velocity dispersion and rheological properties are summarized.

High-frequency approximation ($\omega \gg \omega_0$)

If $\omega \gg \omega_0$, then $Q \gg 1$. A study of the integrand of $I(\omega, \gamma)$ in (9) shows that in this case the term $(z^2 + Q^2)^{-1}$ may be replaced by Q^{-2}. The remaining integral can be found in tables. Then

$$I(\omega, \gamma) = -\frac{1}{Q(\omega)} \cot\left(\gamma \frac{\pi}{2}\right),$$

and the high-frequency approximation of the viscoelastic modulus is

$$M(\omega) = M(\infty) \exp\left\{-\left(\frac{\omega_0}{\omega}\right)^\gamma \left[\cot\left(\gamma \frac{\pi}{2}\right) - i\right]\right\}. \qquad (11)$$

An equivalent form which uses the modulus $M(\omega_r)$ at the reference frequency is

$$M(\omega) = M(\omega_r) \exp\left\{Q^{-1}(\omega_r)\left[\cot\left(\gamma \frac{\pi}{2}\right) - i\right]\left[1 - \left(\frac{\omega_r}{\omega}\right)^\gamma\right]\right\}. \qquad (12)$$

For sufficiently small exponents in (12) a Taylor-series expansion can be restricted to the linear term; then there is agreement (in essence) with results given by Anderson and Minster (1979), Brennan and Smylie (1981) and Smith and Dahlen (1981).

Low-frequency approximation ($\omega \ll \omega_0$)

For $\omega \ll \omega_0$, i.e., for $Q \ll 1$, the term $(z^2 + Q^2)^{-1}$ in (9) is significantly different from zero only for small z. For these z the logarithmic term can be approximated as follows:

$$\ln|1 - z^{-2/\gamma}| = \ln z^{-2/\gamma} = -\frac{2}{\gamma} \ln z.$$

Then we have

$$I(\omega, \gamma) = \frac{2Q}{\pi\gamma} \int_0^\infty \frac{\ln z}{z^2 + Q^2}\, dz = \frac{1}{\gamma} \ln Q.$$

At low frequencies the phase of the viscoelastic modulus is $\pi/2$, as already mentioned. Hence, the low-frequency approximation of the modulus is

$$M(\omega) = iQ(\omega)^{1/\gamma} M(\infty) = i\frac{\omega}{\omega_0} M(\infty). \qquad (13)$$

It is linear in ω and independent of γ. $M(\omega)$ vanishes at frequency zero, and (13) implies that at very low frequencies stress and strain are shifted in phase with respect to each other by 90°.

Exact results for $\gamma = 1/m$ ($m = 1, 2, 3, \ldots$)

In the special case where γ is the reciprocal of a natural number m the integral $I(\omega, \gamma)$ in (9) can be calculated exactly for arbitrary values of frequency. Partial integration leads to another integral,

$$I = \frac{m}{\pi} P \int_{-\infty}^{+\infty} \frac{1}{z} \arctan\frac{z}{Q} \cdot \frac{1}{z^{2m} - 1}\, dz, \qquad (14)$$

which can be calculated by methods of complex calculus. (14) is a principal-value integral with the integration path running along the real axis. The integrand has the first-order poles

$$z_n = e^{i\pi \frac{n}{m}} \quad (n = 0, 1, 2, \ldots, 2m-1),$$

distributed over the unit circle, and two branch points at $z = \pm iQ$. The latter become evident, if the equivalence

$$\arctan\frac{z}{Q} = \frac{1}{2i} \ln\frac{iQ - z}{iQ + z}$$

is taken into account. Then the integration path is extended by small half-circles around the poles $z_0 = 1$ and $z_m = -1$ and subsequently deformed into a half-circle with infinitely large radius in the upper z-half-plane. The poles $z_1, z_2, \ldots, z_{m-1}$ and the branch cut, extending from the branchpoint iQ to $+i\infty$ along the imaginary axis, are circumvented. The integral along the large half-circle vanishes, and the residues and the branch line integral can be calculated exactly. We omit further details of the derivation which is straightforward, but requires some care. The final result for the viscoelastic modulus is:

$$M(\omega) = M(\infty) \frac{Q + i}{(1 + Q^2)^{1/2}} \cdot \left\{\prod_{n=1}^{m} \frac{\left(1 + Q^2 - 2Q \sin\frac{\pi n}{m}\right) Q^4}{\left(1 + Q^2 + 2Q \sin\frac{\pi n}{m}\right)\left(1 + Q^4 - 2Q^2 \cos\frac{\pi p}{m}\right)}\right\}^{1/4} \qquad (15)$$

$$p = \begin{cases} 2n - 1 & \text{for } m = 1, 3, 5, \ldots \\ 2n & \text{for } m = 2, 4, 6, \ldots \end{cases}$$

For $m = 1$ which is the case of a Maxwell rheology (15) simplifies to

$$M(\omega) = M(\infty) \frac{Q(Q + i)}{Q^2 + 1} = M(\infty) \frac{Q}{Q - i} = M(\infty) \frac{\omega}{\omega - i\omega_0}.$$

This is a known result which usually is derived directly from the stress-strain relation of a Maxwell body ($\omega_0 = M(\infty)/\eta$, η = viscosity, $M(\infty)$ = unrelaxed modulus).

Numerical results for the viscoelastic modulus are presented in Figure 1, where $|M(\omega)/M(\omega_r)|$ is displayed as a function of ω/ω_r for different values of $\gamma = 1/m$ and for two values of $Q(\omega_r)$, 200 and 1000. Comparison is made with the high-frequency approximation (12): this approximation performs amazingly well down to the frequency ω_0 where $Q = 1$ (see solid circles on the curves of Fig. 1).

Constant-Q case ($\gamma = 0$)

The case $\gamma = 0$ which is also included in Figure 1 is best treated separately from the beginning, i.e., by starting

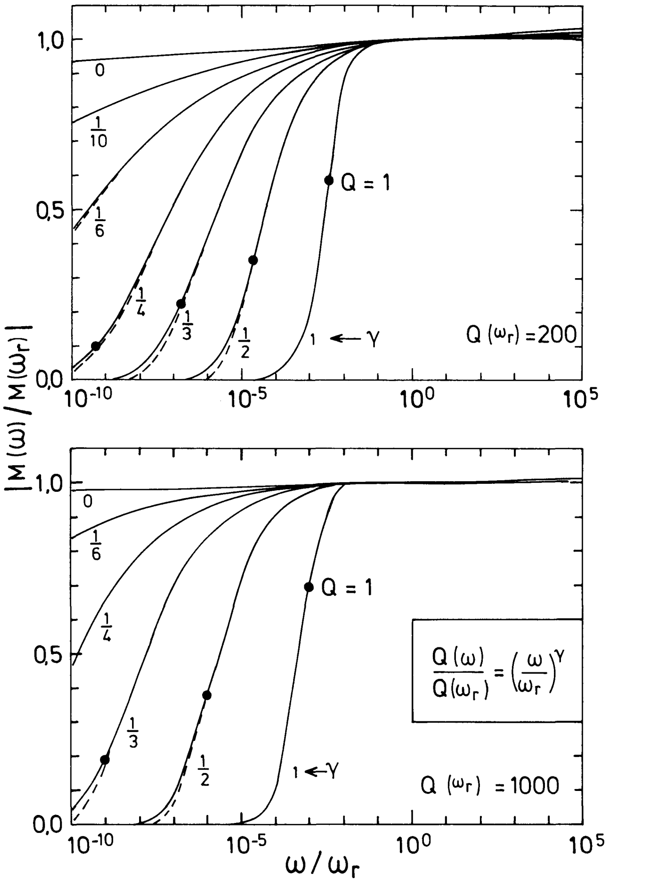

Fig. 1. The viscoelastic modulus $M(\omega)$ as a function of frequency for power-law dependence of Q on frequency and different exponents γ. Calculation with the exact formula (15) (*solid curves*) and with the approximate formula (12) (*dashed curves*)

with the Kramers-Krönig relation (4). The phase φ is $-q$ for negative ω' and q for positive ω', where $q = \arctan \frac{1}{Q}$ with positive, frequency-independent Q. Then we consider a positive frequency ω and have

$$\ln A(\omega) = B + \frac{q}{\pi} \int_{-\infty}^{0} \frac{d\omega'}{\omega' - \omega} - \frac{q}{\pi} P \int_{0}^{+\infty} \frac{d\omega'}{\omega' - \omega}$$
$$= B + \frac{2q}{\pi} \int_{-\infty}^{0} \frac{d\omega'}{\omega' - \omega}.$$

Subtracting a similar expression for the reference frequency ω_r, we obtain

$$\ln \frac{A(\omega)}{A(\omega_r)} = \frac{2q}{\pi} \int_{-\infty}^{0} \left(\frac{1}{\omega' - \omega} - \frac{1}{\omega' - \omega_r} \right) d\omega'$$

$$= \frac{2q}{\pi} \ln \left| \frac{\omega' - \omega}{\omega' - \omega_r} \right|_{\omega' = -\infty}^{\omega' = 0} = \frac{2q}{\pi} \ln \frac{\omega}{\omega_r}.$$

Finally, the viscoelastic modulus is

$$M(\omega) = A(\omega_r) \left(\frac{i\omega}{\omega_r} \right)^{\frac{2}{\pi} \arctan \frac{1}{Q}}, \tag{16}$$

a result which has first been given by Kjartansson (1979). $M(0)$ vanishes as for $\gamma > 0$, but $M(\infty) = \infty$ which is different from the case $\gamma > 0$; therefore $M(\infty)$ cannot be taken as the reference value.

Velocity dispersion

The complex wave velocity v of a dissipating medium follows from the complex modulus M by $v = (M/\rho)^{1/2}$, where ρ is the (real) density. The real phase velocity c of a plane wave in this medium is $(v_1^2 + v_2^2)/v_1$, where v_1 is the real part and v_2 the imaginary part of v. If $Q \gg 1$, as in all potential seismological applications, the modulus of v is a very good approximation for c. From the high-frequency approximation (11) for M we obtain:

$$v(\omega) = v(\infty) \exp \left\{ -\frac{\cot\left(\gamma \frac{\pi}{2}\right)}{2Q(\omega)} + \frac{i}{2Q(\omega)} \right\},$$
$$v(\infty) = [M(\infty)/\rho]^{1/2}.$$

Hence, the phase velocity is

$$c(\omega) = v(\infty) \exp \left\{ -\frac{\cot\left(\gamma \frac{\pi}{2}\right)}{2Q(\omega)} \right\}.$$

Introducing the phase velocity at the reference frequency ω_r, we find

$$c(\omega) = c(\omega_r) \exp \left\{ \frac{\cot\left(\gamma \frac{\pi}{2}\right)}{2Q(\omega_r)} \left[1 - \left(\frac{\omega_r}{\omega} \right)^{\gamma} \right] \right\} \tag{17}$$

and

$$v(\omega) = c(\omega) \exp \left\{ \frac{i}{2Q(\omega_r)} \left(\frac{\omega_r}{\omega} \right)^{\gamma} \right\}. \tag{18}$$

(17) describes the relative dispersion, and (18) additionally includes the attenuation. Expressions for phase-velocity dispersion which are equivalent to (17) have been given, e.g., by Brennan (1980) and Minster (1980).

Figure 2 shows a few numerical results, calculated with (17) for $Q(\omega_r) = 200$ and different values of γ. As expected, dispersion is slight; over 3 decades in frequency below ω_r the phase-velocity decrease does not exceed a few per cent. However, dispersion increases with γ. Group velocity has been included in Figure 2, although it is not yet clear whether, in a weakly dispersive medium it has any special meaning.

The phase velocity in the constant-Q case ($\gamma = 0$) follows from (16) (see also Kjartansson, 1979):

$$c(\omega) = c(\omega_r) \left(\frac{\omega}{\omega_r} \right)^{\frac{1}{\pi} \arctan \frac{1}{Q}}. \tag{19}$$

This is an exact result, valid for all frequencies. For $Q \gg 1$, (19) has the well-known and often used approximation

$$c(\omega) = c(\omega_r) \left(1 + \frac{1}{\pi Q} \ln \frac{\omega}{\omega_r} \right).$$

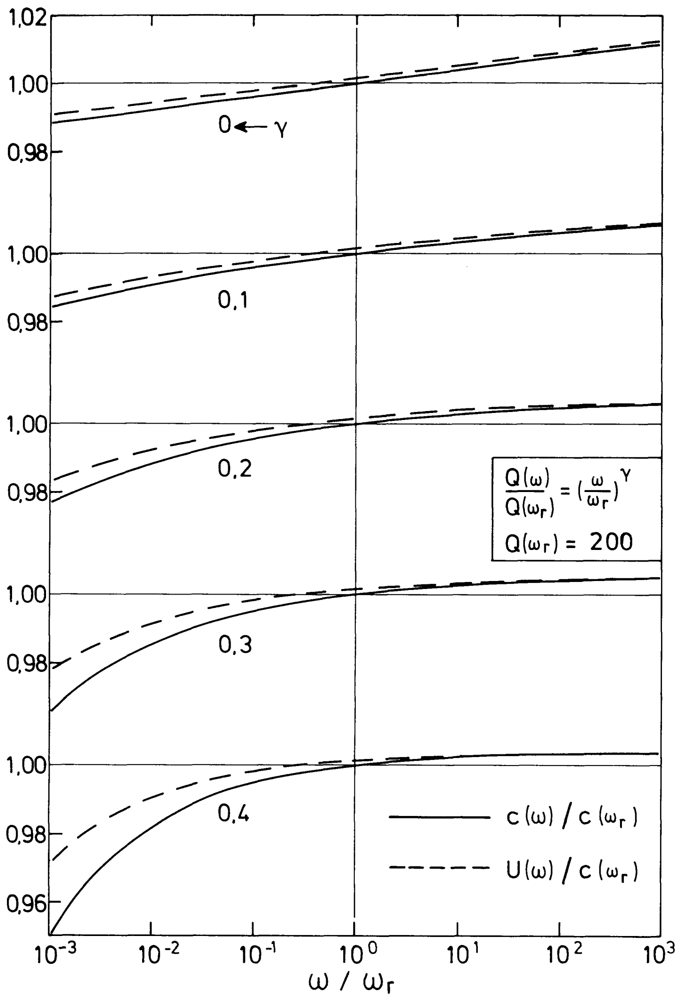

Fig. 2. Relative dispersion for phase velocity c and group velocity U in the case of power laws for Q with exponents $\gamma \geqq 0$

Relaxation and creep functions

The relaxation function $R(t)$ and the creep function $K(t)$ can be calculated from the viscoelastic modulus $M(\omega)$ with the aid of their Fourier transforms $\bar{R}(\omega)$ and $\bar{K}(\omega)$:

$$\bar{R}(\omega)=\frac{M(\omega)}{i\omega}, \qquad \bar{K}(\omega)=\frac{1}{M(\omega)\,i\omega}. \tag{20}$$

Short-time-scale approximations of $R(t)$ and $K(t)$ follow by using the high-frequency approximation (11) for $M(\omega)$ in (20). Inverse transformation into the time domain is possible after a series expansion of $\bar{R}(\omega)$ and $\bar{K}(\omega)$ with the aid of the Taylor series of the exponential function. The results, valid for $t \ll \omega_0^{-1}$, are:

$$R(t)=M(\infty)\left\{1+\sum_{n=1}^{N}\frac{(-1)^n(\omega_0 t)^{n\gamma}}{n!\,\Gamma(1+n\gamma)\sin^n\left(\gamma\frac{\pi}{2}\right)}\right\}H(t), \tag{21}$$

$$K(t)=\frac{1}{M(\infty)}\left\{1+\sum_{n=1}^{N}\frac{(\omega_0 t)^{n\gamma}}{n!\,\Gamma(1+n\gamma)\sin^n\left(\gamma\frac{\pi}{2}\right)}\right\}H(t). \tag{22}$$

Here, $H(t)$ is the unit-step function, $\Gamma(x)$ the gamma function, and N the largest integer less than or equal to $1/\gamma$. Summation in (21) and (22) has been restricted to terms with non-vanishing slope for $t=0$ which contribute most for short times.

The finite jumps of $R(t)$ and $K(t)$ at $t=0$ are determined by the unrelaxed modulus $M(\infty)$, as expected, and the slopes change continuously from infinite values at $t=0$ to finite values afterwards. The dominant terms in the sums, for $n=1$, have a power-law dependence on t with exponent γ. Some of these properties are known, but it seems that the precise forms (21) and (22) of the relaxation and creep functions have not yet been given in the geophysical literature.

Short-time-scale approximations in *closed form* follow from (21) and (22) by observing that the arguments of the gamma function vary from $1+\gamma$ to about 2 and hence $\Gamma(1+n\gamma)\approx 1$. The simple expressions

$$R(t)=M(\infty)\exp\left\{-\frac{(\omega_0 t)^{\gamma}}{\sin\left(\gamma\frac{\pi}{2}\right)}\right\}H(t), \tag{23}$$

$$K(t)=\frac{1}{M(\infty)}\exp\left\{\frac{(\omega_0 t)^{\gamma}}{\sin\left(\gamma\frac{\pi}{2}\right)}\right\}H(t), \tag{24}$$

are obtained. The approximation (23) for the relaxation function appears to be an especially useful form; for $\gamma=1$ it is exact for all times. However, it is uncertain whether, in general, (23) and (24) are better short-time-scale approximations than (21) and (22).

Long-time-scale approximations of $R(t)$ and $K(t)$ follow from the low-frequency approximation (13) of $M(\omega)$. The finite value $\bar{R}(0)$ implies that $R(t)$ decays to zero for $t\to\infty$ which is, of course, expected. The long-time approximation of the creep function, valid for $t \gg \omega_0^{-1}$, is

$$K(t)\simeq\frac{\omega_0 t}{M(\infty)}, \tag{25}$$

i.e., creep is stationary for large times and for all γ from 0 to 1, excluding $\gamma=0$. Under shear deformations, the material behaves as a viscous fluid with viscosity

$$\eta=\frac{M(\infty)}{\omega_0}=\frac{M(\omega_r)}{\omega_r}\,Q(\omega_r)^{1/\gamma}, \tag{26}$$

where $M(\infty)$ is twice the unrelaxed rigidity.

Relaxation and creep functions in the constant-Q case are obtained from (20) and (16). The following expressions, which have already been given by Kjartansson (1979), apply for arbitrary times $t>0$:

$$\left.\begin{aligned}R(t)&=\frac{A(\omega_r)}{\Gamma(1-\alpha)}(\omega_r t)^{-\alpha},\\ K(t)&=\frac{1}{A(\omega_r)\,\Gamma(1+\alpha)}(\omega_r t)^{\alpha},\\ \alpha&=\frac{2}{\pi}\arctan\frac{1}{Q}.\end{aligned}\right\} \tag{27}$$

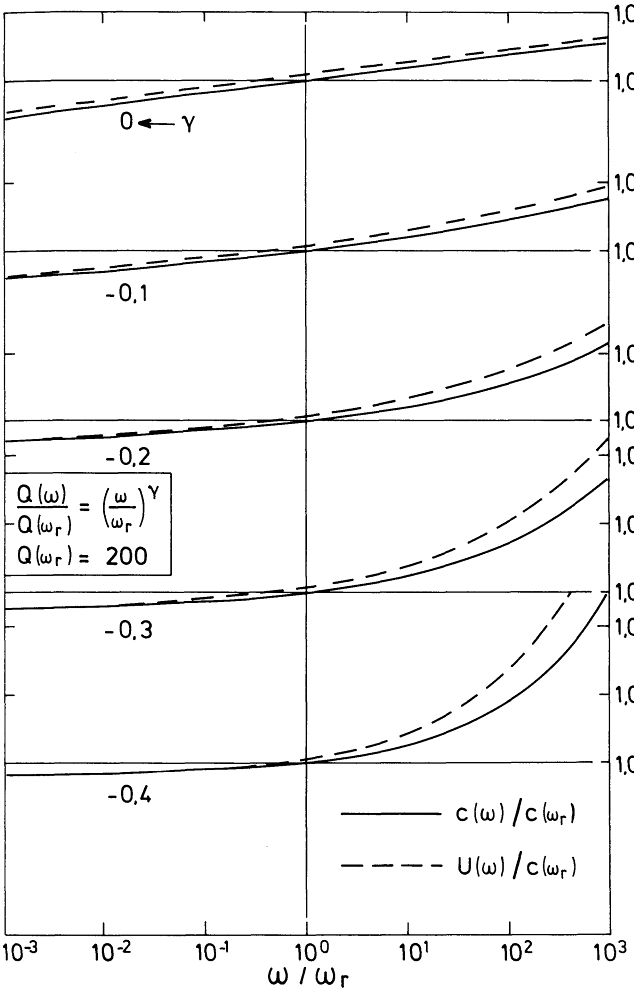

Fig. 3. The same as Fig. 2 for $\gamma \leqq 0$

They are different in character from (21) and (22), since $R(t)$ starts with infinite and $K(t)$ with zero amplitude at $t=0$. Long-time-scale behaviour of the creep function is also different from (25).

Stress relaxation is sometimes described by *relaxation spectra*. It may therefore also be of interest to know these spectra in the case of the rheologies discussed in this paper. In the appendix a general formula is derived which expresses the relaxation spectrum (not to be confused with the spectrum of the relaxation function) in terms of the viscoelastic modulus. For γ with $0<\gamma<1$, which is the most interesting case, the relaxation spectrum represents a relaxation band with a peak at or below the frequency ω_0.

Negative exponents ($-1 \leqq \gamma < 0$)

The power law for Q continues to have the form (7) or (8), but now dissipation is low at low frequencies and high at high frequencies. Elastic behaviour occurs at low frequencies and on long time-scales, whereas for $\gamma>0$ it occurs at high frequencies and on short time-scales. Therefore, the case $\gamma<0$ is largely complementary to the case $\gamma>0$.

This section is a summary of results for $\gamma<0$ whose derivation is quite similar to that for $\gamma>0$; therefore no details will be given. A basic (but expected) difference is that now the viscoelastic modulus at frequency zero, $M(0)=A(0)$, the relaxed modulus, has to be taken as the reference value; $M(0)$ is real. The general form of $M(\omega)$ is (cf. (9) and (10)):

$$M(\omega)=M(0)\exp\{J(\omega,\gamma)+i\arctan Q^{-1}(\omega)\}$$

$$J(\omega,\gamma)=\frac{Q(\omega)}{\pi}\int_0^\infty \frac{1}{z^2+Q^2(\omega)}\ln|1-z^{2/\gamma}|\,dz. \tag{28}$$

The low-frequency approximation of $M(\omega)$ is similar to the high-frequency approximation (11) in the case $\gamma>0$:

$$M(\omega)=M(0)\exp\left\{-\left(\frac{\omega_0}{\omega}\right)^\gamma\left[\cot\left(\gamma\frac{\pi}{2}\right)-i\right]\right\}. \tag{29}$$

It applies for $\omega \ll \omega_0$. The low-frequency approximation, expressed in terms of the modulus $M(\omega_r)$ at the reference frequency, is identical with (12). The high-frequency approximation of $M(\omega)$, valid for $\omega \gg \omega_0$, is

$$M(\omega)=i\frac{\omega}{\omega_0}M(0) \tag{30}$$

and similar to (13). Since according to (29) and (30) $M(\omega)$ increases with ω, a medium with $\gamma<0$ has the peculiar property that at high frequencies it becomes at the same time more dissipative *and* stiffer. For the Kelvin-Voigt body ($\gamma=-1$) this is well-known.

Exact results for $M(\omega)$, similar to (15), can be derived for $\gamma=-1/m$ ($m=1,2,3,\ldots$):

$$M(\omega)=M(0)\frac{Q+i}{(1+Q^2)^{1/2}}$$

$$\cdot\left\{\prod_{n=1}^{m}\frac{\left(1+Q^2-2Q\sin\frac{\pi n}{m}\right)Q^4}{\left(1+Q^2+2Q\sin\frac{\pi n}{m}\right)\left(1+Q^4-2Q^2\cos\frac{\pi p}{m}\right)}\right\}^{-1/4} \tag{31}$$

The conventions for p are as in (15), and for $m=1$ the known results for the Kelvin-Voigt body are reproduced.

The perhaps most important result in the case $\gamma<0$ is that for velocity dispersion. Since, as just mentioned, formula (12) is valid also for $\gamma<0$, namely as a low-frequency and hence high-Q approximation, it is clear that the velocity formulas (17) and (18) also apply in the present case. Figure 3 presents, as an illustration, a few numerical results for phase- and group-velocity dispersion. It is interesting to note that, as in Figure 2, the velocities increase with frequency, although in a different manner. That they increase at all is a consequence of the general increase of $M(\omega)$ with ω.

Short-time-scale approximations for the relaxation and creep functions follow from (20) and (30). For $t \ll \omega_0^{-1}$ we have ($\delta(t)$ = delta function):

$$R(t)=\frac{M(0)}{\omega_0}\delta(t), \qquad K(t)=\frac{\omega_0}{M(0)}tH(t). \tag{32}$$

These approximations imply that all media with γ between 0 and -1 share the known unphysical properties of the Kelvin-Voigt body, that the relaxation function

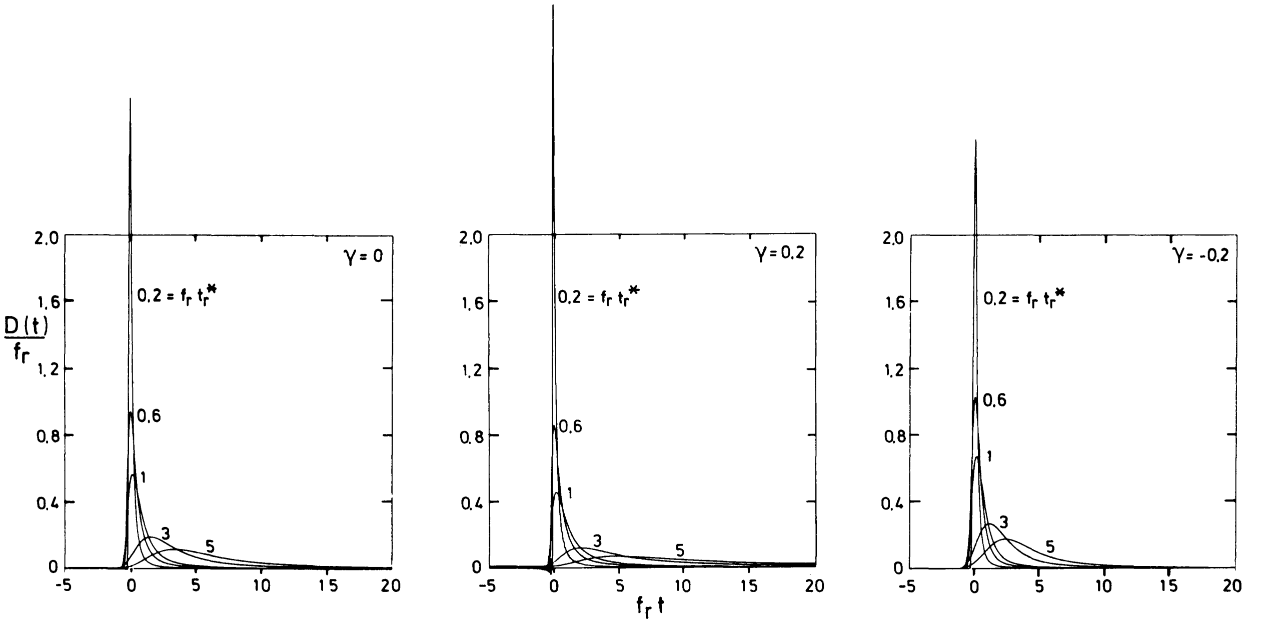

Fig. 4. Time-domain dissipation operators for $\gamma=0$, 0.2 and -0.2. The reference frequency $f_r=\omega_r/2\pi$ is used for normalization. t_r^* is the familiar quantity t^*, taken for the reference frequency. For details see text

has a delta-function contribution and that the creep function starts with zero amplitude.

Long-time-scale approximations for $R(t)$ and $K(t)$ at $t \gg \omega_0^{-1}$ are obtained from (20) and the simplest form of (29), $M(\omega)=M(0)$:

$$R(t) \simeq M(0)\,H(t), \qquad K(t) \simeq \frac{1}{M(0)} H(t). \tag{33}$$

Hence, long-time-scale response is elastic, as mentioned earlier.

Dissipation operators

There are two possibilities to incorporate a power law frequency dependence of Q into the calculation of theoretical seismograms. One is to make the compressional and shear wave velocities complex, using (18) (Müller and Schott, 1981). This procedure has justification through the correspondence principle and is most versatile, since spatial variations of Q can be modelled correctly. The price paid for this is sometimes a considerable increase in computing time, because complex arithmetic must be used to a larger extent than in purely elastic calculations. Moreover, there are computational methods, e.g., generalized ray theory, which cannot accomodate complex wave velocities. Dissipation operators, by which the seismograms of purely elastic calculations are convolved, offer another means to incorporate frequency dependence of Q and are actually a sufficient supplement in many cases; dissipation operators for constant Q are frequently used.

Dissipation operators are wave profiles $u(x,t)$ of a plane wave propagating in x-direction, when the input at $x=0$ corresponds to the delta function:

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[i\omega\left(t-\frac{x}{v(\omega)}\right)\right] d\omega \tag{34}$$

with $v(\omega)$ from (18) and (17). If (34) is written in the form

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{D}(\omega)\exp\left[i\omega\left(t-\frac{x}{c(\omega_r)}\right)\right] d\omega$$
$$= D(t) * \delta\left(t-\frac{x}{c(\omega_r)}\right),$$

i.e., as a convolution of the elastic response, travelling with velocity $c(\omega_r)$, and the dissipation operator $D(t)$, the Fourier transform $\bar{D}(\omega)$ of $D(t)$ is found by inspection:

$$\bar{D}(\omega)=\exp\left\{-\frac{\omega}{2}t_r^*\left[\left(\frac{\omega_r}{\omega}\right)^{\gamma}-i\cot\left(\gamma\frac{\pi}{2}\right)\left(1-\left(\frac{\omega_r}{\omega}\right)^{\gamma}\right)\right]\right\}, \tag{35}$$

$$t_r^*=\frac{x/c(\omega_r)}{Q(\omega_r)}. \tag{36}$$

In the constant-Q case ($\gamma=0$) we have instead of (35):

$$\bar{D}(\omega)=\exp\left\{-\frac{\omega}{2}t_r^*\left(1-\frac{2i}{\pi}\ln\frac{\omega}{\omega_r}\right)\right\}. \tag{37}$$

In seismological applications the dissipation time (36) has to be replaced by the integral along the seismic ray,

$$t_r^*=\int\frac{ds}{c(\omega_r)\,Q(\omega_r)},$$

and the assumption is implicit, that the exponent γ is the same everywhere in the medium. Inverse transformation of (35) or (37) by numerical methods yields the dissipation operator $D(t)$.

Results of calculations are shown in Figure 4 for $\gamma=0$, 0.2 and -0.2. In the calculations and for the display of their results we have used the fact that $\bar{D}(\omega)$ can be written as a function of the *dimensionless* frequency ω/ω_r. Then, parameters in $\bar{D}(\omega)$ are $\vartheta=\omega_r t_r^*$, a dimensionless dissipation time, and γ. As a consequence, $D(t)$ depends on the dimensionless time $\omega_r t$ and on ϑ and γ. In Figure 4 the reference frequency proper, $f_r=\omega_r/2\pi$, has been used instead of ω_r.

The slight acausality which some of the dissipation operators have should pose no problem in applications. This effect is difficult to explain; it may be related to the fact that group velocity is somewhat larger than phase velocity (see Figs. 2 and 3), but it may also simply be a consequence of slight approximations that have been made in the derivation of $\bar{D}(\omega)$. Anyway, Figure 4 shows that dissipation operators determined from (35) or (37) will serve their purpose well.

If in the constant-Q case one is not bound to use a special reference frequency ω_r, the Fourier transform (37) of the dissipation operator can be simplified. Since t_r^* varies only little with ω_r, it can be replaced by a constant value t^*. Then it is evident from (37) that the dissipation operators, corresponding to different ω_r, are time-shifted versions of each other. Therefore, convolution with them has the same low-pass filter effect, but is associated with different time shifts. Then a simple and convenient choice is to identify ω_r with the highest resolvable frequency of the problem under study, i.e., with the Nyquist frequency ω_N:

$$\bar{D}(\omega)=\exp\left\{-\frac{\omega}{2}t^*\left(1-\frac{2i}{\pi}\ln\frac{\omega}{\omega_N}\right)\right\},\quad t^*=\int\frac{ds}{cQ}. \tag{38}$$

The dissipation operators are identical with those in the left part of Figure 4 after sampling with the abscissa increment 0.5 ($f_r=f_N=\omega_N/2\pi$). The simple form (38), which apart from a time shift $2t^*/\pi$ agrees with a result of Frasier and Filson (1972), is sufficient for many practical purposes.

Discussion

The main purpose of this paper has been to give a simple and compact derivation of the properties of a viscoelastic material, having a specific dissipation function Q which strictly obeys a power law for all frequencies. The central role in this derivation is played by the viscoelastic modulus M, whose knowledge as a function of frequency allows the subsequent study of velocity dispersion and rheological properties, i.e., of both high- and low-frequency characteristics. The connection between M and Q is provided by the Kramers-Krönig relations for magnitude and phase of a linear, causal filter. In the case of power laws for Q an exact general solution for M can be given. Approximations come in only afterwards, if at all. This procedure has definite advantages, compared with the more usual treatment based on the wavenumber. It should also be the optimum procedure for other Q laws, in conjunction with numerical Hilbert transformation, if necessary.

The results derived in this paper show that materials with power-law dependence of Q on frequency and exponents γ between 0 and 1 have short- and long-time-scale properties similar to those of a Maxwell body ($\gamma=1$) and hence can be called *generalized Maxwell bodies*. Likewise, materials with $-1<\gamma<0$ can be considered as *generalized Kelvin-Voigt bodies*. Three-dimensional stress-strain relations, valid for arbitrary frequency, as they are known for the Maxwell body and other viscoelastic models (see, e.g., Peltier et al., 1981), can also be constructed for the generalized Maxwell and Kelvin-Voigt bodies in the cases $\gamma=\pm1/2$, $\pm1/3$, $\pm1/4,\ldots$. The simplest procedure would be to assume for the rigidity the exact result for the viscoelastic modulus, (15) or (31), and to neglect bulk dissipation, i.e., to assume a real, frequency-independent bulk modulus. Whether these generalized bodies, notably the generalized Maxwell body, are of practical importance, remains to be seen. It is tempting to speculate that post-glacial-rebound data, which have been successfully interpreted with the Maxwell-body model (Cathles, 1975; Peltier, 1976, 1980), could also be explained by a generalized Maxwell body with $\gamma<1$. In this context it may be of interest to note that for a generalized Maxwell body the relation between the decay time τ of the relaxation function $R(t)$, defined by $R(\tau)=R(0)/e$, and the viscosity follows with sufficient accuracy from (23) and (26):

$$\eta=\left[\sin\left(\gamma\frac{\pi}{2}\right)\right]^{-1/\gamma}M(\infty)\,\tau.$$

This simple formula implies a strong dependence of η on γ. For instance, for $\gamma=1/3$ the inferred viscosity would be 8 times larger than the viscosity of a Maxwell body.

The model of a generalized Maxwell body can be applied to rocks at low pressures and high temperatures in a frequency band of 3 to 4 decades around 1 Hz, according to laboratory studies by Berckhemer et al. (1982). These authors find an exponent γ of about 0.25. It is not clear to what extent this result can also be applied to the rocks of the earth's mantle where there is additional influence on viscoelastic properties by high pressures. A γ value as high as 0.25 cannot apply for *large* frequency ranges in the earth's mantle, as the following example shows. Assuming a $Q(\omega_r)$ of 200 at a period of about 30 s, according to Jordan and Sipkin's (1977) investigation of long-period shear waves of *ScS* type, we can determine from Figure 1 the reduction of the rigidity from this period to the period of the Chandler wobble which is about 435 sidereal days. The relative frequency ω/ω_r is $0.8\cdot10^{-6}$, and for $\gamma=0$, 1/10, 1/6, 1/4 the rigidity is reduced to 0.96, 0.91, 0.84, 0.69 times the rigidity at 30 s. Employing perturbation-theory formulas for the change in Chandler-wobble period due to changes in rigidity, as given, e.g., by Anderson and Minster (1979), one obtains an increase of about 6 days for $\gamma=0$ and 14 days for $\gamma=1/10$. The increase for $\gamma=1/6$ and $\gamma=1/4$ cannot be calculated from perturbation theory, but certainly it would be considerably larger than 14 days. Most of the observed wobble period is explained by elastic earth models, valid for the seismic frequency band, and by the influence of the oceans; only a few days, 3 according to Lambeck (1980, Table 8.1) and 8.5 according to Smith and Dahlen (1981, Fig. 5), respectively, are left for the influence of absorption-related dispersion. This points to γ values between 0 and 0.1 at most, a result which is in principal agreement with one of the cases studied by Smith and Dahlen. Okubo's (1982) γ values,

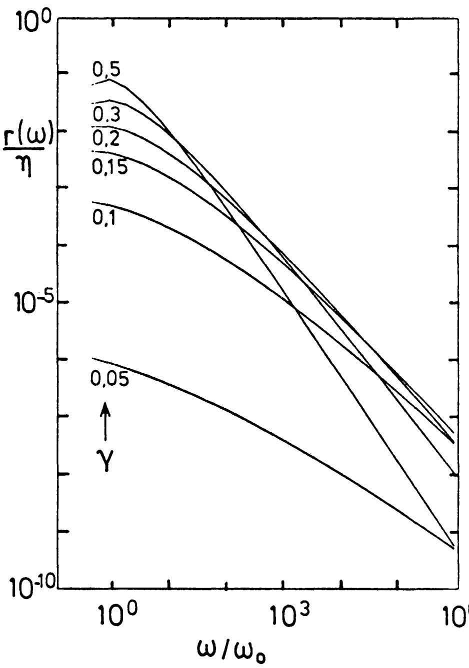

Fig. 5. Normalized relaxation spectra for different values of γ (η = viscosity). Calculation by formula (A.7)

$0.13 \leqq \gamma \leqq 0.2$, were determined from estimates of Q for the Chandler wobble, which is a different (and perhaps less safe) procedure than that based on the wobble period.

The asymptotic, steady-state creep behaviour of a generalized Maxwell body with the properties of the earth's mantle at seismic frequencies and at the frequency of the Chandler wobble is very different from the creep behaviour of the earth's mantle. If (26) is used to determine the viscosity for $\gamma \leqq 0.1$, ω_r and $Q(\omega_r)$ from above and an average rigidity at ω_r of about 2 Mbar, values of 10^{36} Poise and larger are found. These viscosities are far beyond the range that is discussed for the mantle, 10^{21} to 10^{25} Poise. On the other hand, viscosities within this range are found for γ values between about 0.17 and 0.24, but then the Chandler-wobble period remains unexplained. Hence, there is no generalized Maxwell body which adequately describes the complete spectrum of seismological-rheological properties of average mantle material.

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Appendix: Relaxation spectra

The relaxation spectrum $r(\omega)$ is defined by the following representation of the relaxation function $R(t)$, valid for $t \geqq 0$:

$$R(t) = \int_0^\infty r(\omega)\, e^{-\omega t}\, d\omega. \tag{A.1}$$

This is a superposition of elementary exponential relaxation functions with relaxation frequency ω or, equivalently, relaxation time $1/\omega$. In the following, we want to relate $r(\omega)$ to the Fourier transform $\bar{R}(\omega) = M(\omega)/i\,\omega$ of the relaxation function and hence to the viscoelastic modulus $M(\omega)$. We start with the Fourier integral representation of $R(t)$,

$$R(t) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty \bar{R}(\omega)\, e^{i\omega t}\, d\omega,$$

and change the integration variable to $s = -i\,\omega$. This yields

$$\begin{aligned} R(t) &= -\frac{1}{\pi} \operatorname{Im} \int_0^{-i\infty} \bar{R}(i\,s)\, e^{-st}\, ds \\ &= -\frac{1}{\pi} \operatorname{Im} \int_0^{-i\infty} P(\omega)\, e^{-\omega t}\, d\omega. \end{aligned} \tag{A.2}$$

In the second expression we have again written ω instead of s, and $P(\omega)$ is defined by

$$P(\omega) = \bar{R}(i\,\omega) = -\frac{1}{\omega} M(i\,\omega). \tag{A.3}$$

The integration path in (A.2), which runs along the negative imaginary axis in the complex ω plane, is then replaced by an equivalent path along the positive real axis and on a quarter circle with infinitely large radius around the fourth quadrant. If necessary, poles of $P(\omega)$ on the positive real axis or inside the fourth quadrant are circumvented. We obtain

$$R(t) = -\frac{1}{\pi} \operatorname{Im} \left\{ \int_0^\infty P(\omega)\, e^{-\omega t}\, d\omega + \pi\, i \sum_k \varepsilon_k\, e^{-\omega_k t} \right\}. \tag{A.4}$$

In this expression the contribution from the quarter circle has been omitted, because it vanishes, and the following abbreviations have been used:

ω_k = pole of $P(\omega)$, assumed to be of first order.
ε_k = residue of $P(\omega)$ at ω_k, multiplied by 1 for ω_k on the positive real axis and by 2 for ω_k inside the fourth quadrant, respectively.

(A.4) can be written in the form

$$R(t) = \int_0^\infty \left\{ -\frac{1}{\pi} \operatorname{Im} P(\omega) - \sum_k \operatorname{Re} \varepsilon_k \delta(\omega - \omega_k) \right\} e^{-\omega t}\, d\omega.$$

Then comparison with (A.1) yields the relaxation spectrum

$$r(\omega) = \frac{1}{\pi\,\omega} \operatorname{Im} M(i\,\omega) - \sum_k \operatorname{Re} \varepsilon_k\, \delta(\omega - \omega_k). \tag{A.5}$$

Here (A.3) has been used. The relaxation spectrum (A.5) has a continuous part and (occasionally) a discrete part. In the simple case of a Maxwell body, whose modulus has been mentioned in connection with formula (15), only the second part is present, giving $r(\omega) = M(\infty)\,\delta(\omega - \omega_0)$ and $R(t) = M(\infty)\, e^{-\omega_0 t}$, as expected. The continuous spectrum in (A.5) follows also from results of Gross (1947) and MacDonald and Brachman (1956), which were derived in an entirely different manner.

In the case of constant Q ($\gamma=0$) the viscoelastic modulus has been given in (16). The corresponding relaxation spectrum is continuous:

$$r(\omega)=\frac{2A(\omega_r)}{\pi\,\omega_r(Q+Q^{-1})}\left(\frac{\omega_r}{\omega}\right)^{\frac{2}{\pi}\arctan Q} \quad (A.6)$$

This is a power law with a frequency exponent between 0 and -1, depending on Q; however, for $Q \gg 1$ the exponent is practically -1.

In the case of power laws (8) for Q with $0<\gamma<1$ the low-frequency approximation (13) for the modulus implies that $r(\omega)$ tends to zero with ω. The high-frequency approximation (11) for $M(\omega)$ gives the continuous relaxation spectrum

$$r(\omega)=\frac{\eta}{\pi}\left(\frac{\omega_0}{\omega}\right)\exp\left\{\frac{-1}{\sin\left(\gamma\frac{\pi}{2}\right)}\left(\frac{\omega_0}{\omega}\right)^{\gamma}\right\}$$
$$\cdot\sin\left\{2\cos\left(\gamma\frac{\pi}{2}\right)\left(\frac{\omega_0}{\omega}\right)^{\gamma}\right\}. \quad (A.7)$$

This is a valid approximation for $\omega \geqq \omega_0$, since (11) approximates $M(\omega)$ well for these frequencies (see Fig. 1). η is the viscosity defined in (26). Figure 5 shows results for that part of the relaxation spectrum which is described by (A.7). For the larger γ values the decrease of $r(\omega)$ with ω below ω_0 is indicated. The relaxation spectrum is effectively bandlimited with its peak at or below ω_0. For frequencies very much larger than ω_0 $r(\omega)$ follows approximately a power law with the exponent $-(1+\gamma)$. This law would apply in the seismic frequency band. Exact values of the relaxation spectrum, valid at arbitrary frequencies, can be calculated from (15) in the special cases $\gamma=1/2$, $1/3$, $1/4$, ...

The generalized Maxwell body ($0<\gamma<1$) can be considered as a *parallel connection* of classical Maxwell bodies ($\gamma=1$) with relaxation frequencies ω from 0 to ∞. According to (A.1) the (real) elastic modulus of each of these classical Maxwell bodies is $r(\omega)\,d\omega$ and the viscosity $r(\omega)\,d\omega/\omega$.

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