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Calculation of synthetic seismograms for complex subsurface geometries by a combination of finite integral Fourier transforms and finite difference techniques

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Abstract. An algorithm for the calculation of synthetic seismograms for complex geological structures is suggested. Two versions of the algorithm are considered.

The first version is based on a combination of finite integral Fourier transforms with respect to one of the spatial coordinates (e.g. the coordinate corresponding to the epicentral distance) and the finite difference method. In this case the problem reduces to solving a system of equations with partial derivatives with respect to one spatial coordinate (say, the vertical one), with coefficients which are finite Fourier integrals of the elastic parameters varying along the epicentral coordinate. The approach is an extension of the standard techniques of separation of variables to the solution of problems for complex subsurface geometries.

The second version of the algorithm is based on utilization of finite integral Fourier transforms with respect to two spatial coordinates. A Cauchy problem is obtained for a system of ordinary differential equations where the coefficients are double integrals of the parameters of the medium. The problem is solved numerically. The finite Fourier integrals of the parameters of the medium are calculated analytically, if the medium at nonuniform intervals is approximated, say, by linear functions.

Calculated synthetic seismograms are given.

Key words: Synthetic seismograms – Finite differences – Partial separation of variables – Finite integral transforms

Introduction

During recent years, instead of the approximation of space derivative terms by finite-difference expressions a complete set of orthogonal basis functions, whose derivatives are known exactly, has been widely used for the solution of different problems of mathematical physics. Trigonometrical functions are usually suitable and the Fourier transform methods proved efficient in solving many problems (see, for example, Kreiss and Oliger, 1972; Gazdag, 1973, 1981; Fornberg, 1975; Merilees and Orszag, 1979). This approach allows one to calculate spatial derivatives with high accuracy, but requires substantially more computing per degree of freedom. However, the use of the fast Fourier transform (FFT) for the calculation of spatial derivatives makes it possible to overcome this difficulty.

In this paper we will discuss an algorithm for the calculation of synthetic seismograms for complex subsurface geometries, based on a combination of the finite integral Fourier transforms with the finite difference approach. The algorithm was suggested in Mikhailenko (1978, 1979), then it was developed in a series of papers inside the USSR. For the solution of elastodynamic equations, see, for example, Mikhailenko and Korneev (1983). In the following sections we will present some of the results which have been obtained during recent years. Following Mikhailenko (1978, 1979), we will describe two versions of the algorithm.

The first version is based on the combination of finite integral Fourier transforms with respect to one of the spatial coordinates (e.g. the coordinate corresponding to the epicentral distance) and the finite difference approach. In this case the problem reduces to solving a system of equations with partial derivatives with respect to one spatial coordinate (say, the vertical one), with coefficients which are the finite Fourier integrals of the elastic parameters varying along the other coordinate, corresponding to the epicentral distance. For the calculation of the Fourier integrals, at fixed spatial wave numbers, a model of the medium given along the epicentral coordinate is approximated at nonuniform intervals by a linear function. The Fourier integrals of the linear function are analytically calculated within each nonuniform interval. The number of the nonuniform intervals depends on the complexity of the medium along the epicentral coordinate. The calculation of the Fourier integrals of the elastic parameters is performed with the help of a special subroutine, therefore the complexity of the medium along the epicentral coordinate does not practically increase the computation time of the basic program. The order of decrease of the magnitude of the Fourier coefficients can be changed by using splines of different order to approximate the medium. The system of equations obtained with partial derivatives with respect to one spatial coordinate is solved by the finite difference method.

If the parameters of the medium vary along the epicentral coordinate according to some special law (for

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example, the sine or cosine law), the system of equations degenerates. In that part of the medium where the Lamé parameters are independent of the epicentral distance (parameters of the medium vary along the vertical coordinate), we arrive at the classical separation of variables in combination with the finite difference method. This algorithm was developed in a series of papers (see, for example, Mikhailenko, 1973; Alekseev and Mikhailenko, 1976, 1978, 1980).

The second version of the algorithm is based on the utilization of finite Fourier transforms with respect to two spatial coordinates. In addition, the Cauchy problem is obtained for the system of ordinary differential equations where the coefficients are double integrals of the parameters of the medium. For the calculation of the double finite Fourier integrals of the elastic parameters, at fixed spatial wave numbers, the whole domain is partitioned into non-uniform segments, the number of which, and their size and shape, depending on the complexity of subsurface geometries. At each of these segments the parameters of the medium are approximated by linear or bilinear functions. In this case the double finite Fourier integral at each segment is calculated analytically. In this paper we also show an approach where one need not calculate these integrals. The second version of the algorithm described here can be related to the so-called spectral methods. In the recent paper of Kosloff and Baysal (1982) a pseudospectral or the so-called collocation method was suggested for the calculation of synthetic seismograms. The pseudospectral method is an approximation which uses interpolating functions to evaluate derivatives represented on a grid in physical space. It is called a pseudospectral method because the interpolating functions used are the same as in the spectral method. In the pseudospectral method, all operations except for differentiation are carried out in the physical space defined by a grid. In contrast to the familiar spectral method we gain some advantage in computation time since spectral multiplication is not necessary. The price that is paid for this advantage is that the calculations are aliased. The effect of aliasing may not be important but we must keep in mind that aliasing has implications for the stability of a calculation for long intervals of time (Merilees and Orszag, 1979).

When one uses the pseudospectral method the elastic parameters are defined at the points of a uniform grid. The FFT dimension in this case is determined by the number of grid points. To approximate a medium with complex subsurface geometries or a medium containing thin layers it is necessary to use a large number of grid points which leads to an essential increase in computation time.

When one uses the second version of the algorithm described in the following sections most of the computation time is spent in computing double convolution sums by means of the FFT. One can essentially save computation time by using an array processor for the calculation of the double convolution sums. As long as the convolution coefficients, which are double Fourier integrals of the parameters of the medium, are computed beforehand for fixed wave numbers, the complexity of the medium has an insignificant effect on the computer costs of the basic programme.

The dimension of the double convolution sums is determined by the size of the domain where the problem is solved, by the time dependence of the source and by the minimum velocity in the medium.

The two versions of the algorithm described in the paper have been developed in recent years in parallel. The first version is most effective for thin-layer models involving heterogeneities whose amplitude along one coordinate (say, the vertical one) is much less than along the other. These are models of typical petroleum traps (anticlines, reefs, thrust faults, etc.) whose size in the vertical direction is not greater than 8–10 dominant wavelengths. This version of the algorithm is used to calculate Rayleigh waves in two-dimensional heterogeneous media and synthetic seismograms for complex subsurface geometries in a cylindrical coordinate system. In this case one performs the finite Fourier transform along the vertical coordinate and the finite-difference scheme with a non-uniform step is applied along the radial coordinate.

The second version is suitable for use in calculating synthetic seismograms for very complex subsurface geometries including inhomogeneities which are much less than the predominant wavelengths.

Method of solution

SH-wave propagation in complex subsurface geometries

First we consider an elastic half-space in a Cartesian coordinate system, occupying the region $z \geqq 0$, assuming that *SH*-wave velocity $v_s(x,z)$ is an arbitrary piece-wise continuous function of two coordinates. For simplicity the density ρ of the medium is taken constant. *SH*-wave propagation in such a medium from a line source, whose location is the point $x=x_0$, $z=z_0$, is given by the equation

$$\frac{\partial}{\partial x}\left(v_s^2\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial z}\left(v_s^2\frac{\partial U}{\partial z}\right)$$
$$=\frac{\partial^2 U}{\partial t^2}+F_1(x-x_0)F_2(z-z_0)f(t) \tag{1}$$

with the boundary condition

$$\left.\frac{\partial U}{\partial z}\right|_{z=0}=0 \tag{2}$$

and the initial values

$$U|_{t=0}=\left.\frac{\partial U}{\partial t}\right|_{t=0}=0. \tag{3}$$

Here the function $f(t)$ represents the time variation of the source. For a finite source, the functions $F_1(x-x_0)$, $F_2(z-z_0)$ are of the form

$$F_1(x-x_0)=\sqrt{\frac{n_0}{\pi}}\,e^{-n_0(x-x_0)^2},$$

$$F_2(z-z_0)=\sqrt{\frac{m_0}{\pi}}\,e^{-m_0(z-z_0)^2}. \tag{4}$$

For n_0, $m_0 \to \infty$ the functions F_1 and F_2 tend to delta-functions. For the application of the finite integral Fourier transform we introduce the boundary conditions for $x=0$ and $x=b$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0}=\left.\frac{\partial U}{\partial x}\right|_{x=b}=0, \tag{5}$$

and for $z=a$

$$\left.\frac{\partial U}{\partial z}\right|_{z=a}=0. \tag{6}$$

We select sufficiently large distances a and b and consider the wave field up to $t=T$ where T is the minimal time taken for the propagation of the wave front to the reflecting surfaces introduced by conditions (5), (6).

Let us apply the finite integral Fourier transform along the coordinate x from 0 to b:

$$R(z,n,t)=\int_0^b U(z,x,t)\cos\frac{n\pi x}{b}\,dx, \tag{7}$$

$$U(z,x,t)=\frac{1}{b}R(z,0,t)+\frac{2}{b}\sum_{n=1}^{\infty}R(z,n,t)\cos\frac{n\pi x}{b}. \tag{8}$$

Multiplying both parts of Eq. (1) by $\cos\frac{n\pi x}{b}$ and integrating from 0 to b we obtain:

$$\int_0^b \frac{\partial}{\partial x}\left(v_s^2\frac{\partial U}{\partial x}\right)\cos\frac{n\pi x}{b}\,dx+\int_0^b \frac{\partial}{\partial z}\left(v_s^2\frac{\partial U}{\partial z}\right)\cos\frac{n\pi x}{b}\,dx$$
$$=\frac{\partial^2 R}{\partial t^2}+F_2(z-z_0)\,e^{-\frac{n^2\pi^2}{4b^2n_0}}\cos\frac{n\pi x_0}{b}f(t). \tag{9}$$

Here we made use of the following approximate equality:

$$\int_0^b \sqrt{\frac{n_0}{\pi}}\,e^{-n_0(x-x_0)^2}\cos\frac{n\pi x}{b}\,dx$$
$$\approx\int_0^{\infty}\sqrt{\frac{n_0}{\pi}}\,e^{-n_0(x-x_0)^2}\cos\frac{n\pi x}{b}\,dx$$
$$=e^{-\frac{n^2\pi^2}{4b^2n_0}}\cos\frac{n\pi x_0}{b}f(t),$$

which is fulfilled with high accuracy for n_0 sufficiently large.

There is no separation of variables in Eq. (9) since velocity $v_s(x,z)$ is an arbitrary function of two spatial coordinates. Integrating the first term of Eq. (9) by parts and making use of condition (5) we have

$$\int_0^b \frac{\partial}{\partial x}\left(v_s^2\frac{\partial U}{\partial x}\right)\cos\frac{n\pi x}{b}\,dx=\frac{n\pi}{b}\int_0^b v_s^2\frac{\partial U}{\partial x}\sin\frac{n\pi x}{b}\,dx. \tag{10}$$

We integrate this expression once again. Then the first term of Eq. (9) can be written in the form

$$\int_0^b \frac{\partial}{\partial x}\left(v_s^2\frac{\partial U}{\partial x}\right)\cos\frac{n\pi x}{b}\,dx$$
$$=-\left(\frac{n\pi}{b}\right)^2\int_0^b v_s^2 U\cos\frac{n\pi x}{b}\,dx$$
$$-\frac{n\pi}{b}\int_0^b \dot{v}_s^2 U\sin\frac{n\pi x}{b}\,dx. \tag{11}$$

Equation (9) with the account of (11) reduces to

$$\frac{\partial^2 R}{\partial t^2}=-\left(\frac{n\pi}{b}\right)^2\int_0^b v_s^2 U\cos\frac{n\pi x}{b}\,dx$$
$$-\frac{n\pi}{b}\int_0^b \dot{v}_s^2 U\sin\frac{n\pi x}{b}\,dx$$
$$+\int_0^b \frac{\partial}{\partial z}\left(v_s^2\frac{\partial U}{\partial z}\right)\cos\frac{n\pi x}{b}\,dx$$
$$-F_2(z-z_0)\,e^{-\frac{n^2\pi^2}{4b^2n_0}}\cos\frac{n\pi x_0}{b}f(t). \tag{12}$$

Here $\dot{v}_s^2$ is the derivative of the squared velocity with respect to x. Substituting series (8) in (12) instead of the function $U(z,x,t)$ and factoring out the terms independent of x from under the integral sign we obtain the system:

$$\frac{\partial^2 R(z,n,t)}{\partial t^2}=-\left(\frac{n\pi}{b}\right)^2\sum_{m=0}^{\infty}R(z,m,t)\cdot\frac{2}{b}\int_0^b v_s^2(x,z)$$
$$\cdot\cos\frac{m\pi x}{b}\cos\frac{n\pi x}{b}\,dx$$
$$-\left(\frac{n\pi}{b}\right)\sum_{m=0}^{\infty}R(z,m,t)\cdot\frac{2}{b}\int_0^b \dot{v}_s^2(x,z)$$
$$\cdot\cos\frac{m\pi x}{b}\sin\frac{n\pi x}{b}\,dx$$
$$+\sum_{m=0}^{\infty}\frac{2}{b}\int_0^b\frac{\partial}{\partial z}\left[v_s^2(x,z)\frac{\partial R(z,m,t)}{\partial z}\right]$$
$$\cdot\cos\frac{m\pi x}{b}\cos\frac{n\pi x}{b}\,dx$$
$$-F_2(z-z_0)\,e^{-\frac{n^2\pi^2}{4b^2n_0}}\cos\frac{n\pi x_0}{b}f(t). \tag{13}$$

For $m=0$ the factor $1/b$ replaces $2/b$ in this system. The boundary conditions are of the form

$$\left.\frac{\partial R(z,n,t)}{\partial z}\right|_{z=0}=\left.\frac{\partial R(z,n,t)}{\partial z}\right|_{z=a}=0. \tag{14}$$

Problem (13), (14) is solved with zero initial values

$$R(z,n,t)|_{t=0}=\left.\frac{\partial R(z,n,t)}{\partial t}\right|_{t=0}=0. \tag{15}$$

The system has a simpler form if we integrate by parts the first term of Eq. (9) only once. Then the system is of the form

$$\frac{\partial^2 R(z,n,t)}{\partial t^2} = -\frac{n\pi^2}{b^2} \sum_{m=0}^{\infty} R(z,m,t) \cdot m \cdot \frac{2}{b} \int_0^b v_s^2(x,z) \cdot \sin\frac{m\pi x}{b} \sin\frac{n\pi x}{b} dx + \sum_{m=0}^{\infty} \frac{2}{b} \int_0^b \frac{\partial}{\partial z}\left[v_s^2(x,z)\frac{\partial R(z,m,t)}{\partial z}\right] \cdot \cos\frac{m\pi x}{b} \cos\frac{n\pi x}{b} dx - F_2(z-z_0)\, e^{\frac{-n^2\pi^2}{4b^2 n_0}} \cos\frac{n\pi x_0}{b} f(t). \quad (16)$$

As is seen, this system is more convenient for numerical implementation although the factor m is under the summation sign, which affects the character of convergence of our problem. Let us consider the ways of solving system (13) or (16) with boundary conditions (14) and initial values (15). As in the work by Mikhailenko (1978, 1979) we consider two versions of the method. The first version is the use of an explicit difference scheme for solving problem (14)–(16). The second version is based on the application of a finite integral cosine transform in the variable z from 0 to a and the reduction of our problem to the Cauchy problem for a system of ordinary differential equations with constant coefficients.

First version of the method

The first version of the method is based on the finite difference representation of problem (14)–(16). The scheme used here is explicit, with a truncation error of second order with respect to time and space. System (16) in a finite difference form can be written as

$$\frac{1}{\Delta t^2}[R_k^{p+1}(n) - 2R_k^p(n) + R_k^{p-1}(n)] = -\frac{n\pi^2}{b^2} \sum_{m=0}^{\infty} m \cdot R_k^p(m) \cdot D_k(m,n) + \sum_{m=0}^{\infty} \frac{1}{2\Delta z^2} \{[R_{k+1}^p(m) - R_k^p(m)]\, C_{k+1}(m,n) + [R_{k+1}^p(m) - 2R_k^p(m) + R_{k-1}^p(m)]\, C_k(m,n) - [R_k^p(m) - R_{k-1}^p(m)] \cdot C_{k-1}(m,n)\} - F_{2k} \cdot e^{-\frac{n^2\pi^2}{4b^2 n_0}} \cos\frac{n\pi x_0}{b} f^p, \quad (17)$$

where $z = k \cdot \Delta z$, $t = p \cdot \Delta t$,

$$D_k(m,n) = \frac{2}{b}\int_0^b v_{s_k}^2(x) \sin\frac{m\pi x}{b} \sin\frac{n\pi x}{b} dx = \frac{1}{b}\int_0^b v_{s_k}^2(x) \cos\frac{(n-m)\pi x}{b} dx - \frac{1}{b}\int_0^b v_{s_k}^2(x) \cos\frac{(n+m)\pi x}{b} dx = h_k(n-m) - h_k(n+m). \quad (18)$$

$$C_k(m,n) = \frac{2}{b}\int_0^b v_{s_k}^2(x) \cos\frac{m\pi x}{b} \cos\frac{n\pi x}{b} dx = h_k(n-m) + h_k(n+m). \quad (19)$$

The infinite system (17) is replaced by a finite system and is solved numerically. Boundary conditions (14) are approximated in a familiar way. For the nodes k of the difference scheme ($z_k = k \cdot \Delta z$) where the velocity v_{s_k} is constant along the lines parallel to the x axis from 0 to b, system (17) degenerates into one equation with the parameter n:

$$\frac{1}{\Delta t^2}[R_k^{p+1} - 2R_k^p + R_k^{p-1}] = -\frac{n^2\pi^2}{b^2} v_{s_k}^2 R_k^p + \frac{1}{2\Delta z^2}[(R_{k+1}^p - R_k^p) v_{s_{k+1}}^2 + (R_{k+1}^p - 2R_k^p + R_{k-1}^p) v_{s_k}^2 - (R_k^p - R_{k-1}^p) v_{s_{k-1}}^2] - F_{2k}\, e^{\frac{-n^2\pi^2}{4b^2 n_0}} \cos\frac{n\pi x_0}{b} \cdot f^p. \quad (20)$$

Here the velocity v_{s_k} varies from point to point along coordinate z. In this case we have a classical separation of variables as in the method based on a combination of partial separation of variables and finite-difference methods, developed in Mikhailenko (1973), Alekseev and Mikhailenko (1976, 1978, 1980).

In the nodes k of the finite-difference scheme where the velocity $v_{s_k}(x)$ varies arbitrarily along the lines parallel to the x axis from 0 to b the finite system (17) is solved.

In the calculation of the coefficients of the system (17) the quantities

$$h_k(s) = \frac{1}{b}\int_0^b v_{s_k}^2(x) \cos\frac{s\pi x}{b} dx. \quad (21)$$

are needed. At the fixed points k of the scheme, the function $v_s(x)$, arbitrarily varying from 0 to b, is approximated on nonuniform intervals by the linear function $v_{s_l} = v_{0_l}(1 + \beta_l x)$. Here v_{0_l} is the initial velocity at each interval l and β_l is the coefficient of velocity variation. At each interval, the integral (21) is calculated analytically. The coefficients $h_k(s)$ of system (17) are calculated with the help of a special subroutine.

The complexity of subsurface geometries in the direction of the coordinate x does not cause additional computational difficulties in solving system (17). Depending on the complexity the interval of integration between the limits 0 and b in (21) is partitioned into a sufficiently large number of parts in which the function $v_s(x)$ is approximated well by a linear function. It should be noted that the dimension of the finite system (17) is independent of this partitioning. Later we will show that the dimension of the system depends on the spectral width of the signal $f(t)$ in the source as well as on the spatial dimension of the source. Moreover, the decrease of the coefficients $h_k(s)$ with increasing parameter s can be changed if one uses splines of different order for smoothing the function $v_s(x)$. In the numerical solution of system (17) with the explicit difference scheme most of the computing time is taken for the calculation of the convolution type sums

$$\sum_{m=0}^{M-1} R_k^p(m) \cdot [h_k(n-m) \pm h_k(n+m)]. \quad (22)$$

These sums are calculated with the fast Fourier transform (FFT). The use of an array processor allows one to substantially decrease the computer time of synthetic seismograms for complex subsurface geometries.

The convergence of a series of the type of (22) is determined by the behaviour of the function $R_k^p(m)$ with m increasing and the extent of decreasing the coefficients $h_k(n \pm m)$ as well. One can show that in the approximation of the velocity $v_s(x)$ by piecewise linear functions we obtain the order of the decrease of the coefficients $h_k(n \pm m)$ as $1/(n \pm m)^2$. Obviously the decreasing character of the coefficients $h_k(n \pm m)$ with m and n increasing is determined by the smoothness of the function $v_s^2(x)$ according to the well-known theorems of the decrease of the Fourier series. Thus, making use of splines for smoothing the given function $v_s^2(x)$, we obtain an additional convergence of the convolution type sums (22).

Let the interval of integration from 0 to b in integral (21) be partitioned into L non-uniform parts. If for the approximation of the function $v_s^2(x)$ one uses fifth order splines $\tilde{S}(x)$, the expression for the coefficients $h(s)$ takes the form:

$$h(s) = \frac{b^4}{s^5 \pi^5} \left[M_L \sin \frac{s\pi x_4}{b} - M_1 \sin \frac{s \pi x_1}{b} \right] - \frac{b^5}{s^6 \pi^6} \sum_{l=2}^{L} \left(\frac{M_l - M_{l-1}}{x_l - x_{l-1}} \right) \left(\cos \frac{s\pi x_{l-1}}{b} - \cos \frac{s \pi x_l}{b} \right),$$

$$h(0) = \frac{1}{b} \left[\tfrac{1}{120}(M_L x_L^5 - M_1 x_1^5) - \tfrac{1}{720} \sum_{l=2}^{L} (M_l - M_{l-1})(x_l^6 - x_{l-1}^6)/(x_l - x_{l-1}) \right]. \quad (23)$$

Here

$$M_l = \left. \frac{d^4 \tilde{S}(x)}{dx^4} \right|_{x = x_l}, \quad s = (n \pm m). \quad (24)$$

The coefficients $h(s)$ decrease asymptotically as $1/s^6$ (the first term in (23) equals zero).

Let us now consider some models for complex subsurface geometries for which system (16) degenerates. Let for example the velocity be

$$v_s^2(x,z) = v_0(z) \pm \frac{\Delta v(z)}{2} \left(1 - \cos \frac{l \pi x}{b} \right).$$

Take for definiteness

$$v_s^2(x,z) = A(z) - B(z) \cos \frac{l \pi x}{b}, \quad (25)$$

where

$$A(z) = v_0(z) + \Delta v(z)/2, \quad B(z) = \Delta v(z)/2.$$

Figure 1 presents velocity variations given by formula (25) with fixed values A, B and $l = 1, 2, 3, 8$. By changing the values of coefficients A and B along the coordinate z

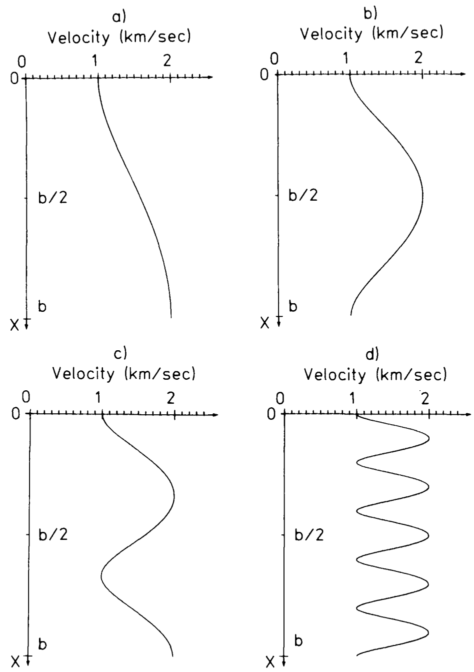

Fig. 1a–d. Variation of the velocity $v_s(x,z)$ given by the formula $v_s^2(x,z) = A(z) + B(z) \cos \frac{l\pi x}{b}$, depending on the epicentral coordinate x, for various values of l: **a** $l=1$, **b** $l=2$, **c** $l=3$, **d** $l=8$. The values of $A(z)$ and $B(z)$ are fixed.

and by fixing l one can obtain sufficiently complex models of media. Substituting the velocity $v_s^2(x,z)$ given by formula (25) in system (16) we obtain:

$$\frac{\partial^2 R(z,n,t)}{\partial t^2} = \frac{\partial}{\partial z} \left[A \frac{\partial R(z,n,t)}{\partial z} \right] - \frac{1}{2} \frac{\partial}{\partial z} \left\{ B \frac{\partial}{\partial z} [R(z,n+l,t) + R(z,n-l,t)] \right\} - A \left(\frac{n\pi}{b} \right)^2 R(z,n,t) + B \frac{n\pi^2}{2b^2} [(n+l) R(z,n+l,t) + (n-l) R(z,n-l,t)] - F_2(z) e^{\frac{-n^2\pi^2}{4b^2 n_0}} \cos \frac{n \pi x_0}{b} f(t). \quad (26)$$

Integrals of the form

$$\frac{2}{b} \int_0^b \cos \frac{(n \pm l)\pi x}{b} \cos \frac{m \pi x}{b} dx = \begin{cases} 1 & \text{for } m = n \pm l, \\ 0 & \text{for } m \neq n \pm l, \end{cases}$$

based on the orthogonality of sine- and cosine-functions in the interval $[0, b]$, have been used and reduce system (16) to (26).

Equation (26) with fixed value of l can be solved by the finite-difference method for $n = 0, 1, 2, 3, \ldots$. Equation (26) is solved for boundary condition (14) and zero initial values (15). The displacement $U(z,x,t)$ is determined by summing up of series (8).

System (16) can be simplified if the velocity $v_s^2(x,z)$ is expanded into the Fourier series

$$v_s^2(x,z)=\frac{d_0(z)}{2}+\sum_{i=1}^{\infty} d_i(z)\cos\frac{i\pi x}{b}, \tag{27}$$

$$d_i(z)=\frac{2}{b}\int_0^b v_s^2(x,z)\cos\frac{i\pi x}{b}\,dx, \tag{28}$$

and we take a small number of terms in (27). One can do it if the function $v_s^2(x)$ is slowly varying along the coordinate x.

One can avoid the calculation of integrals (21) if the velocity $v_s(x)$ is given in the nodes of a uniform grid. In this case the number of nodes should coincide with the number of terms M of convolution type sums (22). Let

$$y(n)=\sum_{m=0}^{M-1} R(m)[h(n-m)+h(n+m)] \tag{29}$$

where

$$h(n\pm m)=\frac{1}{b}\int_0^b v_s^2(x)\cos\frac{(n\pm m)\pi x}{b}\,dx. \tag{30}$$

Sum (29) can be calculated with the help of the FFT. For this purpose we determine the functions $R(m)$ as follows:

$$R(m)=\begin{cases}R(m) & \text{for } m=0,1,2,\dots,M-1\\ 0 & \text{for } m=M,M+1,\dots,2M-1.\end{cases}$$

The function $h(m)$ is represented in a similar way. We then apply the discrete Fourier transform to Eq. (29):

$$Y(k)=\sum_{n=0}^{2M-1}\sum_{m=0}^{2M-1} R(m)\,[h(n-m)+h(n+m)]\,e^{\frac{2\pi i}{2M}nk}. \tag{31}$$

Here

$$Y(k)=\sum_{n=0}^{\infty} y(n)\,e^{\frac{2\pi i}{2M}nk}.$$

Equation (31) is presented in the form:

$$\begin{aligned}Y(k)=&\sum_{n=0}^{2M-1}\sum_{m=0}^{2M-1} R(m)\,e^{\frac{2\pi i}{2M}mk}\cdot h(n-m)\,e^{\frac{2\pi i}{2M}(n-m)k}\\&+\sum_{n=0}^{2M-1}\sum_{m=0}^{2M-1} R(m)\,e^{-\frac{2\pi i}{2M}mk}\,h(n+m)\,e^{\frac{2\pi i}{2M}(n+m)k}\\=&\sum_{m=0}^{2M-1} R(m)\,e^{\frac{2\pi i}{2M}mk}\sum_{n=0}^{2M-1} h(n-m)\,e^{\frac{2\pi i}{2M}(n-m)k}\\&+\sum_{m=0}^{2M-1} R(m)\,e^{\frac{-2\pi i}{2M}mk}\sum_{n=0}^{2M-1} h(n+m)\,e^{\frac{2\pi i}{2M}(n+m)k}\\=&H(k)\cdot\sum_{m=0}^{2M-1} R(m)\cos\frac{2\mathrm{m}k\pi}{2\mathrm{M}}.\end{aligned} \tag{32}$$

Here

$$H(k)=\sum_{m=0}^{2M-1} h(n\pm m)\,e^{\frac{2\pi i}{2M}(n\pm m)k}. \tag{33}$$

Taking the real part of (32) we have:

$$Y(k)=2\,\mathrm{Re}\{\tilde{R}(k)\}\cdot\mathrm{Re}\{H(k)\}. \tag{34}$$

The values of $\tilde{y}(n)$ can be obtained by the inverse discrete Fourier transform:

$$\tilde{y}(n)=\frac{1}{2M}\sum_{k=0}^{2M-1} Y(k)\,e^{-\frac{2\pi i}{2M}nk}, \tag{35}$$

where $Y(k)$ is determined by formula (32). The real part of (35) is obtained in the following way:

$$y(n)=2\,\mathrm{Re}\{\tilde{y}(n)\},\qquad n=0,1,\dots,M-1. \tag{36}$$

If one assumes that in formula (33)

$$\begin{aligned}h(m)=&\frac{2}{b}\int_0^b v_s^2(x)\cos\frac{m\pi x}{b}\,dx\\\approx&\,0\qquad\text{for } m=M,M+1,\dots,\end{aligned}$$

the integral can be replaced by the discrete Fourier series (see, e.g. Gold and Rader, 1969). Hence, from (33) we have:

$$H(k)=v_s^2(k\cdot\Delta x),\qquad k=0,1,2,\dots,M-1, \tag{37}$$

Here

$\Delta x=b/M$.

Thus, it is not necessary to calculate integral (30). Instead, we only need to specify the velocity in nodes of the uniform grid on the interval from 0 to b. The velocity $v_s^2(x)$ is assumed to be a sufficiently smooth function of the coordinate x. For calculating the values of $Y(k)$ and $y(n)$ we make use of the forward and inverse FFT.

Second version of the method

The system of partial differential equations (16) can be reduced to a system of ordinary differential equations if we use the finite integral cosine transform along the coordinate z from 0 to a:

$$W(i,n,t)=\int_0^a R(z,n,t)\cos\frac{i\pi z}{a}\,dz, \tag{38}$$

$$R(z,n,t)=\frac{1}{a}W(0,n,t)+\frac{2}{a}\sum_{i=1}^{\infty} W(i,n,t)\cos\frac{i\pi z}{a}. \tag{39}$$

We multiply (16) by $\cos\frac{i\pi z}{a}$ and integrate by parts from 0 to a. Performing manipulations similar to those mentioned above and making use of condition (14) we obtain:

$$\begin{aligned}\frac{d^2W(i,n,t)}{dt^2}&\\=&-\frac{n\pi}{b}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\delta_{mj}\frac{m\pi}{b}W(j,m,t)\cdot D_1(n,m,i,j)\\&-\frac{i\pi}{a}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\delta_{mj}\frac{j\pi}{a}W(j,m,t)\,D_2(n,m,i,j)\\&-e^{\frac{-n^2\pi^2}{4b^2n_0}}\cdot e^{-\frac{i^2\pi^2}{4a^2i_0}}\cos\frac{n\pi x_0}{b}\cos\frac{i\pi z_0}{b}\cdot f(t),\end{aligned} \tag{40}$$

where

$$\delta_{mj}=\begin{cases}1 & \text{if } m\neq 0,\ j\neq 0\\ \frac{1}{2} & \text{if } m=0,\ j\neq 0 \quad\text{or}\quad m\neq 0, j=0\\ \frac{1}{4} & \text{if } m=j=0\end{cases}$$

$$D_1(n,m,i,j)=\tfrac{1}{4}[h(n-m,i-j)+h(n-m,i+j) -h(n+m,i-j)-h(n+m,i+j)],$$

$$D_2(n,m,i,j)=\tfrac{1}{4}[h(n-m,i-j)-h(n-m,i+j) +h(n+m,i-j)-h(n+m,i+j)],$$

$$h(r,s)=\frac{4}{ab}\int_0^a\int_0^b v_s^2(x,z)\cos\frac{r\pi z}{a}\cos\frac{s\pi x}{b}\,dx\,dz. \tag{41}$$

System (40) is solved with zero initial values

$$W(i,n,t)|_{t=0}=\left.\frac{dW(i,n,t)}{dt}\right|_{t=0}=0. \tag{42}$$

We obtain an analogous system, however without the terms m and j under the summation signs, if we employ the finite integral cosine transform (38), (39) for system (13). In this case, besides coefficients (41) which are double integrals of the velocity $v_s^2(x,z)$, there will be coefficients which are double integrals of the derivatives of the velocity $v_s^2(x,z)$ with respect to x and z. The infinite system (40) is approximated by a finite system and is solved numerically by means of the explicit difference scheme:

$$\frac{1}{\Delta t^2}[W^{p+1}(i,n)-2W^p(i,n)+W^{p-1}(i,n)]$$
$$=-\frac{n\pi}{b}\sum_{m=0}^{M-1}\sum_{j=0}^{J-1}\delta_{mj}\frac{m\pi}{b}W^p(j,m)\,D_1(n,m,i,j)$$
$$-\frac{i\pi}{a}\sum_{m=0}^{M-1}\sum_{j=0}^{J-1}\delta_{mj}\frac{j\pi}{a}W^p(j,m)\,D_2(n,m,i,j)$$
$$-e^{\frac{-n^2\pi^2}{4b^2n_0}}\cos\frac{n\pi x_0}{b}\,e^{\frac{-i^2\pi^2}{4a^2i_0}}\cos\frac{i\pi z_0}{a}f(t) \tag{43}$$

The displacement $U(z,x,t)$ is determined by summing up the series (39) and (8).

For the calculation of integral (41) the integration domain from 0 to a is partitioned into non-uniform segments. In each of these segments the velocity is approximated by the bilinear function $v_{s_{kl}}=a_{kl}+b_{kl}x+c_{kl}z+d_{kl}xz$. The coefficients a_{kl}, b_{kl}, c_{kl}, d_{kl} are determined from the velocity values in the nodes of the rectangle. In this way the value of the integral (41) in each segment is easily calculated analytically. If we sum up the values of the integrals for all segments we will obtain the coefficients $h(r,s)$. Due to the fact that the function $v_s(x,z)$ is continuous at all the boundaries of the segments, the coefficients decrease as $1/(r^2+s^2)$. A discontinuity in the velocity function is approximated well by a gradient layer whose width is much less than the wave-length. The number of segments used for the calculation of the coefficients $h(r,s)$ depends on the complexity of subsurface geometries. Here, the coefficients $h(r,s)$ are calculated with the help of a special subroutine.

In numerical solutions of system (43) double sums of the convolution type

$$\sum_{m=0}^{M-1}\sum_{j=0}^{J-1}\tilde{W}^p(j,m)\cdot h(n\pm m,i\pm j) \tag{44}$$

are calculated with the help of a FFT. The use of an array processor considerably reduces the computer time for synthetic seismograms of complex subsurface geometries. The convergence of the double sums of type (44) is determined by the decrease of the function $\tilde{W}^p(j,m)$ and the coefficients $h(n\pm m,i\pm j)$ while m and j are increasing. The behaviour of the function $\tilde{W}^p(j,m)$, as will be shown, depends on the smoothness of the signal $f(t)$ as well as on the spatial distribution of the source. Moreover, using two-dimensional splines of different order for smoothing the velocity function $v_s(x,z)$ in the integral (41) one can also change the decrease of the coefficient $h(r,s)$ in a similar way as in the one-dimensional case (see formulas (23), (24)).

Note that in calculations of the double sums (44) one can avoid the calculation of integral (41). As in the one-dimensional case it can be done here for the function $v_s(x,z)$ which is given in the nodes of a uniform grid, the number of nodes along the coordinates x and z coinciding with the number of terms to be summed up in (44).

For some particular models of two-dimensional media, e.g. for media given by formulas of type (25), the system (40) degenerates.

Lamb's problem for complex subsurface geometries

Consider Lamb's problem for the elastic inhomogeneous half-space $z\geqq 0$, where P and SV wave velocities are arbitrary piecewise continuous functions spatially varying along the coordinates x and z. For the sake of simplicity we consider the density ρ constant throughout the model. At the boundary $z=0$ a vertical force is applied; then the boundary conditions at the free surface are of the form

$$\begin{aligned}&(v_p^2-2v_s^2)\frac{\partial U}{\partial x}+v_p^2\frac{\partial W}{\partial z}=F(x-x_0)\cdot f(t)\\ &\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}=0 \qquad \text{for } z=0\end{aligned} \tag{45}$$

Here U and W are horizontal and vertical displacements, respectively. As earlier, we take $F(x-x_0)$ in the form

$$F(x-x_0)=\sqrt{\frac{n_0}{\pi}}\,e^{-n_0(x-x_0)^2}.$$

Having selected the parameter n_0 large enough, we will consider a source which is close to a point force. The second order partial differential equations, describing $P-SV$ wave propagation in a two-dimensional medium with rectangular coordinates x and z, can be written as

$$\frac{\partial}{\partial x}\left[v_p^2\frac{\partial U}{\partial x}+(v_p^2-2v_s^2)\frac{\partial W}{\partial z}\right] +\frac{\partial}{\partial z}\left[v_s^2\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right)\right]=\frac{\partial^2 U}{\partial t^2} \tag{46}$$

$$\frac{\partial}{\partial x}\left[v_s^2\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[v_p^2\frac{\partial W}{\partial z}+(v_p^2-2v_s^2)\frac{\partial U}{\partial x}\right]=\frac{\partial^2 W}{\partial t^2} \tag{47}$$

with zero initial values

$$U|_{t=0}=W|_{t=0}=\left.\frac{\partial U}{\partial t}\right|_{t=0}=\left.\frac{\partial W}{\partial t}\right|_{t=0}. \tag{48}$$

Let us introduce new boundary conditions

$$U|_{\substack{x=0\\x=b}}=0, \qquad \left.\frac{\partial W}{\partial x}\right|_{\substack{x=0\\x=b}}=0 \quad \text{for } x=0 \text{ and } x=b. \tag{49}$$

We apply the cosine and sine integral transformations with finite limits to the system of equations (46), (47):

$$R(z,n,t)=\int_0^b W(z,x,t)\cos\frac{n\pi x}{b}dx \tag{50}$$

$$W(z,x,t)=\frac{1}{b}R(z,0,t)+\frac{2}{b}\sum_{n=1}^{\infty}R(z,n,t)\cos\frac{n\pi x}{b} \tag{51}$$

$$S(z,n,t)=\int_0^b U(z,x,t)\sin\frac{n\pi x}{b}dx \tag{52}$$

$$U(z,x,t)=\frac{2}{b}\sum_{n=1}^{\infty}S(z,n,t)\sin\frac{n\pi x}{b}. \tag{53}$$

We multiply Eq. (46) by $\sin\frac{n\pi x}{b}$ and Eq. (47) by $\cos\frac{n\pi x}{b}$ and integrate by parts from 0 to b, making use of conditions (49). Then, substituting series (51), (53) instead of $W(z,x,t)$ and $U(z,x,t)$ in the integral terms of the system obtained, we have

$$\begin{aligned}\frac{\partial^2 S(z,n,t)}{\partial t^2} &= -\frac{n\pi^2}{b^2}\sum_{m=0}^{\infty}m\cdot S(z,m,t)\cdot\frac{2}{b}\int_0^b v_p^2(x,z)\cdot\cos\frac{m\pi x}{b}\cos\frac{n\pi x}{b}dx\\&-\frac{n\pi}{b}\sum_{m=0}^{\infty}\frac{\partial R(z,m,t)}{\partial z}\cdot\frac{2}{b}\int_0^b[v_p^2(x,z)-2v_s^2(x,z)]\cdot\cos\frac{m\pi x}{b}\cos\frac{n\pi x}{b}dx\\&-\frac{\pi}{b}\sum_{m=0}^{\infty}m\cdot\frac{2}{b}\int_0^b\frac{\partial}{\partial z}\left[v_s^2(x,z)R(z,m,t)\right]\cdot\sin\frac{m\pi x}{b}\sin\frac{n\pi x}{b}dx\\&+\sum_{m=1}^{\infty}\frac{2}{b}\int_0^b\frac{\partial}{\partial z}\left[v_s^2(x,z)\frac{\partial S(z,m,t)}{\partial z}\right]\cdot\sin\frac{m\pi x}{b}\sin\frac{n\pi x}{b}dx,\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\partial^2 R(z,n,t)}{\partial t^2} &= -\frac{n\pi^2}{b^2}\sum_{m=0}^{\infty}m\cdot R(z,m,t)\cdot\frac{2}{b}\int_0^b v_s^2(x,z)\cdot\sin\frac{m\pi x}{b}\sin\frac{n\pi x}{b}dx\\&+\frac{n\pi}{b}\sum_{m=1}^{\infty}\frac{\partial S(z,m,t)}{\partial z}\cdot\frac{2}{b}\int_0^b v_s^2(x,z)\cdot\sin\frac{m\pi x}{b}\sin\frac{n\pi x}{b}dx\\&+\sum_{m=0}^{\infty}\frac{2}{b}\int_0^b\frac{\partial}{\partial z}\left[v_p^2(x,z)\frac{\partial R(z,m,t)}{\partial z}\right]\cdot\cos\frac{m\pi x}{b}\cos\frac{n\pi x}{b}dx\\&+\frac{\pi}{b}\sum_{m=1}^{\infty}m\cdot\frac{2}{b}\int_0^b\frac{\partial}{\partial z}[(v_p^2(x,z)-2v_s^2(x,z))\cdot S(z,m,t)]\cos\frac{m\pi x}{b}\cos\frac{n\pi x}{b}dx.\end{aligned} \tag{55}$$

If we take the velocities v_p and v_s independent of the coordinate x close to the free surface, the boundary conditions for $z=0$ will be as follows:

$$\left.\begin{aligned}&v_p^2\frac{\partial R(z,n,t)}{\partial z}+(v_p^2-2v_s^2)\frac{n\pi}{b}S(z,n,t)=e^{-\frac{n^2\pi^2}{4b^2n_0}}\cos\frac{n\pi x_0}{b}f(t) &(56)\\&\frac{\partial S(z,n,t)}{\partial z}-\frac{n\pi}{b}R(z,n,t)=0 &(57)\end{aligned}\right\}\text{ for } z=0$$

Problem (56), (57) is solved with zero initial values

$$S(z,n,t)|_{t=0}=R(z,n,t)|_{t=0}=\left.\frac{\partial S(z,n,t)}{\partial t}\right|_{t=0}=\left.\frac{\partial R(z,n,t)}{\partial t}\right|_{t=0}=0. \tag{58}$$

Problem (54)–(58) is solved in a similar way as the problem of SH wave propagation in compelx subsurface geometries. We will not discuss all the details of the two versions of the method used here, but focus on the main features of the numerical techniques. In the first version the finite difference scheme used to represent the problem, (54)–(58), is explicit with a truncation error of second order. System (54)–(55) degenerates into two equations in the nodes of the finite difference grid, if the velocities v_{p_k} and v_{s_k} are constant form 0 to b along the coordinate x.

Consider a particular case of complex subsurface geometries where system (54)–(55) is of a simple from. Let the velocities $v_p^2(x,z)$ and $v_s^2(x,z)$ be given in the form (25) (see Fig. 1):

$$v_p^2(x,z)=A_p(z)+B_p(z)\cos\frac{l\pi x}{b}, \tag{59}$$

$$v_s^2(x,z)=A_s(z)+B_s(z)\cos\frac{l\pi x}{b}. \tag{60}$$

The difference of the compressional and shear wave velocities is as follows:

$$[v_p^2(x,z)-2v_s^2(x,z)]=A_{ps}(z)-B_{ps}(z)\cos\frac{l\pi x}{b} \tag{61}$$

where

$$A_{ps}(z)=A_p(z)-2A_s(z),$$
$$B_{ps}(z)=B_p(z)-2B_s(z).$$

For this model the system (54), (55) takes the form

$$\begin{aligned}\frac{\partial^2 S(z,n,t)}{\partial t^2}
&=\frac{\partial}{\partial z}\left[A_s\frac{\partial S(z,n,t)}{\partial z}\right]-\frac{n\pi}{b}\frac{\partial}{\partial z}[A_sR(z,n,t)]\\
&-\frac{n\pi}{b}\frac{\partial R(z,n,t)}{\partial z}A_{ps}-\frac{n^2\pi^2}{b^2}A_pS(z,n,t)\\
&-\frac{1}{2}\frac{\partial}{\partial z}\left\{B_s\frac{\partial}{\partial z}[S(z,n+l,t)+S(z,n-l,t)]\right\}\\
&+\frac{\pi(n+l)}{2b}\frac{\partial}{\partial z}[B_s\cdot R(z,n+l,t)]\\
&+\frac{\pi(n-l)}{2b}\frac{\partial}{\partial z}[B_s\cdot R(z,n-l,t)]\\
&+\frac{B_{ps}}{2}\frac{n\pi}{b}\frac{\partial}{\partial z}[R(z,n+l,t)+R(z,n-l,t)]\\
&+B_p\frac{n\pi^2}{2b^2}[(n+l)S(z,n+l,t)\\
&+(n-l)\cdot S(z,n-l,t)]\end{aligned} \tag{62}$$

$$\begin{aligned}\frac{\partial^2 R(z,n,t)}{\partial t^2}
&=\frac{\partial}{\partial z}\left[A_p\frac{\partial R(z,n,t)}{\partial z}\right]+\frac{n\pi}{b}\frac{\partial}{\partial z}[A_{ps}\cdot S(z,n,t)]\\
&+\frac{n\pi}{b}A_s\frac{\partial S(z,n,t)}{\partial z}-\frac{n^2\pi^2}{b^2}A_sR(z,n,t)\\
&-\frac{1}{2}\frac{\partial}{\partial z}\left\{B_p\frac{\partial}{\partial z}[R(z,n+l,t)+R(z,n-l,t)]\right\}\\
&-\frac{\pi(n+l)}{2b}\frac{\partial}{\partial z}[B_{ps}\cdot S(z,n+l,t)]\\
&-\frac{\pi(n-l)}{2b}\frac{\partial}{\partial z}[B_{ps}\cdot S(z,n-l,t)]\\
&-B_{ps}\frac{n\pi}{2b}\frac{\partial}{\partial z}[S(z,n+l,t)+S(z,n-l,t)]\\
&+B_s\frac{n\pi^2}{2b^2}[(n+l)R(z,n+l,t)\\
&+(n-l)R(z,n-l,t)].\end{aligned} \tag{63}$$

Selecting, for example, the parameter $l=1$ in formulas (59)–(61) we obtain a stratified inhomogeneous model where the velocities v_p and v_s also vary smoothly along the epicentral coordinate x.

The second version of the method is the application of finite integral cosine and sine transformations along the coordinate z from 0 to a:

$$Y(i,n,t)=\int_0^a R(z,n,t)\cos\frac{i\pi z}{a}dz, \tag{64}$$

$$R(z,n,t)=\frac{1}{a}Y(0,n,t)+\frac{2}{a}\sum_{i=1}^{\infty}Y(i,n,t)\cos\frac{i\pi z}{a}, \tag{65}$$

$$X(i,n,t)=\int_0^a S(z,n,t)\sin\frac{i\pi z}{a}dz, \tag{66}$$

$$S(z,n,t)=\frac{2}{a}\sum_{i=1}^{\infty}X(i,n,t)\sin\frac{i\pi z}{a}. \tag{67}$$

To apply these transformations we introduce the following boundary condition instead of boundary conditions (45):

$$\left.\frac{\partial W(z,x,t)}{\partial z}\right|_{\substack{z=0\\z=a}}=0,\qquad U(z,x,t)|_{\substack{z=0\\z=a}}=0. \tag{68}$$

The compressional line source is presented in the right-hand side of system (46)–(47) through the components of the vector

$$F=\mathrm{grad}\,[\mathrm{F}_1(\mathrm{x}-\mathrm{x}_0)\cdot\mathrm{F}_2(\mathrm{z}-\mathrm{z}_0)]\,f(t). \tag{69}$$

We multiply Eq. (54) by $\sin\frac{i\pi z}{a}$ and Eq. (55) by $\cos\frac{i\pi z}{a}$ and integrate from 0 to a. Performing manipulations similar to those mentioned above and making use of boundary condition (68), we obtain

$$\begin{aligned}\frac{\partial^2 X(i,n,t)}{\partial t^2}
&=-\frac{n\pi}{b}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{m\pi}{b}Y(j,m,t)\cdot D_1(n,m,i,j)\\
&+\frac{n\pi}{b}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{i\pi}{a}\delta_m\cdot\delta_j\cdot X(j,m,t)\cdot D_2(n,m,i,j)\\
&+\frac{i\pi}{a}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{m\pi}{b}\delta_m\cdot\delta_jX(j,m,t)\cdot D_3(n,m,i,j)\\
&-\frac{i\pi}{a}\sum_{m=1}^{\infty}\sum_{i=1}^{\infty}\frac{j\pi}{a}Y(j,m,t)\cdot D_3(n,m,i,j)\\
&+\frac{4n\pi}{ab^2}\sqrt{\frac{\pi}{n_0}}e^{-\frac{n^2\pi^2}{4b^2n_0}}\sqrt{\frac{\pi}{i_0}}e^{-\frac{i^2\pi^2}{4a^2i_0}}\\
&\cdot\cos\frac{n\pi x_0}{b}\sin\frac{i\pi z_0}{a},\end{aligned} \tag{70}$$

$$\frac{\partial^2 Y(i,n,t)}{\partial t^2}$$
$$= -\frac{n\pi}{b}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{m\pi}{b}\delta_m\delta_j X(j,m,t)\cdot D_3(n,m,i,j)$$
$$+\frac{n\pi}{b}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{i\pi}{a}Y(j,m,t)\cdot D_3(n,m,i,j)$$
$$-\frac{i\pi}{a}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{i\pi}{a}\delta_m\delta_j X(j,m,t)\,D_1(n,m,i,j)$$
$$+\frac{i\pi}{a}\sum_{m=1}^{\infty}\sum_{j=1}^{\infty}\frac{m\pi}{b}Y(j,m,t)\cdot D_2(n,m,i,j)$$
$$-\frac{4i\pi}{a^2 b}\sqrt{\frac{\pi}{n_0}}e^{-\frac{n^2\pi^2}{4b^2 n_0}}\cdot\sqrt{\frac{\pi}{i_0}}e^{-\frac{i^2\pi^2}{4a^2 i_0}}$$
$$\cdot\cos\frac{n\pi x_0}{b}\sin\frac{i\pi z_0}{a}, \tag{71}$$

where

$$D_1(n,m,i,j)=\tfrac{1}{4}[h_1(n-m,i-j)-h_1(n-m,i+j)+h_1(n+m,i-j)-h_1(n+m,i+j)],$$
$$D_2(n,m,i,j)=\tfrac{1}{4}[h_1(n-m,i-j)-h_1(n-m,i+j)+h_1(n+m,i-j)-h_1(n+m,i+j)]-\tfrac{1}{2}[h_2(n-m,i-j)-h_2(n-m,i+j)+h_2(n+m,i-j)-h_2(n+m,i+j)],$$
$$D_3(n,m,i,j)=\tfrac{1}{4}[h_2(n-m,i-j)+h_2(n-m,i+j)-h_2(n+m,i-j)-h_2(n+m,i+j)],$$

$$h_1(r,s)=\frac{4}{ab}\int_0^a\int_0^b v_p^2(x,z)\cos\frac{r\pi z}{a}\cos\frac{s\pi x}{b}\,dx\,dz, \tag{72}$$

$$h_2(r,s)=\frac{4}{ab}\int_0^a\int_0^b v_s^2(x,z)\cos\frac{r\pi z}{a}\cos\frac{s\pi x}{b}\,dx\,dz, \tag{73}$$

$$\delta_{mj}=\begin{cases}1, & \text{if } m\neq 0,\ j\neq 0\\ \frac{1}{2} & \text{if } m=0,\ j\neq 0 \quad\text{or}\quad m\neq 0,\ j=0\\ \frac{1}{4} & \text{if } m=0,\ j=0\end{cases}$$

System (70), (71) is solved with zero initial values

$$X(i,n,t)|_{t=0}=\left.\frac{dX(i,n,t)}{dt}\right|_{t=0}=0, \tag{74}$$

$$Y(i,n,t)|_{t=0}=\left.\frac{dY(i,n,t)}{dt}\right|_{t=0}=0. \tag{75}$$

In the numerical solution of system (70)–(75) the infinite system (70), (71) is replaced by a finite one, and we use an explicit difference scheme with a truncation error of second order. Integrals (72), (73) are calculated, analytically after the velocities $v_p(x,z)$ and $v_s(x,z)$ are approximated on nonuniform segments, e.g. by a linear function. We can avoid calculating these integrals, as mentioned in the foregoing, if the velocities $v_p(x,z)$, $v_s(x,z)$ are given on a uniform rectangular grid and the numbers of mesh elements along x and z coincide with the numbers of terms to be summed up in series (51), (53) and (65), (67).

On some aspects of convergence and accuracy of the method

The convergence of the algorithms described here depends on the following two factors: the smoothness of the function $f(t)$ in the source (i.e. the decrease of its spectrum with increasing frequency) and the extent of spatial distribution of the source. Let us illustrate this on problem (1)–(6) for a homogeneous medium. The exact solution of the problem under study after the application of finite integral cosine transforms (7), (8), (38), (39) with respect to spatial coordinates is presented in the form

$$U(z,x,t)=\frac{4}{ab}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}W(i,n,t)\cos\frac{i\pi z}{a}\cos\frac{n\pi x}{b}, \tag{76}$$

where the function $W(i,n,t)$ is

$$W(i,n,t)=e^{-\frac{n^2\pi^2}{4b^2 n_0}}e^{-\frac{i\pi^2}{4a^2 i_0}}\cos\frac{n\pi x_0}{b}\cos\frac{i\pi z_0}{a}$$
$$\cdot\frac{1}{v}\int_0^t f(\tau)\sin v(t-\tau)\,d\tau. \tag{77}$$

Here

$$v^2=v_s^2\left(\frac{n^2\pi^2}{b^2}+\frac{i^2\pi^2}{a^2}\right).$$

As is seen from formula (77), the convergence of series (76) depends on the values of parameters n_0 and i_0, which characterize the spatial distribution of the source, and also on the smoothness of the function $f(t)$ in the convolution integral of (77).

Select the function $f(t)$ in the source in the form

$$f(t)=-2\gamma(t-t_0)e^{-\gamma(t-t_0)^2}$$

where t_0 is selected such that $f(0)\approx 0$. The parameter γ determines the pulse width. In this case solution (77) is of the form

$$W(i,n,t)=e^{-\frac{n^2\pi^2}{4b^2 n_0}}e^{-\frac{i^2\pi^2}{4a^2 n_0}}\cos\frac{n\pi x_0}{b}\cos\frac{i\pi z_0}{a}$$
$$\cdot\sqrt{\frac{\pi}{\gamma}}e^{-\frac{v^2}{4\gamma}}\cos v(t-t_0) \tag{78}$$

for $t>t_0$. In numerical calculations we selected the parameters n_0 and i_0 large enough. In this case we deal with a source close to a point source, and the convergence of the method is fully dependent on the smoothness of the signal $f(t)$ in the source.

In the numerical solution of system (43) the choice of the step Δt is determined by the condition

$$\left(v_s\sqrt{\frac{n^2\pi^2}{b^2}+\frac{i^2\pi^2}{a^2}}\right)\Delta t<2. \tag{79}$$

The necessary condition of stability (79) is obtained when the velocity v_s is constant.

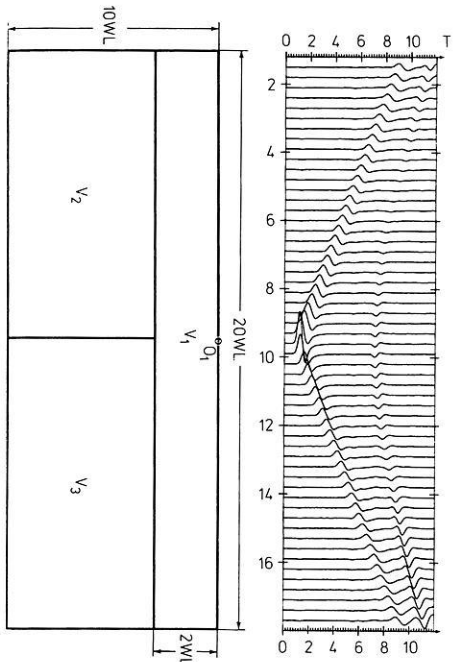

Fig. 2. The model of the medium and theoretical seismograms of SH waves for different epicentral distances. The source empitting SH waves is located at the point O_1. The distances are measured in terms of the predominant wavelength, *WL*, and the time in terms of the predominant period, *T*. ($v_2 = 1.4 v_1$, $v_3 = 2 v_1$).

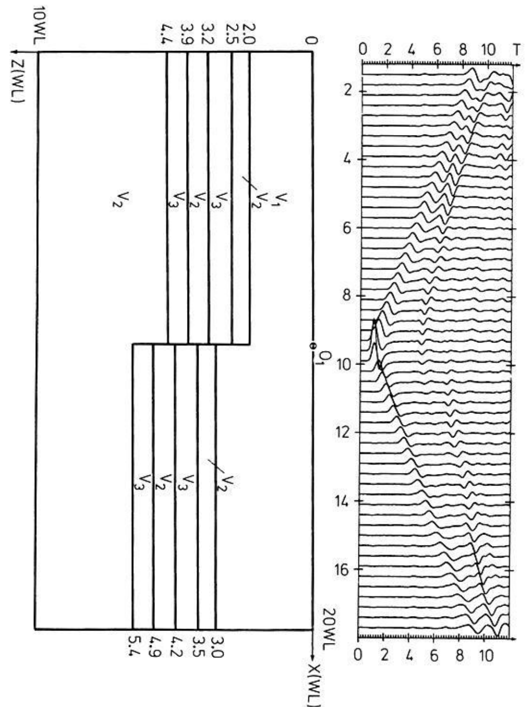

Fig. 3. The model of the medium and theoretical seismograms of SH waves for different epicentral distances. The source empitting SH wave is located at the point O_1. The distances are measured in terms of the predominant wavelength, *WL*, and the time in terms of the predominant period, *T*. ($v_2 = 2 v_1$, $v_3 = 1.5 v_1$).

The total error of the method consists of the truncation error resulting from difference schemes and the error of spatial frequency cut-off in Fourier series. The method was tested against several problems for particular models of inhomogeneous media (see Alford et al., 1974; Alekseev and Mikhailenko, 1980). The results obtained in these papers and the results obtained with the algorithm described here are in satisfactory agreement.

The purpose of this paper is not to study dynamic peculiarities of seismic waves in two-dimensional inhomogeneous media. In what follows we give examples of calculation of synthetic seismograms for some media with non-complex geometries.

Figure 2 shows the model of the medium and theoretical seismograms of SH waves for different epicentral distances at the free surface. The source is located at point O_1. The distances are measured in terms of the predominant wavelength, *WL*, and the time in terms of the period, *T*. The velocities of the medium are $v_2 = 1.4 v_1$ and $v_3 = 2 v_1$. According to Fig. 2, the direct wave from the source is recorded as the first arrivals. After that the wave reflected from the sections with velocities v_2, v_3 is recorded.

Figure 3 shows a more complex model and theoretical seismograms of SH waves for this model and different epicentral distances at the free surface. The source is located at the point O_1. According to Fig. 3, after the direct wave from the source, the waves reflected from the flanks of the fault and later turning into refracted waves are recorded. In addition, here one can also see multiple reflections.

Conclusions

An algorithm for the calculation of synthetic seismograms for complex subsurface geometries by a combination of finite integral Fourier transforms and finite difference techniques has been presented. Two versions of the method for the solution of the SH-wave equation and the elastodynamic wave equation have been described.

One of the advantages of this algorithm is that it allows one to compute synthetic seismograms for maximally complex sub-surface geometries, including inhomogeneities which are much less than the predominant wavelength. At the same time the complexities of the geometry do not substantially increase the computing costs. Moreover, the computing costs could be reduced significantly by performing the computations on an array processor.

References

Alekseev, A.S., Mikhailenko, B.G.: Solution of Lamb's problem for a vertically inhomogeneous half-space (in Russian). Izv. Akad. Nauk SSSR, Fizika Zemli **12**, 11–25, 1976

Alekseev, A.S., Mikhailenko, B.G.: Method of calculation of theoretical seismograms for complex models of media (in Russian). Dokl. Akad. Nauk SSSR **240**, 1062–1065, 1978

Alekseev, A.S., Mikhailenko, B.G.: Solution of dynamic problems of elastic wave propagation in inhomogeneous media by a combination of partial separation of variables and finite difference methods. J. Geophys. **48**, 161–172, 1980

Alford, R.M., Kelly, K.R., Boore, D.M.: Accuracy of finite-difference modeling of acoustic wave equations. Geophysics **39**, 834–841, 1974

Fornberg, B.: On a Fourier method for the integration of hyperbolic equations. SIAM J. Num. Anal. **12**, 509–528, 1975

Gazdag, J.: Numerical convective schemes based on the accurate computation of space derivatives. J. Comput. Phys. **13**, 100–113, 1973

Gazdag, J.: Modeling of the acoustic wave equation with transform methods. Geophysics **46**, 854–859, 1981

Gold, B., Roder, C.M.: Digital processing of signals. McGraw-Hill. New York: Book Company 1969

Kosloff, D., Baysal, E.: Forward modeling by a Fourier method. Geophysics **47**, 1402–1412, 1982

Kreiss, H.O., Oliger, J.: Comparison of accurate methods for the integration of hyperbolic equations. Tellus **24**, 199–215, 1972

Merilees, P.E., Orszag, S.A.: The pseudospectral method; Numerical methods used in atmospheric models. II, Garp Publications Series, no. **17**, 1979

Mikhailenko, B.G.: Numerical solution of Lamb's problem for inhomogeneous half-space (in Russian). In: Mathematical Problems in Geophysics, Vol. 4: pp. 273–297. Computing Center of the Siberian Branch of the Soviet Academy of Sciences, Novosibirsk, 1973

Mikhailenko, B.G.: Calculation of theoretical seismograms for multi-dimensional medium models (in Russian). Uslovno-korrektnye zadachi matematicheskojj fiziki *v* interpretacii geofizicheskikh nabljudenijj, Izv. VTs SO AN SSSR, Novosibirsk, 75–89, 1978

Mikhailenko, B.G.: The method of solution of dynamic seismic problems for two-dimensional inhomogeneous media models (in Russian). Dokl. Akad. Nauk SSSR **246**, 47–51, 1979

Mikhailenko, B.G., Korneev, V.I.: Numerical simulation of seismic fields in two-dimensional inhomogeneous media (in Russian). In: Simulation of wave fields (Mathematical Problems in Geophysics): pp. 273–297. Computing Center of the Siberian Branch of the Soviet Academy of Sciences, Novosibirsk, 1983

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