J Geophys (1984) 54:230–231

Journal of Geophysics

Short communications

A note concerning the interpretation of the $\mathbf{B}_{\|}$-component in geomagnetic pulsation data

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Key words: Geomagnetic micropulsations – MHD wave theory

Introduction

Recent studies on geomagnetic pulsations are frequently based on the comparison of ground based measurements with satellite observations of the magnetic pulsation field near the geomagnetic equator in the outer magnetosphere. It is widely accepted that geomagnetic pulsations are related to magnetohydrodynamic (MHD) waves propagating in the magnetospheric plasma. Therefore, the interpretation of geomagnetic pulsation data is often terms of special modes of the propagation of MHD waves. However, the uncritical transfer of concepts originating in the study of MHD waves under *uniform* conditions to those applying to *non-uniform* conditions sometimes leads to quite obscure results. In particular, a non-vanishing magnetic wave field component $B_{\|}$, parallel to the ambient magnetic field $\mathbf{B}_0$, is often referred to as the so-called "compressional mode" (see, for example, Hillebrand et al. (1982) and references quoted there). It is the aim of this short paper to show that there exist contributions to $B_{\|}$ caused by *purely geometrical effects* in the presence of a non-uniform magnetic (e.g., dipole) background field which have nothing to do with special kinds of propagation modes.

Basic equations

The starting-point of the following analysis is to relate the magnetic wave field $\mathbf{B}$ to the plasma velocity $\mathbf{v}$. In a linearized theory, this can be accomplished by combining the induction law with Ohm's law (for infinite electrical conductivity) yielding (curl $\mathbf{B}_0=0$, $\dot{\mathbf{B}}_0=0$)

$$-i\omega\mathbf{B}=\operatorname{curl}(\mathbf{B}_0\times\mathbf{v}) \tag{2.1}$$

assuming a periodic time dependence $\propto\exp(i\omega t)$. Now, let ($\mathbf{r}$ = position vector)

$$\mathbf{B}_0(\mathbf{r})=B_0(\mathbf{r})\,\hat{\mathbf{t}}(\mathbf{r})$$

where $B_0(\mathbf{r})$ is the magnitude and $\hat{\mathbf{t}}(\mathbf{r})$ is the unit tangential vector with respect to the ambient field $\mathbf{B}_0(\mathbf{r})$. By means of a well known formula of vector analysis, Eq. (2.1) can be performed to obtain

$$-i\omega\mathbf{B}=B_0\operatorname{curl}(\hat{\mathbf{t}}\times\mathbf{v})+\operatorname{grad}B_0\times(\hat{\mathbf{t}}\times\mathbf{v}). \tag{2.2}$$

Dividing Eq. (2.2) by B_0 and expanding the double vector product yields

$$-i\frac{\omega}{B_0}\mathbf{B}=\operatorname{curl}(\hat{\mathbf{t}}\times\mathbf{v})+\hat{\mathbf{t}}\left(\frac{1}{B_0}\operatorname{grad}B_0\cdot\mathbf{v}\right)-\mathbf{v}\left(\frac{1}{B_0}\operatorname{grad}B_0\cdot\hat{\mathbf{t}}\right). \tag{2.3}$$

For a complete description of the vector fields involved let us introduce the principal unit normal vector $\hat{\mathbf{n}}$ and the unit binormal vector $\hat{\mathbf{b}}$ with respect to $\mathbf{B}_0$. Decomposing the velocity field $\mathbf{v}$ into its components with respect to that local triad $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$ (i.e., $\hat{\mathbf{t}}\times\mathbf{v}=-v_b\hat{\mathbf{n}}+v_n\hat{\mathbf{b}}$) and using the same formula leading to Eq. (2.2), Eq. (2.3) reads

$$-i\frac{\omega}{B_0}\mathbf{B}=v_n\operatorname{curl}\hat{\mathbf{b}}-\hat{\mathbf{b}}\times\operatorname{grad}v_n-v_b\operatorname{curl}\hat{\mathbf{n}}+\hat{\mathbf{n}}\times\operatorname{grad}v_b+\kappa v_n\hat{\mathbf{t}}+\xi v_b\hat{\mathbf{t}}-\eta v_n\hat{\mathbf{n}}-\eta v_b\hat{\mathbf{b}} \tag{2.4}$$

with

$$\eta=\frac{1}{B_0}\operatorname{grad}B_0\cdot\hat{\mathbf{t}},\qquad \kappa=\frac{1}{B_0}\operatorname{grad}B_0\cdot\hat{\mathbf{n}},$$
$$\xi=\frac{1}{B_0}\operatorname{grad}B_0\cdot\hat{\mathbf{b}}.$$

For magnetospheric applications it will be sufficient to restrict ourselves to dipole like magnetic fields $\mathbf{B}_0$, i.e., assuming axial symmetry ($\hat{\mathbf{b}}\cdot\operatorname{grad}B_0=0$) and $\hat{\mathbf{n}}$ always lying in meridional planes ($\hat{\mathbf{t}}\cdot\operatorname{curl}\,\hat{\mathbf{n}}=0$, $\hat{\mathbf{n}}\cdot\operatorname{curl}\,\hat{\mathbf{n}}=0$, $\hat{\mathbf{b}}\cdot\operatorname{curl}\hat{\mathbf{b}}=0$). Then, the components of the magnetic wave field are simply obtained from Eq. (2.4):

$$B_{\|}=\hat{\mathbf{t}}\cdot\mathbf{B}=i\frac{B_0}{\omega}[\hat{\mathbf{n}}\cdot\operatorname{grad}v_n+\hat{\mathbf{b}}\cdot\operatorname{grad}v_b+(\kappa+\hat{\mathbf{t}}\cdot\operatorname{curl}\hat{\mathbf{b}})v_n], \tag{2.5a}$$

$$B_n=\hat{\mathbf{n}}\cdot\mathbf{B}=-i\frac{B_0}{\omega}[\hat{\mathbf{t}}\cdot\operatorname{grad}v_n+(\eta-\hat{\mathbf{n}}\cdot\operatorname{curl}\hat{\mathbf{b}})v_n], \tag{2.5b}$$

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$$B_b = \hat{\mathbf{b}} \cdot \mathbf{B} = -i\frac{B_0}{\omega}[\hat{\mathbf{t}} \cdot \text{grad}\, v_b + (\eta + \hat{\mathbf{b}} \cdot \text{curl}\, \hat{\mathbf{n}}) v_b]. \quad (2.5\,\text{c})$$

The last terms of the r.h.s. of Eqs. (2.5a–c), respectively, are due to the non-uniform geometry of the ambient magnetic field $\mathbf{B}_0$. These terms would vanish under uniform conditions yielding the well known relations between the magnetic field $\mathbf{B}$ and the plasma velocity $\mathbf{v}$ with respect to a homogeneous magnetic field $\mathbf{B}_0$.

Applications to a dipole field

Now, let us apply the quite general results of the preceding section to a dipole field. The following relations, referring to spherical polar coordinates (r, θ, ϕ), allow for determination of all terms involved in Eqs. (2.5a–c) $\left(B_0 \propto \frac{1}{r^3}\sqrt{1+3\cos^2\theta}\right)$

$$\frac{1}{B_0}\text{grad}\, B_0 = -\frac{3}{r}\left[\hat{\mathbf{r}} + \frac{\cos\theta \sin\theta}{(1+3\cos^2\theta)}\hat{\boldsymbol{\theta}}\right],$$

$$\hat{\mathbf{t}} = -\frac{1}{\sqrt{1+3\cos^2\theta}}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}),$$

$$\hat{\mathbf{n}} = -\frac{1}{\sqrt{1+3\cos^2\theta}}(\sin\theta\,\hat{\mathbf{r}} - 2\cos\theta\,\hat{\boldsymbol{\theta}}),$$

$$\hat{\mathbf{b}} = -\hat{\boldsymbol{\phi}},$$

$$\text{curl}\, \hat{\mathbf{n}} = \frac{6\cos\theta(1+\cos^2\theta)}{r(1+3\cos^2\theta)^{3/2}}\hat{\boldsymbol{\phi}},$$

$$\text{curl}\, \hat{\mathbf{b}} = -\frac{1}{r\sin\theta}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}).$$

By means of these equations it is possible to rewrite Eqs. (2.5a–c) with respect to a dipole field:

$$B_{\|} = i\frac{B_0}{\omega} \cdot \left[\hat{\mathbf{n}} \cdot \text{grad}\, v_n + \hat{\mathbf{b}} \cdot \text{grad}\, v_b + \frac{2+6\cos^4\theta}{r\sin\theta(1+3\cos^2\theta)^{3/2}} v_n\right], \quad (3.1\,\text{a})$$

$$B_n = -i\frac{B_0}{\omega}\left[\hat{\mathbf{t}} \cdot \text{grad}\, v_n + \frac{6\cos\theta(1+\cos^2\theta)}{r(1+3\cos^2\theta)^{3/2}} v_n\right], \quad (3.1\,\text{b})$$

$$B_b = -i\frac{B_0}{\omega}\left[\hat{\mathbf{t}} \cdot \text{grad}\, v_b + \frac{3\cos\theta}{r\sqrt{1+3\cos^2\theta}} v_b\right]. \quad (3.1\,\text{c})$$

In Eqs. (3.1a–c) only the "structural quantities" depending on the dipole geometry of the field $\mathbf{B}_0$ have been expressed in spherical polar coordinates. The remaining terms have been rewritten in the very distinctive local triad system of reference. In particular, Eq. (3.1a) shows that there is a contribution to $B_{\|}$ even in the equatorial plane $\left(\theta = \frac{\pi}{2}\right)$ of order $\frac{2}{r}\frac{B_0}{\omega} v_n$ whereas the corresponding quantities in Eqs. (3.1b, c) are zero.

To get an idea of the order of magnitude of this "geometrical" contribution to $B_{\|}$ in the equatorial plane, rewrite $\frac{2}{r}\frac{B_0}{\omega} v_n$ as $\sim \frac{TE_b}{20L}$ (by means of $E_b = B_0 v_n$) with T = period in sec, E_b = azimuthal electric field in mV/m, $L = \frac{r}{a}$, a = earth's radius, yielding the corresponding magnetic field in nT. Taking $T = 100$ s, $E_b = 1$ mV/m, $L = 5$, for example, the "geometrical" contribution to $B_{\|}$ is 1 nT. Therefore, the observation of a large $B_{\|}/B_{\perp}$ ratio in geomagnetic pulsation data does not necessarily enforce an interpretation in terms of a special wave mode but can be regarded as a simple geometrical effect caused by the non-uniform background field $\mathbf{B}_0$.

Conclusions

The above result is only one example for a lot of properties which carefully have to be taken into account in the study of MHD waves under non-uniform conditions. In particular, it has been shown that – even in a cold plasma approach – there exists a contribution to the $B_{\|}$-component caused by the curvature of the ambient magnetic field $\mathbf{B}_0$. This result demonstrates how unrealistic is a treatment of MHD wave motions in terms of plane waves ("Alfvén-mode", "compressional mode") alone. A more complete version of MHD wave theory in terms of the local triad system of reference has been given by Siebert (1965).

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Received September 5, 1983; Revised version February 24, 1984
Accepted February 27, 1984