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Conductivity modelling of the Earth using solar and lunar daily magnetic variations

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Abstract. The theory of Chapman and Whitehead (1923) for the modelling of the conductivity of the Earth and oceans using daily magnetic variations is developed to provide the equation:

$$\kappa_s \delta = \frac{2n+1}{2p} \operatorname{Im} \left(1 - \frac{n+1}{n} \frac{I_0}{E_0}\right)^{-1} 5.4 \times 10^{-4}\, S$$

for the conductivity κ_s in an ocean of thickness $a\delta$, where a is the radius of the Earth. The radius qa of a superconducting core can be obtained from:

$$(1 - q^{2n+1})^{-1} = \operatorname{Re} \left(1 - \frac{n+1}{n} \frac{I_0}{E_0}\right)^{-1}$$

independently of any of the parameters for the outer conducting shell. The expression often used is

$$q^{2n+1} = \frac{n+1}{n} |I_0/E_0|.$$

The theory is applied to solar, lunar, lunar elliptic and seasonal variations of the magnetic field.

Key words: Daily magnetic variations - Electric induction in the Earth - Conductivity modelling

Introduction

If I_i and E_i are complex number forms representing the cosine expressions $i_i \cos(mt + \delta_i)$, $e_i \cos(mt + \varepsilon_i)$, with

$$\operatorname{Re} E_i = e_i \cos \varepsilon_i, \quad \operatorname{Im} E_i = e_i \sin \varepsilon_i, \tag{1}$$

then the equation used for modelling the electrical conductivity of the Earth, with a non-conducting shell surrounding a uniformly conducting core of radius qa, $q \leqq 1$, where a is the radius of the Earth, is given by Chapman and Whitehead (1923) as:

$$\frac{I_i}{E_i} = \frac{n}{n+1} q^{2n+1} \left(1 - \frac{R_n}{R_{n-1}}\right), \tag{2}$$

where n is the degree of the spherical harmonic, $m = n - w$, with $w = 0$ for solar magnetic variations and $w = 1/14.7653$ for lunar magnetic variations. The ratio R_n/R_{n-1} is given in the form:

$$\frac{R_{n-1}}{R_n} = \frac{\beta}{2n+1} \left\{ \left(1 + \frac{n}{\beta} + \frac{n(n+1)}{4\beta^2} + \frac{n(n+1)}{4\beta^3} + \dots\right) + i\left(1 - \frac{n(n+1)}{4\beta^2} + \dots\right) \right\} \tag{3}$$

where

$$\beta^2 = 4\pi^2 \kappa p q^2 a^2/(24.60.60) = 136185\, \kappa p q^2. \tag{4}$$

κ is the conductivity of the uniformly conducting core in Siemens, i.e. $(\text{ohm} \cdot \text{m})^{-1}$. Note that $1S = 10^{-11}$ emu. The number of cycles per day made by the magnetic variation is denoted p, whilst $a = 6{,}371{,}000$ m is the radius of the Earth.

It will be seen that the ratio R_{n-1}/R_n is given as a complex number, with real and imaginary parts being functions of the parameters n and β. The "argument", in the complex variable sense, of the ratio I_i/E_i, is the phase angle difference between the internal and external components of the magnetic variation; for a given n it can be used to determine the "argument" of the expression $1 - R_n/R_{n-1}$, and the corresponding value of the parameter β. From the "amplitudes" of the expressions on both sides of Eq. (2), using the value obtained for β, the parameter q, the radius of the uniformly conducting core relative to an Earth of unit radius is obtained. Because β has previously been obtained, the corresponding conductivity κ is then obtained from Eq. (4).

The so-called superconducting approximation (Banks, 1969) is obtained with $\kappa = \infty$, giving $(1 - R_n/R_{n-1}) = 1$, when

$$q^{2n+1} = \frac{n+1}{n} \left|\frac{I_i}{E_i}\right|. \tag{5}$$

This very simple approximation tends to overestimate q and to underestimate the depth to the conducting core, but, given the uncertainties in the spherical harmonic coefficients for I_i and E_i, is still a useful result. Values for q based on Eq. (3) are given in Tables 1-4.

Uniformly conducting shell and core

If now a uniformly conducting shell, thickness $a\delta$, $0 \leqq \delta \leqq 1$, with conductivity κ_s, is introduced, and if E_0

Table 1. Conductivity models from spherical harmonic coefficients

p	P_n^m	I_0/E_0	q, d ($\kappa_s = 0$)	q, d with conducting shell	κ_s
for the solar magnetic variation S (1964–1965) quiet years					
2.0000	P_3^2	1329 pT, 220° / 2922 pT, 197°	0.9310 439 km	0.8868 721 km	5.7 S
3.0000	P_4^3	650 pT, 48° / 1168 pT, 33°	0.9605 252 km	0.9400 382 km	6.6 S
5.0000	P_5^4	135 pT, 244° / 238 pT, 215°	0.9656 219 km	0.9032 616 km	5.7 S
2.0000	P_2^2	416 pT, 269° / 1102 pT, 268°	0.8925 685 km	0.8924 686 km	0.2 S
for the solar magnetic variation S (1958–1959) disturbed years					
2.0000	P_3^2	2500 pT, 198° / 5953 pT, 182°	0.9205 507 km	0.9022 623 km	3.9 S
3.0000	P_4^3	955 pT, 69° / 2144 pT, 16°	0.9370 401 km	0.6947 1945 km	3.5 S
4.0000	P_5^4	195 pT, 232° / 420 pT, 209°	0.9482 330 km	0.9233 489 km	3.6 S
2.0000	P_2^2	539 pT, 295° / 1307 pT, 254°	0.9084 584 km	0.6913 1966 km	3.9 S

Table 2. Conductivity models from spherical harmonic coefficients

p	P_n^m	I_0/E_0	q, d ($\kappa_s = 0$)	q, d with conducting shell	κ_s
for the principal lunar magnetic tide $L(2s-2h)$ 1964–1965 quiet years					
1.9323	P_3^2	214 pT, 291° / 254 pT, 261°	1.0168 ...	−1.4021 ...	11.1 S
2.9323	P_4^3	101 pT, 114° / 134 pT, 93°	0.9934 42 km	−0.7408 ...	13.9 S
3.9323	P_5^4	26 pT, 291° / 42 pT, 272°	0.9733 170 km	0.9399 383 km	7.9 S
0.9323	P_4^1	33 pT, 309° / 117 pT, 245°	0.8906 697 km	0.6907 1971 km	6.5 S
for the principal lunar magnetic tide $L(2s-2h)$ 1958–1959 disturbed years					
1.9323	P_3^2	220 pT, 291° / 497 pT, 273°	0.9274 462 km	0.9019 615 km	5.0 S
2.9323	P_4^3	96 pT, 112° / 189 pT, 101°	0.9508 314 km	0.9420 369 km	4.1 S
3.9323	P_5^4	22 pT, 281° / 39 pT, 266°	0.9651 222 km	0.9494 322 km	5.6 S
0.9323	P_4^1	97 pT, 285° / 151 pT, 270°	0.9759 154 km	0.9418 371 km	36.9 S

Table 3. Conductivity models from spherical harmonic coefficients

p	P_n^m	I_0/E_0	q, d ($\kappa_s = 0$)	q, d with conducting shell	κ_s
for $S^+(h)$, a component of the seasonal change of S (1964–1965)					
1.0027	P_1^1	924 pT, 285° / 2894 pT, 276°	0.8611 885 km	0.8542 987 km	3.5 S
2.0027	P_2^2	523 pT, 150° / 1325 pT, 147°	0.9005 634 km	0.8995 640 km	0.8 S
3.0027	P_3^3	295 pT, 364° / 711 pT, 354°	0.9189 517 km	0.9119 561 km	1.8 S
4.0027	P_4^4	103 pT, 219° / 206 pT, 214°	0.9491 324 km	0.9474 335 km	1.4 S
for $S^-(h)$, a component of the seasonal change of S (1964–1965)					
0.9973	P_1^1	738 pT, 126° / 2535 pT, 117°	0.8350 1051 km	0.8221 1134 km	2.5 S
1.9973	P_2^2	376 pT, 0° / 923 pT, 11°	0.9062 598 km	0.8924 686 km	−2.9 S
2.9973	P_3^3	207 pT, 230° / 534 pT, 224°	0.9079 587 km	0.9057 601 km	0.9 S
3.9973	P_4^4	116 pT, 99° / 248 pT, 95°	0.9421 369 km	0.9411 375 km	0.9 S

Table 4. Conductivity models from spherical harmonic coefficients

p	P_n^m	I_0/E_0	q, d ($\kappa_s = 0$)	q, d with conducting shell	κ_s
for the principal lunar elliptic magnetic tide $L(3s-2h-p)$ 1964–1965					
1.8960	P_3^2	49 pT, 287° / 84 pT, 284°	0.9647 225 km	0.9632 234 km	5.0 S
for the $L(2s-3h)$, a component of the seasonal change of $L(2s-2h)$ 1964–1965					
0.9350	P_1^1	77 pT, 345° / 218 pT, 317°	0.8906 697 km	0.6920 1962 km	7.3 S
1.9350	P_2^2	112 pT, 148° / 268 pT, 148°	0.9108 568 km	0.9108 568 km	0.0 S
2.9350	P_3^2	55 pT, 356° / 122 pT, 334°	0.9299 447 km	0.8901 700 km	3.7 S
1.9350	P_4^2	27 pT, 169° / 67 pT, 162°	0.9266 467 km	0.9243 482 km	1.9 S
2.9350	P_5^3	25 pT, 364° / 41 pT, 333°	0.9720 178 km	0.8804 762 km	8.6 S

and I_0, are the complex representations of magnetic tides as observed at sea level, then the corresponding E_i and I_i, being the complex representations of magnetic tides at the undersurface of the conducting shell, are given by Chapman and Whitehead (1923) in the form:

$$I_i = I_0 + \frac{2i\beta_s^2}{2n+1}\,\delta\left(I_0 - \frac{n}{n+1}E_0\right),$$
$$E_i = E_0 + \frac{2i\beta_s^2}{2n+1}\,\delta\left(\frac{n+1}{n}I_0 - E_0\right), \tag{6}$$

where

$$\beta_s^2 = 4\pi\kappa_s p a^2/(24.60.60) = 136185\,\kappa_s p. \tag{7}$$

By eliminating I_i and E_i, from Eqs. (2) and (6), it is found that the product $\kappa_s\delta$ for a uniformly conducting shell representing the ocean can be determined from:

$$\kappa_s\delta = \frac{2n+1}{2p}\,i\left\{\frac{1}{1-q^{2n+1}\left(1-\frac{R_n}{R_{n-1}}\right)} - \frac{1}{1-\frac{n+1}{n}\frac{I_0}{E_0}}\right\} \cdot 5.4\times10^{-4} \tag{8}$$

in units of Siemens. Only local-time-dependent terms, i.e. terms dependent on $(t+\phi)$, where t is universal time and ϕ is east longitude, will be applied to this theory. Such terms are not able to distinguish non-local time effects, such as those associated with the geographical distribution of land and ocean. Non-local time terms are however available in the magnetic variation analyses of both Malin (1973) and Winch (1981). A uniformly conducting shell 1,000 m thick, corresponding to $\delta = 1{,}000/6{,}371{,}00$, has been used in Tables 1–4 to give numerical values for the conductivity κ_s of such a shell.

If the equation:

$$\frac{I_0}{E_0} = \frac{n}{n+1}\,q^{2n+1}\left(1-\frac{R_n}{R_{n-1}}\right) \tag{9}$$

is satisfied, then it follows from Eq. (8) that κ_s, the conductivity of the shell representing the oceans, is zero, i.e. Eq. (9) is the appropriate equation when the conducting shell is not included.

The expression given in Eq. (8) for κ_s has both real and imaginary components, although only the real component is physically significant. The condition that the imaginary part of κ_s should be zero is:

$$\mathrm{Re}\,\frac{1}{1-q^{2n+1}\left(1-\frac{R_n}{R_{n-1}}\right)} = \mathrm{Re}\,\frac{1}{1-\frac{n+1}{n}\frac{I_0}{E_0}}. \tag{10}$$

Equation (10) for the superconducting core model, corresponding to $(1-R_n/R_{n-1})=1$, reduces to:

$$\frac{1}{1-q^{2n+1}} = \mathrm{Re}\,\frac{1}{1-\frac{n+1}{n}\frac{I_0}{E_0}}. \tag{11}$$

Equation (11), provides values of q, the relative depth of the superconducting core in the presence of a conducting shell without regard to the parameters of the conducting shell, using only the ratio of internal and external magnetic variations.

With the superconducting core approximation, Eq. (8) can be used to give the conductivity of a uniformly conducting shell in the form:

$$\kappa_s\delta = \frac{2n+1}{2p}\,\mathrm{Im}\left(\frac{1}{1-\frac{n+1}{n}\frac{I_0}{E_0}}\right) 5.4\times10^{-4}\,S. \tag{12}$$

Values of q and the corresponding depth to the conducting core are derived from Eqs. (5) and (11), i.e. with and without the conducting shell, and the conductivity κ_s of the conducting shell (assumed to be of thickness 1,000 m, i.e. $\delta = 1{,}000/6{,}371{,}000$) are given in the tables.

Discussion

Tables 1–4 contain results of analyses using Eqs. (5), (11) and (12), applied to spherical harmonic coefficients given by Winch (1981) for solar and lunar magnetic tides, their seasonal variations and the lunar elliptic magnetic tide for years of low sunspot number 1964–1965, and also the spherical harmonic coefficients for solar and lunar magnetic tides given by Malin (1973) for years of high sunspot number 1958–1959.

Daily magnetic variations have always been regarded as difficult to use in induction studies, and indeed the present paper is an attempt to see if the situation might be remedied by developments in the theory. The difficulties still remain, and the 1 cycle per day (cpd) terms remain difficult to use, as the phase angle of the internal induced component is often close to, or in advance of, the phase angle of the external inducing field. At 2 cpd there is also the possibility of direct dynamo action, i.e. tidal movements of solar or lunar origin in the oceans acting directly as a dynamo and contributing directly to the internal component of the magnetic variation. The ocean dynamo effect calculation (Malin, 1977) has been applied to the 2-cpd lunar terms. A difficulty arises with the lunar magnetic tides in that the internal component of the principal lunar magnetic tide $L(2s-2h)$ remains virtually constant from years of high to years of low sunspot number. The effect can be confirmed at individual observatories, by noting that Fourier coefficients for $L(2s-2h)$ are diminished from years of high to years of low sunspot number in a much smaller ratio than the corresponding solar magnetic variation.

In all four tables, the conductivity κ_s has been derived from Eq. (12) for $\kappa_s\delta$ with the assumption that $\delta = 1/6371$ corresponding to an ocean 1 km in depth. The conductivity of sea water varies from 1.5–5 S, depending primarily on temperature (Cox et al. 1970), and it will be seen that the conductivities given are not inconsistent with that range, although some are higher and some are lower.

Table 1 gives conductivity models and depths to the superconducting core based on spherical harmonic coefficients for the internal and external parts of the solar magnetic variation S. The depths to the superconducting core are found to be increased by the use of

Eq. (11) which makes allowance for the presence of the conducting shell, but is independent of the parameters of the conducting shell. Conductivities of the shell given by Eq. (12) are found to be greater in years of high sunspot number.

Table 2 gives the corresponding results based on spherical harmonic coefficients for the principal lunar magnetic tide $L(2s-2h)$. The spherical harmonic coefficients are approximately one-tenth of those for the solar magnetic tide and are therefore more uncertain. The invariance of the internal component with respect to sunspot number makes the interpretation of results for 1964–1965 all the more difficult. Results for conductivity for 1958–1959, for P_3^2, P_4^3, P_5^4, are comparable with corresponding results from the solar magnetic tide.

Table 3 derives models from the equator-symmetric sectorial spherical harmonic coefficients for the annual variation sum S^+ and difference S^- sidebands of the solar magnetic variation S. Conductivities are smaller than those from either S or $L(2s-2h)$, and the 2-cpd result for P_2^2 in S^- is anomalous, and may be associated with an ocean dynamo K_2.

Table 4 gives the results of analysis of the 2-cpd term of the lunar elliptic tide $L(3s-2h-p)$, and also of the seasonal sideband $L(2s-3h)$ of the principal lunar magnetic tide $L(2s-2h)$. Results for the other seasonal sideband $L(2s-h)$ are not included due to difficulties associated with direct dynamo action of the tide O_1. The angular distance of the mean moon from perigee is given by $(s-p)$, and $3s-2h-p=(2s-2h)-(s-p)$, which shows how the tide depends on the lunar angular distance from perigee, and therefore, due to the elliptic orbit, on distance from the Earth. The conductivities and depths to the superconducting core are in general agreement with results given in Tables 1–3.

Note added in proof

The product of the atmospheric tide M_2 in the form:

$$M_2=0.90812\,G_2\cos(2t-2s+2h)$$

and the ionospheric conductivity:

$$\kappa=\sum_{n=0}^{4}\kappa_n\sin(nt+\varepsilon_n)$$

produces the appropriate mathematical form for the time dependence of the magnetic tide or variation which is usually denoted $L(M_2)$ and denoted $L(2s-2h)$ in the present paper. Similarly, $L(O_1)$ and $L(N_2)$ are here denoted $L(2s-h)$ and $L(3s-2h+p)$, respectively.

However, the ionospheric conductivity varies with season, i.e. with h, producing further sum and difference frequencies, which can be denoted systematically, as, for example

$L(2s-3h)$ and $L(2s-h)$ of the magnetic tide $L(2s-2h)$.

Therefore, $L(2s-h)$ as computed is best interpreted as a seasonal component of the lunar magnetic tide $L(2s-2h)$. To denote it $L(O_1)$ as it sometimes is, is quite misleading, because there is only a very small part of $L(2s-h)$ which derives from an ionospheric dynamo associated with the atmospheric tide O_1.

Summary

Spherical harmonic coefficients for daily magnetic variations summarize the results of some millions of observations, and it is somewhat disappointing to find that such coefficients are of limited value in induction studies. The present study has developed the theory of Chapman and Whitehead (1923) to provide estimates of conductivity of a shell representing the oceans, and depths to a superconducting core using spherical harmonics from the standard solar and lunar magnetic variation analyses and also seasonal sidebands of these variations. An expression for depth to the superconducting core has been given which is independent of the parameters of the conducting shell.

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