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The polarization of S waves in a heterogeneous isotropic Earth model

V.F. Cormier

Earth Resources Laboratory, Department of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract. The polarization vector of an S wave propagating in a heterogeneous isotropic medium remains fixed with respect to a vector basis $(\hat{\mathbf{t}}, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$, where $\hat{\mathbf{t}}$ is the unit tangent to the ray at a point on the ray and $\hat{\mathbf{t}}, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ are mutually orthogonal. This vector frame forms the basis of the orthogonal co-ordinate frame, in which the ray is one co-ordinate line. It is not generally equivalent to the Frénet frame $(\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}})$. At any point along a ray in a heterogeneous medium, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ may have rotated about $\hat{\mathbf{t}}$ by an angle Θ with respect to the natural $SH(\hat{\varepsilon}_1)$ and $SV(\hat{\varepsilon}_2)$ directions at that point. In a geocentric, spherical co-ordinate system the derivative of Θ with respect to travel time τ along the ray in a medium with S velocity $v = v(r, \vartheta, \varphi)$ is

$$\frac{d\Theta}{d\tau} = \frac{\cot i}{r}\left[\sin\zeta\,\frac{\partial v}{\partial\vartheta} - \frac{\cos\zeta}{\sin\vartheta}\,\frac{\partial v}{\partial\varphi}\right],$$

where the incidence angle i and azimuthal angle ζ are defined at an instantaneous point on the ray, and where $v(r, \vartheta, \varphi)$ is a sufficiently continuous and slowly varying function for zeroth order ray theory to be applicable. At the source, one may set $\Theta = 0$, $\hat{\mathbf{e}}_1 = \hat{\varepsilon}_1$ and $\hat{\mathbf{e}}_2 = \hat{\varepsilon}_2$. At any other point along the ray, the directions of $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ can be determined from the value of Θ and the directions of $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ at that point. $\hat{\varepsilon}_1, \hat{\varepsilon}_2$ are uniquely defined using the instantaneous tangent vector $\hat{\mathbf{t}}$ and radius vector $\hat{\mathbf{r}}$ to the point. The equations for ray trajectory and Θ are integrated through the three-dimensional velocity perturbation surrounding a descending slab. The results show that realistic three-dimensional structures will rarely generate more than 10° deviations in the orientation of the S polarization vector compared to that observed in a radially homogeneous Earth. This agrees with the general success in the use of S polarizations for focal mechanism solutions. It also suggests that larger deviations and the progressive complexity of the particle motion ellipse observed as time advances in the S waveform are primarily the consequences of phase interference and/or shear wave splitting due to anisotropy.

Key words: S-wave polarization – Ray tracing – Lateral heterogeneity

Introduction

The next generation of whole Earth models will likely include both lateral heterogeneity as well as general anisotropy. The effects of general anisotropy on the splitting and three-component particle motion of S waves are well known (e.g. Crampin, 1981), but the competing effects of lateral heterogeneity in an isotropic Earth model have generally been ignored.

Classical treatments start with Frénet's formulas for the path derivatives of three mutually orthogonal unit vectors $(\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}})$, where $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{b}}$ are respectively the tangent, normal and binormal at a point on a seismic ray. Frénet's formulas show that the S particle motion, as referred to the $(\hat{\mathbf{n}}, \hat{\mathbf{b}})$ directions, rotate around the ray at a rate equal to the local torsion T. Since this rate is equal but of opposite sign to the rate at which the $(\hat{\mathbf{n}}, \hat{\mathbf{b}})$ axes themselves rotate about the ray, it is assumed that the S polarization is not substantially changed through smoothly varying structures. A basic difficulty in discussing the rotation of the Frénet frame, however, is that the rotation only has good meaning if it is referred to some specified orientation. An example derivation and brief discussion of these results is given in Aki and Richards (1980), where they note that the assumption of essentially fixed SH/SV ratios for propagation through smoothly varying structures "has not thoroughly been investigated."

Červený and Hron (1980) have pointed out difficulties in deriving a ray series solution for S waves using the Frénet frame. Determination on $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ requires computation of the ray torsion T and curvature K. Determination of the amplitude coefficients (geometrical spreading) of S waves is difficult because the components of the displacement vector couple into both the $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ directions even in the zeroth order approximation of the asymptotic ray series. Pšenčík (1979) and Červený and Hron instead recommend seeking a ray series solution in a new *ray centred* co-ordinate system $(\hat{\mathbf{t}}, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$. This system has several important advantages. The displacement vector of S waves remains fixed with respect to $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ at all times. In a three-dimensionally varying medium, the orientation of the S displacement can be determined at any point on the ray by simply integrating a differential equation for the orientation of the $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ frame. Although $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ can be expressed in terms of the Frénet frame, one does not actually need to calculate $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ or T and K. In the $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ system the transport equations of both principal components of S waves are independent of one another to zero asymptotic order.

In this paper, the ray tracing equation for the polarization vector of S waves is derived using the $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ frame and a spherical co-ordinate system appropriate for a whole Earth model. An example calculation is given, integrating the S ray tracing equations through the three-dimensional velocity perturbation of a descending slab. This example is chosen because it represents a structure with possibly the largest deviation from radial symmetry in terms of scale length and percent velocity fluctuation likely to be encountered by a teleseismic ray. As such, it can be used to predict the maximum order of deviation of the S polarization vector from its orientation in a radially symmetric Earth.

Ray tracing equation for the polarization of S waves

Following Popov and Pšenčík (1976; 1978) and Červený and Hron (1980), one can introduce unit vectors $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ and $\hat{\mathbf{t}}$. $\hat{\mathbf{t}}$ is tangent to the ray. $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ can be chosen arbitrarily as long as $\hat{\mathbf{t}}$, $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ are mutually orthogonal and form a right-handed system.

The vectors $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ are related to the basis vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ of the ray centred co-ordinate system by rotation through an angle Θ:

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \cos\Theta\,\hat{\varepsilon}_1 - \sin\Theta\,\hat{\varepsilon}_2, \\ \hat{\mathbf{e}}_2 &= \sin\Theta\,\hat{\varepsilon}_1 + \cos\Theta\,\hat{\varepsilon}_2. \end{aligned} \tag{1}$$

The S polarization vector remains fixed with respect to the $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ vectors. The angle Θ describes the rotation of the vector basis $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ with respect to $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ about the central ray as the ray propagates through the heterogeneous medium. Thus the S polarization at any point along the ray can be determined by knowing the initial direction of S polarization, the value of Θ at that point and the orientation of $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ at that point.

The variation of Θ along a ray can be expressed by a differential equation in the co-ordinates commonly used in the tracing of rays through complete models of a spherical earth. These are the fixed geocentric co-ordinates (r, ϑ, φ) shown in Fig. 1 and the incidence angle i and azimuthal angle ζ defined at a local point on the ray. i is the angle between the ray direction and the $\hat{\mathbf{r}}$ direction; ζ is the angle between $\hat{\vartheta}$ and the projection of the ray path into the $\vartheta\varphi$ plane. The vectors $\hat{\mathbf{t}}$, $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ can be expressed by components in the (r, ϑ, φ) system as

$$\begin{aligned} \hat{\mathbf{t}} &= (\cos i, \sin i \cos\zeta, \sin i \sin\zeta), \\ \hat{\varepsilon}_1 &= (0, -\sin\zeta, \cos\zeta), \\ \hat{\varepsilon}_2 &= (\sin i, -\cos i \cos\zeta, -\cos i \sin\zeta). \end{aligned} \tag{2}$$

With this choice, the system $(\hat{\mathbf{t}}, \hat{\varepsilon}_1, \hat{\varepsilon}_2)$ is mutually orthogonal, right-handed, and $\hat{\varepsilon}_1$ is perpendicular to the vector $\hat{\mathbf{r}}$. In a radially symmetric Earth, $\hat{\varepsilon}_1$ is in the SH direction and $\hat{\varepsilon}_2$ is in the SV direction. In a heterogeneous Earth, $\hat{\varepsilon}_1$, $\hat{\varepsilon}_2$ instantaneously define SH and SV directions along a ray with respect to the local tangent to the ray and the radius vector $\hat{\mathbf{r}}$. These directions are not, however, the SH and SV directions inferred from the projection of the ray into the plane defined by the source, receiver and centre of the Earth.

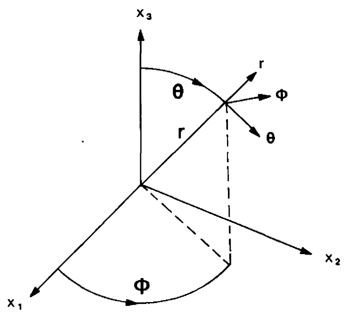

Fig. 1. Definition of the co-ordinate system (r, ϑ, φ) and $\hat{\mathbf{r}}, \hat{\vartheta}, \hat{\varphi}$

An equation for the derivative $\frac{d\Theta}{d\tau}$ with respect to travel time τ along a ray can now obtained by comparing either of two equations for $\frac{d\hat{\mathbf{e}}_1}{d\tau}$ or $\frac{d\hat{\mathbf{e}}_2}{d\tau}$. The first such equation is obtained by differentiating Eq. (1) with respect to τ:

$$\frac{d\hat{\mathbf{e}}_1}{d\tau} = \frac{d\hat{\varepsilon}_1}{d\tau}\cos\Theta - \frac{d\hat{\varepsilon}_2}{d\tau}\sin\Theta - (\hat{\varepsilon}_1 \sin\Theta + \hat{\varepsilon}_2 \cos\Theta)\frac{d\Theta}{d\tau}. \tag{3}$$

The second equation is a property of the ray centred co-ordinate system:

$$\frac{d\hat{\mathbf{e}}_1}{d\tau} = (\nabla v \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{t}}. \tag{4}$$

By equating the right hand side of Eq. (3) to that of Eq. (4), using the rules of differentiation given in Popov and Pšenčík (1976; 1978), the ray tracing equations for $\frac{di}{d\tau}$ and $\frac{d\zeta}{d\tau}$ and relations between spherical and Cartesian geocentric co-ordinates, the following result is obtained:

$$\frac{d\Theta}{d\tau} = \frac{\cot i}{r}\left[\sin\zeta\frac{\partial v}{\partial\vartheta} - \frac{\cos\zeta}{\sin\vartheta}\frac{\partial v}{\partial\varphi}\right]. \tag{5}$$

Equation (5) can now be integrated along with the other five equations (e.g. Julian and Gubbins, 1977; Aki and Richards, 1980) needed to determine the ray trajectory.

Example: the effects of a descending slab

As a test of the effects of laterally heterogeneous structure on the polarization of S waves, Eq. (5) was integrated for sources located in a three-dimensional velocity structure surrounding a descending slab. A 7% positive perturbation was added to the the S velocities of the 1-Hz isotropic PREM model (Dziewonski and Anderson, 1981). The perturbation was taken to be a Gaussian shaped zone centred on a dipping plane. An additional exponential factor controlled its decay with depth. This simple perturbation has been successfully used to predict the order of the deviations of ray paths caused by slab structures (Toksöz et al., 1971). The parameters of the model can be easily modified to closely reproduce the velocity structure predicted by complete thermal models. Figure 2 shows the results of the calculation for the rotation angle Θ for a 540-km-deep earthquake occurring in a slab. The slab strike and dip was chosen to approximately model the Benioff zone dipping beneath the Kuril-Kamchatka arc. The initial take-off angle was held constant and the resulting epicentral distance varied between 85° to 110° for *SKS* waves. The results in Fig. 2 are for a 40-km-thick slab, which penetrates to the centre of the Earth. This experiment was designed to see how much of the *S* wave that leaves the source as *SH* can be rotated into *SV* by propagation through the three-dimensional structure surrounding the slab and then complete its path to the receiver as an *SKS* wave. The motivation for performing this experiment was to explain observations of *SKS* waves on both *SH* and *SV* components at azimuths for which focal mechanism solutions predict only small amounts of *SV* leaving the source (Cormier, 1984).

From Fig. 2 it is seen that this model rotates the initial orientation of the *S* particle motion vector by as much as ± 0.2 rad ($\pm 12°$) for azimuths within 40° of the trend of the strike of the arc on the down-dip side. The rotation vanishes at azimuths perpendicular to the strike of the slab because the partial derivatives $\frac{\partial v}{\partial \vartheta}$ and $\frac{\partial v}{\partial \varphi}$ vanish in this ray direction. A shorter slab, which terminates at 1000 km depth, rotates the polarization angle by only about one-half as much, i.e. ± 0.1 rad ($\pm 6°$).

The complete effects of a slab structure on the apparent *S* polarization must include the deviations it introduces in the ray path as well as the rotation of the *S* polarization vector. The intial *S* radiation that reaches the receiver leaves the source at a take-off angle and azimuth that differs from the angles calculated assuming a radially symmetric Earth. The polarization observed at a teleseismic station will be a sum of these two effects (Fig. 3).

Stated mathematically, consider the *S* wave radiated by a point double couple at initial angles ζ_0, i_0 that reaches a particular receiver in a radially symmetric Earth. This receiver observes an *S* polarization vector described by an angle Φ in a plane perpendicular to the ray, where

$$\Phi = \tan^{-1} \frac{F^{SV}(\zeta_0, i_0)}{F^{SH}(\zeta_0, i_0)}, \tag{6a}$$

with F^{SV} and F^{SH} taken to be the far-field *SV* and *SH* radiation patterns functions of the double couple [e.g. p 115 of Aki and Richards (1980)]. The actual *S* polarization observed by a receiver in a radially heterogeneous Earth will be radiated from the source at different initial angles ζ'_0, i'_0, and this initial *S* polarization will be rotated by an angle Θ during propagation to the receiver. The *S* polarization observed at the receiver is then given by

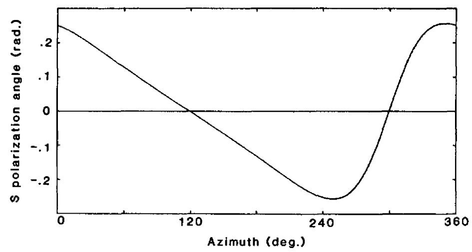

Fig. 2. The orientation of the *S* polarization and the vector basis $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ rotates about the ray as the ray traverses the three-dimensional velocity perturbation of a descending slab. The slab is assumed to have a +7% velocity contrast relative to a reference radially symmetric Earth (PREM). In the vicinity of the slab the velocity is assumed to be given by the form $v_0(r) = v_0(r)\left\{1 + 0.07 \exp\left[-\left(\frac{\xi}{h'}\right)^2 - \frac{z}{d}\right]\right\}$, where ξ is the distance from the axial plane of the slab. The example shown here is for a slab striking at azimuth 40° and dipping 68° to the NW. The initial take-off angle of a 540-km-deep source was held constant and is appropriate for *SKS* waves recorded at a distance of 75° in PREM. $h' = 40$ km and $d = \infty$. For $d = 1000$ km, the maximum deviation of the *S* polarization angle is about one-half as large

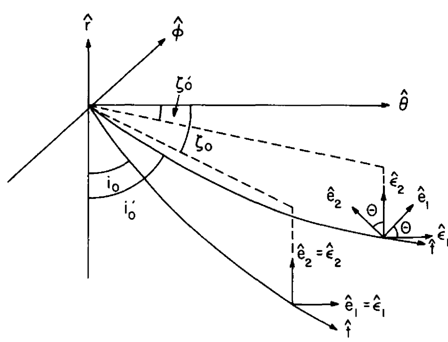

Fig. 3. The *S* polarization observed at a teleseismic receiver is affected in two ways by heterogeneous structure in the vicinity of the source. (1) Since the initial orientation of the *S* polarization vector depends on where the ray leaves the focal sphere, it must be calculated using angles i'_0, ζ'_0 of the ray that has been perturbed by the heterogeneous structure. (2) The initial *S* polarization vector remains fixed with respect to the vector basis $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, but propagation of the ray through the heterogeneous region rotates this vector basis by an angle Θ with respect to the instantaneous *SH* and *SV* directions $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$

$$\Phi = \tan^{-1} \frac{-F^{SH}(\zeta'_0, i'_0)\sin\Theta + F^{SV}(\zeta'_0, i'_0)\cos\Theta}{F^{SH}(\zeta'_0, i'_0)\cos\Theta + F^{SV}(\zeta'_0, i'_0)\sin\Theta}. \tag{6b}$$

Test calculations with slab models show that the effect of path deviation generally does not cancel the effect of rotation of the particle motion vector. Integrated deviations of all angles through the heterogeneous source region are about the same order of magnitude. i_0, ζ_0 and Θ all exhibit a maximum order of deviation of about 10° in the longest slab models. These deviations are too small to grossly bias measurements of the *S* polarization vector. Since the effects of lateral velocity heterogeneity on *S* polarization require either or both strong lateral velocity gradients or lateral velocity gradients maintained over a long portion of the ray path, the slab tests can be used as an estimate of the maximum deviations in S polarization that may be practically observed on the Earth. Thus the small deviations found in the slab tests agree with the general success (Stauder, 1962) of the use of S polarizations in the determination of focal mechanism solutions.

Discussion and conclusions

Although the deviation of ζ_0 can be large (10^0 or more) in a heterogeneous source region, its effect on the apparent azimuth at the receiver is very small (less than several 0.1° s). This effect can be demonstrated by ray tracing and is shown schematically in Fig. 4. It also agrees with the behaviour of azimuth and slowness anomalies observed at large aperture arrays such as LASA and NORSAR (Berteussen, 1976). Thus if heterogeneity primarily exists in the source region and further back along the ray path, S waves can be accurately rotated into SH and SV components using the true great circle azimuth. The observed S polarization vector, however, may deviate as much as 10° from that predicted using the true focal mechanism because of the rotation of the S polarization vector.

Heterogeneity in the receiver region can affect both the apparent azimuth and the orientation of the S polarization vector. In this case, the SH and SV motion can be defined with respect to the local tangent to the ray. S motion can be resolved using the apparent rather than the great circle azimuth. The apparent azimuth may be determined from the three-dimensional orientation of the particle motion ellipse or from fitting a plane wave to the arrival times at a local array.

An important result of numerical experiments with slab structures is that although coherent, long scale length heterogeneities can cause deviations in the orientation of the S polarization vector of up to 10°, it is difficult to generate deviations much larger than this using any reasonable three-dimensional structure. It may generally be expected that deviations will be smaller than this estimated maximum because the estimate is based on zeroth order ray theory, which is strictly only valid at infinite frequency. At finite frequency, a body wave will tend to average the effects of heterogeneities over a wavelength. The small effects of lateral heterogeneity agree with the general success in the use of S wave polarizations for focal mechanism solutions. It does suggest that larger deviations in S polarizations are caused by phase interference and/or shear wave splitting due to anisotropy. These effects may be primarily responsible for the elliptical rather than linear particle motion observed in S waves and the progressive complexity in the particle motion ellipse commonly observed as time advances in the S waveform.

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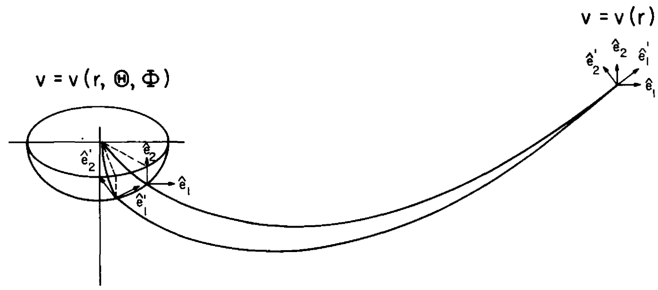

Fig. 4. Heterogeneous structure in the *vicinity of the source* primarily affects S polarization observed at a teleseismic receiver by rotating the vector basis $\hat{e}_1$, $\hat{e}_2$. It does not have much effect on either the apparent azimuth or angle with respect to the vertical observed at the teleseismic station. Reciprocally, heterogeneous structure in the *vicinity of the teleseismic receiver* will affect the S polarization by rotating the vector basis $\hat{e}_1$, $\hat{e}_2$ *and* by generating deviations in both the apparent azimuth and the angle with respect to the vertical

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