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Short communication

On a mixed quadratic invariant of the magnetic susceptibility tensor

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Introduction

Magnetic susceptibility of an anisotropic rock can be described by a second-order symmetric tensor. This tensor has several invariants of which commonly used is only the linear invariant closely associated with the mean susceptibility. The quadratic invariant does not have an explicit physical meaning.

In this paper, a particular mixed quadratic invariant of the susceptibility tensor is discussed which is numerically three times the variance of the principal susceptibilities. Its advantage consists in the fact that it can be determined using components of the susceptibility tensor in an arbitrary coordinate system; principal susceptibilities need not be known. This parameter can be used for characterizing the so called deviatoric component of susceptibility on one hand and statistical testing of the anisotropy of susceptibility on the other hand.

Magnetic susceptibility tensor

For a magnetically linear medium the relation between the intensity of the field $\boldsymbol{H}$ and the induced magnetic polarization $\boldsymbol{J}$ can be in a cartesian system of coordinates expressed by equation

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \mu_0 \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \cdot \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \quad (1)$$

or briefly

$$\boldsymbol{J} = \mu_0 \mathbf{k} \boldsymbol{H}. \quad (2)$$

Constants k_{ij} are components of the symmetric tensor of magnetic susceptibility.

For the sake of brevity we shall hereafter, as a rule, leave out the word "magnetic". Further, we shall not distinguish between the tensor and the matrix representing it.

The linear and the quadratic invariant

The susceptibility tensor may be interpreted geometrically by a quadric in the central position. The quadric is usually an ellipsoid, the so called susceptibility ellipsoid (Nagata 1961). Thus the invariants of quadrics are also invariants of the susceptibility tensor. For the invariants of quadrics see e.g. Rektorys (1969).

The commonly used linear invariant is given by the formula

$$I_1 = k_{11} + k_{22} + k_{33}. \quad (3)$$

It is simply related to the mean susceptibility:

$$\kappa = I_1/3. \quad (4)$$

Further, the quadratic invariant exists,

$$I_2 = \begin{vmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{vmatrix} + \begin{vmatrix} k_{22} & k_{23} \\ k_{23} & k_{33} \end{vmatrix} + \begin{vmatrix} k_{33} & k_{31} \\ k_{31} & k_{11} \end{vmatrix} \quad (5)$$

which has no simple physical meaning.

The mixed quadratic invariant

Let us form a mixed quadratic invariant from the above mentioned invariants,

$$I_M = \tfrac{2}{3} I_1^2 - 2 I_2. \quad (6)$$

After some easy adaptations we get

$$I_M = k_{11}^2 + k_{22}^2 + k_{33}^2 - 3\kappa^2 + 2k_{12}^2 + 2k_{23}^2 + 2k_{31}^2. \quad (7)$$

If the susceptibility tensor is expressed in the system of principal directions, and thus the principal susceptibilities κ_1, κ_2, κ_3 are known, the Eq. (7) may be simplified to

$$I_M = \kappa_1^2 + \kappa_2^2 + \kappa_3^2 - 3\kappa^2 \quad (8)$$

or

$$I_M = (\kappa_1 - \kappa)^2 + (\kappa_2 - \kappa)^2 + (\kappa_3 - \kappa)^2. \quad (9)$$

From Eqs. (8) and (9) it follows that I_M is three times the variance of principal susceptibilities. It may be therefore assumed that it will be possible to assign to the invariant I_M certain practical sense. Before discussing this problem, we shall express the invariant I_M in still another way.

Representation of the mixed quadratic invariant by means of the deviatoric susceptibility tensor

Using the mean susceptibility, we can express the susceptibility tensor by a sum of two tensors

$$\mathbf{k} = \kappa \mathbf{1} + \mathbf{l}, \tag{10}$$

while $l_{11} + l_{22} + l_{33} = 0$. We shall say that the tensor $\kappa\mathbf{1}$ represents the isotropic component and the tensor $\mathbf{l}$ the deviatoric component of the susceptibility tensor $\mathbf{k}$. The latter term is taken over from the theory of stress and strain, where the analogically defined quantity is called "the deviatoric stress" (Ramsay 1967).

It can be easily demonstrated that the mixed quadratic invariant of the tensor $\mathbf{k}$ can be expressed by the components of tensor $\mathbf{l}$,

$$I_M = l_{11}^2 + l_{22}^2 + l_{33}^2 + 2l_{12}^2 + 2l_{23}^2 + 2l_{31}^2 \tag{11}$$

and further,

$$I_M = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{12}$$

where λ_1, λ_2, λ_3 are eigenvalues of the matrix representing the tensor $\mathbf{l}$.

Standard deviatoric susceptibility

It is often essential to express quantitatively the amount of the deviatoric susceptibility component. This is particularly important when measuring the susceptibility anisotropy using instruments which do not provide data on the isotropic component (torque magnetometer, induction anisometer with rotating sample).

For the measure of the amount of the deviatoric component we may take the quantity

$$\tilde{\kappa} = \sqrt{I_M/3}. \tag{13}$$

We shall term it the standard deviatoric susceptibility.

From Eqs. (12) and (13) it is evident that the standard deviatoric susceptibility really characterizes the quantity of the deviatoric component. It is obvious from Eqs. (9) and (13) that it equals the standard deviation of the principal susceptibilities.

The standard deviatoric susceptibility does not belong to the so called anisotropy factors for the very reason that it has the "dimension" of susceptibility. Moreover, it does not satisfy the definition given by Jelínek (1981).

Besides the quantitative description of the deviatoric component, the standard deviatoric susceptibility can be with advantage used for forming the normed tensor of the deviatoric component. The normed tensor is determined by the equation

$$\mathbf{L} = \frac{1}{\tilde{\lambda}} \mathbf{l}, \tag{14}$$

where $\tilde{\lambda} = \tilde{\kappa}$. The mean value of the tensor $\mathbf{L}$ (defined similarly as the mean susceptibility) is zero, while the standard deviatoric value (defined in a similar manner as the standard deviatoric susceptibility) is unity which is very suitable for computation and numerical representation of results.

Test of anisotropy

When measuring the magnetic susceptibility by an A.C. bridge, the changes of the coil inductivity caused by the sample are evaluated (Girdler 1961; Fuller 1967; Jelinek 1973). The sample is inserted in the coil in a certain number of directions, e.g. 9, 15, 18 or more. The choice of the measuring directions is called the design of the experiment. The measured so called directional susceptibilities serve for computing the components of the susceptibility tensor, principal susceptibilities, and principal directions.

Certain differences can always be observed between the directional as well as principal susceptibilities. The purpose of the anisotropy test is to decide whether these differences are caused by the sample anisotropy or whether they are due to measuring errors only.

The test for anisotropy can be performed using the analysis of dispersion. Computations can be considerably simplified if the so called rotatable design of measuring directions (Hext 1963) is used. Hext suggested several rotatable designs, which, however, are not suitable for practical use. One design of 15 measuring directions, easily applicable, is described by Jelínek (1973).

Developing some ideas of Hext (1963), it is possible to construct relatively simply a test for the rotatable design. For the testing statistic it can be derived

$$F = \frac{2n}{75s^2} \hat{I}_M, \tag{15}$$

where n is the number of measuring directions, s^2 the estimate of dispersion in a single direction obtained from the sum of squares of residual errors, $\hat{I}_M$ is the mixed quadratic invariant of the estimate of the susceptibility tensor.

If the sample is isotropic, the statistic F has F-distribution on 5 and $n-6$ degress of freedom. If the statistic F exceeds the critical value on the chosen level of significance, the sample can be considered anisotropic.

Conclusion

A mixed quadratic invariant of the tensor of magnetic susceptibility has been discussed. It equals three times the variance of the principal susceptibilities. From this invariant, the standard deviatoric susceptibility can be computed which characterizes the deviatoric component of susceptibility. The standard deviatoric susceptibility is of particular use when measuring with an instrument which gives no information on the mean susceptibility (torque magnetometer, induction magnetometer with rotating sample). The mixed quadratic invariant also enables derivation of the test of anisotropy and simplifies its execution. It can be assumed that this invariant will simplify some calculations in studying the anisotropy of magnetic susceptibility of rocks.

References

Fuller, M.D.: The A.C. bridge method. In: Methods in palaeomagnetism, D.W. Collinson, K.M. Creer, S.K. Runcorn, eds.: pp. 403–408. Amsterdam-London-New York: Elsevier Pub. Co. 1967

Girdler, R.W.: The measurement and computation of anisotropy of magnetic susceptibility of rocks. Geophys. J. RAS **5**, 34–44, 1961

Hext, G.R.: The estimation of second-order tensors, with related tests and designs. Biometrika **50**, 353–373, 1963

Jelínek, V.: The precision A.C. bridge set for measuring magnetic susceptibility of rocks and its anisotropy. Studia Geophys. Geodaet **17**, 36–48, 1973

Jelínek, V.: Characterization of the magnetic fabric of rocks. Tectonophysics **79**, T63–T67, 1981

Nagata, T.: Rock magnetism. Tokyo: Maruzen 1961

Ramsay, J.G.: Folding and fracturing of rocks. New York: McGraw-Hill 1967

Rektorys, K.: Survey of applicable mathematics. London: Iliffe Books 1969

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