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The inverse scattering problem for reflection of electromagnetic dipole radiation from Earth with vertical variation

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Abstract. The inverse scattering problem of determining the electromagnetic profiles of layers with vertical variation from surface measurements of the electromagnetic field for electric and magnetic dipole radiation is formulated and solved. It is shown that the surface data for dipole radiation, if known at two fixed frequencies and at all transmitter-receiver separations, contain sufficient information for the complete determination of the permittivity, permeability and conductivity distributions of the subterranean layers. The problem is solved by an integral equation formulation and the solution is illustrated with some analytical and numerical examples.

Key words: Inverse scattering method – Electromagnetic dipole radiation

Introduction

The fundamental problem of electromagnetic sounding methods in prospecting work is the determination of the subterranean distribution of the electromagnetic parameters from surface reflection data. This problem is quite complex even in the simplest situation where the subterranean strata have only vertical variation. In the case of reflection of vertically incident plane wave radiation, the inversion problem can, in principle, be solved by the coupled integral equation method of Jaulent and Jean (1972) and Jaulent (1976) or by the less powerful but simpler iterative method of Riska and Vidberg (1983). A drawback of this solution is the fact that the reflection coefficient for plane wave radiation does not, even when known at all frequencies, contain sufficient information for the determination of the permittivity, permeability and conductivity profiles of the subterranean structure. Furthermore, the solution of the inverse scattering problem for plane wave radiation is of limited practical usefulness as most sounding work is carried out with finite-size radiation sources – e.g. dipole antennas. It is, therefore, well worth investigating the corresponding inverse scattering problem for incident dipole radiation.

In this paper we formulate the inverse scattering problem for reflection of electric and magnetic dipole radiation from layers with purely vertical variation. We consider in detail the cases of vertical and horizontal electric and magnetic dipole sources above ground which approximate typical radiation sources used in electrodynamic prospecting work. In contrast to the situation for incident plane wave radiation, the electromagnetic field for dipole radiation depends on position at the surface as well as the radiation frequency. This position- and frequency-dependent electromagnetic field contains sufficient information for the complete determination of the subterranean permittivity, permeability and conductivity profiles. Furthermore, the determination of these profiles is possible from a knowledge of the surface field at only two fixed, but arbitrary, frequencies. Hence no extrapolation into the unmeasurable high-frequency regime is required.

The formulation of the inversion problem for reflection of dipole radiation is mathematically more complicated than that for plane wave radiation, the essential difference being caused by the finite distance to the radiation source. The solution of this inversion problem and the solution algorithms developed in this paper are, on the other hand, simpler. While the inverse scattering problem for plane wave radiation involves a Sturm-Liouville equation with a complex eigenvalue-dependent potential, the corresponding problem for dipole radiation can be formulated in a way that avoids explicit dependence on the eigenvalue in the potential.

In this paper we reduce the inversion problem for dipole radiation to the determination of the complex potential in a Sturm-Liouville equation from the spectrum. The problem is solved by the usual methods of inverse scattering theory. We give a solution based on a Marchenko-type integral equation (Agranovich and Marchenko, 1964), relying on the derivation of the formalism developed by Weidelt (1972). The resulting inverse scattering transform is mathematically very similar to that developed in the work of Coen (1981) on the inverse scattering problem for reflection of elastic waves.

This paper falls into eight sections. Firstly we formulate the inverse scattering problem for a vertical magnetic dipole antenna above ground. Then we solve the problem by means of an inverse scattering integral

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equation. In the subsequent section we discuss the numerical implementation of the solution and then generalize the treatment to the case of a horizontal magnetic dipole source and to the cases of horizontal and vertical electric dipole sources. Following this, we illustrate the solution algorithm for the case of a step function discontinuity which can be handled analytically, and then demonstrate the utility of the solution by some numerical examples. The final section contains a summarizing discussion.

The inversion problem for magnetic dipole radiation

We first consider reflection of the radiation from a vertical magnetic dipole source of strength $me^{-i\omega t}$ at height h from the flat ground surface (Fig. 1). The conductivity, permittivity and permeability distributions below ground are assumed to depend only on depth (z). The situation is then axially symmetric around a vertical axis through the dipole which we choose as the z-axis (we take the downward direction as positive and the position of the dipole as the origin). The electric field is purely azimuthal and depends only on depth (z) and horizontal distance from the dipole (ρ).

The azimuthal electric field amplitude $E(\rho, z)$ satisfies the wave equation

$$\frac{\partial}{\partial z}\frac{1}{\mu(z)}\frac{\partial E}{\partial z}+\frac{1}{\mu(z)}\frac{\partial}{\partial \rho}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E)+\omega^2 K^2(z) E=0, \tag{1}$$

where we have defined the quantity $K(z)$ as

$$K^2(z)=\left[\varepsilon(z)+i\frac{\sigma(z)}{\omega\varepsilon_0}\right]\varepsilon_0\mu_0. \tag{2}$$

Here ω is the angular frequency of the dipolar radiation source and σ, ε and μ the vertically varying conductivity, relative permittivity and permeability parameters. This wave equation may be simplified by introducing the Hankel transform of the electric field:

$$\tilde{E}(z,\lambda)=\int_0^\infty d\rho\,\rho\, J_1(\lambda\rho) E(\rho, z). \tag{3}$$

The transformed field $\tilde{E}$ then satisfies the differential equation

$$\frac{\partial}{\partial z}\frac{1}{\mu(z)}\frac{\partial \tilde{E}}{\partial z}-\frac{\lambda^2}{\mu(z)}\tilde{E}+\omega^2 K^2(z)\tilde{E}=0. \tag{4}$$

Finally, the first-order derivative of $\tilde{E}$ in Eq. (4) may be removed by the field transform

$$\tilde{E}(z,\lambda)=\sqrt{\mu(z)}\,Z(z,\lambda). \tag{5}$$

The transformed field Z then satisfies the generalized Sturm-Liouville equation

$$\frac{d^2 Z}{dz^2}+\{k^2-\lambda^2-V(z,k)\}Z=0, \tag{6}$$

in which k is defined as

$$k=\frac{\omega}{c}, \tag{7}$$

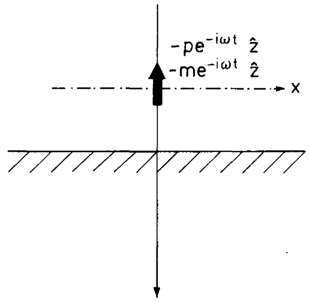

Fig. 1. Vertical dipole source above vertically layered earth

with c being the velocity of light in vacuum. The potential function V in Eq. (6) is defined as

$$V(z,k)=k^2\{1-\varepsilon(z)\mu(z)\}+\sqrt{\mu(z)}\frac{d^2}{dz^2}\frac{1}{\sqrt{\mu(z)}}-ik\frac{\sigma(z)\mu(z)}{\varepsilon_0 c}. \tag{8}$$

The potential function thus defined has support in the region $z>h$, where h is the position of the ground surface, and it is well defined for continuously differentiable permeability profiles.

The differential equation, Eq. (6), has the continuously distributed eigenvalues

$$v=\begin{cases}\sqrt{\lambda^2-k^2}, & k<\lambda\\ -i\sqrt{k^2-\lambda^2}, & k>\lambda.\end{cases} \tag{9}$$

In the following we shall treat the frequency variable k as a fixed parameter and only consider the real branch of the eigenvalue ($\lambda>k$), i.e. only the components of the field with non-oscillatory dependence on depth. By Eq. (3), this implies that the near-field is used in the inversion.

Once k is taken to be fixed, the potential V has no dependence on the variable eigenvalue v and then the differential equation, Eq. (6), may be treated as a standard Sturm-Liouville equation with a complex potential:

$$\frac{d^2 Z}{dz^2}-[v^2+V(z)]Z=0. \tag{10}$$

As the potential V vanishes above ground ($z\leq h$), the general solution to Eq. (10) above ground is a linear combination of $\exp(vz)$ and $\exp(-vz)$. Below ground ($z>h$), the general solution to Eq. (10) may be formed as linear combinations of two linearly independent solutions with exponential behaviour above ground. Such solutions are given by the Jost solutions

$$f_\pm(z)=e^{\pm vz}+\theta(z-h)\int_h^z dz'\frac{\sinh v(z-z')}{v}V(z')f_\pm(z'). \tag{11}$$

These solutions are well defined provided the potential V is constant or decreasing beyond some arbitrary finite depth d. The boundary condition relevant to the geophysical case is that in which the potential becomes constant at large depths beyond the region of prospecting interest.

It is easy to see by forming the Wronskian of the functions (11) that they form a linearly independent pair of solutions to Eq. (10). They may, therefore, be used to express the electric field due to a vertical magnetic dipole of strength m at height h above ground ($z=0$) as

$$E(\rho, z)=\int_0^\infty d\lambda\,\lambda J_1(\lambda\rho)\tilde{E}(z,\lambda), \tag{12}$$

with

$$\tilde{E}(z,\lambda)=i\frac{\mu_0 m\omega}{4\pi}\sqrt{\mu(z)}\,\frac{\lambda}{v}\{f_-(z)+S(v)f_+(z)\}. \tag{13}$$

This expression is valid for $z>0$ (below the dipole). The function $S(v)$ in Eq. (13) can be viewed as the surface reflection coefficient.

The surface reflection coefficient $S(v)$ may be calculated from the electric field at the surface using the form of the Jost solutions above earth as

$$S(v)=-e^{-2vh}-i\frac{4\pi v}{\mu_0 m\omega\lambda}e^{-vh}\tilde{E}(h,\lambda). \tag{14}$$

Knowledge of the electric field at the surface by Eq. (3) thus determines the reflection coefficient for all positive values of λ. The reflection coefficient of course also depends on frequency (k).

The inverse scattering problem is the problem of determining the complex potential function V, Eq. (8), from the known reflection coefficient. The solution of this problem is based on the properties of the Jost functions (11) which carry the connection between the reflection coefficient and the potential function.

A convenient integral expression for the reflection coefficient S can be obtained by considering the following alternative expression for the physical solution to the differential Eq. (10):

$$\xi(z)=e^{-vz}-\frac{1}{2v}\int_h^\infty dz'\,e^{-v|z-z'|}\,V(z')\,\xi(z'). \tag{15}$$

Comparing the behaviour of this solution to the combination of Jost functions in Eq. (13) for $0<z<h$ one obtains the result

$$S(v)=-\frac{1}{2v}\int_h^\infty dz'\,e^{-vz'}\,V(z')\,\xi(z'). \tag{16}$$

In a subsequent section we develop an iterative algorithm for the determination of the potential from S based on this expression and the iterative solution of Eq. (15). In the following section we develop a more powerful integral equation method for the determination of V from S.

If the inverse scattering problem is solved for two frequencies, the frequency-dependent components of the potential (8) may be determined separately from the frequency-independent component. The frequency-dependent terms involve the products $\varepsilon\mu$ and $\sigma\mu$, whereas the frequency-independent component involves only μ. Denoting the frequency-independent (real) component of V as $W(z)$ we have

$$W(z)=\sqrt{\mu(z)}\,\frac{d^2}{dz^2}\frac{1}{\sqrt{\mu(z)}}. \tag{17}$$

The permeability distribution $\mu(z)$ may be determined from $W(z)$ by iterative solution of the integral equation

$$[\mu(z)]^{-1/2}=1+\int_h^z dx\,(z-x)\,W(x)[\mu(x)]^{-1/2}. \tag{18}$$

Here we have assumed that both the permeability and its gradient are continuous on the surface. A discontinuity in μ or $d\mu/dz$ at the surface would lead to a singular potential function $V(z)$. The singular part would have to be handled analytically by extracting the starting values $\mu(h)$ and $\mu'(h)$ from the asymptotic expansion

$$S(v)\,e^{2vh}\to\frac{\mu(h)-1}{\mu(h)+1}+\frac{\mu'(h)}{v[\mu(h)+1]^2},$$
$$|v|\to\infty,\qquad \mathrm{Re}\,v>0. \tag{19}$$

Once $\mu(z)$ has been determined from Eq. (18), $\varepsilon(z)$ may be determined from the frequency-dependent real part of $V(z)$ and $\sigma(z)$ from the imaginary part of $V(z)$.

Integral equation for the inversion problem

The determination of the potential (8) from the reflection coefficient S, Eq. (14), may be carried out by means of a linear integral equation similar to that considered by Weidelt (1972). In order to derive this integral equation we write the Jost solutions, Eq. (11), in the alternative form (Weidelt, 1972)

$$f_\pm(z)=e^{\pm vz}+\theta(z-h)\int_{2h-z}^{z} dz'\,e^{\pm vz'}A(z,z'), \tag{20}$$

where the function $A(z,z')$ is a function independent of v with support in the region $2h-z<z'<z$.

Substituting this representation for the fundamental solutions into the differential Eq. (10) and exploiting the v-independence of $A(z,z')$ gives the conditions

$$A_{zz}(z,z')-A_{z'z'}(z,z')=V(z)A(z,z'), \tag{21a}$$

$$2A_z(z,z)=V(z), \tag{21b}$$

$$A(z,2h-z)=0, \tag{21c}$$

for the function $A(z,z')$. Equation (21b) can be used to determine the potential $V(z)$ once $A(z,z')$ has been determined.

Consider now the function ξ, Eq. (15), which, by Eq. (11), may be expressed as

$$\xi(z)=f_-(z)+S(v)f_+(z). \tag{22}$$

Here $S(v)$ is the reflection coefficient, Eq. (14). From Eq. (20) one then obtains

$$\xi(z)-e^{-vz}=S(v)e^{vz}+\theta(z-h)\int_{2h-z}^{z} dz'\,A(z,z')\,e^{-vz'}$$
$$+\theta(z-h)\int_{2h-z}^{z} dz'\,A(z,z')\,e^{vz'}S(v). \tag{23}$$

Taking the inverse Laplace transform of this equation with respect to v gives

$$\frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} d v\, e^{vt}\{\xi(z)-e^{-vz}\}$$
$$=\tilde{S}(z+t)+\theta(z-h)A(z,t)$$
$$+\theta(z-h)\int_{2h-z}^{z} dz' A(z,z')\tilde{S}(t+z'). \tag{24}$$

Here $\tilde{S}$ is the inverse Laplace transform of the reflection coefficient $S(v)$:

$$S(v)=\int_0^\infty dz\, e^{-vz}\tilde{S}(z). \tag{25}$$

For $t<z$ the left hand side of Eq. (24) vanishes. This can be proven by evaluating the inverse Laplace transform of the successive terms in the iterative expansion of Eq. (15). Note that by the same method one can prove that

$$\tilde{S}(t)=0 \quad \text{for } t<2h, \tag{26}$$

since for $t<2h$ the integral in Eq. (28) may be closed in the right half v-plane and $S(v)$ is analytic for the real part of v sufficiently large and positive. Exploiting the well known analyticity of the Jost solutions, Eq. (20), one can show using Eq. (22) that the iterative expansion of the function ξ, Eq. (15), converges and that hence $S(v)$ is analytic when $\operatorname{Re} v>\sqrt{\sup|V(z)|}$.

For $2h-z<t<z$ one then has the integral equation

$$\tilde{S}(z+t)+A(z,t)+\int_{2h-t}^{z} dz' A(z,z')\tilde{S}(t+z')=0, \tag{27}$$

for the determination of $A(z,t)$. Since both $\tilde{S}$ and A are complex functions, Eq. (27) actually represents two coupled integral equations. The solution of the inverse scattering problem thus involves: (1) determination of $S(v)$ from the surface data at a fixed frequency (k) by Eq. (14); (2) Laplace inversion of the thus determined reflection coefficient, Eq. (25); (3) solution of the integral Eq. (27) and, finally, determination of the potential V from the function $A(z,t)$ by Eq. (21b).

In practice, while the solution of the integral Eq. (27) can be carried out by straightforward numerical techniques the calculation of the inverse Laplace transform of the reflection coefficient can be difficult. Furthermore, the inversion of the Laplace transform is an ill-posed problem and is the origin of the general instability of the solution to the inverse scattering problem.

Numerical implementation of the inverse scattering solution

The integral equation for the kernel function A, Eq. (27), from which the final solution to the inverse scattering problem is obtained, is a Fredholm equation of the second kind and hence may be solved by standard matrix inversion methods. The main problem in the numerical implementation of the inverse scattering solution is associated with the need to evaluate the inverse Laplace transform of the reflection coefficient, Eq. (25). The explicit Mellin formula for the inverse Laplace transform:

$$\tilde{S}(z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} d v\, e^{vz} S(v) \tag{28}$$

requires integration along a line parallel to the imaginary axis in the complex v-plane. As $S(v)$ is calculated from the surface field as a Hankel transform, Eq. (14), that converges only for real λ, use of the Mellin formula, Eq. (28), requires analytic extrapolation of $S(v)$ into the complex v-plane. This is a procedure beset with instability. It should also be noted that the usual methods developed in the literature for numerical solution of the integral transform, Eq. (28), (McWirther and Pike, 1978) only apply for such functions $\tilde{S}(z)$ that vanish as $z\to\infty$, a condition not satisfied in the geophysical case with absorption.

To determine the function $\tilde{S}(z)$ numerically we consider here the simple approach of parametrizing $S(v)$ by a Padé approximant (Baker, 1975)

$$S(v)=\frac{P_M(v)}{Q_N(v)}, \tag{29}$$

where the degree N of the denominator polynomial Q_N is two units larger than the degree M of the numerator polynomial P_M. With the Padé approximant, Eq. (29), the function $\tilde{S}(z)$ may be calculated directly from Eq. (28). The function $\tilde{S}(z)$ will be of the form

$$\tilde{S}(z)=\sum_i A_i \exp(\gamma_i z), \tag{30}$$

where γ_i are the poles of the rational function P/Q and A_i their corresponding residues. In the cases of geophysical interest there will appear poles with positive and negative real parts.

The errors caused by the Padé approximation in the function $\tilde{S}(z)$, which is the input to the inverse scattering integral Eq. (27), will of course cause distortions in the solution for $V(z)$. Nevertheless, we shall demonstrate by numerical examples in a later section that the Padé approximant method appears to yield reliable results for $V(z)$ down to depths of the order of the average skin depth of the subterranean structure studied. This limitation is of course very natural on physical grounds as one cannot in general expect any method of solving the inverse scattering problem to yield information for depths much beyond this average skin depth.

Given the restriction that the Padé approximant method for calculating the function $\tilde{S}(z)$ is reliable only up to the average skin depth, one may be content to solve the inverse scattering integral Eq. (27) iteratively. The solution for the potential $V(z)$, Eq. (8), will then take the form of a series expansion

$$V(z)=\sum_{n=1}^{\infty} V_n(z) \tag{31}$$

with the terms

$$V_1(z)=-2\frac{d}{dz}\tilde{S}(2z), \tag{32a}$$

$$V_n(z)=2\frac{d}{dz}(-1)^n\int\limits_{2h-z}^{z}dy_1\int\limits_{2h-y_1}^{z}dy_2$$
$$\dots\int\limits_{2h-y_{n-2}}^{z}dy_{n-1}\,\tilde{S}(z+y_1)\,\tilde{S}(y_1+y_2)\dots\tilde{S}(y_{n-1}+z). \tag{32b}$$

We have verified up to $n=3$ that this expansion, Eq. (31), is the same as that obtained by using the iterative Jost-Kohn formula (Jost and Kohn, 1952) for solving the inverse scattering problem.

To obtain an estimate of the conditions for convergence of the series (31) we note that the iterative solution $A(z,x)$ of the integral Eq. (27) converges if the norm of the kernel $\tilde{S}$ is less than unity. This condition may be expressed as

$$1>\|\tilde{S}\|=\int\limits_{h}^{z}dy\int\limits_{h}^{z}dx\,|\tilde{S}(x+y)|^2. \tag{33}$$

By Eq. (32a), this condition is satisfied provided

$$\frac{(z-h)^2}{2\sqrt{3}}\sup_{h<u<z}|V_1(u)|<1. \tag{34}$$

This result shows that there will always exist a region in which the integral Eq. (27) may be solved by iteration. Even when condition (34) is satisfied, the potential series (31) which is derived from the iterative expansion of $A(z,x)$ by differentiation [Eq. (21b)] does not necessarily converge. Following the method by Jost and Kohn (1952) one can prove, however, that the series (31) does converge when the norm of S is sufficiently small [the factor $2\sqrt{3}$ in Eq. (34) replaced by $(2\log 2-1)/2$].

In the analytical example treated later we show that the convergence depth is of the same order of magnitude as the skin depth, which, e.g. for granite with $\sigma\sim 10^{-4}$ S/m, exceeds 600 m for frequencies below 1 kHz. From the point of view of prospecting work this will often be quite sufficient. We shall illustrate the convergence properties and the utility of the iterative solution of the inverse scattering integral Eq. (27) by a set of numerical examples with synthetic data using the Padé approximant method discussed above for deriving the inverse Laplace transform of the reflection coefficient.

The inversion problem for radiation from a horizontal magnetic dipole and vertical or horizontal electric dipoles

The inverse scattering problem for reflection of radiation from a vertical magnetic dipole treated above is particularly simple because of the axial symmetry. In general, one may split the electromagnetic field in a vertically varying medium into a transverse electric ($E_z=0$) and a transverse magnetic mode ($B_z=0$). With radiation sources that excite the transverse electric mode one can formulate the inversion problem as an inverse scattering problem with the same potential function as in Eq. (8) by using appropriate field amplitudes.

In the case of a horizontal magnetic dipole source (Fig. 2) the vertical component of the magnetic induction can be written as

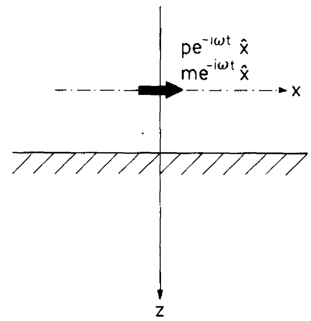

Fig. 2. Horizontal dipole source above vertically layered earth

$$B_z=B(\rho,z)\cos\phi, \tag{35}$$

where the field amplitude $B(\rho,z)$ satisfies the reduced wave equation, Eq. (1). In analogy with the development after Eq. (1), this equation may again be reduced to the Sturm-Liouville form by introducing the transformed field

$$Z(z,\lambda)=\frac{1}{\sqrt{\mu(z)}}\int\limits_0^\infty d\rho\,\rho\,J_1(\lambda\rho)\,B(\rho,z). \tag{36}$$

The transformed field Z then satisfies Eq. (6) with the potential function V given in Eq. (8) and the branches of the eigenvalue v chosen as in Eq. (9).

The physical solution for Z may be expressed as a linear combination of the Jost solutions (11) as

$$Z(z,\lambda)=\frac{\mu_0 m\lambda}{4\pi}\{f_-(z)+S(v)f_+(z)\},\qquad z>0. \tag{37}$$

The reflection coefficient $S(v)$ can finally be calculated from the measured vertical component of the magnetic induction using Eqs. (36) and (37). The recovery of the potential V in this case from the reflection coefficient $S(v)$ is performed as before.

In the case of a horizontal electric dipole source, the vertical component of the magnetic induction has the form

$$B_z=B(\rho,z)\sin\phi, \tag{38}$$

where the field amplitude $B(\rho,z)$ again satisfies the reduced wave Eq. (1). Hence, a transformed field Z that satisfies the Sturm-Liouville Eq. (6) with the potential (8) may again be introduced as in Eq. (36). The physical solution for this transformed field is, in this case,

$$Z(z,\lambda)=-i\frac{p\omega\mu_0}{4\pi}\frac{\lambda}{v}\{f_-(z)+S(v)f_+(z)\}, \tag{39}$$

where p is the strength of the electric dipole. The reflection coefficient $S(v)$ can then be calculated from the measured vertical magnetic field component on the surface, and the recovery of the potential function V from $S(v)$ can be carried out by the method described earlier.

We finally consider the case of a vertical electric dipole source. As the magnetic field in this case is purely azimuthal, only the transverse magnetic mode is excited. This leads to a Sturm-Liouville equation with a potential function that is different from that in the previously considered cases.

In the case of a vertical electric dipole source, the azimuthal magnetic field amplitude $H(\rho, z)$ satisfies the reduced wave equation

$$\frac{\partial}{\partial z}\frac{1}{K^2(z)}\frac{\partial H}{\partial z}+\frac{1}{K^2(z)}\frac{\partial}{\partial \rho}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H)+\omega^2\mu(z)H=0. \quad (40)$$

The function $K(z)$ is defined in Eq. (2). One may introduce a transformed field Z as

$$Z(z,\lambda)=\sqrt{\frac{\varepsilon_0\mu_0}{K^2(z)}}\int_0^\infty d\rho\,\rho\,J_1(\lambda\rho)\,H(\rho,z), \quad (41)$$

which satisfies the Sturm-Liouville equation

$$\frac{d^2 Z}{dz^2}-[\nu^2+\bar{V}(z,k)]\,Z=0. \quad (42)$$

Here the potential function $\bar{V}$ has been defined as

$$\bar{V}(z,k)=k^2[1-\varepsilon(z)\,\mu(z)] -ik\frac{\mu(z)\,\sigma(z)}{\varepsilon_0 c}+K(z)\frac{d^2}{dz^2}\frac{1}{K(z)}. \quad (43)$$

The branches of the eigenvalue ν are chosen as in Eq. (9).

A reflection coefficient $\bar{S}(\nu)$ for this situation may be introduced by expressing the physical solution for Z in terms of Jost solutions $\bar{f}_\pm$, defined as in Eq. (11) but with the potential function $\bar{V}$ in place of the function V:

$$Z(z,\lambda)=-i\frac{\omega p\lambda}{4\pi\nu}\{\bar{f}_-(z)+\bar{S}(\nu)\bar{f}_+(z)\}. \quad (44)$$

This reflection coefficient $\bar{S}(\nu)$ may be calculated from the measured azimuthal magnetic field component on the surface using Eqs. (41) and (44). The potential function $\bar{V}(z,k)$ may then be determined from $\bar{S}(\nu)$ by the method described earlier. In this case one has, however, to require that both the conductivity and the permittivity profiles be continuously differentiable in order that the potential function $\bar{V}$, Eq. (43), be well defined. As these conditions are not usually satisfied, use of a vertical electric dipole radiation source is unattractive in comparison to other orientations. The complex structure of the potential function $\bar{V}(z,k)$, Eq. (43), also makes the disentangling of the conductivity, permittivity and permeability distributions from $\bar{V}$ a complicated task, requiring information for more than the two frequencies which proved sufficient in the previously considered cases.

Analytic treatment of a step discontinuity

In order to demonstrate the utility of the integral equation method for solving the inverse scattering problem and the convergence properties of its iterative solution, we treat the case of a step discontinuity at the surface analytically. More precisely, we consider the case of a vertical magnetic dipole source at height h above ground which is assumed to have a constant conductivity σ and relative permittivity $\varepsilon+1$ and permeability $\mu=1$. Assuming that the electromagnetic properties of air are the same as in vacuum, the potential function V, Eq. (8), will be

$$V(z)=k^2[1-\varepsilon]\,\theta(z-h)-ik\frac{\sigma}{\varepsilon_0 c}\theta(z-h)\equiv U\,\theta(z-h). \quad (45)$$

As the frequency (k) is taken to be fixed, U can be treated as a complex constant.

The Jost solutions, Eq. (11), for the step potential (45) are readily obtained by matching the solutions of Eq. (6) for $z \gtrless h$ using the continuity of the field Z and its vertical derivative at $z=h$:

$$f_\pm(z)=\theta(h-z)\,e^{\pm\nu z}+\theta(z-h)\frac{e^{\pm\nu h}}{2\nu'} \cdot\{(\nu'+\nu)\,e^{\pm\nu'(z-h)}+(\nu'-\nu)\,e^{\mp\nu'(z-h)}\}. \quad (46)$$

Here we have used the notation

$$\nu'=\sqrt{\nu^2+U}. \quad (47)$$

From Eq. (13) one obtains the reflection coefficient $S(\nu)$ as

$$S(\nu)=e^{-2\nu h}\frac{\nu-\nu'}{\nu+\nu'}. \quad (48)$$

The inverse Laplace transform of $S(\nu)$ is, Eq. (25),

$$\tilde{S}(z)=-\frac{2}{z-2h}J_2[(z-2h)\sqrt{U}]\,\theta(z-2h). \quad (49)$$

The kernel function $A(z,t)$ in the representation for the Jost functions, Eq. (20), may be obtained by taking an inverse Laplace transform with respect to ν of Eq. (46):

$$A(z,t)=\frac{(t+z-2h)\sqrt{U}}{2\sqrt{(z-h)^2-(t-h)^2}}I_1\{\sqrt{[(z-h)^2-(t-h)^2]\,U}\}. \quad (50)$$

With the help of the power series expansions of the Bessel functions J_2 and I_1 in the expressions for $\tilde{S}$ and A, it is then possible to demonstrate explicitly that the integral Eq. (27) is satisfied by the expressions (49) and (50). One can also easily verify that $A(z,t)$ does produce the original step potential $V(z)$ in Eq. (45) through Eq. (21 b).

We turn then to the illustration of the utility of the iterative algorithm described earlier. By the convergence criterion (34) one expects that the step potential (45) can be recovered from the reflection coefficient, Eq. (48), by the iterative algorithm for depth values H roughly satisfying the inequality

$$H^2\,|U|\ll 1. \quad (51)$$

If the imaginary part of the potential U (the conductive term) is the one of main importance, which is the case in typical low-frequency prospecting work, it is easy to see that condition (51) is equivalent to

$$H \ll \delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}, \tag{52}$$

where δ is the skin depth. This is a very natural result as one ought not to expect any solution algorithm for the inverse scattering problem to yield a reliable answer for depths much beyond the skin depth where the radiation does not penetrate.

The lowest-order potential (Born approximation) V_1 is readily obtained from the reflection coefficient, Eq. (48) or (49) using Eq. (32a) as

$$V_1(z) = U\{J_0(\rho) - J_4(\rho)\} = U\left\{1 - \frac{\rho^2}{4} + \frac{5}{384}\rho^4 + O(\rho^6)\right\}, \tag{53}$$

where the variable ρ has been defined as

$$\rho = 2(z-h)\sqrt{U}. \tag{54}$$

As the first term in the power series expansion in Eq. (53) gives the exact result, Eq. (45), the higher terms represent the error of the Born approximation. The Born approximation is obviously reliable for $|\rho| \ll 1$, which is equivalent to the skin depth condition, Eq. (51).

The second- and third-order terms in the iterative expansion of the solution may be obtained from Eq. (32b) as

$$V_2(z) = \frac{16U}{\rho^2} J_2^2(\rho) = \frac{U}{4}\rho^2\left[1 - \frac{\rho^2}{6} + O(\rho^4)\right] \tag{55}$$

$$V_3(z) = \frac{8U}{\rho} J_2(\rho)(D_1 + 3D_3) - \frac{4U}{\rho}\sum_{k=1}^{\infty}[(k+2)J_{k+2}(\rho) - (k-2)J_{k-2}(\rho)](D_{1+k} - D_{3+k}) = \frac{11U}{384}\rho^4[1 + O(\rho^2)]. \tag{56}$$

We have used the notation

$$D_n = \begin{cases} \frac{\rho}{4-n^2}(J_1 J_n - J_2 J_{n-1}) - \frac{J_2 J_n}{n+2} & (n \neq 2) \\ \frac{1}{4}(1 - J_0^2) - \frac{1}{2}J_1^2 - \frac{1}{4}J_2^2 & (n = 2), \end{cases} \tag{57}$$

where the argument of the Bessel function is ρ. By adding V_1, V_2 and V_3 as given by these expressions, we obtain a result for the potential with an error that is proportional to ρ^6.

In order to illustrate the convergence of the iterative expansion of the potential we have plotted the first, second and third iterative results for the potential as a function of the dimensionless depth variable $|\rho|$ in Fig. 3. We have chosen U to be purely imaginary, which corresponds to a conductivity step and makes

$$|\rho| = (z-h)\sqrt{f\sigma\{\mathrm{Hz\,Sm^{-1}}\}}/178\ \mathrm{m}.$$

The iterative method is seen to converge well up to depth $|\rho| \leq 1$. With a typical conductivity value $\sigma = 10^{-4}$ S/m corresponding to granite and frequency values $f = 100$ Hz and 1 kHz, the convergence depths with three terms in V are of the order of 1800 and 600 m, respectively.

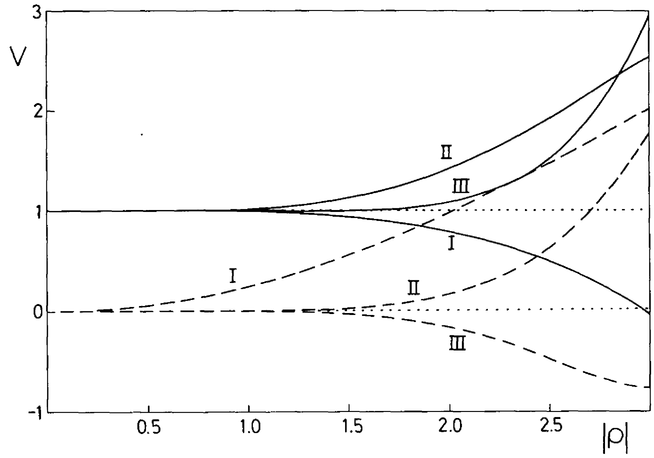

Fig. 3. First (I), second (II) and third (III) iterative results for a conductivity step potential as functions of the depth variable $|\rho|$. The *solid lines* denote the real and the *dashed lines* the imaginary parts of the iterated potential divided by the exact value. The deviation from 1 gives the error in the real and the deviation from 0 the error in the imaginary parts

Numerical examples

To examine the numerical quality of the solution to the inverse scattering problem using the Padé approximant method for obtaining the inverse Laplace transform of the reflection coefficient $S(\nu)$ [Eqs. (29) and (30)] we consider the following examples. We first study numerically the step discontinuity problem considered analytically in the previous section. In the second example we consider a step discontinuity, Eq. (45), with an additional layer of large conductivity, which corresponds to the geophysical situation of a high-conductivity layer in a background of low-conductivity rock. In the third example we consider the case of a staircase conductivity profile chosen to illustrate the solution in the case of an approximately continuously varying conductivity profile.

In all three examples above one can express the reflection coefficient $S(\nu)$, Eq. (16), as a combination of elementary functions. To generate synthetic data for the surface field one can then use these expressions for $S(\nu)$ in Eqs. (12) and (13) and the code of Anderson (1979) for the numerical evaluation of the Hankel transform. Going backwards from the calculated surface field using the same computer code, one can then obtain numerical values for the reflection coefficient that will have errors associated with the double numerical quadrature. In this way we obtain values for $S(\nu)$ at discrete real ν-values (equidistant on a logarithmic scale to save computer time). A small number of these numerical values for $S(\nu)$ are then used to generate Padé approximants to $S(\nu)$ of the form of Eq. (29) by pointwise fitting (Baker, 1975). The inverse Laplace transform of the Padé approximant is finally evaluated analytically as described earlier. Finally, the inverse scattering integral equation, Eq. (27), is solved iteratively using Eqs. (31) and (32) including terms up to third order ($V_1 + V_2 + V_3$).

In Fig. 4 we display the result thus obtained for the potential step, Eq. (45), considered analytically in the

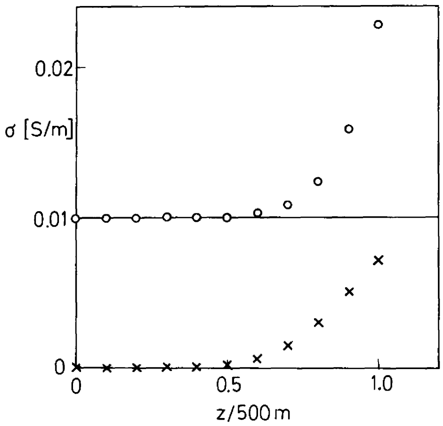

Fig. 4. Real (o) and imaginary (×) parts of the reconstructed uniform conductivity profile as a function of depth, in units of the penetration depth for the frequency 100 Hz

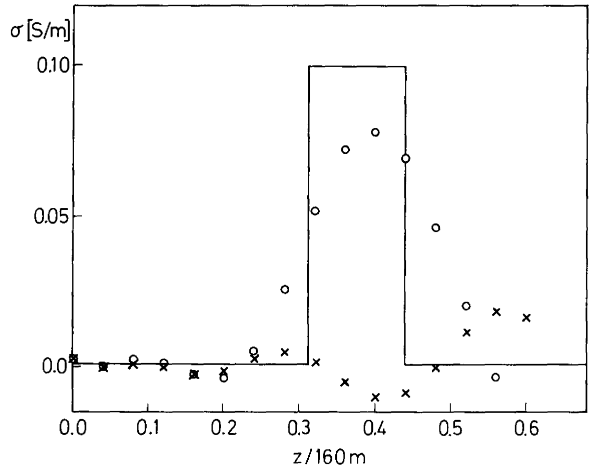

Fig. 5. Reconstructed conductivity profile (o = real part, × = imaginary part) compared with the original profile (*solid*). The length unit is the skin depth in the high-conductivity layer for the frequency 100 Hz

previous section. In the figure the depth coordinate has been divided by the skin depth for a radiation frequency $f = 100$ Hz. In this case a 7-point Padé approximant was used. The numerical results in this case are excellent up to half the skin depth.

In Fig. 5 we show the result of the numerical recovery of the conductivity profile in which the conductivity below ground is taken to be 10^{-3} S/m except in the layer $50\,\mathrm{m} < z < 70\,\mathrm{m}$ where it is taken to be 10^{-1} S/m. The depth scale in the figure has been divided by the skin depth in the high-conductivity layer, which is 160 m at the frequency 100 Hz. Here a 13-point Padé approximant was used. In this case the quality of the numerical recovery is only average: the region of high conductivity is recovered as a smooth bell-shaped distribution. In any case, the average strength of the region of high conductivity is satisfactorily recovered.

In Fig. 6 we show the result of the numerical recovery of a multi-step conductivity profile. The multi-step conductivity profile is superimposed on a uniform background conductivity of 0.025 S/m. The profile contains four conductivity increases of 0.025 S/m and four corresponding decreases of equal magnitude. Again a 13-point Padé approximant was used. The depth scale has again been divided by the minimum skin depth, which in this case is 140 m at 100 Hz. The results show that the quality of the inverse scattering solution is very satisfactory.

The conclusion from these results is that the numerical method based on the Padé approximant is most reliable when the conductivity profile is smooth. When there is a large conductivity contrast, as in the case considered in Fig. 5, the recovery of the original profile by the inverse scattering method is only qualitative, the reason being the inability of a finite Fourier series, Eq. (30), to represent sharp discontinuities. That this is indeed the source of the error can be verified by exact integration based on the Mellin formula, Eq. (28). In that case the reconstructed conductivity profile is in excellent agreement with the original one, in agreement with the convergence criterion, Eq. (34).

The results in Figs. 4–6 also show that the permittivity profile is poorly reconstructed. Although the

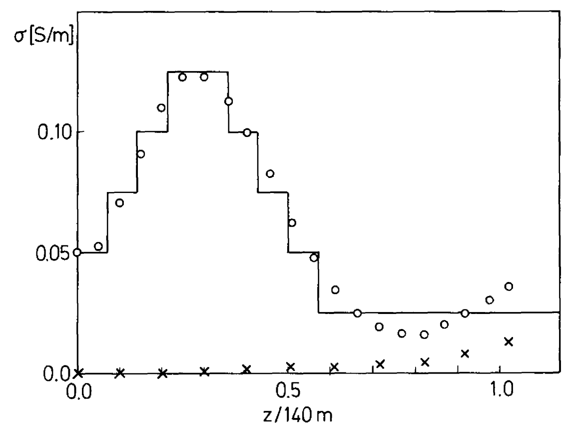

Fig. 6. Reconstruction of a staircase conductivity profile (o = real part, × = imaginary part. *solid* = original profile). The length unit is the skin depth in the high-conductivity layer at 100 Hz

relative permittivity in all the models considered was kept at unity, the numerical results give a non-zero if small imaginary part for the conductivity, which by Eq. (8), corresponds to a non-unitary permittivity value. For a reliable recovery of the permittivity profile, the frequency has to be high enough to make the real and imaginary parts of the potential (8) of the same order of magnitude. This means, in practice, that the permittivity profile can be determined reliably by the inverse scattering method only in strata of very low conductivity as, otherwise, the skin depth will be too small to allow any useful information to be obtained.

Because a staircase variation of the permeability by Eq. (8) would make the potential singular and thus numerically difficult to recover, the permeability was not allowed to vary in these examples. A non-vanishing imaginary conductivity, however, might also be interpreted as a permeability profile with a non-vanishing second derivative. In order to separate the permeability, permittivity and conductivity profiles using the frequency dependence as described at the end of the second section, one might be forced to use impracticably high frequencies. In practice, however, it might be a very good approximation to neglect the permeability variations completely.

Conclusions

The main qualitative result of the present development of the inverse scattering solution to the problem of determining the electromagnetic parameter profiles of a subterranean structure with only vertical variation from the surface reflection coefficient for electromagnetic dipole radiation is that a complete determination may be achieved by measurements at two fixed frequencies. This result substantially increases the utility of the inverse scattering method in interpretation of electromagnetic prospecting data as such data are invariably obtained with field apparatus with a restricted low-frequency range. Previous investigations of the inverse scattering method for electromagnetic prospecting work (Weidelt, 1972; Riska and Vidberg, 1983) dealt with reflection of plane wave radiation for which the solution to the inverse scattering problem requires knowledge of the reflection coefficient over the whole frequency spectrum $0 \leqq \omega \leqq \infty$, and thus had to confront the question of extrapolating the data into the unmeasurable high-frequency range.

The solution to the inverse scattering problem for dipole radiation at fixed frequency is obtained by solving a linear integral equation of the Marchenko-type (Agranovich and Marchenko, 1964), Eq. (27), based on Weidelt's (1972) representation for the Jost solutions, Eq. (20). This integral equation has the inverse Laplace transform of the reflection coefficient as a driving term. While the solution to the integral equation can be obtained by straightforward methods, the determination of the inverse Laplace transform requires a numerical inversion of Eq. (25). To solve the inverse scattering problem numerically we use the method of Padé approximants to determine the inverse Laplace transform of the reflection coefficient and then solve the inverse scattering integral equation by iteration. This method is shown to be adequate for the recovery of the potential function that contains the electromagnetic parameter profiles for depths less than the penetration depth. On physical grounds it should also be obvious that the solution should be unstable for depths beyond the average skin depth of the subterranean structure which are essentially unprobed by the radiation field.

The solution to the inverse scattering problem has been developed here for the cases of horizontal and vertical electric and magnetic dipole radiation sources. The solutions in the case of the magnetic dipole sources and the horizontal electric dipole source are formally similar. In the case of a vertical electric dipole source the potential function containing the electromagnetic parameter profiles is more complicated than in the other cases and the disentangling of these profiles is far more complicated. The simpler cases of a vertical magnetic and horizontal electric dipole source are fortunately those that correspond to practical field equipment used in prospecting work.

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