## Werk

**Jahr:** 1985
**Kollektion:** fid.geo
**Signatur:** 8 Z NAT 2148:56
**Digitalisiert:** Niedersächsische Staats- und Universitätsbibliothek Göttingen
**Werk Id:** PPN1015067948_0056
**PURL:** http://resolver.sub.uni-goettingen.de/purl?PPN1015067948_0056

**LOG Id:** LOG_0041
**LOG Titel:** The method of integral equation in the direct current resistivity method and its accuracy
**LOG Typ:** article

## Übergeordnetes Werk

**Werk Id:** PPN1015067948
**PURL:** http://resolver.sub.uni-goettingen.de/purl?PPN1015067948
**OPAC:** http://opac.sub.uni-goettingen.de/DB=1/PPN?PPN=1015067948



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J Geophys (1985) 56:192–200

Journal of Geophysics

# The method of integral equation in the direct current resistivity method and its accuracy

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**Abstract.** Model calculations of potential are necessary to interpret quantitatively d – c geoelectrical measurements over three-dimensional bodies. The method of integral equation is very suitable for a body buried in a horizontally stratified half-space. In this approach, the integral representation for the surface potential is not valid for completely conducting bodies. On the other hand, the integral representation for the so-called surface charge density is generally valid.

Solving the integral equation numerically, discretization errors appear. These errors are not negligible because the numerical procedure has only a linear order of convergence. To improve the numerical results, an extrapolation can be used.

Model calculations demonstrate the importance of the ratio of the resistivities of the overburden and the substratum. The more conducting the substratum in relation to the overburden, the more significant is the effect which is caused by a conducting body.

**Key words:** D.c. resistivity method - Integral equation - 3*D* modelling - Boundary element method - Numerical accuracy

## Introduction

In the direct current resistivity method, information about the real distribution of resistivity of the underground is gained by measurements of apparent resistivity on the surface of the earth. The apparent resistivity is the quotient of measured potential and introduced current, multiplied by a geometry factor. This factor is defined so that the apparent resistivity is equal to the real resistivity in the case of a homogeneous isotropic ground. The apparent resistivity of the frequently used Schlumberger array (Fig. 1) is formulated as follows:

$$\rho_a = \pi \frac{(AB/2)^2 - (MN/2)^2}{MN} \cdot \{U(\mathbf{r}_M, \mathbf{r}_A) - U(\mathbf{r}_N, \mathbf{r}_A) - U(\mathbf{r}_M, \mathbf{r}_B) + U(\mathbf{r}_N, \mathbf{r}_B)\}/I \tag{1}$$

and analogously for the Schlumberger half-array (Fig. 1)

$$\rho_a = 2\pi \frac{AO^2 - (MN/2)^2}{MN} \{U(\mathbf{r}_M, \mathbf{r}_A) - U(\mathbf{r}_N, \mathbf{r}_A)\}/I. \tag{2}$$

Theoretical values of potential distribution are necessary to interpret field measurements. There are model calculations (e.g. Koefoed, 1979) and inversion procedures (Mundry and Dennert, 1980) for the case of a horizontally stratified earth. The interpretation for more complicated geological structures is more difficult. Generally, potential distribution can be calculated only by numerical methods, e.g. by finite differences (Dey and Morrison, 1979; Mundry, 1984), finite elements (Coggon, 1971) or by the method of integral equations (Dieter et al., 1969; Okabe, 1981; 1982).

The integral equation approach is especially suitable in the case of three-dimensional problems. In this approach, the calculation can be reduced to a two-dimensional problem and its numerical solution can be carried out on a computer with an acceptable computation time.

Although the integral equation approach has been well known for a long time and is also used for elec-

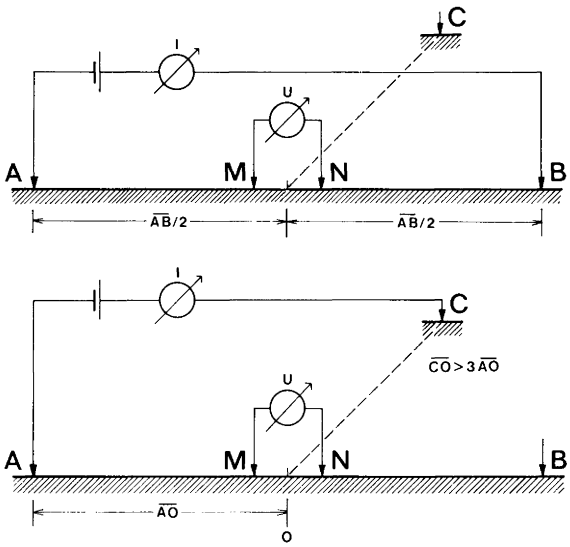


**Fig. 1.** Schlumberger array (*above*) and Schlumberger half-array (*below*). $\overline{AB}/2 = |\mathbf{r}_A - \mathbf{r}_B|/2 = \overline{AO}$; $\overline{MN} = |\mathbf{r}_M - \mathbf{r}_N|$; $\overline{MN} \ll \overline{AB}$

tromagnetics, there are no papers about the numerical accuracy of the method.

### Fundamental solutions of the potential equation

Suppose that a point source of current density $I$ is located at $\mathbf{r}_A$, then the potential equation for an isotropic medium of conductivity $\sigma$ can be written as

$$-\nabla\cdot\{\sigma(\mathbf{r})\nabla U(\mathbf{r},\mathbf{r}_A)\}=I\delta(\mathbf{r}-\mathbf{r}_A), \tag{3}$$

where $\delta$ is the Dirac function. For a homogeneous space of resistivity $\rho_1$, Eq. (3) is reduced to Poisson's equation

$$\Delta U(\mathbf{r},\mathbf{r}_A)=-I\rho_1\delta(\mathbf{r}-\mathbf{r}_A). \tag{4}$$

Its unique solution is given by

$$U(\mathbf{r},\mathbf{r}_A)=\frac{I\rho_1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}_A|}. \tag{5}$$

For a homogeneous half-space the potential can be derived by the method of images

$$U(\mathbf{r},\mathbf{r}_A)=\frac{I\rho_1}{4\pi}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_A|}+\frac{1}{|\mathbf{r}-\tilde{\mathbf{r}}_A|}\right) \tag{6}$$

with $\tilde{\mathbf{r}}_A=(x_A, y_A, -z_A)$ as the image of the source point.

On boundary interfaces $S$, at which the conductivity is discontinuously changing, the normal current density and the potential itself are continuous:

$$\sigma(\mathbf{r}_0^+)\mathbf{n}\cdot\nabla U(\mathbf{r}_0^+,\mathbf{r}_A)-\sigma(\mathbf{r}_0^-)\mathbf{n}\cdot\nabla U(\mathbf{r}_0^-,\mathbf{r}_A)=0, \tag{7}$$

$$U(\mathbf{r}_0^+,\mathbf{r}_A)-U(\mathbf{r}_0^-,\mathbf{r}_A)=0, \tag{8}$$

where $\mathbf{r}_0$ is a point on $S$; $\mathbf{n}$ is the normal unit vector of $S$, which is outwardly directed in the case of closed surfaces or has the direction of the positive coordinates in the case of a plane. The superscripts + and − at $\mathbf{r}_0$ denote the direction of limiting.

In electrostatics, charges are influenced at an uncharged body by a point source. The sum of the influenced charges over the whole body vanishes. The charge influenced at the surface $S$ of the body is determined by the Gauss theorem

$$\hat{\mu}(\mathbf{r}_0)=\varepsilon_0[\mathbf{n}\cdot\mathbf{E}(\mathbf{r}_0^+,\mathbf{r}_A)-\mathbf{n}\cdot\mathbf{E}(\mathbf{r}_0^-,\mathbf{r}_A)],$$

where $\mathbf{r}_0$ is a point on $S$, $\hat{\mu}$ is the surface charge density ($As/m^2$) and $\varepsilon_0$ is the dielectric constant (e.g. Smythe, 1968). The analogue of the Gauss theorem is formulated for stationary currents as follows:

$$\tilde{\mu}(\mathbf{r}_0)=\sigma[\mathbf{n}\cdot\mathbf{E}(\mathbf{r}_0^+,\mathbf{r}_A)-\mathbf{n}\cdot\mathbf{E}(\mathbf{r}_0^-,\mathbf{r}_A)].$$

$\tilde{\mu}$ is now a surface current density ($A/m^2$). In the original paper (Dieter et al., 1969) and in papers following it, the value $\mu=\tilde{\mu}/\sigma$ is used for the derivation of the integral equation; $\mu$ ($V/m$) was denoted as surface charge density. The same notation will be used in this paper.

The normal derivative of potential is discontinuous at interface boundaries, Eq. (7), and the discontinuity is defined as

$$\mu(\mathbf{r}_0)=\mathbf{n}\cdot\nabla U(\mathbf{r}_0^-,\mathbf{r}_A)-\mathbf{n}\cdot\nabla U(\mathbf{r}_0^+,\mathbf{r}_A). \tag{9}$$

In the case of a horizontally stratified half-space the conductivity only depends on the depth $z$ and is constant in each layer:

$$\sigma(z)=\sigma_i;\quad z_{i-1}\leqq z<z_i,\quad i=1,\ldots,n;\quad z_0=0,\quad z_n=\infty.$$

The thickness of the $i$-th layer is denoted by

$$h_i=z_i-z_{i-1}.$$

In order to derive the integral representation and to solve the integral equation, the potential has to be formulated for any arbitrary point in space. The potential can be calculated by a recursive procedure, used also in electromagnetics (Weidelt, 1978). Exponential expressions with positive exponents are avoided in this recursive procedure. In doing this, good stability in numerical calculation is obtained.

The differential equation, Eq. (3), is solved by separation into cylindrical coordinates (e.g. Morse and Feshbach, 1953). The complete solution is given by

$$U(\mathbf{r},\mathbf{r}_A)=\frac{I}{2\pi}\int_0^\infty f(z,z_A,\lambda)J_0(\lambda\bar{r})d\lambda, \tag{10}$$

with $\bar{r}=\sqrt{(x-x_A)^2+(y-y_A)^2}$. $J_0$ is the Bessel function of order zero. The unknown function $f$ has to be derived from the differential equation, Eq. (3), and the boundary conditions, Eqs. (7) and (8). Assuming a fictitious boundary plane $z_k$ for the plane of the point source ($z_k=z_A$) and assuming that the point $\mathbf{r}$ is situated below $r_A$ in the $j$-th layer ($z_A\leqq z_{j-1}\leqq z<z_j$), function $f$ can be given at the boundary $z_{j-1}$ by (Schulz, 1983)

$$f(z_{j-1},z_A,\lambda)=\frac{e^{-\lambda(z_{j-1}-z_A)}}{\sigma(z_A)(T_{k+1}+R_{k+1})}\prod_{i=k+1}^{j-1}\frac{1+T_i}{1+\frac{\sigma_{i+1}}{\sigma_i}T_{i+1}}. \tag{11}$$

Expressions $T_i$ and $R_i$ are calculated by the following recursion formulae:

$$T_i=\begin{cases}1; & i=n\\ \dfrac{\sigma_i\tanh(\lambda h_i)+\sigma_{i+1}T_{i+1}}{\sigma_i+\sigma_{i+1}T_{i+1}\tanh(\lambda h_i)}; & i=n-1,\ldots,k+1\end{cases} \tag{12a}$$

$$R_i=\begin{cases}0; & i=1\\ \dfrac{\sigma_{i-1}}{\sigma_i}\dfrac{\tanh(\lambda h_{i-1})+R_{i-1}}{R_{i-1}\tanh(\lambda h_{i-1})+1}; & i=2,\ldots,k+1.\end{cases} \tag{12b}$$

Now the function $f$, Eq. (11), can be continued for an arbitrary point in the $j$-th layer. If $j=n$, i.e. $\mathbf{r}$ is a point in the last layer, then

$$f(z,z_A,\lambda)=f(z_{n-1},z_A,\lambda)e^{-\lambda(z-z_{n-1})}. \tag{13a}$$

Otherwise, if $j<n$, then

$$f(z,z_A,\lambda)=f(z_{j-1},z_A,\lambda)\{\cosh[\lambda(z-z_{j-1})]-T_j\sinh[\lambda(z-z_{j-1})]\}. \tag{14}$$

In this equation, a difference of high values can appear. For stability, it is useful to reconstruct Eq. (14) in such

a manner that only negative exponents will appear. Equation (12a) yields

$$\frac{\sigma_i+\sigma_{i+1}T_{i+1}}{\sigma_i-\sigma_{i+1}T_{i+1}}e^{\lambda h_i}=\frac{1+T_i}{1-T_i}e^{-\lambda h_i}.$$

Inserting this into Eq. (14) we get, for $z_{j-1}\leqq z<z_j$ with $j<n$,

$$f(z,z_A,\lambda)=1/2\,f(z_{j-1},z_A,\lambda)(1+T_j)\cdot\left[e^{-\lambda(z-z_{j-1})}+e^{-\lambda(z_j-z+h_j)}\frac{\sigma_j-\sigma_{j+1}T_{j+1}}{\sigma_j+\sigma_{j+1}T_{j+1}}\right]. \tag{13b}$$

Equation (13), connected with Eq. (11), is valid for $z\geqq z_A$; while reciprocity can be used for $z<z_A$.

In the special case, of source and measuring point lying at the surface of the earth ($z=z_A=0$), Eq. (11) is reduced to

$$f(0,0,\lambda)=1/\sigma_1 T_1,$$

where $T_1$ is calculated by Eq. (12a). The potential, Eq. (10), becomes

$$U(\mathbf{r},\mathbf{r}_A)=\frac{I}{2\pi\sigma_1}\int_0^\infty\frac{1}{T_1(\lambda)}J_0(\lambda\bar{r})d\lambda.$$

This representation is identical with the formula of Pekeris cited by Koefoed (1979), considering that the kernel $K_i$ of Koefoed is identical to the expression $1/T_i$ of this paper.

The normal derivatives $\mathbf{n}\cdot\nabla U$ and, therefore, the partial derivatives of $U$ are needed for the derivation of the integral representation of the potential of a buried body. Considering $J_0'(r)=-J_1(r)$, we obtain

$$\frac{\partial}{\partial x}U(\mathbf{r},\mathbf{r}_A)=-\frac{I}{2\pi}\frac{x-x_A}{\bar{r}}\int_0^\infty f(z,z_A,\lambda)\lambda J_1(\lambda\bar{r})d\lambda,$$

$$\frac{\partial}{\partial y}U(\mathbf{r},\mathbf{r}_A)=-\frac{I}{2\pi}\frac{y-y_A}{\bar{r}}\int_0^\infty f(z,z_A,\lambda)\lambda J_1(\lambda\bar{r})d\lambda,$$

$$\frac{\partial}{\partial z}U(\mathbf{r},\mathbf{r}_A)=\frac{I}{2\pi}\int_0^\infty\frac{\partial}{\partial z}f(z,z_A,\lambda)J_0(\lambda\bar{r})d\lambda.$$

All the integrals above and the integral of Eq. (10) have the form

$$h(\bar{r})=\int_0^\infty g(\lambda)J_\nu(\lambda\bar{r})d\lambda;\qquad \nu=0,1.$$

Such integrals can be calculated by the theory of linear filters (Ghosh, 1971a; b). Filter coefficients, upper and lower limits of summation depend on the discretization width and the required accuracy. The coefficients in this paper are computed by the method of fast Hankel transforms published by Johansen and Sørensen (1979).

## The method of integral equation

The potential of a buried body (Fig. 2) can be given by an integral representation. The potential outside the body will be denoted by $U^{(1)}$, the potential within the body by $U^{(2)}$. The body $K$ with surface $S$ has conductivity $\sigma_K(\mathbf{r})$; the space outside the body is denoted by $\Omega$. The surface $S$ is formed so that Green's theorems can be applied. The extended derivation of the following formulae can be found in Okabe (1981; 1982) and Schulz (1983).

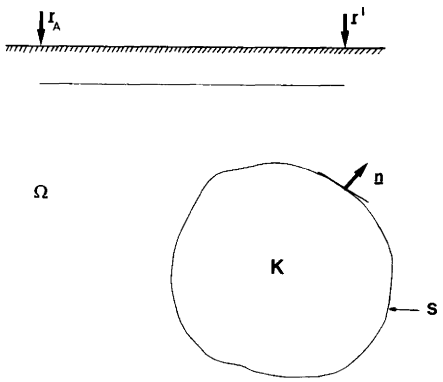


**Fig. 2.** Body buried in an inhomogeneous half-space

It is supposed that fundamental solutions $\Psi^{(1)}$ in the space and $\Psi^{(2)}$ in the body exist, governed by:

$$\begin{aligned}-\nabla\cdot\{\sigma(\mathbf{r})\nabla\psi^{(1)}(\mathbf{r},\mathbf{r}')\}&=\delta(\mathbf{r}-\mathbf{r}');\quad \mathbf{r}\in\Omega\\ -\nabla\cdot\{\sigma_K(\mathbf{r})\nabla\psi^{(2)}(\mathbf{r},\mathbf{r}')\}&=\delta(\mathbf{r}-\mathbf{r}');\quad \mathbf{r}\in K.\end{aligned} \tag{15}$$

Fundamental solutions are, for example

$$\Psi(\mathbf{r},\mathbf{r}')=U(\mathbf{r},\mathbf{r}')/I$$

with the potential $U$ of
- Eq. (5) for a homogeneous space,
- Eq. (6) for a half-space,
- Eq. (10) for a layered half-space.

To apply Green's formulae, fundamental solutions, Eq. (15), and differential expressions of the potential, Eq. (3), are multiplied and integrated over corresponding domains.

Assuming the source point $\mathbf{r}_A$ and the point $\mathbf{r}$ are outside the body, the potential can be formulated as follows (differentiations and integrations – *do* – are taken with respect to the variable $\mathbf{r}$):

$$\begin{aligned}U^{(1)}(\mathbf{r}',\mathbf{r}_A)&=I\psi^{(1)}(\mathbf{r}_A,\mathbf{r}')\\&+\int_S\{U^{(1)}(\mathbf{r},\mathbf{r}_A)[\sigma(\mathbf{r})\mathbf{n}\cdot\nabla\psi^{(1)}(\mathbf{r},\mathbf{r}')-\sigma_K(\mathbf{r})\mathbf{n}\cdot\nabla\psi^{(2)}(\mathbf{r},\mathbf{r}')]\\&+\sigma(\mathbf{r})\mathbf{n}\cdot\nabla U^{(1)}(\mathbf{r},\mathbf{r}_A)[\psi^{(2)}(\mathbf{r},\mathbf{r}')-\psi^{(1)}(\mathbf{r},\mathbf{r}')]\}\,do.\end{aligned} \tag{16}$$

The potential of a point source for a buried body consists of the usual potential and a disturbed potential depending on the unknown surface potential *and* its derivative. Now this integral representation has to be reduced by Green's formula in such a way that the integral will be carried out over the potential itself *or* over its normal derivative.

The second term of Eq. (16)

$$\int_S\sigma(\mathbf{r})\mathbf{n}\cdot\nabla U^{(1)}(\mathbf{r},\mathbf{r}_A)\{\psi^{(2)}(\mathbf{r},\mathbf{r}')-\psi^{(1)}(\mathbf{r},\mathbf{r}')\}\,do \tag{16a}$$

can be rebuilt by boundary condition, Eq. (7), to form the expression

$$\int_S \sigma_K(\mathbf{r})\mathbf{n}\cdot\nabla U^{(2)}(\mathbf{r},\mathbf{r}_A)\{\psi^{(2)}(\mathbf{r},\mathbf{r}')-\psi^{(1)}(\mathbf{r},\mathbf{r}')\}\,do,$$

to which Green's second formula can be applied. This procedure is only valid if the conductivity of the body is constant: $\sigma_K(\mathbf{r})=\sigma_K$. As the potential $U^{(2)}$ and the fundamental solutions $\Psi^{(1)}$ and $\psi^{(2)}$ satisfy the potential equations, Eqs. (3) and (15), within the body, the volume integral vanishes. From Eq. (16a) we obtain

$$\int_S \sigma_K U^{(2)}(\mathbf{r},\mathbf{r}_A)\mathbf{n}\cdot\nabla\{\psi^{(2)}(\mathbf{r},\mathbf{r}')-\psi^{(1)}(\mathbf{r},\mathbf{r}')\}\,do.$$

This term is inserted into Eq. (16) instead of the second integral term and with respect to the continuity of the potential, Eq. (8), the following integral representation is obtained

$$U^{(1)}(\mathbf{r}',\mathbf{r}_A)=I\psi^{(1)}(\mathbf{r}_A,\mathbf{r}')+\int_S U^{(1)}(\mathbf{r},\mathbf{r}_A)[\sigma(\mathbf{r})-\sigma_K]\mathbf{n}\cdot\nabla\psi^{(1)}(\mathbf{r},\mathbf{r}')do. \tag{17}$$

This representation is only valid if

$$\sigma_K<\infty, \quad \text{i.e. } \rho_K\not\equiv 0. \tag{18}$$

The unknown surface potential is determined by an integral equation. If the point $\mathbf{r}'$ tends to a point $\mathbf{r}_0$ of the surface $S$, we obtain

$$U(\mathbf{r}_0^+,\mathbf{r}_A)=I\psi(\mathbf{r}_A,\mathbf{r}_0)+\frac{\sigma(\mathbf{r}_0)-\sigma_K}{2\sigma(\mathbf{r}_0)}U(\mathbf{r}_0,\mathbf{r}_A)+\int_S U(\mathbf{r},\mathbf{r}_A)[\sigma(\mathbf{r})-\sigma_K]\mathbf{n}\cdot\nabla\psi(\mathbf{r},\mathbf{r}_0)do.$$

The integral becomes singular, but convergent; the integration has to be carried out without the singularity at $\mathbf{r}_0$. The indices (1), appearing at potential and fundamental solutions, are omitted.

The equation for the determination of the surface potential is obtained

$$\frac{\sigma(\mathbf{r}_0)+\sigma_K}{2\sigma(\mathbf{r}_0)}U(\mathbf{r}_0,\mathbf{r}_A)=I\psi(\mathbf{r}_A,\mathbf{r}_0)+\int_S U(\mathbf{r},\mathbf{r}_A)[\sigma(\mathbf{r})-\sigma_K]\mathbf{n}\cdot\nabla\psi(\mathbf{r},\mathbf{r}_0)do. \tag{19}$$

Every constant potential is an eigensolution of Eq. (19) in the case of $\sigma_K$ tending to infinity. Therefore, and because of Eq. (18), the case of a highly conducting body, which is very important for the exploration of ore, can not be treated by the integral equation for surface potential. This method produces incorrect results for cases of resistivity $\rho_K$ tending to 0 (see Table 2). It is for this reason that this method, published by Barthes and Vasseur (1978), will not be used further.

The first term in Eq. (16)

$$\int_S U^{(1)}(\mathbf{r},\mathbf{r}_A)\{\sigma(\mathbf{r})\mathbf{n}\cdot\nabla\psi^{(1)}(\mathbf{r},\mathbf{r}')-\sigma_K(\mathbf{r})\mathbf{n}\cdot\nabla\psi^{(2)}(\mathbf{r},\mathbf{r}')\}\,do \tag{16b}$$

will now be rebuilt in such a way that only the derivative of the potential will appear. $U^{(1)}$ is replaced by $U^{(2)}$ because of the continuity of the potential, Eq. (8). In analogy to the derivation above, the following expression is obtained by Green's formula

$$\int_S \mathbf{n}\cdot\nabla U^{(2)}(\mathbf{r},\mathbf{r}_A)\{\sigma(\mathbf{r})\psi^{(1)}(\mathbf{r},\mathbf{r}')-\sigma_K\psi^{(2)}(\mathbf{r},\mathbf{r}')\}\,do.$$

According to the boundary condition, Eq. (7), the integral representation, Eq. (16), can be reconstructed to give

$$U(\mathbf{r}',\mathbf{r}_A)=I\psi(\mathbf{r}_A,\mathbf{r}')+\int_S \sigma(\mathbf{r})\mathbf{n}\cdot\nabla U^{(1)}(\mathbf{r},\mathbf{r}_A)\frac{\sigma(\mathbf{r})-\sigma_K}{\sigma_K}\psi^{(1)}(\mathbf{r},\mathbf{r}')do. \tag{20}$$

The unknown normal derivative of potential is replaced by the also unknown surface charge density. Equations (7) and (9) yield

$$\mathbf{n}\cdot\nabla U^{(1)}(\mathbf{r}_0^+,\mathbf{r}_A)=\sigma_K\mu(\mathbf{r}_0)/\{\sigma(\mathbf{r}_0)-\sigma_K\}.$$

Replacing this in Eq. (20), the integral representation over the surface charge density is obtained

$$U(\mathbf{r}',\mathbf{r}_A)=I\psi(\mathbf{r}_A,\mathbf{r}')+\int_S \sigma(\mathbf{r})\mu(\mathbf{r})\psi(\mathbf{r},\mathbf{r}')do. \tag{21}$$

In this representation the surface charge density $\mu$ is unknown. Differentiating this equation with respect to $\mathbf{r}'$ and letting $\mathbf{r}'$ tend to a point $\mathbf{r}_0$ on the surface of the body, an integral equation for the surface charge density is obtained:

$$\frac{1}{2}\frac{\sigma(\mathbf{r}_0)+\sigma_K}{\sigma(\mathbf{r}_0)-\sigma_K}\mu(\mathbf{r}_0)=I\mathbf{n}\cdot\nabla\psi(\mathbf{r}_A,\mathbf{r}_0)+\int_S \mu(\mathbf{r})\sigma(\mathbf{r})\mathbf{n}\cdot\nabla\psi(\mathbf{r},\mathbf{r}_0)do. \tag{22}$$

Inserting the solution of this integral equation into Eq. (21), the potential $U$ can be calculated for every point $\mathbf{r}'$ outside the body. In the special case of a homogeneous half-space, Eqs. (21) and (22) with the corresponding fundamental solution yield the well known integral representation and equation (Dieter et al., 1969).

Without entering into the particulars, it should be noted that there are other integral representations and methods to solve the problem, see e.g. Lee (1975).

## The numerical treatment of the integral equation

Integral equation, Eq. (22), can generally be solved only by numerical methods. The unknown surface charge density can be approximated by piecewise defined interpolation polynoms, the coefficients of which have to be determined (boundary element method). In this approach, the surface $S$ of the body is completely divided into $N$ boundary elements $S_q$ (Fig. 3). For practical treatment, the boundary element method is used in the simplest way (Riehle, 1979): the unknown function $\mu$ is approximated by a function $\mu_N$, which is constant on every boundary element $S_q$.

$$\mu_N(\mathbf{r})=\sum_{q=1}^{N} d_q(\mathbf{r})\mu_q \quad \text{with } d_q(\mathbf{r})=\begin{cases}1; & r\in S_q\\ 0; & r\notin S_q\end{cases}$$

The integral equation, Eq. (22), is transformed by this approach to a system of linear equations

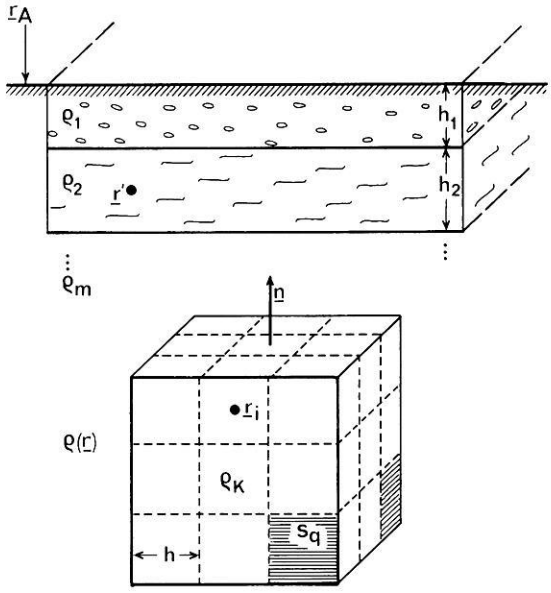


**Fig. 3.** Three-dimensional body (cube) in a horizontally stratified half-space; dissection of the surface into boundary elements

$$\frac{1}{2}\frac{\sigma(\mathbf{r}_i)+\sigma_K}{\sigma(\mathbf{r}_i)-\sigma_K}\mu_i = I\,\mathbf{n}\cdot\nabla\psi(\mathbf{r}_A,\mathbf{r}_i) + \sum_{q=1}^{N}\mu_q\int_{S_q}\sigma(\mathbf{r})\mathbf{n}\cdot\nabla\psi(\mathbf{r},\mathbf{r}_i)\,do; \qquad i=1,\ldots,N. \tag{23}$$

The potential $U$, Eq. (21), is approximately represented, with the solution of Eq. (23), by

$$U_N(\mathbf{r}',\mathbf{r}_A) = I\psi(\mathbf{r}_A,\mathbf{r}') + \sum_{q=1}^{N}\mu_q\int_{S_q}\sigma(\mathbf{r})\psi(\mathbf{r},\mathbf{r}')\,do. \tag{24}$$

The integrals over the fundamental solutions can be exactly calculated only in special cases, e.g. if $\Omega$ is a homogeneous half-space and the boundary elements $S_q$ are planar facets (see Barnett, 1972). But it is sufficient to calculate the integrals numerically. The quadrature formula should be chosen in such a way that the integrals over the approximating functions are calculated exactly. In our case it is sufficient to use the simplest Gauss quadrature formula. Then the linear equations, Eq. (23), have the form

$$\frac{1}{2}\frac{\sigma(\mathbf{r}_i)+\sigma_K}{\sigma(\mathbf{r}_i)-\sigma_K}\bar{\mu}_i = I\,\mathbf{n}\cdot\nabla\psi(\mathbf{r}_A,\mathbf{r}_i) + \sum_{q=1}^{N}\bar{\mu}_q\sigma(\mathbf{r}_q)\mathbf{n}\cdot\nabla\psi(\mathbf{r}_q,\mathbf{r}_i)|S_q|; \qquad i=1,\ldots,N, \tag{25}$$

where $\mathbf{r}_q$ is the centre of gravity and $|S_q|$ is the area of the boundary element $S_q$. The potential, Eq. (24), runs as follows

$$\bar{U}_N(\mathbf{r}',\mathbf{r}_A) = I\psi(\mathbf{r}_A,\mathbf{r}') + \sum_{q=1}^{N}\bar{\mu}_q\sigma(\mathbf{r}_q)\psi(\mathbf{r}_q,\mathbf{r}')|S_q| \tag{26}.$$

The same system of equations and expression for the potential can be obtained by using Gauss simplest

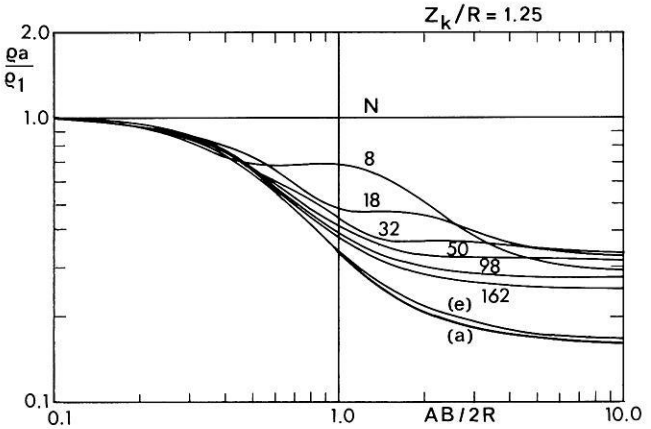


**Fig. 4.** Influence of the numerical discretization, represented by model curves of apparent resistivity of a Schlumberger sounding array for a sphere (radius $R$, depth of the centre $1.25\,R$, resistivity $\rho_K/\rho_1=0$) buried in a homogeneous half-space. Point of measurement: epicentre of the sphere. $N$: number of boundary elements; ($e$): extrapolated values, Eq. (28); ($a$): analytical (exact) values

**Table 1.** Extrapolation table for the numerical solution of the integral equation, calculated for one model value of the apparent resistivity $\rho_a/\rho_1$ of a Schlumberger array ($AB/2 = 100\,R$) for a sphere (radius $R$, depth of the centre $1.25\,R$, resistivity $\rho_K/\rho_1=0$) buried in a homogeneous half-space (resistivity $\rho_1$). Point of measurement: epicentre of the sphere. $N$: number of boundary elements, $h_N$: width of the elements, which are quadratic in bi-spherical coordinates, $\rho_a^{(N)}$: value calculated by Eq. (26), $\rho_a^{(a)}$: analytical value (Snyder and Merkel, 1973), $V_{ik}$, $W_{ik}$: extrapolated values (Stoer, 1972) with

$$V_{ik} = \sum_{j=i-k}^{i} c_{kj}^{(i)} V_{i0} \quad \text{with} \quad c_{kj}^{(i)} = \prod_{\substack{l=i-k \\ l\neq j}}^{i} h_l/(h_l-h_j)$$

$$W_{ik} = 2\,V_{i+1,k} - V_{ik}$$

$(i=2,\ldots,12;\ k=1,2,3)$

| $N$ | $h_N$ | $i=\pi/h_N$ | $\rho_a^{(N)} = V_{i0}$ | $V_{i1}$ | $V_{i2}$ | $V_{i3}$ | $\rho_a^{(a)}$ |
|---|---|---|---|---|---|---|---|
| 8 | 1.57 | 2 | 0.2810 | | | | |
| 18 | 1.05 | 3 | 0.3205 | 0.3995 | | | |
| 32 | 0.78 | 4 | 0.3267 | 0.3454 | 0.2914 | | |
| 50 | 0.63 | 5 | 0.3115 | 0.2508 | 0.1087 | −0.0131 | |
| 72 | 0.52 | 6 | 0.2923 | 0.1962 | 0.0871 | 0.0654 | |
| 98 | 0.45 | 7 | 0.2754 | 0.1739 | 0.1182 | 0.1596 | |
| 128 | 0.39 | 8 | 0.2618 | 0.1668 | 0.1455 | 0.1911 | |
| 162 | 0.35 | 9 | 0.2511 | 0.1651 | 0.1592 | 0.1866 | |
| 200 | 0.31 | 10 | 0.2425 | 0.1649 | 0.1638 | 0.1745 | |
| 242 | 0.29 | 11 | 0.2354 | 0.1648 | 0.1643 | 0.1657 | |
| 288 | 0.26 | 12 | 0.2295 | 0.1646 | 0.1636 | 0.1612 | |
| | | | | | | | 0.1596 |
| | | 11 | 0.2235 | 0.1644 | 0.1628 | 0.1568 | |
| | | 10 | 0.2283 | 0.1647 | 0.1649 | 0.1570 | |
| | | 9 | 0.2338 | 0.1646 | 0.1684 | 0.1624 | |
| | | 8 | 0.2403 | 0.1634 | 0.1729 | 0.1822 | |
| | | 7 | 0.2482 | 0.1597 | 0.1728 | 0.2250 | |
| | | 6 | 0.2585 | 0.1516 | 0.1493 | 0.2539 | |
| | | 5 | 0.2731 | 0.1416 | 0.0654 | 0.1438 | |
| | | 4 | 0.2963 | 0.1560 | −0.0740 | | |
| | | 3 | 0.3330 | 0.2914 | | | |
| | | 2 | 0.3600 | | | | |
| | | $i$ | $W_{i0}$ | $W_{i1}$ | $W_{i2}$ | $W_{i3}$ | $\rho_a^{(a)}$ |

**Table 2.** Comparison of different methods of calculation of apparent resistivity for a buried body. Model: Sphere (radius $R$, resistivity $\rho_K$, depth of the centre $Z_K/R=2$) in a homogeneous half-space (resistivity $\rho_1$). Array: Schlumberger array with $AB/2=100\,R$. Parameters of the model: $X/R$, $Y/R$ centre of the Schlumberger array; $Z_K/R$ depth of the centre of the sphere; $\rho_K/\rho_1$ ratio of the resistivities. Analytical method: see Brass et al. (1981). Integral equation: surface charge density, Eq. (21) and Eq. (26); surface potential, Eq. (17); $N$ = number of boundary elements; Extrap = extrapolated values, Eq. (28); * = values which are not correct, Eq. (18)

| Parameter | | | | Analytical method | Integral equation | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | Surface charge density | | | Surface potential | | |
| $X/R$ | $Y/R$ | $Z_K/R$ | $\rho_K/\rho_1$ | | $N=98$ | $N=162$ | Extrap. | $N=98$ | $N=162$ | Extrap. |
| 0.0 | 0.0 | 2.0 | 0.0 | 0.7560 | 0.7938 | 0.7864 | 0.7605 | 0.7897 | 0.7830 | 0.7595 |
| 0.0 | 0.0 | 2.0 | 0.1 | 0.8160 | 0.8383 | 0.8338 | 0.8179 | 0.8357 | 0.8316 | 0.8174 |
| 0.0 | 0.0 | 2.0 | 0.2 | 0.8592 | 0.8727 | 0.8699 | 0.8600 | 0.8710 | 0.8685 | 0.8597 |
| 0.0 | 0.0 | 2.0 | 0.5 | 0.9379 | 0.9407 | 0.9401 | 0.9379 | 0.9403 | 0.9398 | 0.9379 |
| 0.0 | 0.0 | 2.0 | 2.0 | 1.0502 | 1.0520 | 1.0516 | 1.0502 | 1.0517 | 1.0514 | 1.0502 |
| 0.0 | 0.0 | 2.0 | 10.0 | 1.1082 | 1.1174 | 1.1152 | 1.1078 | 1.1157 | 1.1139 | 1.1079 |
| 0.0 | 0.0 | 2.0 | 9999.0 | 1.1264 | 1.1392 | 1.1362 | 1.1257 | 1.1367 | 1.1343 | 1.1259 |
| 0.5 | 0.0 | 2.0 | 0.0 | 0.8160 | 0.8452 | 0.8395 | 0.8196 | 0.7662 * | 0.7692 * | 0.7794 * |
| 0.5 | 0.0 | 2.0 | 0.1 | 0.8613 | 0.8785 | 0.8751 | 0.8629 | 0.8566 | 0.8583 | 0.8644 |
| 0.5 | 0.0 | 2.0 | 0.2 | 0.8939 | 0.9043 | 0.9022 | 0.8946 | 0.8938 | 0.8945 | 0.8968 |
| 0.5 | 0.0 | 2.0 | 0.5 | 0.9532 | 0.9554 | 0.9549 | 0.9532 | 0.9533 | 0.9535 | 0.9542 |
| 0.5 | 0.0 | 2.0 | 2.0 | 1.0377 | 1.0392 | 1.0388 | 1.0377 | 1.0392 | 1.0387 | 1.0372 |
| 0.5 | 0.0 | 2.0 | 10.0 | 1.0813 | 1.0884 | 1.0868 | 1.0809 | 1.0869 | 1.0854 | 1.0802 |
| 0.5 | 0.0 | 2.0 | 9999.0 | 1.0950 | 1.1049 | 1.1026 | 1.0944 | 1.1025 | 1.1006 | 1.0937 |

quadrature formula in Eqs. (21) and (22). Then $\bar{\mu}_q$ has the meaning of the value of $\mu$ at the centre of gravity, $\mathbf{r}_q$, but $\mu(\mathbf{r}_q)$ is generally not equal to $\bar{\mu}_q$.

## Convergence of the numerical solution

The problem of how the numerical solution $\bar{\mu}_N$ of Eq. (25) converges to the exact solution $\mu$ of Eq. (22) for $N\to\infty$ has not been analysed yet. The difference between numerical and exact solution can be significant, as Fig. 4 shows. The problem can be solved when theorems of convergence for singular integral equations (Michlin and Prößdorf, 1980) are applied.

The solution of the last section can be interpreted as a special case of the Ritz-Galerkin method (Michlin and Prößdorf, 1980, Chapter 18). In the Appendix, Eq. (33), the convergence of the numerical solution to the exact one is proved; the procedure is stable, and the discretization error is estimated by Eq. (35).

In order to get an impression of the dimensions of this error, the estimation is analysed for a special case: it is assumed that the surface $S$ of the body can be divided into boundary elements $S_q$, each of which is a square of length $h$ (see Fig. 3). The error between the numerically calculated potential $U_N$, Eq. (24), and the exact potential $U$, Eq. (21), can be estimated by Eq. (40):

$$|U_N-U|\leqq c\cdot h. \tag{27}$$

$c$ is a constant, which depends on the continuity of the function and generally can not be more exactly determined. If a quadrature formula is used for solving the integral equation, the estimation of the approximation error is not essentially changed, Eq. (41). The error between the exact potential, Eq. (21), and the potential calculated completely numerically, Eq. (26), is estimated by

$$|\bar{U}_N-U|\leqq c\cdot h+d\cdot h^2.$$

The quadrature formula is less important with regard to convergence. The order of convergence is not improved by exact integration. Equation (27) shows that the considered numerical procedure for solving the integral equation has only a linear order of convergence. This is verified by model calculations. Figure 4 shows model curves for a Schlumberger array (see Eq. (1)) for a sphere ($\rho_K/\rho_1=0$) buried in a homogeneous half-space. The curves are calculated by solving the integral equation numerically, Eqs. (25) and (26). Beforehand, the Cartesian coordinates must have been transformed into bi-spherical coordinates and the integral equation must have been discretized in this coordinate system (Schulz, 1983). So the surface of the sphere has not been approximated by a polygon, as in Barnett (1972). Nevertheless, it is obvious that the numerical solution significantly differs from the analytical solution even if the number of boundary elements is relatively high. The analytical solution is given by a sum of Legendre, trigonometric and exponential functions (e.g. Snyder and Merkel, 1973).

The order of convergence, Eq. (27), can be used to increase the accuracy by an extrapolation technique. Table 1 shows extrapolated values, which are calculated by Lagrange's interpolation formula (Stoer, 1972). It is difficult to give an exact mathematical proof of the validity of this extrapolation, but in all cases the extrapolated values approximate the exact ones very well.

The computation time for producing such tables is very high, but it has turned out that it is sufficient to obtain the extrapolated value by two computations only (see also Fig. 4). If $\mu_{(1)}$ and $\mu_{(2)}$ are the solutions of

two systems of equations which are obtained by two different discretization widths $h_1$ and $h_2$, then the extrapolation yields

$$\bar{\mu}_{(e)}=(1/h_1\bar{\mu}_{(1)}-1/h_2\bar{\mu}_{(2)})/(1/h_1-1/h_2). \quad (28)$$

This extrapolation corresponds to the second column of Table 1. Of course, it must be assumed that the discretization steps $h_i$ are sufficiently small, i.e. the number of boundary elements is sufficiently high. Especially in the case of shallow bodies, for example see Fig. 4, should a close-meshed discretization be chosen.

Table 2 shows that the extrapolation, Eq. (28), generally yields good results. In Table 2 some results of model calculations (Schulz, 1983) are summarized. The model body is a sphere of variable resistivity and constant depth (depth of the centre $z_K=2R$, $R$ is the radius), buried in a homogeneous half-space. It is given the apparent resistivity for a Schlumberger array, Eq. (1), with $AB/2=100R$; centre of coordinates is the epicentre of the sphere. For $\rho_K=0$ the analytical solution has a closed form (Snyder and Merkel, 1973) and for $\rho_K>0$ it is computed by a system of equations (Brass et al., 1981).

The results of Table 2 show significantly:
- The method of integral equation for the surface potential is not valid for a completely conducting body.
- The numerical results of the method of integral equations are different from the exact results, even if the number of boundary elements is high.
- The extrapolation, Eq. (28), yields a good correspondence between numerical and analytical results.

## Model curves

Model curves for a body buried in a stratified half-space can be calculated by the described procedure. Studies have shown that the shape of the body influences the model curve only to a low extent. On the other hand, resistivity of the overburden produces significant effects (Schulz, 1983).

As the number of parameters are high, only the fundamental effect is presented by one example (Fig. 5). A completely conducting cube ($\rho_K=\rho_3=0$) of width $b$, buried in a two-layer earth, is chosen as the model. The overburden has thickness $d/b=0.1$ and resistivity $\rho_1$; the substratum has resistivity $\rho_2$. All dimensions of length are normalized by the width $b$ of the cube, all resistivities are normalized by the resistivity $\rho_1$ of the overburden.

In Fig. 5 the depth of the top of the body ($t/b=0.25$, 0.5, 0.75, 1.0) and the resistivity of the substratum ($\rho_2/\rho_1=10$ - well-conducting overburden; $\rho_2/\rho_1=1$ - homogeneous half-space; $\rho_2/\rho_1=0.1$ - poorly conducting overburden) are varied.

The Schlumberger half-array, Eq. (2), is chosen as the electrode array. This array has been used in field measurements, with good results (Brass et al., 1981).

The shape of the curves is changed according to the point of measurement. If this point is situated before the epicentre of the body, as in Fig. 5, the curves can have slopes of more than 45°.

It is worth noting that the asymptotic value of apparent resistivity for great distances can be higher than

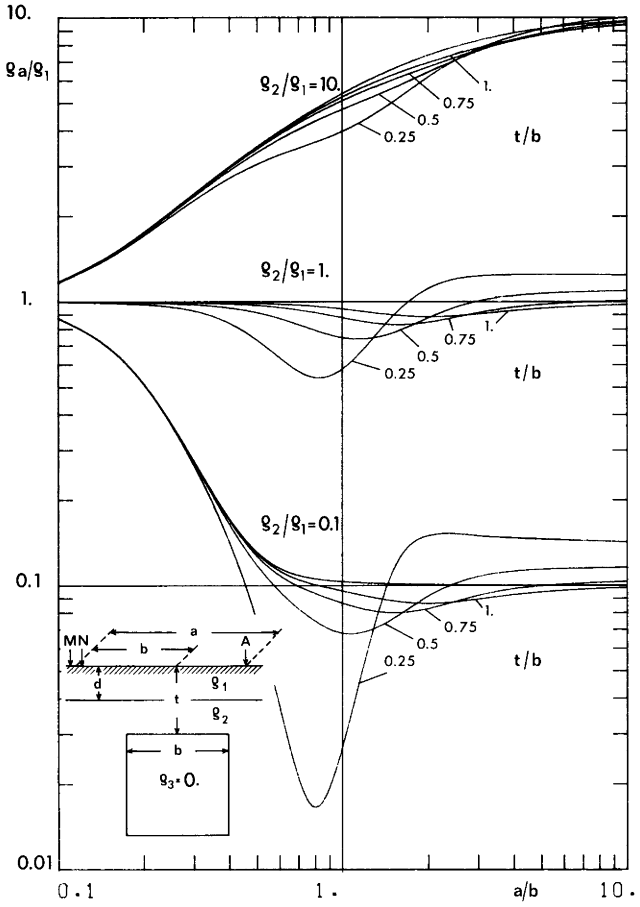


**Fig. 5.** Model curves of apparent resistivity of Schlumberger sounding half-array for a cube (width $b$, $\rho_3=0$) buried in a two-layer earth (thickness of the overburden $d/b=0.1$, resistivity of the overburden $\rho_1$). The curves are plotted for different ratios of resistivity ($\rho_2/\rho_1$) and different depths of the top of the cube ($t/b$); the curve on top of each set represents the two-layer case without a buried body. Point of measurement: $b$ before the epicentre of the cube

the resistivity of the substratum, although the buried body has a vanishing resistivity.

The curves are distinctly marked by the resistivity of the overburden. The lower the ratio $\rho_2/\rho_1$, the more distinct is the shape of the curve. Therefore, the ratio of resistivities of overburden and substratum will influence of depth of investigation for buried bodies.

Field measurements can be interpreted by comparison with such model curves. This method has been applied, with success, in the exploration of graphite (Schulz, 1983).

*Acknowledgements.* The author is greatly indebted to Professor Mundry, Geological Survey of Lower Saxony (NLfB), Hannover, and Professor Weidelt, Federal Institute for Geoscience and Natural Resources (BGR), Hannover, for important suggestions and helpful discussions. The author also wants to thank Professor Rosenbach, Technical University of Clausthal, for many useful hints.

The model calculations have been carried out on Cyber 76 at the Regional Computer Centre of Lower Saxony (RRZN), Hannover. This paper has been supported by the Federal Ministry of Research and Technology (BMFT), Bonn, under project no. NTS 1052-R201 and no. RG 81043.

## Appendix: proof of convergence

The function space, in which the integral equation, Eq. (22), has to be solved, is the space $L_2(S)$ of functions for which the Lebesgue integral of their square exists. The norm of space $L_2(S)$ is defined by

$$\|s\| := \{\int_S [s(\mathbf{r})]^2 do\}^{1/2}; \quad s \in L_2(S). \tag{29}$$

A complete sequence of orthogonal coordinate functions is defined by the functions $d_q(\mathbf{r})$, i.e.

$$(d_p, d_q) = \int_S d_p(\mathbf{r}) d_q(\mathbf{r}) do = \begin{cases} |S_q|; & p = q \\ 0; & p \neq q \end{cases} \tag{30}$$

and

$$\lim_{N\to\infty} \int \left\{ s(\mathbf{r}) - \sum_{q=1}^{N} s_q d_q(\mathbf{r}) \right\}^2 do = 0. \tag{31}$$

$|S_q|$ is the area of the boundary element $S_q$.

The orthogonal projection from the space $L_2(S)$ into the space of piecewise constant functions is noted by $O_N$:

$$O_N(s) = \sum_{q=1}^{N} s(\mathbf{r}_q) d_q(\mathbf{r}), \tag{32}$$

where it is assumed that $\mathbf{r}_q$ is not a singularity of $s$.

With these notes, the following estimation between the exact solution $\mu$ of the integral equation, Eq. (22), and the solution $\mu_N$ of the system of linear equations, Eq. (23), is valid (Michlin and Prößdorf, 1980, Chapter 18)

$$\|\mu_N - \mu\| \leqq (1 + \varepsilon_N) \|(I - O_N)\mu\| \tag{33}$$

with $\lim_{N\to\infty} \varepsilon_N = 0$.

$\mu_N$ converges to the exact solution $\mu$ for $N \to \infty$; also, the procedure is stable.

In the system of equations, Eq. (25), the integrals are numerically calculated. If it is assumed that the functions are differentiable, it can be shown, by expansion in a Taylor series, that

$$\max_{\mathbf{r}\in S} |\bar{\mu}_N(\mathbf{r}) - \mu_N(\mathbf{r})| \leqq c_1 \max |R(\mu, \sigma \mathbf{n} \cdot \nabla \psi)| \tag{34}$$

is valid, where $c_1$ is a constant independent of $\mathbf{r}$ and $R$ is the residual term of the quadrature formula used. Combining Eqs. (33) and (34), we get the solution of the system of equations, Eq. (25), to be:

$$\|\bar{\mu}_N - \mu\| \leqq (1 + \varepsilon_N) \|(I - O_N)\mu\| + \hat{c}_1 \max |R(\mu, \sigma \mathbf{n} \cdot \nabla \psi)|. \tag{35}$$

Applying Schwarz's inequality, it ensues that, for the potential $U_N$, Eq. (24),

$$|U_N - U| \leqq \|\mu_N - \mu\| \, \|\sigma \psi\|. \tag{36}$$

In analogy to Eq. (34), it can be shown that

$$|\bar{U}_N - U_N| \leqq c_2 \max |R(\mu, \sigma \psi)| \tag{37}$$

is valid and, therefore,

$$|\bar{U}_N - U| \leqq \|\mu_N - \mu\| \, \|\sigma \psi\| + \hat{c}_2 \max |R(\mu, \sigma \psi)|. \tag{38}$$

This estimation is analysed for the following special case: the surface $S$ is divided into boundary elements $S_q$, each of which is a square of the same width $h$. It is assumed that all functions of the integral equation and representation are twice continuously differentiable. Then the residual term of the simplest Gauss quadrature formula used can be estimated by Taylor expansion

$$\max_{\mathbf{r}\in S_q} |R| \leqq c_3 h^4.$$

Since the number $N$ of boundary elements is reciprocally proportional to $h^2$, it ensues that

$$\max_{\mathbf{r}\in S} |R| \leqq c_4 h^2 \tag{39}$$

for the whole surface $S$.

The term $\|\mu_N - \mu\|$ is additionally significant for convergence because of Eqs. (38) and (33).

It ensues, from Eq. (32), that

$$\|(I - O_N)\mu\| = \left( \sum_{q=1}^{N} \int_{S_q} (\mu(\mathbf{r}) - \mu(\mathbf{r}_q))^2 do_r \right)^{1/2}.$$

Assuming the differentiability of $\mu$, one gets from Eq. (36)

$$|U_N - U| \leqq c_6 h \tag{40}$$

and from Eq. (37)

$$|\bar{U}_N - U| \leqq c_6 h + c_7 h^2. \tag{41}$$

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Received October 10, 1984; Revised December 21, 1984
Accepted December 21, 1984