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Transient *SH* waves in dipping layers: the buried line-source problem

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Abstract. The theory of generalized rays is applied to analysing transient waves in a layered half-space with non-parallel interfaces. The propagation, transmission, reflection and refraction of *SH* waves generated by a line source which is buried in the underlying half-space (bedrock) of a three-layer model is considered, each of the two overlying layers having a different dip angle.

Generalized ray integrals for multiply refracted rays are formulated by using three rotated coordinate systems, one for each interface. Through a series of transformations of the local slowness and the application of Snell's law, all ray integrals are expressible in a common slowness variable. The arrival time of each ray undergoing multiple transmissions and reflections before reaching an observation point in the top layer is then determined from the stationary value of the phase function with common slowness of the ray integral. Early arrivals of head waves which follow a ray path refracted at a fast bottom are calculated from proper branch points of the Cagniard mapping. Inverse Laplace transformation of these ray integrals is then completed by Cagniard's method.

Key words: Cylindrical waves – Ray integrals – Ray sorting – Arrival times – Head waves – Divergence effect – Synthetic seismograms

Introduction

Recently the theory of generalized ray integrals, originally developed for layered media with parallel surfaces (see e.g. Müller, 1968a, b, c; Wiggins and Helmberger, 1974; Pao and Gajewski, 1977; Kennett, 1980; and, for a comparison with other theories, Aki and Richards, 1980), was extended to account for a single dipping layer by Ishii and Ellis (1970a, b) and Hong and Helmberger (1977). They showed that the exact solution of *SH* waves in a wedge (Hudson, 1963) can be expanded into a series of integrals. One of them represents the radiation from the apex of the wedge and the others can be identified as generalized ray integrals, representing cylindrical waves which are multiply reflected between two plane nonparallel surfaces. These ray integrals are then evaluated by applying the Cagniard-de Hoop method and by using a first motion approximation.

Pao and Ziegler (1982) have shown that by expressing the *SH*-source ray integral in two systems of rotated coordinates, one for each of the two nonparallel surfaces, one can construct successively the ray integrals for multi-reflected waves within the wedge. A comparison with the method of images has been included as far as direct rays are concerned. Their approach was recently generalized to a multilayered medium, Ziegler and Pao (1985), when the line source and the receiver are located in the surface layer.

A report on related site effects has been given by Porceski (1969) and model seismic experiments are described in Drimmel et al. (1973) and in the dissertation by Wiedmann (1983).

We consider a three-layered medium, each of the overlying layers having a different dip angle and non-common apexes. A line source which generates a transient *SH* wave is placed beneath the two top layers. The observational station can be anywhere in the medium, but our interest is confined mainly to that in the surface layer or on the top surface.

Waves emitted by the line source which are incident at the lowest interface (3), see Fig. 1, are represented by $\phi_{\mathrm{inc}}(\mathbf{x}, t)$, the transmitted waves in layer 2 will be denoted $\psi(\mathbf{x}, t)$ and those further transmitted to the top layer 1 are denoted $\phi(\mathbf{x}, t)$. For *SH* waves in solids, ϕ or ψ is the displacement component in the direction of the line source. In a fluid medium, ϕ or ψ is the wave potential, the gradient of which is the velocity. The Laplace-transformed wave potentials are denoted by $\bar{\phi}(\mathbf{x}, s)$ and $\bar{\psi}(\mathbf{x}, s)$. They are solutions to the homogeneous reduced wave equations, except $\bar{\phi}_{\mathrm{inc}}(\mathbf{x}, s)$ which satisfies the inhomogeneous Helmholtz equation

$$\nabla^2 \bar{\phi}_{\mathrm{inc}} - b^2 s^2 \bar{\phi}_{\mathrm{inc}} = -b^2 \bar{f}(s)\,\delta(x)\,\delta(z - z_0),$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$, and $b = c_3^{-1}$ is the slowness (reciprocal of wave speed) of the underlying half-space and $\bar{f}(s)$ the Laplace transform of the time function $f(t)$ of the line source which is located at $x = 0$, $z = z_0$.

In the following section we construct the generalized ray integrals of the waves transmitted through interface (3) into the intermediate layer 2 from the Weyl-Sommerfeld integral representation of the incident

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wave $\bar{\phi}_{\text{inc}}(\mathbf{x}, s)$. These waves are multiply reflected within layer 2 as well as refracted back to the underlying half-space and partially transmitted to the surface layer 1. Multiple reflections of waves transmitted to the top layer are considered in a subsequent section, followed by a discussion of higher order wave trains with more than two transmissions and propagating towards the receiver station in the surface layer.

Application of Cagniard's method (1962) to the inverse Laplace transformation of these ray integrals is then briefly discussed. Special attention is given to the calculation of the stationary point of the phase function and the branch points of the common slowness variable for all rays. The former yields the arrival time of direct rays and the latter is connected with head waves of refracted rays. Receiver stations are assumed at down-dip und up-dip locations.

Synthetic seismograms at those observational points located at the free surface are presented for the simplified case of one dipping top layer and a triangular source time function of a single pulse, thus the apparent source ray is transmitted only once through the interface before reaching the receiver. A comparative study is made to the signals arriving at the surface of a half-space with a parallel top layer of constant thickness equal to that of the dipping layer at source location. The final section contains the conclusions.

Source ray and transmission into the intermediate layer 2

The configuration of the three-layer half-space is shown in Fig. 1, together with three coordinate systems rotated about the common epicentral origin 0. Interface (2) is inclined against the free surface (1) by angle α and interface (3) is inclined against (2) by angle β and, therefore, by $\gamma = (\alpha + \beta)$ against the top surface.

Material of the surface layer has shear modulus μ, slowness $a = c^{-1}$ and vertical thickness h measured along the z-axis through source point S. Material of layer 2 or 3 has shear modulus μ_j $(j = 2, 3)$ and slowness $a_2 = c_2^{-1}$, $b = a_3 = c_3^{-1}$. The vertical thickness along the z-axis of the intermediate layer 2 is h_2.

The axes parallel to the interfaces (1), (2) and (3) are denoted x, x' and x'', respectively. A line source S is located in the underlying half-space 3 at depth $z_0 > H$, measured from the free surface (1), and $H = h + h_2$. The final position of the observation point (x, z) will be in the top layer and a ray path from the source to the receiver passes through all three layers.

The equation of each plane surface is:

$$\begin{aligned} &\text{Free surface (1):} && z = z_1 = 0 \\ &\text{Interface (2)} && z' = z_2' = h \cos\alpha = x_0 \sin\alpha \\ &\text{Interface (3):} && z'' = z_3'' = H \cos\gamma,\ H = h + h_2,\ \gamma = \alpha + \beta. \end{aligned} \tag{1}$$

We shall use the tip-distance measured along interface (2)

$$d' = (h/\sin\alpha) - h_2 \cos\gamma / \sin\beta, \tag{2}$$

see Fig. 1, and

$$x_0'' = -h \sin\gamma + h_2 \cos\gamma \cos\beta / \sin\beta, \tag{3}$$

to be convenient geometric parameters.

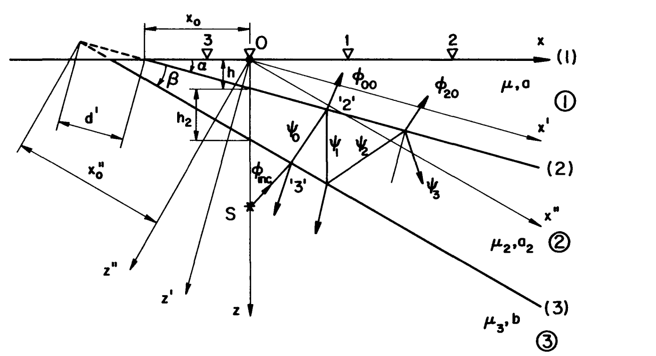

Fig. 1. Geometry of a half-space with two dipping layers. Source S located in the bedrock 3 at $x = 0$ and $z = z_0 > H$, $H = h + h_2$. Also shown are: the source ray ϕ_{inc}, a direct ray path ψ_j, $j = 3$, in the intermediate layer 2 and two apparent source rays in the surface layer, ϕ_{00} and ϕ_{20}. ∇ ... Observation stations on top surface, 0 ... epicentre, 1 and 2 down-dip locations, 3 up-dip location

The angles α and β are positive when rotated clockwise, negative when rotated counter-clockwise. At $z = z_1 = 0$, the shear stress (or pressure for a fluid medium) vanishes. At both interfaces, the stress and displacement (or velocity) are continuous and Hooke's law applies to all three layers.

The source ray

The Weyl-Sommerfeld integral representation of the Laplace-transformed displacement for the waves radiated by the line source into infinite homogeneous and isotropic space is (Aki and Richards, 1980, Chapter 6)

$$\bar{\phi}_{\text{inc}}(s) = \bar{F}(s) \int_{-\infty}^{\infty} S(\xi'') e^{sg(\xi'')} d\xi'' \tag{4}$$

where

$$\begin{aligned} &\bar{F}(s) = b^2 \bar{f}(s)/4\pi, \\ &S(\xi'') = 1/\chi, \quad \chi = (b^2 + \xi''^2)^{\frac{1}{2}} \end{aligned} \tag{5}$$

and the phase function becomes, noting the source location in doubly primed coordinates,

$$g(\xi'') = i\xi''(x'' - z_0 \sin\gamma) - \chi |z'' - z_0 \cos\gamma|. \tag{6}$$

We call $\bar{\phi}_{\text{inc}}$ the source ray and $\bar{F}(s) S(\xi'')$ the source function. The variable of integration, ξ'', is the apparent slowness of waves in the x''-direction, χ that along the z''-axis. The phase function shows the projections of the source ray to a receiver station at the point (x'', z'') in bedrock in the x''- and z''-directions.

The source ray, when incident at interface (3), is subject to partial reflection and transmission into layer 2. The transmitted ray which may be called an apparent source ray in the intermediate layer 2 is given by the generalized ray integral

$$\bar{\psi}_0(s) = \bar{F}(s) \int_{-\infty}^{\infty} S_0(\xi'') e^{sh_0(\xi'')} d\xi'', \tag{7}$$

where we changed the source function $S(\xi'')$ of the incident source ray to the apparent source function of the transmitted ray

$$S_0(\xi'')=S(\xi'')\,T^{(3)}(\xi''), \tag{8}$$

by multiplying with the transmission coefficient of plane SH waves,

$$T^{(3)}=2\mu_3\chi/(\mu_3\chi+\mu_2\zeta_0''), \qquad \zeta_0''=(a_2^2+\xi''^2)^{1/2}. \tag{9}$$

Also, the phase function is changed according to the phase contribution of the incident source ray, $g(\xi'', x_3'', z_3'')$, to a point of refraction at interface (3), say (x_3'', z_3''), and the phase of the emanating ray to an observation point (x'', z'') within layer 2, $z''\leqq z_3''$, Fig. 1,

$$h_0(\xi'')=g(\xi'', x_3'', z_3'')+i\xi''(x''-x_3'')+\zeta_0''(z''-z_3''). \tag{10}$$

Since Snell's law applies to transmission, the apparent slowness ξ'' is common and the unknown coordinate x_3'' cancels. Hence,

$$\begin{aligned} h_0(\xi'')=&-i\xi'' z_0\sin\gamma-\chi(z_0-H)\cos\gamma\\ &-\zeta_0'' z_3''+i\xi'' x''+\zeta_0'' z''. \end{aligned} \tag{11}$$

The apparent source ray $\bar{\psi}_0$ is incident at interface (2) and subject to reflection and to transmission into the top layer. The refraction should be formulated in primed coordinates. Therefore, we transform the phase function $h_0(\xi'')$ to primed coordinates where, in matrix notation,

$$\begin{aligned} &\mathbf{x}''=\mathbf{D}_2\mathbf{x}', \quad \mathbf{D}_2=\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix},\\ &\mathbf{x}'^T=(x', z'), \quad \mathbf{x}''^T=(x'', z'') \end{aligned} \tag{12}$$

is to be substituted and require the phase of the ray to be invariant under coordinate rotation through the angle β:

$$\begin{aligned} h(\xi_0')=h(\xi'')=&-i\xi'' z_0\sin\gamma-\chi(z_0-H)\cos\gamma\\ &-\zeta_0'' z_3''+i\xi_0' x'+\zeta_0' z', \end{aligned} \tag{13}$$

where the pair of apparent slownesses of the ray in primed coordinates is derived from the doubly primed pair of apparent slownesses in the doubly primed coordinates by the transformation

$$\boldsymbol{\xi}_0'^T=\boldsymbol{\xi}_0''^T\mathbf{D}_2, \quad \boldsymbol{\xi}_0''^T=(i\xi'', \zeta_0''), \quad \boldsymbol{\xi}_0'^T=(i\xi_0', \zeta_0'), \tag{14}$$

where ζ_0'' is the irreducible radical of Eq. (9).

Hence, the transformed phase function $h(\xi')$ still contains the same number of two radicals as $h(\xi'')$, namely χ and ζ_0'', which are apparent slownesses of the transmitted ray path in the z''-direction in the two media with common interface (3).

We consider multiple reflections within layer 2 first.

Reflections within the intermediate layer 2

The apparent source ray $\bar{\psi}_0$ is subject to partial reflection at interface (2) and the reflected cylindrical wave is given by the generalized ray integral

$$\bar{\psi}_1(s)=\bar{F}(s)\int_{-\infty}^{\infty} S_0(\xi'')R^{(2)}(\xi_1')e^{sh_1(\xi'')}d\xi'' \tag{15}$$

where, $l=1$ above,

$$R^{(2)}(\xi_l')=[-\mu\eta_l'+\mu_2\zeta_l']/[\mu\eta_l'+\mu_2\zeta_l'], \tag{16}$$

is the reflection coefficient of plane waves at interface (2). Expressed in the local apparent slowness in the primed coordinates:

$$\eta_l'=(a^2+\xi_l'^2)^{1/2} \tag{17}$$

is an irreducible radical and $\zeta_l'=\sqrt{(a_2^2+\xi_l'^2)}$ may be reduced to the pair of slownesses ξ'' and ζ_0'' of the apparent source ray $\bar{\psi}_0$, cf. Eq. (21).

The phase function is changed according to the superposition, $z'>z_2'$,

$$h_1(\xi'')=h_1(\xi_1')=h(\xi', x_2', z_2')+i\xi_1'(x'-x_2')-\zeta_1'(z'-z_2'), \tag{18}$$

where the unknown coordinate x_2' of the point of reflection at interface (2) cancels, since Snell's law holds, see Fig. 1,

$$\xi_1'=\xi'. \tag{19}$$

Hence, the phase function of the once-transmitted and once-reflected ray becomes

$$\begin{aligned} h_1(\xi'')=&-i\xi'' z_0\sin\gamma-\chi(z_0-H)\cos\gamma-\zeta_0'' z_3''\\ &+z_2'(\zeta_0'+\zeta_1')+i\xi_1' x'-\zeta_1' z', \end{aligned} \tag{20}$$

where, Eq. (14),

$$\boldsymbol{\xi}_1'^T=\boldsymbol{\xi}_0'^T=\boldsymbol{\xi}_0''^T\mathbf{D}_2, \tag{21}$$

and (x', z') is an observational point in layer 2.

Requiring invariance of phase of this reflected ray under coordinate rotation through the angle β,

$$i\xi_1' x'-\zeta_1' z'=i\xi_1'' x''-\zeta_1'' z'', \tag{22}$$

the constant part remains unaffected, the phase function can be expressed in doubly primed coordinates, thereby preparing it for the next reflection to take place at interface (3):

$$\begin{aligned} h_1(\xi'')=&-i\xi'' z_0\sin\gamma-\chi(z_0-H)\cos\gamma-\zeta_0'' z_3''\\ &+z_2'(\zeta_0'+\zeta_1')+i\xi_1'' x''-\zeta_1'' z'', \end{aligned} \tag{23}$$

where

$$\boldsymbol{\xi}_2''^T=\boldsymbol{\xi}_1''^T=\zeta_1'^T\mathbf{D}_2=\boldsymbol{\xi}_0''^T\mathbf{D}_2^2. \tag{24}$$

The transformation shows both the diverging effect on a ray pointing down-dip, $\xi''>0$, and the steepening effect during reflection at interface (2) for a ray pointing up-dip, $\xi''<0$.

Partial reflection at interface (3) renders the ray $\bar{\psi}_2$ in layer 2 and the generalized ray integral becomes

$$\bar{\psi}_2(s)=\bar{F}(s)\int_{-\infty}^{\infty} S_0(\xi'')R^{(2)}(\xi_1')R_{(3)}(\xi_2'')e^{sh_2(\xi'')}d\xi''. \tag{25}$$

The reflection coefficient at interface (3) expressed in local slowness ξ_j'', $j=2$ above, is given by

$$R_{(3)}(\xi_j'')=(\mu_2\zeta_j''-\mu_3\chi_j)/(\mu_2\zeta_j''+\mu_3\chi_j), \tag{26}$$

where

$$\chi_j=(b^2+\xi_j''^2)^{1/2} \tag{27}$$

denotes an irreducible radical and $\zeta_j''=\sqrt{(a_2^2+\xi_j''^2)}$ can be reduced together with ξ_j'', cf. Eq. (24) for $j=1$ and see Eq. (35).

The phase function

$$h_2(\xi'')=h_1(\xi_1'', x_3'', z_3'')+i\xi_2''(x''-x_3'')+\zeta_2''(z''-z_3'') \tag{28}$$

becomes, under Snell's condition, $\xi_2''=\xi_1''$, and after the cancellation of the unknown coordinate x_3'' of the point of reflection at interface (3), see Fig. 2,

$$\begin{aligned} h_2(\xi'')= & -i\xi'' z_0 \sin\gamma-\chi(z_0-H)\cos\gamma-z_3''(\zeta_0+\zeta_1+\zeta_2)'' \\ & +z_2'(\zeta_0'+\zeta_1')+i\xi_2''x''+\zeta_2''z''. \end{aligned} \tag{29}$$

A transformation to primed coordinates leaves the constant part unchanged and renders

$$i\xi_2''x''+\zeta_2''z''=i\xi_2'x'+\zeta_2'z', \tag{30}$$

where the pair of local apparent slowness in primed coordinates of the last ray segment is transformed according to

$$\xi_2'^T=\xi_2''^T\mathbf{D}_2=\xi_0''^T\mathbf{D}_2^3. \tag{31}$$

Hence, the cylindrical wave with a ray path showing one transmission and j reflections within layer 2 is given by inference through the generalized ray integral

$$\bar{\psi}_j(s)=\bar{F}(s)\int_{-\infty}^{\infty} S_0(\xi'')\Pi_j e^{sh_j(\xi'')}d\xi''. \tag{32}$$

The product of reflection coefficients is expressed in local apparent slowness in the x'- and x''-axis, respectively, and is given for an even-numbered ray which is subject to partial transmission into the top layer by

$$\Pi_j=R^{(2)}(\xi_1')R_{(3)}(\xi_2'')R^{(2)}(\xi_3')\ldots R_{(3)}(\xi_j''). \tag{33}$$

The phase function expressed in terms of local slowness is

$$\begin{aligned} h_j(\xi'')= & -i\xi'' z_0 \sin\gamma-\chi(z_0-H)\cos\gamma-z_3''\sum_{l=0}^{j}\zeta_l'' \\ & +z_2'\sum_{l=0}^{j-1}\zeta_l'+i\xi_j'x'+\zeta_j'z', \quad j(\text{zero, even}), \end{aligned} \tag{34}$$

and the proper pairs of apparent slowness are related to the pair of slownesses of the apparent source ray ψ_0 through

$$\begin{aligned} & \xi_{l+1}'^T=\xi_l'^T=\xi_0''^T\mathbf{D}_2^{l+1}, \\ & \xi_l''^T=\xi_{l-1}''^T=\xi_0''^T\mathbf{D}_2^l, \qquad l\ (\text{even}). \end{aligned} \tag{35}$$

Snell's law is indicated.

The first train of waves in the surface layer 1

The even-numbered rays $\bar{\psi}_j$ are incident at interface (2) and are partially transmitted into the top layer, thereby forming apparent source rays $\bar{\phi}_{j0}$. The first subscript indicates the even number of reflections within layer 2 and the subscript zero identifies the ray to be just transmitted into the surface layer without further reverberations. See Fig. 1 for those ray paths with two transmission and $j=2$ reflections. The ray $\bar{\phi}_{00}$, therefore, denotes the direct ray path from the source to a receiver in layer 1 without any reflection. The generalized ray integral is given by

$$\bar{\phi}_{j0}(s)=\bar{F}(s)\int_{-\infty}^{\infty} S_{0j}(\xi'')e^{sg_{j0}(\xi'')}d\xi'', \quad j=0,2,4\ldots. \tag{36}$$

The apparent source function becomes

$$S_{0j}(\xi'')=S_0(\xi'')\Pi_j T^{(2)}(\xi_j').$$

when considering the transmission coefficient of plane waves through interface (2):

$$T^{(2)}(\xi_j')=2\mu_2\zeta_j'/[\mu_2\zeta_j'+\mu\eta_{j0}'], \tag{37}$$

where

$$\eta_{j0}'=(a^2+\xi_j'^2)^{1/2} \tag{38}$$

denotes an irreducible radical.

Taking into account that a receiver in the top layer requires $z'<z_2'$, the phase function is expressed in primed receiver coordinates by

$$g_{j0}'(\xi'')=h_j(\xi_j', x_2', z_2')+i\xi_j'(x'-x_2')+\eta_{j0}'(z'-z_2') \tag{39}$$

and, since the unknown coordinate of transmission x_2' cancels, we have in (x, z)-coordinates of the observation point in the top layer,

$$\begin{aligned} g_{j0}'(\xi'')= & -i\xi'' z_0 \sin\gamma-\chi(z_0-H)\cos\gamma-z_3''\sum_{l=0}^{j}\zeta_l'' \\ & +z_2'\left(\sum_{l=0}^{j}\zeta_l'-\eta_{j0}'\right)+i\xi_{j0}x+\eta_{j0}z, \end{aligned} \tag{40}$$

where the proper pair of slownesses of the apparent source ray in unprimed coordinates is changed according to invariance of phase under coordinate rotation through the angle α

$$i\xi_j'x'+\eta_{j0}'z'=i\xi_j x+\eta_{j0}z. \tag{41}$$

Hence, in analogy to Eqs. (14) and (12)

$$\begin{aligned} & \xi_{j0}^T=\xi_{j0}'^T\mathbf{D}, \quad \mathbf{D}=\begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}, \\ & \mathbf{x}'=\mathbf{D}\mathbf{x}, \quad \xi_{j0}'^T=(i\xi_j', \eta_{j0}'), \quad \xi_{j0}=(i\xi_j, \eta_{j0}). \end{aligned} \tag{42}$$

In the following subsection we study the generalized ray integral of a ray $\bar{\phi}_{jq}$ undergoing q successive reflections within the top layer.

The ray $\bar{\phi}_{jq}$ of the first wave train

According to the changes of the generalized ray integrals of the apparent source ray $\bar{\psi}_0$ to the ray with $(j+1)$ reverberations ψ_j, we substitute the product of reflection coefficients

$$\begin{aligned} \Pi_{jq}= & R^{(1)}(\xi_{j1})R_{(2)}(\xi_{j2}')R^{(1)}(\xi_{j3}) \\ & \ldots\begin{cases} R^{(1)}(\xi_{jq}), & q(\text{odd}) \\ R_{(2)}(\xi_{jq}'), & q(\text{even}) \end{cases} \end{aligned} \tag{43}$$

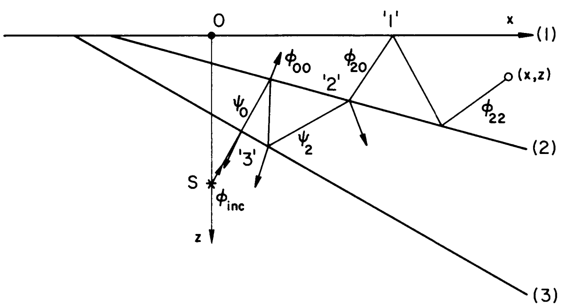

Fig. 2. Direct ray path from source S to observation station (x, z), ϕ_{jq} of first train of waves in surface layer 1. The ray shown is $j=2$, $q=2$

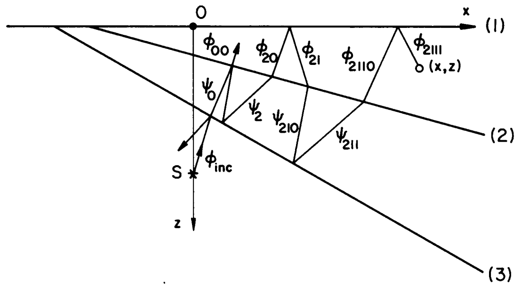

Fig. 3. Direct ray path from source S to observation station (x, z), ϕ_{jqpn} of second train of waves in surface layer 1. The ray shown is $j=2$, $q=1$, $p=1$, $n=1$

where $R_{(2)}=-R^{(2)}$, and change the phase of the apparent source ray $\bar{\phi}_{j0}$ to

$$g_{jq}(\xi'')=-i\xi'' z_0 \sin\gamma-\chi(z_0-H)\cos\gamma-z_3'' \sum_{l=0}^{j} \zeta_l'' + z_2'\left[\sum_{l=0}^{j} \zeta_l' - \sum_{l=0}^{q^*} \eta_{jl}'\right] + i\xi_{jq}x+(-1)^q z\eta_{jq}, \tag{44}$$

where $q^*=q$ (even) and $q^*=q-1$ when q is odd and, hence, find the generalized ray integral

$$\bar{\phi}_{jq}(s)=\bar{F}(s) \int_{-\infty}^{\infty} S_{0j}(\xi'')\Pi_{jq} e^{sg_{jq}(\xi'')} d\xi'', \quad j=0, 2, 4 \dots . \tag{45}$$

We note the transformations of proper slowness pairs within layer 1, see Fig. 2,

$$\boldsymbol{\xi}_{jl}'^{T}=\boldsymbol{\xi}_{j(l-1)}'^{T}=\boldsymbol{\xi}_{j0}'^{T}\mathbf{D}^{l}, \quad l \text{ (even)} \tag{46}$$

$$\boldsymbol{\xi}_{j(q+1)}^{T}=\boldsymbol{\xi}_{jq}^{T}=\boldsymbol{\xi}_{j0}'^{T}\mathbf{D}^{q+1}, \quad q \text{ (even)}. \tag{47}$$

Elimination of the local slowness in Eq. (44) and noting the finite sums

$$2\sum_{l=1}^{j/2} \sin(2l-1)\beta=(1-\cos j\beta)/\sin\beta$$

$$2\sum_{l=1}^{j/2} \cos(2l-1)\beta=\sin j\beta/\sin\beta, \tag{48}$$

renders the final form of the phase function in a matrix multiplicative form

$$g_{jq}(x, z, \xi'')=-\boldsymbol{\xi}''^{T}\mathbf{x}_S''-\boldsymbol{\xi}_0''^{T}\mathbf{D}_2^{j+1}\mathbf{T}d'+\boldsymbol{\xi}_{j0}'^{T}\mathbf{D}^{q^*+1}\mathbf{x}, \tag{49}$$

where slowness vectors of apparent source rays are related to ξ'', the integration variable,

$$\boldsymbol{\xi}''^{T}=(i\xi'', \chi), \quad \boldsymbol{\xi}_0''^{T}=(i\xi'', \zeta_0''), \quad \boldsymbol{\xi}_{j0}'^{T}=(i\xi_j', \eta_{j0}'), \tag{50}$$

and

$$\chi=(b^2+\xi''^2)^{1/2}, \quad \zeta_0''=(a_2^2+\xi''^2)^{1/2}, \quad \eta_{j0}'=(a^2+\xi_j'^2)^{1/2},$$
$$i\xi_j'=\boldsymbol{\xi}_0''^{T}\mathbf{D}_2^{j+1}\mathbf{T}, \quad \mathbf{T}^T=(1, 0). \tag{51}$$

The point vectors are conveniently defined by

$$\mathbf{x}_S''^{T}=[(x_0''+z_0\sin\gamma), (z_0-H)\cos\gamma],$$
$$x_0''=x_0\cos\gamma-d'\cos\beta, \tag{52}$$

d' of Eq. (2) and

$$\mathbf{x}^T=[(x+x_0), -\varepsilon z]. \tag{53}$$

The exponent q^* equals q (even) with direction factor $\varepsilon=-1$ and equals $(q-1)$ when q is odd, $\varepsilon=+1$.

The second train of waves in the top layer 1

In Fig. 3 we show the ray path of a ray $\bar{\phi}_{jqpn}$ with a last segment in layer 1 which is a result of partial transmission of the ray $\bar{\phi}_{jq}$, j (zero, even), q (odd) through interface (2) and, after $(p+1)$ reverberations within the intermediate layer 2, is further transmitted upwards through interface (2), p (odd). The last index n gives the number of successive reflections considered in the top layer before reaching the observation station (x, z). Considering $\bar{\psi}_{jq0}$ as an apparent source ray in layer 2, pointing downward, see Fig. 3, and $\bar{\phi}_{jqp0}$ as the apparent source ray of the second train of waves in the surface layer 1, we find the generalized ray integral

$$\bar{\phi}_{jqpn}(s)=\bar{F}(s) \int_{-\infty}^{\infty} S_{0jqp}(\xi'')\Pi_{jqpn} e^{sg_{jqpn}(\xi'')} d\xi'',$$
$$j=0, 2, 4 \dots, \quad q=1, 3, 5 \dots,$$
$$p=1, 3, 5 \dots, \tag{54}$$

where

$$S_{0jqp}(\xi'')=S_0(\xi'')\Pi_j T^{(2)}(\xi_j')\Pi_{jq} T_{(2)}(\xi_{jq}')\Pi_{jqp} T^{(2)}(\xi_{jqp}') \tag{55}$$

denotes the apparent source function of the second train of waves in layer 1, and the transmission coefficient from the top layer to the intermediate layer 2 becomes

$$T_{(2)}(\xi_{jq}')=2\mu\eta_{jq}'/[\mu\eta_{jq}'+\mu_2\zeta_{jq0}']. \tag{56}$$

$$\zeta_{jq0}'=(a_2^2+\xi_{jq}'^2)^{\frac{1}{2}}, \tag{57}$$

is an irreducible radical. Since the $(j+1)$ reverberations in layer 2 are subsequently followed by further p reflections within the same intermediate layer 2, the product of reflection coefficients is

$$\Pi_{jqp}=R_{(3)}(\zeta_{jq1}'')R^{(2)}(\xi_{jq2}') \dots R_{(3)}(\zeta_{jqp}''), \quad p \text{ (odd)} \tag{58}$$

Similarly, we consider n subsequent reflections in the top layer, hence,

$$\Pi_{jqpn}=R^{(1)}(\xi_{jqp1})R_{(2)}(\xi'_{jqp2}) \\ \ldots\left\langle \begin{array}{ll} R^{(1)}(\xi_{jqpn}), & n\,(\text{odd}) \\ R_{(2)}(\xi'_{jqpn}), & n\,(\text{even}). \end{array}\right. \tag{59}$$

Analogous to Eq. (49), the phase function has the final form

$$g_{jqpn}(\xi'', x, z) = -\xi''^T \mathbf{x}''_S - [\xi''^T_0 \mathbf{D}_2^{j+1} - \xi'^T_{j0}\mathbf{D}^{q+1} \\ + \xi'^T_{jq0}\mathbf{D}_2^{p+1}]\mathbf{T}d' + \xi'^T_{jqp0}\mathbf{D}^{n^*+1}\mathbf{x}, \tag{60}$$

where n^* equals n(even) and $n^*=n-1$ when n is odd.

It contains five irreducible radicals

$$\chi=(b^2+\xi''^2)^{\frac{1}{2}}, \quad \zeta''_0=(a_2^2+\xi''^2)^{\frac{1}{2}}, \\ \eta'_{j0}=(a^2+\xi'^2_j)^{\frac{1}{2}}, \quad \zeta'_{jq0}=(a_2^2+\xi'^2_{jq})^{\frac{1}{2}}, \\ \eta'_{jqp0}=(a^2+\xi'^2_{jqp})^{\frac{1}{2}}, \tag{61}$$

and the local slownesses are expressed in terms of the integration variable ξ'' by successive transformations:

$$\xi'^T_{jqp}=(i\xi'_{jqp}, \zeta'_{jqp})=\xi'^T_{jq0}\mathbf{D}_2^{p+1}, \qquad p(\text{odd}) \tag{62}$$

$$\zeta'_{jq0}=(a^2+\xi'^2_{jq})^{1/2},$$

$$\xi'^T_{jq}=(i\xi'_{jq}, \eta'_{jq})=\xi'^T_{j0}\mathbf{D}^{q+1}, \qquad q(\text{odd}) \tag{63}$$

$$\eta'_{j0}=(a^2+\xi'^2_j)^{1/2}$$

$$\xi'^T_j=(i\xi'_j, \zeta'_j)=\xi''^T_0\mathbf{D}_2^{j+1}, \qquad j(\text{even}) \tag{64}$$

$$\zeta''_2=(a_2^2+\xi''^2)^{1/2},$$

Furthermore, we have the pairs of transformations, including Snell's law:
in layer 1

$$\xi^T_{jqpn}=\xi^T_{jqp(n-1)}=\xi'^T_{jqp0}\mathbf{D}^n, \qquad n(\text{odd}) \tag{65}$$

$$\xi'^T_{jqp(l+1)}=\xi'^T_{jqpl}=\xi'^T_{jqp0}\mathbf{D}^{l+1}, \qquad l(\text{odd}) \tag{66}$$

where apparent source ray slowness is expressed in terms of ξ'' by

$$\xi'^T_{jqp0}=(i\xi'_{jqp}, \eta'_{jqp0}), \quad \eta'_{jqp0}=(a^2+\xi'^2_{jqp})^{1/2}, \\ i\xi'_{jqp}=\xi'^T_{jq0}\mathbf{D}_2^{p+1}\mathbf{T} \\ =\{[(i\xi'', \zeta''_0)\mathbf{D}_2^{j+1}\mathbf{T}, \eta'_{j0}]\mathbf{D}^{q+1}\mathbf{T}, \zeta'_{jq0}\}\mathbf{D}_2^{p+1}\mathbf{T}. \tag{67}$$

Wave trains in the surface layer of order higher than two are constructed in much the same manner in which Eq. (45) was changed to Eq. (54) and are not recorded here.

In the following section we consider the inverse Laplace transform of the generalized ray integrals and discuss the arrival times of individual direct and refracted rays.

Inverse Laplace transform of generalized ray integrals

The generalized ray integrals of wave trains of first and second order observed in layer 1 are given by Eqs. (45) and (54) in terms of the Laplace-transformed displacements:

$$\bar{\phi}_{j\ldots}(s)=\bar{F}(s)\int_{-\infty}^{\infty} S_{0j\ldots}(\xi'')\Pi_{j\ldots}e^{sg_{j\ldots}(\xi'')}d\xi''. \tag{68}$$

The integration can be broken into two parts and from the complex conjugate properties of the reflection and transmission coefficients we can show that

$$\bar{\phi}_{j\ldots}(s)=2\bar{F}(s)\,\mathrm{Re}\int_0^{\infty} S_{0j\ldots}(\xi'')\Pi_{j\ldots}e^{sg_{j\ldots}(\xi'')}d\xi'', \tag{69}$$

where Re denotes "the real part of".

In the method of Cagniard (1962) the phase function is changed to t by the mapping

$$t=-g_{j\ldots}(\xi'') \tag{70}$$

and the integration over the real variable ξ'' is extended to the complex ξ''-plane. Assume that t is real and the inverse mapping exists,

$$\xi''=\hat{\xi}''(t)=g^{-1}_{j\ldots}(t). \tag{71}$$

The ray integral is transformed to

$$\bar{\phi}_{j\ldots}(s)=2\bar{F}(s)\,\mathrm{Re}\int_{t_A}^{\infty}\left\{S_{0j\ldots}[\hat{\xi}''(t)]\Pi_{j\ldots}\frac{d\hat{\xi}}{dt}\right\}e^{-st}dt, \tag{72}$$

where

$$t_A=[-g_{j\ldots}(\xi'')]_{\xi''=0}. \tag{73}$$

When $\bar{f}(s)=1$ in Eq. (5), the inverse Laplace transform of $\bar{\phi}(s)$ is

$$\phi^{\delta}_{j\ldots}(t)=(b^2/2\pi)H(t-t_A)\,\mathrm{Re}\left\{S_{0j\ldots}[\hat{\xi}''(t)]\Pi_{j\ldots}\frac{d\hat{\xi}}{dt}\right\}. \tag{74}$$

$H(t-t_A)$ is the Heaviside step function, vanishing when $t<t_A$. The inverse Laplace transform is thus accomplished by "inspection".

For a general time function $f(t)$, the result is obtained by convolution and the integration may be changed to one over ξ'':

$$\phi_{j\ldots}(t)=(b^2/2\pi)H(t-t_A)\int_0^t \dot{f}(t-\tau) \\ \cdot[\mathrm{Re}\int_{\Gamma} S_{0j\ldots}(\xi'')\Pi_{j\ldots}d\xi'']d\tau. \tag{75}$$

The integration contour Γ in the ξ''-plane, the portion OM is along the imaginary ξ''-axis, is shown in Fig. 4. The point M is at $\xi''=\xi''_M=\pm iB_M$, where B_M is real and ξ''_M is the root of the equation

$$dt/d\xi''=0. \tag{76}$$

The point ξ''_M is a saddle point on the ξ''-plane and a stationary point for the phase functions $g_{j\ldots}$. The upper limit of integration, $\hat{\xi}''(\tau)$, is calculated numerically from Eq. (71) when $t=\tau$. The locus of $\hat{\xi}''(\tau)$ is the contour Γ. Note that the saddle point M can be either below the lowest branch point of the integrand in Eq. (75) as shown in Fig. 4a or above the lowest branch point (Fig. 4b).

Once the wave along each ray path is evaluated, the total response is the summation of rays that have arrived at an observation point (x, z),

$$\phi(t)=\Sigma\Sigma\phi_{jq}(t)+\Sigma\Sigma\Sigma\Sigma\phi_{jqpn}+\ldots. \tag{77}$$

The first term, $j=q=0$, is the ray from the source to the receiver, traversing the layers, and the double sum represents the first train of waves. The second group

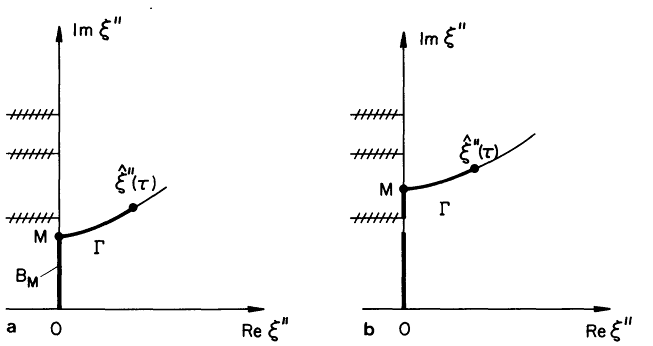

Fig. 4a, b. Cagniard's contour of integration (Γ in ξ''-plane, with branch cuts in the second quadrant, extends into the first quadrant. If $B_M<0$, contour Γ extends into the fourth quadrant)

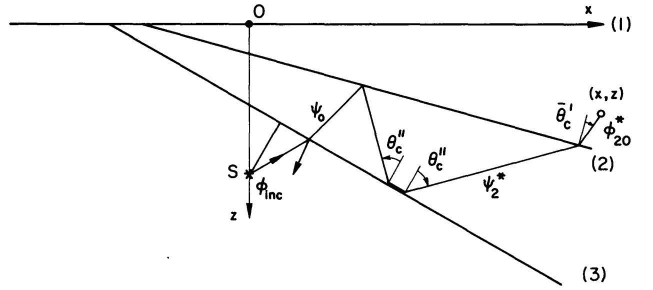

Fig. 5. Refracted ray path ϕ_{20}^*. Refraction of ψ_2^* at fast bottom to interface (3). (No counterpart corresponding to a head wave travelling up-dip is observed by an up-dip receiver point under the conditions assumed in the figure)

represents waves that are transmitted to the surface layer, subsequently transmitted to the intermediate layer 2 and then refracted back to the top layer.

In the following two subsections we derive Eq. (76) explicitly and discuss proper branch points corresponding to early arrivals of head waves.

Arrival times of direct rays

The stationary point of the phase function is determined by solving Eq. (76). To calculate the roots, we must first carry out the differentiation explicitly and then solve this irrational equation numerically. Once the roots $\xi''=\xi_M''=\pm iB_M$ are found, the value of t at this stationary point is the arrival time of a particular ray,

$$t_M=-g_{j\ldots}(\xi_M''). \tag{78}$$

We call t_M the arrival time of a direct ray. By direct ray we mean a ray that is either transmitted or reflected at an interface. The case of a refracted ray, a part of which travels along an interface, is discussed in the next subsection.

In the following we calculate $dt/d\xi''$ for the first two trains of waves in the surface layer.

Differentiation of the phase function, Eq. (49), renders the equation for ξ_M'' for the first wave train,

$$-i\frac{dg_{jq}}{d\xi''}=-\frac{1}{\chi}\boldsymbol{\chi}^T\mathbf{x}_S''-\frac{1}{\zeta_0''}(\boldsymbol{\zeta}_0''^T\mathbf{D}_2^{j+1}\mathbf{T})d' + \frac{1}{\zeta_0''}(\boldsymbol{\zeta}_0''^T\mathbf{D}_2^{j+1}\mathbf{T})\frac{1}{\eta_{j0}'}\boldsymbol{\eta}_{j0}'^T\mathbf{D}^{q^*+1}\mathbf{x}=0, \tag{79}$$

where the vectors

$$\boldsymbol{\chi}^T=(\chi,-i\xi''),\quad \boldsymbol{\zeta}_0''^T=(\zeta_0'',-i\xi'')$$
and
$$\boldsymbol{\eta}_{j0}'^T=(\eta_{j0}',-i\xi_j'), \tag{80}$$

which are orthogonal to the corresponding differentiated slowness vectors, $\boldsymbol{\xi}''$, $\boldsymbol{\xi}_0''$, $\boldsymbol{\xi}_{j0}'$, respectively, have been introduced in the compact matrix notation. Note the scalar factor $\frac{1}{\zeta_0''}(\boldsymbol{\zeta}_0''^T\mathbf{D}_2^{j+1}\mathbf{T})$ when differentiating the higher order slowness vector and the transformation

$$i\xi_j'=\boldsymbol{\xi}_0''^T\mathbf{D}_2^{j+1}\mathbf{T}: \tag{81}$$

$$\begin{aligned}-i\frac{dg_{jqpn}}{d\xi''}=&-\frac{1}{\chi}\boldsymbol{\chi}^T\mathbf{x}_S''-\frac{1}{\zeta_0''}(\boldsymbol{\zeta}_0''^T\mathbf{D}_2^{j+1}\mathbf{T})d'\\&+\frac{1}{\zeta_0''}(\boldsymbol{\zeta}_0''^T\mathbf{D}_2^{j+1}\mathbf{T})\frac{1}{\eta_{j0}'}(\boldsymbol{\eta}_{j0}'^T\mathbf{D}^{q+1}\mathbf{T})d'\\&-\frac{1}{\zeta_0''}(\boldsymbol{\zeta}_0''^T\mathbf{D}_2^{j+1}\mathbf{T})\frac{1}{\eta_{j0}'}(\boldsymbol{\eta}_{j0}'^T\mathbf{D}^{q+1}\mathbf{T})\frac{1}{\zeta_{jq0}'}\\&\cdot(\boldsymbol{\zeta}_{jq0}'^T\mathbf{D}_2^{p+1}\mathbf{T})d'\\&+\frac{1}{\zeta_0''}(\boldsymbol{\zeta}_0''^T\mathbf{D}_2^{j+1}\mathbf{T})\frac{1}{\eta_{j0}'}(\boldsymbol{\eta}_{j0}'^T\mathbf{D}^{q+1}\mathbf{T})\frac{1}{\zeta_{jq0}'}\\&\cdot(\boldsymbol{\zeta}_{jq0}'^T\mathbf{D}_2^{p+1}\mathbf{T})\frac{1}{\eta_{jqp0}'}\boldsymbol{\eta}_{jpq0}'^T\mathbf{D}^{n^*+1}\mathbf{x}=0,\end{aligned} \tag{82}$$

where the additional higher order vectors

$$\boldsymbol{\zeta}_{jq0}'^T=(\zeta_{jq0}',-i\xi_{jq}')$$
and
$$\boldsymbol{\eta}_{jqp0}'^T=(\eta_{jqp0}',-i\xi_{jqp}'), \tag{83}$$

which are orthogonal to $\boldsymbol{\xi}_{jq0}'$ and $\boldsymbol{\xi}_{jqp0}'$, respectively, enter through differentiation. Slownesses are related to source ray slowness by

$$\begin{aligned}i\xi_{jq}'&=\boldsymbol{\xi}_{j0}'^T\mathbf{D}^{q+1}\mathbf{T}=(\boldsymbol{\xi}_0''^T\mathbf{D}_2^{j+1}\mathbf{T}.\,\eta_{j0}')\mathbf{D}^{q+1}\mathbf{T},\\ i\xi_{jqp}'&=\boldsymbol{\xi}_{jq0}'^T\mathbf{D}_2^{p+1}\mathbf{T}\\&=[(\boldsymbol{\xi}_0''^T\mathbf{D}_2^{j+1}\mathbf{T},\eta_{j0}')\mathbf{D}^{q+1}\mathbf{T},\zeta_{jq0}']\mathbf{D}_2^{p+1}\mathbf{T}.\end{aligned} \tag{84}$$

Arrival times of refracted rays

We assume layers with increasing slowness, $b<a_2<a$. Since refraction occurs only during critical reflection at a fast bottom, the lowest order refracted ray is ψ_2^*, see Fig. 5, when the once-transmitted ray ψ_0 after reflection at interface (2), ψ_1, is incident under critical angle condition, $\theta_c''=\sin^{-1}(b/a_2)$, at interface (3). A portion of the refracted ray coincides with interface (3) and the last two segments to an observation point in layer 1 are under critical angle condition, θ_c'', and $\theta_c'=\sin^{-1}(a_2/a)$ is changed to $\bar{\theta}_c'$ due to the divergence through the angle β.

We calculate early arrival times of head waves from the branch points

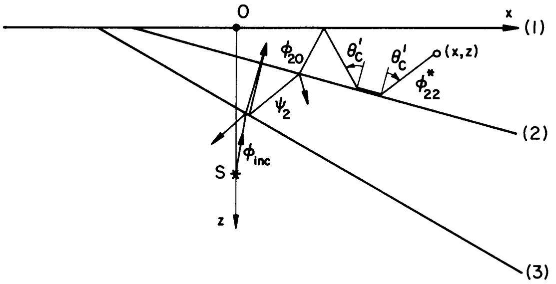

Fig. 6. Refracted ray path ϕ_{22}^*. Refraction of ϕ_{22} at fast bottom to interface (2). First train of waves in surface layer

$$\xi_j'' = {}_{(\pm)} ib, \quad \chi_j'' = 0. \tag{85}$$

The location in the ξ''-plane is calculated by solving the equations

$$\xi_j''^T = \xi_0''^T \mathbf{D}_2^j = -b(1, -\kappa_2),$$
$$\kappa_2 = \left(\frac{a_2^2}{b^2} - 1\right)^{1/2}, \text{ real} \quad j\,(\text{even}). \tag{86}$$

Hence,

$$\xi_0''^T = -b({}_{(\pm)}1, -\kappa_2)(\mathbf{D}_2^j)^T \quad j\,(\text{even}) \tag{87}$$

determines the branch point which may fall below the stationary point. The minus sign $(-)$ applies to the ray $j=1$ only when incident under critical angle condition and ψ_2^* renders a head wave propagating up-dip in the intermediate layer 2.

Head waves of the first train of waves propagating in the surface layer are given by rays ϕ_{jq}^* refracted at interface (2), Fig. 6.

The location of the branch points

$$\xi_{jq}' = {}_{(\pm)} ia_2, \quad \zeta_{jq}' = 0 \tag{88}$$

in the ξ''-plane are calculated by successive solution of the equations

$$\xi_{jq}'^T = \xi_{j0}'^T \mathbf{D}^q = -a_2(1, -\kappa), \qquad \kappa = \left(\frac{a}{a_2^2} - 1\right)^{1/2}, \text{ real}, \tag{89}$$

$$\xi_j'^T = \xi_0''^T \mathbf{D}_2^{j+1} \tag{90}$$

to be at

$$\xi_{j0}'^T = -a_2(1, -\kappa)(\mathbf{D}^q)^T, \quad i\xi_j' = -a_2(1, -\kappa)(\mathbf{D}^q)^T \mathbf{T},$$
$$\zeta_j' = (a_2^2 + \xi_j'^2)^{1/2}, \quad \xi_0''^T = \xi_j'(\mathbf{D}_2^{j+1})^T, \quad i\xi'' = \xi_j'^T (\mathbf{D}_2^{j+1})^T \mathbf{T}. \tag{91}$$

Head waves in the second train of waves may travel in the intermediate layer 2, the branch points being

$$\xi_{jqp}'' = {}_{(\pm)} ib, \quad \chi_{jqp}'' = 0, \tag{92}$$

or in the surface layer. Refraction of the ray ϕ_{jqpn}^* at interface (2) is given by the branch points

$$\xi_{jqpn}' = {}_{(\pm)} ia_2, \quad \zeta_{jqpn}' = 0. \tag{93}$$

The locations in the ξ''-plane are found by successive solutions of proper transformation equations: For even j we have

$$\xi'' = \xi_j'[\cos(j+1)\beta - \sqrt{(ia_2/\xi_j')^2 - 1}\, \sin(j+1)\beta], \tag{94}$$

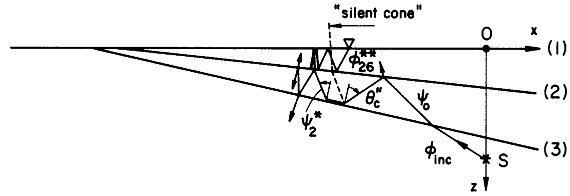

Fig. 7. Refracted ray path ϕ_{26}^{**} corresponding to backscattering of an up-dip travelling head wave ψ_2^*. Note the steepening effect in the direct ray path shown in the surface layer 1

where, in the case of refraction at interface (3),

$$\xi_j' = ib[{}_{(\pm)}\cos p\beta - \sqrt{(a_2/b)^2 - 1}\, \sin p\beta]\,\{\cos(q+1)\alpha$$
$$- \sqrt{[a/a_3({}_{(\pm)}\cos p\beta - \sqrt{(a_2/b)^2 - 1}\, \sin p\beta)]^2 - 1}\, \sin(q+1)\alpha\}. \tag{95}$$

In the case of refraction at interface (2) the local slowness to be inserted in Eq. (94) becomes

$$\xi_j' = \xi_{jq}'[\cos(q+1)\alpha - \sqrt{(ia/\xi_{jq}')^2 - 1}\, \sin(q+1)\alpha], \tag{96}$$

where, with $p\,(\text{odd})$

$$\xi_{jq}' = \xi_{jqp}'[\cos(p+1)\beta - \sqrt{(ia_2/\xi_{jqp}')^2 - 1}\, \sin(p+1)\beta] \tag{97}$$

and with $n\,(\text{even})$

$$\xi_{jqp}' = ia_2[{}_{(\pm)}\cos n\alpha - \sqrt{(a/a_2)^2 - 1}\, \sin n\alpha]. \tag{98}$$

All radicals are assumed real and the branch points located on the imaginary ξ''-axis may fall below the stationary point M, indicating the early arrival of the head wave. The arrival time is calculated by substituting ξ'' of the branch point into Eq. (70).

Observation stations receiving those head waves with a last segment under critical angle condition must be located outside the "silent cone" defined by the conditions

$$\frac{dt}{db} > 0, \quad \frac{dt}{da_2} > 0. \tag{99}$$

Up-dip travelling head waves, corresponding to branch points like $\xi_1'' = -ib$ or $\xi_{j1}' = -ia_2$, are subject to backscattering due to the steepening effect of the wedge. A delayed signal, corresponding to a ray path with incident segment under critical angle condition followed by a ray portion coinciding with the proper interface and a subsequent direct ray path to the observational point, is also received within the silent cone, see Fig. 7.

Numerical results: One dipping layer

We omit the intermediate layer and take the limit $h_2 \to 0$ and $\beta \to 0$ in Eq. (45). The ray integrals of displacement V become

$$\bar{V}_q(s) = \bar{F}(s) \int_{-\infty}^{\infty} S_0(\xi')\Pi_{0q} e^{sg_{0q}(\xi')} d\xi'. \tag{100}$$

Π_{0q} is given by Eq. (43) and $S_0 = ST^{(2)}(\xi')$. The phase function after q reverberations is given in the limit of

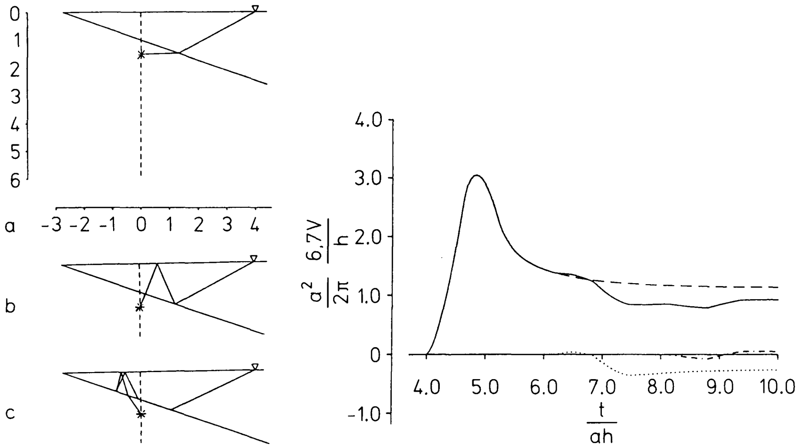

Fig. 8a–c. Dipping surface layer, $\alpha=20°$. Source location in (fast) bedrock at $z_0=1.5h$. Observational point at free surface, epicentral distance $x=4h$. Triangular source time function: $2\Delta=2.4$. Arrival times: **a** source ray-3.995, **b** ray 2-6.1828, **c** ray 4*-8.0847

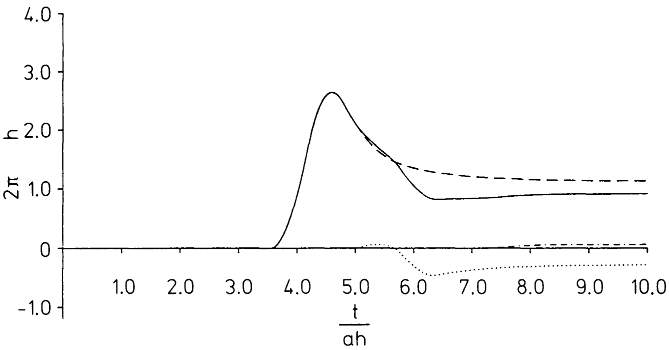

Fig. 9. Parallel surface layer, $\alpha=0°$. Source location in (fast) bedrock at $z_0=1.5h$. Observational point at free surface, epicentral distance $x=|4h|$. Triangular source time function: $2\Delta=2.4$

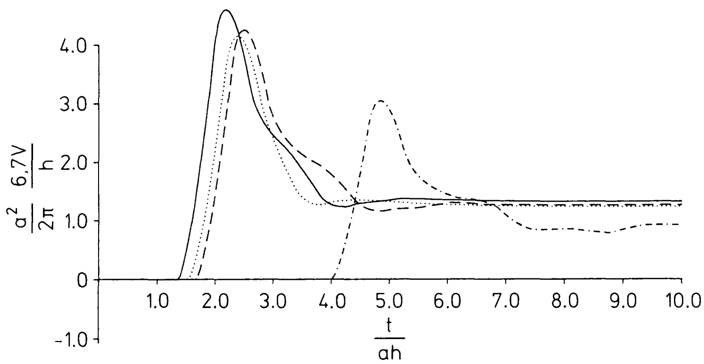

Fig. 10. Signals observed at the free surface of a dipping layer, $\alpha=20°$. Epicentral distances are: $x=0$ (epicentre) ——, $x=-h$ ..., $x=h$ – – –, $x=4h$ —·—·—·—. Source location in bedrock, $z_0=1.5\,h$

Eq. (49)

$$g_{0_q}(\xi')=-\boldsymbol{\xi}'^T\mathbf{D}\mathbf{x}_0+\boldsymbol{\xi}_0'^T\mathbf{D}^{q^*+1}\mathbf{x},\quad \mathbf{x}_0^T=(x_0, z_0) \tag{101}$$

with the newly defined slowness vectors

$$\boldsymbol{\xi}'^T=(i\xi',\zeta'),\quad \boldsymbol{\xi}_0'^T=(i\xi',\eta_0'),\quad \zeta'=(b^2+\xi'^2)^{1/2},$$
$$\eta_0'=(a^2+\xi'^2)^{1/2}. \tag{102}$$

For convenience of integrations in the ξ-plane, we calculate the response to a source time function of triangular shape of duration 2Δ, with equal rise and fall time Δ, when the line source is located at $x=0$, $z_0=1.5h$. Observational points are situated on the free surface at epicentral distances $x=4h$, $\pm h$, respectively.

Material properties of the layer and the underlying bedrock are $a=c^{-1}=1$, $a_2=c_2^{-1}=1/\sqrt{2}$, $\rho_2/\rho=1$. Dipping angle $\alpha=20°$ ($\alpha=0°$ in the parallel layer case). Individual and summed ray integrals are shown in Figs. 8 and 9. The pulse recorded in the parallel layer case of Fig. 9 is much smoother in the less peaked source ray at early times as well as in the tail at later times. Thus Fig. 8, indicating higher accelerations in the seismogram on top of an inclined layer with arrival of a head wave, contributes to the understanding of high damage records of the Skopje (1963) earthquake for structures above sloping parts of the interface, reported by Porceski (1969). Note that only a finite number of rays arrive at a fixed receiver in the dipping layer case.

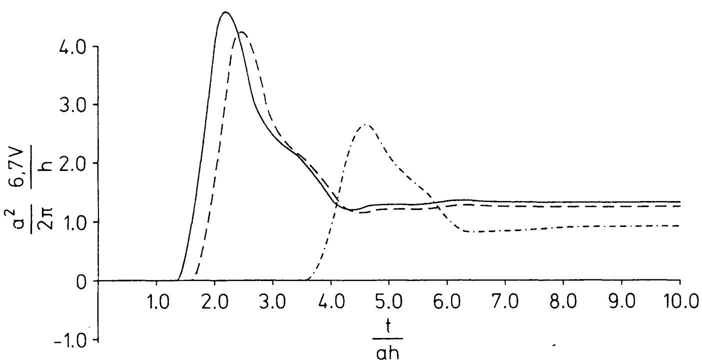

Fig. 11. Signals observed at the free surface of a parallel layer, $\alpha=0°$. Epicentral distances are: $x=0$ (epicentre) ——, $x=|h|$ – – –, $x=|4h|$ -·-·-·-. Source location in bedrock, $z_0=1.5h$

There are two direct rays and a returning and refracted ray shown in Fig. 8a–c, respectively. Within the recording time of Fig. 9 there are also three rays observed, but they are all direct rays. Individual ray signals arriving at later times in the parallel layer case are all weakened by transmission of energy through interface (2). Figures 10 and 11 clearly show the inclination effect and the asymmetry in the synthetic seismograms recorded at $x=\pm h$. Amplification effects and comparatively higher accelerations are much more pronounced at larger down-dip distances of the observation points. Note the critical angle of incidence, at the interface to the fast bedrock, to be 45°.

Conclusions

Cylindrical waves emitted by a non-stationary line source located in bedrock and transmitted through two dipping layers several times up and down, with an arbitrary number of subsequent reflections within those layers considered between transmission, are described in closed form by generalized ray integrals. Through proper transformations of local slowness vectors by rotational matrices, the ray integrals are put in a form suitable for the application of the Cagniard technique. Thus, inversion of the Laplace-transformed solution is the same as in the parallel layer case; hence, becomes a standard procedure. Contrary to the parallel layer case, the number of ray integrals received at a fixed point of observation becomes finite. Generalization of the analytical part of the solution to a multi-layered configuration is rather simple, as long as the source remains in the underlying bedrock.

Divergence effects of a single dipping layer on synthetic seismograms, observed at receivers at the free surface of the half-space, are studied by comparison of individual and summed ray signals to those of the parallel layer case; equal thickness of the layers is understood at source location. Integration in the source ray slowness plane is performed by Gaussian quadrature in both cases, thus the standard procedure of the parallel layer case is directly applicable. A triangular source time shape function was chosen to save computer time on the convolution integrals.

The number of ray integrals to be evaluated in the multi-layered case is dramatically increased together with computer time required for numerical integration. Thus, numerical results had to be restricted to one layer only. The variation of locations of the observational point on top of the half-space and the amplification and increased unevenness of the records observed give a quantitative interpretation of damage results reported on structures above dipping parts of an interface in the Skopje (1963) earthquake, where a single layer has been reported.

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