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Short communication

Velocity-viscosity correlation in convection cells with non-uniform viscosity

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Abstract. The correlation between the logarithm of local viscosity η and local velocity v ($\log |v| \cong a - b \log \eta$) in numerical models of variable viscosity convection is studied. Nineteen selected cases of 2-D stationary convection with different viscosity laws including non-Newtonian rheology, different Rayleigh numbers and different aspect ratios are studied. The quality of correlation is very different, ranging from non-existent to very good. The coefficient b is found to be below 0.5 in almost all cases. Heuristic arguments on the structure of mantle convection which are based on assumptions as $|v| \propto \eta^{-1}$ therefore appear unfounded.

Key words: Convection – Viscosity

Introduction

The rheology of mantle rock is strongly temperature and pressure dependent. Possibly it is also non-Newtonian or stress dependent. Some attempts have been made to determine the influence of non-constant viscosity on the structure of mantle convection by solving the equations of motion and energy transport (e.g. Torrance and Turcotte, 1971; Houston and DeBremaecker, 1975; Schmeling and Jacoby, 1981). However, these efforts have been sporadic and did not attack the problem in a very systematic way. Recently I have studied the heat transport properties of variable viscosity convection in a large number of model cases (Christensen, 1984a, 1985). These models provide the data basis for the present investigation.

The lack of exact systematic solutions for the variable viscosity convection problem has often been bypassed by the use of scaling analysis, heuristic arguments and intuition. For example, it appears reasonable to assume a higher flow velocity v in parts of the convection cell where the viscosity η is lower than in those where η is high. It has been speculated that the thermal state of the mantle may be closer to isoviscous than to adiabatic because the advective heat transport should be less efficient in regions of high viscosity. In order to study this issue by scaling analysis, Fowler (1983) assumed that local velocity and local viscosity correlate like

$$|v| \propto \eta^{-1}. \tag{1}$$

Such a relation, if it could be confirmed, would have important consequences. A viscosity increase by two orders of magnitude from the upper to the lower mantle (Hager, 1984) would reduce the velocity from $O(1\ \mathrm{cm/year})$ to $O(0.1\ \mathrm{mm/year})$ in the lower mantle. Such low velocity would significantly influence ideas about mixing of mantle heterogeneities and tapping of geochemical 'reservoirs'. Although Eq. (1) may appear intuitively appealing, the arguments to support it are rather weak. Given a certain stress level, it is not the velocity but its spatial derivatives which are related to the viscosity. Furthermore, there is no a priori reason to assume the same level of stress in regions of different viscosity within the cell. In a previous comment to Fowler's paper (Christensen, 1983), I have presented some qualitative arguments as to why I expect the relation between v and η to be much weaker than Eq. (1) assumes. However, it appears useful to determine the correlation between local velocity and local viscosity systematically from numerical solutions of variable viscosity convection.

Results

The finite element solutions which are used to study the correlation are described in more detail elsewhere (Christensen, 1984a, 1985). They are for two-dimensional steady state convection in rectangular boxes. In most case studies the boundaries are stress-free, the temperature difference from top to bottom is fixed and there are no internal heat sources. Besides the aspect ratio l, a Rayleigh number Ra_0 based on the viscosity at the top boundary and two rheological parameters (θ and ζ) describing temperature and pressure dependence determine the state of convection:

$$\mathrm{Ra}_0 = \frac{\alpha g \rho \Delta T h^3}{\kappa \eta_0} \tag{2}$$

$$\eta = \eta_0 \exp(-\theta \bar{T} + \zeta \bar{z}). \tag{3}$$

α stands for the coefficient of thermal expansion, g for the gravitational acceleration, ρ for the density, ΔT is

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Table 1. Model parameters, Nusselt-numbers and correlation parameters

Model	Ra_0	l	$e^{\theta-\zeta}$	e^{ζ}	n	Nu	a	b	r
1	1,000	1	1,000	1	1	4.49	1.024	0.383	−0.719
2	10,000	1	1,000	1	1	10.06	1.909	0.289	−0.506
3	100,000	1	1,000	1	1	25.58	2.770	0.219	−0.279
4	1,250	1	64,000	1	1	6.64	0.760	0.518	−0.886
5	3,750	1	64,000	1	1	9.15	1.202	0.488	−0.829
6	1,897	1	250,000	1	1	8.46	0.908	0.519	−0.878
7	10,000	2	1,000	1	1	9.11	2.315	0.019	−0.032
8	10,000	3	1,000	1	1	8.46	2.336	0.000	0.000
9	100,000	2	1,000	1	1	20.46	2.960	0.044	−0.049
10[a]	10,000	1	1,000	1	1	4.73	0.749	0.680	−0.734
11	10,000	1	1,000	64	1	9.31	2.241	0.086	−0.244
12	10,000	1	1,000	1,000	1	8.28	1.942	0.100	−0.354
13	50,000	1	250	1,000	1	10.19	1.920	0.163	−0.606
14	50,000	2	250	1,000	1	8.43	1.936	0.207	−0.723
15	500	1	10^7	1	3	27.10	2.642	0.129	−0.245
16	500	2	10^7	1	3	16.64	2.726	0.043	−0.077
17	1,000	1	10^5	10^5	3	10.98	1.734	0.133	−0.371
18[b]	1.5×10^7	1	7.2×10^{-5}	2×10^7	1	3.44	2.714	0.460	−0.922
19[c]	343,600	1	1	1,000	1	9.10	1.925	0.029	−0.113

[a] Rigid top and bottom boundary
[b] Purely internal heating, zero bottom heat flux. The Nusselt number is obtained by dividing the 'conductive reference temperature' at the bottom by the actual mean bottom temperature
[c] 3/4 internal heating, 1/4 bottom heat flux. For Nu see [b]

the temperature difference across the convective layer of height h, κ is the thermal diffusivity and η_0 the viscosity at the upper boundary. $\bar{T}$ is the temperature normalized to zero on top and one at the bottom, $\bar{z}$ is the vertical coordinate normalized in the same way. In some cases a non-Newtonian third-power law rheology is used with an effective viscosity given by

$$\eta=\eta_0\left(\frac{s}{\eta_0\kappa/h^2}\right)^{-2}\exp(-\theta\bar{T}+\zeta\bar{z}), \tag{4}$$

where s is the second invariant of the deviatoric stress tensor. Details of the numerical method are described in Christensen (1984b). Convergence tests indicate an accuracy of the solutions concerning the Nusselt number of better than 1% and concerning the local velocity of at least better than 10%.

The correlation is studied by 'sampling' the local viscosity and velocity at N random points within the convection cell, where N is typically 500. A correlation of the form

$$|v|\propto\eta^{-b} \tag{5}$$

or

$$\log|v|=a-b\log\eta \tag{6}$$

is assumed. The optimal parameters a and b in Eq. (6) are determined by a least-squares fit and the usual correlation coefficient

$$r=\frac{\sum\log\eta\log|v|-\frac{1}{N}\sum\log\eta\sum\log|v|}{\left\{\left[\sum(\log\eta)^2-\frac{1}{N}(\sum\log\eta)^2\right]\left[\sum\log|v|^2-\frac{1}{N}(\sum\log|v|)^2\right]\right\}^{1/2}} \tag{7}$$

is taken as a measure of the quality of the fit. One complication must be considered. At some points, like the corners of the cell or the centre of the circulation, the velocity is zero while the viscosity may have arbitrary values. The velocity vanishes because of geometrical reasons and not because of high viscosity. To include the vicinity of these points may obscure an intrinsic correlation between η and $|v|$. At these points $v\to0$, whereas either the strain rate $\dot{\varepsilon}$ or the vorticity ω or both remain large. When, on the other hand, $|v|$ becomes small due to high local viscosity, both $\dot{\varepsilon}$ and ω should likewise be small. To get rid of the stagnation points and their immediate vicinity, I thus rejected all points where

$$\frac{v^2}{\dot{\varepsilon}^2+\omega^2}<K^2 \tag{8}$$

with K of the order of 0.04. Typically 2%–5% of the samples were rejected. Varying K within reasonable limits had only a small effect on the values of a, b and r, but taking it as zero significantly deteriorated the correlation in some cases.

In Table 1 the results for all 19 models are listed. Instead of θ and ζ the values $e^{\theta-\zeta}$ and e^{ζ} are given. They are, respectively, the actual ratio of top to bottom viscosity due to the combined effects of temperature and pressure dependence, and the hypothetical increase due to the pressure effect alone. For $n=3$ (non-Newtonian power-law creep with stress exponent 3), the actual viscosity difference is much smaller than $e^{\theta-\zeta}$ because of the moderating influence of the stress dependence (Christensen, 1984b).

The quality of the correlation differs strongly from case to case; r values between 0.00 and −0.92 are found. The coefficient b varies in the range 0.00–0.68. In Fig. 1 some typical examples of correlation diagrams

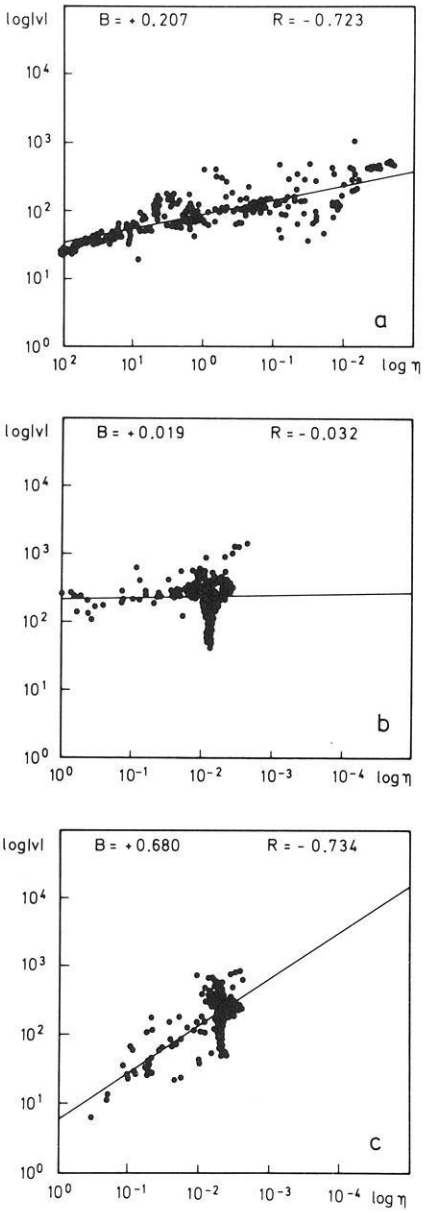

Fig. 1a–c. Correlation diagrams for three convection models. Of 500 points which have been used to determine the correlation, only 200 are plotted. **a** Model number 14 with strong pressure effect on the rheology, good correlation but low b value. **b** Model number 7 with simply temperature-dependent viscosity, aspect ratio 2, no correlation is found. **c** Model number 10 with rigid boundaries and temperature-dependent viscosity, highest b value of all models

are shown. The correlation is good ($|r|>0.6$) for strong enough temperature dependence of viscosity ($e^{\theta}>10^4$), where a sluggishly moving surface layer is formed. The b values are of the order of 0.5 in these cases (models 4–6 in Table 1). The correlation is also good with strong pressure influence (models 13 and 14), but here b is only of the order 0.2. The highest value of $b=0.68$ is obtained for a model with rigid boundaries (model 10). Due to the boundary condition, a stagnant high-viscosity layer is produced near the surface. Given the same viscosity contrast, this layer moves relatively fast with a free upper boundary (leading to lower b). The best correlation ($|r|=0.922$) is obtained in a model with purely internal heating and extremely strong pressure influence on the flow law (model 18, $b=0.46$). In many cases, however, the correlation is quite poor ($|r|<0.4$) and the b values are small.

The influence of changing the aspect ratio is not clear. With purely temperature-dependent Newtonian or non-Newtonian rheology, the correlation deteriorates when increasing l (models 2, 7 and 8 and models 15 and 16). However, with additional pressure influence there is even a slight improvement on increasing l (models 13 and 14). With non-Newtonian rheology the correlation seems slightly weaker than in equivalent cases of Newtonian rheology (I consider those Newtonian cases equivalent which have a similar Nusselt number and about half the θ and ζ values of the non-Newtonian model, cf. Christensen, 1984b).

Different ways of determining the correlation do not lead to qualitatively different results. If, instead of random sampling, the horizontal and vertical boundary layers are sampled four times more frequently than the centre of the cell, the correlation coefficient improves by typically 0.1–0.2, but b increases by no more than 0.05. If horizontally averaged velocities and viscosities are correlated, the result is comparable to that of pointwise sampling.

Discussion

Although only a restricted number of cases was studied, a wide range of possible parameter combinations was covered. I found that there is either no good correlation between local velocity and viscosity or, if there is a good correlation, the coefficient b is much less than unity. Thus it seems that the assumption $v \propto \eta^{-1}$ must be abandoned for variable viscosity convection and that speculations based on it are misleading. The dependence between v and η is strongest when the surface layer forms a stagnant lid. However, the Earth's surface plates are actually moving and the viscosity in the deep mantle is probably not high enough to produce stagnant regions. The parameter b is then expected to be less than 0.5. Recently, the idea that the viscosity of the lower mantle is significantly higher than that of the upper mantle has found more support (Hager, 1984; Christensen, 1984c). The result of the present study suggests that even with a viscosity of $O(10^{24}$ Poise), two orders of magnitude higher than in the upper mantle, the velocity would probably be more than one-tenth of the typical plate velocity. The overturn time of whole mantle convection would then be of the order of 1 billion years.

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