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Spheroidal and torsional global response functions

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Abstract. The response of a spherically symmetric non-rotating elastic isotropic (SNREI) model earth to spheroidal forcing is described by six coefficients, five of which, h_n, k_n, l_n, h'_n and l'_n, are related to earth tide measurements. The sixth parameter, l_n^-, a spheroidal stress coefficient, is completely independent, whereas all other coefficients used to describe the response to spheroidal forcing are linearly dependent on the five coefficients above. The corresponding relations are compiled in this paper.

The response of the SNREI model earth to torsional forcing is completely described by the torsional stress coefficient, l_n^*, which is resonant at the corresponding eigenperiods of the torsional free oscillations of the earth. For $n=1$, the zero frequency mode, ${}_0T_1$, corresponding to a rigid rotation, is established, and can only be excited by external forcing. After attenuation is introduced eigenperiods and quality factors of the fundamental torsional modes, ${}_0T_2$–${}_0T_{10}$, are calculated for the isotropic preliminary reference earth model (PREM). A comparison with the corresponding values originally determined for the PREM yields a small bias. As a probable cause of the bias, differences in the procedures of numerical integration are discussed.

Key words: Earth tides – Torsional free oscillations – Love-numbers – Response functions – Numerical integration – Q determination

Introduction

At spherical coordinates r, ϑ, λ, every physical vector field can be decomposed in spheroidal and torsional fields (Morse and Feshbach, 1953). These constituents in spherical harmonics can be developed for surfaces where r is constant to produce a component representation of the spherical coordinate system. For example, the r, ϑ, λ components of the surface traction, $\tau(r, \vartheta, \lambda)$, acting on spherical surfaces where r is constant can be shown as

$$\tau(r,\vartheta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{q}\begin{cases}U_n^{mq}(r)\,Y_n^{mq}(\vartheta,\lambda)\\ V_n^{mq}(r)\dfrac{\partial Y_n^{mq}}{\partial\vartheta}\\ V_n^{mq}(r)\dfrac{1}{\sin\vartheta}\dfrac{\partial Y_n^{mq}}{\partial\lambda}\end{cases}$$

$$+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{q}\begin{cases}0\\ W_n^{mq}(r)\dfrac{1}{\sin\vartheta}\dfrac{\partial Y_n^{mq}}{\partial\lambda},\\ -W_n^{mq}(r)\dfrac{\partial Y_n^{mq}}{\partial\vartheta}\end{cases}\tag{1}$$

where

$$Y_n^{mq}(\vartheta,\lambda)=P_n^m(\cos\vartheta)\begin{cases}\cos m\lambda & \text{for } q=c\\ \sin m\lambda & \text{for } q=s\end{cases}$$

are spherical harmonics. The first term on the right hand side of Eq. (1) represents the spheroidal and the second term the torsional part of the surface traction, τ.

A corresponding decomposition can be given for the displacement field, $\mathbf{s}(r, \vartheta, \lambda)$:

$$\mathbf{s}(r,\vartheta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{q}\begin{cases}u_n^{mq}(r)\,Y_n^{mq}(\vartheta,\lambda)\\ v_n^{mq}(r)\dfrac{\partial Y_n^{mq}}{\partial\vartheta}\\ v_n^{mq}(r)\dfrac{1}{\sin\vartheta}\dfrac{\partial Y_n^{mq}}{\partial\lambda}\end{cases}$$

$$+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{q}\begin{cases}0\\ w_n^{mq}(r)\dfrac{1}{\sin\vartheta}\dfrac{\partial Y_n^{mq}}{\partial\lambda}\\ -w_n^{mq}(r)\dfrac{\partial Y_n^{mq}}{\partial\vartheta}.\end{cases}\tag{2}$$

Corresponding to the spheroidal and torsional types of vector fields, forced motions and free oscillations of a spherically symmetric non-rotating elastic isotropic (SNREI) earth are described by two separate systems of differential equations with adequate boundary conditions (Alterman et al., 1959; Aki and Richards, 1980; Lapwood and Usami, 1981). Both types of motion, spheroidal and torsional, are forced by corresponding stress fields and are completely decoupled. Any deviation from the SNREI properties will act to couple these fields (Masters et al., 1983).

Forced motions can be induced by volume or surface forces. For example, tidal forces exerted by the moon and the sun, or loading forces at the earth's surface such as the tidally varying ocean load, are volume forces. Their gravitational potential is only involved in the spheroidal system of equations, and therefore earth tides are spheroidal in character. In loading tides, both volume forces (gravitational attraction) and surface forces (pressure) are active. The load pressure squeezing the earth's surface, $r=a$, is a vectorial field, $\tau(r, \vartheta, \lambda)$, of spheroidal character, since it has only a radial component at $r=a$.

Torsional motions are restricted to movements on spherical surfaces and therefore do not change the mass distribution within a spherically symmetric earth. For an SNREI model earth they can only be induced by horizontal traction, e.g. friction caused by winds or ocean currents. In this context two questions arise:

1. Is it possible that both spheroidal and torsional motions are induced by purely horizontal traction at $r=a$?

2. What kind of structure of applied horizontal traction will induce torsional motions alone?

Boundary conditions for stress

These questions can be answered quite generally if the surface traction, $\tau(r, \vartheta, \lambda)$, is decomposed according to Eq. (1) and if the continuity of the stress at $r=a$ is taken into account. With applied traction of $\mathring{\tau}(\vartheta, \lambda)$, the continuity of stress yields

$$\tau(a, \vartheta, \lambda)=\mathring{\tau}(\vartheta, \lambda). \tag{3}$$

The radial component of $\mathring{\tau}(\vartheta, \lambda)$, developed in spherical harmonics, is

$$\mathring{\tau}_r(\vartheta, \lambda)=\sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{q} a_n^{mq} Y_n^{mq}(\vartheta, \lambda), \tag{4}$$

where

$$a_n^{mq}=U_n^{mq}(a) \tag{5}$$

follows from Eqs. (1) and (3). For reasons which will immediately become evident, the horizontal components of $\mathring{\tau}(\vartheta, \lambda)$ are developed in spherical harmonics according to Eq. (1), but are also decomposed in the following way:

$$\frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathring{\tau}_\vartheta)+\frac{\partial\mathring{\tau}_\lambda}{\partial\lambda}\right] =\sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{q} b_n^{mq} Y_n^{mq}(\vartheta, \lambda), \tag{6}$$

$$\frac{1}{\sin\vartheta}\left[\frac{\partial\mathring{\tau}_\vartheta}{\partial\lambda}-\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\mathring{\tau}_\lambda)\right] =\sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{q} c_n^{mq} Y_n^{mq}(\vartheta, \lambda). \tag{7}$$

From Eq. (1), for the stresses σ_{rr}, $\sigma_{r\vartheta}$, $\sigma_{r\lambda}$ at $r=a$, it follows that

$$\frac{1}{\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\sigma_{r\vartheta})+\frac{\partial\sigma_{r\lambda}}{\partial\lambda}\right]_n^{mq} =-n(n+1)\,V_n^{mq}(a)\,Y_n^{mq}(\vartheta, \lambda), \tag{8}$$

$$\frac{1}{\sin\vartheta}\left[\frac{\partial\sigma_{r\vartheta}}{\partial\lambda}-\frac{\partial}{\partial\vartheta}(\sin\vartheta\,\sigma_{r\lambda})\right]_n^{mq} =-n(n+1)\,W_n^{mq}(a)\,Y_n^{mq}(\vartheta, \lambda), \tag{9}$$

with

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y_n^{mq}}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_n^{mq}}{\partial\lambda^2} =-n(n+1)\,Y_n^{mq}(\vartheta, \lambda).$$

From Eqs. (1) and (3), which are valid for each harmonic component specified by a given combination of n, m, q,

$$[\sigma_{r\vartheta}]_n^{mq}=[\mathring{\tau}_\vartheta]_n^{mq},$$
$$[\sigma_{r\lambda}]_n^{mq}=[\mathring{\tau}_\lambda]_n^{mq}.$$

at $r=a$. The following relations are deduced from Eqs. (6)–(9):

$$b_n^{mq}=-n(n+1)\,V_n^{mq}(a), \tag{10}$$

$$c_n^{mq}=-n(n+1)\,W_n^{mq}(a), \tag{11}$$

yielding vanishing coefficients for $n=0$. With the relations of Eqs. (5), (10) and (11), it is obvious that question (1) of the introduction has to be answered with *yes*. Clearly, question (2) is answered by simply stating that the horizontal traction must be torsional in order to induce only torsional motion. A sufficient condition for this is that the applied horizontal traction, $\mathring{\tau}(\vartheta, \lambda)$, is zonal, i.e. $\mathring{\tau}_\vartheta=0$ and $\mathring{\tau}_\lambda$ is independent of λ. In this case

$$\mathring{\tau}_r(\vartheta)=0=\sigma_{rr}(a, \vartheta, \lambda),$$
$$\mathring{\tau}_\vartheta(\vartheta)=0=\sigma_{r\vartheta}(a, \vartheta, \lambda),$$
$$\mathring{\tau}_\lambda(\vartheta)=\sigma_{r\lambda}(a, \vartheta)=-\sum_{n=1}^{\infty} W_n^0(a)\frac{dP_n}{d\vartheta};$$

hence

$$a_n^{mq}=0=b_n^{mq}$$

and

$$U_n^{mq}(a)=0=V_n^{mq}(a).$$

The boundary conditions for the stress field at $r=a$ can be specified according to the character of the forcing field. In the spheroidal case four types of boundary conditions can be formulated (Molodenskiy, 1977; Saito, 1978; Okubo and Saito, 1983; Varga, 1983). For

(a) a forcing external potential (e.g. tidal potential),

$$S_n^{mq}(r, \vartheta, \lambda)=G_n^{mq}\left(\frac{r}{a}\right)^n Y_n^{mq}(\vartheta, \lambda), \qquad 0<r<\infty$$

and then $U_n^{mq}(a)=0=V_n^{mq}(a)$;

(b) a forcing mass-load potential,

$$S_n'^{mq}(r,\vartheta,\lambda) = G_n'^{mq}\, Y_n^{mq}(\vartheta,\lambda) \begin{cases} \left(\frac{r}{a}\right)^n & \text{for } 0<r<a \\ \left(\frac{a}{r}\right)^{n+1} & \text{for } a<r<\infty \end{cases},$$

with $G_n'^{mq} = \frac{4\pi G a}{2n+1} \rho_L \xi_n^{mq}$,

where G is the gravitational constant, ρ_L the mass-load density and ξ_n^{mq} the corresponding coefficient of the mass-load height development in spherical harmonics, so that $\rho_L \xi_n^{mq} Y_n^{mq}(\vartheta,\lambda)$ is the surface-load density (e.g. Zürn and Wilhelm, 1984), and then

$$U_n^{mq}(a) = -\frac{(2n+1)\, g_0}{4\pi G a} G_n'^{mq}$$

$$V_n^{mq}(a) = 0;$$

(c) a forcing pressure $\mathring{p}_n^{mq} Y_n^{mq}(\vartheta,\lambda)$ at $r=a$, then

$$U_n^{mq}(a) = -\mathring{p}_n^{mq} = -\frac{(2n+1)\, g_0}{4\pi G a} G_n^{+mq} \tag{12}$$

$$V_n^{m}(a) = 0,$$

where G_n^{+mq} is defined by Eq. (12), and g_0 is the gravitational force per unit mass at the surface $r=a$;

(d) a forcing horizontal surface traction at $r=a$,

$$\mathring{\tau}_n^{mq}(\vartheta,\lambda) = \begin{cases} 0 \\ \mathring{\tau}_n^{mq} \frac{\partial Y_n^{mq}}{\partial\vartheta} \\ \mathring{\tau}_n^{mq} \frac{1}{\sin\vartheta} \frac{\partial Y_n^{mq}}{\partial\lambda} \end{cases}$$

and then

$$U_n^{mq}(a) = 0,$$

$$V_n^{mq}(a) = \mathring{\tau}_n^{mq} = \frac{2n+1}{n(n+1)} \frac{g_0}{4\pi G a} G_n^{-mq}, \tag{13}$$

with G_n^{-mq} defined by Eq. (13).

The boundary conditions of (b) evolve from superposing the conditions of (a) and (c).

In the torsional case the forcing traction is given by

$$\mathring{\tau}_n^{mq}(\vartheta,\lambda) = \begin{cases} 0 \\ \mathring{\tau}_n^{mq} \frac{1}{\sin\vartheta} \frac{\partial Y_n^{mq}}{\partial\lambda} \\ -\mathring{\tau}_n^{mq} \frac{\partial Y_n^{mq}}{\partial\vartheta} \end{cases} \tag{14}$$

and the boundary condition for the stress yields

$$W_n^{mq}(a) = \mathring{\tau}_n^{mq} = \frac{g_0}{4\pi G a} G_n^{*mq}, \tag{15}$$

with G_n^{*mq} defined by Eq. (15).

Additional boundary conditions

The continuity of the potential at $r=a$ can be achieved by defining a response potential in case (a) by

$$\tilde{S}_n^{mq}(r,\vartheta,\lambda) = G_n^{mq}\, Y_n^{mq}(\vartheta,\lambda) \begin{cases} k_n(r) \left(\frac{r}{a}\right)^n & \text{for } 0<r<a \\ k_n(a) \left(\frac{a}{r}\right)^{n+1} & \text{for } a<r<\infty \end{cases}$$

and a corresponding response potential in cases (b), (c) and (d) by replacing k_n by k_n', k_n^+, k_n^- and G_n^{mq} by $G_n'^{mq}$, G_n^{+mq}, G_n^{-mq}, respectively.

A further boundary condition is a result of the continuity of the expression (Farrell, 1972)

$$(\nabla \tilde{S}_n^{mq} - 4\pi G \rho(r)\, \mathbf{s}_n^{mq}) \cdot \hat{\mathbf{r}},$$

where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. From Eq. (2), it follows for case (a) that

$$\left[\left(\frac{dk_n}{dr} + \frac{2n+1}{r} k_n(r)\right) G_n^{mq} - 4\pi G \rho(r)\, u_n^{mq}(r)\right]_{r=a} = 0, \tag{16}$$

and there is a corresponding relation for the other cases.

Relations between spheroidal coefficients

In the spheroidal state a system of six first-order linear differential equations can be used to describe the forced and free motions of the spherical earth. Alterman et al. (1959) introduced the five variables:

$$y_1(r) = u_n^{mq}(r), \quad y_2(r) = U_n^{mq}(r),$$

$$y_3(r) = v_n^{mq}(r), \quad y_4(r) = V_n^{mq}(r),$$

$$y_5(r) = G_n^{mq} \left(\frac{r}{a}\right)^n (\delta + k_n(r)),$$

and a variable, y_6, which is not used here in its original form but as defined by Saito (1974):

$$y_6 = \frac{dy_5}{dr} + \frac{n+1}{r} y_5 - 4\pi G \rho\, y_1,$$

which is

$$y_6(r) = \left[\frac{dk_n}{dr} + \frac{2n+1}{r}(\delta + k_n(r))\right] G_n^{mq} \left(\frac{r}{a}\right)^n - 4\pi G \rho\, y_1,$$

with $\delta=1$ in cases (a) and (b) and $\delta=0$ in cases (c) and (d). In addition, Saito showed that between two solutions $(y_{11}, \ldots, y_{61})$ and $(y_{12}, \ldots, y_{62})$, for two different sets of boundary conditions, the relation

$$y_{11} y_{22} + n(n+1) y_{31} y_{42} + (4\pi G)^{-1} y_{51} y_{62} = y_{12} y_{21} + n(n+1) y_{32} y_{41} + (4\pi G)^{-1} y_{52} y_{61} \tag{17}$$

holds at $r=a$. While y_2, y_4 and y_6 are determined by boundary conditions, y_1, y_3 and y_5 are linearly related to the forcing function by coefficients and factors according to Table 1 (Okubo and Saito, 1983).

Table 1. Factors and coefficients used to describe the earth's response to spheroidal forcing

	G_n^m	$G_n'^m$	G_n^{+m}	G_n^{-m}
y_1	h_n/g_0	h'_n/g_0	h_n^+/g_0	h_n^-/g_0
y_3	l_n/g_0	l'_n/g_0	l_n^+/g_0	l_n^-/g_0
y_5	$(1+k_n)$	$(1+k'_n)$	k_n^+	k_n^-
y_2	0	$-\frac{(2n+1)\,g_0}{4\pi G a}$	$-\frac{(2n+1)\,g_0}{4\pi G a}$	0
y_4	0	0	0	$\frac{(2n+1)\,g_0}{n(n+1)\,4\pi G a}$
y_6	$(2n+1)/a$	$(2n+1)/a$	0	0

By combining the solutions for two different sets of boundary conditions, six relations between the coefficients are derived (Saito, 1978; Okubo and Saito, 1983; Varga, 1983):

(a), (b): $k'_n = k_n - h_n$

(a), (c): $k_n^+ = -h_n$

(a), (d): $k_n^- = l_n$

(b), (c): $h_n^+ = h'_n + k_n^+ = h'_n - h_n$

(b), (d): $h_n^- = k_n^- - l'_n = l_n - l'_n$

(c), (d): $l_n^+ = -h_n^- = l'_n - l_n$, (18)

and trivially, because of the linearity of the boundary conditions and the differential equations,

$$h'_n = h_n + h_n^+$$
$$k'_n = k_n + k_n^+$$
$$l'_n = l_n + l_n^+.$$

From the twelve coefficients introduced in Table 1 six coefficients can be expressed as linear combinations of h_n, k_n, l_n, h'_n and l'_n, which can therefore be regarded as independent coefficients. However, the stress coefficient l_n^- does not appear in the relations [Eq. (18)], being completely independent. From earth tide measurements and model calculations the first five coefficients are fairly well known, but l_n^- has not yet been investigated. However, a complete knowledge of the reaction of the spherical non-rotating earth to spheroidal forcing fields requires the determination of l_n^-. From the definition in Table 1, it is clear that l_n^- appears in spheroidal motions of the earth that are induced by horizontal frictional forces exerted on the earth's surface, e.g. by winds and ocean currents. It is, however, expected that local disturbances will inhibit the identification of this effect.

Torsional forced motions

In the torsional case there is no disturbing potential and the displacements are purely horizontal everywhere. For a spherical earth with a fluid outer core, only two boundary conditions, at the earth's surface and at the core-mantle boundary, need be taken into account. There are only two differential equations, of first order, for the two variables (Alterman et al., 1959):

$$y_1(r) = w_n^{mq}(r)$$
$$y_2(r) = W_n^{mq}(r), \tag{19}$$

viz.:

$$\frac{dy_1}{dr} = y_1/r + y_2/\mu$$
$$\frac{dy_2}{dr} = ((n^2+n-2)\,\mu/r^2 - \omega^2\rho)\,y_1 - 3y_2/r, \tag{20}$$

where μ is the shear modulus and ρ the density, both depending on the radial distance, r.

The boundary condition at $r=a$ for a given torsional traction field, $\mathring{\tau}_n^{mq}(\vartheta, \lambda)$ according to Eq. (14), is given by Eqs. (15) and (19), yielding

$$y_2(a) = \mathring{\tau}_n^{mq},$$

whereas the displacement is linearly related to the torsional stress by

$$y_1(a) = l_n^*\, G_n^{*mq}/g_0$$

where G_n^{*mq} is given by Eq. (15). Numerical integration of the system [Eq. (20)] yields the torsional stress coefficient, l_n^*.

For torsional motions it is particularly easy to describe an attenuating medium, achieved by introducing a complex shear modulus:

$$\hat{\mu} = \mu(1 + i/Q_\mu),$$

where μ, but not the dissipation parameter Q_μ, is assumed to be frequency-dependent (Kanamori and Anderson, 1977), so that

$$\mu(\omega) = \mu(\omega_r)\,(1 + 2/(\pi Q_\mu)\ln(\omega/\omega_r)),$$

and ω_r is a reference frequency.

Consequently, the stress and the displacement also become complex quantities:

$$\hat{w}_n^{mq} = w_n^{mq} + i\,\bar{w}_n^{mq} = y_1 + i\,\bar{y}_1$$
$$\hat{W}_n^{mq} = W_n^{mq} + i\,\bar{W}_n^{mq} = y_2 + i\,\bar{y}_2.$$

Correspondingly, there are now four differential equations instead of two:

$$\frac{dy_1}{dr} = y_1/r + y_2/(\mu(\omega)\,(1+Q_\mu^{-2})) + \bar{y}_2/(Q_\mu\,\mu(\omega)\,(1+Q_\mu^{-2}))$$
$$\frac{d\bar{y}_1}{dr} = \bar{y}_1/r + \bar{y}_2/(\mu(\omega)\,(1+Q_\mu^{-2})) - y_2/(Q_\mu\,\mu(\omega)\,(1+Q_\mu^{-2}))$$
$$\frac{dy_2}{dr} = y_1((n^2+n-2)\,\mu(\omega)/r^2 - \omega^2\rho) - 3y_2/r - \bar{y}_1(n^2+n-2)\,\mu(\omega)/(r^2 Q_\mu)$$
$$\frac{d\bar{y}_2}{dr} = \bar{y}_1((n^2+n-2)\,\mu(\omega)/r^2 - \omega^2\rho) - 3\bar{y}_2/r + y_1(n^2+n-2)\,\mu(\omega)/(r^2 Q_\mu), \tag{21}$$

and the resulting stress coefficient is also complex:

$$\hat{l}_n^* = l_n^* + i\,\bar{l}_n^*.$$

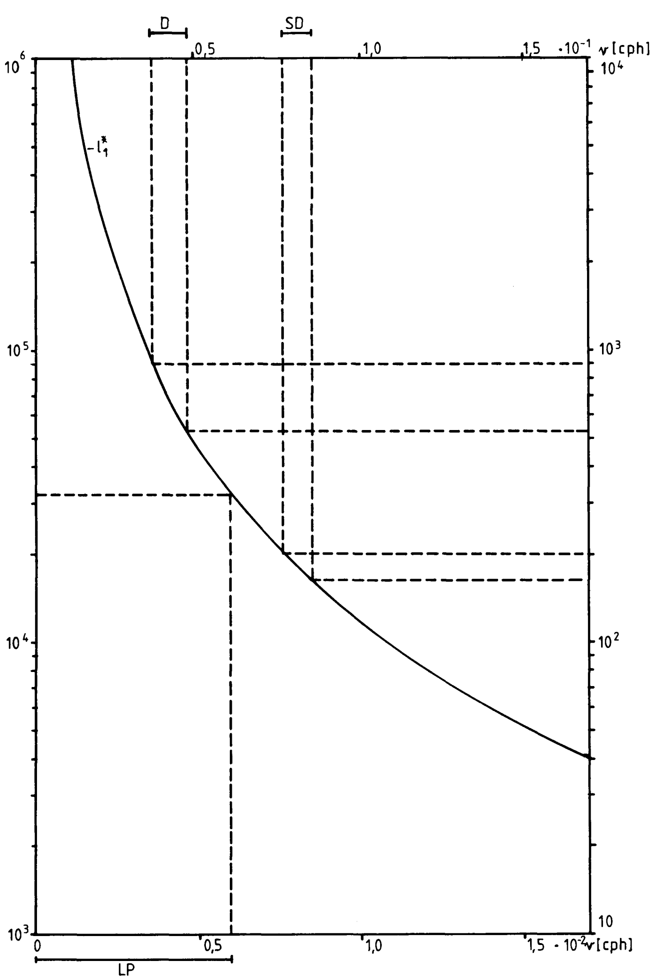

Fig. 1. Negative real part l_1^* of complex torsional stress coefficient $\hat{l}_1^*$ for small frequencies using the preliminary earth reference model (PREM). Two frequency and ordinate scales are used. Long period (*LP*), diurnal (*D*), and semi-diurnal (*SD*) tidal frequency bands are indicated

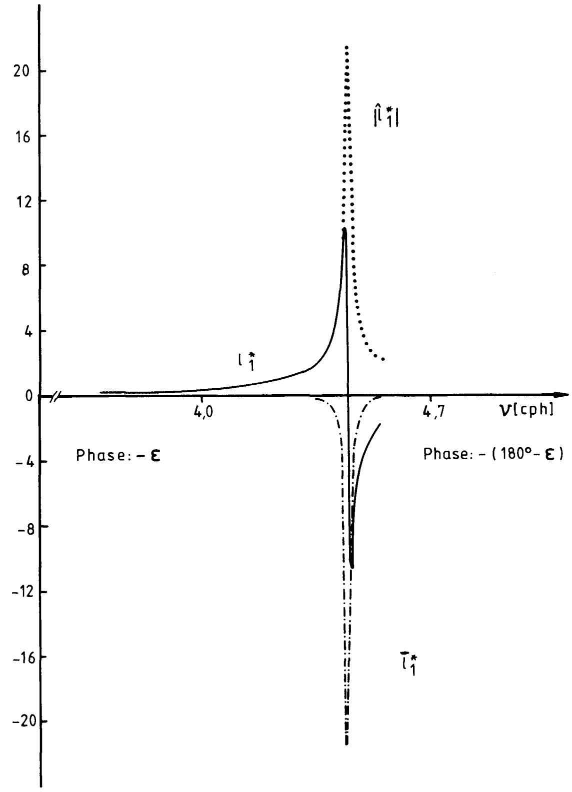

Fig. 2. Frequency dependence of the complex stress coefficient, $\hat{l}_1^*$, in the vicinity of the mode ${}_1T_1$ using the preliminary earth reference model (PREM). The real part, l_1^*, the imaginary part, $\bar{l}_1^*$, and the magnitude, $|\hat{l}_1^*|$, were determined

If the forcing field is assumed to be a real quantity, i.e. $\bar{y}_2(a)=0$, and if $y_2(a)$ is given, the response of the earth, i.e. the displacement and the stress within the earth, can be calculated by numerical integration of the system [Eq. (21)]. At the earth's surface, $r=a$, the coefficient $\hat{l}_n^*$ completely describes the response, since according to

$$\hat{y}_1 = \hat{l}_n^* G_n^{*mq}/g_0 \tag{22}$$

the amplitude and phase of $\hat{l}_n^*$ relates the displacement linearly to the forcing field.

The torsional stress coefficient, $\hat{l}_n^*$, has been calculated for the anelastic isotropic preliminary reference earth model (PREM) (Dziewonski and Anderson, 1981) and a varying circular frequency ω. For $n=1$ and small frequencies, $\hat{l}_n^*$ has an extremely small imaginary part compared to the real part, which is resonant for $\omega \to 0$, whereas the imaginary part becomes zero. This resonance at zero frequency corresponds to the torsional free mode, ${}_0T_1$, which can only be excited by static external forces exerting a torque on the earth. The earth's response to the torque is a rotation which leads to infinite displacements with respect to a non-rotating coordinate system. Figure 1 shows the negative real part of $\hat{l}_1^*$ for the two frequency bands 0–0.017 c/h and 0–0.17 c/h. For small frequencies the phase difference between the displacement and the applied stress is almost 180°, since the imaginary part of $\hat{l}_1^*$ is positive. This corresponds to the behaviour of a damped harmonic oscillator for frequencies greater than the resonance frequency. For $n=2, 3, \ldots, \hat{l}_n^*$ has definite real limits for $\omega \to 0$. In a rotating coordinate system the zero frequency mode is transformed into a free mode, with a period corresponding to the rotation of the inertial frame (Dahlen and Smith, 1975).

At the appropriate frequencies of the applied traction, the torsional vibrations of the earth lead to a resonant behaviour. Figure 2 shows this behaviour for $\hat{l}_1^*$ in the neighbourhood of the eigenfrequency of the ${}_1T_1$ mode. The zero crossing of l_1^* determines the resonance frequency.

By calculating $\hat{l}_n^*$ in the vicinity of the resonances, the eigenperiod ${}_kT_n$ and the quality factor ${}_kQ_n$ of the corresponding torsional oscillation can be determined. For a damped harmonic oscillator with resonance frequency ${}_k\omega_n$ and quality ${}_kQ_n$ the frequency response is given (Ben Menahem and Singh, 1981) by

$$M(\omega) = ({}_k\omega_n^2 - \omega^2 + i\,\omega_n\,\omega_k/{}_kQ_n)^{-1}.$$

The ratio of the real to the imaginary part of $M(\omega)$ is

$$G(\omega) = (\omega/{}_k\omega_n - {}_k\omega_n/\omega)\,{}_kQ_n. \tag{23}$$

Fourier transformation of Eq. (22) yields the corresponding frequency response, $\hat{l}_n^*(\omega)$; the ratio of the real

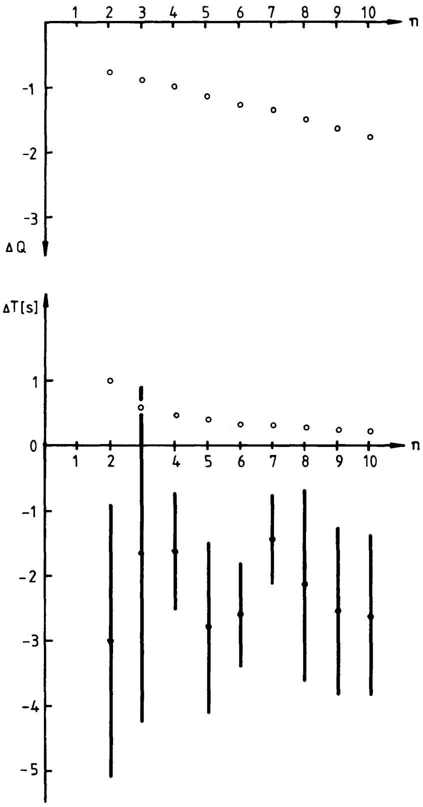

Fig. 3. Differences, $\Delta Q = Q_{nC} - Q_{nDA}$ and $\Delta T = T_{nC} - T_{nDA}$, between the calculated values, Q_{nC} and T_{nC}, from the resonances of $\hat{l}_n^*$ and the values Q_{nDA} and T_{nDA}, given by Dziewonski and Anderson (1981) for ${}_0T_n$ and the preliminary earth reference model (PREM), $n=2 \ldots 10$. Differences of the observed values are given with *error bars*

to the imaginary part of $\hat{l}_n^*$ is well represented by a function like Eq. (23), with ${}_k\omega_n$ and ${}_kQ_n$ as parameters whose values are determined by least squares fitting of the calculated ratio $l_n^*:\bar{l}_n^*$ to the response ratio [Eq. (23)] for different values of ω. The frequencies used for the least squares fit have to be chosen sufficiently close to and symmetrically distributed around the zero crossing value. It happens that the least squares fit is also a robust fit in the calculated numerical examples. In this way eigenperiods and quality factors of torsional oscillations have been determined for the PREM. For the lowest fundamental modes, the differences between ${}_0T_n = 2\pi/{}_0\omega_n$ and ${}_0Q_n$ obtained by this method and the corresponding values given by Dziewonski and Anderson (1981) are shown in Fig. 3. In addition the differences of the observed eigenperiods are shown with error bars. There is a bias of about 0.1% between the corresponding eigenperiods and about 1% between the Q values.

Though this bias is rather small it looks systematic. Since it might have arisen through neglecting the influence of neighbouring eigenperiods on the determination of ${}_k\omega_n$, in the response function up to 12 eigenfrequencies were taken into account instead of one. The effect of the additional resonances is to give a deviation of about 0.01 s for the resonance required, at low degrees of n. For $n=8$ the deviation is 0.02 s for the period ${}_0T_8$, and 0.03 for ${}_0Q_8$. This difference is reached after taking the next four additional resonances into account, and

Table 2. Numerical methods used to integrate the linear system $\mathbf{y}'(r) = \mathbf{A}(r)\,\mathbf{y}(r)$ (Stiefel, 1976) and corresponding results for the fundamental mode ${}_0T_2$: 1. Heun method, with predictor for unequal integration intervals; 2. Runge-Kutta method, with predictor for unequal integration intervals; 3. modified Runge-Kutta method with inversion for unequal integration intervals; 4. Runge-Kutta first method with predictor and linear interpolation for equal integration intervals; 5. Runge-Kutta second method with predictor and linear interpolation for equal integration intervals; 6. calculated values of Dziewonski and Anderson (1981); 7. observed values according to Dziewonski and Anderson (1981)

Procedure	${}_0T_2$ [s]	${}_0Q_2$
1	2640.8	249.61
2	2629.5	250.33
3	2640.4	249.63
4	2640.0	249.63
5	2640.0	249.63
6	2639.4	250.40
7	2636.38 ±2.11	?

the use of additional eigenfrequencies does not change the result further at the five-digit level. Therefore, saturation is reached with a limited number of neighbouring resonances. For growing values of n the number of resonances that have to be respected also increases. This effect, however, cannot be responsible for the bias shown in Fig. 3 because the corresponding differences are much lower than the bias. Only those eigenfrequencies with the same degree n but different node number k are involved in this problem.

Another proposal was that different methods of numerical integration may yield differences in the results. Hence, as a computational example, different methods of integration were applied for the determination of the period and quality of ${}_0T_2$. The result is shown in Table 2 for five different procedures. Lines 6 and 7 show the values calculated by Dziewonski and Anderson (1981) compared with the observed values. Except for method 2, the least squares fitted values agree to within ± 0.01 for ${}_0Q_2$ and to within ± 0.04 s for ${}_0T_2$. The differences between the results are comparable with the differences shown in Fig. 3. Therefore it can be concluded that a possible cause of the bias may be found in different integration procedures. In particular, two inherently different methods (4 and 5; Table 2) yield the same results within five digits, yet differ from the results of the Dziewonski and Anderson method (1981).

Dziewonski and Anderson (1981) calculated the Q of a normal mode with index j according to a procedure described by Sailor and Dziewonski (1978):

$$Q_j^{-1} = \frac{1}{a^3} \int_0^a (\mu(r)\,\tilde{M}_j(r)\,Q_\mu^{-1}(r) + K(r)\,\tilde{K}_j(r)\,Q_k^{-1}(r))\,r^2\,dr,$$

where $\tilde{M}_j(r)$ and $\tilde{K}_j(r)$ are defined by Backus and Gilbert (1967, eqs. 30, 31), and $\mu(r)$ and $K(r)$ are the shear and the bulk modulus and $Q_\mu^{-1}(r)$ and $Q_k^{-1}(r)$ the radially dependent imaginary parts of the corresponding complex moduli (Ben Menahem and Singh, 1981, eqs.

10.130, 10.198). The method finally used in this paper is a combined determination of the eigenfrequency and the quality factor of a normal mode, using the response of a linear oscillator to an external forcing function. If there are $N+1$ resonances the response is given by

$$M(\omega)=\sum_{j=0}^{N}\{(\omega_j^2-\omega^2)/[(\omega_j^2-\omega^2)^2+(\omega\,\omega_j/Q_j)^2] \\ -i(\omega\,\omega_j/Q_j)/[(\omega_j^2-\omega^2)^2+(\omega\,\omega_j/Q_j)^2]\}, \quad (24)$$

where $j=0$ is the resonance in which ω_0 and Q_0 are to be determined and $j=1, 2, \ldots N$ are the resonances which have to be taken into account for the determination of ω_0 and Q_0, and for which preliminary values of sufficient accuracy are inserted in Eq. (24) in order to determine the influence of the other N resonances on ω_0 and Q_0. The ratio of the real to the imaginary part of $M(\omega)$ is $G(\omega;\ \omega_0, Q_0;\ \omega_j, Q_j)$, $j=1 \ldots N$, which is calculated from Eq. (24). By fitting the ratio $l_n^*:\bar{l}_n^*$ to the ratio of the real to the imaginary part of $M(\omega)$ in a linearized least squares procedure, the unknown values of ω_0 and Q_0 are determined.

Conclusions

The response of an SNREI model earth to a forcing spheroidal field is completely described by six coefficients, five of which (h_n, k_n, l_n, h_n' and l_n') can, in principle, be calculated by numerical integration of the corresponding set of linear differential equations, and are related to earth tide measurements. Except one, all other coefficients used to describe the response are linearly composed of these five coefficients. The exception is l_n^-, which is completely independent and difficult to determine by measurements. Until now, it has not received any attention, but it is fundamental. Model calculations should reveal its numerical value as a function of frequency.

The response of an SNREI model earth to a forcing torsional stress field acting at the earth's surface can be completely described by only one coefficient, l_n^*. After introducing attenuation to the earth model, the eigenperiods and quality factors of the torsional free oscillations can be determined by a least squares fit procedure to the response function of a linear oscillator model. For the isotropic PREM (Dziewonski and Anderson, 1981), the calculated values yield differences of 0.1% for the period ${}_0T_n$ and 1% for ${}_0Q_n$ with respect to the values given by Dziewonski and Anderson (1981). Using different integration procedures, differences of the same order can be obtained.

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