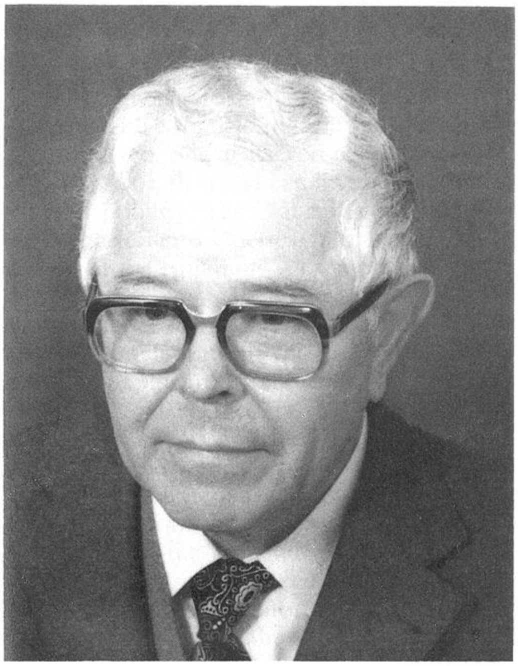
Journal of Geophysics

In memoriam

Herbert Lennartz (1916–1986)



Herbert Lennartz, founder and, until his retirement in 1984, owner of Lennartz-Electronic – manufacturer of seismic instruments, Tübingen – died October 23, 1986, after a few months of illness. He was a pioneer in the design and construction of seismic data-acquisition systems, and his instruments are in use in more than 50 countries all over the world.

Lennartz was born in Lingen (Ems) in 1916 as the son of a pharmacist. At the age of seven, his school-mates wondered if it was possible to climb one of those newly installed high-voltage transmission towers. Young Herbert showed that it was, but he lost his right arm completely and the left arm had to be amputated above the elbow. After long hospital treatment he had a private teacher for one year, but then he was again able to attend a public school. Following his Abitur, which he took in Essen, he went to Berlin to study physics at the Technische Hochschule. He earned his living by writing technical articles for various radio magazines. In addition to his construction articles, he distributed the necessary electronic parts. Possibly he was the first to sell television receiver kits.

During the second World War he was forced to develop electronic measuring devices for military use. With air raids against Berlin getting more severe, he had to transfer his company to a quieter place. He chose Tübingen to be close to a university, and so this town became the home for the rest of his life.

Radio receivers were much in demand in the years after the war. His company "Labor für technische Physik" (L.T.P.), which he owned together with a partner, now produced radios using parts from surplus military equipment. In the late 1940s they belonged to the largest manufacturers of radio receivers in West Germany, employing 700 people. After the dreadful war, people in Germany were strongly prone to romanticism. A best seller of L.T.P. were radio chassis installed in ceramic flower vases. But, alas, a thousand vases turned out to be a few millimeters too small to serve as cabinets for the chassis. Economic difficulties followed, and the company was forced to close. Again, for some time, Herbert Lennartz earned his living writing numerous articles for electronics journals. With no hands available, he had to take the soldering iron in his mouth to assemble the test circuits.

In 1959, his economic situation again allowed the foundation of his own office: "Ingenieurbüro Herbert Lennartz". He used his experience in the design of instrumentation amplifiers and data recording systems made during the war and developed circuits for storing low-frequency and D.C. data on magnetic tape using a frequency modulation multiplex channel system. His system was ideally suited to the requirements of recording seismograms in refraction shooting and, after a testing phase, the German geophysical institutes ordered the first 40 recording stations which later became known as "MARS 66". Performance and reliability of Lennartz equipment soon became proverbial. In 1978, he was able to sell modulator number 500. The name of Lennartz is strongly associated with the beginning of modern long-profile refraction measurements.

He now worked almost exclusively for geophysics, under the new name Lennartz-Electronic. Although most of his life was devoted to analog circuits, he did not miss the train to the digital age. But health problems became more severe, and, when in 1984 the Lennartz 5800 PCM system was ready for series production, he decided to retire and sold his company to leading employees.

Herbert Lennartz was a versatile man: a genius engineer, a clever business man, an active radio amateur and, despite his severe physical handicap, an enthusiastic sports-car driver.

R. Schick

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Original investigations

An analytical approach to the magnetic field of the Earth's crust

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Abstract. A method is introduced that allows calculation of the magnetic crustal field of the Earth from crustal and geomagnetic core field data. If induced magnetization only is considered, and if its vertical extension is neglected on a global scale, the crust can be characterized by an integrated susceptibility. The inducing core field is represented by an expansion into spherical harmonics, and the explicit derivation of the equations is exemplified with the dipole term of this expansion. The application to a recently developed global model of the Earth's crust involves the conversion of the discrete data of the crustal magnetization into the continuous integrated susceptibility function which is then definitive for the model. This is accomplished by means of potential theory and classical spherical harmonic analysis applied to the surface density of magnetization of variable amount and direction. A comparison of the spatial energy density spectrum of the crustal field with previous results demonstrates the usefulness of the method. The analytical approach yields results of comparable accuracy with considerably less numerical effort. Regarding the induced crustal field, it turns out that the inducing core field may satisfactorily be approximated by its zonal dipole term. The angular dependence of the inducing field has only little influence upon the crustal model field.

Key words: Geomagnetic crustal field – Integrated crustal susceptibility – Potential theory – Spherical harmonic analysis – Multipole expansion – Spatial energy density spectrum

Introduction

In recent years there has been a growing interest in the crustal part of the Earth's magnetic field, both under regional and global aspects (Regan et al., 1975; Mayhew, 1979; Alldredge, 1983; Nakagawa et al., 1985). The investigation of the crustal field or *crustal anomaly* has been greatly facilitated by satellite surveys at low altitudes. We mention in particular the mission of the Magnetic Field Satellite (Magsat) that was planned and operated by NASA and the U.S. Geological Survey between October 1979 and June 1980. It was the first satellite designed to provide a global vector survey of the magnetic field.

The measured whole field $\mathbf{B}_w$ is the superposition of the internal geomagnetic field $\mathbf{B}_i$ originating from sources within the Earth's core ($\mathbf{B}_c$) and crust ($\mathbf{B}_a$), and of the external field $\mathbf{B}_e$ that results from ionospheric and magnetospheric currents. In the following we shall be concerned only with the internal part of the field, i.e. in the case of any comparison with experiment we assume that the external contributions have been removed in advance from the measured data. The crustal field is then obtained by subtracting the geomagnetic core field $\mathbf{B}_c$ from the remaining internal field

$$\mathbf{B}_a = \mathbf{B}_i - \mathbf{B}_c = (\mathbf{B}_w - \mathbf{B}_e) - \mathbf{B}_c. \tag{1}$$

The core field cannot be measured independently of the internal field, and commonly it has been subtracted from the latter by adopting a suitable field model. The core field is thought to be characterized by the large-scale or low-degree part of the spherical harmonic expansion of the internal field. Cain (1975) and Langel and Estes (1982) suggested attributing the first n degrees of the expansion to the core field and to represent the crustal field by all higher terms. These authors specified $n=13$ in view of an obvious separation of the energy density spectrum into two distinct straight sections that intersect at approximately this value. This suggestion has raised several objections. Carle and Harrison (1982) note that such a separation does not remove all large-scale oscillations from the crustal field. There are always some, though small in value, that remain. On the other hand, Meyer et al. (1983) have pointed out that also important large-scale variations pertaining to the crustal field are removed from the anomaly charts. Crustal structures of continental size, e.g. the continental margins, cannot be reproduced with the truncated expansion ($n \gtrsim 13$) taken for the crustal field; however, they should clearly be visible if a full expansion is used (Meyer et al., 1985). In view of the deficiencies of the conventional methods these authors made a new approach to the forward analysis of the Magsat anomaly data. On the basis of surface geology and seismic investigations they have developed a global model of the Earth's crust that is consistent with the crustal part of a particular field model derived by Cain et al. (1984) from the Magsat data. The improved version of the crustal model, that is effective for the time being, has been published by Hahn et al. (1984).

The representation of the magnetic field of the new crustal model in terms of isodynamic charts for the vertical component, or of the energy density spectrum, required ex-

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tensive numerical computations. The numerical expenditure and the need for repeated computation with different input parameters suggested elaborating a more economical way in order to study the dependence of the crustal model field on the particulars of the model and of the inducing core field. We present here an analytical method of forward calculation of the crustal field that was originally outlined by one of the authors (M.S.) and further developed and applied to the above-mentioned crustal model by the other author (Nolte, 1985). After formulation of the pertinent equations in the next section, the method is applied to the latest crustal model and evaluated by comparison with previous results. Conclusions will be drawn with respect to the crustal magnetization of the Earth that can be deduced from the model, and to the role of the inducing core field.

Theory

The approach is based on the following assumptions. At the present level of approximation we neglect all deviations of the shape of the terrestrial body from a sphere with equivolumetric radius a ($a = 6371.2$ km). The crustal thickness does not surpass about 0.5% of the Earth's radius. Since we are interested in global aspects of the field, it is admissible to neglect the thickness as compared to the Earth's radius. Taking into account only induced magnetization, the crustal data can be described by an integrated susceptibility that depends upon the angular coordinates of the spherical surface. Spherical coordinates are understood to coincide with the conventional geographic system of radius r, colatitude θ and longitude λ.

Consider a volume element $d\tau'$ of magnetizable material located at the point $\mathbf{r}' = (r', \theta', \lambda')$ within the Earth's crust that is exposed to the magnetic induction $\mathbf{B}_c$ of the geomagnetic core field. The magnetization of $d\tau'$ causes a crustal magnetic induction with a scalar potential dV_a at a fixed external point $\mathbf{r} = (r, \theta, \lambda)$. The underlying geometric situation is illustrated in Fig. 1. If $d\tau'$ is sufficiently small, dV_a can be formulated as the dipole potential

$$dV_a(\mathbf{r}, \mathbf{r}') = \frac{\mu_0}{4\pi}\, d\mathbf{m}(\mathbf{r}') \cdot \nabla' \frac{1}{l}. \tag{2}$$

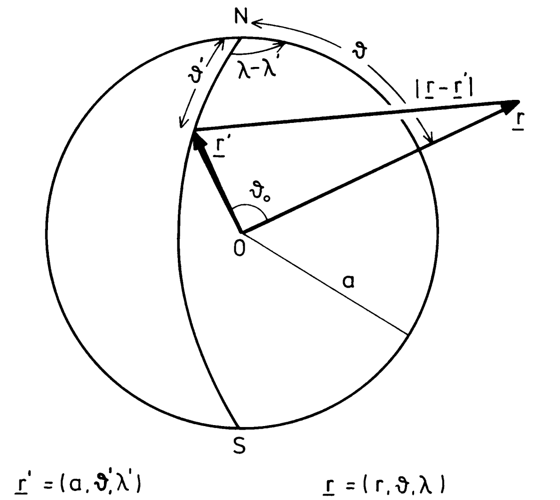
Here μ_0 is the permeability of vacuum, $l = |\mathbf{r} - \mathbf{r}'|$, and the prime with the Nabla operator ∇' denotes differentiation with respect to crustal coordinates. The magnetic dipole moment $d\mathbf{m}$ is the product of $d\tau'$ with the magnetization $\mathbf{M}$. Under the assumptions mentioned above, the latter is the product of the magnetic susceptibility χ of the material and of the inducing field $\mathbf{B}_c$ (or, more strictly, $\mathbf{B}_c/\mu_0$) with a scalar potential V_c outside the Earth's core

$$d\mathbf{m}(\mathbf{r}') = \mathbf{M}(\mathbf{r}')\, d\tau' = -\frac{1}{\mu_0}\, \chi(\mathbf{r}')\, \nabla' V_c(\mathbf{r}')\, d\tau'. \tag{3}$$

To take advantage of the negligible crustal thickness d, we use the integrated susceptibility of the crust in order to define the surface susceptibility $\tilde{\chi}$ by

$$\tilde{\chi}(\theta', \lambda') = \lim_{\substack{d \to 0 \\ \chi \to \infty}} \int_{a-d}^{a} \chi(\mathbf{r}')\, dr' \tag{4}$$

which is related to the corresponding vector of surface magnetization by



$$\cos\vartheta_0 = \cos\vartheta \cos\vartheta' + \sin\vartheta \,\sin\vartheta' \cos(\lambda - \lambda')$$

Fig. 1. Illustration of the geometric relations between some point in space and some point on the spherical Earth's surface

$$\tilde{\mathbf{M}}(\theta', \lambda') = -\frac{1}{\mu_0}\, \tilde{\chi}(\theta', \lambda')\, \nabla' V_c(\mathbf{r}')\big|_{r'=a}. \tag{5}$$

The vertical bar denotes that the variable r' has to be replaced by the radius a after differentiation.

The total potential of the crustal field at the external point is obtained from integrating Eq. (2) over the entire crustal volume. We note that $d\tau'$ is connected with the differential solid angle $d\omega'$ on the spherical surface by $d\tau' = a^2\, dr'\, d\omega'$, where $d\omega' = \sin\theta'\, d\theta'\, d\lambda'$. After introduction of Eqs. (3) and (4) into Eq. (2) the result is

$$V_a(\mathbf{r}) = -\frac{a^2}{4\pi} \int_0^{4\pi} \tilde{\chi}(\theta', \lambda')\, \nabla' V_c(\mathbf{r}') \cdot \nabla' \frac{1}{l}\bigg|_{r'=a} d\omega'. \tag{6}$$

The potential of the geomagnetic core field is expanded into a series of spherical harmonics with reference radius a

$$V_c(\mathbf{r}') = a \sum_{n'=1}^{\infty} \left(\frac{a}{r'}\right)^{n'+1} Y_{n'}(\theta', \lambda'). \tag{7}$$

Due to the missing magnetic monopole, there is no term with degree $n' = 0$. $Y_{n'}$ represents a sum over one or more elementary surface harmonics of the same degree n' but with orders varying between $m' = 0$ and $m' = n'$

$$Y_{n'} = \sum_{m'=0}^{n'} (g_{n'm'} \cos m'\lambda' + h_{n'm'} \sin m'\lambda')\, P_{n'}^{m'}(\theta'). \tag{8}$$

Here the quasi-normalization according to Schmidt is used throughout for Legendre's associated functions $P_{n'}^{m'}$, and all Gaussian coefficients $g_{n'm'}$ and $h_{n'm'}$ are given by an appropriate field model.

There is a well-known expansion of the reciprocal distance between points $\mathbf{r}$ and $\mathbf{r}'$ into spherical harmonics (Hobson, 1955) which reads, for $r' \leq r$,

$$\frac{1}{l} = \frac{1}{r} \sum_{\nu'=0}^{\infty} \left(\frac{r'}{r}\right)^{\nu'} P_{\nu'}(\theta_0). \tag{9}$$

The geocentric angle θ_0 between $\mathbf{r}$ and $\mathbf{r}'$ combines the angular components of both vectors by the cosine relation of spherical trigonometry

$$\cos\theta_0 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda-\lambda'). \tag{10}$$

Provided the integrated susceptibility is differentiable, it can be expanded into a series of spherical surface harmonics with variable degree n. Let

$$\tilde{\chi} = \sum_{n=1}^{\infty} \tilde{\chi}_n(\theta', \lambda'), \tag{11}$$

where $\tilde{\chi}_n$ is a sum of elementary surface harmonics

$$\tilde{\chi}_n = \sum_{m=0}^{n} (p_{nm}\cos m\lambda' + q_{nm}\sin m\lambda') P_n^m(\theta'). \tag{12}$$

Strictly speaking, the expansion in Eq. (11) should start with a term of degree $n=0$ that represents a constant distribution of the integrated susceptibility over the entire surface of the Earth. However, this term is dropped because of a theorem due to Runcorn (1975) stating that a spherical shell of uniform susceptibility does not contribute to the induced magnetic field to be observed outside.

The potential V_a is evaluated by separating Eq. (6) into a sum of two integrals that comprise only differentiations with respect to (i) the radial variable r' and (ii) the angular variables θ' and λ', respectively.

$$V_a(\mathbf{r}) = V_a'(\mathbf{r}) + V_a''(\mathbf{r}), \tag{13}$$

where

$$V_a'(\mathbf{r}) = -\frac{a^2}{4\pi}\int_0^{4\pi} \tilde{\chi}\frac{\partial V_c}{\partial r'}\frac{\partial}{\partial r'}\frac{1}{l}\bigg|_{r'=a} d\omega' \tag{14}$$

and

$$V_a''(\mathbf{r}) = -\frac{a^2}{4\pi}\int_0^{4\pi} \tilde{\chi}\nabla_t' V_c \cdot \nabla_t' \frac{1}{l}\bigg|_{r'=a} d\omega'. \tag{15}$$

The truncated Nabla operator is defined by

$$\nabla_t' = \frac{1}{a}\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta'} + \hat{\boldsymbol{\lambda}}\frac{1}{\sin\theta'}\frac{\partial}{\partial\lambda'}\right), \tag{16}$$

and the unit vectors $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\lambda}}$ point into the directions of increasing colatitude and longitude, respectively.

It is no loss of generality to pick out one representative term each from Eqs. (7) and (11). All operators (integral and differential) that will be encountered are linear, and the most general result can easily be obtained by summing up finally over all pertinent degrees n and n'.

i) After introducing Eq. (9) and Eqs. (7) and (11) (representative terms) into Eq. (14) we obtain, with $r'=a$,

$$V_a'(\mathbf{r}) = \frac{1}{4\pi}\sum_{\nu'=0}^{\infty} \nu'\left(\frac{a}{r}\right)^{\nu'+1}\int_0^{4\pi}(n'+1)\tilde{\chi}_n Y_{n'} P_{\nu'}(\theta_0)\, d\omega'$$
$$(n=1,2,3,\ldots;\; n'=1,2,3,\ldots). \tag{17}$$

Note that by means of Eq. (10) Legendre's polynomials $P_{\nu'}$ depend upon the variables θ' and λ', as do the functions $\tilde{\chi}_n$ and $Y_{n'}$. The product $\tilde{\chi}_n Y_{n'}$ is defined on the surface of the sphere and can be expanded into a series of spherical surface harmonics S_ν with variable degree ν because the latter form a complete and orthogonal set of functions

$$S_\nu(\theta', \lambda') = \sum_{\mu=0}^{\nu} S_\nu^\mu(\theta', \lambda')$$
$$= \sum_{\mu=0}^{\nu}(a_{\nu\mu}\cos\mu\lambda' + b_{\nu\mu}\sin\mu\lambda') P_\nu^\mu(\theta'), \tag{18}$$

with $\nu = 0, 1, 2, \ldots$. With the definition

$$\sum_{\nu=0}^{\infty} S_\nu = (n'+1)\tilde{\chi}_n Y_{n'} \tag{19}$$

we obtain from Eq. (17)

$$V_a'(\mathbf{r}) = \frac{1}{4\pi}\sum_{\nu,\nu'=0}^{\infty}\nu'\left(\frac{a}{r}\right)^{\nu'+1}\int_0^{4\pi} S_\nu(\theta', \lambda') P_{\nu'}(\theta_0)\, d\omega'. \tag{20}$$

ii) The same considerations applied to Eq. (15) lead to

$$V_a''(\mathbf{r}) = -\frac{1}{4\pi}\sum_{\nu'=0}^{\infty}\left(\frac{a}{r}\right)^{\nu'+1}\int_0^{4\pi}\tilde{\chi}_n a\nabla_t' Y_{n'} \cdot a\nabla_t' P_{\nu'}(\theta_0)\, d\omega'. \tag{21}$$

For the sake of brevity we temporarily replace the "prefactor" of this integral by

$$\Gamma := \frac{1}{4\pi}\sum_{\nu'=0}^{\infty}\left(\frac{a}{r}\right)^{\nu'+1}. \tag{21a}$$

The operator ∇_t' fulfils the identity

$$\nabla_t' \cdot (u\mathbf{v}) = u\nabla_t' \cdot \mathbf{v} + \mathbf{v}\cdot\nabla_t' u, \tag{22}$$

where u and $\mathbf{v}$ denote a scalar and a vector field, respectively, of the variables θ' and λ'. Setting $u = P_{\nu'}$ and $\mathbf{v} = \tilde{\chi}_n \nabla_t' Y_{n'}$ this identity allows us to rewrite Eq. (21) as

$$V_a''(\mathbf{r}) = -\Gamma\int_0^{4\pi} a\nabla_t' \cdot (P_{\nu'}\tilde{\chi}_n a\nabla_t' Y_{n'})\, d\omega'$$
$$+\Gamma\int_0^{4\pi} P_{\nu'} a\nabla_t' \cdot (\tilde{\chi}_n a\nabla_t' Y_{n'})\, d\omega'. \tag{23}$$

The first integral on the right-hand side vanishes (see Appendix) and we are left with

$$V_a''(\mathbf{r}) = \Gamma\int_0^{4\pi} P_{\nu'}(\tilde{\chi}_n(a\nabla_t')^2 Y_{n'} + a\nabla_t'\tilde{\chi}_n \cdot a\nabla_t' Y_{n'})\, d\omega'. \tag{24}$$

The square of $(a\,\nabla_t')$ is Legendre's differential operator

$$L := -(a\nabla_t')^2 = -\frac{1}{\sin\theta'}\left(\frac{\partial}{\partial\theta'}\sin\theta'\frac{\partial}{\partial\theta'} + \frac{1}{\sin\theta'}\frac{\partial^2}{\partial\lambda'^2}\right) \tag{25}$$

which has the basic property that any spherical surface harmonic of degree, say, n is an eigenfunction to the eigenvalue $n(n+1)$. The second term of the integrand in Eq. (24) obeys the relation

$$a\nabla_t'\tilde{\chi}_n \cdot a\nabla_t' Y_{n'} = -\tfrac{1}{2}(L\tilde{\chi}_n Y_{n'} - \tilde{\chi}_n L Y_{n'} - Y_{n'} L\tilde{\chi}_n). \tag{26}$$

With the definition of the functions S_ν according to Eq. (19) and with the "prefactor" Γ according to Eq. (21a) we obtain

$$V_a''(\mathbf{r}) = -\frac{1}{4\pi}\sum_{\nu,\nu'=0}^{\infty}\frac{\nu(\nu+1)+n'(n'+1)-n(n+1)}{2(n'+1)} \cdot\left(\frac{a}{r}\right)^{\nu'+1}\int_0^{4\pi} S_\nu(\theta',\lambda')\,P_{\nu'}(\theta_0)\,d\omega'. \tag{27}$$

The integral over the solid angle in Eqs. (20) and (27) is solved by application of an integral-theorem for spherical surface harmonics (Lense, 1954, p. 76)

$$\int_0^{4\pi} S_\nu(\theta',\lambda')\,P_{\nu'}(\theta_0)\,d\omega' = \frac{4\pi}{2\nu+1}\,S_\nu(\theta,\lambda)\,\delta_{\nu\nu'}. \tag{28}$$

$\delta_{\nu\nu'}$ is the Kronecker symbol. Note that the angular coordinates of the fixed external point $\mathbf{r}$ enter as arguments into the function S_ν on the right-hand side. The whole potential of the crustal magnetic field at the external point is the sum of Eqs. (20) and (27). The result is generalized here by allowing for the full expansions Eqs. (7) and (11) of the core field and the susceptibility

$$V_a(\mathbf{r}) = \sum_{n,n'=1}^{\infty}\sum_{\nu=0}^{\infty}\frac{1}{2\nu+1} \cdot\left[\nu-\frac{\nu(\nu+1)+n'(n'+1)-n(n+1)}{2(n'+1)}\right]\left(\frac{a}{r}\right)^{\nu+1} S_\nu(\theta,\lambda). \tag{29}$$

Note that S_ν also depends on the summation indices n and n' according to Eq. (19). Equation (29) is the basic equation of the geomagnetic crustal field. The functions S_ν are specified by definite assumptions about the crustal susceptibility and the core field. A detailed inspection of the recurrence relations for spherical harmonics (Jahnke and Emde, 1945; Kautzleben, 1965), however, reveals that only a finite number of functions S_ν according to Eq. (19) are different from zero. The pertinent degrees vary in steps of two units between $\nu = |n-n'|$ and $\nu = n+n'$.

Then the crustal anomaly is derived from the gradient of the potential V_a

$$\mathbf{B}_a(\mathbf{r}) = -\nabla V_a(\mathbf{r}). \tag{30}$$

To be specific, the geomagnetic core field shall be represented by its dipole term only ($n'=1$). This case is the most practicable one for calculation and, as it turns out later, also the most important one at all

$$Y_1(\theta',\lambda') = g_{10}\,P_1(\theta') + (g_{11}\cos\lambda' + h_{11}\sin\lambda')\,P_1^1(\theta'). \tag{31}$$

The integrated susceptibility of the crust shall be accounted for by the representative term of Eq. (12) with arbitrary, but fixed, degree n and order m. Thus, the twofold sum over indices n and n' in Eq. (29) reduces to exactly one term. With the abbreviations

$$\begin{aligned}
\alpha(\lambda') &= p_{nm}\cos m\lambda' + q_{nm}\sin m\lambda',\\
\beta(\lambda') &= p_{nm}\cos[(m-1)\lambda'+\varepsilon] + q_{nm}\sin[(m-1)\lambda'+\varepsilon],\\
\gamma(\lambda') &= p_{nm}\cos[(m+1)\lambda'-\varepsilon] + q_{nm}\sin[(m+1)\lambda'-\varepsilon],
\end{aligned} \tag{32}$$

and

$$\varepsilon = \arctan(h_{11}/g_{11}),$$

we have

$$\tilde{\chi}\,Y_1 = g_{10}\,\alpha P_1 P_n^m + \tfrac{1}{2}\sqrt{g_{11}^2+h_{11}^2}\,(\beta+\gamma)\,P_1^1 P_n^m. \tag{33}$$

The products of Legendre's associated functions are expanded into a linear combination of the functions by means of the recurrence relations

$$P_1 P_n^m = \frac{1}{2n+1}\left(\sqrt{(n-m)(n+m)}\,P_{n-1}^m + \sqrt{(n-m+1)(n+m+1)}\,P_{n+1}^m\right) \tag{34}$$

and

$$\begin{aligned}
P_1^1 P_n^m &= \frac{\sqrt{\delta_{m-1}}}{2n+1}\left(\sqrt{(n+m-1)(n+m)}\,P_{n-1}^{m-1} - \sqrt{(n-m+1)(n-m+2)}\,P_{n+1}^{m-1}\right)\\
&= \frac{-1}{(2n+1)\sqrt{\delta_m}}\left(\sqrt{(n-m-1)(n-m)}\,P_{n-1}^{m+1} - \sqrt{(n+m+1)(n+m+2)}\,P_{n+1}^{m+1}\right)
\end{aligned} \tag{35}$$

with

$$\delta_k = \begin{cases} 1/2 & \text{for } k=-1\\ 2 & \text{for } k=0\\ 1 & \text{otherwise.}\end{cases}$$

The alternative recurrence relation differs with respect to the order of Legendre's associated functions. The correct choice is made if the order of the expanded form agrees with the argument of the trigonometric functions β and γ, respectively. The prefactors involving the operation δ_k result from the normalization due to Schmidt. Actually, the order assumes non-negative values only. However, the particular case δ_{-1} is introduced formally, because it is needed for the determination of the Gaussian coefficients of the crustal field. For the sake of clarity it is stressed that δ_k should not be confused with the Kronecker $\delta_{\nu\nu'}$.

According to Eq. (19) the product $\tilde{\chi}\,Y_1$ defines the spherical surface harmonics S_ν

$$\tilde{\chi}\,Y_1 = \tfrac{1}{2}(S_{n-1}+S_{n+1}) = \frac{1}{2}\left(\sum_{\mu=0}^{n-1} S_{n-1}^\mu + \sum_{\mu=0}^{n+1} S_{n+1}^\mu\right) \tag{36}$$

with the functions S_ν^μ defined in Eq. (18). For the degrees of this expansion which result from the proper recurrence relations, cf. the remark following Eq. (29). To specify these functions in terms of known quantities we equate the right-hand sides of Eqs. (33) and (36) and use the recurrence relations for the former. The functions are then determined by integrating over the entire surface of the Earth and utilizing fully the orthogonality relations for spherical surface harmonics.

It is obvious from Eqs. (33)–(36) that in the course of this procedure the order of the functions S_ν^μ assumes only the values $\mu = m-1$, m and $m+1$. We obtain, for example,

$$S_{n-1}^{m}=\frac{2g_{10}}{2n+1}\,\alpha\sqrt{(n-m)(n+m)}\,P_{n-1}^{m} \tag{37}$$

and similar but more extended formulae result for the remaining five functions. These functions are introduced into Eq. (29) with n fixed and $n'=1$. The sum over ν has only two terms, $\nu=n-1$ and $\nu=n+1$, and leads to the crustal potential

$$\begin{aligned}V_a(\mathbf{r})=&\frac{3(n-1)}{(2n-1)(2n+1)}\left(\frac{a}{r}\right)^n\Bigg\{g_{10}\,\alpha(\lambda)\sqrt{(n-m)(n+m)}\,P_{n-1}^{m}(\theta)\\&+\frac{\sqrt{g_{11}^2+h_{11}^2}}{2}\Bigg[\sqrt{\delta_{m-1}}\,\beta(\lambda)\sqrt{(n+m-1)(n+m)}\,P_{n-1}^{m-1}(\theta)\\&-\frac{\gamma(\lambda)}{\sqrt{\delta_m}}\sqrt{(n-m-1)(n-m)}\,P_{n-1}^{m+1}(\theta)\Bigg]\Bigg\}\\&+\frac{n}{(2n+1)(2n+3)}\left(\frac{a}{r}\right)^{n+2}\\&\cdot\Bigg\{g_{10}\,\alpha(\lambda)\sqrt{(n-m+1)(n+m+1)}\,P_{n+1}^{m}(\theta)\\&-\frac{\sqrt{g_{11}^2+h_{11}^2}}{2}\Bigg[\sqrt{\delta_{m-1}}\,\beta(\lambda)\sqrt{(n-m+1)(n-m+2)}\,P_{n+1}^{m-1}(\theta)\\&-\frac{\gamma(\lambda)}{\sqrt{\delta_m}}\sqrt{(n+m+1)(n+m+2)}\,P_{n+1}^{m+1}(\theta)\Bigg]\Bigg\}.\end{aligned} \tag{38}$$

The first and fourth terms of this equation represent the potential if only the zonal dipole part of the inducing core field is considered, and the other terms account for the sectorial dipole. With regard to Legendre's associated functions P_{n-1}^{m} and P_{n-1}^{m+1}, it should be mentioned that these functions vanish by definition if the order surmounts the degree, i.e. for $m\geqq n$ and for $m\geqq n-1$, respectively.

It has been emphasized by Meyer et al. (1983) that the spatial spectrum of the mean energy density is in many respects an appropriate measure for the global comparison of the crustal model field and the observed field. The spectrum function itself was first derived by Mauersberger (1956) and Lucke (1957). It is expressed in discrete terms of degree N in the form

$$W(N)=(N+1)\sum_{M=0}^{N}\left[(g_N^M)^2+(h_N^M)^2\right]. \tag{39}$$

Except for a factor $1/2\mu_0$ this quantity is the energy density of the particular field constituent (= field of the 2^N-pole) averaged over the whole Earth. g_N^M and h_N^M are the spherical harmonic expansion coefficients of the field under consideration, i.e. in our case of the crustal potential

$$V_a(\mathbf{r})=a\sum_{N=0}^{\infty}\sum_{M=0}^{N}\left(\frac{a}{r}\right)^{N+1}\left(g_N^M\cos M\lambda+h_N^M\sin M\lambda\right)P_N^M(\theta). \tag{40}$$

In order to express the expansion coefficients in terms of core and crustal parameters, one has to identify the general form of Eq. (40) with V_a given by Eq. (29) and to utilize again the orthogonality relations for spherical surface harmonics. If we approximate the geomagnetic core field by its dipole term only, we obtain from Eqs. (38) and (40), in a way analogous to that described after Eq. (36),

$$\begin{aligned}g_N^M=&\frac{1}{a(2N+1)}\Bigg\{\frac{N-1}{2N-1}\Bigg[g_{10}\sqrt{(N-M)(N+M)}\,p_{N-1,M}\\&+\frac{\sqrt{g_{11}^2+h_{11}^2}}{2}\Big(\sqrt{\delta_{M-1}^{-1}(N+M-1)(N+M)}\\&\cdot(p_{N-1,M-1}\cos\varepsilon-q_{N-1,M-1}\sin\varepsilon)\\&-\sqrt{\delta_M(N-M-1)(N-M)}\\&\cdot(p_{N-1,M+1}\cos\varepsilon+q_{N-1,M+1}\sin\varepsilon)\Big)\Bigg]\\&+\frac{3N}{2N+3}\Bigg[g_{10}\sqrt{(N-M+1)(N+M+1)}\,p_{N+1,M}\\&-\frac{\sqrt{g_{11}^2+h_{11}^2}}{2}\Big(\sqrt{\delta_{M-1}^{-1}(N-M+1)(N-M+2)}\\&\cdot(p_{N+1,M-1}\cos\varepsilon-q_{N+1,M-1}\sin\varepsilon)\\&-\sqrt{\delta_M(N+M+1)(N+M+2)}\\&\cdot(p_{N+1,M+1}\cos\varepsilon+q_{N+1,M+1}\sin\varepsilon)\Big)\Bigg]\Bigg\}.\end{aligned} \tag{41}$$

Again, only the first and the fourth terms remain if the core field is approximated by a zonal dipole field (i.e. $g_{11}=h_{11}=0$). A similar formula results for h_N^M; it differs from Eq. (41) in that the coefficients p and q and the boldface signs are interchanged. It should be mentioned that these coefficients are defined only for non-negative values of the subscripts and that the second subscript must not exceed the first one by value. In all other cases the coefficients have to be set equal to zero.

More complicated analytical expressions have been derived for the crustal potential and the expansion coefficients if the quadrupole and octupole terms of the geomagnetic core field are taken into account (Nolte, 1985). The formulae are not reproduced here because these cases turn out to be less important. Nevertheless, the effects of core field multipoles of higher degrees will be included in the discussion of numerical results below. The pertinent equations are readily evaluated by a digital computer and require only little computation time.

Results

We apply the analytical approach to the specific global model of crustal magnetization established by Hahn et al. (1984). This model subdivides the Earth's crust into blocks of $2°\times 2°$ extension along geographical coordinates and classifies each block on the basis of 16 crustal types. Each block is composed of two or three layers of definite thickness. Within a layer the magnetic susceptibility is assumed to be constant.

The transformation of the model data into the continuous function of the integrated susceptibility proceeds in two steps.

1) For each crustal type the defining Eq. (4) now assumes the form of a sum

$$\tilde{\chi}=\sum_{i=1}^{j}\chi_i d_i \tag{42}$$

where χ_i is the magnetic susceptibility, d_i the thickness of the i-th layer and j the number of layers in the block. Table 1 shows the 16 types that constitute the basis of the crustal

Table 1. Crustal types of the model of Hahn et al. (1984) and integrated susceptibility $\tilde{\chi}$ from Eq. (42)

Code	Crustal type	$\tilde{\chi}$ (unit: m)
Continental crust		
A	Platform	1231.7
K	Platform with 3000-m sedimentary cover	1055.7
B	Shield	1633.7
C	Black Sea	138.3
D	Basin structures (Basin and Range type)	429.9
L	Basin structures with thick sedimentary cover	477.6
E	Mobile belts-Palaeozoic Age; Moho at 30 km	681.2
F	Mobile belts-younger than Palaeozoic; Moho deeper than 30 km	907.4
G	Continental side } of passive continental margin	628.4
H	Oceanic side } of passive continental margin	251.3
J	Marginal sea	75.4
Oceanic crust		
Z	General oceanic crust	75.4
Y	Mid-ocean ridge	50.3
X	Oceanic plateau	716.4
W	Island arcs	590.7
V	Trench	25.1

model together with the integrated susceptibilities from Eq. (42). The first column in the table denotes the code that has been chosen for an easy demonstration of the distribution of crustal types in a world chart.

2) Each crustal block can thus be characterized by a discrete value of the integrated susceptibility which is attributed to the centre of the block surface, viz. to all pairs of coordinates (θ', λ') with $\theta' = 1°, 3°, \ldots, 179°$ and $\lambda' = 1°, 3°, \ldots, 359°$. Conventional spherical harmonic analysis (Chapman and Bartels, 1940) is used to determine the expansion coefficients p_{nm} and q_{nm} of Eq. (12). The computation of the energy density spectrum of the crustal model field up to degree $N = 35$ requires the knowledge of all coefficients up to degree and order $n = m = 38$ if octupole terms of the core field are included in the consideration. The coefficients are then definitive for the crustal model and can be used for repeated computations of the spectrum as well as for other purposes.

The analysis reveals that many of the coefficients are of similar magnitude. No prominent coefficient representing a particular periodicity of the integrated susceptibility can be singled out. The maximum value is found with coefficient p_{00}, as a mere consequence of the fact that $\tilde{\chi}$ can assume non-negative values only. However, this coefficient need not be considered further because it does not contribute to the induced crustal field.

In agreement with previous work the geomagnetic core field is approximated by the IGRF (Peddie, 1982), here for the epoch 1980.0. In favour of this approximation one can argue that it incorporates the effect of demagnetization in the crust which is not built into the theory explicitly. After selection of a specific core field model we have calculated the spatial spectrum of the crustal model field from Eq. (39). Figure 2 shows the resulting spectra for different elementary multipole terms of the core field and the spectrum of the previous numerical computation. Attention is drawn to three facts that show up clearly in the figure. (i) Except for the first three or four terms the spectra are almost constant with degree N or "white", i.e. the mean energy density of the crustal model field is distributed equally among the degrees, at least as far as $N = 35$. (ii) The spectrum by Hahn et al. (1984) is reproduced almost perfectly if the core field is approximated by the zonal dipole term. (iii) Neglecting small deviations, the spectra appear almost as if they were shifted parallel on the logarithmic ordinate scale if other multipole terms of the core field are considered, i.e. they differ only by nearly constant factors.

The first observation has already been discussed in detail in the literature. A white spectrum is regarded as an indication that the reference sphere for expansion into spherical harmonics (here the Earth's surface) coincides roughly with the location of the sources of the field under consideration. The other observations will be discussed in the next section.

Figure 3 shows the energy density spectrum of the crustal model field at different steps of approximating the

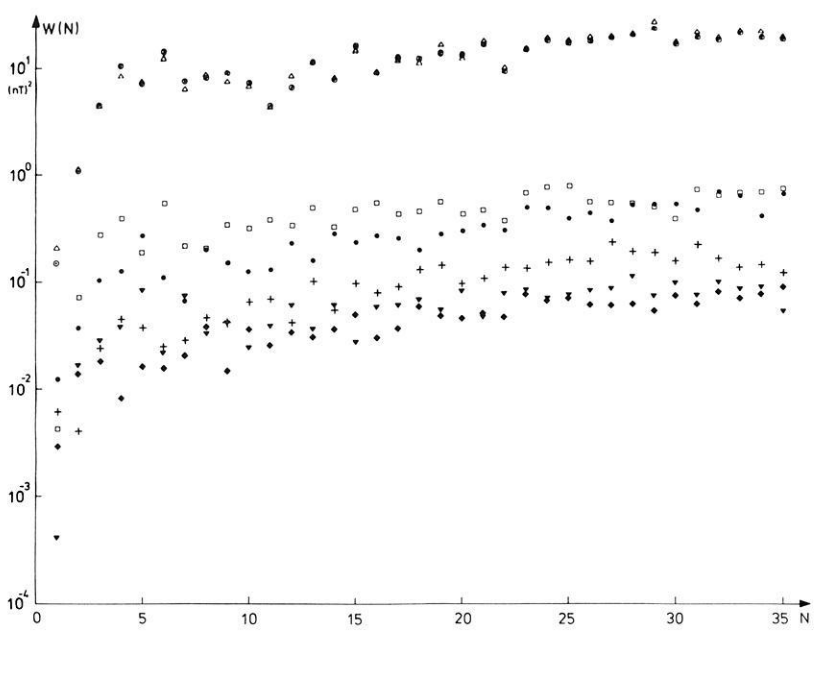


Fig. 2. Spatial energy density spectra of the crustal model field due to various core field models.
⊙ numerical computation (Hahn et al., 1984) with core field approximated by full IGRF of 1980.0;
present investigation with core field approximated by IGRF of 1980.0:
△ zonal, □ sectorial dipole term;
▼ zonal, ● tesseral, ◆ sectorial quadrupole term;
+ tesseral ($n' = 3$, $m' = 1$) octupole term

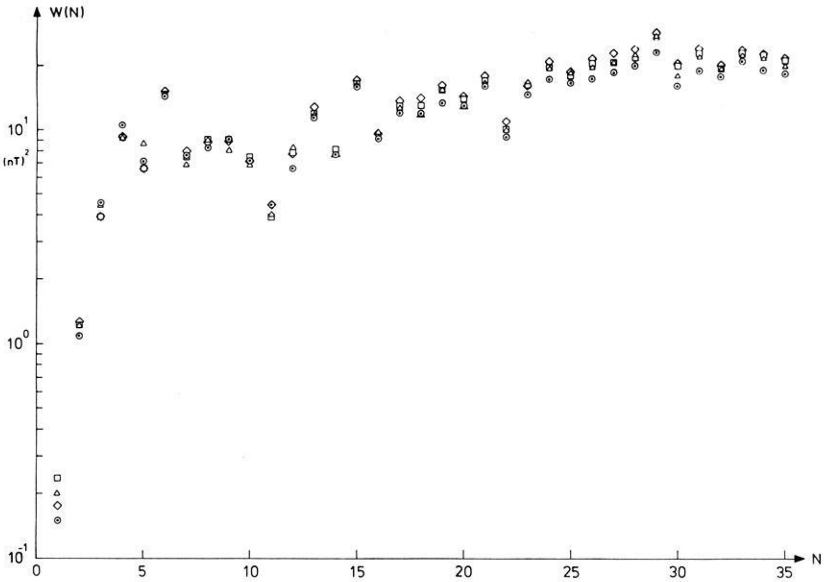


Fig. 3. Spatial energy density spectra of the crustal model field at different steps of approximation of the core field. ⊙ cf. Fig. 2;
present investigation with core field approximated by IGRF of 1980.0:
△ dipole term;
□ dipole and quadrupole terms;
◇ dipole and quadrupole and tesseral ($n'=3$, $m'=1$) octupole terms

geomagnetic core field. Note from Eq. (39) that the spectra in this figure cannot be obtained by simple addition of the spectra in Fig. 2, because the coefficients p_{nm} and q_{nm} pertaining to the individual central multipoles have to be added first. Close inspection of the higher degree terms reveals that the agreement with the spectrum of comparison slightly worsens with a more accurate approximation of the core field. We attribute the deviations to the different treatment of the global crustal model in the original and in the present work.

Discussion

The convenience of the analytical approach to the geomagnetic crustal field is based on two simplifying assumptions that can be justified on a global scale.

1) The approach considers induced magnetization only, i.e. at any crustal point the magnetization is proportional to the inducing field from the core. The crust is then characterized by its magnetic susceptibility which, being a scalar function, allows for simple notation of the relevant formulae. The restriction to consider induced magnetization only and, hence, to disregard any remanent magnetization is justified by the argument that the contributions of successive crustal layers which have been magnetized during different palaeomagnetic epochs cancel each other to a large extent. There remains only the contribution of the most recent magnetization which is in fact proportional to the inducing field. In principle the model can also take into account any remanent magnetization parallel to the induced magnetization. In this case, however, the function χ loses its direct physical meaning and turns into a mere computational quantity.

2) The second simplification concerns the neglect of the radial extension of the crust with respect to global dimensions. Admittedly we lose information about the crustal magnetization, and this approximation certainly is not allowed if we are investigating local anomalies. With the integrated susceptibility function we are able to make extensive use of spherical surface harmonics and to benefit from their orthogonality relations in the formulation of the theory. In the actual computation of the potential we are saved from the time-consuming three-dimensional integration over the crustal volume which proved to be a limiting factor in the previous work.

An essential step is the expansion of the product of spherical surface harmonics into a linear combination of the same kind of functions. Instead of solving this problem in full generality, which is difficult because of a lack of generalized recurrence relations for Legendre's associated functions, we have allowed for a full expansion of the integrated susceptibility; but the multipole expansion of the core field has been restricted to its terms of lowest degree ($n' \leqq 3$) which are the most important ones at the Earth's surface. There is no fundamental difficulty to include multipoles of higher degrees, except for the rapidly increasing length of the resulting formulae and the number of individual terms. Whereas the expression for the crustal potential splits into six different terms in the case of a central dipole, cf. Eq. (38), the corresponding formula splits into $(n'+1)(2n'+1)$ terms if a central multipole of degree n' is considered.

The construction of the function $\tilde{\chi}(\theta', \lambda')$ from discrete data by means of the spherical harmonic analysis has provided further insight into the crustal model of Hahn et al. (1984). Although large connected areas of the Earth's surface appear to be uniform when characterized on the basis of 16 crustal types, a prominent periodicity in the integrated susceptibility cannot be detected. Apparently this function exhibits a random distribution in terms of wavelengths. The individual expansion coefficient p_{nm} or q_{nm}, therefore, has no physical meaning, and only the coefficients altogether describe the magnetic properties of the crust satisfactorily. A rather large expansion into spherical surface harmonics is necessary in order to ensure an adequate description of the crustal model field.

The magnetic field of the model is controlled almost exclusively by the distribution of the magnetizable material in the Earth's crust. The angular dependence of the inducing core field is of little influence. This is concluded from the observation in Fig. 2 that the distribution of energy among the individual terms of the spectra is nearly equal, although the core field components are different. The spectra differ in absolute value, reflecting the field strengths of the core field components at the surface that enter into the function $W(N)$ with their second power. Obviously the zonal dipole term of the core field dominates the inductive effect on the crustal magnetization. This might have been supposed from inspection of the Gaussian coefficients of the IGRF. However, the crustal field results from both the crustal properties and the core field properties, and the minor role of other central multipoles, could not be known a priori; they had to be confirmed by actual computation.

From Figs. 2 and 3 we notice that the spectrum after Hahn et al. (1984) and the result of the analytical approach agree best if the inducing core field is approximated by the zonal dipole only. A more complete expansion of the core field, including octupole terms, even worsens the agreement at degrees above $N=20$. The optimum with the zonal dipole is probably due to the cancellation of small errors that arise from the different treatment of the crust in the previous papers and in the present investigation. Neglect of the finite crustal thickness can partially account for the differences in some of the degrees in Fig. 3. Moreover, simple analytical expressions result if the integration is performed over the entire spherical surface, whereas the original numerical computation of the crustal model field and of the spectrum considers only a limited number of crustal blocks within a geocentric angular distance of 70° (see Meyer et al., 1983) around each external point. This restriction had been introduced to save computation time. It is also supported by the argument that crustal regions which are hidden by the conductive core of the Earth should not be treated in the same way as the regions close to the external point.

The formalism described in this paper provides an effective method of forward analysis of the geomagnetic crustal field under global aspects. Concerning the spectrum of mean energy density, it has confirmed the previous results with high accuracy and with only a fraction of the numerical effort involved. The role of the inducing core field could be evaluated in some detail. The analytical approach can easily be applied to the computation of world charts of large-scale anomalies and to other magnetic phenomena of the crust. It may serve as an appropriate tool for a global comparison between observational data and model conceptions in future studies.

Appendix

For the proof that the first integral in Eq. (23) is zero, we define the vector field $\mathbf{T}(\theta', \lambda')$ tangential to the surface of the sphere

$$\mathbf{T}=\hat{\boldsymbol{\theta}}T_\theta+\hat{\boldsymbol{\lambda}}T_\lambda:=\tilde{\chi}_n\, a\nabla_t'\, Y_{n'}.$$

Then we have

$$\int_0^{4\pi} a\nabla_t'\cdot(P_{v'}\mathbf{T})\,d\omega'$$

$$=\int_0^{4\pi}\left(\hat{\boldsymbol{\theta}}\frac{\partial}{\partial\theta'}+\hat{\boldsymbol{\lambda}}\frac{1}{\sin\theta'}\frac{\partial}{\partial\lambda'}\right)\cdot(\hat{\boldsymbol{\theta}}P_{v'}T_\theta+\hat{\boldsymbol{\lambda}}P_{v'}T_\lambda)\,d\omega'$$

$$=\int_0^{4\pi}\left\{\frac{\partial}{\partial\theta'}(P_{v'}T_\theta)+P_{v'}T_\theta\cot\theta'+\frac{1}{\sin\theta'}\frac{\partial}{\partial\lambda'}(P_{v'}T_\lambda)\right\}d\omega'$$

$$=\int_0^{2\pi}\left\{\int_0^{\pi}\frac{\partial}{\partial\theta'}(\sin\theta' P_{v'}T_\theta)\,d\theta'\right\}d\lambda'$$

$$+\int_0^{\pi}\left\{\int_0^{2\pi}\frac{\partial}{\partial\lambda'}(P_{v'}T_\lambda)\,d\lambda'\right\}d\theta'$$

$$=0,$$

since

$$\int_0^{\pi}\frac{\partial}{\partial\theta'}(\sin\theta' P_{v'}T_\theta)\,d\theta'=\sin\theta' P_{v'}T_\theta|_0^\pi=0,$$

and

$$\int_0^{2\pi}\frac{\partial}{\partial\lambda'}(P_{v'}T_\lambda)\,d\lambda'=P_{v'}T_\lambda|_0^{2\pi}=0.$$

Both $P_{v'}$ and T_λ are 2π-periodical with respect to the variable λ'.

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