J Geophys (1987) 61:168–172

Journal of Geophysics

Simple algorithms for vertical seismic profiling in transversely isotropic media

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Abstract. A set of stable algorithms for the full wave synthesis of vertical seismic profiles in transversely isotropic media is presented. The algorithms are in a form suitable for immediate numerical implementation. The complete seismograms are computed from plane waves by numerical integrations in the wavenumber and frequency domains. Compared with algorithms for surface receivers, algorithms for vertical seismic profiles require larger amount of computer memory and take, at the most, three times the amount of computing time to compute the various quantities needed to define the displacement vector. The computing time for vertical seismic profiles depends on the number of layers spanned by the depth range of the seismic sensors. The computing time required to compose the displacement vectors from the overall reflection and transmission coefficients is also greater for subsurface receivers than that for free surface receivers. The symmetries in the reflection and transmission coefficients for seismic waves simplify the computational scheme and reduce the amount of computation required for synthesizing vertical seismic profiles.

Key words: Vertical seismic profile – Transverse isotropy – Synthetic seismograms

Introduction

The application of vertical seismic profiling techniques to the imaging of the Earth's interior has received a lot of attention in the past decade. It has been the subject of several books and many research papers [see, for example, Gal'perin (1974) and Balch (1984)]. Vertical seismic profiling (VSP) provides additional information that may not be obtainable from other types of data. VSP displays the propagation of seismic waves through the Earth in depth as well as time. This time-depth representation is quite instructive. It offers insights into the wave propagation properties of the medium, as the identification of down and upgoing waves and multiples is facilitated. For example, it is often possible to follow the path along which the seismic pulse propagates through the Earth to form multiples.

One major drawback of the application of VSP techniques is the relative expense involved in the collection of data. The computation of VSP is also relatively more expensive compared with that for the surface receivers. The theoretical development in the past decade has decreased the cost of numerical computation of synthetic seismograms. Recently, Ha (1984a, 1984b, 1986) presented a set of stable algorithms for computing synthetic seismograms in isotropic and transversely isotropic media that are simpler to implement and require up to 30% fewer algebraic operations than the reflection matrix method. The algorithms are for surface receivers only. The stability of these algorithms in the numerical computation of synthetic seismograms results from the non-existence of any growing exponential in the computational scheme. Also, the loss of precision problem associated with high frequencies that is inherent in the propagator matrix approach does not occur in these algorithms. In this paper, the algorithms of Ha are extended to enable the computation of seismograms for a range of receiver depths in transversely isotropic media. The extension is straightforward and forms a natural adjunct to the algorithms for surface receivers. The incorporation of uniaxial anisotropy into the stratification does not entail additional cost to the computation time but allows both isotropic and transversely isotropic materials to be included in the model. The solution procedure for multiple receiver depths follows that of Ha (1986) for surface receivers. The numerical examples presented there demonstrate the validity of the solution procedure on which the algorithms of the present paper are based. The algorithms for computing VSPs are presented in the next section in a form that is suitable for immediate numerical implementation.

Although our algorithms differ from those based on the matrix method, they have certain features in common with other computational schemes reported in the literature. Our computational procedure for *SH* waves is similar to that for acoustic waves presented by Temme and Müller (1982, Appendix). The greater accuracy and shorter computing time of our algorithms than those based on the propagator matrix method were also noted by Temme and Müller. The global matrix method of Schmidt and Tango (1986) also achieves improved accuracy and efficiency over the propagator matrix method for computing synthetic seismograms for stratified isotropic media. The algorithms of this paper shares other important advantages with the global matrix method. The handling of multiple sources is straightforward. The wavefields generated by multiple sources are simply superposed. It is also not necessary for our algorithms to introduce dummy interfaces at the source depths.

Numerical algorithms

The derivation of the algorithms for multiple receiver depths is similar to that presented in Ha (1986) except that the displacement vectors at different receiver depths are now required. The derivation for *SH* waves is presented first, followed by the derivation for *P* and *SV* waves. For each wave type, there are three cases to consider – the receiver layer is above, below or the same as the source layer. Although the source is normally on or near the free surface for VSP in practice, we consider all possible source-receiver configurations for completeness.

A cylindrical coordinate system (r, ϕ, z) is used with the origin at the free surface and the z-axis taken positive downward. The model considered consists of n homogeneous, transversely isotropic (with vertical axis of symmetry) or isotropic layers overlying a half-space. The mth layer of thickness d_m and density ρ_m is bounded by the horizontal planes $z=z_{m-1}$ and $z=z_m$. For each anisotropic layer, five elastic moduli c_{11}, c_{33}, c_{66}, c_{44} and c_{13} in abbreviated subscripts are required to describe the elastic behaviour fully [see, for example, Auld (1973)]. From these elastic moduli, we define the following parameters:

$$\alpha_h=\sqrt{c_{11}/\rho},$$
$$\alpha_v=\sqrt{c_{33}/\rho},$$
$$\beta_h=\sqrt{c_{66}/\rho},$$
$$\beta_v=\sqrt{c_{44}/\rho},$$
$$\gamma=\sqrt{c_{13}/\rho}.$$

These parameters are associated with the velocities of the three body waves that correspond to the *P*, *SV* and *SH* waves in isotropic media. The velocities α_h and α_v are associated with the *P* waves, while β_h and β_v are associated with the *SH* waves. The subscripts h and v denote horizontal and vertical propagation, respectively. For *SV* waves, the horizontal and vertical velocities take the same value β_v. In the following discussion, the source is assumed to be in layer s and the receiver in layer r of the stratification.

SH waves

In the frequency(ω)-wavenumber(k) domain, the azimuthal displacement at depth z of the jth layer can be written as

$$\bar{v}(z)=A_j \exp(-\xi_j z)+B_j \exp(\xi_j z), \tag{1}$$

and

$$\xi_j=\begin{cases} k_{v_j}\sqrt{p^2\beta_{h_j}^2-1} & p>1/\beta_{h_j} \\ ik_{v_j}\sqrt{1-p^2\beta_{h_j}^2} & p<1/\beta_{h_j}\end{cases}$$
$$k_{v_j}=\omega/\beta_{v_j}.$$

Here, p denotes slowness. The two terms on the right of Eq. (1) represent the down and upgoing components of the *SH* wavefield. In the source layer ($j=s$), the appropriate source term of the form $S\exp[-\xi_s|h-z|]$ for a source at depth h needs to be added to the right of Eq. (1).

Applying the solution procedure of Ha (1984a) and working upward from the bottom of the stratification, the following relation is obtained for layer j:

$$B_j \exp(\xi_j z_{j-1})=\Lambda_j^u+\Omega_j^d A_j \exp(-\xi_j z_{j-1}), \tag{2}$$

where

$$D_j^d=1+r_j^d\,\Omega_{j+1}^d \qquad 1\leq j\leq n$$
$$\Omega_j^d=[r_j^d+\Omega_{j+1}^d]\exp(-2\xi_j d_j)/D_j^d \qquad 1\leq j\leq n$$
$$\Lambda_j^u=\begin{cases}0 & \text{if } j>s\\ [r_j^d+\Omega_{j+1}^d]\,S_h^d\exp[-\xi_j(z_j-h)]\exp(-\xi_j d_j)/D_j^d & \text{if } j=s\\ t_j^u\{S_h^u\exp[-\xi_j(h-z_{j-1})]+\Lambda_{j+1}^u\}\exp(-\xi_j d_j)/D_j^d & \text{if } j=s-1\\ t_j^u\Lambda_{j+1}^u\exp(-\xi_j d_j)/D_j^d & \text{if } 1\leq j\leq s-2\end{cases}$$
$$r_j^d=(\rho_j\beta_{v_j}^2\xi_j-\rho_{j+1}\beta_{v_{j+1}}^2\xi_{j+1})/(\rho_j\beta_{v_j}^2\xi_j+\rho_{j+1}\beta_{v_{j+1}}^2\xi_{j+1})$$
$$t_j^u=2\rho_{j+1}\beta_{v_{j+1}}^2\xi_{j+1}/(\rho_j\beta_{v_j}^2\xi_j+\rho_{j+1}\beta_{v_{j+1}}^2\xi_{j+1}).$$

Here, r_j^d and t_j^u denote the jth interfacial reflection and transmission coefficient, respectively. The superscripts u and d indicate that the wave approaches the jth interface from below and above the interface, respectively. S_h^d and S_h^u represent the amplitudes of *SH* waves leaving the source downward and upward, respectively. We make this distinction to allow for possible asymmetry in the source radiation. These source amplitudes for a point horizontal force and for a point shear dislocation are defined in Ha (1986). From Eq. (2), it is clear that the upgoing wave of layer j arises from the upward transmission and downward reflection from the stratification below layer j. When the source layer is above layer j, the first term on the right of Eq. (2) is absent. The upgoing wave of layer j comes from the downgoing wave that has reflected back up from interface j and below. When the source layer is below the jth layer, the source contribution to the upgoing wave of layer j is given by the term Λ_j^u.

Similarly, we obtain the following relation expressing the downgoing wave component of layer j in terms of the upgoing wave component from the transmission and reflection properties of the stratification above layer j:

$$A_j \exp(-\xi_j z_j)=\Lambda_j^d+\Omega_j^u B_j \exp(\xi_j z_j) \tag{3}$$

where

$$D_j^u=1+r_{j-1}^u\,\Omega_{j-1}^u \qquad 1\leq j\leq n$$
$$\Omega_j^u=[r_{j-1}^u+\Omega_{j-1}^u]\exp(-2\xi_j d_j)/D_j^u \qquad 1\leq j\leq n$$
$$\Lambda_j^d=\begin{cases}0 & \text{if } j<s\\ [r_{j-1}^u+\Omega_{j-1}^u]\,S_h^u\exp[-\xi_j(h-z_{j-1})]\exp(-\xi_j d_j)/D_j^u & \text{if } j=s\\ t_{j-1}^d\{S_h^d\exp[-\xi_{j-1}(z_{j-1}-h)]+\Lambda_{j-1}^d\}\exp(-\xi_j d_j)/D_j^u & \text{if } j=s+1\\ t_{j-1}^d\Lambda_{j-1}^d\exp(-\xi_j d_j)/D_j^u & \text{if } j\geq s+2\end{cases}$$
$$r_j^u=-r_j^d$$
$$t_j^d=2\rho_j\beta_{v_j}^2\xi_j/(\rho_j\beta_{v_j}^2\xi_j+\rho_{j+1}\beta_{v_{j+1}}^2\xi_{j+1})$$

Here, r_j^u and t_j^d denote the reflection and transmission coefficients for waves that approach the jth interface from below and above the interface, respectively. The term Λ_j^d is absent from Eq. (3) if the source layer is below layer j. The downgoing wave of layer j comes from the upgoing wave that has reflected back down from interface $(j-1)$ and above. When the source layer is above the jth layer, Λ_j^d gives the source contribution to the downgoing wave of layer j.

The similarity in structure between Eqs. (2) and (3) is not surprising as the formal symmetry between upward and downward propagation is well appreciated [see, for example, Kennett et al. (1978) and Thomson et al. (1986)]. The reciprocals of D_j^d and D_j^u represent infinite series of multiply reflected and transmitted rays that interact with the stratification below and above layer j. Ω_j^d and Ω_j^u are the overall downward and upward reflection coefficients below and above layer j, respectively. For multiple sources, it is clear from Eqs. (2) and (3) that additional source terms of the form Λ_s^u (or Λ_s^d) for sources present in layer j are simply added to Λ_j^u (or Λ_j^d).

In general, we have the following two relations at any receiver depth z from Eqs. (2) and (3):

$$B_r \exp(\xi_r z) = \tilde{\Lambda}_r^u + \tilde{\Omega}_r^d A_r \exp(-\xi_r z), \tag{4a}$$

$$A_r \exp(-\xi_r z) = \tilde{\Lambda}_r^d + \tilde{\Omega}_r^u B_r \exp(\xi_r z). \tag{4b}$$

The subscript r refers to the receiver layer. $\tilde{\Lambda}_r^u$ and $\tilde{\Omega}_r^d$ are obtained from Λ_r^u and Ω_r^d, respectively, by replacing the layer thickness d_r in the exponential term by $(z_r - z)$. $\tilde{\Lambda}_r^d$ and $\tilde{\Omega}_r^u$ are obtained from Λ_r^d and Ω_r^u, respectively, by replacing the layer thickness d_r in the exponential term by $(z - z_{r-1})$. When the source and receiver are in different layers of the stratification, $\tilde{\Lambda}_r^u$ is zero if $r > s$, but $\tilde{\Lambda}_r^d$ is zero if $r < s$. By solving Eqs. (4a) and (4b) simultaneously, we obtain the following result for the azimuthal displacement at the receiver depth z:

$$\bar{v}(z) = \begin{cases} \tilde{\Lambda}_r^u (1 + \tilde{\Omega}_r^u)/\Box_h & \text{if } r < s \\ S \exp[-\xi_r |h - z|] + [\tilde{\Lambda}_r^u (1 + \tilde{\Omega}_r^u) + \tilde{\Lambda}_r^d (1 + \tilde{\Omega}_r^d)]/\Box_h & \text{if } r = s \\ \tilde{\Lambda}_r^d (1 + \tilde{\Omega}_r^d)/\Box_h & \text{if } r > s \end{cases} \tag{5}$$

where

$$\Box_h = 1 - \tilde{\Omega}_r^d \tilde{\Omega}_r^u$$

$$\text{and } S = \begin{cases} S_h^u & \text{if } (h - z) > 0 \\ S_h^d & \text{if } (h - z) < 0. \end{cases}$$

Equation (5) constitutes the VSP algorithm for SH waves. The recurrence relations for the various quantities involved are defined in Eqs. (2) and (3). The relations are in a form suitable for immediate numerical implementation. In Eq. (5), the terms $\tilde{\Lambda}_r^u$ and $\tilde{\Lambda}_r^d \tilde{\Omega}_r^d$ represent the upgoing wave components while $\tilde{\Lambda}_r^d$ and $\tilde{\Lambda}_r^u \tilde{\Omega}_r^u$ represent the downgoing wave components of the SH wavefield at depth z. The multiple interaction of the source pulse with the stratified medium is given by the reciprocal of $\Box_h$.

P and SV waves

In the transformed domain, the horizontal ($\bar{u}$) and vertical ($\bar{w}$) components of displacement for P and SV waves at depth z can be written as follows. For P waves,

$$\bar{u}(z) = \varepsilon_p [A \exp(-\eta z) + B \exp(\eta z)]$$
$$\bar{w}(z) = i \varepsilon_p q_p [A \exp(-\eta z) - B \exp(\eta z)] \tag{6}$$

where $\varepsilon_p = 1/\sqrt{2 \rho q_p}$

$$q_p = -[\beta_v^2 \sigma^2 + 1 - p^2 \alpha_h^2]/[p \sigma (\gamma^2 + \beta_v^2)]$$

$$\sigma = \eta/\omega = \{[-a_1 + (a_1^2 - 4 a_2 a_0)^{1/2}]/2 a_2\}^{1/2} \qquad \mathrm{Re}(\sigma) \geqq 0$$

$$a_0 = (1 - p^2 \alpha_h^2)(1 - p^2 \beta_v^2)$$

$$a_1 = \alpha_v^2 + \beta_v^2 + p^2 [(\gamma^2 + \beta_v^2)^2 - \beta_v^4 - \alpha_v^2 \alpha_h^2]$$

$$a_2 = \alpha_v^2 \beta_v^2.$$

For SV waves,

$$\bar{u}(z) = \varepsilon_s [C \exp(-\xi z) - D \exp(\xi z)]$$
$$\bar{w}(z) = i \varepsilon_s q_s [C \exp(-\xi z) + D \exp(\xi z)] \tag{7}$$

where $\varepsilon_s = 1/\sqrt{2 \rho q_s}$

$$v = \xi/\omega = \{[-a_1^2 - 4 a_2 a_0)^{1/2}]/2 a_2\}^{1/2} \qquad \mathrm{Re}(v) \geqq 0$$

and q_s is similar to q_p except that σ is replaced by v. To refer the above parameters to layer m of the stratification, we shall attach a subscript m to each of them.

The derivation of the algorithm for P and SV waves is analogous to that for SH waves except that there is now the need to incorporate the coupling of P and SV waves when they interact with the interface between two different materials. For any depth z within the receiver layer, we have the following results in terms of the reflection and transmission properties of the stratification below layer r:

$$B_r \exp(\eta_r z) = \Omega_{pp_r}^d A_r \exp(-\eta_r z) + \Omega_{sp_r}^d C_r \exp(-\xi_r z) + \Lambda_{pp_r}^u W_s^u + \Lambda_{sp_r}^u X_s^u \tag{8}$$

$$D_r \exp(\xi_r z) = \Omega_{ps_r}^d A_r \exp(-\eta_r z) + \Omega_{ss_r}^d C_r \exp(-\xi_r z) + \Lambda_{ps_r}^u W_s^u + \Lambda_{ss_r}^u X_s^u$$

where

$$\Omega_{pp_r}^d = [r_{pp_r}^d + \Omega_{pp_{r+1}}^d R_r + \Omega_{ss_{r+1}}^d S_r + \Omega_{sp_{r+1}}^d T_r + J_{r+1} r_{ss_r}^d] \cdot E_{p_r}^u E_{p_r}^u / \Delta_r^d$$

$$\Omega_{sp_r}^d = [r_{sp_r}^d + (\Omega_{pp_{r+1}}^d + \Omega_{ss_{r+1}}^d) R_r^* + \Omega_{sp_{r+1}}^d T_r^* + J_{r+1} r_{sp_r}^d] \cdot E_{s_r}^u E_{p_r}^u / \Delta_r^d$$

$$\Omega_{ps_r}^d = \Omega_{sp_r}^d$$

$$\Omega_{ss_r}^d = [r_{ss_r}^d + \Omega_{pp_{r+1}}^d S_r + \Omega_{ss_{r+1}}^d R_r + \Omega_{sp_{r+1}}^d T_r + J_{r+1} r_{pp_r}^d] \cdot E_{s_r}^u E_{s_r}^u / \Delta_r^d$$

$$E_{p_m}^u = \begin{cases} \exp[-\eta_m (z_m - z)] & m = r \\ \exp(-\eta_m d_m) & m \neq r \end{cases}$$

$$E_{s_m}^u = \begin{cases} \exp[-\xi_m (z_m - z)] & m = r \\ \exp(-\xi_m d_m) & m \neq r \end{cases}$$

$$\Delta_r^d = 1 - r_{pp_r}^u \Omega_{pp_{r+1}}^d - r_{ss_r}^u \Omega_{ss_{r+1}}^d - r_{sp_r}^u \Omega_{ps_{r+1}}^d - r_{ps_r}^u \Omega_{sp_{r+1}}^d + K_r^u J_{r+1}^d$$

$$K_r^u = r_{pp_r}^u r_{ss_r}^u - r_{sp_r}^u r_{ps_r}^u$$

$$J_{r+1}^d = \Omega_{pp_{r+1}}^d \Omega_{ss_{r+1}}^d - \Omega_{sp_{r+1}}^d \Omega_{ps_{r+1}}^d.$$

The physical interpretation of Eq. (8) is identical to that of Eq. (2). The extra terms in Eq. (8) incorporate additional source type and the coupling of P and SV waves. The terms W_s^u and X_s^u represent, respectively, the P and SV source contribution to the upgoing components of the P and SV waves. These source terms are absent if the source layer (layer s) is above the receiver layer (layer r). The coefficients of W_s^u and X_s^u are defined in the following three sets of equations. For $r = s$,

$$\Lambda_{pp_s}^u = \Omega_{pp_s}^d \exp[\eta_s (z_s - z)]$$
$$\Lambda_{sp_s}^u = \Omega_{sp_s}^d \exp[\xi_s (z_s - z)]$$
$$\Lambda_{ps_s}^u = \Omega_{ps_s}^d \exp[\eta_s (z_s - z)]$$
$$\Lambda_{ss_s}^u = \Omega_{ss_s}^d \exp[\xi_s (z_s - z)]. \tag{9}$$

For $r=s-1$,

$$\begin{aligned}
\Lambda^u_{pp_r}&=[t^u_{pp_r}-\Omega^d_{sp_{r+1}}L_r+\Omega^d_{ss_{r+1}}M_r]\,E^u_{p_r}/\Delta^d_r\\
\Lambda^u_{sp_r}&=[t^u_{sp_r}+\Omega^d_{pp_{r+1}}L_r-\Omega^d_{ps_{r+1}}M_r]\,E^u_{p_r}/\Delta^d_r\\
\Lambda^u_{ps_r}&=[t^u_{ps_r}-\Omega^d_{sp_{r+1}}N_r+\Omega^d_{ss_{r+1}}O_r]\,E^u_{s_r}/\Delta^d_r\\
\Lambda^u_{ss_r}&=[t^u_{ss_r}+\Omega^d_{pp_{r+1}}N_r-\Omega^d_{ps_{r+1}}O_r]\,E^u_{s_r}/\Delta^d_r.
\end{aligned}\tag{10}$$

For $1\leqq r\leqq s-2$,

$$\begin{aligned}
\Lambda^u_{pp_r}=[&\Lambda^u_{pp_{r+1}}(t^u_{pp_r}-\Omega^d_{sp_{r+1}}L_r+\Omega^d_{ss_{r+1}}M_r)\\
&+\Lambda^u_{ps_{r+1}}(t^u_{sp_r}+\Omega^d_{pp_{r+1}}L_r-\Omega^d_{ps_{r+1}}M_r)]\,E^u_{p_r}/\Delta^d_r\\
\Lambda^u_{ps_r}=[&\Lambda^u_{pp_{r+1}}(t^u_{ps_r}-\Omega^d_{sp_{r+1}}N_r+\Omega^d_{ss_{r+1}}O_r)\\
&+\Lambda^u_{ps_{r+1}}(t^u_{ss_r}+\Omega^d_{pp_{r+1}}N_r-\Omega^d_{ps_{r+1}}O_r)]\,E^u_{s_r}/\Delta^d_r.
\end{aligned}\tag{11}$$

When $1\leqq r\leqq s-2$, the recurrence relations for Λ^u_{sp} and Λ^u_{ss} are similar to those for Λ^u_{pp} and Λ^u_{ps}, respectively, except that Λ^u_{pp} is replaced by Λ^u_{sp} and Λ^u_{ps} by Λ^u_{ss}. Here, r_{xy_r} and t_{xy_r} denote the rth interfacial reflection and transmission coefficient, respectively. The subscript xy denotes x to y conversion. The superscripts d and u indicate that the wave approaches the rth interface from above and below the interface, respectively. These coefficients as well as the terms R, S, T, R^*, T^*, L, M, N and O are defined in Ha (1986, Appendix).

The source terms W^u_s and X^u_s of Eq. (8) are defined by the following two equations:

$$W^u_s=\begin{cases}S^d_p\exp[-\eta_s(z_s-h)] & \text{if } r=s\\ S^u_p\exp[-\eta_s(h-z_{s-1})]+\Lambda^u_{pp_s}S^d_p\exp[-\eta_s(z_s-h)] & \\ \quad+\Lambda^u_{sp_s}S^d_s\exp[-\xi_s(z_s-h)] & \text{if } r<s\end{cases}$$

$$X^u_s=\begin{cases}S^d_s\exp[-\xi_s(z_s-h)] & \text{if } r=s\\ S^u_s\exp[-\xi_s(h-z_{s-1})]+\Lambda^u_{ps_s}S^d_p\exp[-\eta_s(z_s-h)] & \\ \quad+\Lambda^u_{ss_s}S^d_s\exp[-\xi_s(z_s-h)] & \text{if } r<s\end{cases}$$

Here, S^u_p and S^u_s denote the scaled amplitudes of the P and SV waves that leave the source upward, and S^d_p and S^d_s denote those that leave the source downward. These scaled source amplitudes are defined in Ha (1986, Appendix).

The downgoing components of the wavefield of layer r can be expressed in terms of the upgoing components using the reflection and transmission properties of the stratification above the rth layer:

$$\begin{aligned}
A_r\exp(-\eta_r z)&=\Omega^u_{pp_r}B_r\exp(\eta_r z)+\Omega^u_{sp_r}D_r\exp(\xi_r z)\\
&\quad+\Lambda^d_{pp_r}W^d_s+\Lambda^d_{sp_r}X^d_s\\
C_r\exp(-\xi_r z)&=\Omega^u_{ps_r}B_r\exp(\eta_r z)+\Omega^u_{ss_r}D_r\exp(\xi_r z)\\
&\quad+\Lambda^d_{ps_r}W^d_s+\Lambda^d_{ss_r}X^d_s
\end{aligned}\tag{12}$$

where

$$\begin{aligned}
\Omega^u_{pp_{r+1}}&=[r^u_{pp_r}+\Omega^u_{pp_r}R_r+\Omega^u_{ss_r}S_r+2\Omega^u_{sp_r}R^*_r+J^u_r r^u_{ss_r}]\\
&\quad\cdot E^d_{p_{r+1}}E^d_{p_{r+1}}/\Delta^u_{r+1}\\
\Omega^u_{sp_{r+1}}&=[r^u_{sp_r}+\tfrac{1}{2}(\Omega^u_{pp_r}+\Omega^u_{ss_r})T_r+\Omega^u_{sp_r}T^*_r+J^u_r r^u_{sp_r}]\\
&\quad\cdot E^d_{s_{r+1}}E^d_{p_{r+1}}/\Delta^u_{r+1}\\
\Omega^u_{ps_r}&=\Omega^u_{sp_r}\\
\Omega^u_{ss_{r+1}}&=[r^u_{ss_r}+\Omega^u_{pp_r}S_r+\Omega^u_{ss_r}R_r+2\Omega^u_{sp_r}R^*_r+J^u_r r^d_{pp_r}]\\
&\quad\cdot E^d_{s_{r+1}}E^d_{s_{r+1}}/\Delta^u_{r+1}
\end{aligned}$$

$$E^d_{p_m}=\begin{cases}\exp[-\eta_m(z-z_m)] & m=r\\ \exp(-\eta_m d_m) & m\neq r\end{cases}$$

$$E^d_{s_m}=\begin{cases}\exp[-\xi_m(z-z_m)] & m=r\\ \exp(-\xi_m d_m) & m\neq r\end{cases}$$

$$\begin{aligned}
\Delta^u_{r+1}&=1-r^d_{pp_r}\Omega^u_{pp_r}-r^d_{ss_r}\Omega^u_{ss_r}-r^d_{sp_r}\Omega^u_{ps_r}\\
&\quad-r^d_{ps_r}\Omega^u_{sp_r}+K^d_r J^u_r\\
K^d_r&=r^d_{pp_r}r^d_{ss_r}-r^d_{sp_r}r^d_{ps_r}\\
J^u_r&=\Omega^u_{pp_r}\Omega^u_{ss_r}-\Omega^u_{sp_r}\Omega^u_{ps_r}.
\end{aligned}$$

The coefficients of W^d_s and X^d_s of Eq. (12) take on similar definitions as those of W^u_s and X^u_s of Eq. (8). They are defined by the following three sets of equations. For $r=s$,

$$\begin{aligned}
\Lambda^d_{pp_s}&=\Omega^u_{pp_s}\exp[\eta_s(z-z_{s-1})]\\
\Lambda^d_{sp_s}&=\Omega^u_{sp_s}\exp[\xi_s(z-z_{s-1})]\\
\Lambda^d_{ps_s}&=\Omega^u_{ps_s}\exp[\eta_s(z-z_{s-1})]\\
\Lambda^d_{ss_s}&=\Omega^u_{ss_s}\exp[\xi_s(z-z_{s-1})].
\end{aligned}\tag{13}$$

For $r=s+1$,

$$\begin{aligned}
\Lambda^d_{pp_r}&=[t^d_{pp_{r-1}}-\Omega^u_{sp_{r-1}}O_{r-1}-\Omega^u_{ss_{r-1}}M_{r-1}]\,E^d_{p_r}/\Delta^u_r\\
\Lambda^d_{sp_r}&=[t^d_{sp_{r-1}}+\Omega^u_{pp_{r-1}}O_{r-1}+\Omega^u_{ps_{r-1}}M_{r-1}]\,E^d_{p_r}/\Delta^u_r\\
\Lambda^d_{ps_r}&=[t^d_{ps_{r-1}}+\Omega^u_{sp_{r-1}}N_{r-1}+\Omega^u_{ss_{r-1}}L_{r-1}]\,E^d_{s_r}/\Delta^u_r\\
\Lambda^d_{ss_r}&=[t^d_{ss_{r-1}}-\Omega^u_{pp_{r-1}}N_{r-1}-\Omega^u_{ps_{r-1}}L_{r-1}]\,E^d_{s_r}/\Delta^u_r.
\end{aligned}\tag{14}$$

For $s+2\leqq r\leqq n$,

$$\begin{aligned}
\Lambda^d_{pp_r}=[&\Lambda^d_{pp_{r-1}}(t^d_{pp_{r-1}}-\Omega^u_{sp_{r-1}}O_{r-1}-\Omega^u_{ss_{r-1}}M_{r-1})\\
&+\Lambda^d_{ps_{r-1}}(t^d_{sp_{r-1}}+\Omega^u_{pp_{r-1}}O_{r-1}+\Omega^u_{ps_{r-1}}M_{r-1})]\\
&\cdot E^d_{p_r}/\Delta^u_r\\
\Lambda^d_{ps_r}=[&\Lambda^d_{pp_{r-1}}(t^d_{pp_{r-1}}+\Omega^u_{sp_{r-1}}N_{r-1}+\Omega^u_{ss_{r-1}}L_{r-1})\\
&+\Lambda^d_{ps_{r-1}}(t^d_{ss_{r-1}}-\Omega^u_{pp_{r-1}}N_{r-1}-\Omega^u_{ps_{r-1}}L_{r-1})]\\
&\cdot E^d_{s_r}/\Delta^u_r.
\end{aligned}\tag{15}$$

When $s+2\leqq r\leqq n$, the recurrence relations for Λ^d_{sp} and Λ^d_{ss} are identical to those for Λ^d_{pp} and Λ^d_{ps}, respectively, but for the replacement of Λ^d_{pp} by Λ^d_{sp} and Λ^d_{ps} by Λ^d_{ss}. The source terms W^d_s and X^d_s of Eq. (12) are as follows:

$$W^d_s=\begin{cases}S^u_p\exp[-\eta_s(h-z_{s-1})] & \text{if } r=s\\ S^d_p\exp[-\eta_s(z_s-h)]+\Lambda^d_{pp_s}S^u_p\exp[-\eta_s(h-z_{s-1})] & \\ \quad+\Lambda^d_{sp_s}S^u_s\exp[-\xi_s(h-z_{s-1})] & \text{if } r>s\end{cases}$$

$$X^d_s=\begin{cases}S^u_s\exp[-\xi_s(h-z_{s-1})] & r=s\\ S^d_s\exp[-\xi_s(z_s-h)]+\Lambda^d_{ps_s}S^u_p\exp[-\eta_s(h-z_{s-1})] & \\ \quad+\Lambda^d_{ss_s}S^u_s\exp[-\xi_s(h-z_{s-1})] & \text{if } r>s.\end{cases}$$

Solving Eqs. (8) and (12) simultaneously, we obtain the following solutions for $\bar{u}$ and $\bar{w}$ from Eqs. (6) and (7):

$$\bar{u}=\begin{cases}\bar{u}^-/\square & \text{if } r<s\\ \varepsilon_{p_r}W\exp[-\eta_r|h-z|]+\varepsilon_{s_r}X\exp[-\xi_r|h-z|] & \\ \quad+(\bar{u}^++\bar{u}^-)/\square & \text{if } r=s\\ \bar{u}^+/\square & \text{if } r>s\end{cases}$$

$$\bar{w}=\begin{cases}\bar{w}^-/\square & \text{if } r<s\\ i\varepsilon_{p_r}q_{p_r}W\exp[-\eta_r|h-z|]+i\varepsilon_{s_r}q_{s_r}X\\ \quad\cdot\exp[-\xi_r|h-z|]+(\bar{w}^++\bar{w}^-)/\square & \text{if } r=s\\ \bar{w}^+/\square & \text{if } r>s\end{cases} \quad (16)$$

where

$$\bar{u}^-=\varepsilon_{p_r}(A^-+B^-)+\varepsilon_{s_r}(C^--D^-)$$
$$\bar{w}^-=i[\varepsilon_{p_r}q_{p_r}(A^--B^-)+\varepsilon_{s_r}q_{s_r}(C^-+D^-)]$$
$$A^-=aW^-+cX^-$$
$$B^-=aW_T^u+bX_T^u$$
$$C^-=bW^-+dX^-$$
$$D^-=cW_T^u+dX_T^u$$
$$W^-=\Omega^u_{pp_r}W_T^u+\Omega^u_{sp_r}X_T^u$$
$$X^-=\Omega^u_{ps_r}W_T^u+\Omega^u_{ss_r}X_T^u$$
$$\bar{u}^+=\varepsilon_{p_r}(A^++B^+)+\varepsilon_{s_r}(C^+-D^+)$$
$$\bar{w}^+=i[\varepsilon_{p_r}q_{p_r}(A^+-B^+)+\varepsilon_{s_r}q_{s_r}(C^++D^+)]$$
$$A^+=aW_T^d+cX_T^d$$
$$B^+=aW^++bX^+$$
$$C^+=bW_T^d+dX_T^d$$
$$D^+=cW^++dX^+$$
$$W^+=\Omega^d_{pp_r}W_T^d+\Omega^d_{sp_r}X_T^d$$
$$X^+=\Omega^d_{ps_r}W_T^d+\Omega^d_{ss_r}X_T^d$$
$$a=1-\Omega^u_{sp_r}\Omega^d_{sp_r}-\Omega^u_{ss_r}\Omega^d_{ss_r}$$
$$b=\Omega^u_{pp_r}\Omega^d_{sp_r}+\Omega^u_{sp_r}\Omega^d_{ss_r}$$
$$c=\Omega^d_{pp_r}\Omega^u_{sp_r}+\Omega^u_{ss_r}\Omega^d_{sp_r}$$
$$d=1-\Omega^d_{pp_r}\Omega^u_{pp_r}-\Omega^d_{sp_r}\Omega^u_{sp_r}$$
$$\square=ad-bc$$
$$W_T^d=\Lambda^d_{pp_r}W_s^d+\Lambda^d_{sp_r}X_s^d$$
$$X_T^d=\Lambda^d_{ps_r}W_s^d+\Lambda^d_{ss_r}X_s^d$$
$$W_T^u=\Lambda^u_{pp_r}W_s^u+\Lambda^u_{sp_r}X_s^u$$
$$X_T^u=\Lambda^u_{ps_r}W_s^u+\Lambda^u_{ss_r}X_s^u$$
$$W=\begin{cases}S_p^u & \text{if } (h-z)>0\\ S_p^d & \text{if } (h-z)<0\end{cases}$$

$$X=\begin{cases}S_s^u & \text{if } (h-z)>0\\ S_s^d & \text{if } (h-z)<0.\end{cases}$$

Equations (16) constitutes the algorithm for computing VSPs of P and SV waves for transversely isotropic media. In the expressions for $\bar{u}^+$ and $\bar{w}^+$, the terms B^+ and D^+ represent the upgoing waves and the terms A^+ and C^+ represent the downgoing waves at depth z. For $\bar{u}^-$ and $\bar{w}^-$, the terms B^- and D^- represent the upgoing waves and the terms A^- and C^- represent the downgoing waves. The reciprocal of $\square$ represents the reverberation of the stratified medium.

Discussion

In the previous section, a set of unconditionally stable algorithms for computing the VSPs of P, SV and SH waves are presented. Inverse Hankel and Fourier transforms are applied to transform the displacement vectors back to the time-distance domain. The algorithms of the last section are in a form suitable for immediate numerical implementation. The source terms and the various quantities that are required to define the recurrence relations needed by the algorithms are presented in Ha (1986, Appendix).

The recurrence relations for Ω^u and Ω^d are similar in structure, as are the recurrence relations for Λ^u and Λ^d. Their computation times through the stratification are thus similar. The computation time for VSP will depend on the number of layers and receivers in the depth range spanned by the seismic sensors. Suppose the number of layers in the range of receiver depths is n and that the source is on the free surface, the values of Ω^u, Ω^d and Λ^d are required for each layer of the stratification. In the case of surface source and surface receivers, only the value of Ω^d at $z=0$ is required. Although Λ^d requires less computation time than Ω^u (Ha, 1984b), Λ^d and Ω^u are assumed to require similar computation time for the present discussion. Hence, the computation time for all the quantities required to compose the displacement vectors for VSPs is roughly three times that for surface receivers at most. It is also relatively more expensive to compose the displacement vector at a buried receiver from the overall reflection and transmission coefficients than its counterpart at a free surface receiver. Compared with computer programs for surface receivers, computer programs for VSPs demand larger amounts of computer memory because of the overhead involved to store the various quantities required by Eqs. (5) and (16) for all the layers within the range of the receiver depths. In a separate paper, these times and memory requirements in the numerical implementation of our algorithms will be discussed in more detail together with some numerical examples. The symmetry between upward and downward propagation reduces the number of parameters required to define the recurrence relations. The symmetries of the reflection and transmission coefficients reduce the number of recurrence relations required to define the displacement vector. These reductions translate to shorter computation time for synthesizing VSPs.

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Received January 6, 1987; revised version May 14, 1987
Accepted May 15, 1987