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The canonical decomposition and its relationship to other forms of magnetotelluric impedance tensor analysis

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Abstract. A technique for magnetotelluric (MT) data analysis, known as the canonical decomposition, is developed from first principles. This analysis is based on the canonical decomposition of the impedance tensor **Z** and explicitly parametrizes **Z** in terms of eight physically relevant structural parameters which specify the transfer characteristics of the earth system (i.e. the maximum and minimum principal apparent resistivities and the associated principal phases) as well as the principal or intrinsic coordinate system for **Z** (i.e. the two principal orthogonal electric and magnetic field polarization states). It is shown that the formulation of canonical decomposition in which the polarization descriptors are specified in terms of elliptic parameters results in the MT impedance tensor analysis presented by LaTorraca et al. The relationships between canonical decomposition and several other forms of magnetotelluric data analysis are explored. Specifically, we compare the canonical decomposition with the "conventional" analysis, the maximum coherency analysis, the associate and conjugate directions analysis developed by Counil et al., Eggers' eigenstate analysis and Spitz's rotation analysis. It is shown that canonical decomposition is a natural generalization of the conventional analysis in that both the rotation and ellipticity properties of **Z** are utilized in the definition of a principal coordinate system. A generalization of the maximum coherency analysis is shown to yield the same parameters as those extracted in canonical decomposition. By imposing a specific restriction on the generalized maximum coherency analysis, we next show how to extract the parameters (i.e. the directions of maximum and minimum current and induction and the corresponding electric and magnetic sheet impedances) that were obtained by Counil et al. in their associate and conjugate directions analysis. The relationship between canonical decomposition and Eggers' eigenstate analysis is developed and it is shown that the primary deficiency in the eigenstate formulation resides in the incorporation of an artificial field constraint. Spitz's rotation analysis extracts two analytical rotation angles from the matrix factors in the Cayley factorization of **Z**. It is shown that the Cayley factorization of **Z** is nothing more than the repackaging of the information in canonical decomposition and, as a consequence, Spitz's rotation analysis is not required to extract a principal or intrinsic coordinate system of **Z**.

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Introduction

The magnetotelluric (MT) sounding method involves the determination of the relationship between the natural horizontal electric and magnetic field fluctuations at various physical points on the earth's surface. The primary entity of interest in the MT method is the impedance tensor $Z(\omega)$, the operator that transforms the horizontal magnetic field fluctuations into the horizontal electric or telluric field variations. For a uniform monochromatic, plane-wave source excitation, the horizontal components of the complex MT wave field are related as

$$|E_{x',y'}(\omega)\rangle = \mathbf{Z}(\omega)|H_{x,y}(\omega)\rangle, \tag{1a}$$

where

$$|E_{x',y'}(\omega)\rangle = \begin{pmatrix} E_{x'}(\omega) \\ E_{y'}(\omega) \end{pmatrix} \tag{1b}$$

and

$$|H_{x,y}(\omega)\rangle = \begin{pmatrix} H_x(\omega) \\ H_y(\omega) \end{pmatrix} \tag{1c}$$

are the tangential electric and magnetic field components measured relative to the two pairs of Cartesian axes (x', y') and (x, y), respectively. We follow the usual Dirac notation of representing a vector in a Hilbert space by a ket $|a\rangle$ and its dual by the associated bra $\langle a|$. The inner product between two vectors $|a\rangle$ and $|b\rangle$ is

$$\langle a|b\rangle = \mathbf{a}^\dagger \mathbf{b} = \sum_n a_n^* b_n,$$

where $\dagger$ denotes Hermitian conjugation, $*$ complex conjugation, and the sum extends over the components of **a** and **b**. Note that **a** and **b** are the column matrix representations of $|a\rangle$ and $|b\rangle$ in some selected orthonormal basis in the Hilbert space. From this perspective, the ket vectors

$|E_{x',y'}(\omega)\rangle$ and $|H_{x,y}(\omega)\rangle$ are to be interpreted as two-dimensional complex vectors whose components constitute the phasor representation for the corresponding components of the time-harmonic fields at some fixed spatial position. Furthermore, for the chosen measurement directions for the magnetic and electric field components, the impedance tensor possesses the following matrix representation:

$$\mathbf{Z}(\omega)=\begin{pmatrix} Z_{x'x}(\omega) & Z_{x'y}(\omega) \\ Z_{y'x}(\omega) & Z_{y'y}(\omega) \end{pmatrix}. \tag{2}$$

The subscripts on the various elements of the impedance tensor emphasize the fact that the input magnetic field is defined in the coordinate system (x, y), whereas the output electric field is defined in the coordinate system (x', y'). Hence, the (i', j)th element of $\mathbf{Z}(\omega)$, which can be expressed as

$$Z_{i'j}(\omega)=\left.\frac{E_{i'}(\omega)}{H_j(\omega)}\right|_{H_{i\neq j}(\omega)=0}, \quad (i'=x', y'; j=x, y),$$

can be interpreted as the $j \rightarrow i'$ input-magnetic to output-electric field transformation with the magnetic field linearly polarized only along the j-axis direction, viz. $H_{i\neq j}(\omega)=0$. In practically all MT measurements, the electric and magnetic fields are measured in the same coordinate system, viz. $(x, y)\equiv(x', y')$. For the plane-wave excitation source, the elements of the impedance tensor depend only on the frequency of the excitation source, the choice of the coordinate system for measurement of the magnetic and electric fields, the observation site at the earth's surface and the electrical conductivity distribution reflecting the geoelectric structure of the underlying medium.

It is important to note that elements of the impedance tensor are dependent on the choice of the orientation of the measuring axes for the determination of the electric and magnetic fields. For the purposes of interpretation, it is natural to seek a principal or intrinsic coordinate system in which the impedance tensor reduces to a particularly simple form that is more amenable to interpretation and insight. The most widely used technique for obtaining a principal coordinate system for a given impedance tensor $\mathbf{Z}(\omega)$ is the method developed by Sims and Bostick (1969) which will be referred to as the conventional analysis. In the conventional analysis, the principal-axis directions of $\mathbf{Z}(\omega)$ are obtained from the rotation properties of the tensor. For two-dimensional (2-D) conductivity distributions, the rotation of the impedance tensor results in an anti-diagonal form when the orientation of the coordinate axes is along the strike-dip direction of the structure. As a consequence, in this case, the rotation of $\mathbf{Z}(\omega)$ into an anti-diagonal form provides a natural choice for the principal coordinate axes of the structure. However, for a general three-dimensional (3-D) conductivity distribution, the corresponding impedance tensor cannot be anti-diagonalized for any choice of a real rotation angle ψ. In this case, the conventional analysis involves the choice of rotation angle $\psi=\psi_0$ such that the rotated impedance tensor $\mathbf{Z}'(\psi)$ approximates some anti-diagonal form in some optimum fashion. This is accomplished by choosing $\psi=\psi_0$ either to maximize $|Z'_{xy}(\psi)+Z'_{yx}(\psi)|$ or to minimize $|Z'_{xx}(\psi)-Z'_{yy}(\psi)|$. In point of fact then, this is actually an attempt to approximate the 3-D structure with some 2-D structure. While this type of analysis is suitable for conductivity distributions that are approximately 2-D, it is fairly evident that such a procedure does not produce a natural or intrinsic choice of a principal coordinate system for the general 3-D structure. Indeed for such 3-D geometries, rotation of $\mathbf{Z}$ merely reflects how a change of the orientation of the sensor axes affects the form of the impedance tensor; it certainly does not yield an intrinsic coordinate system. Furthermore, evidence that the conventional analyis is linked to 2-D structures comes from the fact that for 3-D geometries two indicators, the skew index α and the ellipticity index β, have been introduced in a rather *ad hoc* manner. These indicators are basically semi-quantitative measures of the departure of a 3-D structure from some 2-D structure.

A very disturbing aspect of the conventional analysis is that the principal impedances (off-diagonal elements of the rotated tensor) are independent of the trace of $\mathbf{Z}$. Hence, the set of parameters extracted by the conventional analysis for an impedance tensor corresponding to some 3-D structure is incomplete. Eggers (1982) recognized this important fact and, consequently, proposed the eigenstate formulation of the impedance tensor as a technique for the extraction of a complete set of physically meaningful scalar parameters from $\mathbf{Z}$. However, to obtain a complete set of parameters from $\mathbf{Z}$, Eggers imposed a somewhat artificial constraint and only considered those electric and magnetic field states, corresponding to some $\mathbf{Z}$, whose scalar product with each other vanishes. Spitz (1985) proposed the application of the Cayley factorization to the impedance tensor in order to construct two analytical procedures for the determination of two rotation angles which define two complete intrinsic coordinate systems for $\mathbf{Z}$. Although these procedures generalize the conventional analysis, it is not clear which of the two rotation angles is better suited for the analysis and interpretation of MT data. Along the same lines, associate and conjugate directions concepts have been applied to MT impedance analysis by Counil et al. (1986) who introduced the restriction of either linearly polarized output electric or input magnetic fields in order to define certain physically meaningful electric or magnetic sheet impedances. However, it should be noted that linear polarized fields (either electric or magnetic) do not necessarily determine impedances that are any more physically meaningful than those determined by elliptically polarized fields. LaTorraca et al. (1986) presented an analysis of the impedance tensor for 3-D structures based on parameters which describe elliptically polarized fields. As will be shown, this approach is very similar to the elliptic parameters formulation of canonical decomposition.

The purpose of this paper is to show how a unique intrinsic or principal coordinate system can be chosen for $\mathbf{Z}$ without the introduction of any artificial constraints and without regard to the dimensionality of the geoelectric structure from which $\mathbf{Z}$ is derived. A very general decomposition of the impedance tensor will be used which explicitly displays the structural components of the operator and, hence, simplifies its geometric nature. This basic structural representation for $\mathbf{Z}$ will be shown to yield naturally eight readily interpretable scalar parameters that completely characterize the impedance tensor at any frequency. The subsequent development emphasizes the importance of the consideration of the complex MT wave field in relation to the transfer characteristics of the earth structure as embodied in $\mathbf{Z}$. Indeed, the choice of a proper polarization description for representing the input-magnetic and output-electric fields, consistent with some $\mathbf{Z}$, results in a diagonal-form characterization of the impedance tensor. After the formulation of the basic structural representation for $\mathbf{Z}$, the relationships between canonical decomposition and those analyses previously cited will be developed.

Polarization descriptors

Since the polarization information embodied in the complex MT wave field is utilized to develop a structural representation for $\mathbf{Z}$, it is convenient to review briefly descriptors for the polarization of vector waves. A monochromatic uniform plane wave propagating in the z-direction and possessing an arbitrary state of elliptical polarization may be represented mathematically in terms of its complex field vector as

$$\mathbf{p}(z, t) = |p\rangle \exp[i(\omega t - kz)], \tag{3a}$$

where $|p\rangle$ is the complex polarization vector which may be characterized in some selected basis in the horizontal x-y plane as

$$|p\rangle = A \exp(i\alpha) \begin{pmatrix} a_1 \\ a_2 \exp(i\phi) \end{pmatrix}. \tag{3b}$$

Here A is the amplitude of the wave, α is the common phase, ϕ is the relative phase, and a_1 and a_2 are relative amplitudes which verify

$$a_1 = A_1/A \in [0, 1] \tag{3c}$$

and

$$a_2 = A_2/A \in [0, 1] \tag{3d}$$

with $A = (A_1^2 + A_2^2)^{1/2}$. When the amplitude and common phase information about the harmonic oscillation of the vector wave is only of secondary importance, the state of polarization of a plane wave may be specified completely through the complex polarization ratio, P, given by

$$P = \frac{a_2}{a_1} \exp(i\phi) \tag{4}$$

which relays the information concerning the relative amplitude and the relative phase (phase difference) between the component scalar oscillations of the wave measured in two orthogonal directions along the wavefront. As a consequence of the constraint expressed in Eq. (3c), it is clear that a parameter θ may be introduced to parametrize a_1 as

$$a_1 = \cos(\theta) \tag{5a}$$

with $\theta \in [0, \pi/2]$. With this definition, a_2 becomes

$$a_2 = (1 - a_1^2)^{1/2} = \sin(\theta) \tag{5b}$$

and the polarization ratio P now is characterized as

$$P = \tan(\theta) \exp(i\phi). \tag{5c}$$

Hence, the polarization state of a wave can be completely specified by providing the angles θ with $\theta \in [0, \pi/2]$ and ϕ with $\phi \in (-\pi, \pi]$, where θ and ϕ will be referred to as

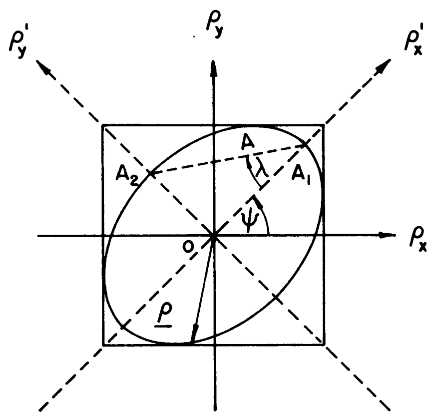

Fig. 1. The ellipse of polarization showing the elliptic parameters that describe the polarization state

the polarization parameters. Since P is a complex number, it can be placed in one-to-one correspondence with a unique point in the complex plane C which thus provides a direct association between the points of C and the states of polarization. In this representation, the points of C at the origin ($\theta = 0$, $\phi = 0$) and at infinity ($\theta = \pi/2$, $\phi = 0$) correspond to waves that are linearly polarized in the x- and y-directions, respectively. All linearly polarized waves with azimuths from 0 to $\pi/2$ are represented by taking $\phi = 0$ and $\theta \in [0, \pi/2]$ and correspond to points on the positive real axis of the complex plane; linear polarization with azimuths from $-\pi/2$ to 0 are represented by taking $\phi = \pi$ and $\theta \in [0, \pi/2]$ and correspond to points on the negative real axis. The two points on the imaginary axis, $P = i(\theta = \pi/4, \phi = \pi/2)$ and $P = -i(\theta = \pi/4, \phi = -\pi/2)$ correspond to right-handed and left-handed circular polarizations, respectively. It may be noted that $\phi > 0$ corresponds to right-handed elliptical polarizations (upper-half-plane) and $\phi < 0$ corresponds to left-handed elliptical polarizations (lower-half-plane). Note also that two modes of polarization specified by the polarization ratios P_1 and P_2 are orthogonal if and only if $P_1 P_2^* = P_1^* P_2 = -1$.

Most MT researchers are probably more familiar with the representation of the polarization state of a vector wave in terms of parameters defining its ellipse of polarization. Two quantities are necessary to uniquely characterize the polarization ellipse proper of the wave: the orientation angle ψ (sometimes referred to as the tilt or azimuthal angle) of the major axis of the ellipse relative to some reference axis (usually the x-axis of the specified rectangular x-y coordinate frame) and the ellipticity ε of the ellipse, defined to be the ratio of the length of the semi-minor axis to the length of the semi-major axis. These parameters are shown in Fig. 1 in relation to the ellipse of polarization. Observe that the shape of the polarization ellipse and its orientation in its plane are determined by giving the ellipticity

$$\varepsilon = A_2/A_1 \tag{6}$$

with $\varepsilon \in [-1, 1]$ and the tilt angle ψ with $\psi \in [0, \pi)$. Note that the sense of rotation of the ellipse is absorbed in the algebraic sign of ε. For a right-handed sense of rotation (i.e. the field vector rotates clockwise when looking back against the direction of propagation of the wave) ε is chosen positive ($\varepsilon > 0$), while a left-handed polarization implies a negative ε ($\varepsilon<0$). It is convenient to introduce the ellipticity angle λ which is defined as

$$\tan\lambda=\varepsilon, \quad \lambda\in[-\pi/4, \pi/4]. \tag{7}$$

The parameters ψ and λ which characterize the shape and orientation of the polarization ellipse in its plane will be referred to as the elliptic parameters. It is straightforward to show that elliptic parameters ψ and λ of the vector wave can be related to the polarization parameters as

$$\tan 2\psi=\tan 2\theta\cdot\cos\phi \tag{8a}$$

and

$$\sin 2\lambda=\sin 2\theta\cdot\sin\phi. \tag{8b}$$

Or, equivalently,

$$\cos 2\theta=\cos 2\lambda\cdot\cos 2\psi \tag{8c}$$

and

$$\tan\phi=\tan 2\lambda\cdot\csc 2\psi. \tag{8d}$$

Generalized apparent resistivity

To motivate the development of a basic structural representation for the impedance tensor, we begin by generalizing Cagniard's definition of the apparent resistivity $\rho_a(\omega)$ (Vozoff, 1972). We define the generalized apparent resistivity as

$$\rho_a(\omega)=\frac{1}{\omega\mu_0}\frac{\||E\rangle\|_2^2}{\||H\rangle\|_2^2}=\frac{1}{\omega\mu_0}\frac{\langle E|E\rangle}{\langle H|H\rangle}, \tag{9}$$

where $\|\cdot\|_2$ denotes the standard Euclidean vector norm, and $|E\rangle$ the electric field output from the impedance tensor $\mathbf{Z}$ for a magnetic field input $|H\rangle$. Note that $|E\rangle$ and $|H\rangle$ can specify any arbitrary polarization state for the electric and magnetic field, respectively, subject to the constraint $|E\rangle=\mathbf{Z}|H\rangle$. For an input magnetic field linearly polarized along the y-axis with the resulting output electric field linearly polarized in an orthogonal direction, viz. along the x-direction (as in the case of an isotropic, 1-D earth), it is evident that Eq. (9) simplifies to the usual Cagniard definition of apparent resistivity.

Since $|E\rangle$ is related to $|H\rangle$ via the impedance tensor as $|E\rangle=\mathbf{Z}|H\rangle$, it follows that the dual relation can be written as $\langle E|=\langle H|\mathbf{Z}^\dagger$. Then $\langle E|E\rangle=\langle H|\mathbf{Z}^\dagger\mathbf{Z}|H\rangle$ and, hence,

$$\rho_a(\omega)=\frac{1}{\omega\mu_0}\frac{\langle H|\mathbf{Z}^\dagger\mathbf{Z}|H\rangle}{\langle H|H\rangle}. \tag{10}$$

Since $\mathbf{Z}^\dagger\mathbf{Z}$ is a non-negative definite Hermitian operator, the spectral theorem assures the existence of an orthonormal basis $\{|h_i\rangle, i=1,2\}$ in C^2 and non-negative numbers σ_1^2 and σ_2^2 such that

$$\mathbf{Z}^\dagger\mathbf{Z}=\sum_{i=1}^{2}\sigma_i^2|h_i\rangle\langle h_i| \tag{11}$$

with $\mathbf{Z}^\dagger\mathbf{Z}|h_i\rangle=\sigma_i^2|h_i\rangle$, ($i=1, 2$). Hence, any input magnetic field vector many be expressed as some linear combination of the basis vectors $\{|h_i\rangle, i=1, 2\}$, viz.

$$|H\rangle=\sum_{i=1}^{2}c_i|h_i\rangle, \tag{12a}$$

where the coordinates of $|H\rangle$ in the orthonormal basis are

$$c_i=\langle h_i|H\rangle, \quad i=1,2. \tag{12b}$$

Hence, from Eqs. (10), (11) and (12), it follows that

$$\rho_a(\omega)=\frac{1}{\omega\mu_0}\frac{\sum_{i=1}^{2}\sigma_i^2|c_i|^2}{\sum_{i=1}^{2}|c_i|^2}. \tag{13}$$

Equation (13) expresses $\rho_a(\omega)$ as a convex combination of the eigenvalues of $\mathbf{Z}^\dagger\mathbf{Z}$ with the weights

$$\left\{|\langle h_i|H\rangle|^2\Big/\sum_{i=1}^{2}|\langle h_i|H\rangle|^2, \quad i=1,2\right\}.$$

Let the eigenvalues of $\mathbf{Z}^\dagger\mathbf{Z}$ be labelled so that $\sigma_1^2\geqq\sigma_2^2$. By observing that

$$\sigma_2^2\sum_{i=1}^{2}|c_i|^2\leqq\sum_{i=1}^{2}\sigma_i^2|c_i|^2\leqq\sigma_1^2\sum_{i=1}^{2}|c_i|^2,$$

we obtain the following result:

$$\frac{1}{\omega\mu_0}\sigma_2^2(\omega)\leqq\rho_a(\omega)\leqq\frac{1}{\omega\mu_0}\sigma_1^2(\omega). \tag{14}$$

In Eq. (14), the dependence of σ_1^2 and σ_2^2 on the frequency has been explicitly indicated, viz. for each frequency, ω, $\sigma_1^2(\omega)$ and $\sigma_2^2(\omega)$ are the eigenvalues of $\mathbf{Z}^\dagger\mathbf{Z}(\omega)$. Equation (14) indicates that the generalized apparent resistivity is bounded above and below by the corresponding maximum and minimum eigenvalues of $\mathbf{Z}^\dagger\mathbf{Z}$ normalized by the factor $1/\omega\mu_0$. Hence, the generalized apparent resistivity $\rho_a(\omega)$ is a point on the line segment $[\sigma_2^2/\omega\mu_0, \sigma_1^2/\omega\mu_0]$ formed from all convex combinations of the two elements $\sigma_2^2/\omega\mu_0$ and $\sigma_1^2/\omega\mu_0$, these two elements being the extreme points of the segment. In view of this, we define $\bar{\rho}_a(\omega)=\sigma_1^2(\omega)/\omega\mu_0$ and $\underline{\rho}_a(\omega)=\sigma_2^2(\omega)/\omega\mu_0$ as the maximum and minimum principal apparent resistivities, respectively, associated with the impedance tensor $\mathbf{Z}(\omega)$. Note that $\rho_a(\omega)$ assumes the value of either the maximum or minimum principal apparent resistivity only when the magnetic field input vector $|H\rangle$ coincides with one of the eigenvectors of $\mathbf{Z}^\dagger\mathbf{Z}$ [i.e. if $|H\rangle=|h_1\rangle$ then $\rho_a(\omega)=\bar{\rho}_a(\omega)$, and if $|H\rangle=|h_2\rangle$ then $\rho_a(\omega)=\underline{\rho}_a(\omega)$].

To expand further the concept of principal apparent resistivity, consider the norm of the operator $\mathbf{Z}$. A natural definition of the operator norm for $\mathbf{Z}$ is given by

$$\|\mathbf{Z}\|=\sup\{\|\mathbf{Z}|H\rangle\|_2/\||H\rangle\|_2 : 0\neq|H\rangle\in C^2\}, \tag{15}$$

where sup denotes supremum. Observe that the operator norm $\|\mathbf{Z}\|$ defined in Eq. (15) depends on the Euclidean norm $\|\cdot\|_2$ used to measure the 'size' of $|H\rangle$ and $|E\rangle=\mathbf{Z}|H\rangle$. Indeed, the ratio $\|\mathbf{Z}|H\rangle\|_2/\||H\rangle\|_2$ with $|H\rangle\neq 0$ can be viewed as the gain or amplification of the impedance tensor $\mathbf{Z}$ for a given input magnetic field $|H\rangle$. Consequently, $\|\mathbf{Z}\|$ must correspond to the maximum amount that the operator $\mathbf{Z}$ can 'stretch' any magnetic input field vector in the sense of the Euclidean norm. However, it is clear that $\|\mathbf{Z}\|^2=\sigma_1^2$ where σ_1^2 is the maximum eigenvalue of $\mathbf{Z}^\dagger\mathbf{Z}$. In light of this, the maximum principal apparent resistivity can be expressed as

$$\bar{\rho}_a(\omega)=\frac{1}{\omega\mu_0}\|\mathbf{Z}\|^2. \tag{16}$$

Equation (16) may be considered to be a generalization to the impedance tensor of the Cagniard apparent resistivity $\rho_a(\omega)=|Z|^2/\omega\mu_0$ valid for a scalar wave impedance. Note that the absolute value of $Z(\omega)$, which measures the 'size' of the scalar wave impedance $Z(\omega)$, essentially has been replaced by the operator norm of $\mathbf{Z}(\omega)$ which measures the 'size' of the tensor $\mathbf{Z}(\omega)$.

Canonical decomposition for the MT impedance tensor: polarization parameter formulation

The principal apparent resistivities were shown to be dependent only on the eigenvalues of an auxiliary operator, namely $\mathbf{Z}^\dagger\mathbf{Z}$. It would be useful to relate these principal apparent resistivities to parameters that may be extracted directly from the operator $\mathbf{Z}$. It is to this end that the singular value decomposition (SVD) of $\mathbf{Z}$ is considered. Recall that the singular values of the operator $\mathbf{Z}$ (Stewart, 1973) are defined to be the non-negative square roots of the eigenvalues of $\mathbf{Z}^\dagger\mathbf{Z}$. Since these were previously labelled as $\{\sigma_i^2, i=1, 2\}$, it is clear that the singular values of $\mathbf{Z}$ are $\{\sigma_i, i=1, 2\}$ with $\sigma_1\geqq\sigma_2$. The SVD of $\mathbf{Z}$ is defined by $\mathbf{Z}=\mathbf{USV}^\dagger$ where $\mathbf{U}\in C^{2\times 2}$ and $\mathbf{V}\in C^{2\times 2}$ are unitary matrices and $\mathbf{S}\in C^{2\times 2}$ is a diagonal matrix with elements $\sigma_i>0$ ($i=1$, 2) along the diagonal. The SVD of $\mathbf{Z}$ satisfies the following properties:

1) The diagonal elements σ_i of $\mathbf{S}$ are the non-negative square roots of the eigenvalues of $\mathbf{Z}^\dagger\mathbf{Z}$ or of $\mathbf{Z}\mathbf{Z}^\dagger$.

2) The columns of $\mathbf{U}$ denoted by $\{|e_i\rangle, i=1, 2\}$ are the eigenvectors of $\mathbf{Z}\mathbf{Z}^\dagger$ corresponding to the eigenvalues $\{\sigma_i^2, i=1, 2\}$ with $\langle e_i|e_j\rangle=\delta_{ij}$, where δ_{ij} denotes the Kronecker delta function.

3) The columns of $\mathbf{V}$ denoted by $\{|h_i\rangle, i=1, 2\}$ are the eigenvectors of $\mathbf{Z}^\dagger\mathbf{Z}$ corresponding to the eigenvalues $\{\sigma_i^2, i=1, 2\}$ with $\langle h_i|h_j\rangle=\delta_{ij}$.

4) $\mathbf{Z}|h_i\rangle=\sigma_i|e_i\rangle, i=1, 2.$

5) $\mathbf{Z}^\dagger|e_i\rangle=\sigma_i|h_i\rangle, i=1, 2.$

At this point, it may be remarked that the principal apparent resistivities are determined entirely by the singular values of $\mathbf{Z}$. From this viewpoint, the sets $\{|h_i\rangle, i=1, 2\}$ and $\{|e_i\rangle, i=1, 2\}$ which form complete orthonormal bases for the input magnetic field space C_H^2 and the output electric field space C_E^2, respectively, can be considered as constituting the principal magnetic and electric field directions for $\mathbf{Z}$. Hence, these sets contribute a natural or intrinsic coordinate system for describing the impedance tensor. The principal magnetic and electric field directions are related through the singular values as in relations (4) and (5). In particular, relation (4) indicates that an input principal magnetic field direction is transformed by the operator $\mathbf{Z}$ into the corresponding output principal electric field direction scaled by the associated singular value of $\mathbf{Z}$. Relation (5) is a dual result and expresses the fact that a principal electric field direction is caused by $\mathbf{Z}^\dagger$ to transform into the corresponding principal magnetic field direction again scaled by its associated singular value. Observe that the singular values of $\mathbf{Z}$, which determine the principal apparent resistivities, reflect the gain properties of the tensor and, hence, they effectively enter as the scaling factors in the determination of the input-output properties of $\mathbf{Z}$ as embodied in the principal magnetic and electric field directions.

From the manner in which the principal apparent resistivities were defined earlier, it is clear that the singular values of $\mathbf{Z}$ can be interpreted as the moduli of the principal impedances associated with the tensor. The problem of how to assign phases to these values to form the principal impedances must now be confronted. To achieve this objective, first observe that since $|h_i\rangle\in C_H^2$ with the normalization $\langle h_i|h_i\rangle=1$, essentially three real parameters are required to completely specify the ket vector $|h_i\rangle$. This suggests that an overall or absolute phase may be extracted from the input principal magnetic field state $|h_i\rangle$ without altering its characteristic direction in the space C_H^2. The same discussion holds for the output principal electric field state $|e_i\rangle$; viz. an absolute phase can be extracted from $|e_i\rangle$ without disturbing its alignment in the space C_E^2. Taken together, this suggests that phases may be assigned to the principal impedance moduli $\{\sigma_i, i=1, 2\}$ by extracting the absolute phases from the corresponding principal input-magnetic and output-electric field states and combining these absolute phases so obtained to generate the phase factors for the principal impedance moduli. Applying this simple idea to the SVD of the impedance tensor provides the following result.

Proposition 1. Let $\mathbf{Z}(\omega)\in C^{2\times 2}$ be a linear mapping $\mathbf{Z}$: $C_H^2\to C_E^2$, parametrized by the frequency ω. For each $\omega\in R$, $\mathbf{Z}(\omega)$ admits to the decomposition

$$\mathbf{Z}(\omega)=\begin{pmatrix}\cos[\theta_E(\omega)] & -\exp[-i\phi_E(\omega)]\sin[\theta_E(\omega)]\\ \exp[i\phi_E(\omega)]\sin[\theta_E(\omega)] & \cos[\theta_E(\omega)]\end{pmatrix}\cdot\begin{pmatrix}\sigma_1(\omega)\exp[i\gamma_1(\omega)] & 0\\ 0 & \sigma_2(\omega)\exp[i\gamma_2(\omega)]\end{pmatrix}\cdot\begin{pmatrix}\cos[\theta_H(\omega)] & -\exp[-i\phi_H(\omega)]\sin[\theta_H(\omega)]\\ \exp[i\phi_H(\omega)]\sin[\theta_H(\omega)] & \cos[\theta_H(\omega)]\end{pmatrix}^\dagger,$$

where $\theta_E(\omega), \theta_H(\omega)\in[0, \pi/2]$; $\phi_E(\omega), \phi_H(\omega)\in(-\pi, \pi]$; $\gamma_1(\omega), \gamma_2(\omega)\in(-\pi, \pi]$ and $\sigma_1(\omega)$ and $\sigma_2(\omega)$ are the singular values of $\mathbf{Z}(\omega)$ with $0<\sigma_2(\omega)\leqq\sigma_1(\omega)$.

A detailed proof of this result can be found in Yee (1985). Henceforth, the decomposition given in Proposition 1 shall be referred to as the canonical decomposition of $\mathbf{Z}$.

Interpretation of parameters in the canonical decomposition

The result embodied in Proposition 1 clearly displays the structure inherent in the impedance tensor $\mathbf{Z}(\omega)$. Observe that $\mathbf{Z}(\omega)$ is completely specified by eight independent real-valued parameters $\theta_E(\omega)$, $\phi_E(\omega)$, $\theta_H(\omega)$, $\phi_H(\omega)$, $\sigma_1(\omega)$, $\sigma_2(\omega)$, $\gamma_1(\omega)$ and $\gamma_2(\omega)$. This is not surprising since $\mathbf{Z}(\omega)$ possesses four complex entries which implies that eight real quantities are needed for their specification.

Recall that the squares of the singular values of $\mathbf{Z}$ determine the principal apparent resistivities associated with the tensor. With this in mind, let us introduce the following definition: the maximum principal impedance and the minimum principal impedance of the operator $\mathbf{Z}(\omega)$ are defined as $\sigma_1(\omega)\exp[i\gamma_1(\omega)]$ and $\sigma_2(\omega)\exp[i\gamma_2(\omega)]$, respectively. Refer to $\gamma_1(\omega)$ and $\gamma_2(\omega)$ as the principal phases of $\mathbf{Z}(\omega)$ and to $\sigma_1(\omega)$ and $\sigma_2(\omega)$ as the principal impedance moduli of $\mathbf{Z}(\omega)$. Note that the principal impedances assume the same role with respect to the determination of the principal apparent resistivities as does the Cagniard scalar impedance with respect to the determination of the Cagniard apparent resistivity. Hence, four of the eight quantities emerging in Proposition 1, namely $\sigma_1(\omega)$, $\sigma_2(\omega)$, $\gamma_1(\omega)$ and $\gamma_2(\omega)$, resolve the principal impedances associated with $\mathbf{Z}(\omega)$ and, in essence, determine the transfer characteristics of the earth system.

The physical significance of the remaining four parameters $\theta_E(\omega)$, $\phi_E(\omega)$, $\theta_H(\omega)$ and $\phi_H(\omega)$ extracted in the canonical decomposition of $\mathbf{Z}$ should be clear in view of the discussion on the representation of polarization states contained in a previous section. The parameters θ_H and ϕ_H completely characterize the states of polarization of the input principal magnetic field vectors; indeed, the orthogonal principal magnetic field states are given by

$$|\bar{h}_1\rangle=\begin{pmatrix}\cos(\theta_H)\\ \exp(i\phi_H)\sin(\theta_H)\end{pmatrix} \tag{17a}$$

and

$$|\bar{h}_2\rangle=\begin{pmatrix}-\exp(-i\phi_H)\sin(\theta_H)\\ \cos(\theta_H)\end{pmatrix}. \tag{17b}$$

Similarly, the parameters θ_E and ϕ_E determine the states of polarization of the output principal electric field vectors. Again there are two principal electric field states, one state described by

$$|\bar{e}_1\rangle=\begin{pmatrix}\cos(\theta_E)\\ \exp(i\phi_E)\sin(\theta_E)\end{pmatrix}, \tag{18a}$$

with the second state occupying an orthogonal mode of polarization, i.e.

$$|\bar{e}_2\rangle=\begin{pmatrix}-\exp(-i\phi_E)\sin(\theta_E)\\ \cos(\theta_E)\end{pmatrix}. \tag{18b}$$

It may be remarked that the input-magnetic and output-electric polarization parameters θ_H, ϕ_H, θ_E and ϕ_E can be thought of as being associated with certain intrinsic directional properties of the earth structure as characterized by the impedance tensor.

To reiterate, if Eqs. (17) and (18) are introduced into the canonical decomposition of $\mathbf{Z}$, the expression

$$\mathbf{Z}=\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{V}}^{\dagger}$$
$$=(|\bar{e}_1\rangle|\bar{e}_2\rangle)\begin{pmatrix}\sigma_1\exp(i\gamma_1) & 0\\ 0 & \sigma_2\exp(i\gamma_2)\end{pmatrix}(|\bar{h}_1\rangle|\bar{h}_2\rangle)^{\dagger} \tag{19}$$

is obtained. Observe that the matrices $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ of Eq. (19), obtained by extracting the absolute phases from the columns of $\mathbf{U}$ and $\mathbf{V}$, are unimodular unitary matrices, viz. $\bar{\mathbf{U}}^{\dagger}\bar{\mathbf{U}}=\bar{\mathbf{U}}\bar{\mathbf{U}}^{\dagger}=\mathbf{I}$ and $\bar{\mathbf{V}}^{\dagger}\bar{\mathbf{V}}=\bar{\mathbf{V}}\bar{\mathbf{V}}^{\dagger}=\mathbf{I}$ with $\det(\bar{\mathbf{U}})=\det(\bar{\mathbf{V}})=1$. Geometrically the unitary transformations $\bar{\mathbf{V}}$ and $\bar{\mathbf{U}}$ can be viewed as pure or rigid frame rotations in the complex two-dimensional input magnetic and output electric field spaces C_H^2 and C_E^2, respectively; as such, these transformations preserve the angle ψ between any two vectors in the unitary spaces C_H^2 and C_E^2. They can be seen to be the generalization to the complex space of the rotation transformations employed in conventional magnetotelluric analysis to determine principal impedances; recall that in the conventional analysis, only rotations in a real (physical) space are considered. The diagonal matrix $\bar{\mathbf{S}}$ expresses the principal transfer characteristics of the earth system and, as such, links together the input magnetic field principal states contained in the $\bar{\mathbf{V}}$ matrix and the output electric field principal states contained in the $\bar{\mathbf{U}}$ matrix. Hence, the canonical decomposition of Eq. (19) represents the impedance tensor $\mathbf{Z}$ as the product of a rotation in C_H^2 followed by a 'stretching' in the form of the amplitude and phase characteristic modification required to transfer from C_H^2 to C_E^2 followed in turn by a rotation in C_E^2. It is emphasized that the rotation in C_E^2 need not be the same as the rotation in C_H^2. It should perhaps be noted that the parameters that emerge in the canonical decomposition of $\mathbf{Z}$ are uniquely determined except when (1) $\bar{\mathbf{U}}$ possesses zero off-diagonal elements in which case ϕ_E is undetermined; (2) $\bar{\mathbf{V}}$ possesses zero off-diagonal elements in which case ϕ_H is undetermined; and (3) $\sigma_1=\sigma_2$ in which case $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are only determined up to a unitary unimodular transformation so that the choice for θ_E, ϕ_E, θ_H and ϕ_H is non-unique.

The canonical decomposition of the impedance tensor given by Eq. (19) can be aptly represented in the form of a block diagram as shown in Fig. 2. The physical structure

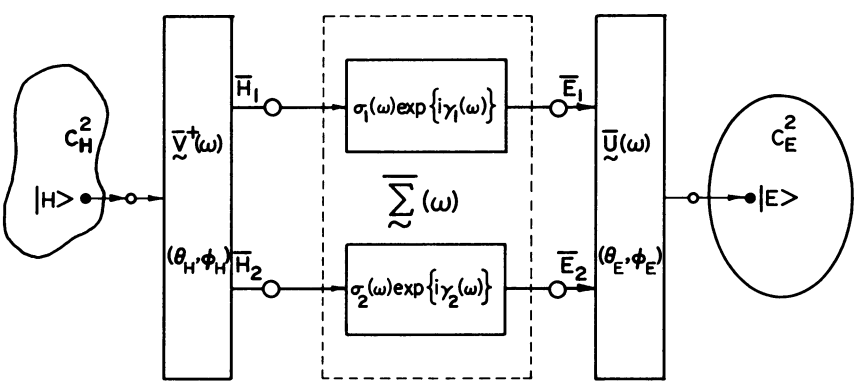

Fig. 2. Block diagram showing the components of the impedance tensor obtained by canonical decomposition

of the impedance tensor decomposes to the cascade of three basic blocks. The central block consists of two decoupled scalar subsystems that exhibit the principal impedances of the structure. These principal impedances provide the amplitude and phase relationships that connect the principal magnetic field vector $|\bar{H}\rangle$ to the principal electric field vector $|\bar{E}\rangle$ as

$$|\bar{E}\rangle=\begin{pmatrix}\bar{E}_1\\ \bar{E}_2\end{pmatrix}=\begin{pmatrix}\sigma_1\exp(i\gamma_1) & 0\\ 0 & \sigma_2\exp(i\gamma_2)\end{pmatrix}\begin{pmatrix}\bar{H}_1\\ \bar{H}_2\end{pmatrix}=\bar{\mathbf{S}}|\bar{H}\rangle.$$

Note that since the principal impedance transfer function matrix $\bar{\mathbf{S}}$ is diagonal, there is no interaction between the two scalar subsystems. Hence, in the principal coordinate system, the impedance tensor assumes a simple diagonal form. The interaction or coupling mechanisms of the system arise from the input magnetic field transformation $\bar{\mathbf{V}}$ and the output electric field transformation $\bar{\mathbf{U}}$. These input and output transformations relate the observed electric and magnetic field vectors $|E\rangle$ and $|H\rangle$ to the principal electric and magnetic field vectors as

$$|E\rangle=\bar{\mathbf{U}}|\bar{E}\rangle$$

and

$$|H\rangle=\bar{\mathbf{V}}|H\rangle.$$

The preceding transformations relating $|E\rangle$ and $|H\rangle$ with $|\bar{E}\rangle$ and $|\bar{H}\rangle$ arise from the fact that the input and output signals are observed in a basis other than the natural input basis $\{|\bar{h}_i\rangle, i=1, 2\}$ and the natural output basis $\{|\bar{e}_i\rangle, i=1, 2\}$. These intrinsic bases are determined primarily by the structural properties of the earth system and its effect on the polarization characteristics of the complex MT wave field. Indeed, if the input and output spaces C_H^2 and C_E^2 are referred to these bases, the impedance tensor reduces to the principal diagonal impedance operator $\bar{\mathbf{S}}$ in which the tensor is essentially replaced by two independent scalar impedance systems.

The dynamic structure of $\mathbf{Z}$ is vividly revealed in Fig. 2 by working through the block diagram from right to left:

$$|E\rangle=\bar{\mathbf{U}}|\bar{E}\rangle=\sum_{i=1}^{2}|\bar{e}_i\rangle\,\bar{E}_i=\sum_{i=1}^{2}|\bar{e}_i\rangle\,\sigma_i\exp(i\gamma_i)\,\bar{H}_i$$
$$=\sum_{i=1}^{2}\sigma_i\exp(i\gamma_i)|\bar{e}_i\rangle\langle\bar{h}_i|H\rangle=\mathbf{Z}|H\rangle.$$

Or,

$$\mathbf{Z}=\sum_{i=1}^{2}\sigma_i\exp(i\gamma_i)|\bar{e}_i\rangle\langle\bar{h}_i|. \quad (20)$$

This result expresses $\mathbf{Z}$ in a vector outer product form from which it is seen that $\mathbf{Z}$ is a linear combination of two matrices $|\bar{e}_i\rangle\langle\bar{h}_i|$, each of rank one, constructed from the input magnetic field and output electric field principal polarization states each weighted by the respective principal impedances. Observe that if the earth sytem is excited by an input magnetic field coresponding to one of the principal magnetic field polarization states, viz. $|H\rangle=|\bar{h}_i\rangle$ $(i=1, 2)$, then the resulting output electric field will be observed in the corresponding principal electric field polarization state $|\bar{e}_i\rangle$ scaled by the corresponding principal impedance. Hence,

$$|E\rangle=\mathbf{Z}|H\rangle=\mathbf{Z}|\bar{h}_i\rangle=\sigma_i\exp(i\gamma_i)|\bar{e}_i\rangle.$$

A reformulation of the canonical decomposition in terms of elliptic parameters

Since most MT researchers are probably more familiar with the use of the elliptic parameters in the specification of states of polarization, it is useful to express the canonical decomposition for $\mathbf{Z}$ contained in Proposition 1 in terms of elliptic parameters. This leads to the following result.

Proposition 2. Let $\mathbf{Z}$: $C_H^2\rightarrow C_E^2$ be the impedance tensor represented by the transfer matrix $\mathbf{Z}(\omega)\in C^{2\times 2}$ with parametric dependence on frequency ω. Then, for each $\omega\in R$, $\mathbf{Z}(\omega)$ can be decomposed as

$$\mathbf{Z}(\omega)=\begin{pmatrix}\cos[\psi_E(\omega)] & -\sin[\psi_E(\omega)]\\ \sin[\psi_E(\omega)] & \cos[\psi_E(\omega)]\end{pmatrix}$$
$$\cdot\begin{pmatrix}\cos[\lambda_E(\omega)] & i\sin[\lambda_E(\omega)]\\ i\sin[\lambda_E(\omega)] & \cos[\lambda_E(\omega)]\end{pmatrix}$$
$$\cdot\begin{pmatrix}\sigma_1(\omega)\exp[i\bar{\gamma}_1(\omega)] & 0\\ 0 & \sigma_2(\omega)\exp[i\bar{\gamma}_2(\omega)]\end{pmatrix}$$
$$\cdot\begin{pmatrix}\cos[\lambda_H(\omega)] & -i\sin[\lambda_H(\omega)]\\ -i\sin[\lambda_H(\omega)] & \cos[\lambda_H(\omega)]\end{pmatrix}$$
$$\cdot\begin{pmatrix}\cos[\psi_H(\omega)] & \sin[\psi_H(\omega)]\\ -\sin[\psi_H(\omega)] & \cos[\psi_H(\omega)]\end{pmatrix},$$

where $\psi_E(\omega)$, $\psi_H(\omega)\in[0, \pi)$; $\lambda_E(\omega)$, $\lambda_H(\omega)\in[-\pi/4, \pi/4]$; $\bar{\gamma}_1(\omega)$, $\bar{\gamma}_2(\omega)\in(-\pi, \pi]$; and $0<\sigma_2(\omega)\leqq\sigma_1(\omega)$ are the singular values of $\mathbf{Z}(\omega)$. Furthermore,

$$\bar{\gamma}_1(\omega)=\gamma_1(\omega)-[\xi_E(\omega)-\xi_H(\omega)]$$

and

$$\bar{\gamma}_2(\omega)=\gamma_2(\omega)+[\xi_E(\omega)-\xi_H(\omega)],$$

where

$$\xi_E(\omega)=\arg\{\cos[\psi_E(\omega)]\cos[\lambda_E(\omega)]-i\sin[\psi_E(\omega)]\sin[\lambda_E(\omega)]\},$$
$$\xi_H(\omega)=\arg\{\cos[\psi_H(\omega)]\cos[\lambda_H(\omega)]-i\sin[\psi_H(\omega)]\sin[\lambda_H(\omega)]\}$$

and $\gamma_1(\omega)$ and $\gamma_2(\omega)$ are the principal phases determined in Proposition 1.

For a detailed proof of these results, see Yee (1985). Again, observe that eight real-valued scalar parameters, namely, $\psi_E(\omega)$, $\lambda_E(\omega)$, $\psi_H(\omega)$, $\lambda_H(\omega)$, $\sigma_1(\omega)$, $\sigma_2(\omega)$, $\bar{\gamma}_1(\omega)$ and $\bar{\gamma}_2(\omega)$ emerge from the canonical decomposition and serve to completely characterize the impedance tensor. The last four parameters determine the principal impedances, whereas the first four parameters describe the polarization states of the input principal magnetic field vectors and the output principal electric field vectors. It is important to note that the elliptic parameters formulation of the canonical decomposition embodied in Proposition 2 is, in essence, the approach for the analysis of the impedance tensor presented by LaTorraca et al. (1986). Furthermore, observe that the principal phases $\bar{\gamma}_1$ and $\bar{\gamma}_2$ extracted in the elliptic parameters formulation are different from the principal phases γ_1 and γ_2 extracted in the polarization parameters formulation. This difference arises from the introduction of auxiliary phase factors ξ_H and ξ_E which are required in order to rewrite the principal electric and magnetic field states [cf. Eqs.(17) and (18)] in the form

$$\exp(i\xi_E)|\bar{e}_1\rangle = \mathbf{r}_1^e + i\mathbf{r}_2^e$$

and

$$\exp(i\xi_H)|\bar{h}_1\rangle = \mathbf{r}_1^h + i\mathbf{r}_2^h,$$

where $\mathbf{r}_1^e$ (respectively, $\mathbf{r}_1^h$) and $\mathbf{r}_2^e$ (respectively, $\mathbf{r}_2^h$) are orthogonal real vectors which specify the directions of the major and minor axes of the principal electric (respectively, magnetic) field polarization ellipse in real space.

Although both sets of principal phases (i.e. γ_1, γ_2 and $\bar{\gamma}_1$, $\bar{\gamma}_2$) can be assigned to the principal impedance moduli, it should perhaps be noted that the phases $\bar{\gamma}_1/\omega$ and $\bar{\gamma}_2/\omega$ can be interpreted physically in the time domain as the phase lead or lag of the output electric field with respect to the corresponding input magnetic field (LaTorraca et al. 1986; Counil et al. 1986). For example, in greater detail, $\bar{\gamma}_1/\omega$ corresponds to the phase lead or lag of the output electric field $\tilde{\mathbf{e}}_1(t)$ relative to the corresponding input magnetic field $\tilde{\mathbf{h}}_1(t)$, where $\tilde{\mathbf{e}}_1(t)$ and $\tilde{\mathbf{h}}_1(t)$ denote the time-harmonic fields of frequency ω associated with the complex vector phasors

$$|\tilde{e}_1\rangle = \sigma_1 \exp(i\bar{\gamma}_1)\exp(i\xi_E)|\bar{e}_1\rangle$$

and

$$|\tilde{h}_1\rangle = \exp(i\xi_H)|\bar{h}_1\rangle,$$

respectively. Along the same vein, it is of interest to point out that $\bar{\gamma}_1$ may be interpreted with reference to the polarization ellipses associated with the time-harmonic fields $\tilde{\mathbf{e}}_1(t)$ and $\tilde{\mathbf{h}}_1(t)$. Accordingly, for an input magnetic field $\tilde{\mathbf{h}}_1(t)$ whose magnetic field vector at $t=0$ is aligned with the major axis of the associated magnetic field polarization ellipse, the corresponding output electric field $\tilde{\mathbf{e}}_1(t)$ has its electric field vector at $t=0$ inclined at the angle $\tan^{-1}[\tan(\lambda_E)\tan(\bar{\gamma}_1)]$ with the major axis of the associated electric field polarization ellipse. Here, λ_E is the ellipticity angle corresponding to the polarization state $|\tilde{e}_1\rangle$. In this sense then, $\bar{\gamma}_1$ may be interpreted as the absolute phase of the output electric field $\tilde{\mathbf{e}}_1(t)$ which determines the position of its initial field vector along the associated ellipse of polarization. Similar remarks apply to $\bar{\gamma}_2$.

Observe from Proposition 2 that the impedance tensor can be reduced to a diagonal form by pre-multiplying and post-multiplying $\mathbf{Z}$ by an operator with the general form

$$\mathbf{S}(\psi,\lambda) = \mathbf{R}(-\psi)\,\mathbf{P}(\lambda) = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}\begin{pmatrix} \cos\lambda & i\sin\lambda \\ i\sin\lambda & \cos\lambda \end{pmatrix}, \tag{21}$$

where ψ and λ are the elliptic parameters corresponding to some polarization state. Indeed with $\mathbf{S}$ determined as in Eq. (21), $\mathbf{Z}$ can be diagonalized as

$$\mathbf{S}^\dagger(\psi_E,\lambda_E)\,\mathbf{Z}\mathbf{S}(\psi_H,\lambda_H) = \begin{pmatrix} \sigma_1 \exp[i\bar{\gamma}_1(\omega)] & 0 \\ 0 & \sigma_2 \exp[i\bar{\gamma}_2(\omega)] \end{pmatrix}. \tag{22}$$

It is important to note that $\mathbf{S}(\psi, \lambda)$ is composed of the product of two operators, namely a rotation operator $\mathbf{R}(-\psi)$ which depends only on the orientation (rotation) angle ψ, and an ellipticity operator $\mathbf{P}(\lambda)$ which depends only on the ellipticity angle λ. Observe that Eq. (22) constitutes a natural generalization of the conventional analysis in which only the rotation operator $\mathbf{R}(-\psi)$ plays a role in the definition of a principal coordinate system. The relationship between the canonical decomposition and other forms of impedance tensor analysis (including the conventional analysis) will be explored after considering some applications.

Applications of the canonical decomposition

In this section, a number of applications of the canonical decomposition for $\mathbf{Z}$ are presented by way of examples. In what follows, it will be assumed that the impedance tensor is measured such that the input magnetic and output electric field frames are referenced to the same coordinate system (x, y). Hence, (x, y) constitutes a pair of orthogonal linear basis states in which the electric and magnetic fields may be expressed and the impedance tensor assumes the usual form, viz.

$$\mathbf{Z} = \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix}.$$

Example 1 (One-dimensional earth). An isotropic earth in which the conductivity distribution varies only in the vertical direction, i.e. a 1-D earth, verifies the conditions $Z_{xx} = Z_{yy} = 0$ and $Z_{xy} = -Z_{yx} = Z_0$, where Z_0 is the scalar Cagniard impedance function measured at the surface of the earth. Hence, the impedance tensor for a 1-D earth reduces to the form

$$\mathbf{Z}_I = \begin{pmatrix} 0 & Z_0 \\ -Z_0 & 0 \end{pmatrix}.$$

The singular values of $\mathbf{Z}_I$ are $\sigma_1 = \sigma_2 = |Z_0|$. Since $\sigma_1 = \sigma_2$ for $\mathbf{Z}_I$, the canonical decomposition for $\mathbf{Z}_I$ is not unique. Physically, this is a manifestation of the observation that for 1-D conductivity structures, $\mathbf{Z}_I$ is independent of the choice of the coordinate system used for its representation (i.e. the choice of the pairs of orthogonal elliptic polarizations to be used as basis states for the input magnetic and output electric field frames). Indeed, as already mentioned earlier, for the case where $\sigma_1 = \sigma_2$, the matrices $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are determined only up to the same unitary unimodular transformation. Accordingly, for some choice of the magnetic field polarization parameters ($\theta_H \in [0, \pi/2]$ and $\phi_H \in (-\pi, \pi]$), the canonical decomposition for $\mathbf{Z}_I$ can be written as

$$\mathbf{Z}_I=\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{V}}^\dagger=\begin{pmatrix}\exp(i\phi_H)\sin(\theta_H) & \cos(\theta_H)\\ -\cos(\theta_H) & \exp(-i\phi_H)\sin(\theta_H)\end{pmatrix}\cdot\begin{pmatrix}|Z_0|\exp(i\gamma) & 0\\ 0 & |Z_0|\exp(i\gamma)\end{pmatrix}\cdot\begin{pmatrix}\cos(\theta_H) & -\exp(-i\phi_H)\sin(\theta_H)\\ \exp(i\phi_H)\sin(\theta_H) & \cos(\theta_H)\end{pmatrix}^\dagger, \tag{23}$$

where $\gamma\equiv\arg(Z_0)$. An interesting observation concerning the input-output behaviour of the transfer properties of an isotropic 1-D earth is obtained from the relation $\mathbf{Z}_I|\bar{h}_i\rangle=Z_0|\bar{e}_i\rangle$. From Eq. (23), it is evident that an input magnetic field polarization state described by the polarization parameters (θ_H, ϕ_H) is transformed by $\mathbf{Z}_I$ into an output electric field vector scaled by Z_0 with the polarization parameters $\theta_E=\pi/2-\theta_H$ and $\phi_E=\pi-\phi_H$. In terms of the elliptic parameters, an isotropic 1-D earth carries an input magnetic field specified by (ψ_H, λ_H) into an output electric field characterized by $(\psi_E=\psi_H+\pi/2, \lambda_E=\lambda_H)$ with the electric field vector scaled by $|Z_0|$ and phase shifted by γ along the polarization ellipse. In other words, the input magnetic field and the corresponding output electric field for a 1-D earth have polarization ellipses which have the same ellipticity and the same sense of rotation, but the major axis of the magnetic field polarization ellipse is perpendicular to the major axis of the electric field polarization ellipse. Recalling that two polarization states are orthogonal if and only if their associated ellipses of polarization possess equal ellipticities, opposite senses of rotation and mutually perpendicular major axes, it should be noted that the output electric field state is, in general, not orthogonal to the input magnetic field state for a 1-D earth. It is only for the special case of a linearly polarized input magnetic field state that mutual orthogonality exists between it and the corresponding output electric field state.

Example 2 (Conductivity distributions possessing a vertical plane of mirror symmetry). This class of conductivity distributions includes the symmetric structures and the 2-D structures (Fischer, 1975). The impedance tensor $\mathbf{Z}_{II}$ corresponding to such structures is traceless and can be anti-diagonalized into the form

$$\mathbf{Z}'_{II}(\psi_0)=\begin{pmatrix}0 & Z_1\\ Z_2 & 0\end{pmatrix}$$

by rotating $\mathbf{Z}_{II}$ with some real rotation angle ψ_0. Although a traceless tensor is parametrized by six parameters, it should be noted that since $\mathbf{Z}_{II}$ can be anti-diagonalized by a pure rotation operation, it must necessarily also be constrained as $\arg(Z_{xy}+Z_{yx})=\arg(Z_{xx}-Z_{yy})$. Hence, $\mathbf{Z}_{II}$ can be parametrized by only five parameters. Accordingly, it is straightforward to show that the canonical decomposition for $\mathbf{Z}_{II}$ is given by

$$\mathbf{Z}_{II}=\begin{pmatrix}\cos\psi_0 & \sin\psi_0\\ -\sin\psi_0 & \cos\psi_0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}|Z_1|\exp(i\gamma_1) & 0\\ 0 & |Z_2|\exp(i\gamma_2)\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\psi_0) & -\sin(\psi_0)\\ \sin(\psi_0) & \cos(\psi_0)\end{pmatrix}, \tag{24}$$

where $\gamma_1\equiv\arg(Z_1)$ and $\gamma_2\equiv\arg(Z_2)+\pi$. Equation (24) should be compared with the elliptic parameters formulation of the canonical decomposition for $\mathbf{Z}$ embodied in Proposition 2. This result contains the following information: (1) the maximum and minimum principal impedances are Z_1 and Z_2, respectively, which coincide with the principal impedances extracted by the conventional impedance tensor method; (2) the principal magnetic field and electric field polarization states are linearly polarized $(\lambda_H=\lambda_E=0)$ along the axes of the conventionally defined principal coordinate system (i.e. along the strike-dip coordinate system for the earth structure); (3) the input principal magnetic field states are orthogonal to the corresponding output electric field states, viz. $\langle\bar{e}_i|\bar{h}_i\rangle=0$ for $i=1, 2$; (4) the canonical decomposition implies that $\mathbf{Z}_{II}$ can be rotated into the usual anti-diagonal form for some real rotation angle ψ_0 since Eq. (24) implies

$$\mathbf{R}(-\psi_0)\,\mathbf{Z}_{II}\,\mathbf{R}(\psi_0)=\begin{pmatrix}0 & Z_1\\ Z_2 & 0\end{pmatrix}=\mathbf{Z}'_{II}(\psi_0).$$

Example 3 (Impedance tensors invariant under a rotation operation). These impedence tensors correspond to conductivity distributions that are completely symmetric about some vertical z-axis (e.g. a sphere or a vertical cylinder embedded in either a homogeneous or horizontally layered earth) (Spitz, 1985). In this case, the impedance tensor verifies the conditions $Z_{xy}=-Z_{yx}=Z_1$ and $Z_{xx}=Z_{yy}=Z_2$ and, hence, assumes the form

$$\mathbf{Z}_s=\begin{pmatrix}Z_2 & Z_1\\ -Z_1 & Z_2\end{pmatrix}.$$

It is clear that $\mathbf{Z}_S$ commutes with the rotation operator $\mathbf{R}(\psi)$ [i.e. $\mathbf{R}(\psi)\,\mathbf{Z}_S=\mathbf{Z}_S\,\mathbf{R}(\psi)$] and, consequently, the earth medium corresponding to $\mathbf{Z}_S$ must be transparent to the azimuthal or orientation angle of the input magnetic and output electric fied polarization states. Only the ellipticity angle of such states should be affected by the conductivity structure and, with this insight, it is intuitively clear that the proper elliptic basis for the representation of the input magnetic and output electric field vectors are the left- and right-circularly polarized states. The left- and right-circular basis states correspond to the most natural description of the rotational symmetry of the conductivity distribution about a vertical z-axis. The canonical decomposition for $\mathbf{Z}_S$ assumes the form

$$\mathbf{Z}_S=\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ i/\sqrt{2} & -i/\sqrt{2}\end{pmatrix}\begin{pmatrix}Z_+ & 0\\ 0 & Z_-\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -i/\sqrt{2}\\ 1/\sqrt{2} & i/\sqrt{2}\end{pmatrix}$$

and supports our intuition of choosing the circular basis states as the natural mode of description for the electric and magnetic fields. Here $Z_\pm=Z_2\pm iZ_1$ are the maximum and minimum principal impedances associated with $\mathbf{Z}_S$.

Example 4 (General three-dimensional conductivity structures). An arbitrary three-dimensional conductivity configuration is characterized by a full impedance tensor without any inherent constraints between any of its elements. The canonical decomposition of an arbitrary $\mathbf{Z}$ can be computed by first determining the SVD using the complex version of an algorithm due to Golub and Reinsch implemented in LINPACK (Dongarra et al., 1979). However, since $\mathbf{Z}$ is a 2×2 matrix, an explicit expression for the parameters of $\mathbf{Z}$ can be obtained. After some lengthy calculations, the parameters of $\mathbf{Z}$ as contained in Proposition 1 are given as follows [for details of the calculation involved, see Yee (1985)]. The squared principal impedance moduli are given by

$$\sigma_1^2=\tfrac{1}{2}\|\mathbf{Z}\|_F^2+\{(\tfrac{1}{2}\|\mathbf{Z}\|_F^2)^2-|\det(\mathbf{Z})|^2\}^{1/2} \tag{25a}$$

and

$$\sigma_2^2=\tfrac{1}{2}\|\mathbf{Z}\|_F^2-\{(\tfrac{1}{2}\|\mathbf{Z}\|_F^2)^2-|\det(\mathbf{Z})|^2\}^{1/2}, \tag{25b}$$

where

$$\|\mathbf{Z}\|_F=(|Z_{xx}|^2+|Z_{xy}|^2+|Z_{yx}|^2+|Z_{yy}|^2)^{1/2}=[\mathrm{tr}(\mathbf{Z}^\dagger\mathbf{Z})]^{1/2}$$

is the Frobenius norm of $\mathbf{Z}$ and

$$\det(\mathbf{Z})=Z_{xx}Z_{yy}-Z_{xy}Z_{yx}.$$

The polarization parameters θ_H and ϕ_H that specify the principal magnetic field states are computed from

$$\tan(\theta_H)=\frac{|Z_{xx}Z_{xy}^*+Z_{yx}Z_{yy}^*|}{\sigma_1^2-(|Z_{xy}|^2+|Z_{yy}|^2)} \tag{25c}$$

and

$$\phi_H=\arg(Z_{xx}Z_{xy}^*+Z_{yx}Z_{yy}^*). \tag{25d}$$

Finally, the polarization parameters θ_E and ϕ_E that specify the principal electric field states and the principal phases γ_1 and γ_2 are determined from

$$\tan(\theta_E)=\left|\frac{Z_{yx}\cos(\theta_H)+Z_{yy}\exp(i\phi_H)\sin(\theta_H)}{Z_{xx}\cos(\theta_H)+Z_{xy}\exp(i\phi_H)\sin(\theta_H)}\right|, \tag{25e}$$

$$\gamma_1=\arg[Z_{xx}\cos(\theta_H)+Z_{xy}\exp(i\phi_H)\sin(\theta_H)], \tag{25f}$$

$$\phi_E=\arg[Z_{yx}\cos(\theta_H)+Z_{yy}\exp(i\phi_H)\sin(\theta_H)]-\gamma_1 \tag{25g}$$

and

$$\gamma_2=\arg[Z_{yy}\cos(\theta_H)-Z_{yx}\exp(-i\phi_H)\sin(\theta_H)]. \tag{25h}$$

Note that the principal magnetic and electric field vectors are general elliptical states for the arbitrary 3-D structure. Indeed, unlike the 1-D and 2-D structures considered in the preceding examples, the direction of the major axis of the principal magnetic field polarization ellipse need no longer be perpendicular to the direction of the major axis of the corresponding principal electric field polarization ellipse. It is important to emphasize that the principal impedance moduli σ_1 and σ_2 depend only on $\|\mathbf{Z}\|_F$ and $\det(\mathbf{Z})$, two quantities which remain invariant under any similarity tranformation of $\mathbf{Z}$ and, in particular, under both rotation and ellipticity transformations. Consequently, the maximum and minimum principal apparent resistivities, which are the two quantities most amenable to physical insight and interpretation, are polarization-invariant and, hence, are more apt to reflect the properties of the earth's geoelectric medium since they are independent of the coordinate system selected to measure $\mathbf{Z}$. Furthermore, note that the principal impedance moduli coincide if and only if the impedance tensor verifies the condition

$$\|\mathbf{Z}\|_F^2=4|\det(\mathbf{Z})|^2.$$

Relationship to conventional analysis

In the conventional analysis of the MT impedance tensor, the principal-axis directions of $\mathbf{Z}$ are obtained from the rotation properties of the tensor. This approach was first formulated by Sims and Bostick (1969) and later described by Vozoff (1972). In this approach, the off-diagonal elements of a suitably rotated impedance tensor are the basic parameters used in the quantitative interpretation while two additional indicators, namely the skew index α and the ellipticity index β, are introduced to provide semi-quantitative measures of the three-dimensional nature of $\mathbf{Z}$.

The comparison between the canonical analysis and the conventional analysis is most easily made with respect to the elliptic parameters formulation of the canonical decomposition as embodied in Proposition 2. As already indicated earlier, the impedance tensor can be diagonalized [cf. Eq. (22)] by pre-multiplying and post-multiplying $\mathbf{Z}$ by $\mathbf{S}(\psi, \lambda)$ [cf. Eq. (21)]. Along this vein, it is convenient at this point to study in more detail the structure of $\mathbf{S}(\psi, \lambda)$. Observe that $\mathbf{S}(\psi, \lambda)$ is the product of two unitary unimodular matrices, namely $\mathbf{R}(-\psi)$ and $\mathbf{P}(\lambda)$, whose operation on some arbitrary polarization state $|\chi\rangle$ can be described as follows. Although in conventional analysis $\mathbf{R}(-\psi)$ is interpreted as a rotation operator which implements a clockwise rotation of a coordinate system about some fixed vertical z-axis through the angle ψ (passive rotation), it is more natural in the present context to interpret $\mathbf{R}(-\psi)$ as an operator that rotates the polarization ellipse of the state $|\chi\rangle$ counter-clockwise about its centre through the angle ψ without altering the ellipticity angle of the state (active rotation). In other words, $\mathbf{R}(-\psi)$ changes the orientation angle for the state $|\chi\rangle$ without affecting the ellipticity angle. From the same point of view, the ellipticity operator $\mathbf{P}(\lambda)$ can be interpreted as that operator whose action on $|\chi\rangle$ alters the ellipticity angle of the associated polarization ellipse by λ without affecting the orientation angle. Hence, if $|\chi\rangle$ is characterized by elliptic parameters ψ_χ and λ_χ, the action of $\mathbf{S}(\psi, \lambda)$ [cf. Eq. (21)] on $|\chi\rangle$ [i.e. $\mathbf{S}(\psi, \lambda)|\chi\rangle$] results in a polarization state whose elliptic parameters are described by $\psi_\chi+\psi$ and $\lambda_\chi+\lambda$.

In view of this, the essential difference between the canonical and conventional analysis of $\mathbf{Z}$ can be clearly seen from Eq. (22). In the conventional analysis, we consider only the rotation properties of the tensor and attempt to select a rotation angle ψ such that the rotated impedance tensor approximates an anti-diagonal form in some optimum manner. Indeed, for two-dimensional (2-D) conductivity distributions, there exists a real angle ψ such that $\mathbf{Z}$ in the rotated coordinate systems is anti-diagonal, viz.

$$\mathbf{R}^\dagger(\psi)\,\mathbf{Z}\mathbf{R}(\psi)=\begin{pmatrix}0 & Z_1\\ Z_2 & 0\end{pmatrix}. \tag{26}$$

However, this procedure breaks down for three-dimensional (3-D) conductivity distributions and in such cases reduces to no more than a rather *ad hoc* approximation procedure. The reason for this is clear from Eq. (22) which suggests that a principal or intrinsic coordinate system in which $\mathbf{Z}$ assumes a simple decoupled form cannot be obtained by restricting attention to only pure rotation operations. For general 3-D conductivity structures, it is necessary to apply both rotation and ellipticity transformations on $\mathbf{Z}$ in order to secure a pair of principal impedances. We emphasize that the rotation and ellipticity operation need not be symmetrical from the left and right, viz. $\mathbf{S}(\psi_E, \lambda_E)$ need not be the same transformation as $\mathbf{S}(\psi_H, \lambda_H)$ since the principal electric and magnetic field elliptic parameters are not identical in general. After all, in the principal coordinate frame in which the impedance tensor assumes a simple decoupled form, the input (magnetic field) and output (electric field) of the earth system need not be expressed relative to the same pair of basis states. The conventional analysis is restrictive in that it only considers orthogonal linear basis states for the expression of the magnetic and electric fields and, moreover, assumes that the input (magnetic field) and output (electric field) must be observed relative to the same pair of linear basis states.

We note further that Eq. (22) subsumes Eq. (26) as a special case and indeed for 2-D conductivity distributions, it has already been shown (cf. Example 2) that the principal impedances and the principal coordinate system obtained in the canonical analysis are identical to those obtained from the conventional analysis. It is interesting to note that for 2-D conductivity structures which possess well-defined longitudinal (strike) and transversal (dip) directions (i.e. horizontal directions of symmetry), canonical decomposition reflects these circumstances by yielding orthogonal linearly polarized basis states as the principal states. However, observe that for rotationally invariant conductivity structures (cf. Example 3) which do not possess well-defined horizontal directions of symmetry, the canonical decomposition reflects this situation in a natural fashion by yielding the orthogonal left- and right-circularly polarized states as the principal states. In this context, it is important to emphasize that canonical decomposition extends the conventional MT analysis, so far as it is valid for the special structures, in a completely natural manner to accommodate the most general conductivity structures. Indeed, the principal states that are selected in the canonical decomposition are the ones that are most descriptive of the geometrical configuration of the underlying conductivity distribution. In particular, if the electric field is observed relative to two orthogonal basis states specified by (ψ_E, λ_E) and $(\psi_E+\pi/2, -\lambda_E)$ and if the magnetic field is observed relative to two orthogonal basis states specified by (ψ_H, λ_H) and $(\psi_H+\pi/2, -\lambda_H)$, then the resulting impedance tensor assumes a simple diagonal form and, in this sense, constitutes the most natural description for $\mathbf{Z}$.

Relationship to maximum coherency analysis

The application of maximum coherency analysis to the determination of a principal coordinate system for $\mathbf{Z}$ was proposed by Reddy and Rankin (1974) and involves rotating the coordinate system of the magnetic field until the coherency between some horizontal component of the magnetic field and the corresponding orthogonal horizontal component of the electric field attains its maximum. For 2-D conductivity structures, it is known that an input magnetic field of fixed power that is linearly polarized along the strike of the structure results in a maximum electric field response (as measured by its power) that is linearly polarized perpendicular to the strike direction. It is clear that for this case, maximum coherency analysis can be utilized to ascertain the principal coordinate system for $\mathbf{Z}$ and indeed, in this coordinate system, there is a complete linear relationship between components of the electric and magnetic field that are respectively linearly polarized along and perpendicular to the strike of the 2-D structure.

As in the conventional analysis, the maximum coherency analysis considers only linearly polarized fields when computing the coherency between the electric and magnetic fields. While this is sufficient for 2-D structures, it is inadequate for the analysis of 3-D structures. For such structures, we consider the generalization of the maximum coherency analysis to encompass general elliptically polarized electric and magnetic fields and in so doing we will demonstrate that such a generalization leads essentially to canonical decomposition.

Firstly, we note that an input magnetic field polarization state of amplitude A_H and absolute phase δ_H, characterized by the complex polarization ratio P_H, may be represented by

$$|H\rangle = \frac{A_{HC}}{(1+|P_H|^2)^{1/2}} \begin{pmatrix} 1 \\ P_H \end{pmatrix},$$

where $A_{HC} \equiv A_H \exp(i\delta_H)$ is the complex magnetic field amplitude. Recall that the complex polarization ratio of some polarization state is defined as the ratio of the two orthogonal oscillating components which determine the state relative to some orthonormal basis. If $|H\rangle$ serves as input to an earth system characterized by the impedance tensor $\mathbf{Z}$, the output electric field is given by

$$|E\rangle = \frac{A_{EC}}{(1+|P_E|^2)^{1/2}} \begin{pmatrix} 1 \\ P_E \end{pmatrix} = \frac{A_H \exp(i\delta_H)}{(1+|P_H|^2)^{1/2}} \begin{pmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{pmatrix} \begin{pmatrix} 1 \\ P_H \end{pmatrix},$$

which implies that

$$P_E = \frac{Z_{yx}+Z_{yy}P_H}{Z_{xx}+Z_{xy}P_H} \tag{27a}$$

and

$$A_{EC} \equiv A_E \exp(i\delta_E) = \left(\frac{1+|P_E|^2}{1+|P_H|^2}\right)^{1/2} (Z_{xx}+Z_{xy}P_H)\,A_{HC}. \tag{27b}$$

From Eqs. (27a) and (27b), it follows that the power or intensity of the output electric field response for an input magnetic field of unit amplitude (i.e. $A_H=1$) may be expressed as

$$I_E = \langle E|E\rangle = |A_{EC}|^2 = \frac{|Z_{xx}+Z_{xy}P_H|^2 + |Z_{yx}+Z_{yy}P_H|^2}{1+|P_H|^2}. \tag{28}$$

Now, let us consider the problem of finding the input magnetic field polarization states of unit amplitude that result in the minimum and maximum electric field response as measured in terms of its power or intensity. To solve this problem, let us write $P_H = x+iy$ and express the numerator and denominator of I_E as contained in Eq. (28) in terms of x and y. This yields

$$1+|P_H|^2 = 1+x^2+y^2 \tag{29a}$$

and

$$|Z_{xx}+Z_{xy}P_H|^2 + |Z_{yx}+Z_{yy}P_H|^2 = a(x^2+y^2)+2bx+2cy+d, \tag{29b}$$

where

$$a=|Z_{xy}|^2+|Z_{yy}|^2, \tag{29c}$$

$$b=Re[Z_{xx}^* Z_{xy}+Z_{yx}^* Z_{yy}], \tag{29d}$$

$$c=Re[i(Z_{xx}^* Z_{xy}+Z_{yx}^* Z_{yy})] \tag{29e}$$

and

$$d=|Z_{xx}|^2+|Z_{yx}|^2. \tag{29f}$$

If we substitute Eqs. (29a) and (29b) into Eq. (28), we obtain a second degree equation in x and y of the form

$$a(x^2+y^2)+2bx+2cy+d-I_E(1+x^2+y^2)=0. \tag{30}$$

Now, if we consider I_E to be a fixed constant in Eq. (30), it follows that the locus of input magnetic field states of unit amplitude that result in a given constant output electric field intensity describes a circle in the complex polarization plane (i.e. the plane obtained by associating the polarization ratios of the possible states of polarization with the points of the complex plane). That this is so is easily seen by writing Eq. (30) in the form

$$\left[x-\frac{b}{I_E-a}\right]^2+\left[y-\frac{c}{I_E-a}\right]^2=\frac{1}{I_E-a}\left[\frac{b^2+c^2}{I_E-a}-(I_E-d)\right] \tag{31a}$$

which describes a circle with centre at

$$(x_c, y_c)=\left(\frac{b}{I_E-a}, \frac{c}{I_E-a}\right) \tag{31b}$$

and radius

$$r=\frac{1}{I_E-a}\cdot[b^2+c^2-(I_E-d)(I_E-a)]^{1/2}. \tag{31c}$$

Now, we observe that Eq. (31a) describes a real circle provided $r>0$; viz., in order for Eq. (31a) to specify a real circle, the value of I_E must be restricted to a certain range that is consistent with the positivity constraint $r \geqq 0$. The values of I_E that result in $r=0$ constitute the extreme points of this admissible range and, hence, provide the maximum and minimum electric field power outputs of the earth system for input unit amplitude magnetic field states. Setting $r=0$ in Eq. (31c) yields

$$I_E^{\max}=\tfrac{1}{2}[(a+d)+\{(a+d)^2-4[ad-(b^2+c^2)]\}^{1/2}] \tag{32a}$$

and

$$I_E^{\min}=\tfrac{1}{2}[(a+d)-\{(a+d)^2-4[ad-(b^2+c^2)]\}^{1/2}]. \tag{32b}$$

Observing that [cf. Eqs. (29c)-(29f)]

$$a+d=\|\mathbf{Z}\|_F^2$$

and

$$ad-(b^2+c^2)=|\det(\mathbf{Z})|^2,$$

it follows that the maximum and minimum output electric field intensities are given by

$$I_E^{\max}, I_E^{\min}=\tfrac{1}{2}\|\mathbf{Z}\|_F^2 \pm \{(\tfrac{1}{2}\|\mathbf{Z}\|_F^2)^2-|\det(\mathbf{Z})|^2\}^{1/2}. \tag{33a}$$

A comparison of Eq. (14a) with Eqs. (4a) and (4b) shows that

$$(I_E^{\max}, I_E^{\min})=(\sigma_1^2, \sigma_2^2), \tag{33b}$$

so that the maximum and minimum output electric field intensities for input magnetic field states of unit intensity coincide, respectively, with the maximum and minimum principal impedance moduli squared.

For I_E equal to either the maximum or minimum output electric field intensity, the circle of intensity transmittance specified by Eq. (31a) has radius zero so that the centre of the circle must determine the polarization state of the input magnetic field that provides the corresponding extremum in intensity. Hence, the input magnetic field polarization states that result in the maximum and minimum output electric field intensities may be obtained from Eq. (31b) by setting $I_E=\sigma_1^2$ and $I_E=\sigma_2^2$, respectively. This results in

$$P_H^{\max}=\frac{b+ic}{\sigma_1^2-a} \tag{34a}$$

and

$$P_H^{\min}=\frac{b+ic}{\sigma_2^2-a}. \tag{34b}$$

Now by virture of Eqs. (29d) and (29e),

$$b+ic=Z_{xx}Z_{xy}^*+Z_{yx}Z_{yy}^*,$$

which on insertion into Eq. (34a) together with Eq. (29c) yields the results of Eqs. (25c) and (25d) since

$$P_H^{\max}\equiv \tan(\theta_H)\exp(i\phi_H).$$

Of course, $P_H^{\max}$ and $P_H^{\min}$ are the complex polarization ratios of the principal magnetic field polarization states extracted in the canonical decomposition and, as such, describe orthogonal states. This fact may be independently verified by observing that by virtue of Eqs. (32a), (32b) and (33b),

$$(\sigma_1^2-a)(\sigma_2^2-a)=-(b^2+c^2),$$

which, in conjunction with Eqs. (34a) and (34b) results in

$$P_H^{\min}(P_H^{\max})^*=\frac{b^2+c^2}{(\sigma_1^2-1)(\sigma_2^2-a)}=-1.$$

The output electric field polarization states corresponding to $P_H^{\min}$ and $P_H^{\max}$ are the electric field states possessing minimum and maximum power and may be obtained by substituting for P_H from Eqs. (34a) and (34b) into Eq. (27a). After some algebra, this yields the results of Eqs. (25e)-(25h). Hence, the structure parameters obtained from the canonical decomposition of $\mathbf{Z}$ can also be extracted from a purely physical point of view by considering the electric field intensity transmittance from the earth system as a function of the polarization state of a variable input test magnetic field of unit amplitude. This, in essence, represents the generalization of the maximum coherency analysis to encompass elliptically polarized electric and magnetic field states and also shows that the generalized maximum coherency analysis is essentially equivalent to canonical decomposition. Note that the generalized maximum coherency analysis reduces to the usual maximum coherency analysis for 2-D conductivity structures.

Relationship to associate and conjugate directions analysis

The method of associate and conjugate directions developed by Counil et al. (1986) can be considered to be a partial generalization of the maximum coherency analysis of Reddy and Rankin (1974). Whereas the method of maximum coherency considers only linearly polarized states in both the output electric and input magnetic field spaces in the selection of principal directions, the method of associate and conjugate directions considers linearly polarized states in either the output electric or input magnetic field spaces in the determination of principal directions. In this context, the method of associate and conjugate directions occupies an intermediate position between the maximum coherency analysis of Reddy and Rankin and the generalized maximum coherency analysis which was shown to be equivalent to canonical decomposition. As a consequence, the procedure utilized in the previous section to relate maximum coherency analysis to canonical decomposition may be applied to develop the relationship between the associate and conjugate directions analysis and canonical decomposition.

Towards this objective, we first consider the directions of maximum and minimum current which are defined to be the real directions (i.e. characterized by linear polarizations) in the output electric field space that provide the maximum and minimum electric field response as gauged in terms of the intensity of the electric field for an arbitrary (i.e. elliptically polarized) magnetic field of unit intensity. Since the output electric field is constrained to be linearly polarized, its polarization ratio must necessarily assume the form $P_E = \tan(\psi_c)$ where ψ_c is the orientation angle (azimuth) of the linear polarization. In view of Eq. (27a), the corresponding input magnetic field is characterized by the complex magnetic field polarizaton ratio

$$P_H = \frac{Z_{xx}\tan(\psi_c) - Z_{yx}}{-Z_{xy}\tan(\psi_c) + Z_{yy}}. \tag{35}$$

Now, insertion of Eq. (35) into Eq. (28) leads to the following expression for the intensity of the linearly polarized electric field response corresponding to an elliptically polarized magnetic field of unit intensity:

$$I_E = \langle E|E\rangle = \frac{|Z_{xx}Z_{yy} - Z_{xy}Z_{yz}|[1+\tan^2(\psi_c)]}{l_1\tan^2(\psi_c) - l_2\tan(\psi_c) + l_3}, \tag{36a}$$

where

$$l_1 = |Z_{xy}|^2 + |Z_{xx}|^2, \tag{36b}$$

$$l_2 = 2\,\mathrm{Re}\,[Z_{xx}^* Z_{yx} + Z_{yy} Z_{xy}^*], \tag{36c}$$

and

$$l_3 = |Z_{yy}|^2 + |Z_{yx}|^2. \tag{36d}$$

Differentiating I_E with respect to ψ_c [cf. Eq. (36a)] and setting the result to zero gives the condition

$$\tan(2\psi_c) = \frac{l_2}{l_1 - l_3} = \frac{2\,\mathrm{Re}\,[Z_{xx}^* Z_{yx} + Z_{yy} Z_{xy}^*]}{|Z_{xy}|^2 + |Z_{xx}|^2 - |Z_{yx}|^2 - |Z_{yy}|^2} \tag{37}$$

whose solution defines the directions of maximum and minimum current. It is important to note that Eq. (37) coincides with Eq. (19) of Counil et al. (1986).

Utilizing the same methodology, let us next consider the determination of the directions of maximum and minimum induction which are defined to be the real directions (i.e. characterized by linear polarizations) in the input magnetic field space that result in the maximum and minimum intensities of the electric field response for a linearly polarized magnetic field of unit intensity. Since the magnetic field is constrained to be a linear polarization, its polarization ratio is given by $P_H = \tan(\psi_i)$ where ψ_i is the azimuth of the linear polarization. Insertion of this value of P_H into Eq. (28) yields the intensity of the output electric field response for a linearly polarized input magnetic field of unit intensity, i.e.

$$I_E = \langle E|E\rangle = \frac{k_1\tan^2(\psi_i) + k_2\tan(\psi_i) + k_3}{1+\tan^2(\psi_i)}, \tag{38a}$$

where

$$k_1 = |Z_{xy}|^2 + |Z_{yy}|^2, \tag{38b}$$

$$k_2 = 2\,\mathrm{Re}\,[Z_{xx} Z_{xy}^* + Z_{yx} Z_{yy}^*], \tag{38c}$$

and

$$k_3 = |Z_{xx}|^2 + |Z_{yx}|^2. \tag{38d}$$

The directions of maximum and minimum induction can be obtained by differentiating I_E [cf. Eq. (38a)] with respect to ψ_i and setting the derivative equal to zero to give

$$\tan(2\psi_i) = \frac{k_2}{k_3 - k_1} = \frac{2\,\mathrm{Re}\,[Z_{xx} Z_{xy}^* + Z_{yx} Z_{yy}^*]}{|Z_{xx}|^2 + |Z_{yx}|^2 - |Z_{xy}|^2 - |Z_{yy}|^2}. \tag{39}$$

We point out that Eq. (39) is identical to Eq. (23) of Counil et al. (1986).

The technique used to derive Eqs. (37) and (39) can be exploited further to relate the responses (impedances) corresponding to the directions of maximum and minimum current and induction to the principal impedances. Along this vein, first observe that the maximum and minimum principal apparent resistivities $\bar{\rho}_a$ and $\underline{\rho}_a$ can be computed from

$$\bar{\rho}_a = \frac{1}{\omega\mu_0} I_E^{\max} \tag{40a}$$

and

$$\underline{\rho}_a = \frac{1}{\omega\mu_0} I_E^{\min}, \tag{40b}$$

where $I_E^{\max}$ and $I_E^{\min}$ are the maximum and minimum electric field intensities for an arbitrary input magnetic field of unit intensity [cf. Eqs. (32) and (33)]. In this context, the generalized apparent resistivity [cf. Eq. (9)] associated with an input magnetic field of unit intensity that is characterized by the complex polarization ratio P_H can then be expressed as

$$\rho_a = \frac{1}{\omega\mu_0} I_E, \tag{40c}$$

where I_E is the electric field intensity transmittance for the given magnetic field input, determined as per Eq. (28). Now, in light of Eqs. (28), (32), (33) and (34), the generalized apparent resistivity of Eq. (40c) can be written in terms of the maximum and minimum principal apparent resistivities of

Eqs. (40a) and (40b) as

$$\rho_a = \frac{m_1}{m_1+m_2}\,\underline{\rho}_a + \frac{m_2}{m_1+m_2}\,\bar{\rho}_a, \tag{41a}$$

where

$$m_1 = |P_H^{\max}|/|P_H^{\min}| \tag{41b}$$

and

$$m_2 = |P_H - P_H^{\max}|/|P_H - P_H^{\min}|. \tag{41c}$$

An immediate physical interpretation of this result is obtained by observing that Eq. (41) expresses the generalized apparent resistivity in terms of the maximum and minimum principal apparent resistivities. The weights $m_1/(m_1+m_2)$ and $m_2/(m_1+m_2)$ correspond to the proportions of the intensity of the input magnetic field state with polarization P_H contained in the two orthogonal polarizations $P_H^{\min}$ and $P_H^{\max}$. In particular, if we set P_H in Eq. (41) equal to the value given by Eq. (35) with ψ_c determined as per Eq. (37), then the generalized apparent resistivities of Eq. (41a) reduce to the maximum and minimum current apparent resistivities ρ_a^{cM} and ρ_a^{cm} as defined in Counil et al. (1986). Hence, we have succeeded in expressing the extremal current apparent resistivities in terms of the maximum and minimum principal apparent resistivities. Similarly, if we let $P_H = \tan(\psi_i)$ in Eq. (41c) with ψ_i determined according to Eq. (39), then the generalized apparent resistivities of Eq. (41a) coincide with the maximum and minimum induction apparent resistivities ρ_a^{iM} and ρ_a^{im}. Note that the apparent resistivities of maximum and minimum current and induction must be bounded above and below by the maximum and minimum principal apparent resistivities, respectively.

Relationship to Eggers' eigenstate analysis

Eggers (1982) proposed an eigenstate formulation for the extraction of physically meaningful scalar parameters from the impedance tensor **Z** that is similar in philosophy to canonical decomposition. Eggers' eigenstate analyis is based on the concept of the generalized eigenproblem. His principal impedances are defined to be the eigenvalues γ^i of the matrix pencil $(\mathbf{Z}-\gamma^i\mathbf{J})$ where **J** is the skew-symmetric matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These eigenvalues are determined from the determinantal equation

$$\det(\mathbf{Z}-\gamma^i\mathbf{J}) = 0. \tag{42}$$

Hence, Eggers' principal impedances are given by

$$\gamma^{1,2} = Z_1 \pm [Z_1^2 - \det(\mathbf{Z})]^{1/2}, \tag{43a}$$

where

$$Z_1 \equiv \tfrac{1}{2}(Z_{xy} - Z_{yx}). \tag{43b}$$

Eggers' magnetic field eigenstates $|H^i\rangle$ verify

$$(\mathbf{Z}-\gamma^i\mathbf{J})|H^i\rangle = 0 \quad (i=1,2)$$

and are found to be

$$|H^i\rangle = \begin{pmatrix} \gamma^i - Z_{xy} \\ Z_{xx} \end{pmatrix} \quad (i=1,2). \tag{43c}$$

The corresponding Eggers' electric field eigenstates $|E^i\rangle$ are determined from

$$|E^i\rangle = \gamma^i\mathbf{J}|H^i\rangle = \gamma^i\begin{pmatrix} Z_{xx} \\ Z_{xy}-\gamma^i \end{pmatrix} \quad (i=1,2). \tag{43d}$$

Eggers' principal impedances differ from the principal impedances extracted by application of the canonical decomposition. This can be seen by comparing Eq. (43a) with Eq. (25). It is only in the special cases of 1-D and 2-D conductivity structures that the Eggers' and the canonical principal impedances coincide. Also, from Eq. (43a), it is clear that since γ^1 and γ^2 depend only on Z_1 and $\det(\mathbf{Z})$, Eggers' principal impedances are invariant under a rotation transformation. However, since Z_1 is not invariant under an ellipticity transformation [cf. Eq. (21)], it is evident that Eggers' principal impedances are not truly polarization-invariant quantities. This implies that the corresponding Eggers' principal apparent resistivities determined as $\rho_a^i = |\gamma^i|^2/\omega\mu_0$ (parameters which are most amenable to physical insight) are not polarization-invariant quantities. However, the canonically determined principal apparent resistivities are invariant under both rotation and ellipticity transformations since they depend only on $\|\mathbf{Z}\|_F$ and $|\det(\mathbf{Z})|$ and hence are true polarization-invariant parameters. In this aspect then, the canonical principal impedances are more fundamentally related to the conducting structure than are Eggers' principal impedances.

We note that for a magnetic field input coinciding with Eggers' magnetic field eigenstate [cf. Eq. (43b)], the generalized apparent resistivity simplifies to

$$\rho_a \equiv \frac{1}{\omega\mu_0}\frac{\langle E|E\rangle}{\langle H|H\rangle} = \frac{1}{\omega\mu_0}\frac{\langle E^i|E^i\rangle}{\langle H^i|H^i\rangle} = \frac{|\gamma^i|^2}{\omega\mu_0} = \rho_a^i. \tag{44}$$

The preceding result shows that the generalized apparent resistivity reduces to an Eggers' principal apparent resistivity whenever the magnetic field input to the earth system coincides with one of the Eggers' magnetic field eigenstates $|H^i\rangle$. Combining this observation with the fact that the canonical principal apparent resistivities constitute the maximum and minimum values for the generalized apparent resistivity, it follows that

$$\underline{\rho}_a(\omega) \equiv \frac{\sigma_2^2}{\omega\mu_0} \leqq \rho_a^i \equiv \frac{|\gamma^i|^2}{\omega\mu_0} \leqq \bar{\rho}_a(\omega) \equiv \frac{\sigma_1^2}{\omega\mu_0} \quad (i=1,2).$$

Hence, Eggers' principal apparent resistivities are bounded above and below by the canonical maximum and minimum principal apparent resistivities. It can be shown (Yee, 1985) that the canonical principal impedances coincide with the Eggers' principal impedance if and only if

$$\bar{\mathbf{U}} = \mathbf{J}\bar{\mathbf{V}} = \mathbf{R}(\pi/2)\,\bar{\mathbf{V}}, \tag{45}$$

where $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are defined as in Eqs. (17)-(19). This condition is fulfilled whenever the principal electric field polarization states $|\bar{e}_i\rangle$ $(i=1, 2)$ are parallel to the corresponding principal magnetic field polarization states $|\bar{h}_i\rangle$ $(i=1, 2)$ coordinate-rotated through $\pi/2$. This condition is verified for

1-D and 2-D conductivity distributions. For 3-D structures, Eq. (45) is not generally satisfied which, of course, leads to the divergence of the canonical from the Eggers' principal impedances.

The condition of Eq. (45) is due primarily to the imposition of the constraint

$$|E^i\rangle^T \cdot |H^i\rangle = E_x^i H_x^i + E_y^i H_y^i = 0 \qquad (46)$$

in the definition of the Eggers' electric and magnetic field eigenstates. This constraint necessarily forces the magnetic field eigenstate polarization ellipses to be perpendicularly oriented with respect to the corresponding electric field eigenstate polarization ellipses. We remark that this condition does not necessarily hold with regard to the principal electric and magnetic field polarization states extracted from the canonical decomposition of **Z**. Eggers' motivation for introducing the constraint of Eq. (46) arises from the observation that this condition is verified in the case of the transverse electromagnetic (TEM) mode of wave propagation in a homogeneous medium. It is Eggers' contention that a physically more-appealing definition of principal impedances can be attained if one extends the TEM mode relationship to apply to general conductivity structures. While this constraint is valid for 1-D and 2-D conductivity structures, it does not necessarily apply to 3-D conductivity structures. Indeed, the effect of a local electrical conductivity inhomogeneity in what otherwise would be a 2-D conductivity distribution, may result in surface charges whose net effect is to distort the electric field so that it is no longer orthogonal to the magnetic field.

Hence, there is no physical reason why the constraint of Eq. (46) should be imposed on the input magnetic and output electric field states. To assume a priori that this constraint is valid for general conductivity structures (as in Eggers' eigenstate analysis) when there is no physical evidence to support such a relationship is tantamount to ignoring the great majority of possible input magnetic and output electric field polarization states consistent with the structure and focusing attention only on some small and unrepresentative subset of them. For this reason, the Eggers' principal apparent resistivities lie somewhere in the interval bounded above and below by the maximum and minimum canonical principal apparent resistivities, respectively. These latter apparent resistivities are the true absolute maximum and minimum principal resistivities since they are obtained by considering all possible electric and magnetic field states consistent with the conductivity structure and not merely those which verify the somewhat artificial TEM relationship. We emphasize that in the canonical decomposition of **Z**, no *ad hoc* constraints are inroduced and only information embodied in **Z** is utilized to extract physically motivated, highly descriptive structure parameters in a totally natural manner. For this reason, canonical decomposition provides a physically more satifactory set of structure parameters than those obtained through Eggers' eigenstate analysis.

Furthermore, it can be shown (Yee, 1985) that the canonical principal impedances and their associated principal electric and magnetic field polarization states are related to Eggers' principal impedances and their associated Eggers' principal electric and magnetic field eigenstates as follows:

$$\gamma^i = \sigma_j \exp(i\gamma_j) \frac{\langle \bar{h}_j | H^j \rangle}{\langle \bar{e}_j | \tilde{H}^j \rangle} \qquad (j=1, 2). \qquad (47\text{a})$$

Or, equivalently,

$$\sigma_j \exp(i\gamma_j) = \gamma^j \frac{\langle \bar{e}_j | \tilde{H}^j \rangle}{\langle \bar{h}_j | H^j \rangle} = \frac{\langle \bar{e}_j | E^j \rangle}{\langle \bar{h}_j | H^j \rangle} \qquad (j=1, 2), \qquad (47\text{b})$$

where

$$|\tilde{H}^j\rangle \equiv \mathbf{J} |H^j\rangle = \mathbf{R}(\pi/2) |H^j\rangle \qquad (j=1, 2).$$

We note in particular that Eq. (47) implies that the canonical principal impedances coincide with the Eggers' principal impedances if and only if the projection of the Eggers' magnetic field eigenstate $|H^j\rangle$ along the canonical principal magnetic field polarization state $|\bar{h}_j\rangle$ is equal to the projection of the eigenstate $|H^j\rangle$, coordinate-rotated by $\pi/2$, along the canonical principal electric field polarization state $|\bar{e}_j\rangle$. From Eq. (45), we see that this occurs when the canonical principal electric field polarization states are aligned parallel to the canonical principal magnetic field polarization states coordinate-rotated by $\pi/2$. In essence then, Eqs. (47a) and (47b) embody the result of Eq. (45) and in addition indicate how the degree of discrepancy between the canonical and Eggers' principal impedances arises from the amount of misalignment of the canonical principal electric and magnetic field states relative to the Eggers' electric and magnetic field eigenstates.

Relationship to Spitz's rotation analysis

Spitz's rotation analysis (Spitz, 1985) is similar in philosophy to the conventional analysis in that it depends primarily on the rotation properties of the impedance tensor. Spitz utilizes the Cayley factorization of the impedance tensor to construct two analytical rotation angles whose associated intrinsic coordinate systems are more complete than that obtained in the conventional analysis. This is so because each of the rotation angles satisfies the criterion that the off-diagonal elements of the rotated tensor as well as the corresponding rotation angle depend on all eight degrees of freedom in **Z**. To obtain the two analytical rotation angles, Spitz applies the Cayley factorization to express the impedance tensor as $\mathbf{Z} = \mathbf{QU}$ where **Q** is a positive definite Hermitian matrix and **U** is a unitary matrix; the conventional analysis is then applied to the matrix factors **Q** and **U** to construct the rotation angles θ_1 and θ_2, respectively. In general, there is no relationship between θ_1 and θ_2, although it is Spitz's contention that each angle determines an intrinsic coordinate system for **Z**. We argue that since it is impossible to decide which of the two coordinate systems is more appropriate for MT data analysis, these coordinate systems (defined by θ_1 and θ_2) cannot truly be termed intrinsic. After all, an intrinsic coordinate system for **Z** must be uniquely determined by the information in the tensor. As we have already remarked, a principal or intrinsic coordinate system for expression of the impedance tensor requires that we consider both rotation and ellipticity transformations of **Z**. We will now proceed to show that the Cayley factorization of **Z** is an alternative expression of canonical decomposition and as such, all the necessary information for the extraction of a principal coordinate system is already explicitly embedded in the matrix factors. Consequently, there is no need to apply the conventional analysis to each of these factors in order to extract real rotation angles for the construction of intrinsic coordinate systems as in Spitz's rotation analysis.

First, observe that in view of Eqs. (17) and (18), the canonical decomposition of **Z** may be expressed as

$$\mathbf{Z}=\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{V}}^{\dagger}, \tag{48}$$

where $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are unitary unimodular matrices defined in Eq. (19) and

$$\bar{\mathbf{S}}=\begin{pmatrix}\sigma_1 \exp(i\gamma_1) & 0\\ 0 & \sigma_2 \exp(i\gamma_2)\end{pmatrix}$$

is the principal impedance tensor, i.e. the impedance tensor that would be observed in the principal coordinate system obtained by changing the input magnetic and the output electric field basis states from the orthogonal (x, y) linear polarizations (sensor or measurement coordinate system) to the orthogonal elliptic states ($|\bar{h}_1\rangle$, $|\bar{h}_2\rangle$) and ($|\bar{e}_1\rangle$, $|\bar{e}_2\rangle$), respectively. Now let us note that Eq. (48) may be rearrangement to read

$$\mathbf{Z}=\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{V}}^{\dagger}=(\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{U}}^{\dagger})(\bar{\mathbf{U}}\bar{\mathbf{V}}^{\dagger})\equiv\mathbf{QU} \tag{49a}$$

or

$$\mathbf{Z}=\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{V}}^{\dagger}=(\bar{\mathbf{U}}\bar{\mathbf{V}}^{\dagger})(\bar{\mathbf{V}}\bar{\mathbf{S}}\bar{\mathbf{V}}^{\dagger})\equiv\mathbf{UP}, \tag{49b}$$

where $\mathbf{U}\equiv\bar{\mathbf{U}}\bar{\mathbf{V}}^{\dagger}$ is the relative phase matrix, $\mathbf{Q}\equiv\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{U}}^{\dagger}$ is the output (electric field) amplitude matrix and $\mathbf{P}\equiv\bar{\mathbf{V}}\bar{\mathbf{S}}\bar{\mathbf{V}}^{\dagger}$ is the input (magnetic field) amplitude matrix of **Z**. Note that Eq. (49) rearranges the information embodied in canonical decomposition and reassembles this information in the matrix factors of the Cayley factorization. Since $\bar{\mathbf{U}}$ and $\bar{\mathbf{V}}$ are unitary unimodular matrices, it follows then that the phase matrix **U** is also unitary and unimodular. Hence, we may write **U** in the following more suggestive form:

$$\mathbf{U}=\bar{\mathbf{U}}\bar{\mathbf{V}}^{\dagger}=\exp(i\mathbf{T}_r)=\sum_{j=1}^{2}\exp(i\theta_r^j)|s_j\rangle\langle s_j|, \tag{50a}$$

where $\{\exp(i\theta_r^j), j=1, 2\}$ $(\theta_r^1=-\theta_r^2)$ are the eigenvalues of **U**, $\{|s_j\rangle, j=1, 2\}$ is a complete orthonormal system of eigenvectors of **U** and

$$\mathbf{T}_r=\sum_{j=1}^{2}\theta_r^j|s_j\rangle\langle s_j| \tag{50b}$$

is a Hermitian matrix. We refer to $\theta_r^1=-\theta_r^2=\theta_r$ as the relative electric/magnetic field alignment angle. In particular, for 2-D structures we find that $\theta_r=\pi/2$. Furthermore, we note that the relative phase matrix relates the principal magnetic and electric field polarization states as

$$|\bar{e}_j\rangle=\exp(i\mathbf{T}_r)|\bar{h}_j\rangle \quad (j=1, 2). \tag{51}$$

In view of Eq. (50a), we may write Eqs. (49a) and (49b) as

$$\mathbf{Z}=\mathbf{Q}\exp(i\mathbf{T}_r)=\exp(i\mathbf{T}_r)\,\mathbf{P}. \tag{52}$$

Also, since

$$\mathbf{Q}\equiv\bar{\mathbf{U}}\bar{\mathbf{S}}\bar{\mathbf{U}}^{\dagger}=\sum_{j=1}^{2}\sigma_j\exp(i\gamma_j)|\bar{e}_j\rangle\langle\bar{e}_j|$$

and

$$\mathbf{P}\equiv\bar{\mathbf{V}}\bar{\mathbf{S}}\bar{\mathbf{V}}^{\dagger}=\sum_{j=1}^{2}\sigma_j\exp(i\gamma_j)|\bar{h}_j\rangle\langle\bar{h}_j|,$$

it is clear that **Q** incorporates the information concerning the principal impedances and the principal electric field polarization states and that **P** incorporates the information concerning the principal impedances and the principal magnetic field polarization states. The relative phase matrix **U** incorporates the information concerning the relationship between the principal electric and magnetic field polarization states and, as a consequence, couples the input magnetic field space to the output electric field space according to the prescription of Eq. (51).

Hence, we have shown that the Cayley factorization of **Z** repackages the information in canonical decomposition and, consequently, all the information required to construct a principal coordinate system for **Z** is already explicitly embedded in the matrix factors. The Spitz rotation analysis is not needed to extract an intrinsic coordinate system for **Z** once the Cayley factorization is obtained.

Conclusions

The canonical decomposition of **Z** parametrizes the tensor in terms of eight physically relevant real scalar parameters that are suitable for quantitative interpretation. Four of these structure parameters (two moduli and two phases) determine the two principal impedances and, hence, serve to characterize the transfer properties of the earth system. The remaining four parameters are polarization parameters which resolve the principal coordinate system for **Z**; two of these parameters specify the principal electric field polarization states, whereas the remaining two parameters specify the principal magnetic field polarization states. These states constitute the proper basis for the expression of the coordinate systems for the input magnetic and output electric field spaces. It is important to emphasize that each of the eight structural parameters that emerge from the canonical decomposition of **Z** can be associated with particular physical characteristics of the earth system and, as such, can be utilized as physically meaningful discriminants in the classification of various features of the conductivity structure. However, before this can be done, it is necessary to perform numerical and analog modelling of 3-D conductivity structures with the objective of studying how the canonical parameters are determined by the nature of the conductivity distribution. It should be noted that an initial step in this direction has already been taken by LaTorraca et al. (1986). Certainly, a deeper understanding of how the canonical parameters relate to certain 3-D features in the underlying conductivity structure would increase the usefulness of canonical decomposition in MT interpretation and analysis.

The conventional analysis, which is based on the rotation properties of the impedance tensor, is inadequate since both rotation and ellipticity transformations are required in general to define a principal coordinate system for **Z**. We have shown that canonical decomposition is a natural extension of the conventional analysis and indeed, in those cases where the conventional analysis is adequate (such as 2-D conductivity structures), canonical decomposition reduces to the conventional analysis. Along the same lines, maximum coherency analysis is restrictive in that it considers only linearly polarized electric and magnetic field states in its formulation. While this is adequate for the analysis of 2-D structures, it is not appropriate for characterizing structures possessing 3-D distortions. We have shown that a generalized form of maximum coherency analysis, which allows for elliptically polarized electric and magnetic field states in its formulation, is essentially identical to canonical decomposition. Furthermore, we have demonstrated that the associate and conjugate directions analysis proposed by Counil et al. (1986) can be viewed as a partial generalization of the maximum coherency analysis in the sense that this form of analysis allows for elliptically polarized fields in either the input magnetic or output electric field spaces with the fields in the opposing space constrained to be linearly polarized.

Eggers' eigenstate analysis, although similar in philosophy to canonical decomposition is not entirely satisfactory since it incorporates a somewhat artitfical a priori constraint in its formulation. In contrast, canonical decomposition introduces no such constraint and utilizes only the information in **Z** to extract a set of structure parameters in a totally natural manner. The Spitz rotation analysis utilizes the Cayley factorization of **Z** to obtain two analytical rotation angles that define two different intrinsic coordinate systems for **Z**. We have shown that the Cayley factorization is nothing more than a repackaging of the information contained in the canonical decomposition of **Z**. All the information required for the definition of a principal coordinate system is already explicitly contained in the Cayley matrix factors and, as a consequence, there is no need to apply the conventional analysis to these factors to extract real rotation angles as in Spitz's rotation analysis.

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References

Counil, J.L., Le Mouël, J.L., Menvielle, M.: Associate and conjugate directions concepts in magnetotellurics. Ann. Geophys. **B 4**, 115–130, 1986

Dongarra, J., Bunch, J.R., Moler, C.B., Stewart, G.W.: LINPACK users guide. Philadelphia: Siam Publications 1979

Eggers, D.E.: An eigenstate formulation of the magnetotelluric impedance tensor. Geophysics **47**, 1204–1214, 1982

Fischer, G.: Symmetry properties of the surface impedance tensor for structures with a vertical plane of symmetry. Geophysics **40**, 1046–1050, 1975

LaTorraca, G.A., Madden, T.R., Korringa, J.: An analysis of the magnetotelluric impedance tensor for three-dimensional conductivity structures. Geophysics **51**, 1819–1829, 1986

Reddy, I.K., Rankin, D.: Coherence functions for magnetotelluric analysis. Geophysics **39**, 312–320, 1974

Sims, W.E., Bostick, F.X., Jr.: Methods of magnetotelluric analysis: Electrical Geophysics Research Laboratory Technical Report No. 58, University of Texas, Austin, 1969

Spitz, S.: The magnetotelluric impedance tensor properties with respect to rotations. Geophysics **50**, 1610–1617, 1985

Stewart, G.W.: Introduction to matrix computations. London: Academic Press 1973

Vozoff, K.: The magnetotelluric method in the exploration of sedimentary basins. Geophysics **37**, 98–141, 1972

Yee, E.: The magnetotelluric impedance tensor – its reconstruction and analysis. Unpublished Ph.D. Thesis, University of Saskatchewan, Saskatoon 1985

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