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Fredholm Theories in von Neumann Algebras. I

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The theory of compact and of Fredholm operators of a Hilbert space has been generalized in various directions by several authors (Cordes [2, 3], Neubauer [12, 13] and others). In the present paper the foundations of a generalization of this theory to von Neumann algebras are laid. This generalization depends heavily on the notion of the finiteness of a projection relative to a von Neumann algebra.

Using the equivalence classes of finite projections of a von Neumann algebra A (of continuous linear operators of a complex Hilbert space H) one can canonically construct an ordered commutative group Γ and a canonical map Dim of the finite projections of A into Γ . The function Dim has all the formal properties of the relative dimension function which Murray and von Neumann [11] defined for factors.

The definition and the existence proof of the null projection N_T and of the range projection R_T of $T \in A$ are straightforward. Call T finite if R_T is finite and compact if it is a limit in the norm of finite elements. For T to be Fredholm it is required that (i) N_T is finite and (ii) there is a finite projection E of A such that the range of $1 - E$ is contained in the range of T . Condition (ii) does not imply that the range of T is closed¹, unless the finiteness of E relative to A implies the finiteness of E relative to the full operator algebra $\mathcal{L}(H)$.² It is easy to see that (ii) implies the finiteness of N_{T^*} , so that the index

$$\text{Index } T = \text{Dim } N_T - \text{Dim } N_{T^*}$$

of every Fredholm element T of A is well-defined as an element of Γ .

The generalized Fredholm alternatives say that $1 - T$ is Fredholm of index zero (relative to A) if T is compact (relative to A). This theorem will be used in a subsequent paper to generalize some classical theorems on Fredholm operators (notably those of Atkinson [1] and Cordes-Labrousse [4]) to Fredholm elements of von Neumann algebras.

This paper concludes with the study of the increasing sequence of the null projections of the elements $(1 - T)^n$, $n = 1, 2, 3, \dots$, where $T \in A$ is compact. A well-known decomposition theorem of F. Riesz for compact operators (see Riesz-Nagy [15, n° 77–78] of Dieudonné [5]) is generalized to a certain extent to von Neumann algebras. An open problem arising in connection with the generalized decomposition theorem is briefly discussed at the end of this paper.

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¹ It is shown that if T is Fredholm, then the equation $T\xi = \eta$ is uniquely solvable for every $\eta \in (1 - E)(H)$, where $E = N_{T^*}$ is as small a finite projection of A as one wants. The $\mathcal{J}$ -Fredholm operators of Cordes [3] also do not necessarily have a closed range.

² E is finite relative to $\mathcal{L}(H)$ iff the linear space $E(H)$ is finite-dimensional in the usual sense.

1. Preliminaries on von Neumann Algebras

Let H be a complex Hilbert space. $\mathcal{L}(H)$ denotes the B^* -algebra of all continuous linear operators of H . If Σ is a subset of $\mathcal{L}(H)$, then its commutant Σ' is defined as the set of all $T \in \mathcal{L}(H)$ satisfying $ST = TS$ for all $S \in \Sigma$. A $*$ -subalgebra A of $\mathcal{L}(H)$ is called a von Neumann algebra, if $A = A''$. In the following A denotes always a von Neumann algebra of continuous linear operators of H . The set of all projections (hermitian idempotents) of A is denoted by $P(A)$. Together with the order relation $\leq$, defined by

$$E \leq F \Leftrightarrow EF = E, \quad (1.1)$$

$P(A)$ is a complete lattice (Kaplansky [9]). The equivalence relation $\sim$ and the order relation $<$ in $P(A)$ are defined as follows

$$E \sim F \Leftrightarrow \text{there is a } U \in A \text{ such that } E = U^*U, F = UU^*, \quad (1.2)$$

$$E < F \Leftrightarrow \text{there is an } E' \in P(A) \text{ such that } E \sim E' \leq F. \quad (1.3)$$

Lemma 1. $E < F$ and $F < E$ imply $E \sim F$.

Lemma 2 (Complete orthogonal additivity). Let $(E_i)_{i \in I}$ and $(F_i)_{i \in I}$ be two equipotent families of elements of $P(A)$. If $E_i E_x = F_i F_x = 0$ for all $i \neq x$ and $E_i \sim F_i$ for all $i \in I$, then $\Sigma E_i \sim \Sigma F_i$.

Lemma 3 (Parallelogram law). For all E, F of $P(A)$ one has

$$E - \inf(E, 1 - F) \sim F - \inf(F, 1 - E). \quad (1.4)$$

Lemma 4 (Comparability). Let $Z = A \cap A'$ be the center of A . For all E, F of $P(A)$ there is a $G \in P(Z)$ satisfying

$$EG < FG \quad \text{and} \quad E(1 - G) > F(1 - G). \quad (1.5)$$

An element $E \in P(A)$ is called finite relative to A , or briefly: finite, if for every $F \in P(A)$ the relations $E \sim F$ and $E \geq F$ imply $E = F$.

Lemma 5. If E, F are finite elements of $P(A)$, then $\sup(E, F)$ is a finite element of $P(A)$.

Lemma 6. Let E, F be finite elements of $P(A)$. Then

- (i) $E \sim F$ iff there is a unitary element U of A such that $U^*EU = F$.
- (ii) If $E \sim F$ and if $G \geq \sup(E, F)$, then there exists a unitary element U of A such that $U^*EU = F$ and $U^*GU = G$.

The proofs of Lemma 1–6 can be found in the literature (e.g. Kaplansky [9]).

Lemma 7. If E, F are finite elements of $P(A)$, then $E \sim F$ implies

$$1 - E \sim 1 - F.$$

This is a trivial consequence of part (i) of Lemma 6.

Lemma 8 (Cancellation law). Let $(E_1, E_2), (F_1, F_2)$ be two pairs of finite projections of A , and let $E_1 E_2 = F_1 F_2 = 0$. Then $E_1 \sim F_1$ and $E_1 + E_2 \sim F_1 + F_2$ imply $E_2 \sim F_2$.

Proof. Lemma 5 implies that $E_1 + E_2$ and $F_1 + F_2$ are finite. Part (i) of Lemma 6 implies that there is a unitary element U of A such that $U^*(E_1 + E_2)U = F_1 + F_2$. Part (ii) of Lemma 6 implies that there is a unitary element $V \in A$ such that $V^*F_1V = U^*E_1U$ and $V^*(F_1 + F_2)V = F_1 + F_2$. It follows that $V^*F_2V = U^*E_2U$. Hence $E_2 \sim F_2$.

The von Neumann algebra A is called finite if its unit element is a finite projection of A .

A is called countably decomposable if every family of pairwise orthogonal projections of A is countable.

For the definitions of finite, semifinite, faithful and normal traces we refer to Dixmier [7]. The following two statements characterize finite and finite countably decomposable von Neumann algebras in terms of traces (Dixmier [7, p. 96–98 and p. 308]).

I. A is finite if and only if for every positive hermitian element T of A there is a finite normal trace φ of A such that $\varphi T \neq 0$.

II. A is finite and countably decomposable if and only if there is a finite normal faithful trace of A .

Lemma 9³. *Let A be a finite von Neumann algebra. Let E, F be elements of $P(A)$. Then*

(i) *If $E < F$, then $\varphi E \leq \varphi F$ for every trace φ of A .*

(ii) *If $\varphi E \leq \varphi F$ for every finite normal trace φ of A , then $E < F$.*

Proof. (i) is a trivial consequence of the elementary properties of traces. To prove (ii) let $(H_i)_{i \in I}$ be a family of pairwise orthogonal projections of the center of A such that

$$1 = \sum_{i \in I} H_i \quad (1.6)$$

and that each of the reduced algebras

$$A_i = H_i A H_i \quad (1.7)$$

is countably decomposable (see Dixmier [7, p. 98]). It follows from statement II above that for each $i \in I$ there is a finite normal trace φ_i of A such that

1. H_i is the support of φ_i .

2. The restriction of φ_i to A_i is a faithful finite normal trace.

According to Lemma 4 choose the projection G_i of the center of A_i such that

$$EG_i > FG_i, \quad E(H_i - G_i) < F(H_i - G_i). \quad (1.8)$$

Let ψ_i be the finite normal trace of A defined by

$$\psi_i T = \varphi_i(TG_i) \quad (1.9)$$

for every positive hermitian element T of A . Part (i) of Lemma 9 and $EG_i > FG_i$ imply $\psi_i E \geq \psi_i F$. On the other hand, $\psi_i E \leq \psi_i F$ by hypothesis. Hence $\varphi_i(EG_i) = \varphi_i(FG_i)$. Choose $F_i \in P(A_i)$ such that $FG_i \sim F_i \leq EG_i$. Then $\varphi_i(EG_i - F_i) = 0$. Hence $EG_i = F_i \sim FG_i$ since φ_i is faithful on A_i . It follows that $EH_i < FH_i$ for all $i \in I$. Using Lemma 2 one gets

$$E = \sum H_i E < \sum H_i F = F. \quad (1.10)$$

³ Compare Dixmier [7, p. 324, exercise 5], where the proof of a similar lemma on projections of semifinite algebras is outlined.

Lemma 10⁴. *Let $E_1 \leq E_2 \leq E_3 \leq \dots$ be a non-decreasing sequence in $P(A)$. If the supremum E_∞ of this sequence is finite, then*

$$\inf(E_\infty, F) = \sup(\inf(E_n, F)) \quad \text{for all } F \in P(A). \quad (1.11)$$

Proof. Lemma 3 implies

$$F - \inf(F, 1 - E_n) \sim E_n - \inf(E_n, 1 - F) \quad (1.12)$$

$$\text{and} \quad F - \inf(F, 1 - E_\infty) \sim E_\infty - \inf(E_\infty, 1 - F). \quad (1.13)$$

$E_n \leq E_\infty$ implies

$$F - \inf(F, 1 - E_n) \leq F - \inf(F, 1 - E_\infty). \quad (1.14)$$

$$\text{Hence} \quad E_n - \inf(E_n, 1 - F) < E_\infty - \inf(E_\infty, 1 - F). \quad (1.15)$$

Since E_∞ is finite and $E_\infty \geq E_n$, Lemma 9 implies that

$$0 \leq \varphi(\inf(E_\infty, 1 - F) - \inf(E_n, 1 - F)) \leq \varphi(E_\infty - E_n) \quad (1.16)$$

for every finite normal trace φ of the reduced algebra A_{E_∞} ⁵. The normality of φ and $\sup E_n = E_\infty$ imply

$$\lim_{n \rightarrow \infty} \varphi(E_\infty - E_n) = 0 \quad (1.17)$$

and consequently

$$\lim_{n \rightarrow \infty} \varphi(\inf(E_\infty, 1 - F) - \inf(E_n, 1 - F)) = 0. \quad (1.18)$$

Hence

$$\varphi \inf(E_\infty, 1 - F) = \lim_{n \rightarrow \infty} \varphi(\inf(E_n, 1 - F)) = \varphi(\sup_n \inf(E_n, 1 - F)) \quad (1.19)$$

for every finite normal trace of A_{E_∞} . This relation and Lemma 9 (applied to A_{E_∞}) imply Lemma 10.

2. The Index Group of a von Neumann Algebra

Let B be an involutive algebra. Let K be a complex Hilbert space. A representation of B in K is a $*$ -homomorphism ϱ of B into $\mathcal{L}(K)$. If ϱ is a representation of B in K , then K is called the representation space of ϱ and is denoted by H_ϱ . Two representations ϱ, σ of B are called equivalent, notation: $\varrho \cong \sigma$, if there is an isomorphism U of H_ϱ onto H_σ such that $U \circ \varrho(x) = \sigma(x) \circ U$ for all $x \in B$. The equivalence class of the representation ϱ of B is denoted by $[\varrho]$. The direct sum of two representations ϱ, σ of B is denoted by $\varrho \oplus \sigma$. If $\varrho_1, \varrho_2, \sigma_1, \sigma_2$ are representations of B , then $\varrho_1 \cong \sigma_1$ and $\varrho_2 \cong \sigma_2$ imply $\varrho_1 \oplus \varrho_2 \cong \sigma_1 \oplus \sigma_2$. Hence $[\varrho_1] + [\varrho_2]$ is well-defined as $[\varrho_1 \oplus \varrho_2]$. The set of all equivalence classes of representations of B , equipped with $+$, is denoted by $M(B)$. It is easy to see that $M(B)$ is a commutative monoid.

⁴ This lemma seems to be known but I lack the reference. Halperin [8] states this lemma for finite factors.

⁵ If $E \in P(A)$, then $A_E = EAE$ is called a reduced algebra. A_E is a von Neumann algebra of operators of $E(H)$ (Dixmier [7, p. 18]).

In the following let A be a von Neumann algebra of continuous linear operators of the complex Hilbert space H . Let E be a projection of A , and let H_E be the range of E . Then the restriction map

$$\pi_E: A' \rightarrow \mathcal{L}(H_E) \quad (2.1)$$

which associates to each $T \in A'$ its restriction $\pi_E T$ to H_E is a representation of A' .

Lemma 11. *Let E, F be elements of $P(A)$. Then $E \sim F$ if and only if $\pi_E \cong \pi_F$.*

Proof. Suppose that $E \sim F$. Choose $U \in A$ satisfying $E = U^*U$, $F = UU^*$. Then $UE = FU$. If $T \in A'$, then $UTE = TFU$ and consequently

$$U \circ (\pi_E T) = (\pi_F T) \circ U. \quad (2.2)$$

Hence $\pi_E \cong \pi_F$. Conversely, suppose that $\pi_E \cong \pi_F$. Then there is an isomorphism U of H_E onto H_F such that (2.2) holds for every $T \in A'$. Obviously $E = U^*U$, $F = UU^*$. One can uniquely extend U to a partial isometry of H satisfying

$$U(1 - E) = 0, \quad (1 - F)U = 0. \quad (2.3)$$

Let $T \in A'$. Then (2.2) implies $UTE = TFU$. Using (2.3) one gets $UT = UET = UTE = TFU = TU$. Hence $U \in A'' = A$ and $E \sim F$.

Lemma 12 (Cancellation law). *Let E_1, E_2, F_1, F_2 be finite projections of A . Then $\pi_{E_1} \cong \pi_{F_1}$ and $\pi_{E_1} \oplus \pi_{E_2} \cong \pi_{F_1} \oplus \pi_{F_2}$ imply $\pi_{E_2} \cong \pi_{F_2}$.*

Proof. Let

$$\mathcal{L}(H \oplus H) = \left\{ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \middle| T_{ij} \in \mathcal{L}(H) \right\} \quad (2.4)$$

$$\text{and} \quad B = \left\{ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \middle| T_{ij} \in A \right\}. \quad (2.5)$$

$$\text{Then} \quad B' = \left\{ \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \middle| T \in A' \right\} \quad (2.6)$$

$$\text{and} \quad B = B'', \quad (2.7)$$

so that B is a Neumann algebra of bounded linear operators of $H \oplus H$. For any $G \in P(A)$ define

$$\tilde{G} = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{\tilde{G}} = \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}. \quad (2.8)$$

Obviously $\tilde{G}, \tilde{\tilde{G}} \in P(B)$ and $\tilde{G}\tilde{\tilde{G}} = 0$. Let

$$\alpha: (H \oplus H)_{\tilde{E}_1 + \tilde{E}_2} \rightarrow H_{E_1} \oplus H_{E_2}, \quad \beta: (H \oplus H)_{\tilde{F}_1 + \tilde{F}_2} \rightarrow H_{F_1} \oplus H_{F_2} \quad (2.9)$$

be the canonical isomorphisms. Then

$$\pi_{\tilde{E}_1 + \tilde{E}_2} \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} = \alpha^{-1} \begin{pmatrix} \pi_{E_1} T & 0 \\ 0 & \pi_{E_2} T \end{pmatrix} \alpha \quad (2.10)$$

$$\text{and} \quad \pi_{\tilde{F}_1 + \tilde{F}_2} \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} = \beta^{-1} \begin{pmatrix} \pi_{F_1} T & 0 \\ 0 & \pi_{F_2} T \end{pmatrix} \beta \quad (2.11)$$

for every $T \in A'$. Let

$$U: H_{E_1} \oplus H_{E_2} \rightarrow H_{F_1} \oplus H_{F_2} \quad (2.12)$$

be an isomorphism satisfying

$$U \circ (\pi_{E_1} \oplus \pi_{E_2}) T = (\pi_{F_1} \oplus \pi_{F_2}) T \circ U \quad (2.13)$$

for every $T \in A'$. Then

$$V = \beta^{-1} U \alpha \quad (2.14)$$

is an isomorphism

$$V : (H \oplus H)_{\tilde{E}_1 + \tilde{E}_2} \rightarrow (H \oplus H)_{\tilde{F}_1 + \tilde{F}_2}. \quad (2.15)$$

For every $T \in A'$ one has

$$\begin{aligned} V \circ \pi_{\tilde{E}_1 + \tilde{E}_2} \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} &= \beta^{-1} U \begin{pmatrix} \pi_{E_1} T & 0 \\ 0 & \pi_{E_2} T \end{pmatrix} \alpha \\ &= \beta^{-1} \begin{pmatrix} \pi_{F_1} T & 0 \\ 0 & \pi_{F_2} T \end{pmatrix} U \alpha \\ &= \pi_{\tilde{F}_1 + \tilde{F}_2} \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \circ V. \end{aligned} \quad (2.16)$$

Hence $\pi_{\tilde{E}_1 + \tilde{E}_2} \cong \pi_{\tilde{F}_1 + \tilde{F}_2}$. Lemma 11 implies $\tilde{E}_1 + \tilde{E}_2 \sim \tilde{F}_1 + \tilde{F}_2$. Similarly one can show $\tilde{E}_1 \sim \tilde{F}_1$. Since $\tilde{E}_1 \tilde{E}_2 = \tilde{F}_1 \tilde{F}_2 = 0$, Lemma 8 implies $\tilde{E}_2 \sim \tilde{F}_2$. Let

$$W = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \quad (2.17)$$

be a partial isometry of B satisfying

$$\tilde{E}_2 = W^* W, \quad \tilde{F}_2 = W W^*. \quad (2.18)$$

An easy calculation shows that $U_{11} = U_{12} = U_{21} = 0$ and that

$$E_2 = U_{22}^* U_{22}, \quad F_2 = U_{22} U_{22}^*. \quad (2.19)$$

Hence $E_2 \sim F_2$. Lemma 11 implies $\pi_{E_2} \cong \pi_{F_2}$.

Let $M(A, A')$ be the submonoid of $M(A')$ generated by the set of all $[\pi_E]$, where $E \in P(A)$ is finite relative to A . Let $\Gamma(A, A')$ be the universal group (Grothendieck group) of the commutative monoid $M(A, A')$, and let

$$\gamma : M(A, A') \rightarrow \Gamma(A, A') \quad (2.20)$$

be the canonical monoid homomorphism⁶.

Proposition 1. γ is a monomorphism.

Proof. If the cancellation law holds in the commutative monoid M , then the canonical map γ of M into its universal group is a monomorphism (Lang [10, p. 43]). Lemma 2 says exactly that the cancellation law holds in $M(A, A')$.

Define the dimension $\dim E$ of a finite projection E of A by the formula

$$\dim E = \gamma[\pi_E]. \quad (2.21)$$

It is easy to see that $\Gamma(A, A')$ can canonically be equipped with an order relation $\leq$ such that $(\Gamma(A, A'), \leq)$ is an ordered commutative group and that

$$\dim E \leq \dim F \Leftrightarrow E < F \quad (2.22)$$

⁶ For the definition of the universal group Γ of a commutative monoid M and of the canonical homomorphism $M \rightarrow \Gamma$ see Lang [10, p. 43].

for any pair E, F of finite projections of A . Proposition 1 shows that $\Gamma(A, A')$ is non-trivial, if A contains non-trivial finite projections. If A is a factor of type I (resp. II), then one can easily show that $\Gamma(A, A')$ is isomorphic to the additive group of integers (resp. real numbers). In these cases the function Dim , defined above, coincides with the restriction of the dimension function of Murray and von Neumann to the set of finite projections of A . Observe that $\Gamma(A, A')$ is always zero, if A is of type III. We call $\Gamma(A, A')$ the index group of the operator algebra A because it contains the indices of the Fredholm elements of A to be defined in the next section.

3. Definition of Finite, Compact and Fredholm Elements Relative to A

Let $T \in A$. Then

$$N_T = \sup \{E \in P(A) \mid TE = 0\} \quad (3.1)$$

is called the null projection of T , and

$$R_T = \inf \{E \in P(A) \mid ET = T\} \quad (3.2)$$

is called the range projection of T . One has

$$TN_T = 0, \quad R_T T = T \quad (3.3)$$

$$\text{and} \quad N_T = 1 - R_{T^*}, \quad N_{T^*} = 1 - R_T. \quad (3.4)$$

$$\text{Let} \quad T = W|T| \quad (3.5)$$

be the polar decomposition of T . Then

$$|T| \in A, \quad W \in A \quad (3.6)$$

$$\text{and} \quad R_T = WW^*, \quad R_{T^*} = W^*W \quad (3.7)$$

(Dixmier [7, p. 226]). Hence

$$R_T \sim R_{T^*}. \quad (3.8)$$

The element $T \in A$ is called finite (or of finite rank) relative to A , if R_T is finite. m_0 denotes the set of finite elements of A . Obviously

$$m_0 = m_0 A. \quad (3.9)$$

Lemma 5 implies

$$\text{If } T \in m_0 \text{ and } S \in m_0, \text{ then } T - S \in m_0. \quad (3.10)$$

(3.9) and (3.10) imply that m_0 is a right ideal of A . (3.8) implies that m_0 is a two-sided ideal of A .

Let m be the norm closure of m_0 . The elements of m are called compact (relative to A). m is the smallest closed two-sided ideal of A containing the finite projections of A (Dixmier [6], see also Dixmier [7, p. 238, Exercise 9]).

The element $T \in A$ is called Fredholm (relative to A), if the following two conditions hold

- (i) N_T is finite.
- (ii) There is a finite projection E of A such that

$$(1 - E)(H) \subseteq T(H).$$

If the finiteness of a projection relative to A implies finite dimensionality of its range space in the usual sense, then (ii) implies that the range of a Fredholm element of A is closed. In general, however, the range of a Fredholm element relative to a von Neumann algebra need not be closed.

Condition (ii) and (3.4) imply that N_{T^*} is finite, if T is Fredholm. This allows us to define the index of a Fredholm element T of A as an element of the index group $\Gamma(A, A')$ by the formula

$$\text{Index}(T) = \text{Dim } N_T - \text{Dim } N_{T^*}. \quad (3.11)$$

Lemma 13. *For every $T \in A$ there is a non-decreasing sequence $E_1 \leq E_2 \leq \dots$ in $P(A)$ satisfying the following two conditions:*

- (i) *For every $k = 1, 2, \dots$ the range of E_k is contained in the range of T .*
- (ii) *R_T is the supremum of the sequence $(E_k)_{k=1, 2, \dots}$.*

Proof. In view of (3.5), (3.6) and (3.7) it is sufficient to prove the lemma for positive hermitian $T \in A$. So let T be given by its spectral decomposition

$$T = \int_0^\infty \lambda dF_\lambda. \quad (3.12)$$

Then
$$F_\lambda \in A \quad \text{for all } \lambda \quad (3.13)$$

(Dixmier [7, p. 3]). Also

$$N_T = F_0, \quad R_T = 1 - F_0. \quad (3.14)$$

The descending continuity of the spectral family (F_λ) implies

$$R_T = \sup_{\varepsilon > 0} (1 - F_\varepsilon). \quad (3.15)$$

The pair $(F_\varepsilon(H), (1 - F_\varepsilon)(H))$ of subspaces of H reduces T and each F_λ . Denote the restrictions of T and F_λ to the space $(1 - F_\varepsilon)(H)$ by T_ε resp. $F_{\lambda, \varepsilon}$. Then

$$T_\varepsilon = \int_\varepsilon^\infty \lambda dF_{\lambda, \varepsilon} \quad (3.16)$$

and
$$T_\varepsilon^{-1} = \int_\varepsilon^\infty \lambda^{-1} dF_{\lambda, \varepsilon}. \quad (3.17)$$

Hence
$$(1 - F_\varepsilon)(H) \subseteq T(H). \quad (3.18)$$

(3.15) and (3.18) imply that the sequence $E_n = 1 - F_{1/n}$, $n = 1, 2, 3, \dots$ satisfies the conditions of the lemma.

Corollary. *For every Fredholm element T of A there is a non-decreasing sequence $E_1 \leq E_2 \leq \dots$ in $P(A)$ such that conditions (i), (ii) of Lemma 13 and*

(iii) $1 - E_1$ is finite relative to A are satisfied.

Proof. Since T is Fredholm there is an $E \in P(A)$ such that $E(H) \subseteq T(H)$ and $1 - E$ is finite. Let $F = R_T - E$. Applying Lemma 13 to FT one gets a sequence $E_1^1 \leq E_2^1 \leq \dots$ such that $E_k^1(H) \subseteq (FT)(H)$ and $\sup E_k^1 = R_{FT}$. Then the sequence $E_k = E + E_k^1$, $k = 1, 2, \dots$ satisfies the conditions of the corollary.

4. Generalized Fredholm Alternatives

Theorem 1. *Let A be a von Neumann algebra. If $T \in A$ is compact relative to A , then $1 - T$ is Fredholm relative to A with index zero.*

Proof. (i) Suppose T is finite relative to A . Then

$$E = \sup(R_T, R_{T^*}) \quad (4.1)$$

is finite. E satisfies the relations

$$(1 - E)(1 - T) = 1 - E, \quad (1 - E)(1 - T^*) = 1 - E \quad (4.2)$$

$$\text{and} \quad (1 - T^*)(1 - E) = 1 - E, \quad (1 - T)(1 - E) = 1 - E. \quad (4.3)$$

(4.3) implies

$$N_{1-T^*} \leq E, \quad N_{1-T} \leq E. \quad (4.4)$$

Hence N_{1-T} , N_{1-T^*} are finite. Since $R_S \sim R_{S^*}$ for every $S \in A$, one has

$$R_{E-T} \sim R_{E-T^*}. \quad (4.5)$$

The relations

$$R_{E-T} = E - N_{1-T^*}, \quad R_{E-T^*} = E - N_{1-T} \quad (4.6)$$

are easy to verify (use (3.4)). Since E is finite, (4.5), (4.6) and Lemma 7 imply

$$N_{1-T} \sim N_{1-T^*}. \quad (4.7)$$

(ii) Let T be compact relative to A . Then there is a finite element $T_0 \in A$ such that

$$S = 1 - (T - T_0) \quad (4.8)$$

is regular. Observe that

$$N_{SF} = 1 - F \quad \text{for every } F \in P(A). \quad (4.9)$$

Together with (3.8) and (3.4) this implies

$$R_{SF} \sim F. \quad (4.10)$$

One has the following equivalences

$$\begin{aligned} (S - T_0)F = 0 &\Leftrightarrow (1 - T_0 S^{-1})SF = 0 \Leftrightarrow (1 - T_0 S^{-1})R_{SF} = 0, \\ (1 - T_0 S^{-1})F = 0 &\Leftrightarrow (1 - T_0 S^{-1})SS^{-1}F = 0 \Leftrightarrow (S - T_0)R_{S^{-1}F} = 0. \end{aligned} \quad (4.11)$$

Substituting F by N_{S-T_0} resp. $N_{1-T_0 S^{-1}}$ the first resp. the second relation (4.11) and (4.10) imply

$$N_{S-T_0} < N_{1-T_0 S^{-1}} \quad \text{and} \quad N_{1-T_0 S^{-1}} < N_{S-T_0}. \quad (4.12)$$

Lemma 1 implies

$$N_{S-T_0} \sim N_{1-T_0 S^{-1}}. \quad (4.13)$$

The relation

$$N_{S^*-T^*} \sim N_{1-(S^*)^{-1}T_0} \quad (4.14)$$

is trivial. Since $T_0 S^{-1}$ is finite, (4.13), (4.14) and part (i) of the proof of this theorem imply that $N_{1-T} = N_{S-T_0}$ is finite and that

$$N_{1-T} \sim N_{1-T^*}. \quad (4.15)$$

(iii) Keep the notation of (ii) and define

$$F = \sup(R_{T_0 S^{-1}}, R_{(S^*)^{-1}T_0}) \quad (4.16)$$

$$\text{and} \quad F_S = S^{-1} F S. \quad (4.17)$$

Then F is finite and the relation

$$(1-T)(1-F_S)S^{-1} = 1-F \quad (4.18)$$

implies that the range of $1-T$ contains the range of $1-F$.

5. A Decomposition Theorem

Theorem 2. *Let T be a compact element of the von Neumann algebra A . Let N_∞ be the supremum of the non-decreasing sequence*

$$N_{1-T} \leq N_{(1-T)^2} \leq N_{(1-T)^3} \leq \cdots, \quad (5.1)$$

and let R_∞ be the infimum of the non-increasing sequence

$$R_{1-T} \geq R_{(1-T)^2} \geq R_{(1-T)^3} \geq \cdots. \quad (5.2)$$

Then

- (i) $N_\infty \sim 1 - R_\infty$.
- (ii) $N_\infty T N_\infty = T N_\infty$, $R_\infty T R_\infty = T R_\infty$.
- (iii) *If T is finite, then N_∞ is finite.*
- (iv) *If N_∞ is finite, then*

$$\inf(N_\infty, R_\infty) = 0, \quad \sup(N_\infty, R_\infty) = 1. \quad (5.3)$$

Proof. Define

$$N_n = N_{(1-T)^n}, \quad R_n = R_{(1-T)^n}. \quad (5.4)$$

(i) Theorem 1 implies

$$N_n \sim 1 - R_n. \quad (5.5)$$

Since all N_n are finite, Lemma 6 implies

$$R_n - R_{n+1} \sim N_{n+1} - N_n. \quad (5.6)$$

Lemma 2 implies

$$N_\infty = N_1 + \sum (N_{n+1} - N_n) \sim 1 - R_1 + \sum (R_n - R_{n+1}) = 1 - R_\infty. \quad (5.7)$$

(ii) The relation

$$(1-T)^n(1-T)N_{n+1} = 0 \quad (5.8)$$

implies that the range projection of $(1-T)N_{n+1}$ is majorized by the null projection of $(1-T)^n$. Hence

$$(1-N_n)(1-T)N_{n+1} = 0. \quad (5.9)$$

Taking the limits in the strong topology of A one gets

$$(1 - N_\infty) T N_\infty = 0 \quad (5.10)$$

which is equivalent to the first relation (ii). Similarly the relation

$$(1 - R_n)(1 - T)(1 - T)^{n-1} = 0 \quad (5.11)$$

implies

$$(1 - R_n)(1 - T)R_{n-1} = 0. \quad (5.12)$$

Taking the limits one gets

$$(1 - R_\infty)(1 - T)R_\infty = 0 \quad (5.13)$$

which is equivalent with the second relation (ii).

(iii) Suppose that T is finite. Using the notation of part (i) of the proof of Theorem 1 it follows from (4.2) that

$$(1 - E)(1 - T)^n = (1 - E) \quad (5.14)$$

for all n . Hence $N_n \leq E$ for all n which implies

$$N_\infty \leq E. \quad (5.15)$$

It follows that N_∞ is finite, since E is finite.

(iv) Define $T_{(k)} \in \mathfrak{M}$ by

$$1 - T_{(k)} = (1 - T)^k. \quad (5.16)$$

Then

$$N_k = N_{1 - T_{(k)}}. \quad (5.17)$$

Define

$$R_n^{(k)} = \inf(R_{nk}, N_k). \quad (5.18)$$

Then

$$\begin{aligned} R_n^{(k)} &= \inf\{E \in P(A) \mid E(1 - T_{(k)})^n(N_{(n+1)k} - N_{nk}) \\ &= (1 - T_{(k)})^n(N_{(n+1)k} - N_{nk})\}. \end{aligned} \quad (5.19)$$

Thus

$$(1 - T_{(k)})^n(N_{(n+1)k} - N_k) \in A_{N_{(n+1)k}}. \quad (5.20)$$

Since $A_{N_{(n+1)k}}$ is a finite von Neumann algebra with $N_{(n+1)k}$ as its unit element, (3.4) implies

$$R_n^{(k)} \sim N_{(n+1)k} - N_{nk}. \quad (5.21)$$

Since N_∞ is finite by hypothesis, the reduced algebra A_{N_∞} is finite. For every finite normal trace φ of A_{N_∞} one has

$$\lim_{n \rightarrow \infty} \varphi(N_{(n+1)k} - N_{nk}) = 0. \quad (5.22)$$

Hence

$$\inf(R_\infty, N_k) = 0. \quad (5.23)$$

The finiteness of N_∞ , Lemma 10 and (5.23) imply

$$\inf(R_\infty, N_\infty) = 0. \quad (5.24)$$

Lemma 3 and (5.24) imply

$$\sup(R_\infty, N_\infty) - R_\infty \sim N_\infty. \quad (5.25)$$

(5.25) and (5.7) imply

$$1 - R_\infty \sim \sup(R_\infty, N_\infty) - R_\infty. \quad (5.26)$$

Obviously

$$1 - R_\infty \geq \sup(R_\infty, N_\infty) - R_\infty. \quad (5.27)$$

(5.26), (5.27) and the finiteness of $1 - R_\infty$ imply

$$1 - R_\infty = \sup(R_\infty, N_\infty) - R_\infty. \quad (5.28)$$

Hence

$$\sup(R_\infty, N_\infty) = 1. \quad (5.29)$$

It remains an open question, whether N_∞ is finite for every compact element T of A . Using the classical theory of compact operators (Dieudonné [5, (11, 3, 3)]) and the theory of direct integrals of von Neumann algebras (Dixmier [7, Chapter II]) one gets the following partial answer to that question.

If A is decomposable as a direct integral

$$A = \int_{\zeta \in Z}^{\oplus} A(\zeta) dv(\zeta) \quad (5.30)$$

such that $A(\zeta)$ is a factor of type I for almost every ζ of the locally compact space Z , then N_∞ is finite for every compact element T of A .

The proof of this statement will be omitted, since it seems likely that N_∞ is finite for every compact T of a von Neumann algebra of type I (decomposable or not) and that there is a proof of it which does not use direct integrals. The question of the finiteness of N_∞ remains completely open for non-finite compact elements of von Neumann algebras of type II_∞ .

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