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Fredholm Theories in von Neumann Algebras. II

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In the first part of this paper (Breuer [2]) the foundations of a generalized theory of compact operators and Fredholm operators relative to a von Neumann algebra A were laid. The index group $I(A)$ of A and the index map

$$\text{Index: } \mathcal{F}(A) \rightarrow I(A)$$

of the space $\mathcal{F}(A)$ of Fredholm elements of A were defined. Also the *generalized Fredholm alternatives* stating that $1 - C$ is Fredholm relative to A of index zero if C is compact relative to A were proved.

In the second part of this paper the generalized Fredholm theory will be further developed. Some of the classical theorems of Atkinson [1], Gohberg-Krein [6] and Cordes-Labrousse [3] will be generalized. These generalizations will characterize the relative Fredholm elements modulo the relative compact elements, state the additivity of the index, and relate the index group $I(A)$ to the group $\pi_0 \mathcal{F}(A)$ of connected components of $\mathcal{F}(A)$. In particular, it will be shown that if A is properly infinite, then the *algebraic* invariant $I(A)$ of A is canonically isomorphic to the *topological* invariant $\pi_0 \mathcal{F}(A)$ of $\mathcal{F}(A)$. The von Neumann algebra analogue of a theorem on the multiplicativity of the index (Palais [9]) is also included, although it is not used in the following.

Of course, the proofs are modeled after the classical proofs. To avoid misunderstandings the following two points should be mentioned. First the compact elements of A (or more precisely, relative to A) are not necessarily compact operators in the usual sense. Second the range of a Fredholm element of A is not necessarily closed. It is mainly the lack of these two properties that required the modification of the classical proofs.

The following notations will be used. A denotes a von Neumann algebra of linear operators of a complex Hilbert space H . The index group of A which was denoted by $\Gamma(A, A')$ in [2] is denoted in this paper by $I(A)$ ¹. The space of the Fredholm elements of A equipped with its norm topology is denoted by $\mathcal{F}(A)$. $\mathfrak{m}$ denotes the two-sided $*$ -ideal of compact elements of A . The quotient algebra $A/\mathfrak{m}$ equipped with its quotient topology is denoted by $\bar{A}$. $\pi: A \rightarrow \bar{A}$ is the canonical homomorphism. $G(A)$, $G(\bar{A})$ denotes the group of regular elements of A , $\bar{A}$. For additional definitions it is referred to [2].

¹ This simplified notation is justified by Proposition 2 of the appendix saying that the index group of A is independent of A' .

1. Characterization of the Fredholm Elements Modulo $\mathfrak{m}$

Lemma 1. *Let S, T be elements of $\mathcal{F}(A)$. Then*

$$N_{ST} - N_T \sim \inf(R_T, N_S). \quad (1.1)$$

Proof. The sequence

$$0 \rightarrow \text{Ker } T \rightarrow \text{Ker } ST \xrightarrow{T} \text{range } T \cap \text{Ker } S \rightarrow 0 \quad (1.2)$$

is exact. Hence the sequence

$$0 \rightarrow \text{Ker } ST \ominus \text{Ker } T \xrightarrow{T(N_{ST} - N_T)} \text{range } T \cap \text{Ker } S \rightarrow 0 \quad (1.2a)$$

is exact. It follows that

$$R_{(N_{ST} - N_T)T^*} = N_{ST} - N_T \quad (1.3)$$

and

$$R_{T(N_{ST} - N_T)} \leq \inf(R_T, N_S). \quad (1.4)$$

Choose the sequence $E_1 \leq E_2 \leq E_3 \leq \dots$ of projections of A according to Lemma 13 of Part I (Breuer [2]) such that the range of each E_n is contained in the range of T and that all projections $1 - E_n$ are finite and that $\sup E_n = R_T$. The exactness of the sequence (1.2a) implies

$$\inf(E_n, N_S) \leq R_{T(N_{ST} - N_T)}. \quad (1.4a)$$

Define $E_0 = \inf(E_1, 1 - N_S)$. Then $1 - E_0$ is finite. The relations $E_0 N_S = 0$ and $E_0 \leq E_n$ imply

$$\inf(E_n, N_S) = \inf(E_n - E_0, N_S), \quad \inf(R_T, N_S) = \inf(R_T - E_0, N_S). \quad (1.4b)$$

Using Lemma 10 of Part I (Breuer [2]) it follows from (1.4b) that

$$\sup \{ \inf(E_n, N_S) \mid n = 1, 2, 3, \dots \} = \inf(R_T, N_S). \quad (1.4c)$$

The relations (1.4), (1.4a) and (1.4c) imply

$$R_{T(N_{ST} - N_T)} = \inf(R_T, N_S). \quad (1.5)$$

The relations (1.3), (1.5) and

$$R_{T(N_{ST} - N_T)} \sim R_{(N_{ST} - N_T)T^*} \quad (1.6)$$

imply (1.1).

Theorem 1. (i) *If A is of finite type, then*

$$\mathfrak{m} = A \quad \text{and} \quad \mathcal{F}(A) = A. \quad (1.7)$$

(ii) *If A is not of finite type, then*

$$\mathfrak{m} \neq A \quad \text{and} \quad \mathcal{F}(A) = \pi^{-1} G(\bar{A}). \quad (1.8)$$

The second relation (1.8) says that $T \in A$ is Fredholm relative to A iff πT is a regular element of $\bar{A}$.

The proof of (i) is obvious. The proof of (ii) follows.

Suppose that T is regular modulo $\mathfrak{m}$. Then there is an $S \in A$ such that

$$ST = 1 - C, \quad TS = 1 - D \quad (1.9)$$

with C, D compact relative to A . The first relation (1.9) implies

$$N_T \subseteq N_{1-C}. \quad (1.10)$$

Since $1 - C$ is Fredholm (Breuer [2, Theorem 1]), N_{1-C} is finite. Hence N_T is finite. Thus the first axiom for T to be a Fredholm element is satisfied.

The second relation (1.9) implies

$$\text{range}(1 - D) \subseteq \text{range } T. \quad (1.11)$$

Since $1 - D$ is Fredholm (Breuer, loc. cit.), there is a finite projection E of A satisfying

$$\text{range}(1 - E) \subseteq \text{range}(1 - D) \quad (1.12)$$

and consequently

$$\text{range}(1 - E) \subseteq \text{range } T. \quad (1.13)$$

Thus the second and last axiom for T to be Fredholm is also satisfied.

Conversely suppose that T is a Fredholm element relative to A . Then there is a finite projection E of A such that (1.13) holds. Lemma 1 implies

$$N_{(1-E)T} - N_T \sim \inf(R_T, E). \quad (1.14)$$

Hence the projection

$$F := N_{(1-E)T} \quad (1.15)$$

is finite. The sequence

$$0 \rightarrow \text{range}(1 - F) \xrightarrow{(1-E)T(1-F)} \text{range}(1 - E) \rightarrow 0 \quad (1.16)$$

is exact. Hence

$$0 \rightarrow \text{range}(1 - E) \xrightarrow{(1-F)T^*(1-E)} \text{range}(1 - F) \rightarrow 0 \quad (1.17)$$

is exact. It follows that $(1 - E)T(1 - F)T^*(1 - E)$ and $(1 - F)T^*(1 - E)T(1 - F)$ are regular elements of the reduced algebras A_{1-E} resp. A_{1-F} . Hence there are elements T', T'' of A such that

$$(1 - E)T(1 - F)T^*(1 - E)T' = 1 - E, \quad T''(1 - F)T^*(1 - E)T(1 - F) = 1 - F. \quad (1.18)$$

Since E, F are finite projections of A it follows from (1.18) that

$$T(T^*T') - 1 \in \mathfrak{m}, \quad (T''T^*)T - 1 \in \mathfrak{m}. \quad (1.19)$$

Hence T is regular modulo $\mathfrak{m}$.

Theorem 1 is the von Neumann algebra analogue of a well-known theorem of Atkinson [1] characterizing the Fredholm operators modulo the compact operators.

Corollary 1. $\mathcal{F}(A)$ is an involutive monoid, i.e.

- (i) $1 \in \mathcal{F}(A)$,
- (ii) $S \in \mathcal{F}(A)$ and $T \in \mathcal{F}(A)$ implies $ST \in \mathcal{F}(A)$,
- (iii) $S \in \mathcal{F}(A)$ implies $S^* \in \mathcal{F}(A)$.

Corollary 2. $\mathcal{F}(A)$ is open in the norm topology.

Corollary 3. If there is an $S \in \mathcal{F}(A)$ such that $S + \lambda \cdot 1 \in \mathcal{F}(A)$ for all $\lambda \in \mathbb{C}$, then $\mathcal{F}(A) = A$ is of finite type.

2. Algebraic Properties of the Index Map

The following theorem is the von Neumann algebra analogue of another well-known theorem of Atkinson [1].

Theorem 2. Let S, T be Fredholm elements of A . Then

$$\text{Index } S^* = -\text{Index } S \quad (2.1)$$

and

$$\text{Index } ST = \text{Index } S + \text{Index } T. \quad (2.2)$$

Proof. (2.1) is trivial. To prove (2.2) observe that Lemma 1 implies

$$N_{ST} - N_T \sim \inf(R_T, N_S) \quad (2.3)$$

and

$$N_{(ST)^*} - N_{S^*} \sim \inf(R_{S^*}, N_{T^*}). \quad (2.4)$$

One has according to the parallelogram law (Dixmier [4, p. 227])

$$N_S - \inf(1 - N_{T^*}, N_S) \sim N_{T^*} - \inf(1 - N_S, N_{T^*}). \quad (2.5)$$

One has also

$$1 - N_{T^*} = R_T, \quad 1 - N_S = R_{S^*}. \quad (2.6)$$

Hence

$$N_S - \inf(R_T, N_S) \sim N_{T^*} - \inf(R_{S^*}, N_{T^*}). \quad (2.7)$$

The formulas (2.3), (2.4) and (2.7) imply

$$\text{Dim } N_S - \text{Dim } (N_{ST} - N_T) = \text{Dim } N_{T^*} - \text{Dim } (N_{(ST)^*} - N_{S^*}) \quad (2.8)$$

which implies (2.2).

Let A_1, A_2 be von Neumann algebras, and let

$$A = A_1 \otimes A_2 \quad (2.9)$$

be their von Neumann algebra tensor product. Let E_1, E_2 be projections of A_1 resp. A_2 . Then $E_1 \otimes E_2$ is a projection of A . Let $\pi_{E_1}, \pi_{E_2}, \pi_{E_1 \otimes E_2}$ be the representations of A'_1, A'_2, A' associated to $E_1, E_2, E_1 \otimes E_2$ as defined in Breuer [2, n° 2]. Define the representation

$$\pi_{E_1} \otimes \pi_{E_2} : A'_1 \otimes A'_2 \rightarrow \mathcal{L}(H_{E_1} \otimes H_{E_2}) \quad (2.10)$$

as usual by the relation

$$(\pi_{E_1} \otimes \pi_{E_2})(T_1 \otimes T_2) = \pi_{E_1} T_1 \otimes \pi_{E_2} T_2 \quad (2.11)$$

for every $T_1 \in A'_1$, $T_2 \in A'_2$.

Lemma 2. *If A_1, A_2 are semifinite, then*

$$(A_1 \otimes A_2)' = A'_1 \otimes A'_2 \quad (2.12)$$

and

$$\pi_{E_1 \otimes E_2} = \pi_{E_1} \otimes \pi_{E_2}. \quad (2.13)$$

Proof. (2.12) is proved in Dixmier [4, p. 102]. It follows from (2.12) that the domains of definition of $\pi_{E_1 \otimes E_2}$ and $\pi_{E_1} \otimes \pi_{E_2}$ are equal. (2.13) then follows from the following trivial calculation

$$\begin{aligned} \pi_{E_1 \otimes E_2}(T_1 \otimes T_2) &= (E_1 \otimes E_2)(T_1 \otimes T_2) = E_1 T_1 \otimes E_2 T_2 \\ &= \pi_{E_1} T_1 \otimes \pi_{E_2} T_2 = (\pi_{E_1} \otimes \pi_{E_2})(T_1 \otimes T_2). \end{aligned}$$

Lemma 3. *If A_1, A_2 are semifinite, then there is a unique Z -bilinear map*

$$\Lambda : I(A_1) \times I(A_2) \rightarrow I(A_1 \otimes A_2) \quad (2.14)$$

satisfying

$$\Lambda(\text{Dim } E_1, \text{Dim } E_2) = \text{Dim}(E_1 \otimes E_2) \quad (2.15)$$

for all finite projections E_1, E_2 of A_1, A_2 .

Proof. Define Λ by (2.15). This makes sense since

(i) $E_1 \otimes E_2$ is finite if E_1, E_2 are finite

and

(ii) $E_1 \sim F_1$ and $E_2 \sim F_2$ implies $E_1 \otimes E_2 \sim F_1 \otimes F_2$.

(i) is shown in Dixmier [4, p. 101] and (ii) is trivial.

The Z -bilinearity follows easily from Lemma 2 and from the well-known distributive laws

$$\begin{aligned} (\pi_{E_1} \oplus \pi_{F_1}) \otimes \pi_{E_2} &\sim (\pi_{E_1} \otimes \pi_{E_2}) \oplus (\pi_{F_1} \otimes \pi_{E_2}) \\ \pi_{E_1} \oplus (\pi_{E_2} \otimes \pi_{F_2}) &\sim (\pi_{E_1} \otimes \pi_{E_2}) \oplus (\pi_{E_1} \otimes \pi_{F_2}). \end{aligned} \quad (2.16)$$

The uniqueness of Λ follows from the Z -bilinearity of Λ and from the fact that the elements of the form $\text{Dim } E_1$ resp. $\text{Dim } E_2$ generate $I(A_1)$ resp. $I(A_2)$.

Lemma 4. *There is a unique homomorphism*

$$\lambda_{A_1, A_2} : I(A_1) \otimes_Z I(A_2) \rightarrow I(A_1 \otimes A_2) \quad (2.17)$$

satisfying

$$\lambda_{A_1, A_2}(\text{Dim } E_1 \otimes \text{Dim } E_2) = \text{Dim } E_1 \otimes E_2. \quad (2.18)$$

Proof. This is a trivial consequence of Lemma 3 and the theory of tensor products.

Let $M_2 = \mathcal{L}(\mathbb{C}^2)$ be the von Neumann algebra of $(2, 2)$ -matrices over $\mathbb{C}$. Then

$$\lambda_{A, M_2} : I(A) \otimes_Z I(M_2) \rightarrow I(A \otimes M_2) \quad (2.19)$$

is well-defined according to Lemma 4. Observe that $I(M_2) = Z$. Hence the Z -linear map

$$\lambda_A : I(A) \rightarrow I(A \otimes M_2) \quad (2.20)$$

is well-defined by

$$\lambda_A(\alpha) = \lambda_{A, M_2}(\alpha \otimes 1). \quad (2.21)$$

Define the Z -linear map

$$\mu_{A_1, A_2} : I(A_1) \otimes_Z I(A_2) \rightarrow I(A \otimes M_2) \quad (2.22)$$

by

$$\mu_{A_1, A_2} = \lambda_A \circ \lambda_{A_1, A_2}. \quad (2.23)$$

Let $S_1 \in A_1$, $S_2 \in A_2$ and define $S_1 \# S_2 \in A \otimes M_2$ by

$$S_1 \# S_2 = \begin{pmatrix} S_1 \otimes 1 & -1 \otimes S_2^* \\ 1 \otimes S_2 & S_1^* \otimes 1 \end{pmatrix}. \quad (2.24)$$

The following theorem is the von Neumann algebra analogue of a theorem of Jänich [7] and Palais [9] on Fredholm operators of a Hilbert space. The proof is a modification of Jänich's proof (see Jänich loc. cit., proof of Lemma 1).

Theorem 3. *Let A_1 , A_2 be semifinite and properly infinite. If $S_1 \in A_1$, $S_2 \in A_2$ are relative Fredholm elements, then $S_1 \# S_2$ is a Fredholm element relative to $A \otimes M_2$ and*

$$\text{Index}(S_1 \# S_2) = \mu_{A_1, A_2}(\text{Index } S_1 \otimes \text{Index } S_2). \quad (2.25)$$

Proof. Let $\mathfrak{m}_1$ resp. $\mathfrak{m}_2$ be the ideal of compact elements of A_1 resp. A_2 . Let $\mathfrak{m}$ be the ideal of compact elements of A . Then

$$\mathfrak{m} \supseteq \mathfrak{m}_1 \otimes \mathfrak{m}_2. \quad (2.26)$$

Theorem 1 implies that $S_1 S_1^*$, $S_1^* S_1$ resp. $S_2 S_2^*$, $S_2^* S_2$ are regular modulo $\mathfrak{m}_1$ resp. $\mathfrak{m}_2$. Hence $S_1^* S_1 \otimes 1$, $S_1 S_1^* \otimes 1$, $1 \otimes S_2^* S_2$, $1 \otimes S_2 S_2^*$ are regular modulo $\mathfrak{m}_1 \otimes \mathfrak{m}_2$. Since these elements are positive, the elements

$$S_1^* S_1 \otimes 1 + 1 \otimes S_2 S_2^*, \quad S_1 S_1^* \otimes 1 + 1 \otimes S_2^* S_2 \quad (2.27)$$

are also regular modulo $\mathfrak{m}_1 \otimes \mathfrak{m}_2$. Together with (2.26) this implies that the elements (2.27) are regular modulo $\mathfrak{m}$.

An easy calculation yields

$$(S_1 \# S_2)^*(S_1 \# S_2) = \begin{pmatrix} S_1^* S_1 \otimes 1 + 1 \otimes S_2^* S_2 & 0 \\ 0 & S_1 S_1^* \otimes 1 + 1 \otimes S_2 S_2^* \end{pmatrix}. \quad (2.28)$$

Therefore, the regularity of the elements (2.27) modulo $\mathfrak{m}$ implies the regularity of (2.28) modulo $\mathfrak{m} \otimes M_2$. Since $\mathfrak{m} \otimes M_2$ is the ideal of compact

elements of $A \otimes M_2$, Theorem 1 implies that $(S_1 \# S_2)^*(S_1 \# S_2)$ is a Fredholm element of $A \otimes M_2$. Similarly one can show that $(S_1 \# S_2)(S_1 \# S_2)^*$ is Fredholm relative to $A \otimes M_2$. Hence $S_1 \# S_2$ is a Fredholm element of $A \otimes M_2$.

To prove (2.25) observe that the identity of $A_1 \otimes A_2$ is the sum of the following pairwise orthogonal projections of $A_1 \otimes A_2$.

$$\begin{aligned} E_0 &= N_{S_1} \otimes N_{S_2}, & E_1 &= N_{S_1} \otimes (1 - N_{S_2}), \\ E_2 &= (1 - N_{S_1}) \otimes N_{S_2}, & E_3 &= (1 - N_{S_1}) \otimes (1 - N_{S_2}). \end{aligned} \quad (2.29)$$

Also observe that

$$R_{(S_1^* S_1 \otimes 1 + 1 \otimes S_2^* S_2) E_i} = \begin{cases} 0 & \text{for } i=0 \\ E_i & \text{for } i \neq 0. \end{cases} \quad (2.30)$$

Hence

$$N_{S_1^* S_1 \otimes 1 + 1 \otimes S_2^* S_2} = N_{S_1} \otimes N_{S_2}. \quad (2.31)$$

Similarly one has

$$N_{S_1 S_1^* \otimes 1 + 1 \otimes S_2 S_2^*} = N_{S_1^*} \otimes N_{S_2^*}. \quad (2.32)$$

Hence

$$N_{S_1 \# S_2} = \begin{pmatrix} N_{S_1} \otimes N_{S_2} & 0 \\ 0 & N_{S_1^*} \otimes N_{S_2^*} \end{pmatrix}. \quad (2.33)$$

Similarly one gets

$$N_{(S_1 \# S_2)^*} = \begin{pmatrix} N_{S_1^*} \otimes N_{S_2} & 0 \\ 0 & N_{S_1} \otimes N_{S_2^*} \end{pmatrix}. \quad (2.34)$$

Hence

$$\begin{aligned} \text{Index } S_1 \# S_2 &= \text{Dim } N_{S_1 \# S_2} - \text{Dim } N_{(S_1 \# S_2)^*} \\ &= \lambda_A(\text{Dim } N_{S_1} \otimes N_{S_2} + \text{Dim } N_{S_1^*} \otimes N_{S_2^*}) \\ &\quad - \lambda_A(\text{Dim } N_{S_1^*} \otimes N_{S_2} + \text{Dim } N_{S_1} \otimes N_{S_2^*}) \\ &= \mu_{A_1, A_2}(\text{Dim } N_{S_1} \otimes \text{Dim } N_{S_2} + \text{Dim } N_{S_1^*} \otimes \text{Dim } N_{S_2^*}) \\ &\quad - \mu_{A_1, A_2}(\text{Dim } N_{S_1^*} \otimes \text{Dim } N_{S_2} + \text{Dim } N_{S_1} \otimes \text{Dim } N_{S_2^*}) \\ &= \mu_{A_1, A_2}((\text{Dim } N_{S_1} - \text{Dim } N_{S_1^*}) \otimes (\text{Dim } N_{S_2} - \text{Dim } N_{S_2^*})) \\ &= \mu_{A_1, A_2}(\text{Index } S_1 \otimes \text{Index } S_2). \end{aligned} \quad (2.35)$$

3. The Connected Components of $\mathcal{F}(A)$

Lemma 5. Suppose that A is not of finite type. Then there is a unique homomorphism

$$v: G(\bar{A}) \rightarrow I(A) \quad (3.1)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{F}(A) & & \\ \pi \downarrow & \searrow \text{Index} & \\ G(\bar{A}) & \xrightarrow{v} & I(A) \end{array} \quad (3.2)$$

is commutative.

Proof. One has to show that $S \in \mathcal{F}(A)$, $T \in \mathcal{F}(A)$ and $\pi S = \pi T$ imply

$$\text{Index } S = \text{Index } T. \quad (3.3)$$

Because of Theorem 1 there is a $Q \in \mathcal{F}(A)$ such that

$$SQ = 1 - C, \quad C \in \mathfrak{m} \quad (3.4)$$

(3.4) and $\pi S = \pi T$ imply

$$TQ = 1 - D, \quad D \in \mathfrak{m}. \quad (3.5)$$

(3.4), (3.5) and the generalized Fredholm alternatives imply

$$\text{Index } SQ = \text{Index } TQ \quad (3.6)$$

(3.6) and Theorem 2 imply (3.3).

Lemma 6. *The index map is locally constant.*

Proof. If A is of finite type, then the index map is zero, hence constant. Suppose that A is not of finite type. Let $T \in G(A)$. Then its index is zero. Hence v is constant on the open set $\pi G(A)$. Since v is also a homomorphism, it follows that it is locally constant. Hence Lemma 5 implies Lemma 6.

Lemma 7. *Let S, T be Fredholm elements of A . If $\text{Index } S = \text{Index } T$, then S, T lie in the same connected component of $\mathcal{F}(A)$.*

Proof. If A is of finite type, then $A = \mathcal{F}(A)$ and consequently $\mathcal{F}(A)$ is connected. Suppose that A is not of finite type.

1. Suppose that $\text{Index } S = 0$. Then it can be shown that S lies in the same connected component of $\mathcal{F}(A)$ as 1. Let $S = U \cdot |S|$ be the polar decomposition of S . Because of $\pi S \in G(\bar{A})$ the elements $\pi U, \pi |S|$ are also in $G(\bar{A})$. It is clear that

$$(1 - t)\pi |S| + t \cdot \pi 1, \quad 0 \leq t \leq 1 \quad (3.7)$$

is a continuous path in $G(\bar{A})$ joining $\pi |S|$ with $\pi 1$. The connectedness of $\mathfrak{m}$ implies that $|S|$ and 1 lie in the same connected component of $\mathcal{F}(A)$. So let

$$f_t, \quad 0 \leq t \leq 1 \quad (3.8)$$

be a continuous path in $\mathcal{F}(A)$ with

$$f_0 = S, \quad f_1 = U. \quad (3.9)$$

Theorem 2 and $\text{Index } S = 0$ imply $\text{Index } U = 0$. Hence $N_U \sim N_{U^*}$. Choose a partial isometry $V \in A$ such that $N_U = V^* V$, $N_{U^*} = V V^*$. Then $W = V + U$ is a unitary element of A . Also $V \in \mathfrak{m}$ because of $N_U \in \mathfrak{m}$. Hence

$$f_t = U + (t - 1)V, \quad 1 \leq t \leq 2 \quad (3.10)$$

is a continuous path in $\mathcal{F}(A)$ with

$$f_1 = U, \quad f_2 = W. \quad (3.11)$$

Let

$$W = \int_0^{2\pi} e^{i\varphi} dE_\varphi \quad (3.12)$$

be the spectral decomposition of the unitary operator W . If the operator $Q \in \mathcal{L}(H)$ satisfies $QW = WQ$, then $QE_\varphi = E_\varphi Q$ for all φ . Hence $E_\varphi \in A'' = A$ for all φ . It follows that

$$f_{2+t} = \int_0^{2\pi} e^{i(1-t)\varphi} dE_\varphi, \quad 0 \leq t \leq 1 \quad (3.13)$$

is a continuous path in $G(A)$ with

$$f_2 = W, \quad f_3 = 1. \quad (3.14)$$

f_t , $0 \leq t \leq 3$, is a continuous path in $\mathcal{F}(A)$ joining S with 1.

2. Let $\text{Index } S = \text{Index } T$. Then $\text{Index } T^*T = 0$ and $\text{Index } TS^* = 0$. By what has already been proved there are continuous paths f_t, g_t , $0 \leq t \leq 1$ in $\mathcal{F}(A)$ such that

$$f_0 = 1, \quad f_1 = T^*T, \quad g_0 = ST^*, \quad g_1 = 1. \quad (3.15)$$

Hence

$$h_t := \begin{cases} Sf_{2t}, & 0 \leq t \leq 1/2 \\ g_{2t-1}T, & 1/2 \leq t \leq 1 \end{cases} \quad (3.16)$$

is a continuous path in $\mathcal{F}(A)$ joining S with T .

Theorem 4. Let $\pi_0 \mathcal{F}(A)$ be the group of connected components of $\mathcal{F}(A)$ and let

$$\kappa : \mathcal{F}(A) \rightarrow \pi_0 \mathcal{F}(A) \quad (3.17)$$

be the canonical map². Then there is a monomorphism

$$\varrho : \pi_0 \mathcal{F}(A) \rightarrow I(A) \quad (3.18)$$

such that the diagram

$$\begin{array}{ccc} \mathcal{F}(A) & & \\ \kappa \downarrow & \searrow \text{Index} & \\ \pi_0 \mathcal{F}(A) & \xrightarrow{\varrho} & I(A) \end{array} \quad (3.19)$$

is commutative.

Proof. This is an obvious consequence of Lemmas 6 and 7.

Lemma 8. Let A be properly infinite and let E be a finite projection of A . Then

$$E < 1 - E \quad (3.20)$$

and

$$1 - E \sim 1. \quad (3.21)$$

² I.e., if $T \in \mathcal{F}(A)$ then κT is defined as the connected component of $\mathcal{F}(A)$ which contains T .

Proof of (3.20). According to the comparability theorem (Dixmier [4, p. 228] or Breuer [2, Lemma 4]) there is a projection Q in the centre of A such that

$$QE \succ Q(1-E), \quad (1-Q)E \prec (1-Q)(1-E). \quad (3.22)$$

The first relation (3.22) together with the finiteness of QE implies that Q is finite. Since A is properly infinite its centre contains no finite projections except zero. Hence $Q=0$. The second relation (3.22) and $Q=0$ imply (3.20).

Proof of (3.21). Using (3.20) one can inductively construct a sequence $(E_i)_{i=1,2,\dots}$ such that

$$E_0 = E, \quad E_i \sim E_k, \quad E_i E_k = 0 \quad \text{for } i \neq k. \quad (3.23)$$

Then

$$\sum_{i=0}^{\infty} E_i \sim \sum_{i=1}^{\infty} E_i. \quad (3.24)$$

Hence

$$1-E = \sum_{i=1}^{\infty} E_i + \left(1 - \sum_{i=0}^{\infty} E_i\right) \sim \sum_{i=0}^{\infty} E_i + \left(1 - \sum_{i=0}^{\infty} E_i\right) = 1. \quad (3.25)$$

Theorem 5. *If A is properly infinite, then ϱ is an isomorphism of $\pi_0 \mathcal{F}(A)$ onto $I(A)$.*

Proof. Let $\alpha \in I(A)$. There are finite projections $E_1, \dots, E_n, F_1, \dots, F_m$ such that

$$\alpha = \sum_{i=1}^n \text{Dim } E_i - \sum_{k=1}^m \text{Dim } F_k. \quad (3.26)$$

According to Lemma 8 there are partial isometries $U_1, \dots, U_n, V_1, \dots, V_m$ of A such that

$$1 = U_i U_i^*, \quad 1 - E_i = U_i^* U_i, \quad 1 = V_k V_k^*, \quad 1 - F_k = V_k^* V_k \quad (3.27)$$

for $i=1, \dots, n, k=1, \dots, m$. Of course the U_i 's and V_k 's are Fredholm elements of A and satisfy

$$\text{Index } U_i = \text{Dim } E_i, \quad \text{Index } V_k = \text{Dim } F_k. \quad (3.28)$$

Corollary 1 of Theorem 1 and Theorem 2 imply that

$$T = U_1 \cdot \dots \cdot U_n \cdot V_1^* \cdot \dots \cdot V_m^* \quad (3.29)$$

is a Fredholm element of A satisfying

$$\text{Index } T = \alpha. \quad (3.30)$$

Hence ϱ is an epimorphism.

The last two theorems are a partial generalization of a result of Cordes-Labrousse [3] (Fredholm case). To get the full generalization of Cordes-Labrousse' result one would have to consider operators which are affiliated with A and semi-Fredholm relative to A .

4. Appendix: The Functor $I: \mathfrak{N} \rightarrow \mathfrak{A}$

Let $\mathfrak{N}$ be the category whose objects are the von Neumann algebras of linear operators of complex Hilbert spaces and whose morphisms are the $\ast$ -homomorphisms which preserve the relative finiteness of projections³. Let $\mathfrak{A}$ be the category whose objects are the ordered abelian groups and whose morphisms are the order preserving homomorphisms.

If A is an object of $\mathfrak{N}$, then the index group $I(A)$ is an object of $\mathfrak{A}$.

Let $\Phi: A \rightarrow B$ be a morphism of $\mathfrak{N}$. Then the morphism $I(\Phi): I(A) \rightarrow I(B)$ of $\mathfrak{A}$ is defined as follows.

Let E be any finite projection of A . Then ΦE is a finite projection of B by hypothesis. Let π_E resp. $\pi_{\Phi E}$ be the corresponding representations of A' resp. B' . Let $M(A, A')$ resp. $M(B, B')$ be the monoids of equivalence classes of representations of A' resp. B' arising from the finite projections of A resp. B , as defined in [2]. It follows from Lemma 11 in [2] that

$$M(\Phi): M(A, A') \rightarrow M(B, B') \quad (4.1)$$

is well-defined by

$$M(\Phi) [\pi_E] = [\pi_{\Phi E}]. \quad (4.2)$$

Let

$$\gamma_A: M(A, A') \rightarrow I(A), \quad \gamma_B: M(B, B') \rightarrow I(B) \quad (4.3)$$

be the canonical imbeddings. It follows from the universal property of the index group that there is a homomorphism

$$I(\Phi): I(A) \rightarrow I(B) \quad (4.4)$$

satisfying

$$I(\Phi) \circ \gamma_A = \gamma_B \circ M(\Phi). \quad (4.5)$$

It is clear that $I(\Phi)$ is order preserving. Thus $I(\Phi)$ is a well-defined morphism of $\mathfrak{A}$.

Proposition 1. $I: \mathfrak{N} \rightarrow \mathfrak{A}$ is a covariant functor.

Proof. Let A, B, C be objects of $\mathfrak{N}$ and let

$$A \xrightarrow{\Phi} B \xrightarrow{\Psi} C \quad (4.6)$$

be morphisms. For every finite projection E of A one has

$$M(\Psi \circ \Phi) [\pi_E] = [\pi_{\Psi(\Phi E)}] = M(\Psi) [\pi_{\Phi E}]. \quad (4.7)$$

Hence

$$M(\Psi \circ \Phi) = M(\Psi) \circ M(\Phi). \quad (4.8)$$

One also has

$$I(\Phi) \circ \gamma_A = \gamma_B \circ M(\Phi), \quad I(\Psi) \circ \gamma_B = \gamma_C \circ M(\Psi) \quad (4.9)$$

³ I.e., the $\ast$ -homomorphism $\Phi: A \rightarrow B$ is a morphism of $\mathfrak{N}$ if the finiteness of any projection E relative to A implies the finiteness of the projection ΦE relative to B .

and

$$I(\Psi \circ \Phi) \circ \gamma_A = \gamma_C \circ M(\Psi \circ \Phi). \quad (4.10)$$

An easy calculation yields

$$I(\Psi \circ \Phi) \circ \gamma_A = I(\Psi) \circ I(\Phi) \circ \gamma_A. \quad (4.11)$$

Since the range of γ_A generates $I(A)$ it follows that

$$I(\Psi \circ \Phi) = I(\Psi) \circ I(\Phi). \quad (4.12)$$

It is obvious that $I(1_A)$ is the identity map of $I(A)$ if 1_A is the identity map of A . Together with (4.12) this implies that I is a covariant functor.

Proposition 2. *If A, B are $*$ -isomorphic von Neumann algebras, then the ordered abelian groups $I(A), I(B)$ are isomorphic.*

Proof. Let Φ be a $*$ -isomorphism of A onto B and let Φ^{-1} be its inverse. Φ, Φ^{-1} are morphisms of $\mathfrak{N}$. Hence $I(\Phi), I(\Phi^{-1})$ are well-defined. Proposition 1 (or (4.12)) imply that $I(\Phi^{-1}) = I(\Phi)^{-1}$. Hence $I(A), I(B)$ are isomorphic.

In the following let A, B be two von Neumann algebras. $\mathfrak{m}_A, \mathfrak{m}_B$ resp. $\text{Index}_A, \text{Index}_B$ denote the corresponding ideals of compact elements resp. index functions.

Proposition 3. *Let A be a subalgebra of B and let the inclusion map*

$$\Phi : A \rightarrow B \quad (4.13)$$

be a morphism of $\mathfrak{N}$. Then

$$\mathcal{F}(A) = A \cap \mathcal{F}(B) \quad (4.14)$$

and

$$\text{Index}_B \Phi T = I(\Phi) \text{Index}_A T \quad (4.15)$$

for every $T \in \mathcal{F}(A)$.

Proof. One has

$$\mathfrak{m}_A = A \cap \mathfrak{m}_B \quad (4.16)$$

since Φ is a morphism of $\mathfrak{N}$. Hence $A/\mathfrak{m}_A$ is a closed $*$ -subalgebra of $B/\mathfrak{m}_B$. The unit element of $B/\mathfrak{m}_B$ is contained in $A/\mathfrak{m}_A$. Hence

$$G(A/\mathfrak{m}_A) = A/\mathfrak{m}_A \cap G(B/\mathfrak{m}_B) \quad (4.17)$$

since $B/\mathfrak{m}_B$ is a B^* -algebra (Dixmier [5, 1.3.10]). It follows that $T \in A$ is regular modulo $\mathfrak{m}_A$ if and only if ΦT is regular modulo $\mathfrak{m}_B$. Together with Theorem 1 this implies (4.14). Observe that the lattice operations in $P(A)$ are the same as in $P(B)$ (Kaplansky [8]). Hence

$$\Phi N_T = N_{\Phi T} \quad (4.18)$$

for all $T \in A$. (4.18) implies (4.15).

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