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Newforms and Functional Equations

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Introduction

Let k be a fixed positive integer. If $F(\tau)$ is a holomorphic function on the upper half plane and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a real matrix with positive determinant, then $F \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right.$ denotes the function on the upper half plane defined by

$$F \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. (\tau) = (ad - bc)^{k/2} (c\tau + d)^{-k} F \left(\frac{a\tau + b}{c\tau + d} \right),$$

and $F \left| \sum_i c_i A_i \right.$ is $\sum_i c_i F|A_i$ where c_i are constants, and A_i are 2×2 real matrices with positive determinant. For positive integers M, N we define

$$\Gamma_0(M, N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \left| \begin{array}{l} M|c, N|b \end{array} \right. \right\}$$

and

$$\Gamma(M, N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M, N) \left| \begin{array}{l} a \equiv d \equiv 1 \pmod{MN} \end{array} \right. \right\}.$$

Thus $\Gamma_0(M, 1) = \Gamma_0(M)$ and $\Gamma(M, 1) = \Gamma_1(M)$ in the usual notation. If $\varepsilon(n)$ is a character of the integers mod MN such that $\varepsilon(-1) = (-1)^k$, we let $C_\varepsilon(M, N)$ denote the space of all cusp forms for $\Gamma(M, N)$ of weight k and character ε , i.e. $F \in C_\varepsilon(M, N)$ if and only if F is holomorphic on the upper half plane, satisfies $F \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. = \varepsilon(d) F$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M, N)$, satisfies the usual regularity conditions at the cusps of $\Gamma(M, N)$, and vanishes at each cusp of $\Gamma(M, N)$. Therefore the space $C(M, N)$ of all cusp forms for $\Gamma(M, N)$ of weight k is $\bigoplus_\varepsilon C_\varepsilon(M, N)$. Hecke operators $T_n = T^{MN}(n)$ are defined on $C_\varepsilon(M, N)$ for each positive integer n with $(n, MN) = 1$ (see [3], Nos. 35 and 36) and such T_n map $C_\varepsilon(M, N)$ into itself (Satz 32 and Satz 33 in [3, No. 36]).

Let $F \in C_\varepsilon(M, N)$ be a common eigenform of all the Hecke operators T_n with $(n, MN) = 1$. If there are no other linearly independent cusp forms

in $C_\varepsilon(M, N)$ which have the same eigenvalue for each T_n , F is called a newform. When ε is the principal character and $N = 1$, the above defines newforms for the group $\Gamma_0(M)$, which coincide with those defined by Atkin and Lehner [1]. In their paper [1], they show how the space of cusp forms for $\Gamma_0(M)$ may be built up of newforms of lower level and newforms of level M . In Section 2, we shall perform an analogous construction for the spaces $C_\varepsilon(M, N)$, and thereby study some properties of newforms. This in particular answers the problem raised in [1] of extending the concept of the newform to the case of $\Gamma(M)$ since, as remarked by Hecke, the space of all cusp forms for the group $\Gamma(M)$ of weight k is simply the direct sum of $C_\varepsilon(M, M)$ over all characters $\varepsilon \bmod M$.

Necessary and sufficient conditions for the associated Dirichlet series of a cusp form in $C_\varepsilon(M, 1)$ to have analytic continuation, satisfy functional equations, etc. are explored in Section 3. A characterization of newforms in $C_\varepsilon(M, 1)$ in terms of functional equations is also given there. In the last section, a theorem of Ogg in [8] is generalized, although not completely.

For convenience, p and q always denote primes; $\bar{z}$ is the complex conjugate of z . It is understood that all the cusp forms mentioned in this paper are of fixed weight k .

1. Basic Lemmas

Besides Hecke operators T_n , we need the following operators. For each positive integer d and $F \in C_\varepsilon(M, N)$, let

$$F|B_d = d^{-k/2} F \begin{vmatrix} d & 0 \\ 0 & 1 \end{vmatrix}$$

$$F|W_d = d^{k/2} F \begin{vmatrix} 1 & 0 \\ 0 & d \end{vmatrix}$$

$$F|U_d^N = d^{k/2-1} F \left| \sum_{u \bmod d} \begin{pmatrix} 1 & uN \\ 0 & d \end{pmatrix} \right| = d^{-1} F|W_d \left| \sum_{u \bmod d} \begin{pmatrix} 1 & uN \\ 0 & 1 \end{pmatrix} \right|.$$

Since $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1/d & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & dy \\ z/d & w \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/d \end{pmatrix} = \begin{pmatrix} x & y/d \\ dz & w \end{pmatrix}$, we have

$$B_d: C_\varepsilon(M, N) \rightarrow \begin{cases} C_\varepsilon(dM, N) \\ C_\varepsilon(dM, N/d) & \text{if } d|N, \end{cases}$$

and

$$W_d: C_\varepsilon(M, N) \rightarrow \begin{cases} C_\varepsilon(M, Nd) \\ C_\varepsilon(M/d, Nd) & \text{if } d|M. \end{cases}$$

It is clear that if $d|N$, $B_d: C_\varepsilon(M, N) \rightarrow C_\varepsilon(dM, N/d)$ is an isomorphism with the inverse map W_d . In particular, $C_\varepsilon(M, N)$ is isomorphic to $C_\varepsilon(MN, 1)$ under B_N .

Lemma 1. *If $q|M$, then $U_q^1: C_\varepsilon(M, 1) \rightarrow C_\varepsilon(M, 1)$. Furthermore, if $q^2|M$ and ε is a character mod M/q , then $U_q^1: C_\varepsilon(M, 1) \rightarrow C_\varepsilon(M/q, 1)$.*

Proof. Let $F \in C_\varepsilon(M, 1)$, then $F|W_q \in C_\varepsilon(M/q, q)$ since $q|M$. $\Gamma_0(M, q)$ [resp. $\Gamma_0(M/q, q)$] is a subgroup of $\Gamma_0(M, 1)$ [resp. $\Gamma_0(M/q, 1)$] with $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, $0 \leq u < q$, as a set of right coset representatives since $q|M$ (resp. $q^2|M$). Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M, 1)$ [resp. $\Gamma_0(M/q, 1)$]. Since $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_{u'} & b_{u'} \\ c_{u'} & d_{u'} \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}$ where $\begin{pmatrix} a_{u'} & b_{u'} \\ c_{u'} & d_{u'} \end{pmatrix} \in \Gamma_0(M, q)$ [resp. $\Gamma_0(M/q, q)$] and $d_{u'} \equiv d \pmod{M}$ (resp. $\pmod{M/q}$); ε is a character mod M (resp. $\pmod{M/q}$), in either case we have

$$\begin{aligned} F|U_q^1 \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. &= q^{-1} F|W_q \left| \sum_{u \pmod{q}} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right| \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. \\ &= q^{-1} F|W_q \left| \sum_{u' \pmod{q}} \begin{pmatrix} a_{u'} & b_{u'} \\ c_{u'} & d_{u'} \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right. = \varepsilon(d) F|U_q^1. \end{aligned}$$

Q.E.D.

Using $U_q^N B_N = B_N U_q^1$, we get $U_q^N: C_\varepsilon(M, N) \rightarrow C_\varepsilon(M, N)$ if $q|MN$.

If we express a form $F \in C_\varepsilon(M, N)$ by its Fourier expansion, namely, $F(\tau) = \sum_{n \geq 1} a(n) z_N^n$ where $z_N = e^{2\pi i \tau/N}$, then

$$F|B_d(\tau) = \sum_{n \geq 1} a(n) z_N^{nd},$$

$$F|W_d(\tau) = \sum_{n \geq 1} a(n) z_N^n,$$

$$F|U_d^N(\tau) = \sum_{n \geq 1} a(nd) z_N^n,$$

$$F|T_p(\tau) = \sum_{n \geq 1} (\varepsilon(p) p^{k-1} a(n/p) + a(np)) z_N^n \quad \text{if } (p, MN) = 1$$

where $a(x) = 0$ if x is not an integer.

From the above expression, we can easily show that $T_p B_q = B_q T_p$, $T_p W_q = W_q T_p$, and $T_p U_q^N = U_q^N T_p$ provided $(p, qMN) = 1$. Also $U_d^N B_d = B_d U_d^N$ if $(d, d') = 1$.

For a positive integer M and prime q , define $V_q^M = \begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix}$ where $\alpha = \text{ord}_q M$, x, y, z are integers such that $q^{2\alpha} x - yMz = q^\alpha$. If $F \in C_\varepsilon(M, 1)$

and ε is a character mod M/q^α , then $F|V_q^M$ is a well-defined function in $C_\varepsilon(M, 1)$ because

$$\begin{aligned} & \begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -y'/q^\alpha \\ -M/q^\alpha z' & x' \end{pmatrix} \\ &= \begin{pmatrix} q^\alpha xa + yc & q^\alpha xb + yd \\ Mza + q^\alpha c & Mzb + q^\alpha d \end{pmatrix} \begin{pmatrix} 1 & -y'/q^\alpha \\ -M/q^\alpha z' & x' \end{pmatrix} \\ &= \begin{pmatrix} * & * \\ * & D \end{pmatrix} \in \Gamma_0(M, 1) \end{aligned}$$

and $D \equiv d \pmod{M/q^\alpha}$ provided $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M, 1)$ and $q^{2\alpha}x - yzM = q^\alpha = q^{2\alpha}x' - y'zM$.

Lemma 2. If $q' | M$ and $(q', q) = 1$, then $B_{q'} V_q^M = V_{q'}^{M/q'} B_{q'}$. If ε is a character mod M/q^α where $\alpha = \text{ord}_q M$, then $V_q^M : C_\varepsilon(M, 1) \rightarrow C_\varepsilon(M, 1)$, $V_q^M T_p = T_p V_q^M$ provided $(p, M) = 1$, and $F|V_q^M|V_q^M = \varepsilon(q^\alpha)F$ for all $F \in C_\varepsilon(M, 1)$. In particular, this shows V_q^M is an automorphism.

Proof. The first statement follows from

$$\begin{pmatrix} q' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix} \begin{pmatrix} 1/q' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q^\alpha x & q' y \\ zM/q' & q^\alpha \end{pmatrix} \text{ and } \text{ord}_q M = \text{ord}_q M/q'.$$

Now assume that ε is a character mod M/q^α where $\alpha = \text{ord}_q M$. The discussion above Lemma 2 says that V_q^M maps $C_\varepsilon(M, 1)$ into itself. If $(p, M) = 1$, we can choose y, z such that $y \equiv z \equiv 0 \pmod{p}$. Let $F \in C_\varepsilon(M, 1)$. Since $(x, p) = 1$, for each $0 \leq t < p$, there exists a unique $0 \leq t' < p$ such that $t \equiv t'x \pmod{p}$, therefore

$$\begin{aligned} & \begin{pmatrix} 1 & tM \\ 0 & p \end{pmatrix} \begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix} \begin{pmatrix} 1 & -t'M/p \\ 0 & 1/p \end{pmatrix} \\ &= \begin{pmatrix} q^\alpha x + tM^2 z & -t'M(q^\alpha x + tM^2 z)/p + y/p + tMq^\alpha/p \\ pMz & -t'M^2 z + q^\alpha \end{pmatrix} \end{aligned}$$

for $0 \leq t < p$ shows

$$F \left| \sum_{t \pmod{p}} \begin{pmatrix} 1 & tM \\ 0 & p \end{pmatrix} \right| V_q^M = F|V_q^M \left| \sum_{t' \pmod{p}} \begin{pmatrix} 1 & t'M \\ 0 & p \end{pmatrix} \right|.$$

(Note that $-t'M^2 z/q^\alpha + 1 \equiv 1 \pmod{M}$.) On the other hand,

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix} \begin{pmatrix} 1/p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q^\alpha x & py \\ Mz/p & q^\alpha \end{pmatrix}$$

implies

$$F \left| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right| V_q^M = F|V_q^M \left| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right|.$$

Thus we have shown $V_q^M T_p = T_p V_q^M$ provided $(p, M) = 1$.

Finally, since

$$\begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix} \begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix} = \begin{pmatrix} q^{2\alpha} x^2 + yMz & q^\alpha xy + yq^\alpha \\ q^\alpha xMz + q^\alpha Mz & Myz + q^{2\alpha} \end{pmatrix} = \begin{pmatrix} q^\alpha A & q^\alpha B \\ q^\alpha C & q^\alpha D \end{pmatrix}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(M, 1)$ and $D \equiv q^\alpha \pmod{M/q^\alpha}$, we get

$$F|V_q^M|V_q^M = F \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| = \varepsilon(q^\alpha) F. \quad \text{Q.E.D.}$$

Two forms $F, G \in C_\varepsilon(M, N)$ are said to be equivalent, denoted by $F \sim G$, iff $F|T_p = \lambda_p \cdot F$ and $G|T_p = \lambda_p \cdot G$ for all $(p, MN) = 1$. So, if $F \in C_\varepsilon(M, N)$ is an eigenform of all T_p with $(p, MN) = 1$, then $F|B_q \sim F$, $F|U_q^N \sim F$ and $F|W_q \sim F$ provided $q|MN$. Further, if $\alpha = \text{ord}_q M$ and ε is a character mod M/q^α , then $F \sim F|V_q^M$.

Lemma 3. Suppose $q|M$, $q^2 \nmid M$ and ε is a character mod M/q . Then

$$U_q^1 + q^{k/2-1} V_q^M : C_\varepsilon(M, 1) \rightarrow C_\varepsilon(M/q, 1).$$

Proof. Note that $V_q^M = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} qx & y \\ Mz/q & 1 \end{pmatrix} = q^{-k/2} W_q \begin{pmatrix} qx & y \\ Mz/q & 1 \end{pmatrix}$.

$\Gamma_0(M/q, q)$ is a subgroup of $\Gamma_0(M/q, 1)$ with $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = R_t$, $0 \leq t < q$, and

$\begin{pmatrix} qx & y \\ Mz/q & 1 \end{pmatrix} = R_q$ as a set of right coset representatives. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M/q, 1)$, then

$$\left\{ R_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid 0 \leq t \leq q \right\} = \left\{ \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} R_t \mid 0 \leq t \leq q \right\}$$

where $\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} \in \Gamma_0(M/q, q)$

and $d_t \equiv d \pmod{M/q}$. Therefore,

$$\begin{aligned} F|(qU_q^1 + q^{k/2} V_q^M) \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= F|W_q \left| \sum_{t=0}^q R_t \right| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = F|W_q \left| \sum_{t=0}^q \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} R_t \right| \\ &= \varepsilon(d) F|W_q \left| \sum_{t=0}^q R_t \right| = \varepsilon(d) F|(qU_q^1 + q^{k/2} V_q^M) \text{ for all } F \in C_\varepsilon(M, 1). \end{aligned}$$

Q.E.D.

Lemma 4. Suppose $F = G|B_d \in C_\varepsilon(M, 1)$ where $G(\tau) = \sum_{n \geq 1} a(n) z_1^n$ and d is a prime.

- i) If $d|M$ and ε is a character mod M/d , then $G \in C_\varepsilon(M/d, 1)$.
- ii) If $d \nmid M$, then $F = 0$.

Proof. $G = F | W_d \in C_\varepsilon(M, d)$ [resp. $C_\varepsilon(M/d, d)$] if $d \nmid M$ (resp. $d | M$). By hypothesis, G is invariant under $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, therefore G is invariant under the group $\Gamma(M, 1)$ [resp. $\Gamma(M/d, 1)$] since d is a prime. $\Gamma_0(M, d)$ [resp. $\Gamma_0(M/d, d)$] is a subgroup of $\Gamma_0(M, 1)$ [resp. $\Gamma_0(M/d, 1)$] with a set of right coset representatives $R_i \in \Gamma(M, 1)$ [resp. $\Gamma(M/d, 1)$]. If $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \Gamma_0(M, 1)$ [resp. $\Gamma_0(M/d, 1)$], then $\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix} R_i$ for some i , where $\begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix} \in \Gamma_0(M, d)$ [resp. $\Gamma_0(M/d, d)$] and $w \equiv w' \pmod{M}$ (resp. $\pmod{M/d}$). Since ε is a character mod M (resp. mod M/d), in either case we have $G \begin{vmatrix} x & y \\ z & w \end{vmatrix} = G \begin{vmatrix} x' & y' \\ z' & w' \end{vmatrix} R_i = \varepsilon(w) \cdot G$. This shows i) and $G \in C_\varepsilon(M, 1)$ if $d \nmid M$. For ii), we use

$$G | B_d^i | T_d = d^{3k/2-1} \varepsilon(d) G | B_d^{i+1} + d^k G | B_d^{i-1} \quad \text{for all } i \geq 1$$

to prove inductively that $F | B_d^i \in C_\varepsilon(M, 1)$ for all $i \geq 1$. This implies $F = 0$ because $C_\varepsilon(M, 1)$ is of finite dimension. Q.E.D.

2. Newforms in $C_\varepsilon(M, N)$

Let $C_\varepsilon^-(M, N)$ be the vector subspace of $C_\varepsilon(M, N)$ generated by all $G | B_d$ and $H | W_{d'}$ where $G \in C_\varepsilon(M', N)$, $H \in C_\varepsilon(M, N')$; M', N' are proper factors of M, N respectively; $d | M/M'$ and $d' | N/N'$. Since T_p commutes with B_d and $W_{d'}$ if $(p, MN) = 1$, we have

$$T_p : C_\varepsilon^-(M, N) \rightarrow C_\varepsilon^-(M, N) \quad \text{for all } (p, MN) = 1.$$

From the following commutative diagrams (true in all possible directions):

$$\begin{array}{ccc} C_\varepsilon(M, N) & \xleftrightarrow[B_N]{W_N} & C_\varepsilon(MN, 1) \\ \uparrow W_{d'} & & \uparrow B_{N/(N', d')} \\ C_\varepsilon(M', N') & \xleftrightarrow[W_{N'}]{B_{N'}} & C_\varepsilon(M'N', 1) \end{array}$$

$$\begin{array}{ccc} C_\varepsilon(M, N) & \xleftrightarrow[B_N]{W_N} & C_\varepsilon(MN, 1) \\ \uparrow B_d & & \uparrow B_{Nd/N'} \\ C_\varepsilon(M', N') & \xleftrightarrow[W_{N'}]{B_{N'}} & C_\varepsilon(M'N', 1) \end{array}$$

where $M' | M$, $N' | N$, $d | M/M'$, and $d' | N/N'$, we see that $C_\varepsilon^-(M, N)$ is mapped isomorphically onto $C_\varepsilon^-(MN, 1)$ by B_N with the inverse map W_N .

Theorem 1. Let $F(\tau) = \sum_{n \geq 1} a(n) z_1^n \in C_\varepsilon(M, 1)$ such that $a(n) = 0$ if $(n, K) = 1$ where K is a fixed positive integer. Then

- i) $a(n) = 0$ if $(n, M) = 1$.
- ii) If $q|M$ and ε is a character mod M/q , set

$$\varphi = \begin{cases} F|U_q^1 & \text{if } q^2|M \text{ or } (K, M) = q \\ ((1 + q^{-1})^{-1} F|(U_q^1 + q^{k/2-1} V_q^M)) & \text{otherwise.} \end{cases}$$

Then $\varphi \in C_\varepsilon(M/q, 1)$ and $F - \varphi|B_q(\tau) = \sum_{n \geq 1} b(n) z_1^n \in C_\varepsilon(M, 1)$ with $b(n) = 0$ for $(n, (K, M)) = 1$ or $n = \text{some power of } q$.

Note that in Case ii), if F is an eigenform of all T_p with $(p, M) = 1$, so are φ and $F - \varphi|B_q$, both equivalent to F .

Proof. (Almost rewrite the proof of Theorem 1 in Atkin and Lehner [1].) For a prime p , introduce the annihilator operator A_p defined by

$$F|A_p = F - F|U_p^1|B_p.$$

Thus $F|A_p \in C_\varepsilon(Mp^2, 1)$. If $p|M$ and ε is a character mod M/p , $F|A_p \in C_\varepsilon(Mp, 1)$ or $F|A_p \in C_\varepsilon(M, 1)$ according to $p|M$ or $p^2|M$ by Lemma 1. Suppose $p_1, \dots, p_r$ are the distinct primes dividing K and say, $p_r \nmid M$. Then from hypothesis, $F|A_{p_1} \dots |A_{p_{r-1}} = \Phi|B_{p_r} \in C_\varepsilon(Mp_1^2 \dots p_{r-1}^2, 1)$ where $\Phi(\tau) = \sum_{n \geq 1} c(n) z_1^n$. Apply Lemma 4, $\Phi = 0$. Therefore $a(n) = 0$ if $(n, p_1 \dots p_{r-1}) = 1$. Inductively, we get i). For ii), it is clear that $F - \varphi|B_q$ satisfies the requirement if $q^2|M$ or $(M, K) = q$. If $q^2 \nmid M$, write

$F = \sum_{i=1}^t \Phi_i|B_{q_i}$ where $q_1, \dots, q_t = q$ are the distinct primes dividing (M, K) , $\Phi_1 = F|U_{q_1}^1$, $\Phi_i = F|A_{q_1} \dots |A_{q_{i-1}}|U_{q_i}^1$ for $i \geq 2$. We may assume $t > 1$. It is clear that $\Phi_i|B_{q_i} \in C_\varepsilon(Mq_1^2 \dots q_{i-1}^2, 1)$ for $t > i \geq 1$ and

$$\Phi_t|B_q = F|A_{q_1} \dots |A_{q_{t-1}} \in C_\varepsilon(Mq_1^2 \dots q_{t-1}^2, 1),$$

so by Lemma 4, $\Phi_t \in C_\varepsilon(Mq_1^2 \dots q_{t-1}^2/q, 1)$. Apply Lemma 3,

$$G = F|(U_q^1 + q^{k/2-1} V_q^M) \in C_\varepsilon(M/q, 1).$$

Note that

$$\text{ord}_q Mq_1^2 \dots q_{t-1}^2 = \text{ord}_q M = \text{ord}_q Mq_1^2 \dots q_{t-1}^2 q_i = 1$$

for $i = 1, \dots, t-1$. So

$$\begin{aligned} \Phi_t|B_q|(U_q^1 + q^{k/2-1} V_q^{Mq_1^2 \dots q_{t-1}^2}) &= \Phi_t + q^{-1} \Phi_t \begin{vmatrix} q & 0 \\ 0 & 1 \end{vmatrix} V_q^{Mq_1^2 \dots q_{t-1}^2} \\ &= \Phi_t + q^{-1} \Phi_t|L \quad \text{with some } L \in \Gamma(Mq_1^2 \dots q_{t-1}^2/q, 1) \\ &= (1 + q^{-1}) \Phi_t \quad \text{since } \Phi_t \in C_\varepsilon(Mq_1^2 \dots q_{t-1}^2/q, 1) \end{aligned}$$

and for $1 \leq i < t$

$$\begin{aligned}\Phi_i | B_{q_i} | (U_q^1 + q^{k/2-1} V_q^M q_1^2 \dots q_{t-1}^2) \\ = \Phi_i | U_q^1 | B_{q_i} + q^{k/2-1} \Phi_i | V_q^M q_1^2 \dots q_{t-1}^{2-1/q_i} | B_{q_i} \quad \text{by Lemma 2} \\ = \Psi_i | B_{q_i} \quad \text{where} \quad \Psi_i = \Phi_i | U_q^1 + q^{k/2-1} \Phi_i | V_q^{(M/q_i)} q_1^2 \dots q_{t-1}^{2-1} \in C_\varepsilon(M^3, 1).\end{aligned}$$

Therefore

$$\begin{aligned}G = F | (U_q^1 + q^{k/2-1} V_q^M q_1^2 \dots q_{t-1}^2) \\ = (1 + q^{-1}) \Phi_t + \sum_{i=1}^{t-1} \Psi_i | B_{q_i} = (1 + q^{-1}) \varphi \in C_\varepsilon(M/q, 1).\end{aligned}$$

This shows $\varphi \in C_\varepsilon(M/q, 1)$ and

$$\begin{aligned}F - \varphi | B_q = \sum_{i=1}^t \Phi_i | B_{q_i} - \Phi_t | B_q - \sum_{i=1}^{t-1} (1 + q^{-1})^{-1} \Psi_i | B_{q_i} \\ = \sum_{i=1}^{t-1} (\Phi_i - (1 + q^{-1})^{-1} \Psi_i) | B_{q_i}, \quad \text{as desired.} \quad \text{Q.E.D.}\end{aligned}$$

More generally, we have

Theorem 1'. Let $F(\tau) = \sum_{n \geq 1} a(n) z_N^n \in C_\varepsilon(M, N)$ such that $a(n) = 0$ if $(n, K) = 1$ where K is a fixed positive integer. Then

- i) $a(n) = 0$ if $(n, MN) = 1$.
- ii) If $q | MN$ and ε is a character mod MN/q , then there exists $\varphi \in C_\varepsilon^-(M, N)$ such that $F - \varphi = \sum_{n \geq 1} b(n) z_N^n$ with $b(n) = 0$ for $(n, (MN, K)) = 1$ or $n = \text{some power of } q$.

Proof. Let $G = F | B_N = \sum_{n \geq 1} a(n) z_1^n \in C_\varepsilon(MN, 1)$. i) Follows immediately from Theorem 1, i). For ii), let

$$\varphi' = \begin{cases} G | U_q^1 & \text{if } q^2 | MN \text{ or } (K, MN) = q \\ (1 + q^{-1})^{-1} G | (U_q^1 + q^{k/2-1} V_q^{MN}) & \text{otherwise.} \end{cases}$$

Put $\varphi = \varphi' | B_q | W_N$. Then $\varphi' \in C_\varepsilon(MN/q, 1)$, $\varphi \in C_\varepsilon^-(M, N)$ and $F - \varphi$ has the required property. Q.E.D.

We now prove the crucial

Theorem 2. Suppose $F(\tau) = \sum_{n \geq 1} a(n) z_N^n \in C_\varepsilon(M, N)$ such that $a(n) = 0$ if $(n, K) = 1$ where K is a fixed positive integer. Then

$$F \in C_\varepsilon^-(M, N).$$

Proof. Let $G(\tau) = F|B_N(\tau) = \sum_{n \geq 1} a(n)z_1^n \in C_\varepsilon(MN, 1)$. Since $G \in C_\varepsilon^-(MN, 1)$ iff $F \in C_\varepsilon^-(M, N)$, we may assume $N = 1$ to begin with. To proceed, we need the following

Lemma 5. Suppose $F(\tau) = \sum_{n \geq 1} a(nq)z_1^n \in C_\varepsilon(M, 1)$, $q|M$ and ε is not a character mod M/q . Then $F = 0$.

Proof. Let $G(\tau) = F|W_q(\tau) = \sum_{n \geq 1} a(nq)z_1^n \in C_\varepsilon(M/q, q)$. It suffices to show $G = 0$. It is clear that $\begin{pmatrix} 1 & q \\ M/q & M+1 \end{pmatrix} \in \Gamma(M/q, q)$ and G is invariant under

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q \\ M/q & M+1 \end{pmatrix} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+uM/q & u'(1+uM/q)+q+uM+u \\ M/q & u'M/q+M+1 \end{pmatrix} \\ = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{for all } u, u' \in \mathbb{Z}.$$

Note that $B \equiv u'(1+uM/q)+u = u(1+u'M/q)+u' \pmod{q}$. Since ε is not a character mod M/q , there is some $u' \in \mathbb{Z}$ such that $u'M/q+1 \not\equiv 0 \pmod{q}$ and $\varepsilon(u'M/q+1) \neq 1$. For this u' , there exists $u \in \mathbb{Z}$ such that $B \equiv 0 \pmod{q}$.

Choosing u, u' as above, we have $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(M/q, q)$ and $G = G \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right. = \varepsilon(u'M/q+1)G$. This shows $G = 0$.

Now we prove Theorem 2. By Theorem 1', working modulo $C_\varepsilon^-(M, 1)$, we may assume $q_1, \dots, q_r$ are the distinct primes dividing M such that ε is not a character mod M/q_i for $i = 1, \dots, r$, and $F(\tau) = \sum_{n \geq 1} a(n)z_1^n$ with

$a(n) = 0$ if $(n, q_1 \dots q_r) = 1$. Let $\Phi_r = F|A_{q_1} \dots A_{q_{r-1}}|U_{q_r}^1$. Then $\Phi_r|B_{q_r} = F|A_{q_1} \dots A_{q_{r-1}} \in C_\varepsilon(Mq_1^2 \dots q_{r-1}^2, 1)$. Since ε is not a character mod $Mq_1^2 \dots q_{r-1}^2/q_r$, by Lemma 5, $\Phi_r = 0$. This shows $a(n) = 0$ if $(n, q_1 \dots q_{r-1}) = 1$. By induction, we see $F = 0$. Q.E.D.

Combining the proof above and Theorem 1, we get

Corollary 1. If $F(\tau) = \sum_{n \geq 1} a(n)z_1^n \in C_\varepsilon(M, 1)$ such that $a(n) = 0$ if

$(n, K) = 1$ where K is a fixed positive integer, then $F = \sum_{i=1}^r \varphi_i|B_{q_i}$ where $q_1, \dots, q_r$ are the primes dividing (M, K) such that ε is a character mod M/q_i and $\varphi_i \in C_\varepsilon(M/q_i, 1)$ for all $i = 1, \dots, r$.

Let F and G be two cusp forms for some principal congruence subgroup $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N, N) \mid a \equiv d \equiv 1 \pmod{N} \right\}$. The "absolute Pe-

tersson inner product" is defined by

$$\langle F, G \rangle = \frac{1}{|\Gamma(1) : \Gamma(N)|} \int_{D(N)} \int F(\tau) \overline{G(\tau)} y^{k-2} dx dy$$

where $\tau = x + iy$ and $D(N)$ is a fundamental domain for $\Gamma(N)$ (cf. Ogg [8]). Further, $\langle F|E, G|E \rangle = \langle F, G \rangle$ for all 2×2 integer matrices E of positive determinant (a fact due to Petersson [9]). Let $C_\varepsilon^+(M, N)$ be the orthogonal complement of $C_\varepsilon^-(M, N)$ in $C_\varepsilon(M, N)$. Since T_p maps $C_\varepsilon^-(M, N)$ into itself and $\langle F|T_p, G \rangle = \varepsilon(p) \langle F, G|T_p \rangle$ for all $(p, MN) = 1$, $F, G \in C_\varepsilon(M, N)$, we know that $C_\varepsilon^+(M, N)$ is invariant under T_p . The formula $\langle F|B_N, G|B_N \rangle = \langle F, G \rangle$ for all $F, G \in C_\varepsilon(M, N)$ shows $F \in C_\varepsilon^+(M, N)$ iff $F|B_N \in C_\varepsilon^+(MN, 1)$.

Definition. A newform $F \in C_\varepsilon(M, N)$ is a non-zero form in $C_\varepsilon^+(M, N)$ which is a common eigenform of all T_p with $(p, MN) = 1$. The form $F(\tau) = \sum_{n \geq 1} a(n) z_N^n \in C_\varepsilon(M, N)$ is normalized if $a(1) = 1$.

Suppose $F(\tau) = \sum_{n \geq 1} a(n) z_N^n \in C_\varepsilon(M, N)$ is a newform such that $F|T_p = \lambda_p F$ for all $(p, MN) = 1$. The formula

$$\varepsilon(p) p^{k-1} a(n/p) + a(np) = \lambda_p a(n) \quad \text{for all } n \geq 1, (p, MN) = 1$$

shows in particular $a(p) = \lambda_p a(1)$. Theorem 2 implies $a(1) \neq 0$. So assume F is normalized. If G is a normalized cusp form in $C_\varepsilon(M, N)$ such that $G|T_p = \lambda_p G$ for almost all primes p with $(p, MN) = 1$, then by Theorem 2 again we get $F - G \in C_\varepsilon^-(M, N)$. This shows F is uniquely determined up to a constant multiple by its eigenvalues, in fact, any other normalized cusp form in $C_\varepsilon(M, N)$ which is a common eigenform of all T_p with $(p, MN) = 1$ must have infinitely many distinct eigenvalues since $C_\varepsilon(M, N)$ has an orthogonal basis consisting of forms which are common eigenforms of all T_p with $(p, MN) = 1$.

A form $G \in C_\varepsilon^-(M, N)$ is called an oldform if $G = H|B_d|W_{d'}$, where H is a newform in $C_\varepsilon(M', N')$ with $M'|M$, $N'|N$, $d|M/M'$, and $d'|N/N'$. By induction on number of prime factors (counting multiplicity) of MN , we see $C_\varepsilon^-(M, N)$ is generated by all oldforms in $C_\varepsilon^-(M, N)$. Thus if $G \in C_\varepsilon^-(M, N)$ and G is a common eigenform of all T_p with $(p, MN) = 1$, then there exists a newform H in $C_\varepsilon(M', N')$ with $M'|M$, $N'|N$ such that $G \sim H$. We also see that any non-zero cusp form in $C_\varepsilon^-(M, N)$ which is a common eigenform of all T_p with $(p, MN) = 1$ cannot be determined up to a constant multiple by its eigenvalues. Thus the definition of newforms in $C_\varepsilon(M, N)$ given above is equivalent to the one given in the introduction.

Now let $F(\tau) = \sum_{n \geq 1} a(n) z_N^n$ be a normalized newform in $C_\varepsilon(M, N)$. Thus $G(\tau) = F|B_N(\tau) = \sum_{n \geq 1} a(n) z_1^n$ is a normalized newform in $C_\varepsilon(MN, 1)$.

Suppose $G|T_p = \lambda_p G$ for all $(p, MN) = 1$, then $\lambda_p = a(p)$ for all $(p, MN) = 1$ and

$$\varepsilon(p) p^{k-1} a(n/p) + a(np) = a(p) a(n) \quad \text{for all } n \geq 1, (p, MN) = 1.$$

Let $q|MN$. Then $G|U_q^1(\tau) = \sum_{n \geq 1} a(nq) z_1^n \sim G$ and

$$G \sim G|U_q^1 - a(q) G \in C_\varepsilon^-(MN, 1)$$

by Theorem 2, hence $G|U_q^1 - a(q) G = 0$. In other words, we have

$$a(nq) = a(n) a(q) \quad \text{for all } n \geq 1, q|MN.$$

Case I. ε is a character mod MN/q .

i) $q^2|MN$. Then $G \sim G|U_q^1 \in C_\varepsilon(MN/q, 1)$ by Lemma 1. This shows $G|U_q^1 = 0$, i.e. $a(q) = 0$.

ii) $q^2 \nmid MN$. Lemma 3 says $G|U_q^1 + G|q^{k/2-1} V_q^{MN} \in C_\varepsilon(MN/q, 1)$.

Apply Lemma 2, $G|V_q^{MN} \sim G$. Therefore

$$G|U_q^1 = a(q) G = -q^{k/2-1} G|V_q^{MN}.$$

Since $G|V_q^{MN}|V_q^{MN} = \varepsilon(q) G$ (Lemma 2), we have $a(q)^2 = \varepsilon(q) q^{k-2}$.

Case II. ε is not a character mod MN/q . We apply Corollary 1 in Ogg [7] to conclude that $|a(q)| = q^{(k-1)/2}$.

To sum up, we state the following

Theorem 3. Let $F(\tau) = \sum_{n \geq 1} a(n) z_N^n$ be a normalized newform in $C_\varepsilon(M, N)$, p a prime with $(p, MN) = 1$, q a prime dividing MN . Then

i) $F|T_p = a(p) F$, $F|U_q^N = a(q) F$ and

$$\begin{cases} \varepsilon(p) p^{k-1} a(n/p) + a(np) = a(p) a(n) & \text{for all } n \geq 1 \\ a(nq) = a(n) a(q) & \text{for all } n \geq 1. \end{cases}$$

Therefore, if $L(s) = \sum_{n \geq 1} a(n) n^{-s}$ is the associated Dirichlet series of F , then $L(s)$ has the Euler product

$$L(s) = \prod_{q|MN} (1 - a(q) q^{-s})^{-1} \prod_{p \nmid MN} (1 - a(p) p^{-s} + \varepsilon(p) p^{k-1-2s})^{-1}.$$

ii) $|a(q)| = q^{(k-1)/2}$ if ε is not a character mod MN/q .

iii) If ε is a character mod MN/q , then $a(q) = 0$ if $q^2|MN$; $a(q)^2 = \varepsilon(q) q^{k-2}$ if $q^2 \nmid MN$. In the latter case, we have

$$-a(q) q^{-k/2+1} F = F|B_N|V_q^{MN}|W_N.$$

Remarks. i) For $N = 1$ and $\varepsilon = 1$, the principal character, the problem is completely solved by Atkin and Lehner in their paper [1].

ii) For $N=1$ and $M=\text{prime } q$, most of these results are found in Hecke's two short papers [3, Nos. 39 and 40].

iii) Ogg in his paper [7] gave the results ii) and part of iii) for the case $N=1$. The case of arbitrary N is an easy generalization.

To continue, we need the following two operators:

$$F|H_M = F \left| \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \right. \quad \text{and} \quad F|K(\tau) = \overline{F(-\bar{\tau})}.$$

It is easy to check that H_M, K map $C_\varepsilon(M, 1)$ to $C_\varepsilon(M, 1)$ with the properties: $H_M^2 = (-1)^k$, $K^2 = 1$, $H_M K = (-1)^k K H_M$, $T_p K = K T_p$, $K B_d = B_d K$, $H_M T_p = \bar{\varepsilon}(p) T_p H_M$ if $(p, M) = 1$; and if $F(\tau) = \sum_{n \geq 1} a(n) z_1^n \in C_\varepsilon(M, 1)$, then $F|K(\tau) = \sum_{n \geq 1} \overline{a(n)} z_1^n$ (cf. Ogg [7]). K maps $C_\varepsilon^-(M, 1)$ isomorphically

onto $C_\varepsilon^-(M, 1)$, so does H_M since $G|B_d|H_M = d^{-k/2} G|H_{M/d}$ and

$$G|H_M = d^{k/2} G|H_{M/d}|B_d \quad \text{for all } G \in C_\varepsilon(M/d, 1)$$

with $d|M$, $d \neq 1$. Furthermore, the fact $\langle F, G \rangle = \langle F|H_M, G|H_M \rangle$ implies H_M is an isomorphism from $C_\varepsilon^+(M, 1)$ onto $C_\varepsilon^+(M, 1)$.

Suppose F, G are two cusp forms for the group $\Gamma(N)$, then

$$\begin{aligned} \langle F|K, G|K \rangle &= \frac{1}{|\Gamma(1) : \Gamma(N)|} \int_{\overline{D(N)}} \int \overline{F(-\bar{\tau})} G(-\bar{\tau}) y^{k-2} dx dy \\ &= \frac{1}{|\Gamma(1) : \Gamma(N)|} \int_{-\overline{D(N)}} \int G(\tau) \overline{F(\tau)} y^{k-2} dx dy. \end{aligned}$$

Obviously, $-\overline{D(N)}$ is a fundamental domain for $\Gamma(N)$. We thus have $\langle F|K, G|K \rangle = \langle G, F \rangle$. This relation in particular implies K maps $C_\varepsilon^+(M, 1)$ isomorphically onto $C_\varepsilon^+(M, 1)$.

Now let F be a normalized newform in $C_\varepsilon(M, 1)$. The above discussion tells us $F|K$ is a normalized newform in $C_\varepsilon(M, 1)$ and $F|H_M = \gamma F|K$ for some constant γ .

If $M'|M$, then $\Gamma_0(M', 1)$ is a subgroup of $\Gamma_0(M, 1)$ with right coset representatives $R_i \in \Gamma(M', 1)$. For $F \in C_\varepsilon(M, 1)$ and ε a character mod M' , we define "Trace" function by:

$$\text{Tr}_{M'}^M F = \sum_i F|R_i.$$

Thus $\text{Tr}_{M'}^M F$ does not depend on representatives and is in $C_\varepsilon(M', 1)$.

Lemma 6. Suppose prime $q|M$, $F \in C_\varepsilon(M, 1)$ and ε is a character mod M/q .

i) If $M|N$ and $\text{ord}_q M = \text{ord}_q N$, then $\text{Tr}_{M/q}^M F = \text{Tr}_{N/q}^N F$.

ii) Let d be a positive integer such that $(d, q) = 1$. Then

$$\text{Tr}_{M^d/q}^{M^d} (F|B_d) = (\text{Tr}_{M/q}^M F)|B_d.$$

iii) If $\text{Tr}_{M/q}^M F = 0 = \text{Tr}_{M/q}^M F|H_M$, then $\langle F, G \rangle = 0$ and $\langle F, G|B_q \rangle = 0$ for all $G \in C_\varepsilon(M/q, 1)$.

iv) If $\text{Tr}_{M/q}^M F = 0 = \text{Tr}_{M/q}^M F|H_M$ and d is a positive integer such that $(d, q) = 1$, then $\langle F|B_d, G \rangle = 0$ and $\langle F|B_d, G|B_q \rangle = 0$ for all $G \in C_\varepsilon(N/q, 1)$ with $1 \leq \text{ord}_q N \leq \text{ord}_q M$.

Proof. i) If $R_i \in \Gamma(N/q, 1)$ are a set of right coset representatives for $\Gamma_0(N, 1)$ in $\Gamma_0(N/q, 1)$, then R_i are also a set of right coset representatives for $\Gamma_0(M, 1)$ in $\Gamma_0(M/q, 1)$ since

$$|\Gamma_0(N/q, 1) : \Gamma_0(N, 1)| = |\Gamma_0(M/q, 1) : \Gamma_0(M, 1)| \quad (\text{ord}_q M = \text{ord}_q N)$$

and $R_i = AR_j$ for some A in $\Gamma_0(M, 1)$ implies A in $\Gamma_0(M, 1) \cap \Gamma(N/q, 1)$, hence in $\Gamma_0(N, 1)$. This proves i).

ii) It suffices to prove the case $d = \text{prime}$, relatively prime to q . If $R_i \in \Gamma(Md/q, 1)$ are a set of right coset representatives for $\Gamma_0(Md, 1)$ in $\Gamma_0(Md/q, 1)$, then $R'_i = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} R_i \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \Gamma(M/q, d)$ are a set of right coset representatives for $\Gamma_0(M, d)$ in $\Gamma_0(M/q, d)$. We claim that R'_i are also a set of right coset representatives for $\Gamma_0(M, 1)$ in $\Gamma_0(M/q, 1)$ for $|\Gamma_0(M/q, d) : \Gamma_0(M, d)| = |\Gamma_0(Md/q, 1) : \Gamma_0(Md, 1)| = |\Gamma_0(M/q, 1) : \Gamma_0(M, 1)|$ since $(q, d) = 1$; and if $R'_i = AR'_j$ for some $A \in \Gamma_0(M, 1)$, then $A \in \Gamma_0(M, d)$. Therefore $\text{Tr}_{Md/q}^{Md} (F|B_d) = \sum_i F|B_d|R_i = \sum_i F|R'_i|B_d = (\text{Tr}_{M/q}^M F)|B_d$.

iii) Let $G \in C_\varepsilon(M/q, 1)$. Then $\langle F, G \rangle = \text{const.} \langle \text{Tr}_{M/q}^M F, G \rangle = 0$. Since $\text{Tr}_{M/q}^M F|H_M = 0$, $F|H_M$ is orthogonal to every element in $C_\varepsilon(M/q, 1)$. Therefore $\langle F, G|B_q \rangle = \langle F|H_M, G|B_q|H_M \rangle = q^{-k/2} \langle F|H_M, G|H_{M/q} \rangle = 0$ since $G|H_{M/q} \in C_\varepsilon(M/q, 1)$.

iv) We may assume $Md|N$. By iii), it suffices to show

$$\text{Tr}_{N/q}^N (F|B_d) = 0 = \text{Tr}_{N/q}^N (F|B_d|H_N).$$

But

$$\text{Tr}_{N/q}^N (F|B_d) = \text{Tr}_{Md/q}^{Md} (F|B_d) \quad (\text{by i)}) = (\text{Tr}_{M/q}^M F)|B_d \quad (\text{by ii)}) = 0$$

and

$$\begin{aligned} \text{Tr}_{N/q}^N (F|B_d|H_N) &= d^{-k/2} \text{Tr}_{N/q}^N (F|H_{N/d}) \\ &= \text{const.} \text{Tr}_{N/q}^N (F|H_M|B_{N/(Md)}) \\ &= \text{const.} (\text{Tr}_{Md/q}^{Md} F|H_M)|B_{N/(Md)} \quad (\text{by ii)}) \\ &= \text{const.} (\text{Tr}_{M/q}^M F|H_M)|B_{N/(Md)} \quad (\text{by i)}) \\ &= 0, \quad \text{as we want.} \end{aligned}$$

Q.E.D.

It was Serre's idea to use Trace function to characterize the space generated by newforms for the group $\Gamma_0(q)$ with $\varepsilon = 1$, $q = \text{prime}$, namely, $F \in C_1(q, 1)$ is in $C_1^+(q, 1)$ iff $\text{Tr}_1^q F = 0 = \text{Tr}_1^q F|H_q$ (cf. Serre [12]). This turns out true in general as follows:

Theorem 4. Let $F \in C_\varepsilon(M, N)$. Then $F \in C_\varepsilon^+(M, N)$ iff for all prime $q | MN$ such that ε is a character mod MN/q ,

$$\text{Tr}_{MN/q}^{MN}(F|B_N) = 0 = \text{Tr}_{MN/q}^{MN}(F|B_N|H_{MN}).$$

(The case $N = 1$, $\varepsilon = 1$ had been proved by Ogg.)

Proof. We may assume $N = 1$.

"Only if" part: We may assume F is a newform. Suppose q is a prime dividing M such that ε is a character mod M/q , then

$$\langle \text{Tr}_{M/q}^M F, \text{Tr}_{M/q}^M F \rangle = |\Gamma_0(M/q, 1) : \Gamma_0(M, 1)| \langle F, \text{Tr}_{M/q}^M F \rangle = 0$$

since $\text{Tr}_{M/q}^M F \in C_\varepsilon(M/q, 1)$ and F is a newform. This shows $\text{Tr}_{M/q}^M F = 0$. Since F is a newform in $C_\varepsilon(M, 1)$, we see that $F|H_M$ is a newform in $C_\varepsilon(M, 1)$ from the discussion above Lemma 6. Therefore $\text{Tr}_{M/q}^M F|H_M = 0$.

"If" part: A direct consequence of Lemma 6 iii) and the definition of $C_\varepsilon^+(M, 1)$. Q.E.D.

Corollary 2. Suppose F is a newform in $C_\varepsilon(M, 1)$, $G \in C_\varepsilon(N, 1)$ and there is a prime q such that $\text{ord}_q M > \text{ord}_q N$. Then $\langle F|B_d, G|B_{d'} \rangle = 0$ provided $\text{ord}_q d' \geq \text{ord}_q d$ and $\text{ord}_q d' - \text{ord}_q d + \text{ord}_q N \leq \text{ord}_q M$.

Proof. Say, $\text{ord}_q d = r$. Since $\langle F|B_d, G|B_{d'} \rangle = 0$ iff

$$\langle F|B_d|W_{q^r}, G|B_{d'}|W_{q^r} \rangle = 0,$$

we may assume $r = 0$, i.e. $(d, q) = 1$ to begin with. If $(d', q) = 1$, $G|B_{d'} \in C_\varepsilon(Nd', 1)$ with $\text{ord}_q Nd' = \text{ord}_q N < \text{ord}_q M$; if $(d', q) = q$, $G|B_{d'/q} \in C_\varepsilon(Nd'/q, 1)$ with $\text{ord}_q Nd' = \text{ord}_q N + \text{ord}_q d' \leq \text{ord}_q M$ by hypothesis; apply Theorem 4 and Lemma 6 iv) to F in both cases. Q.E.D.

Let F, G be normalized newforms in $C_1(M, 1)$, $C_1(N, 1)$ respectively. Atkin and Lehner proved if F and G have the same eigenvalues with respect to T_p for almost all primes p such that $(p, MN) = 1$, then $F = G$ and $M = N$ by using ζ -functions (cf. [1], Theorem 4). In the following we prove this for arbitrary character, using the notion of Trace:

Theorem 5. Let $F(\tau) = \sum_{n \geq 1} a(n) z_1^n$, $G(\tau) = \sum_{n \geq 1} b(n) z_1^n$ be normalized newforms in $C_\varepsilon(M, 1)$, $C_\varepsilon(N, 1)$ respectively such that $a(p) = b(p)$ for almost all primes p . Then $F = G$ and $M = N$.

Proof. Let $F - G(\tau) = \sum_{n \geq 1} c(n) z_1^n$, $L = \text{l.c.m.}(M, N)$ and $K = \text{product of all primes } p \text{ such that } a(p) \neq b(p)$. Then $F - G \in C_\varepsilon(L, 1)$ and $c(n) = 0$ if $(n, K) = 1$. By Corollary 1, we can write $F - G = \sum_{i=1}^r \theta_i |B_{q_i}$, where $q_1, \dots, q_r$ are the distinct primes dividing (L, K) such that ε is a character mod L/q_i and $\theta_i \in C_\varepsilon(L/q_i, 1)$. (If no such q_i exists, by convention, we mean

$F - G = 0$.) If for each $i = 1, \dots, r$, $\text{ord}_{q_i} M = \text{ord}_{q_i} N = \text{ord}_{q_i} L$, applying Lemma 6 iv) to newforms F, G , get $\langle F, \theta_i | B_{q_i} \rangle = 0 = \langle G, \theta_i | B_{q_i} \rangle$ for every $i = 1, \dots, r$. This shows $\langle F - G, F - G \rangle = 0$, i.e. $F = G$. If not, say, $q_1, \dots, q_s$ ($r \geq s \geq 1$) are the primes among q_i such that $\text{ord}_{q_i} M \neq \text{ord}_{q_i} N$, this will lead to a contradiction. Let $f_0 = F, g_0 = G$.

Case I. $r > s$ or $r = s$ and $q_r^2 | L$. Construct non-zero forms f_i, g_i for $i = 1, \dots, s$ inductively such that

$$\begin{aligned} f_i &= \text{linear combination of } f_0 | B_d, \\ g_i &= \text{linear combination of } g_0 | B_d, \end{aligned} \quad d | q_1^2 \dots q_i^2,$$

and $f_0 | B_d$ (resp. $g_0 | B_d$) has nonzero coefficient only if $Md | L$ (resp. $Nd | L$) as follows: Suppose we already have f_{i-1}, g_{i-1} , $s \geq i \geq 1$.

Define

$$h_i = \begin{cases} (f_{i-1} - g_{i-1}) | U_{q_i}^1 & \text{if } q_i^2 | L \\ (1 + q_i^{-1})^{-1} (f_{i-1} - g_{i-1}) | (U_{q_i}^1 + q_i^{k/2-1} V_{q_i}^L) & \text{otherwise.} \end{cases}$$

Since $\text{ord}_{q_i} M \neq \text{ord}_{q_i} N$, we may assume $\text{ord}_{q_i} L = \text{ord}_{q_i} M > \text{ord}_{q_i} N$.

i) $q_i^2 | L$. Then $f_0 | U_{q_i}^1 = 0$ since $q_i^2 | M$, and

$$g_0 | U_{q_i}^1 = \begin{cases} b(q_i) g_0 & \text{if } q_i | N \\ b(q_i) g_0 - \varepsilon(q_i) q_i^{k-1} g_0 | B_{q_i} & \text{if } q_i \nmid N. \end{cases}$$

From $B_d U_{q_i}^1 = U_{q_i}^1 B_d$ if $(d, q_i) = 1$, we know

$$h_i = \begin{cases} -b(q_i) g_{i-1} & \text{if } q_i | N \\ -b(q_i) g_{i-1} + \varepsilon(q_i) q_i^{k-1} g_{i-1} | B_{q_i} & \text{if } q_i \nmid N. \end{cases}$$

ii) $q_i^2 \nmid L$. Then

$$f_0 | (U_{q_i}^1 + q_i^{k/2-1} V_{q_i}^L) = 0 \quad \text{if } M | L' \quad \text{and} \quad \text{ord}_{q_i} M = \text{ord}_{q_i} L'$$

and

$$g_0 | (U_{q_i}^1 + q_i^{k/2-1} V_{q_i}^L) = b(q_i) g_0 \quad \text{if } N | L'' \quad \text{and} \quad \text{ord}_{q_i} L'' = 1.$$

If $f_0 | B_d$ (resp. $g_0 | B_d$) appears in the summation of f_{i-1} (resp. g_{i-1}), we have f_0 (resp. g_0) $| B_d | V_{q_i}^L = f_0$ (resp. g_0) $| V_{q_i}^{L/d} | B_d$ by Lemma 2, and $M | L/d$, $\text{ord}_{q_i} M = \text{ord}_{q_i} L/d$ (resp. $N | L/d$, $\text{ord}_{q_i} L/d = \text{ord}_{q_i} L = 1$) by hypothesis on f_{i-1} (resp. g_{i-1}). Therefore,

$$h_i = -(1 + q_i^{-1})^{-1} b(q_i) g_{i-1}.$$

In either case, let $f_i = f_{i-1}, g_i = g_{i-1} + h_i | B_{q_i}$. From the construction of f_i, g_i , Theorem 1 and Corollary 1, we see inductively that for $s \geq i \geq 1$,

$$\begin{aligned} f_i - g_i &= (f_{i-1} - g_{i-1}) - h_i | B_{q_i} \\ &= \sum_{j>i}^r \text{some form in } C_\varepsilon(L/q_j, 1) | B_{q_j}. \end{aligned}$$

In particular, $f_s - g_s = \sum_{j>s}^r \psi_j |B_{q_j}$ where $\psi_j \in C_\varepsilon(L/q_j, 1)$. If $f_0 | B_d$ (resp. $g_0 | B_d$) appears in the summation of f_s (resp. g_s), then

$$\langle f_0 \text{ (resp. } g_0) | B_d, \psi_j | B_{q_j} \rangle = 0 \quad \text{for every } r \geq j > s$$

by Corollary 2. This shows $\langle f_s - g_s, f_s - g_s \rangle = 0$, i.e. $f_s = g_s$.

Case II. $r = s$ and $q_r | L$, $q_r^2 \nmid L$. Construct $f_0, \dots, f_{s-1}$ as in Case I. Then $f_{s-1} - g_{s-1} = \varphi_s | B_{q_s}$. Therefore $\varphi_s = (f_{s-1} - g_{s-1}) | U_{q_s}^1$. Assume $\text{ord}_{q_s} M = \text{ord}_{q_s} L = 1 > \text{ord}_{q_s} N = 0$. Then $f_0 | U_{q_s}^1 = a(q_s) f_0$ and $g_0 | U_{q_s}^1 = b(q_s) g_0 - \varepsilon(q_s) q_s^{k-1} g_0 | B_{q_s}$. Let

$$f_s = f_{s-1} - f_{s-1} | U_{q_s}^1 | B_{q_s} = f_{s-1} - a(q_s) f_{s-1} | B_{q_s},$$

$$g_s = g_{s-1} - g_{s-1} | U_{q_s}^1 | B_{q_s} = g_{s-1} - b(q_s) g_{s-1} | B_{q_s} + \varepsilon(q_s) q_s^{k-1} g_{s-1} | B_{q_s}^2.$$

Thus $f_s = g_s$, $f_s \neq 0$, and $g_s \neq 0$.

Suppose $f_0 | B_d$, $g_0 | B_{d'}$ appear in the summation of f_s, g_s respectively in Cases I or II. If there exists some prime $q \neq q_1, \dots, q_s$ such that $\text{ord}_q M \neq \text{ord}_q N$, then $(d, q) = 1$, $(d', q) = 1$, $\langle f_0 | B_d, g_0 | B_{d'} \rangle = 0$ by Corollary 2. Now assume for every prime $q \neq q_1, \dots, q_s$, $\text{ord}_q M = \text{ord}_q N$. Then Case II can happen only if $r = s > 1$ (otherwise, we would have $M | N$ or $N | M$, and $M \neq N$, impossible). Consider $\text{ord}_{q_1} M$ and $\text{ord}_{q_1} N$, say $\text{ord}_{q_1} M > \text{ord}_{q_1} N$. Then in either case, we have $(d, q_1) = 1$ and $\text{ord}_{q_1} N d' \leq \text{ord}_{q_1} L = \text{ord}_{q_1} M$. Applying Corollary 2 again, get $\langle f_0 | B_d, g_0 | B_{d'} \rangle = 0$. In any case, we have shown $\langle f_s, g_s \rangle = 0$, i.e. $f_s = g_s = 0$, a contradiction. Therefore, we cannot have $s \geq \frac{1}{2}$, thus $F = G$.

We prove $M = N$ by

Lemma 7. If $F \in C_\varepsilon(M, 1) \cap C_\varepsilon(N, 1)$ and $D = \text{g.c.d.}(M, N)$, then $F \in C_\varepsilon(D, 1)$.

Proof. It suffices to prove the case $N = p^\alpha N'$, $M = p^\beta M'$ with $(p, M'N') = 1$ and $|\alpha - \beta| = 1$, say $\alpha - \beta = 1$, then $F \in C_\varepsilon(p^\beta N', 1)$. [Note that ε is a character mod $\text{g.c.d.}(M, N)$, hence mod $p^\beta N'$.] For this, we have only to find a set of right coset representatives R_i for $\Gamma_0(p^\alpha N', 1)$ in $\Gamma_0(p^\beta N', 1)$ such that $R_i \in \Gamma(p^\beta N', 1) \cap \Gamma_0(p^\beta M', 1)$, which is visibly possible since $(p, M') = 1$.

Remark. Theorem 5 was proved in different formulations by Miyake [6], following Weil [17], and by Casselman [2], using representation method, respectively. To the author's knowledge, the elementary proof given above is new.

Corollary 3. Suppose $F(\tau) = \sum_{n \geq 1} a(n) z_N^n$, $G(\tau) = \sum_{n \geq 1} b(n) z_N^n$ are newforms in $C_\varepsilon(M, N)$, $C_\varepsilon(M', N')$ respectively such that $a(p) = b(p)$ for almost all primes p . Then $a(n) = b(n)$ for all $n \geq 1$ and $M'N' = MN$.

3. Dirichlet Series and Functional Equations

To begin with, we state a theorem due to Weil: (cf. [16]).

Theorem 6 (Weil). Let $F(\tau) = \sum_{n \geq 1} a(n) z_1^n \in C_\varepsilon(M, 1)$;

$$L_\chi(s) = \sum_{n \geq 1} a(n) \chi(n) n^{-s} \quad \text{and} \quad \Lambda_\chi(s) = (m/(2\pi))^s \Gamma(s) L_\chi(s)$$

be functions defined on some half plane $\operatorname{Re}(s) > \sigma > 0$ where χ is a primitive character with conductor $f_\chi = m$ relatively prime to M . Suppose $F = C \cdot i^k \cdot F|H_M$ with $C = \pm 1$. Write

$$C_\chi = C \frac{\mathcal{G}(\chi)}{\mathcal{G}(\bar{\chi})} \chi(-M) \varepsilon(f_\chi)$$

where $\mathcal{G}(\chi) = \text{Gauss sum} = \sum_{y=1}^m \chi(y) e^{2\pi i y/m}$. Then Λ_χ has analytic continuation in the whole s -plane, remains bounded in each vertical strip $\sigma' \leq \operatorname{Re}(s) \leq \sigma''$ and satisfies

$$\Lambda_\chi(s) = C_\chi M^{k/2-s} \Lambda_{\bar{\chi}}(k-s) \quad \text{whenever} \quad (M, m) = 1.$$

Conversely, suppose $F(\tau) = \sum_{n \geq 1} a(n) z_1^n$ is a formal power series with complex coefficients such that $a(n) = O(n^\sigma)$ for some $\sigma > 0$ and the associated Dirichlet series $L(s) = \sum_{n \geq 1} a(n) n^{-s}$ converges absolutely at $s = k - \delta$ for some $k > \delta > 0$. ε is a character mod M . If the above results for Λ_χ hold for almost all primitive characters χ with the conductor $f_\chi = m$, an odd prime not dividing M , and for the principal character $\varepsilon = 1$, then $F \in C_\varepsilon(M, 1)$ and $F = C \cdot i^k \cdot F|H_M$. (Note that the character ε here must be real.)

Further, we prove

Theorem 7. Let $F(\tau) = \sum_{n \geq 1} a(n) z_1^n \in C_\varepsilon(M, 1)$ and $L(s) = \sum_{n \geq 1} a(n) n^{-s}$ its associated Dirichlet series. Suppose $F|H_M = \gamma F$, γ a constant, and L has Euler-product

$$L(s) = \prod_{q|M} (1 - a(q) q^{-s})^{-1} \prod_{p \nmid M} (1 - a(p) p^{-s} + \varepsilon(p) p^{k-1-2s})^{-1}.$$

Then $\varepsilon = 1$, the principal character, and F is a normalized newform in $C_1(M, 1)$.

Proof. $L(s)$ has Euler-product means $F|T_p = a(p)F$ for all $(p, M) = 1$ and $F|U_q^1 = a(q)F$ for all $q|M$. Since $H_M T_p = \bar{\varepsilon}(p) T_p H_M$ for all $(p, M) = 1$, the assumption $F|H_M = \gamma F$ implies for $(p, M) = 1$,

$$\gamma a(p)F = \gamma F|T_p = F|H_M|T_p = \bar{\varepsilon}(p)F|T_p|H_M = \bar{\varepsilon}(p)a(p)F|H_M = \bar{\varepsilon}(p)a(p)\gamma F.$$

$L(s) \neq 0$ says $F \neq 0$. There exists a normalized newform $G \in C_\varepsilon(N, 1)$ with $N|M$ such that $F \sim G$, i.e. $G|T_p = a(p)G$ for all $(p, M) = 1$. Therefore $a(p) \neq 0$ for almost all primes p and the above formula implies $\bar{\varepsilon}(p) = 1$ for almost all primes p . This shows $\varepsilon = 1$, the principal character.

Now write $F = \sum_{d|M/N} c_d \cdot G|B_d$ where c_d are complex constants. Clearly, $c_1 = 1$. $\gamma F = F|H_M$ implies $\gamma^2 = (-1)^k$, in particular, $|\gamma| = 1$. G is a normalized newform in $C_1(N, 1)$, hence is an eigenform of H_N , say $\gamma' G = G|H_N$ with $|\gamma'| = 1$. From

$$\begin{aligned} \gamma F = F|H_M &= \sum_{d|M/N} c_d G|B_d|H_M = \sum_d c_d \cdot d^{-k/2} G|H_{M/d} \\ &= \sum_d c_d \cdot d^{-k/2} (M/(Nd))^{k/2} G|H_N|B_{M/(Nd)} \\ &= \sum_{d|M/N} c_d \cdot d^{-k} \cdot (M/N)^{k/2} \gamma' \cdot G|B_{M/(Nd)}, \end{aligned}$$

by comparing coefficients of $G|B_d$, we get a relation:

$$\gamma c_d = \gamma' c_{M/(Nd)} d^k (M/N)^{-k/2} \quad \text{for all } d|M/N$$

since $G|B_d$ are linearly independent as d runs over all factors of M/N . Taking absolute value of both sides and setting $d = 1$, we get

$$1 = |c_{M/N}| (M/N)^{-k/2}. \quad (1)$$

For $q|M$, $a(q) = 0$ if $q^2|M$; $a(q) = \pm q^{k/2-1}$ if $q^2 \nmid M$ by Theorem 1 in Ogg [8] (cf. Ogg [7, 8]). Suppose $G(\tau) = \sum_{n \geq 1} b(n) z_1^n$. Then

$$\begin{aligned} a(q)F = F|U_q^1 &= \sum_{q \nmid d} c_d G|B_d|U_q^1 + \sum_{q|d} c_d G|B_d|U_q^1 \\ &= \sum_{q \nmid d} c_d G|U_q^1|B_d + \sum_{q|d} c_d G|B_{d/q}. \end{aligned}$$

Case I. $q|M$, $q \nmid N$, $q^2 \nmid M$. Then $G|U_q^1 = b(q)G - q^{k-1}G|B_q$ and

$$\begin{aligned} \sum_d a(q) c_d G|B_d &= a(q)F = \sum_{q \nmid d} c_d b(q) G|B_d - q^{k-1} \sum_{q|d} c_d G|B_{d/q} \\ &\quad + \sum_{q|d} c_d G|B_{d/q}. \end{aligned}$$

Thus

$$\begin{cases} a(q) c_d = b(q) c_d + c_{dq} & \text{for } q \nmid d|M/N. \\ a(q) c_{dq} = -q^{k-1} c_d \end{cases} \quad (2)$$

From (3), $c_q = -a(q)^{-1} q^{k-1} = -a(q)q$ since $a(q)^2 = q^{k-2}$. This together with (2), we get

$$b(q) = a(q) - c_q = a(q)(q^2 - 1)/(q - 1).$$

Using the formula

$$q^{k-1} b(n/q) + b(nq) = b(n) b(q),$$

we obtain inductively that

$$b(q^n) = a(q)^n (q^{n+1} - 1)/(q - 1).$$

This shows $b(q^n)$ is exactly $O((q^n)^{k/2})$ as $n \rightarrow \infty$, contradictory to the fact $b(n) = O(n^{k/2-1/5})$ as $n \rightarrow \infty$ (cf. Selberg [11] and Rankin [10]). Therefore Case I cannot happen.

Case II. $q|M$, $q \nmid N$, $q^2|M$. Then $a(q) = 0$, $G|U_q^1 = b(q)G - q^{k-1}G|B_q$ and

$$0 = a(q)F = \sum_{q \nmid d} c_d b(q) G|B_d - q^{k-1} \sum_{q \nmid d} c_d G|B_q|B_d + \sum_{q|d} c_d G|B_{d/q}.$$

Thus

$$\begin{cases} b(q)c_d + c_{dq} = 0 & \text{for } q \nmid d|M/N \\ -q^{k-1}c_d + c_{qd^2} = 0 & \text{for } q \nmid d|M/N, \\ c_d = 0 & \text{if } q^3|d|M/N. \end{cases} \quad (4)$$

(5)

(1) and (5) imply $q^3 \nmid M/N$.

Case III. $q|N$ and $q|M/N$. Then $a(q) = 0$, $G|U_q^1 = b(q)G$, and

$$0 = a(q)F = \sum_{q \nmid d} c_d b(q) G|B_d + \sum_{q|d} c_d G|B_{d/q}.$$

Thus

$$\begin{cases} c_d b(q) + c_{dq} = 0 & \text{for } q \nmid d|M/N, \\ c_d = 0 & \text{if } q^2|d|M/N. \end{cases} \quad (6)$$

(7)

(1) and (7) imply $q^2 \nmid M/N$.

Therefore we can write $M/N = K_2 K_3$ such that $q|K_2$ iff $q^2|M$, $q \nmid N$; $q|H_3$ iff $q|M/N$, $q|N$. Thus $K_2 = (q'_1 \dots q'_r)^2$, $K_3 = q''_1 \dots q''_s$ where q'_i , q''_j are distinct primes dividing M/N . From (4) and (6), we know

$$c_{M/N} = c_{K_2 K_3} = K_2^{(k-1)/2} (-1)^s b(q''_1) \dots b(q''_s).$$

$c_{M/N} \neq 0$ shows $b(q''_j) \neq 0$ i.e. $|b(q''_j)| = (q''_j)^{k/2-1}$, therefore

$$|c_{M/N}| = K_2^{(k-1)/2} K_3^{k/2-1}$$

and (1) becomes $1 = (M/N)^{-k/2} |c_{M/N}| = K_2^{-\frac{1}{2}} K_3^{-1}$. This shows $K_2 = K_3 = 1$, i.e. $M = N$, hence $F = G$ is a normalized newform in $C_1(M, 1)$. Q.E.D.

Remarks. i) Let E be an elliptic curve defined over $\mathbb{Q}$, the rational field, with the conductor M and the associated "zeta-function"

$$L(s) = \prod_{q|M} (1 - a(q)q^{-s})^{-1} \prod_{p \nmid M} (1 - a(p)p^{-s} + p^{1-2s})^{-1}$$

(for detailed definition, see Ogg [8]). Granting Hasse conjecture, namely, $L(s)$ has the properties stated in Theorem 6, converse part with $k=2$, Weil's theorem says that the associated function $F(\tau) = \sum_{n \geq 1} a(n) z_1^n$ is a cusp form of weight 2 for $\Gamma_0(M, 1)$ and $F|H_M = -C \cdot F$. Theorem 7 says F is in fact a newform in $C_1(M, 1)$. With respect to this newform F , we can construct a Shimura curve E' defined over $\mathbb{Q}$ with zeta-function $L'(s)$ and the associated modular form F' . It follows from the "congruence relations for the modular correspondences" of Eichler and Shimura [15] that $L'(s)$ is the same as $L(s)$, except perhaps for the factors corresponding to the bad primes. Since F and F' are both newforms, we must have $F = F'$ or $L = L'$. We now state

Conjecture (of Weil). Every elliptic curve defined over $\mathbb{Q}$ is a weak Weil curve (cf. Mazur [5]).

ii) Theorem 7 says that the normalized newforms in $C_1(M, 1)$ are exactly those which are eigenforms of H_M and whose associated Dirichlet series have Euler product. A generalization of Weil's theorem to $\varepsilon \neq 1$ is Theorem 8 below, followed by Theorem 9 which gives a criterion for newforms in $C_\varepsilon(M, 1)$ of arbitrary character, similar to Theorem 7.

To begin with, we set up the following conventions: Let M, K be fixed integers, $F(\tau) = \sum_{n \geq 1} a(n) z_1^n$ where $a(n)$ are complex numbers with $a(n) = O(n^\sigma)$ for some $\sigma > 0$, $G(\tau) = F|K(\tau) = \sum_{n \geq 1} \overline{a(n)} z_1^n$. For a primitive character χ with conductor $f_\chi = m$, relatively prime to M , we define

$$\begin{aligned} L_\chi(s) &= \sum_{n \geq 1} a(n) \chi(n) n^{-s}, & \tilde{L}_\chi(s) &= \sum_{n \geq 1} \overline{a(n)} \chi(n) n^{-s} \\ A_\chi(s) &= \left(\frac{m}{2\pi}\right)^s \Gamma(s) L_\chi(s), & \tilde{A}_\chi(s) &= \left(\frac{m}{2\pi}\right)^s \Gamma(s) \tilde{L}_\chi(s) \\ F_\chi(\tau) &= \sum_{n \geq 1} a(n) \chi(n) z_1^n & \text{and} & \quad G_\chi(\tau) = \sum_{n \geq 1} \overline{a(n)} \chi(n) z_1^n = F_\chi|K(\tau) \end{aligned}$$

on the half plane $\operatorname{Re}(s) > \sigma$ and the upper half plane $\operatorname{Im}(\tau) > 0$.

It is known that $A_\chi, \tilde{A}_\chi$ have analytic continuation in the whole s -plane, remain bounded on each vertical strip $\sigma' \leq \operatorname{Re}(s) \leq \sigma''$ and satisfy the functional equation $A_\chi(s) = C_\chi M^{k/2-s} \tilde{A}_\chi(k-s)$ for some constant C_χ [call this condition (A)] iff

$$F_\chi = C_\chi i^k G_\chi |H_{Mm^2} = C_\chi i^k \cdot F_\chi |K |H_{Mm^2}.$$

Since

$$\chi(-1) \cdot F_\chi = F \left| \frac{\varphi(\chi)}{m} \sum_{a \bmod m} \bar{\chi}(a) \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| \quad \text{where } \varphi = \text{Gauss sum},$$

the above says (A) holds iff

$$F \left| \frac{\mathcal{P}(\chi)}{m} \sum_{a \bmod m} \bar{\chi}(a) \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| = C_\chi i^k \cdot F \left| \frac{\mathcal{P}(\chi)}{m} \sum_{a \bmod m} \bar{\chi}(a) \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| K | H_{Mm^2},$$

i.e.

$$\begin{aligned} \frac{\mathcal{P}(\chi)}{m} \sum_{a \bmod m} \bar{\chi}(a) F \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| &= C_\chi i^k \frac{\overline{\mathcal{P}(\chi)}}{m} \cdot \sum_{a \bmod m} \chi(a) \cdot F \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| K | H_{Mm^2} \\ &= C_\chi i^k \frac{\overline{\mathcal{P}(\chi)}}{m} \sum_{a \bmod m} \chi(a) \cdot F | K \left| \begin{pmatrix} m & -a \\ 0 & m \end{pmatrix} \right| H_{Mm^2}, \end{aligned}$$

i.e.

$$\mathcal{P}(\chi) \sum_{a \bmod m} \bar{\chi}(a) F \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| = C_\chi i^k \mathcal{P}(\bar{\chi}) \sum_{a \bmod m} \chi(a) F | K \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| H_{Mm^2}$$

since $\overline{\mathcal{P}(\chi)} = \mathcal{P}(\bar{\chi}) \chi(-1)$, i.e.

$$\mathcal{P}(\chi) \sum_{a \bmod m} \bar{\chi}(a) F \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| = C_\chi i^k \mathcal{P}(\bar{\chi}) \sum_{a \bmod m} \chi(a) \cdot F | K | H_M \gamma(a, b) \begin{pmatrix} m & b \\ 0 & m \end{pmatrix}$$

where

$$\gamma(a, b) = \begin{pmatrix} m & -b \\ -Ma & n \end{pmatrix} \in \Gamma_0(M, 1)$$

and

$$H_M \gamma(a, b) \begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} H_{Mm^2}.$$

Therefore (A) holds if and only if

$$\begin{aligned} \text{(B)} \quad \mathcal{P}(\chi) \sum_{a \bmod m} \bar{\chi}(a) F \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| \\ = C_\chi i^k \mathcal{P}(\bar{\chi}) \overline{\chi(-M)} \sum_{b \bmod m} \bar{\chi}(b) \cdot F | K H_M \gamma(a, b) \begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \end{aligned}$$

since $mn - Mab = 1$ implies $-Mab \equiv 1 \pmod{m}$.

If $F \in C_\varepsilon(M, 1)$ and (A) holds for $\chi = 1$ (therefore $m = 1$), i.e.

$$F = C_1 i^k \cdot F | K | H_M,$$

then for a primitive character χ with conductor $f_\chi = m$ relatively prime to M , (B) reads

$$\begin{aligned} \mathcal{P}(\chi) C_1 i^k \sum_{a \bmod m} \bar{\chi}(a) \cdot F | K | H_M \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right| \\ = C_\chi i^k \mathcal{P}(\bar{\chi}) \overline{\chi(-M)} \overline{\varepsilon(m)} \sum_{b \bmod m} \bar{\chi}(b) \cdot F | K | H_M \left| \begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \right|. \end{aligned}$$

This proves the first part of

Theorem 8. Let $F(\tau) = \sum_{n \geq 1} a(n) z_1^n \in C_\varepsilon(M, 1)$, Λ_χ , $\tilde{\Lambda}_\chi$ as above. Suppose $F = C_1 i^k \cdot F|K|H_M$. Then (A) holds for all primitive characters χ with conductor $f_\chi = m$, relatively prime to M , and with

$$C_\chi = C_1 \frac{\mathcal{G}(\chi)}{\mathcal{G}(\bar{\chi})} \chi(-M) \varepsilon(f_\chi)$$

where $\mathcal{G}(\chi)$ = Gauss sum.

Conversely, suppose $F(\tau) = \sum_{n \geq 1} a(n) z_1^n$ where $a(n)$ are complex numbers with $a(n) = O(n^\sigma)$ for some $\sigma > 0$. Let ε be a character mod M . Assume (A) holds for the principal character $\chi = 1$ and for almost all primitive characters χ with conductor $f_\chi = m$, an odd prime not dividing M , and with

$$C_\chi = C_1 \cdot \frac{\mathcal{G}(\chi)}{\mathcal{G}(\bar{\chi})} \chi(-M) \varepsilon(f_\chi).$$

Then F is a modular form of character ε and weight k on $\Gamma_0(M)$, and

$$F = C_1 \cdot i^k \cdot F|K|H_M.$$

Moreover, if $L(s) = \sum_{n \geq 1} a(n) n^{-s}$ converges absolutely at $s = k - \delta$ for some $k > \delta > 0$, then F is a cusp form (cf. Shimura [14]).

Remark. Note that if a form $0 \neq F \in C_\varepsilon(M, 1)$ is an eigenform of KH_M with eigenvalue α , then $|\alpha| = 1$ since

$$\begin{aligned} F &= (-1)^k F|K^2 H_M^2 = F|KH_M KH_M = (\alpha F)|KH_M = \bar{\alpha}(F|KH_M) \\ &= \bar{\alpha}\alpha F = |\alpha|^2 F. \end{aligned}$$

Therefore $|C_\chi| = 1$.

Proof. It remains to prove the converse part. By hypothesis, $F = C_1 i^k \cdot F|K|H_M$. Let S be the set consisting of all odd primes m such that (A) holds for all primitive characters χ with conductor $f_\chi = m$. Then for $m \in S$, χ a character with conductor $f_\chi = m$ (χ is thus primitive), (B) reads

$$(C) \quad \sum_{a \bmod m} \bar{\chi}(a) F \left| \begin{pmatrix} m & a \\ 0 & m \end{pmatrix} \right. = \varepsilon(m) \sum_{b \bmod m} \bar{\chi}(b) \cdot F| \gamma(a, b) \begin{pmatrix} m & b \\ 0 & m \end{pmatrix}$$

since

$$C_\chi = C_1 \frac{\mathcal{G}(\chi)}{\mathcal{G}(\bar{\chi})} \chi(-M) \varepsilon(f_\chi).$$

We shall use Weil's method in the following proof. Only the main steps will be sketched. For more details, see Weil [16].

Let Ω_F be the algebra over the complex field generated by all $w \in GL_+(2, \mathbb{R})$ such that $F|w = 0$. Then for $m \in S$, χ a character with $f_\chi = m$, (C) is equivalent to

$$(D) \quad (1 - \varepsilon(m) \gamma(a, b)) \begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \equiv (1 - \varepsilon(m) \gamma(a', b')) \begin{pmatrix} m & b' \\ 0 & m \end{pmatrix} \pmod{\Omega_F}$$

for all b, b' with $(b, m) = 1 = (b', m)$.

We first claim $F|\gamma(a, b) = \varepsilon(n) F$, i.e. $1 \equiv \varepsilon(m) \gamma(a, b) \pmod{\Omega_F}$ where

$$\gamma = \gamma(a, b) = \begin{pmatrix} m & -b \\ -Ma & n \end{pmatrix} \in \Gamma_0(M, 1) \text{ with } m, n \in S. \text{ Set } \gamma' = \begin{pmatrix} m & b \\ Ma & n \end{pmatrix}.$$

$$\text{Note that } \gamma^{-1} = \begin{pmatrix} n & b \\ Ma & m \end{pmatrix} \text{ and } \gamma'^{-1} = \begin{pmatrix} n & -b \\ -Ma & m \end{pmatrix}.$$

Apply (D) to $m \in S$:

$$(1 - \varepsilon(m) \gamma) \begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \equiv (1 - \varepsilon(m) \gamma') \begin{pmatrix} m & -b \\ 0 & m \end{pmatrix} \pmod{\Omega_F},$$

i.e.

$$1 - \varepsilon(m) \gamma' \equiv (1 - \varepsilon(m) \gamma) \begin{pmatrix} m & 2b \\ 0 & m \end{pmatrix} \pmod{\Omega_F}. \quad (a)$$

Apply (D) to $n \in S$:

$$(1 - \varepsilon(n) \gamma'^{-1}) \begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \equiv (1 - \varepsilon(n) \gamma^{-1}) \begin{pmatrix} m & -b \\ 0 & m \end{pmatrix} \pmod{\Omega_F},$$

i.e.

$$-\varepsilon(n) (1 - \varepsilon(m) \gamma') \gamma'^{-1} \equiv (1 - \varepsilon(n) \gamma^{-1}) \begin{pmatrix} m & -2b \\ 0 & m \end{pmatrix} \pmod{\Omega_F},$$

i.e.

$$1 - \varepsilon(m) \gamma' \equiv (1 - \varepsilon(m) \gamma) \gamma^{-1} \begin{pmatrix} m & -2b \\ 0 & m \end{pmatrix} \gamma' \pmod{\Omega_F}. \quad (b)$$

From (a) and (b), we get

$$(1 - \varepsilon(m) \gamma) (1 - \mu) \begin{pmatrix} m & 2b \\ 0 & m \end{pmatrix} \equiv 0 \pmod{\Omega_F}$$

or

$$(1 - \varepsilon(m) \gamma) (1 - \mu) \equiv 0 \pmod{\Omega_F}$$

where

$$\mu = \gamma^{-1} \begin{pmatrix} 1 & -2b/m \\ 0 & 1 \end{pmatrix} \gamma' \begin{pmatrix} 1 & -2b/m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2b/m \\ 2Ma/n & 4/(mn) - 3 \end{pmatrix}$$

is elliptic with infinite order. Thus $1 - \varepsilon(m) \gamma \equiv 0 \pmod{\Omega_F}$.

Now let $\gamma = \begin{pmatrix} a & b \\ Mc & d \end{pmatrix} \in \Gamma_0(M, 1)$; we want to show $F|\gamma = \varepsilon(d)F$. For $c = 0$, $a = d = 1$, $F|\gamma = F$ by assumption. For $b = 0$, $a = d = 1$,

$$\begin{pmatrix} 1 & 0 \\ Mc & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}^{-1}, \text{ therefore}$$

$$\begin{aligned} F \left| \begin{pmatrix} 1 & 0 \\ Mc & 1 \end{pmatrix} \right. &= (-1)^k F|H_M \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} H_M = (-1)^k F|K^2 H_M \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} H_M \\ &= F|K|H_M|K \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix} H_M = (C_1 i^k F) \left| \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} K|H_M \right. \\ &= (C_1 i^k F)|KH_M = |C_1 i^k|^2 F = F. \end{aligned}$$

Suppose $b \neq 0$. $ad - Mbc = 1$ implies $(a, Mb) = 1$, and $(d, Mb) = 1$. So there exist integers s, t such that $m = a + Mbs \in S$, $n = d + Mbt \in S$ and hence $\gamma = \begin{pmatrix} 1 & 0 \\ -Mt & 1 \end{pmatrix} \begin{pmatrix} m & b \\ Mb' & n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Ms & 1 \end{pmatrix}$ with $b' = at + c + ns$. This implies $F|\gamma = \varepsilon(n)F = \varepsilon(d)F$. Finally, for the case $b = 0$, $a = d = -1$, we can apply the above results to $\gamma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ with u non-zero integer to get the desired conclusion.

The condition $a(n) = O(n^\sigma)$ for some $\sigma > 0$ implies F is holomorphic at each real cusp; $L(s)$ converges absolutely at $s = k - \delta$ with $k > \delta > 0$ implies F vanishes at each real cusp. This completes the proof. Q.E.D.

When $\varepsilon = 1$, the principal character, $F = F|K$ for every normalized newform F in $C_1(M, 1)$, therefore Theorem 8 coincides with Theorem 6 on the normalized newforms in $C_1(M, 1)$.

Theorem 9. Suppose $F(\tau) = \sum_{n \geq 1} a(n) z_1^n \in C_\varepsilon(M, 1)$ with the associated Dirichlet series $L(s) = \sum_{n \geq 1} a(n) n^{-s}$. If $F|K|H_M = \gamma F$ for some constant γ and $L(s)$ has Euler product, i.e.

$$L(s) = \prod_{q|M} (1 - a(q)q^{-s})^{-1} \prod_{p \nmid M} (1 - a(p)p^{-s} + \varepsilon(p)p^{k-1-2s})^{-1},$$

then F is a normalized newform in $C_\varepsilon(M, 1)$.

Proof. By assumption, F is an eigenform of T_p with $(p, M) = 1$ and U_q^1 with $q|M$, hence there is a normalized newform $G(\tau) = \sum_{n \geq 1} b(n) z_1^n \in C_\varepsilon(N, 1)$ with $N|M$, equivalent to F , such that $F = \sum_{d|M/N} c_d G|B_d$ where c_d are complex constants. Since G is a newform, $G|K|H_N = \gamma' G$ for some

constant γ' . By the remark under Theorem 8, we know $|\gamma| = |\gamma'| = 1$. From

$$\begin{aligned}\gamma F &= F|K|H_M = \sum_{d|M/N} (c_d G|B_d)|K|H_M = \sum_d \bar{c}_d \cdot G|K|B_d|H_M \\ &= \sum_d \bar{c}_d \cdot d^{-k(M/N)^{k/2}} \cdot G|K|H_N|B_{M/(Nd)} \\ &= \sum_d \bar{c}_d \cdot d^{-k(M/N)^{k/2}} \gamma' \cdot G|B_{M/(Nd)},\end{aligned}$$

we get

$$\gamma c_d = \gamma' \bar{c}_{M/(Nd)} d^k (M/N)^{-k/2} \quad \text{for all } d|M/N$$

by comparing the coefficients of $G|B_d$ since $G|B_d$ are linearly independent as d runs over all the factors of M/N . Clearly, $C_1 = 1$. Taking absolute value of both sides and setting $d = 1$, we have

$$1 = |c_{M/N}| (M/N)^{-k/2}. \quad (8)$$

If $q|M/N$, then ε is a character mod M/q .

To go further, we need the following

Lemma 8. *Notations as above. If $q|M/N$ such that $q \nmid N$, $q^2 \nmid M$, then we have $|a(q)|^2 = q^{k-1}$ and $C_{qd} = -\varepsilon(q) \overline{a(q)} c_d$ for all $d|M/N$, $q \nmid d$.*

Proof. $G|U_q^1 = b(q) G - \varepsilon(q) q^{k-1} G|B_q$ by hypothesis. From

$$\begin{aligned}a(q) F &= F|U_q^1 = \sum_{q \nmid d} c_d G|B_d|U_q^1 + \sum_{q|d} c_d G|B_d|U_q^1 \\ &= \sum_{q \nmid d} c_d b(q) G|B_d - \sum_{q \nmid d} c_d \varepsilon(q) q^{k-1} G|B_q|B_d + \sum_{q|d} c_d G|B_{d/q},\end{aligned}$$

we have

$$\begin{cases} a(q) c_d = b(q) c_d + c_{dq} & \text{for } q \nmid d|M/N. \end{cases} \quad (d)$$

$$\begin{cases} a(q) c_{dq} = -\varepsilon(q) q^{k-1} c_d \end{cases} \quad (e)$$

Since $C_1 = 1$, (e) gives $c_q = -a(q)^{-1} \varepsilon(q) q^{k-1}$. Substituting this into (d), get

$$a(q) - b(q) + a(q)^{-1} \varepsilon(q) q^{k-1} = 0 \quad (f)$$

or

$$\begin{aligned}\overline{\varepsilon(q)} a(q)^2 - \overline{\varepsilon(q)} a(q) b(q) &= -q^{k-1} = \text{its conjugate} \\ &= \varepsilon(q) \overline{a(q)}^2 - \varepsilon(q) \overline{a(q)} \overline{b(q)}\end{aligned}$$

or

$$a(q)^2 - a(q) b(q) = \varepsilon(q)^2 \overline{a(q)}^2 - \varepsilon(q) \overline{a(q)} b(q) \quad \text{since } \overline{b(q)} = \overline{\varepsilon(q)} b(q).$$

This shows $a(q) = \varepsilon(q) \overline{a(q)}$ or $a(q) + \varepsilon(q) \overline{a(q)} = b(q)$.

Claim $a(q) = \varepsilon(q) \overline{a(q)}$ is impossible. If so, we must have $a(q) + \varepsilon(q) \overline{a(q)} = b(q)$, together with (f), $a(q) \overline{a(q)} = q^{k-1}$ and hence

$$c_{dq} = -a(q)^{-1} \varepsilon(q) q^{k-1} c_d = -\overline{a(q)} \varepsilon(q) c_d \quad \text{for } q \nmid d|M/N.$$

Suppose on the contrary, i.e. $\overline{a(q)} = \bar{\varepsilon}(q) a(q)$. Recall the operator $V_q^M = \begin{pmatrix} xq & y \\ Mz & q \end{pmatrix}$ with $xq^2 - yzM = q$. For $d \mid M/N$,

$$\begin{aligned} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} xq & y \\ Mz & q \end{pmatrix} &= \begin{pmatrix} xq & yd \\ zM/d & q \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{cases} \begin{pmatrix} x & yd \\ zM/(dq) & q \end{pmatrix} \begin{pmatrix} dq & 0 \\ 0 & 1 \end{pmatrix} & \text{if } q \nmid d \\ \begin{pmatrix} xq & yd/q \\ zM/d & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & q \end{pmatrix} & \text{if } q \mid d. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} F|K|V_q^M &= \sum_{q \mid d} q^{-k/2} \bar{c}_d \cdot G|K|B_{d/q} + \sum_{q \nmid d} \bar{\varepsilon}(q) q^{k/2} \bar{c}_d \cdot G|K|B_{dq} \\ &= F|V_q^M|K. \end{aligned}$$

Now let $g = F|U_q^1 + F|q^{k/2-1} V_q^M = a(q) F + q^{k/2-1} F|V_q^M \in C_\varepsilon(M/q, 1)$ by Lemma 3.

$$\begin{aligned} g|K|H_M &= (a(q) F)|K|H_M + q^{k/2-1} F|V_q^M|K|H_M \\ &= \overline{a(q)} \cdot F|K|H_M + q^{k/2-1} F|K|V_q^M|H_M \\ &= \overline{a(q)} \gamma F + q^{k/2-1} F|K|H_M \begin{pmatrix} q & -z \\ -My & xq \end{pmatrix} \\ &\quad \text{since } V_q^M H_M = H_M \begin{pmatrix} q & -z \\ -My & xq \end{pmatrix} \\ &= \bar{\varepsilon}(q) a(q) \gamma F + q^{k/2-1} \gamma \cdot \overline{\varepsilon(q)} F|V_q^M \\ &\quad \text{since } \overline{a(q)} = \bar{\varepsilon}(q) a(q) \text{ and } F \left| \begin{pmatrix} q & -z \\ -My & xq \end{pmatrix} \right. = \bar{\varepsilon}(q) F|V_q^M \\ &= \bar{\varepsilon}(q) \gamma g. \end{aligned}$$

On the other hand, $g \sim G$, therefore $g = \sum_{d \mid M/(qN)} y_d \cdot G|B_d$ where y_d are complex constants. So

$$\bar{\varepsilon}(q) \gamma \cdot g = g|K|H_M = \sum_{d \mid M/(qN)} \bar{y}_d d^{-k} (M/N)^{k/2} \gamma' \cdot G|B_{M/(Nd)}.$$

This shows $g=0$ for $d \mid M/(Nq)$ implies $q \nmid d$, while $q \mid M/(Nd)$. We then have $a(q)^2 = \varepsilon(q) q^{k-2}$ since $F|V_q^M V_q^M = \varepsilon(q) F$ by Lemma 2. Therefore, $c_q = -a(q)q$ and $b(q) = a(q)(1+q)$. From the formula

$$\varepsilon(q) q^{k-1} b(n/q) + b(nq) = b(n) b(q),$$

we get inductively $b(q^n) = a(q)^n (1 - q^{n+1}) / (1 - q)$. This shows $b(q^n)$ is exactly $O((q^n)^{k/2})$ as $n \rightarrow \infty$, a contradiction (cf. proof of Theorem 7, Case I). Q.E.D.

For the cases $q|M/N$, $q \nmid N$, $q^2|M$; and $q|M/N$, $q|N$, we have $q^2|M$. Apply Lemma 1, $F|U_q^1 = a(q)F \in C_\varepsilon(M/q, 1)$ hence $a(q) = 0$. (If not, then $F \in C_\varepsilon(M/q, 1)$ and $\gamma F = F|K|H_M = q^{k/2}F|K|H_{M/q}|B_q$ would imply $a(1) = 0$.) Using the same method as in the proof of Theorem 7, Cases II and III, we obtain

if $q|M/N$, $q \nmid N$, $q^2|M$,

$$\begin{cases} -\varepsilon(q)q^{k-1}c_d + c_{dq^2} = 0 & \text{if } q \nmid d|M/N, \\ c_d = 0 & \text{if } q^3|d|M/N; \end{cases} \quad (9)$$

if $q|M/N$, $q|N$,

$$\begin{cases} c_d b(q) + c_{dq} = 0 & \text{if } q \nmid d|M/N, \\ c_d = 0 & \text{if } q^2|d|M/N. \end{cases} \quad (11)$$

$$(12)$$

(8), (10), and (12) imply we can write $M/N = K_1 K_2 K_3$ such that $q|K_1$ iff $q|M/N$, $q \nmid N$, $q^2 \nmid M$; $q|K_2$ iff $q|M/N$, $q \nmid N$, $q^2|M$; $q|K_3$ iff $q|M/N$, $q|N$; and $K_1 = q_1 \dots q_t$, $K_2 = (q'_1 \dots q'_r)^2$, $K_3 = q''_1 \dots q''_s$ where q_i , q'_j , q''_n are distinct primes. From Lemma 8 and (9), we know

$$|c_{M/N}| = K_1^{(k-1)/2} K_2^{(k-1)/2} |c_{K_3}|.$$

Substituting into (8), we must have $K_1 = K_2 = 1$ i.e. $M/N = K_3$, and $K_3^{-k/2} |c_{K_3}| = 1$. But this shows $K_3 = 1$, otherwise, for $q|K_3$,

$$|b(q)| = \begin{cases} q^{k/2-1} & \text{if } \varepsilon \text{ is a character mod } N/q \\ q^{(k-1)/2} & \text{if } \varepsilon \text{ is not a character mod } N/q \end{cases} \quad \text{by Theorem 3,}$$

deriving from (11), we would have q^{-1} or $q^{-\frac{1}{2}}$ appear in $K_3^{-k/2} |c_{K_3}|$ which can not be cancelled. Q.E.D.

Suppose F is a nonzero form in $C_\varepsilon(M, 1)$ and F is a common eigenform of all T_p , U_q^1 , $(p, M) = 1$, $q|M$, then there is a newform $G \in C_\varepsilon(M', 1)$ with $M'|M$ such that $F = \sum_{d|M/M'} c_d G|B_d$, where c_d are complex numbers. We claim $c_1 \neq 0$. If not, let $d_0 > 1$ be the smallest integer such that $c_{d_0} \neq 0$. Then there is a prime $q|d_0$. Since $G|B_d|U_q^1 = G|U_q^1|B_d$ if $q \nmid d$, and $G|B_d|U_q^1 = G|B_{d/q}$ if $q|d$, the relation

$$F|U_q^1 = \sum_{q|d} c_d G|B_{d/q} + \sum_{q \nmid d} c_d G|U_q^1|B_d = \text{constant} \cdot F$$

implies d_0/q is an integer less than d_0 with $c_{d_0/q}$ nonzero, a contradiction.

Therefore F can be normalized. Applying Theorem 9, we get another version of newforms, namely, a non-zero form $F \in C_\varepsilon(M, 1)$ is a newform iff it is a common eigenform of T_p , U_q^1 , and KH_M where $(p, M) = 1$, $q | M$. If $F \in C_\varepsilon(M, N)$, then $F|B_N \in C_\varepsilon(MN, 1)$ and it has the same associated Dirichlet series as F . Moreover, F is a normalized newform iff $F|B_N$ is. Therefore corresponding results on forms in $C_\varepsilon(M, N)$ can easily be set up from the results on forms in $C_\varepsilon(MN, 1)$ with suitable modifications, not worth discussing here.

4. Some Result on a Convolution of L -Series

In this section we generalize a theorem of Ogg. Let $F(\tau) = \sum_{n \geq 1} a(n) z_1^n$ and $G(\tau) = \sum_{n \geq 1} b(n) z_1^n$ be two forms in $C_1(M, 1)$. The convolution of Dirichlet series $L_{F,G}$ is defined by

$$L_{F,G}(s) = \zeta_M(2s) \sum_{n \geq 1} a(n) \overline{b(n)} n^{-(s+k-1)}$$

where

$$\zeta_M(s) = \prod_{p \nmid M} (1 - p^{-s})^{-1}.$$

Rankin proved $L_{F,G}(s)$ defined above is entire if $\langle F, G \rangle = 0$, and entire except for a simple pole at $s = 1$ if $\langle F, G \rangle \neq 0$, using the following method (cf. Rankin [10] and Ogg [8]).

First note

$$\begin{aligned} \int_{|x| \leq \frac{1}{2}} F(\tau) \overline{G(\tau)} y^{s-1} dx dy &= \int_0^\infty y^{s-1} dy \int_{-\frac{1}{2}}^{\frac{1}{2}} F(\tau) \overline{G(\tau)} dx \\ &= \int_0^\infty y^{s-1} \sum_{n \geq 1} a(n) \overline{b(n)} e^{-4\pi n y} dy \\ &= (4\pi)^{-s} \Gamma(s) \sum_{n \geq 1} a(n) \overline{b(n)} n^{-s}. \end{aligned}$$

Replacing s by $s + k - 1$, we have

$$(4\pi)^{-(s+k-1)} \Gamma(s+k-1) \sum_{n \geq 1} a(n) \overline{b(n)} n^{-(s+k-1)} = \int_{|x| \leq \frac{1}{2}} y^s \delta(F, G),$$

where $\delta(F, G) = F(\tau) \overline{G(\tau)} y^{k-2} dx dy$. If we let $D_0(M)$ denote a fundamental domain for $\Gamma_0(M, 1)$, then

$$2\zeta_M(2s) \int_{|x| \leq \frac{1}{2}} y^s \delta(F, G) = \sum_{d|M} \frac{\mu(d)}{d^{2s}} \int_{D_0(M)} \delta(F, G) y^s E(M\tau/d, s)$$

where μ is the Möbius function and

$$E(\tau, s) = \sum_{\substack{m, n \\ (m, n) \neq (0, 0)}} |m\tau + n|^{-2s}$$

is an Epstein zeta-function. Let

$$I(\tau, s) = (y/\pi)^s \Gamma(s) E(\tau, s).$$

Then $I(\tau, s)$ is entire except for a simple pole of residue 1 at $s = 1, 0$, and satisfies the functional equations $I(\tau, 1-s) = I(\tau, s)$ and $I(\alpha(\tau), s) = I(\tau, s)$ for every $\alpha \in \Gamma(1, 1)$. To sum up, we have

$$\begin{aligned} \Phi(s) &= (2\pi/\sqrt{M})^{-2s} \Gamma(s) \Gamma(s+k-1) L_{F,G}(s) \\ &= \frac{(4\pi)^{k-1}}{2} \sum_{d|M} \frac{\mu(d)}{d^s} \int_{D_0(M)} \delta(F, G) I(M\tau/d, s). \end{aligned}$$

Ogg's Theorem 6 in [8] can be generalized to

Theorem 10. Let $F \in C_1^+(N_1, 1)$, $G \in C_1^+(N_2, 1)$, $M = \text{l.c.m.}(N_1, N_2)$ and $N = \text{product of primes } q|M \text{ such that } \text{ord}_q N_1 = \text{ord}_q N_2 (= \text{ord}_q M)$. Suppose for every prime $q|N$, $\text{ord}_q N_1 = \text{ord}_q N_2 = 1$ and $F|V_q^M = \xi_q F$, $G|V_q^M = \eta_q G$, $\xi_q^2 = 1 = \eta_q^2$. Then

$$\Phi^+(s) = \Phi(s) \prod_{q|N} (1 - \xi_q \eta_q q^{-s})^{-1} = \Phi^+(1-s)$$

where

$$\Phi(s) = (2\pi/\sqrt{M})^{-2s} \Gamma(s) \Gamma(s+k-1) L_{F,G}(s).$$

Proof. If prime $q|M$ such that $\text{ord}_q N_1 \neq \text{ord}_q N_2$, say $\text{ord}_q N_1 > \text{ord}_q N_2$, then for $dq|M$, $D_0(M) = \bigcup_i R_i D_0(M/q)$ where $R_i \in \Gamma(M/q, 1)$ are right coset representatives for $\Gamma_0(M)$ in $\Gamma_0(M/q)$, and

$$\begin{aligned} \int_{D_0(M)} \delta(F, G) I(M\tau/(dq), s) &= \sum_i \int_{R_i D_0(M/q)} \delta(F, G) I(M\tau/(dq), s) \\ &= \sum_i \int_{D_0(M/q)} \delta(F|R_i, G|R_i) I(M\tau/(dq), s) \circ R_i \\ &= \sum_i \int_{D_0(M/q)} \delta(F|R_i, G) I(M\tau/(dq), s) \quad \text{since } G \in C_1(M/q, 1) \\ &\quad \text{and } \begin{pmatrix} M/(dq) & 0 \\ 0 & 1 \end{pmatrix} R_i \begin{pmatrix} M/(dq) & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \Gamma(1) \\ &= \int_{D_0(M/q)} \delta(\text{Tr}_{M/q}^M F, G) I(M\tau/(dq), s) \\ &= \int_{D_0(M/q)} \delta(\text{Tr}_{N_1/q}^{N_1} F, G) I(M\tau/(dq), s) \quad \text{by Lemma 6, i)} \\ &= 0 \quad \text{since } F \in C_1^+(N_1, 1), \text{Tr}_{N_1/q}^{N_1} F = 0 \quad \text{by Theorem 4.} \end{aligned}$$

Thus

$$\begin{aligned}
 \Phi(s) &= \frac{(4\pi)^{k-1}}{2} \sum_{\substack{d|M \\ q \nmid d}} \frac{\mu(d)}{d^s} \\
 &\quad \cdot \left[\int_{D_0(M)} \delta(F, G) I(M\tau/d, s) - q^{-s} \int_{D_0(M)} \delta(F, G) I(M\tau/(dq), s) \right] \\
 &= \frac{(4\pi)^{k-1}}{2} \sum_{\substack{d|M \\ q \nmid d}} \frac{\mu(d)}{d^s} \int_{D_0(M)} \delta(F, G) I(M\tau/d, s) \\
 &= \frac{(4\pi)^{k-1}}{2} \sum_{d|N} \frac{\mu(d)}{d^s} \int_{D_0(M)} \delta(F, G) I(M\tau/d, s) \\
 &\quad \text{by repeating for all } q|M/N.
 \end{aligned}$$

Now let prime $q|N$. By hypothesis, $\text{ord}_q M = 1$. For $dq|N$,

$$\begin{pmatrix} M/(dq) & 0 \\ 0 & 1 \end{pmatrix} V_q^M \begin{pmatrix} M/d & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \Gamma(1),$$

therefore

$$\begin{aligned}
 \int_{D_0(M)} \delta(F, G) I(M\tau/(dq), s) &= \int_{V_q^M D_0(M)} \delta(F|V_q^M, G|V_q^M) I(M\tau/(dq), s) \circ V_q^M \\
 &= \xi_q \eta_q \int_{D_0(M)} \delta(F, G) I(M\tau/d, s),
 \end{aligned}$$

and

$$\begin{aligned}
 \Phi(s) &= \frac{(4\pi)^{k-1}}{2} \sum_{d|N/q} \frac{\mu(d)}{d^s} \\
 &\quad \cdot \left[\int_{D_0(M)} \delta(F, G) I(M\tau/d, s) - q^{-s} \int_{D_0(M)} \delta(F, G) I(M\tau/(dq), s) \right] \\
 &= \frac{(4\pi)^{k-1}}{2} \prod_{q|N} (1 - \xi_q \eta_q q^{-s}) \int_{D_0(M)} \delta(F, G) I(M\tau, s)
 \end{aligned}$$

by repeating for all $q|N$. Thus $\Phi^+(s) = \Phi^+(1-s)$ since

$$I(M\tau, s) = I(M\tau, 1-s). \quad \text{Q.E.D.}$$

The remaining problem is whether there is a functional equation for $\Phi(s)$ in case there exists a prime q such that $\text{ord}_q N_1 = \text{ord}_q N_2 \geq 2$. If this were solved, it would be very interesting to compare this with Jacquet's representation-theoretic generalization of Rankin's result (cf. [4]) and with Serre's definition for the local zeta-function of a product of elliptic curves (cf. [13]).

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