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A Criterion for Finite Representation Type

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We know by experience that an algebra has infinitely many isomorphism classes of indecomposable modules if and only if it "contains" an algebra which is closely related to the path algebra of an extended Dynkin-quiver. We prove the corresponding result for a certain class of algebras which we call standard. Moreover, a not representation-finite standard algebra of dimension d has infinitely many nonisomorphic indecomposables of dimension $\leqq 4d+30$. This improves the bound announced in [3] considerably.

There is some evidence [8, 14] that all minimal distributive not representation-finite algebras are standard. This would give a numerical version of the theorem of Nazarova-Roiter on the second Brauer-Thrall conjecture.

The proof of the theorem is based on the classification given in [3] and [4] and uses some generalizations of results on coverings from [2] and [7]. In Sect. 1, we recall some important definitions from [7] which we need to state our theorem precisely. Section 2 contains the generalizations of [2] and [7, 2], already mentioned. In Sect. 3 we prove the theorem and illustrate it in Sect. 4 by an example occurring in the classification of representation-finite algebras with three simple modules [8].

This article has been motivated partially by discussions with Fischbacher and Gabriel about practical difficulties which went up in the classification of representation-finite algebras with three simples. I am grateful for this as well as for some useful remarks they made on the preprint of this paper.

We use the notations introduced in [6] and [7]. In particular, k denotes an algebraically closed field, and we consider only finite dimensional right modules. When we use a notion for the first time, we indicate by (x, y) the section y of the article x, where the reader can find a definition.

1. Statement of the Main Result

Let Λ be a Schurian category [7, 1.3]. As in [7, 2.1], we denote by $S_n\Lambda$ the set of sequences $(x_0, x_1, \ldots, x_n)$ of pairwise different objects of Λ such that the composition

$$\Lambda(x_0, x_1) \times \Lambda(x_1, x_2) \times \ldots \times \Lambda(x_{n-1}, x_n) \to \Lambda(x_0, x_n)$$

is non-zero, by $C_n\Lambda$ the free abelian group generated by $S_n\Lambda$ and by $H_n\Lambda$ the homology groups of the corresponding differential complex $C_*\Lambda$.

Definition. A k-category Λ [6, 2.1] is called simply connected if it has the following properties:

a) Λ is locally bounded [6, 2.1] and Schurian.

b) The Gabriel-quiver Q_Λ ([6, 2.1] and [5, Introduction]) of Λ is connected, directed [7, 2.6] and interval-finite [7, 2.6].

c) $H_1\Lambda=0$.

For representation-finite algebras, this definition coincides by [7, 2.12] with the usual one [6, 6]. The same is true for the next definition.

Definition. An algebra Λ is called standard if Λ is finite dimensional, distributive, and basic [5] and if it admits a Galois-covering [9, 3.1] $\tilde{\Lambda} \xrightarrow{\pi} \Lambda$, where $\tilde{\Lambda}$ is simply connected.

Recall that Λ is distributive if and only if $\Lambda(a,b)$ is a uniserial $\Lambda(a,a)-\Lambda(b,b)$-bimodule for all $a,b\in\Lambda$. Given an object s of a locally bounded Schurian category Λ with directed Gabriel-quiver Q_Λ, we denote by Λ_s the set $\{t\in\Lambda|\Lambda(t,s)\neq 0\}$ endowed with the partially ordering

$$t\leqq t' :\Leftrightarrow (t,t',s)\in S_2\Lambda .$$

Dually, we define Λ^s (compare [7, 2.6]). With a partially ordered set S we associate a k-category kS which is defined as S_k in [7, 2.6]. In Fig. 1a we recall the shape of certain graphs.

Definition. A k-category is called $\tilde{A}$-free if it does not contain a full subcategory $B \xrightarrow{\sim} kQ_B$ satisfying $|Q_B|=\tilde{A}_n$, $n\geqq 1$. Here $|K|$ denotes the underlying graph of the quiver K and kK the path category [6, 2.1].

Figure 1b describes 4 families of algebras by quivers and relations. An edge — can be replaced by an arrow in arbitrary orientation. The algebras of type (2) and (3) are defined by the obvious commutativity relations, those of family (4) by the vanishing of the sum of all paths from $n+1$ to 1.

Our main result is as follows:

Theorem. *Let Λ be a standard algebra of dimension d. Denote by $\tilde{\Lambda}\to\Lambda$ a Galois-covering with simply connected $\tilde{\Lambda}$. Then the following two conditions are equivalent:*

1) *Λ is representation-finite.*

2) *$\tilde{\Lambda}$ has the following properties:*

a) *All $k\tilde{\Lambda}_s$ and $k\tilde{\Lambda}^s$, $s\in\tilde{\Lambda}$, are $\tilde{A}$-free.*

b) *$\tilde{\Lambda}$ does not contain a full convex [7, 2.6] subcategory $B \xrightarrow{\sim} kQ_B$ with $|Q_B|=A_{2d+1}$.*

c) *An algebra of the families (1), (2), or (3) with strictly less than $2d+3$ points does not occur as full convex subcategory of $\tilde{\Lambda}$.*

d) *Any full convex subcategory of $\tilde{\Lambda}$ with at most 9 points is representation-finite.*

Moreover, there exist infinitely many isomorphism classes of indecomposables of dimension $\leqq 4d+30$, if Λ is not representation-finite.

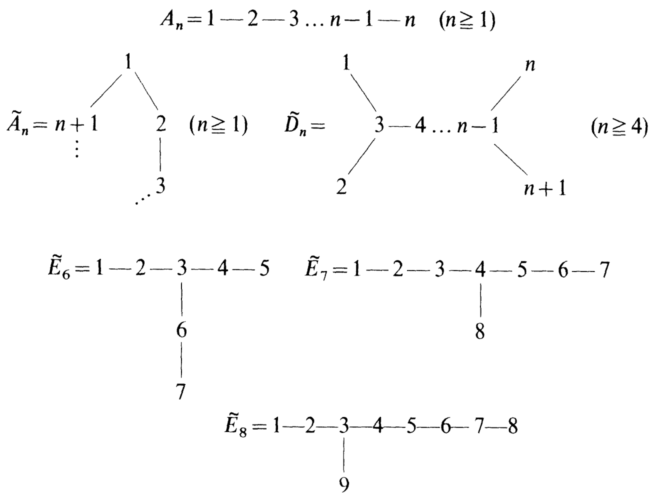

$A_n = 1 — 2 — 3 \ldots n-1 — n \quad (n \geqq 1)$
$\tilde{A}_n =$ (n ≥ 1)
$\tilde{D}_n =$ (n ≥ 4)
$\tilde{E}_6 =$
$\tilde{E}_7 =$
$\tilde{E}_8 =$

Fig. 1a

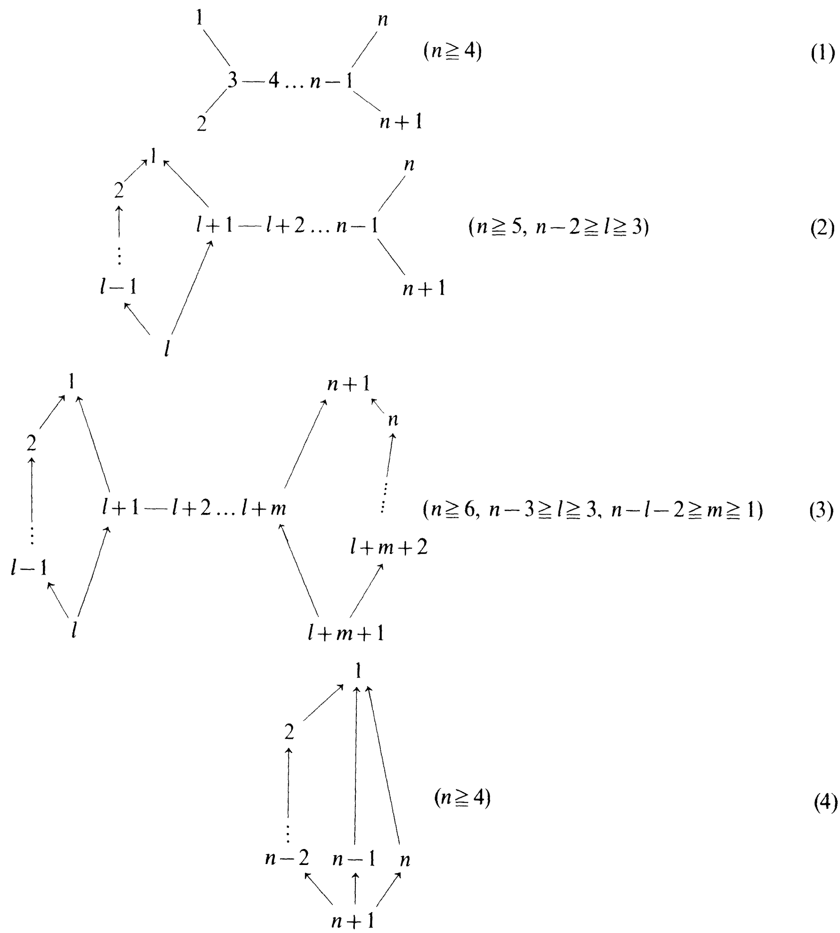

$(n \geqq 4)$ (1)
$(n \geqq 5,\ n-2 \geqq l \geqq 3)$ (2)
$(n \geqq 6,\ n-3 \geqq l \geqq 3,\ n-l-2 \geqq m \geqq 1)$ (3)
$(n \geqq 4)$ (4)

Fig. 1b

Remark. The conditions stated in 2) are equivalent to

a), b), c), and d′) which says:

d′) $\tilde{A}$ does not contain a full convex subcategory which has one of the frames (see [11, 1]) 1–141 in [4, 3.2].

This equivalence is shown in [4]. Condition d′) is much easier to verify in practice (cf. Sect. 4).

2. Homology Groups and the Separation Criterion

2.1. Let Λ be a locally bounded, directed Schurian category. Given a sink s in Q_Λ, we denote by Λ', respectively, $k\Lambda'_s$ the full subcategories of Λ, respectively, $k\Lambda_s$ obtained by forgetting the object s. This gives rise to an exact sequence of differential complexes

$$0 \to C_* k\Lambda'_s \to C_* k\Lambda_s \oplus C_* \Lambda' \to C_* \Lambda \to 0 \qquad (1)$$

and induces a Mayer-Vietoris-sequence

$$\ldots \to H_n k\Lambda'_s \xrightarrow{i_n} H_n k\Lambda_s \oplus H_n \Lambda' \xrightarrow{\pi_n} H_n \Lambda \xrightarrow{\partial_n} H_{n-1} k\Lambda'_s \to \ldots . \qquad (2)$$

This sequence is crucial in [7, 2].

Finally, we recall the definition of a separating point and fix the notations. Choose a point $x \in \Lambda$ and decompose the radical of $\Lambda(?, x)$ into indecomposables so that $\operatorname{rad} \Lambda(?, x) = \bigoplus_{i=1}^{m(x)} R_i(x)$. Denote by $\Lambda(x)$, respectively, $B_i(x)$ the full subcategories of Λ supported by the points $y \not\geq x$ [7, 2.6], respectively, by the connected component in $\Lambda(x)$ defined by the support of $R_i(x)$. The point x is separating in Λ, if $i \neq j$ implies $B_i(x) \neq B_j(x)$ for all i and j.

Lemma. *Let S be a finite partially ordered set with associated k-category $\Lambda = kS$. If Λ is $\tilde{A}$-free, all points are separating.*

Proof. We show the contraposition. Suppose s is not separating. Thus, there are two neighbours u, v of s and two indices $i \neq j$ such that $B_i(s) = B_j(s)$ and $R_i(s)(u) \neq 0 \neq R_j(s)(v)$. Choose such a pair (u, v), which is connected in $\Lambda(s)$ by a walk w containing as few sources as possible. Figure 2 shows such a walk. Of course, $u = q_1$ or $v = q_n$ is not excluded, but n is at least 2.

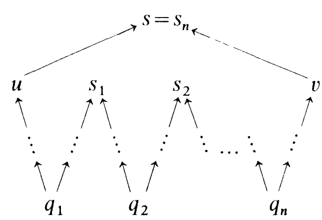
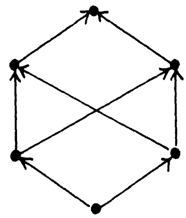


Fig. 2

By minimality the full subcategory supported by the sources and sinks is isomorphic to kQ with $|Q| = \tilde{A}_{2n-1}$. q.e.d.

The lemma cannot be reversed, as is shown by the commutative quiver



2.2. Proposition. *Let S be a partially ordered set of finite cardinality $\# S$. Set $\Lambda = kS$. If Λ is $\tilde{A}$-free, we have $H_n\Lambda = 0$ for all $n \geqq 1$.*

Proof. By induction on $\# S$. If S has only one sink, one can write down an appropriate chain homotopy as in the proof of [7, 2.6], first step, without using the assumption that Λ is $\tilde{A}$-free. In the other case, choose a sink s. The associated categories Λ', $k\Lambda'_s$, and $k\Lambda_s$ are again $\tilde{A}$-free. By induction, the sequence (2) of 2.1 tells us $H_n\Lambda = 0$ for $n \geqq 2$. Moreover, we obtain the exact sequence

$$0 \to H_1\Lambda \to H_0 k\Lambda'_s \xrightarrow{i_0} H_0 k\Lambda_s \oplus H_0\Lambda'.$$

The "connected components" form natural bases of the H_0's and i_0 is injective, because s is separating (cf. [7, 2.9]). q.e.d.

2.3. Now we can generalize Theorem 2.6 of [7] easily.

Theorem. *Let Λ be a locally bounded Schurian category such that Q_Λ is directed and interval-finite and all $k\Lambda_s$ and $k\Lambda^s$, $s \in \Lambda$, are $\tilde{A}$-free. Then we have:*

a) *$H_n\Lambda = 0$ for $n \geqq 2$.*

b) *$H_1\Lambda$ is a free abelian group.*

c) *The inclusion $\bar{\Lambda} \subset \Lambda$ of a full convex subcategory induces an isomorphism from $H_1\bar{\Lambda}$ onto a pure subgroup of $H_1\Lambda$.*

Proof. We can jump to step 5 in the proof of [7, 2.6] and follow the same way. q.e.d.

Remarks. a) Given Λ as in the theorem above, $H^2(\Lambda, k^*) = 0$ [7, 2.1], whence Λ has a multiplicative basis [14 and 7, 2]. This leads to the question whether any algebra has a multiplicative basis which does not "contain" a $k\tilde{A}_n$.

b) Suppose Λ is locally bounded, Schurian, directed and interval-finite. Choose a point $s \in \Lambda$ and consider the subcategory B of Λ supported by Λ_s. For x, y in B we define $B(x, y) = \Lambda(x, y)$, if $(x, y, s) \in S_2\Lambda$, and $B(x, y) = 0$, otherwise. Hence B and $k\Lambda_s$ have the same simplicial frame [7, 2.1]. By the proof of 2.2, we have $H_n k\Lambda_s = 0$ for $n \geqq 1$, whence $H^2(k\Lambda_s, k^*) = 0$ and $B \rightsquigarrow k\Lambda_s$ [7, 2.2]. Extension by 0 on objects and morphisms outside of B identifies the B-modules with a full subcategory of the Λ-modules.

In Theorem 2.6 of [7], Λ is assumed to be locally representation-finite. As shown before, this forces $B \xrightarrow{\sim} k\Lambda_s$ to be representation-finite, whence $\tilde{A}$-free. Therefore, Theorem 2.3 in fact generalizes Theorem 2.6 of [7].

c) Given a finite Λ as in Theorem 2.3, we have $H_1\Lambda = 0$ if and only if each point is separating. Namely, the proof of [7, 2.9] still works.

2.4. Up to the end of this section Λ denotes a finite locally bounded Schurian category with directed Gabriel-quiver Q_Λ. A connected component C of the Auslander-Reiten-quiver [6, 2.2] Γ_Λ of Λ is called preprojective, if C contains no oriented cycle and consists in indecomposables, which are given by $(\mathrm{Tr}\,D)^i P$ for some natural number i and some indecomposable projective P [1]. Equivalently, C contains no oriented cycle and each point in C has only finitely many predecessors. Here U is a predecessor of V provided there is path $U = X_0 \to X_1 \to \ldots \to X_n = V$ in C with $n \geqq 0$.

We generalize a result of Bautista and Larrion [2] and prove, that Λ has a preprojective component if each point of Λ is separating. We use the notations of 2.1. Since $B_i(x)$ is a full convex subcategory of Λ, we identify $\mathrm{mod}\,B_i(x)$ [6, 2.2] with a full subcategory of $\mathrm{mod}\,\Lambda$ by extension with 0, and speak about a $B_i(x)$-module simply. We denote by τ and τ^{-1} (respectively, $\bar{\tau}$ and $\bar{\tau}^{-1}$) $\mathrm{Tr}\,D$ and $D\,\mathrm{Tr}$ with respect to Λ (respectively, $B_i(x)$). If I is injective, we set $\mathrm{Tr}\,DI = 0$.

Lemma. *Suppose, all points of Λ are separating. Choose an $x \in \Lambda$, which is no sink.*

a) *If $U \to \Lambda(?, y)$, $y \geqq x$, is irreducible in* $\mathrm{mod}\,\Lambda$ *for an indecomposable $B_1(x)$-module U, we have $y = x$ and $U \xrightarrow{\sim} R_1(x)$.*

b) *Let C be a preprojective component of $\Gamma_{B_1(x)}$. Choose an $U \in C$ such that $R_1(x)$ is not a proper predecessor. Then:*

i) *Any predecessor of U in Γ_Λ is so in C.*

ii) *If $U \not\xrightarrow{\sim} R_1(x)$, we have $\bar{\tau}U \xrightarrow{\sim} \tau U$.*

Proof. a) The claim holds for $y = x$, because x is separating. Thus, let $y = y_n \leftarrow y_{n-1} \leftarrow \ldots \leftarrow y_1 \leftarrow y_0 = x$ be a path with $n \geqq 1$. Since U is a direct summand of $\mathrm{rad}\,\Lambda(?, y)$, it does not vanish on a neighbour y' of y, which belongs to $B_1(x)$ (see Fig. 3). Because y is separating, we conclude $U(y_{n-1}) \neq 0$, a contradiction.

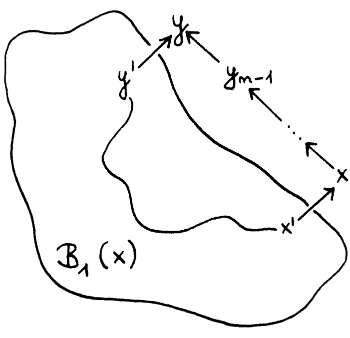


Fig. 3

b) The proof is by induction on the number $a(U)$ of predecessors of U in C. If $a(U) = 1$, U is simple projective in $\mathrm{mod}\,B_1(x)$, whence in $\mathrm{mod}\,\Lambda$ by the definition of $B_1(x)$. This implies i). Moreover, in U start only irreducible maps to projectives, which are $B_1(x)$-modules because of $U \neq R_1(x)$. Part ii) follows.

In the induction-step, we show first, that any (mod $\Lambda -$) irreducible map $X \to U$ starts from a $B_1(x)$-module. If U is projective in mod$B_1(x)$, it is so in modΛ, whence X is a $B_1(x)$-module. In the other case, the claim holds by induction. Namely, $a(\bar{\tau}^{-1}U) < a(U)$ implies $U = \bar{\tau}\bar{\tau}^{-1}U = \tau\bar{\tau}^{-1}U$.

Part i) follows immediately. To prove $\tau U = \bar{\tau}U$, let $U \to Y$ be a (mod$\Lambda -$) irreducible map to an indecomposable Y. If Y is not projective, $\tau^{-1}Y$ is a $B_1(x)$-module, hence $Y = \tau\tau^{-1}Y = \bar{\tau}\tau^{-1}Y$ (observe $a(\tau^{-1}Y) < a(U)$!). If Y is projective, it is a $B_1(x)$-module by part a) of the lemma. Now if U is injective in modΛ, it is so in mod$B_1(x)$ a fortiori. In the other case, look at the almost split sequence [1] $0 \to U \to Z \to \tau U \to 0$ in modΛ. As shown before, Z is a $B_1(x)$-module, and so is τU. Part ii) follows. q.e.d.

2.5. Theorem. *Let Λ be a finite locally bounded Schurian category with directed Gabriel-quiver. If each point of Λ is separating, the Auslander-Reiten-quiver of Λ has a preprojective component C.*

Proof. By induction on dimΛ. There are two cases. First, assume that there is a point x and an index i, such that $R_i(x)$ does not belong to a preprojective component of $\Gamma_{B_i(x)}$. By induction, there exists a preprojective component C of $\Gamma_{B_i(x)}$. It is even a preprojective component of Γ_Λ by part b)i) of Lemma 2.4.

In the other case we paste together a preprojective component of Γ_Λ, i.e. we construct inductively full subquivers C_n of Γ_Λ satisfying:

i) C_n is finite, connected, contains no oriented cycle and is closed under predecessors.

ii) $\mathrm{Tr}DC_n \cup C_n \subsetneqq C_{n+1}$.

Clearly, $\bigcup_{n \in \mathbb{N}} C_n$ is the wanted preprojective component.

Set $C_0 = \{S\}$, where S is a simple projective. To get C_{n+1} from C_n, number the modules $M_1, M_2, \ldots, M_t$ of C_n with $\mathrm{Tr}DM_i \notin C_n$ in such a way, that $i<j$ provided M_i is a predecessor of M_j. If $t=0$ set $C_{n+1} = C_n$. In the remaining case we define full subquivers D_i, $0 \leqq i \leqq t$, of Γ_Λ satisfying $D_0 = C_n$, $D_i \cup \{\mathrm{Tr}DM_{i+1}\} \subsetneqq D_{i+1}$ and condition i) imposed on the C_n. Of course, D_t is our candidate for C_{n+1}.

Take the almost split sequence $0 \to M_{i+1} \to X \to \mathrm{Tr}DM_{i+1} \to 0$ and define D_{i+1} as the full subquiver of Γ_Λ supported by D_i and all predecessors of $\mathrm{Tr}DM_{i+1}$. We have to show, that each indecomposable direct summand Y of X has only finitely many predecessors and does not lie on an oriented cycle. If Y is not projective, $D\mathrm{Tr}\,Y$ belongs to C_n, whence Y to D_i, and we are done. If Y is projective, say $Y = \Lambda(?, x)$, all $R_i(x)$ lie on a preprojective component of $\Gamma_{B_i(x)}$, and our claim follows from Part b)i) of 2.4. q.e.d.

2.6. Given Λ as in Theorem 2.5 with preprojective component C, we define $\Lambda(C)$ as the full subcategory supported by those x such that $\Lambda(?, x)$ belongs to C. Thus, $\Lambda(C)$ is a convex subcategory of Λ and the restriction identifies C with a preprojective component of $\Gamma_{\Lambda(C)}$ containing all indecomposable projectives. The graph G_C associated to C [6, 4.2] is a tree T_C, what follows from the separation property easily by induction on dimΛ. Hence C is a simply connected translation-quiver [6, 4.2] described by an admissible graded tree as in [6, 6.2] and $\Lambda(C)$ is isomorphic to the algebra $A^T = \bigoplus_{p,q} k(R_T)(p, q)$ [6, 6.4].

A Schurian algebra Λ with directed Q_Λ is called *critical* if any proper convex full subcategory is representation-finite, but Λ is not. We cite the following result from [4].

Theorem. *Let (T, g) be an admissible graded tree such that A^T is critical. Then the following holds:*
a) $T \in \{\tilde{D}_n, \tilde{E}_m | n \geqq 4,\ 8 \geqq m \geqq 6\}$.
b) *A^T is isomorphic to an algebra of the list in* [4].

3. Proof of the Theorem

We use the same notation and make the same assumptions as in the theorem.

3.1. Lemma. *If $k\tilde{A}_s$ is not $\tilde{A}$-free for some s, there are infinitely many non-isomorphic $\tilde{A}$-modules of dimension $\leqq 2d$.*

Proof. We choose a full subcategory $D \rightsquigarrow kQ$ with $|Q| = \tilde{A}_n$, $n \geqq 1$, such that the convex hull $\hat{D}$ is as small as possible. Denote the sinks in D by $s_1, s_2, \ldots, s_m$, the sources by $q_1, q_2, \ldots, q_m$. We have $m \geqq 2$ because $k\tilde{A}_s$ is Schurian. Let L be the left-adjoint of the restriction $R: \mathrm{mod}\,\hat{D} \to \mathrm{mod}\,D$. Then $LD(?, s_i) = \hat{D}(?, s_i)$ holds for $1 \leqq i \leqq m$. The well-known family of indecomposable D-modules $U(\xi)$, $\xi \in k^*$, has projective cover $\bigoplus^m D(?, s_i) \to U(\xi) \to 0$. Applying L we get exact sequences $\bigoplus^m \hat{D}(?, s_i) \to LU(\xi) \to 0$ with indecomposable non-isomorphic $LU(\xi)$.

If $m = 2$ we get $\dim LU(\xi) \leqq 2 \cdot \# \hat{D} \leqq 2 \cdot \# \tilde{A}_s \leqq 2d$. If $m \geqq 3$, it is enough to show $\dim LU(\xi)(y) \leqq 1$ for all $y \in \hat{D}$. This follows from $RL \rightsquigarrow \mathrm{id}_{\mathrm{mod}\,D}$ provided y is a source or a sink. In the other case suppose $q_1 \leqq y \leqq s_j$, whence $j = 1$ or $j = m$. If $y \leqq s_1$ and $y \leqq s_m$ the convex hull of the full subcategory supported by $(D \backslash \{q_1\}) \cup \{y\}$ would be smaller than $\hat{D}$. It follows $\dim \left(\bigoplus^m \hat{D}(?, s_i) \right)(y) = 1$.

For any m, the constructed family $LU(\xi)$ of $\hat{D}$-modules can be extended to a family of $\tilde{A}$-modules of the same dimension [see 2.3 Remark b)]. q.e.d.

3.2. Lemma. *Let $B \rightsquigarrow kQ_B$ be a full convex subcategory of $\tilde{\Lambda}$ with $|Q_B| = A_{2d+1}$. Then there are infinitely many non-isomorphic Λ-modules of dimension $\leqq 2d$.*

Proof. We mark all sources and sinks in Q_B and get – up to duality –

$$Q_B = q_1 \to t \to \ldots \to s_1 \leftarrow \ldots u \leftarrow q_i \to v \ldots x \leftarrow q_r \to y \to \ldots \to s_r,$$

where the right part after y can be missing. Writing D for the usual duality $\mathrm{Hom}_k(?, k)$ we have the inequality:

$$2d + 1 = \# Q_B \leqq \sum^r \dim D\tilde{\Lambda}(q_i, ?) = \sum^r \dim D\Lambda(q_i, ?).$$

Therefore, Q_B contains three sources which have the same image under π. Changing notation, we have $\pi(q_1) = \pi(q_i) = \pi(q_r)$. It follows $\pi(t) \neq \pi(u)$ or $\pi(v) \neq \pi(x)$ or $\pi(x) \neq \pi(t)$. In any case B contains a full convex subcategory C with Gabriel-quiver $x_0 \to x_1 \to \ldots \to x_n \leftarrow x_{n+1}$, such that $\pi(x_0) = \pi(x_{n+1})$ and $\pi(x_1)$

$+\pi(x_n)$. Let U be "the" indecomposable $\tilde{\Lambda}$-module with support C. The push-down $F_\lambda U$ [9, 3.2] has a natural basis $e(x_i)$, $0 \leqq i \leqq n+1$. The morphisms $\tilde{\alpha}_i$ associated to the arrows in Q_C yield morphisms $\alpha_i := \pi(\tilde{\alpha}_i)$, $0 \leqq i \leqq n$, so that

$$e(x_i)\alpha_j = \begin{cases} e(x_{i+1}), & \text{if } x_i \leftarrow x_{i+1} \text{ and } \pi(\tilde{\alpha}_j) = \pi(\tilde{\alpha}_i), \\ e(x_{i-1}), & \text{if } x_i \leftarrow x_{i-1} \text{ and } \pi(\tilde{\alpha}_j) = \pi(\tilde{\alpha}_i), \\ 0 & \text{in the remaining cases}. \end{cases}$$

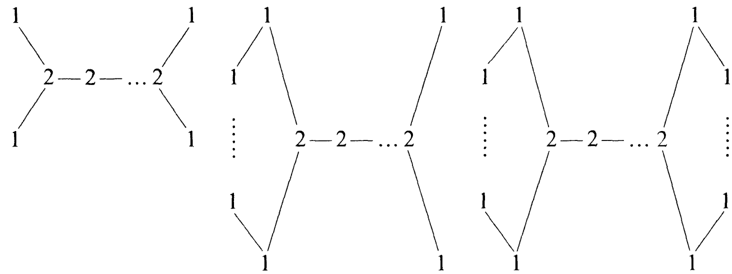
For any $\xi \in k^*$ we take the 1-dimensional submodule $V(\xi)$ of $F_\lambda U$ generated by $\xi e(x_0) + e(x_{n+1})$. The quotient $M(\xi) := F_\lambda U / V(\xi)$ has the images $\bar{e}(x_i)$ of the $e(x_i)$ with $0 \leqq i \leqq n$ as basis.

We claim that the $M(\xi)$ yield infinitely many isomorphism classes. The proof is based on an idea of [10]. We start with some general remarks.

Given a Λ-module M, a k-subspace $U \subseteq M$ and an element $\alpha \in \Lambda$, we define $U\alpha^1$, respectively, $U\alpha^{-1}$ as the image, respectively, the inverse image of U under multiplication with α in M. A word $w = \alpha_0^{\varepsilon_0}\alpha_1^{\varepsilon_1} \dots \alpha_n^{\varepsilon_n}$ of length $n+1$ is a sequence of elements $\alpha_i \in \Lambda$ and signs $\varepsilon_i \in \{+1, -1\}$. By induction, we define $Uw = (U\alpha_0^{\varepsilon_0} \dots \alpha_{n-1}^{\varepsilon_{n-1}})\alpha_n^{\varepsilon_n}$. Looking at the powers w^m, $m \geqq 1$, we find inclusions $Mw^m \supseteq Mw^{m+1}$, $Mw^m \supseteq Ow^m$, and $Ow^{m+1} \supseteq Ow^m$. Set $M(w) = \left(\bigcap_m Mw^m\right) \Big/ \left(\bigcup^m Ow^m\right)$ and denote by $w_i = \alpha_i^{\varepsilon_i} \dots \alpha_n^{\varepsilon_n}\alpha_1^{\varepsilon_1} \dots \alpha_{i-1}^{\varepsilon_{i-1}}$ the cyclic permutations of $w = w_0$, $i \in \mathbb{Z}/(n+1)\mathbb{Z}$. Multiplication by α_i induces a surjective linear map $M(w_i) \to M(w_{i+1})$ if $\varepsilon_i = +1$ and an injective map $M(w_{i+1}) \to M(w_i)$ if $\varepsilon_i = -1$. Walking through the cycle $M(w_0) \to M(w_1) \to \dots \to M(w_n) \to M(w_0)$ and looking at dimensions, we conclude that all maps are bijective. Hence "multiplication" by w induces an automorphism of $M(w_0)$. We have constructed a functor $F(w)$ from $\operatorname{mod}\Lambda$ to $\operatorname{mod}k[T, T^{-1}]$.

In the situation of our proof we set $\varepsilon_i = +1$ if $x_i \leftarrow x_{i+1}$ in C and $\varepsilon_i = -1$, otherwise. We consider the word $w = \alpha_0^{\varepsilon_0}\alpha_1^{\varepsilon_1} \dots \alpha_n^{\varepsilon_n}$. Of course, $\bar{e}(x_0) \in (M(\xi))w^m$ for all $\xi \in k^*$ and all $m \geqq 1$. On the other hand, we obtain inclusions $\left(\bigoplus_{j \neq i} k\bar{e}(x_j)\right)\alpha_i^1 \subseteq \bigoplus_{j \neq i+1} k\bar{e}(x_j)$ in case $x_i \leftarrow x_{i+1}$ and $\left(\bigoplus_{j \neq i+1} k\bar{e}(x_j)\right)\alpha_i^{-1} \subseteq \bigoplus_{j \neq i}^{j \neq i} k\bar{e}(x_j)$, otherwise. It follows $\bar{e}(x_0) \notin Ow^m$, $m \geqq 1$. Clearly, $\bar{e}(x_0)$ yields an eigenvector to the eigenvalue $-\xi$ for the multiplication with w in $(M(\xi))(w)$. Thus the $F(w)M(\xi)$ give rise to infinitely many isomorphism classes, and so do the $M(\xi)$ a fortiori. q.e.d.

3.3. We are ready to prove our theorem and start with the easy direction. So, suppose one of the conditions a), b), c), d′) is not satisfied. We will show that there are infinitely many isomorphism classes of Λ-modules of dimension $\leqq 4d+30$. If b) does not hold, this follows from 3.2. In the other cases it is enough to find infinitely many non-isomorphic $\tilde{\Lambda}$-modules of dimension $\leqq 4d+30$. For Lemma 3.2 of [9] implies that the push-down F_λ does not identify infinitely many isomorphism classes. If a) is not satisfied we are done by 3.1. If *c*) is wrong the well-known families of indecomposables over the algebras of type (1)–(3) do the job. We only describe them by giving the dimensions at the points:



In the last case, a glance at the list in [4] shows, that there is always a family of dimension $\leqq 30$.

To show the other direction, assume that the conditions a), b), c), and d′) hold. We have to prove that Λ is representation-finite, or equivalently, that $\tilde{\Lambda}$ is locally representation-finite [9, 12]. Suppose not. To get a contradiction, we distinguish two cases.

1st Case. There is a finite convex full subcategory B of $\tilde{\Lambda}$ which is not representation-finite.

We can assume that B is critical (2.6). By 2.3, $H_1\tilde{\Lambda}=0$ implies $H_1 B=0$, so that all points of B are separating (2.5). Thus, B is given by a graded tree (T, g) with T of type $\tilde{D}_n$ or $\tilde{E}_n$ (2.6). But $T=\tilde{D}_n$ is excluded by b) and c), and $T=\tilde{E}_n$ by d′).

2nd Case. All finite convex full subcategories of $\tilde{\Lambda}$ are representation-finite.

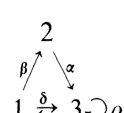
Then there is an indecomposable $\tilde{\Lambda}$-module U with support B containing more than $10d+1000$ points. The convex hull $\hat{B}$ of B is still finite, since $\tilde{\Lambda}$ is interval-finite. Using 2.3, 2.5, and 2.6, one sees that $\hat{B}$ is an extremal algebra in the sense of [3], so that we can apply the classification given there. Looking at the possible algebras of families (1)–(24) and $(1)^{op}$–$(24)^{op}$, we find a contradiction to condition b).

We have shown that 1) is equivalent to 2). The remaining part of the theorem follows from the first part of our proof. q.e.d.

Remarks. a) The proof of Brauer-Thrall 2 for standard algebras (without the bound $4d+30$) does not depend on Theorem 2.6, but uses part of the classification obtained in [3].

b) Analyzing more carefully the situation, one can prove that a not representation-finite standard algebra of dimension d has a $\mathbb{P}_1(k)$-family of indecomposables of dimension $n\leqq 4d+30$. This means that there is a natural number $n\leqq 4d+30$ and a polynomial map f from the projective line over k into the variety of n-dimensional Λ-modules, such that the images $f(x)$, $x\in\mathbb{P}_1(k)$, correspond to non-isomorphic indecomposables.

4. An Example

Look at the algebra Λ of dimension 15 defined by the quiver  and relations $\alpha\beta=\varrho\delta$, $\gamma\alpha=\gamma\delta=\delta\gamma=\varrho^2=0$. We claim that Λ is representation-finite and use our theorem to prove it.

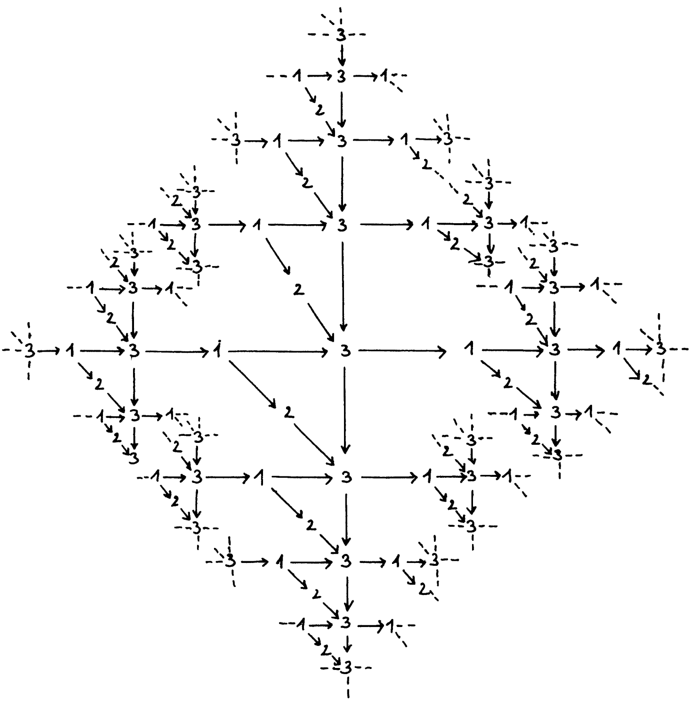


Fig. 4

Using the method described in [7, 3.3], we construct a covering $\tilde{\Lambda} \xrightarrow{\pi} \Lambda$. Figure 4 shows a finite piece of $Q_{\tilde{\Lambda}}$. All points of $\pi^{-1}(x)$ are also denoted by x. $\tilde{\Lambda}$ is defined by the lifted relations, whence locally bounded and Schurian. Moreover, $Q_{\tilde{\Lambda}}$ is connected, directed, and interval-finite. The group G of the Galois-covering $\tilde{\Lambda} \xrightarrow{\pi} \Lambda$ is free in two generators.

To check $H_1 \tilde{\Lambda} = 0$, we verify first that 2a) holds, which is easy. By [7, 2.6], $H_1 \tilde{\Lambda}$ is computed as a limit of some $H_1 C$, where C runs through some finite convex full subcategories of $\tilde{\Lambda}$. Since all points of such a C are separating, we have $H_1 C = 0$ by 2.3.

Next, we verify easily, that conditions b) and c) are satisfied.

To check d′), one clearly has to deal only with algebras of types 1–34 from the list in [4]. All cases are excluded, most of them by obvious reasons.

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Two Reduction Theorems for Threefold Birational Morphisms

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In this paper, we prove two reductions for the factorization of proper birational morphisms of smooth threefolds. A birational map is *factorizable* if it may be written as a composition of smooth blow-ups and blow-downs. An open conjecture is that all proper birational maps of smooth algebraic spaces are factorizable. In [3], Hironaka reduced the conjecture to the case of birational morphisms, that is, given a proper birational map $F: X' \dashrightarrow Y$, F may be modified by a factorizable map (in fact, a composition of smooth blow-ups) $\sigma: X \to X'$ such that $f = F \circ \sigma: X \to Y$ is a birational morphism and the exceptional divisor of f is a union of smooth hypersurfaces meeting normally. For birational morphisms of smooth threefolds, we show, by two reductions, the existence of a modification by a factorizable map such that the resulting morphism is a point modification with only rational surfaces contained in the exceptional divisor.

Let S_f be the subvariety of Y on with f^{-1} is not a morphism and E_f be the exceptional divisor, so $\operatorname{codim}_Y S_f \geqq 2$ and $\operatorname{codim}_X E_f = 1$. Given f as above, where X and Y are smooth threefolds, we prove:

I (Theorem 2.1)

Given $f: X \to Y$ as above, there is $\beta: X \dashrightarrow W$, a factorizable map, such that $g = f \circ \beta^{-1}: W \to Y$ is a birational morphism of smooth threefolds such that E_g is a union of smooth surfaces meeting normally and $\dim S_g = 0$. Thus g satisfies the same conditions as f and g collapses surfaces only to points, not to curves.

II (Theorem 3.1)

Given $g: W \to Y$ as in I above, there is $\gamma: W \dashrightarrow Z$, a factorizable map, such that $h = g \circ \gamma^{-1}$ is a birational morphism of smooth threefolds satisfying the above conditions for g; moreover, $S_h = S_g$ and all components of E_h are rational surfaces. Thus h satisfies the same conditions as g and h collapses only rational surfaces to points.

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To prove these theorems, we first establish a lemma showing an elementary method for birationally modifying a threefold to make an embedded ruled surface minimal ruled. In both reductions, this lemma is used to make a generically contractible ruled surface smoothly contractible.

For the first reduction, we consider a surface normal to a curve in S_f. Knowing the factorization of the induced birational morphism on the surface, we find a suitable surface in E_f which is generically contractible and maps to a curve in S_f. Using the minimizing lemma mentioned above, we are able to smoothly contract the surface. Birational modifications both before and after blowing down this surface ensure that the resulting exceptional divisor will also have normal crossings and smooth components, allowing us to repeat this procedure until there are no surfaces remaining in E_f which map to curves. For the second reduction, we use Frumkin's results on regular resolutions of birational maps [2, pp. 198–199, Proposition 2.2] to find an irrational surface in E_g which is generically contractible and then proceed as in the first reduction.

Kulikov has independently shown the first reduction [6, p. 882, Theorem 2] using essentially the same methods as in our proof of the minimizing lemma plus results of Danilov [1] on the factorization of birational morphisms with fiber dimension zero or one. This coincidence is not entirely accidental since study of Kulikov's earlier work on birational modifications of degenerations of $K3$ surfaces [5] has greatly influenced our approach to birational morphisms of threefolds.

Notation and Basic Facts

In this section, we establish notation and review some basic facts about birational morphisms between smooth threefolds and surfaces and curves in smooth threefolds.

0.1 Total and Proper Transforms. Given a rational map $f: X \dashrightarrow Y$ of smooth algebraic spaces, let Γ_f be the graph of f, that is, the closure of the graph of $f|_U$, where U is the largest open on which $f|_U$ is regular. $\Gamma_f \subset X \times Y$ and, restricting the projection maps, we have

$$\begin{array}{ccc} & \Gamma_f & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & \overset{f}{\dashrightarrow} & Y. \end{array}$$

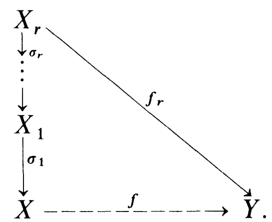
For $Z \subset X$, let $f\{Z\} = \pi_2(\pi_1^{-1}(Z))$, the *total transform* of Z. If $Z \cap U \neq \emptyset$, the *proper transform* of Z is $f[Z] = \overline{f(Z \cap U)}$ the closure of $f(Z \cap U)$ in Y. If f is a morphism, then $f(Z) = f\{Z\} = f[Z]$. Throughout this paper, solid arrows indicate morphisms, while dashed arrows indicate rational maps (which may or may not be morphisms).

0.2 Exceptional Divisors and Fundamental Loci. For a proper birational morphism $f: X \to Y$ of smooth algebraic spaces, the *fundamental locus* of f, denoted S_f, is the subvariety of Y on which the rational map f^{-1} is not regular. The *exceptional divisor* of f, denoted E_f, is the subvariety of X collapsed to S_f; thus

$E_f = f^{-1}\{S_f\}$. By Zariski's Main Theorem [10, p. 154, Satz 5], E_f has pure codimension one, while S_f has codimension at least two. If S_f is a union of points, then f is a *point modification.*

0.3 Smooth Blow-ups and Blow-downs. By *smooth blow-up* we mean the blow-up of a smooth algebraic subspace of a smooth algebraic space. By *smooth blow-down* we mean the inverse map of a smooth blow-up. Under this convention, a blow-down is a morphism, while a blow-up is a mapping.

0.4 Regular Resolutions. For $f: X \dashrightarrow Y$ a rational map of smooth algebraic spaces, a *regular resolution* of f is $\sigma_1 \circ \ldots \circ \sigma_r$, where $\sigma_i^{-1}: X_{i-1} \dashrightarrow X_i$ is a smooth blow-up, $i=1, \ldots, r$, $X_0 = X$, such that



(i) $E_{\sigma_1 \circ \ldots \circ \sigma_i}$ has global normal crossings for all $i=1, \ldots, r$,

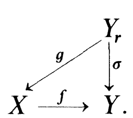
(ii) $f_i = f \circ \sigma_1 \circ \ldots \circ \sigma_i$ is not a morphism for $i=1, \ldots, r-1$; moreover, $S_{\sigma_{i+1}}$ is contained in the locus of indeterminacy of f_i,

(iii) f_r is a morphism, and

(iv) r is minimal with respect to (i), (ii), and (iii).

The existence of regular resolutions (Chow's lemma) is due to Hironaka (for schemes [3, pp. 142–143, Main Theorem II]; for complex spaces [4, p. 504, Corollary 2]).

0.5 Proposition (Frumkin). *If $f: X \to Y$ is a proper birational morphism of smooth threefolds and $\sigma = \sigma_1 \circ \ldots \circ \sigma_r$ is a regular resolution of f^{-1} (so $g = f^{-1} \circ \sigma$ is a birational morphism), then all components of E_g are rational.* For the proof, see Frumkin [2, p. 192, Corollary 2.3 and pp. 198–199, Proposition 2.2].



0.6 Proposition. *If $f: X \to Y$ is a proper birational morphism of smooth algebraic spaces, then each components of E_f is birational to a $\mathbb{P}^\ell$-bundle $(1 \leqq \ell \leqq \dim X - 1)$.*

Proof. Let $\sigma = \sigma_1 \circ \ldots \circ \sigma_r$ be a regular resolution of f^{-1} and let $g = f^{-1} \circ \sigma$. By 0.4(ii), $S_\sigma \subset S_f$; thus $g(E_\sigma) \subset E_f$. On the other hand, $\sigma = f \circ g$, so $E_\sigma = E_{f \circ g} \supset g^{-1}\{E_f\}$. Thus $g(E_\sigma) = E_f$.

Let V be a component of E_f. Since $g(E_\sigma) = E_f$ and $\operatorname{codim}_X(S_f) \geqq 2$, there is a unique component, W, of E_σ such that $g(W) = V$; moreover, $g|_W$ maps W birationally onto V.

Since $W \subset E_\sigma$, there is a j, $1 \leqq j \leqq r$, such that $\dim W_{j-1} < \dim W_j = \dim W$, where $W_i = \sigma_i \circ \ldots \circ \sigma_r(W)$. Thus $\sigma_j \circ \ldots \circ \sigma_r|_W$ maps W birationally onto $W_j = E_{\sigma_{j-1}}$. Since σ_{j-1} is a smooth blow-down, W_j is a $\mathbb{P}^\ell$-bundle over $\sigma_{j-1}(W_j)$, proving the lemma.

0.7 Remark. Let $f: X \to Y$ be a proper birational morphism of smooth threefolds.

(i) Each smooth component of E_f is a ruled surface or $\mathbb{P}^2$, since any smooth surface birational to a $\mathbb{P}^1$-bundle is by definition ruled or $\mathbb{P}^2$.

(ii) If V is a smooth component of E_f and $f(V)$ is a curve, then (in the notation of 0.6) $\sigma|_W$ gives a ruling of W over the normalization of $\sigma(W) = f(V)$. Since $g|_W: W \to V$ is a birational morphism of smooth surfaces over $f(V)$, V is therefore ruled by $f|_V$ over the normalization of $f(V)$.

0.8 Contraction Criteria. Given a divisor in a smooth algebraic space, there is a criterion for when that divisor is the exceptional divisor of a smooth blow-down [7, p. 158, Theorem 2]. For D a divisor in a smooth threefold X, D may be *smoothly contracted*, i.e. D is the exceptional divisor of a smooth blow-down iff

(i) $D \cong \mathbb{P}^2$ and $(E \cdot D)_X = -1$ for E a line in D, or

(ii) D is a smooth minimal ruled surface and $(F \cdot D)_X = -1$ for F a fiber on D.

We say that D is *generically contractible* [9, p. 51] provided D is ruled, smooth and $(F \cdot D)_X = -1$. Thus, if $D \ncong \mathbb{P}^2$, then D is smoothly contractible iff D is generically contractible and minimal ruled.

0.9 Normal Bundles and Blow-ups. Let X be a smooth threefold and V and W divisors in X. Assume that V and W meet normally, $V \cap W = C$, a smooth curve, and V and W are smooth near C. Then

(i) $(C \cdot V)_X = (C^2)_W$,

(ii) if $C \cong \mathbb{P}^1$, then $N_{C/X} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$, where $a = (C^2)_V$ and $b = (C^2)_W$, and

(iii) let $\sigma^{-1}: X \dashrightarrow \tilde{X}$ blow up C. Letting $\tilde{V} = \sigma^{-1}[V]$, $\tilde{W} = \sigma^{-1}[W]$, $Z = \sigma^{-1}\{C\}$, $\tilde{C}_V = \tilde{V} \cap Z$, and $\tilde{C}_W = \tilde{W} \cap Z$, we have
we have

(a) $$Z \cong \mathbb{P}_C(N_{C/X})$$

and

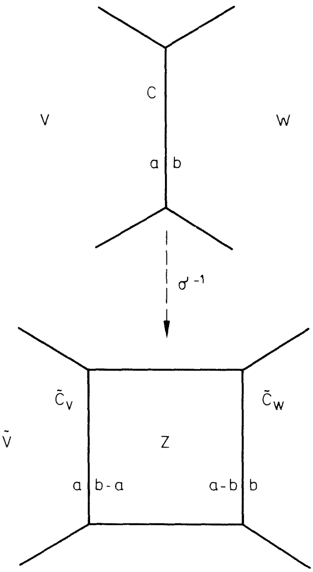
(b) $(\tilde{C}_V^2)_{\tilde{V}} = a$ and $(\tilde{C}_V^2)_Z = b - a$ while $(\tilde{C}_W^2)_{\tilde{W}} = b$ and $(\tilde{C}_W^2)_Z = a - b$.

Proof. From the inclusions $C \hookrightarrow V$ and $V \hookrightarrow X$ and the definition of normal bundle, we have the exact sequence

$$0 \longrightarrow N_{C/V} \xrightarrow{i_*} N_{C/X} \longrightarrow N_{V/X}|_C \longrightarrow 0,$$

where i_* is induced from the inclusion of V in X. Since V and W meet normally along C, $N_{C/X} \cong N_{C/V} \oplus N_{C/W}$; moreover, under this isomorphism the summand $N_{C/V}$ is the image of $N_{C/V}$ under i_*. Thus $N_{V/X}|_C \cong N_{C/W}$. Since C is smooth and V and W are smooth near C, $N_{V/X} \equiv (V \cdot V)_X$, so $N_{V/X}|_C \equiv (C \cdot V)_X$ while $N_{C/W} \equiv (C \cdot C)_W$, giving (i) and (ii).

To show (iii), note that since C is a Cartier divisor in V, $\sigma|_{\tilde{V}}$ maps $\tilde{V}$ isomorphically to V. Thus $(\tilde{C}_V^2)_{\tilde{V}} = a$. On the other hand, by (i), $(\tilde{C}_V^2)_Z$



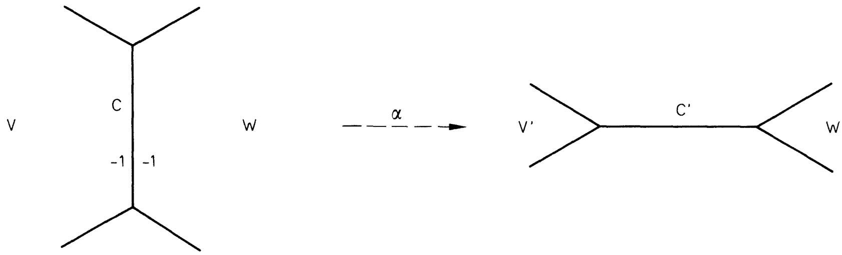
$=(\tilde{C}_V\cdot(\sigma^*(V)-Z))_{\tilde{X}}=(C\cdot V)_X-(\tilde{C}_V\cdot Z)_{\tilde{X}}$. Now $(C\cdot V)_X=(C^2)_W=b$, while $(\tilde{C}_V\cdot Z)_{\tilde{X}}=(\tilde{C}_V^2)_{\tilde{V}}=a$. Similarly for $\tilde{C}_W$.

0.10 Elementary Modifications. If $C\subset X$ is a smooth rational curve in a smooth threefold with $N_{C/X}\cong\mathcal{O}_C(-1)\oplus\mathcal{O}_C(-1)$, then an *elementary modification centered at* C is a birational map $\alpha: X\dashrightarrow X'$ such that

$$\begin{array}{ccc} & \tilde{X} & \\ {\scriptstyle \sigma_C}\swarrow & & \searrow{\scriptstyle \tilde{\sigma}_C} \\ X & \overset{\alpha}{\dashrightarrow} & X', \end{array}$$

where σ_C^{-1} is the blow-up of C in X. If $V=\sigma_C^{-1}\{C\}$, then $V\cong\mathbb{P}^1\times\mathbb{P}^1$; moreover, by 0.8(ii), V is smoothly contractible under either of the rulings given by the projections. Let $\tilde{\sigma}_C$ be the blow-down of V under the other ruling, i.e. such that for all $c\in C$, $\tilde{\sigma}_C\circ\sigma_C^{-1}\{c\}=C'=\tilde{\sigma}_C(V)$ and for all $c'\in C'$, $\sigma_C\circ\tilde{\sigma}_C^{-1}\{c'\}=C$.

0.11 Modifications of Type II [5, p. 969]. A *modification of type II* is an elementary modification centered at C, $C\subset V\cap W$, where V and W are, near C, smooth surfaces meeting normally and $(C^2)_V=(C^2)_W=-1$. By 0.9(ii), $N_{C/X}\cong\mathcal{O}_C(-1)\oplus\mathcal{O}_C(-1)$, so an elementary modification may be performed centered at C. If $\alpha: X\dashrightarrow X'$ is a modification of type II, $V'=\alpha[V]$ and $W'=\alpha[W]$, then $\alpha|_V$ has blow down the exceptional curve C on V (similarly for W) and, near $C'=\alpha\{C\}$ in X, V' does not intersect W'.



0.12 Surfaces. *A) Let f be a proper birational morphism of smooth algebraic surfaces and $E_f = \sum_{i \in I} n_i L_i$, where n_i are the multiplicities given by the pullback of the reduced ideal sheaf of S_f. Then*

(i) E_f has global normal crossings, a negative definite intersection matrix and contains at least one exceptional curve, and

(ii) it is not possible for all exceptional curves L_j in E_f to satisfy both (a) and (b) below:

(a) $L_j \cdot \left(\sum_{i \neq j} L_i\right) = 2$,

(b) if $L_j \cdot L_i > 0$, then $L_i^2 = -N_j$, where N_j depends only on j.

B) If V is a smooth ruled surface with fiber $F = \sum_{i \in I} n_i L_i$ and either

(i) L_j is the unique exceptional curve in F or

(ii) L_j is an exceptional curve in F and $L_j \cdot \left(\sum_{i \neq j} L_j\right) \geqq 2$,

then $n_j \geqq 2$.

Proof. A) The result in (i) is well-known (see Zariski [11, p. 538], and Mumford [8, p. 6]). For (ii), we proceed by contradiction and assume that all exceptional curves in E_f are of the given form. Since E_f is an exceptional divisor, $K \cdot E_f = -1$, where K is the canonical divisor.

Let $\xi_1, \ldots, \xi_h$ be the connected components of the union of all exceptional curves in E_f and all curves in E_f intersecting exceptional curves. By (a) and (b), each ξ_α is a union of exceptional curves and curves with self-intersection $-N_\alpha$, where N_α depends only on α.

Let $I_\alpha = \{i \in I | L_i \subset \xi_\alpha\}$ and $E_\alpha = \sum_{i \in I_\alpha} n_i L_i$, so $\operatorname{Supp} E_\alpha = \xi_\alpha$.

If $L_i \not\subset \operatorname{Supp}(E_1 + \ldots + E_h)$, then $L_i^2 \leqq -2$ and so by the genus formula $K \cdot L_i \geqq 0$. Since $K \cdot E_f = -1$, for some α, $K \cdot E_\alpha < 0$. Let $J_\alpha = \{i \in I_\alpha | L_i^2 = -1\}$ and $K_\alpha = \{i \in I_\alpha | L_i^2 = -N_\alpha\}$, so $I_\alpha = J_\alpha \coprod K_\alpha$. By the genus formula $K \cdot L_i = -2 - L_i^2$, so $K \cdot E_\alpha = -\sum_{j \in J_\alpha} n_j + \sum_{k \in K_\alpha} (N_\alpha - 2) n_k$. For $j \in J_\alpha$, L_j is an exceptional curve meeting only L_k and $L_{k'}$, for some $k, k' \in K_\alpha$, so $n_j = n_k + n_{k'}$. For $k \in K_\alpha$, let $\Delta_k = \{j \in J_\alpha | L_j \cdot L_k \neq 0\}$ and δ_k be the cardinality of Δ_k. Thus $K \cdot E_\alpha = \sum_{k \in K_\alpha} (N_\alpha - (2 + \delta_k)) n_k$. Since $K \cdot E_\alpha < 0$, $2 + \delta_k > N_\alpha$ for some $k \in K_\alpha$.

Assume that L_k has the property that $2+\delta_k > N_\alpha$. Thus $L_k^2 = -N_\alpha$ and L_k intersects at least $N_\alpha + 1$ exceptional curves in E_f. Let $\tilde{E} = L_k + \sum_{j \in \Delta_k} L_j$, so $\tilde{E}^2 = \delta_k - N_\alpha$. By (i), $\tilde{E}^2 < 0$, so $\delta_k < N_\alpha$. Thus $N_\alpha = \delta_k + 1$. If $\delta_k = 1$ or 2, then $\xi_\alpha^2 = 0$, so $\delta_k \geqq 3$. On the other hand, if $N_\alpha = \delta_k + 1$, then if all $L_j, j \in \Delta_k$, were contracted, then L_k would become an exceptional curve, the contraction of which would produce a δ_k-tuple point in E_f, impossible for $\delta_k \geqq 3$ by (i).

B) By the genus formula, $-2 = K_V \cdot F$, so $-2 = \sum_{i \in I} n_i K_V \cdot L_i$. For all i, $L_i^2 \leqq -1$ and so $K_V \cdot L_i \geqq -1$ with equality iff L_i is an exceptional curve. If L_j is the unique exceptional curve in F, then $-2 = -n_j + \sum_{i \neq j} n_i K_V \cdot L_i$, where $K_V \cdot L_i \geqq 0$, so $n_j \geqq 2$.

If L_j is an exceptional curve, then $L_j^2 = -1$, so $0 = L_j \cdot F = -n_j + \sum_{i \neq j} n_i L_i \cdot L_j$. Thus if $L_j \cdot \left(\sum_{i \neq j} L_j \right) \geqq 2$, then $n_j \geqq 2$.

1. The Minimizing Lemma

In this section, we prove a lemma which allows us to modify a normal crossings divisor containing a ruled surface so that the proper transform of the ruled surface is minimal ruled. This modification requires only smooth blow-ups and blow-downs centered at points or rational curves. Under mild conditions on the divisor [see 1.1.4) below] normal crossings are preserved, although smooth rational self-intersection curves may be introduced in the components of the total transform of the divisor.

In Sects. 2 and 3, we will find certain generically contractible surfaces which we will wish to smoothly contract (see 0.8). Using the minimizing lemma, we will also be able to make these surfaces minimal ruled without affecting their generic contractibility, thus allowing them to be smoothly contracted.

1.1 Minimizing Lemma. *Let X be a smooth threefold and $\{V, V_1, \ldots, V_n\}$ be a collection of smooth surfaces in X with normal crossings. If V is ruled, then there is a birational map $\alpha: X \dashrightarrow X'$ such that*

1) *$\alpha[V]$ is smooth and minimal ruled,*

2) *α is an isomorphism away from the reducible fibers on V,*

3) *α is a composition of smooth blow-ups and blow-down with centers in points or rational curves, and*

4) *if $V \cap (V_1 \cup \ldots \cup V_n)$ contains no multi-sections and no more than two sections on V, then $\alpha\{V \cup V_1 \cup \ldots \cup V_n\}$ is a union of surfaces meeting normally and all self-intersection curves are smooth and rational.*

Proof. Let $J = 8(1 - q(V)) - K_V^2$, so J is the number of blow-downs needed to make V minimal ruled. If $J = 0$, there is nothing to show. If $J \geqq 1$, then there exists an exceptional curve L on V which is therefore a component of a reducible fiber. We blow up all components of this reducible fiber which are not double curves of $V \cup V_1 \cup \ldots \cup V_n$. These blow-ups are all centered at smooth rational curves meeting normally. We may therefore assume that all components of this reducible fiber are double curves of $V \cup V_1 \cup \ldots \cup V_n$; in particular, that $L \subset V \cap V_i$ for some i, $1 \leqq i \leqq n$.

Note that L meets one or two other fiber components. Let L_j be one such fiber component. Since V and V_i meet normally and L is rational, by 0.9(ii),

$N_{L/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$, where $a = (L^2)_{V_t}$. If $a = -1$, then a modification of type II centered at L will blow L down on V, reducing J by one. If $a < -1$, then blow up L in X, yielding an $\mathbb{F}_{-a-1}$ as exceptional divisor. Letting σ_L^{-1} be the blow-up of L and $L' = \sigma_L^{-1}\{L\} \cap \sigma_L^{-1}[V]$,

$$N_{L'/\sigma_L^{-1}\{X\}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(a+1).$$

Iterating this procedure $-(a+1)$ times, the new L has the correct normal bundle to perform a modification of type II. If $a > -1$, then we blow up L_j $a+1$ times along V, each time decreasing $(L^2)_{V_t}$ by one; thereby correcting the normal bundle. Thus after blow-ups centered at rational curves and a modification of type II, J is decreased by one. Repeating the above procedure J times, 1.1.1)–1.1.3) are satisfied, where α is the corresponding sequence of blow-ups and modifications of type II; moreover, all surfaces in $\alpha\left\{V \cup \left(\bigcup_{k \in K} V_k\right)\right\}$ meet normally, where $K = \{k \in \mathbb{N} \mid 1 \leq k \leq n \text{ and } V_k \cap V \text{ is not a section or a multi-section on } V\}$.

To verify 1.1.4), we first note that if α consisted only of blow-ups with double curves as centers, $\alpha\{V \cup V_1 \cup \ldots \cup V_n\}$ would be a union of smooth surfaces meeting normally. Thus we need only check 1.1.4) for modifications of type II. If L intersects at most two other double curves, then a modification of type II centered at L preserves normal crossings and smoothness of components unless $L \cdot V_j = 2$ and $L \not\subset V_j$; in which case normal crossings will be preserved but $\alpha[V_j]$ will have a smooth rational self-intersection curve. If L intersects at least three other double curves, then normal crossings will not be preserved by a modification of type II centered at L. We will show that under the conditions 1.1.4), some exceptional curve (not necessarily L) in the fiber containing L intersects at most two other double curves.

We assume that $V \cap (V_1 \cup \ldots \cup V_n)$ contains no multi-sections and no more than two sections on V. If $V \cap (V_1 \cup \ldots \cup V_n)$ contains no sections, then each exceptional curve L meets one or two other fiber components and so only one or two other double curves. We may therefore assume that $V \cap (V_1 \cup \ldots \cup V_n)$ contains either one or two sections on V.

If $V \cap (V_1 \cup \ldots \cup V_n)$ contains exactly one section S on V, then L would intersect more than two other double curves only if L were to intersect S and two other fiber components. This is impossible, though, since by 0.12B)(ii), L would have multiplicity at least two in the fiber and so could not intersect a section.

If $V \cap (V_1 \cup \ldots \cup V_n)$ contains exactly two sections, S_1 and S_2, on V, then, noting 0.12B)(ii) again, L intersects more than two other double curves only if L intersects S_1, S_2 and exactly one other fiber component. Since L intersects a section, L is a reduced component of the fiber on V containing it, and so, by 0.12B)(i), there is an L', another exceptional curve in the fiber. Since S_1 and S_2 only intersect the fiber once, L' intersects only double curves which are fiber components and so intersects at most two other double curves. We may thus use L' instead of L in the algorithm to decrease J.

2. Reduction to Point Modifications

In this section we consider the fundamental locus of a proper birational morphism of smooth threefolds. The fundamental locus, by 0.2, is a union of curves and/or points. In 2.1 below, we will modify the domain by a birational map (which is a composition of smooth blow-ups and blow-downs) so that the composite birational map is still a morphism; moreover, the composite morphism has only points in its fundamental locus. The proof consists essentially in showing that among the surfaces in the exceptional divisor which are mapped to curves, at least one such surface is generically contractible and then applying the minimizing lemma (1.1) in order to smoothly contract it.

2.1 Theorem. *Let $f: X \to Y$ be a proper birational morphism of smooth threefolds such that E_f consists of smooth surfaces meeting normally. Then there is a birational map $\beta: X \dashrightarrow W$ such that:*

$$\begin{array}{ccc} X & \overset{\beta}{\dashrightarrow} & W \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y. & \end{array}$$

1) *$g = f \circ \beta^{-1}$ is a proper birational morphism of smooth threefolds such that $S_g \subset S_f$, $\dim S_g = 0$, and E_g consists of smooth surfaces meeting normally and*

2) *β is a composition of smooth blow-ups and blow-downs.*

Proof. If $\dim S_f = 0$, there is nothing to show. If $\dim S_f = 1$, let C be an irreducible curve obtained in S_f. Let $E_f = \bigcup_{i \in I} V_i$ and $I_C = \{i \in I \mid f(V_i) = C\}$. The main step of the proof consists in reducing the cardinality of I_C by one, creating new components which are mapped only to points on C.

By 0.7(ii), $f|_{V_i}$ gives a ruling of V_i over the normalization of C for each $i \in I_C$. Let P be a smooth point of C such that $f^{-1}(P) \cap V_i$ is an irreducible fiber on V_i if $i \in I_C$ and $f^{-1}(P) \cap V_i = \emptyset$ if $i \notin I_C$. Let E be a surface in Y smooth at P such that E intersects C normally at P. Let $\tilde{E} = f^{-1}[E]$ and $Q \in f^{-1}(P) \cap \tilde{E}$. For some $j \in I_C$, $Q \in V_j$. V_j is a ruled surface, ruled by $f|_{V_j}$, which is minimal near $f^{-1}(P)$, so V_j contains a section S_j through Q. Now we compute $(S_j \cdot \tilde{E})_{X,Q}$. Since

$$f^*(E) = f^{-1}[E] = \tilde{E}, \qquad (S_j \cdot \tilde{E})_{X,Q} = (f_*(S_j) \cdot E)_{Y,P} = n_j (C \cdot E)_{Y,P},$$

where $n_j = \deg_C(f|_{S_j})$. Since S_j is a section on V_j, $n_j = 1$. Since $(C \cdot E)_{Y,P} = 1$, $(S_j \cdot E)_{X,Q} = 1$. Thus E is smooth at Q and so E is smooth near $f^{-1}(P)$; moreover, E intersects each V_j normally (thus E intersects E_f transversely, although not necessarily normally).

Because $f|_{\tilde{E}}: E \to E$ is a proper birational morphism of smooth surfaces (near P), by 0.12A)(i), $\tilde{E} \cap f^{-1}(P)$ contains at least one exceptional curve. If $L \subset \tilde{E} \cap f^{-1}(P)$ is an exceptional curve, then $L \subset V_i$ for some $i \in I_C$ and $(L \cdot V_i)_X = (L^2)_{\tilde{E}} = -1$ since V_i intersects $\tilde{E}$ normally along L. Since L is a fiber on V_i, each such V_i is generically contractible in X.

To finish the proof, we will use the minimizing lemma to make some generically contractible V_i smoothly contractible, contract it and then blow-up any resulting self-intersection curves. To preserve normal crossings, we need a lemma:

2.2 Lemma. *For some $j \in I_C$,*

1) *V_j contains an exceptional curve of $f|_{\tilde{E}}$, and*

2) *$V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ contains no multi-sections and at most two sections on V_j (under*

the ruling given by $f|_{V_j}$*); moreover, if* $V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ *contains two sections,* S_1 *and* S_2*, then* $S_1 \cup S_2 \nsubseteq V_j \cap V_k$ *for any* $k \neq j$.

Proof. Let L be an exceptional curve of $f|_{\tilde{E}}$, so [0.12A)(i)] L intersects at most two other components of $f^{-1}(P)$. If $L \subset V_j$ and L intersects only one other component, $L' \subset V_k$, of $f^{-1}(P)$, then $V_j \cap V_k$ is a section on V_j and $V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ contains no other sections or multi-sections. Similarly, if $L \subset V_j$ and L intersects two components, $L_1 \subset V_k$ and $L_2 \subset V_\ell$, then $V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ contains two sections or one bisection, according to whether $k \neq \ell$ or $k = \ell$, and $V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ contains no other sections or multi-sections.

If each exceptional curve of $f|_{\tilde{E}}$ were to intersect two components of $f^{-1}(P)$, L_1 and L_2, which lie on the same surface V_k, then $(L_1^2)_{\tilde{E}} = L_1 \cdot V_k = L_2 \cdot V_k = (L_2^2)_{\tilde{E}}$ since L_1 and L_2 are both fibers of the same ruling on V_j. By 0.12A)(ii), this would not be possible; thus, for some exceptional curve L of $f|_{\tilde{E}}$, $L \subset V_j$, where $V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ contains no multi-sections and at most two sections of V_j; moreover, if $V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ contains two sections, these sections are double curves of V_j with two other surfaces, as claimed.

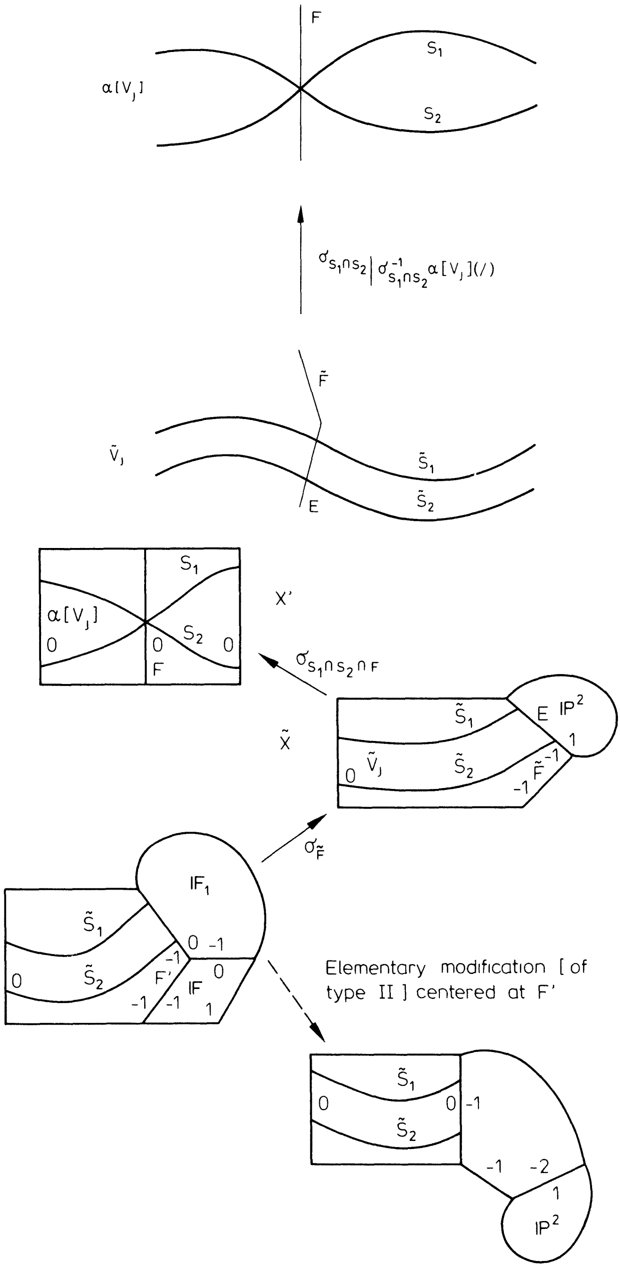
2.3. Returning to the proof of 2.1, let $L \subset V_j$ be an exceptional curve, where $V_j \cap \left(\bigcup_{i \neq j} V_i\right)$ contains no multi-sections and at most two sections of V_j, as in 2.2. As was earlier noted, since L is an exceptional curve of $f|_{\tilde{E}}$, V_j is generically contractible under the ruling given by $f|_{V_j}$. Let α be the map of the minimizing lemma (1.1) such that $\alpha[V_j]$ is minimal ruled [1.1.1)]. Since α is an isomorphism away from the reducible fibers of V_j [1.1.2)] under that same ruling, $\alpha[V_j]$ is generically contractible and so smoothly contractible.

2.4. In order to smoothly blow down $\alpha[V_j]$ while preserving normal crossings and ensuring that all self-intersection curves map to points in Y, we must simply show that if $\alpha[V_j] \cap \left(\alpha\left\{\bigcup_{i \neq j} V_i\right\}\right)$ contains two sections, S_1 and S_2, then, first, S_1 and S_2 are not both contained in $\alpha[V_k]$ for some $k \neq j$, and, second, S_1 and S_2 are disjoint. This first condition is ensured by 2.2.2).

If $S_1 \cap S_2 \neq \emptyset$, then since S_1 and S_2 meet normally in $\alpha[V_j]$, we may separate them by blowing up the points $S_1 \cap S_2$ in $X' = \alpha\{X\}$.

Now $N_{F/X'} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, where F is the fiber on $\alpha[V_j]$ passing through the given point of $S_1 \cap S_2$, so $N_{\tilde{F}/\tilde{X}} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ since a point was blown up in X'. Blowing up $\tilde{F}$ in $\tilde{X}$, we find that $N_{F'/\sigma_{\tilde{F}}^{-1}(\tilde{X})} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, where $F' = \sigma_{\tilde{F}}^{-1}[\tilde{V}_j] \cap \sigma_{\tilde{F}}^{-1}\{\tilde{F}\}$ and so we may perform a modification of type II centered at F'.

The effect on $\alpha[V_j]$ is to perform an elementary modification of ruled surfaces centered at $F \cap S_1 \cap S_2$, i.e. in $\alpha[V_j]$, to blow up $F \cap S_1 \cap S_2$ to obtain a reducible fiber $E + \tilde{F}$, where $\tilde{S}_1 \cap \tilde{S}_2 \cap \tilde{F} = \emptyset$ and then blow down $\tilde{F}$; this separates S_1 and S_2, leaving $\alpha[V_j]$ minimal.



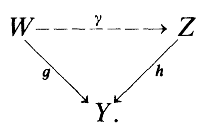
2.5. To conclude the proof of 2.1, let $\beta_1 = \delta \circ \sigma \circ \zeta \circ \alpha$, where α is the map of Lemma 1.1, making V_j minimal (while introducing new surfaces and self-intersection curves mapping to points under $f \circ \alpha^{-1}$), ζ is the composition of sequences of two blow-ups followed by a modification of type II as in 2.4 to separate sections on $\alpha[V_j]$, σ is the smooth blow-down of $\zeta \circ \alpha[V_j]$ and δ is the smooth blow-up of all self-intersection curves introduced by $\sigma \circ \zeta \circ \alpha$. Let $g_1 = f \circ \beta_1^{-1}$, so g_1 is a birational morphism: α and ζ only introduce new surfaces which are mapped by α^{-1} and ζ^{-1} to fibers on V_j, whence to points on Y by f; fibers of σ are mapped by $(\zeta \circ \alpha)^{-1}$ to fibers on V_j which are contained fibers of f; and finally, δ^{-1} is a morphism. Since $E_{g_1} = \beta_1\{E_f\}$, E_{g_1} has normal crossings. By definition of δ, E_{g_1} has no self-intersection curves, so E_{g_1} consists of smooth surfaces meeting normally.

Over each one-dimensional point of S_f, α, ζ, and δ are isomorphisms, since α and ζ are isomorphisms off fibers of V_j (and so fibers of f) and all self-intersection curves introduced by δ are mapped to points [1.1.2) and 2.4]. On the other hand, σ reduces the cardinality of I_C by one. Thus by smooth blow-ups and blow-downs, only new surfaces mapping to points have been created, while on surface, V_j, mapping to C, has been blown down. Iterating this process until the set of surfaces mapping to curves is exhausted, 2.1 is proved.

3. Reduction to Rational Surfaces

In this section, we consider the exceptional divisor of a point modification (see 0.2) of smooth threefolds. By 0.7(i), the exceptional divisor is a union of ruled surfaces and $\mathbb{P}^2$'s. In 3.1 below, we will modify the domain by a birational map (which is a composition of smooth blow-ups and blow-downs) so that the composite birational map is a point modification; moreover, the exceptional divisor of the composite map contains only *rational* ruled surfaces and $\mathbb{P}^2$'s. As in section two, the proof consists essentially in showing that among the surfaces in the exceptional divisor which are irrational at least one such surface is generically contractible and then applying the minimizing lemma (1.1) in order to smoothly contract it.

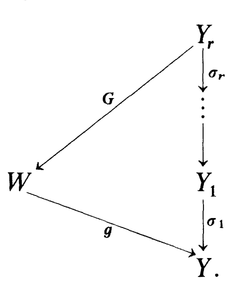
3.1 Theorem. *Let $g: W \to Y$ be a point modification of smooth threefolds such that E_g consists of smooth surfaces meeting normally. Then there is a birational map $\gamma: W \dashrightarrow Z$ such that*



1) *$h = g \circ \gamma^{-1}$ is a point modification of smooth threefolds such that $S_h = S_g$ and E_h consists of smooth rational surfaces meeting normally along rational double curves and*

2) *γ is a composition of smooth blow-ups and blow-downs.*

Proof. Let $\sigma = \sigma_1 \circ \ldots \circ \sigma_r$ be a regular resolution of g^{-1} (see 0.4) and $G = g^{-1} \circ \sigma$.



G is a morphism [0.4(iii)] and $S_\sigma = S_g$ [since $S_\sigma \subset S_g$ by 0.4(ii) and g is a morphism], so $G(E_\sigma) = E_g$; moreover, since $\sigma = g \circ G$, $E_\sigma = G^{-1}\{E_g\} = E_G \cup G^{-1}[E_g]$. By Frumkin's result (0.5), E_G contains only rational surface.

3.2. Assume E_g contains only rational surfaces. If so, then E_σ also contains only rational surfaces, in which case each σ_i^{-1} is the blow-up of a point or smooth rational curve and so E_σ contains only rational surfaces meeting along rational double curves. If C is a double curve in E_g, either $C \subset S_G$ or $C = G(C')$ for C' a double curve of E_σ. Since E_G contains only rational surfaces, S_G contains no irrational curves. Thus if E_g contains only rational surfaces, then all double curves of E_g are also rational.

3.3. Now assume that E_g contains at least one irrational surface and so E_σ also contains at least one irrational surface. Thus at least one σ_i^{-1} is the blow-up of an irrational curve. Let s be the largest integer such that σ_s^{-1} is the blow-up of an irrational curve. E_{σ_s} is a minimal irrational ruled surface smoothly contractible in Y_s; moreover, the double curves of $E_{\sigma_1 \circ \ldots \circ \sigma_s}$ on E_{σ_s} are fibers and one or two sections, since $E_{\sigma_1 \circ \ldots \circ \sigma_s}$ and $E_{\sigma_1 \circ \ldots \circ \sigma_{s-1}}$ have normal crossings [0.4(i)]. Let $\tilde{V} = (\sigma_{s+1} \circ \ldots \circ \sigma_r)^{-1}[E_{\sigma_s}]$ and $V = G(\tilde{V})$. Now $\sigma_{s+1}, \ldots, \sigma_r$ are blow-ups of points or smooth rational curves by definition of s, so since rational curves cannot be sections or multi-sections on an irrational ruled surface, $(\sigma_{s+1} \circ \ldots \circ \sigma_r)^{-1}$ is an isomorphism near a general fiber on E_{σ_s}. Thus $\tilde{V}$ is an irrational ruled surface in Y_r which is generically contractible; moreover, the double curves of E_σ on $\tilde{V}$ are components of fibers and one or two sections. Finally, E_G contains only rational surfaces, so G maps $\tilde{V}$ birationally onto V and G is also an isomorphism near a general fiber on $\tilde{V}$. Thus E_g contains V, an irrational ruled surface which is generically contractible in W; moreover, the double curves of E_g on V are components of fibers and one or two sections.

Let $\alpha: W \dashrightarrow W'$ be the map of Lemma 1.1. Thus $\alpha[V]$ is smoothly contractible in W' [1.1.1)], $\alpha\{E_g\}$ contains no new irrational surfaces [1.1.3)] and has normal crossings with only rational self-intersection curves.

To smoothly blow down $\alpha[V]$ while preserving normal crossings, as in 2.2.2) and 2.4 we must ensure that if the double curves of $\alpha\{E_g\}$ include two sections, S_1 and S_2, of $\alpha[V]$, then $S_1 \cup S_2 \nsubseteq \alpha[V']$ for some $V' \neq V$ and $S_1 \cap S_2 = \emptyset$. Since α, G, $\sigma_r, \ldots, \sigma_{s+2}$, and σ_{s+1} are all isomorphisms in the neighborhood of a general fiber of $\alpha[V]$ and $\sigma_1, \ldots, \sigma_s$ are all smooth blow-ups preserving normal crossings, if $\alpha[V]$ contains two sections which are double curves of $\alpha\{E_g\}$, then they are the image of two sections on E_{σ_s} which are double curves of E_{σ_s} and two distinct surfaces in $\sigma_s^{-1}[E_{\sigma_1 \circ \ldots \circ \sigma_{s-1}}]$. Thus S_1 and S_2 are not contained in a single surface other than $\alpha[V]$. To separate S_1 and S_2, we proceed as in 2.4 (since $\alpha[V]$ is smoothly contractible) to blow up points of $S_1 \cap S_2$, blow up the fibers of $\alpha[V]$ through $S_1 \cap S_2$ and perform modifications of type II, leaving $\alpha[V]$ smoothly contractible.

To conclude, let $\gamma_1 = \delta \circ \sigma \circ \zeta \circ \alpha$ as in 2.5, where α is the map of Lemma 1.1 making V smoothly contractible, ζ separates sections on $\alpha[V]$, σ smoothly contracts $\zeta \circ \alpha[V]$ and δ blows up all self-intersection curves in $\sigma \circ \zeta \circ \alpha\{E_g\}$. Since α, ζ, and δ consist solely of blow-ups of points and smooth rational curves, they preserve the number of irrational surfaces in E_g, while σ decreases that number by one. Since g is a point modification and γ_1 is an isomorphism off E_g, $h_1 = g \circ \gamma_1^{-1}$ is also a point modification over the same points of Y, so $S_{h_1} = S_g$.

3.4. Iterating the procedure of 3.3 until E_{h_n} contains no irrational ruled surfaces, we note (3.2) that E_{h_n} contains only rational double curves, proving Theorem 3.1, where $\gamma = \gamma_n \circ \ldots \circ \gamma_1$ and $h = h_n$.

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Relative Inversion in der Störungstheorie von Operatoren und Ψ-Algebren

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In der Theorie der Pseudodifferentialoperatoren hat sich durch die Ergebnisse von Beals [5], Connes [9, 28], Cordes [10, 11], Dunau [15] und Seeley [50] herausgestellt, daß wichtige Fréchet-Operatorenalgebren eine offene Gruppe invertierbarer Elemente haben. In dieser Arbeit wird zu einem Teil der Störungstheorie für Banachalgebren und topologische Algebren mit stetiger Inversion ein Zugang entwickelt, der sich auf die relative Inversion (in der Literatur auch Pseudoinversion oder verallgemeinerte Inversion genannt) stützt. 4.14 Satz besagt spezialisiert auf Hilberträume E, daß die Zusammenhangskomponenten der stetigen linearen Operatoren mit abgeschlossenem Bild bzgl. der Norm-Öffnungsmetrik [16, 31] analytische Quotienten der Produktgruppe $\mathrm{Gl}(E) \times \mathrm{Gl}(E)$ sind; dies ist für die Anwendung der Ergebnisse von Hayes [29] wichtig. Die Untersuchung von Fréchetalgebren erfolgt insbesondere, da $\mathscr{C}^\infty$-Eigenschaften (pseudolokal, microlokal, hypoelliptisch) bei Vervollständigung von Operatorenringen zu Banach- bzw. C^*-Algebren i.a. nicht erhalten bleiben.

Für Algebren von Pseudodifferentialoperatoren ergibt sich die Frage, in welchem Umfang eine Theorie der stetigen, differenzierbaren bzw. holomorphen Operatorfunktionen entwickelt werden kann, wie sie für Banachräume (vgl. z.B. [17, 37, 38, 58, 20, 23, 25]) in spektraltheoretischer und topologischer (Okasches Prinzip) Hinsicht bekannt ist (vgl. [12]). Auf die Arbeiten von Gohberg, Bart, Cuéllar, Dynin, Kaashoek, Kaballo, Lay, Leiterer, Mennicken u. Sagraloff sowie [58] sei besonders hingewiesen. Während Liftingmethoden und additive Kohomologie [19, 20, 23] hierbei keine großen Schwierigkeiten bereiten, treten dort Probleme auf, wo man „den" Satz über implizite Funktionen (der für Fréchetträume sehr problematisch ist [26, 45, 56]) bzw. ein Iterationsverfahren anwendet (Mannigfaltigkeitsnachweis, Liegruppenzerlegung [49, 1.5], Heftungs-Lemma von H. Cartan). Für einige dieser Fragen entwickeln wir „rationale" Methoden (durch explizite Formeln [16]), die dann manche, für Banachräume bekannte Resultate verschärfen (2.13, 3.9, 4.14, § 5, 6) und auch eine Erweiterung der holomorphen Faktorisierung (§6, Cartan-Lemma) auf spezielle Fréchetalgebren (5.9 bis 5.15) ermöglichen. Ferner wird eine gewisse Abgrenzung durch Gegenbeispiele (§ 6) gegeben. Für die meromorphe Inversion holomorpher Funktionen mit Werten in der Menge der Semi-Fredholmoperatoren von Algebren von Pseudodifferentialoperatoren sei auf Kaballo [30] hingewiesen.

In den Paragraphen 2, 3 und 4 werden idempotente Elemente, projizierte Unterräume und Operatoren mit projizierten Kernen und Bildern unter dem Aspekt der homogenen Räume behandelt (2.13, 3.7, 4.14); dabei verschärft die Spezialisierung auf Banachräume Resultate von Douady [13, 2.], Koschorke [33] und Raeburn [49][1]. Die Paragraphen 1 bis 4 beziehen sich ebenso auf den Ring $\mathscr{B}=\mathscr{C}^\infty(\Omega, \mathscr{L}(E))$, Ω kompakte differenzierbare Mannigfaltigkeit, $\mathscr{L}(E)$ die Algebra der stetigen Endomorphismen eines Hilbertraumes, wobei auch $E=\mathbb{C}^n$ von Interesse ist.

In [49] wird ausgehend von Taylor [53] das Okasche Prinzip für Banachalgebren untersucht und dabei gezeigt, daß die Zusammenhangskomponenten der Menge $\mathscr{P}$ der idempotenten Elemente analytische homogene Räume sind. Die Beweismethode beruht auf Banach-Liegruppentheorie und dem Satz über implizite Funktionen. Für topologische Algebren $\mathscr{B}$ mit stetiger Inversion (insbesondere die Gruppe $\mathscr{B}^{-1}$ offen) folgt aus § 2, daß die Zusammenhangskomponenten von $\mathscr{P}$ lokal-$\mathscr{B}$-rationale homogene Mannigfaltigkeiten sind (zur Definition vgl. 1.10). Unsere Methode scheint auch für die Algebren $\mathscr{C}^k(\Omega)\otimes\mathscr{L}(\mathbb{C}^n)$, $k=0, 1, \ldots, \infty$, Ω kompakt, neu zu sein. Aufgrund der verwendeten „rationalen" Formeln gilt das Verfahren für alle topologischen Algebren mit stetiger Inversion und deren Ideale im Sinne von 1.1.

In § 3 wird insbesondere jeder Fréchetalgebra mit offener Gruppe (bzw. jeder Banachalgebra) eine Graßmann-Mannigfaltigkeit zugeordnet. Daraus ergibt sich z.B., daß die Komponenten von $\mathscr{C}^\infty$-Untervektorraumbündel von $\Omega\times E$, E Hilbertraum, mit der von $\mathscr{B}=\mathscr{C}^\infty(\Omega, \mathscr{L}(E))$ induzierten Topologie lokal-$\mathscr{B}$-rationale (also insbesondere analytische) homogene Fréchet-Mannigfaltigkeiten sind (vgl. R. Palais [1, S. 302]).

Grundlegend für § 4 ist eine von Atkinson [2] entwickelte lokale Pseudoinverse (4.1 Lemma), die in [13, 33, 49, 53] nicht verwendet wird. Bei der relativen Inversion dienen die §§ 2 und 3 als Vorbereitung zu 4.14 Satz, wobei sich speziell die Komponenten von $\mathscr{C}^\infty(\Omega, \Phi_{n,m})$, Ω kompakt, $\Phi_{n,m}$ Menge der Fredholmoperatoren mit Kerndimension n und Bild-Kodimension m, als lokal-$\mathscr{B}$-rationale homogene Mannigfaltigkeiten erweisen. Douady [13, 2] hat mit anderen Methoden schon gezeigt, daß $\Phi_{n,m}$ eine „lokal rationale" Banachmannigfaltigkeit ist. Das Ergebnis 4.14 beruht auf den Hilfssätzen 3.1 und 4.2 sowie auf der „algebraischen" Zerlegungsmethode 4.15 bis 4.22. Diese Methode umgeht einerseits die Anwendung des Satzes über implizite Funktionen und zeigt andererseits die Gültigkeit des Verfahrens für alle Ideale im Sinne von 1.1 in topologischen Algebren mit stetiger Inversion. Die Einschränkung auf eine Algebra ist nicht wesentlich, die Ergebnisse von § 4 gelten auch für Operatoren zwischen verschiedenen Räumen [21] (bzw. Kategorien mit mehreren Objekten), so daß auch abgeschlossene Operatoren als stetige lineare Abbildungen zwischen Banachräumen erfaßt werden.

Motiviert durch Algebren von Pseudodifferentialoperatoren der Ordnung 0 wird in § 5 die einfache Definition 5.1 gegeben: Eine lokalkonvexe Fréchetunteralgebra Ψ einer Banachalgebra $\mathscr{B}$ mit Einselement heißt Ψ-Algebra, wenn für die Gruppen Ψ^{-1} und $\mathscr{B}^{-1}$ der invertierbaren Elemente die Aussage

$$\Psi\cap\mathscr{B}^{-1}=\Psi^{-1}$$

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1 Vgl. J. Raeburn, Pac. J. Math. **81**, 525–535 (1979); J. Raeburn, J. L. Taylor, J. Funct. Anal. **25**, 258–266 (1977)

gilt; ist außerdem $\mathscr{B}$ eine C^*-Algebra und Ψ symmetrisch, dann heißt Ψ eine Ψ^*-Algebra. Auf diese Algebren ist der holomorphe Funktionalkalkül anwendbar (vgl. Waelbroeck [55]). Für die in 5.9 bis 5.17 eingeführten speziellen Ψ-Algebren $\langle E;\mathscr{A},\alpha\rangle$ sind als Grundlage die Arbeiten von Cordes [11] und Connes [9] wichtig; beide Arbeiten gehen von der Idee der W^*- bzw. C^*-dynamischen Systeme aus, und zwar zur Untersuchung bzw. Charakterisierung von Algebren von Pseudodifferentialoperatoren. Die Klasse $\langle E;\mathscr{A},\alpha\rangle$ ergab sich aus dem Versuch, die holomorphe Faktorisierung (Heftungslemma von H. Cartan) für Fréchetalgebren zu beweisen (§6, 6.3, 6.8, 6.9). Ferner enthält §6 ein einfaches Gegenbeispiel zu einer möglicherweise offenen Frage (vgl. Omori [45, S. 140]). Die Fréchetalgebra aller Operatoren der Ordnung 0 hat keine offene Gruppe. Die dazu benützte Methode wurde auch von S. Lojasiewicz, Jr. und E. Zehnder (1979, vgl. §6) in ähnlichem Zusammenhang angewandt.

Durch eine Lokalisierung wird die Untersuchung der Klasse $\langle E;\mathscr{A},\alpha\rangle$ im Zusammenhang mit der Pseudo- und Microlokalität fortgesetzt werden [24]. Die auf Ergebnissen von H. Grauert und K.J. Ramspott beruhenden Resultate zum Okaschen Prinzip für homogene Räume und zur nicht abelschen Kohomologie in [7, 17, 29, 49, 53] sollen für die Ψ-Algebren $\langle E;\mathscr{A},\alpha\rangle$ und ihre Ideale untersucht werden. Die Resultate von [17] lassen sich wegen 6.8 und 6.9 leicht auf die Fréchetalgebren $\langle E;C^\infty,\alpha\rangle$ ausdehnen.

Eine Weiterentwicklung eines Teils der Ergebnisse von [21] bleibt einer folgenden Arbeit vorbehalten, insbesondere Dichte- und Surjektivitätssätze für $\mathscr{R}(\mathscr{B})$ (vgl. §4, §5) im Zusammenhang mit der Homotopie von Operatorfunktionen und Algebren mit Involution.

§1 Bezeichnungen und Vorbemerkungen

1.1. Mit $\mathscr{B}$ bezeichnen wir eine separierte topologische Algebra über $\mathbb{K}$, $\mathbb{K}=\mathbb{R}$ oder $\mathbb{C}$, mit Stetigkeit der Multiplikation in beiden Faktoren und Einselement e. Ferner sei $\mathscr{B}$ mit stetiger Inversion, insbesondere sei die Gruppe $\mathscr{B}^{-1}$ der invertierbaren Elemente offen und $\mathscr{B}^{-1}\ni b\mapsto b^{-1}\in\mathscr{B}$ sei stetig bzgl. der auf $\mathscr{B}$ gegebenen Topologie $\tau(\mathscr{B})$ (vgl. Naimark [41, S. 169] und Waelbroeck [55, S. 87]). Das zweiseitige Ideal $\mathscr{I}$ von $\mathscr{B}$ sei eine topologische Algebra mit einer Topologie $\tau(\mathscr{I})$, die feiner ist als die von $\tau(\mathscr{B})$ induzierte. Ferner seien die beiden kanonischen bilinearen Abbildungen

$$(\mathscr{B},\tau(\mathscr{B}))\times(\mathscr{I},\tau(\mathscr{I}))\to(\mathscr{I},\tau(\mathscr{I}))\,,$$

$(b,c)\to bc$ bzw. cb stetig. Wegen der Theorie der Operatorenideale wird auf Pietsch [48] verwiesen. Wir verzichten auf die Annahme der lokalen Konvexität, da wesentliche Ideale, die bei Sobolevschen Einbettungssätzen benötigt werden, nicht lokal konvex sind; auch das von A. Grothendieck eingeführte Ideal der Operatoren mit schnell fallenden Approximationszahlen ist nicht lokalkonvex [19], aber vollständig metrisch.

1.2. Ist für $\mathscr{I}$ und $\mathscr{B}$ wie in 1.1 das Ideal $\mathscr{I}$ in $\mathscr{B}$ echt, so bildet die lineare Hülle

$$\tilde{\mathscr{I}}:=\{\lambda e+c:\lambda\in\mathbb{K},c\in\mathscr{I}\}$$

versehen mit der Produktraumtopologie von $\mathbb{K} \times \mathcal{I}$ (bzgl. $\tau(\mathcal{I})$) eine topologische Algebra mit stetiger Inversion, und ein Element von $\tilde{\mathcal{I}}$ ist in $\mathcal{B}$ genau dann invertierbar, wenn es in $\tilde{\mathcal{I}}$ invertierbar ist. Die Spektren ($\mathbb{K}=\mathbb{C}$) von $x \in \tilde{\mathcal{I}}$ bzgl. $\mathcal{B}$ und $\tilde{\mathcal{I}}$ sind also identisch. Man beachte die Identitäten $a^{-1}-b^{-1}=a^{-1}(b-a)b^{-1}$ sowie

$$(e-x)^{-1}-(e-y)^{-1}=(e-x)^{-1}(x-y)(e-y)^{-1}, \tag{1.1}$$

$$(e-x)^{-1}=e+x+\ldots+x^{n-1}+x^{n-\nu}(e-x)^{-1}x^{\nu}. \tag{1.1'}$$

Aus (1.1′) ergibt sich auch, daß für die Identität der Spektren bzgl. $\mathcal{B}$ und $\tilde{\mathcal{I}}$ der Elemente $c \in \mathcal{I}$, die Algebra $\mathcal{I}$ nicht einmal als einseitiges Ideal vorausgesetzt werden muß, sondern nur die Annahme $c\mathcal{B}c \subset \mathcal{I}$ für $c \in \mathcal{I}$ zu erfüllen hat. Man vergleiche $\mathcal{C}^{\infty}$-Integraloperatoren auf $L^2(X,\mu)$, X kompakt.

1.3. Unter zusammenhängend verstehen wir bogenweise zusammenhängend; für offene Mengen in topologischen Mannigfaltigkeiten bzw. topologischen Vektorräumen sind „die“ Begriffe identisch. Für das zweiseitige Ideal $\mathcal{I}$ (wie in 1.1) sei

$$G(\mathcal{I})=\{b \in \mathcal{B}^{-1}: b=e+c,\, c \in \mathcal{I}\}$$

versehen mit der von $(\tilde{\mathcal{I}})^{-1}$ (bzw. $\tau(\mathcal{I})$) induzierten Topologie. Der Fall $\mathcal{I}=\mathcal{B}$ ist im folgenden ausdrücklich zugelassen; dies ist die zunächst interessante Situation.

1.4. Mit $G_e(\mathcal{I})$ wird die e enthaltende Zusammenhangskomponente von $G(\mathcal{I})$ bezeichnet; $G_e(\mathcal{I})$ ist Normalteiler in $G(\mathcal{I})$. Für einen Banachraum über $\mathbb{C}$ und $\mathcal{I}$ enthalten in dem Ideal $\mathcal{K}(E)$ der kompakten Operatoren in $\mathcal{B}=\mathcal{L}(E)$ (Banachalgebra aller beschränkten linearen Transformationen von E) hat man bekanntlich [17] aufgrund der Diskretheit des Spektrums $G_e(\mathcal{I})=G(\mathcal{I})$. Man setze etwa $\mathcal{I}=\mathcal{F}(E)$ das Ideal der Operatoren von endlichem Rang. Im Falle $\mathcal{I}=\mathcal{B}$ wird auch die Bezeichnung $\mathcal{B}_e^{-1}$ für $G_e(\mathcal{I})$ verwendet.

Zusammenhangsfragen für $\mathcal{B}^{-1}$ werden von Yuen [57] behandelt[2]. In bezug auf die Gruppen $\mathcal{B}^{-1}/\mathcal{B}_e^{-1}$ gibt es noch eine Reihe von Problemen (vgl. auch Douady [14]).

Um einige Ergebnisse geeignet formulieren zu können, führen wir für die topologische Algebra $\mathcal{B}$, wie in 1.1, den Begriff der lokal $\mathcal{B}$-rationalen Mannigfaltigkeit ein.

1.5. Ist $X \neq \emptyset$ eine Teilmenge des endlichen Produktes $\prod_{j=1}^{n} \mathcal{B}_j$, $\mathcal{B}_j=\mathcal{B}$, und Ω eine Klasse von Abbildungen $f: X \to \mathcal{B}$, so sei $\Omega' := \{f+g,\, f\cdot g : f, g \in \Omega\} \cup \{f^{-1}: f \in \Omega$ und $\forall x \in X \exists f(x)^{-1} \in \mathcal{B}\}$; dabei benützen wir zur Definition die punktweisen Operationen Addition, Multiplikation und Inversion (wenn möglich) im Ring $\mathcal{B}$.

1.6 Definition. Sei X wie in 1.5, pr_j, $j=1,\ldots,n$, die Projektion von X auf die j-te Komponente und $C_b: X \to \mathcal{B}$ die konstante Abbildung mit Wert $b \in \mathcal{B}$. Sei nun $\mathfrak{R}_0=\{pr_j: j=1,\ldots,n\} \cup \{C_b: b \in \mathcal{B}\}$, so setzen wir $\mathfrak{R}_{\nu+1}=\mathfrak{R}'_{\nu}$, $\nu=0,1,\ldots$, und

$$\mathcal{B}\mathrm{r}(X,\mathcal{B}) := \bigcup_{\nu=0}^{\infty} \mathfrak{R}_{\nu}, \tag{1.2}$$

2 Vgl. V. Paulsen, Coll. Math. **47**, 97–100 (1982)

die Familie der $\mathscr{B}$-rationalen Funktionen auf X mit Werten in $\mathscr{B}$. Für $Y \subset \prod_{k=1}^{m} \mathscr{B}_k$, $\mathscr{B}_k = \mathscr{B}$, setzen wir

$$\mathscr{B}\mathrm{r}(X, Y) = \{f : X \to Y : f_k \in \mathscr{B}\mathrm{r}(X, \mathscr{B}),\ k = 1, \ldots, m\}; \tag{1.3}$$

diese Abbildungen nennen wir kurz $\mathscr{B}$-rational.

Der Verfasser dankt Heinz König für ein Gespräch zu dieser Definition. Außerdem sei W. A. J. Luxemburg für seine hilfreiche Kritik gedankt.

1.7 Bemerkung. 1) Ist $f_j \in \mathscr{B}\mathrm{r}(X_j, Y_j)$, $j = 1, 2$, dann gehört die Produktabbildung

$$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$$

zu $\mathscr{B}\mathrm{r}(X_1 \times X_2, Y_1 \times Y_2)$.

2) (Komposition) Aus $f \in \mathscr{B}\mathrm{r}(X, Y)$ und $g \in \mathscr{B}\mathrm{r}(Y, Z)$ folgt $g \circ f \in \mathscr{B}\mathrm{r}(X, Z)$.

3) $\mathscr{B}\mathrm{r}(X, \mathscr{B}) \subset \mathscr{C}(X, \mathscr{B})$ (Raum der stetigen $\mathscr{B}$-wertigen Funktionen auf X).

4) (Fortsetzung) Zu jedem $f \in \mathscr{B}\mathrm{r}(X, \mathscr{B})$ gibt es eine offene Menge $W \supset X$, so daß sich f zu einem $\tilde{f} \in \mathscr{B}\mathrm{r}(W, \mathscr{B})$ fortsetzen läßt.

Sämtliche Eigenschaften sind mit (1.2) direkt nachzuprüfen.

1.8 Bemerkung. Sei $\mathscr{B}$ über $\mathbb{C}$, X offen in $\prod_{j=1}^{n} \mathscr{B}_j$, $\mathscr{B}_j = \mathscr{B}$, und $f \in \mathscr{B}\mathrm{r}(X, \mathscr{B})$; für jedes $b \in X$ gibt es eine Nullumgebung U von $\prod_{j=1}^{n} \mathscr{B}_j$, so daß für $a \in U$ die $\mathscr{B}$-wertige Funktion $h(z) := f(b + za)$ auf $\{z \in \mathbb{C} : |z| < 1\}$ komplex differenzierbar ist. Unter der Annahme der lokalen Konvexität von $\mathscr{B}$ hat man mit 1.7.3)

$$\mathscr{B}\mathrm{r}(X, \mathscr{B}) \subset \mathrm{Hol}(X, \mathscr{B})$$

im Sinne unendlich dimensionaler Holomorphie (vgl. (1.1) u. [43]).

1.9. Ist $X \subset \prod_{j=1}^{n} \mathscr{B}$ kompakt, so bildet $\mathscr{B}\mathrm{r}(X, \mathscr{B})$ versehen mit der Topologie der gleichmäßigen Konvergenz auf X wieder eine topologische Algebra mit stetiger Inversion.

1.10 Definition. Ein topologischer Raum M wird lokal $\mathscr{B}$-rationale (kurz: $\mathrm{l}\mathscr{B}\mathrm{r}$) Mannigfaltigkeit mit einem Atlas $\{(U_\alpha, \varphi_\alpha, V_\alpha),\ \alpha \in A\}$ von Karten genannt, wenn die folgenden Bedingungen erfüllt sind:

0) $\{U_\alpha : \alpha \in A\}$ ist eine offene Überdeckung von M; für jedes $\alpha \in A$ ist V_α eine offene Teilmenge eines topologischen Vektorraumes $(T_\alpha, \tau(T_\alpha))$ und $\varphi_\alpha : U_\alpha \to V_\alpha$ ist eine Homöomorphie.

1) T_α ist für jedes $\alpha \in A$ ein linearer Teilraum von $\Pi_\alpha := \prod_{j=1}^{n_\alpha} \mathscr{B}_j$, $\mathscr{B} = \mathscr{B}_j$, $n_\alpha < \infty$, so daß die von Π_α durch $\tau(\mathscr{B})$ auf T_α induzierte Topologie gröber ist als die Topologie $\tau(T_\alpha)$.

2) Alle Kartenwechsel, α, $\beta \in A$,

$$\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

sind $\mathscr{B}$-rationale Abbildungen.

Wenn $(T_\alpha, \tau(T_\alpha)) = (T, \tau(T))$ für alle $\alpha \in A$ erfüllt ist, so heißt M modelliert über $(T, \tau(T))$.

Bemerkung. 1) in 1.10 ist so gewählt, da die $(T_\alpha, \tau(T_\alpha))$ bei den im folgenden betrachteten Mannigfaltigkeiten abgeschlossene (projizierte) Unterräume eines zweiseitigen Ideals $\mathscr{I}$ (wie in 1.1) von $\mathscr{B}$ sind. Die Rechnungen erfolgen allerdings nicht nur in $\mathscr{I}$, sondern auch in $\mathscr{B}$.

1.11. Produkte von $\mathfrak{l}\mathscr{B}$r-Mannigfaltigkeiten sind wieder von dieser Art; ebenso offene Teilmengen. Wie üblich definiert man lokal $\mathscr{B}$-rationale Untermannigfaltigkeiten mit Hilfe linearer, topologisch direkter Zerlegungen von $(T_\alpha, \tau(T_\alpha))$.

Eine Abbildung $f: M \to M'$ für $\mathfrak{l}\mathscr{B}$r-Mannigfaltigkeiten M und M' heißt eine $\mathfrak{l}\mathscr{B}$r-Abbildung, wenn $\forall p \in M$ Karten (U, φ, V), $p \in U$, und (U', φ', V'), $f(p) \in U'$, existieren, so daß $\varphi' \circ f \circ \varphi^{-1}: V \to V'$ eine $\mathscr{B}$r-Abbildung und stetig ist. f heißt $\mathfrak{l}\mathscr{B}$r-Homöomorphismus, wenn die inverse Abbildung von f existiert und ebenfalls eine $\mathfrak{l}\mathscr{B}$r-Abbildung ist.

Ein Tripel (G, π, M) bestehend aus einer Gruppe G, einer Menge M und einer (surjektiven) Abbildung $\pi: G \times M \to M$ heißt homogener Raum, wenn mit $g \cdot p := \pi(g, p)$ die Bedingung $g_1(g_2 p) = (g_1 g_2)p$ erfüllt ist und $\pi^p: G \to M$, $p \in M$, definiert durch $\pi^p(g) = \pi(g, p)$, surjektiv ist.

1.12 Definition. Ein homogener Raum (G, π, M) heißt lokal $\mathscr{B}$-rationaler (kurz: $\mathfrak{l}\mathscr{B}$r-) homogener Raum und M $\mathfrak{l}\mathscr{B}$r-homogene Mannigfaltigkeit, wenn die folgenden Bedingungen erfüllt sind:

1) G und M sind $\mathfrak{l}\mathscr{B}$r-Mannigfaltigkeiten,
2) die Gruppenoperationen $G \times G \to G$ sind $\mathfrak{l}\mathscr{B}$r-Abbildungen.
3) $\pi: G \times M \to M$ ist eine $\mathfrak{l}\mathscr{B}$r-Abbildung.
4) $\forall p \in M \exists$ eine offene Umgebung U von p und eine $\mathfrak{l}\mathscr{B}$r-Untermannigfaltigkeit $\mathscr{T}_p$ von G, so daß $\pi^p|_{\mathscr{T}_p}: \mathscr{T}_p \to U$ ein $\mathfrak{l}\mathscr{B}$r-Homöomorphismus ist.

1.12′. Statt 4) in 1.12 wäre auch die folgende Bedingung möglich:

4*) Für jedes $p \in M$ gibt eine Umgebung $W(e) \subset G$ und eine Untermannigfaltigkeit $\mathscr{T}_p$ von G und eine $\mathscr{B}$r-Karte $\Psi: W(e) \to W' \subset (T, \tau(T))$ mit einer linear topologisch direkten Zerlegung $T = T_1 \oplus T_2$, so daß für $H_p = \{g \in G : gp = p\}$ und $\mathscr{T}_p$ folgendes erfüllt ist:

i) $\mathscr{T}_p \cdot H_p \supset W(e)$
ii) $\mathscr{T}_p \cap W(e) = \Psi^{-1}(T_1 \cap W')$ und $H_p \cap W(e) = \Psi^{-1}(T_2 \cap W')$.

Für die Theorie Banachanalytischer homogener Räume wird auf Bourbaki [6], Hayes [29] und Raeburn [49] hingewiesen. Man vergleiche auch Steenrod [52, I.7]. Auf rationale homogene Räume in der alg. Geometrie hat N. Kuhlmann (Essen) hingewiesen.

1.13 Bemerkung. Die Gruppen $G(\mathscr{I})$ aus 1.3 sind $\mathfrak{l}\mathscr{B}$r-Mannigfaltigkeiten, denn $X = \{c \in \mathscr{I} : \exists (e-c)^{-1}\}$ ist offen in $\mathscr{I}$, wie man aus $e - c = (e - c_0)\,(e - (e - c_0)^{-1}(c - c_0))$ und (1.1′) sieht. Für eine Karte kann man jede offene Teilmenge in X zulassen. Ferner ist dann die kanonische Abbildung $G(\mathscr{I}) \times G(\mathscr{I}) \to G(\mathscr{I})$ eine $\mathfrak{l}\mathscr{B}$r-Abbildung und ebenso $(e-c) \to (e-c)^{-1}$.

1.14 Bemerkung. In dieser Arbeit betrachten wir geeignete Einschränkungen der Abbildung

$$\bar{\pi}: (\mathscr{B}^{-1} \times \mathscr{B}^{-1}) \times \mathscr{B} \to \mathscr{B},$$

definiert durch

$$\bar{\pi}(g, \dot{g}, b) = gb\dot{g}^{-1}$$

und der Gruppe

$$G = \mathscr{B}^{-1} \times \mathscr{B}^{-1}$$

mit der Verknüpfung $(g_1, \dot{g}_1) \cdot (g_2, \dot{g}_2) = (g_1 g_2, \dot{g}_1 \dot{g}_2)$.

1.15 Beispiele von lokalkonvexen topologischen Algebren mit stetiger Inversion erhält man aus Banachalgebren $\mathscr{L}$ mit Einselement bekanntlich folgendermaßen:

1) $\mathscr{C}^\infty(\Omega, \mathscr{L})$ für eine kompakte differenzierbare Mannigfaltigkeit Ω.

1′) Sei $\mathscr{A}(\Omega)$ eine kommutative Fréchetalgebra mit stetiger Inversion auf dem kompakten Gelfandraum Ω, dann ist das projektive Tensorprodukt $\mathscr{A}(\Omega)\hat{\otimes}\mathscr{L}$ mit stetiger Inversion; häufig führen auch andere Tensorvervollständigungen dazu; $\mathscr{A}(\Omega)$ nuklear, z.B. geeignete Gevrey-Klasse.

2) $K \neq \emptyset$ eine kompakte Teilmenge des $\mathbb{C}^m$ und $\mathscr{H}(K)$ der nukleare induktive Limes der jeweils in einer Umgebung von K holomorphen Funktionen; man bilde $\mathscr{H}(K)\hat{\otimes}\mathscr{L}$ oder $\mathscr{H}(K)\hat{\otimes}\mathscr{L}(\mathbb{C}^n)$.

2′) $\mathfrak{R}(K)$ der Unterraum von $\mathscr{H}(K)$, der aus rationalen Funktionen mit Polen außerhalb K besteht; man betrachte $\mathfrak{R}(K)\otimes\mathscr{L}$ mit der von $\mathscr{H}(K)\hat{\otimes}\mathscr{L}$ induzierten Topologie.

3) Sei Ω ein Gebiet im $\mathbb{R}^n$ oder eine σ-kompakte endlichdimensionale differenzierbare Mannigfaltigkeit, dann ist

$$\mathscr{C}^\infty B(\Omega, \mathscr{L}) = \left\{ a \in \mathscr{C}^\infty(\Omega, \mathscr{L}) : \|a\|_\infty = \sup_{t\in\Omega} \|a(t)\|_{\mathscr{L}} < \infty \right\}$$

eine topologische Algebra mit stetiger Inversion bzgl. $\| \ \|_\infty$ und eine Fréchetalgebra dieser Art bzgl. $\mathscr{C}^\infty$-Topologie zusammen mit $\| \ \|_\infty$.

Es sei hier auch darauf hingewiesen, daß die Sobolev-Hilberträume $H^m(\Omega)$, Ω kompakt, $m > \frac{1}{2}\dim\Omega$, bezüglich punktweiser Multiplikation Banachalgebren sind.

1.16. Die Struktur der Ideale in 1.1 als topologische Vektorräume kann sehr kompliziert sein; vgl. [48, 22, 19].

§ 2 Idempotente Elemente und homogene Räume

Sei $\mathscr{P} = \{p \in \mathscr{B} : p^2 = p\}$ die Menge der idempotenten Elemente (Projektoren) von $\mathscr{B}$.

In [49] hat Raeburn für eine Banachalgebra $\mathscr{B}$ über $\mathbb{C}$ gezeigt, daß für jede Zusammenhangskomponente M von $\mathscr{P}$

$$M \cong G/H_p, \qquad p \in M,$$

im Banachanalytischen Sinn mit $G = \mathscr{B}_e^{-1}$ und

$$H_p = \{g \in G : gpg^{-1} = p\}$$

(Stabilisator von $p \in M$, und gpg^{-1} das Produkt in der Banachalgebra $\mathscr{B}$) erfüllt ist, wobei insbesondere der Satz über implizite Funktionen für Banachräume herangezogen wird. Wegen der Banachanalytischen Lie-Theorie vgl. man Bourbaki [6, Chap. III].

2.1. Sei $p_0 \in \mathscr{P}$ und $\mathscr{I}$ ein zweiseitiges Ideal von $\mathscr{B}$, wie in 1.1 definiert, wobei $\mathscr{I} = \mathscr{B}$ zugelassen ist. Die Menge

$$\mathscr{P}(\mathscr{I}, p_0) = \{p_0 + x \in \mathscr{P} : x \in \mathscr{I}\}$$

wird als (abgeschlossene) Untermenge von $\{p_0 + \mathscr{I}\}$ mit der von $\tau(\mathscr{I})$ induzierten Topologie versehen.

2.2 Lemma. *Sei* $M := M(\mathscr{I}, p_0)$ *die* p_0 *enthaltende Zusammenhangskomponente von* $\mathscr{P}(\mathscr{I}, p_0)$ *und* $G = G_e(\mathscr{I})$ *die Hauptkomponente von* $G(\mathscr{I})$ *(vgl.* 1.4) *und*

$$\pi : G \times M \to M \tag{2.2}$$

definiert durch $\pi(g, p) = gpg^{-1}$, *dann ist für jedes* $p \in M$ *die Abbildung*

$$\pi^p : G \to M , \quad \pi^p(g) := gpg^{-1} \ (\textit{Produkt in } \mathscr{B}) , \tag{2.3}$$

stetig, surjektiv und offen.

Beweis. Bekanntlich gilt für $p, q \in \mathscr{P}$ die Gleichung (vgl. [54])

$$(pq + (e-p)(e-q))q = p(pq + (e-p)(e-q));$$

existiert nun $g := (pq + (e-p)(e-q))^{-1}$, so folgt $q = gpg^{-1}$.

Wegen

$$p(p+x) + (e-p)(e-p-x) = e - (e-2p)x$$

gibt es eine Umgebung $W(p)$ in $\mathscr{B}$, so daß die Abbildung

$$g_p : W(p) \to \mathscr{B}_e^{-1} , \quad g_p(p+x) := (e-(e-2p)x)^{-1}$$

definiert ist, und zwar mit der Eigenschaft $g(p+x) \in G_e(\mathscr{I})$ für $x \in \mathscr{I}$ (vgl. 1.1')

$$f_p : W(p) \to \mathscr{P} , \quad f_p(x+p) := g(p+x)pg(p+x)^{-1} \tag{2.4}$$

ist eine Retraktion von $W(p)$ nach $\mathscr{P}$ wegen $f_p(x+p) = x+p$ für $x+p \in \mathscr{P}$. Demnach ist für $U(p) := W(p) \cap \mathscr{P}$ die Abbildung f_p eingeschränkt auf $U(p)$, die wir mit $s_p : U(p) \to G_e(\mathscr{I})$ bezeichnen, ein lokaler Schnitt der Abbildung $\pi^p : G \to M$:

$$\pi^p \circ s_p = \mathrm{Id}_{U(p)} , \quad U(p) \subset M . \tag{2.5}$$

Aufgrund der Existenz der lokalen Schnitte s_p von π^p für jedes $p \in M$ führt eine wohlbekannte Schlußweise zur Surjektivität von π^p: Sei $w(t)$, $0 \leqq t \leqq 1$, ein stetiger Weg in M mit $w(0) = p$ und $w(1) = p'$, dann führt eine genügend feine Unterteilung $0 = t_0 < t_1 < t_r = 1$ mit den Schnitten $s_{w(t_j)} : U(w(t_j)) \to G_e(\mathscr{I})$, $j = 0, \ldots, r-1$, zu der Gleichung

$$w(1) = gpg^{-1} , \quad g \in G_e(\mathscr{I})$$

mit

$$g = [s_{w(t_{r-1})}(w(t_r))] \cdot \ldots \cdot [s_{w(t_0)}(w(t_1))] . \tag{2.6}$$

Die Existenz der lokalen Schnitte s_p von π^p impliziert bekanntlich auch, daß die Abbildung π^p offen ist: Sei 0 eine offene Menge in G und $q \in \pi^p(0)$. Für die Rechtstranslation $R_{g'}: G \to G$ und $g' \in 0$, $\pi^p(g') = q$, gilt $\pi^p \circ R_{g'} \circ s_q = \mathrm{Id}_{U(q)}$ auf einer Umgebung $U(q) \subset M$, die aber, wenn sie genügend klein gewählt wird, in $\pi^p(0)$ liegen muß (man beachte $g'p = q$ sowie $\pi^q = \pi^p \circ R_{g'}$).

Bemerkung. Ist $\mathscr{B}$ kommutativ, so folgt aus 2.2 die Aussage $M = \{p_0\}$. Der Beweis von 2.2 ist für eine Banachalgebra $\mathscr{B}$ und $\mathscr{B} = \mathscr{I}$ bekannt [54].

Um zu zeigen, daß $M := M(\mathscr{I}, p_0)$ eine l$\mathscr{B}$r-Mannigfaltigkeit ist, wird $G := G_e(\mathscr{I})$ in einer Umgebung des Einselementes mit Hilfe des Stabilisators $(p \in M)$

$$H_p(\mathscr{I}) = \{g \in G : \pi^p(g) = p\} \tag{2.7}$$

für jedes $p \in M$ zerlegt (lokal)

$$G \sim \mathscr{T}_p \times H_p(\mathscr{I}), \tag{2.7'}$$

so daß die Untermannigfaltigkeit $\mathscr{T}_p$ eng mit dem „Tangentialraum“ $T_p(M)$ zusammenhängt.

2.3 Beispiel. Wir berechnen das „Fréchetdifferential“ der Abbildung (2.4): Sei $\delta := (e - 2p)x$,

$$\begin{aligned} f_p(p+x) &= g(p+x)\, p[g(p+x)]^{-1} \\ &= (e-(e-2p)x)^{-1} p(e-(e-2p)x) \\ &= (e-\delta)^{-1} p(e-\delta) \\ &= [e+\delta+\delta^2(e-\delta)^{-1}]p(e-\delta) \\ &= p+\delta p - p\delta - \delta p \delta + \delta^2(e-\delta)^{-1} p(e-\delta) \\ &= p + \delta p - p\delta + R(\delta) \\ &= p + (e-2p)xp - p(e-2p)x + R(\delta) \\ &= p + (e-p)xp + px(e-p) + R(\delta)\,. \end{aligned}$$

2.4 Definition. Für $p \in M = M(\mathscr{I}, p_0)$ sei

$$T_p(M) := \{(e-p)xp + px(e-p) : x \in \mathscr{I}\}\,. \tag{2.8}$$

Durch (2.8) ist offensichtlich ein projizierter Untervektorraum von $\mathscr{I}$ gegeben.

2.5. *Mit $p = p^2$, p fest, $p_1 = p$, $p_2 = e - p$, wählen wir die folgende Darstellung (p-Koordinaten) für $b \in \mathscr{B}$*

$$b = \sum_{j,k=1}^{2} p_j b p_k = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad b_{jk} = p_j b p_k\,.$$

Der Raum $T_p(M)$ hat dann die Darstellung

$$T_p(M) = \left\{ \begin{pmatrix} 0 & c_{12} \\ c_{21} & 0 \end{pmatrix} : c_{jk} \in \mathscr{I} \right\}. \tag{2.9}$$

Sei in p-Koordinaten (p fest)

$$G_p^{(1)}:=\left\{\begin{pmatrix}1 & 0\\ c_{21} & 1\end{pmatrix}: c_{21}=p_2 c p_1, c\in\mathscr{I}\right\}, \tag{2.10}$$

$$G_p^{(2)}:=\left\{\begin{pmatrix}1 & c_{12}\\ 0 & 1\end{pmatrix}: c_{12}=p_1 c p_2, c\in\mathscr{I}\right\}, \tag{2.10'}$$

wobei mit „1" wie üblich p_1 oder p_2 gemeint ist. $G_p^{(j)}$ ist offensichtlich eine Untergruppe von $G=G_e(\mathscr{I})$. Ferner sei

$$\mathscr{T}_p:=\{g_1\cdot g_2: g_j\in G_p^{(j)}, j=1,2\}. \tag{2.11}$$

Nun definieren wir die Abbildung (in p-Koordinaten)

$$\gamma_p: T_p(M)\to\mathscr{T}_p \tag{2.12}$$

durch

$$\gamma_p\left(\begin{pmatrix}0 & c_{12}\\ c_{21} & 0\end{pmatrix}\right):=\begin{pmatrix}1 & 0\\ c_{21} & 1\end{pmatrix}\begin{pmatrix}1 & c_{12}\\ 0 & 1\end{pmatrix},$$

ferner sei

$$\varkappa_p:\mathscr{T}_p\times H_p(\mathscr{I})\to G \tag{2.13}$$

definiert durch $\varkappa_p(c,h)=ch$ (Produkt in $\mathscr{B}$); $\varkappa'_p$ sei die Einschränkung von $\varkappa_p$ auf $\mathscr{T}_p$.

2.6 Lemma (vgl. [49, 4.9]). *In $G(\mathscr{I})$ ($\mathscr{I}=\mathscr{B}$ zugelassen) besteht der Stabilisator $H:=\{g\in G(\mathscr{I}): gpg^{-1}=p\}$ von $p\in\mathscr{P}$ in p-Koordinaten genau aus der Menge*

$$\left\{\begin{pmatrix}h_{11} & 0\\ 0 & h_{22}\end{pmatrix}\in G(\mathscr{I})\right\}. \tag{2.14}$$

Beweis. Jedes Element der Menge (2.14) liegt offensichtlich in H. Andererseits folgt aus $hph^{-1}=p$ die Aussage $hp=ph$ und damit $h_{12}=0=h_{21}$.

2.7 Lemma. *Sei $p\in M:=M(\mathscr{I},p_0)$, dann ist die Abbildung*

$$\pi^p\circ\varkappa'_p\circ\gamma_p: T_p(M)\to M, \tag{2.15}$$

injektiv.

Beweis. Mit c, $d\in T_p(M)$ und

$$(\pi^p\circ\varkappa_p\circ\gamma_p)(c)=(\pi^p\circ\varkappa_p\circ\gamma_p)(d)$$

ergibt sich

$$\gamma_p(c)p\gamma_p(c)^{-1}=\gamma_p(d)p\gamma_p(d)^{-1},$$
$$\gamma_p(d)^{-1}\gamma_p(c)p(\gamma_p(d)^{-1}\gamma_p(c))^{-1}=p.$$

Daraus folgt $\gamma_p(d)^{-1}\gamma_p(c)\in H_p(\mathscr{I})$. Wir zeigen nun $d=c$:

$$\gamma_p(d)^{-1}\gamma_p(c)=\begin{pmatrix}1 & -d_{12}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -d_{21} & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ c_{21} & 1\end{pmatrix}\begin{pmatrix}1 & c_{12}\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}h_{11}=1-d_{12}(c_{21}-d_{21}), & h_{12}=(1-d_{12}(c_{21}-d_{21}))c_{12}-d_{12}\\ h_{21}=c_{21}-d_{21}, & h_{22}=1+(c_{21}-d_{21})c_{12}\end{pmatrix}.$$

Aus 2.6 Lemma folgt nun $c_{21}=d_{21}$ und aus $h_{12}=0$, dann $c_{12}=d_{21}$, so daß sich $c=d$ ergibt.

Das folgende Lemma ersetzt die Anwendung des Satzes über implizite Funktionen zum Nachweis, daß $\mathscr{P}$ für Banachalgebren eine analytische Mannigfaltigkeit ist. Aus 2.5 folgt außerdem: $\mathscr{P}$ ist eine lokal $\mathscr{B}$-rationale Mannigfaltigkeit.

2.8 Lemma ($\mathscr{B}$-rationale Zerlegung). *Für jedes $p\in M(\mathscr{I},p_0)$ gibt es eine Umgebung U des Einselementes e von $G:=G_e(\mathscr{I})$ und eine stetige Abbildung*

$$\zeta^p: U\to G_p^{(1)}\times G_p^{(2)}\times H_p(\mathscr{I})\,(\mathscr{T}_p=G_p^{(1)}\cdot G_p^{(2)}) \tag{2.16}$$

mit $\zeta^p(e)=(e,e)$; ζ^p ist außerdem eine $\mathscr{B}$-rationale Abbildung.

Beweis. Eindeutigkeit: Sei $g=r\cdot h=r'\cdot h'$, r, $r'\in\mathscr{T}_p$, h, $h'\in H_p(\mathscr{I})$, so ergibt sich

$$\pi^p(g)=\pi^p(r)=\pi^p(r'), r, r'\in\mathscr{T}_p,$$

damit folgt aus 2.7 die Aussage $r=r'$ und daher $h=h'$.

Die Existenz von $\zeta^p: U\to G_p^{(1)}\times G_p^{(2)}\times H_p(\mathscr{I})$ folgt aus der Matrixidentität (in p-Koordinaten)

$$g=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix} 1 & 0\\ g_{21}g_{11}^{-1} & 1\end{pmatrix}\begin{pmatrix} 1 & g_{12}a^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix} g_{11} & 0\\ 0 & a\end{pmatrix} \tag{2.17}$$

mit $a=g_{22}-g_{21}g_{11}^{-1}g_{12}$. Dabei bedeutet g_{11}^{-1} die (1,1) Komponente von $(g_{11}+g_{22})^{-1}=e+c$, $c\in\mathscr{I}$; man beachte (1.1'); a^{-1} ist die (2,2)-Komponente von $(p+a)^{-1}$; die Elemente g_{12} und g_{21} sind klein wegen $g\in U$ und liegen in $\mathscr{I}$.

2.9. Die vorangehenden Überlegungen, insbesondere 2.5 bis 2.8 werden in einem Diagramm zusammengefaßt und erweitert, wobei $p\in M:=M(\mathscr{I},p_0)$ fest ist; $G=G_e(\mathscr{I})$.

$$\begin{array}{ccccc}
 & \zeta^p=(\tau^p,h^p) & & & \\
G & \dashrightarrow & \mathscr{T}_p & \times & H_p(\mathscr{I}) \\
s_p\uparrow\downarrow\pi^p & & \gamma^{-1}\downarrow\uparrow\gamma_p & & \Psi_p\downarrow\uparrow\Psi_p^{-1} \\
M & \underset{\varphi_p}{\dashrightarrow} & T_p(M) & \times & T_p^0 \quad (\cong\mathscr{I})
\end{array}
\qquad \left(\begin{array}{l} \text{- - - lokal}\\ \text{—— global}\end{array}\right) \tag{2.18}$$

$$\zeta^p(g)=(c,h),\qquad g=ch, \tau^p(g)=c, h^p(g)=h,\qquad g\in U.$$

Für eine Umgebung $W(e)\subset H_p(\mathscr{I})$ sei $\Psi_p: W(e)\to T_p^0$ für

$$T_p^0=\{(c_{11},c_{22}): c_{11}=p_1c_{11}p_1\in\mathscr{I}, c_{22}=p_2c_{22}p_2\in\mathscr{I}\}$$

definiert durch

$$\Psi_p\left(\begin{pmatrix} h_{11} & 0\\ 0 & h_{22}\end{pmatrix}\right):=(h_{11}-p_1, h_{22}-p_2).$$

2.10. Für eine geeignete Umgebung $W(e)\subset G$ sieht man aus dem Diagramm 2.6, daß

$$(\gamma^{-1}x\Psi_p)\circ\zeta^p: W(e)\to T_p(M)xT_p^0$$

eine Abbildung mit den Eigenschaften in 1.12′ vermittelt; $\mathcal{T}_p$ und $H_p(\mathcal{J})$ sind l$\mathcal{B}$r-Untermannigfaltigkeiten von G.

2.11 Definition einer Karte für $M = M(\mathcal{J}, p_0)$. Aufgrund der Stetigkeit der Abbildungen in (2.18) gibt es eine Umgebung $U(p) \subset M$, so daß

$$\varphi_p := \gamma_p^{-1} \circ \tau^p \circ s_p : U(p) \to T_p(M) \tag{2.19}$$

definiert ist; außerdem ist φ_p eine $\mathcal{B}$-rationale Abbildung.

2.12 Lemma. *Die Abbildung φ_p ist ein Homöomorphismus auf eine Nullumgebung V von $T_p(M)$.*

Beweis. Zunächst gilt

$$(\pi^p \circ \varkappa'_p \circ \gamma_p) \circ (\gamma_p^{-1} \circ \tau^p \circ s_p) = \mathrm{Id}_{U(p)},$$

wie aus der Konstruktion (2.18) folgt. φ_p ist also injektiv. Es bleibt zu zeigen, daß $V := \varphi_p(U(p))$ in $T_p(M)$ offen ist. Da die Abbildung $\Psi := \pi^p \circ \varkappa'_p \circ \gamma_p$ auf ganz $T_p(M)$ stetig und nach 2.7 Lemma injektiv ist, überlegt man sich leicht, daß φ_p eine offene Abbildung ist. $\varphi_p : U(p) \to V$ ist also injektiv und surjektiv auf die Nullumgebung V von $T_p(M)$.

2.13 Satz. *Das Tripel $(G_e(\mathcal{J}), \pi, M(\mathcal{J}, p_0))$ definiert eine lokal $\mathcal{B}$-rationale homogene Mannigfaltigkeit; dabei ist*

$$\pi : G_e(\mathcal{J}) \times M(\mathcal{J}, p_0) \to M(\mathcal{J}, p_0)$$

definiert durch $\pi(g, p) = gpg^{-1}$ (Produkt in $\mathcal{B}$).

Beweis. Wir zeigen zunächst, daß $M := M(\mathcal{J}, p_0)$ eine l$\mathcal{B}$r-Mannigfaltigkeit ist. Seien $\varphi : U \to V$ und $\varphi' : U' \to V'$, $p \in U$, $p' \in U'$ zwei Karten, dann ist

$$\varphi' \cdot \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U')$$

eine stetige $\mathcal{B}$r-Abbildung, womit die Bedingungen von 1.10 wegen 2.9 nachgewiesen sind.

Entsprechend der Definition 1.12 wird nun gezeigt: M ist l$\mathcal{B}$r-homogene Mannigfaltigkeit. Die Bedingungen 1) und 2) aus 1.12 sind mit dem Vorangehenden und 1.13 erfüllt. Zu 3) in 1.12: Aus der Definition der Karten für G und M folgt dies unmittelbar. Die Bedingung 4) in 1.12 ist wegen 2.7 erfüllt.

2.14 Bemerkung. 1) Nach Steenrod [52, I.7], folgt außerdem, daß (G, π, M) ein Faserbündel definiert mit der Faser $F = H_{p_0}(\mathcal{J})$, $p_0 \in M$ fest. 2) Wegen $H_{gp} = gH_pg^{-1}$ und der Eigenschaft

$$T_{gp}(M) = gT_p(M)g^{-1} \quad \text{bzw.} \quad \mathcal{T}_{gp} = g\mathcal{T}_pg^{-1} \tag{2.20}$$

(wobei $gp := \pi(g, p) = g \cdot p \cdot g^{-1}$) sieht man, daß M eine l$\mathcal{B}$r-Mannigfaltigkeit modelliert über dem Vektorraum $T_{p_0}(M)$ ist.

2.15 Bemerkung. 1) Sei $\mathcal{P}_\nu$ die Menge der stetigen ν-dimensionalen $(\nu < \infty)$ Projektoren des lokalkonvexen Vektorraumes E, dann gilt für das Ideal $\mathcal{J} = \mathcal{F}(E)$ der Operatoren von endlichem Rang und jedes $p \in \mathcal{P}_\nu$ die Aussage $\mathcal{P}_\nu = \{gpg^{-1} : g \in G(\mathcal{J})\}$. Entsprechendes hat man für die Menge der ν-codimensionalen Projektoren. 2) Für Banachräume über $\mathbb{C}$ ist die Menge $\mathcal{P}_\nu$ (vgl. 1.4) zusammenhängend. 3) Für die Räume c_0 und ℓ^p, $1 < p < \infty$, ist die Menge $\mathcal{P}_{\infty,\infty}$ der

Projektoren mit ∞-dimensionalem und ∞-codimensionalem Bild zusammenziehbar. Dies folgt aus [44] in Verbindung mit einem Satz von Pelczynski, daß in c_0 und ℓ^p alle projizierten ∞-dimensionalen Unterräume isomorph sind, wenn man die Homotopiesequenz wie in [49, S. 388], auf die exakte Sequenz $0 \to H_p \to \mathrm{Gl}(E) \to \mathscr{P}_{\infty,\infty} \to 0$ anwendet. 4) Im Fall $E = c_0 \oplus \ell^p$ ist $\mathscr{P}_{\infty,\infty}$ nicht zusammenhängend und hat nicht die Form $\{gpg^{-1} : g \in \mathrm{Gl}(E)\}$. 5) Für $\mathscr{B} = \mathscr{L}(E)$, E Hilbertraum, $\mathscr{J}$ das Ideal der Hilbert-Schmidt-Operatoren, ist $M(\mathscr{J}, p_0)$ eine Hilbert-Mannigfaltigkeit.

§ 3 Graßmann-Mannigfaltigkeiten für topologische Algebren mit stetiger Inversion

Für die topologische Algebra $\mathscr{B}$ über $\mathbb{R}$ oder $\mathbb{C}$ mit stetiger Inversion (insbesondere sei $\mathscr{B}^{-1}$ offen) und das zweiseitige Ideal $\mathscr{J}$ seien die Voraussetzungen wie in 1.1, 1.2, 1.3 gegeben. Auf der Menge $\mathscr{P} = \{p \in \mathscr{B} : p^2 = p\}$ definieren wir bezüglich $\mathscr{J}$ die folgende Äquivalenzrelation, wobei auch $\mathscr{J} = \mathscr{B}$ zugelassen ist.

$$p \sim p' \Leftrightarrow pp' = p' \quad \text{sowie} \quad p'p = p \quad \text{und} \quad p - p' \in \mathscr{J}\,. \tag{3.1}$$

Mit X_p wird die Äquivalenzklasse von $p \in \mathscr{P}$ und mit $\Gamma(\mathscr{B}, \mathscr{J})$ die Menge der Äquivalenzklassen von $\mathscr{P}$ bezeichnet, $\Gamma(\mathscr{B}) := \Gamma(\mathscr{B}, \mathscr{B})$. Man prüft leicht nach, daß (3.1) eine Äquivalenzklasse definiert. Eine weitere Äquivalenzrelation erhält man auf $\mathscr{P}$ durch

$$p \dot{\sim} p' \Leftrightarrow pp' = p, p'p = p' \quad \text{und} \quad p - p' \in \mathscr{J}\,. \tag{3.2}$$

Es gilt

$$p \sim p' \Leftrightarrow (e-p) \dot{\sim} (e-p'),$$

wie man direkt nachrechnet.

Die Äquivalenzrelation (3.1) definiert eine Abbildung

$$\gamma_{\mathscr{J}} : \mathscr{P} \to \Gamma(\mathscr{B}, \mathscr{J}) \tag{3.2'}$$

mit der Eigenschaft

$$\gamma_{\mathscr{J}}^{-1}(X_p) = \{p + px(e-p) : x \in \mathscr{J}\}, \tag{3.3}$$

wie man leicht ausrechnet.

Ferner gilt für $X \in \Gamma(\mathscr{B}, \mathscr{J}), p, p', q \in X$ und $y := p - p'$

$$qy(e-q) = y, y^2 = 0, (e-y)^{-1} = e + y, p' = (e+y)p(e-y)\,. \tag{3.4}$$

Die Aussagen (3.3) und (3.4) folgen durch unmittelbares Ausrechnen; insbesondere besagt (3.3), daß jede Faser der Abbildung (3.2') ein translatierter Untervektorraum von $\mathscr{B}$, also insbesondere konvex ist.

Auf $\Gamma(\mathscr{B}, \mathscr{J})$ führen wir die folgende Halbordnung ein: Für $X, Y \in \Gamma(\mathscr{B}, \mathscr{J})$ bedeute die Relation

$$X < Y \tag{3.5}$$

die Aussage: es gibt ein $p \in X$ und ein $q \in Y$ mit $qp = pu.\ p - q \in \mathscr{J}$. Man sieht sofort: $\forall p' \in X$ und $\forall q' \in Y$ gilt dann $q'p' = p'$; außerdem ergibt sich aus $X < Y$ und $Y < Z$ die Relation $X < Z$, und aus $X < Y$ und $Y < X$ folgt $X = Y$.

3.1 Lemma. 1) *Aus* $p \sim p'$ *und* $g \in \mathscr{B}^{-1}$ *folgt* $gpg^{-1} \sim gp'g^{-1}$. 2α) *Seien* $p, q \in \mathscr{P}$, $x := p - q \in \mathscr{I}$ *und* $e \pm x$ *invertierbar; dann gilt* $q' := (e-p+q)\,p(e-p+q)^{-1} \sim q$. 2β) *Ist zusätzlich* $qp = p$ *erfüllt* ($X_p < X_q$), *so folgt* $p \sim q$.

Beweis. 1) $(gpg^{-1})(gp'g^{-1}) = gp'g^{-1}$; $gp'g^{-1} - gpg^{-1} = g(p'-p)g^{-1}$. 2α) Aus $(e-p)p = 0$ folgt $q' = qp(e-p+q)^{-1} \Rightarrow qq' = q'$. Nun zu $q'q = q$:

$$
\begin{aligned}
(e-q)(e-p) &= (e-q)(e-p+q),\\
(e-q+p)(e-p) &= (e-q)(e-p+q),\\
(e+x)(e-p) &= (e-q)(e-x),\\
(e-p)(e-x)^{-1} &= (e+x)^{-1}(e-q)|q,\\
(e-p)(e-x)^{-1}q &= 0,\\
(e-x)^{-1}q &= p(e-x)^{-1}q,\\
q &= (e-x)\,p(e-x)^{-1}q. \qquad (3.6)
\end{aligned}
$$

Wegen $x \in \mathscr{I}$ gilt nach (1.1′) $q' - p \in \mathscr{I}$, also mit $x = p - q \in \mathscr{I}$ auch $q' - q \in \mathscr{I}$. 2): Aus (3.6) folgt mit

$$
\begin{aligned}
q &= (e-p+q)\,p(e-p+q)^{-1}q,\\
q &= qp(e-p+q)^{-1}q,\\
q &= p(e-p+q)^{-1}q,\\
pq &= q.
\end{aligned}
$$

3.1′. Ausgehend von der Ähnlichkeitsabbildung $\mathscr{B}^{-1} \times \mathscr{B} \to \mathscr{B}$ $(g, b) \to gbg^{-1}$ bzw. deren Einschränkung

$$\mathscr{B}^{-1} \times \mathscr{P} \to \mathscr{P} \qquad (3.7)$$

definieren wir aufgrund von 3.1.1) für $\Gamma := \Gamma(\mathscr{B}, \mathscr{I})$ und $G := G_e(\mathscr{I})$ ($\mathscr{I} = \mathscr{B}$ zugelassen) die Abbildung

$$\pi : G \times \Gamma \to \Gamma \qquad (3.8)$$

durch

$$(g, X) \xrightarrow{\pi} gX := X_{gpg^{-1}} \quad \text{für} \quad p \in X,$$

und außerdem

$$\pi^X : G \to \Gamma, \qquad \pi^X(g) = \pi(g, X) = gX. \qquad (3.8')$$

Die auf der topologischen Gruppe G mittels $\tau(\mathscr{I})$ gegebene Topologie induziert auf Γ eine Topologie $\tau(\Gamma)$ in der folgenden Weise: Zu einer Umgebung U von e in G und für $X \in \Gamma$ definieren wir auf Γ eine Umgebung von X

$$U(X) := \{Y \in \Gamma : \exists g \in U \quad \text{mit} \quad gX = Y\}. \qquad (3.9)$$

Durch das entsprechende System aller so definierten Umgebungen überträgt sich die uniforme Struktur von G zu einer solchen auf Γ, wie man leicht nachprüft.

3.2 Lemma. *Der Stabilisator* $H(X)=\{g\in G: X=gX\}\subset G$ *eines Elementes* $X\in\Gamma$ *besteht in „p-Koordinaten“,* $p\in X$*, genau aus allen Elementen der Form*

$$h=\begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{22} \end{pmatrix} \quad \textit{mit} \quad (h_{11}+h_{22})^{-1}\in G. \tag{3.10}$$

Beweis. Aus $g\in G$ mit $gpg^{-1}\sim p$ folgt

α) $(gpg^{-1})p=p, pg^{-1}=g^{-1}p, (e-p)g^{-1}p=0,$

β) $p(gpg^{-1})=gpg^{-1}, pgp=gp, (e-p)gp=0.$

Also hat g die Form (3.10); es bleibt noch $(g_{11}+g_{22})^{-1}\in G$ zu zeigen. Aus

$$e=gg^{-1}=\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}\begin{pmatrix} g'_{11} & g'_{12} \\ 0 & g'_{22} \end{pmatrix}$$

folgt $g_{11}g'_{11}=p(=1)$, womit sich mit $g'_{11}=g_{11}^{-1}$

$$\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}=\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}\begin{pmatrix} 1 & g'_{11}g_{12} \\ 0 & 1 \end{pmatrix} \tag{3.11}$$

ergibt, so daß $(g_{11}+g_{22})^{-1}$ existiert ($g_{12}\in\mathscr{I}$).

Umgekehrt habe h die Form (3.10) mit der Existenz von $(h_{11}+h_{22})^{-1}$. Aus der Matrixdarstellung von h und h^{-1} folgt $hph^{-1}p=hph'_{11}=h_{11}ph'_{11}=p$; ebenso führt die Matrixdarstellung zu $php=hp$, so daß wir $phph^{-1}=hph^{-1}$ erhalten; also $p\sim hph^{-1}$. Damit ist 3.2 gezeigt.

Bemerkung. Bekanntlich haben im ∞-dimensionalen Fall invertierbare 2×2-Dreiecksmatrizen (h_{jk}) (d.h.: $h_{21}=0$) i.a. nicht die Eigenschaft $\exists\ (h_{11}+h_{22})^{-1}$.

Zum Nachweis, daß $\Gamma(\mathscr{B},\mathscr{I})$ eine lokal $\mathscr{B}$-rationale Mannigfaltigkeit ist, benötigen wir (vgl. 2.8 Lemma) zunächst

3.3 Lemma. *Für* $X\in\Gamma(\mathscr{B},\mathscr{I})$, $p\in X$, *gibt es eine Umgebung* $U(e)\subset G(\mathscr{I})$ *mit einer eindeutig bestimmten,* $\mathscr{B}$*-rationalen Zerlegungsabbildung*

$${}^{p}\zeta: U(e)\to G_p^{(1)}\times H(X_p) \quad \textit{mit} \quad {}^{p}\zeta(g)=(r,h), g=r\cdot h$$

($G_p^{(1)}$ *wie in* 2.10).

Beweis. Eindeutigkeit: $g=r\cdot h=r'h'$ impliziert $r^{-1}r'=hh'^{-1}$, so daß sich aus der Matrixstellung $r=r'$ und $h=h'$ ergibt. Die Existenz folgt aus der Identität (in p-Koordinaten)

$$g=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ g_{21}g_{11}^{-1} & 1 \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22}-g_{21}g_{11}^{-1}g_{12} \end{pmatrix}; \tag{3.12}$$

man beachte $g_{12}, g_{21}\in\mathscr{I}$.

3.4. Sei ${}^{p}\zeta=({}^{p}\tau,{}^{p}h)$, ${}^{p}\tau(g)=r$, ${}^{p}h(g)=h$ für ${}^{p}\zeta(g)=(r,h)$; ${}^{p}\kappa: G_p^{(1)}\times H(X_p)\to G$, ${}^{p}\kappa(r,h)=r\cdot h$ und ${}^{p}\kappa'$ die Einschränkung von ${}^{p}\kappa$ auf $G_p^{(1)}$. Ferner sei

$${}_{p}T_X:=\{c\in\mathscr{I}: c=(e-p)cp\}$$

und

$$_p\gamma : {}_pT_X \to G_p^{(1)} \tag{3.13}$$

definiert durch $_p\gamma(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$.

3.4′ Bemerkung. Die Abbildung $(X = X_p \in \Gamma(\mathscr{B}, \mathscr{I}))$

$$\pi^X \circ {}_p\kappa' \circ {}_p\gamma : {}_pT_X \to \Gamma(\mathscr{B}, \mathscr{I}) \tag{3.14}$$

ist stetig und injektiv.

Beweis. Die Stetigkeit ist klar; aus $r_1 X = r_2 X$ folgt $r_1^{-1} r_2 \in H(X)$, und deshalb $r_1 = r_2$.

3.5 Lemma. *Die Voraussetzungen wie in* 3.3 *seien gegeben. Dann existiert für jedes* $p \in X \in \Gamma(\mathscr{B}, \mathscr{I})$ *eine Umgebung* $U_p(X)$ *von* X *in* $\Gamma(\mathscr{B}, \mathscr{I})$ *und ein stetiger (lokaler) Schnitt*

$$_ps_X : U_p(X) \to G_p^{(1)} \subset G(\mathscr{I})$$

der Abbildung $\pi^X : G(\mathscr{I}) \to \Gamma(\mathscr{B}, \mathscr{I})$ *(vgl.* (3.8′); $\pi^X \circ {}_ps_X = \mathrm{Id}_{U_p(X)}$.

Beweis. Die Topologie auf $\Gamma(\mathscr{B}, \mathscr{I})$ ist durch (3.9) definiert. Für eine geeignete Umgebung $U(e) \subset G(\mathscr{I})$ setzen wir vermöge 3.3

$$U_p(X) = \{Y \in \Gamma(\mathscr{B}, \mathscr{I}) : Y = rX, r = {}^p\tau(g), g \in U(e)\} .$$

Die Abbildung $r \to Y_r := r \cdot X (= X_{rpr^{-1}})$ ist nach 3.4′ injektiv und stetig. Die Abbildung $_ps_X$ definiert durch

$$\Gamma(\mathscr{B}, \mathscr{I}) \ni Y_r \to r \in G_p^{(1)} \subset G(\mathscr{I})$$

ist ebenfalls stetig, wie man aufgrund der Stetigkeit von $^p\tau$ und (3.12) sieht; außerdem gilt $\pi^X \circ {}_ps_X = \mathrm{Id}_{U_p(X)}$.

3.6 Lemma. *Sei* M *eine Zusammenhangskomponente von* $\Gamma(\mathscr{B}, \mathscr{I})$ *und* $G := G_e(\mathscr{I})$ *(vgl.* 1.4). *Dann ist die Abbildung*

$$\pi^X : G \to M, X \in M, p \in X, \pi^X(g) = X_{gpg^{-1}}$$

stetig, surjektiv und offen. Es gilt $M \cong G/H(X)$, *für* $X \in M$, *als topologischer homogener Raum, insbesondere operiert* G *transitiv auf* M.

Beweis. Vermöge der lokalen Schnitte $_qs_Y$, $q \in Y$, der Abbildungen π^Y zeigt man, wie in 2.2 Lemma (2.6), daß G auf M transitiv operiert und π^X surjektiv ist. Die Offenheit von π^X folgt ebenfalls aus der Existenz der lokalen Schnitte mit $\pi^{gX} = \pi^X \circ R_g$, wobei R_g die Rechtstranslation auf G ist.

Die schematische Darstellung (2.18) des Beweises von 2.13 Satz läßt sich mit den in 3.3, 3.4 und 3.5 definierten Abbildungen auf $\Gamma(\mathscr{B}, \mathscr{I})$ bzw. eine Zusammenhangskomponente übertragen.

3.7 Satz. *Jede Zusammenhangskomponente* M *des Graßmannraumes* $\Gamma(\mathscr{B}, \mathscr{I})$ *versehen mit der durch* (3.9) *gegebenen Topologie ist mit der Gruppe* $G = G_e(\mathscr{I})$ *und der Abbildung*

$$\pi : G \times M \to M, \qquad \pi(g, X_p) := X_{gpg^{-1}}$$

eine lokal $\mathscr{B}$*-rationale homogene Mannigfaltigkeit.*

Beweis. Zunächst zeigen wir, daß M eine lokal $\mathscr{B}$-rationale Mannigfaltigkeit ist, indem die für jedes feste $X \in M$ und $p \in X$ die Karte

$$_p\varphi_X : U_p(X) \to {}_pT_X,$$

definiert durch

$$_p\varphi_X := {}_p\gamma^{-1} \circ {}^ph \circ {}_ps_X,$$

eingeführt wird; mittels 3.4′ ergibt sich, daß $_p\varphi_X$ eine Homöomorphie auf eine offene Nullumgebung V in $_pT_X$ ist. Wir zeigen nun, daß die Kartenwechsel $\mathscr{B}$-rational sind.

Sei $m_j = X_j \in M$, $j=1,2$; $p_j \in X_j$, $U_j := U_{p_j}(X_j)$ und $\varphi_j := {}_{p_j}\varphi_{X_j}$; $V_j = \varphi_j(U_j)$ Nullumgebung in $_{p_j}T_{X_j}$, $U = U_1 \cap U_2$. Sei $m \in U$, $m = g_j(y_j) \cdot m_j$, $g_j(y_j) \in G_{p_j}^{(1)}$, $y_j \in V_j$. In p_j-Koordinaten hat man

$$g_j(y_j) = \begin{pmatrix} 1 & 0 \\ y_j & 1 \end{pmatrix}.$$

Da G transitiv auf M operiert, existiert $g_0 \in G$ ($g_0 = e$ möglich für $m_1 = m_2$, also $p_1 \sim p_2$) mit $g_0 m_1 = m_2$, so daß aus $g_1(y_1) \cdot m_1 = g_2(y_2) \cdot m_2$ die Gleichung

$$g_1(y_y) g_0^{-1} m_2 = g_2(y_2) m_2$$

folgt und

$$g_2(y_2)^{-1} g_1(y_1) g_0^{-1} = h \in H(m_2) \tag{3.15}$$

erfüllt ist. Den Ausdruck $g_1(y_1) g_0^{-1} = a(y_1)$ rechnet man in p_2-Koordinaten um, so daß sich aus (3.15)

$$a(y_1) = \begin{pmatrix} a_{11}(y_1) & a_{12}(y_1) \\ a_{21}(y_1) & a_{22}(y_1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y_2 & 1 \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ 0 & h_{22} \end{pmatrix}$$

ergibt.

Nach 3.2 existiert $(h_{11} + h_{22})^{-1} \in G(\mathscr{I})$, also auch die (1,1)-Komponente h_{11}^{-1}. Wegen $h_{11} = a_{11}(y_1)$ gilt für y_2 die Gleichung [vgl. (3.12)]

$$y_2 = a_{21}(y_1) a_{11}(y_1)^{-1}.$$

Da alle Operationen rational sind, ergibt sich $\varphi_2 \varphi_1^{-1}: \varphi_1(U) \to \varphi_2(U)$ ist $\mathscr{B}$-rational.

Die Bedingungen der Definition 1.12 sind ebenfalls erfüllt: Zu 3) in 1.12: Die Karten von M werden durch die Gruppen $G_p^{(1)}$ (vgl. (2.10) in G repräsentiert, daraus folgt 3) in 1.12. Zu 4) in 1.12: $G_p^{(1)}$ ist eine Untermannigfaltigkeit von G. Eine geeignete e-Umgebung $\mathscr{T}_X$, $p \in X$, in $G_p^{(1)}$ erfüllt wegen der Definition der Karten $_p\varphi_X$ die Bedingung 4) in 1.12.

3.8 Bemerkung. Aufgrund von Steenrod [52, I.7], definiert die Abbildung $\gamma_{\mathscr{I}}$ (3.2′) eingeschränkt auf eine Zusammenhangskomponente $M = M(\mathscr{I}, p_0)$ von $\mathscr{P}$ ein Faserbündel mit Faser $\gamma_{\mathscr{I}}(X_0)$, $\gamma_{\mathscr{I}}(p_0) = X_0$, mit dem Basisraum $M' \subset \Gamma(\mathscr{B}, \mathscr{I})$, $X_0 \in M'$, M' Zusammenhangskomponente.

3.9 Satz. 1) *Ist die Graßmann-Mannigfaltigkeit $\Gamma(\mathscr{B})$ der topologischen Algebra $\mathscr{B}$ (mit stetiger Inversion) parakompakt, so ist $\Gamma(\mathscr{B})$ homotopieäquivalent zur Mannigfaltigkeit $\mathscr{P}(\mathscr{B})$ aller idempotenten Elemente von $\mathscr{B}$.*

2) *Ist $\Gamma(\mathscr{B}, \mathscr{I})$ parakompakt, so gibt es einen globalen Schnitt der Abbildung $\gamma_{\mathscr{I}}: \mathscr{P} \to \Gamma(\mathscr{B}, \mathscr{I})$ in (3.2'), wobei $\mathscr{P}$ mit der disjunkten Vereinigung der homogenen Mannigfaltigkeiten $M(\mathscr{I}, p)$ (vgl. 2.2) identifiziert wird; ferner ist dann $\mathscr{P}$ homotopieäquivalent zu $\Gamma(\mathscr{B}, \mathscr{I})$.*

Beweis. 1) ist ein Spezialfall, $\mathscr{I} = \mathscr{B}$, von 2): Aufgrund der Konvexität der Fasern von $\gamma_{\mathscr{I}}$ genügt es Zusammenhangskomponenten $P \subset \mathscr{P}$ und $M \subset \Gamma(\mathscr{B}, \mathscr{I})$ zu betrachten mit der Einschränkung γ von $\gamma_{\mathscr{I}}$, $\gamma: \mathscr{P} \to M$. Sei nun $\{U_\alpha : \alpha \in A\}$ eine Überdeckung von M mit $X_\alpha \in U_\alpha$, $p_\alpha \in X$ und $s_\alpha: U_\alpha \to G := G_e(\mathscr{I})$, $(\pi^X \alpha) \circ s_\alpha = \mathrm{Id}_{U_\alpha}$; man kann die kanonischen Schnitte von 3.5 wählen; dann gilt wegen der speziellen Form von $G_p^{(1)}$ die Aussage $s_\alpha(X) \cdot p_\alpha \in X \in U_\alpha \subset \Gamma(\mathscr{B}, \mathscr{I})$. Sei nun $\{\psi_j : j \in J\}$ eine der Überdeckung $\{U_\alpha : \alpha \in A\}$ untergeordnete lokalendliche Zerlegung der Eins, so setzen wir mit $p_j = p_\alpha$ und $s_j = s_\alpha$ für Träger $(\psi_j) \subset U_\alpha$

$$s(X) := \sum_{j \in J} \psi_j(X) s_j(X) \cdot p_j$$

und haben so einen globalen Schnitt von γ erhalten, denn die Fasern von γ sind konvex. Andererseits wird durch

$$H(t,p) := (1-t)p + t(s \circ \gamma)(p), \qquad 0 \leqq t \leqq 1,$$

eine Homotopie zwischen Id_p und $s \circ \gamma$ hergestellt, so daß $\gamma: \mathscr{P} \to M$ eine Homotopieäquivalenz definiert.

Für die Klassifikation von Graßmann-Mannigfaltigkeiten eines separablen Hilbertraumes wird auf [33, S. 109] verwiesen; man vergleiche auch [29] und [32]. 3.9 Satz zeigt die Verbindung zur Menge der Projektoren. 3.9 gibt ebenso einen weiteren Zusammenhang zwischen den Ergebnissen 4.5 und 4.9 aus Raeburn [49] im Spezialfall der Banachalgebren.

Eine Analogie zu Stiefelmannigfaltigkeiten erhält man für $\mathscr{B}$ (wie in 1.1), wenn man den Quotienten $\mathscr{B}^{-1}/H$ betrachtet, wobei H eine Untergruppe (in festen p-Koordinaten) von $\mathscr{B}^{-1}$ bestehend aus allen Elementen der Form $\begin{pmatrix} 1 & g_{12} \\ 0 & g_{22} \end{pmatrix}$ ist.

3.10. Wir spezialisieren 3.7 Satz zu $\mathscr{B} = \mathscr{C}^\infty(\Omega, \mathscr{L}(E))$, Ω kompakte differenzierbare Mannigfaltigkeit, E Hilbertraum, $\mathscr{I} = \mathscr{B}$. Aufgrund von [35, S. 52], wird jedes endlichdimensionale Vektorraumbündel durch ein $p \in \mathscr{P}(\mathscr{B})$ dargestellt. $\Gamma(\mathscr{B})$ ist die Menge aller Untervektorraumbündel von $\Omega \times E$. Aus 3.7 ergibt sich also, daß jede Zusammenhangskomponente von $\Gamma(\mathscr{B})$ mit der von $\mathscr{B}^{-1}$ induzierten Topologie lokal-$\mathscr{B}$-rationale (insbesondere analytische) homogene Fréchetmannigfaltigkeit ist. Für $\mathscr{C}^k$-Bündel folgt das Entsprechende.

§ 4 Relative Inversion und lokal $\mathscr{B}$-rationale homogene Räume

Ein Element a eines Ringes $\mathscr{B}$ heißt regulär (vgl. J. Kaplansky [2]) oder relativ invertierbar, wenn ein $b \in \mathscr{B}$ existiert mit $aba = a$; zu jedem Element a der Menge $\mathscr{R}$ der regulären Elemente von $\mathscr{B}$ gibt es ein $\tilde{a} \in \mathscr{B}$ mit $a\tilde{a}a = a$ und $\tilde{a}a\tilde{a} = \tilde{a}$ (setze

$\tilde{a}=bab$). Die Menge $\mathscr{R}$ des Ringes $\mathscr{L}(E)$ der beschränkten linearen Transformationen eines Banachraumes E wurde seit Atkinson [2] eingehend untersucht, man vergleiche etwa die umfangreichen Literaturangaben in Nashed [42]. Wegen der Verbindung zur Theorie der singulären Integraloperatoren wird auf Gohberg und Krupnik [16] hingewiesen. Douady und Koschorke haben in [13] und [33] die lokal $\mathscr{B}$-rat. ($\mathscr{B}=\mathscr{L}(E)$) Banachmannigfaltigkeit $=\mathscr{R}$ untersucht. Durch unsere Überlegungen folgt, daß die Zusammenhangskomponenten von $\mathscr{R}\subset\mathscr{L}(E)$ mit der von Douady betrachteten Topologie lokal $\mathscr{L}(E)$-rationale homogene Mannigfaltigkeiten sind.

Eine wichtige Anregung zu den folgenden Ergebnissen geht von Atkinson [2] aus, der 4.1 Lemma für $\mathscr{B}=\mathscr{L}(E)$ bewiesen hat. $\mathscr{B}$ sei wie in 1.1 und für $a\in\mathscr{R}$

$$\mathscr{R}_a := \{\tilde{a}\in\mathscr{B} : a\tilde{a}a=a \quad \text{und} \quad \tilde{a}a\tilde{a}=\tilde{a}\}\,. \tag{4.1}$$

4.1 Lemma. *Für $a\in\mathscr{R}$ und $\tilde{a}\in\mathscr{R}_a$ erfüllt die in einer Umgebung $W(a)$ gegebene Funktion $u: W(a)\to\mathscr{B}$, definiert durch*

$$u(b) := \tilde{a}(e+(b-a)\tilde{a})^{-1} \tag{4.2}$$

die Gleichung

$$u(b)bu(b)=u(b)\,, \quad \textit{also} \quad (u(b)b)^2=u(b)b, (bu(b))^2=u(b)b\,. \tag{4.3}$$

Beweis. [2], [20]

$$\tilde{a}b=e+\tilde{a}(b-a)-(e-\tilde{a}a)$$
$$(e+\tilde{a}(b-a))^{-1}\tilde{a}b=e-(e+\tilde{a}(b-a))^{-1}(e-\tilde{a}a)\,.$$

Da $(e+yx)^{-1}$ genau dann existiert, wenn $(e+xy)^{-1}$ existiert und

$$x(e+yx)^{-1}=(e+xy)^{-1}x \tag{4.4}$$

erfüllt ist (vgl. [41, S. 161]), folgt

$$\begin{aligned} u(b)b &= e-(e+\tilde{a}(b-a))^{-1}(e-\tilde{a}a) \\ u(b)bu(b) &= u(b)-(e+\tilde{a}(b-a))^{-1}(e-\tilde{a}a)\tilde{a}(e-(b-a)\tilde{a})^{-1} \\ &= u(b) \quad \text{wegen} \quad (e-\tilde{a}a)\tilde{a}=0 \end{aligned}$$

(man benötigt nur $\tilde{a}a\tilde{a}=\tilde{a}$).

Für das Konzept des homogenen Raumes ist die folgende Aussage als „lokale Liftingidentität“ wichtig (vgl. 1.14)

4.2 Lemma. *Für jedes $a\in\mathscr{R}$ und $\tilde{a}\in\mathscr{R}_a$ gibt es auf einer geeigneten Umgebung $W=W(a)$ definierte $\mathscr{B}$-rationale Funktionen δ, $\dot{\delta}: W\to\mathscr{B}$ mit*

$$bu(b)b=(e-\delta(b))a(e-\dot{\delta}(b)) \tag{4.5}$$

und $\delta(a)=\dot{\delta}(a)=0$; ferner gilt für $b-a\in\mathscr{I}$ ($\mathscr{I}$ wie in 1.1) $\delta(b)$, $\dot{\delta}(b)\in\mathscr{I}$.

Beweis. Setzte

$$e-\dot{\delta}(b)=\dot{g}(b)^{-1}=(e-\tilde{a}a)(e-u(b)b)+\tilde{a}au(b)b$$
$$e-\delta(b)=g(b)=e-(a-b)u(b)b\dot{g}(b)\tilde{a};$$

es gilt 1) $\tilde{a}a\dot{g}(b)^{-1} = \tilde{a}au(b)b = \dot{g}(b)^{-1}u(b)b$

2) $au(b)b = a\dot{g}(b)^{-1}$, 3) $(u(b)b)^2 = u(b)b$.

$$\begin{aligned} g(b)a\dot{g}(b)^{-1} &= (e-(a-b)u(b)b\dot{g}(b)\tilde{a})a\dot{g}(b)^{-1} \\ &= a\dot{g}(b)^{-1} - (a-b)u(b)b\dot{g}(b)\tilde{a}a\dot{g}(b)^{-1} \\ 1) \Rightarrow \quad &= a\dot{g}(b)^{-1} - (a-b)u(b)b\dot{g}(b)\dot{g}(b)^{-1}u(b)b \\ 2) \cup 3) \Rightarrow \quad &= au(b)b-(a-b)u(b)bu(b)b = au(b)b-(a-b)u(b)b \\ &= bu(b)b. \end{aligned}$$

Die letzte Aussage sieht man aufgrund von (1.1′).

Wir werden nun sehen, daß für $b \in \mathscr{B}$ die Gleichung

$$b - bu(b)b = 0, \qquad a \in \mathscr{R}, \tag{4.6}$$

lokal unabhängig von dem speziellen $\tilde{a} \in \mathscr{R}_a$ ist und eine $\mathfrak{l}\mathscr{B}\mathfrak{r}$-Mannigfaltigkeit definiert.

4.3 Lemma. I) $\forall a \in \mathscr{R}\ \forall \tilde{a} \in \mathscr{R}_a\ \forall U(e) \subset \mathscr{B}^{-1}, \exists W(a) \subset (\mathscr{B}, \tau(\mathscr{B}))$ *mit* $\{b \in W(a)\colon b = bu(b)b\} \subset \{ga\dot{g}^{-1}\colon g, \dot{g} \in U(e)\}$.

II) $\forall a \in \mathscr{R}\ \forall \tilde{a} \in \mathscr{R}_a\ \forall W(a) \subset (\mathscr{B}, \tau(\mathscr{B})), \exists U(e) \subset \mathscr{B}^{-1}$ *mit* $\{ga\dot{g}^{-1}\colon g, \dot{g} \in U(e)\} \subset \{b \in W(a)\colon bu(b)b = b\}$.

Beweis. I folgt aus 4.2 wegen der Stetigkeit von δ und $\dot{\delta}$ in (4.5); zu II): Aus $b = ga\dot{g}^{-1}$ folgt $\tilde{b} := \dot{g}\tilde{a}g^{-1} \in \mathscr{R}_b$. Für $p := b\tilde{b}$ und $q := bu(b)$ gilt also $pq = q$. Durch Wahl von $U(e)$ liegen die idempotenten Elemente p und q nahe an $a\tilde{a}$, so daß dann aus 3.1.2β) auch $qp = p$ folgt. Damit haben wir $bu(b)\,b\tilde{b} = b\tilde{b}$, so daß sich $bu(b)\,b = b$ ergibt.

4.4 Definition. Für $b \in \mathscr{R}$ sei $\mathfrak{ker}\, b := X_{e-\tilde{b}b} \in \Gamma(\mathscr{B})$ eine (vgl. 3.1, $\mathscr{I} = \mathscr{B}$) Äquivalenzklasse von Projektoren; ferner sei $\mathfrak{im}\, b := X_{b\tilde{b}} \in \Gamma(\mathscr{B})$.

4.5 Definition. Auf der Menge (Mannigfaltigkeit) $\mathscr{P}$ versehen mit der von $\tau(\mathscr{B})$ induzierten Topologie sei eine lokalkonstante Funktion

$$\mathfrak{di}\colon \mathscr{P} \to \mathfrak{D}, \mathfrak{D} = \mathfrak{D}_0 \cup \mathfrak{D}_\infty, \qquad \mathfrak{D}_0 \cap \mathfrak{D}_\infty = \emptyset,$$

in eine Menge $\mathfrak{D}$ gegeben. Die Abbildung $\mathfrak{di}$ habe die folgenden Eigenschaften

1) Aus $p, q \in \mathscr{P}$ und $qp = p$ sowie $\mathfrak{di}(p) = \mathfrak{di}(q) \in \mathfrak{D}_0$ folgt $pq = q$.

2) Aus $p, q \in \mathscr{P}$ und $qp = p$ sowie $\mathfrak{di}(e-p) = \mathfrak{di}(e-q) \in \mathfrak{D}_0$ folgt $pq = q$.

Dann heißt $\mathfrak{di}\colon \mathscr{P} \to \mathfrak{D}$ eine Dimensionsfunktion.

Da $X \in \Gamma(\mathscr{B})$ konvex ist, haben wir also eine Funktion $\mathfrak{di}\colon \Gamma(\mathscr{B}) \to \mathfrak{D}$.

Man findet leicht Beispiele, daß $\mathfrak{di}$ viel komplizierter als die übliche Dimensionsfunktion mit Werten in $\{0\} \cup \mathbb{N} \cup \{\infty\}$ sein kann.

4.6 Bemerkung. Sei $_{\mathfrak{d}}\mathscr{R} := \{b \in \mathscr{R}\colon \mathfrak{di}(\mathfrak{ker}\, b) = \mathfrak{d}\}$, $^{\mathfrak{d}}\mathscr{R} := \{b \in \mathscr{R}\colon \mathfrak{di}(e - b\tilde{b}) = \mathfrak{d}, \tilde{b} \in \mathscr{R}_b\}$, $\mathscr{R}^{\mathfrak{d}} := \{b \in \mathscr{R}\colon \mathfrak{di}(\mathfrak{im}\, b) = \mathfrak{d}\}$, wobei $\mathfrak{d} \in \mathfrak{D}_0$. Sei M eine der vorangehenden drei Mengen. Dann gibt es für jedes $a \in M$ und $\tilde{a} \in \mathscr{R}_a$ eine Umgebung $W = W(a) \subset (\mathscr{B}, \tau(\mathscr{B}))$, so daß

$$M \cap W = \{b \in W\colon b = bu(b)b\}. \tag{4.7}$$

Beweis. α) Aus $\tilde{b} \in \mathscr{R}_b$ und $(e-u(b)b)(e-\tilde{b}b) = e - \tilde{b}b - u(b)b + u(b)b\tilde{b}b = e - \tilde{b}b$ (wegen 4.5.1) folgt nun $(e-\tilde{b}b)(e-u(b)b) = e - u(b)b$ also $e - u(b)b - \tilde{b}b + \tilde{b}bu(b)b = e - u(b)b$, so daß sich $\tilde{b}b = \tilde{b}bu(b)b$ und deshalb durch Multiplikation von links mit b die Aussage $b = bu(b)b$ ergibt. β) Wenn umgekehrt $b = bu(b)b$ erfüllt ist, hat man wegen 4.1 die Aussage $u(b) \in \mathscr{R}_b$. Für eine genügend kleine Umgebung W gibt es aber einen in $\mathscr{P}$ stetigen Weg von $e - \tilde{a}a$ nach $e - u(b)b$, so daß 4.5 die Aussage $b \in M$ impliziert. Analog schließt man in den übrigen Fällen, bei $\mathscr{R}^b$ unter Verwendung von 4.5.2).

4.7. Wir definieren nun eine *Topologie* (uniforme Struktur) auf $\mathscr{R}$ mittels der Abbildung (Aktion der Gruppe $\mathscr{G} = \mathscr{B}^{-1} \times \mathscr{B}^{-1}$ auf $\mathscr{R}$) (vgl. 1.14)

$$\pi: \mathscr{B}^{-1} \times \mathscr{B}^{-1} \times \mathscr{R} \to \mathscr{R}, \qquad \pi(g, \dot{g}, a) = ga\dot{g}^{-1}, \tag{4.8}$$

die Abbildung π ist wegen „$\tilde{a} \in \mathscr{R}_a$ impliziert $\dot{g}\tilde{a}g^{-1} \in \mathscr{R}_{ga\dot{g}^{-1}}$“ definiert. Für jedes $a \in \mathscr{R}$ definieren wir ein Umgebungssystem folgendermaßen: Sei $U(e)$ eine e-Umgebung in $\mathscr{B}^{-1}$ (bzw. $\mathscr{B}$), so setzen wir

$$U(a) = \{ga\dot{g}^{-1} : g, \dot{g} \in U(e)\}; \tag{4.9}$$

damit erhalten wir auf $\mathscr{R}$ eine Topologie $\tau(\mathscr{R})$, die eine uniforme Struktur definiert. Eine weitere Topologie τ_1 erhält man, indem wir τ_1 als die gröbste Topologie auf $\mathscr{R}$ definieren, so daß die Abbildungen

$$\mathscr{R} \ni b \to b \in (\mathscr{B}, \tau(\mathscr{B})) \quad \text{und} \quad \mathscr{R} \ni b \to \mathfrak{ker}\, b \in \Gamma(\mathscr{B}) \tag{4.10}$$

stetig sind. Die Topologie τ_2 auf $\mathscr{R}$ definieren wir mittels (4.6): Für ein $a \in \mathscr{R}$ und $\tilde{a} \in \mathscr{R}_a$ und eine genügend kleine Umgebung $W(a)$ von a in $(\mathscr{B}, \tau(\mathscr{B}))$ sei

$$\tilde{W}_{\tilde{a}}(a) = \{b \in W(a) : b = bu(b)b\}. \tag{4.11}$$

Aufgrund von 4.3 Lemma ergibt sich die Äquivalenz von τ_2 zu $\tau(\mathscr{R})$.

Wir zeigen nun τ_1 ist äquivalent τ_2: Sei $V(a)$ eine Umgebung von a bzgl. τ_1; dann gilt für eine genügend kleine Umgebung $W(a) \subset (\mathscr{B}, \tau(\mathscr{B}))$ und $b = bu(b)b$, daß $\mathfrak{ker}\, b = X_{e-u(b)b}$ erfüllt ist; $e - u(b)b$ liegt aber in $\mathscr{P}$ mit der von $\tau(\mathscr{B})$ induzierten Topologie nahe an $e - \tilde{a}a$; dies impliziert $\tilde{W}_{\tilde{a}}(a) \subset V(a)$. Sei umgekehrt $\tilde{W}_{\tilde{a}}(a)$ gegeben, so zeigen wir mit der uniformen Struktur von $\Gamma(\mathscr{B})$ und dem folgenden Lemma die Existenz einer Umgebung $V(a)$ bzgl. τ_1 mit

$$V(a) \subset \tilde{W}_{\tilde{a}}(a) \tag{4.12}$$

4.8 Lemma. *Für jedes $X \in \Gamma(\mathscr{B})$, $\Gamma(\mathscr{B})$ versehen mit der von $\mathscr{B}^{-1}$ bzw. $\mathscr{B}_e^{-1}$ induzierten uniformen Struktur (Topologie) (vgl. (3.9)) gibt es eine Umgebung $U(X)$, so daß aus $X < Y$ oder $Y < X$ (vgl. (3.5)), $Y \in U(X)$, die Aussage $X = Y$ folgt.*

Beweis. Sei $Y \in U(X) := \{Y \in \Gamma(\mathscr{B}) : Y = gX, g \in U(e)\}$, wobei für $p \in X$ die Definition $gX := X_{gpg^{-1}}$ gemeint ist. Für $q \in Y$ und $Y = X_{gpg^{-1}}$ gilt $\dot{q} := gpg^{-1} \sim q$. Wir halten nun $p \in X$ fest und wählen $U(e)$ so klein, daß $\forall g \in U(e)$ das Element $(e \pm (p - \dot{q}))$ invertierbar ist. Dann folgt aus 3.1

$$q' := (e - p + \dot{q})\, p\, (e - p + \dot{q})^{-1} \sim \dot{q} \sim q \in X_{gpg^{-1}}.$$

Die Annahme $X < Y$ bedeutet $qp = p$ für $q \in Y$, so daß sich aus $\dot{q}p = \dot{q}(qp) = qp = p$ mit 3.1 nun $\dot{q} \sim p$ und daher $q \sim p$ also $X = Y$ ergibt. Die zweite Aussage beweist man ebenso.

Zu (4.12): Zu einer Umgebung $V'(bu(b)b)$, $a \in \mathscr{R}$, $\tilde{a} \in \mathscr{R}_a$, bezüglich τ_1 gibt es eine Umgebung $V_1(a)$ bzgl. τ_1, so daß aus $b \in \mathscr{R}$ und $b \in V(a)$ sowie $bu(b)b \in V(a)$ die Aussage $b \in V'(bu(b)b)$ folgt. Aus $(e - u(b)b)\,(e - \tilde{b}b) = e - \tilde{b}b$ für $\tilde{b} \in \mathscr{R}_b$ folgt nun, wenn die Umgebung $V'(bu(b)b)$ genügend klein ist, nach 4.8 mit $X = X_{e - \tilde{b}b}$ und $Y = X_{e - u(b)b}$ die Aussage $X = Y$, also $(e - \tilde{b}b)\,(e - u(b)b) = e - u(b)b$, womit sich $b = bu(b)b$ für $b \in V(a)$ ergibt.

4.9 Bemerkung. Die durch (4.9), (4.10) und (4.11) gegebenen Topologien $\tau(\mathscr{R})$, τ_1 und τ_2 sind (auf $\mathscr{R}$) identisch.

4.10 Satz. *Eingeschränkt auf die Mengen* $^{\mathfrak{d}}\mathscr{R}$, $\mathscr{R}^{\mathfrak{d}}$ und $_{\mathfrak{d}}\mathscr{R}$, $\mathfrak{d} \in \mathscr{D}_0$, *stimmt die Topologie* $\tau(\mathscr{B})$ *von* $\mathscr{B}$ *mit* $\tau(\mathscr{R})$ *überein.*

Dies folgt nun aus 4.6, 4.7 und 4.9.

Man kann nun direkt zeigen, daß $(\mathscr{R}, \tau(\mathscr{R}))$ eine lokal $\mathscr{B}$-rationale Mannigfaltigkeit ist, insbesondere folgt dies dann für M aus 4.6.

Für jedes feste $a \in \mathscr{R}$ und jedes feste $\tilde{a} \in \mathscr{R}_a$ und $x \in \mathscr{B}$ hat man mit $p_1 = a\tilde{a} \in \mathscr{P}$, $p_2 := e - p_1$, $\dot{p}_1 = \tilde{a}a \in \mathscr{P}$, $\dot{p}_2 := e - \tilde{a}a$ die direkte Zerlegung

$$\begin{aligned}\mathscr{B} = (p_1 + p_2)\mathscr{B}(\dot{p}_1 + \dot{p}_2) &= \underbrace{[p_1\mathscr{B}\dot{p}_1 \oplus p_2\mathscr{B}\dot{p}_1 \oplus p_1\mathscr{B}\dot{p}_2]}_{T_a} \oplus p_2\mathscr{B}\dot{p}_2 \\ &= \qquad\qquad T_a \qquad\qquad \oplus N_{a,\tilde{a}} \end{aligned} \tag{4.13}$$

$$x = Px + Qx\,. \tag{4.13'}$$

Man sieht nun leicht, daß der direkte Unterraum T_a von $\mathscr{B}$ nicht von der speziellen Wahl $\tilde{a} \in \mathscr{R}_a$ abhängt und aufgrund des folgenden Lemma als „Tangentialraum“ von $\mathscr{R}$ in a aufgefaßt werden kann.

4.11 Lemma. *Sei* $a \in \mathscr{R}$, $\tilde{a} \in \mathscr{R}_a$, $b \in \mathscr{B}$ *und*

$$u(b) := \tilde{a}(e + (b - a)\tilde{a})^{-1}$$

auf der Umgebung $W = W(a) \subset (\mathscr{B}, \tau(\mathscr{B}))$, $b \in W$, *definiert. Dann gilt für* $x := b - a$, $y := Px$ *(vgl.* 4.13′), *und* $f(a + x) := (a + x)u(a + x)(a + x)$

α) $f(a + x) = f(a + y) = a + y + p_2(y(e - \tilde{a}y\tilde{a}a)^{-1}\tilde{a}y)\dot{p}_2$

und

β) *Jedes* b *aus Bild* f *erfüllt die Gleichung*

$$b - bu(b)b = 0\,.$$

Mit 4.11 zeigt man leicht, daß $(\mathscr{R}, \tau(\mathscr{R}))$ eine $\mathfrak{l}\mathscr{B}\mathfrak{r}$-Mannigfaltigkeit ist, denn $\varphi: \tilde{W}_{\tilde{a}}(a) \to T_a$ [vgl. (4.11)] definiert mit $\varphi(b) := P(b - a)$, (vgl. (4.13′)) eine Karte von $(\mathscr{R}, \tau(\mathscr{R}))$; $f(a + \varphi(b)) = b$, $\varphi(f(a + y)) = y$.

Wir verzichten auf den etwas längeren Beweis (vgl. Gramsch [20, 3.18′]) von 4.11, da 4.11 zum Aufbau dieser Arbeit nicht benötigt wird.

4.12 Bemerkung. Für Banachräume E und $\mathscr{B}=\mathscr{L}(E)$ hat man so einen neuen Beweis für das Ergebnis von Douady [12], daß $(\mathscr{R}, \tau(\mathscr{R}))$ in diesem Fall eine $\mathrm{l}\mathscr{B}\mathrm{r}$-Mannigfaltigkeit ist. Man vergleiche dazu auch Koschorke [33, S. 99–101]. In [13] und [32] wird die Funktion $bu(b)b$ nicht verwendet.

4.13. In einem interessanten Spezialfall vereinfachen sich die Überlegungen 4.7, 4.8, 4.9, 4.10 folgendermaßen. Sei $(\mathscr{B}, \tau(\mathscr{B}))$ eine Unteralgebra von $\mathscr{L}(E)$, E Banachraum, und $\tau(\mathscr{B})$ feiner als die von $\mathscr{L}(E)$ induzierte Operatornorm, ferner sei $e=\mathrm{Id}_E$. Dann genügt es zur Definition der Topologie $\tau(\mathscr{R})$ statt der Topologie von $\Gamma(\mathscr{B})$ die Öffnungsmetrik auf der Menge der abgeschlossenen Unterräume zu nehmen, eingeschränkt auf die $\mathscr{B}$-projizierten Unterräume von E. (Vgl. Gohberg u. Krupnik [16], Kato [31], Massera u. Schäffer [40].) Die Schlußweise $\{X, Y\in\Gamma(\mathscr{B}),\ X<Y,\ Y\in U(X)$, impliziert $X=Y\}$, die für die Gültigkeit der Gleichung $0=b-bu(b)b$ verwendet wird, folgt so einfacher mit den Eigenschaften der Öffnungsmetrik.

Wir gehen nun zu dem Beweis des folgenden Hauptsatzes der Arbeit über.

4.14 Satz. *Sei $\mathscr{B}$ eine topologische Algebra mit stetiger Inversion und $\mathscr{I}$ ein zweiseitiges Ideal von $\mathscr{B}$ mit den Eigenschaften wie in* 1.1. *Ferner sei $r\in\mathscr{R}$ und M eine Zusammenhangskomponente von*

$$\{r+\mathscr{I}, \tau(\mathscr{I})\}\cap(\mathscr{R}, \tau(\mathscr{R}))$$

bezüglich der gröbsten Topologie, so daß die Abbildungen

$$M\ni b\mapsto b\in\{r+\mathscr{I}, \tau(\mathscr{I})\} \quad \textit{und} \quad M\ni b\mapsto b\in(\mathscr{R}, \tau(\mathscr{R}))$$

stetig sind. Außerdem sei $G=\dot{G}=G_e(\mathscr{I})$. Dann ist M mit der Abbildung

$$\pi: G\times\dot{G}\times M\to M, \qquad \pi(g, \dot{g}, b):=gb\dot{g} \tag{4.14}$$

eine lokal $\mathscr{B}$-rationale homogene Mannigfaltigkeit.

Natürlich ist in 4.14 der Fall $\mathscr{I}=\mathscr{B}$ zugelassen. 4.14 ist für $\mathscr{B}=\mathscr{L}(\mathbb{K}^n)$, $\mathbb{K}=\mathbb{R}$ oder $\mathbb{C}$, bekannt; die hiesige Beweismethode vielleicht nicht.

In (4.14) wählen wir auf der Produktgruppe $G\times\dot{G}$ als Gruppenoperation $(g_2, \dot{g}_2)\cdot(g_1, \dot{g}_1)=(g_2g_1, \dot{g}_1\dot{g}_2)$, da dies der Fragestellung angemessen erscheint.

4.15 Lemma. *Unter den Voraussetzungen von* 4.14 *ist für $a\in M$ die Abbildung*

$$\pi^a: G\times\dot{G}\to M, \qquad \pi^a(g, \dot{g})=ga\dot{g}$$

surjektiv und offen. Durch die auf einer Umgebung $V=V(a)\subset M$ gegebene Abbildung $s_a: V\to G\times\dot{G}$ definiert durch

$$s_a(b)=(e-\delta(b), \qquad e-\dot{\delta}(b)), \tag{4.15}$$

mit $\delta, \dot{\delta}: V(a)\to G_e(\mathscr{I})$ wie in (4.5), *ist über $V(a)$ ein lokaler Schnitt der Abbildung π^a gegeben:*

$$\pi^a\circ s_a=\mathrm{Id}_V. \tag{4.15'}$$

Beweis. (4.15′) folgt aus 4.2. Die restliche Behauptung ergibt sich aus der Existenz der lokalen Schnitte s_c von π^c für jedes $c\in M$, da M lokal durch die Gleichung „$c=cu(c)c$" charakterisiert ist. Die letzte Behauptung folgt analog zu 2.2.

4.16. Die Idee des Beweises von 4.14 ist ähnlich wie in § 2, 2.5 bis 2.10; sie besteht darin, bei festem $a \in M$ und $\tilde{a} \in \mathcal{R}_a$ in einer Umgebung von (e, e) die Gruppe $G \times \dot{G}$ mit den durch a bzw. $\tilde{a}$ gegebenen „Koordinaten" so zu zerlegen, daß geeignete Teile dieser Zerlegung zu Karten von M führen.

Sei $p_1 = a\tilde{a}$, $p_2 = e - p_1$, $\dot{p}_1 = \tilde{a}a$, $\dot{p}_2 = e - \dot{p}_1$. Für $(g, \dot{g}) \in G \times \dot{G}$ sei $g_{\nu\mu} = p_\nu g p_\mu$, $\dot{g}_{\nu\mu} = \dot{p}_\nu \dot{g} \dot{p}_\mu$, $\nu, \mu = 1, 2$.

Es gilt $g_{11} + g_{22} \in e + \mathcal{I}$, $\dot{g}_{11} + \dot{g}_{22} \in e + \mathcal{I}$ und $g_{\nu\mu}$, $\dot{g}_{\nu\mu} \in \mathcal{I}$ für $\nu \neq \mu$.

Auf diese Weise erhalten wir acht „Koordinaten", die wir kurz $(a, \tilde{a})$-Koordinaten nennen.

4.17 Lemma. *Der Stabilisator H_a von $a \in \mathcal{R}$,*

$$H_a = \{(g, \dot{g}) \in G \times \dot{G} : a = g a \dot{g}\},$$

besteht in $(a, \tilde{a})$-Koordinaten genau aus allen Elementen $(g, \dot{g})$ der Form

$$\left(\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}, \begin{pmatrix} \dot{g}_{11} & 0 \\ \dot{g}_{12} & \dot{g}_{22} \end{pmatrix} \right) \tag{4.16}$$

mit $g_{11} a \dot{g}_{11} = a$, wobei $(g_{11} + g_{22})^{-1}$ und $(\dot{g}_{11} + \dot{g}_{22})^{-1}$ existieren.

Beweis. Aus

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} a \begin{pmatrix} \dot{g}_{11} & \dot{g}_{12} \\ \dot{g}_{21} & \dot{g}_{22} \end{pmatrix} = a$$

folgt

$$(g_{21} + g_{11}) a (\dot{g}_{11} + \dot{g}_{12}) = a,$$

$$g_{21} a \dot{g}_{11} + g_{21} a \dot{g}_{12} + g_{11} a \dot{g}_{11} + g_{11} a \dot{g}_{12} = a.$$

Durch Multiplikationen von links mit p_ν und von rechts mit $\dot{p}_\mu$ erhalten wir

1) $g_{21} a \dot{g}_{11} = 0$,
2) $g_{11} a \dot{g}_{12} = 0$,
3) $g_{21} a \dot{g}_{12} = 0$,
4) $g_{11} a \dot{g}_{11} = a$.

Zum Nachweis der Existenz von g_{11}^{-1} und $\dot{g}_{11}^{-1}$ fassen wir $b \in \mathcal{B}$ als linearen Operator von $\mathcal{B}$ auf. Wir zeigen, $\dot{g}_{11}$ muß eingeschränkt auf das Bild von $\dot{p}_1$ eine Abbildung auf das Bild von $\dot{p}_1$ sein. Wenn dies nicht der Fall ist, existiert ein $x \neq 0$ aus Bild $(\dot{p}_1)$ und $y \in \mathcal{B}$ mit $x = \dot{g}_{12} y$, da $\dot{g}$ surjektiv ist. Da a auf dem Bild $(\tilde{a}a)$ injektiv ist, folgt $z := a \dot{g}_{12} y \neq 0$ und $gz \neq 0$; wegen $z \in$ Bild $(a\tilde{a})$ schließt man aus 2) $g_{11} a \dot{g}_{12} y = 0$, womit wir $g_{21} a \dot{g}_{12} y \neq 0$ erhalten im Widerspruch zu 3). Demnach muß $\tilde{a} g_{11} a$ nicht nur linksinvers zu $\dot{g}_{11}$ eingeschränkt auf Bild $(\tilde{a}a)$ sein, sondern auch rechtsinvers. So haben wir die Existenz von $\dot{g}_{11}^{-1}$ und g_{11}^{-1} erhalten; denn $\dot{g}_{11} \tilde{a} g_{11} a = \dot{p}_1$ impliziert durch Multiplikation von links mit a und von rechts mit $\tilde{a}$ $a \dot{g}_{11} \tilde{a} g_{11} = p_1$, so daß g_{11} eingeschränkt auf Bild $(a\tilde{a})$ auch linksinvertierbar ist. Aus 2) folgt demnach $a \dot{g}_{12} = 0$ und $\tilde{a} a \dot{g}_{12} = \dot{g}_{12} = 0$. Ebenso schließt man aus 1) $g_{21} = 0$. Wegen der Dreiecksgestalt der Matrizen in (4.16) folgt nun leicht die Existenz von $(g_{11} + g_{22})^{-1}$ und $(\dot{g}_{11} + \dot{g}_{22})^{-1}$.

4.18. Wir definieren nun die Untergruppen

$$G_1=\left\{\begin{pmatrix}1 & 0\\ g_{21} & 1\end{pmatrix}: g_{21}\in\mathscr{I}\right\} \text{ von } G \quad \text{und} \quad \dot{G}_1=\left\{\begin{pmatrix}1 & \dot{g}_{12}\\ 0 & 1\end{pmatrix}: \dot{g}_{12}\in\mathscr{I}\right\} \text{ von } \dot{G}$$

sowie

$$G_0=\left\{\begin{pmatrix}g & 0\\ 0 & 1\end{pmatrix}\in G\right\} \text{ von } G,$$

$$H_1=\left\{\begin{pmatrix}h_{11} & h_{12}\\ 0 & h_{22}\end{pmatrix}: \exists(h_{11}+h_{22})^{-1}\right\} \text{ von } G,$$

$$\dot{H}_1=\left\{\begin{pmatrix}\dot{h}_{11} & 0\\ \dot{h}_{21} & \dot{h}_{22}\end{pmatrix}: \exists(\dot{h}_{11}+\dot{h}_{22})^{-1}\right\} \text{ von } \dot{G}.$$

Ferner sei

$$\mathscr{T}_a=G_1\cdot G_0\times\dot{G}_1\subset G\times\dot{G} \tag{4.17}$$

Die Untergruppe (!) $\mathscr{T}_a$ von $G\times\dot{G}$ erfüllt $\mathscr{T}_a\cap H_a=\{(e,e)\}$.

Für $b=ga\dot{g}$ mit $(g,\dot{g})$ aus einer Umgebung W von (e,e) in $G\times\dot{G}$ soll nun eine Zerlegung

$$(g,\dot{g})=(g_1g_0h,\dot{h}\dot{g}_1) \quad \text{mit} \quad g_1\in G_1, g_0\in G_0, \dot{g}_1\in\dot{G}_1$$

und $(h,\dot{h})\in H_a$ gefunden werden.

4.19 Lemma. *Es gibt eine Umgebung W des Einzelelementes von $G\times\dot{G}$ und eine eindeutig bestimmte stetige Abbildung*

$$\zeta^a: W\to\mathscr{T}_a\times H_a,$$

so daß

$$\zeta((e,e))=(e,e) \quad \textit{und} \quad \zeta^a((g,\dot{g}))=(g_1g_0,g_1)\times(h,\dot{h})$$

mit

$$g_1\in G_1, g_0\in G_0, \dot{g}_1\in\dot{G}_1 \quad \textit{und} \quad (h,\dot{h})\in H_a, (g,\dot{g})=(g_1g_0h,\dot{h}\dot{g}_1).$$

Ferner ist ζ^a $\mathscr{B}$-rational.

Beweis. Existenz: Zunächst wird g mit der Identität (3.12) zerlegt und dann $\dot{g}$ mit der Identität

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}a_{11} & 0\\ a_{21} & a_{22}-a_{21}a_{11}^{-1}a_{12}\end{pmatrix}\begin{pmatrix}1 & a_{11}^{-1}a_{12}\\ 0 & 1\end{pmatrix}, \tag{4.18}$$

die in einer Umgebung von e besteht; $a_{12},a_{21}\in\mathscr{I}$, $(a_{11}+a_{22})^{-1}\in G_e(\mathscr{I})$.

Dadurch erhalten wir eine Abbildung

$$\zeta': W\to(G_1\times H_1)\times(\dot{H}_1\times\dot{G}_1),$$

$$\zeta'((g,\dot{g}))=(g_1,h_1,\dot{h}_1,g_1), \quad g=g_1h_1, \dot{g}=\dot{h}_1\dot{g}_1. \tag{4.19}$$

Für

$$h_1=\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ 0 & \alpha_{22}\end{pmatrix},\qquad \dot h_1=\begin{pmatrix}\dot\alpha_{11} & 0\\ \dot\alpha_{21} & \dot\alpha_{22}\end{pmatrix}$$

setzen wir $h_1=g_0 h$, $g_0\in G_0$, $\dot h=\dot h_1$ und erreichen $a=ha\dot h$ durch

$$g_0=\begin{pmatrix}\alpha_{11}a\dot\alpha_{11}\tilde a & 0\\ 0 & 1\end{pmatrix},\qquad h=\begin{pmatrix}a\dot\alpha_{11}^{-1}\tilde a & a\dot\alpha_{11}^{-1}\tilde a\alpha_{11}^{-1}\alpha_{12}\\ 0 & \alpha_{22}\end{pmatrix},$$

wie man sofort sieht; deshalb ergibt sich

$$hah=(a\dot\alpha_{11}^{-1-}\tilde a)\,a(\tilde a a\dot\alpha_{11}\tilde a a)=a \quad\text{wegen}\quad \dot\alpha_{11}^{-1}\dot\alpha_{11}=\tilde a a=\dot p_1\,,$$

also $(h,\dot h)\in H_a$.

Zur Eindeutigkeit: Sei

$$(g,\dot g)=(g_1 g_0 h,\dot h\dot g_1)=(g_1'\cdot g_0'\cdot h',\dot h'\dot g_1')\,.$$

Aus der Matrixdarstellung folgt $\dot h^{-1}\dot h'=\dot g_1(\dot g_1')^{-1}=e$, also $\dot g_1=\dot g'$, $\dot h=\dot h'$. Ebenso ergibt sich mit der Matrixdarstellung aus $g_1(g_0h)=g_1'(g_0'h')$ die Aussage $g_1=g_1'$.

Aus $\dot h=\dot h'$ folgt

$$\begin{aligned} a&=ha\dot h=h'a\dot h\,,\\ a&=h_{11}a\dot h_{11}=h_{11}'a\dot h_{11}\,,\\ a\dot h_{11}^{-1}&=h_{11}a=h_{11}'a\,,\\ a\dot h_{11}^{-1}\tilde a&=h_{11}=h_{11}'\,.\end{aligned}$$

Aus der Matrixdarstellung (in 4.18) für g_1g_0h bzw. $g_1'g_0'h'$ und $g_1=g_1'$ erhalten wir $g_0=g_0'$ und deshalb auch $h=h'$.

4.20. Unter den Voraussetzungen von 4.14 kommen wir zu der Definition einer Karte für eine Umgebung $U\subset M$ von a.

Sei $a\in M$, $\tilde a\in\mathscr{R}_a$, p_ν, $\dot p_\nu$ wie in 4.16; ferner

$$\begin{aligned} &T_1^a:=\{c\in\mathscr{I}:c=p_2cp_1\}; &&\gamma_1:T_1^a\to G_1\,,\\ &T_0^a:=\{c\in\mathscr{I}:c=p_1cp_1\}; &&\gamma_0:V(0)\to G_0,\ V(0)\subset T_0^a\,,\\ &\dot T_1^a:=\{c\in\mathscr{I}:c=\dot p_2c\dot p_1\}; &&\dot\gamma_1:\dot T_1^a\to G_1\,,\end{aligned}$$

$$\gamma_1(c)=\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix},\ \dot\gamma_1(c)=\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix},\ \gamma_0(c)=\begin{pmatrix}p_1+c & 0\\ 0 & p_1\end{pmatrix},$$

$$\gamma:=\gamma_1\times\gamma_0\times\dot\gamma_1,\ \zeta^a=(\tau^a,h^a),\ \tau^a:W\to\mathscr{T}_a,\ h^a:W\to H_a\,,$$

$$\varkappa_a:G_1\cdot G_0\times\dot G_1\times H_a\to G\times\dot G,\ \varkappa(g_1g_0,\dot g_1,h,\dot h)=(g_1g_0h,\dot h\dot g_1)$$

und $\varkappa_a'$ die Einschränkung von $\varkappa_a$ auf $G_1\cdot G_0\times\dot G_1$.

Ferner sei

$$T_a(M):=T_1^a\times T_0^a\times\dot T_1^a\,.$$

Entsprechend (2.18) erhalten wir das folgende Diagramm

$$\begin{array}{ccccccccc}
G\times\dot{G} & \underset{\kappa_a}{\overset{\zeta^a=(\tau^a,h^a)}{\rightleftarrows}} & G_1 & \times & G_0 & \times & \dot{G}_1 & \times & H_a \\
s_a\uparrow\ \downarrow\pi_a & & \gamma_1^{-1}\downarrow\ \uparrow\gamma_1 & & \gamma_0^{-1}\downarrow\ \uparrow\gamma_0 & & \dot{\gamma}_1^{-1}\downarrow\ \uparrow\dot{\gamma}_1 & & \\
M & \xrightarrow{\ \varphi_a\ } & T_1^a & \times & T_0^a & \times & \dot{T}_1^a=T_a(M). & &
\end{array}
\qquad \left(\begin{array}{l}\text{----- lokal}\\ \text{—— global}\end{array}\right) \quad (4.20)$$

Auf einer Umgebung U von a, $U\subset M$, setzt man $\varphi_a=\gamma^{-1}\circ\tau^a\circ s_a$ als Kartenabbildung

4.21 Bemerkung. Sei $W_0=\{x\in T_0^a:\exists(p_1+x+p_2)^{-1}\}$. Dann ist die Abbildung

$$\pi^a\circ\varkappa_a'\circ\gamma: T_1^a\times W_0\times\dot{T}_1^a\to M$$

injektiv, stetig und $\mathscr{B}$-rational.

Beweis. Da γ injektiv ist, genügt es, die Injektivität von $\pi^a\circ\varkappa_a'$ zu zeigen:

$$g_1g_0a\dot{g}_1=v_1v_0a\dot{v}_1\,.$$
$$a=(g_0^{-1}g_1^{-1}v_1v_0)a(\dot{v}_1\dot{g}_1^{-1})$$

impliziert vermöge der Matrixdarstellung von H_a die Aussage $\dot{v}_1=\dot{g}_1$. Aus $a=g_0^{-1}g_1^{-1}v_1v_0a$ folgt mit

$$g_0=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix},\quad g_1=\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix},\quad v_1=\begin{pmatrix}1 & 0\\ c' & 1\end{pmatrix},\quad v_0=\begin{pmatrix}\alpha' & 0\\ 0 & 1\end{pmatrix},$$

$$g_0^{-1}g_1^{-1}v_1v_0=\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -c & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ c' & 1\end{pmatrix}\begin{pmatrix}\alpha' & 0\\ 0 & 1\end{pmatrix},$$

und

$$a=\begin{pmatrix}\alpha^{-1}\alpha' & 0\\ (c'-c)\alpha' & 1\end{pmatrix}a,$$

die Aussage $\alpha=\alpha'$ wegen $p_1=a\tilde{a}=\alpha^{-1}\alpha' p_1=p_1\alpha^{-1}\alpha' p_1$; ferner ergibt sich durch Multiplikation von links mit $e-a\tilde{a}$ die Gleichung $0=p_2(c'-c)\alpha' p_1$, so daß aus der Existenz von $(\alpha')^{-1}$ nun $c'=c$ folgt. Die restlichen Aussagen von 4.21 sind klar.

4.22. Unter den Voraussetzungen von 4.14 gibt es zu jedem $a\in M$, $\tilde{a}\in\mathscr{R}_a$, eine offene Umgebung U von a, $U\subset M$, so daß die Abbildung

$$\varphi_a:=\gamma^{-1}\circ\tau^a\circ s_a: U\to T_1^a\times W_0\times\dot{T}_1^a$$

eine Homöomorphie auf eine offene Nullumgebung V von $T_a(M)$ ist; φ_a ist außerdem $\mathscr{B}$-rational. (U,φ_a,V) ist also eine Karte.

Beweis. Dies ergibt sich nach Konstruktion aus

$$(\pi^a\circ\varkappa_a'\circ\gamma)\circ(\gamma^{-1}\circ\tau^a\circ s_a)=\mathrm{Id}_U$$

mit der Stetigkeit der Abbildungen und der Injektivität (vgl. 4.21) des ersten Faktors; $V:=\varphi_a(U)$ ist offen, wie man sofort sieht.

4.23 Beweis von 4.14. Die Kartenwechsel sind $\mathscr{B}$-rational, wie man aus der Konstruktion 4.15 bis 4.22 unmittelbar sieht, denn wir haben in 4.15 die Charakterisierung (4.11) lokal für M benutzt. Daß M eine lokal $\mathscr{B}$-rationale Mannigfaltigkeit ist, ergibt sich ebenfalls aus der Konstruktion, denn $\mathscr{T}_a$ (vgl. (4.17)) ist eine $\mathfrak{l}\mathscr{B}\mathfrak{r}$-Untermannigfaltigkeit von $G \times \dot{G}$; dies sieht man so: $\gamma^{-1}: \mathscr{T}_a \to T_1^a \times W_0 \times \dot{T}_1^a$ ist ein $\mathscr{B}\mathfrak{r}$-Homöomorphismus. Mit der Zerlegungsmethode erhält man einen $\mathscr{B}\mathfrak{r}$-Homöomorphismus Ψ einer e-Umgebung $U' \subset H^a$ auf eine offene Teilmenge des Vektorraumes

$$E_a := (p_1 \mathscr{I} p_2 \oplus p_2 \mathscr{I} p_2) \oplus (\dot{p}_1 \mathscr{I} \dot{p}_1 \oplus \dot{p}_2 \mathscr{I} \dot{p}_1 \oplus \dot{p}_2 \mathscr{I} \dot{p}_2)$$

[$p_\nu, \dot{p}_\nu$ wie in (4.13)], $\Psi: U' \to V' \subset E_a$. Die Abbildung [vgl. 4.19, (2.18)]

$$f := (\gamma^{-1} \times \Psi) \circ \zeta^a : W \to W' \subset T \oplus E_a,$$

$$T := T_a(M),$$

hat dann die Eigenschaften $f^{-1}(W' \cap T) = \mathscr{T}_a$ und $f^{-1}(W' \cap E_a) = U'((e,e)) \subset H_a$. Für $S = \mathscr{T}_a \cap W$, W geeignete Umgebung des Einselementes von $G \times \dot{G}$, ist $\pi^a: S \to U \subset M$, $U := \pi^a(S)$ ein $\mathscr{B}\mathfrak{r}$-Homöomorphismus auf eine geeignete Umgebung U von a in M; Dies folgt unmittelbar aus (4.17), 4.21 und 4.22.

4.24 Bemerkung. Sei M wie in 4.14. Da $M \cong (G \times \dot{G})/H_a$ und $H_a \subset H_1 \times \dot{H}_1$ (vgl. 4.18) gilt, erhält man wegen

$$(G \times \dot{G})/(H_1 \times \dot{H}_1) = (G/H_1) \times (\dot{G}/\dot{H}_1) \quad \text{mit} \quad \Gamma := G/H_1 \quad \text{und} \quad \dot{\Gamma} := \dot{G}/\dot{H}_1$$

eine Abbildung $\delta: M \to \Gamma \times \dot{\Gamma}$, die ein Faserbündel definiert (vgl. Steenrod [52, I.7.4]).

4.25 Bemerkung. 1) Die Mengen $\mathscr{R}^{\mathfrak{d}}$, ${}_{\mathfrak{d}}\mathscr{R}$ und ${}^{\mathfrak{d}}\mathscr{R}$ aus 4.6 mit $\mathfrak{d} \in \mathfrak{D}_0$ sind Untermannigfaltigkeiten von $\mathscr{B}$. 2) Für einen ∞-dimensionalen Hilbertraum E ist die Menge $\mathscr{R}_{\infty,\infty} = \{a \in \mathscr{L}(E)\colon \dim(\mathfrak{ker}\, a) = \infty, \dim(\mathfrak{im}\, a) = \infty, \operatorname{codim}(\mathfrak{im}\, a) = \infty\}$ *keine* Untermannigfaltigkeit von $\mathscr{L}(E)$.

Beweis. 1) folgt aus (4.7) in 4.6 in Verbindung mit 4.11 Lemma. 2) Darauf wird in [33, S. 101], hingewiesen.

4.26. Zu $a \in \mathscr{C}^\infty(\Omega, \mathscr{B})$, Ω differenzierbare Mannigfaltigkeit, $\dim(\Omega) < \infty$, mit Werten in einer der Mengen $M = \mathscr{R}^{\mathfrak{d}}, {}_{\mathfrak{d}}\mathscr{R}, {}^{\mathfrak{d}}\mathscr{R}, \mathfrak{d} \in \mathfrak{D}_0$ (wie in 4.25 bzw. 4.6) gibt es ein $\tilde{a} \in \mathscr{C}^\infty(\Omega, \mathscr{B})$ mit

$$a(t)\tilde{a}(t)a(t) = a(t) \quad \text{und} \quad \tilde{a}(t)a(t)\tilde{a}(t) = \tilde{a}(t)$$

für alle $t \in \Omega$.

Beweis. Für jedes $t_0 \in \Omega$ gibt es zu $a(t_0)$ ein $\tilde{a}_0 \in \mathscr{R}_{a_0}$; nun setzen wir auf einer geeigneten Umgebung $V(t_0) \subset \Omega$

$$u(a(t)) := \tilde{a}_0(e + (a(t) - a_0)\tilde{a}_0)^{-1}.$$

Nach 4.6 gilt auf $V(t_0)$ die Gleichung $a(t) \cdot u(a(t)) \cdot a(t) = a(t)$. Mit einer geeigneten C^∞-Zerlegung $\{\varphi_j\}$ der Eins bzgl. $\{V_j(t_j)\}$ erhält man mit der Funktion $v(t) = \sum \varphi_j(t) u_j(a(t))$ die Gleichung $ava = a$, so daß $\tilde{a} := vav$ die beiden Gleichungen in 4.26 erfüllt.

4.26′. Man beachte insbesondere den Spezialfall $\mathscr{B}=\mathscr{L}(\mathbb{C}^n)$, der für die Symbolalgebra bei Systemen von Pseudodifferentialoperatoren interessant ist.

4.27 Bemerkung. 1) Sei E ein lokalkonvexer Fréchetraum und $\Phi_{n,m}=\{a\in\mathscr{L}(E)$: $\dim(\ker a)=n$, $\operatorname{codim}(\operatorname{im} a)=m\}$, n, $m<\infty$; dann gilt für jedes $c\in\Phi_{n,m}$

$$\Phi_{n,m}=\{gc\dot{g}: g, \dot{g}\in\mathscr{L}(E)^{-1}\}.$$

2) $\mathscr{R}_{\infty,\infty}$ ist entsprechend 2.15.4) für die Räume $E=c_0$, l^p, $0<p<\infty$ (vgl. 4.25.1)) zusammenziehbar, wie für $E=l^2$ in [33] gezeigt wurde.

Mit denselben Methoden wie in diesem Paragraphen lassen sich relativinvertierbare Abbildungen zwischen verschiedenen Räumen ($\mathscr{R}\subset\mathscr{L}(E,F)$) behandeln. Dazu betrachtet man statt einer Algebra $\mathscr{B}$ ein Vier-Tupel $\{\mathscr{B}_{jk}: j,\ k=1,2\}$ mit entsprechenden Verknüpfungen [22, 21]. Dadurch erfaßt man dann auch die Räume $\mathscr{C}^\infty(\Omega, \mathscr{L}(\mathbb{C}^n, \mathbb{C}^m))$, Ω kompakt.

Aus 4.14 Satz zusammen mit den Ergebnissen von Hayes [29] und Raeburn [49] folgt

4.28 Satz. *Sei Ω eine Steinsche Mannigfaltigkeit und E ein Banachraum, dann gilt für den Raum der holomorphen bzw. stetigen Funktionen auf Ω mit Werten in $\Phi_{n,m}$ bezüglich der natürlichen Inklusion*

$$\pi_0(\mathscr{H}(\Omega, \Phi_{n,m}))\cong\pi_0(\mathscr{C}(\Omega, \Phi_{n,m}))$$

(für die Zusammenhangskomponenten).

Vermutlich besteht zwischen $\mathscr{H}(\Omega, \Phi_{n,m})$ und $\mathscr{C}(\Omega, \Phi_{n,m})$ eine Homotopieäquivalenz, indem man mit $\mathscr{C}(S^n)$, $n=1,2,\ldots$, tensoriert.

§ 5 Ψ-Algebren

In der Theorie der singulären Integraloperatoren, Pseudodifferentialoperatoren und Toeplitzoperatoren spielen spezielle exakte Sequenzen

$$0\to\mathscr{J}\xrightarrow{j}\Psi\xrightarrow{\theta}\Psi/\mathscr{J}\to 0 \tag{5.1}$$

für zweiseitige Ideale $\mathscr{J}$ und Algebren Ψ eine wichtige Rolle. Ist $\mathscr{J}$ ein abgeschlossenes Ideal der topologischen Algebra Ψ, dann hat θ in vielen Fällen eine stetige bzw. stetige lineare Rechtsinverse

$$m: \Psi/\mathscr{J}\to\Psi \tag{5.1′}$$

nach Resultaten von E. Michael, wenn Ψ ein lokalkonvexer Fréchetraum ist, und von T.B. Andersen, wenn Ψ eine C^*-Algebra ist und $\Psi/\mathscr{J}$ die beschränkte Approximationseigenschaft hat und separabel ist.

5.1 Definition. Sei Ψ eine Teilalgebra einer Banachalgebra $\mathscr{B}$ mit Einselement e und $e_\Psi=e$; außerdem sei Ψ mit einer Topologie $\tau(\Psi)$, die feiner ist als die von $\mathscr{B}$ induzierte Normtopologie, eine lokalkonvexe Fréchetalgebra (insbesondere topologische Algebra). 1) Dann heißt Ψ eine Ψ-Algebra in $\mathscr{B}$, wenn für die Gruppen $\mathscr{B}^{-1}$ und Ψ^{-1} der invertierbaren Elemente von $\mathscr{B}$ und Ψ die Aussage

$$\mathscr{B}^{-1}\cap\Psi=\Psi^{-1} \tag{5.2}$$

erfüllt ist. 2) Ψ heißt Ψ_0-Algebra in $\mathscr{B}$, wenn ein $\varepsilon>0$ existiert, so daß die Eigenschaft

$$\{b\in\mathscr{B}: \|b-e\|<\varepsilon\}\cap\Psi\subset\Psi^{-1}$$

vorliegt. 3) Ist $\mathscr{B}$ eine C^*-Algebra (mit Einselement) und Ψ eine in $\mathscr{B}$ enthaltene symmetrische Ψ-Algebra, dann heißt Ψ eine Ψ^*-Algebra.

5.2 Bemerkung. 1) Da für jede Fréchetalgebra mit offener Gruppe invertierbarer Elemente die Inversion stetig ist [55, S. 115], ist die Inversion in Ψ- und Ψ_0-Algebren stetig. 2) Aufgrund des Graphensatzes ist die *-Operation auf Ψ^*-Algebren stetig. 3) Abzählbare Durchschnitte von Ψ_0-, Ψ- und Ψ^*-Algebren sind wieder derartige Algebren in $\mathscr{B}$. 4) Die Banachalgebra $\mathscr{B}$ in 5.1 kann man durch $\mathscr{L}(E)$, E Banach- oder Hilbertraum ersetzen (linke reguläre Darstellung von $\mathscr{B}$ in $\mathscr{L}(\mathscr{B})$, bzw. Gelfand-Naimark).

5.3 Lemma. *Sei $(\mathscr{B},\tau(\mathscr{B}))$ eine topologische Algebra mit stetiger Inversion und $(\Psi,\tau(\Psi))$ eine Teilalgebra von $\mathscr{B}$ mit demselben Einselement e, ferner sei $\tau(\Psi)$ feiner als die von $\tau(\mathscr{B})$ induzierte Topologie. Mit Ψ_e^{-1} bezeichnen wir die e enthaltende Wegkomponente von Ψ^{-1} und mit $[\Psi\cap\mathscr{B}^{-1}]_e$ die e enthaltende Wegkomponente von $\Psi\cap\mathscr{B}^{-1}$ bzgl. der Topologie $\tau(\mathscr{B})$. Wenn nun eine e-Umgebung $U\subset\mathscr{B}^{-1}$ (bzgl. $\tau(\mathscr{B})$) mit der Eigenschaft $\Psi\cap U\subset\Psi^{-1}$ existiert, dann gilt als Mengengleichheit*

$$[\Psi\cap\mathscr{B}^{-1}]_e=\Psi_e^{-1}. \tag{5.3}$$

Ist $\mathscr{B}$ ferner eine Algebra über $\mathbb{C}$, dann sind die unbeschränkten Komponenten der Spektren von $a\in\Psi$ bzgl. $\mathscr{B}$ und Ψ identisch. Ist außerdem Ψ dicht in $\mathscr{B}$, so liegt $\Psi\cap\mathscr{B}^{-1}=\Psi^{-1}$ vor.

Beweis. Es bleibt $[\Psi\cap\mathscr{B}^{-1}]_e\subset\Psi_e^{-1}$ zu zeigen: Es gibt eine Nullumgebung $V\subset\mathscr{B}$ mit $e+V\subset U$ und $\alpha V\subset V$, $|\alpha|\leqq 1$.

Sei $g: [0,1]\to\mathscr{B}^{-1}$ ein stetiger Weg mit $g(0)=e$ und $g(t)\in\Psi$, $0\leqq t\leqq 1$. Nun gibt es wegen der gleichmäßigen Stetigkeit von g eine Zerlegung $0=t_0<t_1<\ldots<t_{r-1}<t_r=1$, so daß

$$e+x_{\nu+1}:=g(t_\nu)^{-1}g(t_{\nu+1})\in e+V, \qquad \nu=0,\ldots,r-1,$$
$$g(1)=(e+x_1)\ldots(e+x_r).$$

Wegen $g(0)=e$ und $g(t_1)\in\Psi^{-1}$, $g(t_1)^{-1}g(t_2)\in\Psi\cap U\subset\Psi^{-1}$, folgt $g(t_2)\in\Psi^{-1}$ und schließlich $g(1)\in\Psi^{-1}$, da $e+x_\nu\in\Psi^{-1}$ erfüllt ist. $w(t):=(e+tx_1)\ldots(e+tx_r)$, $0\leqq t\leqq 1$, ist ein in Ψ^{-1} stetiger Weg mit $w(0)=e$ und $w(1)=g(1)$. Die beiden zusätzlichen Aussagen von 5.3 sind klar.

Für eine symmetrische Fréchetunteralgebra Ψ einer C^*-Algebra $\mathscr{B}$ ist Ψ eine Ψ^*-Algebra in $\mathscr{B}$, wenn eine der folgenden Bedingungen erfüllt ist:

$$\text{Aus}\quad A\in\Psi \quad\text{und}\quad a^*a\in\mathscr{B}^{-1} \quad\text{folgt}\quad a^*a\in\Psi^{-1}, \tag{5.4}$$

$$\exists\varepsilon>0,\ \text{so daß}\ \{x=e+y\in\mathscr{B}: y^*y=yy^*,\ \|y\|<\varepsilon\}\cap\Psi\subset\Psi^{-1}. \tag{5.4'}$$

Dies ist unmittelbar nachzuprüfen.

5.4 Bemerkung. Sei Ψ eine topologische Algebra mit stetiger Inversion mit einem abgeschlossenen zweiseitigen Ideal $\mathscr{J}\neq\Psi$. Dann gilt

1) $\Psi/\mathscr{J}$ ist topologische Algebra mit stetiger Inversion.

2) Die Einschränkung $\theta': \Psi_e^{-1} \to (\Psi/\mathscr{J})_e^{-1}$ von $\theta: \Psi \to \Psi/\mathscr{J}$ (vgl. (5.1)) ist surjektiv; dabei bezeichnet $(\ )_e^{-1}$ die Hauptkomponente.

3) Für $a \in \Psi$, $a' \in \Psi/\mathscr{J}$, $\theta(a) = a'$, ist die Einschränkung $\theta_1: M \to M'$ von θ in den folgenden Fällen surjektiv.

i) $M = \{ga : g \in \Psi_e^{-1}\}, M' = \{h'a' : h' \in (\Psi/\mathscr{J})_e^{-1}\}$

ii) $M = \{gag^{-1} : g \in \Psi_e^{-1}\}, M' = \{ha'h^{-1} : h \in (\Psi/\mathscr{J})_e^{-1}\}$

iii) $M = \{ga\dot{g} : g, \dot{g} \in \Psi_e^{-1}\}, M' = \{ha'\dot{h} : h, \dot{h} \in (\Psi/\mathscr{J})_e^{-1}\}$

Beweis. 1) Die Abbildung θ ist offen. 2) Man wählt eine Zerlegung von [0, 1] wie im Beweis zu 5.3 und liftet die Faktoren $(e' + x_j')$, $j = 1, 2, \ldots, r$; e', $x_j' \in \Psi/\mathscr{J}$. 3) folgt unmittelbar aus 2).

5.5 Satz. *Für eine lokalkonvexe Fréchetalgebra Ψ mit offener Gruppe und abgeschlossenem zweiseitigen Ideal $\mathscr{J}$, $\theta: \Psi \to \Psi/\mathscr{J}$, M eine Zusammenhangskomponente im metrischen Raum $(\mathscr{R}(\Psi), \tau(\mathscr{R}))$ der relativinvertierbaren Elemente von Ψ, M' eine Zusammenhangskomponente von $(\mathscr{R}(\Psi/\mathscr{J}), \tau(\mathscr{R}))$, wobei $\theta(A) \in M'$ für ein $A \in M$, definiert die Einschränkung $\theta_1: M \to M'$ von θ ein Faserbündel mit Faser $F_a = \theta^{-1}(a)$, $a := \theta(A)$.*

Bemerkung. Wegen Faserbündel und der damit zusammenhängenden Homotopie-Lifting-Eigenschaft vgl. man [51, 52] und [58, Theorem 2.6].

Beweis von 5.5. Wir verwenden die Aussage 4.2. Für eine genügend kleine Umgebung $U = U(a) \subset (\mathscr{R}(\Psi/\mathscr{J}), \tau(\mathscr{R}))$, $a \in M'$, $e' = \theta(e)$, gilt mit den $\mathscr{R}$-rationalen Funktionen $\delta(b)$ und $\dot{\delta}(b)$, $b \in U(a)$,

$$b = (e' - \delta(b))\, a(e' - \dot{\delta}(b)), \qquad \delta(a) = \dot{\delta}(a) = 0\,.$$

Mit der Michael-Abbildung $m: \Psi/\mathscr{J} \to \Psi$, $m(0) = 0$, $\theta \circ m = \mathrm{Id}_{\Psi/\mathscr{J}}$, m stetig, definieren wir

$$\text{für } B \in \theta_1^{-1}(U), \theta(A) = a, \theta_1(B) = b, e \in \Psi\,,$$

$$\begin{aligned} h_U(A, b) :&= [e - m(\delta(\theta(B)))]\, A [e - m(\dot{\delta}(\theta(B)))]\,. \\ &= \quad g \quad \cdot A \quad \cdot \dot{g} \end{aligned} \tag{5}$$

Durch (5.5) wird ein Homöomorphismus h der Faser $F_a = \{A \in M : \theta(A) = a\}$ auf die Faser $F_b = \{B \in M : \theta(B) = b\}$, $b \in U = U(a)$, definiert, wenn g und $\dot{g}$ invertierbar sind; dies kann wegen der Stetigkeit von δ und $\dot{\delta}$ durch Wahl von U erreicht werden; die Surjektivität folgt so: Aus $B \in F_b$, $b \in U(a)$, $C := g^{-1} B \dot{g}^{-1}$, erhält man $\theta(C) = (e' - \delta(b))^{-1}\ b(e - \dot{\delta}(b))^{-1} = a$, so daß $C \in F_a$ und $B = gC\dot{g}$ erfüllt ist. Wir erhalten also eine Homöomorphie

$$\varphi_a: \theta_1^{-1}(U) \to U \times F_a$$

definiert durch

$$\varphi_a(B) = (\theta(B), (e - m(\delta(\theta(B))))^{-1} B (e - m(\dot{\delta}(\theta(B))))^{-1})\,.$$

Da M' zusammenhängend ist, läßt sich $\theta_1: M \to M'$ zu einem Faserbündel (M, M', F, θ_1) machen, wobei $F = F_a$ gewählt werden kann für irgend ein $a \in M'$ (man vgl. hierzu [51, S. 96]).

Bemerkung. Natürlich definiert $\theta' : \theta^{-1}(M') \to M'$ ebenfalls ein Faserbündel über dem Basisraum M'. Ein zu 5.5 analoges Ergebnis gilt auch für Projektoren.

5.6 Satz. *Sei Ψ eine Ψ^*-Algebra in der C^*-Algebra $\mathcal{B}$ und $\mathcal{P}_\perp(\Psi) = \{p \in \Psi : p = p^2 = p^*\}$ die Menge der orthogonalen Projektoren in Ψ versehen mit der von $\tau(\Psi)$ induzierten Topologie.*

Dann gelten die folgenden Aussagen.

1) *Jede Äquivalenzklasse $X \in \Gamma(\Psi)$* (vgl. § 3) *enthält genau ein Element von $\mathcal{P}_\perp(\Psi)$.*

2) *$\mathcal{P}_\perp(\Psi)$ ist starker Deformationsretrakt von $\mathcal{P}(\Psi)$.*

3) *Jede Zusammenhangskomponente von $\mathcal{P}_\perp(\Psi)$ ist lokal-Ψ-rationale homogene Mannigfaltigkeit; insbesondere kann $\mathcal{P}_\perp(\Psi)$ zu einer (komplex-) analytischen Fréchetmannigfaltigkeit gemacht werden.*

Beweis. 1) Eindeutigkeit: Aus $q_j \in X_p$ und $q_j = q_j^*$ folgert man $q_2 = q_1 \cdot q_2 = (q_1 \cdot q_2)^* = q_2^* q_1^* = q_2 q_1 = q_1$. Existenz: Für genügend kleines $\varepsilon(p) > 0$, $p \in \mathcal{P}$ und $a = (e-p)^*(e-p)$ ist

$$q = f(p) := \frac{1}{2\pi i} \int_{|z| = \varepsilon(p)} (z e - a)^{-1} dz \sim p \tag{5.6}$$

und aus $\mathcal{P}_\perp(\Psi)$ wegen Kern $(e-p) =$ Bild p und der Abgeschlossenheit des Bildes von $(e-p)$ sowie $\sigma(a) \subset \{0\} \cup \{\delta \leqq t < \infty\}$ für geeignetes $\delta > 0$.

2) Die Abbildung $f : \mathcal{P}(\Psi) \to \mathcal{P}_\perp(\Psi)$ erfüllt $f(p) \sim p$; f ist stetig, da die $*$-Operation auf Ψ stetig und die Integration in (5.6) ebenfalls stetig ist ($\varepsilon(p)$ kann variieren). Wir setzen

$$H(t,p) = (1-t)p + t f(p) \; (\in X_p)$$

und erhalten wegen $H(p,1) = f(p)$, $H(p,0) = p$ und $H(t, f(p)) = f(p)$, $0 \leqq t \leqq 1$, die gewünschte Homotopie.

Zu 3): Wegen 2.13. Satz genügt es, einen Homöomorphismus $h : \Gamma(\Psi) \to \mathcal{P}_\perp(\Psi)$ anzugeben; h sei die Abbildung, die jedem $X \in \Gamma(\Psi)$ das eindeutig bestimmte Element aus $\mathcal{P}_\perp(\Psi)$ zuordnet. h^{-1} ist stetig, wie man sofort sieht. Umgekehrt sei $X = X_p \in \Gamma(\Psi)$ und U eine e-Umgebung in Ψ^{-1}, dann erhält man mit $V(X) := \{Y = X_{gpg^{-1}} : g \in U\}$ eine Umgebung von X. Wegen $h(Y) = f(gpg^{-1})$ und der Stetigkeit von $f(gpg^{-1})$ in g bei festem p, gelangt man zur Stetigkeit von h.

5.7 Bemerkung. Für eine in $\mathcal{L}(E)$, E Hilbertraum, enthaltene Ψ^*-Algebra gilt:

$$\Psi \cap \mathcal{R}(\mathcal{L}(E)) = \mathcal{R}(\Psi)$$

und

$$\Psi \cap \Phi^{l,r}(\mathcal{L}(E)) = \Phi^{l,r}(\Psi, \mathcal{J})$$

wobei $\Phi^{l,r}(\mathcal{L}(E))$ die Menge der Semi-Fredholmoperatoren von $\mathcal{L}(E)$ bezeichnet (links- bzw. rechtsinvertierbar modulo des Ideals $\mathcal{K}$ der kompakten Operatoren), $\mathcal{J} := \Psi \cap \mathcal{K}$ und $\Phi^{l,r}(\Psi, \mathcal{J})$ die Familie der modulo $\mathcal{J}$ links- bzw. rechtsinvertierbaren Operatoren von Ψ.

Der Beweis erfolgt z.B. unter Verwendung von

$$(p + a^* a)^{-1} a^* a, \tag{5.7}$$

wobei p der durch das Cauchyintegral dargestellte orthogonale Projektor auf den Kern von a ist (vgl. Cordes [10a, § 7]).

5.8 Satz. *Sei Ψ eine Ψ^*-Algebra in $\mathscr{L}(E)$, E Hilbertraum, die der folgenden Bedingung genügt:*

Wenn $p_j \in \mathscr{P}_\perp(\Psi)$, $j=1,2$, von endlichem Rang ist, und $x_j \in$ Bild p_j, dann ist $\bar{x}_1 \otimes x_2 \in \Psi$, d.h. $A \in \mathscr{L}(E)$, definiert durch $Ax = \langle x, x_1 \rangle x_2$, liegt in Ψ.

(5.8)

Für die Menge

$$\Phi_{n,m}(\Psi) := \{a \in \Psi : \dim(\ker a) = n, \operatorname{codim}(\operatorname{im} a) = m\}$$

gelten dann die Aussagen

1) $\forall c \in \Phi_{n,m}(\Psi)$
$\Phi_{n,m}(\Psi) = \{gc\dot{g} : g, \dot{g} \in \Psi^{-1}\}$.
2) *$\Phi_{n,m}(\Psi)$ ist eine Fréchetanalytische homogene Mannigfaltigkeit.*

Beweis. Seien $b_j \in \Phi_{n,m}(\Psi)$, $N_j =$ Kern b_j, $R_j =$ Bild b_j, $j=1,2$. Es gibt orthogonale Projektoren q_j, $\dot{p}_j \in \Psi$ mit Bild $\dot{p}_j = N_j$ und Bild $q_j = R_j$, denn mit einem genügend kleinen Weg $w := \{z : |z| = \varepsilon\}$ gilt

$$p_j = \frac{1}{2\pi i} \int_w (ze - b_j^* b)^{-1} dz \quad \text{und} \quad e - q_j = \frac{1}{2\pi i} \int_w (ze - b_j b_j^*)^{-1} dz\,.$$

Sei $E = R_j \oplus^\perp Y_j$, $F := Y_1 + Y_2$ ist endlich dimensional und daher abgeschlossen; ferner gilt $E = F \oplus^\perp (R_1 \cap R_2)$. Nach der Annahme (5.8) gehören alle linearen Abbildungen von E, die auf $R_1 \cap R_2$ verschwinden und Werte in F haben zu Ψ. Wir erhalten nun

$$E = Y_1 \oplus^\perp (F \ominus Y_1) \oplus^\perp (R_1 \cap R_2)$$
$$E = Y_2 \oplus^\perp (F \ominus Y_2) \oplus^\perp (R_1 \cap R_2)\,,$$

wie man mit endlichen Orthogonalisierungsprozessen sieht. Sei $\{e_k\}$ eine Orthonormalbasis in F, $s := \sum \bar{e}_k \otimes e_k$ die orthogonale Projektion auf $F (\bar{e}_k \otimes e_k \in \Psi)$ und $q := e - s$. Wegen der Gleichheit der Dimensionen gibt es ein $r \in \Psi$, das Y_1 auf Y_2 abbildet und auf $(F \ominus Y_1) \oplus^\perp (R_1 \cap R_2)$ verschwindet, ebenso gibt es $l \in \Psi$, das $(F \ominus Y_1)$ auf $(F \ominus Y_2)$ abbildet und auf $Y_1 \oplus^\perp (R_1 \cap R_2)$ verschwindet. Das Element $g_1 := r + l + q \in \Psi$ ist also in Ψ invertierbar und bildet R_1 auf R_2 ab. Analog konstruiert man $\dot{g}_1 \in \Psi^{-1}$, so daß $\dot{g}_1$ den Unterraum N_2 auf N_1 abbildet (betrachte $M = N_1 + N_2$, $E = N_j \oplus^\perp (M \ominus N_j) \oplus^\perp (N_1^\perp \cap N_2^\perp)$). Mit $b := g_1 b \dot{g}_1$ haben wir also ein Element aus Ψ erhalten mit Kern $(b) =$ Kern (b_2) und Bild $(b) =$ Bild (b_2). Sei

$$p = \frac{1}{2\pi i} \int_w (ze - b^* b)^{-1} dz\,, \qquad w = \{z \in \mathbb{C} : |z| = \varepsilon(b)\}$$

der orthogonale Projektor auf Kern (b) und $\tilde{b} := (p + b^* b)^{-1} b^*$; dann gilt $b\tilde{b}b = b$ und $\tilde{b}b\tilde{b} = \tilde{b}$ sowie Bild $(\tilde{b}) = (\text{Kern}\,(b))^\perp$. Setzen wir nun $g = ((e - b\tilde{b}) + b_2 \tilde{b})\, g_1$ und $\dot{g} = \dot{g}_1$, so gelangen wir zu $b_2 = g b_1 \dot{g} (g, \dot{g} \in \Psi^{-1})$ wegen

$$((e - b\tilde{b}) + b_2 \tilde{b}) g_1 b_1 \dot{g}_1 = (e - b\tilde{b}) g_1 b_1 \dot{g}_1 + b_2 \tilde{b} g_1 b \dot{g}_1$$
$$= (e - b\tilde{b}) b + b_2 \tilde{b} b = 0 + b_2\,.$$

Erzeugung von Ψ-Algebren durch parametrisierte Automorphismen von Banachräumen

Die Frage nach der Gültigkeit des Heftungslemma von H. Cartan für Funktionen mit Werten in Fréchetoperatoralgebren (vgl. [7, 12]) mit stetiger Inversion führte zur im folgenden dargestellten Klasse von Ψ-Algebren [7]. Ebenso hängt dies mit der Möglichkeit multiplikativer Zerlegungen in Ψ-Algebren zusammen. Ein wichtiger Ausgangspunkt ist die Arbeit von Cordes [11]. Die Idee zur Definition dieser speziellen Ψ-Algebren hängt auch mit den Untersuchungen von Connes [9] und mit C^*-dynamischen Systemen [47] zusammen [vgl. R. T. Moore, Mem. AMS **78** (1968); E. Zehnder, Commun. Pure Appl. Math. **29**, 49–111 (1976)].

5.9. Sei $\mathscr{A}=\mathscr{A}(\Omega)$ eine lokalkonvexe Fréchetalgebra komplexwertiger Funktionen (punktweise Addition und Multiplikation) auf einer Menge $\Omega(\neq\emptyset)$ mit Einselement 1, so daß die Topologie $\tau(\mathscr{A})$ feiner ist als die punktweise Konvergenz auf Ω. Jedem Banachraum E über $\mathbb{C}$ sei ein lokalkonvexer Fréchetraum $\mathscr{A}(\Omega, E)$ bestehend aus E-wertigen Funktionen (punktweise Addition) zugeordnet (z.B. vermöge geeigneter Tensorprodukte), der E als konstante Funktionen enthält und dessen Topologie feiner ist als die der punktweisen Konvergenz auf Ω; es gelte $\mathscr{A}(\Omega,\mathbb{C})=\mathscr{A}$. Die Zuordnung $E\mapsto\mathscr{A}(\Omega, E)$ habe die folgenden Eigenschaften A_1 und A_2.

A_1) Für Banachräume $E_j, j=1,2,3$, und jede stetige lineare Abbildung $u: E_1\to E_2$ und jede stetige bilineare Abbildung $v: E_1\times E_2\to E_3$ sowie $f_j\in\mathscr{A}(\Omega, E_j)$, $j=1,2$, sind die Aussagen

$$u\circ f_1\in\mathscr{A}(\Omega, E_2) \quad\text{und}\quad v\circ(f_1, f_2)\in\mathscr{A}(\Omega, E_3)$$

erfüllt.

Daraus ergibt sich, daß $\mathscr{A}(\Omega, E)$ für jede Banachalgebra E eine Fréchetalgebra bzgl. punktweiser Multiplikation ist, wie man mit dem Graphensatz sieht, da $\tau(\mathscr{A}(\Omega, E))$ feiner ist als die punktweise Konvergenz.

A_2) Für jede Banachalgebra E mit Einselement ist

$$\mathscr{A}(\Omega, E)\ \textit{invers abgeschlossen},$$

d.h. für jede Funktion $c\in\mathscr{A}(\Omega, E)$, die punktweise auf Ω in E invertierbar ist, existiert ein $d\in\mathscr{A}(\Omega, E)$ mit $d(t)\,c(t)=e$ für alle $t\in\Omega$.

5.10 Definition. Für einen festen Banachraum E sei

$$\alpha:\Omega\to\mathscr{L}(E)^{-1}, \qquad t\mapsto\alpha_t,$$

eine punktweise definierte Abbildung in die Automorphismengruppe von E. Mit dem Tripel $\langle E;\mathscr{A},\alpha\rangle$ bezeichnen wir die Familie der Elemente $x\in E$, für die die E-wertige Funktion

$$\{\Omega\ni t\mapsto\alpha_t(x)\in E\} \quad\text{in}\quad \mathscr{A}(\Omega, E)$$

liegt; $\langle E;\mathscr{A},\alpha\rangle$ wird versehen mit der von $\tau(\mathscr{A}(\Omega, E))$ induzierten Topologie, d.h. für eine Nullumgebung $U\subset\mathscr{A}(\Omega, E)$ ist

$$\tilde{U}=_{\text{def}}\{x\in\langle E;\mathscr{A},\alpha\rangle:\{\Omega\ni t\mapsto\alpha_t(x)\}\in U\}$$

eine Nullumgebung in $\langle E;\mathscr{A},\alpha\rangle$. (Der Vektorraum $\langle E;\mathscr{A},\alpha\rangle$ ist offensichtlich ein Fréchetraum, wie sich aus der Vollständigkeit von $\mathscr{A}(\Omega, E)$ ergibt)

Der Schar $\{\alpha_t : t \in \Omega\}$ von Automorphismen des Banachraumes E ist die Abbildung

$$\check{\alpha} : \Omega \to [\mathscr{L}(\mathscr{L}(E))]^{-1}$$

in die Automorphismengruppe von $\mathscr{L}(E)$ zugeordnet, wobei

$$\check{\alpha}_t(a) := \alpha_t a \alpha_t^{-1}, \qquad a \in \mathscr{L}(E), \tag{5.10}$$

gesetzt wird. Offensichtlich gilt für $a, b \in \mathscr{L}(E)$, $x \in E$,

$$\begin{aligned} \check{\alpha}_t(ab) &= \check{\alpha}_t(a)\check{\alpha}_t(b) \\ \alpha_t(ax) &= \check{\alpha}_t(a)\alpha_t(x). \end{aligned} \tag{5.11}$$

5.11 Satz. *Sei E eine Banachalgebra mit Einselement und $\alpha : \Omega \to \mathscr{L}(E)^{-1}$ mit der Eigenschaft $\alpha_t(xy) = \alpha_t(x)\,\alpha_t(y)$, $x, y \in E$.*

1) *Dann ist der Unterraum $\Psi := \langle E; \mathscr{A}, \alpha\rangle$ von E eine Ψ-Algebra in E; insbesondere gilt $\Psi \cap E^{-1} = \Psi^{-1}$.* 2) *Für jede abgeschlossene Teilalgebra $\mathscr{B}$ von E mit demselben Einselement wie E ist $\Psi \cap \mathscr{B}$ in $\mathscr{B}$ eine Ψ-Algebra bzgl. der von $\mathscr{A}(\Omega, E)$ induzierten Topologie.*

Beweis. 1) Da $\mathscr{A}(\Omega, E)$ ein Fréchetalgebra E-wertiger Funktionen auf Ω ist, muß die Teilmenge $\langle E; \mathscr{A}, \alpha\rangle$ von E wegen $\alpha_t(xy) = \alpha_t(x)\,\alpha_t(y)$ ebenfalls eine Fréchetalgebra sein. Nun zur Eigenschaft $\Psi \cap E^{-1} = \Psi^{-1}$: Für $a \in \Psi$ ist die Funktion

$$\{\Omega \ni t \mapsto \alpha_t(a)\} \in \mathscr{A}(\Omega, E);$$

wenn nun a^{-1} in E existiert, so folgt $e = \alpha_t(a)\,\alpha_t(a^{-1})$, $t \in \Omega$; nach Voraussetzung A_2) muß die Funktion $\Omega \ni t \mapsto \alpha_t(a^{-1})$ ebenfalls in $\mathscr{A}(\Omega, E)$ liegen. Nach Waelbroeck [55, S. 115], ist für Fréchetalgebren mit offener Gruppe die Inversion stetig. 2) beweist man analog.

5.12 Bemerkung. Der Fréchetunterraum $\langle E; \mathscr{A}, \alpha\rangle$ des Banachraumes E wird durch die Elemente der in $\mathscr{L}(E)$ enthaltenen Ψ-Algebra $\Psi := \langle \mathscr{L}(E); \mathscr{A}, \check{\alpha}\rangle$ stetig in sich abgebildet. Die durch $(a, x) \mapsto ax$, $a \in \Psi$, $x \in \langle E; \mathscr{A}, \alpha\rangle$ gegebene bilineare Abbildung

$$\Psi \times \langle E; \mathscr{A}, \alpha\rangle \to \langle E; \mathscr{A}, \alpha\rangle$$

ist stetig.

Beweis. Wegen $\alpha_t(ax) = \check{\alpha}_t(a)\,(\alpha_t(x))$, $a \in \mathscr{L}(E)$, $x \in E$, folgt mit der Eigenschaft A_1) die Aussage $ax \in \langle E; \mathscr{A}, \alpha\rangle$ für $x \in \langle E; \mathscr{A}, \alpha\rangle$ und $a \in \Psi$. Die komponentenweise Anwendung des Graphensatzes führt zur Stetigkeit der bilinearen Abbildung.

5.12. Für einen abgeschlossenen Unterraum F von E bezeichnen wir mit $\mathscr{L}(F \subset E)$ die abgeschlossene Teilalgebra von $\mathscr{L}(E)$, deren Elemente F invariant lassen. Dann ist

$$D := \langle E; \mathscr{A}, \alpha\rangle \cap F \text{ abgeschlossener Unterraum von } \langle E; \mathscr{A}, \alpha\rangle$$

und

$$\Psi := \langle \mathscr{L}(E); \mathscr{A}, \check{\alpha}\rangle \cap \mathscr{L}(F \subset E) \text{ eine } \Psi\text{-Algebra in } \mathscr{L}(F \subset E).$$

Ferner hat man $ax \in D$ für $a \in \Psi$ und $x \in D$, so daß eine stetige bilineare Abbildung $\Psi \times D \to D$ vorliegt.

Wir kommen nun zu einer Permanenzeigenschaft der hier definierten Ψ-Algebren, die in einigen Konstruktionen als Ersatz dafür dienen kann, daß für eine Banachalgebra E mit e $\mathscr{L}(E)$ wieder eine Algebra mit stetiger Inversion ist und mittels regulärer Darstellungen E enthält [41, Chap. 2, § 7].

Für die Banachalgebra E mit e ist durch L_a, R_a, $a \in E$. $x \in E$, $L_a = ax$, $R_a x = xa$ linke bzw. rechte reguläre Darstellung von E in $\mathscr{L}(E)$ gegeben. Für $\alpha: \Omega \to \mathscr{L}(E)^{-1}$ mit $\alpha_t(xy) = \alpha_t(x)\,\alpha_t(y)$, $x, y \in E$, erhält man

$$\check{\alpha}_t(L_a) = L_{\alpha_t(a)}, \qquad \check{\alpha}_t(R_a) = R_{\alpha_t(a)}, \tag{5.12}$$

$$\check{\alpha}_t(T^{-1}) = [\check{\alpha}_t(T)]^{-1}, \qquad T \in \mathscr{L}(E)^{-1}, \tag{5.13}$$

als $\mathscr{L}(E)$-wertige Funktion von $t \in \Omega$.

Beispiel. $\alpha_t(x) = u_t \times u_t^{-1}$, $x \in E$, für $u: \Omega \to E^{-1}$.

5.13 Satz. *Sei E eine Banachalgebra mit Einselement e und $\alpha: \Omega \to \mathscr{L}(E)^{-1}$ mit $\alpha_t(xy) = \alpha_t(x)\,\alpha_t(y)$, $x, y \in E$. Dann ist*

$$\mathscr{L}\Psi := \langle \mathscr{L}(E); \mathscr{A}, \check{\alpha} \rangle \text{ eine } \Psi\text{-Algebra in } \mathscr{L}(E), \tag{5.14}$$

die alle Elemente L_a, R_a, $a \in \Psi := \langle E; \mathscr{A}, \alpha \rangle$, enthält.

Sei ferner F eine abgeschlossene Teilalgebra von E mit $e_F = e$, dann ist

$$\mathscr{L}\Psi_1 := \langle \mathscr{L}(E); \mathscr{A}, \check{\alpha} \rangle \cap \mathscr{L}(F \subset E) \text{ eine } \Psi\text{-Algebra in } \mathscr{L}(F \subset E)$$

für $\Psi_1 := \langle E; \mathscr{A}, \alpha \rangle \cap F$, die alle Elemente L_a, R_a, $a \in \Psi_1$, enthält.

Beweis. Wir zeigen $L_a \in \mathscr{L}\Psi$ für $a \in \Psi$. Aus (5.12) und $\alpha_t(a) \in \mathscr{A}(\Omega, E)$ mit der Eigenschaft A_1) und der Injektion $u: E_1 := E \to \mathscr{L}(E)$ ergibt sich vermöge linksregulärer Darstellung $L_a \in \mathscr{L}\Psi$ wegen 5.12. Aus 5.12 folgt: $\mathscr{L}\Psi$ ist eine Ψ-Algebra in $\mathscr{L}(E)$. Die zweite Behauptung folgt ebenso mit 5.12′.

Es besteht natürlich die Frage: Welche Elemente außer L_a, R_b und den aufgrund der Ψ-Algebra-Eigenschaft existierenden Inversen sind in $\mathscr{L}\Psi$ enthalten? In erster Linie die mit α_t vertauschbaren Operatoren $T \in \mathscr{L}(E)$, d.h. $\alpha_t(Tx) = T(\alpha_t(x))$. $x \in E$, $\check{\alpha}_t(T) = T$. Diese Situation wird leicht durch nukleare Algebren $\mathscr{A}(\Omega)$, $\mathscr{A}(\Omega, E) := \mathscr{A}(\Omega) \hat{\otimes} E$ und $E = W \hat{\otimes} \mathscr{B}$ realisiert, wenn W und $\mathscr{B}$ Banachalgebren sind und „α_t“ $= \mathrm{Id}_w \otimes \alpha_t$ nur auf die zweite Komponente wirkt, während „T“ $= T \otimes \mathrm{id}_{\mathscr{B}}$ nur W transformiert.

5.14. Neben der Erweiterung von Ψ in 5.13 zu $\mathscr{L}\Psi$ ist für Anwendungen eine Erweiterung durch Tensorprodukte wichtig. Dazu spezialisieren wir die Annahmen in 5.9: Die Zuordnung $E \mapsto \mathscr{A}(\Omega, E)$ sei durch das projektive Tensorprodukt $\mathscr{A}(\Omega) \hat{\otimes} E$ gegeben; die Bedingungen von 5.9 besagen also insbesondere, daß $\mathscr{A}(\Omega) \hat{\otimes} E$ ein Raum E-wertiger Funktionen auf Ω ist. Wenn wir annehmen, daß die kommutativen Fréchetalgebren $\mathscr{A}_j = \mathscr{A}_j(\Omega_j)$, $j = 1, 2$, die Eigenschaft haben, daß $\mathscr{A} = \mathscr{A}(\Omega) := \mathscr{A}_1(\Omega_1) \hat{\otimes} \mathscr{A}_2(\Omega_2)$, $\Omega = \Omega_1 \times \Omega_2$ die Forderungen in 5.9 wieder erfüllen (dies gilt z.B. für $\mathscr{A}_j = \mathscr{C}^\infty(\Omega_j)$, Ω_j endlichdimensionale differenzierbare Mannigfaltigkeit), dann ist mit $F_j := \langle E_j; \mathscr{A}_j, \alpha^{(j)} \rangle$; $j = 1, 2$,

$$F_1 \hat{\otimes} F_2 \tag{5.15}$$

ein abgeschlossener Unterraum von

$$\langle E_1 \hat{\otimes} E_2; \mathscr{A}_1 \hat{\otimes} \mathscr{A}_2, \alpha^{(1)} \otimes \alpha^{(2)} \rangle, \tag{5.16}$$

denn aufgrund der projektiven Entwicklung $u = \sum \lambda_k x_k^{(1)} \otimes x_k^{(2)}$, $\sum |\lambda_k| < \infty$, $x_k^{(j)} \to 0$ in F_j, liegt

$$(\alpha_{t_1}^{(1)} \otimes \alpha_{t_2}^{(2)})(u) = \sum \lambda_k \alpha_{t_1}^{(1)}(x_k^{(1)}) \otimes \alpha_{t_2}^{(2)}(x_k^{(2)})$$

in $(\mathscr{A}_1 \hat{\otimes} \mathscr{A}_2) \hat{\otimes} (E_1 \hat{\otimes} E_2) = (\mathscr{A}_1 \hat{\otimes} E_1) \hat{\otimes} (\mathscr{A}_2 \hat{\otimes} E_2)$, da nach Grothendieck das projektive Tensorprodukt assoziativ und kommutativ ist. Wenn nun die Faktoren in (5.15) dicht in E_1 bzw. E_2 sind, dann liegt die topologische Isomorphie der Ausdrücke (5.15) und (5.16) vor. Im Falle von Banachalgebren E_j und multiplikativen $\alpha^{(j)}$ sind die Fréchetalgebren in (5.15) und (5.16) wohldefiniert.

Wir betrachten den Spezialfall von vektorwertigen Fourierreihen.

5.15 Satz. *Sei W die Wieneralgebra der skalaren absolut konvergenten Fourierreihen, dann gilt*

1) $W \hat{\otimes} \langle E; \mathscr{A}, \alpha \rangle = \langle W \hat{\otimes} E; \mathscr{A}, \mathrm{id}_w \otimes \alpha \rangle$
auch im Sinne von Fréchetalgebren, wenn E eine Banachalgebra und α *multiplikativ ist.*

2) *Sei ferner* $E_1 = W$, $\mathscr{A}_1 = \mathscr{C}^\infty(S^1)$, $S^1 = \Omega_1$, $(\alpha_{t_1}^{(1)}(w))(\varphi) = w(\varphi + t_1)$, $w \in W$, $0 \leqq \varphi$, $t_1 \leqq 2\pi$, *und* $\alpha^{(2)}: \Omega_2 \to \mathscr{L}(E_2)^{-1}$, $\mathscr{A}_2(\Omega_2, E) = \mathscr{A}_2(\Omega_2) \hat{\otimes} E$ *in* 5.9 *sowie* $\alpha^{(2)}$ *multiplikativ. Dann gilt für die Fréchetalgebren* $\Psi := \langle E_2; \mathscr{A}_2, \alpha^{(2)} \rangle$ *und* $F_1 := \langle E_1; \mathscr{A}_1, \alpha^{(1)} \rangle$ *die topologische Isomorphie*

$$F_1 \hat{\otimes} \Psi = \mathscr{C}^\infty(S^1, \Psi) = \langle E_1 \hat{\otimes} E_2; \mathscr{A}_1 \hat{\otimes} \mathscr{A}_2, \alpha^{(1)} \otimes \alpha^{(2)} \rangle \tag{5.17}$$

und zwar auch im Sinne von Fréchetalgebren.

Beweis. 1) Aufgrund von 5.14 bleibt nur noch die Inklusion $\supset$ zu zeigen. Dazu verwendet man die kanonische absolute Basis $\{e_\nu\}$ von W: $u \in W \hat{\otimes} E$, $u = \sum e_\nu \otimes d_\nu$; für $u \in \langle W \hat{\otimes} E;\ \mathscr{A}, \mathrm{id}_w \otimes \alpha \rangle$ erhält man andererseits

$$(\mathrm{id}_w \otimes \alpha_t)(u) = \sum w_k \otimes a_k(t), \sum \|w_k\| < \infty, a_k \to 0 \text{ in}$$

$\mathscr{A}(\Omega, E)$. Mit $w_k = \sum_\nu w_{k,\nu} e_\nu$, $\|w_k\| = \sum_\nu |w_{k,\nu}|$ folgt nun durch Umrechnung $u = \sum e_\nu \otimes d_\nu \in W \hat{\otimes} \Psi$.

2) Zunächst gilt $F_1 = C^\infty(S^1)$. Jedes Element der rechten Seite von (5.17) hat eine Fourierentwicklung $b = \sum e_\nu \otimes b_\nu$. Es genügt nun zu zeigen, daß für jede Halbnorm p_m auf $\mathscr{A}_2(\Omega_2, E_2)$ und für jedes $r > 0$

$$\sup_\nu \{(1 + |\nu|)^r p_m(\alpha_{t_2}^{(2)}(b_\nu))\} < \infty$$

erfüllt ist. Dies kann man z.B. direkt durch Abschätzen der Fourierkoeffizienten b_ν von b zeigen.

5.16. Beispiele zu 5.9 und 5.10. 1) Ω eine offene Teilmenge des $\mathbb{R}^n$, $\alpha: \Omega \to \mathscr{L}(E)^{-1}$, $\mathscr{A} = \mathscr{C}^\infty(\Omega)$.

2) $\{G_j : j = 1, 2, \ldots\}$ sei eine Familie endlichdimensionaler Liegruppen und U_j jeweils eine Umgebung des Einselementes von G_j. $T_j: G_j \to \mathscr{L}(E)^{-1}$ seien Darstel-

lungen auf dem Hilbertraum E;

$$\Omega_k = U_{j_1} \times \ldots \times U_{j_{r(k)}} \ni t_k = (g_{j_1}, \ldots, g_{j_{r(k)}})$$
$$\alpha_{t_k}^{(k)} : \Omega_k \to \mathscr{L}(E)^{-1} \text{ definiert durch}, \quad t_k \in \Omega_k,$$
$$\alpha_{t_k}^{(k)} = T_{j_1}(g_{j_1}) T_{j_2}(g_{j_2}) \ldots T_{j_{r(k)}}(g_{j_{r(k)}});$$

Ω = disjunkte Vereinigung der Ω_k, $\alpha : \Omega \to \mathscr{L}(E)^{-1}$, $\alpha_t := \alpha_{t_k}^{(k)}$ für $t \in \Omega_k$.

Wenn die Darstellungen T_j unitär sind, erhält man eine Ψ^*-Algebra $\langle \mathscr{L}(E); \mathscr{C}^\infty(\Omega), \alpha \rangle$.

5.17 Bemerkung. Betrachtet man in 5.15. 2) statt S^1 eine kompakte Liegruppe S, $E_1 = C(S)$ oder die durch den Sobolev-Hilbertraum $H^m(S)$ ($m > \frac{1}{2}\dim S$) gegebene Banachalgebra, $\mathscr{A}_1(S) := \mathscr{C}^\infty(S)$, $(\alpha_t^{(1)}(w))(\mu) := w(\mu \cdot t)$, $\mu \in S$, $t \in S$, dann gilt ebenfalls $\langle E_1; \mathscr{A}_1, \alpha^{(1)} \rangle = \mathscr{C}^\infty(S)$. Aufgrund der linear topologischen Isomorphie von $\mathscr{C}^\infty(S)$ zum Schwartzschen Folgenraum s erhält man auch in dieser allgemeineren Situation die Aussage (5.17) bzw. 5.15, 2).

§ 6 Gegenbeispiele in der Fréchetalgebra der Operatoren mit der Ordnung 0. Holomorphe Faktorisierung

Nun wird gezeigt, daß die Gruppe der invertierbaren Elemente der Fréchetalgebra $\mathscr{L}_0$ aller Operatoren der Ordnung 0 auf der Kreislinie nicht offen in $\mathscr{L}_0$ ist[3]. Damit wird eine Frage von Omori [45, S. 140] beantwortet. Der Raum $\mathscr{L}_0$ wurde im Prinzip schon von E. Michael [Mem. AMS **11**, 66–67 (1952)] eingeführt. In 6.2 wird ein Operator definiert, der einem von S. Lojasiewicz, Jr. und E. Zehnder [J. Funct. Anal. **33**, 165–174 (1979)] angewandten Operator sehr ähnlich ist. Ein weiteres Beispiel wird durch den Operator $T: C^\infty(S^1) \to C^\infty(S^1)$, $(Tf)(\tau) = f(2\tau)$ mit demselben Beweis wie in 6.2 erbracht.

6.1. Sei E^k, $k \in \mathbb{Z}$, der Hilbertraum der Folgen $x = \{x_\nu \in \mathbb{C} : \nu \in \mathbb{Z}\}$ mit

$$\|x\|_k^2 := \sum_{\nu \in \mathbb{Z}} \varrho_\nu^{(k)} |x_\nu|^2 < \infty, \varrho_0^{(k)} = 1, \varrho_\nu^{(k)} = \nu^{2k}, \nu \neq 0. \tag{6.1}$$

Für die kanonische Orthonormalbasis $\{e_\nu : \nu \in \mathbb{Z}\}$ von E^0 gilt

$$\begin{gathered} \|e_0\|_k = 1 \quad \text{und} \quad \|e_\nu\|_k = |\nu|^k, \nu \neq 0 \\ \ldots \subset E^1 \subset E^0 \subset E^{-1} \subset \ldots . \end{gathered} \tag{6.2}$$

$\mathscr{L}(E^k)$ sei die Algebra der beschränkten linearen Transformationen von E^k versehen mit der Operatornorm $\| \ \|_k$. Mit $\mathscr{L}_0$ wird die Algebra der Elemente von $\mathscr{L}(E^0)$ bezeichnet, die sich zu Elementen von $\mathscr{L}(E^k)$, $k \in \mathbb{Z}$, fortsetzen bzw. einschränken lassen. In diesem Sinn ist

$$\mathscr{L}_0 = \bigcap_{k \in \mathbb{Z}} \mathscr{L}(E^k) \tag{6.3}$$

versehen mit dem System $\{\| \ \|_k : k \in \mathbb{Z}\}$ submultiplikativer Normen eine Fréchetalgebra mit Einselement I.

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3 Der Verfasser hat diese Aussage 1982 im Anschluß an ein Gespräch mit H. O. Cordes in etwas anderer Form als 6.2 bewiesen

6.2 Beispiel. Sei $T \in \mathscr{L}(E^0)$ definiert durch $T(e_\nu)=0$ für $\nu \leqq 0$ und $T(e_\nu)=e_{2\nu}$ für $\nu \geqq 1$; d.h. $T=\sum_{\nu \geqq 1} \bar{e}_\nu \otimes e_{2\nu}$.

Es gilt $T \in \mathscr{L}_0$ mit $\|T\|_k=2^k$, $k \in \mathbb{Z}$.

Behauptung. *Für kein $\mu \in \mathbb{R}$ (oder $\mathbb{C}$), $\mu \neq 0$, existiert $(I-\mu T)^{-1}$ in $\mathscr{L}_0$.*

Beweis. Wegen $\|e_{2\nu}\|_k=2^k\|e_\nu\|_k$ $(\nu \neq 0)$ liegt T in $\mathscr{L}_0$. Für $|\mu|<2^{-k}$ existiert

$$(I-\mu T)^{-1}=\sum_{m=0}^{\infty} \mu^m T^m \quad \text{in} \quad \mathscr{L}(E^k)\,. \tag{6.4}$$

Es gilt für $|\mu|>2^{-k}$

$$(I-\mu T)^{-1}(e_1) \notin E^k, \qquad \left(e_1 \in \bigcap_k E^k\right):$$

$T^m(e_1)=e_{2^m}$, $\|e_{2^m}\|_k=2^{mk}$,

$$\|(I-\mu T)^{-1}(e_1)\|_k^2=\left\|\sum_{m=0}^{\infty} \mu^m e_{2^m}\right\|_k^2=\sum_{m=0}^{\infty} \mu^{2m} 2^{2mk}=\sum_{m=0}^{\infty} (\mu 2^k)^{2m}\,. \tag{6.5}$$

Damit ist gezeigt, daß $\mathscr{L}_0^{-1}$ nicht offen in $\mathscr{L}_0$ ist.

Die Beweisidee $T(e_\nu)=e_{\sigma(\nu)}$, $\sigma: \mathbb{N} \to \mathbb{N}$, kann auf andere Folgenräume in Abhängigkeit von der Belegungsfolge $\varrho_\nu^{(k)}$ verallgemeinert werden. Es sei bemerkt, daß für den Operator $S=\sum_{\nu \geqq 1} \bar{e}_{2\nu} \otimes e_{4\nu+1}$ wegen $S^2=0$ die Inverse $(I-\mu S)^{-1}\,\forall \mu \in \mathbb{C}$ in $\mathscr{L}_0$ existiert.

Wir benötigen für die folgenden Gegenbeispiele sowie für 6.7 und 6.8

6.3 Lemma. *Sei $\mathscr{B}$ eine Banachalgebra mit Einselement e und $\mathscr{L}(\mathscr{B})$ die Banachalgebra der stetigen linearen Transformationen von $\mathscr{B}$ mit Einselement I. Durch $\mathscr{B} \ni b \mapsto L_b \in \mathscr{L}(\mathscr{B})$, $L_b x=bx$, sei $\mathscr{B}$ in $\mathscr{L}(\mathscr{B})$ eingebettet. Ferner seien $S, T \in \mathscr{L}(\mathscr{B})$ mit $T+S=I$.*

Dann hat die Gleichung

$$e-y=(e-Sx)(e+Tx)^{-1} \tag{6.6}$$

für jedes $y \in V:=\{y \in \mathscr{B}: \|y\|\,\|T\|<\frac{1}{2}\}$ genau eine Lösung $x \in U:=\{x: \|Tx\|<1\}$. Die Lösung $x=g(y)$ hat die Form

$$g(y)=(I-L_y T)^{-1}(y)\,. \tag{6.7}$$

Beweis. Aus (6.6) folgt

$$\begin{aligned}
(e-y)(e+Tx)&=e-Sx\\
e+Tx-y-L_yTx&=e-Sx\\
(S+T)x-L_yTx&=y\\
(I-L_yT)x&=y\\
x&=(I-L_yT)^{-1}(y)\,.
\end{aligned} \tag{6.8}$$

Aus (6.8) folgt (6.6) mit $y \in V$, wegen

$$\|x\|<\frac{1}{1-\|y\|\,\|T\|}\|y\| \quad \text{und} \quad \|Tx\| \leqq \frac{\|y\|\,\|T\|}{1-\|y\|\,\|T\|}<1, y \in V.$$

6.4 Bemerkung. $g: V \to g(V) \subset U$ ist lokale analytische Inverse der Abbildung $f: U \to \mathscr{B}$,

$$f(x) = e - (e - Sx)(e + Tx)^{-1} = e - (e - Sx)(e - Tx + (Tx)^2 (e + Tx)^{-1}) = x + \varphi(x).$$

Für das Fréchetdifferential gilt $df(o) = I$ und $d\varphi(o) = 0$. Die Faktoren in (6.6) sind für $y \in V \cap \{y : \|y\| < 1\}$ invertierbar.

Das Lemma 6.3 ist eine Vereinfachung und möglicherweise interessante Erweiterung von Ideen von Douady [13], P. Masani (vgl. [37, 2.4]), N. Wiener, I. Gohberg, M.G. Krein [AMS Transl. Monographs **24** (1970), Chap. IV], G. Baxter, F.V. Atkinson und J. Leiterer [37] zum Beweis des Cartan-Lemma (vgl. [7, 2.1]) für holomorphe Matrizen und der multiplikativen Faktorisierung in Banachalgebren. Derartige Faktorisierungen sind für die Theorie der singulären Integraloperatoren [16] und Toeplitzoperatoren wichtig. Die einfache „rationale" Form der Lösung (6.7) ist nützlich. Andere speziellere „rationale" Formeln findet man z.B. in [16, Theorem 4.2].

6.5. Sei S^1 die Kreislinie, $e_\nu = e^{i\nu\varphi}$, $\nu \in \mathbb{Z}$, $\varrho_\nu^{(k)} = 1$ für $\nu = 0$ und $\varrho_\nu^{(k)} = |\nu|^k$, $k = 0, 1, 2, \ldots$

$$W^k = \left\{ x = \sum_{\nu \in \mathbb{Z}} x_\nu e_\nu : \|x\|_k = \sum_{\nu \in \mathbb{Z}} \varrho_\nu^{(k)} |x_\nu| < \infty, x_\nu \in \mathbb{C} \right\}.$$

In dieser Notation ist W^0 die Wieneralgebra (punktweise Multiplikation der Fourierreihen) und $\mathscr{C}^\infty(S^1) = \bigcap_{k=0}^{\infty} W^k$. Sei $T \in \bigcap_k \mathscr{L}(W^k)$ definiert durch $T(e_\nu) = 0$ für $\nu \leqq 0$ und $T(e_\nu) = e_{2\nu - 1}$ für $\nu \geqq 1$, $\|T\|_0 = 1$. Wir zeigen: die Lösung $x \in U \cap W^0$ (6.7) der Gleichung (6.6) mit T und $y = \mu e_1$, $0 < |\mu| < 1$, $e_1 = e^{i\varphi}$, liegt nicht in $\mathscr{C}^\infty(S^1)$. Dazu genügt es, die Divergenz von

$$x = \sum_{m=0}^{\infty} (L_y T)^m (y)$$

in W^k, $|\mu| > 2^{-k}$, nachzuweisen. Wegen $(L_y T)(e_\nu) = \mu e_{2\nu}$, $\nu \geqq 1$, erhält man

$$(L_y T)^m (\mu e_1) = \mu^{m+1} e_{2m},$$

so daß sich

$$\|x\|_k = |\mu| \sum_{m=0}^{\infty} (|\mu| 2^k)^m$$

ergibt.

6.6 Beispiel. Es gibt ein $T = T^2 \in \bigcap_k \mathscr{L}(W^k)$ und ein $h \in C^\infty(S^1)$, so daß die Gleichung (6.6) für *kein* $y = \mu h$, $0 < \mu < 1$, in $\mathscr{C}^\infty(S^1) \cap U$, $U \subset W^0$, (vgl. 6.3) eine Lösung hat.

Beweis. 6.5 wird modifiziert. Sei $P \in \bigcap_k \mathscr{L}(W^k)$ definiert durch $P(e_\nu) = e_\nu$ für ν ungerade und $P(e_\nu) = 0$ für ν gerade; es gilt $\|P\|_k = 1$, $k = 0, 1, \ldots$; sei $T_0 \in \bigcap_k \mathscr{L}(W^k)$ mit $T_0(e_\nu) = 0$ für $\nu \leqq 0$ und für ν gerade; ferner sei $T_0(e_\nu) = e_{2\nu}$ für ungerade ν, $\nu > 0$. Für $T = P + T_0$ erhalten wir $T^2 = T$ wegen $PT_0 = 0 = T_0^2$ und $T_0 P = T_0$. Sei nun $h = e_1 = e^{it}$, $y = \mu e_1$; dann liegt $g(y)$ [vgl. (6.7)] nicht in allen W^k, denn ähnlich wie im vorangehenden Beispiel zeigt man nun die Ungleichung

$$\|g(y)\|_k \geqq \mu \sum_{m=0}^{\infty} (\mu 2^k)^m .$$

Damit haben wir im Fall der multiplikativen Zerlegung (6.6) einfache Gegenbeispiele zum „Satz über implizite Funktion“ in der Fréchetalgebra $\mathscr{C}^\infty(S^1)$ erhalten (vgl. 6.5 u. [26]).

Bemerkung. Für kommutative folgenvollständige lokalkonvexe Algebren $\mathscr{B}$ und S, $T \in \mathscr{L}(\mathscr{B})$, $S(\mathscr{B}) \subset \mathscr{B}_1$ und $T(\mathscr{B}) \subset \mathscr{B}_2$ (vgl. 6.3), wobei $\mathscr{B}_1$ und $\mathscr{B}_2$ submultiplikative folgenvollständige lokalkonvexe Teilalgebren von $\mathscr{B}$ sind, deren Topologien feiner als die von $\mathscr{B}$ induzierten sind, gibt es immer eine Nullumgebung $V \subset \mathscr{B}$, so daß die Faktorisierung

$$(e-y) = (e-x_1)(e-x_2), \qquad x_j \in \mathscr{B}_j, \qquad j=1,2,$$

für jedes $y \in V$ besteht, da man dann die log- und exp-Funktionen anwenden kann; $\exp(S+T)u) = (\exp Su)(\exp Tu)$.

Nun wird eine Anwendung von 6.3 auf die in 5.9 bis 5.17 behandelten speziellen Ψ-Algebren für Fourierreihen gegeben.

6.7. Sei $\mathscr{D}$ eine Banachalgebra mit Einselement e und W die Wieneralgebra der absolutkonvergenten Fourierreihen auf S^1.

$P \in \mathscr{L}(W)$ sei definiert durch

$$Pf = \sum_{\nu=0}^{\infty} c_\nu e^{i\nu\varrho} \quad \text{für} \quad f = \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu\varrho} .$$

Auf $\mathscr{B} = W \hat{\otimes} \mathscr{D}$ sei durch $T := P \otimes Id_{\mathscr{D}}$ die Toeplitz-Projektion gegeben, natürlich gilt $T \in \mathscr{L}(\mathscr{B})$ mit $\|T\| = 1$. Sei $\Psi_2 := \langle \mathscr{D}; \mathscr{A}_2, \alpha^{(2)} \rangle$, $\alpha_t^{(2)}(ab) = (\alpha_t^{(2)}(a))(\alpha_t^{(2)}(b))$, $a, b \in \mathscr{D}$, ferner setzen wir $\mathscr{A}_2(\Omega_2, E) = \mathscr{A}_2(\Omega_2) \otimes E$ (projektives Tensorprodukt) in 5.9; zum Beispiel $\mathscr{A}_2(\Omega_2) = \mathscr{C}^\infty(\Omega_2)$ für eine endlichdimensionale differenzierbare Mannigfaltigkeit.

6.8 Satz.* *Sei y eine Fourierreihe aus $\mathscr{V} := W \hat{\otimes} \Psi_2$ mit $\|y\|_{\mathscr{B}} < \frac{1}{2}$, $\mathscr{B} := W \hat{\otimes} \mathscr{D}$. Dann gibt es x_+ und $x_- \in \mathscr{V}$, wobei x_+ bzw. x_- auf das Innere bzw. das Äußere des Einheitskreises als Ψ_2-wertige Funktion holomorph fortsetzbar ist, so daß auf S^1 die Gleichung*

$$e - y = (e - x_-)(e - x_+)$$

erfüllt ist und beide Faktoren der rechten Seite invertierbar sind.

Ist y zusätzlich eine $\mathscr{C}^\infty$-Funktion auf S^1 mit Werten in Ψ_2, dann gilt außerdem $x_-, x_+ \in \mathscr{C}^\infty(S^1, \Psi_2)$.

Beweis. Wir verbinden 5.12, 5.13, 5.14, 5.15 und 6.3 zu dem folgenden Beweisprinzip: Zunächst sei $\Psi_1 = W$; wir betrachten die “Tensorerweiterung” $\Psi = \Psi_1 \hat{\otimes} \Psi_2 \ni y$; da T mit $\alpha^{(2)}$ vertauschbar ist, gilt $T \in \mathscr{L}\Psi$ und deshalb $I - L_y T \in \mathscr{L}\Psi$; wegen $I - L_y T \in \mathscr{L}(\mathscr{B})^{-1}$ erhalten wir die Existenz von $(I - L_y T)^{-1}$ in $\mathscr{L}\Psi$ (vgl. 5.13) aus der Voraussetzung $\|y\| < \frac{1}{2}$ wegen $\|T\| = 1$. Mittels 5.12 ergibt sich $\mathscr{L}\Psi \times \Psi \to \Psi$, so daß $(I - L_y T)^{-1}(y)$ in Ψ liegt; aus $\Psi \cong \Psi_1 \hat{\otimes} \Psi_2$ folgt nun die

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* Vergl. P. D. Lax: Commun. Pure Appl. Math. **29**, 683–688 (1976)

erste Behauptung. In 6.3 setzen wir dabei $S=I-T$, $x_-=Sx$, $x_+=e-(e+Tx)^{-1}$. Die zusätzliche Behauptung ergibt sich mit $\Psi_1=\mathscr{C}^\infty(S^1)$ aufgrund von 5.15 mit demselben Beweisgedanken:

$$(\#) \quad \Psi_1\hat{\otimes}\Psi_2\subset\Psi\hookrightarrow\mathscr{L}\Psi; \qquad \mathscr{L}\Psi\times\Psi\to\Psi\subset\Psi_1\hat{\otimes}\Psi_2.$$

Nun wird das Heftungslemma von H. Cartan für die speziellen Ψ-Algebren (S. 9 bis 5.17) auf 6.8 zurückgeführt (vgl. Gohberg u. Leiterer [17] für Banachalgebren).

6.9 Satz. *Sei Ψ die Fréchetalgebra $\langle E;\mathscr{A},\alpha\rangle$, wobei E eine Banachalgebra über $\mathbb{C}$ mit Einselement und α multiplikativ ist; ferner sei in 5.9 $\mathscr{A}(\Omega,E)=\mathscr{A}(\Omega)\hat{\otimes}E$. Seien Q_j $(j=1,2)$, $Q:=Q_1\cap Q_2$, $V:=Q_1\cup Q_2$ kompakte Quader im $\mathbb{C}^n$ und $h:Q\to\Psi^{-1}$ eine auf einer Umgebung $U(Q)$ definierte holomorphe Funktion, $h\in\mathscr{H}(Q,\Psi)$. Dann gibt es auf geeigneten Umgebungen $U_j(Q_j)$, $j=1,2$, definierte holomorphe Funktionen $h_j\in\mathscr{H}(U_j,\Psi)$,*

$$h_j: U_j\to\Psi^{-1},$$

so daß $\forall z\in Q$

$$h(z)=h_1(z)h_2(z)$$

erfüllt ist.

Beweis. Wegen der auch für holomorphe Funktionen mit Werten in Ψ-Algebren gültigen Approximation von h durch invertierbare holomorphe Funktionen auf $\mathbb{C}^n$ (vgl. [7, vor 2.2]), genügt es, 6.9 für h nahe an e zu beweisen, $h(z)=e-c(z)$. Ferner können wir annehmen, daß die Q_j $(j=1,2)$ in $\mathbb{C}$ liegen, indem die restlichen $n-1$ Koordinaten als holomorphe Parameter betrachtet werden; diese zusätzlichen holomorphen Parameter lassen sich mit der Schlußweise $(\#)$ im Beweis von (6.8) vereinbaren. Wie in [17] wählt man einen genügend großen Kreis, der durch Q läuft und $Q_1\backslash Q$ im Innern enthält. Dann wählt man eine C^∞-Funktion β, die auf einer Umgebung $\tilde{U}(Q)$ identisch 1 ist und Träger in $U(Q)$ hat. Die Funktion $y(z)=\beta(z)\cdot c(z)$ liegt also in $\mathscr{C}^\infty(S^1)\otimes\Psi$. Nach 6.8 erhalten wir $(e-y)=(e-x_-)(e-x_+)$. Wegen $(e-x_-)=(e-y)(e-x_+)^{-1}$ ist $e-x_-$ auf $U_2(Q_2)$ holomorph fortsetzbar und entsprechend $e-x_+$ auf $U_1(Q_1)$. Damit ist 6.9 bewiesen.

6.10 Bemerkung. Wenn man den zur Faktorisierung in 6.8 und 6.9 benützten Ausdruck (6.7) aus 6.3 Lemma in der Form [vgl. (1.1')], $n\in\mathbb{N}$,

$$(I-L_yT)^{-1}(y)=\left[\sum_{j=0}^{n-1}(L_yT)^j+(L_yT)^r(I-L_yT)^{-1}(L_yT)^{n-r}\right](y)$$

darstellt, ergibt sich eine Anwendung der Faktorisierung auf Funktionen mit Werten in ein- und zweiseitigen Idealen, und zwar insbesondere für die 1^p-Ideale, die bei Sobolevschen Einbettungssätzen eine Rolle spielen [18, 19]. Ebenso kann man das Ideal der Operatoren der Ordnung $-\omega$, $\omega>0$, bzw. $-\infty$, in Algebren von Pseudodifferentialoperatoren behandeln. Ferner beachte man in 6.3. Lemma den Spezialfall $T^2=T$ und $T(\mathscr{B})$ eine Teilalgebra von $\mathscr{B}$.

6.11 Bemerkung. Einen weiteren Zugang zur Faktorisierung für spezielle Fréchet-Liegruppen erhält man durch Eigenschaften der Abbildung

$$f(x):=\log[(\exp Sx)(\exp Tx)] \quad \text{(vgl. [6], Chap. II, §6, 7)}$$

auf einer geeigneten Nullumgebung der Lie-Algebra E einer Banach-Liegruppe G, $S, T \in \mathscr{L}(E)$, $S+T=Id_E$, wenn man spezielle Familien $\{\alpha_t : t \in \Omega\}$ von Automorphismen von E betrachtet, die $\alpha_t\ (f(x)) = f(\alpha_t(x))$ erfüllen (vgl. $\langle E, C^\infty, \alpha \rangle$, 5.10–5.16). E ist dabei ein Tensorprodukt $E = E_1 \hat{\otimes} E_2$, wobei E_1 die Lie-Algebra einer Banach-Liegruppe G_1 ist und E_2 eine halbeinfache kommutative Banachalgebra mit Approximationseigenschaft. Der Operator T ist dabei so gewählt, daß die auf dem Raum M der maximalen Ideale von E_2 definierten Vektorfunktionen Tx, $(I-T)x$, $x \in E$, sich jeweils linear in x und beschränkt zu Vektorfunktionen auf größere Mengen M' und M'' mit $M' \cap M'' = M$ fortsetzen lassen.

Bezüglich Fréchet-Liegruppen vgl. man T. Ratin u. R. Schmid, Math. Z. **177**, 81–100 (1981) sowie H. Omori et al., Tokyo J. Math. **5**, 365–398 (1982) und [26, 45].

7. Schlußbemerkungen

1) In dieser Arbeit wird "der" Satz über implizite Funktionen für ∞-dimensionale Räume in einigen Schlußweisen durch explizite "algebraische" Formeln umgangen, wobei sich dann eine Allgemeinheit ergibt, die wesentlich über Banachalgebren und Fréchetträume hinausgeht (vgl. 2.13, 3.7, 4.14, 6.8). Untersuchungen zum Satz über implizite Funktionen in Fréchetträumen findet man z. B. in Hamilton [26], insbesondere S. 171–186, Omori [45] und Yamamuro [56]. A. Hertle hat mich 1980 auf eine Arbeit von F. Sergeraert [26] aufmerksam gemacht. Die „rationalen“ Resultate in den §§ 2 bis 6 geben möglicherweise einen Hinweis auf weitere Sätze über implizite Funktionen. Die multiplikative Struktur der Problemstellungen muß stärker berücksichtigt werden in Verbindung mit topologischen Algebren. Außerdem sollten Verbindungen zwischen den Ergebnissen von [45, 26] und [56] zum Satz über implizite Funktionen genauer analysiert werden. Eine auf § 6 und 5.9 bis 5.15 basierende Untersuchung zum Satz über implizite Funktionen soll später folgen; Fréchetalgebren von Pseudodifferentialoperatoren in Verbindung mit der (holomorphen) Faktorisierung und nichtabelscher Kohomologie zeigen eine Reihe von Problemen auf (z. B.: [7, 17, 37, 38]). Bei den Ergebnissen der §§ 2, 3, 4 spielt der Grundkörper im wesentlichen keine Rolle.

2) Eine Erweiterung des Divisionssatzes von S. Lojasiewicz auf reellanalytische Fredholmfunktionen und Operatordistributionen [25] kann für Funktionen mit Werten in Ψ-Algebren durchgeführt werden.

3) Sei Ψ eine Ψ^*-Algebra in $\mathscr{L}(E)$, E Hilbertraum, $\mathscr{J} = \Psi \cap \mathscr{K}$, $\mathscr{K}$ das Ideal der kompakten Operatoren von E, und $\vartheta : \Psi \to \Psi/\mathscr{J}$ der kanonische Homomorphismus; dann ist die Einschränkung $\vartheta_M : M(\Psi) \to M(\Psi/\mathscr{J})$ in den folgenden Fällen surjektiv [21]: M die Menge der Projektoren, der orthogonalen Projektoren, der relativ invertierbaren Elemente bzw. der partiell unitären Elemente; darauf wird zusammen mit einer Erweiterung auf Ψ-Algebren noch eingegangen werden. Speziell gilt für C^*-Algebren Ψ, $\mathscr{J} = \mathscr{K}(E) \cap \Psi$ und $\Psi/\mathscr{J} = \mathscr{C}(\Omega) \otimes \mathscr{L}(\mathbb{C}^n)$, Ω kompakt, die folgende (bekannte) Aussage: Zu jedem $a \in \mathscr{C}(\Omega, \mathscr{L}(\mathbb{C}^n))$ mit konstantem Rang gibt es einen Operator A mit abgeschlossenem Bild, so daß $\vartheta(A) = a$ erfüllt ist.

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Mathematische Annalen

Monotone Functions on Formally Real Jordan Algebras

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To Max Koecher on his sixtieth birthday

Let $\mathfrak{A}$ be a formally real Jordan algebra. It is well known that the set of all squares in $\mathfrak{A}$ is a convex cone. There is, therefore, a natural partial order in $\mathfrak{A}$: One writes $x \leqq x'$ if $x' - x$ is a square, and $x < x'$ if $x' - x$ is in the interior of the cone of squares. Let e denote the unit element of $\mathfrak{A}$. Given a real interval (a, b) we denote by $\mathfrak{A}(a, b)$ the set of all x in $\mathfrak{A}$ such that $ae < x < be$. If f is a real-valued function on (a, b), one can in a natural way define $f(x)$ for all x in $\mathfrak{A}(a, b)$. We denote the map $\mathfrak{A}(a, b) \to \mathfrak{A}$ obtained in this way by $f^{\mathfrak{A}}$ (but, as customary, we keep writing $f(x)$ instead of $f^{\mathfrak{A}}(x)$). It is shown in [1, Chap. XI] that if f is monotone on (a, b) then $f^{\mathfrak{A}}$ is a one-to-one map.

In this note we consider the question whether $f^{\mathfrak{A}}$ is monotone with respect to the partial order of $\mathfrak{A}$, i.e. whether it is true that $f(x) \leqq f(x')$ whenever $x \leqq x'$. We shall say that f is $\mathfrak{A}$-monotone if it has this property.

In the special case where $\mathfrak{A}$ is $\mathfrak{S}_n$, the Jordan algebra of real symmetric n-by-n matrices, this question was answered by Löwner [5] (see also [2]). He proved that f is $\mathfrak{S}_n$-monotone if and only if for every choice of numbers $x_1, \ldots, x_n$ in (a, b) the n-by-n matrix with entries

$$[x_j, x_k]_f = \frac{f(x_j) - f(x_k)}{x_j - x_k} \tag{1}$$

is non-negative definite. (When $j = k$ the quotient is understood as a derivative; differentiability of f is part of the necessary and sufficient condition when $n \geqq 2$.)

We shall prove that this condition is also necessary and sufficient for $\mathfrak{A}$-monotonity for any fixed simple $\mathfrak{A}$ whose rank equals n. The proof presents no serious difficulty, it consists of an adaptation of known arguments from the symmetric matrix case [2, 5] and of an application of various facts about the structure of Jordan algebras [1]. Still, it is not completely trivial and deserves, perhaps, to be on record.

We recall that for every x in $\mathfrak{A}$ there exists a complete system $\{e_i\}$ of orthogonal primitive idempotents such that $x = \sum x_i e_i$ with some $x_i \in \mathbb{R}$. $f(x)$ is then given by

$\sum f(x_i)e_i$. The system $\{e_i\}$ induces a Peirce decomposition of $\mathfrak{A}$ into a vector space direct sum

$$\mathfrak{A}=\sum_i \mathbb{R}e_i+\sum_{j<k}\mathfrak{A}_{jk} \tag{2}$$

(see [1, Chap. VIII]). Given any $h\in\mathfrak{A}$, we write its corresponding decomposition as

$$h=\sum_i h_i e_i+\sum_{j<k} h_{jk} \tag{3}$$

(here $h_i\in\mathbb{R}$ and $h_{jk}\in\mathfrak{A}_{jk}$).

We shall denote the norm corresponding to the canonical inner product [1, p. 321] of $\mathfrak{A}$ by $\|\ \|$.

Lemma. *If $f\in C^1(a,b)$ then $f^{\mathfrak{A}}\in C^1(\mathfrak{A}(a,b))$ and the differential of $f^{\mathfrak{A}}$ at $x=\sum x_i e_i$ is given, for any h in $\mathfrak{A}$, by*

$$(df^{\mathfrak{A}})_x(h)=\sum_i [x_i,x_i]_f h_i e_i+\sum_{j<k}[x_j,x_k]_f h_{jk}. \tag{4}$$

Proof. First we consider the special case where $f(x_i)=f'(x_i)=0$ for each i. The right hand side of (4) as well as $f(x)$ is then zero, so the statement to prove is

$$\|f(x+h)\|=o(\|h\|). \tag{5}$$

We have $x+h=\sum(x_i+\varepsilon_i)e_i'$ with some orthogonal system $\{e_i'\}$. As well known, the "eigenvalues" of an element are the roots of a generalized characteristic equation which has only real roots. Hence, when appropriately ordered, they depend analytically on the element (see e.g. [6, Hilfssatz 2]). It follows that $\varepsilon_i=o(1)$. It is also known that there exists an automorphism u of $\mathfrak{A}$ such that $e_i'=u\cdot e_i$ $(1\leqq i\leqq n)$. By choosing a local cross-section in the automorphism group over the stabilizer of x the choice of u can be made unique, and we also have $u=o(1)$. Now

$$h=u\cdot\Big(\sum_i(x_i+\varepsilon_i)-\sum_i x_i e_i\Big)+\Big(u\cdot\sum_i x_i e_i-\sum_i x_i e_i\Big). \tag{6}$$

It is shown in [4, p. 184] that the orbit of any element of the subalgebra $\mathfrak{a}=\sum\mathbb{R}e_i$ under the automorphism group is orthogonal to $\mathfrak{a}$ and hence also to $u\cdot\mathfrak{a}$. Thus the two terms on the right of (6) are orthogonal up to terms of order two. It follows that

$$\sum\varepsilon_i^2=\Big\|u\cdot\Big(\sum_i(x_i+\varepsilon_i)\,e_i-\sum_i x_i e_i\Big)\Big\|\leqq 2\|h\|^2$$

for small $\|h\|$.

The mean value theorem now gives, with some ε_i' between 0 and ε_i,

$$\begin{aligned}\|f(x+h)\|^2&=\Big\|\sum_i f(x_i+\varepsilon_i)\,e_i'-\sum_i f(x_i)\,e_i'\Big\|^2\\&=\sum_i\varepsilon_i^2 f'(x_i+\varepsilon_i')^2\leqq 2\|h\|^2\sum_i f'(x_i+\varepsilon_i')^2\end{aligned}$$

and (5) follows.

In the case of a general C^1-function f, we can find a polynomial p such that p and p' coincide with f and f' at the points x_i. Then, by what we just proved, the

Lemma holds for $f-p$; it only remains to prove that it also holds for p. For this, it will clearly suffice to prove the Lemma for the functions $x^m (m \in \mathbb{N})$.

Explicitly, what remains to prove is that

$$(x+h)^m - x^m = \sum_i m x_i^{m-1} h_i e_i + \sum_{j<k} \frac{x_j^m - x_k^m}{x_j - x_k} h_{jk} + o(\|h\|) \tag{7}$$

for all $m \in \mathbb{N}$. We prove this by induction on m. By power-associativity we have

$$(x+h)^{m+1} - x^{m+1} = (x+h)\left[(x+h)^m - x^m\right] + h x^m .$$

Using the induction hypothesis (7) this equals

$$(x+h)\left\{\sum_i m x_i^{m-1} h_i e_i + \sum_{j<k} \frac{x_j^m - x_k^m}{x_j - x_k} h_{jk}\right\} + h x^m + o(\|h\|) .$$

In this expression we write x and h in terms of the Peirce decomposition and use the rules $e_i e_j = \delta_{ij} e_i$, $e_i h_{jk} = \frac{1}{2}(\delta_{ij} + \delta_{ik}) h_{jk}$ (cf. [1, Chap. VIII]). After a simple computation it turns out to be equal to

$$\sum_i (m+1) x_i^m h_i e_i + \sum_{j<k} \frac{x_j^{m+1} - x_k^{m+1}}{x_j - x_k} h_{jk} + o(\|h\|)$$

finishing the proof.

Theorem. *Let $\mathfrak{A}$ be a simple formally real Jordan algebra of rank n and let f be a real-valued function defined on an interval (a, b). Then f is $\mathfrak{A}$-monotone if and only if for every choice of $x_1, \ldots, x_n$ in (a, b) the matrix $[x_j, x_k]_f$ is non-negative definite.*

Proof. Suppose that f is $\mathfrak{A}$-monotone. It is known [1, Chap. VIII, Lemma 4.2] that $\mathfrak{A}$ contains a subalgebra $\tilde{\mathfrak{A}}$ isomorphic with $\mathfrak{S}_n$. It is clear that $x \leqq x'$ with respect to the partial order of $\tilde{\mathfrak{A}}$ is equivalent with $x \leqq x'$ in $\mathfrak{A}$ (in fact, if $x' - x = u^2$ for some $u \in \mathfrak{A}$, then $x' - x$ also has a square root in the subalgebra $\mathbb{R}[x' - x]$ generated by $x' - x$, hence in $\tilde{\mathfrak{A}}$). It is also clear that for x in $\tilde{\mathfrak{A}}$ the element $f(x)$ is in $\mathbb{R}[x]$, hence in $\tilde{\mathfrak{A}}$. So f is $\tilde{\mathfrak{A}}$-monotone, and the "only if" part of the Theorem follows from Löwner's results.

The converse statement is trivial in the case $n = 1$. When $n \geqq 2$, the condition implies that the derivative of f is continuous ([5, p. 187] or [2, p. 73]). Therefore, by the Lemma, it will suffice to prove that if $h \geqq 0$ has the expansion (3) with respect to some Peirce decomposition and if (p_{jk}) is a non-negative definite n-by-n matrix then it follows that

$$\sum_i p_{ii} h_i e_i + \sum_{j<k} p_{jk} h_{jk} \geqq 0 . \tag{8}$$

Since every non-negative matrix is a sum of matrices of form $(q_j q_k)$, it is enough to prove that

$$\sum_i q_i^2 h_i e_i + \sum_{j<k} q_j q_k h_{jk} \geqq 0$$

for any $q_1, \ldots, q_n$ in $\mathbb{R}$. Now the left hand side here is $P(\sum q_i e_i) h$ (cf. [3]), where P denotes the quadratic representation of $\mathfrak{A}$. By [1, Chap. XI, Satz 4.1] we know that $P(a)$, for any a in $\mathfrak{A}$, preserves the cone of squares. This concludes the proof.

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On the Tensor Stability of s-Number Ideals

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Let $S_{p,q}$ denote the class of operators having approximation numbers belonging to the Lorentz sequence space $l_{p,q}$, $0<p<\infty$, $0<q\leqq\infty$. We show that these ideals are unstable under suitable tensor norms τ for p,q with $1/q<1/p+1/2$, in particular for $p=q$; i.e. there are $S_{p,q}$-operators S, T between Banach spaces such that the tensor product operator $S\hat{\bigotimes}_\tau T$ does not belong to $S_{p,q}$, answering a question of Pietsch. These operator ideals $S_{p,q}$ are, however, "almost" stable under arbitrary tensor norms, namely stable up to logarithmic factors. For Hilbert space operators, $S, T\in S_{p,q}(H,H)$ implies $S\hat{\bigotimes}_\tau T\in S_{p,q}(H\hat{\bigotimes}_\tau H,\ H\hat{\bigotimes}_\tau H)$, by contrast, if $q\geqq p$. Using this, factorization theorems and bilinear interpolation we show that $S_{p,q}$-operators are stable under tensor products, if $1/q\geqq 1/p+1$ and the operators act between dual cotype 2 and cotype 2 Banach spaces. If the cotype conditions are not satisfied, one can still conclude that the Weyl-numbers of the tensor product operator belong to $l_{p,q}$, at least for certain tensor norms.

For the standard notions of Banach space theory we refer to Lindenstrauss-Tzafriri [7]. Concerning the theory of operator ideals we refer to Pietsch [10]. In particular, we will use well-known facts about the absolutely p-summing operators (Π_p, π_p) like the factorization theorem, cf. [10] or [7]. Standard interpolation theory notations are given in Bergh and Löfström [1].

1. Tensor Norm stability of Operator Ideals

A *cross-norm* τ is a norm defined simultaneously on all algebraic tensor products $X\otimes Y$ of Banach spaces X and Y such that

$$\tau(x\otimes y)=\|x\|\,\|y\| \quad \text{for all} \quad x\in X, \quad y\in Y.$$

We denote the space $X\otimes Y$ equipped with τ by $X\bigotimes_\tau Y$ and its completion by $X\hat{\bigotimes}_\tau Y$. The algebraic tensor product $T_1\otimes T_2$ of two continuous linear operators $T_j\in L(X_j, Y_j)$, $j=1,2$ is the linear operator from $X_1\otimes X_2$ into $Y_1\otimes Y_2$ defined uniquely by

$$(T_1 \otimes T_2)\left(\sum_{i=1}^{n} x_i^1 \otimes x_i^2\right) := \sum_{i=1}^{n} T_1 x_i^1 \otimes T_2 x_i^2, \quad x_i^1 \in X_1, \quad x_i^2 \in X_2.$$

A cross-norm is called a *tensor norm* provided that all such maps are again continuous with respect to τ and

$$\|T_1 \otimes T_2 : X_1 \underset{\tau}{\otimes} X_2 \to Y_1 \underset{\tau}{\otimes} Y_2\| \leqq \|T_1\| \, \|T_2\|$$

holds. We abbreviate this by writing $\|T_1 \underset{\tau}{\otimes} T_2\| \leqq \|T_1\| \, \|T_2\|$. The map $T_1 \underset{\tau}{\otimes} T_2$

All standard norms as ε and π or the norms d_p, g_p, and w_p of Saphar [16] and

$\left\|T_1 \underset{\tau}{\hat{\otimes}} T_2\right\| \leqq \|T_1\| \, \|T_2\|$.

All standard norms as ε and π or the norms d_p g_p and w_p of Saphar [16] and Lapresté [6] are tensor norms. The property means that the class of all continuous linear operators is stable with respect to τ. The stability of other classes of operators under tensor norms has been studied e.g. by Holub [3] and Pietsch [11]. Holub showed that the p-summing operators are stable under ε with

$$\pi_p\left(T_1 \underset{\varepsilon}{\hat{\otimes}} T_2\right) \leqq \pi_p(T_1)\pi_p(T_2),$$

whereas Pietsch considered certain approximation number ideals.

Recently, estimates of this type were successfully applied by Pietsch [14] to find optimal constants in certain eigenvalue estimates. To state his simple lemma giving the connection we need two notions. First, a quasi-normed operator ideal $(\mathfrak{A}, \alpha)$ in the sense of Pietsch [10, 6.1] is called *c-stable* ($c > 0$) with respect to a tensor norm τ provided that $T_j \in \mathfrak{A}(X_j, Y_j)$ ($j = 1, 2$; X_j, Y_j Banach spaces) always implies $T_1 \underset{\tau}{\hat{\otimes}} T_2 \in \mathfrak{A}\left(X_1 \underset{\tau}{\hat{\otimes}} X_2, Y_1 \underset{\tau}{\hat{\otimes}} Y_2\right)$ with

$$\alpha\left(T_1 \underset{\tau}{\hat{\otimes}} T_2\right) \leqq c\alpha(T_1)\alpha(T_2). \tag{1}$$

Secondly, a quasi-normed operator ideal $(\mathfrak{A}, \alpha)$ is *of Riesz type* l_r iff some power $\mathfrak{A}^m$ of $\mathfrak{A}$ ($m \in \mathbb{N}$) is contained in the compact operators $\mathscr{K}$ and the eigenvalues of any $T \in \mathfrak{A}(X, X)$ are r^{th}-power summable. Denoting the eigenvalues by $(\lambda_n(T))_{n \in \mathbb{N}}$ – counted according to their multiplicity and ordered non-increasingly in absolute value – it is easily seen that there is $d > 0$ such that for any Banach space X and any $T \in \mathfrak{A}(X, X)$

$$\|(\lambda_n(T))\|_{l_r} \leqq d\alpha(T). \tag{2}$$

Lemma (Pietsch [14]). *If a quasi-normed operator ideal* $(\mathfrak{A}, \alpha)$ *is c-stable with respect to some tensor norm* τ *and of Riesz type* l_r, *one has the estimate*

$$\|(\lambda_n(T))\|_{l_r} \leqq c\alpha(T) \quad \textit{for all} \quad T \in \mathfrak{A}(X, X).$$

That is, $d \leqq c$ for the optimal constants (1) and (2). Pietsch used this very successfully for the r-summing operators. Other standard ideals of Riesz type l_r are the ideals of r^{th}-power summable approximation- or Weyl-numbers. We will study the problem of tensor stability of such s-number ideals now.

The *approximation numbers* $a_n(T)$ of a map $T \in L(X, Y)$ are given by

$$a_n(T) := \inf\{\|T - T_n\| \mid \operatorname{rank} T_n < n\},$$

the Gelfand-numbers $c_n(T)$ by

$c_n(T) := a_n(iT)$, where i is an isometric imbedding of Y into some $L_\infty(\mu)$-space, the *Weyl-numbers* $x_n(T)$ by

$$x_n(T) := \sup\{a_n(TA) \mid \|A : l_2 \to X\| \leqq 1\}, \quad n \in \mathbb{N}.$$

Clearly, these sequences are non-increasing with $x_n(T) \leqq c_n(T) \leqq a_n(T)$ and $x_1(T) = a_1(T) = \|T\|$.

Given $0 < p < \infty$, $0 < q \leqq \infty$, the Lorentz sequence space $l_{p,q}$ is defined by

$$l_{p,q} := \left\{(x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|(x_n)\|_{p,q} := \left(\sum_{n\in\mathbb{N}} x_n^{*q} n^{q/p-1}\right)^{1/q} < \infty\right\}.$$

Here (x_n^*) denotes the decreasing rearrangement of $(|x_n|)$. For $q = \infty$ the requirement is supposed to mean $\|(x_n)\|_{p,\infty} := \sup_{n\in\mathbb{N}} x_n^* n^{1/p} < \infty$. The spaces are quasi-normed by $\|\cdot\|_{p,q}$. Clearly $l_p = l_{p,p}$.

We consider the quasi normed operator ideals

$$S_{p,q} := \{T \in L \mid \sigma_{p,q}(T) := \|(a_n(T))_{n\in\mathbb{N}}\|_{p,q} < \infty\},$$

$$S_{p,q}^s := \{T \in L \mid \sigma_{p,q}^s(T) := \|(s_n(T))_{n\in\mathbb{N}}\|_{p,q} < \infty\}, \qquad s \in \{c, x\}$$

of maps with approximation (Gelfand-, Weyl-) numbers belonging to $l_{p,q}$. We let $S_p = S_{p,p}$ and $S_p^s = S_{p,p}^s$.

The spaces $l_{p,q}$ as well as the ideals $S_{p,q}$ are ordered lexicographically, as easily seen, i.e.

$$0 < p_1 < p_2 < \infty, 0 < q_1, q_2 \leqq \infty \;\Rightarrow\; l_{p_1,q_1} \subsetneqq l_{p_2,q_2}$$

$$0 < p < \infty, 0 < q_1 < q_2 \leqq \infty \;\Rightarrow\; l_{p,q_1} \subsetneqq l_{p,q_2}.$$

Pietsch [11] showed that the ideals $S_{p,q}$ are not stable under tensor norms for $p < q$ since the sequence spaces $l_{p,q}$ are not tensor-stable in that case. We consider the remaining case $p \geqq q$, in particular $p = q$. The ideals $S_{p,q}$ are, however, "almost stable" under tensor norms τ since by [11]

$$T_1 \in S_{p_1}, T_2 \in S_{p_2}, p_1 > p_2 \;\Rightarrow\; T_1 \hat{\otimes}_\tau T_2 \in S_{p_1}. \tag{3}$$

We start with a slight improvement of (3) showing that tensor stability holds up to logarithmic factors. The method is similar to the one in [11].

Proposition 1. *For any* $0 < p < \infty$ *there is* $c_p > 0$ *such that for all Banach spaces* X_1, X_2, Y_1, Y_2, *all operators* $T \in S_p(X_1, Y_1)$, $S \in S_p(X_2, Y_2)$ *and all tensor norms* τ, $T \hat{\otimes}_\tau S \in L\left(X_1 \hat{\otimes}_\tau X_2, Y_1 \hat{\otimes}_\tau Y_2\right)$ *satisfies*

$$\left(\sum_{n\in\mathbb{N}} a_n\left(T \hat{\otimes}_\tau S\right)^p \Big/ \ln(n+1)^{\max(1,p)}\right)^{1/p} \leqq c_p \sigma_p(T) \sigma_p(S)$$

Proof. In the following estimates, constants $c_1, c_2, \dots$ may depend on p but nothing else. It is well known that $T \in S_p(X_1, Y_1)$ can be written as $T = \sum_{k=0}^{\infty} T_k$ with $T_k \in L(X_1, Y_1)$, $rank(T_k) \leqq 2^k$ and $\left(\sum_{k=0}^{\infty} 2^k \|T_k\|^p\right)^{1/p} \leqq c_1 \sigma_p(T)$, cf. [12]. Similarly, $S = \sum_{l=0}^{\infty} S_l$, $\operatorname{rank}(S_l) \leqq 2^l$, $\left(\sum_{l=0}^{\infty} 2^l \|S_l\|^p\right)^{1/p} \leqq c_1 \sigma_p(S)$. Let $N \in \mathbb{N}$. Since $T \hat{\bigotimes}_\tau S = \sum_{n=0}^{\infty} \sum_{k+l=n} T_k \hat{\bigotimes}_\tau S_l$ and $U_n := \sum_{k+l \leqq N-1} T_k \hat{\bigotimes}_\tau S_l$ has $\operatorname{rank} U_n \leqq \sum_{n=0}^{N-1} (n+1) 2^n < N 2^N =: f(N)$, we get

$$a_{f(N)}\left(T \hat{\bigotimes}_\tau S\right) \leqq \left\| \sum_{n=N}^{\infty} \sum_{k+l=n} T_k \hat{\bigotimes}_\tau S_l \right\| \leqq \sum_{n=N}^{\infty} \sum_{k+l=n} \|T_k\| \, \|S_l\| .$$

Using the monotonicity of the a_n's, this yields

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} a_n\left(T \hat{\bigotimes}_\tau S\right)^p \Big/ \ln(n+1)\right)^{1/p} \\
&= \left(\sum_{N=1}^{\infty} \sum_{n=f(N)}^{f(N+1)-1} a_n\left(T \hat{\bigotimes}_\tau S\right)^p / \ln(n+1)\right)^{1/p} \\
&\leqq \left(\sum_{N=1}^{\infty} f(N+1)/(N \ln 2) \cdot a_{f(N)}\left(T \hat{\bigotimes}_\tau S\right)^p\right)^{1/p} \\
&\leqq c_2 \left(\sum_{N=1}^{\infty} 2^N a_{f(N)}\left(T \hat{\bigotimes}_\tau S\right)^p\right)^{1/p} \\
&\leqq c_2 \left(\sum_{N=1}^{\infty} 2^N \left\{\sum_{n=N}^{\infty} \left(\sum_{k+l=n} \|T_k\| \, \|S_l\|\right)\right\}^p\right)^{1/p}
\end{aligned}$$

which by the lemma in [12] can be estimated

$$\leqq c_3 \left(\sum_{N=1}^{\infty} 2^N \left(\sum_{k+l=N} \|T_k\| \, \|S_l\|\right)^p\right)^{1/p} .$$

For $p \leqq 1$, this is bounded by

$$\begin{aligned}
&\leqq c_3 \left(\sum_{N=1}^{\infty} 2^N \sum_{k+l=N} \|T_k\|^p \|S_l\|^p\right)^{1/p} \\
&\leqq c_3 \left(\sum_{k=0}^{\infty} 2^k \|T_k\|^p\right)^{1/p} \left(\sum_{l=0}^{\infty} 2^l \|S_l\|^p\right)^{1/p} \\
&\leqq c_3 c_1^2 \sigma_p(T) \sigma_p(S) .
\end{aligned}$$

For $p>1$ we estimate similarly

$$\left(\sum_{n=2}^{\infty} a_n\left(T\hat{\bigotimes_\tau} S\right)^p \Big/ \ln(n+1)^p\right)^{1/p}$$
$$\leqq c_4\left(\sum_{N=1}^{\infty}(N+1)^{1-p}2^N a_{f(N)}\left(T\hat{\bigotimes_\tau} S\right)^p\right)^{1/p}$$
$$\leqq c_5\left(\sum_{N=1}^{\infty}(N+1)^{1-p}2^N\left(\sum_{k+l=N}\|T_k\|\,\|S_l\|\right)^p\right\}^{1/p}$$
$$\leqq c_5\left\{\sum_{N=1}^{\infty}2^N\left(\sum_{k+l=N}\|T_k\|^p\,\|S_l\|^p\right)\right\}^{1/p},$$

where the last estimate follows by Hölder's inequality. This, as before is bounded by

$$\leqq c_5 c_1^2 \sigma_p(T)\sigma_p(S). \quad \square$$

Corollary 1. *For any two operators $T\in L(X_1,Y_1)$, $S\in L(X_2,Y_2)$ of rank $(T)\leqq n$, rank $(S)\leqq n$ and any tensor norm τ, one has*

$$\sigma_p\left(T\hat{\bigotimes_\tau} S\right)\leqq c_p \ln(n+1)^{\max(1,1/p)}\sigma_p(T)\sigma_p(S).$$

Logarithmic factors of this sort are actually necessary for the ideals S_p, or more generally for $S_{p,q}$ with $1/q<1/p+1/2$: The following example shows that these ideals are unstable under certain tensor norms, answering a question in [11] negatively.

Proposition 2. *Let $0<p<\infty$, $0<q<\infty$ with $1/q<1/p+1/2$. Then there are diagonal operators $T:l_2\to l_1$ in $S_{p,q}(l_2,l_1)$ such that $T\hat{\bigotimes_\varepsilon} T$ is not in $S_{p,q}\left(l_2\hat{\bigotimes_\varepsilon} l_2, l_1\hat{\bigotimes_\varepsilon} l_1\right)$.*

Proof. Let $1/r:=1/p+1/2$. Given a decreasing positive sequence $\sigma\in l_2$, the diagonal operator $D_\sigma: l_2\to l_1$, $(x_n)\to(\sigma_n x_n)$ has approximation numbers $a_n(D_\sigma)=\left(\sum_{j=n}^{\infty}\sigma_j^2\right)^{1/2}$, cf. Pietsch [10, 11.11].

Calculation shows that $\left\|\left(\left(\sum_{j=n}^{\infty}\sigma_j^2\right)^{1/2}\right)_{n\in\mathbb{N}}\right\|_{p,q}$ is equivalent to $\|\sigma\|_{r,q}$ and thus $D_\sigma\in S_{p,q}(l_2,l_1)$ iff $\sigma\in l_{r,q}$, $0<q\leqq\infty$; we actually only need the easy part $\sigma_{p,q}(D_\sigma)\leqq c_{p,q}\|\sigma\|_{r,q}$ of this equivalence. Thus let $\sigma\in l_{r,q}$. To estimate the approximation numbers $a_n\left(D_\sigma\hat{\bigotimes_\varepsilon} D_\sigma\right)$ from below, we first show that the formal identity map $J: l_1\hat{\bigotimes_\varepsilon} l_1\to l_2(\mathbb{N}^2)$ is 2-summing: Let hs denote the Hilbert tensor product, thus for measure spaces (K,μ) we have $L_2(K,\mu)\hat{\bigotimes_{hs}} L_2(K,\mu)=L_2(K\times K,\mu\times\mu)$, in particular $l_2\hat{\bigotimes_{hs}} l_2=l_2(\mathbb{N}^2)$. Since the identity map $I: l_1\to l_2$ is 2-summing with $\pi_2(I)\leqq\sqrt{2}$ (cf. [10]), there is a factorization $I=BjA$ with $A: l_1\to C(K)$, $j: C(K)$

$\to L_2(K,\mu)$, $B: L_2(K,\mu)\to l_2$ and $\|B\|\,\|A\|\leqq\sqrt{2}$. Here (K,μ) is an appropriate probability measure space and j the formal identity map. Tensoring these factorizations, we have

$$J: l_1 \hat{\bigotimes}_\varepsilon l_1 \xrightarrow{A\hat{\otimes}_\varepsilon A} C(K)\hat{\bigotimes}_\varepsilon C(K) = C(K\times K) \xrightarrow{\text{id}} L_2(K\times K, \mu\times\mu)$$
$$= L_2(K,\mu)\hat{\bigotimes}_{hs} L_2(K,\mu) \xrightarrow{B\hat{\otimes}_{hs} B} l_2 \hat{\bigotimes}_{hs} l_2 = l_2(\mathbb{N}^2),$$

where $\pi_2(\text{Id})=1$. Thus J is 2-summing with $\pi_2(J)\leqq 2$. This basically is Holub's argument [3]. Given $\delta>0$, choose $T_n: l_2\hat{\bigotimes}_\varepsilon l_2 \to l_1 \hat{\bigotimes}_\varepsilon l_1$ of rank $(T_n)<n$ such that

$$\delta + a_n\left(D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma\right) \geqq \left\|D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma - T_n : l_2 \hat{\bigotimes}_\varepsilon l_2 \to l_1 \hat{\bigotimes}_\varepsilon l_1\right\|$$
$$\geqq 1/2\pi_2\left(J\left(D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma - T_n\right): l_2\hat{\bigotimes}_\varepsilon l_2 \to l_2(\mathbb{N}^2)\right)$$
$$\geqq 1/2\pi_2\left(J\left(D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma - T_n\right)\text{Id}: l_2(\mathbb{N}^2)\to l_2(\mathbb{N}^2)\right)$$

using $\pi_2(J)\leqq 2$ and $\left\|\text{Id}: l_2(\mathbb{N})\to l_2\hat{\bigotimes}_\varepsilon l_2\right\|\leqq 1$. For any Hilbert space operator $U: H \to H$, however, the π_2- and σ_2-norms coincide [10]. Thus, on $l_2(\mathbb{N}^2)$,

$$\pi_2\left(J\left(D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma - T_n\right)\text{Id}\right)$$
$$= \left(\sum_{j\in\mathbb{N}} a_j\left(J\left(D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma - T_n\right)\text{Id}\right)^2\right)^{1/2}$$
$$\geqq \sqrt{n}\, a_n\left(J\left(D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma - T_n\right)\text{Id}\right)$$
$$\geqq \sqrt{n}\, a_{2n}\left(D_\sigma \hat{\bigotimes}_{hs} D_\sigma : l_2(\mathbb{N}^2)\to l_2(\mathbb{N}^2)\right)$$

using the monotonicity of the a_n's as well as

$$a_{2n}(D_\sigma \hat{\otimes} D_\sigma) \leqq a_n(D_\sigma\hat{\otimes} D_\sigma - T_n) + a_n(T_n) = a_n(D_\sigma \hat{\otimes} D_\sigma - T_n).$$

These estimates yield

$$\left\|a_n\left(D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma : l_2 \hat{\bigotimes}_\varepsilon l_2 \to l_1 \hat{\bigotimes}_\varepsilon l_1\right)\right\|_{p,q}$$
$$\geqq 1/2\left(\sum_{n=1}^{\infty} a_{2n}\left(D_\sigma \hat{\bigotimes}_{hs} D_\sigma\right)^q n^{q/2+q/p-1}\right)^{1/q}$$
$$\geqq 2^{-(1+1/r+1/q)}\left\|\left(a_n\left(D_\sigma \hat{\bigotimes}_{hs} D_\sigma\right): l_2(\mathbb{N}^2)\to l_2(\mathbb{N}^2)\right)\right\|_{r,q}$$
$$= 2^{-(1+1/r+1/q)}\|\sigma\otimes\sigma\|_{r,q};$$

the last equality holds since the operator $D_\sigma \hat{\bigotimes}_{hs} D_\sigma$ on $l_2(\mathbb{N}^2)$ is the diagonal operator induced by the double-indexed sequence $\sigma\otimes\sigma = (\sigma_i\sigma_j)_{i,j\in\mathbb{N}}$. Hence if

$D_\sigma \in S_{p,q}$ would imply $D_\sigma \hat{\bigotimes}_\varepsilon D_\sigma \in S_{p,q}$ we would get

$$\|\sigma \otimes \sigma\|_{r,q} \leqq c \|\sigma\|_{r,q}{}^2$$

with some c depending only on p, q and r. This is false for $r<q$, i.e. $1/q<1/p+1/2$ and correct for $r \geqq q$, cf. [11]. The essential reason for the last statement becomes clear looking at the special sequence $\sigma = (n^{-1/r})_{n \in \mathbb{N}}$ in $l_{r,\infty}$: Since for large $n \in \mathbb{N}$

$$\{(i,j) \in \mathbb{N}^2 \mid i \cdot j \leqq n\}$$

has cardinality proportional to $n \ln n$, the decreasing rearrangement of $\sigma \otimes \sigma$ satisfies $(\sigma \otimes \sigma)^*(n \ln n) \asymp n^{-1/r}$ and thus

$$\|\sigma \otimes \sigma\|_{r,\infty} = \sup_{m \in \mathbb{N}} (\sigma \otimes \sigma)^*(m) m^{1/r} = \sup_{n \in \mathbb{N}} (\ln n)^{1/r} = \infty . \quad \square$$

Remarks. i) The argument also works for any tensor norm τ stronger than ε such that $\mathrm{Id}: l_2 \hat{\bigotimes}_{hs} l_2 \to l_2 \hat{\bigotimes}_\tau l_2$ is continuous.

ii) Probably the proposition can be strengthened that there are operators in $S_{p,q}$ such that their ε-tensor products are not in $S_{p,q}$ even for $1/q < 1/p + 1$ instead of only $1/q < 1/p + 1/2$, maybe for operators from l_∞ to l_1; these estimates, however, do not yield this. Let $e_n(T)$ denote the entropy numbers of some $T \in L(X, Y)$, cf. [2] or [10], and for

$$0 < p < \infty : S^e_{p,q}(X, Y) := \{T \in L(X, Y) \mid \sigma^e_p(T) := \|e_n(T)\|_p < \infty\} .$$

Whereas the approximation number ideals S_p are "almost" stable under any tensor norm τ, the other well-known s-number ideals like S^x_p, S^c_p or S^e_p are remarkably unstable under some tensor norms, e.g. under π:

Lemma 1. *Let $0<p<2$, $0<q<\infty$ and suppose that $T, S \in S^s_p(l_1, l_2)$ always implies $T \hat{\bigotimes}_\pi S \in S^s_q\left(l_1 \hat{\bigotimes}_\pi l_1, l_2 \hat{\bigotimes}_\pi l_2\right)$ for either $s = x$, c or e. Then q necessarily satisfies $1/q \leqq 1/p - 1/4$.*

Proof. The assumption implies that there is $c_1 > 0$ such that for all T, $S \in S^s_p(l_1, l_2)$

$$\sigma^s_q\left(T \hat{\bigotimes}_\pi S\right) \leqq c_1 \sigma^s_p(T) \sigma^s_p(S) , \qquad s \in \{x, c, e\} .$$

By Carl [2] there is $c_2 > 0$ such that for all $R \in S^c_q(X, Y)$

$$\sigma^e_q(R) \leqq c_2 \sigma^c_q(R) .$$

Let $T = S = \mathrm{Id}_n : l^n_1 \to l^n_2$. By Kashin [5] and Mityagin [8],

$$c_j(\mathrm{Id}_n : l^n_1 \to l^n_2) \leqq c_3 \ln(n+1)^{3/2} / \sqrt{j} .$$

Thus for $s \in \{c, e\}$

$$\sigma^s_q\left(\mathrm{Id} : l^n_1 \hat{\bigotimes}_\pi l^n_1 \to l^n_2 \hat{\bigotimes}_\pi l^n_2\right) \leqq c_1 \sigma^s_p(T) \sigma^s_p(S) \leqq c_4 \sigma^c_p(T) \sigma^c_p(S) \leqq c_5 \ln(n+1)^3 n^{2/p-1} .$$

Thus, in these cases, it suffices to estimate $e_j\left(T \hat{\bigotimes_\pi} S\right)$ from below. Standard volume estimates yield for $j=1, \ldots, n^2$

$$e_j\left(\mathrm{Id}: l_1^{n^2} \to l_2^n \hat{\bigotimes_\pi} l_2^n\right) \geqq c_6\left(\mathrm{vol}\, B(l_1^{n^2})/\mathrm{vol}\, B\left(l_2^n \hat{\bigotimes_\pi} l_2^n\right)\right)^{1/n^2},$$

where $\mathrm{vol}\, B(X)$ denotes the volume of the unit ball of X. By Schütt [17], the right side is of order $n^{-1/2}$. Thus

$$c_7 n^{2/q-1/2} \leqq \sigma_q^e\left(\mathrm{Id}: l_1^{n^2} \to l_2^n \hat{\bigotimes_\pi} l_2^n\right) \leqq c_5 \ln(n+1)^3 n^{2/p-1},$$

yielding $1/q \leqq 1/p - 1/4$. In the case of Weyl-numbers $s=x$ the estimate is even more direct since

$$x_j\left(\mathrm{Id}: l_1^{n^2} \to l_2^n \hat{\bigotimes_\pi} l_2^n\right) \geqq x_j(\mathrm{Id}: l_1^{n^2} \to l_2^{n^2}) \geqq 1/\sqrt{j}$$

implies $\sigma_q^x\left(\mathrm{Id}: l_1^{n^2} \to l_2 \hat{\bigotimes_\pi} l_2\right) \geqq c_8 n^{2/q-1/2}$ as well. □

Remark. The lemma also holds for the Kolmogorov-number ideals, if operators from l_2 to l_∞ are considered.

2. Cotype 2 and Tensor Stability

The example of proposition 2 does not exclude the possibility that the ideals $S_{p,q}$ with $1/q \geqq 1/p + 1/2$ or $1/q \geqq 1/p + 1$ are stable under tensor norms. We prove a positive result of this type under restrictive assumptions on the Banach spaces considered.

Theorem 1. *Let $0 < p < \infty$ and $0 < q < \infty$ with $1/q \geqq 1/p + 1$. Let X_i, Y_i, $i=1,2$ be Banach spaces such that X_i^* and Y_i are of cotype 2, $i=1,2$. Let τ be any tensor norm. Then $T \in S_{p,q}(X_1, Y_1)$, $S \in S_{p,q}(X_2, Y_2)$ imply $T \hat{\bigotimes_\tau} S \in S_{p,q}\left(X_1 \hat{\bigotimes_\tau} X_2, Y_1 \hat{\bigotimes_\tau} Y_2\right)$ with*

$$\sigma_{p,q}\left(T \hat{\bigotimes_\tau} S\right) \leqq c\,\sigma_{p,q}(T)\,\sigma_{p,q}(S),$$

where c depends only on p, q and the cotype 2-constants of the spaces X_i^* and Y_i.

Recall that a Banach space X is of cotype 2 iff there is $c > 0$ such that for all $x_1, \ldots, x_n \in X$

$$\left(\sum_{i=1}^n \|x_i\|^2\right)^{1/2} \leqq c\left(\underset{\pm}{\mathrm{Average}} \left\|\sum_{i=1}^n \pm x_i\right\|^2\right)^{1/2};$$

The best constant c here is called the cotype 2-constant of X, $K_2(X)$. The proof of Theorem 1 will be reduced to the Hilbert space case using

Proposition 3. *Let $0 < q \leqq p < \infty$ and H be a Hilbert space. Then $S, T \in S_{p,q}(H, H)$ implies for any tensor norm τ that*

$$T \hat{\bigotimes_\tau} S \in S_{p,q}\left(H \hat{\bigotimes_\tau} H, H \hat{\bigotimes_\tau} H\right)$$

with

$$\sigma_{p,q}\left(T \hat{\bigotimes_\tau} S\right) \leqq c_{p,q}\sigma_{p,q}(T)\sigma_{p,q}(S).$$

Proof. (i) It suffices to prove the proposition for diagonal operators in l_2: This is because of the polar decomposition theorem for arbitrary $S, T \in S_{p,q}(H, H)$: there are partial isometries $U_S, U_T: H \to l_2$, $V_S, V_T: l_2 \to H$ and diagonal maps $D_\sigma, D_\mu: l_2 \to l_2$, σ, μ decreasing, with $S = V_S D_\sigma U_S$, $T = V_T D_\mu U_T$. Here $\sigma = (a_n(S))_{n\in\mathbb{N}} \in l_{p,q}$ and $\mu \in l_{p,q}$, cf. [10, Chap. 11.3]. Thus if $D_\sigma, D_\mu \in S_{p,q}(l_2, l_2)$ implies $D_\sigma \hat{\bigotimes_\tau} D_\mu \in S_{p,q}(l_2, l_2)$ and

$$\sigma_{p,q}\left(D_\sigma \hat{\bigotimes_\tau} D_\mu\right) \leqq c_{p,q}\sigma_{p,q}(D_\sigma)\sigma_{p,q}(D_\mu),$$

$$S \hat{\bigotimes_\tau} T = \left(V_S \hat{\bigotimes_\tau} V_T\right)\cdot\left(D_\sigma \hat{\bigotimes_\tau} D_\mu\right)\cdot\left(U_S \hat{\bigotimes_\tau} U_T\right) \in S_{p,q}\left(H \hat{\bigotimes_\tau} H, H \hat{\bigotimes_\tau} H\right)$$

with

$$\sigma_{p,q}\left(S \hat{\bigotimes_\tau} T\right) \leqq \sigma_{p,q}\left(D_\sigma \hat{\bigotimes_\tau} D_\mu\right)$$
$$\leqq c_{p,q}\sigma_{p,q}(D_\sigma)\sigma_{p,q}(D_\mu) = c_{p,q}\sigma_{p,q}(S)\sigma_{p,q}(T).$$

(ii) We now prove the claim for diagonal maps $S = D_\sigma$, $T = D_\mu$ in l_2 with σ, $\mu \in l_{p,q}$ positive and decreasing. We may further assume $\sigma_1, \mu_1 \leqq 1$. For any $n \in \mathbb{N} \cup \{0\}$ let

$$Z_n(S) := \left\{j \in \mathbb{N} \mid 2^{-\frac{n+1}{p}} < a_j(S) \leqq 2^{-\frac{n}{p}}\right\}, \quad M_n(S) := \# Z_n(S)$$

$$Z_n(T) := \left\{j \in \mathbb{N} \mid 2^{-\frac{n+1}{p}} < a_j(T) \leqq 2^{-\frac{n}{p}}\right\}, \quad M_n(T) := \# Z_n(T).$$

Here $a_j(S) = \sigma_j$, $a_j(T) = \mu_j$, of course. Define sequences

$$\sigma^n = (\sigma_j^n), \qquad \mu^n = (\mu_j^n)$$

where

$$\sigma_j^n = \begin{cases} \sigma_j & j \in Z_n(S) \\ 0 & j \notin Z_n(S) \end{cases} \qquad \mu_j^n = \begin{cases} \mu_j & j \in Z_n(T) \\ 0 & j \notin Z_n(T) \end{cases}$$

and consider the diagonal operators $S_n := D_{\sigma^n}$, $T_n := D_{\mu^n}$ in l_2. Then clearly $S = \sum_{n=0}^{\infty} S_n$, $T = \sum_{n=0}^{\infty} T_n$ with rank $(S_n) = M_n(S)$, rank$(T_n) = M_n(T)$. In view of σ, $\mu \in l_{p,q}$, these series as well as $S \hat{\bigotimes_\tau} T = \sum_{k,l\in\mathbb{N}\cup\{0\}} S_k \hat{\bigotimes_\tau} T_l$ converge in the operator norm. We now approximate $S \hat{\bigotimes_\tau} T$ by

$$U_N = \sum_{0 \leqq k+l \leqq N} S_k \hat{\bigotimes_\tau} T_1, \qquad N \in \mathbb{N} \cup \{0\}.$$

Let

$$f_n := \sum_{0 \leqq k+l \leqq N} M_k(S)M_l(T).$$

Then rank $U_N \leqq f_N$, hence

$$a_{f_N+1}\left(S \hat{\underset{\tau}{\otimes}} T\right) \leqq \left\| S \hat{\underset{\tau}{\otimes}} T - U_N \right\| \leqq \sum_{n=N}^{\infty} \left\| \sum_{k+l=n} S_k \hat{\underset{\tau}{\otimes}} T_l \right\|$$

We will show below in part (iii) that

$$\left\| \sum_{k+l=n} S_k \hat{\underset{\tau}{\otimes}} T_l \right\| = \max_{k+l=n} \|S_k\| \, \|T_l\| \tag{4}$$

holds, which by definition of S_k, T_l, and $Z_n(S)$, $Z_n(T)$ can be estimated by $\max\limits_{k+l=n} 2^{-k/p} \cdot 2^{-l/p} = 2^{-n/p}$. Thus

$$a_{f_N+1}\left(S \hat{\underset{\tau}{\otimes}} T\right) \leqq \sum_{n=N}^{\infty} 2^{-n/p} \leqq c_1 2^{-N/p}; \tag{5}$$

constants $c_1, c_2 \ldots$ depend again only on p and q.

We now show that $S \hat{\underset{\tau}{\otimes}} T$ is in $S_{p,q}$: By (5)

$$\begin{aligned}
\sigma_{p,q}\left(S \hat{\underset{\tau}{\otimes}} T\right) &= \left(\sum_{j \in \mathbb{N}} a_j\left(S \hat{\underset{\tau}{\otimes}} T\right)^q j^{q/p-1}\right)^{1/q} \\
&= \left(\sum_{N=0}^{\infty} \sum_{j=f_N+1}^{f_{N+1}} a_j\left(S \hat{\underset{\tau}{\otimes}} T\right)^q j^{q/p-1}\right)^{1/q} \\
&\leqq \left(\sum_{N=0}^{\infty} a_{f_N}\left(S \hat{\underset{\tau}{\otimes}} T\right)^q \sum_{j=f_N+1}^{f_{N+1}} j^{q/p-1}\right)^{1/q} \\
&\leqq c_2\left(\sum_{N=0}^{\infty} 2^{-Nq/p} \cdot f_{N+1}{}^{q/p}\right)^{1/q} \\
&\leqq c_3\left(\sum_{n=0}^{\infty} (f_n/2^n)^{q/p}\right)^{1/q}
\end{aligned}$$

with $f_0 := 0$, using the monotonicity of the approximation numbers.
Thus

$$\begin{aligned}
\sigma_{p,q}\left(S \hat{\underset{\tau}{\otimes}} T\right) &\leqq c_3\left(\sum_{n=0}^{\infty}\left(\sum_{k+l \leqq n} M_k(S) M_l(T)\right)^{q/p} 2^{-nq/p}\right)^{1/q} \\
&\leqq c_3\left(\sum_{n=0}^{\infty}\left(\sum_{k+l \leqq n} M_k(S)^{q/p} M_l(T)^{q/p}\right) 2^{-nq/p}\right)^{1/q} \qquad (q \leqq p) \\
&= c_3\left(\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left(\sum_{k+l=m} M_k(S)^{q/p} M_l(T)^{q/p}\right) 2^{-nq/p}\right)^{1/q} \\
&\leqq c_4\left(\sum_{m=0}^{\infty} 2^{-mq/p} \sum_{k+l=m} M_k(S)^{q/p} M_l(T)^{q/p}\right)^{1/q} \\
&= c_4\left(\sum_{k=0}^{\infty} 2^{-kq/p} M_k(S)^{q/p}\right)^{1/q}\left(\sum_{l=0}^{\infty} 2^{-lq/p} M_l(T)^{q/p}\right)^{1/q}
\end{aligned}$$

with $c_4 = c_3\left(\sum\limits_{n=0}^{\infty} 2^{-nq/p}\right)^{1/q}$. We now show that the last two terms are basically the $\sigma_{p,q}$-norms of S and T. Since

$$\left(\sum_{j \in Z_k(S)} j^{q/p-1}\right) \underset{\cap}{\cup} \frac{p}{q}\left(\left(\sum_{j=0}^{k} M_j(S)\right)^{q/p} - \left(\sum_{j=0}^{k-1} M_j(S)\right)^{q/p}\right),$$

we find, putting $l=k-1$ in the calculation,

$$\left(\sum_{k=0}^{\infty} 2^{-kq/p}\mathrm{M}_k(\mathrm{S})^{q/p}\right)^{1/q} \leqq \left(\sum_{k=0}^{\infty} 2^{-kq/p}\left(\sum_{j=0}^{k} \mathrm{M}_j(\mathrm{S})\right)^{q/p}\right)^{1/q}$$

$$= \left(2\sum_{k=0}^{\infty} 2^{-kq/p}\left(\sum_{j=0}^{k} \mathrm{M}_j(\mathrm{S})\right)^{q/p} - \sum_{l=0}^{\infty} 2^{-lq/p}\left(\sum_{j=0}^{l} M_j(S)\right)^{q/p}\right)^{1/q}$$

$$= \left[(2-2^{q/p})\left(\sum_{k=0}^{\infty} 2^{-kq/p}\left(\sum_{j=0}^{k} M_j(S)\right)^{q/p}\right)\right.$$

$$\left. + 2^{q/p}\left(\sum_{k=0}^{\infty} 2^{-kq/p}\left\{\left(\sum_{j=0}^{k} M_j(S)\right)^{q/p} - \left(\sum_{j=0}^{k-1} M_j(S)\right)^{q/p}\right\}\right)\right]^{1/q}.$$

This yields with $c_5=(2^{q/p}/(2^{q/p}-1))^{1/q}$

$$\left(\sum_{k=0}^{\infty} 2^{-kq/p}M_k(S)^{q/p}\right)^{1/q}$$

$$\leqq c_5\left(\sum_{k=0}^{\infty} 2^{-kq/p}\left(\sum_{j\in Z_k(S)} j^{q/p-1}\right)\right)^{1/q}$$

$$\leqq c_6\left(\sum_{k=0}^{\infty} \sum_{j\in Z_k(S)} a_j(S)^q j^{q/p-1}\right)^{1/q} = c_6\sigma_{p,q}(S)<\infty\,.$$

For $p=q$, this argument can be simplified, of course. Thus $\sigma_{p,q}\left(S\hat{\bigotimes_\tau} T\right) \leqq c_{p,q}\sigma_{p,q}(S)\sigma_{p,q}(T)$.

(iii) It remains to prove (4). Let τ be any tensor norm on $l_2\hat{\bigotimes_\tau} l_2$. Any element $A\in l_2\hat{\bigotimes_\tau} l_2$ can be identified with some operator $\tilde{A}\in L(l_2,l_2)$. The set $\tilde{\tau}(l_2,l_2)$ of these operators forms an operator ideal on Hilbert spaces; the ideal property comes from the tensor norm property. Thus, by Pietsch [10, Chap. 15.3], there is a symmetric norm $|||\cdot|||$ determined by τ such that

$$\|A\|_{l_2\hat{\otimes}_\tau l_2} = |||(a_n(\tilde{A}))_{n\in\mathbb{N}}|||\,.$$

Let P_k (and Q_k) be mutually orthogonal projections in l_2, $k=0,\ldots,n$. Since the approximation numbers of some $\tilde{A}$ are just the roots of the eigenvalues of $\tilde{A}^*\tilde{A}$, the approximation number sequence of $\sum_{k=0}^{n} P_kAQ_k$ is just the decreasing rearrangement of all approximation numbers of all P_kAQ_k-maps. Thus, identifying A and $\tilde{A}$ as above, we find for the operators S_k and T_l, which are mutually orthogonally supported,

$$\left\|\sum_{k=0}^{n} S_k \hat{\bigotimes_\tau} T_{n-k}\right\|_{l_2\hat{\otimes}_\tau l_2\to l_2\hat{\otimes}_\tau l_2}$$

$$= \sup_{0\neq A\in l_2\hat{\otimes}_\tau l_2} \frac{\left\|\sum_{k=0}^{n} T_{n-k}\tilde{A}S_k\right\|_{\tilde{\tau}(l_2,l_2)}}{\|\tilde{A}\|_{\tilde{\tau}(l_2,l_2)}}$$

$$= \sup_{0\neq A\in l_2\hat{\otimes}_\tau l_2} \frac{|||(a_1(T_n\tilde{A}S_0),\ldots,a_1(T_{n-1}\tilde{A}S_1),\ldots,a_1(T_0\tilde{A}S_n)\ldots)|||}{\|\tilde{A}\|_{\tilde{\tau}(l_2,l_2)}}$$

The sequences of approximation numbers of the single operators $T_{n-k}\tilde{A}S_k$ satisfy, however,

$$a_j(T_{n-k}\tilde{A}S_k) \leqq \|T_{n-k}\| \, \|S_k\| a_j(P_{n-k}\tilde{A}Q_k)$$

where P_k (or Q_k) are the orthogonal projections onto the vectors with support $Z_k(T)$ (or $Z_k(S)$). Since $|||\cdot|||$ is monotone and since

$$|||(a_1(P_n\tilde{A}Q_0), \ldots, a_1(P_{n-1}\tilde{A}Q_1), \ldots, a_1(P_0\tilde{A}Q_n)\ldots)|||$$
$$\leqq |||(a_n(\tilde{A}))_{n\in\mathbb{N}}||| = \|\tilde{A}\|_{\tilde{\tau}(l_2,l_2)}$$

by Theorem 1.19 of Simon [18], the proof of which also works for $\sum_{k=0}^{n} P_k A Q_{n-k}$ instead of $\sum_{k=0}^{n} P_k A P_k$, we get

$$\left\| \sum_{k=0}^{n} S_k \hat{\bigotimes_{\tau}} T_{n-k} \right\|_{l_2 \hat{\otimes}_{\tau} l_2 \to l_2 \hat{\otimes}_{\tau} l_2} \leqq \max_{0\leqq k\leqq n} \|S_k\| \, \|T_{n-k}\| \, .$$

Clearly, here we have even equality. This proves (4) and thus Proposition 3. □

The crucial property (4) of the last proof can be verified for the tensor norm τ also in a Banach space situation:

Lemma 2. *Let X_1, X_2 be Banach spaces, $S \in L(l_2, X_1)$, $T \in L(l_2, X_2)$, P_k and Q_k be mutually orthogonal projections in l_2 $(k=0, \ldots n)$. Then*

$$\left\| \sum_{k=0}^{n} SP_k \hat{\bigotimes_{\pi}} TQ_k \right\|_{l_2 \hat{\otimes}_{\tau} l_2 \to X_1 \hat{\otimes}_{\tau} X_2} \leqq \sup_{0\leqq k\leqq n} \|SP_k\| \, \|TQ_k\| \, .$$

Proof. Since the extreme points of the unit ball of $l_2 \hat{\bigotimes}_{\pi} l_2$ have the form $x \otimes y$ with $\|x\|_2 = \|y\|_2 = 1$, we find

$$\begin{aligned}
\left\| \sum_{k=0}^{n} SP_k \hat{\bigotimes_{\pi}} TQ_k \right\| &= \sup_{\|x\|_2=\|y\|_2=1} \left\| \sum_{k=0}^{n} SP_k x \otimes TQ_k y \right\|_{X_1 \hat{\otimes}_{\pi} X_2} \\
&\leqq \sup_{\|x\|_2=\|y\|_2=1} \sum_{k=0}^{n} \|SP_k x\| \, \|TQ_k y\| \\
&\leqq \left(\max_{0\leqq k\leqq n} \|SP_k\| \, \|TQ_k\| \right) \\
&\quad \cdot \sup_{\|x\|_2=\|y\|_2=1} \left(\sum_{k=0}^{n} \|P_k x\|^2 \right)^{1/2} \left(\sum_{k=0}^{n} \|Q_k y\|^2 \right)^{1/2} \\
&\leqq \max_{0\leqq k\leqq n} \|SP_k\| \, \|TQ_k\| \, .
\end{aligned}$$

As a consequence of Lemma 2 and the proof of Proposition 3 we get

Corollary 2. *Let $0 < q \leqq p < \infty$, X_1, X_2 be Banach spaces, $S \in S_{p,q}(l_2, X_1)$ and $T \in S_{p,q}(l_2, X_2)$. Then $S \hat{\bigotimes}_{\pi} T \in S_{p,q}\left(l_2 \hat{\bigotimes}_{\pi} l_2, X_1 \hat{\bigotimes}_{\pi} X_2\right)$ with*

$$\sigma_{p,q}\left(S \hat{\bigotimes_{\pi}} T\right) \leqq c_{p,q} \sigma_{p,q}(S) \sigma_{p,q}(T) \, .$$

Proof. It is well-known that for any $S: l_2 \to X$ there is an orthonormal system $(e_j)_{j\in\mathbb{N}}$ such that $a_j(S) = \|Se_j\|$, cf. [13]. Let $Z_n(S)$ be as in (ii) of the proof of Proposition 3. Let P_n be the orthogonal projection onto $[e_j]_{j\in Z_n(S)}$ and define $S_n := SP_n$. Similarly define T_n. Replacing (4) by Lemma 2, the same arguments as in (ii) of the proof of Proposition 3 apply. □

Remark. As Proposition 2 showed, Corollary 2 does not hold for the ε-tensor-norm (for $p=q$, $X_1 = X_2 = l_1$).

To prove Theorem 1, we need the bilinear interpolation theorem of Karadčov

Theorem [4]. *Let $\bar{X} = (X_0, X_1)$, $\bar{Y} = (Y_0, Y_1)$, $\bar{Z} = (Z_0, Z_1)$ be couples of quasi-normed spaces, Z_i be r_i-normed ($i = 0, 1$; $0 < r_0$, $r_1 \leqq 1$). Let $0 < q_0$, $q_1 \leqq \infty$, $0 < \theta < 1$, and $1/r = (1-\theta)/r_0 + \theta/r_1$. Let T be a continuous bilinear operator mapping $T: X_i \times Y_i \to Z_i$ ($i = 0, 1$). Then T defines a continuous bilinear map*

$$T: \bar{X}_{\theta,q_0} \times \bar{Y}_{\theta,q_1} \to \bar{Z}_{\theta,q}$$

where $1/q := 1/q_0 + 1/q_1 - 1/r$ if $q_0, q_1 \geqq r$ and $q := \max(q_0, q_1)$ if $q_0 < r$ or $q_1 < r$.

For the notions and the theory of real interpolation spaces we refer to Bergh and Löfström [1]. Since [4] does not provide a proof, we sketch a proof for the convenience of the reader.

Proof. Assume $\|T: X_i \times Y_i \to Z_i\| \leqq 1$ ($i = 0, 1$). We show that there is $c > 0$ depending only on the r_i, q_i, θ such that

$$\|T(x,y)\|_{\bar{Z}_{\theta,q}} \leqq c\|x\|_{\bar{X}_{\theta,q_0}} \|y\|_{\bar{Y}_{\theta,q_1}}; \qquad x \in \bar{X}_{\theta,q_0}, \qquad y \in \bar{Y}_{\theta,q_1}.$$

If $(A, \|\cdot\|)$ is a quasinormed space, we denote by A^s the space A equipped with $\|\cdot\|^s$ ($s > 0$). Let $\eta := r\theta/r_1$ (< 1). Then $r = (1-\eta)r_0 + \eta r_1$. By the power theorem [1] we thus have

$$\bar{X}_{\theta,q_0}{}^r = (X_0^{r_0}, X_1^{r_1})_{\eta,q_0/r}, \qquad \bar{Y}_{\theta,q_1}{}^r = (Y_0^{r_0}, Y_1^{r_1})_{\eta,q_1/r}$$

and $\bar{Z}_{\theta,q}{}^r = (Z_0^{r_0}, Z_1^{r_1})_{\eta,q/r}$. Using the J-method, we may thus assume decompositions $x = \sum_{k\in\mathbb{Z}} x_k$, $y = \sum_{l\in\mathbb{Z}} y_l$ of x in $X_0 + X_1$ and y in $Y_0 + Y_1$ respectively with

$$\|(\lambda_k(x))_{k\in\mathbb{Z}}\|_{q_0/r} \leqq c_1 \|x\|_{\bar{X}_{\theta,q_0}}, \qquad \|(\mu_l(y))_{l\in\mathbb{Z}}\|_{q_1/r} \leqq c_1 \|y\|_{\bar{Y}_{\theta,q_1}}$$

where

$$\lambda_k(x) := 2^{-\eta k} J(2^k, x_k; X_0^{r_0}, X_1^{r_1})$$

and

$$\mu_l(y) := 2^{-\eta l} J(2^l, y_l; Y_0^{r_0}, Y_1^{r_1}) \qquad (k, l \in \mathbb{Z}).$$

In $Z_0 + Z_1$, $T(x,y) = \sum_{n\in\mathbb{Z}} T_n(x,y)$ with $T_n(x,y) := \sum_{k+l=n} T(x_k, y_l)$. Elementary estimates show

$$\begin{aligned} &J(2^n, T_n(x,y); Z_0^{r_0}, Z_1^{r_1}) \\ &\leqq \sum_{k+l=n} J(2^k, x_k; X_0^{r_0}, X_1^{r_1}) J(2^l, y_l; Y_0^{r_0}, Y_1^{r_1}) \end{aligned}$$

and thus again by the power theorem

$$\begin{aligned} \|T(x,y)\|_{\bar{Z}_{\theta,p}}^r &\leqq c_2 \|(2^{-\eta n} J(2^n, T_n(x,y); Z_0^{r_0}, Z_1^{r_1})\|_{q/r} \\ &\leqq c_2 \|\lambda(x) * \mu(y)\|_{q/r}. \end{aligned}$$

By Young's inequality this is bounded by

$$\leqq c_2 \|(\lambda_k(x))_{k\in\mathbb{Z}}\|_{q_0/r} \|(\mu_l(y))_{l\in\mathbb{Z}}\|_{q_1/r}$$

if $q_0/r \geqq 1$, $q_1/r \geqq 1$. In the case $q_0 < r$ or $q_1 < r$ this holds as well with $q = \max(q_0, q_1)$. Thus

$$\|T(x,y)\|^r_{Z_{\theta,q}} \leqq c_2 c_1^2 \|x\|^r_{X_{\theta,q}} \; \|y\|^r_{Y_{\theta,q}}$$

Proof of Theorem 1

a) Let $0<q<1/2$ and $1/r := 1/q - 2$. We first show that any $T \in S_{p,q}(X_1, Y_1)$ factors as $T = SRQ$ with $Q \in L(X_1, l_2)$, $R \in S_{p,r}(l_2, l_2)$ and $S \in L(l_2, Y_1)$. To see this, we decompose T as $T = \sum_{n\in\mathbb{N}} T_n$, $\operatorname{rank}(T_n) \leqq 2^n$, $\left(\sum_{n\in\mathbb{N}} \|T_n\|^q 2^{nq/p}\right)^{1/q} \leqq c_1 \sigma_{p,q}(T)$; this is possible by [12]. Since X_1^* and Y_1 are of cotype 2, Pisier's factorization theorem [15] says that the finite rank maps T_n factor over Hilbert space as

$$T_n = S_n U_n, \quad U_n \in L(X_1, l_2), \quad S_n \in L(l_2, Y_1),$$
$$\operatorname{rank}(U_n) \leqq 2^n, \quad \operatorname{rank}(S_n) \leqq 2^n$$

with $\|U_n\| \cdot \|S_n\| \leqq c \|T_n\|$ with uniform constant $c>0$. Actually c may be taken [15] as $c = 3(c_2(X_1^*) c_2(Y_1))^{3/2}$, where $c_2(X_1^*)$ and $c_2(Y_1)$ denote the cotype 2-constants of X_1^* and Y_1. We may normalize S_n to $\|S_n\| = \|T_n\|^q 2^{nq/p}$. Moreover, we split some multiple of a 2^n-dim. Hilbert space identity from U_n: Let $\beta_n := \|T_n\|^{1-2q} 2^{-2nq/p}$, P_n be the orthogonal projection onto $U_n(X_1) \subseteqq l_2$ and $R_n : l_2 \to l_2$ be given by $R_n = \beta_n P_n$. Let $Q_n := \beta_n^{-1} U_n$. Then $\|R_n\| = \beta_n$ and

$$\|Q_n\| \leqq \beta_n^{-1} \|U_n\| \leqq c\beta_n^{-1} \|T_n\| / \|S_n\| = c \|T_n\|^q 2^{nq/p}.$$

Thus $\sum_{n\in\mathbb{N}} \|Q_n\| < \infty$, $\sum_{n\in\mathbb{N}} \|S_n\| < \infty$ and hence

$$Q := \sum_{n\in\mathbb{N}} \oplus\, Q_n : X_1 \to \Big(\sum_{n\in\mathbb{N}} \oplus\, l_2\Big)_2, \qquad S := \Big(\sum_{n\in\mathbb{N}} \oplus\, l_2\Big)_2 \to Y_1$$

$$x \mapsto (Q_n x)_{n\in\mathbb{N}} \qquad (y_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} S_n y_n$$

define bounded operators. On the Hilbert space sum, we have an operator $R : \Big(\sum_{n\in\mathbb{N}} \oplus\, l_2\Big)_2 \to \Big(\sum_{n\in\mathbb{N}} \oplus\, l_2\Big)_2$ given by $(y_n)_{n\in\mathbb{N}} \mapsto (R_n y_n)_{n\in\mathbb{N}}$. Since $r = q/(1-2q)$,

$$\sum_{n\in\mathbb{N}} \|R_n\|^r 2^{nr/p} = \sum_{n\in\mathbb{N}} \beta_n^r 2^{nr/p} = \sum_{n\in\mathbb{N}} \|T_n\|^q (2^n)^{q/p} < \infty$$

states that R is not only continuous but belongs to $S_{p,r}(l_2, l_2)$, if we identify $l_2 = \Big(\sum_{n\in\mathbb{N}} \oplus\, l_2\Big)_2$. Clearly $T = SRQ$. Thus we derived the factorization of the claim with

$$\|S\| \sigma_{p,r}(R) \, \|Q\| \leqq c \Big(\sum_{n\in\mathbb{N}} \|T_n\|^q 2^{nq/p}\Big)^{2+1/r} \leqq c c_1 \sigma_{p,q}(T).$$

b) We now show that $T \in S_{p,q}(X_1, Y_1)$, $S \in S_{p,q}(X_2, Y_2)$ implies $T \hat{\otimes}_\tau S \in S_{p,r}\Big(X_1 \hat{\otimes}_\tau X_2, Y_1 \hat{\otimes}_\tau Y_2\Big)$ if $1/q \geqq 1/p + 2$ and $1/r := 1/q - 2$. By part a) we find factorizations of T and S

$$T=T_3T_2T_1\,,\quad T_1\in L(X_1,l_2)\,,\quad T_2\in S_{p,r}(l_2,l_2)\,,\quad T_3\in L(l_2,Y_1)$$
$$S=S_3S_2S_1\,,\quad S_1\in L(X_2,l_2)\,,\quad S_2\in S_{p,r}(l_2,l_2)\,,\quad S_3\in L(l_2,Y_2)\,.$$

By Proposition 3, $T_2\hat{\bigotimes}_\tau S_2\in S_{p,r}\left(l_2\hat{\bigotimes}_\tau l_2,\, l_2\hat{\bigotimes}_\tau l_2\right)$ since $r\leqq p$. Hence also

$$T\hat{\bigotimes}_\tau S=\left(T_3\hat{\bigotimes}_\tau S_3\right)\left(T_2\hat{\bigotimes}_\tau S_2\right)\left(T_1\hat{\bigotimes}_\tau S_1\right)\in S_{p,r}\left(X_1\hat{\bigotimes}_\tau X_2,\, Y_1\hat{\bigotimes}_\tau Y_2\right)$$

by the ideal property of $S_{p,r}$. The quantitative estimate

$$\sigma_{p,r}\left(T\hat{\bigotimes}_\tau S\right)\leqq c\sigma_{p,q}(T)\sigma_{p,q}(S)$$

holds with c depending only on p, q and the cotype 2-constants of X_i^* and Y_i $(i=1,2)$.

c) We now improve the index r by bilinear interpolation. Let $0<q<p<\infty$, $1/q\geqq 1/p+1$ as in the formulation of Theorem 1 and choose $0<p_0<p<p_1<\infty$ and define θ and q_0, q_1 by

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\qquad \frac{1}{q_0}=\frac{1}{p_0}+2\,,\qquad \frac{1}{q_1}=\frac{1}{p_1}+2\,.$$

By part b), the bilinear map $A:(T,S)\to T\hat{\bigotimes}_\tau S$ induces continuous bilinear operators

$$A:S_{p_i,q_i}(X_1,Y_1)\times S_{p_i,q_i}(X_2,Y_2)\to S_{p_i,\infty}\left(X_1\hat{\bigotimes}_\tau X_2,\, Y_1\hat{\bigotimes}_\tau Y_2\right)$$

for $i=0,1$; the image is even contained in $S_p\left(X_1\hat{\bigotimes}_\tau X_2,\, Y_1\hat{\bigotimes}_\tau Y_2\right)$. The spaces $S_{p,q}(X,Y)$ interpolate well for arbitrary Banach spaces X and Y, by Peetre and Sparr [9] or [13]

$$(S_{p_0,q_0}(X,Y),S_{p_1,q_1}(X,Y))_{\theta,q}=S_{p,q}(X,Y)$$

independently of the particular values of q_0 and q_1 (which might also be ∞). The quasi-triangle inequality in $S_{p,\infty}(X,Y)$ reads

$$\sigma_{p,\infty}(R_1+R_2)=\sup_{n\in\mathbb{N}}a_n(R_1+R_2)n^{1/p}\leqq 2^{1/p}(\sigma_{p,\infty}(R_1)+\sigma_{p,\infty}(R_2))$$

for $R_1, R_2\in S_{p,\infty}(X,Y)$. Thus by Pietsch [10, 6.2.], $S_{p,\infty}(X,Y)$ is a s-normed space with $1/s:=1/p+1$. Since with $1/r_i=1/p_i+1$ $(i=0,1)$, $1/r:=(1-\theta)/r_0+\theta/r_1=1/p+1$, Karadčov's bilinear interpolation theorem yields the continuity of

$$A:S_{p,q}(X_1,Y_1)\times S_{p,q}(X_2,Y_2)\to S_{p,q}\left(X_1\hat{\bigotimes}_\tau X_2,\, Y_1\hat{\bigotimes}_\tau Y_2\right)$$

using $1/q\geqq 1/p+1$, i.e. $q\leqq r$. This proves Theorem 1. □

Remark. The cotype condition in Theorem 1 was used in a technical way of Hilbert space factorizations. I have been unable to decide whether this is necessary or whether the ideals $S_{p,q}$ with $1/q\geqq 1/p+1$ are stable with respect to any tensor norm (on the class of all Banach spaces). An example to check might be $S=T=\mathrm{Id}_n : l_1^n \to l_\infty^n$. Independently of the cotype assumption, one can show in a similar fashion at least

Corollary 3. *Let $0<p<\infty, 0<q<\infty$ with $1/q \geqq 1/p+1$. Let X_i, Y_i be Banach spaces, for $i=1,2$ and τ be a tensor norm which on $l_2 \otimes l_2$ is weaker than the Hilbert-Schmidt norm. Then $T \in S_{p,q}(X_1, Y_1)$, $S \in S_{p,q}(X_2, Y_2)$ imply*

$$T \hat{\bigotimes}_\tau S \in S_{p,q}^x \left(X_1 \hat{\bigotimes}_\tau X_2, Y_1 \hat{\bigotimes}_\tau Y_2\right) \quad \textit{with}$$

$$\sigma_{p,q}^x \left(T \hat{\bigotimes}_\tau S\right) \leqq c_{p,q} \sigma_{p,q}(T) \sigma_{p,q}(S).$$

Sketch of the Proof

a) Let $1/q \geqq 1/p+3/2$ and $p<2$ first and let $1/r := 1/q-2$, $1/s := 1/p-1/2$. Absolutely 2-summing operators factor through Hilbert space. Using the same idea as in the proof of Theorem 1, but replacing Pisier's factorization theorem by the fact that rank-2^n-operators T_n have 2-summing norm with $\pi_2(T_n) \leqq 2^{n/2} \|T_n\|$, cf. [10, Chap. 17.5], we find that any operator $T \in S_{p,q}(X_1, Y_1)$ factors as $T=SRQ$ with $Q \in \Pi_2(X_1, l_2)$, $R \in S_{s,r}(l_2, l_2)$, $S \in L(l_2, Y_1)$. The difference is that the approximation numbers of R only belong to $l_{s,r}$ (not $l_{p,r}$).

b) Factoring $T, S \in S_{p,q}$ as $T=T_3T_2T_1$, $S=S_3S_2S_1$ with $T_1, S_1 \in \Pi_2$, $T_2, S_2 \in S_{s,r}$ and $T_3, S_3 \in L$, we find with $r \leqq s$

$$T \hat{\bigotimes}_\tau S = \left(T_3 \hat{\bigotimes}_\tau S_3\right)\left(T_2 \hat{\bigotimes}_\tau S_2\right)\left(T_1 \hat{\bigotimes}_\tau S_1\right) \in L \circ S_{s,r} \circ \Pi_2$$

using Proposition 3 again and the fact that $T_1 \hat{\bigotimes}_\tau S_1 \in \Pi_2$ which follows by tensoring the 2-summing factorizations using the assumption on τ. But $S_{s,r} \subseteqq S_{s,r}^x$ and $\Pi_2 \subseteqq S_{2,\infty}^x$, cf. [13]. Hence

$$T \hat{\bigotimes}_\tau S \in S_{s,r}^x \circ S_{2,\infty}^x \left(X_1 \hat{\bigotimes}_\tau X_2, Y_1 \hat{\bigotimes}_\tau Y_2\right) \subseteqq S_{p,\infty}^x \left(X_1 \hat{\bigotimes}_\tau X_2, Y_1 \hat{\bigotimes}_\tau Y_2\right)$$

because $\dfrac{1}{p} = \dfrac{1}{s} + \dfrac{1}{2}$.

c) The last statement means that $A : (T, S) \to T \hat{\bigotimes}_\tau S$ defines continuous bilinear maps

$$A : S_{p,q}(X_1, Y_1) \times S_{p,q}(X_2, Y_2) \to S_{p,\infty}^x \left(X_1 \hat{\bigotimes}_\tau X_2, Y_1 \hat{\bigotimes}_\tau Y_2\right).$$

Now bilinear interpolation yields Corollary 3. The spaces $S_{p,\infty}^x(X, Y)$ are again r-normed with $1/r = 1/p+1$, and although they do not interpolate as well as the $S_{p,\infty}(X, Y)$-spaces, they satisfy

$$(S_{p_0,\infty}^x(X, Y), S_{p_1,\infty}^x(X, Y))_{\theta,q} \subseteqq S_{p,\infty}^x(X, Y)$$

for $0<q \leqq \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $0<\theta<1$, cf. [13]. This is sufficient for our purpose. □

Remarks. (1) There is a third version of Theorem 1: If γ_2 denotes the Hilbert factorization norm, i.e. for $T: X \to Y$

$$\gamma_2(T) = \inf\{\|S\|\,\|R\| \mid T = SR, R: X \to l_2, S: l_2 \to Y\},$$

and

$$s_n(T) := \inf\{\gamma_2(T - T_n) \mid \operatorname{rank} T_n < n\},$$

then the ideals $S_{p,q}^{\gamma_2} := \{T \in L \mid \|(s_n(T))_{n \in \mathbb{N}}\|_{p,q} < \infty\}$ are stable under any tensor norm τ provided that $1/q \geqq 1/p + 1$ (no assumption of cotype 2 is necessary, the Hilbert space factorization is inherent in γ_2).

(2) The constants $c_{p,q}$ in Corollary 3 can be bounded by $c_{p,q} \leqq A d^{1/q}$ for some absolute constants A, d. Similarly, the constant c in Theorem 1 can be bounded by $c \leqq A d^{1/q} \cdot \max\limits_{i=1,2} (K_2(X_i^*), K_2(Y_i))^3$.

(3) Since $L_\infty \hat{\otimes}_\varepsilon L_\infty = L_\infty$, Corollary 3 means for T and S in $S_{p,q}^c$ with $1/q \geqq 1/p + 1$ that $T \hat{\otimes}_\varepsilon S$ is in $S_{p,q}^x$.

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Spektrale Starrheit gewisser Drehflächen

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1. Einleitung

Ziel dieser Note ist der Beweis, daß eine zu S^2 homöomorphe Drehfläche mit zusätzlicher Symmetrieebene senkrecht zur Drehachse innerhalb dieser Klasse von Drehflächen eindeutig durch ihr Spektrum bestimmt ist. Genauer zeigen wir Folgendes.

Theorem. *Seien $M, \tilde{M}$ zwei Riemannsche Mannigfaltigkeiten isometrisch zu (S^2, g) bzw. $(S^2, \tilde{g})$, wobei $g, \tilde{g}$ unter den Drehungen um die z-Achse und der Spiegelung an der (x, y)-Ebene invariante Metriken sind. Dann sind die folgenden Aussagen äquivalent:*

(i) *$M, \tilde{M}$ sind isometrisch,*
(ii) *$M, \tilde{M}$ haben gleiches Spektrum,*
(iii) *$M, \tilde{M}$ haben gleiches S^1-invariantes Spektrum.*

Dabei ist das Spektrum von M die Folge der (nach der Größe und mit Vielfachheiten angeordneten) Eigenwerte des Laplace-Beltrami-Operators; das S^1-invariante Spektrum ist die Teilfolge der Eigenwerte, die zu unter den Drehungen um die z-Achse (S^1-Aktion) invarianten Eigenfunktionen gehören (ebenfalls mit Vielfachheiten).

Unsere wesentliche Beobachtung ist die Implikation (ii) $\Rightarrow$ (iii), anders ausgedrückt: man kann das invariante Spektrum aus dem ganzen Spektrum „heraushören". Damit können wir das Isometrieproblem auf ein singuläres inverses Sturm-Liouville-Problem reduzieren, das von Marčenko gelöst wurde.

2. Vorbemerkungen und der Beweis von (ii) $\Rightarrow$ (iii)

In diesem Paragraphen sei $M = (S^2, g)$, g eine nur unter den Drehungen um die z-Achse invariante Metrik. Die minimalen Geodätischen vom Nordpol zum

Südpol (Meridiane) haben alle die gleiche Länge, die wir mit L bezeichnen. Wie das Gaußlemma zeigt, stehen diese Geodätischen senkrecht auf den S^1-Orbits (den Breitenkreisen). Daher ist die Abbildung

$$\pi : M \to [0, L], \qquad p \mapsto d(n, p),$$

n der Nordpol, d der Riemannsche Abstand, eine Riemannsche Submersion. $[0, L]$ können wir mit dem Orbitraum M/S^1 identifizieren.

Sei $h(t) := \frac{1}{2\pi} \operatorname{vol} \pi^{-1}(t)$, $t \in [0, L]$. Dann gilt $h(0) = h(L) = 0$, $h'(0) = -h'(L) = 1$. Zwei solche Flächen $M, \tilde{M}$ sind isometrisch wenn $L = \tilde{L}$ und $h(t) = \tilde{h}(t)$ oder $h(t) = \tilde{h}(L-t)$ für $t \in [0, L]$. Denn in Polarkoordinaten (r, θ) um den Nordpol hat die Metrik die Form $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & (h \circ \pi)^2 \end{pmatrix}$. Aus dieser Darstellung der g_{ij} ergibt sich auch die Darstellung des Laplace-Beltrami-Operators:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{h \circ \pi} \frac{\partial h \circ \pi}{\partial r} \frac{\partial f}{\partial r} + \frac{1}{(h \circ \pi)^2} \frac{\partial^2 f}{\partial \theta^2}. \tag{1}$$

Insbesondere ist eine S^1-invariante Funktion $f \in C^\infty(M)$ Eigenfunktion zum Eigenwert λ genau dann, wenn für die heruntergedrückte Funktion $\bar{f} : [0, L] \to \mathbb{R}$, $\bar{f} \circ \pi = f$, gilt:

$$\bar{f}'' + \frac{h'}{h} \bar{f}' + \lambda \bar{f} = 0. \tag{2}$$

Setzen wir $u := h^{1/2} \bar{f}$, so ist (2) äquivalent mit

$$u'' + (\lambda - q) u = 0, \tag{3}$$

wobei

$$q = \frac{(h^{1/2})''}{h^{1/2}} = \frac{2hh'' - h'^2}{4h^2}. \tag{4}$$

Diese Überlegungen gelten natürlich alle nur in dem offenen Intervall $(0, L)$; $\frac{h'}{h}$ und q werden in den Randpunkten singulär und sind nicht einmal integrierbar über $[0, L]$.

Satz 1. *Das S^1-invariante Spektrum von M stimmt mit dem Spektrum des Operators $u \mapsto -u'' + qu$ in $L^2([0, L])$ bei geeignet gewählten Randbedingungen überein.*

Beweis. Sei $Tu := -u'' + qu$. Ist $\varphi \in C^\infty(M)$ eine invariante Eigenfunktion von Δ, so ist nach (3) $\bar{\varphi} \cdot h^{1/2}$ eine Eigenfunktion von T mit demselben Eigenwert, $\bar{\varphi}$ die heruntergedrückte Funktion. Die Linearkombinationen dieser Eigenfunktionen von T liegen dicht in $L^2([0, L])$, da die Linearkombinationen invarianter Eigenfunktionen dicht liegen in $L^2(M)^{S^1}$ und wir $L^2(M)^{S^1}$ vermöge der Abbildung $f \to \bar{f} \cdot h^{1/2}$ mit $L^2([0, L])$ identifizieren können (vgl. [4]). Als Definitionsbereich $\mathscr{D}$ von T können wir daher jeden Unterraum von $L^2([0, L])$ wählen, auf dem sich T definieren läßt, der diese Eigenfunktionen enthält und auf dem T symmetrisch ist.

Denn dann ist T mit diesem Definitionsbereich wesentlich selbstadjungiert und sein Spektrum besteht genau aus den Eigenwerten zu den Eigenfunktionen $\bar{\varphi} \cdot h^{1/2}$. Wir wählen

$$\mathscr{D} := \{u \in L^2([0, L]) / u \in C^2((0, L)),\ Tu \in L^2([0, L]), (h^{-1/2}u), (h^{-1/2}u)' \text{ beschränkt}\}.$$

Es bleibt nur zu prüfen, daß T auf $\mathscr{D}$ symmetrisch ist. Das ist aber klar wegen

$$\begin{aligned}\int_0^L (uTv - vTu)(x) &= \int_0^L (uv'' - u''v)(x)dx = \lim_{x\to L} (uv' - vu')(x) - \lim_{x\to 0} (uv' - vu')(x) \\ &= \lim_{x\to L} h(x)(h^{-1/2}u(h^{-1/2}v)' - (h^{-1/2}u)'h^{-1/2}v)(x) \\ &\quad - \lim_{x\to 0} h(x)(h^{-1/2}u(h^{-1/2}v)' - (h^{-1/2}u)'h^{-1/2}v)(x) \\ &= 0. \quad \square\end{aligned}$$

Sei E_λ der Eigenraum des Laplace-Operators zum Eigenwert λ und $E_\lambda^{S^1}$ der Unterraum der S^1-invarianten Eigenfunktionen.

Satz 2. *Das S^1-invariante Spektrum ist einfach, d.h.*

$$\dim E_\lambda^{S^1} \leqq 1$$

für alle $\lambda \geqq 0$.

Beweis. Die Gleichung (2) ist eine Differentialgleichung mit regulär singulären Randpunkten (vgl. Bôcher [2]), da z.B. in der Nähe von Null $\frac{h'}{h}$ von der Form $\frac{1}{x} + g(x)$ mit einer Funktion $g \in C^\infty([0, L))$ ist. Die zugehörige Indexgleichung lautet:

$$\nu(\nu - 1) + \nu = 0$$

mit den Wurzeln $\nu_1 = \nu_2 = 0$. Also [2] gibt es bei festem λ ein Fundamentalsystem der Form

$$f_1, f_2 + f_1 \cdot \log(x)$$

mit in $[0, L)$ stetigen f_1, f_2 und $f_i(0) \neq 0$ für $i = 0, 1$. Daher ist der Lösungsraum der in den Endpunkten beschränkten Funktionen höchstens eindimensional. $\square$

Satz 3. *Das Spektrum bestimmt das invariante Spektrum, insbesondere also die Länge L der Meridiane.*

Beweis. S^1 operiert in kanonischer Weise auf den Eigenräumen E_λ, die sich daher als direkte Summe S^1-irreduzibler Unterräume schreiben lassen. Da die nichttrivialen irreduziblen reellen Darstellungen von S^1 alle zweidimensional sind, gehört λ nach Satz 2 genau dann zum invarianten Spektrum, wenn $\dim E_\lambda$ ungerade ist. Aus der Verallgemeinerung der Weylschen asymptotischen Beziehung auf das invariante Spektrum [6, 4] folgt schließlich, daß das invariante Spektrum das Volumen des Orbitalraumes bestimmt, in unserem Fall also L. $\square$

Satz 4. *Die Länge L der Meridiane und die Funktion q aus Gleichung* (4) *bestimmen M.*

Beweis. Es ist nur zu zeigen, daß sich die Volumenfunktion h aus q rekonstruieren läßt. Wegen $q=\frac{(h^{1/2})''}{h^{1/2}}$ ist $h^{1/2}$ Lösung von $u''-qu=0$.

Da $q(x)=-\frac{1}{4x^2}+g(x)$ mit $g\in C^\infty([0,L))$ ist dies eine regulär singuläre Gleichung mit der Indexgleichung $v(v-1)+\frac{1}{4}=0$. Wieder nach Bocher [2] ist der Unterraum der Lösungen u mit „$x^{-1/2}u(x)$ beschränkt in der Nähe von Null" eindimensional. Daher ist h bis auf einen Faktor eindeutig bestimmt. Wegen $h'(0)=1$ folgt die Behauptung. □

3. Der Satz von Marčenko und der Beweis von (iii) ⇒ (i)

Borg [3] (s. auch Levinson [9]) hatte für das reguläre Sturm-Liouville Problem, $u\mapsto -u''+qu$ mit $q\in L^1([0,L])$, gezeigt, daß zwei Spektren des Operators, die zu geeigneten Randbedingungen gehören, die Funktion q eindeutig bestimmen und daß bei symmetrischem q, $q(L-x)=q(x)$, ein Spektrum genügt. Obwohl q aus (4) nicht integrierbar ist, läßt sich aber der Levinsonsche Beweis auch auf diese Situation übertragen. Das ist allerdings mit erheblichen technischen Problemen verbunden. Bequemer ist es, folgende Verallgemeinerung der Borgschen Resultate zu benutzen:

Satz 5 (Marčenko [10, Satz 2.3.2]). *Seien L_1, L_2 Differentialoperatoren auf* $(0,L)$ *der Form $L_iu=-u''+q_iu$, $i=1,2$, mit q_i integrierbar auf jedem Teilintervall* $[0,l]$, *$l<L$. Seien die Spektren von L_1, L_2 diskret, d.h. diskret für ein Paar von Randbedingungen (und dann diskret für jedes) und seien* (α) *bzw.* (β) *die Randbedingungen*

$$(\alpha)\quad u(0)\cos\alpha+u'(0)\sin\alpha=0$$

und, wenn notwendig, eine Randbedingung bei L (d.h. im Grenzkreisfall).

$$(\beta)\quad u(0)\cos\beta+u'(0)\sin\beta=0$$

und dieselbe Randbedingung bei L wie in (α).

Ist $\cot(\alpha-\beta)\neq\infty$ *und stimmen die Spektren von L_1 und L_2 sowohl unter den Randbedingungen* (α) *als auch* (β) *überein, so gilt $q_1=q_2$ fast überall.*

Wir betrachten von nun an die noch etwas eingeschränktere Klasse von Flächen $M=(S^2,g)$, wobei g eine unter der S^1-Aktion (Drehung um die z-Achse) und der Spiegelung an der (x,y)-Ebene invariante Metrik ist. Es gilt in diesem Fall $h(L-x)=h(x)$ für die Volumenfunktion und $q(L-x)=q(x)$ für die Funktion aus (4), $x\in(0,L)$.

Um den Satz von Marčenko anwenden zu können, benötigen wir

Satz 6. *Es seien $0=\lambda_0'<\lambda_1'<\ldots<\lambda_n'\to\infty$ die S^1-invarianten Eigenwerte, $\varphi_i\in C^\infty(M)$ eine zu λ_i' gehörige S^1-invariante Eigenfunktion und $\bar{\varphi}_i\in C^\infty[0,L])$ die heruntergedrückte Funktion, $\varphi_i=\varphi_i\circ\pi$. Dann gilt:*

(i) $\bar{\varphi}_i$ *hat genau i Nullstellen in* $[0, L]$, *und diese liegen in* $(0, L)$;
(ii) *zwischen zwei Nullstellen von* $\bar{\varphi}_i$ *liegt genau eine Nullstelle von* $\bar{\varphi}_{i+1}$;
(iii) $\bar{\varphi}_{2i}(L-x)=\bar{\varphi}_{2i}(x)$, $\bar{\varphi}_{2i+1}(L-x)=-\bar{\varphi}_{2i+1}(x)$ *für* $x\in[0,L]$ *und* $i\in\mathbb{Z}_+$;
(iv) $\bar{\varphi}'_{2i}(L/2)=0$, $\bar{\varphi}_{2i+1}(L/2)=0$ *für* $i\in\mathbb{Z}_+$.

Beweis. (i) $\bar{\varphi}_i$ erfüllt (2), so daß nach [2] (vgl. den Beweis von Satz 2) $\bar{\varphi}_i$ in 0 und L nicht verschwinden kann. Die Aussage über die Anzahl der Nullstellen im Inneren beweist man analog zum regulären Fall (s. z. B. [5, S. 395]). Wir haben nämlich die Darstellung

$$\lambda'_i = \inf_{\substack{\varphi\in C^\infty(M)^{S^1},\varphi\neq 0\\ \varphi\perp\langle\varphi_0,\dots,\varphi_{i-1}\rangle}} \frac{\int_M |\operatorname{grad}\varphi|^2}{\int_M \varphi^2}.$$

Da die Knotengebiete der Funktionen φ_i S^1-invariant sein müssen, überträgt sich Courants Satz über die Anzahl der Knotengebiete [5, S. 393] sofort auf diesen Fall: φ_i hat höchstens $i+1$ Knotengebiete, $\bar{\varphi}_i$ also höchstens i Nullstellen. Ist K ein Knotengebiet von φ_i, so haben wir mit $n:=$ äußere Normale an K

$$(\lambda'_i-\lambda'_{i+1})\int_K \varphi_i\varphi_{i+1} = \int_K (\varphi_i\Delta\varphi_{i+1}-\Delta\varphi_i\varphi_{i+1}) = -\int_{\partial K}\frac{\partial\varphi_i}{\partial n}\varphi_{i+1}\,.$$

Also muß φ_{i+1} Nullstellen im Innern von K haben und $\bar{\varphi}_{i+1}$ hat immer mindestens eine Nullstelle zwischen zwei Nullstellen von $\bar{\varphi}_i$, aber auch mindestens eine in $(0, x_i)$ und (y_i, L), wenn x_i und y_i die kleinste bzw. größte Nullstelle von $\bar{\varphi}_i$ in $(0, L)$ bezeichnet. Durch Induktion folgt dann, daß $\bar{\varphi}_i$ auch mindestens i, also genau i Nullstellen in $(0, L)$ hat. Damit ist auch Behauptung (ii) bewiesen.

(iii) Da Δ mit der Spiegelung an der (x, y)-Ebene vertauscht und das invariante Spektrum einfach ist, gilt

$$\bar{\varphi}_i(L-x)^2=\bar{\varphi}_i(x)^2\,,\qquad x\in[0,L]\,,\qquad i\in\mathbb{Z}_+\,.$$

Aus $\bar{\varphi}_i(L-x)=\bar{\varphi}_i(x)$ folgt $\bar{\varphi}'_i(L/2)=0$, also $\bar{\varphi}_i(L/2)\neq 0$, während aus $\bar{\varphi}_i(L-x) = -\bar{\varphi}_i(x)$ folgt $\bar{\varphi}_i(L/2)=0$.

Wegen der Symmetrie der Nullstellen gilt also $\bar{\varphi}_i(L-x)=-\bar{\varphi}_i(x)$ genau dann, wenn die Anzahl der Nullstellen ungerade ist. (iv) ist ebenfalls bewiesen. □

Beweis von (iii) ⇒ (i) *des Theorems*. Der letzte Satz zeigt, daß die Spektren von $u\to -u''+qu$ in $L^2([0, L/2])$ mit den Randbedingungen

(α) $u(L/2)=0$
(β) $u'(L/2)=0$

und der alten Randbedingung in 0, d. h. „$(h^{-1/2}u)$ und $(h^{-1/2}u)'$ beschränkt" durch das S^1-invariante Spektrum und damit durch das Spektrum von M bestimmt sind. Nach dem Satz von Marčenko genügen diese Spektren, um q zu bestimmen. Damit folgt die Behauptung aus Satz 4.

4. Abschließende Bemerkungen

a) Es liegt nahe, eine Verallgemeinerung unserer Resultate auf solche G-Mannigfaltigkeiten ins Auge zu fassen, für die M/G ein Intervall ist. Tatsächlich überlegt man sich leicht, daß auch in diesem Fall das invariante Spektrum die Mannigfal-

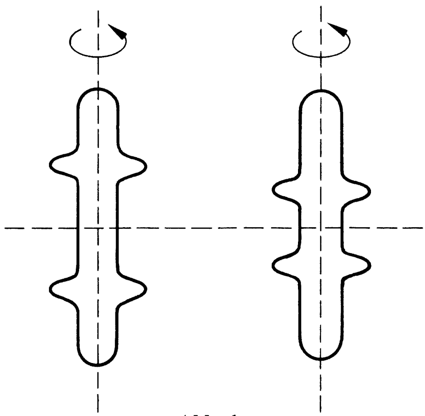

Abb. 1

tigkeiten bestimmt. Es scheint aber nicht möglich, auf einfache Weise das invariante Spektrum aus dem ganzen „herauszuhören“.

b) Für Drehflächen homöomorph zu $S^1 x S^1$ versagt unsere Methode vollständig. Denn im allgemeinen ist dort das invariante Spektrum nicht einfach, es bestimmt aber auch die Funktion h nicht. Daß in dieser Situation viel weniger spektrale Starrheit zu erwarten ist, ergibt sich auch daraus, daß periodische Lösungen der Kortweg-de Vries-Gleichung einparametrige Familien isospektraler Potentiale q_t auf S^1 liefern [8].

Diese q_t kommen alle von S^1-invarianten Metriken auf $S^1 x S^1$ her, nämlich von den “warped products” $S^1 x_{h_t} S^1$, wobei $h_t^{1/2}$ die periodische (nichtverschwindende) Lösung von $-u'' + q_t u = 0$ ist.

c) Viele Ergebnisse über spektrale Starrheit werden mittels der asymptotischen Entwicklung von Minakshisundaram-Pleijel bewiesen. In unserem Fall ist das nicht möglich, wie das folgende Beispiel zeigt (vgl. Abb. 1). M sei eine zylindrische Drehfläche, an die zwei „Kappen“ und zwei „Wülste“ so angesetzt werden, daß M symmetrisch bleibt bzgl. der Spiegelung an der (x, y)-Ebene. Verschiebt man nun die Wülste unter Wahrung der zusätzlichen Symmetrie, so entsteht eine Familie von Drehflächen mit stets derselben asymptotischen Entwicklung für die Spur des heat kernel. Es gilt sogar noch mehr: bildet man die Spur nur über die zu einer festen Darstellung gehörigen Eigenwerte, so sind immer noch die asymptotischen Entwicklungen identisch. Dies beruht darauf, daß man je zwei Drehflächen M_1, M_2 dieser Schar so in gleich viele Teilgebiete D_i^1, D_i^2 zerlegen kann, daß je zwei strikt enthalten sind in zueinander isometrischen Teilgebieten $\bar{D}_i^1, \bar{D}_i^2$. Das zu entwickelnde Integral über D_i^1 bzw. D_i^2 ist aber asymptotisch gleich dem Integral über den heat kernel des Dirichletproblems in $\bar{D}_i^1$ bzw. $\bar{D}_i^2$. Diese Bemerkung zeigt, daß die Methode von Kac [7] im vorliegenden Fall nicht zum Erfolg führt. Unsere Ergebnisse lassen sich auch mit [1] vergleichen, wo stärkere Voraussetzungen gemacht werden.

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Holomorphic Approximation in C^m-Norms on Totally Real Compact Sets in $\mathbb{C}^n$*

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1. Introduction and Statement of Results

In this paper we deal with approximation in the scale of Hölder norms on a compact, totally real set X in C^n. By this we mean a compact $X \subset \mathbb{C}^n$ for which there exists a non-negative, C^2-strictly plurisubharmonic function r defined in a neighbourhood of X such that $X = \{z : r(z) = 0\}$. This terminology is justified by the fact, which follows from results in [9] and [10], that such function r exists if and only if X is locally contained in totally real, C^1-submanifolds of $\mathbb{C}^n$.

Let $H(X)$ denote the space of all functions which are holomorphic in some neighbourhood of X, and let $C(X)$ denote the space of continuous functions on X. For $0 < s < 1$, we denote by $\mathrm{Lip}(s, X)$, $\mathrm{lip}(s, X)$ the space of functions on X with modulus of continuity $\omega(\delta) = O(\delta^s)$, $\omega(\delta) = o(\delta^s)$, respectively. We consider these spaces endowed with their natural topologies. Our first result is:

Theorem 1. *$H(X)$ is dense in $C(X)$ and in* $\mathrm{lip}(s, X)$, $0 < s < 1$.

For a positive integer m we will consider the Whitney definition of $C^m(X)$ [13] which we now formulate in its complex form. More generally, for $0 \leqq s < 1$, let $C^{m,s}(X)$ denote the space of jets $F = (u_{\alpha,\beta})$, where α, β are n-multiindexes with $|\alpha| + |\beta| \leqq m$ and $u_{\alpha,\beta} \in C(X)$, such that

$$|R_z^{m-|\alpha|-|\beta|} u_{\alpha,\beta}(w)| = o(|w-z|^{m+s-|\alpha|-|\beta|})$$

for $w, z \in X$ as $|w - z| \to 0$. Here

$$R_z^{m-|\alpha|-|\beta|} u_{\alpha,\beta}(w) = u_{\alpha,\beta}(w) - \sum_{|\gamma|+|\delta| \leqq m-|\alpha|-|\beta|} \frac{u_{\alpha+\gamma,\beta+\delta}(z)}{\gamma!\delta!}(w-z)^\gamma(\bar{w}-\bar{z})^\delta$$

[so in our notation $C^m(X)$ is $C^{m,o}(X)$]. $C^{m,s}(X)$ becomes a Banach space when provided with the norm

$$\|F\|_{C^{m,s}(X)} = \sum_{|\alpha|+|\beta| \leqq m} \left\{ \|u_{\alpha,\beta}\|_{C(X)} + \sup_{\substack{z,w \in X \\ z \neq w}} \frac{|R_z^{m-|\alpha|-|\beta|} u_{\alpha,\beta}(w)|}{|z-w|^{m+s-|\alpha|-|\beta|}} \right\}$$

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The extension theorem of Whitney states that for $F \in C^{m,s}(X)$ there exists u in $C^{m,s}(\mathbb{C}^n)$, the subspace of $C^m(\mathbb{C}^n)$ of all functions whose m-th order derivatives are in $\mathrm{lip}(s, \mathbb{C}^n)$, such that $D^\alpha \bar{D}^\beta u = u_{\alpha,\beta}$ on X (see [13] and also [20]). Thus we can think in $C^{m,s}(X)$ as consisting of $C^{m,s}$ functions u defined on a neighbourhood of X, with the topology defined by $\| \ \|_{C^{m,s}(X)}$ (in fact the quotient topology). Our second theorem states that the obvious necessary condition for u to be in the closure of $H(X)$ is also sufficient:

Theorem 2. *The closure of $H(X)$ in $C^{m,s}(X)$ is the subspace of all $u \in C^{m,s}(X)$ such that $\bar{\partial} u = 0$ on X up to order $m-1$.*

This kind of approximation problems have been treated along the last sixteen years in a series of papers. The result for uniform approximation was first obtained by Hörmander and Wermer [12] for compact sets contained in totally real submanifolds with some smoothness assumptions, and subsequently refined by Čirka [5] (see also [13]; alternative proofs of the Hörmander-Wermer theorem appear in [2, 21, 22]). For generalizations of Carleman's theorem see [15, 19]. For approximation in Lipschitz norms see [6–8]. Another proof of the result on $\mathrm{Lip}(s, x)$ approximation in Theorem 1, using duality arguments, appears in [8].

The approximation in C^m-norms, $m \geqq 1$, is more delicate. Nirenberg and Wells proved in [14] that if M is a C^∞-totally real submanifold of $\mathbb{C}^n$, then holomorphic functions in a neighbourhood of M are dense in $C^\infty(M)$ and Harvey and Wells [9] proved density in $C^{m-1}(M)$ if M is of class C^m. Finally, Range and Siu [13] proved in this case the density in $C^m(M)$. Now, it is well-known that if M is a totally real submanifold of class C^m and $u \in C^m(M)$ then there exists $v \in C^m(M)$ such that $v = u$ on M and $\bar{\partial} v = 0$ on M up to order $m-1$ (this uses Whitney's extension theorem once more, see [9]).

Hence our results remove all previous smoothness assumptions and the conclusions are strengthened. But, in our oppinion, our main contribution is the naturality and relative simplicity of the method, which we think suitable to treat even a more general situation (as for instance that in [11]). Also, it is reasonable to expect that our method, combined with the duality arguments of [7] and [8], may be as well useful to handle more general sets X having a "singular" part Y on which they are not totally real and to show that the holomorphic approximation in $C^{m,s}$-norms is then localized at Y.

We use in our proofs the so called Henkin-Ramirez kernels with weights recently introduced by Berndtsson and Anderson in [1]. This considerably simplifies some technical aspects of the development, specially those involving derivatives. The reason is that, as we show in Sect. 2, the general construction of these kernels can be also used to obtain in a very simple way the formulas for the derivatives of the solution of the $\bar{\partial}$-problem involved. The weights are then constructed in Sect. 3 and the resulting kernels estimated in Sect. 4. Finally in Sect. 5 we give the proof of the theorems.

As a final remark on notation, we denote by c most of the constants and also use $\simeq$ to mean that two variables have the same order of growth. Also we use D^i, $\bar{D}^i$ instead of $\partial/\partial z_i$, $\partial/\partial \bar{z}_i$, respectively, and $D^\alpha, \bar{D}^\alpha$ instead of $\partial^{|\alpha|}/\partial z^\alpha$, $\partial^{|\alpha|}/\partial \bar{z}^\alpha$. The Lebesgue measure is denoted dm.

2. The Henkin-Ramirez Formulas with Weights

2.1

First we recall the main result of [1]. Let U be a bounded domain in $\mathbb{C}^n$ with C^1-boundary and let $s: \bar{U} \times \bar{U} \to \mathbb{C}^n$ be of class C^1 such that

$$|s(\zeta, z)| \leqq c|\zeta - z|, \qquad \langle s, \zeta - z\rangle \neq 0 \quad \text{for} \quad \zeta \neq z$$
$$|\langle s, \zeta - z\rangle| \geqq c_L|\zeta - z|^2, \qquad \zeta \in \bar{U},\ z \in L \text{ compact in } U$$

$\left(\text{where } \langle \xi, \eta\rangle = \sum_{i=1}^{n} \xi_i \eta_i\right)$. Let $Q: \bar{U} \times \bar{U} \to \mathbb{C}^n$ be also of class C^1, holomorphic in z, and finally let G be a holomorphic function of one complex variable in a neighbourhood of the image of $\bar{U} \times \bar{U}$ by the map $(\zeta, z) \to 1 + \langle Q, z - \zeta\rangle$, with $G(1) = 1$. Let also s and Q denote, respectively, the 1-forms $s = \sum s_j d(\zeta_j - z_j)$, $Q = \sum Q_j d(\zeta_j - z_j)$. Consider the kernels

$$K = c_n \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} G^{(k)}(\langle Q, z - \zeta\rangle + 1) \frac{s \wedge (dQ)^k \wedge (ds)^{n-k-1}}{\langle s, \zeta - z\rangle^{n-k}}$$

$$P = -\frac{c_n}{n} G^{(n)}(\langle Q, z - \zeta\rangle + 1)(dQ)^n, \qquad c_n = (2\pi i)^{-n}/(n-1)!$$

The kernel P has continuous coefficients in $\bar{U} \times \bar{U}$, K is continuous outside the diagonal and $d_{\zeta,z} K = P$ there (in the sense of currents). Moreover, all coefficients of K are integrable in $\zeta \in \bar{U}$, uniformly for z in a compact set $L \subset U$. All these properties are the basis of the following:

Koppelman Formulas. *Let $K_{p,q}$ be the component of K of bidegree (p,q) in z, $(n-p, n-q-1)$ in ζ and let $P_{p,q}$ be the component of P of bidegree (p,q) in z, $(n-p, n-q)$ in ζ. Then, if f is a (p,q) form with coefficients in $C^1(\bar{U})$ one has*

$$f(z) = \int_{bU} f(\zeta) \wedge K_{p,q}(\zeta, z) + (-1)^{p+q+1} \cdot \left\{\int_U \bar{\partial} f(\zeta) \wedge K_{p,q}(\zeta, z) - \bar{\partial}_z \int_U f(\zeta) \wedge K_{p,q-1}(\zeta, z)\right\}$$

if $q > 0$ (in the sense of currents in U) and

$$f(z) = \int_{bU} f(\zeta) \wedge K_{p,0}(\zeta, z) + (-1)^{p+1} \int_U \bar{\partial} f(\zeta) \wedge K_{p,0}(\zeta, z) - \int_U f(\zeta) \wedge P_{p,0}(\zeta, z)$$

if $q = 0$ (as continuous forms in U).

In fact this is stated in [1] under the additional assumption that G is defined in a simply connected domain, so that the proof reduces to the case that G is a polynomial. But, since it is clearly enough to prove the formulas for f with small support, this condition is really not necessary. Also we point out that the form-valued integrals are defined as follows: if $M = bU$ or U and $\alpha(\zeta, z)$ is a (p,q)-form in z and of total degree $2n-1$ or $2n$ in ζ, respectively, then $\beta(z) = \int_M \alpha(\zeta, z)$ is the (p,q)-form in $z \in U$ defined by the equality

$$\int_U \phi(z) \wedge \beta(z) = \int_{U \times M} \phi(z) \wedge \alpha(\zeta, z),$$

for all $(n-p, n-q)$ forms ϕ with compact support in U. It should be noticed that this definition may differ in sign with the also used convention of putting the dz's and $d\bar{z}$'s to the far right side when integrating in $\zeta \in M$.

Corollary 2.1. *If, further, $K_{p,q}(\zeta, z)|_{\zeta \in bU}$ is zero for $q \geqq 1$ (in particular if $s(\zeta, z)$ is holomorphic in z for $\zeta \in bU$), then*

$$Tf = (-1)^{p+q} \int_u f \wedge K_{p,q-1} \tag{1}$$

satisfies $\bar{\partial} Tf = f$ if $\bar{\partial} f = 0$.

2.2

We will be primarily interested in the case $p = q = 0$ of Koppelman's formula. For a regular function u on $\bar{U}$ we set

$$Cu(z) = \int_{bU} u(\zeta) K_{0,0}(\zeta, z) - \int_U u(\zeta) P_{0,0}(\zeta, z), \qquad z \in U, \tag{2}$$

and for a (0, 1) form f we define Tf as in (1). Thus one has the basic decomposition formula $u = Cu + T(\bar{\partial} u)$. In the conditions of Corollary 2.1, Cu is holomorphic in U. Later on (in proving Theorem 2) we will be interested in having formulas for the derivatives of $T(\bar{\partial} u)$. Now we will obtain such formula in a rather simple way, using again the general properties of the kernels K, P.

To do so we must consider *all* components of K and P (those considered before are just the ones which have total degree n in the $d\zeta_i$'s and the dz_i's together). Let $K_{p,q,k}$ denote the component of K of bidegree $(n-p, n-q-1)$ in ζ and $(p+k, q-k)$ in z and let $P_{p,q,k}$ denote the one which is $(n-p, n-q)$ in ζ and $(p+k, q-k)$ in z, the $K_{p,q}$'s being those corresponding to $k=0$. In fact we only need to consider $K_{0,0}$, $P_{0,0}$, $K_{1,0}$, $P_{1,0}$, $K_{0,1,1}$, and $P_{0,1,1}$. Thus $K_{1,0}$ and $K_{0,1,1}$ are all the components of K of degree $(1,0)$ in z and total degree $2n-2$ in ζ and $P_{1,0}$, $P_{0,1,1}$ are the ones of P of bidegree $(1,0)$ in z and total degree $2n-1$ in ζ.

Lemma 2.2. *For $f \in C^\infty_{(0,1)}(\bar{U})$ let $Tf(z) = -\int_U f(\zeta) \wedge K_{0,0}(\zeta, z)$ as above. Then,*

$$\partial Tf = -\int_U \partial f \wedge K_{1,0} + \int_U f \wedge P_{0,1,1} - \int_{bU} f \wedge (K_{1,0} + K_{0,1,1}) - \int_U \bar{\partial} f \wedge K_{0,1,1}\,. \tag{3}$$

Proof. Let $\phi(z)$ be a $(n-1, n)$ form with compact support in U and consider, with $U_\varepsilon = \bar{U} \times \bar{U} \setminus \{(\zeta, z) : |\zeta - z| \leqq \varepsilon\}$,

$$I_\varepsilon = \int_{bU_\varepsilon} \phi(z) \wedge f(\zeta) \wedge K(\zeta, z)\,.$$

By Stoke's theorem, using $dK = P$ and making $\varepsilon \to 0$ we get

$$\begin{aligned} \lim_{\varepsilon \searrow 0} I_\varepsilon &= \int_{U \times U} \phi(z) \wedge f(\zeta) \wedge P(\zeta, z) + \int_{U \times U} d(\phi(z) \wedge f(\zeta)) \wedge K(\zeta, z) \\ &= \int_{U \times U} \phi \wedge f \wedge P_{0,1,1} + \partial\phi \wedge f \wedge K_{0,0} - \phi \wedge \partial f \wedge K_{1,0} \\ &\qquad - \phi \wedge \bar{\partial} f \wedge K_{0,1,1}\,. \end{aligned}$$

(all other terms are zero for bidegree reasons). If ε is sufficiently small we can also write, if $M_\varepsilon = \{(\zeta, z) : |\zeta - z| = \varepsilon\}$

$$I_\varepsilon = \int_{(z,\zeta) \in U \times bU} \phi \wedge f \wedge K - \int_{M_\varepsilon} \phi \wedge f \wedge K$$

It is well known that the limit as $\varepsilon \to 0$ of the last integral is $\int_U \phi(z) \wedge f(z)$ (see [1]) which here is zero for bidegree reasons. The first one is

$$\int_{U \times bU} \phi \wedge f \wedge (K_{1,0} + K_{0,1,1}).$$

The equality of both expressions gives then (3). □

The interest of formula (3) consist in that it avoids the singular integral that would appear if ∂_z is applied coefficientwise to $K_{0,0}$ (for $n=1$, this would be the Beurling transform, convolution with z^{-2}). Also note that the first integral in (3) is $-T\partial f$ and that if $\bar{\partial} f = 0$, then the last one dissapears. Under the conditions of Corollary 2.1, we have thus found simple formulas for the holomorphic derivatives of the integral solution operator T and shown that $\partial T + T\partial$ is an integral operator of order zero (acting on $\bar{\partial}$-closed forms).

2.3

Now we will write (3) in an equivalent way, using partial derivatives, in order to iterate the result. We need the following general property of the kernels $K_{p,q}$:

Lemma 2.3. *If* $\theta(\zeta) = \sum_{i,j} \theta_{i,j}(\zeta) d\zeta_i \wedge d\bar{\zeta}_j$ *is a* (1.1) *form, the component of* dz_i *in* $0(\zeta) \wedge K_{1,0}(\zeta, z)$ *is* $\theta^i(\zeta) \wedge K_{0,0}(\zeta, z) \wedge dz_i$, *where* $\theta^i = \sum_j \theta_{i,j} d\bar{\zeta}_j$.

Proof. As follows from the development in Sect. 2 of [1], the kernel K contains $\omega(\zeta - z) = d(\zeta_1 - z_1) \wedge \ldots \wedge d(\zeta_n - z_n)$ as a factor, i.e. it can be written $K = R \wedge \omega(\zeta - z)$, with R of total degree $n-1$ in ζ, z. Hence if R_0 is the component of R of bidegree $(0, n-1)$ in ζ and $\omega_1(\zeta, z)$ is the component of $\omega(\zeta - z)$ of type $(n-1, 0)$ in ζ and $(1,0)$ in z, it is clear that $K_{0,0} = R_0 \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_n$, $K_{1,0} = R_0 \wedge \omega_1(\zeta, z)$. The assertion of the lemma follows then immediately from these expressions of $K_{0,0}$ and $K_{1,0}$. □

Let now $P^i_{0,1,1}$, $K^i_{1,0}$, and $K^i_{0,1,1}$ denote the coefficients of dz_i in $P_{0,1,1}$, $K_{1,0}$, and $K_{0,1,1}$, respectively. We obtain then from (3), using Lemma 2.3

$$D^i Tf = -\int_U D^i f \wedge K_{0,0} + \int_U f \wedge P^i_{0,1,1} + \int_{bU} f \wedge (K^i_{1,0} + K^i_{0,1,1}).$$

Now observe that the first term on the right is $T(D^i f)$ (the operator D^i acting coefficientwise on forms) and so we have shown that $D^i T - TD^i$ is an integral operator of order zero. The second term can always be differentiated in z under the integral sign (for P is holomorphic in z) and so is the third if $K(\zeta, z)$ is C^∞ in z for $\zeta \in bU$ [this is the case, for instance, when $s(\zeta, z)$ is holomorphic in z when $\zeta \in bU$]. In this situation, it is then clear that we can iterate and obtain that for every multiindex α with $|\alpha| = m$, $D^\alpha T - TD^\alpha$ is an integral operator of order $\leqq m-1$ (on $\bar{\partial}$-closed forms). More concretely we obtain

$$D^\alpha Tf = TD^\alpha f + \Sigma' \int_U D^\gamma f \wedge D^\beta_z P^i_{0,1,1} + \Sigma' \int_{bU} D^\gamma f \wedge D^\beta_z (K^i_{1,0} + K^i_{0,1,1}), \tag{4}$$

where in the last terms γ and β are multiindexes with $|\gamma|+|\beta|=|\alpha|-1$ and $i=1,\dots,n$.

We incidentally remark that using (4) the C^m-estimates for T are easily reduced to the case $m=0$. Also, in the unit ball, formula (4) simplifies further and it can be shown that $D^\alpha Tf$ only depends on $D^\alpha f$ (see details in [3]).

3. Construction of the Weights

3.1

From now on U will denote a fixed bounded neighbourhood of X and r a non-negative C^2 strictly plurisubharmonic function defined on a neighbourhood of $\bar{U}$ such that $U=\{z:r(z)<1\}$, $X=\{z:r(z)=0\}$. We will write $U_\delta=\{z:r(z)<\delta\}$. Our proofs of Theorems 1 and 2 will be based on the basic decomposition formula $u=Cu+T\bar{\partial}u$ on the domain U_δ for small δ. In this section our aim is to construct the weights so that they have a good behaviour as δ goes to zero. The first step is a modification, as in [4], of a well-known lemma of Henkin:

Lemma 3.1. *There exist positive constants* $\varepsilon_0, \delta_0, c_0$ *and* $\phi\in C^1(\bar{U}_{\delta_0}\times\bar{U}_{\delta_0})$ *holomorphic in* z *such that*

(a) $|\phi|\geqq c_0$ *if* $|\zeta-z|\geqq\varepsilon_0$

(b) $2\operatorname{Re}\phi(\zeta,z)\geqq r(\zeta)-r(z)+c_0|\zeta-z|^2$ *if* $|\zeta-z|<\varepsilon_0$

(c) $d_\zeta\phi(\zeta,z)|_{\zeta=z}=-d_z\phi(\zeta,z)|_{\zeta=z}=\partial r(z)$.

Proof. The first step of the construction is well known. Set

$$g(\zeta,z)=\sum_{j=1}^n\frac{\partial r}{\partial\zeta_j}(\zeta)(\zeta_j-z_j)+\frac{1}{2}\sum_{j,k=1}^n\frac{\partial^2 r}{\partial\zeta_j\partial\zeta_k}(\zeta)(\zeta_j-z_j)(\zeta_k-z_k).$$

By the strict plurisubharmonicity of r in $\bar{U}$ it follows that there exists $\varepsilon>0$ such that

$$2\operatorname{Re}g(\zeta,z)\geqq r(\zeta)-r(z)+c|\zeta-z|^2,\qquad \zeta,z\in\bar{U},\qquad |\zeta-z|<\varepsilon. \tag{5}$$

Now, if a_{jk} are C^∞ functions on $\bar{U}$ which are close enough to $\partial^2 r/\partial\zeta_j\partial\zeta_k$, the function

$$F(\zeta,z)=\sum_{j=1}^n\frac{\partial r}{\partial\zeta_j}(\zeta)(\zeta_j-z_j)+\frac{1}{2}\sum_{j,k=1}^n a_{jk}(\zeta)(\zeta_j-z_j)(\zeta_k-z_k)$$

is C^1 in ζ and (5) holds with g replaced by F. Now if $\zeta,z\in\bar{U}$, and $\varepsilon/2<|\zeta-z|<\varepsilon$, it follows that $2\operatorname{Re}F(\zeta,z)\geqq r(\zeta)-r(z)+c\varepsilon^2/4$ is positive if $r(z)<\delta_1\overset{\text{def}}{=}c\varepsilon^2/8$. Let χ denote a C^∞ function on the real line, $0\leqq\chi\leqq1$ such that $\chi(t)=0$ for $t\leqq\varepsilon/2$ and $\chi(t)=1$ for $t\geqq2\varepsilon/3$. With $\zeta\in\bar{U}_{\delta_1}$ fixed, we define for z in U_{δ_1}

$$B(\zeta,z)=\begin{cases}\bar{\partial}_z\dfrac{\log F(\zeta,z)}{F^2}\chi(|\zeta-z|), & z\in\operatorname{sup}\operatorname{grad}_z\chi(|\zeta-z|)\\ 0 & \text{outside}.\end{cases}$$

Then $\bar{\partial}_z B=0$ and, since U_{δ_1} is pseudoconvex, there exists $C(\zeta,z)$ such that $\bar{\partial}_z C(\zeta,z)=B(\zeta,z)$. In $\{z\in U_{\delta_1}:|\zeta-z|<\varepsilon\}$ consider $\varphi(\zeta,z)=C(\zeta,z)-F^{-2}(\log F)\chi(|\zeta-z|)$, which is holomorphic in z. Then define

$$\phi=\begin{matrix} F\exp F^2\varphi & \text{if} & |\zeta-z|<\varepsilon \\ \exp F^2 C & \text{if} & |\zeta-z|\geqq\varepsilon \end{matrix}$$

The function ϕ is then holomorphic in $z\in U_{\delta_1}$ and by a well known argument the whole construction can be made smoothly on $\zeta\in\bar{U}_{\delta_1}$, so that $|\phi|\geqq c$ if $|\zeta-z|\geqq\varepsilon$, ζ, $z\in\bar{U}_{\delta_1}$ (shrinking δ_1 if necessary). In $|\zeta-z|<\varepsilon$, $\phi-F=F(1-\exp F^2\varphi)$ and so $|\phi-F|\leqq M|\zeta-z|^3$ there. This clearly implies (c). It also implies that

$$2\operatorname{Re}\phi(\zeta,z)\geqq r(\zeta)-r(z)+c|\zeta-z|^2-M|\zeta-z|^3$$

for $\zeta,z\in\bar{U}_{\delta_1}$, $|\zeta-z|<\varepsilon$. Now we reduce ε to $\varepsilon_0\overset{\text{def}}{=}\min(\varepsilon,c/2M)$ so that (b) will hold. In $\varepsilon_0\leqq|\zeta-z|\leqq\varepsilon$, ϕ has the same zeroes as F and there $2\operatorname{Re}F(\zeta,z)\geqq -r(z)+c\varepsilon_0^2$. Hence it is enough to take $\delta_0\overset{\text{def}}{=}\min\left(\delta_1,\frac{c}{2}\varepsilon_0^2\right)$. □

In spite of the triviality of the example, we think it is worthwhile to point out that the function $\phi(\zeta,z)$ in Lemma 3.1 plays with respect to X the role that $\phi(\zeta,z)=|\zeta|^2-\bar{\zeta}\cdot z$ plays with respect to $X=\{0\}$, $r(z)=|z|^2$.

We apply now to $\phi(\zeta,z)$ the familiar division procedure with C^1 dependence on ζ, to obtain $P_j\in C^1(\bar{U}_{\delta_0}\times\bar{U}_{\delta_0})$, $j=1,\ldots,n$ holomorphic in z such that

$$\phi(\zeta,z)=\sum_j P_j(\zeta,z)(\zeta_j-z_j)\,.$$

The property (c) of Lemma 3.1 implies then that $P_j(z,z)=0$ for $z\in X$.

3.2

Let $\delta<\delta_0$ so that U_δ is a regular domain i.e., $\operatorname{grad} r\neq 0$ on bU_δ (this is true for almost all δ). To define the weights on U_δ we will imitate the procedure in example 1 of Sect. 2 in [1]. For $\eta>\delta$ define $Q^{\delta,\eta}:\bar{U}_\delta\times\bar{U}_\delta\to\mathbb{C}^n$ by

$$Q_j^{\delta,\eta}(\zeta,z)=\frac{P_j(\zeta,z)}{r(\zeta)-\eta}.$$

Then $Q^{\delta,\eta}\in C^1(\bar{U}_\delta\times\bar{U}_\delta)$, is holomorphic in z and

$$\langle Q^{\delta,\eta},z-\zeta\rangle+1=\frac{\eta-r(\zeta)+\phi(\zeta,z)}{\eta-r(\zeta)}.$$

The numerator of this expression does not vanish in $\bar{U}_\delta\times\bar{U}_\delta$ (taking δ_0 smaller if necessary) for if $|\zeta-z|<\varepsilon_0$, then, by (b) in Lemma 3.1.

$$2\operatorname{Re}(\eta-r(\zeta)+\phi(\zeta,z))\geqq 2\eta-r(\zeta)-r(z)+c_0|\zeta-z|^2 \tag{6}$$

and if $|\zeta-z|\geqq\varepsilon_0$, then by (a) in Lemma 3.1,

$$|\eta-r(\zeta)+\phi(\zeta,z)|\geqq c_0-\eta\,. \tag{7}$$

Hence we may choose $G(w)=w^{-1}$ and obtain the Koppelman formulas for the kernels

$$K^{\delta,\eta}=c_n(n-1)!\sum_{k=0}^{n-1}(-1)^k\left[\frac{\eta-r(\zeta)}{\eta-r(\zeta)+\phi(\zeta,z)}\right]^{k+1}\frac{s\wedge(dQ^{\delta,\eta})^k\wedge(ds)^{n-k-1}}{\langle s,\zeta-z\rangle^{n-k}}$$

$$P^{\delta,\eta}=(-1)^{n+1}c_n(n-1)!\left[\frac{\eta-r(\zeta)}{\eta-r(\zeta)+\phi(\zeta,z)}\right]^{n+1}(dQ^{\delta,\eta})^n. \tag{8}$$

Since

$$(dQ^{\delta,\eta})^k=\left[\frac{1}{r(\zeta)-\eta}d\sum_j P_jd(\zeta_j-z_j)-\frac{1}{(r(\zeta)-\eta)^2}dr(\zeta)\wedge\sum_j P_jd(\zeta_j-z_j)\right]^k \tag{9}$$

all the coefficients of this form are $O((\eta-r(\zeta))^{-k-1})$. Therefore one can let $\eta\to\delta$ and obtain kernels K^δ, P^δ for which the Koppelman formulas hold. Moreover, the restriction of K^δ to bU_δ is zero, because $dr(\zeta)=0$ there and all coefficients of $(dQ^{\delta,\eta})^k$ are then $O(\eta-r(\zeta))^{-k})$. Thus the formulas do not contain integrals over the boundary.

3.3

From now on we will write P_0^δ, K_0^δ instead of $P_{0,0}^\delta, K_{0,0}^\delta$. Thus one has the decomposition formula, for u regular, $u=C^\delta u+T^\delta\bar\partial u$, with

$$C^\delta u(z)=-\int_{U_\delta}u(\zeta)P_0^\delta(\zeta,z),\qquad T^\delta f(z)=-\int_{U_\delta}f(\zeta)\wedge K_0^\delta(\zeta,z) \tag{10}$$

and $C^\delta u$ is holomorphic in U_δ. Now there is no need at all to take $s(\zeta,z)$ holomorphic in z for $\zeta\in bU_\delta$ and so we can simply take $s(\zeta,z)=\bar\zeta-\bar z$, independently of δ. The kernels $K^{\delta,\eta}$ are then C^∞ in $z\in U_\delta$ and so formula (4) holds for the operator $T^{\delta,\eta}$ associated to $K^{\delta,\eta}$. The inequalities (6) and (7) imply that the convergence of $T^{\delta,\eta}f$ towards $T^\delta f$, as well as that of the right member of (4), is uniform on compact sets of U_δ as $\eta\searrow\delta$. Writing now simply P_i^δ to denote the coefficient of dz_i in $P_{0,1,1}^\delta$, we get for $f\in C^\infty_{(0,1)}(\bar U)$, $\bar\partial f=0$

$$D^\alpha T^\delta f=T^\delta D^\alpha f+\Sigma'\int_{i,\beta,\gamma U_\delta}D^\gamma f\wedge D_z^\beta P_i^\delta,\, i=1,\dots,n,\, |\beta|+|\gamma|=|\alpha|-1\,. \tag{11}$$

The decomposition $u=C^\delta u+T^\delta\bar\partial u$, with C^δ, T^δ defined by (10) and the formula (11) will be our main tools. This has been obtained if bU_δ is regular, but, since now no integral over the boundary appears, it is clear that this holds true for all $\delta<\delta_0$.

4. Estimates

4.1

We begin by writing down K^δ and P^δ more explicitely. We use the notations

$$A_\delta(\zeta,z)=\delta-r(\zeta)+\phi(\zeta,z),\qquad \omega=\sum_j P_j(\zeta,z)d(\zeta_j-z_j)\,.$$

Recall that K^δ, P^δ are obtained from (8) and (9) making $\eta=\delta$. From (9),

$$(dQ^\delta)^k=(r(\zeta)-\delta)^{-k}(d\omega)^k-k(r(\zeta)-\delta)^{-k-1}(d\omega)^{k-1}\wedge dr(\zeta)\wedge\omega,$$

and inserted in (8) gives

$$K^\delta(\zeta,z)=c_n(n-1)!\sum_{k=0}^{n-1}\frac{\delta-r(\zeta)}{A_\delta(\zeta,z)^{k+1}}\frac{s\wedge(d\omega)^k\wedge(ds)^{n-k-1}}{|\zeta-z|^{2(n-k)}}$$
$$+c_n(n-1)!\sum_{k=0}^{n-1}\frac{k}{A_\delta(\zeta,z)^{k+1}}\frac{s\wedge(d\omega)^{k-1}\wedge dr(\zeta)\wedge\omega\wedge(ds)^{n-k-1}}{|\zeta-z|^{2(n-k)}}. \quad (12)$$

In the same manner we obtain

$$P^\delta=-\frac{c_n(n-1)!}{A_\delta(\zeta,z)^{n+1}}\{(\delta-r(\zeta))(d\omega)^n+n(d\omega)^{n-1}\wedge dr(\zeta)\wedge\omega\}. \quad (13)$$

Note that all the dependence in δ is contained in $\delta-r(\zeta)$ and $A_\delta(\zeta,z)$.

4.2

We now proceed to evaluate $|K_0^\delta|$ and $|D_z^\beta P_i^\delta|$ in terms of the basic quantity A_δ. Let $d(\zeta,X)$ denote the distance from ζ to X. Recall that $P_j\in C^1(\bar U_{\delta_0}\times\bar U_{\delta_0})$ and that $P_j(z,z)=0$ for $z\in X$. Also recall that $\operatorname{grad} r=0$ on X. Therefore

$$|P_j(\zeta,z)|\leqq c(d(z,X)+|\zeta-z|)\overset{\text{def}}{=}c\varrho(\zeta,z)$$
$$|w|=\sum_j|P_j|\leqq c\varrho(\zeta,z)$$
$$|dr(\zeta)|\leqq cd(\zeta,X)\leqq c\varrho(\zeta,z)$$

and using (12) we get

$$|K_0^\delta|\leqq c\sum_{k=0}^{n-1}\frac{\delta+\varrho(\zeta,z)^2}{|A_\delta|^{k+1}|\zeta-z|^{2(n-k)-1}},\qquad \zeta,z\in U_\delta. \quad (14)$$

Exactly in the same way we obtain

$$|P_i^\delta|\leqq c\frac{\delta+\varrho(\zeta,z)^2}{|A_\delta|^{n+1}},\qquad \zeta,z\in U_\delta.$$

The estimate of $D_z^\beta P_i^\delta$ is more involved. First observe that

$$D_z^iA_\delta(\zeta,z)=D_z^i\phi(\zeta,z)=-P_i(\zeta,z)+\sum_j D_z^iP_j(\zeta,z)(\zeta_j-z_j).$$

The functions $P_j(\zeta,z)$ being C^1 in $\bar U_{\delta_0}\times\bar U_{\delta_0}$ and holomorphic in z, will have all derivatives with respect to z bounded in $\bar U_{\delta_0}\times\bar U_{\delta_0}$ (reducing δ_0 somewhat). Hence

$$|D_z^iA_\delta(\zeta,z)|\leqq c\varrho(\zeta,z) \quad (15)$$

and all derivatives of order greater than 1 of A_δ are uniformly bounded. Then it is easily proved by induction that a derivative of order k of $A_\delta^{-(n+1)}$ is a sum of terms

$$\frac{h_{i,j}(\zeta,z)}{A_\delta(\zeta,z)^{n+j+1}}$$

where $h_{i,j}(\zeta,z)$ is a product of derivatives of order $\leqq k$ of A_δ, containing i derivatives of order 1, and the indexes i,j satisfy $i\leqq j\leqq k$, $k+i\geqq 2j$. Therefore, (15) gives, if $|\gamma|=k$

$$|D_z^\gamma A_\delta^{-(n+1)}|\leqq c\sum_{i,j}\frac{\varrho(\zeta,z)^i}{|A_\delta(\zeta,z)|^{n+j+1}}\overset{\text{def}}{=}cB_k(\zeta,z)$$

with $i\leqq j\leqq k$, $k+i\geqq 2j$. Now, in (13), all derivatives with respect to z of the coefficients of $d\omega$, ω are uniformly bounded, for they are continuous on $\bar U_{\delta_0}\times\bar U_{\delta_0}$ and holomorphic in z, and $|\omega|\leqq c\varrho(\zeta,z)$. In conclusion we get from (13), by Leibniz formula, for $|\beta|=p$

$$|D_z^\beta P_i^\delta(\zeta,z)|\leqq c\delta\sum_{k=0}^{p}B_k(\zeta,z)+c\varrho(\zeta,z)\left(\sum_{k=0}^{p-1}B_k(\zeta,z)+B_p(\zeta,z)\varrho(\zeta,z)\right). \qquad (16)$$

The inequalities (14) and (16) complete the estimate of $|K_0^\delta|$ and $|D_z^\beta P_i^\delta|$ in terms of $|A_\delta|$ and $\varrho(\zeta,z)$. We will combine them with the following one, which follows from (6) and (7)

$$\begin{aligned} 2\operatorname{Re}A_\delta(\zeta,z)&\geqq\delta-r(z)+c_0|\zeta-z|^2 \quad \text{if} \quad |\zeta-z|\leqq\varepsilon_0, \quad \zeta,z\in U_\delta\\ |A_\delta(\zeta,z)|&\geqq c \quad \text{if} \quad |\zeta-z|\geqq\varepsilon_0, \quad \zeta,z\in U_\delta\end{aligned} \qquad (17)$$

for small δ.

4.3

In order to obtain precise estimates of the integrals of K_0^δ, $D_z^\beta P_i^\delta$ over U_δ we need to choose appropriate local coordinates in the neighbourhood of each point of X.

Lemma 4.3. *There exist $\varepsilon_0>0$ (which we assume to be equal to the one in (17)) with the following property: for $z\in X$ one can choose local coordinates $x_j(\zeta)$, $y_j(\zeta)$, $j=1,\dots,n$ in $B(z,\varepsilon_0)$ such that*

$$r(\zeta)\geqq c\sum_{j=1}^{n}x_j^2, \qquad |\zeta-z|^2\simeq\sum_{j=1}^{n}x_j^2+y_j^2, \qquad \zeta\in B(z,\varepsilon_0)$$

and the Jacobian of the transformation $\zeta\to x_j(\zeta)$, $y_j(\zeta)$ is bounded above and below by constants independent of z.

Proof. From Lemma 3 in [10] it follows that for $z\in X$ there exist complex linear coordinates $w_1(\zeta),\dots,w_n(\zeta)$, $w_j(z)=0$, such that if $w_j=u_j+iv_j$, the second order term in the Taylor expansion of r at z is

$$\sum_{j=1}^{n}(1+\lambda_j)u_j^2+(1-\lambda_j)v_j^2$$

with $\lambda_j\geqq 0$. As showed in [10], this implies that

$$M=\left\{w:\frac{\partial r}{\partial u_j}(w)=0, j=1,\dots,n\right\}$$

is a C^1 totally real submanifold in a neighbourhood of z, containing X there, and whose tangent space at 0 is $u_j=0, j=1,\dots,n$. By the implicit function theorem, the equations of M near 0 are $u_j=g_j(v_1,\dots,v_n)$ with g_j of class C^1, $g_j(0)=dg_j(0)=0$.

For small $w=(u,v)$ let $w_0=(g(v),v)$ denote its "projection" onto M. Since the Hessian of r in the u direction is bounded below if w is small, the Taylor development of r at w_0 gives, for small w

$$r(w)\geqq r(w_0)+c\sum_{j=1}^{n}(u_j-g_j(v))^2\geqq c\sum_{j=1}^{n}(u_j-g_j(v))^2$$

[this last term is of the same order of dist $(w,M)^2$]. Hence, $x_j=u_j-g_j(v)$, $y_j=v_j$ are coordinates that near z, say in $B(z,\varepsilon_0)$, will satisfy $r(\zeta)\geqq c\sum_{j=1}^{n}x_j^2$ and

$$|\zeta-z|^2\simeq|w|^2=\sum_{j=1}^{n}u_j^2+v_j^2\simeq\sum_{j=1}^{n}x_j^2+y_j^2\quad[\text{because } g_j(0)=dg_j(0)=0]\,.$$

The complex Jacobian of the transformation $(\zeta_1,\ldots,\zeta_n)\to(w_1(\zeta),\ldots,w_n(\zeta))$ is [9] the square root of the determinant of the Levi form of r at z. On the other hand, ε_0 and all constants involved depend on the bounds of the derivatives of r and their modulus of continuity and so can be chosen independent of $z\in X$. This proves the lemma. □

Corollary. $m(D_\delta)=O(\delta^{n/2})$.

4.4

Now we will state the estimates needed for the proof of the theorems. We will write $\eta=\eta(\delta)=\delta^{1/2}$. Let $V_\delta=\{z:d(z,X)\leqq\eta/2\}$; we may assume, without loss of generality that $r(z)\leqq d(z,X)^2$ and so $V_\delta\subset U_{\delta/4}$.

Lemma 4.4. *With the notations above, the following estimates hold as* $\delta\searrow O$,

(a) $$\sup_{z\in V_\delta}\int_{U_\delta}d(\zeta,x)^\varrho|K_0^\delta(\zeta,z)|dm(\zeta)=O(\eta^{1+\varrho})\,,\qquad 0\leqq\varrho<1\,,$$

(b) $$\sup_{z\in V_\delta}\int_{U_\delta}d(\zeta,x)^\varrho|D_z^\beta P_i^\delta(\zeta,z)|dm(\zeta)=O(\eta^{\varrho-|\beta|})\,,\qquad \varrho<|\beta|+1$$

Proof. Clearly we may assume that δ is small enough so that $\eta\leqq\varepsilon_0/4$, ε_0 being as in Lemma 4.3 and (17). Let $z_0\in X$ such that $|z-z_0|=d(z,X)$. We denote by I and II the integrals in (a) and (b), respectively. We break both I and II into three parts I_i, II_i, $i=1,2,3$, corresponding respectively to the regions $R_1=U_\delta\cap B(z,\eta)$, $R_2=(U_\delta\cap B(z_0,\varepsilon_0/2))\backslash B(z,\eta)$, and $R_3=U_\delta\backslash B(z_0,\varepsilon_0/2)$. Note that $R_1\cup R_2\subset B(z,\varepsilon_0)$ and so by (17) one has for $\zeta\in R_1\cup R_2$

$$2\operatorname{Re}A_\delta(\zeta,z)\geqq\delta-r(z)+c_0|\zeta-z|^2\geqq3\delta/4+c_0|\zeta-z|^2\,.$$

Observe now that $\varrho(\zeta,z)\leqq\eta+|\zeta-z|$ for $z\in V_\delta$, and so $\varrho(\zeta,z)\leqq c|A_\delta|^{1/2}$ using the estimate above. Hence, by (14)

$$|K_0^\delta(\zeta,z)|\leqq c|\zeta-z|^{1-2n}\,,\qquad \zeta\in R_1\cup R_2\,,\qquad z\in V_\delta\,.$$

In the same manner, we see that in this case $|B_k|\leqq c|A_\delta|^{-n-1-\frac{k}{2}}$ and so by (16)

$$|D_z^\beta P_i^\delta(\zeta,z)|\leqq c|A_\delta(\zeta,z)|^{-n-\frac{|\beta|}{2}}\leqq c(\delta+|\zeta-z|^2)^{-n-\frac{|\beta|}{2}}\,.$$

For $\zeta \in R_1$, $d(\zeta, x) \leqq c\eta$ and hence

$$I_1 \leqq c\eta^{\varrho} \int_{B(z,\eta)} |\zeta - z|^{1-2n} dm(\zeta) = c\eta^{1+\sigma}$$

$$II_1 \leqq c\eta^{\varrho}\delta^{-n-\frac{|\beta|}{2}} m(R_1) \leqq c\eta^{\varrho - |\beta|}$$

For $\zeta \in R_2$, $|\zeta - z_0| \leqq |\zeta - z| + \eta/2 \leqq 2|\zeta - z|$. Therefore

$$I_2 \leqq c \int_{U_\delta \cap B(z_0, \varepsilon_0/2)} |\zeta - z_0|^{\varrho + 1 - 2n} dm(\zeta)$$

$$II_2 \leqq c \int_{U_\delta \cap B(z_0, \varepsilon_0/2)} \frac{|\zeta - z_0|^{\varrho} dm(\zeta)}{(\delta + |\zeta - z_0|^2)^{n + \frac{|\beta|}{2}}} \leqq c \int_{U_\delta \cap B(z_0, \varepsilon_0)} (\delta + |\zeta - z_0|^2)^{-n - \frac{|\beta| - \varrho}{2}} dm(\zeta).$$

We evaluate these in the coordinates of Lemma 4.3. Writing $|x|^2 = \sum_{j=1}^{n} x_j^2$, $|y|^2 = \sum_{j=1}^{n} y_j^2$, $dx = dx_1 \dots dx_n$, $d_y = dy_1 \dots dy_n$ we have to verify that

$$\int_{\substack{|x| \leqq \eta \\ |y| \leqq c}} (|x|^2 + |y|^2)^{\frac{\varrho + 1 - 2n}{2}} dx\, dy = O(\eta^{1+\varrho})$$

$$\int_{\substack{|x| \leqq \eta \\ |y| \leqq c}} (\delta + |x|^2 + |y|^2)^{-n - \frac{t}{2}} dx\, dy = O(\eta^{-t})$$

which follow by elementary changes of variable.

Finally, for $\zeta \in R_3$ one has $|\zeta - z| \geqq \varepsilon_0/4$ and so $|K_0^\delta|$, $|D_z^\beta P_i^\delta|$ are bounded by c. Then I_3 and II_3 are dominated by $\int_{U_\delta} d(\zeta, x)^{\varrho} dm(\zeta)$, which is easily seen to satisfy the required estimates. □

5. Proof of the Theorems

5.1. Proof of Theorem 1

Clearly it is enough to approximate any $u \in C^\infty(\mathbb{C}^n)$ and since $u = C^\delta u + T^\delta \bar{\partial} u$ on U_δ, with $C^\delta u$ holomorphic in U_δ, it is sufficient to show that $T^\delta \bar{\partial} u \to 0$ in $C(X)$ and in $\mathrm{Lip}(x, X)$ as $\delta \searrow O$. By part (a) of Lemma 4.4 (with $\varrho = 0$), $\|T^\delta \bar{\partial} u\|_{C(X)} = O(\eta)$, and this proves the theorem for $C(X)$. In case of $\mathrm{Lip}(s, X)$ we have to show that, moreover

$$\sup_{\substack{z, w \in X \\ z \neq w}} \frac{|T^\delta \bar{\partial} u(w) - T^\delta \bar{\partial} u(z)|}{|z - w|^s} \to o \quad \text{as} \quad \delta \searrow O. \tag{18}$$

Write $\varepsilon = |z - w|$; if $\varepsilon \geqq \eta/4$, the quotient above is $O(\eta^{1-s})$ because we have already proved that $|T^\delta \bar{\partial} u| = O(\eta)$ on X. If $\varepsilon \leqq \eta/4$, then $B(z, 2\varepsilon) \subset V_\delta \subset U_\delta$ and, by the mean value theorem, the quotient above is bounded by $c\eta^{1-s} |\mathrm{grad}\, T^\delta \bar{\partial} u(w_0)|$ for some $w_0 \in V_\delta$. Now, $\bar{\partial}^i T^\delta \bar{\partial} u = \bar{\partial}^i u$ is uniformly bounded in δ and by formula (11)

$$|D^i T^\delta \bar{\partial} u(w_0)| \leqq \int_{U_\delta} |D^i \bar{\partial} u(\zeta)| |K_0^\delta(\zeta, w_0)| dm(\zeta) + \int_{U_\delta} |\bar{\partial} u(\zeta)| |P_i^\delta(\zeta, w_0)| dm(\zeta).$$

By the estimates in Lemma 4.4, this is also uniformly bounded in δ. Hence the sup in (18) is $O(\eta^{1-s})$. This ends the proof.

5.2. Proof of Theorem 2

Let $u \in C^{m,s}(\mathbb{C}^n)$ be such that $\bar\partial u = 0$ up to order $m-1$ on X, $0 \leqq s < 1$. We will show that there exist $g_\delta \in H(X)$ such that $g_\delta \to u$ in $C^{m,s}(X)$. Here it is not convenient to use the $C^\delta u$ as approximating functions. The reason is that, when estimating $D^\alpha T^\delta \bar\partial u$ on X, we could use (11) only if $|\alpha| \leqq m-1$ and we would obtain just a C^{m-1}-approximation result (of course this would work if $m=\infty$). Instead we will use a procedure similar to the one giving the Mergelyan theorem in one complex variable (see for instance [17]).

We set $\omega(\tau) = \sup\{|D^\alpha \bar\partial u(\zeta)| : d(\zeta, X) \leqq \tau, |\alpha| = m-1\}$. Then $\omega(\tau) = \omega_1(\tau)\tau^s$ with $\omega_1(\tau) \to 0$ as $\tau \to 0$ and for $|\alpha| \leqq m-1$

$$|D^\alpha \bar\partial u(\zeta)| \leqq c\omega_1(d(\zeta, X)) d(\zeta, X)^{m+s-1-|\alpha|}. \tag{19}$$

Let φ be a fixed, positive radial C^∞-function with support in the unit ball and such that $\int \varphi = 1$. Let $\eta = \eta(\delta) = \delta^{1/2}$ and set $\varphi_\eta(w) = \eta^{-2n}\varphi(w/\eta)$. We define:

$$u_\delta(\zeta) = (u * \varphi_\eta)(\zeta) = \int_{\mathbb{C}^n} u(w)\varphi_\eta(\zeta - w)dm(w) = \int_{\mathbb{C}^n} u(\zeta - w)\varphi_\eta(w)dm(w)$$

We write $\tau = \tau(\delta) = \sup\{d(\zeta, X) : \zeta \in U_\delta\}$. Note that $\tau \geqq \eta$, for $r(\zeta) \leqq d(\zeta, X)^2$. We write $f_\delta = \bar\partial u_\delta$. Then u_δ and f_δ satisfy

$$\|u - u_\delta\|_{C^{m,s}(\mathbb{C}^n)} \to 0 \quad \text{as} \quad \delta \searrow 0, \tag{20}$$

$$|D^\alpha f_\delta(\zeta)| \leqq c\omega_1(2\tau)(\eta^{m+s-1-|\alpha|} + d(\zeta, X)^{m+s-1-|\alpha|}), \quad |\alpha| \leqq m-1, \quad \zeta \in U_\delta, \tag{21}$$

$$|D^\alpha f_\delta(\zeta)| \leqq c\frac{\omega_1(2\tau)}{\eta}(\eta^s + d(\zeta, X)^s), \qquad |\alpha| = m, \qquad \zeta \in U_\delta. \tag{22}$$

The first is well known. If $|\alpha| \leqq m-1$, we can differentiate under the integral sign; if $\zeta \in U_\delta$ and $|w| < \eta$, then $d(\zeta - w, X) \leqq d(\zeta, X) + \eta \leqq 2\tau$ and (19) implies (21). If $|\alpha| = m$, the last derivative is applied to φ_η and, since $\int |\text{grad}\,\varphi_\eta(w)| dm(w) \leqq c\eta^{-1}$, we get (22).

We apply now the basic decomposition to u_δ in $U_\delta : u_\delta = C^\delta u_\delta + T^\delta f_\delta$. We claim that $g_\delta \overset{\text{def}}{=} C^\delta u_\delta$ approaches u in $C^{m,s}(X)$ as $\delta \searrow 0$. Since only holomorphic derivatives need to be considered, and using (20) this amounts to prove, for $|\alpha| \leqq m$

$$\lim_{\delta \searrow 0} \|D^\alpha T^\delta f_\delta\|_{C(X)} = 0 \tag{23}$$

$$\lim_{\delta \searrow 0} \sup_{\substack{z, w \in X \\ z \neq w}} \frac{|R_z^{m-|\alpha|}(D^\alpha T^\delta f_\delta)(w)|}{|z - w|^{m+s-|\alpha|}} = 0 \tag{24}$$

[recall that $R_z^k v(w)$ denotes the error term in the Taylor development of $v(w)$ at z up to order k]. To prove (23), we use formula (11) (which is legitim since u_δ is C^∞) to obtain

$$|D^\alpha T^\delta f_\delta(z)| \leqq \int_{U_\delta} |D^\alpha f_\delta(\zeta)||K_0^\delta(\zeta, z)| dm(\zeta) + \sum_{i,\beta,\gamma}{}' \int_{U_\delta} |D^\gamma f_\delta(\zeta)||D_z^\beta P_i^\delta(\zeta, z)| dm(\zeta),$$

where $|\beta| + |\gamma| = |\alpha| - 1$. If $|\alpha| = m$, (21), (22) and the estimates in Lemma 4.4 [for $\varrho = 0$ and $\varrho = s$, in part (a), $\varrho = 0$ and $\varrho = |\beta| + s$ in part (b)] give for $z \in X$

$$\begin{aligned} |D^\alpha T^\delta f_\delta(z)| &\leqq c\omega_1(2\tau)\eta^{s-1}O(\eta) + c\omega_1(2\tau)\eta^{-1}O(\eta^{1+s}) \\ &\quad + c\sum_{i,\beta,\gamma}{}' \omega_1(2\tau)\{\eta^{m+s-1-|\gamma|}O(\eta^{-|\beta|}) + O(\eta^s)\} = c\omega_1(2\tau)\eta^s. \end{aligned}$$

In general, for $|\alpha|\leqq m$ we would obtain in the same manner

$$|D^\alpha T^\delta f_\delta(z)|\leqq c\omega_1(2\tau)\eta^{m+s-|\alpha|}, \qquad z\in X. \tag{25}$$

Hence (23) is proved.

For (24) we use the same method as in 5.1. First observe that, since the estimates in Lemma 4.4 hold for $z\in V_\delta$, (25) will hold for $z\in V_\delta$. Now, if β and γ are multiindexes, $|\beta|+|\gamma|\leqq m$, $\gamma_i\neq 0$, and $\tilde\gamma$ is defined by $\tilde\gamma_j=\gamma_j$, $j\neq i$, $\tilde\gamma_i=\gamma_i-1$, then $D^\beta\bar D^\gamma T^\delta f_\delta(z)=D^\beta\bar D^{\tilde\gamma}\bar D^i u_\delta(z)$, because $\bar\partial T^\delta f_\delta=f_\delta$, and so by (21), in all cases we obtain

$$|D^\beta\bar D^\gamma T^\delta f_\delta(z)|\leqq c\omega_1(2\tau)\eta^{m+s-|\beta|-|\gamma|}, \qquad z\in V_\delta, \qquad |\beta|+|\gamma|\leqq m. \tag{26}$$

Now fix α, $|\alpha|\leqq m$. Recall that

$$R_z^{m-|\alpha|}(D^\alpha T^\delta f_\delta)(w)=D^\alpha T^\delta f_\delta(w)-\sum_{|\beta|+|\gamma|\leqq m-|\alpha|}\frac{D^\beta\bar D^\gamma D^\alpha T^\delta f_\delta}{\beta!\gamma!}(z)(w-z)^\beta(\bar w-\bar z)^\gamma.$$

Hence this is bounded, using (26), by

$$c\omega_1(2\tau)\eta^{m+s-|\alpha|}+\sum_{|\beta|+|\gamma|\leqq m-|\alpha|}c\omega_1(2\tau)\eta^{m+s-|\beta|-|\gamma|-|\alpha|}|w-z|^{|\beta|-|\gamma|}.$$

Consequently the quotient in (24) is bounded by $c\omega_1(2\tau)$ in case $\varepsilon\overset{\text{def}}{=}|z-w|\geqq\eta/4$. If $\varepsilon\leqq\eta/4$, then $B(z,2\varepsilon)\subset V_\delta\subset U_\delta$ and by Taylor's formula (up to order $m-|\alpha|+1$), the same ratio is bounded by $c\eta^{1-s}$ times the $(m+1)^{\text{th}}$ gradient of $T^\delta f_\delta$ at some point $w_0\in V_\delta$. Now we will see that

$$|D^\beta\bar D^\gamma T^\delta f_\delta(w_0)|\leqq c\omega_1(2\tau)\eta^{s-1}, \qquad |\beta|+|\gamma|=m+1, \qquad w_0\in V_\delta. \tag{27}$$

This will prove that the sup in (24) is $O(\omega_1(2\tau))$ and the proof of the theorem will be finished. If some $\gamma_i\neq 0$, (27) follows from (22). In case $\gamma=0$ we use the formula (11); it is easily seen that

$$|D^\beta f_\delta(\zeta)|\leqq c\frac{\omega_1(2\tau)}{\eta^2}(\eta^s+d(\zeta,X)^s), \qquad |\beta|=m+1, \qquad \zeta\in U_\delta.$$

Then the estimates of Lemma 4.4 give

$$|D^\beta T^\delta f_\delta(w_0)|\leqq c\omega_1(2\tau)\Big\{\eta^{s-2}O(\eta)+\eta^{-2}O(\eta^{1+s})+\sum_{\substack{|\beta_1|+|\beta_2|=m\\|\beta_1|<m}}(\eta^{m+s-1-|\beta_1|}\eta^{-|\beta_2|}+O(\eta^{s-1}))+\sum_{|\beta_1|=m}\eta^{s-1}O(1)+\eta^{-1}O(\eta^s)\Big\}=c\omega_1(2\tau)\eta^{s-1}. \quad \square$$

Remark. In case $r(\zeta)\simeq d(\zeta,X)^2$ then $\eta\simeq\tau$ and the proofs are somewhat simpler. It is clear that in general one cannot choose such r, keeping the strict plurisubharmonicity at X. One can do so in case X is a totally-real submanifold [9].

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Singularities of Elliptic Equations with an Exponential Nonlinearity

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Introduction

This paper is concerned with the existence and the description of the isolated singularities of the solutions of the equation

$$-\Delta u + g(u) = f, \tag{0.1}$$

in $\Omega' = \Omega - \{0\}$ where Ω is an open subset of $\mathbb{R}^2$ containing 0, g a continuous non-decreasing real valued function defined on $\mathbb{R}$ and f a continuous function.

When $g(u) = u|u|^{q-1}$, $q > 1$, and $f = 0$ many results concerning the singularities of the solutions of (0.1) have been given by Veron [11]. In particular he gave a full description of the isolated singularities when $q \geqq 3$:

i) either $u(x)/\mathrm{Log}(1/|x|)$ converges as $x \to 0$ to some constant c which can take any value, and u satisfies

$$-\Delta u + u|u|^{q-1} = 2\pi c \delta_0 \text{ in } \mathscr{D}'(\Omega), \tag{0.2}$$

ii) or $|x|^{2(q-1)}u(x)$ converges to a constant which can take only the two values $\pm(2/(q-1))^{2/(q-1)}$.

In this paper we are interested in the solutions u of (0.1) satisfying the limiting growth condition

$$\lim_{x \to 0} xu(x) = 0, \tag{0.3}$$

and we first prove the following isotropy result: *for any solution of* (0.1) *satisfying* (0.3), *the following limit*

$$\lim_{x \to 0} u(x)/\mathrm{Log}(1/|x|), \tag{0.4}$$

exists in $\mathbb{R} \cup \{-\infty, +\infty\}$. *Moreover if the limit* (0.4) *is zero then u is nonsingular in* Ω.

In order to give a full description of u near 0 we suppose that g satisfies the following technical condition

$$\left.\begin{array}{l} \text{for any } a > 0, \ \lim_{r \to +\infty} e^{-ar} g(r) \text{ and } \lim_{r \to -\infty} e^{ar} g(r) \\ \text{exist in } \mathbb{R} \cup \{+\infty, -\infty\}, \end{array}\right\} \tag{0.5}$$

and we introduce the exponential orders of growth of g [9]

$$a_g^+ = \sup\left\{a \in \mathbb{R}^+ : \lim_{r \to +\infty} a^{-ar} g(r) = +\infty\right\} = \inf\left\{a \in \mathbb{R}^+ : \lim_{r \to +\infty} e^{-ar} g(r) = 0\right\}, \quad (0.6)$$

$$a_g^- = \sup\left\{a \in \mathbb{R}^+ : \lim_{r \to -\infty} e^{ar} g(r) = -\infty\right\} = \inf\left\{a \in \mathbb{R}^+ : \lim_{r \to -\infty} e^{ar} g(r) = 0\right\}. \quad (0.7)$$

Under the assumptions (0.3), (0.4) *the limit c obtained in* (0.4) *satisfies*

$$-2/a_g^- \leqq c \leqq 2/a_g^+ , \quad (0.8)$$

and if c is finite, u is the unique weak solution in some appropriate sense of

$$\left.\begin{aligned} &-\Delta v + g(v) = 2\pi c \delta_0 + f, \\ &v_{|\partial\Omega} = u_{|\partial\Omega} \end{aligned}\right\} \quad (0.9)$$

If $a_g^+ = 0$ (resp. $a_g^- = 0$) there may exist solutions of (0.1) such that $c = +\infty$ (resp. $c = -\infty$).

If $a_g^- = a_g^+ = \infty$ then $c = 0$ and 0 is a removable singularity for any solution of (0.1). In fact we prove a quite more general result extending to the exponential case some results of Brezis and Veron [4] and Veron [12]:

Suppose Ω is an open subset of $\mathbb{R}^N$, $N \geqq 2$, Σ a C^1 submanifold of Ω of dimension $N-2$ and g a continuous real valued function on $\mathbb{R} \times \bar{\Omega}$. If g satisfies

$$\left.\begin{aligned} &\liminf_{r \to +\infty} e^{-ar} g(x, r) = +\infty, \quad \limsup_{r \to -\infty} e^{ar} g(x, r) = -\infty, \\ &\text{uniformly on } \bar{\Omega}, \text{ for any } a > 0; \end{aligned}\right\} \quad (0.10)$$

then any function $u \in C^1(\Omega - \Sigma)$ satisfying

$$-\Delta u + g(x, u) = 0, \quad (0.11)$$

in $\mathscr{D}'(\Omega - \Sigma)$ can be extended to Ω into a C^1 function satisfying (0.11) *in $\mathscr{D}'(\Omega)$.*

The contents of the article is the following: 1. Singular solutions, 2. Removable singularities, 3. Appendix.

1. Singular Solutions

In this section Ω is an open subset of $\mathbb{R}^2$ containing 0 and $\Omega' = \Omega - \{0\}$. Our first result describes the isotropic behaviour of any solution of (0.1).

Theorem 1.1. *Suppose g is a continuous nondecreasing real valued function and $f \in C^0(\Omega)$. If u is twice continuously differentiable in Ω' and satisfies* (0.3) *and*

$$-\Delta u + g(u) = f, \quad (1.1)$$

in $\mathscr{D}'(\Omega')$; then the following limit

$$\lim_{x \to 0} u(x)/\mathrm{Log}(1/|x|), \quad (1.2)$$

exists in $\mathbb{R} \cup \{+\infty, -\infty\}$. Moreover if the limit is zero u is nonsingular and satisfies (1.1) *in $\mathscr{D}'(\Omega)$.*

Remark 1.2. The condition (0.3) is fulfilled as soon as g satisfies the condition

$$\lim_{r\to+\infty} r\left\{\int_{-\infty}^{-r}(sg(s))^{-1/2}ds+\int_{r}^{+\infty}(sg(s))^{-1/2}ds\right\}=0\,, \tag{1.3}$$

which is the case if $\lim_{|r|\to+\infty} g(r)r^{-3}=+\infty$ [10].

Without loss of generality we suppose $\Omega=\{x:|x|<2\}$, we set (r,θ) the polar coordinates in $\mathbb{R}^2$, $r\geqq 0$, $\theta\in S^1$, and

$$\bar{u}(r)=\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)d\theta\,,\qquad \bar{f}(r)=\frac{1}{2\pi}\int_0^{2\pi}f(r,\theta)d\theta\,,$$

and we consider the change of variable

$$\left.\begin{aligned}&t=\operatorname{Log}\frac{1}{r}\,,\qquad t\geqq 0\,;\qquad v(t,\theta)=u(r,\theta)\,,\qquad \bar{v}(t)=\bar{u}(r)\\&h(t,\theta)=f(r,\theta)\,,\qquad \bar{h}(t)=\bar{f}(r)\,.\end{aligned}\right\} \tag{1.4}$$

The functions v and h are 2π-periodic with respect to θ and they satisfy

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial^2 v}{\partial\theta^2}=e^{-2t}(-h+g(v))\quad\text{on}\quad(0,+\infty)\times S^1\,. \tag{1.5}$$

Lemma 1.3. *Under the hypotheses of Theorem 1.1, the function* $t\mapsto\|v(t,\cdot)-\bar{v}(t)\|^2_{L^\infty(S^1)}$ *is integrable on* $[0,+\infty)$.

Proof. The function $\bar{v}$ satisfies

$$\frac{d^2\bar{v}}{dt^2}=e^{-2t}(-\bar{h}+\overline{g(v)})\,, \tag{1.6}$$

with $\overline{g(v)}=\frac{1}{2\pi}\int_0^{2\pi}g(v)d\theta$. As we have $\int_0^{2\pi}\frac{\partial^2}{\partial\theta^2}(v-\bar{v})(v-\bar{v})d\theta\leqq-\int_0^{2\pi}(v-\bar{v})^2d\theta$ and

$$\int_0^{2\pi}(g(v)-\overline{g(v)})(v-\bar{v})d\theta=\int_0^{2\pi}(g(v)-g(\bar{v}))(v-\bar{v})d\theta\geqq 0\,,$$

we deduce from (1.5) and (1.6):

$$\int_0^{2\pi}\frac{\partial^2}{\partial t^2}(v-\bar{v})(v-\bar{v})d\theta-\int_0^{2\pi}(v-\bar{v})^2d\theta\geqq e^{-2t}\int_0^{2\pi}(\bar{h}-h)(v-\bar{v})d\theta\,. \tag{1.7}$$

If $\omega=\{t\geqq 0:\|v(t,\cdot)-\bar{v}(t)\|_{L^2(S^1)}>0\}$, ω is open and we have

$$\frac{d^2}{dt^2}\|v-\bar{v}\|_{L^2(S^1)}-\|v-\bar{v}\|_{L^2(S^1)}\geqq -e^{-2t}\|h-\bar{h}\|_{L^2(S^1)}\,, \tag{1.8}$$

on ω. Take (σ,τ) a connected component of ω. If $\tau<+\infty$, we deduce from the maximum principle that $\|v-\bar{v}\|_{L^2(S^1)}$ is majorized on $[\sigma,\tau]$ by the solution of

$$\left.\begin{aligned}&\frac{d^2X}{dt^2}-X=-e^{-2t}\|h-\bar{h}\|_{L^\infty(\sigma,\tau;L^2(S^1))}\,,\\&X(\sigma)=\|v(\sigma,\cdot)-\bar{v}(\sigma)\|_{L^2(S^1)}\,,\qquad X(\tau)=0\,.\end{aligned}\right\} \tag{1.9}$$

If $\sigma>0$, $X(\sigma)=0$, so in any case $X(\sigma)\leqq\|v(0,\cdot)-\bar{v}(0)\|_{L^2(S^1)}$. From the uniqueness of the solution of (1.9), this solution is majorized on $(0,\tau)$ by the solution of

$$\left.\begin{aligned}&\frac{d^2X}{dt^2}-X=-e^{-2t}\|h-\bar{h}\|_{L^\infty(0,+\infty;L^2(S^1))},\\&X(0)=\|v(0,\cdot)-\bar{v}(0)\|_{L^2(S^1)},\qquad \lim_{t\to+\infty}X(t)=0,\end{aligned}\right\}\tag{1.10}$$

which is

$$X_0(t)=e^{-t}\|v(0,\cdot)-\bar{v}(0)\|_{L^2(S^1)}+\tfrac{1}{3}(e^{-t}-e^{-2t})\,\|h-\bar{h}\|_{L^\infty(0,+\infty;L^2(S^1))}.$$

If $\tau=+\infty$, we consider for $\varepsilon>0$ the solution X_ε of

$$\left.\begin{aligned}&\frac{d^2X}{dt^2}-X=-e^{-2t}\|h-\bar{h}\|_{L^\infty(0,+\infty;L^2(S^1))},\\&X(0)=\|v(0,\cdot)-\bar{v}(0)\|_{L^2(S^1)},\qquad \lim_{t\to+\infty}e^{-t}X(t)=\varepsilon.\end{aligned}\right\}\tag{1.11}$$

As from (0.3), we have $\|v(\cdot,t)-\bar{v}(t)\|_{L^2(S^1)}=o(e^t)$ when $t\to+\infty$ we deduce, from the maximum principle, that, for any $\varepsilon>0$ and $t\geqq0$, $\|v(t,\cdot)-\bar{v}(t)\|_{L^2(S^1)}\leqq X_\varepsilon(t)$. If we compute X_ε and make $\varepsilon\downarrow0$ we deduce $\|v(t,\cdot)-v(t)\|_{L^2(S^1)}\leqq X_0(t)$ where X_0 is given above.

From estimate (1.7) the function

$$t\mapsto\tfrac{1}{2}\|v-\bar{v}\|^2_{L^2(S^1)}(t)+\int_0^t\int_0^s\int_0^{2\pi}e^{-2\sigma}(h-\bar{h})(v-\bar{v})d\theta\,d\sigma\,ds=H(t)\tag{1.12}$$

is convex and $H(t)/t$ is bounded. As $H'(t)$ is increasing, $\lim_{t\to+\infty}H'(t)$ is finite. From (1.5) and (1.6) we have

$$\int_0^{2\pi}\frac{\partial^2}{\partial t^2}(v-\bar{v})(v-\bar{v})d\sigma-\int_0^{2\pi}\left(\frac{\partial}{\partial\theta}(v-\bar{v})\right)^2d\theta\geqq e^{-2t}\int_0^{2\pi}(h-\bar{h})(v-\bar{v})d\theta\tag{1.13}$$

which implies

$$\frac{d^2}{dt^2}H(t)\geqq\int_0^{2\pi}\left(\frac{\partial}{\partial\theta}(v-\bar{v})\right)^2d\theta.\tag{1.14}$$

From Sobolev inequality, $\int_0^{2\pi}\left(\frac{\partial}{\partial\theta}(v-\bar{v})\right)^2d\theta+\int_0^{2\pi}(v-\bar{v})^2d\theta\geqq c\|v-\bar{v}\|^2_{L^\infty(S^1)}$ and we have, for some constant $c>0$,

$$\frac{d^2}{dt^2}H(t)+\int_0^{2\pi}(v-\bar{v})^2d\theta\geqq c\|v-\bar{v}\|^2_{L^2(S^1)}.\tag{1.15}$$

Integrating (1.15) we get

$$\frac{d}{dt}H(t)-\frac{d}{dt}H(0)+\int_0^t\int_0^{2\pi}(v-\bar{v})^2d\theta\,ds\geqq c\int_0^t\|v-\bar{v}\|^2_{L^\infty(S^1)}ds.\tag{1.16}$$

From (1.10) $\int_0^{+\infty}\int_0^{2\pi}(v-\bar{v})^2d\theta\,ds<+\infty$, then

$$c\int_0^{+\infty}\|v-\bar{v}\|_{L^\infty(S^1)}dt\leqq\lim_{t\to+\infty}\frac{dH}{dt}(t)-\frac{dH}{dt}(0)+\int_0^{+\infty}\int_0^{2\pi}(v-\bar{v})^2d\theta\,dt<+\infty.\tag{1.17}$$

Lemma 1.4. *Under the hypotheses of Theorem 1.1,* $\|v(t,\cdot)-\bar{v}(t)\|_{L^\infty(S_1)}$ *converges to* 0 *as* t *tends to* $+\infty$.

Proof. Step 1. We claim that we can construct a sequence $\{t_n\}$ such that

i) $\{t_n\}$ is increasing and $\lim_{n\to+\infty} t_n=+\infty$,
ii) $\lim_{n\to+\infty} \|v(t_n,\cdot)-\bar{v}(t_n)\|_{L^\infty(S^1)}=0$,
iii) $1\leqq t_{n+1}-t_n\leqq 3$.

To see that set $\alpha_n=\min\{\|v(t,\cdot)-v(t)\|_{L^\infty(S^1)}: t\in[2n,2n+1]\}$ and we define t_n by $t_n\in[2n,2n+1]$ and $\alpha_n=\|v(t_n,\cdot)-\bar{v}(t_n)\|_{L^\infty(S^1)}$. From Lemma 1.3 the sequence $\{t_n\}$ satisfies i), ii), and iii).

Step 2. For any $0<\alpha<\beta$ and $\tilde{h}\in L^\infty((\alpha,\beta)\times S^1)$ set $\tilde{v}$ the solution of

$$\left.\begin{aligned}&\frac{\partial^2\tilde{v}}{\partial t^2}+\frac{\partial^2\tilde{v}}{\partial\theta^2}=e^{-2t}(g(\tilde{v})-\tilde{h}) \text{ in } (\alpha,\beta)\times S^1,\\ &\tilde{v}(\alpha,\cdot) \text{ and } \tilde{v}(\beta,\cdot) \text{ given in } C^2(S^1).\end{aligned}\right\}\tag{1.18}$$

For any $p<+\infty$, $\tilde{v}\in W^{2,p}((\alpha,\beta)\times S^1)\cap C^1([\alpha,\beta]\times S^1)$. Comparing (1.5) and (1.18) we deduce classically from the monotonicity of g and the accretivity of $-\dfrac{\partial^2}{\partial\theta^2}$ in $L^p(S^1)$:

$$\frac{d^2}{dt^2}\left\{\|v-\tilde{v}\|_{L^p(S^1)}(t)+\int_\alpha^t\int_\alpha^s e^{-2\sigma}\|h-\tilde{h}\|_{L^p(S^1)}d\sigma\, ds\right\}\geqq 0,\tag{1.19}$$

on the open subset of $[\alpha,\beta]$ where $\|v-\tilde{v}\|_{L^p(S^1)}>0$. Hence the function

$$t\mapsto\|v-\tilde{v}\|_{L^p(S^1)}(t)+\int_\alpha^t\int_\alpha^s e^{-2\sigma}\|h-\tilde{h}\|_{L^p(S^1)}d\sigma\, ds$$

is convex. Going to the limit as $p\to+\infty$ we have the convexity of

$$\|v-\tilde{v}\|_{L^\infty(S^1)}+\int_\alpha^t\int_\alpha^s e^{-2\sigma}\|h-\tilde{h}\|_{L^\infty(S^1)}d\sigma\, ds$$

and then

$$\begin{aligned}\|v-\tilde{v}\|_{L^\infty((\alpha,\beta)\times S^1)}\leqq\max\Big\{&\|(v-\tilde{v})(\alpha)\|_{L^\infty(S^1)},\ \|v-\tilde{v})(\beta)\|_{L^\infty(S^1)}+\dots\\ &+\int_\alpha^\beta\int_\alpha^s e^{-2\sigma}\|h-\tilde{h}\|_{L^\infty(S^1)}d\sigma\, ds\Big\}.\end{aligned}\tag{1.20}$$

Step 3. We fix $\varepsilon>0$. There exists n_0 such that for $n\geqq n_0$

i) $\|v(t_n,\cdot)-\bar{v}(t_n)\|_{L^\infty(S^1)}<\varepsilon$,
ii) $\int_{t_n}^{t_{n+1}}\int_{t_n}^s e^{-2\sigma}\|h\|_{L^\infty(S^1)}d\sigma\, ds<\varepsilon$.

If ψ_n is the solution of

$$\left.\begin{aligned}&\frac{\partial^2\psi_n}{\partial t^2}+\frac{\partial^2\psi_n}{\partial\theta^2}=e^{-2t}(g(\psi_n)-\|h\|_{L^\infty(S^1)}) \text{ in } (t_n,t_{n+1})\times S^1,\\ &\psi_n(t_n,\cdot)=\bar{v}(t_n)+\varepsilon,\qquad \psi_n(t_{n+1},\cdot)=\bar{v}(t_{n+1})+\varepsilon,\end{aligned}\right\}\tag{1.21}$$

ψ_n is independent of θ; from the monotonicity of g and i), it follows that $\psi_n \geqq v$. If we apply (1.20) we get

$$0 \leqq \psi_n - v \leqq 3\varepsilon \quad \text{in} \quad (t_n, t_{n+1}) \times S^1 . \tag{1.22}$$

If we average (1.22) over S^1 we get, since ψ_n does not depend on θ,

$$0 \leqq \psi_n - \bar{v} \leqq 3\varepsilon \quad \text{in} \quad (t_n, t_{n+1}) \times S^1 , \tag{1.23}$$

which yields

$$\|v - \bar{v}\|_{L^\infty(t_n, t_{n+1}) \times S^1)} \leqq 3\varepsilon , \tag{1.24}$$

and $\limsup_{t \to +\infty} \|v(t, \cdot) - \bar{v}(t)\|_{L^\infty(S^1)} \leqq 3\varepsilon$, for any $\varepsilon > 0$, which ends the proof.

Proof of Theorem 1.1. First we remark that if $\{\bar{u}(r)\}$ is bounded for $0 \leqq r \leqq 1$, it is the same for $\{u(x)\}$ from Lemma 1.4 and $\lim_{x \to 0} u(x)/\mathrm{Log}(1/|x|) = 0$.

Step 1. Suppose $\{\bar{u}(r)\}$ *is unbounded.* There exists a sequence $\{r_n\}$ such that $\lim_{n \to +\infty} r_n = 0$ and $\lim_{n \to +\infty} \bar{u}(r_n) = +\infty$ (or $-\infty$ in the same way). *We claim that* $\lim_{x \to 0} u(x) = +\infty$. From Lemma 1.4, there exists a strictly increasing unbounded sequence α_n such that $u(r_n, \cdot) \geqq \alpha_n$. Set $\tilde{g}(s) = g(s) - g(0)$ and ψ_n the solution of

$$\left.\begin{aligned} &-\Delta \psi_n + \tilde{g}(\psi_n) = -(\|f\|_{L^\infty(\Omega)} + |g(0)|) \text{ on } [0, r_n) \times S^1 , \\ &\psi_n(r_n, \cdot) = \alpha \text{ (fixed real number)} . \end{aligned}\right\} \tag{1.25}$$

ψ_n is rotationally symmetric and nondecreasing as a function of r, so it satisfies

$$\left.\begin{aligned} &\frac{d}{dr}\left(r \frac{d}{dr} \psi_n\right) \leqq r(\tilde{g}(\alpha) + \|f\|_{L^\infty(\Omega)} + |g(0)|) , \\ &\psi_n(r_n) = \alpha , \quad \frac{d}{dr} \psi_n(0) = 0 . \end{aligned}\right\} \tag{1.26}$$

If we integrate (1.26) twice we get

$$\psi_n(0) \geqq \alpha - \frac{r_n^2}{4} (\tilde{g}(\alpha) + \|f\|_{L^\infty(\Omega)} + |g(0)|) . \tag{1.27}$$

For a fixed $\alpha > 0$ there exists $n_0 \in N$ such that for $n \geqq n_0$

i) $\alpha_n \geqq \alpha$,

ii) $\frac{r_n^2}{4} (\tilde{g}(\alpha) + \|f\|_{L^\infty(\Omega)} + |g(0)|) < \alpha/2$.

From the maximum principle and the monotonicity of $\tilde{g}$, u is minorized on any shell $[r_n, r_{n_0}] \times S^1$ by ψ_{n_0}; so $u \geqq \psi_{n_0}(0) \geqq \alpha/2$ and

$$\liminf_{x \to 0} u(x) \geqq \alpha/2 , \tag{1.32}$$

for any $\alpha > 0$, so $\lim_{x \to 0} u(x) = +\infty$.

Step 2. Suppose $\lim_{x \to 0} u(x) = +\infty$ *(or* $-\infty$ *in the same way), then we claim that*

$$\lim_{x \to 0} u(x)/\mathrm{Log}(1/|x|)$$

exists in $\mathbb{R}^+ \cup \{+\infty\}$. With the change of variable (1.4) we have $\lim_{t\to+\infty} v(t,\cdot) = +\infty$ uniformly on S^1. As g is nondecreasing and f is continuous at 0 there exists $\lambda \in \mathbb{R} \cup \{+\infty\}$ such that $\lim_{t\to+\infty} \overline{(g(v)-h)}(t) = \lambda$. In order to prove that $u(x)/\mathrm{Log}(1/|x|)$ admits a limit at 0 it is sufficient to prove that $\bar{v}(t)/t$ admits a limit at infinity as $\lim_{t\to+\infty} \|\bar{v}(t) - v(t,\cdot)\|_{L^\infty(S^1)} = 0$. We shall distinguish two cases

i) $\lambda = +\infty$: As $\bar{v}$ satisfies (1.6) we deduce that $\bar{v}$ is convex on $[T, +\infty)$ for some $T>0$ and $(\bar{v}(t)-\bar{v}(T))/(t-T)$ is increasing, so it admits a limit in $\mathbb{R}^+ \cup \{+\infty\}$. As $\lim_{t\to+\infty} \bar{v}(t) = +\infty$, $\bar{v}(t)/t$ admits the same limit.

ii) λ is finite: From (1.6) $\frac{d^2\bar{v}}{dt^2}$ is integrable on $[0, +\infty)$ and $\frac{d\bar{v}}{dt}$ admits a finite limit as t tends to $+\infty$. As $\frac{\bar{v}(t)}{t} = \frac{\bar{v}(0)}{t} + \frac{1}{t}\int_0^t \frac{d\bar{v}}{ds}\,ds$, $\bar{v}(t)/t$ admits the same limit.

Step 3. Suppose $\lim_{x\to 0} u(x)/\mathrm{Log}(1/|x|) = 0$ and set μ the solution of

$$\left.\begin{aligned} -\Delta\mu &= (f-g(0))^+ && \text{in } \{x : |x| \leqq 1\}, \\ \mu(x) &= u^+(x) && \text{for } |x| = 1. \end{aligned}\right\} \tag{1.33}$$

As $\mu \geqq 0$ and $\tilde{g}(s) = g(s) - g(0)$ is nondecreasing and vanishes at 0, μ is a supersolution of (1.1) and for any $\varepsilon > 0$ we deduce from classical comparison principles that for $|x| \leqq 1$, $u(x)$ is majorized by $\mu(x) + \varepsilon\,\mathrm{Log}(1/|x|)$. Letting $\varepsilon \downarrow 0$ we get $u(x) \leqq \mu(x)$. In the same way u is minorized in $\{x : |x| \leqq 1\}$ by a fixed continuous function. As u is bounded we deduce from Theorem 11 of Serrin [6] that u can be extended to Ω as a C^1 function which satisfies (1.1) in $\mathscr{D}'(\Omega)$, which ends the proof.

In what follows we characterize u when the limit (1.2) is finite.

Theorem 1.5. *Suppose g is a continuous nondecreasing real valued function satisfying* (0.5) *and $f \in C^0(\Omega)$. If $u \in C^2(\Omega')$ satisfies* (0.3) *and* (1.1), *then the limit c of $u(x)/\mathrm{Log}(1/|x|)$ is such that*

$$-2/a_g^- \leqq c \leqq 2/a_g^+ . \tag{1.34}$$

Moreover, if c is finite, $g(u) \in L^1_{\mathrm{loc}}(\Omega)$ and u satisfies

$$-\Delta u + g(u) = 2\pi c \delta_0 + f, \tag{1.35}$$

in $\mathscr{D}'(\Omega)$.

We suppose again that $\Omega = \{x : |x| < 2\}$. We have the following a priori estimate

Lemma 1.6. *Suppose $v \in C^2(\Omega')$ satisfies*

$$-\Delta v + a e^{\alpha v} \leqq c \quad \text{on} \quad \{x \in \Omega' : v(x) > 0\}, \tag{1.36}$$

for some constants a, α, and $c > 0$; then

$$v(x) \leqq \frac{2}{\alpha}\mathrm{Log}(1/|x|) + B \quad \textit{for} \quad 0 < |x| \leqq 1, \tag{1.37}$$

where B depends on a, α, and c.

Proof. We choose x_0 such that $0<|x_0|\leqq 1$ and we set

$$\psi(x)=\lambda\,\mathrm{Log}(1/(R^2-|x-x_0|^2))+\mu$$

for $0\leqq|x-x_0|<R$ where $R<|x_0|$ and λ and μ are to be chosen such that $-\Delta\psi+ae^{\alpha\psi}\geqq c$. For the sake of simplicity set $\psi(r)=\lambda\,\mathrm{Log}(1/(R^2-r^2))+\mu$; then

$$\frac{\partial\psi}{\partial r}=\frac{2\lambda r}{R^2-r^2},$$

$$\frac{\partial^2\psi}{\partial r^2}=2\lambda\left\{\frac{1}{R^2-r^2}+\frac{2r^2}{(R^2-r^2)^2}\right\},$$

$$\Delta\psi=\frac{4\lambda R^2}{(R^2-r^2)^2},$$

$$-\Delta\psi+ae^{\alpha\psi}=-\frac{4\lambda R^2}{(R^2-r^2)^2}+a\frac{e^{\alpha\mu}}{(R^2-r^2)^{\alpha\lambda}}.$$

we take $\lambda=2/\alpha$ and $\mu=\frac{1}{\alpha}\mathrm{Log}\{(\alpha c+8)R^2/a\alpha\}$ and we have

$$-\Delta\psi+ae^{\alpha\psi}\geqq c\quad\text{for}\quad 0\leqq|x-x_0|<R. \tag{1.38}$$

As $\lim_{r\to R}\psi(r)=+\infty$ and v is bounded in $\{x:0\leqq|x-x_0|<R\}$, we deduce, from Kato's inequality and the maximum principle as in Lemma 3 of [4] that $v(x)\leqq\psi(x)$ for $0\leqq|x-x_0|<R$. In particular

$$v(x_0)\leqq\psi(x_0)\leqq\frac{2}{\alpha}\mathrm{Log}(1/R)+\frac{1}{\alpha}\mathrm{Log}(\alpha c+8)/a\alpha). \tag{1.39}$$

If we make $R\to|x_0|$, we get (1.37) with $B=\frac{1}{\alpha}\mathrm{Log}((\alpha c+8)/a\alpha)$.

Proof of Theorem 1.5. If $a_g^+=0$ (resp. $a_g^-=0$) we set $2/a_g^+=+\infty$ (resp. $-2/a_g^-=-\infty$) and there exists no finite upper bound for c (resp. finite lower bound).

Step 1. Proof of (1.34). We suppose $a_g^+>0$. For any α, $0<\alpha<a_g^+$, there exist $a>0$, $c>0$ such that $g(s)\geqq ae^{\alpha s}-c$ for any s. Hence

$$-\Delta u+ae^{\alpha u}\leqq c\ \text{in}\ \Omega'. \tag{1.40}$$

From Theorem 1.1 and (1.37), $\lim_{x\to 0}u(x)/\mathrm{Log}(1/|x|)\leqq 2/\alpha$. If we make $\alpha\uparrow a_g^+$ we get the right hand side of (1.34). We prove the inequality $\lim_{x\to 0}u(x)/\mathrm{Log}(1/|x|)\geqq -2/a_g^-$ in the same way.

Step 2. Proof of (1.35). Set $\tilde u$ the solution of

$$\left.\begin{aligned}&-\Delta\tilde u+g(\tilde u)=2\pi c\delta_0+f &&\text{in } \mathscr{D}'(\{x:|x|\leqq 1\}),\\ &\tilde u=u &&\text{for } |x|=1.\end{aligned}\right\} \tag{1.41}$$

Such a solution exists (see Appendix) and satisfies $g(\tilde u)\in L^1(\{x:|x|\leqq 1\})$ and

$$\lim_{x\to 0}\tilde u(x)/\mathrm{Log}(1/|x|)=c.$$

We set

$$h=\begin{cases}(g(u)-g(\tilde{u}))/(u-\tilde{u}) & \text{if } u\neq\tilde{u}\\ 0 & \text{if } u=\tilde{u}\end{cases}\qquad h\geqq 0,$$

and for any $\varepsilon>0$ let w_ε be the solution of

$$\left.\begin{aligned}-\Delta w_\varepsilon&=2\pi\varepsilon\delta_0 && \text{in } \mathscr{D}'(\{x:|x|\leqq 1\}),\\ w_\varepsilon&=0 && \text{for } |x|=1.\end{aligned}\right\}\tag{1.42}$$

The function $u-\tilde{u}$ is in $W^{2;p}_{\text{loc}}(\{x:0<|x|\leqq 1\})$ for any $p<+\infty$ and

$$\left.\begin{aligned}-\Delta(u-\tilde{u})+h(u-\tilde{u})&=0 && \text{in } \{x:0<|x|\leqq 1\},\\ u-\tilde{u}&=0 && \text{for } |x|=1.\end{aligned}\right\}\tag{1.43}$$

Moreover, as $w_\varepsilon\geqq 0$, we have

$$-\Delta w_\varepsilon+hw_\varepsilon\geqq 0 \quad \text{in} \quad \{x:0<|x|\leqq 1\}.\tag{1.44}$$

As $\lim_{x\to 0} w_\varepsilon(x)/\mathrm{Log}(1/|x|)=\varepsilon$, we have $\lim_{x\to 0}(u(x)-\tilde{u}(x)-w_\varepsilon(x))=-\infty$. Let p be a bounded Lipschitz continuous nondecreasing real valued function vanishing on $(-\infty,0]$, then the function $p(u-\tilde{u}-w_\varepsilon)$ vanishes in some neighbourhood of 0 and for $|x|=1$. We deduce from (1.43) and (1.44)

$$\int_{|x|\leqq 1}|\nabla(u-\tilde{u}-w_\varepsilon)|^2p'(u-\tilde{u}-w_\varepsilon)dx\leqq 0,\tag{1.45}$$

so $j(u-\tilde{u}-w_\varepsilon)=0$ a.e., where $j(s)=\int_0^s\sqrt{p'(\sigma)}d\sigma$. As a consequence

$$u-\tilde{u}\leqq w_\varepsilon \quad \text{in} \quad \{x:0<|x|\leqq 1\}.\tag{1.46}$$

If $\varepsilon\downarrow 0$, $w_\varepsilon\downarrow 0$, and $u-\tilde{u}\leqq 0$. In the same way $\tilde{u}-u\leqq 0$ and finally $u=\tilde{u}$, which ends the proof.

Remark 1.7. As a consequence of Theorem 1.5, if $a_g^-=a_g^+=+\infty$ then $c=0$ and u can be extended to Ω as a C^1 function. It means that if g has a "superexponential" growth an isolated singularity is always removable. In the next section we shall see a generalization of that result without any assumption of monotonicity on g.

Remark 1.8. When a_g^+ and a_g^- are finite and nonzero, which means that g is truly of exponential type, then there exists *only one type of singularities* for the solutions of (1.1) which are called the *weak singularities*, that is $u(x)/\mathrm{Log}(1/|x|)$ admits a finite limit. When g satisfies

$$\int_a^{+\infty}(sg(s))^{-1/2}ds+\int_{-\infty}^{-a}(sg(s))^{-1/2}ds<+\infty\tag{1.47}$$

for some $a>0$ and $a_g^+=a_g^-=0$, then there exist two types of positive (negative) singularities:

i) *weak singularities*, for example a solution of (1.41),

ii) *strong singularities* which means that the limit of $u(x)/\mathrm{Log}(1/|x|)$ is not finite.

In order to have a solution of (1.1) with a strong singularity at 0 we proceed as follows: for any $n \in \mathbb{N}$ set u_n the solution of

$$\left.\begin{array}{ll} -\Delta u_n + g(u_n) = 2\pi n \delta_0 + f, & \text{in } \mathscr{D}'(\{x : |x| < 1\}) \\ u_n = h & \text{for } |x| = 1, \end{array}\right\} \tag{1.48}$$

h being a fixed function. The function u_n satisfies

$$\lim_{x \to 0} u_n(x)/\mathrm{Log}(1/|x|) = n$$

and from Vazquez a priori estimate [10] for any compact subset K of $\{x : 0 < |x| \leqq 1\}$ there exists a constant C independant of n such that

$$u_n(x) \leqq C, \quad \forall x \in K. \tag{1.49}$$

Moreover the sequence $\{u_n\}$ is nondecreasing, so $\{u_n\}$ converges, as n goes to $+\infty$, to some function u which satisfies

$$\left.\begin{array}{ll} -\Delta u + g(u) = f & \text{in } \mathscr{D}'(\{x : 0 < |x| < 1\}), \\ u = h & \text{for } |x| = 1, \\ \lim_{x \to 0} u(x)/\mathrm{Log}(1/|x|) = +\infty. & \end{array}\right\} \tag{1.50}$$

When g is a power there exists only one type of strong singularities ([11] and [13] for some generalisations) and it can be obtained with the previous construction; it is unlikely that such a result still holds with (1.47) and $a_g^+ = a_g^- = 0$.

Remark 1.9. Following Theorem 1 of [7] the limit (1.2) of Theorem 1.1 is $< +\infty$ (resp. $> -\infty$) if $g(r)/r$ is bounded for $r > 1$ (resp. $r < -1$).

2. Removable Singularities

In this section Ω is an open subset of $\mathbb{R}^N$, $N \geqq 2$, and Σ a C^1 compact submanifold of Ω of dimension $N-2$. Our result which extends to the critical dimension $N-2$ a previous theorem of Veron [12] (see also [1] for a nice recent extension) is the following

Theorem 2.1. *Suppose g is a continuous real valued function on $\mathbb{R} \times \bar{\Omega}$ satisfying the growth condition* (0.10) *and $u \in C^1(\Omega - \Sigma)$ satisfies*

$$-\Delta u + g(\cdot, u) = 0, \tag{2.1}$$

in $\mathscr{D}'(\Omega - \Sigma)$. Then u coincides in $\Omega - \Sigma$ with a function $\tilde{u} \in C^1(\Omega)$ which satisfies (2.1) *in $\mathscr{D}'(\Omega)$.*

Without any loss of generality we can suppose that Ω is relatively compact and u is bounded in $\Omega - \mathscr{S}$ where $\mathscr{S}$ is any neighbourhood of Σ in Ω. For any $x \in \Omega$ we set $d(x) = \inf\{|x - y| : y \in \Sigma\}$.

Lemma 2.2. *Under the hypotheses of Theorem 2.1, for any $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that*

$$|u(x)| \leqq \varepsilon \,\mathrm{Log}(1/d(x)) + B_\varepsilon, \tag{2.2}$$

for any $x \in \Omega - \Sigma$.

Proof. The neighbourhood $\mathscr{S}$ of Σ can be taken so that for any $y \in \mathscr{S} - \Sigma$ the ball $\{x : 0 \leqq |x-y| < d(y)\}$ is included in Ω. As g satisfies (0.10), for any $\varepsilon > 0$ there exist a and $C > 0$ such that

$$g(x,s) \geqq a e^{2s/\varepsilon} - C. \tag{2.3}$$

Let x_0 be in $\mathscr{S} - \Sigma$. If we apply Lemma 1.6 in $\{x : 0 \leqq |x - x_0| < d(x_0)\}$ we get

$$u(x_0) \leqq \varepsilon \operatorname{Log}(1/d(x)) + \frac{\varepsilon}{2} \operatorname{Log}(2C + 8\varepsilon)/2a). \tag{2.4}$$

We obtain the lower bound for u in $\mathscr{S}$ in the same way. As u is bounded in $\Omega - \mathscr{S}$ we get (2.2).

Lemma 2.3. *Suppose $N > 2$ and define μ in $\Omega - \Sigma$ in the following way*

$$\mu(x) = \int_\Sigma \frac{d\sigma}{|x-\sigma|^{N-2}}. \tag{2.5}$$

Then μ is harmonic in $\Omega - \Sigma$ and there exists two positive constants C and D such that

$$\mu(x) \geqq C \operatorname{Log}(1/d(x)) - D, \tag{2.6}$$

for any $x \in \Omega - \Sigma$.

Proof. The first assertion is not difficult as μ is the convolution between the kernel $|x|^{2-N}$ and the measure δ_Σ defined by $\langle \delta_\Sigma, \varphi \rangle = \int_\Sigma \varphi(\sigma) d\sigma$ for any φ continuous with compact support in Ω. So μ solves

$$-\Delta \mu = (N-2)|S^{N-1}| \delta_\Sigma, \tag{2.7}$$

in $\mathscr{D}'(\Omega)$.

In order to prove (2.6) we fix a point $a \in \Sigma$ and set $B_\eta = \{x : |x-a| < \eta\}$ and $\Sigma_\eta = \Sigma \cap B_\eta$ $(\eta > 0)$. We have

$$\mu(x) = \int_{\Sigma_\eta} \frac{d\sigma}{|x-\sigma|^{N-2}} + \int_{\Sigma - \Sigma_\eta} \frac{d\sigma}{|x-\sigma|^{N-2}}. \tag{2.8}$$

If we take $x \in B_{\eta/2}$ then $|x - \sigma| \geqq \eta/2$ for any $\sigma \in \Sigma - \Sigma_\eta$, hence

$$\int_{\Sigma - \Sigma_\eta} \frac{d\sigma}{|x-\sigma|^{N-2}} \leqq (2/\eta)^{N-2} \int_\Sigma 1 d\sigma. \tag{2.9}$$

There exists a local diffeomorphism ψ from an open subset $G \subset \mathbb{R}^N$ onto B_η such that $\psi(0) = a$ and if $\omega = G \cap \mathbb{R}^{N-2}$, $\psi(\omega) = \Sigma_\eta$. The restriction $\tilde{\psi}$ of ψ to ω is a parametrization of Σ_η. If $u = (\tilde{u}, x) \in \mathbb{R}^{N-2} \times \mathbb{R}^2$, $\tilde{\psi}(\tilde{u}) = \psi(\tilde{u}, 0)$.

We have classically

$$\int_{\Sigma_\eta} \frac{d\sigma}{|x-\sigma|^{N-2}} = \int_\omega \sqrt{\left|\det\left(\frac{\partial \tilde{\psi}}{\partial \tilde{u}_i} \middle| \frac{\partial \tilde{\psi}}{\partial \tilde{u}_j}\right)\right|} \frac{d\tilde{u}}{|x - \tilde{\psi}(\tilde{u})|^{N-2}}. \tag{2.10}$$

As $\tilde{\psi}$ is a parametrization, $\sqrt{\left|\det\left(\frac{\partial \tilde{\psi}}{\partial \tilde{u}_i} \middle| \frac{\partial \tilde{\psi}}{\partial \tilde{u}_j}\right)\right|} \geqq \lambda > 0$ on ω. Moreover ψ and ψ^{-1} can be taken as uniformly Lipschitz continuous; so

$$\int_{\Sigma_\eta} \frac{d\sigma}{|x-\sigma|^{N-2}} \geqq C \int_\omega \frac{d\tilde{u}}{|y - \tilde{u}|^{N-2}}, \tag{2.11}$$

where $y=\psi^{-1}(x)$. If $\omega\supset\{\tilde{u}:|\tilde{u}|<\alpha\}$ and if we set $y=(\tilde{y},d)$ with $\tilde{y}\in\mathbb{R}^{N-2}$ and $d\in\mathbb{R}^2$, we have

$$\int_{\Sigma_\eta}\frac{d\sigma}{|x-\sigma|^{N-2}}\geqq C\int_{|\tilde{u}|\leqq\alpha}\frac{d\tilde{u}}{(|d|^2+|\tilde{y}-\tilde{u}|^2)^{(N-2)/2}}. \tag{2.12}$$

and the last integral is $\int_{|\tilde{u}-\tilde{y}|<\alpha}\frac{d\tilde{u}}{(|d|^2+|\tilde{u}|^2)^{(N-2)/2}}$. If we take x close enough to a, y is close to 0, $\{\tilde{u}:|\tilde{u}-\tilde{y}|<\alpha\}\supset\{\tilde{u}:|\tilde{u}|<\alpha/2\}$, so

$$\int_{|\tilde{u}-\tilde{y}|<\alpha}\frac{d\tilde{u}}{(|d|^2+|\tilde{u}|^2)^{(N-2)/2}}\geqq\int_{|\tilde{u}|<\alpha/2}\frac{d\tilde{u}}{(|d|^2+|\tilde{u}|^2)^{(N-2)/2}}=\int_0^{\alpha/2}\frac{r^{N-3}dr}{(|d|^2+r^2)^{(N-2)/2}}.$$

If we set $\varphi=\operatorname{argsh}(r/|d|)$, the last integral is $\int_0^{\operatorname{argsh}(\alpha/2|d|)} th^{N-3}\varphi d\varphi$, so

$$\int_{\Sigma_\eta}\frac{d\sigma}{|x-\sigma|^{N-2}}\geqq C\left\{th(1/2)\operatorname{argsh}(\alpha/2|d|)+\int_0^{1/2} th^{N-3}\varphi d\varphi\right\}. \tag{2.13}$$

But $d=\operatorname{dist}(y,\mathbb{R}^{N-2}\times\{0\})\geqq cd(x)$ as ψ is Lipshitz continuous and

$$\operatorname{argsh}(\alpha/2|d|)=\operatorname{Log}((\alpha/2+\sqrt{\alpha^2/4+|d|^2})/|d|),$$

then

$$\mu(x)\geqq C_\alpha\operatorname{Log}(1/d(x))-D_a, \tag{2.14}$$

for any x in a small neighbourhood $\mathcal{O}_a$ of $a\in\Sigma$ where C_a and D_a are positive and depend on $\mathcal{O}_a$. Using the compactness of Σ, (2.14) remains valid in some relatively compact neighbourhood $\mathcal{S}'$ of Σ. As u is bounded in $\Omega-\mathcal{S}'$, we get (2.6) (with different constants).

Proof of Theorem 2.1

First Case: $N\geqq 3$. As $g(u)\in L^\infty_{\text{loc}}(\Omega-\Sigma)$, $u\in W^{2,p}_{\text{loc}}(\Omega-\Sigma)$ for any $p<\infty$ and from (0.10)

$$-\Delta u+ae^u\leqq C\quad\text{a.e. on }\{x:u(x)\geqq 0\}. \tag{2.15}$$

We fix $\eta>0$ and $k>0$ such that $ae^k\geqq C$ and we set

$$v_\eta(x)=\eta\mu(x)+k+\|u^+\|_{L^\infty(\partial\Omega)}. \tag{2.16}$$

The function v_η satisfies

$$-\Delta v_\eta+ae^{v_\eta}\geqq C\quad\text{in}\quad\Omega-\Sigma, \tag{2.17}$$

hence, from Kato's inequality as in [4]

$$\Delta(u-v_\eta)^+\geqq 0\quad\text{in}\quad\mathcal{D}'(\Omega-\Sigma). \tag{2.18}$$

But $(u-v_\eta)^+$ vanishes in some neighbourhood of $\partial\Omega$ and in some neighbourhood of Σ (from Lemmas 2.2 and 2.3) so $(u-v_\eta)^+=0$ and

$$u(x)\leqq v_\eta(x)\quad\text{in}\quad\Omega-\Sigma. \tag{2.19}$$

If we make $\eta\downarrow 0$ we deduce $\|u^+\|_{L^\infty(\Omega)}\leqq k+\|u^+\|_{L^\infty(\partial\Omega)}$.

In the same way u^- is bounded in Ω. From Theorem 11 of Serrin [6] we deduce that u can be extended to Σ as a solution of (2.1) in $\mathscr{D}'(\Omega)$ and the extension $\tilde{u}$ is in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $p<\infty$.

Second Case: $N=2$. Σ is just the union of a finite number of points $\{a_1, \ldots, a_q\}$. We then set (for $\eta>0$ and k large enough)

$$v_\eta(x)=\eta \sum_{i=1}^{q} \operatorname{Log}(1/|x-a_i|)+k+\|u^+\|_{L^\infty(\partial\Omega)}, \tag{2.20}$$

which is harmonic in $\Omega-\Sigma$. As $\lim\limits_{x\to a_i} |u(x)|/\operatorname{Log}(1/|x-a_i|)=0$, we conclude as in the first case.

3. Appendix

In this section we study the following Dirichlet problem

$$\left.\begin{array}{ll} -\Delta u+g(u)=f+2\pi c\delta_0 & \text{in } \Omega, \\ u|_{\partial\Omega}=h & \text{on } \partial\Omega, \end{array}\right\} \tag{3.1}$$

where Ω is a bounded open subset of $\mathbb{R}^2$ containing 0, with a C^1 boundary $\partial\Omega$.

We set $d(\cdot)=\operatorname{dist}\cdot(\cdot,\partial\Omega)$. If f is a measurable function such that $d\cdot f\in L^1(\Omega)$ and $h\in L^1(\partial\Omega)$ we say that u solves (3.1) if $u\in L^1(\Omega)$, $d\cdot g(u)\in L^1(\Omega)$ and

$$\left.\begin{array}{l} \int\limits_{\Omega}(-u\Delta\zeta+g(u)\zeta)dx=2\pi c\zeta(0)+\int\limits_{\Omega} f\zeta dx-\int\limits_{\partial\Omega} h\dfrac{\partial\zeta}{\partial\nu}d\sigma, \\ \text{for any } \zeta\in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega). \end{array}\right\} \tag{3.2}$$

The problem (3.1) is studied without any measure in [2] and in $\mathbb{R}^2$ in [9]. Our result is the following

Theorem 3.1. *Suppose g is a continuous nondecreasing function satisfying the condition* (0.5). *Then for any $f\in L^r(\Omega)$, $r>1$, any $h\in L^\infty(\partial\Omega)$, and any c such that $-2/a_g^-\leqq c\leqq 2/a_g^+$ there exists a unique $v\in L^1(\Omega)$ with $g(v)\in L^1(\Omega)$ solution of* (3.1). *Moreover, for any $\varepsilon>0$, v is bounded in $\{x\in\Omega:|x|\geqq\varepsilon\}$, and*

$$\lim_{x\to 0} v(x)/\operatorname{Log}(1/|x|)=c. \tag{3.3}$$

Proof. Let $\{\varrho_n\}$ be the sequence of functions with value n^2/π for $|x|\leqq 1/n$ and 0 for $|x|>1/n$, so $\int\varrho_n dx=1$. We call v_n the solution of

$$\left.\begin{array}{ll} -\Delta v_n+g(v_n)=f+2\pi c\varrho_n & \text{in } \Omega, \\ v_{n|\partial\Omega}=h & \text{on } \partial\Omega. \end{array}\right\} \tag{3.4}$$

Such a solution exists in the following sense (see [2])

$$\left.\begin{array}{l} \int\limits_{\Omega}(-v_n\Delta\zeta+g(v_n)\zeta)dx=2\pi c\int\limits_{\Omega}\varrho_n\zeta dx+\int\limits_{\Omega} f\zeta dx-\int\limits_{\partial\Omega} h\dfrac{\partial\zeta}{\partial\nu}d\sigma, \\ \text{for any } \zeta\in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega). \end{array}\right\} \tag{3.5}$$

The following estimates hold [2]

$$\left.\begin{array}{l}\text{i) } \|v_n\|_{L^1(\Omega)} \leqq K(2\pi|c|\,\|d\cdot\varrho_n\|_{L^1(\Omega)} + \|d\cdot f\|_{L^1(\Omega)} + \|h\|_{L^1(\partial\Omega)}),\\ \text{ii) } \|d\cdot g(v_n)\|_{L^1(\Omega)} \leqq K(2\pi|c|\,\|d\cdot\varrho_n\|_{L^1(\Omega)} + \|d\cdot f\|_{L^1(\Omega)} + \|h\|_{L^1(\partial\Omega)}).\end{array}\right\} \tag{3.6}$$

We shall assume that $c \neq 0$ in the sequel otherwise the problem (3.1) falls into the scope of [2], and for example $c>0$. Moreover we can take $g(0)=0$.

Part I. Existence when $c<2/a_g^+$

Step 1. $|v_n(x)| \leqq c\,\mathrm{Log}(1/|x|)+C_1$ where C_1 is independent of n. For that we set φ_n the solution of

$$\left.\begin{array}{ll}-\Delta\varphi_n = f^+ + 2\pi c\varrho_n & \text{in } \Omega,\\ \varphi_{n|\partial\Omega} = h^+ & \text{on } \partial\Omega,\end{array}\right\} \tag{3.7}$$

and φ the solution of

$$\left.\begin{array}{ll}-\Delta\varphi = f^+ + 2\pi c\delta_0 & \text{in } \Omega,\\ \varphi_{|\partial\Omega} = h^+ & \text{on } \partial\Omega.\end{array}\right\} \tag{3.8}$$

It is known from classical potential theory that $\left(\varphi(x) - c\,\mathrm{Log}\frac{1}{|x|}\right)$ remains bounded in Ω. Moreover we have

$$\left.\begin{array}{ll}-\Delta(\varphi-\varphi_n) = 2\pi c(\delta_0 - \varrho_n) & \text{in } \Omega,\\ (\varphi-\varphi_n)_{|\partial\Omega} = 0 & \text{on } \partial\Omega.\end{array}\right\} \tag{3.9}$$

Hence $\varphi - \varphi_n$ is explicitely given by

$$(\varphi-\varphi_n)(x) = \begin{cases} c\left\{\mathrm{Log}(1/|x|) - \mathrm{Log}\,n - \frac{1}{2} + \frac{n^2|x|^2}{2}\right\} & \text{if } |x| \leqq \frac{1}{n},\\ & \text{if } |x| \geqq \frac{1}{n}.\end{cases} \tag{3.10}$$

Hence for any $x \neq 0$, $0 \leqq \varphi_n(x) \leqq c\mathrm{Log}(1/|x|) + C_1$. We also have from (3.7) and (3.5)

$$\int_\Omega (-(v_n-\varphi_n)\Delta\zeta + g(v_n)\zeta)dx \leqq 0, \tag{3.11}$$

for any $\zeta \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$, $\zeta \geqq 0$. But $v_n - \varphi_n \in W_{\mathrm{loc}}^{2,r}(\Omega)$ [2] then

$$\int ((\nabla(v_n-\varphi)|\nabla\zeta) + g(v_n)\zeta)dx \leqq 0, \tag{3.12}$$

for any $\zeta \geqq 0$, $\zeta \in W_0^{1,q}(\Omega)$ for any q such that $\frac{1}{q} \geqq \frac{1}{2} - \frac{1}{r}$. Set p a bounded Lipschitz continuous nondecreasing real valued function vanishing on $(-\infty, 0]$, then $p(v_n-\varphi_n)$ is an admissible test function. As g is nondecreasing we have

$$\int_\Omega |\nabla(v_n-\varphi_n)|^2 p'(v_n-\varphi_n)dx \leqq 0; \tag{3.13}$$

hence $v_n \leqq \varphi_n$. As v_n is minorized by a bounded function, we get

$$-c' \leqq v_n(x) \leqq c\,\mathrm{Log}(1/|x|) + C_1, \tag{3.14}$$

for $x \neq 0$.

Step 2. $\{g(v_n)\}$ *is uniformly integrable on* Ω. Consider α such that $\frac{2}{c} > \alpha > a_g^+$. There exist two constants A and B such that

$$g(s) \leqq A e^{\alpha s} + B, \tag{3.15}$$

which implies with (3.14)

$$g(-c') \leqq g(v_n(x)) \leqq A e^{c} |x|^{-\alpha c} + B. \tag{3.16}$$

But $\alpha c < 2$, so $|x|^{-\alpha c} \in L^1(\Omega)$ and $\{g(v_n)\}$ is uniformly integrable on Ω.

Step 3. $\{v_n\}$ *is relatively compact in* $L^1(\Omega)$. We set $f_n = 2\pi c \varrho_n + f - g(v_n)$ and let w_n be the solution of

$$\left.\begin{aligned} -\Delta w_n &= f_n \quad \text{in } \Omega, \\ w_{n|\partial\Omega} &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.17}$$

From the Step 2 $\{f_n\}$ is bounded in $L^1(\Omega)$. We deduce from a result of Stampacchia [8] that $\{w_n\}$ remains bounded in $W_0^{1,q}(\Omega)$ for any $q < 2$. If $\tilde{\varphi}_1$ is harmonic in Ω with boundary data h, $v_n = w_n + \tilde{\varphi}_1$ and $\{v_n\}$ is bounded in $W^{1,q}(\Omega)$, $q < 2$, hence relatively compact in $L^1(\Omega)$.

Step 4. From the step 2 and Dunford-Pettis criteria $\{g(v_n)\}$ is relatively weakly compact in $L^1(\Omega)$. Hence, extracting some subsequences, there exists $v \in L^1(\Omega)$ with $g(v) \in L^1(\Omega)$ such that

$$\left.\begin{aligned} &\text{i) } v_{n'} \to v \quad && \text{in } L^1_{\text{loc}}(\Omega) \text{ and a.e. in } \Omega, \\ &\text{ii) } g(v_{n'}) \to g(v) \quad && \text{in } L^1(\Omega)\text{-weak and a.e. in } \Omega. \end{aligned}\right\} \tag{3.18}$$

Using (3.14) and Lebesgue's theorem we can go to the limit in (3.5) and get

$$\int_\Omega (-v\Delta\zeta + g(v)\zeta)dx = 2\pi c \zeta(0) + \int_\Omega f\zeta dx - \int_{\partial\Omega} h \frac{\partial\zeta}{\partial\nu} d\sigma. \tag{3.19}$$

Part II. Existence when $c = 2/a_g^+$

We consider a sequence $\{c_n\}$ such that $c_n \uparrow c$ when $n \to +\infty$. With the method used in the step 1, the sequence $\{v_n\}$ of solutions of (3.1) with right hand side $f + 2\pi c_n \delta_0$ is nondecreasing. It is convergent thanks to (3.14) to some $v \in L^1_{\text{loc}}(\Omega)$. Moreover $\{g(v_n)\}$ is a nondecreasing sequence converging to $g(v)$ and $d \cdot g(v) \in L^1(\Omega)$. With Lebesgue's theorem $v_n \to v$ in $L^1(\Omega)$. Set B a ball containing 0, $\bar{B} \subset \Omega$. Thanks to (3.16) $\{g(v_n)\}$ is bounded on $\Omega - B$, so $g(v_n) \to g(v)$ in $L^1(\Omega - B)$. From (3.6) ii) $\left\{\int_B g(v_n)dx\right\}$ remains bounded. If we use Beppo-Levi's theorem $g(v_n) \to g(v)$ in $L^1(B)$. Hence $g(v_n) \to g(v)$ in $L^1(\Omega)$. If we go to the limit in

$$\int_\Omega (-v_n\Delta\zeta + g(v_n)\zeta)dx = 2\pi c_n \zeta(0) + \int_\Omega f\zeta dx - \int_{\partial\Omega} h \frac{\partial\zeta}{\partial\nu} d\sigma, \tag{3.20}$$

where $\zeta \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$, we get (3.19). Moreover $g(v) \in L^1(\Omega)$.

Part III. Uniqueness

It follows the proof of the uniqueness in [2]. If v and $\hat{v}$ are two solutions of (3.1), we have as in [2], Lemma 1,

$$-\int |v-\hat{v}|\Delta\zeta dx \leqq -\int |g(v)-g(\hat{v})|\zeta dx, \tag{3.21}$$

for any $\zeta \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$, $\zeta \geqq 0$. If ζ is the solution of

$$\left.\begin{aligned} -\Delta\zeta &= 1 \quad \text{in } \Omega, \\ \zeta_{|\partial\Omega} &= 0, \end{aligned}\right\} \tag{3.22}$$

then $\zeta \geqq 0$, $\zeta \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$, hence $v=\hat{v}$.

Part IV. Proof of (3.3)

Let $\tilde{v}$ be the solution of

$$\left.\begin{aligned} -\Delta\tilde{v}+g(\tilde{v}) &= 2\pi c\delta_0 \quad \text{in } \Omega, \\ \tilde{v}_{|\partial\Omega} &= h \qquad\quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{3.23}$$

and set

$$\lambda = \begin{cases} (g(v)-g(\tilde{v}))/(v-\tilde{v}) & \text{if } v \neq \tilde{v} \\ 0 & \text{if } v = \tilde{v}; \end{cases}$$

$\lambda \geqq 0$ and $w = v - \tilde{v}$ satisfies

$$-\Delta w + \lambda w = f \quad \text{in } \mathscr{D}'(\Omega), \tag{3.24}$$

and w is bounded in $\{x : |x| > \varepsilon\}$. So for k large enough w is majorized in Ω by $\psi = \frac{1}{2\pi}\mathrm{Log}(1/|\cdot|) * \tilde{f}^+ + k$, where $\tilde{f}^+$ is the extension of f^+ by 0 outside Ω. But $f^+ \in L^p(\Omega)$, so ψ is locally bounded in $\mathbb{R}^2$. We minorize $v-\tilde{v}$ in the same way and $v-\tilde{v}$ belongs to $L^\infty(\Omega)$. If we apply Theorem 1.1 to $\tilde{v}$ (we no longer need the hypothesis (0.3) as $|\tilde{v}(x)| \leqq c\,\mathrm{Log}(1/|x|) + D$) we deduce that $\tilde{v}(x)/\mathrm{Log}(1/|x|)$ admits a limit $\tilde{c}$ when $x \to 0$ and $0 \leqq \tilde{c} \leqq c$. Suppose now $0 \leqq \tilde{c} < c$. As we should have $|g(\tilde{v})| \leqq K|x|^{-\alpha\tilde{c}} + K'$ for some $\frac{2}{\tilde{c}} > \alpha > a_g^+$, $g(\tilde{v})$ would belong to some $L^q(\Omega)$ for some $q > 1$. If we set ω the solution of

$$\begin{aligned} -\Delta\omega &= 2\pi c\delta_0 \quad \text{in} \quad \mathscr{D}'\{x : |x| < \varepsilon\}, \\ \omega_{|\{x:|x|=\varepsilon\}} &= \tilde{v}_{|\{x:|x|=\varepsilon\}}. \end{aligned} \tag{3.25}$$

Then $\omega - \tilde{v} \in L^\infty\{x : |x| < \varepsilon\}$ from regularity results, $\lim_{x\to 0} \omega(x)/\mathrm{Log}(1/|x|) = c$ and $\lim_{x\to 0} \tilde{v}(x)/\mathrm{Log}(1/|x|) = c$, which would contradict $\tilde{c} < c$. So

$$\lim_{x\to 0} \tilde{v}(x)/\mathrm{Log}(1/|x|) = c, \tag{3.26}$$

and we get (3.3).

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Moduli of Polarized Kähler Manifolds

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A way to solve the classification problem for compact manifolds is the construction of a (coarse) moduli space – a reduced space, whose points correspond to isomorphism classes of varieties under consideration such that a variation of the algebraic or complex analytic structure yields a unique morphism of the parameter space to the moduli space. In algebraic geometry coarse moduli spaces could be constructed for polarized abelian varieties and certain classes of polarized, non-ruled varieties: Popp [PO] showed the existence under the assumption of vanishing irregularity and more generally Mumford and Fogarty [M–F] in the case of discrete automorphism groups. Polarized varieties come up with natural embeddings into projective spaces and a coarse moduli space is essentially a quotient of a Hilbert scheme by an étale equivalence relation, which gives an algebraic space in the sense of M. Artin.

In analytic geometry this approach has to be replaced by transcendental methods. The distinction of varieties by polarizations is indispensable for the Hausdorff property of the moduli space, as an example of families of $K3$-surfaces by Atiyah [AT] shows. If one interprets a polarization as an integer-valued Kähler class, it is natural to transfer this notion to arbitrary Kähler manifolds by just assigning fixed Kähler classes. For polarized manifolds with vanishing first Chern class a coarse moduli space was constructed in [SCH 2]. We first show a criterion that guarantees the existence of the coarse moduli space for any class of polarized manifolds (i) stable under small deformations with the property (ii) that given two families $X \to S$ and $Y \to S$ of manifolds the canonical holomorphic mapping from the space $\mathrm{Isom}_S^\lambda(X, Y)$ of all isomorphisms of fibers X_s to Y_s to the base S is proper. In this situation one has universal Kähler deformations, where the universality rather than versality follows from the above properness. In a universal family there are usually still isomorphic fibers. The identification yields an analytic equivalence relation, whose graph is the orbit of the finite group $\mathrm{Aut}^\lambda(X_0)/\mathrm{Aut}^0(X_0)$, where X_0 is the central fiber and $\mathrm{Aut}^\lambda(X_0)$ consists of all automorphisms that preserve the polarization. The coarse moduli space is patched together from the above "local" quotient spaces. As an application we obtain:

Theorem. *There exists the coarse moduli space of all non-ruled, polarized Kähler manifolds, whose small Kähler deformations are still non-ruled.*

(A ruled manifold is by definition bimeromorphic equivalent to a holomorphic fiber space with $\mathbb{P}_1$ as general fiber.)

In particular, in the algebraic case (over $\mathbb{C}$) we can do without the discreteness of automorphism groups. One assumes that at least the class of those algebraic or Kähler manifolds resp. with non-negative Kodaira dimension is stable under small deformations. For the smaller class of manifolds, whose m-canonical bundle is generated by global sections for some m this is true after a recent result of Levine [LE], who shows that the plurigenera are constant in a deformation. For these manifolds, in particular for all non-ruled polarized surfaces, we give a short proof of the statement of the Matsusaka-Mumford theorem [M–M]. It follows from a relative version (with no polarizations, shown by means of the Barlet space [BA]) and the Calabi-Yau theorem. In the general case the above condition (ii) can be verified by Fujiki's theorem on bimeromorphic maps of Kähler manifolds [FU 2]. The coarse moduli space of non-ruled polarized, projective varieties is an algebraic space.

1. General Theorem

(1.1) *Polarized manifolds* are by definition compact, complex manifolds X_0 equipped with Kähler classes $\lambda_{X_0} \in H^1(X_0, \Omega^1_{X_0})$. In order to define families of polarized manifolds we have to consider *Kähler mappings*. These are proper, smooth, holomorphic mappings $f: X \to S$ of reduced complex spaces together with sections $\lambda_{X/S} \in (R^1 f_* \Omega^1_{X/S})(S)$, so-called relative Kähler classes, whose restrictions to arbitrary fibers are Kähler classes. One can see that relative Kähler classes are represented by relative Kähler forms on sufficiently small open subspaces of S. With respect to an appropriate open covering $\{U_i\}$ of X, these are given as $i \cdot \partial_{/S} \bar{\partial}_{/S} p_i$, where p_j are real differentiable functions such that $p_j - p_k$ are harmonic on $U_j \cap U_k$, and where $\partial_{/S}$ and $\bar{\partial}_{/S}$ denote derivatives in the direction of the fibers. Although in a family all neighboring fibers of a Kähler fiber are still Kähler by the stability theorem of Kodaira and Spencer [K–S], in general the Kähler classes cannot be chosen to depend holomorphically on the parameter. Thus a *family* (X_s, λ_{X_s}); $s \in S$ of *polarized manifolds* is given by a Kähler map $(f: X \to S, \lambda_{X/S})$ such that $X_s = f^{-1}(s)$ and $\lambda_{X_s} = \lambda_{X/S} | X_s$. Isomorphisms of families of polarized manifolds have to take care of Kähler classes. Let $(g: Y \to S, \lambda_{Y/S})$ be a further family, then an *isomorphism* is a biholomorphic map $\varphi: X \to Y$ over S such that $\varphi^* \lambda_{Y/S} = \lambda_{X/S}$. A holomorphic map $\alpha: R \to S$ of reduced complex spaces induces a pull-back $(f_R: X_R = X \times_S R \to R, f_R^* \lambda_{X/S})$. Consider the set-valued functor $Isom^\lambda_S(X, Y)$ on the category of reduced spaces over S, which assigns to a space R over S the set of all isomorphisms of the pull-backs. It was shown in [SCH 1, (2.5)] that this functor can be represented by a complex space $I = \mathrm{Isom}^\lambda_S(X, Y)$ with countably many irreducible components.

(1.2) The *moduli functor* $\mathfrak{M}$ with respect to a given collection $\mathfrak{K}$ (with necessary set-theoretic precautions) of polarized manifolds assigns to a complex space S the set $\mathfrak{M}(S)$ of isomorphism classes of families of polarized manifolds from $\mathfrak{K}$ over S. The map $\mathfrak{M}(S) \to \mathfrak{M}(R)$ induced by $R \to S$ is defined by base change. A *coarse moduli space* M for the moduli functor $\mathfrak{M}$ is a reduced complex space, which is characterized uniquely up to isomorphism by the following conditions:

(i) there exists a morphism of functors $\Phi: \mathfrak{M} \to h^M$

(ii) if p denotes the reduced point, then $\Phi(p)$ is bijective

(iii) if $\Psi: \mathfrak{M} \to h^M$ is another morphism, then there is a unique holomorphic mapping $F: M \to N$ with $\Psi = h(F) \circ \Phi$.

By (ii) points of M correspond to isomorphism classes in $\mathfrak{K}$; by (i) for any family of polarized manifolds the map, which assigns to a point $s \in S$ the point of M corresponding to the isomorphism class of X_s is holomorphic and compatible with base changes.

(1.3) **Theorem.** *Let $\mathfrak{K}$ be a collection of compact polarized Kähler manifolds. Then there exists a coarse moduli space for $\mathfrak{K}$, if*

(i) *$\mathfrak{K}$ is stable under small Kähler deformations*

(ii) *given Kähler families $(X \to S, \lambda_{X/S})$ and $(Y \to S, \lambda_{Y/S})$ then the holomorphic mapping* $\mathrm{Isom}_S^\lambda(X, Y) \to S$ *is proper.*

(1.4) The second assumption in (1.3) though not necessary (consider the collection just consisting of $\mathbb{P}_n$) seems to be essential. One may consider as well the functors $\mathfrak{N}$ where $\mathfrak{N}(S)$ is the set of equivalence classes of families with respect to isomorphy over local irreducible components of the base. Then (ii) implies by a Baire argument that $\mathfrak{N} \to h^M$ is a monomorphism.

(1.5) *Definition.* Let (X_0, λ_0) be a polarized manifold. A deformation of (X_0, λ_0) over the reduced complex space S with a distinguished base point $s_0 \in S$ is a family of polarized manifolds $(X \to S, \lambda_{X/S})$ together with an isomorphism $(X_0, \lambda_0) \xrightarrow{\sim} (X_{s_0}, \lambda_{X_{s_0}})$. The usual notions of deformation theory carry over literally.

(1.6) **Proposition.** *Any polarized manifold possesses a versal Kähler deformation (with respect to reduced spaces).*

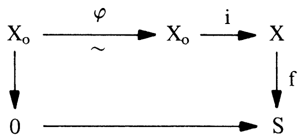
Proof. The non-reduced case was treated in a more special situation in [SCH2]. Let $f: X \to S$ be a smooth, proper holomorphic map, $s_0 \in S$ and λ_0 a Kähler class on X_{s_0}. It is sufficient, to show that there exists a unique maximal germ of a subspace $R \subset S$ through s_0, such that λ_0 can be extended to a section of $(R^1 f_* \Omega^1_{X_T/T})_{\mathbb{R}}$ with $X_T = X \times_S T$, where $(R^1 f_* \Omega^1_{X_T/T})_{\mathbb{R}}$ is the subsheaf of the (locally) free sheaf $R^1 f_* \Omega^1_{X_T/T}$ corresponding to the fibers $H^{1,1}_{\mathbb{R}}(X_t)$. We chose S as Stein and contractible. Since all sections of $(R^1 f_* \Omega^1_{X/S})_{\mathbb{R}}$ are of the form $i \cdot \partial_{/S} \bar{\partial}_{/S} p_j$ with $p_j - p_k$ harmonic as in (1.1), one can see that this sheaf is isomorphic to $R^1 f_* \mathscr{H}_X$, where $\mathscr{H}_X = \mathcal{O}_X / i \cdot \mathbb{R}_X$. Consider the exact sequences $0 \to \mathbb{R}_X \to \mathcal{O}_X \to \mathscr{H}_X \to 0$ and $(R^1 f_* \Omega^1_{X/S})_{\mathbb{R}}(S) \to (R^2 f_* \mathbb{R}_X)(S) \to (R^2 f_* \mathcal{O}_X)(S)$. The Kähler class λ_0 can be uniquely extended to a section $\tilde{\lambda}$ of the constant sheaf $R^2 f_* \mathbb{R}_X$. Since $R^2 f_* \mathcal{O}_X$ is a free $\mathcal{O}_S$-module, the image of $\tilde{\lambda}$ determines a couple of holomorphic functions on S. Its common zero set T is the maximal subspace, such that $\tilde{\lambda} | T$ comes from an $\lambda \in (R^1 f_* \Omega^1_{X_T/T})_{\mathbb{R}}(T)$. In some neighborhood of s_0 it is still positive definite on the fibers.

(1.7) *Definition.* Let $(X \to S, \lambda_{X/S})$ be a family of polarized manifolds. We call $r, s \in S$ equivalent: "$r \sim s$", if the corresponding polarized manifolds are isomorphic. Let $\Gamma = \{(r, s) \in S \times S;\ r \sim s\}$ be the graph of $\sim$.

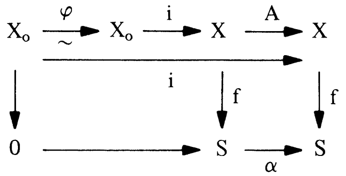
Under the assumptions of (1.3) the map $I = \mathrm{Isom}^\lambda_{S \times S}(X \times S, S \times X) \to S \times S$ is proper and its image Γ is analytic.

(1.8) **Proposition.** *Any polarized Kähler manifold* (X_0, λ_0) *from* $\mathfrak{K}$ *as in* (1.3) *possesses a universal Kähler deformation.*

Proof. We start with a *versal* Kähler deformation as in (1.6). We have to show that $\dim \mathrm{Aut}(X_s) = \mathrm{const}$. Consider the map $p: \Gamma \to S$ induced by the projection of $S \times S$ onto the first factor. After shrinking S we can assume that $p^{-1}(s_0) = \{(s_0, s_0)\}$, and that p is finite. In particular there are no positive dimensional local analytic sets in S over which the family $X \to S$ is trivial. However, this is not sufficient for the effectivity of this deformation, because trivial loci might be non reduced points. Any element $\varphi \in \mathrm{Aut}^{\lambda_0}(X_0)$ induces a further deformation:



This differs from the given one by an (a priori not uniquely determined) base change α.



If ψ is another automorphism, then it yields an isomorphic deformation, if and only if $\varphi\psi^{-1}$ can be extended to an automorphism of X over S. Yet there are only finitely many choices of α, since these are sections of the finite map p. So $\dim \mathrm{Aut}(X_0)$ cannot be larger than $\dim \mathrm{Aut}(X_s)$ for $s = s_0$, s_0 the distinguished point.

(1.9) By (1.8) the map $\mathrm{Aut}^0(X/S) \to S$ is flat and we have an operation of the finite group $G = \mathrm{Aut}^{\lambda_0}(X_0)/\mathrm{Aut}^0(X_0)$ on the base S of a universal Kähler deformation, and after shrinking S, the graph Γ is equal to

$$\bigcup_{g \in G} \{(s, g \cdot s); s \in S\}$$

(see [SCH2, (3.3) and (3.4)]). So $S/\sim$ equals S/G, which is a separated, reduced complex space. The local moduli spaces $S/\sim$ can be glued together by (uniquely determined biholomorphic mappings), the Hausdorff property is shown as follows: It is sufficient to prove that the union of two quotients, say $S/\sim$ and $R/\sim$, is Hausdorff (with families $X \to S$ and $Y \to R$), a fact, which follows directly from the properness of $\mathrm{Isom}_{S\times R}(X\times R, S\times Y)\to S\times R$. (Note that the stated smoothness of the base of a universal deformation in [SCH 2] is in general not clear, however, it is not necessary.) Property (iii) of (1.2) follows as in [SCH 2, (3.5)].

2. Main Theorem

(2.1) **Theorem.** *There exists the coarse moduli space for the class $\mathfrak{K}$ of all non-ruled, polarized manifolds, whose small deformations still have this property.*

(2.2) **Corollary.** *There exists the coarse moduli space of all polarized manifolds, whose m-canonical bundles are generated by global sections for some m.*

(2.3) **Corollary.** *There exists the coarse moduli space of all non-ruled, polarized surfaces.*

The first corollary follows from the theorem, since by a result of Levine [LE, 1.10] the m-canonical genus is constant in a deformation. If $f: X\to S$ is a family $s_0\in S$ and $m\cdot K_{X_{s_0}}$ globally generated, then any set of generators can be extended to a set of sections, which generate the locally free sheaf $f_*(m\cdot K_{X/S})$ in a neighborhood of s_0. The second corollary follows from the first one or from (2.1) by a theorem of Iitaka.

(2.4) We reduce the proof of (2.1) to a particular situation. Given families $(X\to S, \lambda_{X/S})$ and $(Y\to S, \lambda_{Y/S})$, the closure $\bar{I}$ of $I=\mathrm{Isom}_S^\lambda(X, Y)$ in the (relative) Barlet space $B(X\times_S Y/S)$ is an analytic space, which is mapped to the base properly after [FU 1, 5.2]. We have to show $I=\bar{I}$. Take a small curve C in $\bar{I}$ that intersects the analytic set $\bar{I}\backslash I$ in at most one point c_0. This yields a bimeromorphic map $X\times_S C\to Y\times_S C$ that is biholomorphic over $C\backslash\{c_0\}$. In order to show that it is biholomorphic one may introduce a smooth parameter for C and consider the pullbacks. So it is sufficient to show the following:

(*) *Let $S=\{s\in\mathbb{C}; |s|<1\}$ and $(f: X\to S, \lambda_{X/S})$, $(g: Y\to S, \lambda_{Y/S})$ be families in $\mathfrak{K}$. Then any bimeromorphic map $\varphi: X\to Y$ over S, which is biholomorphic over $S'=S\backslash\{0\}$ is biholomorphic over all of S.*

This is exactly the statement of Fujiki's theorem on bimeromorphic mappings [FU 2, Theorem 4.3]. For polarized manifolds, whose m-canonical bundle is globally generated, we can give a very short proof. For this we show a relative version of the Matsusaka-Mumford theorem, which is not contained in Fujiki's result.

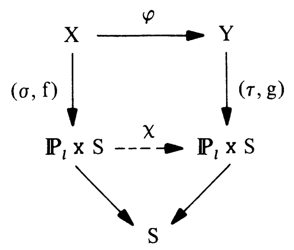
(2.5) **Theorem.** *Let $S:=\{s\in\mathbb{C}; |s|<1\}$ and $f: X\to S$, $g: Y\to S$ be smooth, proper, holomorphic maps, whose fibers X_0 and Y_0 have globally generated m-canonical bundles. If $\varphi: X\to Y$ is a meromorphic map over S, which is biholomorphic over $S\backslash 0$; then (after shrinking S) one can choose m-canonical mappings $\sigma: X\to\mathbb{P}_l$ and $\tau: Y\to\mathbb{P}_l$ such that*

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ & {\scriptstyle(\sigma,f)}\searrow \quad \swarrow{\scriptstyle(\tau,g)} & \\ & \mathbb{P}_l\times S & \end{array}$$

commutes.

Proof. We first consider arbitrary mappings (σ, f) and (τ, g) as above. Let X_0 and Y_0 resp. be the distinguished fibers of f and g resp. Let $\Gamma \subset X \times_S Y$ be the graph of φ and Γ_0 its intersection with $X_0 \times Y_0$. It decomposes into $\Gamma_1 \cup \Gamma_2$, where Γ_1 is the irreducible component, which is mapped by the canonical projection q_0: $X_0 \times Y_0 \to Y_0$ onto all of Y_0, and the rest Γ_2 goes to a proper subspace of Y_0. One needs that Γ_1 is mapped by $p_0: X_0 \times Y_0 \to X_0$ onto X_0 as well. Assume $p_0(\Gamma_1) = A \subseteq X_0$. Consider the modification $p: \Gamma \to X$, where p is induced by the projection $X \times Y \to X$. Then A is the critical set. We regard $\Gamma_1 \to A$ and its Stein factorization $\Gamma_1 \to \tilde{A} \to A$. The generic fiber dimension r of $\Gamma_1 \to A$ is the codimension of A in X minus one. Since X is smooth the generic fiber is the image of some $\mathbb{P}_r$ under a finite map (argue by proper transforms of smooth curves). So the Kodaira dimension of the general fiber is $-\infty$ and $\kappa(Y_0) = \kappa(\Gamma_1) = -\infty$ by [UE, 6.10, 6.13]. One can also show that Γ_1 is ruled.

Consider the following diagram:



The aim is to construct a holomorphic map χ such that the diagram commutes. It already exists over S'. We extend the dualized map to $\chi\hat{}: |mK_{X/S}| \to |mK_{Y/S}|$ of (relative) Chow varieties. The proof depends on the proof of [M–M, Theorem 2]. The m-canonical relative linear systems are closed subspaces of the Barlet spaces $B_{n-1}(X)$ and $B_{n-1}(Y)$ resp. where n denotes the dimension of the fibers of f and g. Let $U \subset B_{n-1}(X_0)$ be the open set consisting of those cycles Z_0, for which $|Z_0| \times Y_0$ intersects Γ_0 properly and $q_0((|Z_0| \times Y_0) \cap \Gamma_0) \subset Y_0$ is purely $(n-1)$-dimensional. Then the map induced by intersection and direct image of cycles [BA, p. 141, Theorem 10 and p. 109, Theorem 6] from U to $B_{n-1}(Y_0)$ is holomorphic. Let $U_0 := U \cap |mK_{X_0}|$. Since any m-canonical divisor Z_0 can be extended to a family of divisors Z_s in X_s, and the image of Z_s is a divisor for $s \neq 0$, the image of Z_0 in Y_0 will at any point be at least $(n-1)$-dimensional. The case of dimension equal to n cannot occur. Now $Z_0 \times Y_0$ intersects Γ_0 properly, if Z_0 does not contain any image of an irreducible component Γ_j of Γ_0 under p_0. Taking a point x_j in all $p(\Gamma_j)$ then $\emptyset \neq |mK_{X_0}| \setminus \bigcup_j |mK_{X_0}|_{x_j} \subset U_0$, where $|mK_{X_0}|_{x_j}$ are the hyperplanes of all divisors containing x_j. Since any divisor from U can be reconstructed from its image in $|mK_{Y_0}|$, the above map is injective. It is also linear, because the analogue construction for neighboring fibers yields linear maps. So it determines the desired extension $\chi\hat{}: |mK_{X/S}| \to |mK_{Y/S}|$.

(2.6) *Remark*. In the above situation the image $\tilde{X}$ of (σ, f) in $\mathbb{P}_l \times S$ can be provided with a reduced structure $\dot{X}$, such that the canonical map to S is flat and has *reduced* fibers.

Proof. The space $\tilde{X}$ as well as $\tilde{X}_s := \tilde{X} \cap (\mathbb{P}_l \times \{s\})$ is irreducible. Let $A := H^0(S, \mathcal{O}_S)$. We consider the A-scheme $\mathcal{X}$ corresponding to X according to the relative Chow-theorem by Hakim [HA]. To be on the safe side, we introduce compact Stein subsets $L \subset S$ and the noetherian ring $A_L = H^0(L, \mathcal{O}_S)$, and the A_L-scheme $\mathcal{X}_L = \mathcal{X} \otimes_A A_L$ (for properties see Bingener [BI]). Let $\hat{\mathcal{X}}_L$ be the normalization and $\hat{X}_L$ the corresponding complex analytic space. The construction is compatible with inclusions $L \subset L' \subset S$. Thus we get a reduced complex space $\hat{X} \to \tilde{X} \to S$ such that $\hat{X} \to \tilde{X}$ is a holomorphic homeomorphism. In order to show that the fibers of $\hat{X} \to S$ are reduced, it is sufficient to show the reducedness of the fibers of $\hat{\mathcal{X}}_L \to \mathrm{Spec}(A_L)$. Let $0 \in S$ and $\hat{\mathcal{X}}_0$ be the fiber of 0. Let $U \subset \mathcal{X}_L$ be an affine subspace and $s \in A_L$ the coordinate function. We have to show the following: Denote by $\hat{\sigma}: X \to \hat{X}$ the map induced by σ. Given $h \in \mathcal{O}_{\mathrm{alg}}(U)$ such that for an $r \geq 1$ we have $h^r \in s \cdot \mathcal{O}_{\mathrm{alg}}(U)$ then $h \in s \cdot \mathcal{O}_{\mathrm{alg}}(U)$. The holomorphic map $(h \circ \hat{\sigma})^r$ is in $f \cdot \mathcal{O}(\hat{\sigma}^{-1}(U))$. Since the fibers of f are reduced by assumption, we conclude $h \circ \hat{\sigma} \in f \cdot \mathcal{O}(\hat{\sigma}^{-1}(U))$. Thus the meromorphic map h/s is continuous. Its polar set must be contained in its zero set. Since $\hat{\mathcal{O}}_L$ is normal, $h/s \in \mathcal{O}_{\mathrm{alg}}(U)$ and $\hat{X}_0$ is reduced.

(2.7) Let L be a line bundle on a compact manifold X_0, which is generated by global sections. If $\sigma_0, \ldots, \sigma_l \in H^0(X_0, L)$ is a basis, then one can construct a hermitian metric ϱ on L as follows: Let $v_x \in L_x$, $x \in X_0$, consider all possible representations $v_x = \sum \alpha_i \sigma_i(x)$ and set $\varrho(v_x)^2 = \inf(\sum |\alpha_i|^2)$. Then

$$\varrho(v_x)^2 = |v_x|^2 / \sum |\sigma_i(x)|^2$$

by abuse of notation (the expression only makes sense in local coordinates).

(2.8) In the situation (*) of (2.4) we get

Proposition. *There are relative volume forms $\varrho_{X/S}$ and $\varrho_{Y/S}$ resp. on X and Y resp. that*

(i) $\varrho_{X/S} \in \lambda^n_{X/S}$, $\varrho_{Y/S} \in \lambda^n_{Y/S}$ *with* $n = \dim(X_0) = \dim(Y_0)$

(ii) $\varphi^* \varrho_{Y/S} = \varrho_{X/S}$ *over* $S \backslash \{0\}$.

Proof. Choose sections of the relative m-canonical bundles as in (2.5), consider the hermitian metrics as in (2.7) and take m^{th} roots. Then (ii) holds. Normalize by a real, smooth, positive function depending upon s to get (i).

(2.9) *Proof of* (2.2). We have to show that in the situation (*) the map φ is bijective. By Yau's solution of the Calabi problem [YAU, see also AU, BOU] there exist Kähler forms $\omega_{X_s} \in \lambda_{X_s}$ and $\omega_{Y_s} \in \lambda_{Y_s}$ with $\omega^n_{X_s} = \varrho_{X_s}$ and $\omega^n_{Y_s} = \varrho_{Y_s}$, which depend continuously on s by [SCH 1, (1.4)]. For $s \in S \backslash 0$ we have $\varphi^*(\omega_{Y_s})^n = \varphi^* \varrho_{Y_s} = \varrho_{X_s} = \varphi^n_{X_s}$. By the uniqueness of the solutions $\varphi^* \omega_{Y_s} = \omega_{X_s}$. So φ induces isometries on the fibers for $s \neq 0$ with metrics defined for all s. Hence the meromorphic map φ sends disjoint pieces of smooth curves transversal to X_0 to disjoint pieces of curves in Y, and φ is actually biholomorphic.

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Note added in proof. It was brought to the author's attention that A. Fujiki proved essentially the same result in a forthcoming paper.

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Analytic Hypo-Ellipticity and Propagation of Regularity for a System of Microdifferential Equations with Non-Involutory Characteristics

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Contents

Introduction

The present paper is mainly concerned with microdifferential (= analytic pseudo-differential) operators of the form $P=P_1P_2I_m+Q$; here P_1 and P_2 are scalar microdifferential operators such that the Poisson bracket of their principal symbols never vanishes, Q is an $m\times m$ matrix of operators of lower order, and I_m is the unit matrix of degree m. We study the regularity of solutions of the system $Pu=0$ of microdifferential equations. First, we give a result on the propagation of micro-analyticity of solutions of $Pu=0$ when the principal symbol of P_1 is real. Secondly, we give a sufficient condition for P to be micro-locally analytic hypo-elliptic when the real characteristic variety of P_1 is symplectic. In the course of the argument, we essentially rely on the theory of positivity due to Schapira [21] (see also Melin and Sjöstrand [12, 13]).

The system $Pu=0$ is a special case of more general (possibly overdetermined) systems whose structure in the complex domain has been studied in our previous paper [16]. We study the regularity of solutions of such systems in Sect. 5.

Now we shall sketch main results of this paper. In Sect. 2, we study operators of the form $P=P_1P_2I_m+Q$ mentioned above when the principal symbol of P_1 is real. Let p be a point of $\sqrt{-1}T^*\mathbb{R}^n-\mathbb{R}^n$ such that $\sigma(P_1)(p)=\sigma(P_2)(p)=0$, where σ denotes the principal symbol. Let P_j be of order l_j $(j=1,2)$ and set $l=l_1+l_2$. Then Q is a matrix of microdifferential operators of order $\leqq l-1$. We denote by $\lambda_1,\dots,\lambda_m$

the eigenvalues of the matrix

$$\sigma_{l-1}(Q)(p)/\{\sigma(P_1), \sigma(P_2)\}(p);$$

here σ_{l-1} denotes the symbol of order $l-1$, and $\{f, g\}$ denotes the Poisson bracket of holomorphic functions f and g defined in an open subset of $T^*\mathbb{C}^n$. Since $\sigma(P_1)$ is real valued on $\sqrt{-1}T^*\mathbb{R}^n$, the real bicharacteristic $b_1(p)$ of P_1 through p can be defined. Assume that $\lambda_j \neq 0, 1, 2, \ldots$ for $j=1, \ldots, m$. Under these assumptions, if f is a column vector of m microfunctions defined in a neighborhood of p such that $Pf=0$ and that f vanishes on $b_1(p)-\{p\}$, then f vanishes in a neighborhood of p (Theorem 2.1). We generalize this result to overdetermined system in Sect. 5 (Theorem 5.1). In Sect. 3 of [16], we have studied the branching of singularities of solutions of $Pu=0$ (or of more general systems) when $\sigma(P_2)$ is also real. Theorem 5.1 generalizes Theorem 3.5 of [16] to the case where $\sigma(P_2)$ is not necessarily real. To prove Theorems 2.1 and 5.1, we use the canonical form of the system in the complex domain which was given in Theorem 2.5 of [16] and the theory of micro-local boundary value problems developed by Kataoka [10, 11].

In Sect. 3, we assume a condition (H) concerning $\sigma(P_1)$, or equivalently that under a real quantized contact transformation, P_1 is equivalent to the operator $D_1 + \sqrt{-1}x_1^k D_n$ with a positive odd integer k in a neighborhood of $(0, \sqrt{-1}dx_n)$; here we write $D_j = \partial/\partial x_j$. We assume also $\lambda_j \neq 0, 1, 2, \ldots$ for $j=1, \ldots, m$. Under these assumptions, we prove that P is micro-locally analytic hypo-elliptic at p (Theorem 3.1). Note that (H) is satisfied with $k=1$ if $\{\sigma(P_1), \sigma(P_1)^c\}<0$ holds, where $\sigma(P_1)^c$ denotes the complex conjugate of the holomorphic function $\sigma(P_1)$ (cf. Sect. 1). When $k=1$, we can generalize Theorem 3.1 to overdetermined systems (Theorem 5.2). These results extend a part of those of Kashiwara et al. [8] (cf. also [7]).

Treves [25] has studied the analytic hypo-ellipticity of operators of the form $AI_m + B$; here A is a scalar pseudodifferential operator with non-negative principal symbol which vanishes precisely to the second order on a real symplectic submanifold, B is an $m \times m$ matrix of operators of lower order. The operator P discussed in Sect. 3 is contained in this class if $\sigma(P_2) = \sigma(P_1)^c$. Recently, the analytic hypo-ellipticity of closely related operators has been studied also by Tartakoff [24] and Métivier [14, 15] (see also Grušin [5] for an earlier result). The C^∞ hypo-ellipticity of similar operators was studied by, for instance, Boutet de Monvel and Treves [2, 3], Sjöstrand [22], Boutet de Monvel [1], Hörmander [6], Grušin [4] In the above mentioned works concerning analytic hypo-ellipticity, the real characteristic variety is assumed to be a symplectic submanifold. However, since we make no assumption concerning the real characteristic variety of P_2, that of the operator P in Theorem 3.1 is not necessarily symplectic or non-singular (see Example 3.6).

To prove Theorem 3.1, we transform P so that $\sigma(P_1)$ becomes real by means of a quantized contact transformation associated with a suitable complex contact transformation. Then Theorem 3.1 follows from Theorem 2.1 in view of Théorème 3.2 of Schapira [21]. When $k=1$, we have sketched in [17] another proof of Theorem 3.1.

In Sect. 4, we construct microfunction solutions of $Pu=0$ with minimal supports in two cases. First, we treat the case where $\sigma(P_1)$ is real and P_2 is equivalent to the operator $D_1-\sqrt{-1}x_1^kD_n$ at $p=(0,\sqrt{-1}dx_n)$ with a positive odd integer k; we find a microfunction solution of $Pu=0$ whose support in $\sqrt{-1}S^*\mathbb{R}^n$ is $b_1^+(p)\cup\{p\}$, where $b_1^+(p)$ is one of the connected components of $b_1(p)-\{p\}$ (Theorem 4.1). Secondly, we treat the case where the real characteristic variety of P is symplectic and $\{\sigma(P_1), \sigma(P_1)^c\}$, $\{\sigma(P_2), \sigma(P_2)^c\}$ are both positive; we find a microfunction solution of $Pu=0$ whose support in $\sqrt{-1}S^*\mathbb{R}^n$ consists of one point (Theorem 4.2).

Main results in Sect. 2 and Sect. 3 have been announced in [18].

1. Preliminaries

Let M be an n-dimensional real analytic manifold and X be its complexification. Let $z=(z_1,\dots,z_n)$ be a local coordinate system of X around a point of M with $z_j=x_j+\sqrt{-1}y_j$ $(x_j, y_j\in\mathbb{R})$ such that z_j are real valued on M. Then $(z,\zeta)=(z,\langle\zeta,dz\rangle)$ denotes a point of the cotangent bundle T^*X of X with $\zeta=(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$, and

$$(x,\sqrt{-1}\eta)=(x,\sqrt{-1}\langle\eta,dx\rangle)=(x,\sqrt{-1}\langle\eta,dz\rangle)$$

denotes a point of the conormal bundle T_M^*X $(\subset T^*X)$ of M in X with $x=(x_1,\dots,x_n)$, $\eta=(\eta_1,\dots,\eta_n)\in\mathbb{R}^n$. We denote by $\mathscr{E}_X$ the sheaf on T^*X of microdifferential operators of finite order and by $\mathscr{C}_M$ the sheaf on T_M^*X of microfunctions [19]. We often fix a local coordinate system z of X as above and identify X with an open subset of $\mathbb{C}^n$ and M with an open subset of $\mathbb{R}^n$. We use the notation $D=(D_1,\dots,D_n)$ with $D_j=\partial/\partial z_j$. We write also $D_j=\partial/\partial x_j$ when it operates on microfunctions or hyperfunctions.

Let $\mathcal{O}(j)$ be the sheaf on T^*X of holomorphic functions homogeneous of degree j with respect to the fiber coordinates. We denote by $\mathscr{E}(j)$ the sheaf of microdifferential operators of order at most j. Then there is a natural homomorphism

$$\sigma_j:\mathscr{E}(j)\to\mathcal{O}(j)\cong\mathscr{E}(j)/\mathscr{E}(j-1).$$

If $P\in\mathscr{E}(j)-\mathscr{E}(j-1)$, we set $\sigma(P)=\sigma_j(P)$ and call it the principal symbol of P.

Let ω be the fundamental 1-form on T^*X. We have $\omega=\zeta_1dz_1+\dots+\zeta_ndz_n$ in the local coordinate system (z,ζ) of T^*X. A holomorphic mapping φ of an open subset of T^*X to T^*X is called a *homogeneous canonical transformation* or a *contact transformation* if $\varphi^*\omega=\omega$, and φ is called a *real contact transformation* if, in addition, $\varphi(T_M^*X)=T_M^*X$ holds.

Let f and g be holomorphic functions defined in an open subset of T^*X. Then we set

$$H_f=\sum_{j=1}^n\left(\frac{\partial f}{\partial\zeta_j}\frac{\partial}{\partial z_j}-\frac{\partial f}{\partial z_j}\frac{\partial}{\partial\zeta_j}\right),$$

which is called the *Hamilton vector field* of f, and denote by $\{f,g\}=H_fg$ the *Poisson bracket* of f and g. If f is holomorphic in a neighborhood of a point of T_M^*X, we denote by f^c the complex conjugate of f with respect to T_M^*X; i.e., f^c is the unique holomorphic function whose value on T_M^*X is the complex conjugate of that of f.

For a homogeneous (= conic) analytic subset V of T^*X, we denote by I_V the sheaf on T^*X of the holomorphic functions which vanish on V and set $V^c = \{(z,\zeta) \in T^*X; f^c(z,\zeta)=0 \text{ for any } f \in I_V\}$. We say that V is *involutory* if $\{f,g\} \in I_V$ holds for any $f, g \in I_V$, and say that V is *regular involutory* if, in addition, V is non-singular and the pull back of ω to it vanishes nowhere. On the other hand, V is called a *symplectic submanifold* if V is non-singular and the pull-back of $d\omega$ to V is non-degenerate. Let V be a d-codimensional homogeneous regular involutory submanifold of T^*X such that $V = V^c$. Then the system $\{H_f; f \in I_V\}$ of vector fields can be regarded as one defined on V, and is completely integrable. The intersection of T_M^*X and a maximal integral manifold (in V) of this system is called a *real bicharacteristic* of $V^{\mathbb{R}} = V \cap T_M^*X$, which is a d-dimensional real analytic submanifold of $V^{\mathbb{R}}$.

Let $V = \{(z,\zeta);\ f_1(z,\zeta) = \ldots = f_d(z,\zeta) = 0\}$ be a submanifold of T^*X, where $f_1, \ldots, f_d$ are homogeneous holomorphic functions defined in a neighborhood of $p \in V \cap (T_M^*X - M)$ with linearly independent differentials. Then the Hermitian form

$$\sum_{i,j=1}^{d} \{f_1, f_j^c\}(p)\xi_i\bar{\xi}_j$$

is called the *generalized Levi form* of V at p [19, Chap. III]. Note that its signature is independent of the choice of f_j and is invariant under real contact transformations.

For a subset S of T^*X, we write

$$\mathbb{R}^+S = \{cp;\ c \in \mathbb{R},\ c > 0,\ p \in S\},$$

$$\mathbb{C}^\times S = \{cp;\ c \in \mathbb{C},\ c \neq 0,\ p \in S\}.$$

2. Propagation of Regularity

In this and the subsequent sections, we treat operators of the form $P = P_1P_2I_m + Q$ mentioned in the introduction. To be more precise, let M be an n-dimensional real analytic manifold and X be its complexification, and fix a point p_0 of $T_M^*X - M$. Let P_1 and P_2 be microdifferential operators of order l_1 and l_2 respectively, defined in a neighborhood (in T^*X) of p_0. Set $l = l_1 + l_2$ and let $Q = (Q_{ij})$ be an $m \times m$ matrix of microdifferential operators of order $\leq l-1$ defined in a neighborhood of p_0. We assume

(C.1) $$\sigma(P_1)(p_0) = \sigma(P_2)(p_0) = 0,$$

(C.2) $$\{\sigma(P_1), \sigma(P_2)\}(p_0) \neq 0.$$

Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of the matrix

$$(\sigma_{l-1}(Q_{ij})(p_0)/\{\sigma(P_1), \sigma(P_2)\}(p_0))_{1 \leq i,j \leq m}.$$

Then we assume also

(C.3) $$\lambda_j \notin \{0, 1, 2, \ldots\} \quad \text{for} \quad j = 1, \ldots, m.$$

Set $V_j = \{(z, \zeta) \in T^*X;\ \sigma(P_j)(z, \zeta) = 0\}$ for $j = 1, 2$. In this section, we assume, in addition,

(R) $$V_1 = V_1^c.$$

In view of (C.2), V_1 is regular involutory. Let $b_1(p_0)$ be the real bicharacteristic of $V_1^{\mathbb{R}} = V_1 \cap T_M^*X$ through p_0.

Theorem 2.1. *Set $P = P_1 P_2 I_m + Q$, where P_1, P_2, Q are as above satisfying* (C.1)–(C.3) *and* (R). *Let f be a column vector of m microfunctions defined in a neighborhood of p_0 such that $Pf = 0$ and that f vanishes on $b_1(p_0) - \{p_0\}$. Then f vanishes on $b_1(p_0)$.*

Proof. Since the situation is micro-local, we may set $X = \mathbb{C}^n \ni z = (z_1, \ldots, z_n)$ and $M = \mathbb{R}^n \ni x = (x_1, \ldots, x_n)$ with $z_j = x_j + \sqrt{-1}\, y_j$ $(x_j, y_j \in \mathbb{R})$. Note that (C.2) implies that P_1 is of simple characteristics. Thus we may assume that $P_1 = z_1$ and $p_0 = (0, \sqrt{-1}\, dz_n)$ by using a real quantized contact transformation in view of (R). Set $N = \{x \in M;\ x_1 = 0\}$ and $Y = \{z \in X;\ z_1 = 0\}$. We define the map

$$\varrho : T_M^*X \underset{M}{\times} N \to T_N^*Y$$

by $\varrho(0, x', \sqrt{-1}\eta) = (x', \sqrt{-1}\eta')$, where $x' = (x_2, \ldots, x_n)$ and $\eta' = (\eta_2, \ldots, \eta_n)$, and put $p_0' = \varrho(p_0)$.

Since $\{z_1, \sigma(P_2)\}(p_0) \neq 0$, there are microdifferential operators A and R such that

$$P_2 = (D_1 - A)R, \quad [z_1, A] = 0,$$

the order of A is at most 1 with $\sigma_1(A)(p_0) = 0$, and R is invertible at p_0 (cf. Theorem 2.2.2 of [19, Chap. II]). Then we have

$$P = (z_1(D_1 - A)I_m + QR^{-1})R,$$

and the order of each component of QR^{-1} is at most 0. In view of the structure theorem in the complex domain (Theorem 2.5 of [16]) and (C.3), the system $Pu = 0$ is equivalent to the direct sum $\oplus_{\lambda \in \Lambda} \mathcal{N}_\lambda$ of systems $\mathcal{N}_\lambda$; here $\Lambda = \{\lambda \in \mathbb{C};\ -1 < \operatorname{Re}\lambda \leqq 0,\ \lambda - \lambda_j \in \mathbb{Z}$ for some $j \in \{1, \ldots, m\}\}$, and $\mathcal{N}_\lambda$ is of the form

$$\mathcal{N}_\lambda : (z_1(D_1 - A)I_{m_\lambda} - B_\lambda)u_\lambda = 0,$$

where $m_\lambda = \#\{j \in \{1, \ldots, m\};\ \lambda_j - \lambda \in \mathbb{Z}\}$, B_λ is an $m_\lambda \times m_\lambda$ matrix of microdifferential operators commuting with z_1 and $D_1 - A$, and of order $\leqq 0$, such that the eigenvalues of $\sigma_0(B_\lambda)(p_0)$ are all λ. We can write

$$B_\lambda = (B_\lambda)_0 + z_1(B_\lambda)_1$$

with matrices $(B_\lambda)_0$ and $(B_\lambda)_1$ of operators of order $\leqq 0$ such that

$$[z_1, (B_\lambda)_0] = [D_1, (B_\lambda)_0] = [z_1, (B_\lambda)_1] = 0.$$

Hence we have only to prove the theorem for the operator

$$P_\lambda = z_1((D_1 - A)I_{m_\lambda} - (B_\lambda)_1) - (B_\lambda)_0.$$

Since $b_1(p_0) = \varrho^{-1}(p_0')$ in this case, we have

$$\operatorname{supp} f \cap \varrho^{-1}(p_0') = \{p_0\}.$$

Thus the mapping

$$\varrho : \operatorname{supp} f \cap \varrho^{-1}(U') \to U'$$

is proper for a sufficiently small neighborhood U' of p'_0. In particular, f can be regarded as a section of $(\varrho_! \mathscr{C}_M)^m$ defined on $U' \subset T_N^* Y$, where $\varrho_!$ denotes the direct image with proper supports with respect to ϱ [19, Chap. I]. Thus it suffices to prove the following proposition.

Proposition 2.2. *Set*

$$P = z_1(D_1 I_m - A) - B;$$

here $A = (A_{ij})$ and $B = (B_{ij})$ are $m \times m$ matrices of microdifferential operators defined in a neighborhood (in T^*X) of $\varrho^{-1}(p'_0)$ (where ϱ and p'_0 are as in the proof of Theorem 2.1) such that A_{ij} are of order $\leqq 1$ and B_{ij} are of order $\leqq 0$ and that

$$[z_1, A] = [z_1, B] = [D_1, B] = 0.$$

Let $\mu_1, \ldots, \mu_m$ be the eigenvalues of $\sigma_0(B)(p_0)$. Then we assume that

$$\mu_i \notin \mathbb{Z} \text{ and } \mu_i - \mu_j \notin \mathbb{Z} - \{0\} \quad \textit{for} \quad 1 \leqq i, j \leqq m, \tag{2.1}$$

or else,

$$\mu_1 = \ldots = \mu_m \in \{-1, -2, -3, \ldots\}. \tag{2.2}$$

Under these assumptions, the homomorphism

$$P : (\varrho_! \mathscr{C}_M)^m_{p'_0} \to (\varrho_! \mathscr{C}_M)^m_{p'_0}$$

is injective.

Proof. Let $f(x) \in (\varrho_! \mathscr{C}_M)^m_{p'_0}$. Then $Y(x_1) f(x)$ is well-defined as a section of $(\varrho_* \mathscr{C}_M)^m$ defined on a neighborhood of p'_0, where $Y(x_1)$ denotes the Heaviside function. Suppose $Pf = 0$. Then we have

$$P(Y(x_1) f(x)) = (x_1 D_1 Y(x_1)) f(x) + Y(x_1) P f(x) = 0$$

as a section of $(\mathscr{C}_M)^m$. From now on, we use methods of micro-local boundary value problems developed by Kataoka [10, 11]. Especially, the sheaf $\mathscr{C}_{M_+|X}$ and the theory of mild hyperfunctions are essential to our argument.

Set

$$T^*_{M_+} X = \{(x, \sqrt{-1}\eta) \in T^*_M X;\, x_1 \geqq 0\} \cup \{(0, x', \zeta_1, \sqrt{-1}\eta') \in T^*_N X;\, \operatorname{Re} \zeta_1 \geqq 0\}$$

and define the mapping

$$\iota^+ : T^*_{M_+} X \underset{M_+}{\times} N \to T^*_N Y$$

by $\iota^+(0, x', \zeta_1, \sqrt{-1}\eta') = (x', \sqrt{-1}\eta')$. We can regard $Y(x_1) f(x)$ as a section of $((\iota^+)_* \mathscr{C}_{M_+|X})^m$ defined on a neighborhood of p'_0. (For the definition and properties of the sheaf $\mathscr{C}_{M_+|X}$, see Sect. 4 of [10].) Put

$$I = \{(0, x', \zeta_1, \sqrt{-1}\eta') \in T^*_N X;\, \operatorname{Re} \zeta_1 = 0\}.$$

Since $\mathscr{C}_{M_+|X}|_I$ is a subsheaf of $\mathscr{C}_M|_I$, and sections of $\mathscr{C}_{M_+|X}$ have the unique continuation property along each fiber of ι^+,

$$P(Y(x_1)f(x))=0 \tag{2.3}$$

holds as a section of $((\iota^+)_*\mathscr{C}_{M_+|X})^m$. We define the mapping

$$\iota: T_N^*X \to T_N^*Y$$

by $\iota(0,x',\zeta_1,\sqrt{-1}\eta')=(x',\sqrt{-1}\eta')$. Set

$$U'_\varepsilon=\{(x',\sqrt{-1}\eta')\in T_N^*Y; |x_i|<\varepsilon\ (2\leqq i\leqq n),$$
$$|\eta_j|<\varepsilon\eta_n\ (2\leqq j\leqq n-1), \eta_n>1/2\}$$

for $\varepsilon>0$ and put $U_\varepsilon=\iota^{-1}(U'_\varepsilon)$, $\tilde{U}_\varepsilon=\{\zeta_1\in\mathbb{C}\}\times U'_\varepsilon$. Then there is a sheaf isomorphism

$$\beta: \mathscr{C}_{N|X}|_{U_\varepsilon}\to\mathscr{C}\mathscr{O}|_{\tilde{U}_\varepsilon},$$

where $\mathscr{C}\mathscr{O}$ denotes the sheaf on $\mathbb{C}\times T_N^*Y$ of microfunctions having ζ_1 as a holomorphic parameter (cf. Proposition 1.1.4 of [11]). Furthermore, β induces a quantized contact transformation of microdifferential operators, which is defined by the following relations:

$$\beta\circ D_{x_1}\circ\beta^{-1}=-\sqrt{-1}\zeta_1D_{x_n}, \qquad \beta\circ D_{x_i}\circ\beta^{-1}=D_{x_i}\ (2\leqq i\leqq n),$$
$$\beta\circ x_1\circ\beta^{-1}=-\sqrt{-1}D_{\zeta_1}D_{x_n}^{-1}, \qquad \beta\circ x_j\circ\beta^{-1}=x_j\ (2\leqq j\leqq n-1),$$
$$\beta\circ x_n\circ\beta^{-1}=x_n+D_{\zeta_1}\zeta_1D_{x_n}^{-1}.$$

Now we set

$$v(\zeta_1,x')=\beta(Y(x_1)f(x)).$$

Choosing sufficiently small $\varepsilon>0$ and sufficiently large $r>0$, we see that v can be extended to a section of $(\mathscr{C}\mathscr{O})^m$ defined on $\{\zeta_1\in\mathbb{C}; \mathrm{Re}\,\zeta_1>0 \text{ or } |\zeta_1|>r\}\times U'_\varepsilon$ in view of the theory of mild hyperfunctions (see Sect. 2.1 of [11]). Put $Q=\beta\circ P\circ\beta^{-1}$. Then from (2.3) we get

$$Qv(\zeta_1,x')=0. \tag{2.4}$$

We can write

$$P=x_1D_1I_m+\sum_{\alpha_1,\alpha_n\geqq 0}P_{\alpha_1,\alpha_n}(x'',D')x_n^{\alpha_n}x_1^{\alpha_1}$$

with $x''=(x_2,\dots,x_{n-1})$ and $D'=(D_2,\dots,D_n)$. Then we have

$$Q=-D_{\zeta_1}\zeta_1I_m+\sum_{\alpha_1,\alpha_n\geqq 0}P_{\alpha_1,\alpha_n}(x'',D')$$
$$\cdot(x_n+D_{\zeta_1}\zeta_1D_n^{-1})^{\alpha_n}(-\sqrt{-1}D_{\zeta_1}D_n^{-1})^{\alpha_1}.$$

Setting $\tau=\zeta_1^{-1}$, we have

$$Q=(\tau D_\tau-1)I_m+\sum_{\alpha_1,\alpha_n\geqq 0}P_{\alpha_1,\alpha_n}(x'',D')$$
$$\cdot(x_n+(1-\tau D_\tau)D_n^{-1})^{\alpha_n}(\sqrt{-1}\tau^2D_\tau D_n^{-1})^{\alpha_1}.$$

Here we note that

$$B=-\sum_{\alpha_n\geqq 0} P_{0,\alpha_n}(x'',D')x_n^{\alpha_n}. \tag{2.5}$$

Put

$$\begin{aligned} A' &= -\sqrt{-1}\sum_{\alpha_1\geqq 1,\alpha_n\geqq 0} P_{\alpha_1,\alpha_n}(x'',D') \\ &\quad \cdot (x_n+(1-\tau D_\tau)D_n^{-1})^{\alpha_n}(\sqrt{-1}\tau^2 D_\tau D_n^{-1})^{\alpha_1-1}D_n^{-1}, \\ B'(\tau D_\tau, x', D') &= -\sum_{\alpha_n\geqq 0} P_{0,\alpha_n}(x'',D') \\ &\quad \cdot (x_n+(1-\tau D_\tau)D_n^{-1})^{\alpha_n}+I_m. \end{aligned} \tag{2.6}$$

Then we have

$$Q=(I_m-A'\tau)\left(\tau D_\tau I_m-\sum_{i=0}^{\infty}(A'\tau)^i B'(\tau D_\tau, x', D')\right)$$

on $\{(\tau,\varrho)\in T^*\mathbb{C};\ |\tau|, |\varrho|<\delta\}\times U'_\varepsilon$ for some $\delta, \varepsilon>0$. Noting that A' can be written in the form

$$A'=\sum_{i,j\geqq 0} A'_{ij}(x',D')\tau^i(\tau D_\tau)^j,$$

we can find $m\times m$ matrices T and $C=C(\tau,x',D')$ of microdifferential operators of order $\leqq 0$ such that

$$\tau D_\tau I_m-\sum_{i=0}^{\infty}(A'\tau)^i B'=T(\tau D_\tau I_m-C(\tau,x',D')) \tag{2.7}$$

holds on $\{(\tau,\varrho);\ |\tau|, |\varrho|<\delta\}\times U'_\varepsilon$ for sufficiently small $\delta, \varepsilon>0$, and that $\sigma_0(T)$ is invertible there. (To find such T and C, we can use the same argument as in the proof of Proposition 2.4 of [16].) Note that $\sigma_0(T)=I_m$ since $\tau\varrho I_m=\sigma_0(T)\tau\varrho$. Since

$$Q=(I_m-A'\tau)T(\tau D_\tau I_m-C) \tag{2.8}$$

holds and $(I_m-A'\tau)T$ is invertible, we have

$$(\tau D_\tau I_m-C(\tau,x',D'))v(\tau^{-1},x')=0 \tag{2.9}$$

on $\{\tau\in\mathbb{C};\ 0<|\tau|<\delta\}\times U'_\varepsilon$ for sufficiently small $\delta, \varepsilon>0$. From (2.7) we get

$$C=(I_m-T^{-1})\tau D_\tau+T^{-1}\sum_{i=0}^{\infty}(A'\tau)^i B'. \tag{2.10}$$

Set

$$B'(\tau D_\tau, x', D')=\sum_{j=0}^{\infty} B'_j(x',D')(\tau D_\tau)^j. \tag{2.11}$$

In view of (2.5), (2.6), and (2.11), it follows from (2.10) that

$$\begin{aligned} \sigma_0(C)(0,x',\sqrt{-1}\eta') &= \sigma_0(B')(0,x',\sqrt{-1}\eta') \\ &= \sigma_0(B'_0)(x',\sqrt{-1}\eta') \\ &= -\sum_{\alpha_n\geqq 0}\sigma_0(P_{0,\alpha_n})(x'',\sqrt{-1}\eta')x_n^{\alpha_n}+I_m \\ &= \sigma_0(B)(x',\sqrt{-1}\eta')+I_m. \end{aligned}$$

In particular, the eigenvalues of $\sigma_0(C)(0,0,\ldots,\sqrt{-1})$ are $\mu_1+1,\ldots,\mu_m+1$. Since $\mu_i-\mu_j\notin\mathbb{Z}-\{0\}$ for $1\leqq i,j\leqq m$, there exists a unique $m\times m$ matrix $R=R(\tau,x',D')$ of microdifferential operators of order $\leqq 0$ defined on $\{\tau\in\mathbb{C};|\tau|<\delta\}\times U'_\varepsilon$ with some $\delta,\ \varepsilon>0$ such that

$$\begin{aligned}\tau D_\tau I_m-C(\tau,x',D')=R(\tau,x',D')(\tau D_\tau I_m-C(0,x',D'))R(\tau,x',D')^{-1},\\ R(0,x',D')=I_m.\end{aligned}\tag{2.12}$$

(See Theorem 2.1.24 of Tahara [23].) Combining this with (2.9), we see that v can be written in the form

$$v(\tau^{-1},x')=R(\tau,x',D')\tau^{C(0,x',D')}a(x')\tag{2.13}$$

on $\{\tau\in\mathbb{C};0<|\tau|<\delta\}\times U'_\varepsilon$ with a column vector $a(x')$ of m microfunctions defined on U'_ε for sufficiently small $\delta,\ \varepsilon>0$. Since $v(\tau^{-1},x')$ is single-valued, we have

$$(\exp(2\pi\sqrt{-1}C(0,x',D'))-I_m)a(x')=0.\tag{2.14}$$

In the case of (2.1), we get $a(x')=0$ from (2.14). Thus $Y(x_1)f(x)=0$ holds in this case.

Now assume (2.2). We may assume $\mu_1=\ldots=\mu_m=-1$. In fact, if $\mu_1=\ldots=\mu_m=-l$ with $l\in\mathbb{Z}$ such that $l\geqq 2$, then $f'=x_1^{l-1}f$ satisfies

$$(x_1(D_1I_m-A)-B-(l-1)I_m)f'=0,$$

and the eigenvalues of $\sigma_0(B)(p_0)+(l-1)I_m$ are all -1. If we can show $f'=0$, then it follows that $f=0$ since real bicharacteristics of $\{(x,\sqrt{-1}\eta);x_1=0\}$ are fibers of ϱ, and since $f\in(\varrho_!\mathscr{C}_M)^m_{p'_0}$. Hence, from now on, we assume $\mu_1=\ldots=\mu_m=-1$. This means that $\sigma_0(C_0)(p'_0)$ is nilpotent, where $C_0=C(0,x',D')$. So it follows from (2.14) that

$$C_0(x',D')a(x')=0.\tag{2.15}$$

Thus we have

$$\begin{aligned}v(\tau^{-1},x')&=R(\tau,x',D')\tau^{C(0,x',D')}a(x')\\&=R(\tau,x',D')a(x').\end{aligned}\tag{2.16}$$

We use the notation $\partial_\tau R(\tau,x',D')=[D_\tau,R]$. Then

$$(\tau\partial_\tau I_m-C(\tau,x',D'))(R(\tau,x',D')\tau^{C(0,x',D')})=0$$

holds, and hence we have

$$Q(\zeta_1,x',\partial_{\zeta_1},D')(R(\zeta_1^{-1},x',D')\zeta_1^{-C(0,x',D')})=0\tag{2.17}$$

on $\{\zeta_1\in\mathbb{C};|\zeta_1|>r\}\times U'_\varepsilon$ for sufficiently large r.

We can write

$$Q(\zeta_1,x',D_{\zeta_1},D')=\sum_{j=0}^{\infty}Q_j(x',D',\zeta_1D_{\zeta_1})D_{\zeta_1}^j,$$

$$Q_j(x',D',\zeta_1D_{\zeta_1})=\sum_{\nu=0}^{\infty}Q_{j,\nu}(x',D')(\zeta_1D_{\zeta_1})^\nu,$$

$$R(\tau,x',D')=\sum_{k=0}^{\infty}R_k(x',D')\tau^k.$$

Then (2.17) is equivalent to the relations

$$\sum_{\nu=0}^{\infty} Q_{0,\nu} R_j(-C_0-j)^\nu$$
$$= -\sum_{k=1}^{j} \sum_{\mu=0}^{\infty} Q_{k,\mu} R_{j-k}(-C_0-j)^\mu(-C_0-j+1)\dots(-C_0-j+k) \qquad (2.18)$$

for $j \geqq 0$ [when $j=0$, the right hand side of (2.18) is zero]. Suppose that S_j (j = 1, 2, ...) satisfy

$$\sum_{\nu=0}^{\infty} Q_{0,\nu} S_j(-C_0-j)^\nu$$
$$= -\sum_{k=1}^{j-1} \sum_{\mu=0}^{\infty} Q_{k,\mu} S_{j-k}(-C_0-j)^\mu(-C_0-j+1)\dots(-C_0-j+k)$$
$$+ \sum_{\mu=0}^{\infty} Q_{j,\mu}(-C_0-j)^\mu(-C_0-j+1)\dots(-C_0-1). \qquad (2.19)$$

Then (2.18) is satisfied if we set $R_0 = I_m$ and $R_j = S_j C_0$ for $j \geqq 1$. By the uniqueness of R satisfying (2.12) and (2.17), we see that $R_j = S_j C_0$ holds for $j \geqq 1$ if S_j satisfy (2.19). Now we shall show that there exist unique S_j which satisfy (2.19) and that

$$S(\tau, x', D') = \sum_{j=1}^{\infty} S_j(x', D')\tau^j$$

converges on $\{\tau \in \mathbb{C};\ |\tau| < \delta\} \times U'_\varepsilon$ for sufficiently small δ, $\varepsilon > 0$. Set

$$F(\zeta_1^{-1}, x', D') = \sum_{j=0}^{\infty} F_j(x', D')\zeta_1^{-1}$$
$$= \sum_{j=1}^{\infty} (Q_j(x', D', \zeta_1\partial_{\zeta_1}) (\partial_{\zeta_1})^{j-1}(\zeta_1^{-C_0-1}))\zeta_1^{C_0+1}.$$

Then it is easy to see that (2.19) is equivalent to

$$Q(\zeta_1, x', \partial_{\zeta_1}, D') (S(\zeta_1^{-1}, x', D')\zeta_1^{-C(0,x',D')}) = F(\zeta_1^{-1}, x', D')\zeta_1^{-C(0,x',D')}, \quad (2.20)$$

where $S(\zeta_1^{-1}, x', D')$ is regarded as a formal power series in ζ_1^{-1}. By (2.8) we see that (2.20) is equivalent to

$$(\tau\partial_\tau I_m - C(\tau, x', D'))(S(\tau, x', D')\tau^{C(0,x',D')}) = G(\tau, x', D')\tau^{C(0,x',D')+1}, \quad (2.21)$$

where
$$G(\tau, x', D') = \sum_{j=0}^{\infty} G_j(x', D')\tau^j$$
$$= (T(\tau, x', \partial_\tau, D')^{-1}(I_m - A'(\tau, x', \partial_\tau, D')\tau)^{-1}$$
$$\cdot (F(\tau, x', D')\tau^{C(0,x',D')+1}))\tau^{-C(0,x',D')-1}.$$

The relation (2.21) is equivalent to

$$(\tau\partial_\tau - C(\tau, x', D'))\, S(\tau, x', D') + S(\tau, x', D')C_0(x', D') = G(\tau, x', D')\tau,$$

and this relation is equivalent to the relations

$$(j - C_0)S_j + S_j C_0 = G_{j-1} + C_1 S_{j-1} + \dots + C_{j-1} S_1 \qquad (2.22$$

for $j \geqq 1$, where we set

$$C(\tau, x', D') = \sum_{j=0}^{\infty} C_j(x', D')\tau^j .$$

Since $\sigma_0(C_0)(p_0')$ is nilpotent, $S_j (j \geqq 1)$ are uniquely determined by induction on j. Furthermore, it is easy to see that $S(\tau, x', D') = \sum_{j=0}^{\infty} S_j(x', D')\tau^j$ converges as a matrix of microdifferential operators on $\{\tau \in \mathbb{C}; |\tau| < \delta\} \times U_\varepsilon'$ for sufficiently small ε, $\delta > 0$ in view of (2.22). Thus R can be written in the form

$$R(\tau, x', D') = I_m + S(\tau, x', D')C_0(x', D') .$$

Hence we have

$$v(\zeta_1, x', D') = R(\zeta_1^{-1}, x', D')a(x') = a(x')$$

in view of (2.16). Thus we have proved that $Y(x_1)f(x)$ is a section of $(\iota_* \mathscr{C}_{N|X})^m$ defined on U_ε'.

In the same way, we can show that $Y(-x_1)f(x)$ is also a section of $(\iota_* \mathscr{C}_{N|X})^m$ defined on U_ε' for some $\varepsilon > 0$. Hence

$$f(x) = Y(x_1)f(x) + Y(-x_1)f(x)$$

is a section of $(\iota_* \mathscr{C}_{N|X})^m$ defined on U_ε'. Since sections of $\mathscr{C}_{N|X}$ have the unique continuation property along each fiber of ι (see Sect. 4 of [10]), and since $f(x)$ is a section of $(\varrho_! \mathscr{C}_M)^m$, it follows that $f(x) = 0$. This completes the proof of Proposition 2.2 and at the same time, the proof of Theorem 2.1.

Remark 2.3. Conditions (2.1) or (2.2) in Proposition 2.2 can be replaced by a weaker condition that $\mu_j \notin \{0, 1, 2, \dots\}$ for $j = 1, \dots, m$.

Corollary 2.4. *Let P_1 and P_2 be as in the beginning of this section satisfying* (C.1), (C.2) *and* (R) *at $p_0 \in T_M^* X - M$. Set*

$$P = P_2^{m'} P_1^m + A_1 P_2^{m'-1} P_1^{m-1} + \dots + A_{m'} P_1^{m-m'} + A_{m'+1} P_1^{m-m'-1} + \dots + A_m ;$$

here A_j is a microdifferential operator of order $\leqq \min(j, m')l_2 + j(l_1 - 1)$ defined in a neighborhood of p_0, and m and m' are integers such that $0 \leqq m' \leqq m$. Set $c = \{\sigma(P_1), \sigma(P_2)\}$ (p_0) and let $\lambda = 0, \dots, m - m' - 1$, $\lambda_1, \dots, \lambda_{m'}$ be the roots of the equation

$$\lambda(\lambda - 1)\dots(\lambda - m + 1) + c^{-1}\sigma_{l-1}(A_1)(p_0)\lambda(\lambda - 1)\dots(\lambda - m + 2) + \dots + c^{-m'}\sigma_{m'(l-1)}(A_{m'})(p_0)\lambda(\lambda - 1)\dots(\lambda - m + m' + 1) = 0 .$$

Assume that $\lambda_j \notin \{-1, -2, -3, \dots\}$ for $j = 1, \dots, m'$. Under these assumptions, if f is a microfunction defined in a neighborhood of p_0 such that $Pf = 0$ and that f vanishes on $b_1(p_0) - \{p_0\}$, then f vanishes on $b_1(p_0)$.

Proof. The equation $P_2^{m-m'} Pu = 0$ is equivalent to a system of the form $(P_1 P_2 I_m + Q)v = 0$ mentioned in Theorem 2.1 (see the proof of Corollary 3.9 of [16]). Thus this corollary is an immediate consequence of Theorem 2.1.

Example 2.5. Putting $x=(x_1, x_2)\in\mathbb{R}^2$ and $D_j=\partial/\partial x_j$, we consider the operator

$$P=D_1(D_1-\sqrt{-1}x_1D_2)+a_1(x)D_1+a_2(x)D_2+b(x);$$

here $a_1(x)$, $a_2(x)$, $b(x)$ are real analytic functions defined in an open subset U of $\mathbb{R}^2$. Assume that $a_2(0,x_2)\notin\{0,\ -\sqrt{-1},\ -2\sqrt{-1},\dots\}$ for any $(0,x_2)\in U$. Let f be a hyperfunction defined in U such that Pf is real analytic in U and that f is real analytic in $\{x\in U;\ x_1\neq 0\}$. Then f is also real analytic in U. In general, the above condition concerning $a_2(0,x_2)$ is necessary. In fact, set

$$f_k(x)=D_1^k\left(\left(x_2+\frac{\sqrt{-1}}{2}x_1^2+\sqrt{-1}0\right)^{-1}\right)\qquad(k=0,1,2,\dots).$$

Then $f_k(x)$ satisfies

$$(D_1(D_1-\sqrt{-1}x_1D_2)-k\sqrt{-1}D_2)f_k(x)=0,$$

but is not real analytic at $0\in\mathbb{R}^2$.

3. Analytic Hypo-Ellipticity

Let P_1, P_2, Q be as in the beginning of Sect. 2 satisfying (C.1)–(C.3) at $p_0\in T_M^*X-M$. Set

$$V_j=\{(z,\zeta)\in T^*X;\ \sigma(P_j)(z,\zeta)=0\}$$

for $j=1,2$. Then, instead of (R), we assume in this section, (H) $V_1\cap V_1^c$ is a symplectic submanifold of T^*X in a neighborhood of p_0, and there are a positive odd integer k and a complex number α such that $F=\alpha\sigma(P_1)$ satisfies

$$(H_{(F-F^c)})^j(F+F^c)(p_0)=0\quad\text{for}\quad 0\leqq j\leqq k-1,$$
$$(H_{(F-F^c)})^k(F+F^c)(p_0)<0,$$
$$dF(p_0)-dF^c(p_0)\neq 0.$$

Theorem 3.1. *Under* (C.1)–(C.3) *and* (H), *the homomorphism*

$$P_1P_2I_m+Q:(\mathscr{C}_M)_{p_0}^m\to(\mathscr{C}_M)_{p_0}^m$$

is injective; i.e., $P=P_1P_2I_m+Q$ *is micro-locally analytic hypo-elliptic at* p_0.

Proof. Condition (H) guarantees that there exists a real contact transformation ψ defined in a neighborhood of p_0 such that

$$\psi(V_1)=\{(z,\zeta)\in T^*X;\ \zeta_1+\sqrt{-1}z_1^k\zeta_n=0\}$$

in a neighborhood of $\psi(p_0)=(0,\sqrt{-1}\,dz_n)\in T_M^*X$ (see Sato et al. [20]). Thus we may assume, from the beginning, that

$$P_1=D_1+\sqrt{-1}z_1^kD_n$$

and $p_0=(0,\sqrt{-1}\,dz_n)$. Setting

$$S(z,\theta)=\langle z,\theta\rangle-\sqrt{-1}\frac{z_1^{k+1}}{k+1}\theta_n,$$

let $(w,\theta)=\varphi(z,\zeta)$ be the complex contact transformation defined by

$$w=\mathrm{grad}_\theta S(z,\theta), \qquad \zeta=\mathrm{grad}_z S(z,\theta).$$

Then $(\varphi^{-1}(T_M^*X), \mathbb{C}^\times T_M^*X)$ is positive at p_0 in the sense of Schapira [21] by virtue of Théorème 2.4 of [21]. Moreover we have

$$L=T_M^*X\cap\varphi(T_M^*X)=\{(x,\sqrt{-1}\eta)\in T_M^*X;\ x_1=0\}.$$

Let

$$\Phi:\varphi^{-1}\mathscr{E}_X\tilde{\to}\mathscr{E}_X$$

be a quantized contact transformation associated with φ. Then the sheaf $\varphi^{-1}\mathscr{C}_M$ on $\varphi^{-1}(T_M^*X)$ can be endowed with the structure of an $\mathscr{E}_X$-module by

$$A(\varphi^{-1}u)=\varphi^{-1}(\Phi^{-1}(A)u)$$

for $u\in\mathscr{C}_M$ and $A\in\mathscr{E}_X$.

By virtue of Théorème 3.2 of [21], there exists an injective sheaf homomorphism of $\mathscr{E}_X$-modules,

$$\Psi:\mathscr{C}_M|_L\to\Gamma_L(\varphi^{-1}\mathscr{C}_M).$$

Now let $f\in(\mathscr{C}_M)_{p_0}^m$ satisfy $Pf=0$ and define $g\in(\mathscr{C}_M)_{p_0}^m$ by $\Psi(f)=\varphi^{-1}g$. Then we have

$$\varphi^{-1}(\Phi^{-1}(P)g)=P(\varphi^{-1}g)=P(\Psi(f))=\Psi(Pf)=0.$$

From this, it follows that $\Phi^{-1}(P)g=0$.

Since Ψ is injective, it suffices to show that $g=0$. Note that

$$\sigma(\Phi^{-1}(P_1))=(\varphi^{-1})^*\sigma(P_1)=\zeta_1.$$

Thus $\Phi^{-1}(P)$ satisfies (C.1)–(C.3) and (R) at $\varphi(p_0)$, and the support of g is contained in L in a neighborhood of $\varphi(p_0)=p_0=(0,\sqrt{-1}dz_n)$. Since the intersection of L and the real bicharacteristic of $\{(x,\sqrt{-1}\eta);\ \eta_1=0\}$ through p_0 is $\{p_0\}$, it follows from Theorem 2.1 that $g=0$ in a neighborhood of p_0. This completes the proof.

Remark 3.2. (i) If we assume that P_2 satisfies (H) instead of P_1 and that $\lambda_j\notin\{-1,-2,-3,\dots\}$ for $j=1,\dots,m$, then the conclusion of Theorem 3.1 also holds.

(ii) When $k=1$, (H) is equivalent to the condition $\{\sigma(P_1),\sigma(P_1)^c\}(p_0)<0$.

Corollary 3.3. *Let P_1 and P_2 be as in Theorem 3.1 satisfying* (C.1), (C.2) *and* (H) *at $p_0\in T_M^*X-M$. Set*

$$P=P_2^{m'}P_1^m+A_1P_2^{m'-1}P_1^{m-1}+\dots+A_{m'}P_1^{m-m'}$$
$$+A_{m'+1}P_1^{m-m'-1}+\dots+A_m;$$

here A_j is a microdifferential operator of order $\leqq \min(j, m')l_2 + j(l_1 - 1)$ defined in a neighborhood of p_0, and m and m′ are integers such that $0 \leqq m' \leqq m$. Set $c = \{\sigma(P_1), \sigma(P_2)\}(p_0)$ and let $\lambda = 0, \ldots, m - m' - 1, \lambda_1, \ldots, \lambda_{m'}$ be the roots of the equation

$$\lambda(\lambda-1)\ldots(\lambda-m+1) + c^{-1}\sigma_{l-1}(A_1)(p_0)\lambda(\lambda-1)\ldots(\lambda-m+2) + \ldots + c^{-m'}\sigma_{m'(l-1)}(A_{m'})(p_0)\lambda(\lambda-1)\ldots(\lambda-m+m'+1) = 0.$$

Assume that $\lambda_j \notin \{-1, -2, -3, \ldots\}$ for $j = 1, \ldots, m'$. Then the homomorphism

$$P : (\mathscr{C}_M)_{p_0} \to (\mathscr{C}_M)_{p_0}$$

is injective.

If a partial differential operator with real analytic coefficients can be written in the form mentioned in Theorem 3.1 and satisfies (C.1)–(C.3) and (H) at each point of its real characteristic variety (we interchange the roles of P_1 and of P_2 if necessary), then the operator becomes analytic hypo-elliptic in the usual sense. Now we give some examples of partial differential operators which are analytic hypo-elliptic in view of Theorem 3.1. For the sake of simplicity, we treat only the case of $m = 1$. We use the notation $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $D_j = \partial/\partial x_j$.

Example 3.4. Set

$$P = D_1^2 + x_1^2(D_2^2 + \ldots + D_n^2) + \sum_{j=1}^{n} a_j(x)D_j + b(x);$$

here a_j and b are real analytic in an open subset U of $\mathbb{R}^n$. Assume that

$$\sum_{j=2}^{n} a_j(0, x')\eta_j \notin \{(2l+1)\sqrt{-1};\ l \in \mathbb{Z}\}$$

for any $(0, x') = (0, x_2, \ldots, x_n) \in U$ and any $(\eta_2, \ldots, \eta_n) \in \mathbb{R}^{n-1}$ such that $\eta_2^2 + \ldots + \eta_n^2 = 1$. Then P is analytic hypo-elliptic in U; i.e., if f is a hyperfunction defined in an open subset U' of U such that Pf is real analytic in U', then f itself is real analytic in U'.

This example is essentially contained in the results of Treves [25] (cf. also [5, 8, 15, 24]) although he studies the analytic hypo-ellipticity in the framework of distributions. However, we believe that the following two examples are essentially new.

Example 3.5. Putting $n = 2$, we consider the operator

$$P = (D_1 + \sqrt{-1}x_1^k D_2)(D_1^l - \sqrt{-1}x_1 D_2^l) + \sum_{\alpha \geqq 0, |\alpha| \leqq l} a_\alpha(x) D_1^{\alpha_1} D_2^{\alpha_2};$$

here k, l are positive odd integers, $|\alpha| = \alpha_1 + \alpha_2$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ with $\alpha_1, \alpha_2 \geqq 0$ and $a_\alpha(x)$ are real analytic in an open subset U of $\mathbb{R}^2$. Assume that $a_{0,l}(0, x_2) \notin \sqrt{-1}\mathbb{Z}$ for any $(0, x_2) \in U$. Then P is analytic hypo-elliptic in U.

Example 3.6. Set

$$P = (D_1 + \sqrt{-1}x_1 D_2)(D_1 - \sqrt{-1}(x_1 + x_2)D_2) + a_1(x)D_1 + a_2(x)D_2 + b(x),$$

where $a_1(x)$, $a_2(x)$, $b(x)$ are real analytic in a neighborhood of $0=(0,0)\in\mathbb{R}^2$. Assume that $a_2(0)\notin 2\sqrt{-1}\mathbb{Z}$. Then P is analytic hypo-elliptic at 0; i.e., if f is a hyperfunction defined in a neighborhood of 0 such that Pf is real analytic in a neighborhood of 0, then f itself is real analytic in a neighborhood of 0.

The condition (H) can be slightly weakened. In fact, we have the following

Theorem 3.7. *Let P_1, P_2, Q be as in the beginning of Sect. 2 satisfying* (C.1)–(C.3) *at $p_0\in T_M^*X-M$. Assume that there is a real contact transformation ψ defined in a neighborhood of p_0 such that*

$$\sigma(P_1)(\psi^{-1}(z,\zeta))=q(z,\zeta)(\zeta_1+\sqrt{-1}\,a(z,\zeta'))$$

in a neighborhood of $\psi(p_0)=(0,\sqrt{-1}\,dz_n)$; here q is a homogeneous holomorphic function which does not vanish at $\psi(p_0)$, and a is of the form

$$a(z,\zeta')=\sum_{j=2}^{n}a_j(z)\zeta_j$$

with a_j holomorphic in a neighborhood of $z=0$. In addition, we assume that

$$x_1a(x,\eta')>0$$

for $x\in\mathbb{R}^n$, $|x_i|<\varepsilon$ $(1\leqq i\leqq n)$, $x_1\neq 0$, $\eta'=(\eta_2,\dots,\eta_n)\in\mathbb{R}^{n-1}$, $|\eta_j|<\varepsilon$ $(2\leqq j\leqq n-1)$, $|\eta_n-1|<\varepsilon$ with sufficiently small $\varepsilon>0$. Then the homomorphism

$$P_1P_2I_m+Q:(\mathscr{C}_M)^m_{p_0}\to(\mathscr{C}_M)^m_{p_0}$$

is injective.

Proof. We may assume, from the beginning, that

$$P_1=D_1+\sqrt{-1}\sum_{j=2}^{n}a_j(z)D_j$$

and $p_0=(0,\sqrt{-1}\,dz_n)$. Let T be the solution of the following Cauchy problem:

$$\frac{\partial T}{\partial z_1}(z,\theta')+\sqrt{-1}\sum_{j=2}^{n}a_j(z)\frac{\partial T}{\partial z_j}(z,\theta')=0\,, \tag{3.1}$$

$$T(0,z',\theta')=\langle z',\theta'\rangle\,, \tag{3.2}$$

where $z'=(z_2,\dots,z_n)$ is sufficiently near to $(0,\dots,0)$, and $\theta'=(\theta_2,\dots,\theta_n)$ sufficiently near to $(0,\dots,0,\sqrt{-1})$. Then we have from (3.1) and (3.2)

$$\operatorname{Re}T(x,\sqrt{-1}\,\eta')=-\operatorname{Im}T(x,\eta')\geqq 0$$

for $x\in\mathbb{R}^n$, $|x_i|<\delta$ $(1\leqq i\leqq n)$, $\eta'=(\eta_2,\dots,\eta_n)\in\mathbb{R}^{n-1}$, $|\eta_j|<\delta$ $(2\leqq j\leqq n-1)$, $|\eta_n-1|<\delta$ with sufficiently small $\delta>0$ since

$$\frac{\partial}{\partial x_1}\operatorname{Im}T(x,\eta')=-\sum_{j=2}^{n}a_j(x)\frac{\partial}{\partial x_j}\operatorname{Re}T(x,\eta')\,,$$

and $\operatorname{grad}_{x'} \operatorname{Re} T(x, \eta')$ is sufficiently near to $(0, \ldots, 0, 1)$. Setting

$$S(z, \theta) = z_1 \theta_1 + T(z, \theta'),$$

we define a contact transformation $(w, \theta) = \varphi(z, \zeta)$ in a neighborhood of p_0 by

$$w = \operatorname{grad}_\theta S(z, \theta), \qquad \zeta = \operatorname{grad}_z S(z, \theta). \tag{3.3}$$

Then by Théorème 2.4 of [21], $(T_M^* X, \mathbb{C}^\times \varphi(T_M^* X))$ is positive at p_0. From (3.1) and (3.3) we have

$$\theta_1 = \zeta_1 + \sqrt{-1} \sum_{j=2}^{n} a_j(z) \zeta_j = \sigma(P_1)(z, \zeta).$$

In view of (3.1) and (3.2), S can be written in the form

$$S(z, \theta) = \langle z, \theta \rangle + z_1^2 S_1(z, \theta').$$

Hence we see that φ fixes each point of

$$L = \{(x, \sqrt{-1}\eta) \in T_M^* X;\ x_1 = 0\},$$

and that

$$\varphi(T_M^* X) \cap T_M^* X = L$$

since $\operatorname{Re} \theta_1 = -a(x, \eta') \neq 0$ for $z = x \in \mathbb{R}^n$ with $x_1 \neq 0$ and $\zeta = \sqrt{-1}\eta$ with $\eta \in \mathbb{R}^n$. Thus, as in the proof of Theorem 3.1, we can prove the analytic hypo-ellipticity of $P_1 P_2 I_m + Q$ using Théorème 3.2 of [21] and Theorem 2.1.

Remark 3.8. For the operator P_1 in Theorem 3.7, the homomorphism

$$P_1 : (\mathscr{C}_M)_{p_0} \to (\mathscr{C}_M)_{p_0}$$

is injective. This fact follows from the above proof.

Example 3.9. Set

$$P = (D_1 + \sqrt{-1} x_1 (x_1^2 + x_2^2) D_2)(D_1 - \sqrt{-1} x_1 D_2) \\ + a_1(x) D_1 + a_2(x) D_2 + b(x);$$

where $a_1(x)$, $a_2(x)$, $b(x)$ are real analytic functions defined in an open subset U of $\mathbb{R}^2$. Assume that

$$(1 + x_2^2)^{-1} a_2(0, x_2) \notin \sqrt{-1}\mathbb{Z}$$

for any $(0, x_2) \in U$. Then P is analytic hypo-elliptic in U.

4. Solutions with Minimal Singularities

In this section, we treat operators of the form studied in preceding sections; we find microfunction solutions with minimal supports under certain conditions. First, in Theorem 4.1, we treat the case where $\sigma(P_1)$ is real and $\sigma(P_2)^c$ satisfies (H). Secondly, in Theorem 4.2, we treat the case where the generalized Levi forms of $\sigma(P_1)$ and of $\sigma(P_2)$ are both positive. In particular, operators treated in these theorems are not analytic hypo-elliptic.

Theorem 4.1. *Set $P=P_1P_2I_m+Q$ with P_1, P_2, Q being as in the beginning of Sect. 2 satisfying* (C.1) *and* (C.2) *at $p_0\in T_M^*X-M$. Assume*

$$V_2\cap V_2^c\subset V_1=V_1^c.$$

In addition, assume that $V_2\cap V_2^c$ is a symplectic submanifold of T^*X and that there are a positive odd integer k and a complex number α such that $F=\alpha\sigma(P_2)$ satisfies

$$(H_{(F-F^c)})^j(F+F^c)(p_0)=0 \quad for \quad 0\leqq j\leqq k-1,$$
$$(H_{(F-F^c)})^k(F+F^c)(p_0)>0,$$
$$dF(p_0)-dF^c(p_0)\neq 0.$$

Let $b_1(p_0)$ be the real bicharacteristic of $V_1^{\mathbb{R}}$ and denote by $b_1^+(p_0)$ and $b_1^-(p_0)$ the connected components of $b_1(p_0)-\{p_0\}$ in a neighborhood of p_0. Under these assumptions, we can find a neighborhood U of p_0 in T_M^*X, and a column vector f of m microfunctions defined in U such that $Pf=0$ and that

$$\mathbb{R}^+\{p_0\}\cap U\subset \operatorname{supp} f\subset \mathbb{R}^+(b_1^+(p_0)\cup\{p_0\})\cap U.$$

Moreover, if $\lambda_j\notin\{0,1,2,\dots\}$ for some $j\in\{1,\dots,m\}$, then we can choose f so that

$$\operatorname{supp} f=\mathbb{R}^+(b_1^+(p_0)\cup\{p_0\})\cap U.$$

Proof. We may assume that

$$V_1=\{(z,\zeta)\in T^*X;\ z_1=0\},$$
$$V_2\cap V_2^c=\{(z,\zeta)\in T^*X;\ z_1=\zeta_1=0\}$$

in a neighborhood of $p_0=(0,\sqrt{-1}\,dz_n)$ by using a real contact transformation. Let f_2 be a holomorphic function homogeneous of degree 1 with respect to ζ defined in a neighborhood of p_0 such that $V_2=\{f_2=0\}$ and $df_2(p_0)\neq 0$. Since $\{z_1,f_2\}(p_0)\neq 0$, we can choose f_2 so that it takes the form

$$f_2=\zeta_1+a(z,\zeta')$$

with a homogeneous of degree 1 with respect to $\zeta'=(\zeta_2,\dots,\zeta_n)$. Put

$$a(z,\zeta')=a_1(z,\zeta')+\sqrt{-1}\,a_2(z,\zeta')$$

with $a_j^c=-a_j (j=1,2)$. Then we have

$$\{(z,\zeta);\ \zeta_1+a_1(z,\zeta')=a_2(z,\zeta')=0\}=\{(z,\zeta);\ z_1=\zeta_1=0\}.$$

In particular, $a_j(0,z',\zeta')=0$ holds for $j=1,2$. Noting that $\{\zeta_1+a_1,z_1\}=1$, we can find a complex homogeneous canonical coordinate system (x,ξ) around p_0 such that

$$\xi_1=\zeta_1+a_1(z,\zeta'),\qquad x_1=z_1,$$
$$(x(p_0),\xi(p_0))=(0,0,\dots,0,\sqrt{-1})$$

and that x and $\sqrt{-1}\xi$ are real valued on T_M^*X. Since $a_1(0, z', \zeta')=0$, we have

$$\begin{aligned} V_1 &= \{(x, \xi) \in T^*X;\ x_1 = 0\}, \\ V_2 \cap V_2^c &= \{(x, \xi) \in T^*X;\ x_1 = \xi_1 = 0\}. \end{aligned} \tag{4.1}$$

On the other hand, we also have

$$V_2 \cap V_2^c = \{(x, \xi);\ \xi_1 = a_2 = 0\}. \tag{4.2}$$

Since $\{x_1, a_2\} = \{z_1, a_2\} = 0$, we can write $a_2 = a_2(x, \xi')$ by abuse of notation. So from (4.1) and (4.2) we get

$$\{(x, \xi');\ a_2(x, \xi') = 0\} = \{(x, \xi');\ x_1 = 0\}.$$

Hence we can write

$$a_2(x, \xi') = x_1^k b(x, \xi'),$$

where k is a positive integer, and b is a holomorphic function homogeneous of degree 1 such that $b^c = -b$ and $b(p_0) \neq 0$. Then we have

$$V_2 = \{(x, \xi) \in T^*X;\ \xi_1 + \sqrt{-1}\, x_1^k b(x, \xi') = 0\}.$$

In view of the proof of Theorem of Sato et al. [20], we see that there is a complex homogeneous canonical coordinate system (y, η) around p_0 such that y and $\sqrt{-1}\,\eta$ are real valued on T_M^*X and that

$$V_1 = \{(y, \eta) \in T^*X;\ y_1 = 0\},$$

$$V_2 = \{(y, \eta);\ \eta_1 + \varepsilon\sqrt{-1}\, y_1^k \eta_n = 0\}$$

in a neighborhood of $(y(p_0), \eta(p_0)) = (0, 0, \ldots, 0, \sqrt{-1})$ with $\varepsilon = \pm 1$. By the assumption in the theorem, we see, $\varepsilon = -1$ and k is odd.

Thus, from now on, we assume

$$V_1 = \{(z, \zeta) \in T^*X;\ z_1 = 0\},$$

$$V_2 = \{(z, \zeta);\ \zeta_1 - \sqrt{-1}\, z_1^k \zeta_n = 0\}$$

in a neighborhood of $p_0 = (0, \sqrt{-1}\, dz_n)$. Let $(w, \theta) = \varphi(z, \zeta)$ be the complex contact transformation defined by

$$w = \operatorname{grad}_\theta S(z, \theta), \qquad \zeta = \operatorname{grad}_z S(z, \theta),$$

where

$$S(z, \theta) = \langle z, \theta \rangle - \sqrt{-1}\, \frac{z_1^{k+1}}{k+1}\, \theta_n.$$

We endow the sheaf $\varphi_* \mathscr{C}_M$ on $\varphi(T_M^*X)$ with the structure of an $\mathscr{E}_X$-module by

$$A(\varphi_* u) = \varphi_*(\Phi(A) u)$$

for $A \in \mathscr{E}_X$ and $u \in \mathscr{C}_M$, where

$$\Phi : \varphi^{-1}\mathscr{E}_X \tilde{\to} \mathscr{E}_X$$

is a quantized contact transformation associated with φ. Then by Théorème 3.2 of [21], there is an injective homomorphism of sheaves of $\mathscr{E}_X$-modules,

$$\Psi:(\varphi_*\mathscr{C}_M)|_L \to \Gamma_L(\mathscr{C}_M),$$

where $L=\{(x,\sqrt{-1}\eta)\in T_M^*X;\ x_1=0\}$. Note that

$$\varphi^{-1}(V_1)=\{(z,\zeta)\in T^*X;\ z_1=0\},$$
$$\varphi^{-1}(V_2)=\{(z,\zeta)\in T^*X;\ \zeta_1=0\}.$$

In particular, the characteristic variety of

$$\Phi(P)=\Phi(P_1)\Phi(P_2)I_m+\Phi(Q)$$

is real. Hence by Theorem 3.6 of [16], there is a column vector f of m microfunctions defined in a neighborhood of p_0 such that $\Phi(P)f=0$ and that

$$\mathbb{R}^+ b_2'(p_0)\subset \operatorname{supp} f\subset \mathbb{R}^+(b_1^+(p_0)\cup b_2'(p_0))$$

in a neighborhood of p_0, where $b_2'(p_0)$ is the real bicharacteristic of $\{(x,\sqrt{-1}\eta)\in T_M^*X;\ \eta_1=0\}$ through p_0; if $\lambda_j\notin\{0,1,2,\ldots\}$ for some $j\in\{1,\ldots,m\}$, then we can choose f so that

$$\operatorname{supp} f=\mathbb{R}^+(b_1^+(p_0)\cup b_2'(p_0))$$

in a neighborhood of p_0. We get

$$P(\Psi(\varphi_*f))=\Psi(P(\varphi_*f))=\Psi(\varphi_*\Phi(P)f)=0.$$

Since Ψ is injective, we have

$$\operatorname{supp}(\Psi(\varphi_*f))=\operatorname{supp}(\varphi_*f)\cap L=\varphi(\operatorname{supp} f)\cap L$$

in a neighborhood of p_0. Noting that

$$\mathbb{R}^+(b_1^+(p_0)\cup b_2'(p_0))\cap L=\mathbb{R}^+(b_1^+(p_0)\cup\{p_0\}),$$

we see that $\Psi(\varphi_*f)$ has the properties mentioned in the theorem. This completes the proof.

Theorem 4.2. *Set $P=P_1P_2I_m+Q$ with P_1, P_2, Q being as in the beginning of Sect. 2 satisfying* (C.1) *and* (C.2) *at $p_0\in T_M^*X-M$. We assume also*

$$\{\sigma(P_1),\sigma(P_1)^c\}(p_0)>0,\qquad \{\sigma(P_2),\sigma(P_2)^c\}(p_0)>0$$

and that $V_1\cap V_1^c=V_2\cap V_2^c$. Then there exist a neighborhood (in T_M^*X) U of p_0 and a column vector f of m microfunctions defined in U such that $Pf=0$ and that

$$\operatorname{supp} f=\mathbb{R}^+\{p_0\}\cap U.$$

Proof. Let f_j $(j=1,2)$ be holomorphic functions homogeneous of degree 1 defined in a neighborhood of p_0 such that $V_j=\{f_j=0\}$ and $df_j(p_0)\neq 0$ $(j=1,2)$. In view of the proof of Theorem 2 of Kashiwara et al. [9], we can find a homogeneous function h of degree 0 defined in a neighborhood of p_0 such that

$$f_2=f_1^c+hf_1 \tag{4.3}$$

and $h^c = h$. Since $\{f_2, f_2^c\} = (h^2 - 1)\{f_1, f_1^c\}$ on $W = V_1 \cap V_1^c$, we have $h^2 > 1$ on $W \cap T_M^* X$. Noting that (4.3) implies

$$(1 - h^2) f_1 = f_2^c - h f_2,$$

we can assume $h > 1$ on $W \cap T_M^* X$ by interchanging f_1 and f_2 if necessary. In view of Theorem 2.3.2 of [19, Chap. III], we may assume, from the beginning, that

$$f_1 = \zeta_1 - \sqrt{-1}\, z_1 \zeta_n$$

and $p_0 = (0, \sqrt{-1}\, dz_n)$. Then we get

$$f_2 = (h - 1)\zeta_1 - \sqrt{-1}\, z_1 (h + 1)\zeta_n .$$

Hence we can choose f_2 so that

$$f_2 = \zeta_1 - \sqrt{-1}\, z_1 \frac{h+1}{h-1} \zeta_n .$$

Let φ be the contact transformation defined in the proof of Theorem 4.1 with $k = 1$. Then we have

$$\varphi^{-1}(V_1) = \{(z, \zeta) \in T^*X;\ \zeta_1 = 0\},$$
$$\varphi^{-1}(V_2) = \{(z, \zeta);\ \zeta_1 - \sqrt{-1}\, z_1 a(z, \zeta)\zeta_n = 0\},$$

where a is a holomorphic function defined in a neighborhood of p_0 and homogeneous of degree 0 such that

$$a|_L = \left.\frac{h+1}{h-1}\right|_L - 1 > 0$$

with $L = \{(x, \sqrt{-1}\, \eta) \in T_M^* X;\ x_1 = 0\}$. Hence, setting $g_2 = \zeta_1 - \sqrt{-1}\, z_1 a \zeta_n$, we get

$$\{g_2, g_2^c\}(p_0) = a(p_0) + a^c(p_0) > 0 .$$

Moreover, it is easy to see that

$$\varphi^{-1}(V_2) \cap (\varphi^{-1}(V_2))^c = \{z_1 = \zeta_1 = 0\} \subset \varphi^{-1}(V_1) .$$

Thus by Theorem 4.1, we can find a column vector f of m microfunctions defined in a neighborhood of p_0 such that $\Phi(P) f = 0$ and that

$$\mathbb{R}^+\{p_0\} \subset \operatorname{supp} f \subset \mathbb{R}^+(b_1^+(p_0) \cup \{p_0\})$$

in a neighborhood of p_0; here Φ is a quantized contact transformation associated with φ, and we set

$$b_1^+(p_0) = \{(x, \sqrt{-1}\, \eta) \in T_M^* X;\ x_1 > 0, x_2 = \dots = x_n = \eta_1 = \dots = \eta_{n-1} = 0, \eta_n = 1\} .$$

Let Ψ be as in the proof of Theorem 4.1. Then we get

$$P\Psi(\varphi_* f) = \Psi(\varphi_* \Phi(P) f) = 0,$$
$$\operatorname{supp}(\Psi(\varphi_* f)) = \varphi(\operatorname{supp} f) \cap L = \mathbb{R}^+\{p_0\} .$$

This completes the proof.

5. Generalization to Overdetermined Systems

In this section, we extend some of the results in Sect. 2 and Sect. 3 to more general systems whose structure in the complex domain has been studied in Ôaku [16].

Let M be an n-dimensional real analytic manifold and X be its complexification. Let $V=V_1\cup V_2$ be a homogeneous involutory analytic subset of an open set $\Omega\subset T^*X-X$. We set $I_V(j)=I_V\cap\mathcal{O}(j)=\{f\in\mathcal{O}(j); f|_V=0\}$. Then $\mathcal{O}_V(0)=\mathcal{O}(0)/I_V(0)$ is a coherent sheaf of rings on V. We denote by $\mathcal{E}_V$ the subring of $\mathcal{E}_X$ generated by $\{P\in\mathcal{E}(1);\ \sigma_1(P)\in I_V(1)\}$.

Now let $\mathcal{M}$ be a coherent $\mathcal{E}_X$-module (i.e. a system of microdifferential equations) defined on Ω. We assume the following conditions concerning $\mathcal{M}$ and V.

(A.1) V_1 and V_2 are d-codimensional homogeneous regular involutory submanifolds of Ω, and $V_0=V_1\cap V_2$ is non-singular.

(A.2) V_1 and V_2 intersect normally; i.e., $T_pV_1\cap T_pV_2=T_pV_0$ for any $p\in V_0$.

(A.3) $\dim V_1=\dim V_2=\dim V_0+1$.

(A.4) The rank of the pull-back of $d\omega$ to V_0 is $2(n-d)$ at each point of V_0.

(A.5) $\mathcal{M}$ has regular singularities along V; i.e., any coherent sub-$\mathcal{E}_V$-module of $\mathcal{M}$ that is defined on an open subset of Ω is coherent over $\mathcal{E}(0)$.

Writing $V_j^{\mathbb{R}}=V_j\cap T_M^*X$ $(j=0,1,2)$, we fix a point p_0 of $V_0^{\mathbb{R}}\cap\Omega$. We can find a neighborhood $U\subset\Omega$ of p_0 and a coherent sub-$\mathcal{E}_V$-module $\mathcal{M}_0$ of $\mathcal{M}|_U$ such that $\mathcal{E}_X\mathcal{M}_0=\mathcal{M}|_U$. In view of (A.5), we see that $\bar{\mathcal{M}}_0=\mathcal{M}_0/\mathcal{E}(-1)\mathcal{M}_0$ is a coherent $\mathcal{O}_V(0)$-module. Then we assume also

(A.6) $\bar{\mathcal{M}}_0$ is a locally free $\mathcal{O}_V(0)$-module of rank m.

Let p be an arbitrary point in $V_0\cap U$. Then (A.6) ensures that there exist generators $u_1,\dots,u_m$ of $\mathcal{M}_0$ over $\mathcal{E}(0)$ in a neighborhood of p whose residue classes are free generators of $\bar{\mathcal{M}}_0$ over $\mathcal{O}_V(0)$. In view of (A.1)–(A.4), we can find two microdifferential operators P_1 and P_2 in a neighborhood of p such that $\sigma(P_j)=0$ on $V_j(j=1,2)$ and that $\{\sigma(P_1),\ \sigma(P_2)\}\neq 0$ on V_0. Let P_j be of order l_j and set $l=l_1+l_2$. Assumption (A.5) guarantees the existence of $A_{ij}\in\mathcal{E}(l-1)$ $(i,j=1,\dots,m)$ defined in a neighborhood of p such that

$$P_1P_2u_i=\sum_{j=1}^{m}A_{ij}u_j \qquad (i=1,\dots,m).$$

Setting

$$a_{ij}(p)=\sigma_{l-1}(A_{ij})(p)/\{\sigma(P_1),\ \sigma(P_2)\}(p),$$

we define a polynomial e_{12} in λ by

$$e_{12}(\lambda,p,\mathcal{M}_0)=\det(\lambda I_m+(a_{ij}(p))_{1\leqq i,j\leqq m}).$$

It is easy to see that e_{12} is independent of the choice of operators P_1 and P_2, and generators $u_1,\dots,u_m$ of $\mathcal{M}_0$ mentioned above. We denote by $\lambda=\lambda_1,\dots,\lambda_m$ the roots of the equation $e_{12}(\lambda,\ p_0,\ \mathcal{M}_0)=0$ in λ. Then we make the additional assumption

(A.7) $$\lambda_j\notin\{0,1,2,\dots\} \quad \text{for} \quad j=1,\dots,m.$$

In the following theorem we also assume

(R) $$V_1=V_1^c.$$

We denote by $b_1(p_0)$ the real bicharacteristic of $V_1^{\mathbb{R}}$ through p_0.

Theorem 5.1. *Under* (A.1)–(A.7) *and* (R), *there is an open neighborhood* U_0 *of* p_0 *in* T_M^*X *which satisfies the following: if* f *is a microfunction solution of* $\mathcal{M}$ *(i.e. a section of the sheaf* $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$*) defined on an open subset* U' *of* U_0 *such that* $U' - \operatorname{supp} f$ *intersects each connected component of* $(b_1(p_0) - V_2^{\mathbb{R}}) \cap U'$, *then* f *vanishes in a neighborhood of* $b_1(p_0) \cap U'$.

Proof. Let U_0 be an open neighborhood of p_0 in $T_M^*X \cap U$ such that $e_{12}(j, p, \mathcal{M}_0) \neq 0$ for any $j = 0, 1, 2, \ldots$ and any $p \in V_0 \cap U_0$. Let f and U' be as in the theorem. First note that $\mathcal{M}$ is isomorphic to the direct sum of m partial de Rham systems with support V_1 in a neighborhood of each point of $b_1(p_0) - V_2^{\mathbb{R}}$. Thus we have

$$\operatorname{supp} f \cap b_1(p_0) \cap U' \subset V_2^{\mathbb{R}} \cap b_1(p_0) \cap U'.$$

Now let p be an arbitrary point in $b_1(p_0) \cap V_2^{\mathbb{R}} \cap U'$. Using (A.1)–(A.4) and (R), we can find microdifferential operators P_1 and P_2 in a neighborhood of p such that $\sigma(P_j) = 0$ on V_j for $j = 1, 2$ and that

$$\{\sigma(P_1), \sigma(P_2)\}(p) \neq 0, \qquad \sigma(P_1)^c = \sigma(P_1).$$

Set $l = \operatorname{ord} P_1 + \operatorname{ord} P_2$. In view of (A.5), there exist $Q_{ij} \in \mathcal{E}(l-1)$ $(1 \leqq i, j \leqq m)$ such that

$$P_1 P_2 u_i + \sum_{j=1}^{m} Q_{ij} u_j = 0 \qquad (1 \leqq i \leqq m);$$

here $u_1, \ldots, u_m$ are generators of $\mathcal{M}_0$ over $\mathcal{E}(0)$ in a neighborhood of p. Set $f_i = f(u_i)$ for $i = 1, \ldots, m$ in a neighborhood of p. Since f is a homomorphism of $\mathcal{E}_X$-modules, we have

$$P_1 P_2 f_i + \sum_{j=1}^{m} Q_{ij} f_j = 0 \qquad (1 \leqq i \leqq m)$$

in a neighborhood of p.

Denote by $b_1'(p)$ the real bicharacteristic of $\{(x, \sqrt{-1}\eta) \in T_M^*X;\ \sigma(P_1)(x, \sqrt{-1}\eta) = 0\}$ through p. Since $\{\sigma(P_1), \sigma(P_2)\} \neq 0$, we have

$$b_1'(p) \cap \{(z, \zeta) \in T^*X;\ \sigma(P_2)(z, \zeta) = 0\} = \{p\}.$$

So we get $b_1'(p) \cap V_2^{\mathbb{R}} = \{p\}$, and hence f vanishes on $b_1'(p) - \{p\}$. The operator $P = P_1 P_2 I_m + (Q_{ij})$ satisfies (C.1)–(C.3) and (R) of Sect. 2. Thus f_i $(i = 1, \ldots, m)$ vanish in a neighborhood of p by virtue of Theorem 2.1. From this, it follows that f vanishes in a neighborhood of p. Since p is an arbitrary point in $b_1(p_0) \cap V_2^{\mathbb{R}} \cap U'$. we see that f vanishes on $b_1(p_0) \cap V_2^{\mathbb{R}} \cap U'$. This completes the proof.

In the next theorem, we assume, instead of (R),

$(\mathrm{H})_1$ The generalized Levi form of V_1 has at least one negative eigenvalue at p_0.

Theorem 5.2. *Under* (A.1)–(A.7) *and* $(\mathrm{H})_1$,

$$\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)_{p_0} = 0$$

holds; i.e., $\mathcal{M}$ *is micro-locally analytic hypo-elliptic at* p_0.

Remark 5.3. This is a partial generalization of the results of Kashiwara et al. [8] Note that we do not assume that the generalized Levi form of V_1 is non-degenerate and that we make no assumption concerning the generalized Levi form of V_2.

Proof of Theorem 5.2. By virtue of Proposition 2.3 of [16], there is a complex homogeneous canonical coordinate system (w, θ) around p_0 such that $(w(p_0), \theta(p_0)) = (0, 0, \ldots, 0, \sqrt{-1})$ and that

$$V_1 = \{(w, \theta);\ w_1 = \theta_2 = \ldots = \theta_d = 0\},$$
$$V_2 = \{(w, \theta);\ \theta_1 = \theta_2 = \ldots = \theta_d = 0\}.$$

In view of $(\mathrm{H})_1$, there is a holomorphic function f_1 homogeneous of degree 1 defined in a neighborhood of p_0 such that $f_1 = 0$ on V_1 and that $\{f_1, f_1^c\}(p_0) < 0$. In particular, f_1 can be written in the form

$$f_1 = c_1 w_1 + c_2 \theta_2 + \ldots + c_d \theta_d$$

with holomorphic functions $c_1, \ldots, c_d$ such that c_1 is homogeneous of degree 1, and $c_2, \ldots, c_d$ are homogeneous of degree 0.

First, suppose $c_1(p_0) \neq 0$. Setting $f_2 = \theta_1$, let P_1 and P_2 be microdifferential operators defined in a neighborhood of p_0 such that $\sigma_1(P_j) = f_j$ for $j = 1, 2$. In view of (A.5), there are $Q_{ij} \in \mathscr{E}(1)$ $(1 \leqq i, j \leqq m)$ defined in a neighborhood of p_0 such that

$$P_1 P_2 u_i + \sum_{j=1}^{m} Q_{ij} u_j = 0 \qquad (1 \leqq i, j \leqq m);$$

here $u_1, \ldots, u_m$ are generators of $\mathscr{M}_0$ over $\mathscr{E}(0)$ in a neighborhood of p_0. Let f be a microfunction solution of $\mathscr{M}$ defined in a neighborhood of p_0. Then we can regard f as a column vector consisting of microfunctions $f_i = f(u_i)$ $(i = 1, \ldots, m)$ and get

$$P_1 P_2 f_i + \sum_{j=1}^{m} Q_{ij} f_j = 0 \qquad (1 \leqq i, j \leqq m).$$

Since the operator $P_1 P_2 I_m + (Q_{ij})$ satisfies (C.1)–(C.3) and (H) with $k = 1$, we see that $f = 0$ holds in a neighborhood of p_0 by virtue of Theorem 3.1.

Finally suppose that $c_1(p_0) = 0$ and set

$$g = c_2 \theta_2 + \ldots + c_d \theta_d .$$

Then we have $\{g, g^c\}(p_0) < 0$ and $g = 0$ on V by the above assumptions. Set

$$V' = \{(z, \zeta) \in T^*X;\ g(z, \zeta) = 0\}.$$

Then V' is a 1-codimensional homogeneous regular involutory submanifold of T^*X, and its generalized Levi form is negative at p_0. Noting that the support V of $\mathscr{M}$ is contained in V', we get

$$\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}_M)_{p_0} = 0$$

by virtue of Theorem 2.3.10 of Sato et al. [19, Chap. III]. This completes the proof.

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Harmonic 4-Spaces

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A *harmonic space* is a pseudo-Riemannian manifold such that in a neighbourhood of each point x_0 there exists a harmonic function $f(r)$ depending only on the distance $r(x, x_0)$. The curvature tensor of such a space has to satisfy an infinite sequence H_α of conditions, the first three being

$$\mathrm{H}_1 \qquad R^i{}_{abi} = k_1 g_{ab}$$

$$\mathrm{H}_2 \qquad \mathfrak{S}(R^i{}_{abj} R^j{}_{cdi}) = k_2 \mathfrak{S}(g_{ab} g_{cd})$$

$$\mathrm{H}_3 \qquad \mathfrak{S}(9R^i{}_{abj} R^j{}_{cdk} R^k{}_{efi} - 32 R^i{}_{abj;c} R^j{}_{dei;f}) = k_3 \mathfrak{S}(g_{ab} g_{cd} g_{ef})$$

[9]. Repeated indices represent contractions with the metric g, $\mathfrak{S}$ denotes symmetrization over all free indices, and k_α is a constant.

It has been conjectured that a harmonic space with positive definite metric is flat or locally rank one symmetric. In dimensions 2 and 3, this follows from only H_1; it is also valid for a 4-manifold, but the proofs to date [2; 9] use H_1, H_2, and H_3 and rely heavily on the constancy of k_α. The present work provides an alternative approach to the problem in 4 dimensions by using group representations to study the pointwise significance of the H_α. *Consequently, our results make no assumption that the scalar functions k_α be constant.*

Now H_1 is simply the Einstein condition, so restricting to a pseudo-Riemannian Einstein 4-manifold, we consider separately H_2 and H_3. The idea of searching for manifolds satisfying just some of the H_α has its origin in [4]. Having first discussed curvature, we are able to relate H_2 to the notion of self-duality. With the aid of some classical invariant theory, we also show that H_1 and H_3 are sufficient to conclude that M is flat or locally rank one symmetric, except when the metric has signature 0.

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0. Preliminaries

Throughout we suppose that M is a pseudo-Riemannian 4-manifold with connected structure group $SO(4)$, $SO_0(3,1)$, or $SO_0(2,2)$, and use $sgn(M)$ to denote the signature of the metric, equal to 4, 2, 0 respectively. Complexifying, we consider the group $SO(4,\mathbb{C})$ of unimodular transformations of $\mathbb{C}^4$ preserving the form ${z_1}^2+{z_2}^2+{z_3}^2+{z_4}^2$, which is the determinant of the matrix

$$Z=\begin{pmatrix} z_1+iz_2 & -z_3+iz_4 \\ z_3+iz_4 & z_1-iz_2 \end{pmatrix}.$$

If $A, B\in SL(2,\mathbb{C})$, the mapping $Z\mapsto AZB^{-1}$ defines an element of $SO(4,\mathbb{C})$, and so a double covering

$$SL(2,\mathbb{C})\times SL(2,\mathbb{C})\to SO(4,\mathbb{C}) \tag{0.1}$$

which can also be described in terms of representations. Using the symbols $+$, $-$ to distinguish the two factors in (0.1), let V_+, V_- be the complex 2-dimensional $SL(2,\mathbb{C})$-modules determined by matrix multiplication on column vectors. Then the basic 4-dimensional $SO(4,\mathbb{C})$-module is given by

$$T=V_+\otimes_{\mathbb{C}}V_-\,, \tag{0.2}$$

and elements of V_+, V_- are called *spinors.*

The following real forms of (0.1) will be important:

$$\begin{gathered} SL(2,\mathbb{R})\times SL(2,\mathbb{R})\to SO_0(2,2)\\ SU(2)\times SU(2)\to SO(4)\\ SL(2,\mathbb{C})\to SO_0(3,1)\,. \end{gathered} \tag{0.3}$$

The first is obtained by restricting the entries of the matrices A, B, Z to be real, so that $\det Z={x_1}^2-{x_2}^2+{x_3}^2-{x_4}^2$, $x_i\in\mathbb{R}$. Then (0.2) is simply the complexification of a real tensor product; otherwise said $SL(2,\mathbb{R})$ commutes with an antilinear involution $\sigma_\pm$ of $V_\pm$, and the fixed points of $\sigma_+, \sigma_-, \sigma_+\otimes\sigma_-$ are real vector spaces.

Taking $z_i=x_i\in\mathbb{R}$ exhibits the quaternions $\mathbb{H}$ as a real subalgebra of the space of 2×2 complex matrices (for more details see e.g. Zelobenko [12; §11]). Moreover $SU(2)$ corresponds to the group Sp(1) of unit quaternions, and $Z\mapsto AZB^{-1}$, $A, B\in SU(2)$, defines an element of $SO(4)$. In this case the modules V_+, V_- are really quaternionic, and $SU(2)$ commutes with an antilinear transformation $j_\pm$ of $V_\pm$ with $j_\pm^2=-1$. Then $j_+\otimes j_-$ has square $+1$ and endows T with the structure of a real vector space, namely the basic $SO(4)$-module.

The third real form of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ is the subgroup $SL(2,\mathbb{C})$ consisting of elements $(A,(A^*)^{-1})$, where $A^*=\bar{A}^T$ is the adjoint matrix. If Z is anti-Hermitian, $\det Z=-{x_1}^2+{x_2}^2+{x_3}^2+{x_4}^2$, $x_i\in\mathbb{R}$, and $Z\mapsto AZA^*$ is a transformation in $SO_0(3,1)$. Unlike in the real and quaternionic cases, $V_\pm$ is no longer self-conjugate. Indeed $V_+=V$ is the basic module of the real form $SL(2,\mathbb{C})$, and $V_-\cong\bar{V}$ is its conjugate, so that (0.2) becomes $T\cong V\otimes_{\mathbb{C}}\bar{V}$.

The complexified tangent bundle of M is associated to the representation (0.2), so it makes sense to study the spin modules V_+, V_-. Let $S^rV_\pm$ denote the submodule of the r-fold tensor product of $V_\pm$ consisting of totally symmetric tensors, and put

$$S^{p,q}=S^pV_+\otimes_{\mathbb{C}}S^qV_-\,.$$

Then $S^{p,q}$ is an irreducible representation of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ or any of its real forms [12; §37, 42, 43]. Because matrices in $SL(2,\mathbb{C})$ have unit determinant, the module Λ^2V_+ is trivial; accordingly let $\{x,y\}$ be a basis of V_+ with $\varepsilon=x\otimes y-y\otimes x$ canonical, and let $\{w,z\}$ be such a basis of V_-. It is sometimes convenient to identify $S^{p,q}$ with the space of polynomials homogeneous of degree p in the symbols x, y, and of degree q in w, z. In particular, $\dim_{\mathbb{C}}(S^{p,q})=(p+1)(q+1)$.

Contraction with $\varepsilon\in\Lambda^2V_+\cong\Lambda^2V_+^*$ followed by a symmetrization gives a homomorphism of $S^mV_+\otimes S^pV_+$ onto each of the modules $S^{m+p}V_+$, $S^{m+p-2}V_+,\ldots,S^{m-p}V_+$ $(m\geqq p)$. Schur's lemma [12; §20] and a dimension count then imply that $S^mV_+\otimes S^pV_+$ is isomorphic to the direct sum of these modules. Similarly for V_-, so

$$S^{m,n}\otimes S^{p,q}\cong\bigotimes_{\substack{j=0,\ldots,\min(m,p)\\ k=0,\ldots,\min(n,q)}}S^{m+p-2j,\,n+q-2k}\,. \tag{0.4}$$

In particular the $SO(4,\mathbb{C})$-module $\otimes^rT=\otimes^rS^{1,1}$ decomposes into a sum of various $S^{p,q}$ with $p+q$ even. When $\mathrm{sgn}(M)=4$ or 0, these are all (complexifications of) real spaces, but for $\mathrm{sgn}(M)=2$, only combinations of $S^{p,p}$ and $S^{p,q}+S^{q,p}$, $p\neq q$, are real. (Here, as in the sequel, "+" denotes direct sum.) For example

$$T\otimes T\cong S^{1,1}\otimes S^{1,1}\cong S^{2,2}+S^{2,0}+S^{0,2}+S^{0,0}\,,$$

with

$$S^2T\cong S^{2,2}+S^{0,0}\,,\qquad \Lambda^2T\cong S^{2,0}+S^{0,2}\,.$$

Then Λ^2T is irreducible over $\mathbb{R}$ for the Lorentzian group $SO_0(3,1)$.

The splitting of Λ^2T can also be detected using a complex $*$-operator defined by the equation

$$g(\alpha,\beta)=\alpha\wedge *\beta\,,\qquad \alpha,\beta\in\Lambda^2T\,,$$

where g denotes the induced metric on Λ^2T, and the trivial $SO(4,\mathbb{C})$-module Λ^4T has been identified with $\mathbb{C}$. Then $*^2=1$, so we can take $S^{2,0}$, $S^{0,2}$ to be the $+1$, -1 eigenspaces Λ^2_+, Λ^2_- respectively of $*$. Let $\{e_1,e_2,e_3,e_4\}$ be any complex basis of T with $g(e_i,e_j)=\delta_{ij}$, $e_1\wedge e_2\wedge e_3\wedge e_4=1$, and put

$$\begin{aligned}
\phi_1&=e_1\wedge e_2+e_3\wedge e_4 &\quad \psi_1&=e_1\wedge e_2-e_3\wedge e_4\\
\phi_2&=e_1\wedge e_3+e_4\wedge e_2 &\quad \psi_2&=e_1\wedge e_3-e_4\wedge e_2\\
\phi_3&=e_1\wedge e_4+e_2\wedge e_3 &\quad \psi_3&=e_1\wedge e_4-e_2\wedge e_3
\end{aligned}\tag{0.5}$$

Then $\{\phi_i\}$, $\{\psi_i\}$ form orthogonal bases of Λ^2_+, Λ^2_- respectively, and the correspondence $\{e_i\}\mapsto(\{\phi_i\},\{\psi_i\})$ defines a double-covering

$$SO(4,\mathbb{C})\to SO(3,\mathbb{C})\times SO(3,\mathbb{C})\,.$$

Comparing this with (0.3) also gives 2:1 homomorphisms

$$SL(2,\mathbb{R})\to SO_0(2,1)\,,\qquad SU(2)\to SO(3)\,,$$
$$SL(2,\mathbb{C})\to SO(3,\mathbb{C})\,.$$

To summarize, of the three types of metric on a pseudo-Riemannian 4-manifold, a Lorentzian metric seems to be the odd one out, because its corresponding structure group $SL(2,\mathbb{C})$ is simple. However we shall see that as far as harmonic spaces are concerned, the signature 0 case is in many ways special.

1. The Curvature Tensor

Let $\mathfrak{R}$ denote the $SO(4,\mathbb{C})$-module consisting of tensors sharing the same symmetries as the curvature R of a pseudo-Riemannian 4-manifold M. We work with complexified tensor spaces in order to cover all signatures. The decomposition of $\mathfrak{R}$ into irreducible components is described in [10]; the following summarizes this in our notation.

Proposition 1.1. $\mathfrak{R}\cong S^{4,0}+S^{0,4}+S^{2,2}+S^{0,0}$.

Proof. The metric g determines an identification $T\cong T^*$ of tangent and cotangent spaces. The curvature R of M is then at each point a self-adjoint linear transformation of $\Lambda^2 T$, so

$$\begin{aligned}\mathfrak{R}\subset S^2(\Lambda^2 T)&=S^2(\Lambda^2_+ +\Lambda^2_-)\\ &\cong S^2(S^{2,0})+S^2(S^{0,2})+S^{2,2}\\ &\cong S^{4,0}+S^{0,4}+2S^{0,0}+S^{2,2}\,,\end{aligned}$$

where $2S^{0,0}$ means the trivial module $S^{0,0}+S^{0,0}$. The first Bianchi identity forces the component of R in the subspace $2S^{0,0}$ to be a multiple of the invariant

$$A=\frac{1}{24}\sum_{i=1}^{3}(\phi_i\otimes\phi_i+\psi_i\otimes\psi_i)$$

[notation of (0.5)]. The proposition follows. □

In terms of the above decomposition, at each point we may write

$$R=W_+ +W_- +B+tA\,.$$

The sum of $W_+\in S^{4,0}$ and $W_-\in S^{0,4}$ is the Weyl conformal tensor, $B\in S^{2,2}$ represents the trace-free Ricci curvature, and $t\in\mathbb{R}$ is the scalar curvature. When $\operatorname{sgn}(M)=2$, $W_+=\bar{W}_-$, but otherwise all components are real. In particular when $\operatorname{sgn}(M)=4$ or 0, we say that M is *self-dual* if $W_-\equiv 0$ and *anti-self-dual* if $W_+\equiv 0$ [1], the distinction being a matter of orientation. In all cases, M is *Einstein* iff $B\equiv 0$.

Let $\mathfrak{D}$ denote the $SO(4,\mathbb{C})$-module of tensors having the same symmetries as the covariant derivative ∇R of the curvature tensor of M with respect to the Levi-Civita connection. The irreducible components of $\mathfrak{D}$ are less well-known, but are readily determined with the aid of Proposition 1.1. For some related results, see [6].

Proposition 1.2. $\mathfrak{D} \cong S^{5,1} + S^{1,5} + S^{3,3} + S^{3,1} + S^{1,3} + S^{1,1}$.

Proof. The space $\mathfrak{D}$ is the kernel of the skewing map

$$\alpha : \mathfrak{R} \otimes T \hookrightarrow \Lambda^2 T \otimes \Lambda^2 T \otimes T \to \Lambda^2 T \otimes \Lambda^3 T$$

corresponding to the second Bianchi identity. Now

$$\begin{aligned}\mathfrak{R} \otimes T &\cong (S^{4,0} + S^{0,4} + S^{2,2} + S^{0,0}) \otimes S^{1,1} \\ &\cong S^{5,1} + S^{1,5} + 2S^{3,1} + 2S^{1,3} + S^{3,3} + 2S^{1,1},\end{aligned}$$

whereas

$$\Lambda^2 T \otimes \Lambda^3 T \cong (S^{2,0} + S^{0,2}) \otimes S^{1,1} \cong S^{3,1} + S^{1,3} + 2S^{1,1}.$$

In the notation of (0.5), put

$$\left.\begin{aligned}\beta &= (\phi_1 \otimes \phi_1 + \psi_1 \otimes \psi_1) \otimes e_3 \\ \gamma &= (\phi_1 \otimes \psi_1 + \psi_1 \otimes \phi_1) \otimes e_3\end{aligned}\right\} \subset \mathfrak{R} \otimes T.$$

Then

$$\alpha(\beta) = \alpha(\gamma) = (\phi_1 + \psi_1) \otimes (e_1 \wedge e_2 \wedge e_3)$$

has non-zero $S^{3,1}$, $S^{1,3}$, and $S^{1,1}$ components in $\Lambda^2 T \otimes \Lambda^3 T$. But β and γ project to linearly independent elements in $2S^{1,1} \subset \mathfrak{R} \otimes T$, so by Schur's lemma

$$\operatorname{Im} \alpha \cong S^{3,1} + S^{1,3} + S^{1,1}.$$

Finally $\mathfrak{D}$ is computed using $\operatorname{Im} \alpha \cong \mathfrak{R} \otimes T / \mathfrak{D}$. □

Suppose now that M is a 4-dimensional Einstein manifold. The Einstein assumption simplifies particularly the form of the covariant derivative of the curvature. To see this, let $\mathfrak{R}_E \subset \mathfrak{R}$ denote the space of Einstein curvature tensors, and $\mathfrak{D}_E \subset \mathfrak{D}$ the space of covariant derivatives of Einstein curvature tensors.

Proposition 1.3. $\mathfrak{R}_E \cong S^{4,0} + S^{0,4} + S^{0,0}$, $\mathfrak{D}_E \cong S^{5,1} + S^{1,5}$.

Proof. The form of $\mathfrak{R}_E$ is an immediate consequence of the definition of an Einstein manifold. As for

$$\mathfrak{D}_E = (\mathfrak{R}_E \otimes T) \cap \mathfrak{D},$$

we have

$$\begin{aligned}\mathfrak{R}_E \otimes T &\cong (S^{4,0} + S^{0,4} + S^{0,0}) \otimes S^{1,1} \\ &\cong S^{5,1} + S^{3,1} + S^{1,5} + S^{1,3} + S^{1,1}.\end{aligned}$$

Referring to the previous proof, it suffices by Schur's lemma to find an element of $\mathfrak{R}_E \otimes T$ whose image under α has non-zero components in the submodules of $S^{3,1}$, $S^{1,3}$, $S^{1,1}$ of $\Lambda^2 T \otimes \Lambda^3 T$. But β does this job. □

If the curvature R of a pseudo-Riemannian 4-manifold M is regarded as a 2-form with values in the vector bundle $\Lambda^2 T^* \cong \Lambda^2 T$, the second Bianchi identity is equivalent to this 2-form being closed. On the other hand, it is easily verified that R is coclosed iff

$$\nabla R \in S^{5,1} + S^{1,5} + S^{3,3},$$

these being precisely the submodules of $\mathfrak{D}$ that do not appear in $\Lambda^2 T \otimes T$. When this is satisfied, M is said to have *harmonic curvature*. From Proposition 1.3, there is obviously a close relationship between the Einstein and harmonic curvature conditions; any Einstein manifold has harmonic curvature, and in either case the absence of $S^{1,1}$ in ∇R means that the scalar curvature t is constant. In the opposite direction, Bourguignon [3] has shown that a compact oriented Riemannian 4-manifold with harmonic curvature and signature $\tau = b_+^2 - b_-^2$ non-zero is necessarily Einstein.

2. The Condition H_2

We begin by decomposing the symmetric powers of T into irreducible components.

Lemma 2.1. $S^r T \cong \bigoplus_{j=0}^{[\frac{1}{2}r]} S^{r-2j,r-2j}$.

Proof. Let $\xi: S^{r-2} T \otimes S^2 T \to S^r T$, $r \geqq 2$, be the symmetrization map. If $g \in S^2 T^* \cong S^2 T$ denotes the metric, $\omega \mapsto \xi(\omega \otimes g)$ defines an injection $S^{r-2} T \hookrightarrow S^r T$, whose cokernel has dimension

$$\binom{3+r}{3} - \binom{3+r-2}{3} = (r+1)^2 = \dim(S^{r,r}).$$

But

$$S^r T \cong S^r(V_+ \otimes V_-) \supset S^r V_+ \otimes S^r V_- = S^{r,r},$$

and the result follows by induction. □

If $r = 2m$ is even, $S^r T$ contains the trivial submodule $S^{0,0}$ spanned by the invariant $\mathfrak{S}(g^m)$, where $\mathfrak{S}$ here denotes the symmetrization $S^m(S^2 T) \to S^{2m} T$.

From now on, M will always be a pseudo-Riemannian 4-manifold satisfying H_1, i.e. an Einstein manifold. The expression

$$\mathfrak{S}(R^i{}_{abj} R^j{}_{cdi})$$

determines a homomorphism $\Omega: S^2 \mathfrak{R} \to S^4 T$ of $SO(4, \mathbb{C})$-modules, and M satisfies H_2 at a given point iff

$$\Omega(R \otimes R) = k_2 \mathfrak{S}(g^2), \tag{2.1}$$

where k_2 is a constant.

Theorem 2.2. *Let M be a pseudo-Riemannian 4-manifold satisfying H_1 and H_2. Then at each point at least one of W_+, W_- is zero; in particular if* $\operatorname{sgn}(M) = 2$, *$M$ has constant curvature.*

Proof. At a fixed point of M, the curvature tensor has the form $R = W_+ + W_- + tA \in \mathfrak{R}_E$. Proposition 1.1 and lemma 2.1 give

$$S^2\mathfrak{R}_E \cong S^2(S^{4,0} + S^{0,4} + S^{0,0}),$$
$$S^4T \cong S^{4,4} + S^{2,2} + S^{0,0}.$$

Use of (0.4) implies that $S^2\mathfrak{R}_E$ (indeed $\otimes^2\mathfrak{R}_E$) has no submodule isomorphic to $S^{2,2}$, so by Schur's lemma

$$\Omega(R \otimes R) \in S^{4,4} + S^{0,0}. \tag{2.2}$$

Now $S^2\mathfrak{R}_E$ has a unique isomorphic to $S^{4,4}$ which contains the term $W_+ \otimes W_- + W_- \otimes W_+$ from $R \otimes R$. The restriction of Ω to this submodule is either zero or an isomorphism; that it is not zero follows by checking that $\tilde{R} = \phi_1 \otimes \phi_1 + \psi_1 \otimes \psi_1$ [see (0.5)] does not satisfy (2.1). Thus (2.1) implies that $W_+ \otimes W_- + W_- \otimes W_+ = 0$, i.e. $W_+ = 0$, or $W_- = 0$.

When $\operatorname{sgn}(M) = 2$, $W_+ = \bar{W}_-$ and consequently $R = tA$. □

In fact any harmonic n-space M with $\operatorname{sgn}(M) = n-2$ has constant curvature [8]; in 4 dimensions our approach links this result with [2, Lemma 6.73].

It is easiest to find metrics satisfying the hypotheses of Theorem 2.2 when $\operatorname{sgn}(M) = 0$. The crucial point is that then Λ^2_+ and Λ^2_- both contain simple 2-vectors. For example in [9, Chapt. 5] it is shown that the metric

$$g = f(dx^1)^2 + dx^2 dx^3 + dx^1 dx^4,$$

where $f = f(x^1, x^2)$ is any function of x^1 and x^2, actually gives rise to a harmonic space M. With the appropriate orientation, $\omega = dx^1 \wedge dx^2 \in \Lambda^2_+$ is null, and the curvature $R = \frac{1}{2} f_{22}\omega \otimes \omega$ is self-dual with zero Ricci tensor; moreover $k_2 \equiv 0$. In general M is not locally symmetric, but it is recurrent, which means that $\nabla R = R \otimes \sigma$ for some covector σ. All rank one symmetric spaces are harmonic [4] but in the Riemannian case, as we shall see, these furnish the only non-flat 4-dimensional examples. However there do exist the "non-linear gravitons" constructed by Hitchin [7] which are self-dual Ricci-flat positive definite metrics, as is the Calabi-Yau metric on a $K3$ surface [11]. These non-harmonic spaces probably all have k_2 non-constant.

As a corollary of the proof of Theorem 2.2, observe that (2.2) translated into classical notation is the assertion that for any Einstein 4-manifold

$$R^{ij}{}_{la} R^{l}{}_{ijb} = k g_{ab},$$

where k is a scalar function. This is so-called "super-Einstein" condition [2]. In view of the strength of the Einstein condition in 4 dimensions, it is natural to ask whether there exist metrics which satisfy H_2 but not H_1 at each point. In this case, in contrast to above, both W_+ and W_- are forced to be non-zero.

3. The Condition H_3

The expressions

$$\mathfrak{S}(R^i{}_{abj}R^j{}_{cdk}R^k{}_{efi}), \qquad \mathfrak{S}(R^i{}_{abj;c}R^j{}_{dei;f})$$

define respectively homomorphisms $\Phi: S^3\mathfrak{R} \to S^6 T$, $\Psi: S^2\mathfrak{D} \to S^6 T$, where $R^i_{abj;c}$ denotes the components of $\nabla R \in \mathfrak{R} \otimes T$. M satisfies H_3 at a given point iff

$$32\Phi(R\otimes R\otimes R) - 9\Psi(\nabla R\otimes \nabla R) = k_3\mathfrak{S}(g^3), \tag{3.1}$$

where k_3 is a constant. Note that Ψ is also the composition

$$S^2\mathfrak{D} \xrightarrow{\gamma} S^2\mathfrak{R}\otimes S^2 T \xrightarrow{\Omega\otimes 1} S^4 T\otimes S^2 T \xrightarrow{\xi} S^6 T, \tag{3.2}$$

where γ is the map given by symmetrizing the indices c, f.

Theorem 3.1. *Let M be a pseudo-Riemannian 4-manifold satisfying H_1 and H_3, with metric of signature 4 or 2. Then M is flat or locally isometric to a rank one symmetric space.*

Proof. Fix any $x \in M$; if $R = W_+ + W_- + tA$ is the curvature tensor, we shall show that $\nabla R = \nabla W_+ + \nabla W_- = 0$ at x. Then M is locally symmetric, and our theorem follows by applying [4, Theorem 1.1], and similar methods when $\operatorname{sgn}(M) = 2$ in which case M will again have constant curvature.

Propositions (1.1), (1.3), and Lemma (2.1) give

$$S^3\mathfrak{R}_E \cong S^3(S^{4,0} + S^{0,4} + S^{0,0})$$
$$S^2\mathfrak{D}_E \cong S^2(S^{5,1} + S^{1,5})$$
$$S^6 T \cong S^{6,6} + S^{4,4} + S^{2,2} + S^{0,0}.$$

First (0.4) implies that $S^3\mathfrak{R}_E$ (indeed $\otimes^3\mathfrak{R}_E$) has no submodule isomorphic to $S^{6,6}$ or $S^{2,2}$, so by Schur's lemma,

$$\Phi(R\otimes R\otimes R) \in S^{4,4} + S^{0,0},$$

and by (3.1),

$$\Psi(\nabla R\otimes \nabla R) \in S^{4,4} + S^{0,0}. \tag{3.3}$$

Regarding elements of $S^{p,q}$ as homogeneous polynomials as in §0, polynomial multiplication defines a homomorphism $\mathfrak{m}: S^{5,1}\otimes S^{1,5} \to S^{6,6}$. Now $S^2\mathfrak{D}_E$ has a unique submodule isomorphic to $S^{6,6}$ arising from $\mathfrak{m}$, the restriction of Ψ to which is non-zero (use (3.2) and the fact that Ω maps *onto* $S^{4,4} \subset S^4 T$). From (3.3), and Schur's lemma again,

$$\mathfrak{m}(\nabla W_+ \otimes \nabla W_-) = 0,$$

i.e. $\nabla W_+ = 0$ or $\nabla W_- = 0$. Both must vanish when $\operatorname{sgn}(M) = 2$, so from now on we can assume without loss of generality that M is Riemannian and $\nabla R = \nabla W_+ \in S^{5,1}$.

Fix an $SL(2,\mathbb{C})$-basis $\{x, y\}$ of the module $V=V_+$ as in §0, and treat elements of S^kV as polynomials. There is a natural homomorphism $\mathfrak{p}: S^5V \otimes S^5V \to S^2V$ defined by

$$\mathfrak{p}(X^5 \otimes Y^5) = (\alpha\delta - \beta\gamma)^4 XY, \tag{3.4}$$

whenever $X = \alpha x + \beta y$, $Y = \gamma x + \delta y \in V$. Since $\mathfrak{p}$ is symmetric in X, Y,

$$S^2(S^{5,1}) \cong S^2(S^5V) \otimes S^2V_- + \Lambda^2(S^5V)$$

has a unique submodule isomorphic to $S^{2,2}$. The restriction of Ψ to this submodule is an isomorphism, for if

$$\begin{aligned} D = &(\phi_1 \otimes \phi_1 - \phi_2 \otimes \phi_2) \otimes e_3 \\ &+ (\phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1) \otimes e_2 \in S^{5,1} \subset \mathfrak{R} \otimes T, \end{aligned}$$

$\Psi(D \otimes D)$ has a non-zero $S^{2,2}$-component (see (0.5), (3.2)). Setting $j = \otimes^5 j_+$,

$$\nabla R = Fw + (jF)z$$

for some polynomial $F \in S^5V$, and some basis $\{w, z = j_- w\}$ of V_-. Then

$$\mathfrak{p}(F \otimes F)w^2 + 2\mathfrak{p}(F \otimes jF)wz + \mathfrak{p}(jF \otimes jF)z^2 = 0$$

by (3.3). The proof of the theorem is completed by

Lemma 3.2. *Let $V \cong \mathbb{C}^2$ denote the basic $SU(2)$-module. If $F \in S^5V$ satisfies $\mathfrak{p}(F \otimes F) = 0 = \mathfrak{p}(F \otimes jF)$, then $F = 0$.*

Proof. We shall use the so-called "symbolic method" from invariant theory; see [5] for the underlying ideas. Take X, Y as above and write

$$\begin{aligned} X^5 &= a_0x^5 + a_1x^4y + a_2x^3y^2 + a_3x^2y^3 + a_4xy^4 + a_5y^5 \\ Y^5 &= b_0x^5 + b_1x^4y + b_2x^3y^2 + b_3x^2y^3 + b_4xy^4 + b_5y^5 . \end{aligned}$$

Using (3.4), with $\alpha, \beta, \gamma, \delta$ expressed in terms of the a_i, b_i gives

$$\begin{aligned} 50\mathfrak{p}(X^5 \otimes Y^5) = &(10a_0b_4 - 4a_1b_3 + 3a_2b_2 - 4a_3b_1 + 10a_4b_0)x^2 \\ &+ (50a_0b_5 - 6a_1b_4 + a_2b_3 + a_3b_2 - 6a_4b_1 + 50a_5b_0)xy \\ &+ (10a_1b_5 - 4a_2b_4 + 3a_3b_3 - 4a_4b_2 + 10a_5b_1)y^2 . \end{aligned}$$

The importance of this formula is that it must remain valid when X^5, Y^5 are replaced by arbitrary elements of S^5V. Now unless $F = X^5$ for some $X \in V \backslash 0$, which would imply that $\mathfrak{p}(F \otimes jF) \neq 0$, it is possible to choose α, β, γ, δ such that

$$F = c_1X^4Y + c_2X^3Y^2 + c_3X^2Y^3 + c_4XY^4$$

and $\alpha\delta - \beta\gamma = 1$. Because $\mathfrak{p}$ is invariant under $SL(2,\mathbb{C})$, we can take $a_i = b_i = c_i$ (with $c_0 = 0 = c_5$) to get

$$\begin{aligned} 0 = 50\mathfrak{p}(F \otimes F) = &(-8c_1c_3 + 3c_2{}^2)X^2 \\ &+ (-12c_1c_4 + 2c_2c_3)XY + (-8c_2c_4 + 3c_3{}^2)Y^2 . \end{aligned}$$

Solving, $c_2=0=c_3$, *and* $c_1=0$ or $c_4=0$. Thus $F=c_1X^4Y$ or $F=c_4XY^4$, but in either case F must be zero, for otherwise putting $Y=\lambda X+\mu jX$, $\lambda, \mu\in\mathbb{C}$, again gives $\mathfrak{p}(F\otimes jF)\neq 0$. □

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Convex Hulls of Complete Minimal Surfaces

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1. Introduction

An immersed surface in $\mathbb{R}^3$ is said to be minimal if its mean curvature vanishes identically. There is a strong link between simply connected minimal surfaces and analytic functions, one that has been exploited profitably over the years. These surfaces are parametrized in the following way [7, 8]. Given three holomorphic functions ψ_i satisfying $\sum_{i=1}^{3} \psi_i'^2 = 0$, $\sum_{i=1}^{3} |\psi_i'|^2 \neq 0$, the formulas $I_i(z) = a_i + \operatorname{Re}\psi_i(z)$ define a minimal surface and, conversely, every such surface arises in this way. The conditions on the ψ_i's say that the immersion is harmonic and conformal. This representation can be refined by writting

$$I_1(z) = a_1 + \operatorname{Re}\frac{1}{2}\int^z f(1-g^2), \quad I_2(z) = a_2 + \operatorname{Re}\frac{i}{2}\int^z f(1+g^2), \quad I_3(z) = a_3 + \operatorname{Re}\int^z fg,$$

where f is analytic, g is meromorphic and the zeros of f occur precisely at the poles of g, their orders as zeros being exactly twice their orders as poles. The metric is given by $\lambda^2(z)|dz|^2$, $\lambda(z) = \frac{|f|}{2}(1+|g|^2)$. In this representation (named after Weierstrass) the function g is, up to composition with stereographic projection, simply the Gauss map of the surface. From the point of view of global differential geometry the interesting objects are the complete minimal surfaces namely, those for which the geodesics are defined for all times. Equivalently, divergent paths must have infinite length. Classical examples are the catenoid, the helicoid, Enneper's surface, Scherk's surface etc. ....

An outstanding problem in this subject, formulated by Calabi is to decide about the existence of complete minimal surfaces that are bounded subsets of $\mathbb{R}^3$. Although this may seem strange at first we point out that there are examples of complete *embedded* surfaces of non-positive curvature that are bounded subsets of $\mathbb{R}^3$. One takes a copy of the unit dis with infinitely many discs removed from it, the

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holes accumulating at the boundary, and connects these holes by tubes to the corresponding holes of another copy of the perforated disc, situated on a parallel plane. Adjusting the several parameters involved (density and shape of the holes, etc. ...) one gets the right object (we are indebted to David Hoffman for showing us this construction). Even though there is a strong feeling that complete minimal surfaces in $\mathbb{R}^3$ are unbounded, no proof has been produced so far. It turns out, however, that one can construct examples that are bounded in one direction. This was done in [3] using Runge's theorem. Due to the non-constructive character of this result it is not apparent how the curvature behaves. The result in this paper implies that the curvature of these examples must be unbounded. We remark, incidentally, that very often in Riemannian geometry one has to impose that the curvature be bounded from below in order to avoid pathologies. To show that complete minimal surfaces of bounded curvature are unbounded is a much easier task. In fact, several techniques are available to deal with the more general situation of complete submanifolds whose scalar and mean curvatures are bounded. In this respect we refer the reader to [4] and [5].

In this paper we study the placement of complete minimal surfaces in $\mathbb{R}^3$ by looking at their convex hulls. The following is a sharp version of Theorem 3 in [4].

Theorem. *The convex hull of a complete non-flat minimal surface of bounded Gaussian curvature is* $\mathbb{R}^3$.

The example in [3] shows, of course, that this theorem is no longer true if we drop the assumption on the curvature.

Having stated our geometric results we now turn to the description of their proofs. Our theorem is a consequence of the Marcinkiewicz-Zygmund-Spencer theorem (M-Z-S, for short); see [11, p. 207]. This is a classical result in the theory of boundary behaviour of analytic functions in the unit disc, also known as the area theorem. We recall that the local form of Fatou's theorem [11, p. 199] asserts that the non-tangential limits of an analytic function on the unit disc exist a.e. if and only if the function is a.e. non-tangentially bounded. The M-Z-S theorem provides another criterion for the existence of these limits. They exist a.e. if and only if for almost every point it is possible to choose a triangular neighborhood with vertex at the point whose image by the function has finite area. In actuality, Spencer [9] proves that finiteness of $\int_{T_\theta} \frac{|h'|^2}{(1+|h|^2)^2}$ for a.e. θ and h *meromorphic* implies existence a.e. of the non-tangential limits. This integral represents, up to a factor, the area in the Riemann sphere of the image of the triangular neighborhood T_θ. An important point of Spencer's argument is to uniformize the situation by making a certain geometric construction that produces a simply connected domain of rectifiable boundary. Composition with a conformal map between this domain and the unit disc gives a function whose characteristic is bounded. The result then follows at once by an appeal to a classical result of Nevanlinna.

Closely related to the problem treated here is a question posed by Chern [1] and solved by Jones [2] using a gap series and Fefferman's theorem on the duality between H^1 and BMO (see also [10, p. 690]). As a consequence, it is possible to find complete minimal surfaces that are bounded in $\mathbb{R}^4$.

Finally, we would like to record that work on this paper was begun at the Institute for Advanced Study during the academic year 1981–1982.

2. Proof of the Theorem

We shall need the following lemma [4, p. 79]

Lemma. *Let M, $\bar{M}$ be Riemannian manifolds, M complete and $I: M \to \bar{M}$ an isometric immersion with bounded second fundamental form. Then, for each $p \in \bar{M}$ there exists a closed ball $B(p)$ such that all connected components of $I^{-1}(B(p))$ are compact.*

Suppose, by way of contradiction, that the theorem is false. This means that there exists a surface $I: M \to \mathbb{R}^3$ of the type described, with $I(M)$ contained in a half-space. There is no loss of generality in supposing that the half-space is the region $z > 0$. By passing to the universal covering it may be assumed that M is simply connected and, as explained in the introduction, that the immersion is conformal and harmonic. By the uniformization theorem M is either $\mathbb{C}$ or D. Since I_3 is a positive non-constant harmonic function the second alternative must prevail. The curvature σ can be computed in terms of the functions f and g in Weierstrass representation by the formula [8, p. 76]

$$\sigma = -\left[\frac{4|g'|}{|f|(1+|g|^2)^2}\right]^2$$

By hypothesis we have for some $C > 0$

$$\frac{4|g'|^2|g|^2}{(1+|g|^2)^4} \leqq C|f|^2|g|^2$$

Setting $h = g^2$ and T_θ a triangular region with vertex at $e^{i\theta}$ as in the M-Z-S theorem (see the introduction) we have

$$\int_{T_\theta} \frac{|h'|^2}{(1+|h|^2)^2} \leqq 4C \int_{T_\theta} |f|^2|g|^2$$

By the M-Z-S theorem, h will have non-tangential limits a.e. provided the integral on the right is finite a.e. Again, this follows from M-Z-S since $I_3 = a_3 + \operatorname{Re}\int fg$ is positive. Next, we shall prove that the non-tangential limits exist for I_1 and I_2. The argument for I_1 runs as follows. It is required to show $\int_{T_\theta} |f|^2|1-g^2|^2 < \infty$ a.e. Finiteness of this integral is equivalent to that of $\int_{T_\theta} |f|^2|g|^2$ at the points θ where g has a non-tangential limit other than $0, 1, \infty$ and -1. It follows from what has been said about h and the Privalov uniqueness theorem [6, p. 84/85], [11, p. 203] that the last set has full measure (g constant means that the surface is a plane). The function I_2 can be treated similarly. From this point on the proof follows that of theorem 3 in [4]. Let then θ be a point for which $\lim_{r\to 1} I_j(re^{i\theta}) = a_j$ exists $(j = 1, 2, 3)$ and let $p = (a_1, a_2, a_3)$. It is not possible to find a ball $B_\varepsilon(p)$ satisfying the conclusions of the lemma. Indeed, no matter how small ε is, the set $I^{-1}(B_\varepsilon(p))$ contains a segment $\{re^{i\theta}, b \leqq r < 1\}$. In particular, the connected component containing it is non-compact. This concludes the proof.

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On the Defining Equations of Points in General Position in $\mathbb{P}^n$

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Let $V \subset \mathbb{P}_K^n$ (with $n \geqq 3$, K an algebraically closed field) be a reduced irreducible nondegenerate variety of dimension $d > 0$ and of degree s, and let L be a generic linear subspace of dimension $m = n - d$ of $\mathbb{P}^n$. Then, it is well known (see, e.g. [7, (2.13)]) that the section $V \cap L$ consists of s points in general position in L, i.e. such that no $m+1$ of them lie on a hyperplane of L. Moreover, if V is arithmetically Cohen-Macaulay (i.e. such that its homogeneous coordinate ring is Cohen-Macaulay) and the graded ideal $I(V \cap L)$ of $V \cap L$ is generated by its homogeneous components of degree $\leqq t$, then the ideal $I(V)$ is also generated by its homogeneous components of degree $\leqq t$ (see, e.g. [4, Lemma 3.6(c)] or [1, Lemma 5.1]). So, in order to get bounds for the degrees of the defining equations of V, one is led to study the ideal of a finite set of points in general position in $\mathbb{P}^n$. Now, following this approach and in particular by using an old argument essentially due to Petri [15] and developed by Saint-Donat [16, 17], recently Treger has shown [20, Theorem 1.1] that "*the ideal of any s points in general position in* $\mathbb{P}^n$, *with* $s \leqq tn$ ($t \geqq 2$), *is always generated by forms of degree* $\leqq t$".

The aim of this paper is to extend and improve the above-mentioned result. This is done by our Theorem 1.2 (whose proof isolates the essence of some classical arguments used in [20]), which moreover enables us to give some new results about certain classes of arithmetically Cohen-Macaulay varieties.

All varieties are reduced irreducible algebraic K-schemes, where K is an algebraically closed field. We are forced to assume char $K = 0$, since no proof of the validity in characteristic $p > 0$ of the General Position Lemma (see, e.g. [7, (2.13)]) seems to be available. Also, we shall write "V is a *CM* variety" in place of "V is an arithmetically Cohen-Macaulay variety". Finally, if $P_1, \ldots, P_s$ are any s distinct points in $\mathbb{P}_K^n$ with homogeneous coordinate ring, say, $A = \oplus A_i$ ($i \geqq 0$), we shall denote by $\{b_i\}_{i \geqq 0}$, with $b_i = \dim_K A_i$, the Hilbert function of A or equivalently, the postulation of $P_1, \ldots, P_s$.

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1. The Main Result

We first recall a well-known fact (see, e.g. [5, Proposition 1.1(3)]).

Lemma 1.1. *Let $A = \oplus A_i$ $(i \geqq 0)$ be a 1-dimensional graded Cohen-Macaulay ring, where $A_0 = K$ is any infinite field and A is finitely generated as a K-algebra by A_1. So, we can write, say, $A = K[X_1, \ldots, X_{n+1}]/I$, where I is an unmixed homogeneous ideal of height n. Then, if t denotes the least integer for which $\dim_K A_t = \dim_K A_{t-1}$, the ideal I is generated by its homogeneous components of degree $\leqq t$.*

Theorem 1.2. *Let Z be a finite set of s points in general position of $\mathbb{P}^n_K$ $(s > n \geqq 2)$ and let $\mathcal{I}_Z$ denote the ideal sheaf of Z in $\mathbb{P}^n_K$. Let t denote the least integer > 1 such that:*
 (i) *$H^1(\mathbb{P}^n, \mathcal{I}_Z(t)) = 0$, and*
 (ii) *$H^1(\mathbb{P}^n, \mathcal{I}_U(t-1)) = 0$ for every subset U of Z consisting of $s-(n-1)$ points.*
Then the ideal $I(Z)$ of Z in the polynomial ring $K[X_1, \ldots, X_{n+1}]$ is generated by forms of degree $\leqq t$.

Before giving the proof of our main result, we shall deduce from it Treger's statement mentioned above.

Corollary 1.3 [20, Theorem 1.1]. *Let Z be a finite set of s points in general position of $\mathbb{P}^n_K$ $(s > n \geqq 2)$, with $s \leqq tn$, $t \geqq 2$. Then the ideal $I(Z)$ of Z is generated by forms of degree $\leqq t$.*

Proof. Start from the exact sheaf sequence

$$0 \to \mathcal{I}_Z(t) \to \mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{O}_Z(t) \to 0,$$

and consider the exact cohomology sequence:

$$0 \to H^0(\mathcal{I}_Z(t)) \to H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_Z(t)) \to H^1(\mathcal{I}_Z(t)) \to 0.$$

Since $\dim_K H^0(\mathcal{O}_Z(t)) = s$, we get that $H^1(\mathcal{I}_Z(t)) = 0$ if, and only if, the points of Z impose exactly s independent conditions to the linear system $|\mathcal{O}_{\mathbb{P}^n}(t)|$. Hence, condition 1.2 (i) is satisfied (see, e.g. [7, (2.13)] or [12, Lemma 2.1]). Similarly, since in our case $s-(n-1) \leqq (t-1)n+1$, we get that condition 1.2 (ii) is also satisfied. So the corollary easily follows from Theorem 1.2.

We now turn to our main result.

Proof (of Theorem 1.2). First of all, from condition (i) and Lemma 1.1 above, it follows that the ideal $I(Z)$ is generated by forms of degree $\leqq t+1$. So, in order to prove the theorem, it is enough to show that every form of degree $t+1$ in $I(Z)$ lies in the ideal generated by all forms of degree t vanishing at Z.

Write $Z = \{P_1, \ldots, P_{n+1}, Q_1, \ldots, Q_r\}$, with $r \geqq 1$ (note that the ideal of any set of $n+1$ points in general position of $\mathbb{P}^n$ is always generated by forms of degree 2, in view of Lemma 1.1) and choose homogeneous coordinates $(X_1, \ldots, X_{n+1})$ in $\mathbb{P}^n$ so that $P_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ for $i = 1, \ldots, n+1$. Then, viewing the elements of $H^0(\mathcal{O}_Z(t))$ as maps from Z to K, we choose r elements Y_α $(1 \leqq \alpha \leqq r)$ of $H^0(\mathcal{O}_Z(t))$ such that

$$Y_\alpha(Q_\alpha) = 1,\ Y_\alpha(Q_\beta) = 0 \quad \text{for} \quad \beta \neq \alpha,\ Y_\alpha(P_i) = 0 \quad \text{for} \quad 1 \leqq i \leqq n+1.$$

So we may consider $X_1^t, \ldots, X_{n+1}^t$, $Y_1, \ldots, Y_r$ as a base of the K-vector space $H^0(\mathcal{O}_Z(t))$.

Now, put $X_i(Q_\alpha) = c_{i\alpha}$ for $i = 1, \ldots, n+1$ and $\alpha = 1, \ldots, r$ (note that $c_{i\alpha} \neq 0$ for all i, α since the points of Z are in general position). Then, for each α $(1 \leqq \alpha \leqq r)$ and for any two integers k, j with $1 \leqq k, j \leqq n+1$ and $k \neq j$, we define the following linear forms

$$L(k, j, \alpha) = c_{k\alpha} X_j - c_{j\alpha} X_k .$$

Next we choose forms $R_\alpha \in H^0(\mathcal{O}_{\mathbb{P}^n}(t))$, $1 \leqq \alpha \leqq r$ such that $\phi(R_\alpha) = Y_\alpha$, where ϕ denotes the natural map $H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(\mathcal{O}_Z(t))$ [note that ϕ is surjective, in view of condition (i)].

Then, for any t-tuple $(i_1, \ldots, i_t)$ with $1 \leqq i_1, \ldots, i_t \leqq n+1$ and $(i_1, \ldots, i_t) \neq (q, q, \ldots, q)$, we have, say: $X_{i_1} \cdot \ldots \cdot X_{i_t} = \sum_\alpha c_{i_1\alpha} \cdot \ldots \cdot c_{i_t\alpha} Y_\alpha$. Hence we consider the following forms of degree t:

$$f(i_1, \ldots, i_t) = X_{i_1} \cdot \ldots \cdot X_{i_t} - \sum_\alpha c_{i_1\alpha} \cdot \ldots \cdot c_{i_t\alpha} R_\alpha .$$

It is easy to check that these forms vanish on Z. Moreover, they generate $H^0(\mathcal{I}_Z(t))$. In fact, any form $F \in H^0(\mathcal{I}_Z(t))$ can be written:

$$F = \sum_{i=1}^{n+1} \beta_i X_i^t + \sum_{i_1, \ldots, i_t} \gamma(i_1, \ldots, i_t) f(i_1, \ldots, i_t) + \sum_{\alpha=1}^{r} \delta_\alpha R_\alpha .$$

Then, since $F = 0$ on Z, we get $\beta_i = 0$ $(1 \leqq i \leqq n+1)$ and $\delta_\alpha = 0$ $(1 \leqq \alpha \leqq r)$.

Also, for each $\alpha = 1, \ldots, r$ and for any two integers k, j with $1 \leqq k, j \leqq n+1$ and $k \neq j$, we put: $g(k, j, \alpha) = L(k, j, \alpha) R_\alpha$.

Claim 1. The sets of forms $\{f(i_1, \ldots, i_t)\}$ and $\{g(k, j, \alpha)\}$ defined above generate the ideal of all forms of degree $\geqq t$ vanishing on Z.

Proof (of Claim 1). Clearly, it is enough to show that if we take any form $G \in H^0(\mathcal{I}_Z(t+1))$, then G belongs to the ideal generated by the f's and g's. Now, since $G(P_i) = 0$ for $1 \leqq i \leqq n+1$, we can write G as follows:

$G = \Sigma_\gamma G_\gamma L_\gamma$, where $L_\gamma \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, $G_\gamma \in H^0(\mathcal{O}_{\mathbb{P}^n}(t))$ and $G_\gamma(P_i) = 0$ for $1 \leqq i \leqq n+1$. Also, since $\phi(R_\alpha) = Y_\alpha$, we can choose suitable elements $\tau_{\alpha\gamma}$ in K such that the forms $G_\gamma - \sum_\alpha \tau_{\alpha\gamma} R_\alpha$ lie in $H^0(\mathcal{I}_Z(t))$. Hence we get:

$$G = \sum T(i_1, \ldots, i_t) \cdot f(i_1, \ldots, i_t) + \sum_\alpha S_\alpha R_\alpha ,$$

where $T(i_1, \ldots, i_t), S_\alpha \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ and $S_\alpha(Q_\alpha) = 0$ for $1 \leqq \alpha \leqq r$. On the other hand, it is easy to check that, for each α, the set $\{L(k, j, \alpha)\}_{k,j}$ generates the vector space of all linear forms vanishing at Q_α (since the $c_{i\alpha}$'s are all $\neq 0$), which proves our claim.

Claim 2. With the above notation, every form $g(k, j, \alpha)$ belongs to the ideal generated by the set of forms $\{f(i_1, \ldots, i_t)\}$.

Proof (of Claim 2). Fix any two integers i_k, i_j with $1 \leqq i_k, i_j \leqq n+1$ and $i_k \neq i_j$. Then, for each $(t+1)$-tuple $(i_1, \ldots, i_k, \ldots, i_j, \ldots, i_{t+1})$ with $1 \leqq i_1, \ldots, i_{t+1} \leqq n+1$ and $(i_k, \ldots, i_k) \neq (i_1, \ldots, \hat{i}_k, \ldots, \hat{i}_j, \ldots, i_{t+1}) \neq (i_j, \ldots, i_j)$, we have the following relation:

$$\begin{aligned} X_{i_j} \cdot f(i_1, \ldots, \hat{i}_j, \ldots, i_{t+1}) - X_{i_k} \cdot f(i_1, \ldots, \hat{i}_k, \ldots, i_{t+1}) \\ = X_{i_1} \cdot \ldots \cdot X_{i_{t+1}} - \Sigma_\alpha c_{i_1\alpha} \cdot \ldots \cdot \hat{c}_{i_j\alpha} \cdot \ldots \cdot c_{i_{t+1}\alpha} X_{i_j} R_\alpha \\ - X_{i_1} \cdot \ldots \cdot X_{i_{t+1}} + \Sigma_\alpha c_{i_1\alpha} \cdot \ldots \cdot \hat{c}_{i_k\alpha} \cdot \ldots \cdot c_{i_{t+1}\alpha} X_{i_k} R_\alpha \\ = -\Sigma_\alpha c_{i_1\alpha} \cdot \ldots \cdot \hat{c}_{i_k\alpha} \cdot \ldots \cdot \hat{c}_{i_j\alpha} \cdot \ldots \cdot c_{i_{t+1}\alpha} \cdot g(i_k, i_j, \alpha). \end{aligned}$$

Let us consider the $\left[\binom{n+t-1}{n} - 2\right] \times r$ matrix whose entries are the coefficients of the g's in the above relations [corresponding to all possible $(t+1)$-tuples], say:

$$\mathbf{M}(i_k, i_j) = (c_{i_1\alpha} \cdot \ldots \cdot \hat{c}_{i_k\alpha} \cdot \ldots \cdot \hat{c}_{i_j\alpha} \cdot \ldots \cdot c_{i_{t+1}\alpha}).$$

$\left[\text{Note that, in view of condition (ii), we have } \binom{n+t-1}{n} - 2 \geqq r.\right]$ Now the claim will follow if we can show that the matrix $\mathbf{M}(i_k, i_j)$ contains a non-trivial $r \times r$ minor. We shall prove the existence of such a minor by induction on $r \geqq 1$, the case $r = 1$ being obvious. So we may assume, by induction, that the first $r-1$ rows of our matrix are linearly independent. Then, suppose that all $r \times r$ minors are trivial and look for a contradiction. At this point we observe that the dependence relations among any r rows including the first $r-1$ rows can be interpreted as (vanishing relations of) forms of degree $t-1$ in $n+1$ variables of the following type, say:

$$M_i = a_{i,1} M_1 + a_{i,2} M_2 + \ldots + a_{i,r-1} M_{r-1} \left(r \leqq i \leqq \binom{n+t-1}{n} - 2 \right),$$

where $M_1, M_2, \ldots, M_{r-1}, M_i$ are distinct monomials of degree $t-1$ in $n+1$ variables, for any i and moreover the M_i's are all distinct. Hence such forms are linearly independent over K and furthermore they vanish on the set $U = U(i_k, i_j) = \{P_{i_k}, P_{i_j}, Q_1, \ldots, Q_r\}$. Therefore we get: $\dim_K H^0(\mathscr{I}_U(t-1)) \geqq \binom{n+t-1}{n} - 2 - (r-1) = \binom{n+t-1}{n} - (s-n)$. On the other hand, from hypothesis (ii) we have: $\dim_K H^0(\mathscr{I}_U(t-1)) = \binom{n+t-1}{n} - (s-n) - 1$. So we get a contradiction, which shows our claim and also concludes the proof of Theorem 1.2.

Remark 1.4. It is easy to check that conditions 1.2(i) and 1.2(ii) are mutually independent. Moreover, they are by no means necessary conditions too, as the following example shows. Let Z denote the generic plane section of the irreducible *CM* curve $C \subset \mathbb{P}^3_K$ given parametrically by the following equations: $X_1 = s^{13}$, $X_2 = s^4 t^9$, $X_3 = s t^{12}$, $X_4 = t^{13}$. Then, applying the method used in the proof of [3, Theorem 3], we get that the ideal $I(C)$ is generated by the following quartic forms: $X_2^4 - X_1 X_3^3$, $X_3^4 - X_2 X_4^3$, $X_2^3 X_3 - X_1 X_4^3$. Hence one easily gets: $H^1(\mathscr{I}_Z(4)) \neq 0$ and $H^1(\mathscr{I}_U(3)) \neq 0$ for any subset U of Z consisting of 12 points.

2. Some Consequences

Our first application of Theorem 1.2 concerns some special sets of points in $\mathbb{P}^n_K$ ($n \geqq 2$).

Definition 2.1 [2, 5, 6]. A set Z of s points in $\mathbb{P}^n$ is said to be in *uniform position* if it has the following property:

(*) if $\{b_i\}_{i \geqq 0}$ denotes the postulation of any subset of Z consisting, say, of r points ($r \leqq s$), we have:

$$b_i = \min\left\{r, \binom{i+n}{n}\right\} \quad \text{for all} \quad i \geqq 0 .$$

It is worth observing that the above notion turns out to be a special case of the more general notion coming out from Harris' Uniform Position Lemma [10, p. 197]. Also, points in uniform position have been studied in [5] in relation to the following problem: "Does there exist a non-empty Zariski open set in $(\mathbb{P}^n)^s$ consisting of s-tuples of points in $\mathbb{P}^n$ for which the minimal number of generators of the corresponding homogeneous ideal is a constant, explicitly computable in terms of n and s?" An affirmative answer to this problem has been given in [5] for $n=2$, while for $n \geqq 3$ the problem is still open, in general. The following corollary, among others, provides a solution to the above problem in some special cases (see also [12]).

Corollary 2.2. *Let Z be a set of s points in uniform position in $\mathbb{P}^n$ ($n \geqq 2$), with* $\binom{d-1+n}{n} \leqq s \leqq \binom{d-1+n}{n} + (n-1)$, $d \geqq 2$.

Then the ideal $I(Z)$ is generated by forms of degree d.

Proof. The corollary is an immediate consequence of Theorem 1.2. The only thing one has to observe is that for any $n, d \geqq 2$ we have: $\binom{d-1+n}{n} - \binom{d-2+n}{n} \geqq n$.

The next two corollaries contain some improvements of Treger's result [20, Theorem 1.1], under some additional assumptions.

Corollary 2.3. *Let $V \subset \mathbb{P}^n_K$ ($n \geqq 3$) be a nondegenerate CM variety of codimension $h \geqq 2$ and of degree $s = th+1$, with $t \geqq 3$. Suppose that the generic section $V \cap L$ of V with a linear subspace L of $\mathbb{P}^n$ of dimension h does not lie on a rational normal curve of L.*

Then the ideal $I(V)$ of V is generated by forms of degree $\leqq t$.

Proof. Since V is CM, it is enough to show that the ideal $I(V \cap L)$ of $V \cap L$ is generated by forms of degree $\leqq t$. To this end, we apply Theorem 1.2 to the s points of $V \cap L$ (which are certainly in general position, in view of the General Position Lemma). Indeed, condition 1.2(i) is clearly satisfied (see, e.g. [7, (2.13)] or [12, Lemma 2.1]), while condition 1.2(ii) easily follows from Harris' Uniform Position Lemma (cf., e.g. [11, Lemma (3.4)]) together with Bertini's Theorem (cf., e.g. [18, p. 147, Lemma 11]), and from Castelnuovo's lemma (cf., e.g. [11, Lemma (3.9)]).

Remark 2.4. Corollary 2.3 above actually improves Treger's result [20, Theorem 1.1], as is shown by taking, for instance, any nondegenerate CM curve of degree 7 in $\mathbb{P}^3$, lying on an irreducible cubic surface.

Corollary 2.5. *Let $V \subset \mathbb{P}^n_K$ ($n \geqq 3$) be a nondegenerate CM variety of codimension 2 and degree s, and let $[(s-1)/2]$ denote the integral part of the rational number $(s-1)/2$. Then:*

(1) If V does not lie on a quadric hypersurface and $s > 8$, the ideal $I(V)$ is generated by forms of degree $\leqq [(s-1)/2]$.

(2) If V does not lie on a cubic hypersurface and $s > 16$, the ideal $I(V)$ is generated by forms of degree $\leqq [(s-1)/2]-2$.

Proof. (1) Clearly, we are led to consider the ideal of a suitable set of s points in general position of $\mathbb{P}^2$, not lying on a conic. Let $\{b_i\}_{i \geqq 0}$ denote the postulation of such points. We distinguish two cases:

(1a) $s = 2t+1$. In this case, we get (see, e.g. [12, Lemma 2.1 and Theorem 2.3 (1)]): $b_t = s$, $b_{t-1} \geqq 2(t-1)+2 = s-1$. Now, if $b_{t-1} = s$, we are done, in view of Lemma 1.1. Otherwise, if $b_{t-1} = s-1$, the conclusion follows immediately from Theorem 1.2, in view of the Uniform Position Lemma (cf. [11, Lemma (3.4)]).

(1b) $s = 2t$ ($t \geqq 5$). In this case, we get (see, e.g. [12, Lemma 2.1 and Theorem 2.3 (1)]): $b_{t-1} = s$ and moreover $b_{t-2} \geqq 2(t-2)+3 = s-1$, since our s points ($s \geqq 10$) are indeed a plane section of V and consequently they lie on at most one plane cubic. So, by arguing as in case (1a) above, we are done. This completes the proof of statement (1).

(2) We have to consider now the ideal of a suitable set of s points in general position of $\mathbb{P}^2$, not lying on a cubic. Let $\{b_i\}_{i \geqq 0}$ denote the postulation of such points. We distinguish two cases:

(2a) $s = 2t+1$. In this case, since $b_4 \geqq 14$ (note that $s > 16$), we get (see, e.g. [12, Lemma 2.1 and Theorem 2.3 (1)]): $b_{t-2} = s$ and $b_{t-3} \geqq 2(t-3)+6 = s-1$. So, arguing as in case (1a) above, we are done.

(2b) $s = 2t$ ($t > 8$). In this case, we have again: $b_4 \geqq 14$.
Also (see [12, Lemma 2.1 and Theorem 2.3 (1)]) we get: $b_{t-3} = s$ and $b_{t-4} \geqq 2(t-4) + 6 = s-2$. Now, we may assume $b_4 = 14$ (if not, we have necessarily $b_{t-4} \geqq s-1$, which concludes the proof, as before) and also $b_{t-4} = s-2$. Hence, from [11, Corollary (3.5)] we have: $t \leqq 9$. So, it remains to consider only the case of 18 points with postulation:

$$1 \quad 3 \quad 6 \quad 10 \quad 14 \quad 16 \quad 18 \quad 18 \quad \ldots$$

But this case is easily ruled out, either using the so-called Cayley-Bacharach theorem (see, e.g. [8]) which implies that $b_5 = 17$ for any such set of 18 points or also by liaison (see, e.g. [14, Lemme 3.5]), since our 18 points are linked to two distinct points. This completes the proof of Corollary 2.5.

Remark 2.6. Perhaps, it is worth observing that if C is any nondegenerate curve in $\mathbb{P}^3$, then, in view of some work done by Castelnuovo (see [9]), the ideal $I(C)$ of C is generated by forms of degree $\leqq \deg C - 1$. Also, E. Davis pointed out to us a purely algebraic proof of Corollary 2.5.

We shall end the paper by stating a problem which is motivated, among others, by some recent work by Ooishi [13].

Problem. Let $V \subset \mathbb{P}_K^n$ $(n \geqq 3)$ be a nondegenerate arithmetically Buchsbaum variety and let $i = i(A)$ denote the invariant of the local Buchsbaum ring A of the affine cone over V at the vertex. Is then Theorem 1.2 still valid, by replacing the upper bound t by $t+i$?

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On the Second L-Functions Attached to Hilbert Modular Forms

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Introduction

We present new proofs of properties of the symmetric-square zeta functions attached to Hilbert modular forms by using Fourier expansions of the Poincaré series and sums of Kloosterman sums. This is an affirmative answer to Zagier's question [23, pp. 141–142] [see the remark after (0.2) below].

We sketch our result in the elliptic modular case. Let

$$f(z)=\sum_{n=1}^{\infty} a(n)\mathbf{e}(nz)$$

be a normalized Hecke eigen cusp form of weight k with respect to $\mathrm{SL}(2,\mathbf{Z})$. Here k is a positive even integer, $\mathbf{Z}$ the ring of rational integers, $\mathbf{e}(x)=\exp(2\pi ix)$, and z a variable on the upper half plane $\mathfrak{H}$. For a complex variable s with $\mathrm{Re}(s)>k$, we put

$$L_2(s,f)=\prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1},$$

where p runs over all prime numbers, and complex numbers α_p, β_p are taken such that $\alpha_p+\beta_p=a(p)$, $\alpha_p\beta_p=p^{k-1}$. By Shimura [16] and Zagier [23], $L_2(s,f)$ is holomorphically continued to the whole s-plane. For elliptic modular forms g and h of weight k such that gh is a cusp form, we put

$$(g,h)=\int_{\mathrm{SL}(2,\mathbf{Z})\backslash\mathfrak{H}} g(z)\overline{h(z)}y^k\frac{dx\,dy}{y^2}\quad (z=x+iy)$$

(the Petersson inner product). By Sturm [21] and Zagier [23], the values of $L_2(m,f)/\pi^{2m-k+1}(f,f)$ for even integers m with $k\leqq m\leqq 2k-2$ belong to $\mathbf{Q}(f)$, the totally real algebraic number field generated by the eigenvalues of all Hecke operators on f over the rational number field $\mathbf{Q}$, i.e. $\mathbf{Q}(f)=\mathbf{Q}(a(n)|n\geqq 1)$.

To prove these results, Zagier [23] calculated explicitly the Fourier coefficients of the unique cusp form $\tilde{\Phi}_s$ with the property

$$(\tilde{\Phi}_s,f)=C_k\frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}}L_2(s+k-1,f)$$

for all normalized eigen cusp forms f of weight k. Here $C_k = \frac{(-1)^{k/2}\pi}{2^{k-3}(k-1)}$ and $\mathrm{Re}(s) > 1$. As a by-product, the explicit form of the Fourier expansion of $\tilde{\Phi}_s$ yields a formula for the traces of Hecke operators (Zagier [23]).

Now let

$$G_r(z) = \frac{1}{2} \sum_{\substack{(c,d)\in \mathbf{Z}^2 \\ (c,d)=1}} (cz+d)^{-k} \mathbf{e}\left(r \cdot \frac{a_0 z + b_0}{cz+d}\right)$$

be the Poincaré series with $0 < r \in \mathbf{Z}$ and $z \in \mathfrak{H}$, where a_0, b_0 are any integers satisfying $a_0 d - b_0 c = 1$. By

$$(f, G_r) = \frac{(k-2)!}{(4\pi r)^{k-1}} a(r)$$

and

$$L_2(s, f) = \zeta(2s-2k+2) \sum_{n=1}^{\infty} a(n^2) n^{-s},$$

we have

$$\tilde{\Phi}_s = C_k \frac{\Gamma(s+k-1)}{(4\pi)^s \Gamma(k-1)} \zeta(2s) \sum_{n=1}^{\infty} n^{k-1-s} G_{n^2}. \tag{0.1}$$

The right-hand side of (0.1) converges uniformly and absolutely on any compact subset of $\{(s,z) | \mathrm{Re}(s) > 1, z \in \mathfrak{H}\}$. Here the Fourier expansion of G_r is given as follows:

$$G_r(z) = \sum_{m=1}^{\infty} \left\{ \delta_{r,m} + 2\pi(-1)^{k/2} \left(\frac{m}{r}\right)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} \frac{1}{c} K_c(r,m) J_{k-1}\left(\frac{4\pi}{c}\sqrt{rm}\right) \right\} \mathbf{e}(mz), \tag{0.2}$$

where $\delta_{r,m}$ is the Kronecker delta, J_{k-1} the Bessel function of order $k-1$, and

$$K_c(r,m) = \sum_{\substack{x \bmod c \\ (x,c)=1}} \mathbf{e}\left(\frac{rx + mx^{-1}}{c}\right)$$

[x^{-1} denotes an integer such that $xx^{-1} \equiv 1 \pmod{c}$], a Kloosterman sum.

Zagier [23, pp. 141–142] asked whether it is possible to obtain the above results on $L_2(s,f)$ (holomorphy, special values) by calculating the Fourier coefficients of $\tilde{\Phi}_s$ directly from (0.1) and (0.2). In this paper we show that this can actually be done; so we give new proofs, not depending on the "Rankin-Selberg method", of these properties of $L_2(s,f)$. Main ingredients of our computations are:

(a) Investigation of properties of the integral:

$$\int_0^{\infty} \cos(2\pi l x) x^{-s} J_{k-1}(4\pi\sqrt{mx})\,dx.$$

(b) The following equality containing sums of Kloosterman sums:

$$\sum_{c=1}^{\infty} c^{-1-s} \sum_{r \bmod c} K_c(r^2, m) \mathbf{e}\left(\frac{lr}{c}\right) = \zeta(2s)^{-1} L(s, l^2 - 4m)$$

with $l, m \in \mathbf{Z}$. Here [23],

$$L(s, l^2-4m)=\begin{cases}\zeta(2s-1) & \text{if } \ l^2=4m\,,\\ L\left(s,\left(\frac{D}{*}\right)\right)\sum\limits_{\substack{d|\mathfrak{f}\\ d>0}}\mu(d)\left(\frac{D}{d}\right)d^{-s}\sigma_{1-2s}(\mathfrak{f}d^{-1}) & \text{if } \ l^2\neq 4m\,;\end{cases}$$

in the latter case, we write $l^2-4m=D\mathfrak{f}^2$ with a positive integer $\mathfrak{f}$ and a discriminant D of $\mathbf{Q}(\sqrt{D})$; $\left(\frac{D}{*}\right)$ is the Kronecker symbol, $L\left(s,\left(\frac{D}{*}\right)\right)$ the associated Dirichlet L-function, μ the Möbius function, and $\sigma_\nu(m)=\sum\limits_{\substack{d|m\\ d>0}} d^\nu$ $(0<m\in\mathbf{Z},\ \nu\in\mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers).

Our method also applies to Hilbert modular forms for a totally real number field of class number one in the narrow sense, so we treat Hilbert modular cases hereafter. (A generalization of the method of Zagier [23] to Hilbert modular cases is treated in Takase [22].)

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1. Hilbert Modular Forms: Statement of Results

Let F be a totally real number field of degree g over $\mathbf{Q}$ with the class number one in the narrow sense. Let $F^{(1)}, \ldots, F^{(g)}$ be the conjugates of F over $\mathbf{Q}$ with $F^{(1)}=F$. The image of an element $a\in F$ under $F\to F^{(j)}$ is denoted by $a^{(j)}$. For $a, b\in F$, we write $a\gg b$ [or $a\geqq b$] if $a^{(j)}>b^{(j)}$ [or $a^{(j)}\geqq b^{(j)}$] for $j=1,\ldots,g$. Let $\mathcal{O}$ be the ring of integers in F, $\mathfrak{d}$ the different of $F/\mathbf{Q}$, and $d(F)$ the discriminant of $F/\mathbf{Q}$. By the assumption we have $\mathfrak{d}=(\delta)$ with $\delta\gg 0$. We denote by $\mathcal{O}^\times$ the group of units in $\mathcal{O}$, and put $\mathcal{O}_+^\times=\{\lambda\in\mathcal{O}\,|\,\lambda\gg 0\}$. Note that

$$\mathcal{O}_+^\times=(\mathcal{O}^\times)^2=\{\lambda^2\,|\,\lambda\in\mathcal{O}^\times\} \tag{1.1}$$

since the class number of F is one in the narrow sense. For a positive integer k, let $M_k(\mathrm{SL}(2,\mathcal{O}))$ [or $S_k(\mathrm{SL}(2,\mathcal{O}))$] be the $\mathbf{C}$-vector space of the Hilbert modular [or cusp] forms of weight k with respect to $\mathrm{SL}(2,\mathcal{O})$.

Let

$$f(z)=\sum_{0\ll\nu\in\mathfrak{d}^{-1}} a((\nu)\mathfrak{d})\mathbf{e}(\mathrm{tr}(\nu z)) \tag{1.2}$$

be a normalized Hecke eigenform in $S_k(\mathrm{SL}(2,\mathcal{O}))$, where $z=(z_1,\ldots,z_g)$ is a variable on $\mathfrak{H}^g$, the product of g-copies of $\mathfrak{H}$, and $\mathrm{tr}(\nu z)=\sum\limits_{j=1}^{g}\nu^{(j)}z_j$; so $a(\mathcal{O})=1$ and $T(\mathfrak{m})f=a(\mathfrak{m})f$ for all non-zero integral ideals $\mathfrak{m}$ of $\mathcal{O}$. By Gundlach [6],

$$a(\mathfrak{m})=O\left(N(\mathfrak{m})^{\frac{k}{2}-\frac{1}{4}+\varepsilon}\right) \tag{1.3}$$

for any $\varepsilon>0$ as $N(\mathfrak{m})\to\infty$. For $g, h\in M_k(\mathrm{SL}(2,\mathcal{O}))$ such that gh is a cusp form, we put

$$(g,h)=\int_{\mathrm{SL}(2,\mathcal{O})\backslash\mathfrak{H}^g} g(z)\overline{h(z)}\,\mathrm{Im}(z)^k d\mu(z)\,,$$

$$\text{where } \mathrm{Im}(z)=\prod_{j=1}^{g} y_j \text{ and } d\mu(z)=\prod_{j=1}^{g}\frac{dx_j dy_j}{y_j^2} \quad \text{if} \quad z=(z_1,\ldots,z_g)$$

and $z_j=x_j+iy_j$.

For $\mathrm{Re}(s) > k + \frac{1}{2}$ we put

$$L_2(s, f) = \prod_{\mathfrak{p}} (1 - \alpha_{\mathfrak{p}}^2 N(\mathfrak{p})^{-s})^{-1} (1 - \alpha_{\mathfrak{p}} \beta_{\mathfrak{p}} N(\mathfrak{p})^{-s})^{-1} (1 - \beta_{\mathfrak{p}}^2 N(\mathfrak{p})^{-s})^{-1},$$

where the product is over all non-zero prime ideals in $\mathcal{O}$, and $\alpha_{\mathfrak{p}}, \beta_{\mathfrak{p}} \in \mathbf{C}$ are taken so that $\alpha_{\mathfrak{p}} + \beta_{\mathfrak{p}} = a(\mathfrak{p})$ and $\alpha_{\mathfrak{p}} \beta_{\mathfrak{p}} = N(\mathfrak{p})^{k-1}$.

Now we state our results. Our method of proofs is entirely different from the known ones, and gives an affirmative answer to the question raised by Zagier [23, pp. 141–142].

Theorem 1. *The notation being as above, suppose $f \in S_k(\mathrm{SL}(2, \mathcal{O}))$ is a normalized Hecke eigenform of even weight $k \geqq 4$. Put*

$$\Lambda_2(s, f) = d(F)^{3s/2} \left(2^{-s} \pi^{-3s/2} \Gamma(s) \Gamma\left(\frac{s-k+2}{2} \right) \right)^g L_2(s, f).$$

Then, $\Lambda_2(s, f)$ has a holomorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda_2(s, f) = \Lambda_2(2k-1-s, f).$$

Theorem 2. *Let f be as in Theorem 1. Then, for each even integer m such that $k \leqq m \leqq 2k-2$, we have:*

$$[L_2(m, f)/\pi^{g(2m-k+1)} (f, f)]^{\sigma} = L_2(m, f^{\sigma})/\pi^{g(2m-k+1)} (f^{\sigma}, f^{\sigma})$$

for all $\sigma \in \mathrm{Aut}(\mathbf{C})$. In particular, $L_2(m, f)/\pi^{g(2m-k+1)}(f, f)$ belongs to $\mathbf{Q}(f)$.

Here $\mathrm{Aut}(\mathbf{C})$ denotes the group of all ring automorphisms of $\mathbf{C}$. Each $\sigma \in \mathrm{Aut}(\mathbf{C})$ acts on f with a Fourier expansion (1.2) by

$$f^{\sigma}(z) = \sum_{0 \ll \nu \in \mathfrak{d}^{-1}} a((\nu)\mathfrak{d})^{\sigma} \mathbf{e}(\mathrm{tr}(\nu z)).$$

We denote by $\mathbf{Q}(f)$ the totally real algebraic number field generated over $\mathbf{Q}$ by the eigenvalues of all Hecke operators on f.

Theorem 3. *The notation being as above, let k be an even integer $\geqq 4$. For each $0 \ll \nu \in \mathcal{O}$, the trace of $T((\nu))$ on $S_k(\mathrm{SL}(2, \mathcal{O}))$ is given as follows.*

(1) *If $F \neq \mathbf{Q}$, then:*

$$\mathrm{tr}(T((\nu))) = (-1)^g \sum_{\substack{l^2 \ll 4\nu \\ l \in \mathcal{O}}} N_{F/\mathbf{Q}}(p_{k,1}(l, \nu)) \frac{h(F(\sqrt{l^2 - 4\nu}))}{w(F(\sqrt{l^2 - 4\nu}))} \sum_{\mathfrak{c} | \mathfrak{f}} N(\mathfrak{c}) \prod_{\mathfrak{p} | \mathfrak{c}} (1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-1})$$

$$+ \begin{cases} 2N_{F/\mathbf{Q}}(\nu)^{\frac{k-2}{2}} d(F)^{\frac{3}{2}} \left(\dfrac{k-1}{(2\pi)^2} \right)^g \zeta_F(2) \\ \quad \textit{if } (\nu) \textit{ is a square of an integral ideal of } F, \\ 0 \quad \textit{otherwise}. \end{cases} \tag{1.4}$$

Here $N_{F/\mathbf{Q}}: F \to \mathbf{Q}$ is the norm map, $p_{k,1}(l, \nu)$ the coefficient of x^{k-2} in $(1 - lx + \nu x^2)^{-1}$; $h(F(\sqrt{l^2 - 4\nu}))$ the class number (in the wide sense) of the number field $F(\sqrt{l^2 - 4\nu})$; $w(F(\sqrt{l^2 - 4\nu}))$ the number of roots of unity in $F(\sqrt{l^2 - 4\nu})$; ζ_F the

Dedekind zeta function of F; if we write $(l^2-4\nu)=d(F(\sqrt{l^2-4\nu})/F)\mathfrak{f}^2$ *with an integral ideal* $\mathfrak{f}$ *where* $d(F(\sqrt{l^2-4\nu})/F)$ *is the relative discriminant of* $F(\sqrt{l^2-4\nu})/F$ *(existence of such* $\mathfrak{f}$ *is proved in § 3 below),* $\mathfrak{c}$ *runs over all integral ideals of F dividing* $\mathfrak{f}$, *and* $\mathfrak{p}$ *runs over all prime ideals of F dividing* $\mathfrak{c}$; χ *is the Hecke character associated with the quadratic extension* $F(\sqrt{l^2-4\nu})/F$.

(2) *If* $F=\mathbf{Q}$, *then:*

$$\begin{aligned}\operatorname{tr}(T(\nu))=&-\sum_{\substack{l^2<4\nu\\ l\in\mathbf{Z}}} p_{k,1}(l,\nu)\frac{h(\mathbf{Q}(\sqrt{l^2-4\nu}))}{w(\mathbf{Q}(\sqrt{l^2-4\nu}))}\sum_{\substack{c\mid\mathfrak{f}\\ 0<c\in\mathbf{Z}}} c\prod_{\substack{p\mid c\\ p:\text{prime}}}(1-\chi(p)p^{-1})\\ &-\frac{1}{2}\sum_{\substack{dd'=\nu\\ 0<d,d'\in\mathbf{Z}}}\min(d,d')^{k-1}+\begin{cases}\nu^{\frac{k-2}{2}}\dfrac{k-1}{12} & \text{if } \nu \text{ is a square},\\ 0 & \text{otherwise}.\end{cases}\end{aligned}\tag{1.5}$$

Theorem 3(1) is a special case of the formula of Selberg-Shimizu [15], and (2) is the formula of Eichler-Selberg-Zagier (cf. [23]).

The rest of this paper is devoted to the proofs of Theorems 1–3. We note that our method does not use the "Rankin-Selberg method", on which most of the known proofs (of Theorems 1 and 2) are based.

Remark 1. (i) A representation theoretic proof (using the Rankin-Selberg method) of Theorem 1 is done in Gelbart and Jacquet [5].

(ii) Theorem 2 for $m=2k-2$ was obtained by the author [13] as an application of the explicit formula for Fourier coefficients of the Eisenstein series $[f]$ of degree two attached to f in the sense of Langlands and Klingen for Hilbert-Siegel modular groups.

2. Sums of Poincaré Series

For an even integer $k\geqq 4$, the Poincaré series are defined by

$$G_\mu(z)=\frac{1}{2}\sum_{\substack{(c,d)\in\mathcal{O}^2\\ (c,d)=1}} N(cz+d)^{-k}\mathbf{e}\left(\operatorname{tr}\left(\frac{\mu}{\delta}\cdot\frac{a_0z+b_0}{cz+d}\right)\right)$$

with $0\ll\mu\in\mathcal{O}$, $z=(z_1,\dots,z_g)\in\mathfrak{H}^g$; here $a_0,b_0\in\mathcal{O}$ are chosen such that $a_0d-b_0c=1$, the summation is over all last rows of the matrices in $\mathrm{SL}(2,\mathcal{O})$,

$$N(cz+d)=\prod_{j=1}^{g}(c^{(j)}z_j+d^{(j)}),$$

and

$$\frac{a_0z+b_0}{cz+d}=\left(\frac{a_0^{(1)}z_1+b_0^{(1)}}{c^{(1)}z_1+d^{(1)}},\dots,\frac{a_0^{(g)}z_g+b_0^{(g)}}{c^{(g)}z_g+d^{(g)}}\right)\in\mathfrak{H}^g.$$

For $c,d\in\mathcal{O}$, we mean by $(c,d)=1$ that $\mathcal{O}c+\mathcal{O}d=\mathcal{O}$. We note that our G_μ is the half of $G_{-k}(z,1,1,\mathrm{SL}(2,\mathcal{O}),\mu/\delta)$ in Maaß [11] and Gundlach [6], and belongs to $S_k(\mathrm{SL}(2,\mathcal{O}))$. As in Maaß [11, § 5],

$$(f,G_\mu)=N_{F/\mathbf{Q}}(\mu)^{1-k}d(F)^{k-(1/2)}((4\pi)^{1-k}\Gamma(k-1))^g a((\mu)).\tag{2.1}$$

[Note that this value is different from that of Maaß [11, (152)], since E_k in [11, (16)] is not contained in $\mathrm{SL}(2,\mathcal{O})$.] The Fourier expansion of G_μ is obtained by Gundlach [7, Satz 3] (with some errors; the correct form in our case is as follows):

$$G_\mu(z)=\sum_{0\ll\nu\in\mathcal{O}}\left[\delta_{(\mu),(\nu)}+(-1)^{gk/2}2^{g-1}\pi^g d(F)^{-1/2}\sum_{0\neq\alpha\in\mathcal{O}}N(\alpha)^{-1}\right.$$
$$\left.\cdot N\left(\frac{\nu}{\mu}\right)^{\frac{k-1}{2}}K_\alpha(\nu,\mu)\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\sqrt{\mu^{(j)}\nu^{(j)}}}{\delta^{(j)}\alpha^{(j)}}\right)\right]\mathbf{e}\left(\mathrm{tr}\left(\frac{\nu}{\delta}z\right)\right), \tag{2.2}$$

where $\delta_{(\mu),(\nu)}=1$ or 0 according as $(\mu)=(\nu)$ or $(\mu)\neq(\nu)$, $N=N_{F/\mathbf{Q}}$, and

$$K_\alpha(\nu,\mu)=\sum_{\substack{x \bmod (\alpha)\\ (x,\alpha)=1\\ x\in\mathcal{O}}}\mathbf{e}\left(\mathrm{tr}\left(\frac{\nu x+\mu x^{-1}}{\delta\alpha}\right)\right).$$

[x^{-1} is an element of $\mathcal{O}$ such that $xx^{-1}\equiv 1(\mathrm{mod}(\alpha))$.]

Concerning the infinite sum with respect to $0\neq\alpha\in\mathcal{O}$ in the right-hand side of (2.2), we have

Lemma 1. *For any $\varepsilon>0$, there exists a positive constant C depending only on F, k, and ε such that*

$$\sum_{0\neq\alpha\in\mathcal{O}}\left|N(\alpha)^{-1}N\left(\frac{\nu}{\mu}\right)^{\frac{k-1}{2}}K_\alpha(\nu,\mu)\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\sqrt{\mu^{(j)}\nu^{(j)}}}{\delta^{(j)}\alpha^{(j)}}\right)\right|$$
$$<CN(\nu)^{\frac{k}{2}-\frac{1}{4}+\varepsilon}N(\mu)^{\frac{3}{4}-\frac{k}{2}+\varepsilon}. \tag{2.3}$$

Proof. The left-hand side of (2.3) is invariant if we replace μ by $\mu\lambda^2$ with $\lambda\in\mathcal{O}^\times$ [cf. (1.1)], since one checks easily that

$$K_{\alpha\lambda^{-1}}(\nu,\mu)=K_\alpha(\nu,\mu\lambda^2). \tag{2.4}$$

So we may assume $C_1^{-1}N(\mu)^{1/g}\leqq\mu^{(j)}\leqq C_1N(\mu)^{1/g}$ with some constant $C_1>0$ for $j=1,\dots,g$ as in Gundlach [6, (36)]. Next, in the proof of Satz 2 in Gundlach [6, § 2], replace Σ_1,Σ_2 there by the summations over $N(\mathfrak{c}_1)\leqq N(\mu\nu)^{1/2}$, $N(\mathfrak{c}_1)>N(\mu\nu)^{1/2}$, respectively. Then the rest of the proof is similar to the proof of Satz 2 in Gundlach [6, § 2].

We put

$$\Psi_s(z)=\sum_{\substack{(\beta)\\ 0\ll\beta\in\mathcal{O}}}N(\beta)^{k-1-s}G_{\beta^2}(z), \tag{2.5}$$

where the summation is over all non-zero integral ideals (β) with $0\ll\beta\in\mathcal{O}$. By (2.3), the right-hand side of (2.5) converges absolutely and uniformly on any compact subset of $\{(s,z)|\mathrm{Re}(s)>\frac{3}{2}, z\in\mathfrak{H}^g\}$, so $\Psi_s(z)$ belongs to $S_k(\mathrm{SL}(2,\mathcal{O}))$ for each s with $\mathrm{Re}(s)>\frac{3}{2}$. [This fact may also be proved as follows: Let $f_1,\dots,f_m$ be a basis for $S_k(\mathrm{SL}(2,\mathcal{O}))$. By (2.1) we have

$$G_\mu=d(F)^{k-(1/2)}((4\pi)^{1-k}\Gamma(k-1))^g\sum_{j=1}^{m}\frac{\overline{a_j((\mu))}}{N(\mu)^{k-1}}\frac{1}{(f_j,f_j)}f_j,$$

where $a_j((\mu))$ is the Fourier coefficient of f_j at (μ). Combining this with the estimation (1.3) which is valid for all cusp forms in $S_k(\mathrm{SL}(2,\mathcal{O}))$, we have the above assertion on the convergence of (2.5).] For $\mathrm{Re}(s)>\frac{3}{2}$, let

$$\Psi_s(z)=\sum_{0\ll v\in\mathcal{O}} b(v,s)\mathbf{e}\left(\mathrm{tr}\left(\frac{v}{\delta}z\right)\right) \tag{2.6}$$

be the Fourier expansion of $\Psi_s(z)$. By (2.2) and (2.5) we have

$$b(v,s)=\sum_{\substack{(\beta)\\0\ll\beta\in\mathcal{O}}} N(\beta)^{k-1-s}\Bigg[\delta_{(\beta^2),(v)}+(-1)^{gk/2}2^{g-1}\pi^g d(F)^{-1/2}$$
$$\cdot\sum_{0\neq\alpha\in\mathcal{O}}\left\{N(\alpha)^{-1}N\left(\frac{v}{\beta^2}\right)^{\frac{k-1}{2}}K_\alpha(v,\beta^2)\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\beta^{(j)}\sqrt{v^{(j)}}}{\delta^{(j)}\alpha^{(j)}}\right)\right\}\Bigg].$$

By (2.3), we interchange the order of summation to obtain

$$b(v,s)=(-1)^{gk/2}2^{g-1}\pi^g d(F)^{-1/2}N(v)^{\frac{k-1}{2}}$$
$$\cdot\sum_{0\neq\alpha\in\mathcal{O}}\sum_{\substack{(\beta)\\0\ll\beta\in\mathcal{O}}}\left\{N(\beta)^{-s}N(\alpha)^{-1}K_\alpha(v,\beta^2)\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\beta^{(j)}\sqrt{v^{(j)}}}{\delta^{(j)}\alpha^{(j)}}\right)\right\}$$
$$+\begin{cases}N(v)^{\frac{k-1-s}{2}} & \text{if } (v) \text{ is a square of an integral ideal of } F,\\ 0 & \text{otherwise}.\end{cases} \tag{2.7}$$

Observe that that the statement "(v) is a square of an integral ideal of F" is equivalent to "$v=a^2$ for some $a\in\mathcal{O}$", by (1.1). So, hereafter we say that (v) (or v) is a square if this condition is satisfied.

We put $\alpha=\alpha_1\lambda$, where (α_1) runs over all non-zero principal ideals in $\mathcal{O}$ with $0\ll\alpha_1\in\mathcal{O}$ and λ over all the elements of $\mathcal{O}^\times$. Then by (2.4) we obtain

$$\sum_{0\neq\alpha\in\mathcal{O}}\sum_{\substack{(\beta)\\0\ll\beta\in\mathcal{O}}} N(\beta)^{-s}N(\alpha)^{-1}K_\alpha(v,\beta^2)\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\beta^{(j)}\sqrt{v^{(j)}}}{\delta^{(j)}\alpha^{(j)}}\right)$$
$$=\sum_{\substack{(\alpha)\\0\ll\alpha\in\mathcal{O}}}\sum_{0\neq\beta\in\mathcal{O}} N((\beta))^{-s}N(\alpha)^{-1}K_\alpha(v,\beta^2)\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi|\beta^{(j)}|\sqrt{v^{(j)}}}{\delta^{(j)}\alpha^{(j)}}\right). \tag{2.8}$$

We write the right-hand side of (2.8) as S. For $x\in\mathcal{O}$ we put

$$A(x)=|N(x)|^{-s}\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\sqrt{v^{(j)}}|x^{(j)}|}{\delta^{(j)}\alpha^{(j)}}\right). \tag{2.9}$$

Now suppose that

$$\tfrac{3}{2}<\mathrm{Re}(s)<k-1 \quad\text{or}\quad s=k-1. \tag{2.10}$$

Then

$$A(0)=\begin{cases}0 & \text{if } \mathrm{Re}(s)<k-1,\\ \left(\dfrac{(2\pi)^{k-1}}{\Gamma(k)}\right)^g N\left(\dfrac{\sqrt{v}}{\delta\alpha}\right)^{k-1} & \text{if } s=k-1,\end{cases}$$

and

$$S=\sum_{\substack{(\alpha)\\0\ll\alpha\in\mathcal{O}}}\sum_{\beta\in\mathcal{O}}N(\alpha)^{-1}K_\alpha(v,\beta^2)A(\beta)-\sum_{\substack{(\alpha)\\0\ll\alpha\in\mathcal{O}}}N(\alpha)^{-1}K_\alpha(v,0)A(0)$$

$$=\sum_{\substack{(\alpha)\\0\ll\alpha\in\mathcal{O}}}\sum_{\gamma\bmod(\alpha)}N(\alpha)^{-1}K_\alpha(v,\gamma^2)\sum_{\substack{\beta\equiv\gamma(\bmod(\alpha))\\\beta\in\mathcal{O}}}A(\beta)$$

$$-\delta_{s,k-1}\left(\frac{(2\pi)^{k-1}}{\Gamma(k)}\right)^g d(F)^{1-k}N(v)^{\frac{k-1}{2}}\sum_{\substack{(\alpha)\\0\ll\alpha\in\mathcal{O}}}N(\alpha)^{-k}K_\alpha(v,0) \qquad (2.11)$$

for s in the region (2.10).

Lemma 2. *For each $0\ll v\in\mathcal{O}$ we have:*

$$\sum_{\substack{(\alpha)\\0\ll\alpha\in\mathcal{O}}}N(\alpha)^{-k}K_\alpha(v,0)=\frac{\sigma_{1-k}((v))}{\zeta_F(k)}, \qquad (2.12)$$

where $\sigma_s((v))=\sum_{\mathfrak{a}|(v)}N(\mathfrak{a})^s$ with $\mathfrak{a}$ running over all integral ideals of F dividing (v).

Proof. Note that

$$K_\alpha(v,0)=\sum_{\substack{x\bmod(\alpha)\\(x,\alpha)=1}}\mathbf{e}\left(\operatorname{tr}\left(\frac{vx}{\delta\alpha}\right)\right) \qquad (2.13)$$

is multiplicative with respect to α in the following sense (cf. Proof of Proposition 2 in § 3 below):

$$\sum_{\substack{(\alpha)\\0\ll\alpha\in\mathcal{O}}}N(\alpha)^{-k}K_\alpha(v,0)=\prod_{\mathfrak{p}}\left(\sum_{m=0}^{\infty}N(\mathfrak{p})^{-mk}K_{\varrho^m}(v,0)\right). \qquad (2.14)$$

Here $\mathfrak{p}=(\varrho)$ runs over all non-zero prime ideals of $\mathcal{O}$. For each $\mathfrak{p}=(\varrho)$, let b be the $\mathfrak{p}$-order of v, i.e. $\mathfrak{p}^b|v$ and $\mathfrak{p}^{b+1}\nmid v$. Then by (2.13) we have

$$K_{\varrho^m}(v,0)=\begin{cases}\varphi(\mathfrak{p}^m) & \text{for } m\leqq b,\\ -N(\mathfrak{p})^b & \text{for } m=b+1,\\ 0 & \text{for } m\geqq b+2,\end{cases}$$

where φ is the Euler's function for F. Substituting these values in the right-hand side of (2.14), we obtain (2.12).

Next we compute $\sum_{\substack{\beta\equiv\gamma(\bmod(\alpha))\\\beta\in\mathcal{O}}}A(\beta)$ with γ,α fixed. By (2.9),

$$\sum_{\substack{\beta\equiv\gamma(\bmod(\alpha))\\\beta\in\mathcal{O}}}A(\beta)=|N(\alpha)|^{-s}\sum_{\mu\in\mathcal{O}}\left|N\left(\mu+\frac{\gamma}{\alpha}\right)\right|^{-s}\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\sqrt{v^{(j)}}}{\delta^{(j)}}\left|\mu^{(j)}+\frac{\gamma^{(j)}}{\alpha^{(j)}}\right|\right). \qquad (2.15)$$

This summation converges absolutely and uniformly for $\frac{3}{2}<\mathrm{Re}(s)<k-1$, and converges absolutely also for $s=k-1$. This follows from the fact that (2.15) is a partial summation $[+A(0)$ if $\gamma\equiv 0(\bmod(\alpha))]$ of (2.8). Putting

$$B(x)=\sum_{\mu\in\mathcal{O}}|N(\mu+x)|^{-s}\prod_{j=1}^{g}J_{k-1}\left(\frac{4\pi\sqrt{v^{(j)}}}{\delta^{(j)}}\left|x_j+\mu^{(j)}\right|\right) \qquad (2.16)$$

with a variable $x=(x_1, \ldots, x_g)$ on $\mathbf{R}^g$ (the product of g-copies of the field of real numbers $\mathbf{R}$), the right-hand side of (2.15) is expressed as

$$|N(\alpha)|^{\quad s}B(\gamma^{(1)}/\alpha^{(1)}, \ldots, \gamma^{(g)}/\alpha^{(g)}).$$

For each s in the region (2.10), the right-hand side of (2.16) converges absolutely and uniformly on $\mathcal{S}$, a compact subset of $\mathbf{R}^g$ whose elements form a system of representatives for $\mathbf{R}^g/\mathcal{O}$ [as an additive group; we consider $\mathcal{O} \hookrightarrow \mathbf{R}^g$ via $\alpha \mapsto (\alpha^{(1)}, \ldots, \alpha^{(g)})$]. For the proof, see Lemma 3 below. Let

$$B(x)=\sum_{l \in \mathcal{O}} c_l \mathbf{e}\left(\operatorname{tr}\left(\frac{l}{\delta}x\right)\right)$$

be the Fourier expansion of $B(x)$. The Poisson summation formula yields

$$c_l=\frac{1}{\sqrt{d(F)}} \prod_{j=1}^{g} I_j(l, v, s) \tag{2.17}$$

with

$$\begin{aligned} I_j(l, v, s) &= \int_{-\infty}^{+\infty} \mathbf{e}\left(-\frac{l^{(j)}}{\delta^{(j)}}x_j\right)|x_j|^{-s} J_{k-1}\left(4\pi\frac{\sqrt{v^{(j)}}}{\delta^{(j)}}|x_j|\right)dx_j \\ &=2\int_0^{\infty} \cos\left(\frac{2\pi l^{(j)}}{\delta^{(j)}}t\right)t^{-s}J_{k-1}\left(4\pi\frac{\sqrt{v^{(j)}}}{\delta^{(j)}}t\right)dt. \end{aligned} \tag{2.18}$$

These procedures are jsutified by the following

Lemma 3. (1) *Let $\mathcal{S}$ be as above. Then the right-hand side of* (2.16) *converges absolutely and uniformly on* $\{(x, s) | x \in \mathcal{S}, \frac{1}{2} < \operatorname{Re}(s) < k-1\}$. *Moreover, for* $s=k-1$, *it converges absolutely and uniformly on* $\mathcal{S}$.

(2) *The summation* $\sum_{l \in \mathcal{O}} \prod_{j=1}^{g} I_j(l, v, s)$ *converges absolutely and uniformly for* $\frac{1}{2} < \operatorname{Re}(s) < k-1$, *and converges absolutely also for* $s=k-1$.

Proof. (1) Let $\omega_1, \ldots, \omega_g$ be an integral basis of F, i.e., $\mathcal{O}=\bigoplus_{j=1}^{g} \mathbf{Z}\omega_j$. Writing

$$x_j=\sum_{i=1}^{g} y_{ij}\omega_i^{(j)}$$

for each $x=(x_1, \ldots, x_g) \in \mathcal{S}$, we may suppose $|y_{ij}| < C_1$ for all i,j with a positive constant C_1. We put

$$h(t_1, \ldots, t_g; x)=\prod_{j=1}^{g}\left[\left|\sum_{i=1}^{g}(t_i+y_{ij})\omega_i^{(j)}\right|^{-s} J_{k-1}\left(\frac{4\pi\sqrt{v^{(j)}}}{\delta^{(j)}}\left|\sum_{i=1}^{g}(t_i+y_{ij})\omega_i^{(j)}\right|\right)\right],$$

so

$$B(x)=\sum_{m_1, \ldots, m_g \in \mathbf{Z}} h(m_1, \ldots, m_g; x).$$

Fix any $0<\sigma<1$ and put

$$r(u)=\begin{cases} 1 & \text{for } |u| \leqq 1, \quad u \in \mathbf{R}, \\ |u|^{-1-\frac{\sigma}{2}} & \text{for } |u|>1, \quad u \in \mathbf{R}. \end{cases}$$

Since

$$x^{-s}J_{k-1}(ax)=O(x^{-\mathrm{Re}(s)-(1/2)}) \quad \text{as} \quad x\to\infty\,,$$

and

$$x^{-s}J_{k-1}(ax)=O(x^{k-1-\mathrm{Re}(s)}) \quad \text{as} \quad x\to+0 \quad (a>0)\,,$$

there exists a constant $C_2>0$ with the following property: If $m_j\leqq t_j\leqq m_j+1$ $(j=1,\dots,g)$ and $x\in\mathcal{S}$, we have

$$|h(m_1,\dots,m_g;x)|\leqq C_2\prod_{j=1}^{g} r\left(\sum_{i=1}^{g} t_i\omega_i^{(j)}\right)$$

for $\frac{1}{2}+\sigma\leqq\mathrm{Re}(s)\leqq k-1-\sigma$ or $s=k-1$. For such s, we have

$$\sum_{m_1,\dots,m_g\in\mathbf{Z}}|h(m_1,\dots,m_g;x)|\leqq C_2\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{j=1}^{g} r\left(\sum_{i=1}^{g} t_i\omega_i^{(j)}\right)dt_1\dots dt_g\,. \tag{2.19}$$

Changing the variables by $u_j=\sum_{i=1}^{g} t_i\omega_i^{(j)}$, we see that the right-hand side of (2.19) is equal to

$$\frac{C_2}{\sqrt{d(F)}}\prod_{j=1}^{g}\int_{-\infty}^{+\infty} r(u_j)du_j<\infty\,.$$

This proves the assertion (1).

(2) Putting $I_1(l,v,s)=I(l,v,s)$, we have $I_j(l,v,s)=I(l^{(j)},v^{(j)},s)$. For $\frac{1}{2}<\mathrm{Re}(s)<k$, by Erdélyi et al. [3, 1.12, (13)],

$$I(l,v,s)=\begin{cases}\dfrac{2^{1-s}\left(4\pi\dfrac{\sqrt{v}}{\delta}\right)^{s-1}\Gamma\left(\dfrac{k-s}{2}\right)}{\Gamma\left(\dfrac{k+s}{2}\right)}F\left(\dfrac{k-s}{2},1-\dfrac{k+s}{2};\dfrac{1}{2};\dfrac{l^2}{4v}\right) & \text{if} \quad l^2\leqq 4v\,,\\[2ex] \dfrac{2\left(2\pi\dfrac{\sqrt{v}}{\delta}\right)^{k-1}\left(2\pi\dfrac{|l|}{\delta}\right)^{s-k}\Gamma(k-s)\cos\left(\dfrac{\pi}{2}(k-s)\right)}{\Gamma(k)} \\ \quad\cdot F\left(\dfrac{k-s}{2},\dfrac{k-s+1}{2};k;\dfrac{4v}{l^2}\right) & \text{if} \quad l^2\geqq 4v\,,\end{cases} \tag{2.20}$$

where $F(a,b;c;z)$ is the hypergeometric function. Thus $I(t,v,s)$ as a function of t is bounded for all $t\in\mathbf{R}$ and $I(t,v,s)=O(|t|^{\mathrm{Re}(s)-k})$ as $|t|\to\infty$. (To obtain these estimations we do not need (2.20), but we use (2.20) later; cf. §4 below.) Suppose $\frac{1}{2}+\sigma\leqq\mathrm{Re}(s)\leqq k-1-\sigma$ with $0<\sigma<1$. Then, if $m_j\leqq t_j\leqq m_j+1$ $(j=1,\dots,g)$ we have

$$\prod_{j=1}^{g}\left|I\left(\sum_{i=1}^{g} m_i\omega_i^{(j)},v^{(j)},s\right)\right|\leqq C_3\prod_{j=1}^{g} r\left(\sum_{i=1}^{g} t_i\omega_i^{(j)}\right)$$

with some constant $C_3>0$, in the notation of the proof of (1). The assertion (2) for $\frac{1}{2}<\mathrm{Re}(s)<k-1$ follows from this as in the proof of (1). For $s=k-1$, we see from (2.20) that $\prod_{j=1}^{g} I_j(l,v,k-1)=0$ unless $l^2\ll 4v$. This completes the proof of Lemma 3.

By (2.15) and (2.17) we obtain

$$\sum_{\substack{\beta \equiv \gamma (\mathrm{mod}\, (\alpha)) \\ \beta \in \mathcal{O}}} A(\beta) = d(F)^{-1/2} |N(\alpha)|^{-s} \sum_{l \in \mathcal{O}} \left\{ \prod_{j=1}^{g} I_j(l, v, s) \right\} \mathbf{e}\left(\mathrm{tr}\left(\frac{l\gamma}{\delta\alpha} \right) \right). \tag{2.21}$$

Note that each $I_j(l, v, s)$ is independent of α, γ. Thus in (2.11) we next compute

$$\sum_{\substack{(\alpha) \\ 0 \ll \alpha \in \mathcal{O}}} N(\alpha)^{-1-s} \sum_{\gamma \bmod (\alpha)} K_\alpha(v, \gamma^2) \mathbf{e}\left(\mathrm{tr}\left(\frac{l\gamma}{\delta\alpha} \right) \right). \tag{2.22}$$

In the next section we shall prove that (2.22) is expressed by the zeta function of the field $F(\sqrt{l^2 - 4v})$.

3. Zeta Functions Associated with Quadratic Forms over F

Let F and $\mathcal{O}$ be as above. We put

$$\mathcal{D} = \{ b^2 - 4ac | a, b, c \in \mathcal{O} \}.$$

For $\Delta \in \mathcal{D}$, we put

$$\mathcal{Q}(\Delta) = \left\{ \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \middle| \begin{matrix} a, b, c \in \mathcal{O}; \\ b^2 - 4ac = \Delta \end{matrix} \right\}.$$

The group $\Gamma = \mathrm{SL}(2, \mathcal{O})$ acts on $\mathcal{Q}(\Delta)$ by

$$T \circ M = {}^t M T M \quad (M \in \Gamma, T \in \mathcal{Q}(\Delta)),$$

where ${}^t M$ is the transpose of M. Next, let

$$X = \left\{ \begin{pmatrix} m \\ n \end{pmatrix} \middle| m, n \in \mathcal{O}, \begin{pmatrix} m \\ n \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \Big/ \sim ,$$

where the equivalence relation $\begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \sim \begin{pmatrix} m_2 \\ n_2 \end{pmatrix}$ is defined as $\begin{pmatrix} m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} \lambda m_2 \\ \lambda n_2 \end{pmatrix}$ for some $\lambda \in \mathcal{O}^\times$. The group Γ acts also on X by

$$x \circ M = M^{-1} x \quad (M \in \Gamma, x \in X),$$

where the product on the right-hand side means the usual product as matrices. Following Zagier [23] (where the case $F = \mathbf{Q}$ is treated), we define $\zeta_F(s, \Delta)$ as follows:

$$\zeta_F(s, \Delta) = \sum_{T : \mathcal{Q}(\Delta)/\Gamma} \sum_{\substack{x : X/\Gamma_T \\ T[x] \gg 0}} N(T[x])^{-s}. \tag{3.1}$$

Here T runs over a complete system of representatives for $\mathcal{Q}(\Delta)/\Gamma$, and x over a complete system of representatives for X/Γ_T such that $T[x] = {}^t x T x$ is totally positive where $\Gamma_T = \{ M \in \Gamma | T \circ M = T \}$, and s is a complex variable with $\mathrm{Re}(s) > 1$.

Proposition 1. *Let the notation be as above. Then:*

(1) $$\zeta_F(s, \Delta) = \zeta_F(2s) \sum_{\mathfrak{a}} n(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

where $\mathfrak{a}$ *runs over all non-zero integral ideals of* F, *and*

$$n(\mathfrak{a}) = \#\{x \bmod (2)\mathfrak{a} \mid x^2 \equiv \Delta (\bmod (4)\mathfrak{a})\} \ (\# \text{ denoting the cardinality}).$$

(2) $\zeta_F(s, \Delta)$ *has a meromorphic continuation to the whole s-plane.*

(3) *Suppose* $\Delta \neq 0$. *Writing* $n_1 = \#\{j \in \mathbf{Z} \mid 1 \leqq j \leqq g, \Delta^{(j)} > 0\}$ *and*

$$n_2 = \#\{j \in \mathbf{Z} \mid 1 \leqq j \leqq g, \Delta^{(j)} < 0\},$$

put

$$\gamma_F(s, \Delta) = |N_{F/\mathbf{Q}}(\Delta)|^{s/2} d(F)^s \left(\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)\right)^{2n_1} ((2\pi)^{-s} \Gamma(s))^{n_2}.$$

Then $\zeta_F(s, \Delta)$ *satisfies the functional equation*

$$\gamma_F(s, \Delta) \zeta_F(s, \Delta) = \gamma_F(1-s, \Delta) \zeta_F(1-s, \Delta).$$

(4) *For* $\Delta = 0$, *we have* $\zeta_F(s, \Delta) = \zeta_F(s) \zeta_F(2s-1)$. *For* $0 \neq \Delta \in \mathscr{D}$, *there exists an integral ideal* $\mathfrak{f}$ *of* F *such that* $(\Delta) = d(F(\sqrt{\Delta})/F)\mathfrak{f}^2$ *where* $d(F(\sqrt{\Delta})/F)$ *is the relative discriminant of* $F(\sqrt{\Delta})/F$. *With this* $\mathfrak{f}$, *we have*

$$\zeta_F(s, \Delta) = \zeta_F(s) L_F(s, \chi) \sum_{\mathfrak{a} \mid \mathfrak{f}} \mu(\mathfrak{a}) \chi(\mathfrak{a}) N(\mathfrak{a})^{-s} \sigma_{1-2s}(\mathfrak{f}\mathfrak{a}^{-1}).$$

Here χ *is the Hecke character associated with the extension* $F(\sqrt{\Delta})/F$ *(so* $\chi = 1$ *if* Δ *is a square),* $L_F(s, \chi)$ *the Hecke L-function associated with* χ, μ *the Möbius function for* F, *and* $\mathfrak{a}$ *runs over all integral ideals of* F *dividing* $\mathfrak{f}$.

(5) *For each* $\Delta \in \mathscr{D}$, *put* $L_F(s, \Delta) = \zeta_F(s, \Delta)/\zeta_F(s)$. *Then* $L_F(s, \Delta)$ *is an entire function if* Δ *is not a square. If* Δ *is a square,* $L_F(s, \Delta)$ *is holomorphic except for a simple pole (of residue* $2^{g-2} R(F)/\sqrt{d(F)}$ *if* $\Delta = 0$ *and* $2^{g-1} R(F)/\sqrt{d(F)}$ *if* $\Delta \neq 0$, *where* $R(F)$ *is the regulator of* F*) at* $s = 1$.

The assertion (1) is proved exactly as in Zagier [23, p. 131], and (2), (3), (5) follow from (4). [Note that $d(K) = N_{F/\mathbf{Q}}(d(K/F)) d(F)^2$ for any quadratic extension K/F]. The equality $\zeta_F(s, 0) = \zeta_F(s) \zeta_F(2s-1)$ follows directly from the definition (3.1). So we prove the assertion (4) for the case $\Delta \neq 0$; we give a proof which will be convenient for explicit numerical computations of $\mathfrak{f}$. We need some lemmas.

Lemma 4. (1) *Let* $n(\mathfrak{a})$ *be as in Proposition 1(1). Then* $n(\mathfrak{a})$ *is multiplicative, i.e., if integral ideals* $\mathfrak{a}, \mathfrak{b}$ *of* F *satisfy* $\mathfrak{a} + \mathfrak{b} = \mathcal{O}$, *then* $n(\mathfrak{a}\mathfrak{b}) = n(\mathfrak{a}) n(\mathfrak{b})$.

(2) *For each non-zero prime ideal* $\mathfrak{p}$ *of* $\mathcal{O}$ *and* $0 \neq \Delta \in \mathscr{D}$, *we put*

$$b = \operatorname{ord}_{\mathfrak{p}}(\Delta) \quad (\text{i.e. } 0 \leqq b \in \mathbf{Z} \text{ with } \mathfrak{p}^b \mid \Delta \text{ and } \mathfrak{p}^{b+1} \nmid \Delta),$$

$$l = \operatorname{ord}_{\mathfrak{p}}(2),$$

and

$$c = \max\left\{ t \in \mathbf{Z} \,\middle|\, \begin{array}{l} x^2 \equiv \Delta \ (\bmod \mathfrak{p}^{t + 2[b/2]}) \text{ is solvable;} \\ t \leqq 2l \end{array} \right\}.$$

Then, for each $m \geqq 0$, *the values of* $n(\mathfrak{p}^m)$ *are as follows:*

m	$n(\mathfrak{p}^m)$
$0 \leqq m \leqq b-2l$	$N(\mathfrak{p})^{[m/2]}$
$b-2l<m \leqq b-2l+c$	$\delta(b/2)N(\mathfrak{p})^{[m/2]}$
$b-2l+c<m \leqq b$	$\delta(b/2)\chi(\mathfrak{p})^2 N(\mathfrak{p})^{[m/2]}$
$m>b$	$\delta(b/2)\chi(\mathfrak{p})(1+\chi(\mathfrak{p}))N(\mathfrak{p})^{b/2}$

Here [ ] *is the Gauß symbol, and* $\delta(x)=1$ *or* 0 *according as* x *is an integer or not. (Note that* $l=c=0$ *if* $\mathfrak{p}\nmid 2$.*)*

Proof. The assertion (1) follows immediately from the definition of $n(\mathfrak{a})$. We prove the assertion (2) in the case $\mathfrak{p}|2$; proof in the case $\mathfrak{p}\nmid 2$ is much easier. If $m \leqq b-2l$, then $n(\mathfrak{p}^m)=\#\{x \bmod \mathfrak{p}^{l+m} | x^2 \equiv 0 (\bmod\, \mathfrak{p}^{2l+m})\}=N(\mathfrak{p})^{[m/2]}$. Next suppose that $m>b-2l$. Then

$$n(\mathfrak{p}^m)=\delta(b/2)\cdot \#\{x \bmod \mathfrak{p}^{l+m-(b/2)} | x^2 \equiv \varrho^{-b}\Delta (\bmod\, \mathfrak{p}^{2l+m-b})\}, \tag{3.2}$$

where $\mathfrak{p}=(\varrho)$.

(i) The case $m>b$

By (3.2), $n(\mathfrak{p}^m) \neq 0$ implies that b is even and $\chi(\mathfrak{p})=1$ (see Hilbert [8, Satz 8]). If so, all the solutions modulo $\mathfrak{p}^{l+m-(b/2)}$ of $x^2 \equiv \varrho^{-b}\Delta$ $(\bmod\, \mathfrak{p}^{2l+m-b})$ are of the form $\pm x_0+\varrho^{l+m-b}y$, $y \bmod \mathfrak{p}^{b/2}$, with a fixed x_0. Thus

$$n(\mathfrak{p}^m)=\delta(b/2)\chi(\mathfrak{p})(1+\chi(\mathfrak{p}))N(\mathfrak{p})^{b/2}.$$

(ii) The case $b-2l<m \leqq b$

The congruence $x^2 \equiv \varrho^{-b}\Delta$ $(\bmod\, \mathfrak{p}^{2l})$ is solvable if and only if $\chi(\mathfrak{p}) \neq 0$ (Hilbert [8, Satz 4]). So if b is even and $\chi(\mathfrak{p}) \neq 0$ (in this case $c=2l$), all the solutions modulo $\mathfrak{p}^{l+m-(b/2)}$ of $x^2 \equiv \varrho^{-b}\Delta$ $(\bmod\, \mathfrak{p}^{2l+m-b})$ are of the form $x_0+\varrho^{[(2l+m-b+1)/2]}y$, $y \bmod \mathfrak{p}^{[m/2]}$, with a fixed x_0. Thus $n(\mathfrak{p}^m)=N(\mathfrak{p})^{[m/2]}$. If b is even and $\chi(\mathfrak{p})=0$, then by the definition of c, we have $n(\mathfrak{p}^m)=N(\mathfrak{p})^{[m/2]}$ as above for $2l+m-b \leqq c$ and $n(\mathfrak{p}^m)=0$ for $2l+m-b>c$.

By Lemma 4 we have (for $0 \neq \Delta \in \mathcal{D}$):

$$\zeta_F(s,\Delta)=\zeta_F(s)L_F(s,\chi)\prod_{\mathfrak{p}|\Delta} M_{\mathfrak{p}}(s) \tag{3.3}$$

where the product is over all prime ideals of F dividing Δ and

$$M_{\mathfrak{p}}(s)=(1-\chi(\mathfrak{p})N(\mathfrak{p})^{-s})\sum_{m=0}^{j(\mathfrak{p})} N(\mathfrak{p})^{m(1-2s)}+\chi(\mathfrak{p})^2 N(\mathfrak{p})^{(j(\mathfrak{p})+1)(1-2s)} \tag{3.4}$$

where $j(\mathfrak{p})=[(b-2l+c-1)/2]$ in the same notation as in Lemma 4(2). Here we used the identity

$$\sum_{m=0}^{t} N(\mathfrak{p})^{[m/2]}N(\mathfrak{p})^{-ms}=(1+N(\mathfrak{p})^{-s})\sum_{m=0}^{[(t-1)/2]} N(\mathfrak{p})^{m(1-2s)}+\delta(t/2)N(\mathfrak{p})^{t(1-2s)/2}$$

for each $0 \leqq t \in \mathbf{Z}$. Note that $c=2l$ if $\chi(\mathfrak{p}) \neq 0$, as above.

Lemma 5. *Let F and $\mathcal{O}$ be as above. For a square-free $\Delta_1 \in \mathcal{O}$ (i.e. Δ_1 not divisible by the square of any proper ideals of $\mathcal{O}$), put $K=F(\sqrt{\Delta_1})$ and denote by $\mathcal{O}_K$ the ring of integers in K. For each non-zero prime ideal $\mathfrak{p}$ in $\mathcal{O}$, put $l=\mathrm{ord}_{\mathfrak{p}}(2)$ and*

$$\kappa(\mathfrak{p})=\kappa=\min\left\{0\leqq t\in\mathbf{Z}\,\middle|\,\begin{array}{l}x^2\equiv\Delta_1(\mathrm{mod}\,\mathfrak{p}^{2(l-t)})\\ \textit{is solvable}.\end{array}\right\}.$$

(Note that $l=\kappa=0$ for $\mathfrak{p}\nmid 2$.) Take a $\xi\in\mathcal{O}$ satisfying $(\xi)=\prod_{\substack{\mathfrak{p}\mid 2\\ \mathfrak{p}:\,\text{prime in }\mathcal{O}}}\mathfrak{p}^{\kappa(\mathfrak{p})}$ and an element $x_0\in\mathcal{O}$ such that $x_0^2\equiv\Delta_1\ (\mathrm{mod}\,4\xi^{-2})$. (Such x_0 exists by the definition of κ.) Then we have:

$$\mathcal{O}_K=\mathcal{O}\oplus\mathcal{O}\frac{x_0+\sqrt{\Delta_1}}{2\xi^{-1}}. \tag{3.5}$$

In particular,

$$\mathrm{ord}_{\mathfrak{p}}d(K/F)=\mathrm{ord}_{\mathfrak{p}}(\Delta_1)+2\kappa(\mathfrak{p}). \tag{3.6}$$

Proof. The equality (3.6) is a simple consequence of (3.5):

$$\det\begin{pmatrix}1 & \dfrac{x_0+\sqrt{\Delta_1}}{2\xi^{-1}}\\ 1 & \dfrac{x_0-\sqrt{\Delta_1}}{2\xi^{-1}}\end{pmatrix}^2=\xi^2\Delta_1.$$

So we prove (3.5). For $\alpha=u+v\sqrt{\Delta_1}$ with $u,v\in F$, we have

$$\alpha^2-2u\alpha+(u^2-v^2\Delta_1)=0.$$

So $\alpha\in\mathcal{O}_K$ is equivalent to $2u\in\mathcal{O}$ and $u^2-v^2\Delta_1\in\mathcal{O}$. If this condition is satisfied, then $(2u)^2-(2v)^2\Delta_1\in\mathcal{O}$ implies $(2v)^2\Delta_1\in\mathcal{O}$, so $2v\in\mathcal{O}$ since Δ_1 is square free. Thus (putting $2u=m$ and $2v=n$),

$$\mathcal{O}_K=\left\{\frac{m+n\sqrt{\Delta_1}}{2}\,\middle|\,\begin{array}{l}m,n\in\mathcal{O};\\ m^2-n^2\Delta_1\equiv 0\ (\mathrm{mod}\,4)\end{array}\right\}.$$

For each $\alpha=(m+n\sqrt{\Delta_1})/2\in\mathcal{O}_K$ and a prime $\mathfrak{p}$ in $\mathcal{O}$ dividing 2, put $r=\mathrm{ord}_{\mathfrak{p}}(n)$. If $r\geqq l$, then $m^2\equiv\Delta_1 n^2\ (\mathrm{mod}\,4)$ implies $\mathfrak{p}^l\mid m$. If $r<l$, then $\mathrm{ord}_{\mathfrak{p}}(m)=r$ since $m^2\equiv\Delta_1 n^2$ $(\mathrm{mod}\,4)$ and $\mathfrak{p}^2\nmid\Delta_1$, so $(m/\varrho^r)^2\equiv\Delta_1(n/\varrho^r)^2\ (\mathrm{mod}\,\mathfrak{p}^{2(l-r)})$ where $\mathfrak{p}=(\varrho)$. Hence $r\geqq\kappa$ by the definition of κ. Thus we have always $\mathfrak{p}^\kappa\mid m$, $\mathfrak{p}^\kappa\mid n$. Therefore

$$\alpha=(m'+n'\sqrt{\Delta_1})/2\xi^{-1}$$

with $m',n'\in\mathcal{O}$ such that $(m')^2\equiv(n')^2\Delta_1\ (\mathrm{mod}(2\xi^{-1})^2)$. On the other hand, $x_0^2\equiv\Delta_1$ $(\mathrm{mod}(2\xi^{-1})^2)$ yields $(n'x_0)^2\equiv(n')^2\Delta_1\ (\mathrm{mod}(2\xi^{-1})^2)$. So $(m')^2\equiv(n'x_0)^2$ $(\mathrm{mod}(2\xi^{-1})^2)$, and one checks easily that this implies $m'\equiv n'x_0\ (\mathrm{mod}\,2\xi^{-1})$. Hence

$$\alpha=\frac{m'-n'x_0+n'(x_0+\sqrt{\Delta_1})}{2\xi^{-1}}\in\mathcal{O}\oplus\frac{x_0+\sqrt{\Delta_1}}{2\xi^{-1}}\mathcal{O}.$$

Since $(x_0+\sqrt{\Delta_1})/2\xi^{-1}\in\mathcal{O}_K$ is obvious, Lemma 5 is proved.

Lemma 6. *For each non-zero $\Delta \in \mathcal{D}$, take $\Delta_1 \in \mathcal{O}$ such that* $\mathrm{ord}_{\mathfrak{p}}(\Delta_1) = \mathrm{ord}_{\mathfrak{p}}(\Delta) - 2[(\mathrm{ord}_{\mathfrak{p}}(\Delta))/2]$ *for each non-zero prime ideal $\mathfrak{p}$ in $\mathcal{O}$ (i.e., Δ_1 is a "square-free part" of Δ). For this Δ_1, let $\kappa(\mathfrak{p})$ be as in Lemma 5. Then $\kappa(\mathfrak{p}) \leqq [(\mathrm{ord}_{\mathfrak{p}}(\Delta))/2]$ for each $\mathfrak{p}$. In particular, $(\Delta) = d(F(\sqrt{\Delta})/F)\mathfrak{f}^2$ with an integral ideal $\mathfrak{f}$ of F.*

Proof. There exist $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\Delta = \beta^2 - 4\alpha\gamma$, so the congruence $x^2 \equiv \Delta \pmod 4$ is always solvable. Thus in the notation of Lemmas 4 and 5, we have $\kappa \leqq [b/2]$ for each prime $\mathfrak{p}$ of F. From this follows the second assertion, since

$$\mathrm{ord}_{\mathfrak{p}}(\Delta) - \mathrm{ord}_{\mathfrak{p}} d(F(\sqrt{\Delta})/F) = b - (b - 2[b/2] + 2\kappa) = 2([b/2] - \kappa),$$

which is a non-negative even integer.

Lemma 7. *The notation being as in Proposition 1(4), suppose $0 \neq \Delta \in \mathcal{D}$. For each non-zero prime ideal $\mathfrak{p}$ in $\mathcal{O}$, let $j(\mathfrak{p})$ be as in (3.4). Then:*

$$j(\mathfrak{p}) + \chi(\mathfrak{p})^2 = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{f}).$$

Proof. In the notation of Lemmas 4 and 5, the above equality is also written as

$$[(b - 2l + c - 1)/2] + \chi(\mathfrak{p})^2 = [b/2] - \kappa. \tag{3.7}$$

For $\mathfrak{p} \nmid 2$, by Hilbert [8, Satz 4], $\chi(\mathfrak{p}) \neq 0$ if and only if b is even. Since $l = c = \kappa = 0$ in this case, (3.7) is valid for $\mathfrak{p} \nmid 2$. Next suppose $\mathfrak{p} | 2$. If $\chi(\mathfrak{p}) \neq 0$, then b must be even from Hilbert [8, Satz 4], and $\kappa = 0$, $c = 2l$. Thus (3.7) is valid. Suppose $\mathfrak{p} | 2$ and $\chi(\mathfrak{p}) = 0$. If b is odd, then $c = 1$ and $\kappa = 1$, so (3.7) holds. Suppose finally that $\mathfrak{p} | 2$, $\chi(\mathfrak{p}) = 0$, and b even. In this case $c < 2l$ by Hilbert [8, Satz 4], and c is an odd integer. In fact: Suppose there exists some $y_0 \in \mathcal{O}$ such that $y_0^2 \equiv \Delta \pmod{\mathfrak{p}^{2u+b}}$ with $0 \leqq u < l$ ($u \in \mathbf{Z}$). So $x_0^2 \equiv \varrho^{-b}\Delta \pmod{\mathfrak{p}^{2u}}$ for some $x_0 \in \mathcal{O}$, where $(\varrho) = \mathfrak{p}$. We assert that $x^2 \equiv \varrho^{-b}\Delta \pmod{\mathfrak{p}^{2u+1}}$ is also solvable. Put $x_0^2 - \varrho^{-b}\Delta = \varrho^{2u}\eta$. If $\eta \equiv 0 \pmod{\mathfrak{p}}$, then there is nothing to prove. Suppose $\eta \not\equiv 0 \pmod{\mathfrak{p}}$. Then there is an $\alpha \in \mathcal{O}$ such that $\alpha^2 \equiv -\eta \pmod{\mathfrak{p}}$ since $(\mathcal{O}/\mathfrak{p})^\times$ is a cyclic group of odd order. Thus $(x_0 + \varrho^u\alpha)^2 \equiv \varrho^{-b}\Delta \pmod{\mathfrak{p}^{2u+1}}$. Thus we have proved: If $x^2 \equiv \Delta \pmod{\mathfrak{p}^{2u+b}}$ is solvable for $0 \leqq u < l$, then $x^2 \equiv \Delta \pmod{\mathfrak{p}^{2u+1+b}}$ is also solvable. Hence c is an odd integer $< 2l$. Using this, we have $\kappa = l - (c-1)/2$ from the definition of κ. Thus (3.7) is valid also in this case. Lemma 7 is proved.

Applying Lemma 7 for (3.3)(3.4), we obtain the remaining assertion of Proposition 1(4), so this completes the proof of Proposition 1.

Now we return to the calculation of the series (2.22).

Proposition 2. *In the notation of Proposition 1 and (2.22), we have:*

$$\sum_{\substack{(\alpha) \\ 0 \ll \alpha \in \mathcal{O}}} N(\alpha)^{-1-s} \sum_{\gamma \bmod (\alpha)} K_\alpha(v, \gamma^2)\mathbf{e}\left(\mathrm{tr}\left(\frac{l\gamma}{\delta\alpha}\right)\right) = \zeta_F(2s)^{-1} L_F(s, l^2 - 4v).$$

Proof. From the definition of the Kloosterman sums,

$$\sum_{\gamma \bmod (\alpha)} \mathbf{e}\left(\mathrm{tr}\left(\frac{l\gamma}{\delta\alpha}\right)\right) K_\alpha(v, \gamma^2) = \sum_{\substack{x \bmod (\alpha) \\ (x, \alpha) = 1}} \sum_{\gamma \bmod (\alpha)} \mathbf{e}\left(\mathrm{tr}\left(\frac{1}{\delta\alpha}(l\gamma + \gamma^2 x^{-1} + vx)\right)\right).$$

In the inner summation, we replace γ by γx to obtain

$$\sum_{\gamma \bmod (\alpha)} \mathbf{e}\left(\operatorname{tr}\left(\frac{l\gamma}{\delta\alpha}\right)\right) K_\alpha(v, \gamma^2) = \sum_{\substack{x \bmod (\alpha) \\ (x, \alpha) = 1}} \sum_{\gamma \bmod (\alpha)} \mathbf{e}\left(\operatorname{tr}\left(\frac{x}{\delta\alpha}(\gamma^2 + l\gamma + v)\right)\right). \tag{3.8}$$

We note that the right-hand side of (3.8) is multiplicative with respect to (α). In fact, if $(\alpha) = (\alpha_1)(\alpha_2)$ with $(\alpha_1, \alpha_2) = 1$, then putting $x = \alpha_1 x_2 + \alpha_2 x_1$ and $\gamma = \alpha_1 \gamma_2 + \alpha_2 \gamma_1$ we have

$$\begin{aligned} &\sum_{\substack{x \bmod (\alpha) \\ (x, \alpha) = 1}} \sum_{\gamma \bmod (\alpha)} \mathbf{e}\left(\operatorname{tr}\left(\frac{x}{\delta\alpha}(\gamma^2 + l\gamma + v)\right)\right) \\ &\quad = \sum_{\substack{x_j \bmod (\alpha_j) \\ (x_j, \alpha_j) = 1 \\ j = 1, 2}} \sum_{\substack{\gamma_j \bmod (\alpha_j) \\ j = 1, 2}} \mathbf{e}\left(\operatorname{tr}\left(\frac{x_1}{\delta\alpha_1}((\alpha_2\gamma_1)^2 + l\alpha_2\gamma_1 + v)\right.\right. \\ &\qquad \left.\left. + \frac{x_2}{\delta\alpha_2}((\alpha_1\gamma_2)^2 + l\alpha_1\gamma_2 + v)\right)\right) \\ &\quad = \prod_{j=1}^{2} \sum_{\substack{x_j \bmod (\alpha_j) \\ (x_j, \alpha_j) = 1}} \sum_{\gamma_j \bmod (\alpha_j)} \mathbf{e}\left(\operatorname{tr}\left(\frac{x_j}{\delta\alpha_j}(\gamma_j^2 + l\gamma_j + v)\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{\substack{(\alpha) \\ 0 \ll \alpha \in \mathcal{O}}} N(\alpha)^{-1-s} \sum_{\gamma \bmod (\alpha)} K_\alpha(v, \gamma^2) \mathbf{e}\left(\operatorname{tr}\left(\frac{l\gamma}{\delta\alpha}\right)\right) \\ &\quad = \prod_{\mathfrak{p}} \left(\sum_{m=0}^{\infty} N(\mathfrak{p}^m)^{-1-s} \sum_{\gamma \bmod \mathfrak{p}^m} K_{\varrho^m}(v, \gamma^2) \mathbf{e}\left(\frac{l\gamma}{\delta\varrho^m}\right)\right), \end{aligned} \tag{3.9}$$

where $\mathfrak{p} = (\varrho)$ runs over all non-zero prime ideals of $\mathcal{O}$. For a prime $\mathfrak{p} = (\varrho)$ and $0 < m \in \mathbf{Z}$, by (3.8) we have:

$$\sum_{\gamma \bmod (\varrho^m)} \mathbf{e}\left(\operatorname{tr}\left(\frac{l\gamma}{\delta\varrho^m}\right)\right) K_{\varrho^m}(v, \gamma^2) = N(\mathfrak{p})^m (n(\mathfrak{p}^m) - n(\mathfrak{p}^{m-1})) \tag{3.10}$$

with $n(\mathfrak{p}^m) = \#\{\gamma \bmod \mathfrak{p}^m | \gamma^2 + l\gamma + v \equiv 0(\mathfrak{p}^m)\}$. Using

$$4(\gamma^2 + l\gamma + v) = (2\gamma + l)^2 + 4v - l^2,$$

we see that this $n(\mathfrak{p}^m)$ coincides with the previous one defined in Proposition 1(1) with $\Delta = l^2 - 4v$. By (3.9) and (3.10) we have:

$$\sum_{\substack{(\alpha) \\ 0 \ll \alpha \in \mathcal{O}}} N(\alpha)^{-1-s} \sum_{\gamma \bmod (\alpha)} K_\alpha(v, \gamma^2) \mathbf{e}\left(\operatorname{tr}\left(\frac{l\gamma}{\delta\alpha}\right)\right) = \prod_{\mathfrak{p}} \left(\sum_{m=0}^{\infty} N(\mathfrak{p}^m)^{-s} \sum_{\mathfrak{b} | \mathfrak{p}^m} \mu(\mathfrak{b}) n(\mathfrak{p}^m / \mathfrak{b})\right), \tag{3.11}$$

where $\mathfrak{b}$ runs over all integral ideals dividing $\mathfrak{p}^m$ for each $\mathfrak{p}^m$, and μ is the Möbius function for F. Thus (3.11) is equal to:

$$\begin{aligned} \sum_{\mathfrak{a}} \left(\sum_{\mathfrak{b} | \mathfrak{a}} \mu(\mathfrak{b}) n(\mathfrak{a}/\mathfrak{b})\right) N(\mathfrak{a})^{-s} &= \left(\sum_{\mathfrak{a}} n(\mathfrak{a}) N(\mathfrak{a})^{-s}\right)\left(\sum_{\mathfrak{c}} \mu(\mathfrak{c}) N(\mathfrak{c})^{-s}\right) \\ &= \left(\sum_{\mathfrak{a}} n(\mathfrak{a}) N(\mathfrak{a})^{-s}\right) \zeta_F(s)^{-1}, \end{aligned}$$

where $\mathfrak{a}$ and $\mathfrak{c}$ run over all non-zero integral ideals of F. From Proposition 1(1), this is equal to:

$$\zeta_F(s, l^2-4\nu)\zeta_F(2s)^{-1}\zeta_F(s)^{-1} = \zeta_F(2s)^{-1} L_F(s, l^2-4\nu).$$

4. Proof of Theorem 1

We recall that S is the right-hand side of (2.8). By (2.11), (2.21), and Proposition 2 in § 3, we obtain

$$\zeta_F(2s)\cdot S = d(F)^{-1/2} \sum_{l\in\mathcal{O}} \left\{\prod_{j=1}^{g} I_j(l, \nu, s)\right\} L_F(s, l^2-4\nu) \tag{4.1}$$

for $\frac{3}{2} < \operatorname{Re}(s) < k-1$, and in this region this converges uniformly and absolutely by Lemma 3(2) in § 2. Thus (2.7) yields

$$\begin{aligned}\zeta_F(2s)b(\nu, s) = (-1)^{gk/2} d(F)^{-1} N(\nu)^{\frac{k-1}{2}} 2^{g-1}\pi^g \cdot \sum_{l\in\mathcal{O}} \left\{\prod_{j=1}^{g} I_j(l, \nu, s)\right\} L_F(s, l^2-4\nu)\\ + \begin{cases} N(\nu)^{\frac{k-1-s}{2}} \zeta_F(2s) & \text{if } (\nu) \text{ is a square},\\ 0 & \text{otherwise},\end{cases}\end{aligned} \tag{4.2}$$

for $\frac{3}{2} < \operatorname{Re}(s) < k-1$. The functions $I_j(l, \nu, s)$ and $L_F(s, l^2-4\nu)$ are meromorphically continued to the whole s-plane by (2.20) and Proposition 1(5).

Lemma 8. *For each $l\in\mathcal{O}$ such that $l^2 \neq 4\nu$ (ν fixed), the meromorphic function*

$$Z_l(s) = d(F)^{3s/2}\Gamma_{\mathbf{R}}(s+1)^g \Gamma_{\mathbf{C}}(s+k-1)^g \left\{\prod_{j=1}^{g} I_j(l, \nu, s)\right\} L_F(s, l^2-4\nu)$$

satisfies the functional equation

$$Z_l(s) = Z_l(1-s).$$

Here

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2) \quad \text{and} \quad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Proof. Using the relation

$$\begin{aligned}F(a, b; c; z) &= (1-z)^{-a} F(a, c-b; c; z/(z-1))\\ &= (1-z)^{-b} F(c-a, b; c; z/(z-1)) \quad \text{for} \quad z\in\mathbf{R} \quad \text{with} \quad 0 \leqq z < 1\end{aligned}$$

(see Erdélyi et al. [2, 2.1.4, (22)]), from (2.20) we obtain:

$$I_j(l, \nu, s) = \begin{cases} 2^{k-1}(\nu^{(j)})^{\frac{k-1}{2}} \left(\dfrac{\pi}{\delta^{(j)}}\right)^{s-1} (4\nu^{(j)} - (l^{(j)})^2)^{\frac{s-k}{2}} \Gamma\left(\dfrac{k-s}{2}\right)\Gamma\left(\dfrac{k+s}{2}\right)^{-1}\\ \quad \cdot F\left(\dfrac{k-s}{2}, \dfrac{k+s-1}{2}; \dfrac{1}{2}; \dfrac{(l^{(j)})^2}{(l^{(j)})^2 - 4\nu^{(j)}}\right) \quad \text{if} \quad 4\nu^{(j)} > (l^{(j)})^2,\\ 2^s(\nu^{(j)})^{\frac{k-1}{2}} \left(\dfrac{\pi}{\delta^{(j)}}\right)^{s-1} ((l^{(j)})^2 - 4\nu^{(j)})^{\frac{s-k}{2}} \Gamma(k-s)\Gamma(k)^{-1}\\ \quad \cdot \cos\left(\dfrac{\pi}{2}(k-s)\right) F\left(\dfrac{k-s}{2}, \dfrac{k+s-1}{2}; k; \dfrac{4\nu^{(j)}}{4\nu^{(j)} - (l^{(j)})^2}\right)\\ \quad \text{if} \quad 4\nu^{(j)} < (l^{(j)})^2,\end{cases} \tag{4.3}$$

for each $j=1, \dots, g$. If $(l^{(j)})^2 < 4v^{(j)}$, (4.3) shows that

$$\left(\frac{\pi}{\delta^{(j)}}\right)^{-s}(4v^{(j)}-(l^{(j)})^2)^{-s/2}\Gamma\left(\frac{k+s}{2}\right)\Gamma\left(\frac{k-s}{2}\right)^{-1} I_j(l, v, s)$$

is invariant under the substitution $s \mapsto 1-s$, since so is

$$F\left(\frac{k-s}{2}, \frac{k+s-1}{2}; \frac{1}{2}; \frac{(l^{(j)})^2}{(l^{(j)})^2-4v^{(j)}}\right).$$

From

$$\Gamma\left(\frac{k-s}{2}\right)^{-1} = \frac{1}{\pi}\sin\left(\frac{\pi}{2}(k-s)\right)\Gamma\left(\frac{s-k+2}{2}\right),$$

we know that

$$(\delta^{(j)})^s(4v^{(j)}-(l^{(j)})^2)^{-s/2}\Gamma_{\mathbf{R}}(s+k)\Gamma_{\mathbf{R}}(s-k+2)\sin\left(\frac{\pi}{2}(k-s)\right)I_j(l, v, s)$$

is invariant under $s \mapsto 1-s$, if $(l^{(j)})^2 < 4v^{(j)}$. Similarly,

$$(\delta^{(j)})^s((l^{(j)})^2-4v^{(j)})^{-s/2}\Gamma_{\mathbf{C}}(s+k-1)\sin\left(\frac{\pi}{2}(k-s)\right)I_j(l, v, s)$$

is invariant under $s \mapsto 1-s$ if $(l^{(j)})^2 > 4v^{(j)}$. On the other hand, by Proposition 1(4)(5),

$$|N_{F/\mathbf{Q}}(l^2-4v)|^{s/2}d(F)^{s/2}\Gamma_{\mathbf{R}}(s)^{n_1}\Gamma_{\mathbf{R}}(s+1)^{n_2}L_F(s, l^2-4v)$$

is invariant under $s \mapsto 1-s$ if $l^2 \neq 4v$, where

$$n_1 = \#\{j \in \mathbf{Z} \mid 1 \leqq j \leqq g, (l^{(j)})^2 > 4v^{(j)}\},$$

and

$$n_2 = \#\{j \in \mathbf{Z} \mid 1 \leqq j \leqq g, (l^{(j)})^2 < 4v^{(j)}\}.$$

[Note that $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$.] Hence

$$d(F)^{3s/2}\Gamma_{\mathbf{R}}(s)^{n_1}\Gamma_{\mathbf{R}}(s+1)^{n_2}\Gamma_{\mathbf{R}}(s+k)^{n_2}\Gamma_{\mathbf{R}}(s-k+2)^{n_2}\Gamma_{\mathbf{C}}(s+k-1)^{n_1}$$
$$\cdot\left(\sin\left(\frac{\pi}{2}(k-s)\right)\right)^g\left\{\prod_{j=1}^{g} I_j(l, v, s)\right\}L_F(s, l^2-4v)$$

is invariant under $s \mapsto 1-s$ if $l^2 \neq 4v$. Moreover,

$$\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+k-1}{2}\right)} \cdot \frac{\Gamma\left(\frac{s-k+2}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \prod_{u=0}^{\frac{k}{2}-2}\left(\frac{s+1}{2}+u\right)^{-1} \cdot \prod_{v=1}^{\frac{k}{2}-1}\left(\frac{s}{2}-v\right)^{-1}$$

is invariant under $s \mapsto 1-s$, so

$$d(F)^{3s/2}\Gamma_{\mathbf{R}}(s)^{n_1}\Gamma_{\mathbf{R}}(s+k-1)^{n_2}\Gamma_{\mathbf{R}}(s+k)^{n_2}\Gamma_{\mathbf{R}}(s)^{n_2}\Gamma_{\mathbf{C}}(s+k-1)^{n_1}$$

$$\cdot\left(\sin\left(\frac{\pi}{2}(k-s)\right)\right)^g\left\{\prod_{j=1}^{g} I_j(l,v,s)\right\}L_F(s,l^2-4v)$$

$$=d(F)^{3s/2}\Gamma_{\mathbf{R}}(s)^g\Gamma_{\mathbf{C}}(s+k-1)^g\left(\sin\left(\frac{\pi}{2}(k-s)\right)\right)^g$$

$$\cdot\left\{\prod_{j=1}^{g} I_j(l,v,s)\right\}L_F(s,l^2-4v) \tag{4.4}$$

is invariant under $s \mapsto 1-s$ if $l^2 \neq 4v$.

Let $F(s)$ be a meromorphic function on $\mathbf{C}$ such that $\Gamma_{\mathbf{R}}(s)\sin\left(\frac{\pi}{2}(k-s)\right)F(s)$ is invariant under $s \mapsto 1-s$. Then by $\Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(2-s)=(\sin(\pi s/2))^{-1}$, it is easy to see that $\Gamma_{\mathbf{R}}(s+1)F(s)$ is invariant under $s \mapsto 1-s$. Using this fact repeatedly for (4.4), we obtain Lemma 8.

Now we prove that

$$\sum_{\substack{l \in \mathcal{O} \\ l^2 \neq 4v}}\left\{\prod_{j=1}^{g} I_j(l,v,s)\right\}L_F(s,l^2-4v) \tag{4.5}$$

converges absolutely and uniformly for $2-k<\mathrm{Re}(s)<k-1$. From Lemma 2(2), the right-hand side of (4.1) converges absolutely and uniformly for $\frac{1}{2}<\mathrm{Re}(s)<k-1$. [Note that $I_j(l,v,s)$ has a zero at $s=1$ if $(l^{(j)})^2 \geqq 4v^{(j)}$ by (2.20).] For $l^2 \neq 4v$, the equality (2.20) is valid for $\mathrm{Re}(s)>-\frac{1}{2}$ (cf. Erdélyi et al. [3, 1.12, (13)]), so the same proof as in Lemma 2(2) shows that $\sum_{\substack{l \in \mathcal{O} \\ l^2 \neq 4v}}\prod_{j=1}^{g} I_j(l,v,s)$ is absolutely and uniformly convergent for $-\frac{1}{2}<\mathrm{Re}(s)<k-1$. So the same is true for (4.5). This, together with Lemma 8, implies that (4.5) converges absolutely and uniformly for $2-k<\mathrm{Re}(s)<k-1$.

Next, by (4.2) and Proposition 1(4), for $\frac{3}{2}<\mathrm{Re}(s)<k-1$ we have:

$$\zeta_F(2s)b(v,s)=(-1)^{gk/2}d(F)^{-1}N(v)^{\frac{k-1}{2}}2^{g-1}\pi^g$$

$$\cdot\sum_{\substack{l \in \mathcal{O} \\ l^2 \neq 4v}}\left\{\prod_{j=1}^{g} I_j(l,v,s)\right\}L_F(s,l^2-4v)+H(s) \tag{4.6}$$

with $H(s)=0$ if (v) is not a square,

$$H(s)=(-1)^{gk/2}d(F)^{-1}N(v)^{\frac{k-1}{2}}2^g\pi^g2^{g(s-1)}N(v)^{\frac{s-1}{2}}d(F)^{1-s}$$

$$\cdot\left(\pi^{s-(1/2)}\frac{\Gamma\left(\frac{k-s}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(\frac{k+s}{2}\right)\Gamma\left(\frac{k+s-1}{2}\right)\Gamma\left(\frac{1-k+s}{2}\right)}\right)^g\zeta_F(2s-1)+N(v)^{\frac{k-1-s}{2}}\zeta_F(2s)$$

if (v) is a square. Here we used (2.20) and

$$F(a,b;c;1)=\Gamma(c)\Gamma(c-a-b)[\Gamma(c-a)\Gamma(c-b)]^{-1}$$
$$(a,b,c\in\mathbf{C};c\neq 0,-1,-2,\ldots;\mathrm{Re}(c-a-b)>0),$$

cf. Erdélyi et al. [2, 2.1.3, (14)]. As we have noted, the summation over $l\in\mathcal{O}$ such that $l^2\neq 4v$ in the right-hand side of (4.6) converges absolutely and uniformly for $2-k<\mathrm{Re}(s)<k-1$, so $\zeta_F(2s)b(v,s)$ has a meromorphic continuation to this region. Moreover, if (v) is a square, from the functional equation of ζ_F we obtain the following:

$$\begin{aligned}
&d(F)^{3s/2}\Gamma_{\mathbf{R}}(s+1)^g\Gamma_{\mathbf{C}}(s+k-1)^g H(s)\\
&=2^{gs}d(F)^{3(1-s)/2}N(v)^{(k-2+s)/2}\pi^{-3g(1-s)/2}\\
&\cdot\left(\Gamma\left(\frac{2-s}{2}\right)\Gamma(k-s)\right)^g\zeta_F(2-2s)\\
&+2^{g(1-s)}d(F)^{3s/2}N(v)^{(k-1-s)/2}\pi^{-3gs/2}\\
&\cdot\left(\Gamma\left(\frac{s+1}{2}\right)\Gamma(s+k-1)\right)^g\zeta_F(2s),
\end{aligned}\tag{4.7}$$

which is invariant under $s\mapsto 1-s$. Combining this with Lemma 8 and (4.6), we see that:

$$d(F)^{3s/2}\Gamma_{\mathbf{R}}(s+1)^g\Gamma_{\mathbf{C}}(s+k-1)^g\zeta_F(2s)b(v,s)$$

is a meromorphic function in $2-k<\mathrm{Re}(s)<k-1$, which is invariant under the substitution $s\mapsto 1-s$, for each $0\ll v\in\mathcal{O}$.

By (2.1) and (2.5) we have:

$$\zeta_F(2s)\,(\Psi_s,f)=\left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right)^g d(F)^{k-(1/2)}L_2(s+k-1,f)\tag{4.8}$$

for $\mathrm{Re}(s)>\frac{3}{2}$; note that

$$L_2(s,f)=\zeta_F(2s-2k+2)\sum_{\mathfrak{a}}a(\mathfrak{a}^2)N(\mathfrak{a})^{-s}$$

for $\mathrm{Re}(s)>k+\frac{1}{2}$, where $\mathfrak{a}$ runs over all non-zero integral ideals of F. This implies that

$$d(F)^{3s/2}\Gamma_{\mathbf{R}}(s+1)^g\Gamma_{\mathbf{C}}(s+k-1)^g L_2(s+k-1,f)\tag{4.9}$$

is meromorphically continued to $2-k<\mathrm{Re}(s)<k-1$ and is invariant under $s\mapsto 1-s$. On the other hand, the Euler product defining $L_2(s,f)$ converges absolutely and uniformly for $\mathrm{Re}(s)>k+\frac{1}{2}$, so the function (4.9) is holomorphic for $\mathrm{Re}(s)>\frac{3}{2}$. Hence the functional equation we have just proved implies that (4.9) is also holomorphically continued to $\mathrm{Re}(s)<-\frac{1}{2}$. Hence, to prove the holomorphy of (4.9) it is sufficient to see that $\zeta_F(2s)b(v,s)$ is holomorphic in the strip $-\frac{3}{4}<\mathrm{Re}(s)<\frac{7}{4}$. In this region $I_j(l,v,s)$ is holomorphic if $l^2\neq 4v$ [see (4.3)], and $L_F(s,l^2-4v)$ is holomorphic if l^2-4v is not a square [Proposition 1(5)]. So $\prod_{j=1}^{g} I_j(l,v,s)L_F(s,l^2-4v)$ is holomorphic in $-\frac{3}{4}<\mathrm{Re}(s)<\frac{7}{4}$ unless $l^2\geqq 4v$. If $l^2\gg 4v$.

then $L_F(s, l^2-4v)$ has at most a simple pole at $s=1$, but $\prod_{j=1}^{g} I_j(l, v, s)$ has a zero of order g at $s=1$ by (4.3) (as we have noted before), so $\prod_{j=1}^{g} I_j(l, v, s)L_F(s, l^2-4v)$ is holomorphic in $-\frac{3}{4}<\mathrm{Re}(s)<\frac{7}{4}$ whenever $l^2 \neq 4v$. In (4.7), the poles of $\zeta_F(2-2s)$ and $\zeta_F(2s)$ at $s=\frac{1}{2}$ cancel, so $H(s)$ is also holomorphic in this region. Thus (4.6) implies that $\zeta_F(2s)b(v, s)$ is holomorphic in $-\frac{3}{4}<\mathrm{Re}(s)<\frac{7}{4}$ for each $0 \ll v \in \mathcal{O}$. This completes the proof of Theorem 1.

5. Proofs of Theorems 2 and 3

From now on suppose r is an odd integer such that $3 \leqq r \leqq k-1$. Then by (2.20) we have $\prod_{j=1}^{g} I_j(l, v, r)=0$ unless $l^2 \ll 4v$. If $l^2 \ll 4v$, by Erdélyi et al. [2, p. 176, (5)] [2, p. 64, (22)],

$$I_j(l, v, r)=(-1)^{\frac{k-r-1}{2}}\left(2\pi\frac{\sqrt{v^{(j)}}}{\delta^{(j)}}\right)^{r-1} 2^{2r-1}\Gamma(k-r)\Gamma(r)\Gamma(r+k-1)^{-1} \cdot\left(1-\frac{(l^{(j)})^2}{4v^{(j)}}\right)^{r-(1/2)} C^r_{k-r-1}\left(\frac{l^{(j)}}{2\sqrt{v^{(j)}}}\right),$$

where C^r_{k-r-1} is the Gegenbauer polynomial. In the notation of Zagier [23], this may also be written:

$$I_j(l, v, r)=(-1)^{\frac{k-r-1}{2}}\left(\frac{2\pi}{\delta^{(j)}}\right)^{r-1} (v^{(j)})^{\frac{1-k}{2}}\Gamma(k-r)\Gamma(r)\Gamma(r+k-1)^{-1} \cdot (4v^{(j)}-(l^{(j)})^2)^{r-(1/2)} p_{k,r}(l^{(j)}, v^{(j)}), \tag{5.1}$$

where $p_{k,r}(a, b)$ is the coefficient of x^{k-r-1} in $(1-ax+bx^2)^{-r}$. On the other hand, by Proposition 1 in § 3, we have

$$L_F(r, l^2-4v)=N(4v-1^2)^{(1/2)-r}d(F)^{(1/2)-r} \cdot\left((-1)^{\frac{r-1}{2}}\pi^r 2^{r-1}\Gamma(r)^{-1}\right)^g L_F(1-r, l^2-4v)$$

if $l^2 \ll 4v$. Hence

$$\prod_{j=1}^{g} I_j(l, v, r)L_F(r, l^2-4v)=\left((-1)^{\frac{k-2}{2}} 2^{2r-2}\pi^{2r-1}\frac{\Gamma(k-r)}{\Gamma(r+k-1)}\right)^g d(F)^{(3/2)-2r} \cdot N(v)^{\frac{1-k}{2}} N(p_{k,r}(l, v))L_F(1-r, l^2-4v)$$

if $l^2 \ll 4v$.

Let

$$E_k(z)=\sum_{\{c,d\}} N(cz+d)^{-k} \quad (z \in \mathfrak{H}^g)$$

be the Eisenstein series of weight k with respect to $\mathrm{SL}(2, \mathcal{O})$, where (c, d) runs over the last rows of matrices which form a complete system of representatives for

$\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\middle|\, \lambda\in\mathcal{O}^{\times}\right\}\Big\backslash \mathrm{SL}(2,\mathcal{O})$. Let

$$E_k(z)=1+\sum_{0\ll v\in\mathcal{O}} a(v,E_k)\mathbf{e}\left(\mathrm{tr}\left(\frac{v}{\delta}z\right)\right)$$

be the Fourier expansion. As is well-known, in (2.12) we have

$$(-1)^{gk/2}\left(\frac{(2\pi)^k}{\Gamma(k)}\right)^g d(F)^{(1/2)-k}\frac{\sigma_{k-1}((v))}{\zeta_F(k)}=a(v,E_k).$$

Thus by (2.11), (2.12), (2.21), and Proposition 2 we obtain:

$$\begin{aligned} b(v,r)=&(-1)^g 2^{(2r-1)g-1}\pi^{2rg}\left(\frac{\Gamma(k-r)}{\Gamma(r+k-1)}\right)^g d(F)^{(1/2)-2r}\\ &\cdot\sum_{l^2\ll 4v} N(p_{k,r}(l,v))\frac{L_F(1-r,l^2-4v)}{\zeta_F(2r)}-\delta_{r,k-1}\cdot\frac{1}{2}a(v,E_k)\\ &+\begin{cases} N(v)^{\frac{k-1-r}{2}} & \text{if } (v) \text{ is a square},\\ 0 & \text{otherwise}.\end{cases}\end{aligned}$$

Using

$$N(p_{k,r}(\pm 2\sqrt{v},v))=N(v)^{\frac{k-1-r}{2}}\left(\frac{\Gamma(k+r-1)}{\Gamma(k-r)\Gamma(2r)}\right)^g$$

if v is a square, we have

$$\begin{aligned} b(v,r)=&(-1)^g 2^{(2r-1)g-1}\pi^{2rg}\left(\frac{\Gamma(k-r)}{\Gamma(k+r-1)}\right)^g d(F)^{(1/2)-2r}\\ &\cdot\sum_{l^2\leqq 4v} N(p_{k,r}(l,v))\frac{L_F(1-r,l^2-4v)}{\zeta_F(2r)}\\ &-\delta_{r,k-1}\cdot\frac{1}{2}a(v,E_k),\quad \text{for each}\quad 0\ll v\in\mathcal{O}. \end{aligned}\tag{5.2}$$

We note that (5.2) is valid also for $v=0$ with $b(0,r)=0$ and $a(0,E_k)=1$ from the functional equation of ζ_F. Hence by (5.2) and (2.6),

$$\begin{aligned}\Psi_r=&(-1)^g 2^{(2r-1)g-1}\pi^{2rg}\left(\frac{\Gamma(k-r)}{\Gamma(k+r-1)}\right)^g d(F)^{(1/2)-2r}\\ &\cdot\zeta_F(2r)^{-1}\mathscr{C}_{k,r}-\delta_{r,k-1}\cdot\frac{1}{2}E_k,\end{aligned}\tag{5.3}$$

where

$$\mathscr{C}_{k,r}(z)=\sum_{0\leqq v\in\mathcal{O}}\left\{\sum_{\substack{l\in\mathcal{O}\\ l^2\leqq 4v}} N(p_{k,r}(l,v))L_F(1-r,l^2-4v)\right\}\mathbf{e}\left(\mathrm{tr}\left(\frac{v}{\delta}z\right)\right)\tag{5.4}$$

with $z\in\mathfrak{H}^g$. In particular we have:

Proposition 3. *The notation being as above, let k be an even integer* $\geqq 4$ *and let* $\mathscr{C}_{k,r}$ *be as in* (5.4) *for each odd integer r with* $3 \leqq r \leqq k-1$. *Then* $\mathscr{C}_{k,r}$ *is a modular form of weight k with respect to* $\mathrm{SL}(2,\mathcal{O})$. *For* $r<k-1$, *it is a cusp form.*

Remark 2. This proposition for $F=\mathbf{Q}$ was obtained by Cohen [1] and Zagier [23] by different methods.

Observe that the Fourier coefficients of $\mathscr{C}_{k,r}$ and Ψ_r (for odd integers r such that $3 \leqq r \leqq k-1$) are rational numbers, since

$$L_F(1-r, l^2-4v) \in \mathbf{Q} \quad \text{for} \quad l^2 \leqq 4v,$$

and

$$\frac{d(F)^{1/2}\pi^{mg}}{\zeta_F(m)} \in \mathbf{Q} \quad \text{for even} \quad m \geqq 2, \tag{5.5}$$

by Siegel [19, pp. 545–546] [20], Klingen [9], and Shintani [18]. Hence, by Shimura [17, Proposition 4.15], for each normalized eigen cusp form $f \in S_k(\mathrm{SL}(2,\mathcal{O}))$ with even $k \geqq 4$ we have

$$\left[\frac{(f,\Psi_r)}{(f,f)}\right]^\sigma = \frac{(f^\sigma,\Psi_r)}{(f^\sigma,f^\sigma)} \quad \text{for all} \quad \sigma \in \mathrm{Aut}(\mathbf{C}). \tag{5.6}$$

By (4.8),

$$L_2(r+k-1,f) = \omega_r \pi^{(k-1+2r)g}(f,\Psi_r) \tag{5.7}$$

with

$$\omega_r = \left(\frac{2^{2k-2}}{\pi^{2r}\Gamma(k-1)}\right)^g d(F)^{(1/2)-k}\zeta_F(2r).$$

Note that $\omega_r \in \mathbf{Q}$ by (5.5). So (5.6) and (5.7) imply Theorem 2 for $k<m \leqq 2k-2$, m even. Theorem 2 for $m=k$ is a direct consequence of the classical

$$L_2(k,f) = \frac{1}{2}\left(\frac{2^{2k}\pi^{k+1}}{\Gamma(k)}\right)^g d(F)^{-1-k}(f,f), \tag{5.8}$$

which is proved by the usual Ranking-Selberg method.

As in Zagier [23], letting $s=1$ in (4.6) we obtain Theorem 3; for reader's convenience we include a proof here. Let $f_1, \ldots, f_m$ be a basis of $S_k(\mathrm{SL}(2,\mathcal{O}))$ over $\mathbf{C}$ such that each f_j is a normalized eigen cusp form. Let

$$f_j(z) = \sum_{0 \ll v \in \mathcal{O}} a_j((v))\mathbf{e}\left(\mathrm{tr}\left(\frac{v}{\delta}z\right)\right)$$

be the Fourier expansion of f_j ($j=1,\ldots,m$). Comparing the Fourier coefficient at (v) of

$$\Psi_s = \sum_{j=1}^m \frac{(\Psi_s, f_j)}{(f_j, f_j)} f_j$$

for each $0 \ll \nu \in \mathcal{O}$, we have

$$\zeta_F(2s)b(\nu, s) = \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right)^g d(F)^{k-(1/2)} \sum_{j=1}^{m} \frac{L_2(s+k-1, f_j)}{(f_j, f_j)} a_j((\nu)) \tag{5.9}$$

by (4.8). As we have seen, the both sides of (5.9) are holomorphic function in $2-k < \mathrm{Re}(s) < k-1$. Putting $s=1$, by (5.8) we have

$$\zeta_F(2)b(\nu, 1) = \frac{1}{2}\left(\frac{(2\pi)^2}{k-1}\right)^g d(F)^{-3/2} \sum_{j=1}^{m} a_j((\nu)),$$

i.e.,

$$\mathrm{tr}(T((\nu))) = 2\left(\frac{k-1}{(2\pi)^2}\right)^g d(F)^{3/2}\zeta_F(2)b(\nu, 1). \tag{5.10}$$

(1) The case $F \neq \mathbf{Q}$.

By the assumption $g > 1$, from (2.20) we have

$$\left\{\prod_{j=1}^{g} I_j(l, \nu, 1)\right\} L_F(1, l^2 - 4\nu) = 0$$

unless $l^2 \ll 4\nu$. Thus by (4.6),

$$\zeta_F(2)b(\nu, 1) = (-1)^{gk/2} d(F)^{-1} N(\nu)^{\frac{k-1}{2}} 2^{g-1}\pi^g \sum_{\substack{l \in \mathcal{O} \\ l^2 \ll 4\nu}} \left\{\prod_{j=1}^{g} I_j(l, \nu, 1)\right\} L_F(1, l^2 - 4\nu)$$

$$+ \begin{cases} N(\nu)^{\frac{k-2}{2}} \zeta_F(2) & \text{if } (\nu) \text{ is a square}, \\ 0 & \text{otherwise}. \end{cases} \tag{5.11}$$

By (5.1) which is also valid for $r=1$, we have

$$\prod_{j=1}^{g} I_j(l, \nu, 1) = (-1)^{\left(\frac{k}{2}-1\right)^g} N(\nu)^{\frac{1-k}{2}} \left(\frac{1}{k-1}\right)^g N(4\nu - l^2)^{1/2} N(\mathfrak{p}_{k,1}(l, \nu)). \tag{5.12}$$

The residue formula for the Dedekind zeta functions and the relation

$$d(F(\sqrt{l^2-4\nu})) = N(d(F(\sqrt{l^2-4\nu})/F))\mathfrak{f}(F)^2$$

yields, as in Zagier [23],

$$L_F(1, l^2 - 4\nu) = (2\pi)^g d(F)^{-1/2} N(4\nu - l^2)^{-1/2} \frac{h(F(\sqrt{l^2-4\nu}))}{w(F(\sqrt{l^2-4\nu}))}$$

$$\cdot \sum_{\mathfrak{c} | \mathfrak{f}} N(\mathfrak{c}) \prod_{\mathfrak{p} | \mathfrak{c}} (1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-1}) \tag{5.13}$$

for $l^2 \ll 4\nu$. Here we used the equality $R(F(\sqrt{l^2-4\nu}))/R(F) = 2^{g-1}$ which follows from (1.1), where $R(L)$ denotes the regulator of a number field L. Substituting (5.12) and (5.13) into (5.11), we obtain (1.4) by (5.10).

(2) The case $F = \mathbf{Q}$.

In this case, by (4.6) we have:

$$\zeta(2)b(v,1) = [\text{the right-hand side of (5.11) with } F=\mathbf{Q}]$$

$$+(-1)^{k/2}2\pi^{3/2}v^{\frac{k-1}{2}}\frac{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{k}{2}\right)}\lim_{s\to 1}\frac{\zeta(2s-1)}{\Gamma\left(\frac{1-k+s}{2}\right)}$$

$$+\lim_{s\to 1}\sum_{\substack{l\in\mathbf{Z}\\ l^2-4v=u^2\\ 0<u\in\mathbf{Z}}} I_1(l,v,s)L_{\mathbf{Q}}(s,l^2-4v)\,. \tag{5.14}$$

In (4.3), we note that

$$F\left(\frac{k-1}{2},\frac{k}{2};k;\left(1-\frac{l^2}{4v}\right)^{-1}\right)=\left(\frac{u+|l|}{2}\right)^{1-k}u^{k-1}=v^{1-k}u^{k-1}\left(\frac{|l|-u}{2}\right)^{k-1},$$

if $l^2-4v=u^2$ with $0<u\in\mathbf{Z}$ (see Erdélyi et al. [2, 2.8, (6)]). Since

$$\lim_{s\to 1}\frac{\zeta(2s-1)}{\Gamma\left(\frac{1-k+s}{2}\right)}=-(-1)^{k/2}\frac{\Gamma\left(\frac{k}{2}\right)}{4},$$

we obtain (see also Proposition 1(5) in § 3 above)

$$\zeta(2)b)b(v,1)=[\text{the right-hand side of (5.11) with } F=\mathbf{Q}]$$

$$-\frac{\pi^2}{k-1}\sum_{\substack{l\in\mathbf{Z}\\ l^2-4v=u^2\\ 0\leq u\in\mathbf{Z}}}\left(\frac{|l|-u}{2}\right)^{k-1}.$$

The last summation may also be written: $\sum_{\substack{dd'=v\\ 0<d,d'\in\mathbf{Z}}}\min(d,d')^{k-1}$ (cf. Zagier [23]). This, together with (5.10), yields (1.5).

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On the Singularities of the Solution to the Cauchy Problem with Singular Data in the Complex Domain

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Introduction

This paper is a subsequent one to the author's [8]. The primary objective here is to study the holomorphy of an integral over a certain relative cycle depending on parameters for applications to the study of the singularities of the solution to the Cauchy problem with singular data.

Let $a(x,\partial)$ $(x=(x_0,\ldots,x_n)\in\mathbb{C}^{n+1})$ be a linear differential operator with holomorphic coefficients and the initial plane $x_0=0$ be non-characteristic. We consider the Cauchy problem near the origin of $\mathbb{C}^{n+1}$ with singular data:

$$\begin{cases} a(x,\partial)u(x)=0\,, \\ \partial_0^k u(0,x')=w_k(x')\,, \quad 0\leqq k\leqq \operatorname{ord} a\,, \quad (x'=(x_1\ldots x_n)) \end{cases} \tag{0.1}$$

where w_k's have poles along a non-singular hypersurface in the initial plane, say $x_0=x_1=0$.

For an operator with involutive characteristics (the multiplicity is arbitrary) we showed in [8] that the solution of (0.1) is expressed by a finite sum of functions of the following form:

$$J(x)=\int_0^{x_0} dt_m \int_0^{t_m} dt_{m-1}\ldots\int_0^{t_2} dt_1 F(t,x)\,, \quad (t=(t_1,\ldots,t_m)\in\mathbb{C}^m)\,, \tag{0.2}$$

where m $(\geqq 0)$ is smaller than the multiplicity and the integrand F is many-valued and holomorphic outside the zeros of a multi-phase function $\varphi=\varphi(t,x)$, which is holomorphic near the origin of $\mathbb{C}^{m+n+1}$ and $\varphi(0,0,x')=x_1$. [When $m=0$, (0.2) means $J(x)=F(x)$ and we do not need to consider any more.] The same result was originally proved by Hamada and Nakamura [6] for an operator of multiplicity at most double.

It is clear that the iterated integral (0.2) defines a germ of holomorphic function at a point $(0,p')\in\mathbb{C}^{n+1}$ with p_1 $(=\varphi(0;0,p'))\neq 0$. Indeed, choose a small open polydisk in $\mathbb{C}^{m+n+1}$ centered at $(0;0,p')$ such that φ never vanishes on it and then integrate iteratively there. We deal with the question where the germ at $(0,p')$ is analytically continued near the origin of $\mathbb{C}^{n+1}$.

For an operator of constant multiplicity (a special case of involutive characteristics) it immediately follows from the expression (0.2) that $J(x)$ has singularities only along a characteristic surface issuing from $x_0 = x_1 = 0$, because the multi-phase function φ is then independent of the variables t. This yields the results of [4] and [9] (see also [5] for systems of constant multiplicity).

Let us see another simple case when $m=1$, that is, the integral in (0.2) is a contour integral:

$$J(x) = \int_0^{x_0} F(t_1, x) dt_1 . \tag{0.3}$$

Obviously $J(x)$ is holomorphic as long as the contour from $t_1 = 0$ to $t_1 = x_0$ can be continuously deformed escaping the singularities of the integrand F. Therefore the singularities of $J(x)$ come from the following three types of obstructions:

a) (end point singularities) Either one of the end points of the contour is a singular point of the integrand F.

b) (pinching singularities) The contour is pinched between two singular points of the integrand F.

c) (pinching singularities at the infinity) The contour is pinched between a singular point of the integrand F and the infinity (or the boundary of the domain of F).

In [6], a)–c) correspond to K^+ or K^- (usual characteristic surfaces), K_0 (spanned by 2-families of bicharacteristics) and the exceptional cases (cf. [6, Examples 5.5–5.7]), respectively.

We shall show that the singularities of $J(x)$ come from essentially the same obstructions as above even if m is greater than one. More precisely, let Δ denote the standard m-simplex

$$\{s = (s_1, \ldots, s_m) \in \mathbb{R}^m | 0 \leqq s_i, s_1 + \ldots + s_m \leqq 1\} .$$

We define a singular m-simplex $\alpha(x) : \Delta \to \mathbb{C}^{m+n+1}$ by

$$\alpha(x)(s) = (s_1 x_0, (s_1 + s_2) x_0, \ldots, (s_1 + \ldots + s_m) x_0, x) \in \mathbb{C}^m \times \mathbb{C}^{n+1}$$

where x are parameters varying in a small neighborhood of $(0, p')$. Then in place of the iterated integral in (0.2), the germ at $(0, p')$ is given by the multiple integral

$$J(x) = \int_{\alpha(x)} F(t, x) dt_1 \wedge \ldots \wedge dt_m \left(= \int_\Delta \alpha(x)^*(F dt_1 \wedge \ldots \wedge dt_m) \right). \tag{0.4}$$

Note here that the singular m-simplex $\alpha(x)$ above satisfies

1) $\alpha(x)$ varies continuously in x.

2) Image $\alpha(x)$ does not intersect the singularities of the integrand F and is contained in the fibre over x of the natural projection $\mathbb{C}^m \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$.

3) For every $i = 1, \ldots, m+1$, Image $\alpha(x)|_{\dot{\Delta}_i}$ is confined in $W_i = \{t_i = t_{i-1}\}$ $(t_0 = 0, t_{m+1} = x_0)$ where $\dot{\Delta}_i = \{s \in \Delta | s_i = 0\}$ for $i = 1, \ldots, m$ and $\dot{\Delta}_{m+1} = \{s \in \Delta | s_1 + \ldots + s_m = 1\}$.

Conversely we shall show in Sect. 1 that the germ at $(0, p')$ can be analytically continuable as long as $\alpha(x)$ can be deformed satisfying the above conditions. This is analogous to the studies of Fotiadi et al. [1] and Pham [14]. There, integrals over absolute cycles were studied by using the first isotopy lemma of Thom. To apply the lemma they assumed that projections were proper. Integrals over relative cycles concerned here are somewhat delicate than those over absolute cycles. Moreover, in our situation the projection is not proper in general. So we use, instead of Thom's isotopy lemma, a local and relative version of it [3]. We recall briefly the lemmas in Sect. 2. The main results are stated in Sect. 3 (Theorem 3.2). The solution of (3.1) has singularities only along $\Sigma_0 \cup \Sigma_\infty$ in a neighborhood of the origin of $\mathbb{C}^{n+1}$. Σ_0 and Σ_∞ correspond to a) or b), and c) respectively. Unfortunately the geometrical properties of Σ_∞ are very complicated and are open as yet. However, Σ_0 is closely related to bicharacteristics. In Sect. 3 the simplest and basic relations between them are also stated.

Recently Schiltz, Vaillant, and Wagschal [15] have obtained the similar results for the Cauchy problem with ramified initial data. They assume that the characteristics are involutive and its multiplicity is at most triple. They study the singularities of the solution under some conditions on the multi-phase functions by following the method of Leray [10], which is different from ours.

1. Analyticity of an Integral over a Relative Cycle

Let $T \subset \mathbb{C}^m$ and $X \subset \mathbb{C}^{n+1}$ be two domains and S a closed subset of the Cartesian product $T \times X$. We put $Y = T \times X - S$ and assume Y to be connected. Let W_i, $i = 1, \ldots, m+1$, be the hyperplanes in Y defined by

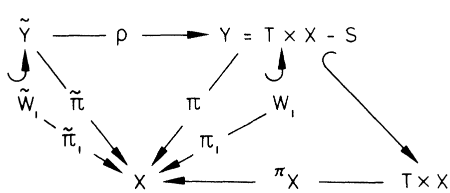
$$W_i = \{(t, x) \in Y \mid t_i - t_{i-1} = 0\},$$

where $t_0 = 0$ and $t_{m+1} = x_0$ (from now on we use this notation).

Let $(\tilde{Y}, \varrho)$ be a covering space of Y where $\varrho : \tilde{Y} \to Y$ is the covering map [i.e., for any point $p \in Y$ there is an open neighborhood U of p such that every connected component of $\varrho^{-1}(U)$ is homeomorphic to U]. Then $\tilde{Y}$ becomes naturally a complex manifold of complex dimension $m+n+1$ and $\varrho : \tilde{Y} \to Y$ is locally biholomorphic. Put $\tilde{t}_i = t_i \circ \varrho$, $i = 0, \ldots, m+1$, $\tilde{x}_j = x_j \circ \varrho$, $j = 0, \ldots, n$, (which are holomorphic functions on $\tilde{Y}$) and $\{\tilde{t}_1, \ldots, \tilde{t}_m, \tilde{x}_0, \ldots, \tilde{x}_n\}$ forms a local coordinate system at any point of $\tilde{Y}$. Let us put

$$\tilde{W}_i = \varrho^{-1}(W_i), \quad i = 1, \ldots, m+1,$$

which are complex submanifolds of $\tilde{Y}$ of complex codimension one with defining equation $\tilde{t}_i - \tilde{t}_{i-1} = 0$. We put $\tilde{\pi} = \pi \circ \varrho : \tilde{Y} \to X$ and $\tilde{\pi}_i = \tilde{\pi}|_{W_i} : \tilde{W}_i \to X$ where $\pi : Y \to X$ is the restriction of the natural projection $\pi_X : T \times X \to X$ to Y. We refer the reader to the following commutative diagram:



(1.1)

The fibre $\tilde{Y}_x = \tilde{\pi}^{-1}(x)$ is a complex manifold of complex dimension m and

$$\tilde{W}_{i,x} = \tilde{W}_i \cap \tilde{Y}_x = \tilde{\pi}_i^{-1}(x), \quad i = 1, \ldots, m+1,$$

are complex submanifolds of $\tilde{Y}_x$ of complex codimension one for every $x \in X$ (we assume that $\tilde{Y}_x$ and $\tilde{W}_{i,x}$'s are not empty for every $x \in X$), for $\tilde{\pi}: \tilde{Y} \to X$ and $\tilde{\pi}_i: \tilde{W}_i \to X$ are holomorphic submersions. The same is true for $Y_x = \pi^{-1}(x)$ and $W_{i,x} = W_i \cap Y_x = \pi_i^{-1}(x)$.

Let ω be a holomorphic m-form on $\tilde{Y}$ of the following form:

$$\omega = F \cdot d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_m (= F \cdot \varrho^*(dt_1 \wedge \ldots \wedge dt_m)), \tag{1.2}$$

where F is a single-valued holomorphic function on $\tilde{Y}$ and the superscript "*" means "pull back". Put $\omega_x = \omega|_{Y_x}$ (the restriction, i.e., pull back, of ω to $\tilde{Y}_x$). Then comparing the degree of $d\omega_x$ (resp. ω_x) with the dimension of $\tilde{Y}_x$ (resp. $\tilde{W}_{i,x}$) we have for every $x \in X$

$$d\omega_x (= d\omega|_{Y_x}) = 0, \qquad \omega_x|_{\tilde{W}_{i,x}} = 0, \qquad i = 1, \ldots, m+1. \tag{1.3}$$

Before going into details we recall a result from the homology theory. Let Δ_q denote the standard q-simplex:

$$\Delta_q = \{(s_1, \ldots, s_q) \in \mathbb{R}^q | 0 \leqq s_i, s_1 + \ldots + s_q \leqq 1\}.$$

When M is a smooth manifold, a singular q-simplex $\sigma: \Delta_q \to M$ is said to be smooth if there is an open set $U \subset \mathbb{R}^q$ containing Δ_q and an extension $\sigma': U \to M$ of σ which is smooth. The smooth singular simplexes form a chain subcomplex $C^s_\cdot(M)$ of the total singular chain complex $C_\cdot(M)$. We denote by $H^s_*(\tilde{Y}_x, \{\tilde{W}_x\})$ the holology group of the chain complex $C^s_\cdot(\tilde{Y}_x)/C^s_\cdot\{\tilde{W}_x\}$, where $\tilde{W}_x = \bigcup_{i=1}^{m+1} \tilde{W}_{i,x}$ and

$$C^s_q\{\tilde{W}_x\} = C^s_q(\tilde{W}_{1,x}) + \ldots + C^s_q(\tilde{W}_{m+1,x}) (\subset C^s_q(\tilde{Y}_x) \cap C_q(\tilde{W}_x)).$$

Since the intersections $\bigcap_{i \in A} \tilde{W}_{i,x}$ are smooth submanifolds of $\tilde{Y}_x$ for all subsets $A \subset \{1, \ldots, m+1\}$ (and for every $x \in X$), we know from the classical homology theory that the inclusion map of chain complexes,

$$C^s_\cdot(\tilde{Y}_x)/C^s_\cdot\{\tilde{W}_x\} \to C_\cdot(\tilde{Y}_x)/C_\cdot(\tilde{W}_x)$$

induces an isomorphism of homology groups,

$$H^s_*(\tilde{Y}_x, \{\tilde{W}_x\}) \simeq H_*(\tilde{Y}_x, \tilde{W}_x), \tag{1.4}$$

where the right side is the usual relative singular homology. The above statement is also true for any coefficient groups. In what follows only the homology groups with coefficient group $\mathbb{C}$ will be used, so we will still denote by $H^s_*(\tilde{Y}_x, \{\tilde{W}_x\}), \ldots$, the homology groups with coefficient group $\mathbb{C}$.

The isomorphism (1.4) means that we can take a smooth relative cycle as a representative of a relative homology class. More precisely, let $h_q(x) \in H_q(\tilde{Y}_x, \tilde{W}_x)$ be a relative homology class, then there is a smooth relative cycle $\beta^s_q(x) \in C^s_q(\tilde{Y}_x)$ which is a representative of $h_q(x)$, and if a relative cycle $\gamma^s_q(x) \in C^s_q(\tilde{Y}_x)$ is another representative of $h_q(x)$, then

$$\beta^s_q(x) - \gamma^s_q(x) \in \partial C^s_{q+1}(\tilde{Y}_x) + C^s_q\{\tilde{W}_x\}.$$

Consequently, combining these facts and (1.3), given a relative homology class

$$h_m(x) \in H_m(\tilde{Y}_x, \tilde{W}_x),$$

we can define the integral of ω over $h_m(x)$ by

$$\int_{h_m(x)} \omega_x = \int_{\beta^s_m(x)} \omega_x \left(= \int_{\Delta_m} \beta^s_m(x)^* \omega \right),$$

which is independent of the choice of the representatives $\beta^s_m(x) \in C^s_m(\tilde{Y}_x)$ of $h_m(x)$. For a relative cycle $\beta_m(x)$ of $(\tilde{Y}_x, \tilde{W}_x)$ we write $\int_{\beta_m(x)} \omega_x$ in place of $\int_{h_m(x)} \omega_x$ where $h_m(x) \in H_m(\tilde{Y}_x, \tilde{W}_x)$ is the homology class of $\beta_m(x)$.

Now we restrict ourselves to a certain family of homology classes which are related to the iterated integral (0.2). Let Δ $(=\Delta_m)$ be the standard m-simplex and $\dot{\Delta}_i$, $i=1, \dots, m+1$, the i-faces of Δ:

$$\dot{\Delta}_i = \begin{cases} \{s \in \Delta | s_i = 0\}, & i=1, \dots, m, \\ \{s \in \Delta | s_1 + \dots + s_m = 1\}, & i = m+1. \end{cases}$$

Let $f: M \to N$ be a map and $\{A_j\}$ and $\{B_j\}, j=1, \dots, \bar{j}$, families of subsets of M and N respectively. If f maps A_j into B_j for all $j=1, \dots, \bar{j}$, we write $f:(M, A_j) \to (N, B_j)$ for simplicity. Two maps $f, g:(M, A_j) \to (N, B_j)$ are said to be $(A_j; B_j)$-homotopic, denoted by $f \simeq g(M, A_j; N, B_j)$, if there is a homotopy $h: I \times M \to N$ between f and g such that $h:(I \times M, I \times A_j) \to (N, B_j)$ where I is the closed interval $[0, 1]$.

Let $\tilde{\alpha}(x): \Delta \to \tilde{Y}$, $x \in U$, be a family of singular m-simplexes where U is an open subset of X. Suppose that $\tilde{\alpha}(x)$ satisfies for every $x \in U$,

i) *$\tilde{\alpha}(x)$ maps Δ into the fibre $\tilde{Y}_x$,*

ii) *$\tilde{\alpha}(x)$ maps $\dot{\Delta}_i$ into the fibre $\tilde{W}_{i,x}$ for all $i=1, \dots, m+1$.*

Namely $\tilde{\alpha}(x):(\Delta, \dot{\Delta}_i) \to (\tilde{Y}_x, \tilde{W}_{i,x})$ for all $x \in U$. In particular $\tilde{\alpha}(x) \in C_m(\tilde{Y}_x)$ is a relative cycle of $(\tilde{Y}_x, \tilde{W}_x)$, hence

$$J(x) = \int_{\tilde{\alpha}(x)} \omega_x \left(= \int_{\tilde{\alpha}(x)} \omega \right) \tag{1.5}$$

is well-defined as a function on U. We show that $J(x)$ is holomorphic if $\tilde{\alpha}(x)$ varies continuously in x.

Lemma 1.1. *Suppose that a family of singular m-simplexes $\tilde{\alpha}(x): \Delta \to \tilde{Y}$, $x \in U$, satisfies the conditions* i) *and* ii) *above and is continuous in x, that is,*

iii) *$\tilde{\alpha}: U \times \Delta \ni (x, s) \to \tilde{\alpha}(x)(s) \in \tilde{Y}$ is continuous in* (x, s). *Then the function $J(x)$ defined by* (1.5) *is holomorphic on U.*

Proof. For the moment we assume that $\tilde{\alpha}$ is piecewise smooth, that is, there is a subdivision $\Delta = \bigcup_{k=1}^{\bar{k}} \Delta(k)$ where $\Delta(k)$'s are m-subsimplexes of Δ such that the restrictions of $\tilde{\alpha}$ to $U \times \Delta(k)$ are smooth for all $k=1, \dots, \bar{k}$.

By Hartog's theorem, it suffices to prove that $J(x)$ is holomorphic in each variable separately. We show that $J(x)$ is holomorphic in x_0. Let $x \in X$ be fixed and $z \in \mathbb{C}$ small enough. Define a singular m-prism $\tilde{\beta} = \tilde{\beta}_z:(I \times \Delta, I \times \dot{\Delta}_i) \to (\tilde{Y}, \tilde{W}_i)$ by $\tilde{\beta}(r, s) = \tilde{\alpha}(x_0 + rz, x')(s)$, which is piecewise smooth and $\tilde{\beta}(0, \cdot) = \tilde{\alpha}(x)$ and $\tilde{\beta}(1, \cdot) = \tilde{\alpha}(x_0 + z, x')$. Note that from the definition of ω and $\tilde{W}_i$, $\omega|_{\tilde{W}_i} = 0$ for $i = 1, \dots, m$. Now applying Stoke's theorem we have

$$\int_{\tilde{\beta}} d\omega \left(= \int_{\partial \tilde{\beta}} \omega \right) = J(x_0 + z, x') - J(x) + (-1)^m \int_{\tilde{\gamma}} \omega, \tag{1.6}$$

where $\tilde{\gamma}=\tilde{\gamma}_z: I \times \Delta_{m-1} \to \tilde{W}_{m+1}$ is a singular m-prism given by

$$\tilde{\gamma}(r, s') = \tilde{\beta}(r, s', 1-s_1-\ldots-s_{m-1}) \quad (s'=(s_1, \ldots, s_{m-1})),$$

which is also piecewise smooth. First we calculate the left side of (1.6). By the condition i) we have $\tilde{\beta}^* d\tilde{x}_0 = z\,dr$ and $\tilde{\beta}^* d\tilde{x}_i = 0$ for $i=1, \ldots, n$, hence

$$\int_{\tilde{\beta}} d\omega = z \int_{I \times \Delta} dr \wedge \tilde{\beta}^* \left(\frac{\partial F}{\partial \tilde{x}_0} d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_m \right).$$

Since $\tilde{\beta}=\tilde{\beta}_z$ converges to $\tilde{\alpha}(x)$ as z tends to 0, integrating with respect to dr we obtain

$$\int_{\tilde{\beta}} d\omega = z \int_{\tilde{\alpha}(x)} \frac{\partial F}{\partial \tilde{x}_0} d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_m + O(|z|^2). \tag{1.7}$$

On the other hand we have $\tilde{\gamma}^* d\tilde{t}_m = z\,dr$ by the condition ii), which yields

$$\int_{\tilde{\gamma}} \omega = (-1)^{m-1} \int_{I \times \Delta'} z\,dr \wedge \tilde{\gamma}^* (F\,d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_{m-1}),$$

where $\Delta' = \Delta_{m-1}$. Define a family of singular $(m-1)$-simplexes $\tilde{\alpha}'(x): \Delta' \to \tilde{W}_{i,x}$ by

$$\tilde{\alpha}'(x)(s') = \tilde{\alpha}(x)(s', 1-s_1-\ldots-s_{m-1}).$$

Then $\tilde{\gamma}=\tilde{\gamma}_z$ converges to $\tilde{\alpha}'(x)$ as z tends to 0, hence integrating with respect to dr we obtain

$$\int_{\tilde{\gamma}} \omega = (-1)^{m-1} z \int_{\tilde{\alpha}'(x)} F\,d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_{m-1} + O(|z|^2). \tag{1.8}$$

Combining (1.6), (1.7), and (1.8) we finally obtain

$$\frac{\partial J}{\partial x_0}(x) = \int_{\alpha(x)} \frac{\partial F}{\partial \tilde{x}_0} d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_m + \int_{\alpha'(x)} F\,d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_{m-1}.$$

Similarly we can prove the equalities

$$\frac{\partial J}{\partial x_j}(x) = \int_{\tilde{\alpha}(x)} \frac{\partial F}{\partial \tilde{x}_j} d\tilde{t}_1 \wedge \ldots \wedge d\tilde{t}_m, \quad j=1, \ldots, n.$$

Now only the first part remains to be verified. To do this, it is enough to show that for an arbitrary point $p \in X$ there is an open neighborhood U of p and a family of singular m-simplexes $\tilde{\beta}(x): (\Delta, \dot{\Delta}_i) \to (\tilde{Y}_x, \tilde{W}_{i,x})$, $x \in U$, such that $\tilde{\beta}(x)$ is piecewise smooth and is $(\Delta_i; \tilde{W}_{i,x})$-homotopic to $\tilde{\alpha}(x)$ for all $x \in U$ [*hence* $\tilde{\alpha}(x)$ and $\tilde{\beta}(x)$ belong to the same homology class of $(\tilde{Y}_x, \tilde{W}_x)$].

Note that since $\varrho: \tilde{Y} \to Y \ (\subset \mathbb{C}^{n+m+1})$ is locally biholomorphic, for two points $\tilde{p}, \tilde{q} \in \tilde{Y}$ close enough to each other the segment $\{(1-r)\tilde{p} + r\tilde{q}, r \in I\}$ joining $\tilde{p}$ and $\tilde{q}$ is meaningful. Further $\tilde{Y}_x$ and $\tilde{W}_{i,x}$'s are convex with respect to this operation for all $x \in X$. Therefore fixing a point $p \in X$ and using barycentric subdivision, we can take a singular m-simplex $\tilde{\beta}^0: (\Delta, \dot{\Delta}_i) \to (\tilde{Y}_p, \tilde{W}_{i,p})$ which is piecewise smooth in $s \in \Delta$ and is $(\dot{\Delta}_i; \tilde{W}_{i,p})$-homotopic to $\tilde{\alpha}(p)$. Then the proof is obvious by (i) and (ii) of the following lemma.

Lemma 1.2. (i) *Given a singular m-simplex* $\tilde{\alpha}^0:(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_p,\tilde{W}_{i,p})$ *with some point* $p\in X$, *then there exist an open neighborhood U of p and an extension* $\tilde{\alpha}(x):(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_x,\tilde{W}_{i,x})$ *of* $\tilde{\alpha}^0$ *(i.e.,* $\tilde{\alpha}(p)=\tilde{\alpha}^0$*) which is continuous in x. Further, if* $\tilde{\alpha}^0$ *is piecewise smooth in* $s\in\Delta$, *then we can take* $\tilde{\alpha}:U\times\Delta\ni(x,s)\to\tilde{\alpha}(x)(s)\in\tilde{Y}$ *to be also piecewise smooth in x and s.*

(ii) *Suppose that two families of singular m-simplexes*

$$\tilde{\alpha}(x),\tilde{\beta}(x):(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_x,\tilde{W}_{i,x}),\qquad x\in U,$$

is continuous in x and that $\tilde{\alpha}(p)$ *and* $\tilde{\beta}(p)$ *are* $(\dot{\Delta}_i;\tilde{W}_{i,p})$*-homotopic to each other with some point* $p\in U$. *Then* $\tilde{\alpha}(x)$ *and* $\tilde{\beta}(x)$ *are* $(\dot{\Delta}_i;\tilde{W}_{i,x})$*-homotopic for all x close to p.*

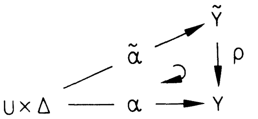
Proof. Put

$$\alpha(x,s)=(s_1(x_0-p_0),\\ (s_1+s_2)(x_0-p_0),\dots,(s_1+\dots+s_m)\times(x_0-p_0),x-p)+\alpha^0(s)\,(\in\mathbb{C}^{n+m+1}),$$

and

$$\alpha(x)=\alpha(x,\cdot)$$

where $p=(p_0,\dots,p_n)\,(\in\mathbb{C}^{n+1})$ and $\alpha^0=\varrho\circ\tilde{\alpha}^0:(\Delta,\dot{\Delta}_i)\to(Y_p,W_{i,p})$. Since $\alpha(p)=\alpha^0$ maps Δ into Y and Δ is compact, if an open set U containing p is sufficiently small α maps $U\times\Delta$ into Y, hence $\alpha(x):(\Delta,\dot{\Delta}_i)\to(Y_x,W_{i,x})$ for all $x\in U$. Since we may assume $U\times\Delta$ to be simply connected, there is a unique lifting $\tilde{\alpha}:U\times\Delta\to\tilde{Y}$ of α such that $\tilde{\alpha}(p,0)=\tilde{\alpha}^0(0)$ and the following diagram commutes:



Then $\tilde{\alpha}(x)=\tilde{\alpha}(x,\cdot):(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_x,\tilde{W}_{i,x})$ is a desired extension. From the construction of $\tilde{\alpha}(x)$ the second part of (i) is clear.

(ii) First we assume that $\tilde{\alpha}(p)=\tilde{\beta}(p)$ in place of $\tilde{\alpha}(p)\simeq\tilde{\beta}(p)\,(\Delta,\dot{\Delta}_i;\tilde{Y}_p,\tilde{W}_{i,p})$. Then, when an open neighborhood U of p is sufficiently small, for $(x,s)\in U\times\Delta$, $\tilde{\beta}(x)(s)$ is close enough to $\tilde{\alpha}(x)(s)$ so that the segment $\{(1-r)\tilde{\alpha}(x)(s)+r\tilde{\beta}(x)(s),r\in I\}$ joining $\tilde{\alpha}(x)(s)$ and $\tilde{\beta}(x)(s)$ is well-defined. Thus a map

$$I\times\Delta\ni(r,s)\to(1-r)\tilde{\alpha}(x)(s)+r\tilde{\beta}(x)(s)\in\tilde{Y}_x$$

is a $(\dot{\Delta}_i;\tilde{W}_{i,x})$-homotopy between $\tilde{\alpha}(x)$ and $\tilde{\beta}(x)$.

Next let $\tilde{\beta}(p)$ be $(\dot{\Delta}_i;\tilde{W}_{i,p})$-homotopic to $\tilde{\alpha}(p)$ and

$$h_r^0:(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_p,\tilde{W}_{i,p}),\qquad r\in I,$$

a homotopy between $\tilde{\alpha}(p)$ and $\tilde{\beta}(p)$ $(h_0^0=\tilde{\alpha}(p),h_1^0=\tilde{\beta}(p))$. For all $r\in I$ we extend h_r^0 to $h_r(x):(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_x,\tilde{W}_{i,x})$ in the same way as in the proof of (i). Since r varies on compact set I, we may assume that $h_r(x)$ are defined on the same open set U $(\ni p)$ for all $r\in I$. It is clear that $h_r(x)$ is a $(\dot{\Delta}_i;\tilde{W}_{i,x})$-homotopy between $h_0(x)$ and $h_1(x)$. Since both $\tilde{\alpha}(x)$ and $h_0(x):(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_x,\tilde{W}_{i,x})$ are continuous extensions of $\tilde{\alpha}(p)$ $(=h_0^0)=h_0(p)$, $\tilde{\alpha}(x)$, and $h_0(x)$ are $(\dot{\Delta}_i;\tilde{W}_{i,x})$-homotopic for all x close to p. The same is true for the pair $\{h_1(x), \tilde{\beta}(x)\}$. Consequently we have

$$\tilde{\alpha}(x) \simeq h_0(x) \simeq h_1(x) \simeq \tilde{\beta}(x)\,(\Delta, \dot{\Delta}_i;\, \tilde{Y}_x, \tilde{W}_{i,x})$$

for all x close to p. Q.E.D.

By Lemmas 1.1 and 1.2, given a singular m-simplex $\tilde{\alpha}^0 : (\Delta, \dot{\Delta}_i) \to (\tilde{Y}_p, \tilde{W}_{i,p})$ with a point $p \in X$, we can define at p a germ of holomorphic function by

$$\int_{\tilde{\alpha}(x)} \omega_x \left(= \int_{\tilde{\alpha}(x)} \omega \right), \tag{1.9}$$

which is independent of the choice of continuous extensions

$$\tilde{\alpha}(x) : (\Delta, \dot{\Delta}_i) \to (\tilde{Y}_x, \tilde{W}_{i,x})$$

of $\tilde{\alpha}^0$. So we denote the germ simply by $\int_{\tilde{\alpha}^0} \omega_p$. We are concerned with the analytic continuation of the germs given by (1.9). Now the next theorem is almost obvious.

Theorem 1.3. *Let a singular m-simplex $\tilde{\alpha}^0$ be $\tilde{\alpha}^0 : (\Delta, \dot{\Delta}_i) \to (\tilde{Y}_p, \tilde{W}_{i,p})$ with some point $p \in X$. Let $l : I \to X$ be a path with initial point p and terminal point $q = l(1)$. Suppose that $\tilde{\alpha}^0$ can be continuously deformed along the path l, namely there exists a family of singular m-simplexes $\tilde{\alpha}_r : (\Delta, \dot{\Delta}_i) \to (\tilde{Y}_{l(r)}, \tilde{W}_{i,l(r)})$ such that $\tilde{\alpha}_0 = \tilde{\alpha}^0$ and $I \times \Delta \ni (r, s) \to \tilde{\alpha}_r(s) \in Y$ is continuous. Then the germ of holomorphic function $\int_{\tilde{\alpha}^0} \omega_p$ at p is analytically continuable along the path to q and the germ at q is given by $\int_{\tilde{\alpha}_1} \omega_q$.*

Proof. For $r \in I$ let $\tilde{\alpha}_r(x) : (\Delta, \dot{\Delta}_i) \to (\tilde{Y}_x, \tilde{W}_{i,x})$ be a continuous extension of $\tilde{\alpha}_r$ $(\tilde{\alpha}_r(l(r)) = \tilde{\alpha}_r)$ where x varies in a small neighborhood of $l(r)$. To prove the theorem, it is enough to show that for a fixed $r^* \in I$ there is a constant $\varepsilon = \varepsilon(r^*) > 0$ such that for every $r \in I$, $|r - r^*| < \varepsilon$, $\tilde{\alpha}_r(x)$ is $(\dot{\Delta}_i, \tilde{W}_{i,x})$-homotopic to $\tilde{\alpha}_{r^*}(x)$ for all x close to $l(r)$. Since both $\tilde{\alpha}_r$ and $\tilde{\alpha}_{r^*}(l(r))$ vary continuously in r and $\tilde{\alpha}_{r^*} = \tilde{\alpha}_{r^*}(l(r^*))$, $\tilde{\alpha}_r(l(r))\ (= \tilde{\alpha}_r)$ is $(\dot{\Delta}, \tilde{W}_{i,l(r)})$-homotopic to $\tilde{\alpha}_{r^*}(l(r))$ for all r close to r^*. Now the proof is clear by (ii) of Lemma 1.2. Q.E.D.

Corollary 1.4. *The same notation as in Theorem 1.3. Put*

$$\alpha^0 = \varrho \circ \tilde{\alpha}^0 : (\Delta, \dot{\Delta}_i) \to (Y_p, W_{i,p}).$$

Suppose that there exists a family of singular m-simplexes

$$\alpha_r : (\Delta, \dot{\Delta}_i) \to (Y_{l(r)}, W_{i,l(r)})$$

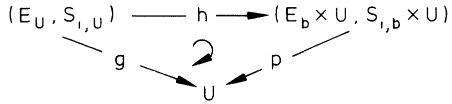
such that $\alpha_0 = \alpha^0$ and $I \times \Delta \ni (r, s) \to \alpha_r(s) \in Y$ is continuous. Then the germ of holomorphic function $\int_{\tilde{\alpha}^0} \omega_p$ is analytically continuable along the path l to q.

Proof. Put $\alpha(r, s) = \alpha_r(s)$, $(r, s) \in I \times \Delta$. Since $I \times \Delta$ is contractible, there is a lifting $\tilde{\alpha} : I \times \Delta \to \tilde{Y}$ of α such that $\tilde{\alpha}(0, 0) = \tilde{\alpha}^0(0)$ and $\alpha = \varrho \circ \tilde{\alpha}$. Then $\tilde{\alpha}_r = \tilde{\alpha}(r, \cdot)$ satisfies the conditions in Theorem 1.3. Q.E.D.

2. Isotopy Lemmas

In Sect. 1 we reduced the analytic problem of analytic continuation to a topological one concerning continuous deformation of singular simplexes. In this section we review topological lemmas which will be very powerful to study the reduced topological problem. First we recall the first isotopy lemma of Thom. To do this we introduce a notion of locally trivial fibration.

Definition. Let E and N be two topological spaces, $S_1, \ldots, S_k$ subsets of E and $g: E \to N$ a continuous map. We say that $g; E \to N$ is a *locally trivial fibration relative to* $S_1, \ldots, S_k$ if for any point $b \in N$ there is an open set $U \subset N$ containing b and a homeomorphism $h: E_U \to E_b \times U$ such that $h(S_{i,U}) = S_{i,b} \times U$ for all $i = 1, \ldots, k$ and the following diagram commutes:



where $E_U = g^{-1}(U)$, $E_b = g^{-1}(b)$, $S_{i,U} = E_U \cap S_i$, $S_{i,b} = E_b \cap S_i$, and p is the natural projection.

In what follows, for simplicity, E denotes a subset of $\mathbb{R}^d$ and N a C^∞-manifold.

The First Isotopy Lemma of Thom. *Let $\mathcal{S}(E)$ be a stratification of E and $\mathcal{S}(N) = \{N\}$ be the trivial one of N. If $g: E \to N$ is a proper stratified map with respect to $\mathcal{S}(E)$ and $\mathcal{S}(N)$ and if $S_1, \ldots, S_k$ are stratified subsets of E with respect to $\mathcal{S}(E)$, then $g: E \to N$ is a locally trivial fibration relative to $S_1, \ldots, S_k$.*

We briefly describe the terminologies used in the above theorem. For the details we refer to Mather [11] and Fukuda [2].

A continuous map $g: E \to N$ is said to be *proper* if the inverse image of any compact subset $K \subset N$, $g^{-1}(K)$, is compact. Let M and N be C^∞-manifolds in $\mathbb{R}^d$ and b a point in $\bar{M} \cap N$ ($\bar{M}$ denotes the closure of M in $\mathbb{R}^d$). The pair (M, N) is said to satisfy the *Whitney condition (b)* at b if for any sequence of pairs $(a_i, b_i) \in M \times N$ $(a_i \neq b_i)$ such that $(a_i, b_i, \widehat{a_i b_i}, T_{a_i}M)$ converges to a point $(b, b, l, \tau) \in \bar{M} \times N \times \mathbb{P}^{d-1} \times G_{d,r}$, we have $l \subset \tau$. Here $\widehat{a_i b_i}$ is the line through the origin of $\mathbb{R}^d$ which is parallel to the one joining a_i and b_i, and $G_{d,r}$ ($r = \dim M$) is the Grassmann manifold of r-planes in $\mathbb{R}^d$. We say the pair (M, N) satisfies the Whitney condition (b) if it satisfies the Whitney condition (b) at every point $b \in \bar{M} \cap N$.

Definition (stratification). A (Whitney) stratification of a subset $E \subset \mathbb{R}^d$ is a partition $\mathcal{S}(E) = \{M_\mu\}$ of E into connected manifolds satisfying the following:

1) The partition $E = \bigcup_\mu M_\mu$ is disjoint and locally finite.
2) All the pairs (M_μ, M_ν) of $\mathcal{S}(E)$ satisfy the Whitney condition (b).
3) If $\bar{M}_\mu \cap M_\nu \neq \emptyset$, then $M_\nu \subset \bar{M}_\mu - M_\mu$ $(\mu \neq \nu)$.

A member of $\mathcal{S}(E)$ is called a *stratum*. A subset S of E is said to be a *stratified subset* with respect to the stratification $\mathcal{S}(E)$ if S is a union of strata of $\mathcal{S}(E)$. If E is a manifold, then the stratification $\mathcal{S}(E) = \{E\}$ consisting of only one stratum E is called the *trivial stratification*. We say that a continuous map $g: E \to \mathbb{R}^h$ is smooth if g extends to a smooth map on a neighborhood of E.

Definition (stratified map). Let $\mathscr{S}(E)$ be a stratification of E, $N \subset \mathbb{R}^h$ a manifold and $\mathscr{S}(N)$ the trivial stratification. A smooth map $g: E \to N$ is called a stratified map with respect to $\mathscr{S}(E)$ and $\mathscr{S}(N)$ if the restriction of g on each stratum $M \in \mathscr{S}(E)$, $g|_M: M \to N$ is a submersion.

Let $\alpha^0: \Delta_{k-1} \to E$ be a singular $(k-1)$-simplex such that α^0 maps Δ (resp. $\dot{\Delta}_i$) into E_b (resp. $S_{i,b}$) with a point $b \in N$, i.e., $\alpha^0: (\Delta_{k-1}, \dot{\Delta}_i) \to (E_b, S_{i,b})$ where $\dot{\Delta}_i$, $i = 1, \ldots, k$, are faces of Δ_{k-1}. Suppose the hypotheses of the first isotopy lemma are satisfied for $(g, E, N, S_1, \ldots, S_k)$. Then α^0 can be deformed along any path l with initial point p, namely, there is a family of singular $(k-1)$-simplexes $\alpha_r: (\Delta_{k-1}, \dot{\Delta}_i) \to (E_{l(r)}, S_{i,l(r)})$, $r \in I = [0, 1]$, which is continuous in r and $\alpha_0 = \alpha^0$. Indeed, because $g: E \to N$ is a locally trivial fibration relative to $S_1, \ldots, S_k$, taking a homeomorphism $h: (E_U, S_{i,U}) \to (E_b \times U, S_{i,b} \times U)$ where U is a neighborhood of b, we put $\alpha_r(s) = h^{-1}(\alpha'(s), l(r))$ where $\alpha'(s)$ is the first component of $h(\alpha^0(s)) \subset E_b \times U$. Then α_r is a desired extension for $r \in I$ with $l(r) \in U$. Since I is compact, we arrive at the terminal point of l after a finite number of above steps.

The map $\pi_X: T \times X \to X$ in (1.1) is, however, not proper, so we can not apply the first isotopy lemma directly. Instead we give a local version of it. Let $U \subset \mathbb{R}^{d-h}$ and $N \subset \mathbb{R}^h$ be two open sets, $E = U \times N$ and $p_N: E \times N$ the natural projection. Note that p_N is not a proper map. Let $S_1, \ldots, S_k$ be subsets of E. We assume that there is a stratification $\mathscr{S}(E)$ of E such that $S_1, \ldots, S_k$ are stratified subsets with respect to $\mathscr{S}(E)$ and that $p_N: E \to N$ is a stratified map with respect to $\mathscr{S}(E)$ and the trivial stratification $\mathscr{S}(N)$. Let us fix an arbitrary non-negative smooth proper function $\delta_0: U \to \mathbb{R}$ and let $\delta: E \to \mathbb{R}$ be the function given by $\delta(u, b) = \delta_0(u)$ $((u, b) \in E = U \times N)$. We put for a positive number $r \geqq \min_{u \in U} \delta_0(u)$,

$$\left.\begin{aligned} &\bar{B}(r) = \{u \in U | \delta_0(u) \leqq r\}, \qquad B(r) = \{u \in U | \delta_0(u) < r\}, \\ &S(r) = \bar{B}(r) - B(r) = \{u \in U | \delta_0(u) = r\}. \end{aligned}\right\} \tag{2.1}$$

Obviously the restriction of p_N on $\bar{B}(r) \times N: \bar{B}(r) \times N \to N$ is proper. For a point $b \in N$ and a stratum $M \in \mathscr{S}(E)$ we put

$$\begin{aligned} &D_M(b) = \{\text{the critical values of } \delta|_{M \cap p_N^{-1}(b)}\}, \\ &D(b) = \bigcup_{M \in \mathscr{S}(E)} D_M(b). \end{aligned} \tag{2.2}$$

[In particular, if $\dim(M \cap p_N^{-1}(b)) = 0$, then $D_M(b) = \delta(M \cap p_N^{-1}(b))$.] Remember that since $p_N|_M: M \to N$ is a submersion, $M \cap p_N^{-1}(b)$ is a manifold for every $b \in N$ and $M \in \mathscr{S}(E)$. By Sard's theorem $D_M(b)$, hence $D(b)$, is measure zero in $\mathbb{R}$ [we have assumed that the number of strata of $\mathscr{S}(E)$ is at most countable].

Theorem 2.1 (T. Fukuda). *Suppose that $p_N: E = U \times N \to N$ is a stratified map with respect to the stratifications $\mathscr{S}(E)$ and $\mathscr{S}(N)$ and that $S_1, \ldots, S_k$ are stratified subsets of E. Then for an arbitrary point $b \in N$ and a positive number $r \notin D(b)$, $r \geqq \min \delta_0$, there exists an open neighborhood V of b such that*

$$p_N|_{\bar{B}(r) \times V}: \bar{B}(r) \times V \to V$$

is a stratified map with respect to the stratifications $\mathscr{S}(\bar{B}(r)\times V)$ *and* $\mathscr{S}(V)$ *given by*

1) $\mathscr{S}(V)=\{V\}$,

2) $\mathscr{S}(\bar{B}(r)\times V)=\{M\cap(B(r)\times V),\ M\cap(S(r)\times V)|M\in\mathscr{S}(E)\}$, *and*

and
$$S_i\cap(\bar{B}(r)\times V),\quad i=1,\dots,k,$$
are stratified subsets of $\bar{B}(r)\times V$.

Remark. $\bigcup_{b\in N}\{b\}\times D(b)\subset N\times\mathbb{R}$ is a closed subset of $N\times\mathbb{R}$.

Corollary 2.2. $p_N|_{\bar{B}(r)\times V}:\bar{B}(r)\times V\to V$ *is a locally trivial fibration relative to* $S_1\cap(\bar{B}(r)\times V),\dots,S_k\cap(\bar{B}(r)\times V)$.

See Fukuda and Kobayashi [3] for the proof of Theorem 2.1.

3. The Main Results

We begin with to state the results of [8]. To save notations we simplify the situation here, which loses, however, no essential points to study the singularities of the solution.

Let $a(x,\partial)$ be a linear differential operator of order $m+1$ with holomorphic coefficients in a neighborhood of the origin of $\mathbb{C}^{n+1}$, the initial plane $x_0=0$ non-characteristic, $p(x,\xi)$ the characteristic polynomial of a and $\lambda_i(x,\xi')$, $i=1,\dots,m+1$, the characteristic roots:

$$p(x,\xi)=a_0(x)\sum_{i=1}^{m+1}(\xi_0-\lambda_i(x,\xi')),$$

here $a_0(x)$ is the coefficient of $(\partial/\partial x_0)^{m+1}$. We consider the Cauchy problem with meromorphic initial data:

$$\left.\begin{aligned}&a(x,\partial)u(x)=0,\\&\partial_0^k u(0,x')=w_k(x'),\qquad k=0,\dots,m,\end{aligned}\right\}\tag{3.1}$$

where w_k's have poles along $x_0=x_1=0$.

In this section we always assume the followings, unless otherwise stated (see [8] for the details):

(A.1) $\lambda_i(x,\xi')$'s are holomorphic at $(x;\xi')=(0;1,0\dots0)$,

(A.2) λ_i's are involutive, that is, for every pair of integers (i,j), $1\leqq i,j\leqq m+1$, the Poisson bracket $\{\xi_0-\lambda_i,\xi_0-\lambda_j\}$ is a linear combination of $\xi_0-\lambda_i$ and $\xi_0-\lambda_j$ with coefficients holomorphic at $(x;\xi')=(0;1,0\dots0)$.

For sequences of integers $I_l=(i_1,\dots,i_l)$, $1\leqq i_1<\dots<i_l\leqq m+1$, we define multi-phase functions $\varphi^{I_l}=\varphi^{I_l}(t_1,\dots,t_{l-1};x)$ as follows:

$$\left.\begin{aligned}&\left.\begin{aligned}&\frac{\partial\varphi^{I_l}}{\partial x_0}-\lambda_{i_l}(x,\nabla_{x'}\varphi^{I_l})=0,\\&\varphi^{I_l}(t_1,\dots,t_{l-1};x)|_{x_0=t_{l-1}}=\varphi^{I_{l-1}}(t_1,\dots,t_{l-2};x)|_{x_0=t_{l-1}},\end{aligned}\right\}\\&\left.\begin{aligned}&\frac{\partial\varphi^{i}}{\partial x_0}-\lambda_i(x,\nabla_{x'}\varphi^i)=0,\\&\varphi^i(x)|_{x_0=0}=x_1,\end{aligned}\right\}\end{aligned}\right.\tag{3.2}$$

where $I_{l-1}=(i_1,\dots,i_{l-1})$ and i is an integer, $1\leqq i\leqq m+1$.

Theorem 3.1 [8]. *Under the hypotheses* (A.1), (A.2), *the solution* $u(x)$ *of the Cauchy problem* (3.1) *is expressed as follows:*

$$u(x)=F_0(x)+\sum_{i=1}^{m}\int_0^{x_0}dt_i\int_0^{t_i}dt_{i-1}\ldots\int_0^{t_2}dt_1F_i(t_1,\ldots,t_i;x)+H(x). \tag{3.3}$$

Here $H(x)$ *is holomorphic in a neighborhood of* $x=0$ *and* $F_i(t_1,\ldots,t_i;x)$, $i=0,\ldots,m$, *are many-valued holomorphic functions in a neighborhood of* $(t_1,\ldots,t_i;x)=(0,\ldots,0;0)$ *except* $\varphi^{1\ldots i+1}(t_1,\ldots,t_i;x)=0$, *respectively.*

Consequently, the singularities of the solution $u(x)$ are determined by studying the ones of the integrals in (3.3). It is enough to deal with the integral for $i=m$ in (3.3):

$$J(x)=\int_0^{x_0}dt_m\int_0^{t_m}dt_{m-1}\ldots\int_0^{t_2}dt_1F(t,x), \tag{3.4}$$

where $t=(t_1,\ldots,t_m)\in\mathbb{C}^m$ and $F=F_m$, which is exactly the one discussed in Sect. 1. We may assume that the multi-phase function $\varphi=\varphi^{1\ldots m+1}$ is holomorphic in a sufficiently small domain $T\times X\subset\mathbb{C}^m\times\mathbb{C}^{n+1}$ containing the origin of $\mathbb{C}^{m+n+1}$ and that F is many-valued and holomorphic in $Y=T\times X-S$ where S is the hypersurface of the zeros of φ. We identify the many-valued holomorphic function F with the single-valued holomorphic one in the universal covering space $(\tilde{Y},\varrho)$ of Y and denote it by the same letter F. In the sequel we shall use the same notations as in Sect. 1 without explicitly saying so. Let $p=(0,x')$ be a point of X such that $\varphi(0;0,x')\,(=x_1)\neq 0$. As stated in the introduction, the integral (3.4) defines a germ of holomorphic function at p. Moreover, let $\alpha^0:(\Delta,\dot{\Delta}_i)\to(Y_p,W_{i,p})$ be a constant singular m-simplex given by $\alpha^0(s)=(0,p)\in Y$ and $\tilde{\alpha}^0:(\Delta,\dot{\Delta}_i)\to(\tilde{Y}_p,\tilde{W}_{i,p})$ a lifting of α^0, then the germ at p given by (3.4) is equal to $\int_{\tilde{\alpha}^0}\omega_p$ where $\omega=F\,d\tilde{t}_1\wedge\ldots\wedge d\tilde{t}_m$. [There is a one to one correspondence between fixing a branch of F and fixing a base point $\tilde{\alpha}^0(0)\in\varrho^{-1}(0,p)$.] By Corollary 1.4 the germ at p is analytically continuable as long as α^0 can be continuously deformed satisfying the conditions in Corollary 1.4.

Now we apply the results of Sect. 2 to determine the obstruction of deformation. Since the problem concerned is a local one near the origin, we may assume that T and X are product domains, $T=T_0\times\ldots\times T_0$ (m times), $X=T_1\times X'$, where $T_1\subset T_0\subset\mathbb{C}$, $X'\subset\mathbb{C}^n$ are domains containing the origin. Then $W_{i,x}$'s are not empty for all $x\in X$. We denote by I^* the set of all sequences $I_l=(i_1,\ldots,i_l)$ with $1\leqq i_1<\ldots<i_l\leqq m+1$, $1\leqq l\leqq m+1$. For $I_l\in I^*$ we define,

$$\Sigma^{I_l}=\Big\{x\in X\,|\,{}^\exists(t)_{l-1}=(t_1,\ldots,t_{l-1})\ \text{s.t.}\ \varphi^{I_l}((t)_{l-1},x)$$
$$=\frac{\partial\varphi^{I_l}}{\partial t_1}((t)_{l-1},x)=\ldots=\frac{\partial\varphi^{I_l}}{\partial t_{l-1}}((t)_{l-1},x)=0\Big\},$$
$$(\Sigma^i=\{x\in X\,|\,\varphi^i(x)=0\}\supset\{x_0=x_1=0\}), \tag{3.5}$$

and

$$\Sigma_0=\bigcup_{I_l\in I^*}\bar{\Sigma}^{I_l}\ (\text{the closure of }\Sigma^{I_l}). \tag{3.6}$$

For a subset $K=\{k_1,\dots,k_j\}\subset\{0,1,\dots,m+1\}$, we put $W_K=W_{k_1}\cap\dots\cap W_{k_j}\subset T\times X$, where $W_0=S$ and W_k's denote, by abuse of notations, the subsets of $T\times X$ with defining equation $t_k-t_{k-1}=0$. We put for a non-negative smooth proper function $\delta_0:T\to\mathbb{R}$, a point $x\in X-\Sigma_0$ and a subset $K\subset\{0,1,\dots,m+1\}$,

$$D_K(x)=\{\text{the critical values of } \delta|_{W_K\cap\pi_X^{-1}(x)}\}, \tag{3.7}$$

and

$$D(x)=D(\delta_0:x)=\bigcup_K D_K(x), \tag{3.8}$$

where $\pi_X:T\times X\to X$ is the natural projection and $\delta:T\times X\to\mathbb{R}$ is given by $\delta(t,x)=\delta_0(t)$.

Now we define a closed subset Σ_∞ of $X-\Sigma_0$ as follows: A point $x\in X-\Sigma_0$ does not belong to Σ_∞ if and only if there exist a non-negative smooth proper function $\delta_0:T\to\mathbb{R}$ and an open neighborhood $U\subset X-\Sigma_0$ of x such that for any $r_0>0$ we can take a positive number $r>r_0$ such that $r\notin D(\delta_0:y)$ for all $y\in U$. With these notations we state the main results.

Theorem 3.2. *Let an open neighborhood X of the origin of $\mathbb{C}^{n+1}$ be small enough and $p=(0,x')$ a point in $X-(\Sigma_0\cup\Sigma_\infty)$. Then the germ of holomorphic function at p which is the solution of the Cauchy problem* (3.1) *can be analytically continuable along any path in $X-(\Sigma_0\cup\Sigma_\infty)$ with initial point p, namely, the solution of* (3.1) *in X has singularities only along $\Sigma_0\cup\Sigma_\infty$.*

Remark 1. Σ^i $(i=1,\dots,m+1)$, $\bar{\Sigma}^{I_l}$ $(l\geqq 2)$ and Σ_∞ correspond to end point singularities, pinching singularities and pinching singularities at the infinity for contour integrals, respectively.

Remark 2. When $m=1$, that is, the multiplicity of the equation is at most double, if $\varphi(t_1;0)\not\equiv 0$ [the assumption (B) in [6]], by shrinking X if necessary, we can make Σ_∞ empty. For the zeros of $\varphi(t_1;x)=0$ with respect to t_1 can be made arbitrary small by letting x small.

Remark 3. The initial branch of the solution does not have singularities along Σ_∞.

Remark 4. Although a point $x\in X-(\Sigma_0\cup\Sigma_\infty)$ may become a singularity of some branch, no such branch can be reached by paths lying exclusively in $X-(\Sigma_0\cup\Sigma_\infty)$.

Proof of the Theorem. Let α^0 be the constant singular m-simplex given by $\alpha^0(s)=(0,p)\in T\times X-S$, $s\in\Delta$. Given an arbitrary path $\gamma:I=[0,1]\to X-(\Sigma_0\cup\Sigma_\infty)$ with initial point p, we denote by J the set of all $\tau\in I$ such that α^0 can be deformed along the path γ to $\gamma(\tau)$ satisfying the conditions in Corollary 1.4. By this corollary the proof finishes if we show $J=I$. Since J contains 0 and is open in I by Lemma 1.2, and I is connected, it is enough to verify the closedness of J.

Since $\varphi=x_1$ when $(t,x_0)=(0,0)$ and the defining functions of W_i's are t_i-t_{i-1}, by shrinking $T\times X$ if necessary, we may assume that $W_0=S$, $W_1,\dots,W_{m+1}$ are in general position in $T\times X$. Then $W_0,\dots,W_{m+1}$ become stratified subsets of $T\times X$ with respect to the following canonical stratification $\mathscr{S}^*$ of $T\times X$:

$$\mathscr{S}^*=\{Z_K\subset T\times X|K\subset\{0,1,\dots,m+1\}\},$$

where

$$Z_K=\begin{cases} W_{k_1}\cap\ldots\cap W_{k_j}-\bigcup_{k\neq k_i} W_{k_1}\cap\ldots\cap W_{k_j}\cap W_k, & K=\{k_1,\ldots,k_j\}\neq\emptyset,\\ T\times X-(W_0\cup\ldots\cup W_{m+1}), & K=\emptyset.\end{cases}$$

Note that Σ^{I_l}, $I_l\in I^*$, is the set of all critical values of the projection restricted on Z_K, $\pi_X|_{Z_K}:Z_K\to X$ where $K=\{0,1,\ldots,m+1\}-\{i_1,\ldots,i_l\}$. Therefore the projection restricted on $Z_K\cap(T\times(X-\Sigma_0))$,

$$\pi_X|_{Z_K\cap(T\times(X-\Sigma_0))}:Z_K\cap(T\times(X-\Sigma_0))\to X-\Sigma_0$$

is submersive for every subset $K\subset\{0,1,\ldots,m+1\}$ except $K=\{1,2,\ldots,m+1\}$.

Let a sequence $\tau_i\in J$ converges to $\tau_0\in I$ as i tends to ∞. To prove the closedness of J, there are two cases to see:

a) $\gamma(\tau_0)\in\{x=(x_0,x')|x_0\neq0\}\cap(X-(\Sigma_0\cup\Sigma_\infty))$.

b) $\gamma(\tau_0)\in\{x=(x_0,x')|x_0=0\}\cap(X-(\Sigma_0\cup\Sigma_\infty))$.

Case a). Because $\gamma(\tau_0)\notin\Sigma_\infty$, there are an open neighborhood $U\subset X-(\Sigma_0\cup\Sigma_\infty)$ of $\gamma(\tau_0)$ and a non-negative smooth proper function $\delta_0:T\to\mathbb{R}$ such that for any $r_0>0$ we can take a positive number $r>r_0$ such that $r\notin D(x)$ $(=D(\delta_0:x))$ for all $x\in U$. We may also assume that U does not intersect $\{x_0=0\}$, which implies $Z_{\{1,\ldots,m+1\}}\cap T\times U=\emptyset$. Consequently the projection restricted on $T\times U$, $\pi_X|_{T\times U}:T\times U\to U$ is a stratified map with respect to the stratifications $\mathscr{S}(U)=\{U\}$ and $\mathscr{S}(T\times U)=\{Z_K\cap(T\times U)|Z_K\in\mathscr{S}^*\}$. Take a point $\gamma(\tau_{i0})$ in U and a sufficiently large number r such that $\bar B(r)\times\{\gamma(\tau_{i0})\}$ contains $\alpha_{\tau_i}^0(\Delta)$ and that $r\notin D(x)$ for all $x\in U$ [see (2.1) for the definition of $B(r)$]. Then, by Corollary 2.2, $\pi_X|_{\bar B(r)\times U}:\bar B(r)\times U\to U$ is a locally trivial fibration relative to $S\cap(\bar B(r)\times U)$, $W_i(\bar B(r)\times U)$, $i=1,\ldots,m+1$, hence we can deform $\alpha_{\tau_{i0}}$ in U keeping the conditions in Corollary 1.4. This shows $\tau_0\in J$.

Case b). Let $\gamma(\tau_0)=(0,q')\in X-(\Sigma_0\cup\Sigma_\infty)$. Since $\varphi(0;0,q')\neq0$, we can choose a small positive number ε and a neighborhood U of $(0,q')$ such that $\varphi(t;x)$ never vanishes on $\bar B_0(2\varepsilon)\times U$ where $\bar B_0(2\varepsilon)$ is the closed ball in $\mathbb{C}^m$ with radius 2ε. By the same argument as in Case a), there exists a positive number r such that for $\gamma(\tau_{i0})\in U$, $\alpha_{\tau_{i0}}(\Delta)\subset\bar B(r)\times\{\gamma(\tau_{i0})\}$ and that

$$\pi_X|_{(\bar B(r)-B_0(\varepsilon))\times U}:(\bar B(r)-B_0(\varepsilon))\times U\to U$$

is a locally trivial fibration relative to

$$S\cap((\bar B(r)-B_0(\varepsilon))\times U,\quad W_i\cap((\bar B(r)-B_0(\varepsilon))\times U,\quad i=1,\ldots,m+1.$$

On the other hand, in $\bar B_0(2\varepsilon)\times U$, since $\bar B_0(2\varepsilon)\times U$ does not meet the hypersurface S of the singularities of F and W_i's are given explicitly, we can easily give a deformation. Patching the two deformations in $(\bar B(r)-B_0(\varepsilon))\times U$ and $B_0(2\varepsilon)\times U$ with a suitable partition of unity, we obtain the desired deformation at τ_0. Thus $\tau_0\in J$. Now the proof of the theorem is completed.

Remark. Let $\alpha_\tau:(\Delta,\dot\Delta_i)\to(Y_{\gamma(\tau)},W_{i,\gamma(\tau)})$, $\tau\in[0,1[$, be a continuous deformation of α^0 along a path γ and let γ lie in $X-\Sigma_0$ in place of $X-(\Sigma_0\cup\Sigma_\infty)$. If the T-components of $\alpha_\tau(\Delta)$, $\tau \in [0, 1[$, are uniformly bounded, then an obvious modification of the above proof implies that the germ at p is analytically continuable along the path γ to the terminal point $\gamma(1)$. The initial branch of the solution is a trivial example of this (cf. Remark 3).

We now state the simplest geometrical properties of Σ^{I_l}, the connections with bicharacteristics. When $m=1$, these have been closely studied in [6, Sect. 5].

Consider the Hamiltonian's canonical systems associated to (3.2)

$$\left.\begin{aligned} \frac{dX^{I_l}}{dx_0} &= (1, -\nabla_{\xi'}\lambda_{i_l}(X^{I_l}, \Xi'^{I_l})), \\ \frac{d\Xi^{I_l}}{dx_0} &= \nabla_x\lambda_{i_l}(X^{I_l}, \Xi'^{I_l}), \end{aligned}\right\} \tag{3.9}$$

with the initial condition

$$\begin{aligned} &(X^{I_l}; \Xi^{I_l})((t)_{l-1}; x_0, y')|_{x_0=t_{l-1}} \\ &\quad = (X^{I_{l-1}}; \lambda_{i_l}(X^{I_{l-1}}, \Xi'^{I_{l-1}}), \Xi'^{I_{l-1}})|_{x_0=t_{l-1}}, \qquad l \geqq 2, \\ &(X^i; \Xi^i)(x_0, y')|_{x_0=0} = (0, y'; \lambda_i(0, y'; 1, 0 \dots 0), 1, 0 \dots 0), \end{aligned} \tag{3.10}$$

where $\xi' = (\xi_1, \dots, \xi_n)$, $y' = (y_1, \dots, y_n)$ and so on. Then it is clear that for each I_l we have $X_0^{I_l} = x_0$ and a family of biholomorphic mappings

$$(x_0, y') \to X^{I_l}((t)_{l-1}; x_0, y'),$$

which depend holomorphically on $(t)_{l-1}$.

Lemma 3.3. *Without the hypothesis of involutiveness,* (A.2), *we have the following equalities:*

$$\varphi^{I_l}((t)_{l-1}, x) = y_1, \tag{3.11}$$

$$\nabla_x\varphi^{I_l}((t)_{l-1}, x) = \Xi^{I_l}((t)_{l-1}; x_0, y'), \tag{3.12}$$

$$\frac{\partial\varphi^{I_l}}{\partial t_k}((t)_{l-1}, x) = (\lambda_{i_k} - \lambda_{i_{k+1}})(X^{I_k}, \Xi'^{I_k}), \qquad 1 \leqq k \leqq l-1, \tag{3.13}$$

where $(X^{I_k}, \Xi'^{I_k}) = (X^{I_k}, \Xi'^{I_k})((t)_{k-1}; t_k, y')$ *and* y' *is given by the relation,* $x = X^{I_l}((t)_{l-1}; x_0, y')$.

Proof. The first two equalities for usual phase functions are well-known. Then we can easily prove these for multi-phase functions by induction l. So we only show (3.13).

Differentiating $(\varphi^{I_l} - \varphi^{I_{l-1}})|_{x_0=t_{l-1}} = 0$ with respect to t_{l-1}, and then substituting $x = X^l((t)_{l-1}; x_0, y')$, we have by (3.2), (3.10), and (3.12)

$$\left.\frac{\partial\varphi^{I_l}}{\partial t_{l-1}}((t)_{l-1}, X^{I_l})\right|_{x_0=t_{l-1}} = (\lambda_{i_{l-1}} - \lambda_{i_l})(X^{I_{l-1}}, \Xi'^{I_{l-1}}).$$

For all $k=1, \dots, l-1$, $F_k(x_0) = (\partial\varphi^{I_l}/\partial t_k)((t)_{l-1}, X^{I_l})$ are independent of x_0, because the derivatives dF_k/dx_0 vanish by (3.2) and (3.12). Hence (3.13) is true for $k = l-1$.

Next applying the above result to the pair $(\varphi^{I_{l-1}}, (l-1)-1)$, we have

$$\left.\frac{\partial\varphi^{I_l}}{\partial t_{l-2}}\right|_{x_0=t_{l-1}} = \left.\frac{\partial\varphi^{I_{l-1}}}{\partial t_{l-2}}\right|_{x_0=t_{l-1}} = (\lambda_{i_{l-2}} - \lambda_{i_{l-1}})(X^{I_{l-2}}, \Xi'^{I_{l-2}}).$$

The left side is constant along the bicharacteristic curve X^{I_l}, which proves (3.13) for $k=l-2$. In this way the proof goes on by induction on

$$k=l-1, l-2, \ldots, 1. \quad \text{Q.E.D.}$$

The condition (A.2) implies that

$$\{\xi_0-\lambda_i, \xi_0-\lambda_j\}=c_{ij}(\lambda_j-\lambda_i) \tag{3.14}$$

with some $c_{ij}=c_{ij}(x,\xi')$ which is holomorphic at $(x;\xi')=(0;1,0\ldots0)$, because the left side is independent of ξ_0. The next lemma easily follows from (3.14) and the uniqueness of the solution to ordinary differential equations (cf. [8, Lemma 2.6]).

Lemma 3.4. *Let* $(X^{I_j}, \Xi'^{I_j})=(X^{I_j}, \Xi'^{I_j})((t)_{j-1}; t_j, y')$. *If* $(\lambda_i-\lambda_{i_{k+1}})(X^{I_k}, \Xi'^{I_k})=0$, *then* $(\lambda_i-\lambda_{i_{k+1}})(X^{I_{k+1}}, \Xi'^{I_{k-1}})=0$, *and vice versa.*

We put for $I_l=(i_1,\ldots,i_l)\in I^*$

$$\Omega^{I_l}=\{(0,x'')|\lambda_{i_1}=\ldots=\lambda_{i_l} \quad \text{at} \quad (0,0,x'';1,0\ldots0)\}, \qquad (x''=(x_2,\ldots,x_n)). \tag{3.15}$$

Theorem 3.5. Σ^{I_l} *is generated by l-families of bicharacteristics issuing from* Ω^{I_l}:

$$\Sigma^{I_l}=\{X^{I_l}((t)_{l-1};x_0,y')|y'\in\Omega^{I_l}\}. \tag{3.16}$$

Moreover, let $x=X^{I_l}((t)_{l-1};x_0,y')\in\Sigma^{I_l}$, *then we have*

$$\frac{\partial\varphi^{I_l}}{\partial x_0}((t)_{l-1},x)-\lambda_{i_k}(x;\nabla_{x'}\varphi^{I_l}((t)_{l-1},x))=0, \qquad i_k\in I_l. \tag{3.17}$$

Proof. Let $x\in\Sigma^{I_l}$, then there is $(t)_{l-1}$ such that

$$\varphi^{I_l}=\partial\varphi^{I_l}/\partial t_1=\ldots=\partial\varphi^{I_l}/\partial t_{l-1}=0 \quad \text{at} \quad ((t)_{l-1},x).$$

Choose y' so that $x=X^{I_l}((t)_{l-1};x_0,y')$. By (3.11), $y_1=\varphi^{I_l}((t)_{l-1},X^{I_l})=0$. We also have by (3.13)

$$(\lambda_{i_k}-\lambda_{i_{k+1}})(X^{I_k},\Xi'^{I_k})=0, \qquad k=1,\ldots,l-1.$$

By applying Lemma 3.4 successively we finally obtain

$$\lambda_{i_1}(X^{I_1},\Xi'^{I_1})=\ldots=\lambda_{i_l}(X^{I_1},\Xi^{I_1}) \quad \text{at} \quad t_1=0,$$

and

$$\lambda_{i_1}(X^{I_l},\Xi'^{I_l})=\ldots=\lambda_{i_l}(X^{I_l},\Xi^{I_l}),$$

which imply (3.16) and (3.17), respectively. Q.E.D.

Proposition 3.6. *If* $\Omega^{I_l}=X'\cap\{x_1=0\}$, *then there is a holomorphic function* $a=a((t)_{l-1},x)$ *such that*

$$\varphi^{I_l}((t)_{l-1},x)=a((t)_{l-1},x)\cdot\varphi^{i_1}(x), \tag{3.18}$$

where $a\equiv1$ *when* $((t)_{l-1},x_0)=(0,0)$. *In particular the zeros of* φ^{I_l} *is independent of* $(t)_{l-1}$.

To prove the proposition we quote a lemma due to Ōuchi.

Lemma 3.7 [13, Appendix]. *Consider the following first order differential equation*

$$\frac{\partial f}{\partial x_0}-\mu(x,\nabla_{x'}f)=0, \tag{3.19}$$

where $\mu(x,\xi')$ is holomorphic at $(x;\xi')=(0;1,0\dots0)$ and homogeneous in ξ' of degree one. If a holomorphic function Ψ (resp. ψ) satisfies (3.19) *on $\Psi=0$ (resp. identically) and if $\Psi=\psi$ on $x_0=\tau$, then there is a holomorphic function b such that $b\equiv 1$ on $x_0=\tau$ and $\Psi=b\cdot\psi$. Namely $b^{-1}\Psi$ satisfies* (3.19) *identically.*

Proof of Proposition 3.6. If $\varphi^{I_l}((t)_{l-1},x)=0$, then $x\in\Sigma^{I_l}$. In fact, let

$$x=X^{I_l}((t)_{l-1};x_0,y'),$$

then $y_1=0$ by (3.11) and $y'\in\Omega^{I_l}$ by the hypothesis. Now this is clear by (3.16). By (3.17) the pair $(\Psi,\psi)=(\varphi^{I_l},\varphi^{I_{l-1}})$ satisfies the hypotheses in Lemma 3.7, hence we can take a holomorphic function $a_l=a_l((t)_{l-1},x)$ such that $a_l\equiv 1$ on $x_0=t_{l-1}$ and

$$\varphi^{I_l}((t)_{l-1},x)=a_l((t)_{l-1},x)\cdot\varphi^{I_{l-1}}((t)_{l-2},x).$$

Since $\Omega^{I_l}\subset\Omega^{I_{l-1}}\subset\dots\subset\Omega^{I_2}$, the above argument is also valid for all $k=2,3,\dots,l$. Hence we have

$$\varphi^{I_k}((t)_{k-1},x)=a_k((t)_{k-1},x)\cdot\varphi^{I_{k-1}}((t)_{k-2},x)$$

with $a_k\equiv 1$ on $x_0=t_{k-1}$. Then $a=\prod_{k=2}^{l} a_k$ obviously has the required properties. Q.E.D.

Corollary 3.8. *If $\Omega^{1,2\dots m+1}=X'\cap\{x_1=0\}$, then $\Sigma_0=\{\varphi^1(x)=0\}$ and $\Sigma_\infty=\emptyset$ in a neighborhood of the origin of $\mathbb{C}^{n+1}$.*

Finally we give some examples. The first three were given in [6] as examples of the origin being "exceptional".

Example 1. $p=\partial_0^2+x_2\partial_0\partial_1$, $\varphi^{12}=x_1-x_2t_1$, $\Sigma^1: x_1-x_2x_0=0$, $\Sigma^2: x_1=0$, $\Sigma^{12}: x_1=x_2=0$, $\Sigma_0=\Sigma^1\cup\Sigma^2$, $\Sigma_\infty: x_2=0$.

Example 2. $p=\partial_0^2-x_2^2(\partial_1^2+\dots+\partial_n^2)$, $\varphi^1=x_1+\frac{1}{2}x_2(e^{x_0}-e^{-x_0})$, $\varphi^2=x_1-\frac{1}{2}x_2\cdot(e^{x_0}-e^{-x_0})$, $\varphi^{12}=x_1+\frac{1}{2}x_2(e^{2t_1-x_0}-e^{x_0-2t_1})$, $\Sigma^1:\varphi^1=0$, $\Sigma^2:\varphi^2=0$, $\Sigma^{12}: x_2^2+x_3^2=0$, $\Sigma_0=\Sigma^1\cup\Sigma^2\cup\Sigma^{12}$, $\Sigma_\infty: x_2=0$.

In this example Σ^{12} is peculiar to the Cauchy problem in the complex domain.

Example 3. $p=\partial_0^2+2x_2\partial_0\partial_1+x_3\partial_0\partial_2$, $\varphi^1=x_1-2x_2x_0+x_3x_0^2$, $\varphi^2=x_1$, $\varphi^{12}=x_1-2x_2t_1+x_3t_1^2$, $\Sigma^1:\varphi^1=0$, $\Sigma^2:\varphi^2=0$, $\Sigma^{12}: x_1=x_2=x_3=0\cup x_3\neq 0, x_1x_3-x_2^2=0$, $\Sigma_0=\Sigma^1\cup\Sigma^2\cup\overline{\Sigma^{12}}$, $\Sigma_\infty: x_3=0$.

Note that Σ^{I_l} are, in general, not closed.

Example 4.

$$p=\partial_0(\partial_0+2x_2\partial_1+\partial_2)(\partial_0+2x_3\partial_1+\partial_3),$$
$$\varphi^{123}=x_1-2x_2t_2+t_2^2+2(x_2-x_3-t_2)t_1+2t_1^2,$$
$$\Sigma^1: x_1-2x_3x_0+x_0^2=0,\quad \Sigma^2: x_1-2x_2x_0+x_0^2=0,\quad \Sigma^3: x_1=0,$$
$$\Sigma^{12}: x_0^2-2(x_2+x_3)x_0-(x_2-x_3)^2+2x_1=0,\quad \Sigma^{23}: x_1-x_2^2=0,$$
$$\Sigma^{13}: x_1-x_3^2=0,\quad \Sigma^{123}: x_1-x_2^2-x_3^2=0,$$
$$\Sigma_0=\Sigma^1\cup\Sigma^2\cup\Sigma^3\cup\Sigma^{12}\cup\Sigma^{13}\cup\Sigma^{23}\cup\Sigma^{123},\quad \Sigma_\infty=\emptyset.$$

Example 5 (cf. Corollary 3.8). $p=\partial_0(\partial_0+x_1\partial_1)(\partial_0+x_1^2\partial_1)$, $\varphi^1=x_1(1+x_0x_1)^{-1}$, $\varphi^2=x_1e^{-x_0}$, $\varphi^3=x_1$, $\varphi^{13}=x_1(1+x_1t_1)^{-1}$, $\varphi^{12}=x_1(e^{x_0-t_1}+x_1t_1)^{-1}$, $\varphi^{23}=x_1e^{-t_2}$, $\varphi^{123}=x_1(e^{t_2-t_1}+x_1t_1)^{-1}$, $\Sigma^1=\Sigma^2=\Sigma^3=\Sigma^{12}=\Sigma^{13}=\Sigma^{23}=\Sigma^{123}: x_1=0$, $\Sigma_0=\Sigma^1$, $\Sigma_\infty=\emptyset$.

The Poisson bracket $\{\xi_0+x_1\xi_1, \xi_0+x_1^2\xi_1\}=x_1^2\xi_1=\dfrac{x_1}{1-x_1}\times(x_1\xi_1-x_1^2\xi_1)$, which does not vanish identically.

Example 6. $p=\partial_0(\partial_0-x_2\partial_1)(\partial_0-x_3\partial_1)$, $\varphi^{12}=x_1+x_2x_0+(x_3-x_2)t_1$, $\varphi^{13}=x_1+x_3t_1$, $\varphi^{23}=x_1+x_2t_2$, $\varphi^{123}=x_1+x_2t_2+(x_3-x_2)t_1$, $\Sigma^1: x_1+x_3x_0=0$, $\Sigma^2: x_1+x_2x_0=0$, $\Sigma^3: x_1=0$, $\Sigma^{13}: x_1=x_3=0$, $\Sigma^{23}: x_1=x_2=0$, $\Sigma^{12}: x_3-x_2=x_1+x_2x_0=0$, $\Sigma^{123}: x_1=x_2=x_3=0$, $\Sigma_0=\Sigma^1\cup\Sigma^2\cup\Sigma^3$, $\Sigma_\infty: x_2x_3=0$.

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The L_1 Structure of Weak L_1

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1. Introduction and Technical Preliminaries

The space weak L_1, also known as $L(1, \infty)$, arose as a substitute for L_1 when key operators of harmonic analysis were found to map L_1 continuously into weak L_1 [21, Theorem 1(b), p. 5; § 2.4, p. 30] but not into L_1 [21, § (i), p. 5; § (c), p. 26]. The celebrated L_0 factorization problem – Does every continuous linear operator from a Banach space into L_0 factor through L_1? – has an affirmative answer with weak L_1 in place of L_1 [16, Theorem 3, p. 797]. The problem itself remains open (but see [9, § 2]).

This paper will examine the extent to which weak L_1, denoted WL_1, resembles L_1. We focus primarily upon the normed envelope $\widehat{WL_1}$ of WL_1 [see (1.4)]. In Sect. 2 we display isometric, order isomorphic copies in $WL_1{}^*$ of the $L_\infty = L_1{}^*$ space of the underlying (nonatomic probability) measure. The linear span of these constitutes a norming, and hence a weak* dense, subspace of $WL_1{}^*$. The results of Sect. 2 serve vitally in Sect. 3, where we show that $\widehat{WL_1}$ is in a certain sense covered by isometric, order isomorphic, complemented copies of the L_1 spaces (and indeed of the l_1 direct sums of $2^{\aleph_0}$ disjoint copies of the L_1 spaces) of both counting measure and Lebesgue measure on $[0, 1]$. In Sect. 4 it is seen that $\widehat{WL_1}$ does not, however, embed in an L_1 space: Subspaces isometric and order isomorphic to l_∞ and to $c_0[0, 1]$ also abound. Section 5 presents an order isomorphic embedding of l_∞ into WL_1 itself. This embedding provides a WL_1-valued operator which does not factor through any L_1 space, and it also provides an L_0-valued operator which cannot, in any natural way, be factored through an L_1 subspace of $\widehat{WL_1}$.

We conclude this section with some technical preliminaries. Let a *nonatomic* probability measure space (X, Σ, μ) be fixed throughout the paper. Unless otherwise stated, all functions (f, g, etc.) will be real-valued and Σ-measurable. We shall frequently consider such sets as $(a \leqq f \leqq b) = \{x \in X : a \leqq f(x) \leqq b\}$ or $(a \leqq f \leqq b, g > c) = (a \leqq f \leqq b) \cap (g > c)$. For these and other cumbersomely denoted

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sets, we shall write χE in place of the more usual χ_E to denote the characteristic function of E. We define the *distribution* of a measurable function f to be the image measure μ_f of μ with respect to f. Thus, if B is a Borel subset of $\mathbb{R}$, we define

$$\mu_f(B) = \mu(f^{-1}(B)).$$

We shall need the elementary identity

$$(1.1) \qquad \int \psi(f) d\mu = \int \psi d\mu_f,$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is μ_f-integrable, and where $\psi(f)$ denotes the composition of ψ and f.

For $0 < p < \infty$, the space $WL_p = WL_p(\mu)$ is defined to be the vector space of functions f (strictly speaking, of equivalence classes of functions f modulo μ-null functions) for which the expression

$$\varphi_p(f) = \sup_{t>0} t\mu(|f| > t)^{1/p}$$

is finite. It is easily checked that φ_p constitutes a *quasinorm* on WL_p and hence determines a metric vector topology on WL_p such that a sequence $\{f_n\}$ converges to a function f in this topology if and only if $\varphi_p(f_n - f) \to 0$ (see [19, pp. 45–46]). We shall need the following facts, both easily verified.

(1.2) WL_p is *complete* – that is, every φ_p-Cauchy sequence converges to a unique element in WL_p.

(1.3) φ_p is a *lattice quasinorm* – that is, we have $\varphi_p(f) \leqq \varphi_p(g)$ whenever $|f| \leqq |g|$.

Henceforth we specialize to the case $p = 1$ (see the introduction to Sect. 2).

The dual of every quasinormed space coincides with the dual of a canonical normed space. The case of WL_1 is typical: Define φ to be the Minkowski functional [11, p. 15] of the convex hull of the unit ball $\{f \in WL_1 : \varphi_1(f) \leqq 1\}$ of WL_1. The fact that φ is a seminorm on WL_1 is most readily seen from the alternate formulation $\varphi(f) = \inf \sum_{i=1}^{n} \varphi_1(f_i)$, where $f \in WL_1$, and where the infimum is taken over the finite subsets $\{f_1, \ldots, f_n\}$ of WL_1 which sum to f. Note that $\varphi(f) \leqq \varphi_1(f)$. We divide out by the subspace $N = \{f \in WL_1 : \varphi(f) = 0\}$ to obtain the quotient space

$$(1.4) \qquad WL_1^\wedge = WL_1/N,$$

upon which φ acts as a norm. We refer to $WL^\wedge$ as the *normed envelope* (and to its completion as the *Banach envelope* [20, p. 116]) of WL_1. Clearly N is a (closed) ideal in WL_1 [13, p. 3], so that $WL_1^\wedge$ inherits the natural vector lattice structure of WL_1. It is also clear that WL_1 and $WL_1^\wedge$ share the same continuous linear functionals, and we denote their common dual space by $WL_1{}^*$.

We have two reasons for requiring that the underlying measure μ be nonatomic. The first is that, in this case, the seminorm φ is given by an explicit integral-like formula (see [5] and [6]). For $f \in WL_1$, and for $0 < a < b$, let

$$(1.5) \qquad I_a^b(f) = \frac{1}{\ln(b/a)} \int |f| \chi(a \leqq |f| \leqq b) d\mu$$

and define

(1.6) $$I(f) = \lim_{n\to\infty}\left(\sup\left\{I_a^b(f): \frac{b}{a} \geqq n\right\}\right).$$

Then $I(f) = \varphi(f)$ by [6, § 1], whence:

(1.7) $$N = \{f \in WL_1 : I(f) = 0\}.$$

(1.8) $$N \supset L_1(\mu).$$

[And in Corollary 5.2 below we shall see that the containment in (1.8) is strict.]

If f is any measurable function, and if $0 < a < b$, then the identity

$$\int |f|\chi(a \leqq |f| \leqq b) = \int_a^b \mu(|f| > t)dt + a\mu(|f| \geqq a) - b\mu(|f| > b),$$

which is a routine consequence of Fubini's theorem, implies that

(1.9) $$I_a^b(f) \leqq \left(1 + \frac{2}{\ln(b/a)}\right)\varphi_1(f),$$

and that

(1.10) $$I(f) = \lim_{n\to\infty}\left(\sup\left\{D_a^b(f): \frac{b}{a} \geqq n\right\}\right),$$

where

(1.11) $$D_a^b(f) = \frac{1}{\ln(b/a)} \int_a^b \mu(|f| > t)dt.$$

Formula (1.10), in turn, gives easy access to the following facts.

(1.12) I is lattice seminorm.

(1.13) $\liminf_{t\to\infty} t\mu(|f| > t) \leqq I(f) \leqq \limsup_{t\to\infty} t\mu(|f| > t)$.

Hence, finally:

(1.14) $I(f) = \lim_{t\to\infty} t\mu(|f| > t)$ whenever this limit exists.

For example, let f have the Cauchy distribution with parameter $\delta > 0$, so that

$$\mu(f \leqq t) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{t}{\delta}$$

for all $t \in \mathbb{R}$. Since the function $t \mapsto t\mu(|f| > t)$ increases strictly, as $t \to \infty$, to the limit $2\delta/\pi$, it follows from (1.14) that

(1.15) $$I(f) = \varphi_1(f) = 2\delta/\pi.$$

A similar statement applies to any function whose distribution coincides with that of $\frac{1}{x}$. It will be convenient to give a name to such functions.

(1.16) *Definition.* A function $f \in WL_1$ is said to be $\frac{1}{x}$*-like* if $f \geqq 0$, and if we have $\mu(f>t)=1/t$ for every number $t \geqq 1$. □

The second reason for requiring that the measure μ be nonatomic is that, in this case, there is an isomorphic (that is, a measure preserving) embedding of the measure algebra of Lebesgue measure λ on [0, 1] into the measure algebra of μ [18, Theorem 2, p. 321]. The corresponding canonical mapping of (equivalence classes of) measurable functions, as discussed in [12, Lemma 6.10, pp. 46–48], preserves the natural lattice structure of measurable functions, it preserves a.e. convergence, and hence it preserves the joint distributions of finite families of measurable functions. These facts will be vital for the L_1 embedding results of Sect. 3.

Strictly speaking, $W\hat{L_1}$ consists of equivalence classes of measurable functions under the equivalence relation $f \equiv g$ if and only if $I(f-g)=0$. As with equivalence modulo null functions, we shall abusively treat the elements of $W\hat{L_1}$ (and of WL_1) as functions. The following definition provides a simple condition on two functions $f, g \in WL_1$ which ensures that the equivalence classes, $\bar{f}$ and $\bar{g}$ of these functions are lattice disjoint (i.e. that $|\bar{f}| \wedge |\bar{g}|=0$).

(1.17) *Definition.* Two functions f and g are said to be *essentially disjointly supported* if there exists a constant $t>0$ such that the sets $(|f|>t)$ and $(|g|>t)$ are disjoint. □

When there is an interplay between essential disjoint support and ordinary disjoint support (i.e. $fg=0$), we shall refer to the latter as *strict* disjoint support.

Although a number of familiar *complete* spaces will be shown to embed in $W\hat{L_1}$, the space itself is not complete, as we now show (cf. [6, § 4]).

(1.18) **Proposition.** *The space $W\hat{L_1}$ is not complete with respect to the I norm.*

Proof. For $f \in WL_1$, define

$$\hat{\varphi}_1(f)=\inf\{\varphi_1(f+g): g \in N\}.$$

Then $\hat{\varphi}_1$ determines a *quotient quasinorm* on $W\hat{L_1}$ in the same way that a quotient norm is determined, and, as a consequence of (1.2), $W\hat{L_1}$ will be complete with respect to $\hat{\varphi}_1$ (cf. [19, Theorem 1, p. 47]). If $W\hat{L_1}$ were also complete with respect to I, then, by the Open Mapping Theorem, the $\hat{\varphi}_1$ and I topologies on $W\hat{L_1}$ would coincide. Consequently there would exist constant $K \geqq 1$ such that $\hat{\varphi}_1(f)<KI(f)$ for every function $f \in WL_1 \backslash N$. We shall now demonstrate that such a constant cannot exist.

To this end, set $\alpha=e^{8K}>2$, and deduce from the nonatomicity of μ that there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint measurable subsets of X such that $\mu(E_n)=\alpha^{-n}$ for all n. Define $f=\sum_{n=1}^{\infty} \alpha^n \chi_{E_n}$. By examining the powers of α which are closest to any given points a and b, one may easily verify that

$$I(f)=\lim\left\{I_a^b(f): a \geqq 1; \frac{b}{a} \to \infty\right\}=\frac{1}{\ln \alpha}=\frac{1}{8K}. \tag{1.18.1}$$

By assumption, there exists a function $g \in N$ such that $\varphi_1(f+g) \leqq KI(f) = 1/8$. Let i be a positive integer, and let $t \in [\alpha^i/4, \alpha^i/2)$. Then we have

$$\begin{aligned}\mu(|g|>t) &\geqq \mu(f>2t) - \mu(|f+g|>t) \\ &\geqq \frac{1}{\alpha^{i-1}(\alpha-1)} - \frac{1}{8t} \geqq \frac{1}{2\alpha^i}.\end{aligned}$$

Now let n be any positive integer, let $a=1$, and let $b=\alpha^n$. Then, for $i=1,2,\ldots,n$, the intervals $[\alpha^i/4, \alpha^i/2)$ are pairwise disjoint, and they are also contained in the interval $[a,b]$, so that, recalling (1.11), we have

$$D_a^b(g) \geqq \frac{1}{n \ln \alpha} \sum_{i=1}^{n} \frac{1}{2\alpha^i}\left(\frac{\alpha^i}{2} - \frac{\alpha^i}{4}\right) = \frac{1}{64K}.$$

Since n was arbitrary, it follows from (1.10) that $I(g)>0$, and this contradicts (1.7). □

(1.19) *Question.* Is it possible to represent the completion of $W\hat{L_1}$ as a space of equivalence classes of measurable functions f under the equivalence relation $f \equiv g$ if and only if $I(f-g)=0$? □

2. The L_∞ Structure of $WL_1{}^*$

For $0<p<1$, the space WL_p has trivial dual [3, Theorem 1, p. 50], whereas, for $1<p<\infty$, it is already a Banach space [4, p. 82], and the structure of its dual space has been extensively analyzed (in [4]). This leaves the dual of WL_1, about which little is known except for the fact that it is nontrivial (see [7, Theorem 6, p. 784] and [5, p. 153]). In the principal result of this section (Theorem 2.8) we show that the isometric, order isomorphic copies of $L_\infty(\mu)$ span (at least) a weak* dense subspace of $WL_1{}^*$. The linear functionals which we uncover in this way will be vital to the L_1 embedding results of Sect. 3, particularly the complementation results.

Our basic idea is quite simple: We convert the nonlinear limit superior expression (1.6) for $I(f)$ into a linear limit expression by directing the $I_a^b(f)$ in some fashion. It will be convenient to accomplish this directing by means of an ultrafilter $\mathscr{U}$. The limit of the $I_a^b(f)$ along $\mathscr{U}$ will determine a canonical integral-like linear functional $I_{\mathscr{U}} \in WL_1{}^*$, and $I_{\mathscr{U}}$, in turn, will determine both a canonical embedding $T_{\mathscr{U}}: L_\infty(\mu) \to WL_1{}^*$ and an L_1-like seminorm $\|f\|_{\mathscr{U}} = I_{\mathscr{U}}(|f|)$ on WL_1 (or on $W\hat{L_1}$) which is (strictly) weaker than I.

We begin with the discussion of $\mathscr{U}$. For $n=1,2,\ldots$, let

$$F_n = \left\{(a,b): 1 \leqq a < b; \frac{b}{a} \geqq n\right\},$$

and then define

(2.1) $$\mathscr{F} = \{F_n : n \geqq 1\}.$$

Treating $\mathscr{F}$ as a filter of subsets of the set $S=[1,\infty)\times[1,\infty)$, we obtain from Zorn's lemma an ultrafilter $\mathscr{U}$ of subsets of S such that $\mathscr{F} \subseteqq \mathscr{U}$. Consider $\mathscr{U}$ now to be fixed. The significance of the ultrafilter property lies in the fact that, for every function $f \in WL_1$, and for every integer n sufficiently large, the set $\{I_a^b(f) : (a,b) \in F_n\}$ is bounded, so that the limit $l = \lim_{\mathcal{U}} I_a^b(f)$ always exists. [Recall that l is defined by the requirement: For every number $\varepsilon > 0$, there is a set $U \in \mathcal{U}$ such that $|I_a^b(f) - l| < \varepsilon$ whenever $(a,b) \in U$. We have stipulated that $a \geqq 1$ in the definition of the sets F_n in order to avoid choices of $\mathcal{U}$ for which the limit uselessly vanishes on WL_1.]

Next we define the "ersatz integral" $I_{\mathcal{U}}$ for every *nonnegative* function $f \in WL_1$ by

(2.2) $$I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} I_a^b(f) = \lim_{\mathcal{U}} D_a^b(f)$$

[cf. (1.10)].

(2.3) **Key Lemma.** *Let $f, g \in WL_1$ be nonnegative, and let $r > 0$. Then we have:*

(2.3.1) $I_{\mathcal{U}}(rf) = rI_{\mathcal{U}}(f)$.

(2.3.2) $I_{\mathcal{U}}(f+g) = I_{\mathcal{U}}(f) + I_{\mathcal{U}}(g)$.

(2.3.3) *If $f \leqq g$, then $I_{\mathcal{U}}(f) \leqq I_{\mathcal{U}}(g)$.*

(2.3.4) $I_{\mathcal{U}}(f) \leqq I(f)$.

Proof. To prove (2.3.1), it is sufficient to show that the expression

$$\int f\chi(a \leqq rf \leqq b) - \int f\chi(a \leqq f \leqq b)$$

is bounded on the set $\left\{(a,b) : 0 < a < b; \frac{b}{a} > r\right\}$. For such a pair (a,b), we may rewrite this as

$$\int f\chi\left(\frac{a}{r} \leqq f < a\right) - \int f\chi\left(\frac{b}{r} < f \leqq b\right).$$

The first term is bounded by $a\mu\left(f \geqq \frac{a}{r}\right) \leqq r\varphi_1(f)$, and the second term is likewise bounded (in absolute value) by $r\varphi_1(f)$.

To prove (2.3.2), it is sufficient to show that the expression

$$\int (f+g)\chi(a \leqq f+g \leqq b) - \int f\chi(a \leqq f \leqq b) - \int g\chi(a \leqq g \leqq b)$$

is bounded (for $0 < a < b$), and to this end we need consider only the expression

$$\int f\chi(a \leqq f+g \leqq b) - \int f\chi(a \leqq f \leqq b).$$

Rewriting this as

$$\left[\int f\chi(a \leqq f+g \leqq b) - \int f\chi\left(\frac{a}{2} \leqq f \leqq b\right)\right] + \int f\chi\left(\frac{a}{2} \leqq f < a\right),$$

we notice at once that the second term above is bounded by $2\varphi_1(f)$. The first term is bounded in absolute value by the integral of f over the symmetric difference of the sets $(a \leqq f+g \leqq b)$ and $\left(\frac{a}{2} \leqq f \leqq b\right)$. We partition this symmetric difference into five sets. Each is listed below together with a suggestive superset and a bound on the integral of f over either set:

$$\left(a \leqq f+g \leqq b, f<\frac{a}{2}\right) \subseteqq \left(f<\frac{a}{2}, g \geqq \frac{a}{2}\right): \varphi_1(g).$$

$$(a \leqq f+g \leqq b, f>b)=\emptyset: 0.$$

$$\left(f+g<a, \frac{a}{2} \leqq f \leqq b\right) \subseteqq \left(\frac{a}{2} \leqq f<a\right): 2\varphi_1(f).$$

$$\left(f+g>b, \frac{a}{2} \leqq f \leqq \frac{b}{2}\right) \subseteqq \left(f \leqq \frac{b}{2}, g>\frac{b}{2}\right): \varphi_1(g).$$

$$\left(f+g>b, \frac{b}{2}<f \leqq b\right) \subseteqq \left(\frac{b}{2}<f \leqq b\right): 2\varphi_1(f).$$

Thus the proof of (2.3.2) is complete.

Condition (2.3.3) follows at once from (2.3.2), while (2.3.4) follows simply because the limit superior always dominates the limit. □

We define $I_{\mathscr{U}}(f)$ for an arbitrary function $f \in WL_1$ by

$$I_{\mathscr{U}}(f)=I_{\mathscr{U}}(f^{+})-I_{\mathscr{U}}(f^{-}). \tag{2.4}$$

The usual familiar arguments will yield the following facts.

(2.5) $I_{\mathscr{U}}$ is linear.

(2.6) $|I_{\mathscr{U}}(f)| \leqq I(f)$ for all $f \in WL_1$.

(2.7) $I_{\mathscr{U}}$ vanishes on N and hence determines a well defined, bounded linear functional on $WL_1^{\wedge}$ [recall (1.4) and (1.7)].

We are now prepared to embed $L_\infty(\mu)$ into $WL_1{}^*$.

(2.8) **Theorem.** *Define a linear operator $T_{\mathscr{U}}: L_\infty(\mu) \rightarrow WL_1{}^*$ by*

$$T_{\mathscr{U}}(m): f \mapsto I_{\mathscr{U}}(mf)$$

for all $m \in L_\infty(\mu)$, and for all $f \in WL_1$. Then $T_{\mathscr{U}}$ constitutes an isometric order isomorphism of $L_\infty(\mu)$ into $WL_1{}^*$. Moreover, the linear span of the subspaces $T_{\mathscr{U}}(L_\infty(\mu))$, as $\mathscr{U}$ ranges over the collection of ultrafilters (of subsets of S) which contain $\mathscr{F}$ (see (2.1)), constitutes a norming, and hence a weak dense, subspace of $WL_1{}^*$.*

Note. In particular, $I_{\mathscr{U}}$ itself corresponds to the function $m \equiv 1$, and so $\|I_{\mathscr{U}}\|=1$.

Proof. The linearity of $T_{\mathscr{U}}$ follows from that of $I_{\mathscr{U}}$, while the fact that

$$T_{\mathscr{U}}(m_1 \vee m_2)=T_{\mathscr{U}}(m_1) \vee T_{\mathscr{U}}(m_2)$$

for all $m_1, m_2 \in L_\infty(\mu)$ is clear from the identity $(m_1 \vee m_2) f=m_1 f \chi(m_1 \geqq m_2)+m_2 f \chi(m_1<m_2)$. It remains only to show that $\|T_{\mathscr{U}}(m)\|=\|m\|_\infty$ for all $m \in L_\infty(\mu)$,

where $\|\cdot\|_\infty$ denotes the essential supremum norm. By (2.6) and (1.12), we have

$$|T_{\mathscr{U}}(m)(f)|=|I_{\mathscr{U}}(mf)|\leqq I(mf)=I(|m||f|)\leqq I(\|m\|_\infty|f|)=\|m\|_\infty I(f)$$

for all $f\in WL_1$, whence $\|T_{\mathscr{U}}(m)\|\leqq\|m\|_\infty$. To establish the reverse inequality, it is sufficient to show that $\|T_{\mathscr{U}}(m)\|\geqq r$ whenever $0<r<\|m\|_\infty$. It follows from the definition of essential supremum that $\mu(|m|\geqq r)>0$, and it clearly suffices to assume that $\mu(m\geqq r)>0$. Set $E=(m\geqq r)$. Then, as discussed in Sect. 1, we embed the measure algebra of Lebesgue measure on $[0,1]$ into the measure algebra of the nonatomic probability measure $v=\mu(E)^{-1}\mu|_E$, and we thereby produce a $\frac{1}{x}$-like function f on E (recall Definition 1.16). Treating f as a measurable function on X which vanishes on E^c, we observe at once that $D_a^b(f)=\mu(E)$ whenever $1\leqq a<b$. Thus, by (1.10) and (2.2), we have $I(f)=I_{\mathscr{U}}(f)=\mu(E)>0$. Since $mf\geqq rf\geqq 0$, it follows from (2.3.3) and (2.3.1) that $I_{\mathscr{U}}(mf)\geqq I_{\mathscr{U}}(rf)=rI_{\mathscr{U}}(f)=rI(f)$. Hence $\|T_{\mathscr{U}}(m)\|\geqq I_{\mathscr{U}}(mf/I(f))\geqq r$, as desired.

We complete the proof of the theorem by showing that the space

$$V=\operatorname{Span}\{T_{\mathscr{U}}(L_\infty(\mu)):\mathscr{U}\text{ is an ultrafilter; }\mathscr{U}\supset\mathscr{F}\}$$

is *norming*, which means that we have

$$I(f)=\sup\{|v(f)|:v\in V;\ \|v\|\leqq 1\}$$

for all $f\in WL_1$. First we select an ultrafilter $\mathscr{U}\supset\mathscr{F}$ such that $I(f)=I_{\mathscr{U}}(|f|)$. [For there is clearly a sequence $\{(a_n,b_n)\}_{n=1}^\infty$ such that $1\leqq a_n<b_n$ for all n, such that $b_n/a_n\to\infty$, and such that $I(f)=\lim_{n\to\infty} I_{a_n}^{b_n}(f)$. We may then incorporate the sets F_n and $G_n=\{(a_i,b_i):i\geqq n\}$ into $\mathscr{U}$.] Next we define $m=\chi(f\geqq 0)-\chi(f<0)$, and set $v=T_{\mathscr{U}}(m)$. Then

$$|v(f)|=|I_{\mathscr{U}}(mf)|=I_{\mathscr{U}}(|f|)=I(f)\,, \tag{2.8.1}$$

as desired. □

The standard integral not only constitutes a distinguished linear functional, but it also serves to define the L_1 norm. Likewise our "ersatz integral" $I_{\mathscr{U}}$ determines an L_1-like seminorm on WL_1 [and, by (2.7), on $WL_1^\wedge$]. Thus, for $f\in WL_1$, define

$$\|f\|_{\mathscr{U}}=I_{\mathscr{U}}(|f|)\,. \tag{2.9}$$

Lemma 2.3 translates into the following properties.

(2.10) $\|\cdot\|_{\mathscr{U}}$ is a lattice seminorm on WL_1.

(2.11) $\|f+g\|_{\mathscr{U}}=\|f\|_{\mathscr{U}}+\|g\|_{\mathscr{U}}$ whenever f and g are nonnegative.

And by (2.8.1), we have

(2.12) $I(f)=\sup\{\|f\|_{\mathscr{U}}:\mathscr{U}$ is an ultrafilter; $\mathscr{U}\supset\mathscr{F}\}$ for all $f\in WL_1$,

so that the seminorm I is now realized as the supremum of a family of L_1-like seminorms.

Again we convert $\|\cdot\|_{\mathscr{U}}$ into a norm by forming the ideal $N_{\mathscr{U}}=\{f\in WL_1 : \|f\|_{\mathscr{U}}=0\}$ and then the quotient vector lattice

$$(2.13) \qquad WL_1(\mathscr{U})=WL_1/N_{\mathscr{U}},$$

on which $\|\cdot\|_{\mathscr{U}}$ acts as a lattice norm. [And of course $WL_1(\mathscr{U})$ may also be regarded as a quotient of $W\hat{L_1}$.] Later on (in Corollary 3.13) we shall see that $WL_1(\mathscr{U})$ is never separable. For the moment we note that it (essentially) contains every function f such that $I(f)=\|f\|_{\mathscr{U}}$. In particular, it contains every function f which satisfies the following definition.

(2.14) *Definition.* A function $f\in WL_1$ is said to be a *limit norm function* if we have

$$I(f)=\lim\left\{I_a^b(f) : 1\leqq a<b;\frac{b}{a}\to\infty\right\}. \quad \square$$

As usual [cf. (1.10)], we also have

$$I(f)=\lim\left\{D_a^b(f) : 1\leqq a<b;\frac{b}{a}\to\infty\right\}.$$

Examples of limit norm functions include the $\frac{1}{x}$-like functions (Definition 1.16), the Cauchy distributed functions [see (1.14) and (1.15)], and, by (1.18.1), the function f which was defined in the proof of Proposition 1.18. Because of this latter, a slight finesse to the proof of Proposition 1.18 will yield the following fact.

(2.15) The space $WL_1(\mathscr{U})$ is not complete with respect to the $\|\cdot\|_{\mathscr{U}}$ norm.

Apart from this shortcoming, it is clear from (2.10) and (2.11) that $WL_1(\mathscr{U})$ satisfies every (other) axiom of an abstract L_1 space [13, Definition 1.b.1, p. 14], and so by [13, Theorem 1.b.2, p. 15] [applied to the completion of $WL_1(\mathscr{U})$]:

(2.16) The space $WL_1(\mathscr{U})$ is isometric and order isomorphic to a dense subspace of $L_1(\nu)$ for some measure ν.

It is natural to hope for some sort of relationship between μ and ν, and we shall now remark upon what is obvious in this connection.

Define an operator $\Phi : WL_1\to L_\infty(\mu)^*$ by

$$\Phi(\mathrm{f})=\hat{\mathrm{f}} : \mathrm{m}\mapsto I_{\mathscr{U}}(mf)$$

for all $f\in WL_1$, and for all $m\in L_\infty(\mu)$. It is easily checked that $\|\Phi(f)\|=\|f\|_{\mathscr{U}}$, and so:

(2.17) Φ determines an isometric, order isomorphic embedding of $WL_1(\mathscr{U})$ into the space $L_\infty(\mu)^*=L_1(\mu)^{**}$.

[The fact that $\Phi(f_1\vee f_2)=\Phi(f_1)\vee\Phi(f_2)$ follows from the identity $(f_1\vee f_2)m=f_1m\chi(f_1\geqq f_2)+f_2m\chi(f_1<f_2)$.]

(2.18) **Proposition.** *Let $\pi : L_1(\mu)\to L_1(\mu)^{**}$ denote the canonical embedding of $L_1(\mu)$ into its second dual. Then the ranges of π and Φ are disjoint.*

Proof. Given a function $f \in WL_1$ such that $\|f\|_{\mathscr{U}} > 0$, we are to show that there cannot exist any function $\varphi \in L_1(\mu)$ such that $\hat{f} = \hat{\varphi} = \pi(\varphi)$. We assume without loss of generality that $I_{\mathscr{U}}(f^+) > 0$. For $k = 1, 2, \ldots$, define $m_k = \chi(f > k) \in L_\infty(\mu)$. Then, by (1.8) and (2.7), we have $I_{\mathscr{U}}(fm_k) = I_{\mathscr{U}}(f^+) > 0$ for all k. If we did have $\hat{f} = \hat{\varphi}$ for some function $\varphi \in L_1(\mu)$, then $\left(\text{since } \bigcap_{k=1}^{\infty} (f > k) = \emptyset\right)$ the dominated convergence theorem would yield the contradiction

$$I_{\mathscr{U}}(fm_k) = \hat{f}(m_k) = \hat{\varphi}(m_k) = \int m_k \varphi \, d\mu \to 0. \qquad \square$$

In spite of this situation, there is a somewhat closer relationship between $WL_1(\mathscr{U})^*$ and $L_1(\mu)^* = L_\infty(\mu)$. A second look at the proof of Theorem 2.8, and in particular at the arguments leading up to statement (2.8.1), will reveal that:

(2.19) The operator $T_{\mathscr{U}}$ of Theorem 2.8 determines an isometric, order isomorphic embedding of $L_\infty(\mu)$ into $WL_1(\mathscr{U})^*$. Moreover, the range of this embedding is norming, and hence weak* dense, in $WL_1(\mathscr{U})^*$.

So $L_\infty(\mu)$ almost fills out the dual of $WL_1(\mathscr{U})$. But it *never* comprises the entire dual, as we now illustrate. (Note the parallel with Proposition 2.18.)

(2.20) **Example.** Fix any nonnegative function $f \in WL_1$ such that $\|f\|_{\mathscr{U}} = 1$. Let $\{D_k\}_{k=1}^{\infty}$ be any decreasing sequence of measurable sets with null intersection such that f is bounded on D_k^c for all k. For $k = 1, 2, \ldots$, define $m_k = \chi_{D_k}$. We identify $\hat{m}_k = T_{\mathscr{U}}(m_k)$ as a positive linear functional on $WL_1(\mathscr{U})$ with norm $= 1$. By Alaoglu's theorem, there is a subnet $\{\hat{m}_{k_\alpha}\}$ which converges in the weak* topology to a (positive!) linear functional ϕ on $WL_1(\mathscr{U})$ with norm $\leqq 1$. Since $\hat{m}_k(f) = I_{\mathscr{U}}(m_k f) = I_{\mathscr{U}}(f) = 1$ for all k, it follows that $\phi(f) = \|\phi\| = 1$, and so, in particular, $\phi \neq 0$. Suppose that we had $\phi = \hat{m}$ for some (necessarily nonnegative) function $m \in L_\infty(\mu)$. Then $\|m\|_\infty = \|\phi\| = 1$, so that $\mu(m > \frac{1}{2}) > 0$. Choose an index k so large that the set $E = D_k^c \cap (m > \frac{1}{2})$ has positive measure. As in the proof of Theorem 2.8, let g be a function on E which is $\frac{1}{x}$-like with respect to the measure $\mu(E)^{-1}\mu|_E$. Then we have $I_{\mathscr{U}}(g) = \mu(E) > 0$, whereas clearly $\phi(g) = 0$. But now $mg \geqq g/2 \geqq 0$, so that we are led to the contradiction $\phi(g) = I_{\mathscr{U}}(mg) \geqq I_{\mathscr{U}}(g/2) > 0$. $\square$

Finally, a routine juggling of stars and hats will reveal the following relationship between $T_{\mathscr{U}}$ and the adjoint operator Φ^* on $L_\infty(\mu)^{**}$.

(2.21) Let $\pi: L_\infty(\mu) \to L_\infty(\mu)^{**}$ denote the canonical embedding of $L_\infty(\mu)$ into its second dual. Then we have $T_{\mathscr{U}} = \Phi^* \pi$.

(2.22) *Question.* What is the relationship between the $\|\cdot\|_{\mathscr{U}}$ norm on $WL_1(\mathscr{U})$ and the quotient norm $\hat{I}$ of I? [Of course we have $\|f\|_{\mathscr{U}} \leqq \hat{I}(f)$, but are they equal?] $\square$

In summary, we have identified a number of quotients of WL_1 (and of $\widehat{WL_1}$) which are sufficient [by (2.12)] to determine the I topology, and all of which embed naturally in the second dual of $L_1(\mu)$. The structure of this space is fairly well understood [8, Theorem 16, p. 296], and so it holds out some promise as a vehicle to the further understanding of WL_1.

3. The L_1 Structure of $WL_1^{\hat{}}$

We have just seen [via (2.12)] that the I topology on $WL_1^{\hat{}}$ constitutes the supremum of a family of weaker vector topologies, each of which is determined by an L_1-like seminorm. In this section we identify a number of subspaces of $WL_1^{\hat{}}$ on which the I topology is itself L_1-like. Indeed, these subspaces will constitute isometric and order isomorphic copies of the familiar spaces $l_1[0,1]$ (Theorem 3.5), $L_1[0,1]$ (Theorem 3.7), and even the l_1 direct sum of $2^{\aleph_0}$ disjoint copies of $L_1[0,1]$ (Theorem 3.11). And we shall also show, in Theorems 3.6, 3.9, and 3.12, that they are complemented in $WL_1^{\hat{}}$. (These last results rely heavily upon the various linear functionals which were exhibited in Sect. 2.) Moreover, the copies of $l_1[0,1]$ and of $L_1[0,1]$ will "cover" $WL_1^{\hat{}}$ in the sense that every nonnegative function $h \in WL_1^{\hat{}}$ with $I(h)=1$ may serve as the image of a basis element, say $\chi\{0\}$, in $l_1[0,1]$, or it may serve as the image of the function $\equiv 1$ in $L_1[0,1]$.

Our strategy will be to fix an ultrafilter $\mathcal{U} \supset \mathcal{F}$ such that $I(h)=\|h\|_{\mathcal{U}}$ [see (2.1), (2.9), and the proof of Theorem 2.8]. We shall then see that the set

(3.1) $$L(\mathcal{U})=\{f \in WL_1 : I(f)=\|f\|_{\mathcal{U}}\},$$

on which I shares the vital L_1 additivity property (2.11) of $\|\cdot\|_{\mathcal{U}}$, is big enough to contain a (natural) copy of the given L_1 space [which is therefore also embedded in $WL_1(\mathcal{U})$ recall (2.13))].

So for the remainder of the section, we take a nonnegative function $h \in WL_1^{\hat{}}$ with $I(h)=1$ to be fixed, and we take an ultrafilter $\mathcal{U} \supset \mathcal{F}$ with $I(h)=\|h\|_{\mathcal{U}}=I_{\mathcal{U}}(h)$ to be fixed. Every unadorned L_p symbol will always refer to $L_p[0,1]$ with Lebesgue measure λ.

In order to "cover" $WL_1^{\hat{}}$ with complete spaces, we shall repeatedly use the partial completeness property in the following theorem. There are two prefatory lemmas.

(3.2) **Lemma.** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of disjointly supported measurable functions, and let f denote the pointwise sum of the series $\sum f_n$. Then, for $0<a<b$, we have*

$$I_a^b(f)=\sum_{n=1}^{\infty} I_a^b(f_n).$$

(recall (1.5)).

Proof. This follows at once from the set identity $(a \leqq |f| \leqq b) = \bigcup_{n=1}^{\infty} (a \leqq |f_n| \leqq b)$. □

(3.3) **Lemma.** *Let $f_1, \dots, f_n \in L(\mathcal{U})$ be pairwise essentially disjointly supported (see (3.1) and Definition 1.17). Then $\sum_{i=1}^{n} f_i \in L(\mathcal{U})$, and we have $I\left(\sum_{i=1}^{n} f_i\right)=\sum_{i=1}^{n} I(f_i)$.*

Proof. By (1.8) and (2.7), we may assume with no loss of generality that the f_i are strictly disjointly supported. Then Lemma 3.2, the addition rule for (ultrafilter) limits, and (2.3.4) combine to yield the following relations:

$$\begin{aligned}\sum_{i=1}^{n} I(f_i) &= \sum_{i=1}^{n} \|f_i\|_{\mathscr{U}} = \lim_{\mathscr{U}} I_a^b\left(\sum_{i=1}^{n} f_i\right) \\ &= \left\|\sum_{i=1}^{n} f_i\right\|_{\mathscr{U}} \leqq I\left(\sum_{i=1}^{n} f_i\right) \\ &\leqq \sum_{i=1}^{n} I(f_i)\,.\end{aligned}$$

There is consequently equality throughout. □

(3.4) **Theorem.** *Let $\{f_n\}_{n=1}^{\infty}$ be a pairwise essentially disjointly supported sequence in $L(\mathscr{U})$ such that:*

(3.4.1) $$\sum_{n=1}^{\infty} I(f_n) < \infty\,.$$

(3.4.2) *There exist positive constants k and K such that, whenever $0 < a < b$ and $b/a \geqq k$, we have $I_a^b(f_n) \leqq K I(f_n)$ simultaneously for all n.*

Then there exists a function $f \in L(\mathscr{U})$ such that

(3.4.3) $$I\left(f - \sum_{i=1}^{n} f_i\right) \to 0\,.$$

(3.4.4) $$I(f) = \sum_{n=1}^{\infty} I(f_n)\,.$$

Moreover, if the f_n are strictly disjointly supported, then f may be taken to be the pointwise sum of the series $\sum f_n$.

Proof. Since I is a seminorm, and since $I\left(\sum_{i=1}^{n} f_i\right) = \sum_{i=1}^{n} I(f_i)$ by Lemma 3.3, it follows that any function f which satisfies (3.4.3) will also satisfy (3.4.4). The basic idea is to *disjointize* the f_n in the following manner: For each index n, write $f_n = g_n + h_n$, where $I(g_n) = 0$, where g_n and h_n are strictly disjointly supported, and where the entire sequence $\{h_n\}$ is strictly disjointly supported. We shall obtain the g_n and the h_n in the next paragraph. (And we may clearly take $g_n = 0$ when the f_n are strictly disjointly supported to begin with.) By Lemma 3.2, we have $I_a^b(h_n) \leqq I_a^b(g_n) + I_a^b(h_n) = I_a^b(f_n)$, so that (3.4.2) will apply to the h_n in place of the f_n. We now define f to be the pointwise sum of the series $\sum h_n$. Let n be an arbitrary positive integer, and let $b/a \geqq k$. By Lemma 3.2 and then (3.4.2), we have $I_a^b\left(\sum_{i=n}^{\infty} h_i\right) = \sum_{i=n}^{\infty} I_a^b(h_i) \leqq K \sum_{i=n}^{\infty} I(h_i) = K \sum_{i=n}^{\infty} I(f_i)$. It is now clear from the definition of I [see (1.6)] and from (3.4.1) that $I\left(\sum_{i=n}^{\infty} h_i\right) \to 0$, so that (3.4.3) and hence (3.4.4) follow easily. In view of (3.4.4), the fact that $f \in L(\mathscr{U})$ follows from a straightforward "$\varepsilon/3$" argument. This argument makes critical use of condition (3.4.2) as it applies to the h_n.

It remains to carry out the disjointization of the f_n as described in the last paragraph. This we accomplish by induction. Set $E_1 = X$, set $h_1 = f_1\chi_{E_1} = f_1$, and set $g_1 = 0$. Suppose that sets $E_1, E_2, \ldots, E_{n-1}$ have been determined, and let $i < n$. Since f_n and f_i are essentially disjointly supported, there is a constant t_i such that f_i is bounded on the set $(|f_n| > t_i)$. Since the sets $(|f_n| > t)\downarrow\emptyset$ as $t\uparrow\infty$, the dominated convergence theorem ensures a choice of t_i for which $\int |f_i|\chi(|f_n| > t_i)d\mu \leqq 1/2^n$. Let $t = \max_{i<n} t_i$, and define $E_n = (|f_n| > t)$. Then we have

$$\int_{E_n} |f_i| d\mu \leqq 1/2^n \text{ for every index } i < n. \tag{3.4.5}$$

With the induction now complete, we define

$$h_n = f_n\chi\left(E_n \setminus \bigcup_{i=n+1}^{\infty} E_i\right) \quad \text{and} \tag{3.4.6}$$

$$g_n = f_n - h_n .$$

The h_n are strictly disjointly supported because, for $i<j$, we have $h_j \equiv 0$ *outside* of E_j, whereas $h_i \equiv 0$ *inside* E_j. It remains to check that $I(g_n) = 0$, and to this end it is sufficient, by (1.8), to establish that g_n is μ-integrable. Write $g_n = f_n\chi\left(E_n^c \cup \left(\bigcup_{i=n+1}^{\infty} E_i\right)\right)$. Then we have $\int |g_n| \leqq \int_{E_n^c} |f_n| + \sum_{i=n+1}^{\infty} \int_{E_i} |f_n|$, and this is finite by (3.4.5), and by the fact that f_n is bounded on E_n^c. □

The following two theorems deal with the embedding of $l_1[0,1]$ into $WL_1^{\hat{}}$.

(3.5) **Theorem.** *Let h be a nonnegative function in $L(\mathscr{U}) \subset WL_1^{\hat{}}$ (see* (3.1)*) such that $I(h) = 1$. Then there is an isometric, order isomorphic linear map T from $l_1[0,1]$ into $WL_1^{\hat{}}$ such that $T(\chi\{0\}) = h$.*

Note. We shall ensure that the range of T is $\subset L(\mathscr{U})$, so that T determines a corresponding embedding of $l_1[0,1]$ into $WL_1(\mathscr{U})$. A hint of this result, in a special case, appears in [5, p. 153].

Proof. We proceed immediately to set up an application of Theorem 3.4. Set $K = 2$, and let k be chosen sufficiently large that, whenever $0 < a < b$ and $b/a \geqq k$, we have

$$I_a^b(h) \leqq 2 = 2I(h) . \tag{3.5.1}$$

If $r > 0$, then we clearly have $rh \in L(\mathscr{U})$, and $I_a^b(rh) = rI_{a/r}^{b/r}(h)$, which is $\leqq 2r = 2I(rh)$ because $\dfrac{b/r}{a/r} = \dfrac{b}{a} \geqq k$. A similar statement is valid when $r < 0$. To set the stage further, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of (nonnegative) pairwise essentially disjointly supported functions each one of which has the same distribution as h [so that each is necessarily in $L(\mathscr{U})$]. To appreciate the significance of this requirement, we apply identity (1.1) with the function $\psi(t) = t\chi_{[a,b]}(t)$ to obtain

$$I_a^b(h) = \frac{1}{\ln(b/a)} \int t\chi_{[a,b]}(t) d\mu_h(t) .$$

It is now clear that if f_n has distribution measure μ_n for all n, then, with no change to the constant k, the inequality (3.5.1) will be simultaneously valid for all n with f_n in place of h. In this way condition (3.4.2) is effectively satisfied. Condition (3.4.1) may be satisfied by specifying a sequence $\{r_n\}_{n=1}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty} |r_n| < \infty$. Then Theorem 3.4 yields a function $f = \sum_{n=1}^{\infty} r_n f_n$ in $L(\mathscr{U})$ whose I-norm is precisely $\sum_{n=1}^{\infty} |r_n|$. This observation will be central to the definition of T. And to see clearly that T possesses the stated properties, it will perhaps be easiest to let $\{h_n\}$ be the disjointization of $\{f_n\}$ which was described in the proof of Theorem 3.4, so that the equivalence class of f is also represented by the *pointwise* sum $\sum_{n=1}^{\infty} h_n$.

The implication of these statements is that, in order to embed the space $l_1[0, 1]$ into $WL_1^{\wedge}$, we must determine a family $\{f_\alpha\}_{\alpha\in[0,1]}$ of pairwise essentially disjointly supported functions each of which has the same distribution as h. To this end, we rely heavily upon the nonatomicity of μ to produce a set $A_1 \in \Sigma$ such that $\mu(A_1) = 1/2$, and such that h is bounded on A_1^c. After $n-1$ steps, we produce a set $A_n \in \Sigma$ such that $A_n \subseteqq A_{n-1}$, such that $\mu(A_n) = 1/2^n$, and such that h is bounded on A_n^c. Next we determine a sequence of finite partitions $\mathscr{P}_n$ of X. Set $\mathscr{P}_1 = \{A_1, A_1^c\}$. At the n^{th} step, choose a partition $\mathscr{P}_n$ with the following properties: (i) $\mathscr{P}_n$ refines $\mathscr{P}_{n-1}$; (ii) $\mathscr{P}_n$ contains 2^n sets, each of μ-measure $1/2^n$; and (iii) $A_n \in \mathscr{P}_n$. Define a *spine* to be a decreasing sequence $S = \{S_n\}_{n=1}^{\infty}$ of sets such that $S_n \in \mathscr{P}_n$ for all n. Declare a function f to *live* on the spine S if f is bounded on S_n^c for all n. (For example, the function h lives on the spine $\{A_n\}$.) The key step is to produce such a function f which also has the same distribution as h. Clearly two functions which live on distinct spines are essentially disjointly supported. Since there are $2^{\aleph_0}$ distinct spines (which may consequently be indexed by the elements of [0, 1]), we thereby obtain the desired family $\{f_\alpha\}$.

We turn to the construction of a function f which lives on the spine S, and which has the same distribution as h. Set $H = A_1^c$, and set $F = S_1^c$. Consider the distribution of h with respect to the measure $\nu_H = \mu(H)^{-1}\mu|_H = 2\mu|_H$. There exists at least one Borel measurable function on [0, 1] whose distribution with respect to Lebesgue measure λ coincides with the distribution of h with respect to the measure ν_H. The canonical example of such a function is the *decreasing rearrangement* h^* of h. This is defined, for $0 < t \leqq 1$, by

$$h^*(t) = \min\{r \geqq 0 : \nu_H(h > r) \leqq t\}$$

[and it satisfies the set identity $(h^* > r) = (0, \nu_H(h > r))$ for every number $r \geqq 0$]. We embed the measure algebra of λ into the measure algebra of the nonatomic probability measure $\nu_F = \mu(F)^{-1}\mu|_F = 2\mu|_F$, and we thereby transfer h^* to a nonnegative, bounded function f on F whose distribution with respect to ν_l coincides with the distribution of h^* with respect to λ, and hence also of h with respect to ν_H. Thus, if B is a Borel subset of $\mathbb{R}$, we have $\nu_H(h^{-1}(B)) = \nu_F(f^{-1}(B))$. In terms of μ, the identity becomes $\mu(h^{-1}(B) \cap H) = \mu(f^{-1}(B) \cap F)$. This completes the definition of the function f on the set $F = S_1^c$. [Notice that the equality of $\mu(H)$ and $\mu(F)$ is critical to this argument.] In identical fashion, we determine f on the set

$S_n \backslash S_{n+1}$, for all $n \geqq 1$, by the requirement that

$$\mu(h^{-1}(B) \cap (A_n \backslash A_{n+1})) = \mu(f^{-1}(B) \cap (S_n \backslash S_{n+1}))$$

whenever B is a Borel subset of $\mathbb{R}$. This determines f on all of X except for the null set $\bigcap_{n=1}^{\infty} S_n$ on which f may be set $=0$. It is clear that f lives on the spine S and has the same distribution (with respect to μ) as h.

Therefore there does indeed exist a family $\{f_\alpha\}_{\alpha \in [0,1]}$ of pairwise essentially disjointly supported (nonnegative) functions on X each of which has the same distribution as h. For $u \in l_1[0, 1]$, define

$$T(u) = \sum_\alpha u(\alpha) f_\alpha .$$

Then, with suitable changes of notation, the remarks in the first paragraph of this proof will establish that T constitutes a well defined operator on $l_1[0, 1]$ which has all of the stated properties. We may ensure that $T(\chi\{0\}) = h$ simply by letting 0 be the index of the spine $\{A_n\}$, and by setting $f_0 = h$. □

(3.6) **Theorem.** *The range of the operator* T *in* Theorem 3.5 *is complemented in* $WL_1^{\hat{}}$.

Note. The proof below also shows that the corresponding image of $l_1[0, 1]$ in $WL_1(\mathscr{U})$ is complemented in $WL_1(\mathscr{U})$.

Proof. Let $\beta \in [0, 1]$, let $S = \{S_n\}_{n=1}^\infty$ be the spine which we have indexed by β in the proof of Theorem 3.5, and let f_β be the corresponding function which lives on S. Let S play the role of the sequence $\{D_k\}$, and let f_β play the role of the function f in Example 2.20. Let $\phi_\beta = \phi$ be the linear functional which was exhibited in that example (and which may clearly be regarded as a linear functional on $WL_1^{\hat{}}$), so that $\phi(f_\beta) = \|\phi_\beta\| = 1$. Then, for an arbitrary function $f \in WL_1^{\hat{}}$, the number $\phi_\beta(f)$ constitutes the limit of a subnet of the sequence $\{I_{\mathscr{U}}(\chi_{S_n} f)\}$, and so it is readily verified that $\phi_\beta(f_\alpha) = 0$ whenever $\alpha \neq \beta$. Fix an $\alpha \neq \beta$, let $R = \{R_k\}_{k=1}^\infty$ be the spine which is indexed by α, and let f_α be the corresponding function which lives on R. Then we have $R_k \cap S_n = \emptyset$ for all indices k and n sufficiently large. Fix such a pair of indices k and n, and fix a function $f \in WL_1$. Let $r = \operatorname{sgn} I_{\mathscr{U}}(\chi_{R_k} f)$, let $s = \operatorname{sgn} I_{\mathscr{U}}(\chi_{S_n} f)$, and define $m = r\chi_{R_k} + s\chi_{S_n}$ so that $\|m\|_\infty = 1$ (barring the trivial case $r = s = 0$). Then we have $|I_{\mathscr{U}}(\chi_{R_k} f)| + |I_{\mathscr{U}}(\chi_{S_n} f)| = I_{\mathscr{U}}(mf) \leqq \|m\|_\infty \|f\|_{\mathscr{U}} = \|f\|_{\mathscr{U}} \leqq I(f)$. By the addition rule for nets (which is a simple special case of [8, Lemma 6, p. 28]), we conclude that, in the limit,

$$|\phi_\alpha(f)| + |\phi_\beta(f)| \leqq \|f\|_{\mathscr{U}} \leqq I(f) .$$

A straightforward elaboration of this observation will establish that the operator P defined, for all $f \in WL_1^{\hat{}}$, by

$$P(f) = \sum_\alpha \phi_\alpha(f) f_\alpha$$

constitutes a norm one projection of $WL_1^{\hat{}}$ onto the range of T. □

The following two theorems deal with the embedding of $L_1 = L_1(\lambda)$ into $WL_1^{\hat{}}$.

(3.7) **Theorem.** *Let h be a nonnegative function in $L(\mathscr{U}) \subset W\hat{L_1}$ (recall* (3.1)*) such that $I(h) = 1$. Then there is an isometric, order isomorphic linear map T from L_1 into $W\hat{L_1}$ such that $T(\mathbf{1}) = h$.*

Note. Again we shall ensure that the range of T is $\subset L(\mathscr{U})$, so that T determines a corresponding embedding of L_1 into $WL_1(\mathscr{U})$.

Proof. It will be remembered that the function h serves only as a representative of an equivalence class in $W\hat{L_1}$. Hitherto an arbitrary choice of representative has sufficed, but in this proof we shall want to select h in such a way that its distribution has no atoms. If the given function h satisfies $\mu(h = r) > 0$ for some constant r, then, as usual, we embed the measure algebra of λ into the measure algebra of normalized μ-measure on the set $(h = r)$, and we replace h on this set by the image of, say, the function $\psi(t) = t + r$. Since there are at most countably many points $r \geqq 0$ for which $\mu(h = r) > 0$, the performance of such a replacement for each of these points will change h by at most a bounded measurable function. Hence we may assume with no loss of generality that h is everywhere strictly positive, and that its distribution μ_h is nonatomic.

Of vital importance will be a measure $v = v_h$ which we define, for every Borel subset B of $(0, \infty)$, by

$$v(B) = \int_B t\, d\mu_h(t) = \int h\chi_B(h)\, d\mu$$

[see (1.1)]. Clearly v is a σ-finite nonatomic measure which is finite on every finite interval [but *never* finite on all of $(0, \infty)$].

Let $\{p_n\}_{n=-\infty}^{\infty}$ be any strictly increasing sequence of positive constants such that $\lim_{n\to\infty} p_n = \infty$, such that $\lim_{n\to-\infty} p_n = 0$, and such that the successive ratios p_{n+1}/p_n are uniformly bounded by some constant $\kappa > 1$. (For example, the sequence $p_n = 2^n$ will do.) For each integer n, consider the restriction of the measure v to the interval $J_n = [p_n, p_{n+1}]$. Barring the case $v(J_n) = 0$ (which we may ignore), we let τ_n be an isomorphic (that is, a measure preserving) embedding of the measure algebra of λ into the measure algebra of the normalized measure $v(J_n)^{-1} v|_{J_n}$. The extension of τ_n to (equivalence classes of) measurable functions, as discussed in Sect. 1, preserves integrals, and so we have

$$\int_{J_n} \tau_n(\varphi)\, dv = v(J_n) \int \varphi\, d\lambda \tag{3.7.1}$$

for every function $\varphi \in L_1$. Treating $\tau_n(\varphi)$ as a function (rather than as an equivalence class) which is $\equiv 0$ on J_n^c, we define

$$\hat{\varphi} = \tau(\varphi) = \sum_{n=-\infty}^{\infty} \tau_n(\varphi)$$

(as a pointwise sum) for all $\varphi \in L_1$. Then τ shares with each τ_n the properties of being linear and of preserving the lattice operations.

We are now ready to define the operator $T: L_1 \to W\hat{L_1}$, for all $\varphi \in L_1$, by

$$T(\varphi) = h\hat{\varphi}(h), \tag{3.7.2}$$

where $\hat{\varphi}(h)$ denotes the composition of $\hat{\varphi}$ and h. It remains only to show that $T(\varphi) \in L(\mathscr{U})$, and that $I(T(\varphi)) = \|\varphi\|_1$, for the other stated properties of T follow at once from the corresponding properties of τ.

We begin with $\varphi = \chi_A$, so that $\hat{\varphi}$ may take the form χ_B for some Borel set $B \subseteqq (0, \infty)$. By (3.7.1), we have

$$v(J_n \cap B) = \int_{J_n} \chi_B dv = \int_{J_n} \tau_n(\chi_A) dv = v(J_n)\lambda(A). \tag{3.7.3}$$

Now take $0 < a < b$ to be fixed. Apply identity (1.1) with the function

$$\psi(t) = t\chi_B(t)\chi_{[a,b]}(t)$$

to obtain

$$\begin{aligned} \int h\chi_B(h)\chi(a \leqq h\chi_B(h) \leqq b) &= \int h\chi_B(h)\chi(a \leqq h \leqq b) \\ &= \int_a^b t\chi_B(t) d\mu_h(t). \end{aligned} \tag{3.7.4}$$

(The symbol $\int_a^b$ is unambiguous here because μ_h is nonatomic.) Let p_i be the largest of the p_n such that $p_i \leqq a$, and let p_j be the smallest of the p_n such that $b \leqq p_j$. Note that $a/p_i \leqq p_{i+1}/p_i \leqq \kappa$, and likewise $p_j/b \leqq \kappa$. Then, by (1.1) and (1.9), we have

$$\begin{aligned} \int_{p_i}^a t\chi_B(t) d\mu_h(t) \leqq \int_{p_i}^a t\, d\mu_h(t) &= \int h\chi(p_i \leqq h \leqq a) \\ &\leqq (2 + \ln \kappa)\varphi_1(h). \end{aligned}$$

Likewise, we have $\int_b^{p_j} t\chi_B(t) d\mu_h(t) \leqq (2 + \ln\kappa)\varphi_1(h)$. Upon writing

$$\int_a^b = \int_{p_i}^{p_j} - \int_{p_i}^a - \int_b^{p_j},$$

and upon taking (3.7.3) into account, we obtain

$$\int_a^b t\chi_B(t) d\mu_h(t) = \lambda(A) \int_{p_i}^{p_j} t\, d\mu_h(t) - \delta(a, b), \tag{3.7.5}$$

where $0 \leqq \delta(a, b) \leqq 2(2 + \ln\kappa)\varphi_1(h)$. An identical operation will convert $\int_{p_i}^{p_j}$ back into $\int_a^b$, and so, upon dividing both sides by $\ln(b/a)$, we obtain an identity of the form

$$I_a^b(h\chi_B(h)) = \lambda(A) I_a^b(h) + \frac{\lambda(A)\gamma(a, b) - \delta(a, b)}{\ln(b/a)},$$

where, again, $0 \leqq \gamma(a, b) \leqq 2(2 + \ln\kappa)\varphi_1(h)$. We now take the limit of both sides of this identity as $b/a \to \infty$ along $\mathscr{U}$, and we obtain $\|h\chi_B(h)\|_{\mathscr{U}} = \lambda(A)\|h\|_{\mathscr{U}} = \lambda(A)$. But since we also have

$$I_a^b(h\chi_B(h)) \leqq \lambda(A) I_a^b(h) + \frac{4(2 + \ln\kappa)}{\ln(b/a)} \varphi_1(h),$$

it follows that $I(h\chi_B(h)) \leqq \lambda(A)$. Hence $h\chi_B(h) \in L(\mathscr{U})$, and

$$I(h\chi_B(h)) = \lambda(A) = \|\chi_A\|_1 .$$

The next case to consider is $\varphi = r\chi_A$, so that $\hat{\varphi} = r\chi_B$ for some Borel set $B \subsetneq (0, \infty)$. It is already clear in this case that $T(\varphi) \in L(\mathscr{U})$, and that $I(T(\varphi)) = |r|\lambda(A) = \|\varphi\|_1$. However, we shall need to derive an approximation for $I_a^b(T(\varphi))$ which will enable us to apply Theorem 3.4 when we come to the case of an elementary measurable function φ. We shall assume that $r > 0$ in what follows, but the argument is valid for a general constant r if we replace the r below by $|r|$. Let $0 < a < b$ be fixed, let p_i be the largest of the p_n such that $p_i \leqq a/r$, and let p_j be the smallest of the p_n such that $b/r \leqq p_j$. Then, by arguing exactly as in (3.7.4) and (3.7.5) above, we obtain the relations

$$\int rh\chi_B(h)\chi(a \leqq rh\chi_B(h) \leqq b) \leqq r \int h\chi_B(h)\chi(p_i \leqq h\chi_B(h) \leqq p_j) = r\lambda(A) \int h\chi(p_i \leqq h \leqq p_j) . \tag{3.7.6}$$

We are now in a position to specify the positive constant k of condition (3.4.2) by requiring $k > 1$ to be so large that $I_a^b(h) \leqq I(h) + 1 = 2$ whenever $b/a \geqq k$. We then define $K = 2 + 4\ln\kappa/\ln k$. With note of the fact that $b/a \leqq p_j/p_i \leqq \kappa^2 b/a$, and in view of (3.7.6), we obtain

$$\begin{aligned} I_a^b(rh\chi_B(h)) &\leqq r\lambda(A)\frac{\ln(p_j/p_i)}{\ln(b/a)} I_{p_i}^{p_j}(h) \\ &\leqq r\lambda(A) \cdot K \\ &= KI(rh\chi_B(h)) . \end{aligned} \tag{3.7.7}$$

It is important to emphasize that k and K have been determined with no reference whatsoever either to r or to A.

The next case to consider is an elementary function $\varphi = \sum_{n=1}^{\infty} r_n\chi_{A_n}$, where the A_n are pairwise disjoint. Here we may set $\hat{\varphi} = \sum_{n=1} r_n\chi_{B_n}$, where the B_n comprise pairwise disjoint Borel subsets of $(0, \infty)$. Write $h\hat{\varphi}(h) = \sum_{n=1}^{\infty} r_n h\chi_{B_n}(h)$, and observe that the summands are disjointly supported. We propose to apply Theorem 3.4 with $f_n = r_n h\chi_{B_n}(h)$. Condition (3.4.1) is satisfied because $\sum_{n=1}^{\infty} I(f_n) = \sum_{i=1}^{\infty} |r_n|\lambda(A_n) = \|\varphi\|_1 < \infty$, while condition (3.4.2) follows at once from (3.7.7). We conclude that $T(\varphi) = h\hat{\varphi}(h) \in L(\mathscr{U})$, and that $I(T(\varphi)) = \|\varphi\|_1$.

We come finally to the case of an arbitrary function $\varphi \in L_1$. Let $\{\varphi_n\}$ be a sequence of elementary measurable functions in L_1 which converges *uniformly* to φ. Since the τ_n, and hence also τ, preserve order, it is clear that $\hat{\varphi}_n \to \hat{\varphi}$ uniformly on $(0, \infty)$ (except possibly on a ν-null set which we eliminate by altering $\hat{\varphi}_n$ and $\hat{\varphi}$ to be $= 0$ on that set). Therefore, for any number $\varepsilon > 0$, we have

$$|T(\varphi_n) - T(\varphi)| = h|\hat{\varphi}_n(h) - \hat{\varphi}(h)| \leqq \varepsilon h$$

whenever n is sufficiently large. Since I is a lattice norm and $\|\cdot\|_{\mathscr{U}}$ a lattice seminorm on $W\hat{L_1}$ [see (1.12) and (2.10)], it follows that $T(\varphi_n) \to T(\varphi)$ with respect to I and also with respect to $\|\cdot\|_{\mathscr{U}}$. From this it is clear that $T(\varphi)\in L(\mathscr{U})$, and, since $\|\varphi_n\|_1\to\|\varphi\|_1$, that $I(T(\varphi))=\|\varphi\|_1$. Thus the proof of the isometry of T, and hence of the entire theorem, is complete. □

(3.8) *Remark*. This result strengthens the case for considering the linear functional $I_{\mathscr{U}}$ to be an "ersatz integral" [recall (2.4)]. It is easily checked that we have $I_{\mathscr{U}}(T(\varphi))=I_{\mathscr{U}}(h\hat{\varphi}(h))=\int\varphi d\lambda$ for all $\varphi\in L_1$. □

(3.9) **Theorem.** *The range of the operator* T *in* Theorem 3.7 *is complemented in* $W\hat{L_1}$.

Note. The proof below, which owes a clear debt to [13, Lemma 1.b.9, pp. 19–20], also shows that the corresponding image of L_1 in $WL_1(\mathscr{U})$ is complemented in $WL_1(\mathscr{U})$.

Proof. It will be sufficient to obtain a norm one operator $P: W\hat{L_1}\to L_1$ whose restriction to the range of T coincides with T^{-1}. The operator TP then constitutes the desired projection.

Recall that, by identifying a function $\varphi\in L_1$ with its indefinite integral $\varphi d\lambda: A\mapsto\int_A\varphi d\lambda$, we may realize L_1 as a natural subspace of the space $C[0,1]^*$. And in view of the Lebesgue decomposition theorem, it is complemented in $C[0,1]^*$. (See [13, p. 20] for further details.) Therefore it will be sufficient to obtain a norm one operator $P: W\hat{L_1}\to C[0,1]^*$ whose restriction to the range of T coincides with T^{-1}.

To this end, let $\mathscr{A}=\{A_1,\ldots,A_n\}$ be a partition of $[0,1]$ into λ-nonnull Borel sets. We make the collection of all such partitions into a directed set by *a.e. refinement* – that is, $\mathscr{B}$ refines $\mathscr{A}$ *(a.e.)* if, for every set $B\in\mathscr{B}$, there is a set $A\in\mathscr{A}$ such that $\lambda(B\backslash A)=0$. Given $\mathscr{A}$ as above, given $f\in W\hat{L_1}$, and recalling the function h of Theorem 3.7, we define

$$P(f|\mathscr{A})=\sum_{i=1}^{n} I_{\mathscr{U}}(f\hat{\chi}_{A_i}(h))\frac{1}{\lambda(A_i)}\chi_{A_i}.$$

Then $P(\cdot|\mathscr{A})$ is linear because $I_{\mathscr{U}}$ is linear [recall (2.5)], and the identity $\|f\|_{\mathscr{U}}=I_{\mathscr{U}}(|f|)=\sum_{i=1}^{n} I_{\mathscr{U}}(|f|\hat{\chi}_{A_i}(h))$, in conjunction with (2.6), shows at once that we have $\|P(f|\mathscr{A})\|_1\leqq\|f\|_{\mathscr{U}}\leqq I(f)$. (Of course we are also treating $P(f|\mathscr{A})$ as a member of $C[0,1]^*$.) Now let $\{P(f|\mathscr{A}_\zeta)\}_{\zeta\in Z}$ be a *universal* subnet of the net $\{P(f|\mathscr{A})\}$ (see [10, Exercise J, p. 81]). Then, by Alaoglu's theorem, the weak* limit

$$P(f)=\lim_{\zeta} P(f|\mathscr{A}_\zeta)$$

exists in $C[0,1]^*$ for all $f\in W\hat{L_1}$, and we also have $\|P(f)\|\leqq\|f\|_{\mathscr{U}}\leqq I(f)$ (where $\|\cdot\|$ denotes the norm in $C[0,1]^*$). Moreover, because of the crucial fact that the directed set Z which determines the universal subnet may be chosen entirely independently of f, it is clear that P is linear. So it remains only to check that $P(T(\varphi))=P(h\hat{\varphi}(h))=\varphi$ for all $\varphi\in L_1$. But in view of Remark 3.8, we see that $P(h\hat{\varphi}(h)|\mathscr{A})=\sum_{i=1}^{n}\left(\frac{1}{\lambda(A_i)}\int_{A_i}\varphi d\lambda\right)\chi_{A_i}$, and a brief exercise in elementary measure theory reveals that this is already norm convergent (and hence weak* convergent) to φ. □

The existence of a complemented copy of L_1 in $WL_1^{\wedge}$ has been independently established by Kalton.

Let $\mathfrak{L}$ denote the l_1 direct sum of $2^{\aleph_0}$ disjoint copies of the space L_1. The following lemma and theorems deal with the embedding of $\mathfrak{L}$ in $WL_1^{\wedge}$ [and in $WL_1(\mathcal{U})$].

(3.10) **Lemma.** *Let T be the operator of* Theorem 3.7. *Then there exist positive constants k and K such that, for every function f in the range of T, we have $I_a^b(f) \leqq KI(f)$ whenever $0<a<b$ and $b/a \geqq k$.*

Proof. We take for k and K the constants defined just prior to (3.7.7). Then (3.7.7) in conjunction with Lemma 3.2 implies the result whenever f is the T-image of an elementary function in L_1. Let $\varphi \in L_1$ be arbitrary, and let $\{\varphi_n\}$ be a sequence of elementary functions in L_1 which converges uniformly to φ. Set $f_n = |T(\varphi_n)|$, and set $f = |T(\varphi)|$. Then, as discussed in the proof of Theorem 3.7, we have $f_n \to f$ pointwise, and we also have $I(f_n) \to I(f)$. It is clear that $f_n\chi(a<f_n<b) \to f\chi(a<f<b)$ pointwise, and so, whenever $b/a \geqq k$, it follows from the dominated convergence theorem that

$$\frac{1}{\ln(b/a)} \int f\chi(a<f<b) \leqq KI(f).$$

But then

$$I_a^b(f) = \lim_{t\to 0^+} \frac{1}{\ln\left(\dfrac{b+t}{a-t}\right)} \int f\chi(a-t<f<b+t),$$

and so the result follows. □

The next theorem is an amalgamation and generalization of Theorems 3.5 and 3.7.

(3.11) **Theorem.** *There exists an isometric, order isomorphic embedding of the space $\mathfrak{L}$ into $WL_1^{\wedge}$.*

Note. As usual, the range of this map will be $\subset L(\mathcal{U})$, so that $\mathfrak{L}$ embeds in $WL_1(\mathcal{U})$ as well.

Proof. The first step is to adjust h as in the proof of Theorem 3.7, and then to create, exactly as in the proof of Theorem 3.5, a family $\{f_\alpha\}_{\alpha\in[0,1]}$ of pairwise essentially disjointly supported, everywhere strictly positive functions each of which has the same (nonatomic!) distribution as h. We then use each f_α, exactly as in the proof of Theorem 3.7, to create an isometric, order isomorphic copy of L_1 in $WL_1^{\wedge}$. Notice that the measure v is independent of α, and let the constants p_n and the maps τ_n also be determined independently of α. We shall use these copies as the basis for our construction of a copy of $\mathfrak{L}$ in $WL_1^{\wedge}$ [and in $WL_1(\mathcal{U})$].

We first verify that, for $\alpha \neq \beta$, the two copies of L_1 corresponding to α and β are indeed disjoint. To this end, set $f = f_\alpha$, set $g = f_\beta$, and let φ and γ be arbitrary nonzero elements of L_1. Then we are to show that the functions $f\hat{\varphi}(f)$ and $g\hat{\gamma}(g)$ represent distinct elements of $WL_1^{\wedge}$.

To this end, let $t>0$ be arbitrary, and fix any index n such that $p_n \geqq t$. Then by (1.1) and (3.7.1), we have

$$(3.11.1) \qquad \int f|\hat{\varphi}(f)|\chi(f \leqq t)d\mu = \int_0^t s|\hat{\varphi}(s)|d\mu_f(s) = \int_0^t |\hat{\varphi}(s)|dv(s)$$
$$\leqq \sum_{i=-\infty}^{n-1} \int_{J_i} \tau_i(|\varphi|)dv = \sum_{i=-\infty}^{n-1} v(J_i) \int |\varphi| d\lambda$$
$$= v(0, p_n] \, \|\varphi\|_1 < \infty \, .$$

It now follows from (1.7) and (1.8) that the functions $f\hat{\varphi}(f)$ and $f\hat{\varphi}(f)\chi(f>t)$ are I-equivalent. Since t may be chosen so that the sets $(f>t)$ and $(g>t)$ are disjoint, it is clear that $f\hat{\varphi}(f)$ and $g\hat{\gamma}(g)$ represent lattice disjoint, and hence distinct, elements of $WL_1^{\hat{}}$.

Secondly we investigate the infinite sums of elements taken from the various copies of L_1. A typical element of a copy of $\mathfrak{L}$ in $WL_1^{\hat{}}$ which *might* be determined by these copies of L_1 would have the form $\sum_{n=1}^{\infty} f_n \hat{\varphi}_n(f_n)$, where $f_n = f_{\alpha_n}$ (and the α_n are all distinct), and where $\{\varphi_n\}_{n=1}^{\infty}$ is a sequence in L_1 such that $\sum_{n=1}^{\infty} \|\varphi_n\|_1 < \infty$. Let $\{h_n\}_{n=1}^{\infty}$ be the disjointization of the f_n which was described in the proof of Theorem 3.4. Recall that the number t_i (for $i<n$) was picked so large that $\int f_i \chi(f_n > t_i)d\mu \leqq 1/2^n$. In this situation, let us *also* select t_i so large that

$$(3.11.2) \qquad \int f_i |\hat{\varphi}_i(f_i)| \chi(f_n > t_i) d\mu \leqq 1/2^n \, .$$

This is possible because of the fact that f_i is bounded on the set $(f_n > t_i)$, because of (3.11.1), and because of the dominated convergence theorem. In all other respects the construction of the h_n is exactly as described in Theorem 3.4. The additional requirement (3.11.2) ensures that, with h_n and g_n as defined in (3.4.6), we have $I(g_n) = I(g_n \hat{\varphi}_n(f_n)) = 0$, so that the functions $f_n \hat{\varphi}_n(f_n)$ and $h_n \hat{\varphi}_n(f_n)$ are I-equivalent. [And the $h_n \hat{\varphi}_n(f_n)$ are strictly disjointly supported.] Notice from the proof of Theorem 3.7, and from the fact that the f_α are identically distributed, that the constants k and K of Lemma 3.10 are precisely the same for *each one* of our $2^{\aleph_0}$ copies of L_1 in $WL_1^{\hat{}}$. Therefore, Lemma 3.2 and Lemma 3.10 combine with the fact that $I(f_n \hat{\varphi}_n(f_n)) = \|\varphi_n\|_1$ (for all n) to make Theorem 3.4 applicable to the sequence $\{h_n \hat{\varphi}_n(f_n)\}$. We conclude that the pointwise sum $f = \sum_{n=1}^{\infty} h_n \hat{\varphi}_n(f_n)$ belongs to $L(\mathscr{U})$, and that

$$I(f) = \sum_{n=1}^{\infty} I(h_n \hat{\varphi}_n(f_n)) = \sum_{n=1}^{\infty} I(f_n \hat{\varphi}_n(f_n)) = \sum_{n=1}^{\infty} \|\varphi_n\|_1 < \infty \, .$$

It is now clear that the collection of (equivalence classes of) functions f as above constitutes an isometric, order isomorphic copy of $\mathfrak{L}$ in the space $WL_1^{\hat{}}$ [and in the space $WL_1(\mathscr{U})$]. □

(3.12) **Theorem.** *The image of* $\mathfrak{L}$ *in* $WL_1^{\hat{}}$ *which was exhibited in* Theorem 3.11 *is complemented in* $WL_1^{\hat{}}$.

Note. And, as usual, the corresponding image of $\mathfrak{L}$ in $WL_1(\mathscr{U})$ is complemented in $WL_1(\mathscr{U})$.

Proof. The argument relies upon and elaborates that of Theorem 3.9. Let $\{f_\alpha\}_{\alpha\in[0,1]}$ be the family of functions which was used in the proof of Theorem 3.11 to obtain the image of $\mathfrak{L}$. Such a family also appears in the proofs of Theorems 3.5 and 3.6, and we let $\{\phi_\alpha\}_{\alpha\in[0,1]}$ be the family of linear functionals on $WL_1^{\hat{}}$ which was introduced in the proof of Theorem 3.6. A careful look at the definition of ϕ_α (see also Example 2.20), in conjunction with (3.11.1) and Remark 3.8, will reveal the following facts.

(3.12.1) $\quad \phi_\alpha(f_\alpha\hat{\varphi}(f_\alpha)) = I_{\mathscr{U}}(f_\alpha\hat{\varphi}(f_\alpha)) = \int \varphi d\lambda$ for all $\varphi\in L_1$.

(3.12.2) $\quad \phi_\alpha(f_\beta\hat{\varphi}(f_\beta)) = 0$ for all $\varphi\in L_1$, and for all $\beta\neq\alpha$.

With α fixed, we repeat the construction in the proof of Theorem 3.9 with ϕ_α in place of $I_{\mathscr{U}}$ to produce a norm one operator $P_\alpha: WL_1^{\hat{}} \to C[0,1]^*$ such that $\|P_\alpha(f)\| \leqq \phi_\alpha(|f|)$ for all $f\in WL_1^{\hat{}}$. It is interesting to note that, because of (3.12.1), P_α determines a projection of $WL_1^{\hat{}}$ onto L_1 (or, more precisely, onto the copy of L_1 determined by f_α) in exactly the manner described by the proof of Theorem 3.9. But it is somehow a "tighter" projection in the sense that we have the inequalities

(3.12.3) $\quad \Sigma_\alpha\|P_\alpha(f)\| \leqq \Sigma_\alpha\phi_\alpha(|f|) \leqq \|f\|_{\mathscr{U}} \leqq I(f)$

directly from the proof of Theorem 3.6. Let Q denote the canonical (norm one) projection of $C[0,1]^*$ onto $L_1 \subset C[0,1]^*$. Define an operator $P: WL_1^{\hat{}} \to \mathfrak{L}$ by declaring that, for all $f\in WL_1^{\hat{}}$, the α^{th} component of $P(f)$ is to be precisely $Q(P_\alpha(f))$. Then it is clear from (3.12.3) that P does indeed map into $\mathfrak{L}$, and that $\|P\| \leqq 1$. To see that P determines a norm one projection of $WL_1^{\hat{}}$ onto the copy of $\mathfrak{L}$ in $WL_1^{\hat{}}$ (in the manner of Theorem 3.9), it remains to check that, if $f = \Sigma_\beta f_\beta\hat{\varphi}_\beta(f_\beta)$ is an arbitrary element of this copy of $\mathfrak{L}$ (where $\varphi_\beta\in L_1$ and $\Sigma_\beta\|\varphi_\beta\|_1 < \infty$), then the α^{th} component of $P(f)$ is precisely φ_α. To this end, recall that $P_\alpha(f)$ is the limit of a net of numbers of the form

$$P_\alpha(f|\mathscr{A}) = \sum_{i=1}^{n} \phi_\alpha(f\hat{\chi}_{A_i}(f_\alpha))\frac{1}{\lambda(A_i)}\chi_{A_i},$$

where $\mathscr{A} = \{A_1, \ldots, A_n\}$ is a partition of $[0,1]$ into λ-nonnull Borel sets. By both (3.12.1) and (3.12.2), and by both the continuity and the positivity of ϕ_α, we have $P_\alpha(f|\mathscr{A}) = \sum_{i=1}^{n}\left(\frac{1}{\lambda(A_i)}\int_{A_i}\varphi_\alpha d\lambda\right)\chi_{A_i}$, and this, as before, does indeed converge to $\varphi_\alpha = Q(\varphi_\alpha)$. □

(3.13) **Corollary.** *The space $WL_1(\mathscr{U})$ (recall (2.13)) is not separable.*

Proof. By Theorem 3.11, it contains the nonseparable space $\mathfrak{L}$. □

(3.14) **Corollary.** *The space $L_1(\mu)^{**}$ contains an isometric, order isomorphic copy of $\mathfrak{L}$ (and hence also of $l_1[0,1]$).*

Proof. This follows from the corresponding fact about $WL_1(\mathscr{U})$ and from (2.17). □

The case $\mu=\lambda$ seems worthy of special mention.

(3.15) **Corollary.** *The second dual L_1^{**} of the space L_1 (of Lebesgue measure on the unit interval) contains an isometric, order isomorphic copy of the l_1 direct sum of $2^{\aleph_0}$ disjoint copies of L_1.*

Remark. And by Proposition 2.18 there is a copy of this l_1 direct sum in L_1^{**} which is disjoint from the canonical image of L_1 in L_1^{**}! □

In view of the relations between $L_1(\mu)$ and $WL_1(\mathscr{U})$ [see (2.17) and (2.19)], and in view of the unlikelihood of Corollary 3.15 providing a property which is exclusive to Lebesgue measure, we ask:

(3.16) *Question.* Does $L_1(\mu)$ embed in $W\hat{L_1}$? [Therefore: Does $L_1(\mu)^{**}$ contain the l_1 direct sum of $2^{\aleph_0}$ disjoint copies of $L_1(\mu)$?] □

We conclude this section with a lighthearted look at the "unpleasant" operator $T: l_1 \to L_0(\mu)$ of [17, Lemma 2.4, p. 231]. This operator also maps into WL_1, and it illustrates that, very occasionally, the quasinorm φ_1 really does try to behave in an "L_1-like" fashion.

(3.17) **Proposition.** *There exists a linear map $T: l_1 \to WL_1$ such that $\varphi_1(T(u)) = \|u\|_1$ for every point $u \in l_1$. Moreover, the restriction of the quotient map (of WL_1 onto $W\hat{L_1}$) to the range of T constitutes an isometric isomorphism.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed measurable functions on X each of which has the Cauchy distribution with parameter $\delta = 1$. (Such a sequence can be explicitly constructed on $[0, 1]$, as indicated, for example, in [17, Lemma 2.2, p. 231], and then it may be transferred to X via the usual measure algebra embedding.) An elementary exercise in probability [2, Exercise 20.20, p. 235] shows that, for every choice of scalars $r_1, \ldots, r_n$, the function $\sum_{i=1}^{n} r_i f_i$ has the Cauchy distribution with parameter $\sum_{i=1}^{n} |r_i|$. Therefore, if $\{r_n\}_{n=1}^{\infty}$ is an infinite sequence of scalars such that $\sum_{n=1}^{\infty} |r_n| < \infty$, then it follows from (1.15) that the sequence of partial sums $\sum_{i=1}^{n} r_i f_i$ is φ_1-Cauchy. Hence, by (1.2), the φ_1 limit $f = \sum_{n=1}^{\infty} r_n f_n$ is well defined in WL_1. Since φ_1 convergence implies convergence in μ-measure, a standard result in probability [2, Theorem 2.5.2, p. 284] implies easily that f has the Cauchy distribution with parameter $\sum_{n=1}^{\infty} |r_n|$. It is now an immediate consequence of (1.15) that the operator $T: l_1 \to WL_1$ defined, for each point $u \in l_1$, by

$$T(u) = \frac{\pi}{2} \sum_{n=1}^{\infty} u(n) f_n$$

satisfies all of the stated requirements. □

4. The L_∞ Structure of $W\hat{L_1}$

This section will attempt to balance the picture of $W\hat{L_1}$ by indicating in an explicit way that the L_1 structure of $W\hat{L_1}$ does not tell the whole story about this space. It also contains natural copies of l_∞ (Theorem 4.2) and of $c_0[0, 1]$ (Theorem 4.4).

Underpinning these results is the following construction: Let h be a fixed nonnegative function in $WL_1^\wedge$ such that $I(h)=1$. Recalling from (1.8) that the functions h and $h\chi(h>t)$ are I-equivalent for all $t>0$, we see easily that there exists a sequence $\{[a_n, b_n]\}_{n=1}^{\infty}$ of *pairwise disjoint* closed intervals such that $b_n/a_n \to \infty$, and such that $I_{a_n}^{b_n}(h) \geqq I(h) - n^{-1}$ for all n. For $S \subseteq \mathbb{N}$, define $B_S = \bigcup_{n \in S} [a_n, b_n]$. Then it is easily checked that we have

$$(4.1) \qquad I(h\chi_{B_S}(h)) = I(h) = 1 \text{ whenever } S \text{ is } \textit{infinite}$$

[where, as usual, $\chi_{B_S}(h)$ denotes the composition of χ_{B_S} and h].

(4.2) **Theorem.** *There is an isometric, order isomorphic embedding of the space* l_∞ *into* $WL_1^\wedge$.

Proof. Write $\mathbb{N} = \bigcup_{i=1}^{\infty} S_i$, where the S_i are infinite and pairwise disjoint. It follows that the sets $B_i = B_{S_i}$ are pairwise disjoint, and therefore that the functions $h\chi_{B_i}(h)$ are (strictly) disjointly supported. Given a point $u \in l_\infty$, define

$$T(u) = \sum_{i=1}^{\infty} u(i) h\chi_{B_i}(h)$$

(as a pointwise sum). Clearly T is linear and preserves the lattice operations. To see that T is isometric, fix a positive integer i and note that $|u(i)| h\chi_{B_i}(h) \leqq |T(u)| \leqq \|u\|_\infty h$. By (1.12) and (4.1), we have $|u(i)| \leqq I(T(u)) \leqq \|u\|_\infty$, and the result follows. □

(4.3) **Corollary.** *The space* $WL_1^\wedge$ *is not* σ*-order continuous* [13, p. 7] *with respect to the I norm.* □

Observe that we now have three distinct ways to embed the space l_1 into $WL_1^\wedge$: via Theorem 3.5, via Proposition 3.17, and via Theorem 4.2 in conjunction with the fact that every separable normed space embeds in l_∞.

(4.4) **Theorem.** *There is an isometric, order isomorphic embedding of the space* $c_0[0,1]$ *into* $WL_1^\wedge$.

Proof. This time let $\{S_\alpha\}_{\alpha \in [0,1]}$ be an *almost* pairwise disjoint family of infinite subsets of $\mathbb{N}$ [1, Lemma 5.2, p. 136]. (This means that the set $S_\alpha \cap S_\beta$ is finite whenever $\alpha \neq \beta$.) It follows that the sets $B_\alpha = B_{S_\alpha}$ intersect pairwise in at most a finite union of the intervals $[a_n, b_n]$. Carrying on from Theorem 4.2, we should like a typical element of a copy of $c_0[0,1]$ in $WL_1^\wedge$ to have the form $\sum_{i=1}^{\infty} r_i h\chi_{B_i}(h)$, where $B_i = B_{\alpha_i}$ (and the α_i are all distinct), and where $\{r_i\}_{i=1}^{\infty}$ is a sequence of scalars such that $r_i \to 0$. To achieve this, we disjointize the B_i in the usual way: Set $D_1 = B_1$, and for $i>1$, set $D_i = B_i \backslash (B_1 \cup \ldots \cup B_{i-1})$. Notice that D_i is obtained from B_i by removing at most finitely many of the intervals $[a_n, b_n]$ from B_i. Therefore the functions $h\chi_{B_i}(h)$ and $h\chi_{D_i}(h)$ are I-equivalent. Since the D_i are pairwise disjoint, we have

$$I\left(\sum_{i=1}^{\infty} r_i h\chi_{D_i}(h)\right) = \max_i |r_i|$$

exactly as in the proof of Theorem 4.2. A similar remark applies to sums of the form $\sum_{i=n}^{\infty} r_i h\chi_{D_i}(h)$, where $n=1,2,\ldots$. Since $r_i \to 0$, it is clear that the original series $\sum r_i h\chi_{B_i}(h)$ is (unconditionally) convergent in $WL_1^\wedge$, and that the collection of sums of series of this form constitutes an isometric, order isomorphic copy of the space $c_0[0,1]$ in $WL_1^\wedge$. □

(4.5) *Question.* Let h be a fixed (nonnegative) function in $WL_1^\wedge$. We have seen (in Theorem 3.7) that there exist nontrivial sets $E \in \Sigma$ such that $I(h)=I(h\chi_E)+I(h\chi_{E^c})$, and we have seen (in the discussion prior to Theorem 4.2) that there exist nontrivial sets $E \in \Sigma$ such that $I(h)=\max\{I(h\chi_E), I(h\chi_{E^c})\}$. What can be said about $h\chi_E$ for an arbitrary set $E \in \Sigma$ (or at least for a set E in the σ-algebra generated by h)? □

The I norm on $WL_1^\wedge$, as defined by (1.6), contains an "L_1 part" (that is, an integral) and an "L_∞ part" (that is, a supremum). These features seem to be reflected in the structure of $WL_1^\wedge$. What we are really asking in Question 4.5 is: How far would a complete understanding of L_1 and of L_∞ spaces take us toward a complete understanding of $WL_1^\wedge$?

One particular instance of this general problem seems to be worthy of mention. Let us point out that:

(4.6) *Remark.* One may create $2^{\aleph_0}$ disjoint copies of $L_1=L_1(\lambda)$ in $WL_1^\wedge$ not only by means of a family of $2^{\aleph_0}$ pairwise essentially disjointly supported copies of the function h as in Theorem 3.11, but also by subdividing h in the manner of Theorem 4.4. To see this, we choose h to be everywhere strictly positive, to have nonatomic distribution, and to have $I(h)=1$ as usual. Recall the sequence $\{p_n\}$, the measure $\nu=\nu_h$, and the linear map $\tau: \varphi \mapsto \hat{\varphi}$ which were used in Theorem 3.7 to obtain the embedding $T: \varphi \mapsto h\hat{\varphi}(h)$ of L_1 into $WL_1^\wedge$. Let us now fix a set $E \in \Sigma$ such that $I(h\chi_E)=1$. We propose to apply Theorem 3.7 to the function $h\chi_E$. To do this, we must modify $h\chi_E$ on the set $E^c=(h\chi_E=0)$ in such a way that the new function, call it h_E, assumes values on E^c in, say, the interval $(0,1]$, and also has a nonatomic distribution measure. With the p_n unchanged, but with a new measure $\nu_E=\nu_{h_E}$ and a new linear mapping $\tau_E: \varphi \mapsto \bar{\varphi}$, the proof of Theorem 3.7 yields an isometric, order isomorphic linear map $T_E: L_1 \to WL_1^\wedge$ which is defined, for all $\varphi \in L_1$, by $T_E(\varphi)=h_E\bar{\varphi}(h_E)$. Since the functions h and h_E coincide on the set E, and since $E^c \subseteqq (h_E \leqq 1)$, it follows as in (3.11.1) that the functions $h_E\bar{\varphi}(h_E)$ and $h_E\bar{\varphi}(h_E)\chi_E = h\bar{\varphi}(h)\chi_E$ are I-equivalent. Therefore, since the values of T_E comprise equivalence classes and not functions, we are free to write $T_E(\varphi)=h\bar{\varphi}(h)\chi_E$.

Armed with this information, let us re-examine the proof of Theorem 4.4. Recalling all of the notation in that proof, let us define, for every index $\alpha \in [0,1]$, the set $E_\alpha=h^{-1}(B_\alpha)$ [so that $h\chi_{E_\alpha}=\chi_{B_\alpha}(h)$], and then let us define the operator $T_\alpha=T_{E_\alpha}$ exactly as above. In this way we obtain $2^{\aleph_0}$ copies of L_1 in $WL_1^\wedge$ [of the form $T_\alpha(L_1)$ for $\alpha \in [0,1]$], and the disjointness of these spaces follows, via (3.11.1), just as in the proof of Theorem 3.11. □

(4.7) *Problem.* Characterize the closed linear span in $WL_1^\wedge$ of the set $\bigcup_\alpha T_\alpha(L_1)$. □

The only thing we are able to say for certain about this closed linear span is that, unlike the corresponding closed linear span in Theorem 3.11, it is *not* an l_1

direct sum. It is *perhaps* something like an l_∞ direct sum when a natural choice is made of the maps $\tau_\alpha = \tau_{E_\alpha}$. We illustrate this statement as follows: Let R and S be disjoint infinite subsets of $\mathbb{N}$, let $B = \bigcup_{n \in R} [a_n, b_n]$ (with the a_n and b_n as discussed at the beginning of this section), let $E = h^{-1}(B)$, let $C = \bigcup_{n \in S} [a_n, b_n]$, and let $F = h^{-1}(C)$. For maximum clarity, let us assume (as we may) that $a_n > 1$ for all n, and that the constants p_n in the proof of Theorem 3.7 are chosen to include all of the a_n and b_n. Observe in this case that the measure ν_E coincides with the measure $\nu = \nu_h$ on the Borel subsets of B. Therefore we are free (and it seems natural) to make the linear maps τ and τ_E coincide on every interval $J_n = [p_n, p_{n+1}] \subseteqq B$, so that $\tau_E(\varphi)\chi_B = \tau(\varphi)\chi_B = \hat{\varphi}\chi_B$ for every $\varphi \in L_1$. And we do likewise for τ_F. Now let $\varphi, \gamma \in L_1$ be arbitrary and consider the sum

$$\begin{aligned} f &= T_E(\varphi) + T_F(\gamma) \\ &= h\tau_E(\varphi)(h)\chi_E + h\tau_F(\gamma)(h)\chi_F \\ &= h\hat{\varphi}(h)\chi_E + h\hat{\gamma}(h)\chi_F . \end{aligned}$$

In the special case that $0 \leqq \varphi \leqq \gamma$, an argument as in the proof of Theorem 4.2 will quickly establish that $I(f) = \|\gamma\|_1$. But if $\varphi, \gamma \geqq 0$ are arbitrary, the most we are able to say at present is that $\max\{\|\varphi\|_1, \|\gamma\|_1\} \leqq I(f) \leqq \|\varphi \vee \gamma\|_1$. We have been unable to pin down the value of $I(f)$ even when φ and γ are the characteristic functions of (disjoint) measurable sets. (But then this is an instance of Question 4.5.)

5. A Remark About the L_0 Factorization Problem

It might be hoped that an $L(\mu)$-valued operator (on a Banach space) which does not already map into $L_1(\mu)$ might somehow be "large" enough that its Nikisin factorization through WL_1 (see [16, Theorem 3, p. 797] and [9, § 2]) would induce a natural factorization through one of the $WL_1(\mathscr{U})$ spaces [recall (2.13) and (2.16)]. The principal result of this section, which follows immediately, serves inter alia to destroy that hope.

(5.1) **Theorem.** *Let $\delta > 0$. Then there is an order isomorphic linear map $T: l_\infty \to WL_1$ such that, for all $u \in l_\infty$, we have*

(5.1.1) $$\|u\|_\infty \leqq \varphi_1(T(u)) \leqq (1+\delta)\|u\|_\infty .$$

Moreover, the range of this map may be taken to lie within the subspace N (recall (1.7)).

Proof. For $n = 1, 2, \ldots$, define $\varepsilon_n = \delta^n / e^{n^2}$. We may and shall assume that $\delta \leqq 1$, so that the ε_n are strictly decreasing. Note that $\sum_{n=1}^{\infty} \varepsilon_n \leqq \sum_{n=1}^{\infty} e^{-n} = (e-1)^{-1} < 1$, and, for each positive integer n, that

(5.1.2) $$\sum_{j=n+1}^{\infty} \varepsilon_j / \varepsilon_n = \sum_{k=1}^{\infty} \delta^k / e^{2nk+k^2} \leqq \delta\left(\sum_{k=1}^{\infty} e^{-k}\right) < \delta .$$

Since μ is nonatomic, there exists a sequence of pairwise disjoint sets E_n such that $\mu(E_n)=\varepsilon_n$ for all n. Define $f_n=\varepsilon_n^{-1}\chi_{E_n}$, and then, for $u\in l_\infty$, define

$$T(u)=\sum_{n=1}^{\infty} u(n)f_n$$

(as a pointwise sum). Also let $f=\sum_{n=1}^{\infty} f_n$.

Clearly T is linear and preserves the lattice operations. We now verify condition (5.1.1) (with many thanks to the referee for deflating our original argument). Fix a positive integer i and note that $|u(i)|f_i\leqq|T(u)|\leqq\|u\|_\infty f$. By (1.3), we have

$$|u(i)|\varphi_1(f_i)\leqq\varphi_1(T(u))\leqq\|u\|_\infty\varphi_1(f).$$

An easy calculation will establish the identities $\varphi_1(f_i)=1$ and $\varphi_1(f)=\sup_n\left(\sum_{j=n}^{\infty}\varepsilon_j/\varepsilon_n\right)$, so that (5.1.1) follows from (5.1.2).

Many choices of ε_n will enable l_∞ to be embedded into WL_1 in this way. We now show that the range of T is $\subseteqq N$ for the particular choice of ε_n above. By (1.12), it is sufficient to verify that $I(f)=0$. This we accomplish by showing that, for every number $\varepsilon\in(0,1)$, we have $I_a^b(f)<\varepsilon$ whenever $a>\varepsilon_1^{-1}$ and $\ln(b/a)\geqq 4/\varepsilon^2$. Let k be the largest index such that $\varepsilon_k^{-1}<a$, and let $n\geqq 0$ be the largest index such that $\varepsilon_{k+n}^{-1}\leqq b$. Note then that $I_a^b(f)=n/\ln(b/a)$. The desired inequality $I_a^b(f)<\varepsilon$ is clearly valid when $n=0$ or 1, and so we assume that $n\geqq 2$. On the one hand, we have

$$\begin{aligned}4/\varepsilon^2&\leqq\ln(b/a)<\ln(\varepsilon_{k+n+1}^{-1}/\varepsilon_k^{-1})\\&=\ln\left(\frac{1}{\delta^{n+1}}e^{(k+n+1)^2-k^2}\right)\\&=(n+1)(n+2k+1-\ln\delta),\end{aligned}$$

and so it follows that the larger of the two factors, namely $K=n+2k+1-\ln\delta$, is $>2/\varepsilon$. But on the other hand, we have

$$\begin{aligned}\ln(b/a)&\geqq\ln(\varepsilon_{k+n}^{-1}/\varepsilon_{k+1}^{-1})\\&=(n-1)\ln\left(\frac{1}{\delta}\right)+((n-1)+(k+1))^2-(k+1)^2\\&=K(n-1)>\frac{2}{\varepsilon}(n-1),\end{aligned}$$

and so it follows that

$$I_a^b(f)=\frac{n}{\ln(b/a)}<\frac{n}{n-1}\cdot\frac{\varepsilon}{2}\leqq\varepsilon. \qquad \square$$

(5.2) **Corollary.** *The space $L_1(\mu)$ is a proper subset of the space N (recall (1.8)).*

Proof. If T is the operator of Theorem 5.1, then, for all $u\in l_\infty$, we have $T(u)\in L_1(\mu)$ if and only if $u\in l_1$. $\square$

(5.3) **Corollary.** *The space WL_1 is not σ-order continuous* [13, p. 7] *with respect to the φ_1 quasinorm.* $\square$

(5.4) **Corollary.** *There exists a Banach space B and a continuous linear operator $T: B \to WL_1$ such that T does not factor through the L_1 space of any measure.*

Proof. If the operator T of Theorem 5.1 factored through some L_1 space, then it would necessarily create an isomorphism of $c_0 \subset l_\infty$ into that space. This, however, is impossible, as is well known (and clear from [13, Theorem 1.c.4, p. 34]). □

It remains only to point out that T constitutes the first factor, and the identity injection of $WL_1(\mu)$ into $L_0(\mu)$ the second factor, of an $L_0(\mu)$-valued operator on l_∞ which factors in this natural way through $WL_1(\mu)$. Since the range of T disappears into thin air when we pass to $WL_1^\frown$, any factorization of this operator through an L_1 subspace of $WL_1^\frown$ would seem to be a matter of pure coincidence. (But it does factor through $L_1(\mu)$, and even through $L_2(\mu)$. This has been explicitly noted in [14, p. 163], and it follows readily from [15, Théorème 2, p. 41].)

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Quotients of Green Functions on $\mathbb{R}^n$

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0. Introduction

Consider an open subset X of $\mathbb{R}^n (n \geqq 3)$ and a uniformly elliptic differential operator

$$Lu = \sum_{i,j} a_{ij} D_{ij} u + \sum_i b_i D_i u + cu$$

with sufficiently smooth coefficients and a Green function G_L^X on X. Let

$$\Delta u = \sum_i D_{ii} u \text{ be the Laplacien.}$$

For any subset Y of X consider the following property $C(Y)$: There is a constant $c > 0$ such that

$$c^{-1} G_\Delta^X \leqq G_L^X \leqq c G_\Delta^X \quad \text{on} \quad Y \times Y.$$

It is well known that $C(K)$ is true for any compact subset K of X (see [2] for example). More recently it has been shown that $C(X)$ is true if X is a bounded open set with sufficiently smooth boundary [10, 11]. A global inequality $C(X)$ contains a lot of information about the behaviour of solutions of $Lu = -f$ $(f \geqq 0)$ at the boundary of X.

In this paper we prove that $C(\mathbb{R}^n)$ holds if the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ of L tend in a certain way to those of Δ for $|x|$ tending to ∞ (Corollary 3.5). This condition is derived from a weaker sufficient condition, namely:

The operator $\tilde{L}$ defined on $\mathbb{R}^n \backslash \{0\}$ by

$$G_{\tilde{L}}^{\mathbb{R}^n \backslash \{0\}}(x, y) = |x|^{n-2} |y|^{n-2} G_L^{\mathbb{R}^n}\left(\frac{x}{|x|^2}, \frac{y}{|y|^2}\right)$$

extends to an elliptic operator with locally Hölder continuous coefficients on $\mathbb{R}^n$. We also show that the following condition is necessary in order that $C(\mathbb{R}^n)$ holds:

The sheaf on $\mathbb{R}^n \backslash \{0\}$ of solutions u of $L'u = 0$ extends to a sheaf of harmonic functions on $\mathbb{R}^n$ in the sense of axiomatic potential theory (see [4] for definitions).

Here

$$L'u = s\tilde{L}(tu) \quad \text{for certain functions } s, t.$$

Operators $L = \sum_{i,j} a_{ij} D_{ij}$ with constant coefficients are shown to provide examples for $C(\mathbb{R}^n)$ where the above sufficient condition fails to hold.

Now operators L in divergence form can be treated in a similar way since $C(K)$ for compact subsets K of $\mathbb{R}^n$ has been shown in [8]. The theory presented in [8] allows singularities of the coefficients of L, and the corresponding sufficient condition for $C(\mathbb{R}^n)$ one finds requires in fact only a certain integrability of the coefficients of $\tilde{L}$ instead of Hölder-continuity near 0. These integrability conditions – basic assumptions for [8] – again fail to hold for $\tilde{L}$ and L' in the case of $L = \sum a_{ij} D_{ij}$ with constants a_{ij}.

The main idea behind our arguments is Kelvin's transformation which allows us to treat infinity as if it were a finite point in a $\mathscr{C}^3$-manifold. In this way parts of our paper (especially the last paragraph) may be considered as a continuation of discussions in [5].

1. Glueing Together

We begin with a proposition, which is more or less a simple consequence of a well known result of axiomatic potential theory. Let X denote a locally compact connected metric space and assume that $\mathscr{H}_1, \mathscr{H}_2$ are two harmonic sheafs on X such that $(X, \mathscr{H}_i)$ are $\mathscr{P}$-Brelot spaces for $i = 1, 2$ in the sense of [4]. Assume that for $i = 1, 2$ there are given functions $G_i : X \times X \to \mathbb{R}$ such that for fixed y the function $x \to G_i(x, y)$ is an $\mathscr{H}_i$-potential with support $\{y\}$, and for fixed x the function $y \to G_i(x, y)$ is continuous in $X \setminus \{x\}$. If the axiom of proportionality is fulfilled such functions exist and are uniquely determined up to multiplication with a strictly positive continuous function $f_i = f_i(y)$ [7, Théorème 18.1].

Proposition 1.1. *Let $K \subset X$ be compact, and assume that for every point $x \in K$ there exists a neighbourhood V and a constant $C_V > 0$ such that*

$$C_V^{-1} G_2 \leqq G_1 \leqq C_V G_2$$

holds on $V \times V$. Then there exists a constant $C > 0$ such that

$$C^{-1} G_2 \leqq G_1 \leqq C G_2$$

holds on $K \times K$.

Proof. There exist $V_1, \ldots, V_n$ such that $\bigcup_{i=1}^{n} V_i \supset K$, and $C_i^{-1} G_2 \leqq G_1 \leqq C_i G_2$ holds on $V_i \times V_i$ with suitable constants C_i. Since G_1 and G_2 are strictly positive functions which are continuous off the diagonal [7, Proposition 18.1], there exists $C_0 > 0$ such that $C_0^{-1} \leqq G_1/G_2 \leqq C_0$ holds on $(K \times K) \setminus \bigcup_{i=1}^{n} (V_i \times V_i)$. Now the desired inequality follows with $C := \max_{0 \leqq i \leqq n} C_i$. □

This way of glueing together local estimates for Green functions gives no information about the constants and is restricted to elliptic spaces. Another way of glueing together is given by the following proposition. We assume that the axiom of proportionality is fulfilled.

Proposition 1.2. *Let $U, V \subset X$ be open such that $\partial U \subset V$, and let $\alpha > 0$ such that $\alpha^{-1} G_1 \leqq G_2 \leqq \alpha G_1$ holds on $(U \times U) \cup (V \times V)$. Then $\alpha^{-3} G_1 \leqq G_2 \leqq \alpha^3 G_1$ holds on $(U \cup V) \times (U \cup V)$.*

Proof. Let $y \in U$. By continuity we have $\alpha^{-1} G_1(\cdot, y) \leqq G_2(\cdot, y) \leqq \alpha G_1(\cdot, y)$ on $\bar{U}$. Let

$$p_1 := {}^{\mathscr{H}_1}\hat{R}^{X \setminus U}_{G_1(\cdot, y)},$$

where $\mathscr{H}_1$ indicates that the balayage is taken with respect to the harmonic sheaf $\mathscr{H}_1$. Then p_1 is an $\mathscr{H}_1$-potential which coincides with $G_1(\cdot, y)$ in $X \setminus \bar{U}$ and is harmonic in U. Hence there exists a measure μ which is supported by ∂U [7, Théorème 18.2] such that

$$p_1 = \int G_1(\cdot, z)\mu(dz).$$

Let

$$p_2 := \int G_2(\cdot, z)\mu(dz).$$

Since $\partial U \subset V$ we have $\alpha^{-1} p_1(x) \leqq p_2(x) \leqq \alpha p_1(x)$ for all $x \in V$. Furthermore $\partial U \subset V$ implies

$$\begin{aligned}\liminf_{\substack{z \to x_0 \\ z \in X \setminus \bar{U}}} (\alpha^2 p_2(z) - G_2(z, y)) &\geqq \liminf_{\substack{z \to x_0 \\ z \in X \setminus \bar{U}}} \alpha^2 p_2(z) - G_2(x_0, y) \\ &\geqq \liminf_{\substack{z \to x_0 \\ z \in X \setminus \bar{U}}} \alpha p_1(z) - \alpha G_1(x_0, y) \\ &= \liminf_{\substack{z \to x_0 \\ z \in X \setminus \bar{U}}} \alpha(p_1(z) - G_1(z, y)) = 0\end{aligned}$$

for all $x_0 \in \partial(X \setminus \bar{U})$. Hence by a general minimum principle [1, Korollar 2.4.3] we have $\alpha^2 p_2 \geqq G_2(\cdot, y)$ in $X \setminus \bar{U}$. Thus for $x \in V \cap (X \setminus \bar{U})$ we get

$$G_2(x, y) \leqq \alpha^2 p_2(x) \leqq \alpha^3 p_1(x) = \alpha^3 G_1(x, y).$$

Hence we have $G_2(x, y) \leqq \alpha^3 G_1(x, y)$ for all

$$x \in \bar{U} \cup (V \cap (X \setminus \bar{U})) = \bar{U} \cup V.$$

Now let $y \in V \setminus U$. Again we have $\alpha^{-1} G_1(\cdot, y) \leqq G_2(\cdot, y) \leqq \alpha G_1(\cdot, y)$ on $\bar{V}$. Let $W := U \setminus \bar{V}$, and let

$$q_1 := {}^{\mathscr{H}_1}\hat{R}^{W}_{G_1(\cdot, y)}.$$

Then q_1 is an $\mathscr{H}_1$-potential which coincides with $G_1(\cdot, y)$ on W. As above there exists a measure ν carried by ∂W such that

$$q_1 = \int G_1(\cdot, z)\nu(dz).$$

Let

$$q_2 := \int G_2(\cdot, z)\nu(dz).$$

Since $\partial W \subset U \cap \partial V$ we have $q_1(x) \leqq \alpha q_2(x) \leqq \alpha^2 q_1(x)$ for all $x \in U$, and as above we get $\alpha^2 q_2 \geqq G_2(\cdot, y)$ on W. This implies

$$G_2(x,y) \leqq \alpha^2 q_2(x) \leqq \alpha^3 q_1(x) = \alpha^3 G_1(x,y)$$

for all $x \in W \cup \bar{V} = U \cup V$.

Altogether we have shown that $G_2 \leqq \alpha^3 G_1$ holds on $(U \cup V) \times (U \cup V)$, and since $G_1 \leqq \alpha^3 G_2$ obviously may be deduced in the same way the proof is finished. □

Note that this proposition together with its proof is valid for a more general class of harmonic spaces, which for example includes the harmonic space associated with the heat equation. The interested reader should look at [12, Theorem 3.4].

2. $\mathscr{L}^+(M)$ for a Manifold M

Let U be an open subset of $\mathbb{R}^n$. We say that an operator L of the form

$$L = \sum_{i,j=1}^{n} a_{ij} D_{ij} + \sum_{i=1}^{n} b_i D_i + c$$

belongs to the class $\mathscr{L}^+(\lambda, \alpha, U)$ with $\lambda \geqq 1$ and $\alpha \in]0,1[$, if the coefficients of L are functions on U, such that the following conditions are fulfilled:

(i) $\lambda^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \leqq \lambda |\xi|^2$ and $a_{ij}(x) = a_{ji}(x)$ for all $x \in U$ and all $\xi \in \mathbb{R}^n$

(ii) $$\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| + \sum_{i=1}^{n} |b_i(x) - b_i(y)| + |c(x) - c(y)| \leqq \lambda |x-y|^\alpha$$

and

$$\sum_{i,j=1}^{n} |a_{ij}(x)| + \sum_{i=1}^{n} |b_i(x)| + |c(x)| \leqq \lambda$$

for all $x, y \in U$

(iii) There exists a function $s \in \mathscr{C}_+^{2,\alpha}(U)$ with $Ls < 0$ on U.

Let M be an n-dimensional manifold ($n \geqq 3$) of class $\mathscr{C}^3$ with a countable base. We fix an atlas $(V_i, f_i)_{i \in I}$ of M with the following property: For each $i \in I$ the mapping $f_i: V_i \to \mathbb{R}^n$ is a homeomorphism with $f_i, f_i^{-1} \in \mathscr{C}^3$ such that $M = \bigcup_{i \in I} f_i^{-1}(B_2(0))$.

We say that a linear partial differential operator L of second order on M lies in $\mathscr{L}^+(M)$ if there exists $s \in \mathscr{C}_+^{2,\alpha}(M)$ with $Ls < 0$ on M, and for every $i \in I$ the operator $f_i(L)$ defined by $(f_i(L))(u) = (L(u \circ f_i)) \circ f_i^{-1}$ lies in $\mathscr{L}^+(\lambda_i, \alpha_i, B_3(0))$ for suitable λ_i, α_i. It is well known that the sheaf $\mathscr{H}_L$ of solutions of $Lu = 0$ is a harmonic sheaf which gives rise to a Brelot space [2, 4, 7]. Every open subset Ω of M has a Green function G_L^Ω, which can be characterized by the following properties:

(i) For each $y \in \Omega$ the function $x \to G_L^\Omega(x,y)$ is an L-potential on Ω which is L-harmonic in $\Omega \setminus \{y\}$.

(ii) If (V_i, f_i) is a chart which contains $y \in \Omega$, and the operator $L_i := f_i(L)$ has the form

$$L_i u(z) = \sum_{i,j=1}^{n} a_{ij}(z) D_{ij} u(z) + \sum_{i=1}^{n} b_i(z) D_i u(z) + c(z) u(z)$$

then

$$\lim_{x\to y} \frac{G_L^\Omega(x,y)}{|f_i(x)-f_i(y)|_{B(f_i(y))}^{2-n}}=1.$$

Here $B(z)$ denotes the inverse of the matrix $(a_{ij}(z))_{ij}$, and $|\tilde{z}|_{B(z)}=\sqrt{\tilde{z}B(z)\tilde{z}^t}$.

Note that the normalization in (ii) is well defined, because it behaves well under changing the charts. In other words: If $T:\Omega\to T(\Omega)$ is a $\mathscr{C}^2$-diffeomorphism and $T(L)$ is defined by $(T(L))(u)=(L(u\circ T))\circ T^{-1}$ then we have $G_{T(L)}^{T(\Omega)}(T(x),T(y)) = G_L^\Omega(x,y)$.

Proposition 2.1. *Let M be a compact connected manifold of class $\mathscr{C}^3$ and let $L_1, L_2\in\mathscr{L}^+(M)$. Then there exists a constant $C>0$ such that*

$$C^{-1}G_{L_1}^M \leqq G_{L_2}^M \leqq CG_{L_1}^M$$

holds on $M\times M$.

The proof of this corollary follows easily from Proposition 1.1 together with Proposition 35.1 from [7]. Using Proposition 12 from [8] one could derive an analogous statement for operators in divergence form. Furthermore one can show that the optimal constant C in Proposition 2.1 tends to one if the coefficients of L_1 tend uniformly against the corresponding coefficients of L_2. The proof of this fact is a simplification of case 1 of the proof of Theorem 5.1 in [11].

3. Sufficient Conditions for Comparability on $\mathbb{R}^n$

Now we want to apply Proposition 1.2 to a special situation, namely we assume $X=\mathbb{R}^n (n\geqq 3)$ and $G_1(x,y)=G_\Delta(x,y)=|x-y|^{2-n}$, that means $\mathscr{H}_1$ is the harmonic sheaf associated to the Laplace operator Δ. By $\mathscr{H}_2$ we denote a harmonic sheaf on $\mathbb{R}^n$ which fulfills the axiom of proportionality. We fix a Green function G_2 for $(X,\mathscr{H}_2)$.

Proposition 3.1. *The following relations* (i), (ii) *are equivalent:*

(i) *There is a constant $c>0$ such that*

$$c^{-1}G_\Delta \leqq G_2 \leqq cG_\Delta$$

holds on $\mathbb{R}^n\times\mathbb{R}^n$.

(ii) *There are constants $c>0$ and $r>1$ such that*

$$c^{-1}G_\Delta(x,y)\leqq G_2(x,y)\leqq cG_\Delta(x,y) \tag{1}$$

and

$$c^{-1}G_\Delta(x,y)\leqq |x|^{2-n}|y|^{2-n}G_2\left(\frac{x}{|x|^2},\frac{y}{|y|^2}\right)\leqq cG_\Delta(x,y) \tag{2}$$

holds for $|x|, |y|\leqq r$.

Proof. Since $G_\Delta(x,y)=|x|^{2-n}|y|^{2-n}G_\Delta\left(\frac{x}{|x|^2},\frac{y}{|y|^2}\right)$, condition (ii)(2) holds for all $|x|, |y|<r$ if and only if

$$c^{-1}G_\Delta(u,v)\leqq G_2(u,v)\leqq cG_\Delta(u,v)$$

is fulfilled for all $|u|, |v| \geqq \frac{1}{r}$. With $U = \{x \in \mathbb{R}^n, |x| < r\}$ and $V = \left\{x \in \mathbb{R}^n, |x| > \frac{1}{r}\right\}$ the statement follows immediately from Proposition 1.2. □

Proposition 3.2. (1) *If* $G_2 = G_L^{\mathbb{R}^n}$ *for some* $L \in \mathscr{L}^+(\mathbb{R}^n)$ *then property* (ii)(1) *of 3.1 holds.*

(2) *If* $|x|^{2-n}|y|^{2-n} G_2\left(\frac{x}{|x|^2}, \frac{y}{|y|^2}\right) = G_L^{\mathbb{R}^n}(x, y)$ *holds for all* $x, y \in \mathbb{R}^n \setminus \{0\}$ *and some* $L \in \mathscr{L}^+(\mathbb{R}^n)$, *then property* (ii)(2) *of 3.1 holds.*

Proof. Both (1) and (2) follow easily from [7, Proposition 35.1]. □

We now suppose $G_2 = G_L^{\mathbb{R}^n}$ for some $L \in \mathscr{L}^+(\mathbb{R}^n)$, and try to find conditions on the coefficients of L which guarantee that (2) of 3.2 holds.

Define $\gamma, m: \mathbb{R}^n \to \mathbb{R}^+$ by $\gamma(x) = |x|^{2-n}$ and $m(x) = |x|^4$. We define multiplication operators Γ, M by $(\Gamma u)(x) = \gamma(x) u(x)$ and $(Mu)(x) = m(x) u(x)$ for any function u for which this operations make sense. Furthermore let $\kappa_n: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ denote the inversion on the boundary of the unit sphere, that means $\kappa_n(x) = |x|^{-2} x$. Then there is a uniquely determined $\mathscr{K}(L) \in \mathscr{L}^+(\mathbb{R}^n \setminus \{0\})$ which fulfills

$$(\mathscr{K}(L))u = (\kappa_n(M\Gamma^{-1} L \Gamma))u = ((M\Gamma^{-1} L\Gamma)(u \circ \kappa_n)) \circ \kappa_n^{-1}$$

for all $u \in \mathscr{C}_0^2(\mathbb{R}^n \setminus \{0\})$.

We call $\mathscr{K}(L)$ the Kelvin-transform of L. Note that $\mathscr{K}(\Delta) = \Delta$.

If $Lu = \sum a_{ij} D_{ij} u + \sum b_i D_i u + cu$, and $(\mathscr{K}(L))u = \sum \tilde{a}_{kl} D_{kl} u + \sum \tilde{b}_k D_i u + \tilde{c} u$, then

$$\tilde{a}_{k,l}(y) = \sum_{i,j=1}^{n} a_{ij}\left(\frac{y}{|y|^2}\right) |y|^{-4} (\delta_{ik}|y|^2 - 2y_i y_k)(\delta_{jl}|y|^2 - 2y_j y_l)$$

$$\tilde{b}_k(y) = \sum_{i=1}^{n} b_i\left(\frac{y}{|y|^2}\right) |y|^{-4} (\delta_{ik}|y|^2 - 2y_i y_k)$$

$$+ \sum_{i,j=1}^{n} a_{ij}\left(\frac{y}{|y|^2}\right) |y|^{-4} (4n y_i y_j y_k - 2n y_i |y|^2 \delta_{jk} - 2y_k |y|^2 \delta_{ij}),$$

and

$$\tilde{c}(y) = c\left(\frac{y}{|y|^2}\right) |y|^{-4}$$

$$+ \sum_{i=1}^{n} b_i\left(\frac{y}{|y|^2}\right) (2-n) y_i |y|^{-4}$$

$$+ \sum_{i,j=1}^{n} a_{ij}\left(\frac{y}{|y|^2}\right) ((2-n)|y|^{-2} \delta_{ij} + n(n-2) y_i y_j |y|^{-4}).$$

The reader who wants to check this formulas should first compute an expression for $L \circ \Gamma$, and then he might use the general formula in [14] on p. 43 to compute the coefficients of $\kappa_n(M\Gamma^{-1} L \Gamma)$.

Proposition 3.3. *We have*

$$G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}(x, y) = |x|^{2-n} |y|^{2-n} G_L^{\mathbb{R}^n}\left(\frac{x}{|x|^2}, \frac{y}{|y|^2}\right)$$

for all $x, y \in \mathbb{R}^n \setminus \{0\}$.

Proof. Let $x, y \in \mathbb{R}^n \backslash \{0\}$. Since $\{0\}$ is polar we have

$$G_L^{\mathbb{R}^n \backslash \{0\}}(x,y) = G_L^{\mathbb{R}^n}(x,y).$$

The equality $G_{L \circ \Gamma}^{\mathbb{R}^n \backslash \{0\}}(x,y) = (\gamma(x))^{-1}(\gamma(y))^{\frac{n}{2}} G_L^{\mathbb{R}^n \backslash \{0\}}(x,y)$ is easy to check, and hence we have

$$\begin{aligned} G_{M\Gamma^{-1}L\Gamma}^{\mathbb{R}^n \backslash \{0\}}(x,y) &= \left(\frac{m(y)}{\gamma(y)}\right)^{\frac{n}{2}-1} (\gamma(x))^{-1}(\gamma(y))^{\frac{n}{2}} G_L^{\mathbb{R}^n \backslash \{0\}}(x,y) \\ &= |x|^{n-2}|y|^{n-2} G_L^{\mathbb{R}^n \backslash \{0\}}(x,y). \end{aligned}$$

We get

$$\begin{aligned} G_{\mathscr{K}(L)}^{\mathbb{R}^n \backslash \{0\}}(\kappa_n(x), \kappa_n(y)) &= G_{\kappa_n(M\Gamma^{-1}L\Gamma)}^{\kappa_n(\mathbb{R}^n \backslash \{0\})}(\kappa_n(x), \kappa_n(y)) \\ &= G_{M\Gamma^{-1}L\Gamma}^{\mathbb{R}^n \backslash \{0\}}(x,y) = |x|^{n-2}|y|^{n-2} G_L^{\mathbb{R}^n \backslash \{0\}}(x,y). \end{aligned}$$

Now the assertion follows if we replace x and y by $\kappa_n(x)$ and $\kappa_n(y)$, and use $\kappa_n \circ \kappa_n = \mathrm{Id}$. □

Theorem 3.4. *Suppose that there is an operator $\tilde{\mathscr{L}} \in \mathscr{L}^+(\mathbb{R}^n)$ which extends $\mathscr{K}(L)$ into zero. Then there exists a constant $c > 0$ such that*

$$c^{-1} G_\Delta^{\mathbb{R}^n} \leqq G_L^{\mathbb{R}^n} \leqq c G_\Delta^{\mathbb{R}^n}$$

holds on $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. The statement follows easily from the preceding three propositions. □

The conditions of this theorem are obviously fulfilled if $L \in \mathscr{L}^+(\mathbb{R}^n)$ coincides with Δ in the complement of a compact set. The explicit formulas for the coefficients of $\mathscr{K}(L)$ give us weaker conditions on the coefficients of $L = \sum a_{ij} D_{ij} + \sum b_i D_i + c$, which guarantee that the hypothesis of the theorem is fulfilled:

Corollary 3.5. *If the following functions – defined to be zero for $y = 0$ –*

$$y \to |y|^{-2}\left(a_{ij}\left(\frac{y}{|y|^2}\right) - \delta_{ij}\right); \quad y \to |y|^{-3} b_i\left(\frac{y}{|y|^2}\right);$$

$$y \to |y|^{-4} c\left(\frac{y}{|y|^2}\right);$$

are Hölder continuous with exponent α in a neighbourhood of zero, then $\mathscr{K}(L)$ is the restriction to $\mathbb{R}^n \backslash \{0\}$ of some $\tilde{L} \in \mathscr{L}^+(\mathbb{R}^n)$.

Proof. First of all note, that it is enough to show that the coefficients $\tilde{a}_{kl}, \tilde{b}_k, \tilde{c}$ of $\mathscr{K}(L)$ are Hölder continuous with exponent α in $U \backslash \{0\}$ for a suitable neighbourhood U of zero. For then $\mathscr{K}(L)$ has a unique Hölder continuous extension $\tilde{L}$ into zero. Using $\mathscr{K}(\Delta) = \Delta$ it is easy to deduce $\tilde{a}_{kl}(0) = \delta_{kl}$, and hence $\tilde{L}$ is uniformly elliptic in a neighbourhood of zero. The existence of $s \in \mathscr{C}_+^{2,\alpha}(\mathbb{R}^n)$ with $\tilde{L}s < 0$ is equivalent to the $\mathscr{P}$-harmonicity of the harmonic space $(\mathbb{R}^n, \mathscr{H}_L)$, and this follows from $\tilde{L} \in \mathscr{L}^+(\mathbb{R}^n \backslash \{0\})$ because $\{0\}$ is polar.

Now we shall only indicate how one can see that the conditions of the corollary imply that $\tilde{c}$ is Hölder continuous with exponent α in a neighbourhood of zero. Assume that z, y are points near zero and without loss of generality let $1 > |z| \geqq |y|$.

Let $d_{ij}(x) := |x|^{-2}\left(a_{ij}\left(\frac{x}{|x|^2}\right) - \delta_{ij}\right)$. With

$$I_1 := (2-n) \sum_{i,j=1}^{n} \delta_{ij}(d_{ij}(y) - d_{ij}(z))$$

$$I_2 := n(n-2) \sum_{i,j=1}^{n} (d_{ij}(y) - d_{ij}(z)) \frac{y_i y_j}{|y|^2}$$

$$I_3 := n(n-2) \sum_{i,j=1}^{n} d_{ij}(z) \left(\frac{y_i y_j}{|y|^2} - \frac{z_i z_j}{|z|^2}\right)$$

$$I_4 := (2-n) \sum_{i=1}^{n} \left(|y|^{-3} b_i\left(\frac{y}{|y|^2}\right) - |z|^{-3} b_i\left(\frac{z}{|z|^2}\right)\right) \frac{y_i}{|y|}$$

$$I_5 := (2-n) \sum_{i=1}^{n} |z|^{-3} b_i\left(\frac{z}{|z|^2}\right)\left(\frac{y_i}{|y|} - \frac{z_i}{|z|}\right)$$

and

$$I_6 := |y|^{-4} c\left(\frac{y}{|y|^2}\right) - |z|^{-4} c(|z|^{-2} z)$$

we have

$$c(y) - c(z) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 .$$

The estimates $|I_1|$, $|I_2|$, $|I_4|$, $|I_6| \leqq C|y-z|^\alpha$ with a suitable constant C are easy to derive. We restrict ourselves to estimate $|I_3|$ because $|I_5|$ may be estimated in the same way. It is enough to show

$$H := \left| d_{ij}(z)\left(\frac{y_i y_j}{|y|^2} - \frac{z_i z_j}{|z|^2}\right)\right| \leqq C|y-z|^\alpha$$

for i,j fixed. Since d_{ij} is Hölder continuous with exponent $\alpha \in]0,1[$ and $d_{ij}(0) = 0$ we have

$$\begin{aligned} H &\leqq C|z|^\alpha \left|\frac{y_i y_j}{|y|^2} - \frac{z_i z_j}{|z|^2}\right| \\ &\leqq C|z|^\alpha \left(\left|\frac{y_i}{|y|}\left(\frac{y_j}{|y|} - \frac{z_j}{|z|}\right)\right| + \left|\frac{z_j}{|z|}\left(\frac{y_i}{|y|} - \frac{z_i}{|z|}\right)\right|\right) \\ &\leqq C|z|^\alpha \left|\frac{y_j}{|y|} - \frac{z_j}{|z|}\right| + c|z|^\alpha \left|\frac{y_i}{|y|} - \frac{z_j}{|y|}\right| \\ &\leqq 2C\,|z^\alpha| \left|\frac{y}{|y|} - \frac{z}{|z|}\right| . \end{aligned}$$

If $\frac{1}{2}|z| \leqq |z-y|$ we have $H \leqq 2C|z|^\alpha \cdot 2 \leqq 4C|y-z|^\alpha$. If $\frac{1}{2}|z| > |z-y|$ we have $\langle z, y\rangle \geqq 0$, and using $|z| \geqq |y|$ we get $\left|z - \frac{|z|}{|y|} y\right| \leqq |z-y|$. Hence in this case we have

$$\begin{aligned} H &\leqq 2C|z|^\alpha |z|^{-1} \left|z - \frac{|z|}{|y|} y\right| \\ &\leqq 2C|z|^{\alpha-1}|z-y| \\ &\leqq 2C|z|^{\alpha-1}|z-y|^{1-\alpha}|z-y|^\alpha \\ &\leqq 2^{\ \alpha} C|z-y|^\alpha . \end{aligned}$$

So we have shown that $\tilde{c}$ is indeed Hölder continuous. To derive Hölder continuity of the other coefficients our assumptions about the a_{ij}, b_i, and c could be weakened. In this way it is possible to improve our result if one uses the local comparability result for the Green function [5, Theorem 1] or [9, § 2] in its full generality. The condition that $\mathscr{K}(L)$ may be extended to some $\tilde{\mathscr{L}} \in \mathscr{L}^+(\mathbb{R}^n)$ has a nice natural interpretation: Let $f: \mathbb{R}_+ \cup \{0\} \to [\frac{1}{4}, \infty[$ be an infinitely often differentiable function which fulfills $f(x) = x$ for $|x| > \frac{1}{2}$. Define γ_f, m_f, Γ_f, and M_f by $\gamma_f(x) = (f(|x|))^{2-n}$, $m_f(x) = (f(|x|))^4$, $\Gamma_f u = \gamma_f u$, and $M_f u = m_f u$. The operators $\mathscr{K}(L)$ and $\mathscr{K}_f(L) := \kappa_n(M_f \Gamma_f^{-1} L \Gamma_f)$ coincide in a neighbourhood of zero. Hence $\mathscr{K}(L)$ is extendable into zero if and only if this is the case for $\mathscr{K}_f(L)$. We identify the n-dimensional sphere S_n with $\{x \in \mathbb{R}^{n+1} \mid |x - (0, \dots, \frac{1}{2})| = \frac{1}{2}\}$ and $\mathbb{R}^n$ with $\{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$. $N := (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ is the northpole of S_n and $S = (0, \dots, 0)$ the southpole. We define $R: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $R(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n, 1 - x_{n+1})$ and hence the mappings

$$\varphi_1 := R\kappa_{n+1}|_{S_n \setminus \{S\}} \quad \text{and} \quad \varphi_2 := R\kappa_{n+1} R|_{S_n \setminus \{N\}}$$

define a $\mathscr{C}^\infty$-atlas of S_n. Now the conditions "$L \in \mathscr{L}^+(\mathbb{R}^n)$" and "$\mathscr{K}_f(L)$ is extendable to some $\tilde{L} \in \mathscr{L}^+(\mathbb{R}^n)$" are equivalent to the condition that $\varphi_1^{-1}(M_f \Gamma_f^{-1} L \Gamma_f)$ is extendable to an operator in $\mathscr{L}^+(S_n)$. This follows easily from $\varphi_2 \circ \varphi_1^{-1} = R\kappa_{n+1} R\kappa_{n+1} R|_{\mathbb{R}^n \setminus \{0\}} = \kappa_n$. In this way our theorem would also follow from Proposition 2.1.

4. Necessary Conditions for Comparability on $\mathbb{R}^n$

Our Theorem 3.4 does not include the very simple case that L is the image of Δ under a nonsingular linear transformation T, that means $L = \sum_{i,0=1}^{n} a_{ij} D_{ij}$ with a positive definite matrix $A = (a_{ij})_{ij}$, which is different from the unit matrix. It is easy to see, that in this case the principal coefficients of $\mathscr{K}(L)$ may behave badly near zero. Nevertheless if B denotes the matrix for which we have $T(x) = xB$ then

$$\begin{aligned} G_L^{\mathbb{R}^n}(x, y) &= G_{T\Delta}^{T\mathbb{R}^n}(x, y) = G_\Delta^{\mathbb{R}^n}(T^{-1}x, T^{-1}y) \\ &= ((x-y)(B^t B)^{-1}(x-y)^t)^{\frac{2-n}{2}} = ((x-y)A^{-1}(x-y)^t)^{\frac{2-n}{2}} \end{aligned}$$

implies

$$C^{\frac{2-n}{2}} G_A^{\mathbb{R}^n}(x,y) \leqq G_L^{\mathscr{R}^n}(x,y) \leqq C^{\frac{n-2}{2}} G_A^{\mathscr{R}^n}(x,y)$$

for any constant C which satisfies $c^{-1} \leqq \lambda \leqq c$ for every eigenvalue λ of A.

This example is still worse than already explained: Not only $\mathscr{K}(L)$ has no good extension to zero; even the harmonic space associated to $\mathscr{K}(L)$ has no reasonable extension into zero. In fact $G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}$ has no extension to $\mathbb{R}^n \times \mathbb{R}^n$ which is continuous off the diagonal if A has two different eigenvalues $\lambda_1 < \lambda_2$. To see this let $e_1 A = \lambda_1 e_1$ and $e_2 A = \lambda_2 e_2$ with $|e_i| = 1$. By Proposition 3.3 we have

$$G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}\left(\frac{x}{|x|^2}, \frac{y}{|y|^2}\right) = |x|^{n-2}|y|^{n-2} G_L^{\mathbb{R}^n}(x,y).$$

Fix y and put $x_\mu := y + \mu e_i$. Then $\lim_{\mu \to \infty} \kappa_n(x_\mu) = 0$ and

$$\lim_{\mu \to \infty} G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}(\kappa_n(x_\mu), \kappa_n(y)) = \lim_{\mu \to \infty} \frac{|y + \mu e_i|^{n-2}|y|^{n-2}}{(\mu e_i A^{-1} \mu e_i^t)^{\frac{n-2}{2}}}$$

$$= \lambda_i^{\frac{n-2}{2}} |y|^{n-2} \quad (i = 1, 2).$$

This discontinuity for $x \to 0$ however disappears if $G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}$ is normalized in the following way: put

$$G(x,y) = G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}(x,y) \, (G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}(x_0, y) G_{\mathscr{K}(L)}^{\mathbb{R}^n \setminus \{0\}}(x, y_0))^{-1}$$

for some $x_0, y_0 \in \mathbb{R}^n \setminus \{0\}$. Then G has a good extension to zero, as is easily seen. In fact this is a consequence of a theorem, which we are going to explain in the sequel.

Since the reasoning may have some interest in itself we do it in the axiomatic framework. So we consider the following situation:

X is a locally compact connected topological space with a countable base, $\mathscr{H}_1$ and ${}^d\mathscr{H}_1$ are harmonic sheafs on X which correspond to $\mathscr{P}$-Brelots spaces which satisfy the axiom of proportionality. We assume that $\mathscr{H}_1$ and ${}^d\mathscr{H}_1$ are in duality with respect to a Green function $G_1 : X \times X \to \mathbb{R}_+$, that means especially that for fixed y the function $z \to G_1(z,y)$ is an $\mathscr{H}_1$-potential with support $\{y\}$, and for fixed x the function $z \to G_1(x,z)$ is an ${}^d\mathscr{H}_1$-potential with support $\{x\}$. In addition assume that $\mathscr{H}_2^0$ and ${}^d\mathscr{H}_2^0$ are harmonic sheafs on $X_0 := X \setminus \{0\}$ – where 0 denotes some fixed point of X – such that these sheafs correspond to Brelot spaces which are in duality with respect to a Green function G_2^0, and such that the axiom of proportionality is fulfilled.

We call G_2^0 normalized, if there are nonnegative measures ϱ, μ with compact support in X_0 such that for all $x, y \in X_0$ we have $\varrho G_2^0(y) = 1 = G_2^0 \mu(x)$. Here we used the notational conventions $\varrho G_2^0(y) = \int G_2^0(z,y)\varrho(dz)$ and $G_2^0\mu(x) = \int G_2^0(x,z)\mu(dz)$.

Lemma 4.1. *Suppose*

(i) *there is a constant* $C > 0$ *such that* $C^{-1} G_1 \leqq G_2^0 \leqq C G_1$ *on* $X_0 \times X_0$,

(ii) G_2^0 *is normalized.*

(iii) $G_1(0,0) = \infty$.

Then $\lim_{z\to 0} G_2^0(z,y)=:G_2(0,y)$ *and* $\lim_{z\to 0} G_2^0(x,z)=:G_2(x,0)$ *exist for all* $x,y\in X_0$. *In this case put* $G_2(x,y):=G_2^0(x,y)$ *for all* $x,y\in X_0$ *and* $G_2(0,0)=\infty$.

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X_0 with $\lim_{n\to\infty} y_n=0$. By (i) the sequence $(G_2^0(\cdot,y_n))_n$ is locally uniformly bounded on X_0. By Theorem 11.1.1 of [4] a certain subsequence converges locally uniformly in X_0 to an $\mathscr{H}_2^0$-harmonic function h. Obviously we have $\varrho(h)=1$, where ϱ is the measure with $\varrho G_2^0\equiv 1$, and

$$C^{-1}G_1(\cdot,0)\leqq h\leqq CG_1(\cdot,0) \text{ on } X_0.$$

By (iii) the set $\{0\}$ is polar, and hence $G_1^0:=G_1|_{X_0\times X_0}$ is a Green function for $\mathscr{H}_1$ restricted to X_0. By the same reasoning the restriction of $G_1(\cdot,0)$ to X_0 is an extremal $\mathscr{H}_1$-harmonic function on X_0. As a consequence of the relations

$$C^{-1}G_1^0\leqq G_2^0\leqq CC_1^0 \quad \text{on} \quad X_0\times X_0$$
$$C^{-1}G_1(\cdot,0)\leqq h\leqq CG_1(\cdot,0) \text{ on } X_0$$

and [13, Satz 2.4] the $\mathscr{H}_2^0$-harmonic function h is extremal on X_0. To show that h does not depend on the choice of the subsequence of $(y_n)_{n\in\mathbb{N}}$ let $(y_n')_{n\in\mathbb{N}}$ be another sequence with $\lim_{n\to\infty} y_n=0$ and $\lim_{n\to\infty} G_2^0(\cdot,y_n')=h'$ locally uniformly in X_0. Then

$$C^{-1}G_1(\cdot,0)\leqq \tfrac{1}{2}h+\tfrac{1}{2}h'\leqq CG_1(\cdot,0) \text{ on } X_0,$$

and as above we get that $\frac{1}{2}h+\frac{1}{2}h'$ is an extremal $\mathscr{H}_2^0$-harmonic function on X_0. But this implies $\beta h=h'$ for some $\beta>0$, and since we have $\varrho(h)=1=\varrho(h')$ we get $\beta=1$.

So far we proved that $\lim_{y\to 0} G_2^0(\cdot,y)$ exists, and even is an $\mathscr{H}_2^0$-harmonic function on X_0. To prove the same for $\lim_{x\to 0} G_2^0(x,\cdot)$ one only has to apply the same reasoning to the adjoint sheafs. □

Note that every extremal $\mathscr{H}_2^0$-harmonic function is the limit of extremal $\mathscr{H}_2^0$-potentials. Hence the proof above shows that up to multiplication the function $G_2(\cdot,0)$ is the only extremal $\mathscr{H}_2^0$-harmonic function on X_0. The same holds for the dual space.

Remark 4.2. Suppose condition (i) of 4.1 is satisfied by G_1 and G_2^0 but (ii) is not satisfied. In this case we modify G_1 and G_2^0 in the following way:

Take measures ϱ_0,μ_0 with compact support such that $\varrho_0 G_1$, $\varrho_0 G_2^0$, $G_1\mu_0$, and $G_2^0\mu_0$ are continuous (it can be shown that such measures exist as a consequence of (i), at least if points are polar for $\mathscr{H}_1$; see [13, Satz 2.3]).

Put

$$\tilde{G}_1(x,y)=\frac{G_1(x,y)}{G_1\mu_0(x)\varrho_0 G_1(y)} \quad \text{and} \quad \tilde{G}_2^0(x,y)=\frac{G_2^0(x,y)}{G_2^0\mu_0(x)\varrho_0 G_2^0(y)}.$$

Then the harmonic sheafs $(G_1\mu_0)^{-1}\mathscr{H}_1$ and $(\varrho_0 G_1)^{-1}{}^d\mathscr{H}_1$ are in duality with respect to $\tilde{G}_1$, and the harmonic sheafs $(G_2^0\mu_0)^{-1}\mathscr{H}_2^0$ and $(\varrho_0 G_2^0)^{-1}{}^d\mathscr{H}_2^0$ are in duality with respect to $\tilde{G}_2^0$. Obviously we have

$$C^{-3}\tilde{G}_1\leqq \tilde{G}_2^0\leqq C\tilde{G}_1 \quad \text{on} \quad X_0\times X_0.$$

Put $\varrho = (G_2^0 \mu_0)\varrho_0$ and $\mu = (\varrho_0 G_2^0)\mu_0$. Then

$$\varrho \tilde{G}_2^0(y) = \int \tilde{G}_2^0(z,y) G_2^0 \mu_0(z) \varrho_0(dz) = \int \frac{G_2^0(z,y)}{\varrho_0 G_2^0(y)} \varrho_0(dz) = 1$$

and similarly $\tilde{G}_2^0 \mu(x) = 1$ for $x, y \in X_0$.

Hence $\tilde{G}_2^0$ is normalized (as well as $\tilde{G}_1$). If X is a manifold of class $\mathscr{C}^3$ and $G_1 = G_L^X$ for some $L \in \mathscr{L}^+(X)$ then we can compute the differential operator $\tilde{L}$ which belongs to $\tilde{G}_1$, at least if $G_1 \mu_0$ and $\varrho_0 G_1$ are twice continuously differentiable:

Denote by Q, $Q^{\frac{2}{2-n}}$ and $P^{\frac{2}{2-n}}$ the multiplication operators which are defined by

$$(Qu)(z) = (G_1 \mu_0(z)) u(z), \quad \left(Q^{\frac{2}{2-n}} u\right)(z) = (G_1 \mu_0(z))^{\frac{2}{2-n}} u(z)$$

and

$$\left(P^{\frac{2}{2-n}} u\right)(z) = (\varrho_0 G_1(z))^{\frac{2}{2-n}} u(z).$$

Then

$$\tilde{L} = P^{\frac{2}{2-n}} Q^{\frac{n}{2-n}} L Q.$$

Note that we have already seen an operation of this sort: With the notations of § 3 the operator $M\Gamma^{-1}\Delta\Gamma$ is nothing but the normalization of the Laplace operator with the singular potential $G_\Delta^{\mathbb{R}^n} \mu_0(z) = \varrho_0 G_\Delta^{\mathbb{R}^n}(z) = |z|^{2-n}$ which belongs to $\mu_0 = \varrho_0 = \varepsilon_0$.

Theorem 4.3. *Suppose*

(i) *there exists a constant* $C > 0$ *such that* $C^{-1} G_1 \leqq G_2^0 \leqq C G_1$ *holds on* $X_0 \times X_0$.

(ii) G_2^0 *is normalized.*

(iii) $G_1(0,0) = \infty$.

Then there are harmonic sheafs $\mathscr{H}_2$ *and* ${}^d\mathscr{H}_2$ *on* X *which coincide with* $\mathscr{H}_2^0$ *and* ${}^d\mathscr{H}_2^0$ *on* X_0. *Both spaces are* $\mathscr{P}$*-Brelot spaces which satisfy the axiom of proportionality and are in duality with respect to a Green function* G_2, *which satisfies* $G_2 = G_2^0$ *on* $X_0 \times X_0$.

Proof. Define $\mathscr{H}_2$ by $\mathscr{H}_2(U) := \mathscr{H}_2^0(U \setminus \{0\}) \cap \mathscr{C}(U)$ for any open U in X. Obviously $\mathscr{H}_2$ is a harmonic sheaf on X. Because of (ii) constants are $\mathscr{H}_2^0$-harmonic in $U \setminus \{0\}$ for each sufficiently small neighbourhood of 0. Let G_2 denote the function which we defined in 4.1. Since $C^{-1} G_1 \leqq G_2 \leqq C G_1$ on $X \times X$, the sets $U_\alpha := \{x \in X \mid G_2(x,0) > \alpha\}$ form a base of neighbourhoods of 0, and for sufficiently large α all points of ∂U_α are $\mathscr{H}_2^0$-regular because $G_2(\cdot, 0) - \alpha$ is a barrier. Let $\mu_x^{U_\alpha}$ denote the harmonic measure associated to $U_\alpha \setminus \{0\}$, x, and $\mathscr{H}_2^0$. Then for α large and $f \in \mathscr{C}(\partial U_\alpha)$ the function $\tilde{f}(x) = \mu_x^{U_\alpha}(f)$ is in

$$\mathscr{C}(\bar{U}_\alpha \setminus \{0\}) \cap \mathscr{H}_2^0(U_\alpha \setminus \{0\}).$$

We shall show that $\tilde{f}(0):=\lim_{x\to 0}\tilde{f}(x)$ exists, hence $\tilde{f}\in\mathscr{H}_2(U_\alpha)\cap\mathscr{C}(\bar{U}_\alpha)$. Since constants are $\mathscr{H}_2^0$-harmonic in $U_\alpha\backslash\{0\}$, and differences of continuous potentials are dense in $\mathscr{C}(\partial U_\alpha)$ we may assume that f is the restriction of a continuous $\mathscr{H}_2^0$-potential p. In this case we have $\tilde{f}=\hat{R}_p^{\partial U}$ on $U_\alpha\backslash\{0\}$. There exists a measure ν which is carried by ∂U_α such that $\hat{R}_p^{\partial U_\alpha}=\int G_2^0(\cdot,z)\nu(dz)$. Let K be a compact neighbourhood of zero with $K\subset U_\alpha$. Since $G_2^0\leqq CG_1$ and G_1 is continuous on $K\times\partial U_\alpha$ there is a constant $\gamma>0$ such that $G_2^0\leqq\gamma$ on $(K\backslash\{0\})\times\partial U_\alpha$. Hence we may use Lebesgue's dominated convergence theorem and get

$$\lim_{x\to 0}\tilde{f}(x)=\lim_{x\to 0}\int G_2^0(x,z)\nu(dz)=\int\lim_{x\to 0}G_2^0(x,z)\nu(dz)=\int G_2(0,z)\nu(dz).$$

Now let $\tilde{g}$ be another function in $\mathscr{H}_2(U_\alpha)\cap\mathscr{C}(\bar{U}_\alpha)$ which extends f. Then $\tilde{g}-\tilde{f}$ is a bounded function in $\mathscr{H}_2^0(U_\alpha\backslash\{0\})$, and using $\lim_{x\to 0}G_2(x,0)=\infty$ the minimum principle implies $\tilde{g}-\tilde{f}+\varepsilon G_2(\cdot,0)\geqq 0$ for every $\varepsilon>0$. Hence $\tilde{g}\geqq\tilde{f}$ and in the same way we get also $\tilde{f}\geqq\tilde{g}$. Hence $\tilde{f}$ is the unique function in $\mathscr{H}_2(U_\alpha)\cap\mathscr{C}(\bar{U}_\alpha)$ which extends f. Altogether we have shown that X has a base of $\mathscr{H}_2$-regular open sets.

Now we shall show that the sheaf $\mathscr{H}_2$ satisfies Brelot's convergence property. It is enough to show that if U is any relatively compact connected open neighbourhood of 0 on which constants are $\mathscr{H}_2$-harmonic and if $(h_n)_{n\in\mathbb{N}}$ is an increasing sequence of functions in $\mathscr{H}_2(U)$, then $h:=\lim_{n\in\mathbb{N}}h_n$ is either in $\mathscr{H}_2(U)$ or equal to ∞ in U. Assume $h(x)<\infty$ for some $x\in U\backslash\{0\}$. The set $U\backslash\{0\}$ is connected because $\{0\}$ is $\mathscr{H}_1$-polar. Since $(X_0,\mathscr{H}_2^0)$ is a Brelot space we get $h\in\mathscr{H}_2^0(U\backslash\{0\})$ and $h_n\to h$ uniformly on ∂U_α for α large. As above we find

$$h(0)=\lim_{n\to\infty}h_n(0)=\lim_{n\to\infty}\lim_{x\to 0}\mu_x^{U_\alpha}(h_n|_{\partial U_\alpha})=\lim_{x\to 0}\mu_x^{U_\alpha}(h|_{U_\alpha}),$$

and thus $h\in\mathscr{H}_2(U)$. If $h\equiv\infty$ in $U\backslash\{0\}$ then for every given $\gamma>0$ we can find an $n_0\in\mathbb{N}$ such that $h_{n_0}\geqq\gamma$ on ∂U_α. By the minimum principle we find $h(0)\geqq\gamma$, and hence $h(0)=\infty$. This shows that Brelot's convergence axiom holds.

Hence $(X,\mathscr{H}_2)$ is a $\mathscr{P}$-Brelot space. Obviously we may apply the same arguments to the dual space and get a Brelot space $(X,{}^d\mathscr{H}_2)$. The remaining assertions are clear. □

Now we can translate this theorem into the situation $X=\mathbb{R}^n$, $G_1=G_\Delta^{\mathbb{R}^n}$, and $G_2=G_{\mathscr{K}(L)}^{\mathbb{R}^n\backslash\{0\}}$ for some $L\in\mathscr{L}^+(\mathbb{R}^n)$. We get that

$$C^{-1}G_\Delta^{\mathbb{R}^n}\leqq G_L^{\mathbb{R}^n}\leqq CG_\Delta^{\mathbb{R}^n}$$

holds on $\mathbb{R}^n\times\mathbb{R}^n$ if and only if there are twice continuously differentiable functions p,q on $\mathbb{R}^n\backslash\{0\}$ such that the harmonic space on $\mathbb{R}^n\backslash\{0\}$ which belongs to $Q\mathscr{K}(L)P$ has a good extension into zero. Here P and Q denote the multiplication operators with p and q: $(Q\mathscr{K}(L)P)u=q((\mathscr{K}(L))(pu))$. That we can choose p and q twice differentiable follows immediately from Remark 4.2: We choose $\varrho_0=\mu_0$ in such a way, that ϱ_0 has a Hölder continuous density with respect to the Lebesgue measure on $\mathbb{R}^n$.

Another necessary condition for comparability of $G_\Delta^{\mathbb{R}^n}$ and $G_L^{\mathbb{R}^n}$ is uniform ellipticity:

Proposition 4.4. *Let* $L=\sum_{i,j=1}^{n} a_{ij}D_{ij}+\sum_{i=1}^{n} b_i D_i+c\in\mathscr{L}^+(\mathbb{R}^n)$ *and suppose that*

$$C^{-1}G_{\Delta}^{\mathbb{R}^n}\leqq G_L^{\mathbb{R}^n}\leqq CG_{\Delta}^{\mathbb{R}^n}$$

holds on $\mathbb{R}^n\times\mathbb{R}^n$ *for some constant* C.

Then we have

$$C^{\frac{2-n}{2}}|\xi|^2\leqq\sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j\leqq C^{\frac{n-2}{2}}|\xi|^2 \quad \text{for all} \quad y,\xi\in\mathbb{R}^n.$$

Proof. Fix $y\in\mathbb{R}^n$, and let $B(y):=(a_{ij}(y))^{-1}$. It suffices to show that for any eigenvalue λ of $B(y)$

$$C^{-1}\leqq\lambda^{\frac{n-2}{2}}\leqq C$$

holds. Let η be an eigenvector for $B(y)$ with eigenvalue λ, and put $x=y+\delta\eta$ with $\delta>0$. Given $\varepsilon>0$ we can choose δ so small that

$$1-\varepsilon\leqq\frac{G_L^{\mathbb{R}^n}(x,y)}{|x-y|_{B(y)}^{2-n}}=\frac{G_L^{\mathbb{R}^n}(x,y)}{|x-y|^{2-n}}\lambda^{\frac{n-2}{2}}\leqq 1+\varepsilon$$

(see our definition of a Green function in § 1). Since $G_{\Delta}^{\mathbb{R}^n}(x,y)=|x-y|^{2-n}$ we get

$$(1-\varepsilon)C^{-1}\leqq\lambda^{\frac{n-2}{2}}\leqq(1+\varepsilon)C,$$

hence

$$C^{-1}\leqq\lambda^{\frac{n-2}{2}}\leqq C$$

because $\varepsilon>0$ was arbitrary. □

To close let us take up the example from the beginning of this paragraph. Let

$$L=\sum_{i,j=1}^{n} a_{ij}D_{ij}$$

with $a_{11}=2$ and $a_{ij}=\delta_{ij}$ otherwise. If

$$\mathscr{K}(L)=\sum_{i,j=1}^{n}\tilde{a}_{ij}D_{ij}+\sum_{i=1}^{n}\tilde{b}_i D_i+\tilde{c}$$

then we have $\tilde{a}_{23}(y)=4y_1^2y_2y_3|y|^{-4}$, and this shows that $\tilde{a}_{23}$ has no continuous extension to zero. One might suppose that this discontinuity disappears after normalization, but this is not the case. Since $G_{\mathscr{K}(L)}^{\mathbb{R}^n\setminus\{0\}}(x,y)=G_{\mathscr{K}(L)}^{\mathbb{R}^n\setminus\{0\}}(y,x)$ we may normalize with $\mu_0=\varrho_0$, and furthermore we may assume that the $\mathscr{K}(L)$-potential $G_{\mathscr{K}(L)}^{\mathbb{R}^n\setminus\{0\}}\mu_0$ coincides with $p=G_{\mathscr{K}(L)}^{\mathbb{R}^n\setminus\{0\}}(\cdot,y_0)$ in $U\setminus\{0\}$ for some neighbourhood U of zero and some $y_0\in\mathbb{R}^n$ with $|y_0|=1$. In this case the normalized operator L' is given by

$$L'=P^{\frac{2+n}{2-n}}\mathscr{K}(L)P$$

in $U\setminus\{0\}$, where P is the multiplication operator which belongs to p (cf Remark 4.2).

If $L' = \sum_{i,j=1}^{n} a'_{ij} D_{ij} + \sum_{i=1}^{n} b'_i D_i + c'$ then we have

$$a'_{ij} = p^{\frac{4}{2-n}} \tilde{a}_{ij}$$

$$b'_i = p^{\frac{4}{2-n}} \left(\tilde{b}_i + \sum_{k=1}^{n} 2p^{-1} \tilde{a}_{ik} D_k p \right)$$

$$c' = p^{\frac{2+n}{2-n}} \mathscr{K}(L)(p)$$

in $U \backslash \{0\}$. Since $G_{\mathscr{K}(L)}^{\mathbb{R}^n \backslash \{0\}}$ is bounded above and below by positive multiples of $G_{\Delta}^{\mathbb{R}^n}$ the function p has strict positive upper and lower bounds in a neighbourhood of zero. This shows that

$$a'_{23}(y) = (p(y))^{\frac{4}{2-n}} 4 y_1^2 y_2 y_3 |y|^{-4}$$

has no continuous extension to zero.

Now with $\beta_j := b'_j - \sum_{k=1}^{n} D_k a'_{kj}$ the operator L' may be written in divergence form:

$$L' = \sum_{i,j=1}^{n} D_i (a_{ij} D_j) + \sum_{j=1}^{n} \beta_j D_j + c.$$

Obviously in $U \backslash \{0\}$ the operator L' fits into the framework of [8]. If we define $a_{ij}(0) = \delta_{ij}$, $\beta_j(0) = 0$ and $c(0) = 0$ then the coefficients a_{ij} and c fit into the framework of [8] on the whole of U. In [8] the requirement for β_j is the following: There is $s > n$ such that $|\beta_j|^s 1_K$ is integrable for any compact $K \subset \mathbb{R}^n$. By a straightforward computation we find ($n=3$):

$$\beta_1 = 6p^{-5} \sum_{k=1}^{3} \tilde{a}_{k1} D_k p.$$

Fixing $y_0 = (1,0,0)$, and using the expressions from the beginning of this paragraph we can compute $D_k p$, and after lengthy calculations we find

$$\beta_1(y) = -3(p(y))^{-2} [-2y_1 |y|^{-2} (1 + y_1^2 |y|^{-2})^2 + M(x)]$$

with a bounded measurable function M. Since p is bounded above and below near zero with positive constants, this shows that $|\beta_1|^s$ is locally integrable near zero if and only if $y \to |y_1 |y|^{-2}|^s$ is locally integrable near zero. But this is impossible for $s > n = 3$. This shows that there exists an operator L' in divergence form on $\mathbb{R}^n$ which does not satisfy the requirements of [8] but nevertheless there is a harmonic space on $\mathbb{R}^n$ which belongs to L'.

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On Prealgebraic Groups

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B a prealgebraic group, we shall mean a subgroup of a real linear algebraic group of finite index. The class of prealgebraic groups include connected linear semisimple Lie groups and also the (topological) identity component of any real linear algebraic group. In the actual application of algebraic group theory to the study of analytic groups, we often find it convenient to deal with prealgebraic groups rather than algebraic groups.

In this paper, we are mainly interested in the class of real analytic groups G which admit faithful finite-dimensional analytic representations ϕ such that $\phi(G)$ is prealgebraic. We call groups in this class prealgebraically representable, and we characterize such groups completely in Theorem 1. We next prove that the isomorphism classes of real algebraic Lie algebras are in bijective correspondence with the isomorphism classes of prealgebraically representable analytic groups with simply connected centers (Theorem 2). This is a real version of the result of Hochschild [4], which states that, over an algebraically closed field of characteristic 0, the isomorphism classes of algebraic Lie algebras are in bijective correspondence with the isomorphism classes of affine algebraic groups with unipotent centers. Finally, we present an example which limits the scope of Theorem 2.

Notation and *Convention*. All representations of topological groups in this paper are assumed to be continuous and finite-dimensional. For any Lie group G, $\mathscr{L}(G)$ denotes the Lie algebra of G, while the topological identity component of G is denoted by G_0 and the center of G by $Z(G)$. Throughout this paper, algebraic groups are defined in the sense of Chevalley [1]. Thus a real algebraic group, for example, is a subgroup of $GL(V)$ (for some finite-dimensional real linear space V) which consists of the zeros of a set of real polynomials.

I. Prealgebraically Representable Groups

A subgroup H of $GL(V)$, where V is a finite-dimensional real vector space, is called *prealgebraic*, if there exists an algebraic subgroup $A \leqq GL(V)$ such that H is a subgroup of A of finite index. The (topological) 1-component of an algebraic group is prealgebraic, and, more generally, it is easy to see that a closed subgroup H of $GL(V)$ is prealgebraic if and only if H/H_0 is finite and $\mathscr{L}(H)$ is algebraic.

For any linear group $G \leqq GL(V)$, we define a rational representation of G to be a representation ϕ of G which can be extended to a rational representation of an algebraic group L such that $G \leqq L \leqq GL(V)$. If G is a prealgebraic group and if ϕ is a rational representation of G, then $\phi(G)$ is prealgebraic. The standard decomposition of algebraic groups carries over to prealgebraic groups. Thus if $G \leqq GL(V)$ is prealgebraic, then G can be written as a semidirect product $G = UF$, where U is the maximal normal subgroup consisting of unipotent elements of $GL(V)$ and F is a maximal reductive subgroup of G. The subgroup U will be called the unipotent radical of G, and we denote it by G_u. For more detailed discussion on prealgebraic groups, we refer to [2].

Definition. A (real) analytic group G is said to be *prealgebraically representable* if G admits a faithful finite-dimensional continuous representation ϱ such that $\varrho(G)$ is prealgebraic.

Lemma 1. *If Q is a finite normal subgroup of a prealgebraically representable analytic group G, then G/Q is prealgebraically representable.*

Proof. Let ϱ be a faithful representation of G such that $\varrho(G)$ is prealgebraic, and let $\varrho(G)^*$ denote the algebraic hull of $\varrho(G)$. Then $\varrho(Q)$ is normal in $\varrho(G)^*$, and hence by a result of Chevalley [1, Vol. III, Proposition 11, p. 119], there exists a rational representation ξ of $\varrho(G)^*$ such that $\mathrm{Ker}(\xi) = \varrho(Q)$. Then $\xi \circ \varrho(G)$ is a prealgebraic group and $\mathrm{Ker}(\xi \circ \varrho) = Q$. Thus $\xi \circ \varrho$ induces a faithful representation ϕ of G/Q such that $\phi(G/Q) = \xi \circ \varrho(G)$ is prealgebraic. Hence G/Q is prealgebraically representable.

II. The Holomorph of Nilpotent Groups

Let N be a simply connected nilpotent analytic group. Then the exponential map $\exp: \mathscr{L}(N) \to N$ is an isomorphism of analytic manifolds. The canonical map $\mathrm{Aut}(N) \to \mathrm{Aut}(\mathscr{L}(N))$, sending each $\alpha \in \mathrm{Aut}(N)$ to its differential α^0, is an isomorphism, under which we often identify $\mathrm{Aut}(N)$ with algebraic group $\mathrm{Aut}(\mathscr{L}(N))$. We consider the holomorph of N, the semidirect product of N by $\mathrm{Aut}(N)$. We denote it by $N \circledS \mathrm{Aut}(N)$.

The main portion of the following lemma is proven in Hochschild [5] in a more general setting.

Lemma 2. *If N is a simply connected nilpotent analytic group, then there exists a faithful finite-dimensional analytic representation ϕ of the holomorph $N \circledS \mathrm{Aut}(N)$ of N such that $\phi(N)$ is unipotent and the restriction of ϕ to* $\mathrm{Aut}(N)$ *is rational.*

Proof. Pick a faithful representation ϱ of N such that $\varrho(N)$ is unipotent [3, Theorem 3.1, p. 219], and let ϱ^0 denote the differential of ϱ. If $(0) = V_{d+1} < V_d < \ldots < V_0 = V$ is a composition series for the representation space V of ϱ, then the elements of $\varrho^0(\mathscr{L}(N))$ map each V_k into V_{k+1}, and hence every monomial in elements of $\varrho^0(\mathscr{L}(N))$ having more than d factors is equal to 0. Let $y_1, y_2, \ldots, y_n$ be a basis of $\mathscr{L}(N)$ and let $z=\sum_j t_j y_j \in \mathscr{L}(N)$ be any. Let $\alpha^0(y_j)=\sum_{k=1}^{n} c_{jk}(\alpha)y_k$, and $B_j=\varrho^0(y_j)$. Then $\varrho^0 \circ \alpha^0(z)=\sum_{i,j} c_{ij}(\alpha)t_i B_j$, and

$$\varrho \circ \alpha(\exp z)=\operatorname{Exp}(\varrho^0 \circ \alpha^0(z))=\sum_{t=0}^{d}(\varrho^0 \circ \alpha^0(z))^t/t! = \sum_t \Big(\sum_{ij} c_{ij}(\alpha)t_i B_j\Big)^t/t!$$

Let $p_1, \ldots, p_s$ be all the monomials of degree $\leqq d$ in n^2 variables ($n=\dim N$). Collecting terms appropriately after expanding the above expression of $\varrho \circ \alpha(\exp z)$, we see that there are maps $g_1, \ldots, g_s$ of $\mathscr{L}(N)$ into the endomorphism ring $\operatorname{End}(V)$ of V such that

$$\varrho \circ \alpha(\exp z)=\sum_{r=1}^{s} p_r(\alpha) g_r(\alpha), \tag{1}$$

where $p_r(\alpha)$ denotes $p_r(\ldots, c_{ij}(\alpha), \ldots)$.

Now let $[\varrho]$ denote the real linear space of representative functions on G that are associated with ϱ (see [3, p. 22]). As $[\varrho]$ is finite-dimensional, we can find linear functionals $\gamma_1, \ldots, \gamma_t$ on $\operatorname{End}(V)$ such that $f_1=\gamma_1 \circ \varrho, \ldots, f_1=\gamma_t \circ \varrho$ form a basis of $[\varrho]$. Let W denote the linear space spanned by $[\varrho] \circ \operatorname{Aut}(N)$. For any x in N, we have from the expression (1) above that

$$\varrho \circ \alpha(x)=\varrho \circ \alpha(\exp \log x)=\sum_{r=1}^{s} p_r(\alpha)(g_r \circ \log)(x). \tag{2}$$

This shows that W is contained in the subspace W' spanned by the functions $\gamma_i \circ g_r \circ \log$, $1 \leqq i \leqq 1$ and $1 \leqq r \leqq s$, and hence W is, in particular, finite-dimensional.

Define $\psi: N \circledS \operatorname{Aut}(N) \to GL(W)$ by $\psi(x, \alpha)(w)=x \cdot (w \cdot \alpha^{-1})$, where, for $x \in N$ and any real valued-function f on N, the left translate $x \cdot f$ of f is defined by $(x \cdot f)(y)=f(yx)$. Then ψ is an analytic representation, and its restriction to N is faithful. Since $\varrho(N)$ is unipotent, it follows that $\psi(N)$ is unipotent. We now show that the restriction of ψ to $\operatorname{Aut}(N)$ is rational. Choose a basis $h_1, \ldots, h_q$ of W' from the list $\{\gamma_i \circ f_r \circ \log\}_{i,r}$. For $f \in [\varrho]$ and $\alpha \in \operatorname{Aut}(N)$, we have from the expression (2) that

$$f \circ \alpha=\sum_{r=1}^{q} Q_r(\alpha, f) h_r,$$

where each $Q_r(\alpha, f)$ depends only on α and f, and $\alpha \to Q_r(\alpha, f)$ is a polynomial function on $\operatorname{Aut}(N)$. Now choose $\alpha_1, \ldots, \alpha_m \in \operatorname{Aut}(N)$ so that $w_1=f_1 \circ \alpha_1, \ldots, w_m=f_m \circ \alpha_m$ form a basis of W and supplement these elements with elements $w_{m+1}, \ldots, w_q$ which are chosen from the list $h_1, \ldots, h_q$ so that $w_1, \ldots, w_q$ form a new basis of W'. Let $h_r=\sum_{k=1}^{q} a_{rk} w_k$ with $a_{rk} \in R$. Thus for $1 \leqq i \leqq m$ and $\beta \in \operatorname{Aut}(N)$, we have

$$\begin{aligned}
\psi(1, \beta)(w_i) &= (f_1 \circ \alpha_i) \circ \beta^{-1} \\
&= f_i \circ (\alpha_i \circ \beta^{-1}) \\
&= \sum_{r=1}^{q} Q_r(\alpha_i \circ \beta^{-1}, f_i) h_r \\
&= \sum_k Q'_{ik}(\beta) w_k,
\end{aligned}$$

where $Q'_{ik}(\beta) = \sum_r Q_r(\alpha_i \circ \beta^{-1}, f_i)a_{rk}$. For any i and k, $\beta \to Q'_{ik}(\beta)$ is a rational function from the algebraic group $\mathrm{Aut}(N)$ to R, and, consequently, $\beta \to \psi(1, \beta)$ is a rational representation of $\mathrm{Aut}(N)$. Let $\xi : N \circledS \mathrm{Aut}(N) \to GL(\mathscr{L}(N))$ be the composite map of the projection $N \circledS \mathrm{Aut}(N) \to \mathrm{Aut}(N)$ with the canonical embedding $\mathrm{Aut}(N)$ into $GL(\mathscr{L}(N))$, and we define ϕ to be the direct sum of the representations ξ and ψ. Then ϕ is a desired representation.

III. Main Result-I

In the proof of Theorem 1, we use the notion of reductivity of analytic groups. An analytic group G is called reductive if G is faithfully representable (i.e., G admits a faithful representation) and every continuous finite-dimensional representation of G is semisimple. It is known (e.g. [3, Theorem 4.4, p. 224]) that a faithfully representable analytic group G is reductive if and only if $Z(G)$ is compact and $G/Z(G)$ is semisimple.

Theorem 1. *Suppose that G is a faithfully representable analytic group. Then G is prealgebraically representable if and only if G satisfies the following properties:*
(i) $\mathrm{Ad}(G)$ *is a prealgebraic subgroup of* $GL(\mathscr{L}(G))$, *and*
(ii) $Z(G)_0$ *is of finite index in* $Z(G)$.

Proof. Suppose G admits a faithful representation ϱ such that $\varrho(G)$ is prealgebraic. Thus $\mathrm{Ad}(\varrho(G))$ is prealgebraic, and we have $\varrho^0 \circ \mathrm{Ad}(x) = \mathrm{Ad}(\varrho(x)) \circ \varrho^0$ for all $x \in G$. Hence the rational isomorphism $\theta \to \varrho^{0-1} \circ \theta \circ \varrho^0$ from $GL(\varrho^0(\mathscr{L}(G)))$ to $GL(\mathscr{L}(G))$ maps $\mathrm{Ad}(\varrho(G))$ onto $\mathrm{Ad}(G)$. Hence $\mathrm{Ad}(G)$ is prealgebraic. To show that $Z(G)_0$ is of finite index in $Z(G)$, we may assume that G itself is prealgebraic. As a prealgebraic group, G is contained in an algebraic group as a subgroup of finite index, and hence G is of finite index in its algebraic hull G^*. Since G is connected, G is contained in the topological identity component G_0^* of G^*. Consequently, G is of finite index in the connected group G_0^*, and we have $G = G_0^*$. (Note, in general, that any subgroup Q of finite index in a connected Lie group L coincides with L. See, for example, [2, p. 265].) Now since $Z(G^*)$ is a real algebraic subgroup, its topological identity component $Z(G^*)_0$ is of finite index in $Z(G^*)$. Clearly, $Z(G) \leqq Z(G^*)$, and $Z(G^*)_0 \leqq G_0^* = G$. Thus $Z(G^*)_0 = Z(G)_0$, and we see that $Z(G)_0$ is of finite index in $Z(G)$, proving (ii).

Next we assume that G satisfies the conditions (i) and (ii). Since G is faithfully representable, we have a semidirect product $G = B \cdot H$, where B is a simply connected solvable closed normal subgroup and H is a maximal reductive analytic subgroup [3, Theorem 4.3, p. 223]. We first reduce our consideration to the case in which $Z(G)_0$ is a vector group. In fact, let C denote the maximal torus of $Z(G)_0$. Then C is contained in H, and since H is a reductive analytic group, we can find a closed normal analytic subgroup H_1 of H such that $H = H_1 \cdot C$ and $H_1 \cap C$ is finite. Put $G_1 = B \cdot H_1$. Then G_1 is a closed subgroup of G and $G = G_1 \cdot C$ with $G_1 \cap C$ finite. Thus G is of the form $G_1 \times C/\Delta$, where Δ is a finite central subgroup of $G_1 \times C$. Since C is prealgebraically representable as a compact group, G becomes prealgebraically representable by virtue of Lemma 1 as soon as we show that G_1 is prealgebraically representable. Since G_1 clearly satisfies the conditions (i) and (ii) and since $Z(G_1)_0$ is a vector group, we may assume, replacing G by G_1 if necessary, that G itself has the property that $Z(G)_0$ is a vector group.

Let N denote the nilradical of G and let $\mathcal{M}=\mathcal{L}(N)$. Since $Z(G)_0$ is a vector group, N is simply connected. In the decomposition $G=B\cdot H$, we may select B so that $N\subseteqq B$. Let K denote the prealgebraic group $\mathrm{Ad}(G)$, and we consider the canonical decomposition $K=K_u\cdot P$, where P is a maximal reductive prealgebraic subgroup of K such that $\mathrm{Ad}(H)\subseteqq P$. If ad denotes the adjoint representation of $\mathcal{L}(G)$, then we have the semidirect sum $\mathrm{ad}(\mathcal{L}(G))=\mathcal{L}(K_u)+\mathcal{L}(P)$, and $\mathrm{ad}^{-1}(\mathcal{L}(K_u))=\mathcal{M}$. Let $P_1=\mathrm{Ad}(B)\cap P$. Then $\mathcal{L}(P_1)=\mathrm{ad}(\mathcal{L}(B))\cap\mathcal{L}(P)$ and $\mathrm{ad}^{-1}(\mathcal{L}(P_1))=\mathcal{L}(B)\cap\mathrm{ad}^{-1}(\mathcal{L}(P))$. Then $\mathcal{L}(B)=\mathcal{M}+\mathrm{ad}^{-1}(\mathcal{L}(P_1))$ and $\mathcal{M}\cap\mathrm{ad}^{-1}(\mathcal{L}(P_1))=Z(\mathcal{L}(G))$, the centre of $\mathcal{L}(G)$. Since P leaves $\mathrm{ad}^{-1}(\mathcal{L}(P_1))$ and $Z(\mathcal{L}(G))$ invariant, $\mathrm{ad}^{-1}(\mathcal{L}(P_1))$ contains a P-invariant subspace $\mathcal{A}$, which is a complement to $Z(\mathcal{L}(G))$ in $\mathrm{ad}^{-1}(\mathcal{L}(P_1))$. Then $\mathcal{A}$ is $\mathcal{L}(P)$-invariant, and $[\mathcal{A},\mathcal{A}]=\mathrm{ad}(\mathcal{A})(\mathcal{A})\subseteqq\mathcal{L}(P)(\mathcal{A})\subseteqq\mathcal{A}$, showing that $\mathcal{A}$ is a subalgebra of $\mathcal{L}(G)$. It is now clear that $\mathcal{L}(B)=\mathcal{M}+\mathcal{A}$ is a semidirect sum, and $\mathcal{A}$ is abelian since B is solvable. Let A denote the analytic subgroup of B with $\mathcal{L}(A)=\mathcal{A}$. Then $B=NA$. The homomorphism N Ⓢ $A\to B$ sending (n,a) to na is a covering homomorphism, and since B is simply connected, this map is an isomorphism. Consequently, $B=N\cdot A$ is a semidirect product. Let $Q=A\cdot H$. Since $\mathrm{Ad}(H)\subseteqq P$ and since $\mathcal{A}$ is P-invariant, it follows that Q is a subgroup of G. Since $Z(H)$ is compact, Q is closed in G. It is now clear that $G=N\cdot Q$ is a semidirect product decomposition, and $\mathrm{Ad}(Q)=P$.

Now consider the homomorphism $f\colon G=N\cdot Q\to N$ Ⓢ $\mathrm{Aut}(N)$, which is given by $f(nx)=(n,I_x|N)$, $n\in N$ and $x\in Q$. $\mathrm{Ker}(f)\subseteqq Q$ and, in fact, $\mathrm{Ker}(f)=Z_Q(N)$, the centralizer of N in Q. Let $K=(\mathrm{Ker}\,f)_0$. Since $N\cap\mathrm{Ker}(f)=\{1\}$, the radical of K is compact. But since G has no central torus, K is semisimple. Consequently, $G=K\cdot G_2$, where G_2 is the identity component of the centralizer of K in G, and $K\cap G_2$ is finite. Since G is isomorphic to a quotient of $K\times G_2$ by a finite subgroup, G will be prealgebraically representable as soon as we show that G_2 is prealgebraically representable (Lemma 1). Let $f_2=f|G_2$. Then $f_2\colon G_2\to N$ Ⓢ $\mathrm{Aut}(N)$ and $\mathrm{Ker}(f_2)$ is discrete and it is even finite because $Z(G)_0$ is of finite index in $Z(G)$. Now let ϕ be a faithful representation of N Ⓢ $\mathrm{Aut}(N)$ such that $\phi(N)$ is unipotent and that $\phi|\mathrm{Aut}(N)$ is rational (Lemma 2). Clearly, $G_2=N\cdot(Q\cap G_2)$ and $f_2(Q\cap G_2)=f(Q)\subseteqq\mathrm{Aut}(N)$. Identifying the inner automorphism group $\mathrm{Int}(G)$ of G with $\mathrm{Ad}(G)$, let $r\colon\mathrm{Int}(G)=\mathrm{Ad}(G)\to\mathrm{Aut}(N)$ be the restriction map. Then $f(Q)=\{1\}\times r(P)$. Since P is prealgebraic, $f_2(Q\cap G_2)=\{1\}\times r(P)$ is a prealgebraic subgroup of $\{1\}\times\mathrm{Aut}(N)$. Hence $f_2(G_2)=f_2(N)f_2(Q\cap G_2)=(N\times\{1\})f_2(Q\cap G_2)$, and $(\phi\cdot f_2)(G_2)=\phi(N\times\{1\})\cdot\phi(\{1\}\times r(P))$ is prealgebraic. Since $\mathrm{Ker}(\phi\circ f_2)$ is finite, G_2 is prealgebraically representable by Lemma 1.

Corollary (to the proof of Theorem 1). *Suppose G is faithfully representable and satisfies the conditions (i) and (ii) of Theorem 1. If the nilradical N of G is simply connected, then G admits a faithful representation ϱ such that $\varrho(G)$ is prealgebraic and $\varrho(N)$ is unipotent.*

IV. Main Result-II

A linear Lie algebra is said to be algebraic if it is isomorphic with the Lie algebra of an algebraic group.

Theorem 2. *Let $\mathscr{L}$ be a finite-dimensional Lie algebra over $\mathbb{R}$ such that the adjoint algebra $ad(\mathscr{L})$ is an algebraic subalgebra of $\mathrm{gl}(\mathscr{L})$. Then there exists a prealgebraically representable analytic group G such that $\mathscr{L}(G) \cong \mathscr{L}$ and $Z(G)$ is simply connected. Moreover, if H is another prealgebraically representable analytic group with $\mathscr{L}(H) \cong \mathscr{L}$ and $Z(H)$ is simply connected, then H is isomorphic with G as analytic groups.*

Proof. Since $\mathrm{ad}(\mathscr{L})$ is algebraic, the adjoint group K [=the subgroup of $GL(\mathscr{L})$ generated by $\mathrm{Exp}(\mathrm{ad}(\mathscr{L}))$] is a prealgebraic subgroup of $GL(\mathscr{L})$ whose Lie algebra is $\mathrm{ad}(\mathscr{L})$. Consider the semidirect decomposition $K = K_u \cdot P$, where P is a maximal reductive subgroup of K. Let $\gamma: \mathscr{L} \to \mathscr{L}(K)$ denote the surjective homomorphism defined by the adjoint representation of $\mathscr{L}$, and let $\mathscr{M} = \gamma^{-1}(\mathscr{L}(K_u))$. Clearly, $\mathscr{M}$ is the nilradical of $\mathscr{L}$, and we have $\mathscr{L} = \mathscr{M} + \gamma^{-1}(\mathscr{L}(P))$. $\mathscr{M} \cap \gamma^{-1}(\mathscr{L}(P)) = Z(\mathscr{L})$, the center of $\mathscr{L}$. Since $\gamma^{-1}(\mathscr{L}(P))$ and $Z(\mathscr{L})$ are invariant under the reductive Lie algebra $\mathscr{L}(P)$, $\gamma^{-1}(\mathscr{L}(P))$ contains an $\mathscr{L}(P)$-invariant subspace $\mathscr{B}$ such that $\gamma^{-1}(\mathscr{L}(P)) = Z(\mathscr{L}) \oplus \mathscr{B}$. Then $[\mathscr{B}, \mathscr{B}] = \mathrm{ad}(\mathscr{B})(\mathscr{B}) = \mathscr{L}(P)(\mathscr{B}) \subseteq \mathscr{B}$, showing that $\mathscr{B}$ is a subalgebra of $\mathscr{L}$. It is clear that $\mathscr{L} = \mathscr{M} + \mathscr{B}$ is a semidirect sum and γ maps $\mathscr{B}$ isomorphically onto $\mathscr{L}(P)$.

As K_u is normal in K, the ideal $\mathscr{M}$ is P-invariant. Let M be a simply connected analytic group such that $\mathscr{L}(M) = \mathscr{M}$. The action of P on $\mathscr{M}$ induces a continuous homomorphism $\alpha: P \to \mathrm{Aut}(M)$ such that $t(u) = \alpha(t)^0(u)$ for $t \in P$ and $u \in \mathscr{M}$, where $\alpha(t)^0$ denotes the differential of $\alpha(t)$. Using α, we form a semidirect product M Ⓢ P. Clearly, $\mathscr{L}(M$ Ⓢ $P) \cong \mathscr{M} + \mathscr{B} \cong \mathscr{L}$. Let Z denote the center of M Ⓢ P. Since the center of $\mathscr{L}$ is contained in $\mathscr{M}$, Z_0 is contained in $M \times \{1\}$, and since M is a simply connected nilpotent analytic group, we have $Z_0 = Z \cap (M \times \{1\})$. In particular, Z_0 is simply connected. Now let $(a, t) \in Z$ be arbitrary. Then a simple computation shows that $t \in Z(P)$ and $\alpha(t)(m) = a^{-1}ma$ for all $m \in M$, and hence $\alpha(t)^0 = \mathrm{Ad}_M(a)$, where Ad_M denotes the adjoint representation of M. But $\alpha(t)^0$ is semisimple while $\mathrm{Ad}_M(a)$ is unipotent on $\mathscr{M}$. Hence $\alpha(t)^0 = 1 = \mathrm{Ad}_M(a)$, which shows $(1, t)$ centralizes $M \times \{1\}$. Consequently, $(1, t) \in Z$ and $(a, 1) = (a, t)(1, t)^{-1} \in Z \cap (M \times \{1\}) = Z_0$. Let $\pi: M$Ⓢ$P \to P$ denote the projection, and let $D = \pi(Z)$. By what we have shown above, $Z \cap P = \{1\} \times D$, and $Z = Z_0(\{1\} \times D)$ is a direct product. Let $G = M$ Ⓢ $P/\{1\} \times D$. Since D is discrete, $\mathscr{L}(G) \cong \mathscr{L}$ and the quotient map M Ⓢ $P \to G$ is a covering homomorphism. Hence $Z(G) \cong Z/\{1\} \times D \cong Z_0$ is simply connected. Hence G satisfies the conditions (i) and (ii) of Theorem 1. Thus in order to show that G is prealgebraically representable, it remains to show that G is faithfully representable (Theorem 1). Since $D \leqq \mathrm{Ker}(\alpha)$, α induces a continuous homomorphism $P/D \to \mathrm{Aut}(M)$ and we form the semidirect product M Ⓢ (P/D) using this homomorphism. Then it is clear that $G \cong M$ Ⓢ (P/D) as analytic groups. Let R (resp. S) denote the radical (resp. a maximal semisimple analytic subgroup) of P, and let $\sigma: P \to P/D$ be the quotient map. Then $\sigma(R)$ is the radical and $\sigma(S)$ is a maximal semisimple analytic subgroup of P/D. Since P is a reductive prealgebraic group, R is central in P, and hence $\sigma(R)$ is faithfully representable as an abelian analytic group. On the other hand, S is prealgebraically representable as a semisimple analytic group and $\sigma(S) \cong S/S \cap D$. Since $S \cap D$ is finite, $\sigma(S)$ is faithfully (even prealgebraically) representable by Lemma 1. Now P/D is faithfully representable by [3, Theorem 4.2, p. 221]. Since M is simply connected, M Ⓢ (P/D) (and hence G) is faithfully representable by [3, Theorem 4.3, p. 223].

In order to prove the uniqueness, we adopt the argument which was used in [4]. Let G and H be as in the theorem, and let $\varrho: \mathscr{L}(G) \to \mathscr{L}(H)$ be an isomorphism. We want to show that ϱ is the differential of some analytic isomorphism $\sigma: G \to H$. By the Corollary to Theorem 1, we may assume that G and H are prealgebraic groups with $Z(G)$ and $Z(H)$ unipotent. If $\mathscr{L}$ is semisimple, the assertion becomes clear. For, in this case, the centers of G and H are both trivial, and σ is the composite of the isomorphisms

$$G \cong \mathrm{Ad}(G) \to \mathrm{Ad}(H) \cong H,$$

where $\mathrm{Ad}(G) \to \mathrm{Ad}(H)$ is given by $\alpha \to \varrho \circ \alpha \circ \varrho^{-1}$. Now for the general case, we first note that any Lie algebra $\mathscr{L}$ can be written as a direct sum $\mathscr{L} = \mathscr{S} \oplus \mathscr{T}$, where $\mathscr{S}$ is the maximum semisimple ideal of $\mathscr{L}$ and $\mathscr{T}$ is the centralizer of $\mathscr{S}$ in $\mathscr{L}$. Let $\mathscr{L}(G) = \mathscr{S}' \oplus \mathscr{T}'$ and $\mathscr{L}(H) = \mathscr{S}'' \oplus \mathscr{T}''$ be the similar decompositions, and let $\mathscr{S}'$ and T' be the analytic subgroups of G corresponding to $\mathscr{S}'$ and $\mathscr{T}'$, respectively. Then $G = S' \cdot T'$ and $S' \cap T'$ is finite and central in G. Since $Z(G)$ is simply connected, $S' \cap T' = \{1\}$ and $G = S' \times T'$. Similarly, $H = S'' \times T''$, where S'' and T'' are the analytic subgroups belonging to $\mathscr{S}''$ and $\mathscr{T}''$, respectively. Clearly, $\mathscr{S}' \cong \mathscr{S}''$ and $\mathscr{T}' \cong \mathscr{T}''$, and since we have already shown the assertion for the semisimple case, we may assume that $\mathscr{L}$ has no semisimple ideals. Consider the semidirect product decomposition $G = G_u \cdot P$, where P is a connected prealgebraic reductive subgroup of G. $\mathscr{L}(G_u)$ and $\mathscr{L}(H_u)$ are the nilradicals of $\mathscr{L}(G)$ and $\mathscr{L}(H)$, respectively. Thus the isomorphism ϱ maps $\mathscr{L}(G_u)$ onto $\mathscr{L}(H_u)$, and hence there is an analytic (even a rational) isomorphism $\sigma_u: G_u \cong H_u$, whose differential is $\varrho_u = \varrho | \mathscr{L}(G_u)$. Let Q denote the analytic subgroup of H corresponding to the maximal reductive subalgebra $\varrho(\mathscr{L}(P))$. Then Q is a maximal reductive subgroup of H and we have a semidirect product $H = H_u \cdot Q$. As $\mathscr{L}(G)$ contains no semisimple ideals and $Z(G) \subseteq G_u$, it follows that the adjoint representation $\alpha: P \to GL(\mathscr{L}(G_u))$ is faithful and hence $\alpha: P \cong \alpha(P)$. Similarly, the adjoint representation $\beta: Q \to GL(\mathscr{L}(H_u))$ is faithful and $\beta: Q \cong \beta(Q)$. Using $\varrho_u \circ \mathrm{ad}(y) = \mathrm{ad}(\varrho(y)) \circ \varrho_u$ for $y \in \mathscr{L}(P)$, we can easily obtain

$$\beta(Q) = \varrho_u \circ \alpha(P) \circ \varrho_u^{-1}.$$

Define $\sigma_r: P \to Q$ by $\sigma_r(x) = \beta^{-1}(\varrho_u \circ \alpha(x) \circ \varrho_u^{-1})$, $x \in P$, and finally, define $\sigma: G = G_u \cdot P \to H = H_u \cdot Q$ by $\sigma(ax) = \sigma_u(a)\sigma_r(x)$. Then σ is a desired analytic isomorphism.

V. Example

We give an example which shows that G and H in Theorem 2 may not be rationally isomorphic, when G and H are prealgebraic groups with $Z(G)$ and $Z(H)$ unipotent.

Let k denote a fixed integer >1, and let L denote the group of all 3×3 matrices

$$\begin{pmatrix} \alpha^k & 0 & z \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha\in\mathbb{C}^* \quad \text{and} \quad z\in\mathbb{C}.$$

L is a complex algebraic group and its real points form a (real) algebraic group L_R which consists of all 3×3 real matrices in L. We note here that L is isomorphic with the semidirect product of the vector group $\mathbb{C}$ by the 1-dimensional algebraic torus $\mathbb{C}^*$, where $\mathbb{C}^*$ acts on $\mathbb{C}$ by $(\alpha, z)\to\alpha^k\cdot z$, $\alpha\in\mathbb{C}^*$ and $z\in\mathbb{C}$. A simple calculation shows that the center $Z(L)$ of L consists of all matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\alpha^k=1$. In particular, $Z(L)$ is nontrivial and finite. Clearly, $Z(L_R)$ is trivial if k is odd and $Z(L_R)=\{-1,1\}$ if k is even. Let G denote the topological identity component $(L_R)_0$ of L_R. Then G is a connected prealgebraic group with a trivial center. Let Ad denote the adjoint representation of L, and let $H=\mathrm{Ad}(G)$. Then $G\cong H$ as analytic groups. However, G and H are not rationally isomorphic. In fact, if they were rationally isomorphic, then the Zariski closure L of G would be isomorphic with the Zariski closure $\mathrm{Ad}(L)$ of $H=\mathrm{Ad}(G)$ in $GL(\mathscr{L}(L))$, which is impossible because $\mathrm{Ad}(L)$ has the trivial center while $Z(L)\neq\{1\}$.

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Shimura Varieties and Twisted Orbital Integrals*

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In [L2, L3] Langlands has drawn attention to a combinatorial problem that arises when one tries to express the zeta function of a Shimura variety in terms of automorphic L-functions. In this paper we will reduce the combinatorial problem to a standard problem in local harmonic analysis: that of establishing the spherical function identities needed for base change [K1, L4, L5].

1. Points on S_K Over Finite Fields

1.1

We consider the Shimura variety S_K associated to a connected reductive group G over $\mathbb{Q}$, a $G(\mathbb{R})$-conjugacy class X_∞ of homomorphisms $h: \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to G_{\mathbb{R}}$, and a sufficiently small compact open subgroup K of $G(\mathbb{A}_f)$ (see [D]). For technical reasons we assume that the derived group of G is simply connected. Recall that

(1.1.1) $$S_K(\mathbb{C}) = G(\mathbb{Q}) \backslash (X_\infty \times (G(\mathbb{A}_f)/K)).$$

As in [D], any $h \in X_\infty$ gives rise to a homomorphism $\mu_h: \mathbb{G}_m \to G_{\mathbb{C}}$, and the $G(\mathbb{C})$-conjugacy class $M_{\mathbb{C}}$ of μ_h is independent of $h \in X_\infty$. Recall that any $\mu \in M_{\mathbb{C}}$ satisfies the following property:

(1.1.2) The representation $Ad \circ \mu$ of $\mathbb{G}_m$ on $\mathrm{Lie}(G_{\mathbb{C}})$ has no weights other than $1, 0, -1$.

For any field F containing $\mathbb{Q}$, we write $\mathcal{M}(F)$ for the set of $G(F)$-conjugacy classes of homomorphisms $\mathbb{G}_m \to G_F$. Of course $M_{\mathbb{C}}$ is an element of $\mathcal{M}(\mathbb{C})$.

(1.1.3) **Lemma.** *Let F be a field containing $\mathbb{Q}$, and let $\bar{F}$ be an algebraic closure of F.*

(a) *Assume that G is quasi-split over F. Then $\mathcal{M}(F) = \mathcal{M}(\bar{F})^{\mathrm{Gal}(\bar{F}/F)}$. Furthermore, for any maximal F-split torus S of G_F, with relative Weyl group Ω_F, we have $X_*(S)/\Omega_F \xrightarrow{\sim} \mathcal{M}(F)$.*

(b) *Let F' be an algebraically closed field containing $\bar{F}$. Then $\mathcal{M}(\bar{F}) = \mathcal{M}(F')$.*

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First we prove the second statement of (a). The surjectivity of $X_*(S)\to\mathcal{M}(F)$ follows from the fact that any two maximal F-split tori in G_F are conjugate under $G(F)$. To prove injectivity we consider $\mu, \nu\in X_*(S)$ such that μ, ν are conjugate under $G(F)$. Choose $g\in G(F)$ such that $\mathrm{Int}(g)\circ\mu=\nu$. Then $\mathrm{Int}(g)(S)$ and S are both maximal F-split tori in the centralizer M of $\nu(\mathbb{G}_m)$ in G. The group M is connected and reductive. Hence there exists $m\in M(F)$ such that $\mathrm{Int}(mg)S=S$. Let $g'=mg\in G(F)$. Then g' normalizes S, and $\mathrm{Int}(g')\circ\mu=\nu$. Therefore μ, ν are in the same Ω_F-orbit.

Next we prove the first statement of (a). Let T be the centralizer of S in G; since G is quasi-split over F, T is a maximal F-torus of G. Using what we have already shown, we are reduced to proving that

$$X_*(S)/\Omega_F \xrightarrow{\sim} [X_*(T)/\Omega]^{\mathrm{Gal}(\bar{F}/F)}, \tag{1.1.3.1}$$

where Ω denotes the absolute Weyl group of T in G. Choose a Borel F-subgroup B containing T, and let C denote the B-positive Weyl chamber of $X_*(T)\otimes\mathbb{R}$. To prove the surjectivity of the map (1.1.3.1), we consider $\mu\in X_*(T)$ whose Ω-orbit is stable under $\mathrm{Gal}(\bar{F}/F)$. Without loss of generality we may assume that $\mu\in\bar{C}$. Let $\sigma\in\mathrm{Gal}(\bar{F}/F)$. Then $\sigma\mu\in\bar{C}$ [since $\mathrm{Gal}(\bar{F}/F)$ preserves C], and $\sigma\mu, \mu$ are in the same Ω-orbit. Since $\bar{C}$ is a fundamental domain for the action of Ω on $X_*(T)\otimes\mathbb{R}$, it follows that $\sigma\mu=\mu$. Therefore μ belongs to $X_*(S)$, and we have proved surjectivity.

To prove injectivity, we consider $\mu, \nu\in X_*(S)$ for which there exists $w\in\Omega$ with $w\cdot\mu=\nu$. Let $C_0=C\cap(X_*(S)\otimes\mathbb{R})$; then C_0 is a Weyl chamber in $X_*(S)\otimes\mathbb{R}$. Without loss of generality we may assume that $\mu, \nu\in\bar{C}_0$. Then $\mu, \nu\in\bar{C}$, and since $\bar{C}$ is a fundamental domain for the action of Ω on $X_*(T)\otimes\mathbb{R}$, we see that $w=1$, and hence that $\mu=\nu$. This proves injectivity.

Finally, part (b) follows from part (a).

1.2

We write $\bar{\mathbb{Q}}$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $M_{\bar{\mathbb{Q}}}$ be the element of $\mathcal{M}(\bar{\mathbb{Q}})$ corresponding to $M_{\mathbb{C}}$ under the canonical bijection $\mathcal{M}(\bar{\mathbb{Q}})\xrightarrow{\sim}\mathcal{M}(\mathbb{C})$. The *reflex field* E of (G, X_∞) is the subfield of $\bar{\mathbb{Q}}$ such that $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$ is equal to the stabilizer of $M_{\bar{\mathbb{Q}}}$ in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$; the field E is a finite extension of $\mathbb{Q}$. Shimura's conjecture is now known to be true [M], so that we have a canonical E-structure on S_K. We consider only those G for which the derived group G_{der} is anisotropic over $\mathbb{Q}$; then S_K is projective (and smooth) over E. As explained in [L2, L3], one hopes to express the zeta function of S_K in terms of automorphic L-functions for the endoscopic groups of G. In order to calculate the zeta function, we look at points mod p.

1.3

Let p be a rational prime and let v be a place of E that lies over p. Assume that G is unramified at p and that K is of the form $K^p\cdot K_p$, where K_p is the stabilizer in $G(\mathbb{Q}_p)$ of a hyperspecial point x_0 in the building of G over $\mathbb{Q}_p$, and K^p is a compact open subgroup of $G(\mathbb{A}_f^p)$ (we use $\mathbb{A}_f^p$ to denote the ring of $\mathbb{Q}$-adeles with trivial components at p and ∞). Let $\bar{E}_v$ be an algebraic closure of E_v. Let $\mathbb{Q}_p^{un}$ denote the maximal unramified extension of $\mathbb{Q}_p$ contained in $\bar{E}_v$, and let L denote the completion of $\mathbb{Q}_p^{un}$. Since G is unramified over $\mathbb{Q}_p$, the field E_v is contained in $\mathbb{Q}_p^{un}$.

The residue field $\bar{k}$ of the valuation ring $\mathfrak{o}_L$ of L is an algebraic closure of the residue field k of the valuation ring $\mathfrak{o}_v$ of E_v.

Langlands [L1] conjectures that S_K can be given a smooth, proper $\mathfrak{o}_v$-structure for which the $\mathrm{Gal}(\bar{k}/k)$-set $S_K(\bar{k})$ has a specified description in terms of the group G. We will assume the truth of this conjecture.

The description of $S_K(\bar{k})$ requires the choice of an embedding $\iota:\bar{\mathbb{Q}}\to\bar{E}_v$ that extends the inclusion map $E\to E_v$. Langlands conjectures that $S_K(\bar{k})$ is a disjoint union of subsets of the form

(1.3.1) $$I(\mathbb{Q})\backslash(X_p\times X^p).$$

The group I is a connected reductive $\mathbb{Q}$-group and comes with an embedding over $\mathbb{A}_f^p$

(1.3.2) $$I\to G.$$

The set X^p is equal to $G(\mathbb{A}_f^p)K^p$.

The set X_p is obtained from an element $b\in G(L)$. For any field F between $\mathbb{Q}_p$ and L we write $V(F)$ for the $G(F)$-orbit of x_0 in the building of G over F. We write σ for the Frobenius automorphism of L over $\mathbb{Q}_p$. There is an action on $V(L)$ of the semidirect product of $G(L)$ and $\langle\sigma\rangle$, the infinite cyclic group generated by σ. Using ι, we get from $M_{\bar{\mathbb{Q}}}$ an element of $\mathcal{M}(\bar{E}_v)$. Since $M_{\bar{\mathbb{Q}}}$ is fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$, this element is independent of the choice of ι and is fixed by $\mathrm{Gal}(\bar{E}_v/E_v)$, hence corresponds by Lemma 1.1.3 to an element $M_v\in\mathcal{M}(E_v)$.

The orbits of $G(L)$ in $V(L)\times V(L)$ are in one-to-one correspondence with the double cosets of $\mathrm{Stab}_{G(L)}(x_0)$ in $G(L)$, and these are in one-to-one correspondence with $\mathcal{M}(L)$ [Cartan decomposition of $G(L)$]. In other words, there is a canonical surjection

(1.3.3) $$\mathrm{inv}: V(L)\times V(L)\to\mathcal{M}(L)$$

whose fibers are the orbits of $G(L)$ in $V(L)\times V(L)$. Now we can say how the element b is used to produce X_p. We take X_p to be the set

(1.3.4) $$X_p=\{x\in V(L)|\mathrm{inv}(b\sigma x,x)=M_v\}.$$

Here we are regarding M_v as an element of $\mathcal{M}(L)$. Note that M_v is fixed by $\sigma^{[E_v:\mathbb{Q}_p]}$; it follows that X_p is stable under $(b\sigma)^{[E_v:\mathbb{Q}_p]}$. There is a connected reductive group J over $\mathbb{Q}_p$ such that

(1.3.5) $$J(\mathbb{Q}_p)=\{g\in G(L)|(b\sigma)g=g(b\sigma)\}.$$

It is clear that X_p is stable under $J(\mathbb{Q}_p)$. The group I also comes with an embedding over $\mathbb{Q}_p$

(1.3.6) $$I\to J.$$

The quotient (1.3.1) is formed by using (1.3.6), (1.3.2), the action of $J(\mathbb{Q}_p)$ on X_p, and the action of $G(\mathbb{A}_f^p)$ on X^p. The Frobenius element of $\mathrm{Gal}(\bar{k}/k)$ preserves the subset (1.3.1) of $S_K(\bar{k})$; it acts through the automorphism $(b\sigma)^{[E_v:\mathbb{Q}_p]}$ of X_p [this automorphism commutes with the action of $J(\mathbb{Q}_p)$ and hence induces an automorphism of the quotient (1.3.1)].

Let Z denote the center of G; Z is also a subgroup of the center of I and the center of J. We write Z_K for $Z(\mathbb{A}_f)\cap K$ and $Z(\mathbb{Q})_K$ for $Z(\mathbb{Q})\cap K$. For sufficiently small K, the following two conditions [L3, pp. 1171–1172] hold:

(1.3.7) If $h\in I(\mathbb{Q})$ fixes a point of $X_p\times X^p$, then $h\in Z(\mathbb{Q})_K$.

(1.3.8) Let $h, g\in I(\mathbb{Q})$, $z\in Z(\mathbb{Q})_K$, and suppose that $ghg^{-1}=hz$. Then $z=1$.

1.4

We will assume that K is small enough so that (1.3.7) and (1.3.8) are satisfied. Let k' be a finite field between k and $\bar{k}$. Let $n=[k':\mathbb{F}_p]$ and $\Phi=(b\sigma)^n$. Then the part of $S_K(k')$ coming from the subset (1.3.1) of $S_K(\bar{k})$ is given by the set of fixed points $[I(\mathbb{Q})\backslash(X_p\times X^p)]^\Phi$ of Φ in $I(\mathbb{Q})\backslash(X_p\times X^p)$. We want to find an expression for the cardinality of $[I(\mathbb{Q})\backslash(X_p\times X^p)]^\Phi$. For this purpose it is convenient to introduce the set A, which consists of all $(h,x,y)\in I(\mathbb{Q})\times X_p\times X^p$ such that $(\Phi x, y)=(hx, hy)$. There is an obvious projection map

$$A\to X_p\times X^p. \tag{1.4.1}$$

The groups $I(\mathbb{Q})$ and $Z(\mathbb{Q})_K$ act on A:

(a) $$g\cdot(h,x,y)=(ghg^{-1}, gx, gy) \quad \text{for} \quad g\in I(\mathbb{Q}),$$

(b) $$z\cdot(h,x,y)=(zh,x,y) \quad \text{for} \quad z\in Z(\mathbb{Q})_K.$$

The two actions commute with each other and with the map (1.4.1) (for the trivial $Z(\mathbb{Q})_K$ action on $X_p\times X^p$). The assumption (1.3.7) on K implies that (1.4.1) induces a bijection from

$$(Z(\mathbb{Q})_K\times I(\mathbb{Q}))\backslash A \tag{1.4.2}$$

to $[I(\mathbb{Q})\backslash X_p\times X^p]^\Phi$.

Now our problem is to calculate the cardinality of the set (1.4.2). The projection map $A\to I(\mathbb{Q})$ commutes with the actions of $I(\mathbb{Q})$ and $Z(\mathbb{Q})_K$ [$I(\mathbb{Q})$ acts on $I(\mathbb{Q})$ by conjugation, and $Z(\mathbb{Q})_K$ acts on $I(\mathbb{Q})$ by translation]. The quotient of $I(\mathbb{Q})$ by the action of $I(\mathbb{Q})\times Z(\mathbb{Q})_K$ is the set of conjugacy classes in $I(\mathbb{Q})/Z(\mathbb{Q})_K$. Using assumption (1.3.8), we see that the stabilizer of $h\in I(\mathbb{Q})$ in $I(\mathbb{Q})\times Z(\mathbb{Q})_K$ is $I_h(\mathbb{Q})$, where I_h denotes the centralizer of h in I. Therefore the set (1.4.2) is equal to

$$\coprod_h I_h(\mathbb{Q})\backslash A_h, \tag{1.4.3}$$

where h runs through a set of representatives for the conjugacy classes in $I(\mathbb{Q})/Z(\mathbb{Q})_K$ and A_h denotes the fiber of $A\to I(\mathbb{Q})$ over h. We have

$$A_h=Y_p\times Y^p, \tag{1.4.4}$$

where $Y_p=\{x\in X_p|h^{-1}\Phi x=x\}$ and $Y^p=\{y\in X^p|hy=y\}$.

We now fix $h\in I(\mathbb{Q})$ and calculate the cardinality of $I_h(\mathbb{Q})\backslash A_h$. If Y_p or Y^p is empty, then the cardinality of $I_h(\mathbb{Q})\backslash A_h$ is 0. So we assume that Y_p and Y^p are both non-empty or, in other words, we assume that

(1.4.5) $h^{-1}\Phi$ fixes some element of X_p,

(1.4.6) h is conjugate in $G(\mathbb{A}_f^p)$ to an element of K^p.

These conditions should imply that the homomorphism (1.3.2) induces an isomorphism

$$I_h \xrightarrow{\sim} G_h \text{ over } \mathbb{A}_f^p \tag{1.4.7}$$

and that the homomorphism (1.3.6) induces an isomorphism

$$I_h \xrightarrow{\sim} J_h \text{ over } \mathbb{Q}_p, \tag{1.4.8}$$

but we will not stop to check this, for reasons that will become clear in 1.6.

(1.4.9) **Lemma.** *Let Ψ be an element of the semidirect product of $\langle\sigma\rangle$ and $G(L)$, and assume that Ψ projects to a non-trivial element of $\langle\sigma\rangle$. Then Ψ is conjugate under $G(L)$ to an element of $\langle\sigma\rangle$ if and only if Ψ fixes some element of $V(L)$.*

If $\Psi = c\sigma^m c^{-1}$ for some $c \in G(L)$, then Ψ fixes $cx_0 \in V(L)$. Conversely, let us assume that Ψ fixes an element of $V(L)$. Conjugating Ψ by an element of $G(L)$, we may assume that Ψ fixes x_0. Possibly replacing Ψ by its inverse, we may further assume that $\Psi = g\tau$ for some positive power τ of σ and some $g \in G(L)$. Since τ fixes x_0, so does g, which means that g belongs to $G(\mathfrak{o}_L)$ for the smooth $\mathbb{Z}_p$-structure on G determined by x_0 (see [T]). Since x_0 is hyperspecial, the reduction modulo p of G is connected, and by a result of Greenberg [G] there exists $c \in G(\mathfrak{o}_L)$ such that $g = c\tau(c^{-1})$. It follows immediately that $\Psi = c\tau c^{-1}$.

Now we continue with the calculation of the cardinality of $I_h(\mathbb{Q})\backslash A_h$. By Lemma 1.4.9 there exists $c \in G(L)$ such that

$$ch^{-1}\Phi c^{-1} = \sigma^n. \tag{1.4.10}$$

Let F denote the fixed field of the automorphism σ^n of L; then $[F:\mathbb{Q}_p] = n$ and the residue field of $\mathfrak{o}_F$ is k'. We define an element $\delta \in G(L)$ by the equation

$$c(b\sigma)c^{-1} = \delta\sigma. \tag{1.4.11}$$

Since h and $b\sigma$ commute, chc^{-1} and $\delta\sigma$ also commute; thus (1.4.10) implies that σ^n and $\delta\sigma$ commute, which means that $\delta \in G(F)$. We claim also that $\mathrm{Int}(c)$ induces an isomorphism

$$J_h(\mathbb{Q}_p) \xrightarrow{\sim} G_\delta^\sigma(\mathbb{Q}_p), \tag{1.4.12}$$

where G_δ^σ is the σ-centralizer of $\delta \in G(F)$ [recall that G_δ^σ is a $\mathbb{Q}_p$-group and that $G_\delta^\sigma(\mathbb{Q}_p) = \{g \in G(F) | g(\delta\sigma) = (\delta\sigma)g\}$]. Indeed, $G_\delta^\sigma(\mathbb{Q}_p)$ is equal to the set of $g \in G(L)$ such that g commutes with σ^n and $\delta\sigma$, which by (1.4.10) is the same as the set of $g \in G(L)$ such that g commutes with chc^{-1} and $\delta\sigma = c(b\sigma)c^{-1}$, while $J_h(\mathbb{Q}_p)$ is equal to the set of $g \in G(L)$ such that g commutes with h and $b\sigma$.

We further claim that the automorphism $x \mapsto cx$ of $V(L)$ induces a bijection from Y_p to

$$\{x \in V(F) | \mathrm{inv}(\delta\sigma x, x) = M_v\}. \tag{1.4.13}$$

Indeed, this follows from (1.4.10), (1.4.11) and the fact that $V(F)$ is the set of fixed points of σ^n in $V(L)$. Let us write Y_p' for the set (1.4.13). Then there is a bijection from $I_h(\mathbb{Q})\backslash A_h$ to

$$I_h(\mathbb{Q})\backslash(Y_p' \times Y^p). \tag{1.4.14}$$

1.5

Let ch_p (resp. ch^p) denote the characteristic function of the subset Y_p' (resp. Y^p) of $V(F)$ (resp. X^p). Let y_0 denote the image of the identity element of $G(\mathbb{A}_f^p)$ in X^p, and let $K_p(F)$ denote the stabilizer of x_0 in $G(F)$. The cardinality of $I_h(\mathbb{Q})\backslash A_h$ is equal to

(1.5.1) $$\sum_{(g_1, g_2)} ch_p(g_1 x_0) ch^p(g_2 y_0),$$

where the sum runs over

(1.5.2) $$(g_1, g_2) \in I_h(\mathbb{Q})\backslash G(F) \times G(\mathbb{A}_f^p)/K_p(F) \times K^p.$$

Now $g_1 x_0$ belongs to Y_p' if and only if $g_1^{-1}\delta\sigma(g_1)$ belongs to the double coset D of $K_p(F)$ in $G(F)$ corresponding to M_v. Similarly, $g_2 y_0$ belongs to Y^p if and only if $g_2^{-1} h g_2$ belongs to K^p. Let ϕ_p (resp. f^p) denote the characteristic function of D in $G(F)$ [resp. of K^p in $G(\mathbb{A}_f^p)$]. The cardinality of $I_h(\mathbb{Q})\backslash A_h$ is equal to

(1.5.3) $$\int \phi_p(g_1^{-1}\delta\sigma(g_1)) f^p(g_2^{-1} h g_2) \frac{dg_1 dg_2}{dx},$$

where the integral is taken over $I_h(\mathbb{Q})Z_K\backslash G(F) \times G(\mathbb{A}_p^f)$, dg_1 (resp. dg_2) is the Haar measure on $G(F)$ [resp. $G(\mathbb{A}_f^p)$] that gives $K_p(F)$ (resp. K^p) measure 1, and dx is the Haar measure on $I_h(\mathbb{Q})Z_K$ that gives Z_K measure 1. To verify (1.5.3) one uses that $I_h(\mathbb{Q})Z_K$ is closed in $G(F) \times G(\mathbb{A}_p^f)$ and that the intersection of $I_h(\mathbb{Q})Z_K$ with any conjugate of $K_p(F) \times K^p$ in $G(F) \times G(\mathbb{A}_p^f)$ is equal to Z_K (this follows from 1.3.7).

We can rewrite (1.5.3) as

(1.5.4) $$a \cdot TO_\delta(\phi_p) \cdot O_h(f^p),$$

where

(1.5.5) $$a = \operatorname{meas}(I_h(\mathbb{Q})Z_K\backslash I_h(\mathbb{A}_f)),$$

(1.5.6) $$TO_\delta(\phi_p) = \int_{G_\delta^\sigma(\mathbb{Q}_p)\backslash G(F)} \phi_p(g^{-1}\delta\sigma(g)),$$

(1.5.7) $$O_h(f^p) = \int_{G_h(\mathbb{A}_f^p)\backslash G(\mathbb{A}_f^p)} f^p(g^{-1} h g).$$

To get (1.5.5) we used (1.4.7), (1.4.8) and (1.4.12) to replace $G_\delta^\sigma(\mathbb{Q}_p) \times G_h(\mathbb{A}_f^p)$ by $I_h(\mathbb{A}_f)$.

1.6

Assuming Langlands' conjecture on the structure of $S_K(\bar{k})$, we have shown that $\operatorname{Card}(S_K(k'))$ can be expressed as a sum of terms of the form (1.5.4). To understand $\operatorname{Card}(S_K(k'))$ completely we would need to determine which pairs (δ, h) appear (and find their multiplicities). No doubt this additional information can be derived from Langlands' conjecture, but there seems to be a simpler way.

Consider the case in which G is the group of symplectic similitudes and S_K is the moduli space for principally polarized abelian varieties with level structure of type K. Langlands' partition of $S_K(\bar{k})$ into a disjoint union of subsets (1.3.1) corresponds to grouping abelian varieties over $\bar{k}$ according to $\bar{k}$-isogeny classes. But it is also possible to group abelian varieties over k' according to k'-isogeny classes. This gives a partition of $S_K(k')$ into disjoint subsets, and these subsets are none other than the subsets $I_h(\mathbb{Q})\backslash A_h$ appearing in (1.4.3). This means that the number (1.5.4) has a simple interpretation: it is the cardinality of some k'-isogeny class of principally polarized abelian varieties with level structure. I will discuss all of this in more detail in another paper.

1.7

Langlands [L2, L3] has outlined a method for showing that the local factor at p of the zeta function of S_K is a product of powers of Euler factors coming from automorphic representations of endoscopic groups H of G. The method involves expressing $\mathrm{Card}(S_K(k'))$ in terms of the stable trace formulas for the groups H, applied to certain functions f_H on $H(\mathbb{A})$. It is important not to misunderstand the notation f_H: this family of functions is not obtained from a function f on $G(\mathbb{A})$ by means of the correspondences $f \to f^H$ of [L5].

Now we come to the main point of this paper, which is to explain how the spherical function identities for base change [K1, L4, L5] can be used to compare (1.5.4) with terms in the stable trace formulas for the groups H. What is of interest to us is the factor $TO_\delta(\phi_p)$. The function ϕ_p is a $K_p(F)$-spherical function on $G(F)$. According to the conjectural spherical function identities, the twisted orbital integral $TO_\delta(\phi_p)$ can be written as a linear combination of orbital integrals of spherical functions ϕ_p^H on the unramified endoscopic groups of G. Similarly, it should be possible [L5] to write $O_h(f^p)$ as a linear combination of stable orbital integrals of functions $(f^p)^H$ on the endoscopic groups H of G. Thus it should be possible to write (1.5.4) as a linear combination of stable orbital integrals of functions on the endoscopic groups H of G [see [L2, L3] to find out how the factor (1.5.5) of (1.5.4) disguises stable orbital integrals of functions $f_{\infty,H}$ on $H(\mathbb{R})$].

It is now clear how these considerations tie in with Langlands' outlined method: granting the assumptions we have made along the way, we see that $\mathrm{Card}(S_K(k'))$ has the right form to be compared with the stable trace formulas for the functions $(f^p)^H \cdot (\phi_p^H) \cdot f_{\infty,H}$ on the adelic endoscopic groups $H(\mathbb{A})$ of G. By making further assumptions in local and global harmonic analysis, as in [K3, L5], we could say much more about the details of the comparison. I will return to this in another paper.

2. Satake Transform of ϕ_p

In this section we determine the Satake transform of the spherical function ϕ_p; this will certainly be needed for the comparison mentioned in 1.7. In fact, we could predict the Satake transform by seeing what the comparison requires, but there would be little point to this, since we can actually prove what we want.

2.1

It is natural to formulate our result for any unramified connected reductive group G over any non-archimedean local field F. Let $\bar{F}$ be an algebraic closure of F and let Γ (resp. W_F) denote the Galois (resp. Weil) group of $\bar{F}/F$.

Let F^{un} denote the maximal unramified extension of F contained in $\bar{F}$, and let W_F^{un} denote the Weil group of F^{un}/F [the infinite cyclic group generated by the Frobenius element $\sigma_F \in \mathrm{Gal}(F^{un}/F)$]. Let q denote the cardinality of the residue field of F.

Let x_0 be a hyperspecial point in the building of G over F. Let K denote the stabilizer of x_0 in $G(F)$, and let $\mathscr{H}(G(F), K)$, or simply $\mathscr{H}$, denote the corresponding Hecke algebra. For $\mu: \mathbb{G}_m \to G$ we write f_μ for the characteristic function of the double coset of K in $G(F)$ that correspond to the $G(F)$-conjugacy class of μ (by Lemma 1.1.3 this is just another way of looking at the Cartan decomposition). Our goal is to determine the Satake transform of f_μ for μ satisfying the following condition:

(2.1.1) The representation $\mathrm{Ad} \circ \mu$ of $\mathbb{G}_m$ on $\mathrm{Lie}(G_{\bar{F}})$ has no weights other than $1, 0, -1$.

(2.1.2) **Lemma.** *For any $\mu: \mathbb{G}_m \to G$ there exists a representation r_μ of ${}^LG$, unique up to isomorphism, satisfying the following two properties:*

(a) *As a $\hat{G}$-representation, r_μ is irreducible with extreme weight μ.*

(b) *Let y be a splitting of $\hat{G}$ and assume that y is fixed by Γ. Then the subgroup W_F of ${}^LG$ acts trivially on the highest weight space of r_μ corresponding to y.*

Here we are using $\hat{G}$ instead of the more usual notation ${}^LG^0$. Given any maximal torus $\hat{T}$ of $\hat{G}$, the element μ determines an orbit of the Weyl group of $\hat{T}$ acting on $X^*(\hat{T})$, and thus it makes sense to say that r_μ is an irreducible $\hat{G}$-representation with extreme weight μ.

The uniqueness part of the assertion is clear. It is also easy to see that if we choose a splitting y_0 of $\hat{G}$ that is fixed by Γ, then there exists a representation r_μ of the quotient $W_F^{un} \ltimes \hat{G}$ of ${}^LG$ which satisfies (a) and also satisfies (b) for the particular splitting y_0. We must show that (b) holds for any splitting y that is fixed by Γ. By Corollary 1.7 of [K3] there exists $g \in \hat{G}^\Gamma$ such that $gy_0 = y$. Then $r_\mu(g)$ induces a W_F-equivariant isomorphism from the highest weight space corresponding to y_0 to the highest weight space corresponding to y. Therefore (b) holds for y.

Now we are ready to state what the Satake transform of f_μ is, for μ satisfying (2.1.1). We take the point of view that the Satake transform of a function $f \in \mathscr{H}$ is the function $\pi \mapsto \mathrm{tr}\,\pi(f)$ on the set of irreducible K-spherical representations π of $G(F)$ (taken up to isomorphism). For an unramified admissible homomorphism $\varphi: W_F \to {}^LG$ we denote by π_φ the corresponding irreducible K-spherical representation of $G(F)$. Finally, we choose a maximal torus T of G and an order of its root system so that μ is a dominant coweight of T, and we write δ for half the sum of the positive roots of T.

(2.1.3) **Theorem.** *Suppose that $\mu: \mathbb{G}_m \to G$ satisfies* (2.1.1). *Let f_μ be the corresponding function in the Hecke algebra $\mathscr{H}(G(F), K)$, and let r_μ be as in the previous lemma. Then for any unramified admissible homomorphism $\varphi: W_F \to {}^LG$ we have*

$$\mathrm{tr}\,\pi_\varphi(f_\mu) = q^{\langle \delta, \mu \rangle} \mathrm{tr}(r_\mu(\varphi(\sigma_F))).$$

There is a slight abuse of notation here, since the Frobenius element σ_F belongs to W_F^{un}, not W_F. But $r_\mu \circ \varphi$ factors through W_F^{un}, because φ is unramified and r_μ factors through $W_F^{un} \ltimes \hat{G}$.

2.2

Before proving the theorem, let us see how it fits with what we did in § 1. The function ϕ_p is equal to f_μ, where $\mu: \mathbb{G}_m \to G_F$ represents the class M_v (see 1.3). It follows from (1.1.2) that μ satisfies (2.1.1). The number $\langle \delta, \mu \rangle$ is equal to $\frac{1}{2}\dim S_K$, and the factor $q^{\langle \delta, \mu \rangle}$ in the Satake transform of ϕ_p is a reflection of the translation by $\frac{1}{2}\dim S_K$ that occurs in the argument of the L-functions in [L2].

Since G is quasi-split over $\mathbb{Q}_p$, one of its endoscopic groups (over $\mathbb{Q}_p$) is G itself. The corresponding conjectural spherical function identity relates the stable twisted orbital integrals of ϕ_p to the stable orbital integrals of ϕ_p^G, where ϕ_p^G is the image of ϕ_p under the base change homomorphism $b: \mathcal{H}(G(F), K_p(F)) \to \mathcal{H}(G(\mathbb{Q}_p), K_p)$. This homomorphism can be characterized by the following property:

$$\operatorname{tr} \pi_\varphi(b(f)) = \operatorname{tr} \pi_\psi(f),$$

where $f \in \mathcal{H}(G(F), K_p(F))$, φ is any unramified admissible homomorphism $W_{\mathbb{Q}_p} \to {}^L G$, and ψ is the restriction of φ to W_F. From Theorem 2.1.3 we see that

$$\operatorname{tr} \pi_\varphi(\phi_p^G) = p^{nd/2} \operatorname{tr} r_\mu(\varphi(\sigma^n)) \tag{2.2.1}$$

for any unramified admissible homomorphism $\varphi: W_{\mathbb{Q}_p} \to {}^L G$, where $n = [F:\mathbb{Q}_p]$ and $d = \dim S_K$.

To make clear the connection with the point of view in [L2], we note first that the representation r_μ is independent of the extension F of E_v, in the sense that all representations are obtained by restriction from the representation r_μ in the case $F = E_v$. Next we consider the representation r_v of the L-group of G over $\mathbb{Q}_p$ induced from the representation r_μ of the L-group of G over E_v. Langlands' representation r of the L-group of G over $\mathbb{Q}$, when restricted to the L-group of G over $\mathbb{Q}_p$, becomes a direct sum of the representations r_v, where v runs over the places of E dividing p. In terms of r_v, Eq. (2.2.1) becomes

$$\operatorname{tr} \pi_\varphi(\phi_p^G) = [E_v:\mathbb{Q}_p]^{-1} p^{nd/2} \operatorname{tr} r_v(\varphi(\sigma))^n. \tag{2.2.2}$$

Thus we see that our ϕ_p^G is equal to $[E_v:\mathbb{Q}_p]^{-1}$ times Langlands' $f_{\mathfrak{p}}^{(n)}$ (our v is his $\mathfrak{p}$). The factor $[E_v:\mathbb{Q}_p]^{-1}$ seems puzzling at first, since Langlands uses $f_{\mathfrak{p}}^{(n)}$ in the trace formula for G, while we are suggesting that ϕ_p^G be used. This seeming discrepancy occurs because Langlands uses n to denote two different quantities, and the conversion factor between the two is $[E_v:\mathbb{Q}_p]$.

2.3

Now we come to the proof of Theorem 2.1.3. In 2.1 we regarded the Satake transform of $f \in \mathcal{H}$ as a function on the set of isomorphism classes of irreducible K-spherical representations of $G(F)$. In proving the theorem, it is useful to keep in mind a second point of view: we choose a maximal F-split torus S of G through which μ factors and regard the Satake transform of $f \in \mathcal{H}$ as an element of $\mathbb{C}[X_*(S)]^{\Omega(F)}$, where $\Omega(F)$ denotes the (relative) Weyl group of S in G. To see what Theorem 2.1.3 looks like from this viewpoint, we need to choose a Γ-invariant splitting y of $\hat{G}$ and determine the character of r_μ on elements of ${}^L G$ of the form

$\sigma \ltimes t$, where t belongs to the torus $\hat{T}$ given by the splitting y. We are simplifying notation by writing σ instead of σ_F.

The centralizer T of S in G is a maximal torus of G. Choose a Borel subgroup B of G, containing T, for which μ is a dominant coweight. The choice of B (and the Borel subgroup $\hat{B}$ given by y) determines a Γ-isomorphism $X_*(T) \xrightarrow{\sim} X^*(\hat{T})$. This isomorphism allows us to identify the Weyl group of T with the Weyl group of $\hat{T}$; we use Ω to denote both of them. The relative Weyl group $\Omega(F)$ may be identified with Ω^Γ. The isomorphism $X_*(T) \xrightarrow{\sim} X^*(\hat{T})$ also allows us to regard μ as a Γ-invariant dominant weight of $\hat{T}$.

The restriction of r_μ to $\hat{G}$ has highest weight μ. Any weight of r_μ belongs to the set $\Sigma(\mu)$ of elements $\nu \in X^*(\hat{T})$ satisfying the following two properties:

(2.3.1) $\mu - \nu$ belongs to the $\mathbb{Z}$-span of the set of roots of $\hat{T}$,

(2.3.2) $\mu \geqq w\nu$ for all $w \in \Omega$ (this means that $\mu - w\nu$ is a linear combination of simple roots with non-negative coefficients).

The first property just says that μ, ν have the same restriction to the center of $\hat{G}$, which is obviously the case for a weight ν of r_μ. The second property is well-known (see [Se], for example).

At this point we need a result about root systems. Let R be a root system, and let C be a chamber of R. We write $P(R)$ for the group of weights of R and $Q(R)$ for the subgroup of $P(R)$ generated by R. We write W for the Weyl group of R. Let $p \in P(R)$ be a dominant weight, and let $\Sigma(p)$ be the set of elements $p' \in P(R)$ such that

(i) $p - p' \in Q(R)$,
(ii) $p \geqq wp'$ for all $w \in W$.

(2.3.3) **Lemma.** *The following conditions on p are equivalent:*
(a) $\langle p, \alpha^\vee \rangle = -1, 0, 1$ *for all* $\alpha \in R$.
(b) $\Sigma(p) = W \cdot p$.

By considering the root system obtained from R by throwing away all $\alpha \in R$ such that $\frac{1}{2}\alpha \in R$, we reduce to the case of a reduced root system R. There is also a trivial reduction to the case in which R is irreducible and p is non-zero. Then condition (a) of the lemma is equivalent to the statement that p is a *poids minuscule*, and by combining several of the exercises (§ 1, Ex. 23, 24 and § 2, Ex. 5d) at the end of Ch. VI of [B], we obtain the lemma.

Now we return to the proof of the theorem. Let $\nu \in \Sigma(\mu)$. The lemma implies that there exists $w \in \Omega$ such that ν, $w\mu$ have the same restriction to $\hat{T} \cap (\hat{G})_{\mathrm{der}}$. It follows from (2.3.1) that ν, $w\mu$ agree on the center of $\hat{G}$. Therefore $\nu = w\mu$, and we have shown that $\Sigma(\mu) = \Omega \cdot \mu$.

Since extreme weights of r_μ occur in r_μ with multiplicity one, the restriction of r_μ to $\hat{T}$ is a direct sum $\bigoplus_{\nu \in \Omega \cdot \mu} \mathbb{C}_\nu$, where $\mathbb{C}_\nu$ denotes $\mathbb{C}$ with $\hat{T}$ acting through the character ν. The Frobenius element σ preserves this decomposition and carries $\mathbb{C}_\nu$ into $\mathbb{C}_{\sigma\nu}$. The character of r_μ on $\sigma \ltimes t$ is therefore equal to

$$\mathrm{tr}(\sigma \ltimes t; \oplus_\nu \mathbb{C}_\nu),$$

where the direct sum is taken over the set of $\nu \in \Omega \cdot \mu$ such that $\sigma\nu = \nu$. It follows from Lemma 1.1.3 that this set is equal to $\Omega(F) \cdot \mu$. It follows from the second property of r_μ given in Lemma 2.1.2 that σ acts trivially on $\mathbb{C}_\nu$ for $\nu \in \Omega(F) \cdot \mu$, since for any such ν we can find a Γ-invariant splitting of $\hat{G}$ so that $\mathbb{C}_\nu$ is the highest weight space of r_μ (for that splitting). Our conclusion is that

$$\mathrm{tr}\, r_\mu(\sigma \ltimes t) = \sum_{\nu \in \Omega(F) \cdot \mu} \nu(t). \tag{2.3.4}$$

Therefore the conclusion of Theorem 2.1.3 is equivalent to the statement that the Satake transform of f_μ is the element $q^{\langle \delta, \mu \rangle} \sum_{\nu \in \Omega(F) \cdot \mu} \nu$ of the algebra $\mathbb{C}[X_*(S)]^{\Omega(F)}$.

Our goal is now to prove this reformation of Theorem 2.1.3. One obvious approach is to use Macdonald's formula, which is, after all, an explicit formula for the Satake transform of f_μ for any $\mu \in X_*(S)$. This approach works, but there is a better one, suggested to me by Casselman.

Let $p: G_{sc} \to G$ be the simply connected covering of the derived group of G, and let S_{sc} be the unique maximal F-split torus of G_{sc} such that $p(S_{sc}) \subset S$. Let $\Sigma_F(\mu)$ be the set of $\nu \in X_*(S)$ such that

$$\mu - \nu \in \mathrm{im}[X_*(S_{sc}) \to X_*(S)], \tag{2.3.5}$$

$$\mu \geqq w\nu \quad \text{for all} \quad w \in \Omega(F). \tag{2.3.6}$$

Let S_{ad} denote the image of S in the adjoint group G_{ad} of G; the torus S_{ad} is a maximal F-split torus in G_{ad}. We are going to apply Lemma 2.3.3 to the root system R consisting of the relative coroots in $X_*(S_{\mathrm{ad}})$, with the aim of showing that $\Sigma_F(\mu) = \Omega(F) \cdot \mu$. It is clear that $P(R) = X_*(S_{\mathrm{ad}})$, since the relative roots in $X^*(S_{\mathrm{ad}})$ form a basis of $X^*(S_{\mathrm{ad}})$. Furthermore, $X_*(S_{sc})$ is generated by elements $\sum \beta^\vee$ where the sum is taken over a Γ-orbit of simple absolute coroots $\beta^\vee \in X_*(T_{sc})$ (T_{sc} denotes the centralizer of S_{sc} in G_{sc}), and it follows from a simple calculation that the images of these generators in $X_*(S_{\mathrm{ad}})$ are the simple coroots in $X_*(S_{\mathrm{ad}})$. Therefore $Q(R) = \mathrm{im}[X_*(S_{sc}) \to X_*(S_{\mathrm{ad}})]$.

Let $\nu \in \Sigma_F(\mu)$. We want to show that $\nu \in \Omega(F) \cdot \mu$. Lemma 2.3.3 implies that there exists $w \in \Omega(F)$ such that $\nu, w\mu$ have the same image in $X_*(S_{\mathrm{ad}})$. But $\nu - w\mu$ belongs to $\mathrm{im}[X_*(S_{sc}) \to X_*(S)]$, and the canonical map $X_*(S_{sc}) \to X_*(S_{\mathrm{ad}})$ is injective. Therefore $\nu = w\mu$.

To finish the proof of Theorem 2.1.3, all we need is the following well-known fact about Satake transforms, valid for any dominant $\mu \in X_*(S)$.

(2.3.7) **Lemma.** *For $\nu \in X_*(S)$ let $c(\nu)$ denote the coefficient of ν in the Satake transform of f_μ.*

(a) *If $c(\nu) \neq 0$, then $\nu \in \Sigma_F(\mu)$.*

(b) $c(\mu) = q^{\langle \delta, \mu \rangle}$.

If G is semisimple and simply connected, then a complete proof can be obtained simply by combining [Sa] and [B–T, 4.4.4]. In the general case we apply [B–T, 4.4.4] to the adjoint group of G and use the homomorphism $\lambda: G(F) \to X^*(Z(\hat{G}))^\Gamma$ constructed in § 3.

3. Construction of the Homomorphism λ

In this section F is a non-archimedean local field, and $\bar{F}, F^{un}, \Gamma$ have the same meaning as in §2. For any unramified F-torus T there is a surjective homomorphism

$$T(F) \to X_*(T)^\Gamma, \tag{3.1}$$

obtained by tensoring the normalized valuation $(F^{un})^\times \to \mathbb{Z}$ with $X_*(T)$ and taking invariants under Γ. But $X_*(T)^\Gamma = X^*(\hat{T})^\Gamma$, and thus we can regard (3.1) as homomorphism

$$\lambda: T(F) \to X^*(\hat{T})^\Gamma, \tag{3.2}$$

functorial in T. To prevent confusion we sometimes write λ_T instead of λ. Now consider the two functors $G \mapsto G(F)$ and $G \mapsto X^*(Z(\hat{G}))^\Gamma$ from the category of unramified connected reductive F-groups and normal homomorphisms to the category of groups (we are using notation and terminology from [K3]).

(3.3) **Lemma.** *There exists a unique extension of* (3.2) *to a homomorphism of functors*

$$\lambda: G(F) \to X^*(Z(\hat{G}))^\Gamma. \tag{3.3.1}$$

For all G the homomorphism λ_G is surjective. For any unramified maximal F-torus T of G, the following diagram commutes:

$$\begin{array}{ccc} T(F) & \longrightarrow & G(F) \\ {\scriptstyle\lambda_T}\downarrow & & \downarrow{\scriptstyle\lambda_G} \\ X^*(\hat{T})^\Gamma & \longrightarrow & X^*(Z(\hat{G}))^\Gamma \end{array} \tag{3.3.2}$$

The restriction of λ_G to any special maximal compact subgroup K of $G(F)$ is trivial.

We extend the homomorphism of functors in two stages. At the first stage we extend it to unramified groups G such that G_{der} is simply connected. Consider such a group G and let $D = G/G_{\mathrm{der}}$. Since $\hat{D} = Z(\hat{G})$, the requirement of functoriality, applied to $G \to D$, forces the definition of λ_G upon us. It is clear that the homomorphisms we get are functorial in G, and that they are surjective. At the second stage we extend (3.3.1) to all unramified groups. Given an unramified group G, we choose a z-extension $H \to G$ with H unramified [K2]. Let Z denote the kernel of $H \to G$; note that Z is an unramified F-torus. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(F) & \longrightarrow & H(F) & \longrightarrow & G(F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & X^*(\hat{Z})^\Gamma & \longrightarrow & X^*(Z(\hat{H}))^\Gamma & \longrightarrow & X^*(Z(\hat{G}))^\Gamma & \longrightarrow & 1 \end{array}$$

in which the two vertical arrows are provided by the first stage of the extension. The two rows are exact; the only non-obvious point is the surjectivity of $X^*(Z(\hat{H}))^\Gamma \to X^*(Z(\hat{G}))^\Gamma$, which follows from the vanishing of $H^1(F, X^*(\hat{Z}))$. We define λ_G to be the unique homomorphism $G(F) \to X^*(Z(\hat{G}))^\Gamma$ making the diagram above commute. The surjectivity of λ_G is obvious. It is easy to check, using the obvious unramified variant of Lemma 2.4.4 of [K3], that the homomorphism λ_G is independent of the choice of $H \to G$ and is functorial in G.

Now consider the commutativity of (3.3.2). Since $T \to G$ is not a normal homomorphism (unless G is a torus), the functoriality of λ does not apply to this situation. However, if G_{der} is simply connected, we let $D = G/G_{\mathrm{der}}$, and obtain the desired commutativity from the functoriality of λ for $T \to D$. Then, in the general case, we choose a z-extension $H \to G$ and reduce to the case just treated.

Finally, we prove that λ is trivial on K. The group K is the stabilizer in $G(F)$ of some special point x_0 in the building $\mathscr{B}$ of G. Choose a maximal F-split torus S of G whose apartment contains x_0, and let T denote the centralizer of S in G. Let $p: G_{sc} \to G$ be the simply connected cover of the derived group of G.

Our first step is to show that

$$G(F) = p(G_{sc}(F)) \cdot T(F). \tag{3.3.3}$$

Using a z-extension, we reduce to the case in which G_{der} is simply connected. In that case, (3.3.3) follows from the surjectivity of $T(F) \to D(F)$, where $D = G/G_{\mathrm{der}}$, which in turn follows from the vanishing of $H^1(F, T_{sc})$ (see the proof of Case 2 of Theorem 4.1 in [K2]).

Let K_{sc} denote the stabilizer of x_0 in $G_{sc}(F)$. Our next step is to show that

$$K = p(K_{sc}) \cdot T(F)_0, \tag{3.3.4}$$

where $T(F)_0$ denotes $T(F) \cap K$. It is obvious that the right side of (3.3.4) is contained in the left side. Let $k \in K$. By (3.3.3) it is possible to write k as $k = p(g)^{-1}t$ for some $g \in G_{sc}(F)$ and some $t \in T(F)$. Therefore $p(g)x_0 = tx_0$, and this point belongs to the apartment A of S, since $T(F)$ preserves A. Let N denote the normalizer of T_{sc} in G_{sc}. Now $N(F)$ acts transitively on the set of chambers of A, and hence there exists $w \in N(F)$ such that $p(w)x_0, tx_0$ belong to the same closed chamber $\bar{C}$ of A. Since x_0 is a special point, we have $N(F) \cdot x_0 = T_{sc}(F) \cdot x_0$ (see [T, 1.9]), and hence there exists $t_1 \in T_{sc}(F)$ such that $p(t_1)x_0, tx_0$ belong to $\bar{C}$. Since $\bar{C}$ is a fundamental domain for the action of $G_{sc}(F)$ on $\mathscr{B}$ [B–T, 2.1.6], and since $tx_0 = p(g)x_0$, we must have that $p(t_1)x_0 = tx_0 = p(g)x_0$. It is immediate that $g^{-1}t_1 \in K_{sc}$, that $p(t_1)^{-1}t \in T(F)_0$, and that $K = p(g^{-1}t_1) \cdot (p(t_1)^{-1}t)$.

Now we conclude the proof that λ is trivial on K. By (3.3.4) it is enough to show that λ is trivial on $p(K_{sc})$ and on $T(F)_0$. The homomorphism $\lambda_{G_{sc}}$ is trivial, and from the functoriality of λ it follows that λ_G is trivial on $p(K_{sc})$. By the commutativity of (3.3.2) the restriction of λ_G to $T(F)$ coincides with λ_T. It is easy to see that the kernel of λ_T is $T(F)_0$, and this concludes the proof.

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An Algebraic Model for G-Simple Homotopy Types

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1. Introduction

Let G be a finite group and let X be a finite G-CW-complex (G-complex). Illman [I] and Rothenberg [R] introduced a notion of generalized Whitehead torsion for non-free actions of G and the corresponding notion of G-simple homotopy type for G-complexes. The object of this paper is to construct an algebraic model for G-complexes and use it to classify G-simple homotopy types up to finite ambiguity. This classification is a generalization and refinement to the simple homotopy problem of results due to Quillen [Q], Sullivan [S] and Triantafillou [T_1], [T_3].

Given a G-complex X, the model we construct (in Sect. 5), called a simple model $\mathcal{S}$ for X, can be roughly described as follows: $\mathcal{S}$ is a free differential graded Lie algebra over $\mathbb{Q}$ generated by the G-set of cells of X such that the space of indecomposables $Q\mathcal{S}=\mathcal{S}/[\mathcal{S},\mathcal{S}]$ is isomorphic to the rational chain complex of X and the homology of $\mathcal{S}$ is isomorphic to the homotopy $\pi_*(X)\otimes\mathbb{Q}$. In fact $\mathcal{S}$ contains the rational chain complex and the rational homotopy of X^H for all subgroups $H\subseteq G$ in a sense to be made precise. Moreover, these is a particular sort of map from $\mathcal{S}$ to a functorial construction $L(X)$. We specify the notions of torsion bound and integral structure of $\mathcal{S}$ and then we prove the following finiteness type classification result in Sect. 5:

Theorem 1.1. *Consider finite G-complexes X such that X^H is nonempty, connected and simply connected for every subgroup $H\subseteq G$. Then there are only finitely many G-simple homotopy types of such G-complexes of given torsion bound N and given simple model $\mathcal{S}$ with integral structure (Z,Z',Z'').*

The functorial construction $L(X)$ above is an equivariant adaptation of Quillen's functor from spaces to differential graded Lie algebras. In Sect. 2, we give an equivariant version of (part of) Quillen's rational homotopy theory [Q] and discuss the algebraic notion of G-homotopy.

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In Sect. 3 we construct an "equivariant minimal Lie model" Λ for G-complexes which generalizes the minimal model of Baues and Lemaire [BL] and Neisendorfer [N] for $G=\{e\}$. This construction is in a certain sense dual to the equivariant minimal model $\mathcal{M}$ of Triantafillou [T_3] and classifies rational G-homotopy types.

In Sect. 4, we develop an algebraic equivariant obstruction theory and study the notion of Postnikov tower in this context. Then we prove a finiteness type classification result for integral G-homotopy types of given torsion bound N and given equivariant minimal Lie model Λ with integral structure (Z, Z'). The same result with Λ replaced by $\mathcal{M}$ is proved in [T_3] for more general G-complexes. Here we basically show that we can refer to that proof. We need this classification result in terms of Λ rather than $\mathcal{M}$ in order to prove Theorem 1.1 above.

As in earlier finiteness type classification results using minimal model theory ([S], [T_3]), elements of the theory of algebraic and arithmetic groups play a crucial role in the proof of Theorem 1.1.

In the last section we prove finiteness theorems for G-homotopy types and G-simple homotopy types of $\mathbb{Z}_{p}n$ actions on spheres and of semilinear finite group actions utilizing machinery that we have developed. These are sample results which are accessible through rather elementary calculations. Certainly more refined calculations with this machinery would yield further results. The results on G-homotopy types generalize and refine earlier results of tom Dieck and Petrie [DP] about semilinear actions although they are not as sharp as theirs.

This paper is part of a lengthier project the purpose of which is to classify G-diffeomorphism types of G-manifolds, up to finite ambiguity, in terms of algebraic invariants thus generalizing Sullivan's result ([S]) for the group $G=\{e\}$. This project breaks up into two stages. The first, essentially completed in this paper, classifies the G-simple homotopy types up to finite ambiguity. The second stage is to classify G-diffeomorphism types within a given simple homotopy type. This is feasible with the development of the equivariant surgery exact sequence in [DR] under certain dimensionality and condimensionality restrictions. We shall take this up in a second paper.

As a last remark we mention that we hope to be able to replace the assumption $X^G \neq \emptyset$ for the construction of the simple model by a less restrictive assumption – a nilpotency condition. That some such restriction is necessary is evident by considering the "classical" case where the actions are free.

2. Equivalence of Certain Categories

In this chapter we construct algebraic models for the rational homotopy category of G-spaces in the sense of Quillen [Q].

G is always a finite group. We consider the following categories:

$\mathcal{J}$: The category of simply connected pointed spaces and base point preserving continuous maps.

$\underline{\mathcal{J}}$: The category of G-spaces X with a base point in $X^G \neq \emptyset$ and with fixed point sets X^H connected and simply connected for all subgroups $H \subseteq G$. Morphisms are the base point preserving equivariant maps.

Definition 2.1. A map $f: X \to Y$ in $\underline{\mathcal{J}}$ is said to be a weak equivalence if

$$(f^H)_*: \pi_*(X^H) \otimes \mathbb{Q} \to \pi_*(Y^H) \otimes \mathbb{Q}$$

is an isomorphism for all subgroups $H \subseteqq G$.

Let Ho $\underline{\mathcal{J}}$ denote the category which results from localizing $\underline{\mathcal{J}}$ with respect to weak equivalences. We will show that the category Ho $\underline{\mathcal{J}}$ is equivalent to the category Ho DGL_G (of algebraic objects) which is defined as follows:

Let $\mathcal{O}_G$ be the category of canonical orbits of G; its objects are the quotient spaces G/H for H a subgroup of G, and its morphisms are the G-maps between them. Here G acts on G/H from the left by multiplication.

DLG_G is the category of contravariant functors from $\mathcal{O}_G$ into the category of differential graded Lie algebras L with $L_0 = 0$. The morphisms are natural transformations between such functors. An object of DGL_G is called a *system of* DG *Lie algebras*. A map f in DGL_G is said to be a weak equivalence if $f(G/H)$ is an ordinary homology isomorphism for every G/H. Ho DGL_G is the localized category with respect to weak equivalences.

As in [Q], there is a sequence of intermediate categories with which $\underline{\mathcal{J}}$ and DGL_G are compared. In the following definitions the term functor means contravariant functor.

$\mathcal{J}_G$: The category of functors from $\mathcal{O}_G$ into $\underline{\mathcal{J}}$.

SS: The category of simplicial sets K with K_0 and K_1 consisting of a single point.

$\underline{\mathrm{SS}}$: The category of G-simplicial sets K with $\emptyset \neq K^H \in \mathrm{SS}$.

SS_G: The category of functors from $\mathcal{O}_G$ into SS.

$\mathcal{S}_G$: The category of functors from $\mathcal{O}_G$ into the category of simplical groups Γ with $\Gamma_0 = \{e\}$.

sCHA_G: The category of functors from $\mathcal{O}_G$ into the simplicial complete Hopf algebras A with $A_0 = \mathbb{Q}$.

sLA_G: The category of functors from $\mathcal{O}_G$ into the simplicial Lie algebras ℓ with $\ell_0 = 0$.

DGC_G: The category of functors from $\mathcal{O}_G$ into the differential graded coalgebras C over $\mathbb{Q}$ such that $\bar{C}_0 = 0 = \bar{C}_1$; here $\bar{C}$ is the kernel of the augmentation.

Vec_G: The category of functors from $\mathcal{O}_G$ into the rational vector spaces. Note that this is an abelian category.

The objects of the sub G categories are called *systems* of spaces, simplicial sets etc.

If we consider $G = \{e\}$ then the above categories become Quillen's original categories which we denote with the same symbols without the sub G.

Quillen proved that SS, $\mathcal{S}$, sCHA, sLA, DGL, and DGC are closed model categories. By a result of Bousfield and Kan [BK], the functor categories SS_G, $\mathcal{S}_G$, sCHA_G, sLA_G, DGL_G, and DGC_G are closed model categories as well. A map f in one of the sub G categories is a weak equivalence or a fibration if $f(G/H)$ is a weak equivalence or a fibration as in [Q] respectively for every $G/H \in \mathcal{O}_G$. A map f is a cofibration if it has the left lifting property with respect to maps which are both fibrations and weak equivalences.

There is a sequence of adjoint functors

$$\mathscr{I}_G \underset{E_2\mathrm{Sing}}{\overset{||}{\rightleftarrows}} \mathrm{SS}_G \underset{\bar{W}}{\overset{\Gamma}{\rightleftarrows}} \mathscr{S}_G \underset{\gamma}{\overset{\hat{\mathbb{Q}}}{\rightleftarrows}} \mathrm{sCHA}_G \underset{\mathscr{P}}{\overset{\hat{U}}{\leftrightarrows}} \mathrm{sLA}_G \underset{N}{\overset{N^*}{\leftrightarrows}} \mathrm{DGL}_G \underset{\mathscr{C}}{\overset{\mathscr{L}}{\leftrightarrows}} \mathrm{DGC}_G$$

induced from the non-equivariant case, where the upper arrows always denote the left adjoint functors.

Localize each of the above categories with respect to the weak equivalences. All the right adjoint functors respect localizations because they do so for every $G/H \in \mathcal{O}_G$. Let

$$\mathrm{Ho}\,\mathscr{I}_G \xrightarrow[G(E_2\mathrm{Sing})^\sim]{}$$

$$\sim \mathrm{Ho}\,\mathrm{SS}_G \xleftarrow[\tilde{W}]{} \mathrm{Ho}\,\mathscr{S}_G \xleftarrow[\tilde{\gamma}]{} \mathrm{Ho}\,\mathrm{CHA}_G \xrightarrow[\tilde{\mathscr{P}}]{} \mathrm{Ho}\,\mathrm{sLA}_G \xrightarrow[\tilde{N}]{} \mathrm{Ho}\,\mathrm{DGL}_G \xrightarrow[\tilde{C}]{} \mathrm{Ho}\,\mathrm{DGC}_G$$

be the induced functors on the localized categories. It is easy to prove the following result:

Theorem 2.2. *The functors $(E_2\,\mathrm{Sing})^\sim$, $\tilde{\bar{W}}$, $\tilde{\gamma}$, $\tilde{\mathscr{P}}$, $\tilde{N}$, and $\tilde{\mathscr{C}}$ are equivalences of categories.*

Proof. Each of the adjoint functor pairs $(||, E_2\,\mathrm{Sing})$, $(\Gamma, \bar{W})$, $(\mathscr{L}, \mathscr{C})$ has the property that each functor carries the localizing family of its source into the localizing family of its target, and the property that the adjunction morphisms are in the localizing families because this is the case for each $G/H \in \mathcal{O}_G$. Consequently the induced $\tilde{}$ functors on the localized categories are equivalences of categories.

For the other pairs of adjoint functors we do not know that the left adjoint functors preserve the localizing families. So, as in the non-equivariant case we will need Proposition 2.3, p. 214 of [Q] to get a quasi-inverse to $\tilde{\gamma}$, $\tilde{\mathscr{P}}$, and $\tilde{N}$. It is trivial to check the assumptions of this proposition for each G/H by referring to [Q].

The relation between the above categories on the one hand and the category of G-spaces $\underline{\mathscr{I}}$ on the other follows from a result due to Elmendorf [E].

Theorem 2.3 (Elmendorf). *The inclusion $\phi: \underline{\mathscr{I}} \to \mathscr{I}_G$ induces an equivalence of categories $\tilde{\phi}: \mathrm{Ho}\,\mathscr{I} \to \mathrm{Ho}\,\mathscr{I}_G$.*

If X is a G-space, $\phi(X)$ is defined by

$$\phi(X)(G/H) = X^H.$$

Elmendorff constructed a functor $C: \mathscr{I}_G \to \underline{\mathscr{I}}$ and a natural transformation $\eta: \phi C \to id$ which is a homotopy equivalence for each G/H. Then McClure [E] proved that C is right adjoint to ϕ. It is easy to see that ϕ and C preserve the localizing families respectively and that the adjunction morphisms are in the localizing families.

Remark. We know that the localized categories in the last two theorems are equivalent as *homotopy categories* [Q]. A proof of this by the second named author will appear elsewhere.

Combining the above results we summarize:

Theorem 2.4. *There exist equivalences of categories*

$$\mathrm{Ho}\,\mathscr{I} \xrightarrow{\lambda} \mathrm{Ho}\,\mathrm{DGL}_G \xrightarrow{\tilde{\mathscr{C}}} \mathrm{Ho}\,\mathrm{DGC}_C .$$

Moreover, there are canonical isomorphisms of functors

$$\underline{\pi}_n(X)\otimes\mathbb{Q} \cong H_{n-1}(\lambda(X)),$$
$$\underline{H}_r(X)\otimes\mathbb{Q} \cong H_n(\tilde{\mathscr{C}}\lambda(X))$$

from $\mathrm{Ho}\,\mathscr{I}$ to GL_G and GC_G respectively for every n.

We explain the notation: Here $\lambda \equiv \tilde{N} \circ \tilde{\mathscr{P}} \circ (\tilde{\mathbb{Q}} \circ \Gamma)^{\sim} \circ (E_2\,\mathrm{Sing})^{\sim} \circ \tilde{\phi}$. The symbols GL_G and GC_G denote the categories of functors from $\mathscr{O}_G$ into the graded Lie algebras and the graded coalgebras respectively. The functors $H_*(L)$, $L \in \mathrm{DGL}_G$, and $H_*(C)$, $C \in \mathrm{DGC}_G$, are defined by

$$H_*(L)(G/H) = H_*(L(G/H))$$

and

$$H_*(C)(G/H) = H_*(C(G/H))$$

respectively on objects of $\mathscr{O}_G$. We also define ([Br])

$$\underline{\pi}_*(X)(G/H) \equiv \pi_*(X^H)$$

and

$$\underline{H}_*(X)(G/H) \equiv H_*(X^H;\mathbb{Z}) \tag{2.5}$$

on objects of $\mathscr{O}_G$.

Next, we describe certain objects of DGL_G which are cofibrant and then we give an explicit notion of homotopy between morphisms from cofibrant objects. In order to do this we need to recall some results on projective systems of rational vector spaces (in Vec_G) from $[T_1]$. As we mentioned, Vec_G is an abelian category and the projective objects are described as follows: Let S be a G-set and let $\mathbb{Q}(S)$ be the rational vector space with basis S. Define $\underline{\mathbb{Q}}(S)$ in Vec_G by

$$\underline{\mathbb{Q}}(S)(G/H) \equiv \mathbb{Q}(S^H). \tag{2.6}$$

$\underline{\mathbb{Q}}(S)$ is called a *free system of vector spaces.* In [Br] it is shown that $\underline{\mathbb{Q}}(S)$ is projective for any G-set S.

Another way to construct a projective object in Vec_G is the following. Fix a subgroup $H \subseteq G$ and let V_H be a $\mathbb{Q}(\mathrm{NH}/H)$-representation, where NH is the normalizer of H in G. Define $\underline{V}_H \in \mathrm{Vec}_G$ by

$$\underline{V}_H(G/K) \equiv \mathbb{Q}((G/H)^{K})\otimes_{\mathbb{Q}(NH/H)} V_H, \tag{2.7}$$

$G/K \in \mathscr{O}_G$. It turns out ($[T_1]$) that $\underline{V}_H$ is projective for any $H \subseteq G$ and any $\mathbb{Q}(NH/H)$-representation and that any projective in Vec_G [in particular $\underline{\mathbb{Q}}(S)$] is a direct sum of $\underline{V}_H$'s. Moreover, any map f from $\underline{V}_H$ is determined by its restriction $f|V_H \equiv \mathbb{Q}((G/H)^H)\otimes_{\mathbb{Q}(\mathrm{NH}/H)} V_H$. Let $p_{K,K'}: \underline{V}_H(G/K') \to \underline{V}_H(G/K)$ be the map induced by the projection $G/K \to G/K'$, where $K \subseteq K'$. We observe that $p_{K,K'}$ is an injective map.

Let V be a system of graded vector spaces which is projective in the graded sense, i.e. $V_n \in \mathrm{Vec}_G$ is projective for all n. We denote by $F[V]$ the system of free graded Lie algebras

$$F[V](G/H) \equiv F[V(G/H)]$$

generated by $V(G/H)$, $G/H \in \mathcal{O}_G$.

Definition 2.8. A system of DG Lie algebras (L, d) is said to be *free* if $L \cong F[V]$ by neglect of the differential, where $V \in \mathrm{Vec}_G$ is projective in the graded sense.

Proposition 2.9. *Any free system of* DG *Lie algebras L is cofibrant.*

Proof. We have to construct a lift $\tilde{g}$ of g in the diagram

$$\begin{array}{ccc} & & N \\ & \overset{\tilde{g}}{\nearrow} & \downarrow p \\ L & \xrightarrow{g} & K \end{array},$$

where p is a fibration (epimorphism) and a weak equivalence in DGL_G and g is an arbitrary map. This is done inductively on degree and on the subgroups of G. Suppose $\tilde{g}$ is constructed on $F\left[\bigoplus_{i \leqq n-1} V_i\right]$ and write $V_n = \bigoplus \underline{V}_H$ where the sum runs over a collection of subgroups containing at most one subgroup from each conjugacy class. We can construct a lift, say g'_H, of $g|V_H$ as in the non-equivariant case for every V_H. We average each g'_H to make it NH/H equivariant and define $\tilde{g}|V_n$ by

$$\tilde{g}|V_H = \frac{1}{|NH/H|} \sum_{h \in NH/H} h g'_H h^{-1}.$$

This process extends the given lift to $F\left[\bigoplus_{i \leqq n} V_i\right]$. This completes the proof.

As in the non-equivariant case [BL], we construct an explicit cylinder object $L \times I$ for any free system of DG Lie algebras $L = (F[V], d)$.

$$L \times I = (F[V' \oplus V'' \oplus sV], d),$$

where V' and V'' are two copies of V and s means suspension. The differential of $L \times I$ restricted on V' and V'' is as in L and on sV is defined as follows: Let $S: L \to L \times I$ be a map of degree $+1$ defined by the two conditions

(1) $$S(G/H)(v) = sv, \qquad v \in V(G/H)$$

and

(2) $$S(G/H)[x, y] = [S(G/H)x, y] + (-1)^{|x|}[x, S(G/H)y], \qquad x, y \in L(G/H).$$

Then d is defined on sV by

$$d(sv) = v'' - v' - S(G/H)dv, \qquad v \in V(G/H).$$

We have two obvious canonical inclusions i', $i'': L \to L \times I$ which are chain homotopic through S, i.e.

$$i'' - i' = Sd + dS .$$

Moreover, i' and i'' are chain homotopy equivalences (with the obvious definition of chain homotopy equivalence for systems of chain complexes).

Definition 2.10. Two maps $f, g: L \to M$ from a free system of DG Lie algebras to an arbitrary system in DGL_G are said to be homotopic if there exists a map $H: L \times I \to M$ such that $H \circ i' = f$ and $H \circ i'' = g$.

If f and g are homotopic then they are chain homotopic through $H \circ S$. Moreover, the maps induced on the indecomposables Qf, $Qg: QL \to QM$ are chain homotopic through $QH \circ QS$. Here, we define QL by

$$QL(G/H) = L(G/H)/[L(G/H), L(G/H)] .$$

$L \times I$ is a cylinder object in the sense of Quillen, ([Q], p. 234) because, just as in Proposition 2.9, the inclusion $(i', i''): L \vee L \to L \times I$ can be proved to be a cofibration. Then the notion of homotopy above is the same as the notion of left homotopy in [Q] which is an equivalence relation between maps from a cofibration object. A universal lifting property, up to homotopy, of maps from cofibrant objects with respect to weak equivalences holds in our context as well:

Proposition 2.11. *Given a weak equivalence h and an arbitrary map f in the diagram*

$$\begin{array}{ccc} & & M' \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow h \\ L = (F[V], d) & \xrightarrow[f]{} & M \end{array} ,$$

there exists a map $\tilde{f}$ such that the diagram commutes up to homotopy. Moreover, the homotopy class of $\tilde{f}$ is unique.

The proof is entirely similar to the proof of Proposition 2.9 and will be omitted.

3. Equivariant Minimal Lie Models

In order to describe the minimal models in the category DGL_G we need to recall certain facts about projective systems of vector spaces Vec_G from $[T_1]$.

Every system of rational vector spaces A has a projective cover, i.e. there exists a surjective map from a projective object onto A. In fact, A has a unique minimal projective cover $P \overset{\tau}{\twoheadrightarrow} A$ such that for any projective cover $P' \overset{\tau'}{\twoheadrightarrow} A$ there exists a surjection q such that the diagram

$$\begin{array}{ccc} P & & \\ & \searrow^{\tau} & \\ q \big\uparrow & & A \\ & \nearrow_{\tau'} & \\ P' & & \end{array}$$

commutes. We describe P: Let $a_{H,H'}: A(G/H') \to A(G/H)$ be the map induced by the projection $G/H \to G/H'$(morphism in $\mathcal{O}_G$), where H is a proper subgroup of H'. The set $\sum_{H' \supsetneq H} \operatorname{im} a_{H,H'} \subseteqq A(G/H)$ is a $\mathbb{Q}(NH/H)$-module. Let V_H be a $\mathbb{Q}(NH/H)$-equivariant complement of this subspace of $A(G/H)$. Then $P \equiv \bigoplus_{(H)} \underline{V}_H$; here the sum is over a collection of subgroups, exactly one group in each conjugacy class. Moreover, $\tau(G/H)|V_H$ is the inclusion. Obviously

$$\ker \tau(G/H) \subseteqq \sum_{H' \supsetneq H} \operatorname{im} p_{H,H'}. \tag{3.1}$$

We recall that a DG Lie algebra L is said to be minimal if (a) it is free as a graded Lie algebra and (b) the induced differential on $QL \equiv L/[L, L]$ is zero.

Definition 3.2. Let Λ be a system of DG Lie algebras. Λ is said to be minimal if

(a) $\Lambda = F[V] =$ system of free graded Lie algebras generated by the projective system of vector spaces $V \in \mathrm{Vec}_G$ (by neglect of the differential) and

(b) the system of graded differential vector spaces $Q\Lambda$ has the following property:

$$d(Q\Lambda(G/H)) \subseteqq \sum_{H' \supsetneq H} \operatorname{im} \lambda_{H,H'}.$$

Here we use $\lambda_{H,H'}$ for the maps $\Lambda(G/H') \to \Lambda(G/H)$ and $Q\Lambda(G/H') \to Q\Lambda(G/H)$ indiscriminately. Note that $Q\Lambda \equiv V$ by neglect of the differential. The main result of this section is the following:

Theorem 3.3. *For every system of* DG *Lie algebras L (in* DGL*) there exists a minimal system of* DG *Lie algebras Λ and a weak equivalence $\alpha: \Lambda \to L$.*

Proof. By assumption $L_0 = 0$. Consider

$$\begin{array}{ccc} L_1 & \twoheadrightarrow & H_1(L) \\ & \nwarrow^{\alpha_1} & \uparrow \tau_1 \\ & & P_1 \end{array},$$

where P_1 is the minimal projective cover of $H_1(L)$. By definition of projectivity, τ_1 lifts to α. Then

$$\Lambda^{(1)} \equiv F[P_1] \xrightarrow{\alpha_1} L$$

is an epimorphism in one dimensional homology. $\Lambda^{(1)}$ is minimal. Moreover,

$$\ker \alpha_1(G/H) \subseteqq \ker \tau_1(G/H) \subseteqq \sum_{H' \supsetneq H} \operatorname{im} p_{H,H'}$$

for every G/H.

Inductively, suppose $\Lambda^{(n)} \xrightarrow{\alpha^{(n)}} L$ is an isomorphism (epimorphism) in homology in dimension $\leqq n-1 (=n)$, where $\Lambda^{(n)}$ is minimal.

Furthermore, suppose that

$$Q_j(\Lambda^{(n)})=0\,, \qquad j \geqq n+1\,,$$

and that for every G/H the kernel $N(G/H)$ of the composite map

$$Z_n(\Lambda^{(n)}(G/H)) \twoheadrightarrow H_n(\Lambda^{(n)}(G/H)) \twoheadrightarrow H_n(L(G/H))$$

consists of elements which are sums of decomposable elements and elements in the image of the maps induced by the projection $G/H \to G/H'$ [see (3.1)].

Let $K=\ker H_n(\Lambda^{(n)} \to L)$. Consider the diagram

$$\begin{array}{ccccc}
K & \longrightarrow & H_n(\Lambda^{(n)}) & \twoheadrightarrow & H_n(L) \\
\uparrow{\scriptstyle \tau'} & & \uparrow & & \uparrow{\scriptstyle p} \\
P'_{n+1} & \xrightarrow{d} & Z_n(\Lambda^{(n)}) & \xrightarrow{\alpha(n)} & Z_n(L) \\
 & {\scriptstyle d}\searrow & \cup & & \cup \\
 & & N & & \\
 & {\scriptstyle \alpha'}\searrow & L_{n+1} & \xrightarrow{d_{n+1}} & B_n(L)\,.
\end{array}$$

Here P'_{n+1} is a minimal projective cover of K, and d is a lift of τ'. Obviously, d takes values in N. We also observe that $\alpha^{(n)} \circ d$ takes values in the boundaries B_nL, since $p \circ \alpha^{(n)} \circ d=0$. P'_{n+1} being projective, there is a lift α' to L_{n+1}.

We consider the free product

$$\mathcal{M}^{(n+1)} \equiv \Lambda^{(n)} \vee F[P'_{n+1}]$$

having differential which restricts on $\Lambda^{(n)}$ to the differential of $\Lambda^{(n)}$ and is equal to the lift d on P'_{n+1}. $\mathcal{M}^{(n+1)}$ is minimal by the construction of $d|P'_{n+1}$. We define the map $\bar{\alpha}: \mathcal{M}^{(n+1)} \to L$ by $\bar{\alpha}|\Lambda^{(n)}=\alpha^{(n)}$ and $\bar{\alpha}|P'_{n+1}=\alpha'$. Obviously, this map induces an isomorphism on homology in dimension $\leqq n$ for every $G/H \in \mathcal{O}_G$.

Next we consider the following diagram

$$\begin{array}{ccccc}
H_{n+1}(\mathcal{M}^{(n+1)}) & \longrightarrow & H_{n+1}(L) & \overset{\varrho}{\twoheadrightarrow} & C \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \tau''} \\
Z_{n+1}(\mathcal{M}^{(n+1)}) & \longrightarrow & Z_{n+1}(L) & \underset{\alpha''}{\leftarrow} & P''_{n+1}
\end{array},$$

where $C=\operatorname{coker} H_{n+1}(\mathcal{M}^{(n+1)} \to L)$ and P''_{n+1} is a minimal projective cover of C. Since P''_{n+1} is projective, a lift a'' of τ'' exists. Let

$$\Lambda^{(n+1)} \equiv \mathcal{M}^{(n+1)} \vee F[P''_{n+1}]\,,$$

with $d|P''_{n+1}=0$. There is a map $\alpha^{(n+1)}: \Lambda^{(n+1)} \to L$ such that $\alpha^{(n+1)}|\mathcal{M}^{(n+1)}=\bar{\alpha}$ and $\alpha^{(n+1)}|P''_{n+1}=\alpha$.

$\Lambda^{(n+1)}$ is obviously minimal and $\alpha^{(n+1)}$ induces an isomorphism (epimorphism) in dimension $\leqq n$ $(=n+1)$. Moreover $Q_j\Lambda^{(n+1)}=0$, $j \geqq n+2$. It remains to show the inductive assumption about the kernel of

$$Z_{n+1}(\Lambda^{(n+1)}(G/H)) \xrightarrow{q} H_{n+1}(\Lambda^{(n+1)}(G/H)) \xrightarrow{\alpha_*} H_{n+1}(L(G/H))$$

for every G/H.

Let $K(G/H)$ be the kernel of the last composite map and let $z \in K(G/H)$. Since there is no ambiguity, we will omit the G/H from now on. We observe that

$$Z_{n+1}(\Lambda^{(n+1)}) = Z_{n+1}(\Lambda^{(n)} \vee P'_{n+1}) \oplus P''_{n+1},$$

and therefore $z = z' + z''$ with $z' \in Z_{n+1}(\Lambda^{(n)} \vee P'_{n+1})$ and $z'' \in P''_{n+1}$. We have $\alpha_*([z]) = 0$ and hence $[\alpha(z)] = 0$, i.e. $[\alpha(z')] = -[\alpha(z'')]$. Since $\varrho([\alpha(z')]) = 0$, $\varrho([\alpha(z'')]) = 0$, and therefore $\tau''(z'') = 0$. This means that z'' lies in the image of the maps induced by the projections $G/H \to G/H'$ for larger subgroups H'.

On the other hand, $z' = z_1 + z_2$ where $z_1 \in \Lambda^{(n)}$ and $z_2 \in P'_{n+1}$. Then $dz' = 0$ implies $dz_1 = -dz_2$ and therefore $z_2 \in \ker(P'_{n+1} \to H_n(\Lambda^{(n)}))$. This again means that z_2 lies in the image of the maps induced by the projections. The last term z_1 of z is decomposable since it lies in $\Lambda^{(n)}$ in degree $n+1$.

This completes the proof of Theorem 3.3.

Theorem 3.4. *Let $\alpha : \Lambda \to \Lambda'$ be a weak equivalence between two minimal systems of* DG *Lie algebras. Then α is an isomorphism.*

Proof. Since the DG Lie algebras $\Lambda(G/H)$ and $\Lambda'(G/H)$ are free as algebras $\alpha(G/H)$ is a homology isomorphism if and only if $Q\alpha(G/H) : Q\Lambda(G/H) \to Q\Lambda'(G/H)$ is a homology isomorphism [BL]. The systems of indecomposables $Q \equiv Q\Lambda$ and $Q' \equiv Q\Lambda'$ are systems of differential graded vector spaces and they are projective in Vec_G.

It suffices to prove that $\alpha \equiv Q\alpha : Q \to Q'$, being a homology isomorphism, is an isomorphism. We will prove this by induction on degree and on the subgroups of G. We observe that $\alpha(G/G) : Q(G/G) \to Q'(G/G)$ is an isomorphism because $\Lambda(G/G)$ and $\Lambda'(G/G)$ are minimal DG Lie algebras and we know that homology isomorphism between minimal DG Lie algebras are isomorphisms ([BL]).

Let $G/H \in \mathcal{O}_G$. Assume inductively that $\alpha : Q_i \to Q'_i$ is an isomorphism for $i \leqq n-1$ and for all $G/K \in \mathcal{O}_G$ (starting from $Q_0 = Q'_0 = 0$). Moreover assume that $\alpha : Q_n(G/H') \xrightarrow{\cong} Q_n(G/H')$ for all $H' \supsetneqq H$. We will prove that $\alpha : Q_n(G/H) \to Q_n(G/H)$ is an isomorphism.

Injectivity. Let $x \in Q_n(G/H)$ such that $\alpha(x) = 0$. Consider $dx \in Q_{n-1}(G/H)$. Because α is an isomorphism in dimension $n-1$, $dx = 0$. Since $\alpha(x) = 0$, and α is a homology isomorphism, x must be a boundary. By the minimality condition, x must lie in $\sum_{H' \supsetneqq H} \mathrm{im}\, q_{H,H'}$. Therefore, it suffices to prove that $\alpha|\sum \mathrm{im}\, q_{H,H'}$ is injective. For this we need the following lemma. We consider the set of subgroups of G which contain a fixed subgroup H as a proper subgroup. This is a directed set where $H' < H''$ if $H' \supseteqq H''$. Then the set of vector spaces $Q(G/H')$ and maps $q_{H'',H'}$ for $H'' \subseteqq H'$ forms a direct system of vector spaces. Let D be the direct limit. Then there is a natural surjection $p : D \to \sum_{H' \supsetneqq H} \mathrm{im}\, q_{H,H'}$.

Lemma 3.5. *If Q is projective then $p : D \to \sum_{H' \supsetneqq H} \mathrm{im}\, q_{H,H'}$ is an isomorphism.*

Proof. If Q is projective then an easy consequence of (2.7) is that p is injective whence bijective.

An immediate consequence of this is

Corollary 3.6. *If* $\alpha: Q \to Q'$ *is a map of projective systems of vector spaces and* $\alpha: Q(G/H') \to Q'(G/H')$ *is an isomorphism for all* $H' \supsetneq H$ *for a fixed* H *then* $\alpha: \sum_{H' \supsetneq H} \operatorname{im} q_{H,H'} \to \sum_{H' \supsetneq H} \operatorname{im} q'_{H,H'}$ *is an isomorphism.*

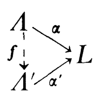
This proves the injectivity of α.

Surjectivity. Let $y \in Q'_n(G/H)$. Then $dy \in \sum_{H' \supsetneq H} \operatorname{im} q'_{H,H'}$. There is exactly one element $z \in Q_{n-1}(G/H)$ such that $\alpha(z) = dy$ because α is an isomorphism in this dimension. The element z must be a boundary since α is a homology isomorphism. So $z = dx$, $x \in Q_n(G/H)$. Consider $\alpha(x) - y$. It is a cycle in $Q'_n(G/H)$ therefore there exists a cycle $u \in Q_n(G/H)$ such that

$$\alpha(u) = \alpha(x) - y + dv, \quad v \in Q'_{n+1}(G/H).$$

But $dv \in \sum_{H' \supset H} \operatorname{im} q'_{H,H'}$, so it has a unique element ω in the preimage, $\alpha(\omega) = du$. Hence $y = \alpha(x - u + \omega)$. This completes the proof.

Theorem 3.7 (uniqueness). *Let* Λ *and* Λ' *be two minimal systems of* DG *Lie algebras and L an arbitrary one. Let* $\alpha: \Lambda \to L$ *and* $\alpha': \Lambda' \to L$ *be two weak equivalences. Then there is an isomorphism* $f: \Lambda \to \Lambda'$ *such that the diagram*



commutes up to homotopy.

Proof. By the lifting property 2.11, f exists. Moreover f is a homology isomorphism because α and α' are such. By Theorem 3.4, f must be an isomorphism.

Definition 3.8. Let L be a system of DG Lie algebras and let $\Lambda \xrightarrow{\alpha} L$ be a weak equivalence where Λ is a minimal system of DG Lie algebras. Then Λ is called a minimal model of L.

By the theorem above, Λ is unique up to isomorphism.

Now let X be a G-space in $\mathcal{I}$, and let $\lambda(X) \in \mathrm{DGL}_G$ be the system of DG Lie algebras constructed in Sect. 2. Let $\Lambda_X \in \mathrm{DGL}_G$ be the minimal model of $\lambda(X)$.

Definition 3.9. Λ_X is called an *equivariant minimal Lie model* of the G-space X.

The model $L(X)$ (3.10). Let X be a CW-complex which is of the form K/K', where K is a simplical complex and K' is a subcomplex containing the 1-skeleton of K. Such an X gives rise to a simplicial set S with $S_0 = S_1 = pt$ and such that there is exactly one non-degenerate n-simplex of S for every n-cell of $X([M])$. Let

$$L(X) \equiv \mathcal{L}\mathcal{C}N\mathcal{P}\mathbb{Q}\Gamma(S).$$

By construction $L(X)$ is free as a graded Lie algebra. There is the adjunction map $S \to E_2 \operatorname{Sing}|S|$ which induces a weak equivalence

$$L(X) \to \mathscr{L}\mathscr{C}\lambda(|S|) \to \lambda(|S|). \tag{3.11}$$

Moreover, there is a canonical epimorphism

$$QL(X) \twoheadrightarrow s^{-1}\bar{C}(X)$$

which is a weak equivalence; here $\bar{C}(X)$ is the kernel of the augmentation of the rational chain complex of X and s^{-1} means desuspension. This follows from a combination of facts from [Q]. First of all we have a composition of surjective maps induced by the adjunctions

$$\hat{U}N^*\mathscr{L}\mathscr{C}N\mathscr{P}\hat{\mathbb{Q}}\Gamma(S) \to \hat{U}N^*N\mathscr{P}\hat{\mathbb{Q}}\Gamma(S) \to \hat{U}\mathscr{P}\hat{\mathbb{Q}}\Gamma(S) \to \hat{\mathbb{Q}}\Gamma(S).$$

We pass to the indecomposables and then to the non-degenerate elements

$$N(Q(\hat{U}N^*\mathscr{L}\mathscr{C}N\mathscr{P}\hat{\mathbb{Q}}\Gamma(S))) \to N(Q(\hat{\mathbb{Q}}\Gamma(S))).$$

It follows from [Q], p. 265 and p. 218 that

$$N(Q(\hat{\mathbb{Q}}\Gamma(S))) = N(Q(\mathbb{Q}\Gamma(S))) = N(\Gamma_{ab}(S) \otimes \mathbb{Q}),$$

where $\mathbb{Q}\Gamma$ is the group ring of a group Γ and $\Gamma_{ab} = \Gamma/[\Gamma,\Gamma]$. But $N(\Gamma_{ab}(S) \otimes \mathbb{Q}) \cong s^{-1}\bar{C}(X)$ ([M]).

On the other hand, if L is a free DG Lie algebra we have

$$N(Q(\hat{U}N^*L)) = N(Q(\mathscr{P}\hat{U}N^*L) = N(Q(N^*L)) = QL,$$

where the equalities follow from [Q], pp. 273, 216, and 223 respectively. This shows that $N(Q(\hat{U}N^*\mathscr{L}\mathscr{C}N\mathscr{P}\hat{\mathbb{Q}}\Gamma(S))) = Q\mathscr{C}(X)$ and completes the argument about the above epimorphism.

It is an easy observation following from the definitions that $L(X) \subseteq L(X')$ if $X \subseteq X'$.

Now, consider X as above with an action of the finite group G. Applying the same construction to all the fixed point sets X^H, we get a homology isomorphism of systems of chain complexes

$$q: QL(X) \twoheadrightarrow s^{-1}\bar{C}(X). \tag{3.12}$$

Here, by abuse of notation, $L(X)$ and $C(X)$ are the systems defined by

$$L(X)(G/H) = \Lambda(X^H)$$

and

$$C(X)(G/H) = C(X^H)$$

respectively.

Because of (3.11), a minimal model Λ_X is the minimal model of $L(X)$ as well. The advantage of $L(X)$ over $\lambda(X)$ is, besides being free, that it is related by (3.12) to the chain complex of X instead of the much larger singular complex. This is important in the next chapter, where the finiteness of X is crucial.

4. Models for G-Homotopy Types

By the results so far, the invariant Λ_X classifies G-complexes up to rational G-homotopy type. In this section we shall use Λ_X to get a finiteness type classification result for G-complexes up to (integral) G-homotopy type. For this purpose we need to develop an algebraic obstruction theory in DGL_G analogous to the geometric obstruction theory.

Consider the full subcategory FDGL_G of DGL_G consisting of free systems of DG Lie algebras (see Definition 2.8). Let L be a system in FDGL_G. Then $L=F[U]$ by neglect of the differential where $U=\bigoplus_{i=0}^{\infty} U_i \in \mathrm{Vec}_G$ is projective in the graded sense.

Definition 4.1. A free system N of DG Lie algebras is called an *F-subsystem* of L if $N=F[W]$ and W is a direct summand of U.

We will call such a pair (L, N) an *F-pair*. We observe that $(L, L^{(n)})$ is an F-pair, where $L^{(n)}=F\left[\bigoplus_{i \leqq n} U_i\right]$. We can identify $U=QL$ as systems of graded vector spaces. The QN is a direct summand of QL and QL/QN is a projective system in Vec_G.

We define the relative Bredon cohomology $H_G^*(Q(L,N); A)$ of the indecomposables of an F-pair (L, N) with coefficients $A \in \mathrm{Vec}_G$ by

$$H_G^*(Q(L,N); A) \equiv H^*(\mathrm{Hom}(QL/QN, A)); \tag{4.2}$$

here Hom(,) means the morphisms in Vec_G, and $\mathrm{Hom}(QL/QN, A)$ is a differential graded vector space over $\mathbb{Q}$ its differential being induced by the differential of QL/QN.

If N' and N'' are subsystems of L we denote by $N' \vee N''$ the subsystem generated by N' and N''.

Obstructions to extending maps (4.3). Let (L, N) be an F-pair of DGL's, and let $f: N \vee L^{(n)} \to L'$ be a map, where L' is an arbitrary system of DG Lie algebras. In the diagram

$$\begin{array}{ccccccccc}
L_{n+2} & \xrightarrow{d_{n+2}} & L_{n+1} & \xrightarrow{d_{n+1}} & Z_n(N \vee L^{(n)}) & \xrightarrow{f} & Z_n(L') & \xrightarrow{p} & H_n(L') \\
\downarrow & & \downarrow & & & & & \nearrow{\scriptstyle c} & \\
Q_{n+2}(L) & \xrightarrow{\bar{d}_{n+2}} & Q_{n+1}(L) & \dashrightarrow & & & & &
\end{array}$$

the map $p \circ f \circ d_{n+1}$ factors uniquely via $Q_{n+1}(L)$. Here L_n is the system of vector spaces which contains the elements of L of degree n. Since $d_{n+1} \circ d_{n+2}=0$ and the vertical maps are surjective, we have $c \circ \bar{d}_{n+2}=0$. Further, since

$$p \circ f \circ d_{n+1} | N_{n+1} = p \circ d_{L'} \circ f | N_{n+1} = 0,$$
$$c \circ Q(f): Q_{n+1}(N) \to Q_{n+1}(L) \to H_n(L')$$

is the zero map. Therefore, c defines a unique obstruction cocycle

$$0(f) \in Z^{n+1}(\mathrm{Hom}(QL/QN, H_n(L'))).$$

Obstructions to homotopy (4.4). Let $f_1, f_2 : N \vee L^{(n)} \to L'$ be two maps such that

$$f_1|N \vee L^{(n-1)} = f_2|N \vee L^{(n-1)}.$$

We can factor the map $p \circ (f_1 - f_2)$

$$\begin{array}{ccccc} L_n & \xrightarrow{f_1 - f_2} & Z_n(L') & \xrightarrow{p} & H_n(L') \\ \downarrow & & & \nearrow_{c} & \\ Q_n(L) & & & & \end{array}$$

because $f_1 - f_2$ vanishes on the decomposable elements. Again c restricted to $Q_n(N)$ is zero and therefore it defines a unique cochain

$$\Delta(f_1, f_2) \in (\mathrm{Hom}(QL/QN, H_n(L')))^n$$

We have

$$0(f_1) - 0(f_2) = d(\Delta(f_1, f_2)).$$

Assuming finiteness and using the projectivity of the systems L and N we can prove easily

(i) $f : N \vee L^{(n)} \to L'$ extends to $N \vee L^{(n+1)}$ if and only if $0(f) = 0$.

(ii) Given f as above and $\alpha \in Z^n(\mathrm{Hom}(QL/QN, H_n(L')))$ there exists $f_1 : N \vee L^{(n)} \to L'$ such that $f_1|N \vee L^{(n-1)} = f|N \vee L^{(n-1)}$ and $\Delta(f, f_1) = \alpha$.

From (i) and (ii) we get

(iii) Given $f : N \vee L^{(n)} \to L'$, the class $\overline{0(f)} \in H_G^{n+1}(Q(L, N); H_n(L'))$ vanishes if and only if there exists $f_1 : N \vee L^{(n+1)} \to L'$ which extends $f|N \vee L^{(n+1)}$.

Note. Using these facts we can prove the usual Hurewicz and Whitehead theorems in the context of systems of DG Lie algebras. Also we can construct Postnikov towers for systems of DG Lie algebras.

Definition 4.5. A commutative diagram of systems of DG Lie algebras

$$\begin{array}{c} L \\ \swarrow \quad \swarrow \quad \searrow \quad \searrow \\ L_{(0)} \longleftarrow L_{(1)} \longleftarrow \cdots L_{(n-1)} \longleftarrow L_{(n)} \longleftarrow \cdots \end{array}$$

is called a *Postnikov tower* of L if

a) $H_i(L) \cong H_i(L_{(n)})$ for $i \leqq n$ and

b) $H_i(L_{(n)}) = 0$ for $i > n$.

Example: Let X be a G-space and

$$\begin{array}{c} X \\ \swarrow \quad \swarrow \quad \searrow \quad \searrow \\ X_0 \longleftarrow X_1 \longleftarrow \cdots X_{n-1} \longleftarrow X_n \longleftarrow \cdots \end{array}$$

its equivariant Postnikov tower ([T_1]). By applying Quillen's functor λ to the spaces and maps of this diagram we get a Postnikov tower of systems of DG Lie algebras for $\lambda(X) \in \mathrm{DGL}_G$.

By imitating the standard killing of homology groups construction and using the existence of projective covers we can construct Postnikov towers of a particular sort for free systems of DG Lie algebras, namely

Definition 4.6. A commutative diagram of free systems of DG Lie algebras (in FDGL_G)

$$\begin{array}{c} L \\ \swarrow \quad \swarrow \quad \searrow \quad \searrow^{j_n} \\ L_{(0)} \longleftarrow L_{(1)} \longleftarrow \dots L_{(n-1)} \xleftarrow[p_n]{} L_{(n)} \longleftarrow \dots \end{array}$$

is called an *F-Postnikov tower* of L if

a) The map j_n is an inclusion and in particular $(L_{(n)}, L)$ is an F-pair for every n,

b) $L^{(n+1)} = L_{(n)}^{(n+1)}$ for every n, i.e. L and $L_{(n)}$ differ only in degrees higher than $n+1$.

c) $H_j(L_{(n)}) = 0$ for $j > n$.

Proposition 4.7. *Let $f: L \to L'$ be a map between systems of* DG *Lie algebras, where L is a free system. Let $\{L_{(n)}\}_{n \geqq 0}$ and $\{L'_{(n)}\}_{n \geqq 0}$ be two Postnikov towers for L and L′ respectively, where $\{L_{(n)}\}_{n \geqq 0}$ is an F-Postnikov tower. Then there exist maps $f_n: L_{(n)} \to L'_{(n)}$ which extend f, are unique up to homotopy and are such that the diagram*

$$\begin{array}{ccc} L_{(n)} & \xrightarrow{f_n} & L'_{(n)} \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p'_n} \\ L_{(n-1)} & \xrightarrow[f_{n-1}]{} & L'_{(n-1)} \end{array}$$

commutes up to homotopy.

Proof. The obstructions to constructing f_n such that the diagram

$$\begin{array}{ccc} L & \longrightarrow & L' \\ {\scriptstyle j_n}\downarrow & & \downarrow{\scriptstyle j'_n} \\ L(n) & \xrightarrow[f_n]{} & L'_n \end{array}$$

commutes lie in $H_G^{k+1}(Q(L_{(n)}, L); H_k(L'_{(n)}))$ for $k \geqq n+1$. But these groups are zero since $H_k(L'_{(n)}) = 0$ for $k \geqq n+1$.

The obstruction to constructing a homotopy between $f_{n-1} \circ p_n$ and $p'_n \circ f_n$ lies in

$$H^k(Q(L_{(n)}, L); H_k(L'_{(n-1)}))$$

for $k \geqq n+2$ which is again zero.

If f_n and f'_n exist, the obstruction to deforming f_n to f'_n again lie in a zero group.

Now let X be a G-complex in $\mathscr{J}$ such that $\pi_n(X^H)$ is a finitely generated abelian group for all n and all $H \subseteq G$, and let $\{X_n\}_{n \geqq 0}$ be an equivariant Postnikov tower of X. Let $\Lambda_X \xrightarrow{\alpha} \lambda(X)$ be an equivariant minimal Lie model of X and let $\{\Lambda_{(n)}\}_{n \geqq 0}$ be an F-Postnikov tower of Λ_X. The map α induces isomorphisms

$$H_n(\Lambda_X) \cong H_n(\lambda(X)) \cong \underline{\pi}_{n+1}(X) \otimes \mathbb{Q} \tag{4.7}$$

and

$$H_n(QA_X) \cong \underline{H}_{n+1}(X; \mathbb{Q}) \tag{4.8}$$

for every n. The map α induces homology isomorphisms

$$\alpha_n : A_{(n)} \to \lambda(X_{n+1})$$

[the dimensions here are explained by the shift in dimensions in (4.7)] such that

$$(\alpha_n)_* : H_i(QA_{(n)}) \cong \underline{H}_{i+1}(X_{n+1}; \mathbb{Q})$$

for every i and n. This last isomorphism and (4.7) yield an isomorphism on Bredon cohomology

$$H_G^{n+2}(QA_{(n)}; H_{n+1}(A_X)) \cong H_G^{n+3}(X_{n+1}; \underline{\pi}_{n+2}(X)) \otimes \mathbb{Q} \tag{4.9}$$

for every n. This can be seen by comparing the E_2-terms of the Bredon spectral sequences [Br]

$$E_2^{p,q} = \mathrm{Ext}^p(\underline{H}_q(QA_{(n)}); H_{n+1}(A_X)) \cong \mathrm{Ext}^p(\underline{H}_{q+1}(X_{n+1}; \mathbb{Q}); \underline{\pi}_{n+2}(X) \otimes \mathbb{Q}) \equiv E_2'^{p,q+1}$$

which converge to the above cohomologies.

The isomorphisms (4.7) and (4.8) define lattices (i.e. finitely generated subgroups of maximal rank) in $H_n(A_X)(G/H)$ and $H_G^{n+2}(QA_{(n)}; H_{n+1}(A_X))$ for every n and every $H \subseteq G$ which reflect the integral homotopy and equivariant cohomology of X_n modulo torsion. In fact the first isomorphism defines a system of lattices in the following sense:

Definition 4.10. Let $U \in \mathrm{Vec}_G$ and let Z be a system of finitely generated free abelian groups (i.e. a functor from $\mathcal{O}_G$ into the finitely generated free abelian groups). Let $i : Z \to U$ be an inclusion. Z is called a system of lattices for U if $i \otimes \mathrm{id} : Z \otimes \mathbb{Q} \to U \otimes \mathbb{Q} = U$ is an isomorphism.

Definition 4.11. Let A be a minimal system of DG Lie algebras. An *integral structure* (Z, Z') of A consists of a system of lattices $Z = \bigoplus_{n \geq 0} Z_n$ in $\bigoplus_{n \geq 0} H_n(A)$ and a lattice

$$Z' = \bigoplus_{n \geq 0} Z'_n \quad \text{in} \quad \bigoplus_{n \geq 0} H_G^{n+2}(QA_{(n)}; H_{n+1}(A))$$

where $\{A_{(n)}\}$ is an F-Postnikov tower of A.

We want to discuss the question of how many G-homotopy types there are with given minimal model A with integral structure (Z, Z'). For this purpose we consider two kinds of G-complexes:

(A) finite G-complexes of dimension N such that X^H is nonempty, connected and simply connected for every $H \subseteq G$, or

(B) G-complexes such that X^H is as in (A) and $\pi_n(X^H) = 0$ for all $H \subseteq G$ and all n larger than a fixed integer N.

By Theorem 5.6, G-complexes of either type can be approximated by G-complexes of the type K/K' where K is a G-simplicial complex and K' is a G-subcomplex containing the 1-skeleton. This enables us to use the system of free DG Lie algebras $L(X)$ instead of $\lambda(X)$ (cf. 3.10).

Definition 4.12. An integer M is called a torsion bound for a G-space X of type (A) or (B) if

$$\sum_{H \subseteq G} \sum_{n=0}^{N} |T(\pi_n(X^H))| \leqq M,$$

where $T(A)$ is the torsion subgroup of a group A and $|A|$ is its order.

Now we can state the main result of this section.

Theorem 4.13. *Let Λ be a minimal system of* DG *Lie algebras with integral structure (Z, Z') and let M be an integer. Then there are only finitely many G-homotopy types of G-complexes X [of type* (A) *or* (B)] *of torsion bound M having equivariant minimal Lie model Λ such that the map $\alpha: \Lambda \to L(X)$ induces isomorphisms between the lattices Z and Z' on the one hand and the lattices of integral homotopy and Bredon cohomology of the Postnikov tower of X respectively on the other.*

Proof. This statement is a corollary of a similar statement in [T_3] for equivariant minimal models of Sullivan type. Let X be a G-complex of type (A) or (B) which satisfies the conditions of the theorem. As mentioned above, we can assume that $X = |S|$, where $S \in \underline{SS}$. We have a weak equivalence $\alpha: \Lambda \to L(X)$ and induced maps $\alpha_{n-1}: \Lambda_{(n-1)} \to L(X_n)$ which on homotopy and equivariant cohomology map Z and Z' isomorphically onto the natural lattices of $\underline{\pi}_*(X) \otimes \mathbb{Q}$ and

$$\bigoplus_{n \geqq 0} H_G^{n+2}(X_n; \underline{\pi}_{n+1}(X)) \otimes \mathbb{Q}$$

respectively.

The map $\alpha: \Lambda \to \mathscr{L}\mathscr{C}N\mathscr{P}\hat{\mathbb{Q}}\Gamma(S) \equiv L(X)$ induces by adjunction a sequence of maps in a canonical way:

$$\Lambda \to N\mathscr{P}\hat{\mathbb{Q}}\Gamma(S),$$
$$N^*\Lambda \to \mathscr{P}\hat{\mathbb{Q}}\Gamma(S),$$
$$\hat{U}N^*\Lambda \to \hat{\mathbb{Q}}\Gamma(S),$$
$$\gamma\hat{U}N^*\Lambda \to \gamma\hat{\mathbb{Q}}\Gamma(S),$$
$$\bar{W}\gamma\hat{U}N^*\Lambda \to \bar{W}\gamma\hat{\mathbb{Q}}\Gamma(S)$$

and

$$\bar{\alpha}: |C\bar{W}\gamma\hat{U}N^*\Lambda| \to |C\bar{W}\gamma\hat{\mathbb{Q}}\Gamma(S)|.$$

We call the last two G-complexes Y and X' respectively and we have G-maps

$$Y \xrightarrow{\bar{\alpha}} X' \xleftarrow{f} X,$$

where f is induced by the adjunction map $S \to C\bar{W}\gamma\hat{\mathbb{Q}}\Gamma(S)$. From the properties of the functors above ([Q]) follows that Y and X' are rational G-complexes and the maps $\bar{\alpha}$ and f are G-rational homotopy equivalences. Applying the same procedure to the maps $\alpha_{n-1}: \Lambda_{(n-1)} \to L(X_n)$ we get commutative diagrams

$$\begin{array}{ccccc} Y & \xrightarrow{\bar{\alpha}} & X' & \xleftarrow{f} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y_n & \xrightarrow{\bar{\alpha}_n} & X'_n & \longleftarrow & X_n \end{array}$$

and homotopy commutative diagrams

$$\begin{array}{ccccc} Y_n & \xrightarrow{\bar{\alpha}_n} & X'_n & \xleftarrow{f_n} & X_n \\ \downarrow & & \downarrow & & \downarrow \\ Y_{n-1} & \xrightarrow{\bar{\alpha}_{n-1}} & X'_{n-1} & \xleftarrow{f_{n-1}} & X_{n-1} \end{array}$$

for every n.

In $\underline{\pi}_*(Y)$ and $\bigoplus_{n\geqq 0} H_G^{n+2}(Y_n; \underline{\pi}_{n+1}(Y))$ we have the lattices induced by Z and Z' respectively, and in $\underline{\pi}_*(X')$ and

$$\bigoplus_{n\geqq 0} H_G^{n+2}(X'_n; \underline{\pi}_{n+1}(X'))$$

we have the lattices induced by f and the f_n's. By construction $\bar{\alpha}$ and the $\bar{\alpha}_n$'s preserve the lattices isomorphically.

Now consider the induced maps on the systems of de Rham complexes

$$\begin{array}{ccccc} \underline{\mathscr{E}}_Y & \xleftarrow{\tilde{\alpha}} & \underline{\mathscr{E}}_{X'} & \xrightarrow{\tilde{f}} & \underline{\mathscr{E}}_X \\ \uparrow & & \uparrow & & \uparrow \\ \underline{\mathscr{E}}_{Y_n} & \xleftarrow{\tilde{\alpha}_n} & \underline{\mathscr{E}}_{X_n} & \xrightarrow{\tilde{f}_n} & \underline{\mathscr{E}}_{X_n} \end{array},$$

where $\underline{\mathscr{E}}_X$ is the system of DG algebras defined in $[T_1]$ by

$$\mathscr{E}_X(G/H) \equiv \underline{\mathscr{E}}_{X^H};$$

here $\mathscr{E}_X$ is the Sullivan-de Rham complex of PL forms on X. Let $\mathscr{M} \to \underline{\mathscr{E}}_Y$ be an equivariant minimal model of Y as in $[T_1]$, and let $\bar{Z}$ and $\bar{Z}'$ be the lattices on $\pi_*(\mathscr{M})$ and $\bigoplus_{n\geqq 0} H_G^{n+2}(\mathscr{M}(n); Q^{n+1}\mathscr{M})$ induced by the ones in $\underline{\pi}_*(Y)$ and $\bigoplus H_G^{n+2}(Y_n; \underline{\pi}_{n+1}(Y))$ respectively. We have lifts up to homotopy

$$\begin{array}{ccccc} \underline{\mathscr{E}}_Y & \xleftarrow{\tilde{\alpha}} & \underline{\mathscr{E}}_{X'} & \xrightarrow{\tilde{f}} & \underline{\mathscr{E}}_X \\ \uparrow & \nearrow & & \nearrow & \\ \mathscr{M} & & & & \end{array},$$

where all maps involved preserve the lattices.

This means that all spaces X which have the same equivariant minimal Lie model Λ and lattices Z and Z' also have the same equivariant minimal model $\mathscr{M}$ and lattices $\bar{Z}$ and $\bar{Z}'$. By $[T_3]$, there are only finitely many G-homotopy types having this property.

5. Simple Models

In this section we construct simple models for G-complexes and use them to classify simple homotopy types.

We recall that a system of rational vector spaces (system of abelian groups) M is said to be *free* if $M=\underline{\mathbb{Q}}(S)$ $(=\underline{\mathbb{Z}}(S))$, where

$$\underline{\mathbb{Q}}(S)(G/H)=\mathbb{Q}(S^H) \quad (\underline{\mathbb{Z}}(S)(G/H)=\mathbb{Z}(S^H)).$$

Here S is a G-set and $\mathbb{Q}(S)$ is the rational vector space [$\mathbb{Z}(S)$ is the free abelian group] with basis S [cf. (2.6)]. Such a system is always projective.

For example, the system of chain complexes $C(X)$ is free in the graded sense for any G-complex X.

Let C and C' be two systems of rational (integral) chain complexes such that C_n and C'_n are free generated by finite G-sets. Let $f: C \to C'$ be a chain map which is a homology isomorphism for every $G/H \in \mathcal{O}_G$. Rothenberg [R_1] constructed an invariant

$$\tau(f) \in Wh(G;\mathbb{Q}) \quad (\tau(f) \in Wh(G;\mathbb{Z})),$$

the generalized Whitehead torsion; here

$$Wh(G;\mathbb{Z}) = \sum_{(H)} Wh_{\mathbb{Z}}(N(H)/H)$$

$$\left(Wh(G;\mathbb{Q}) = \sum_{(H)} Wh_{\mathbb{Q}}(N(H)/H)\right),$$

where $Wh_{\mathbb{Z}}(\ )$ is the classical Whitehead group [$Wh_{\mathbb{Q}}(\ )$ is the rational Whitehead group [W]].

Now let $h: X \to Y$ be a G-map between two finite G-complexes which is a G-homotopy equivalence. The generalized torsion $\tau(h)$ of h is by definition the torsion of the induced chain map $C(h): C(X) \to C(Y)$ in the above sense. In analogy to the classical case we have the notions of G-simple homotopy equivalence and G-simple homotopy type.

Definition 5.1. A system of DG Lie algebras L is said to be (finite) set-free if $L \cong L[\underline{\mathbb{Q}}(S)]$ as a system of graded Lie algebras, where S is a graded (finite) G-set.

Definition 5.2. A morphism $f: L \to L'$ between two finite set-free systems of DG Lie algebras is said to be a simple homotopy equivalence if $\tau(Qf)=0$ in $Wh(G;\mathbb{Q})$ where Qf is the map induced on the indecomposables.

Definition 5.3. A finite set-free system of DG Lie algebras $\mathscr{L}$ is called a *geometric model* for a finite G-complex X if there is a map $\alpha: \mathscr{L} \to L(X)$ such that the induced map

$$q \circ Q\alpha: Q\mathscr{L} \to QL(X) \twoheadrightarrow s^{-1}\bar{C}(X)$$

is an isomorphism and a G-simple homotopy equivalence.

Remark. It follows that α is a homology equivalence and that $\mathscr{L} \cong L[s^{-1}\bar{C}(X)]$ as a system of graded Lie algebras.

Definition 5.4. A finite set-free system of DG Lie algebras $\mathscr{S}$ is called a simple model for a finite G-complex X if there is a map $\beta: \mathscr{S} \to L(X)$ such that

$$q \circ Q\beta: Q\mathscr{S} \to QL(X) \to s^{-1}\bar{C}(X)$$

is a simple homotopy equivalence.

Then β is a homology equivalence as well.

Note that a geometric model for X is also a simple model for X.

We shall construct geometric models for G-CW-complexes which have a particular form namely

Definition 5.5. A G-CW-complex X is called a G-S-complex if $X=K/K'$, where K is a finite G-simplicial complex and K' is a G-subcomplex of K containing its one-skeleton.

The advantage of considering G-S-complexes is that they give rise in the obvious way to G-simplicial sets S in $\underline{SS}$ $(S_0=S_1=pt)$ which are needed for the Quillen constructions. This does not lead to loss of generality since we can prove the following:

Theorem 5.6. *Let X be a finite G-CW-complex such that X^H is nonempty connected and simply connected for all subgroups $H \subseteqq G$. Then X has the same G-simple homotopy type as a G-S-complex K/K', where K' is a collapsible subcomplex of K containing its one-skeleton.*

We defer the proof to the end of the section.

Theorem 5.7. *Every G-S-complex X has a geometric model.*

Theorem 5.8 (Uniqueness). *Let $\alpha: \mathscr{L} \to L(X)$ and $\alpha': \mathscr{L}' \to L(X)$ be two geometric models. Then there exists a morphism $i: \mathscr{L} \to \mathscr{L}'$ such that*

(a) *the diagram*

$$\begin{array}{ccc} \mathscr{L} & \overset{\alpha}{\searrow} & \\ {\scriptstyle i}\big\downarrow & & L(X) \\ \mathscr{L}' & \underset{\alpha'}{\nearrow} & \end{array}$$

commutes up to homotopy and

(b) *i is a simple homotopy equivalence.*

Theorem 5.9 ("Functoriality"). *Let $f: X \to X'$ be a cellular G-map which is a rational simple homotopy equivalence $[\tau(f)=0$ in $Wh(G;\mathbb{Q})]$. Let $(\mathscr{L},\alpha)$ and $(\mathscr{L}',\alpha')$ be two geometric models of X and X' respectively. Then there is a morphism $\tilde{f}: \mathscr{L} \to \mathscr{L}'$ such that*

(a) *the diagram*

$$\begin{array}{ccc} L(X) & \xrightarrow{L(f)} & L(X') \\ {\scriptstyle \alpha}\big\uparrow & & \big\uparrow{\scriptstyle \alpha'} \\ \mathscr{L} & \overset{\tilde{f}}{\dashrightarrow} & \mathscr{L}' \end{array}$$

commutes up to homotopy and

(b) *$\tilde{f}$ is a simple homotopy equivalence.*

The main result of this chapter is the following classification of (integral) simple homotopy types of finite G-complexes up to finite ambiguity. First

Definition 5.10. Let $\mathscr{S}$ be a set-free system of DG Lie algebras and let $\{\mathscr{S}_{(n)}\}$ be an F-Postnikov tower of $\mathscr{S}$. Let Z be a system of lattices in $\bigoplus_{n \geqq 0} H_n(\mathscr{S})$, Z' a lattice in

$$\bigoplus_{n \geqq 0} H_G^{n+2}(Q\mathscr{S}_{(n)}; H_{n+1}(\mathscr{S}))$$

and Z'' a system of lattices in $\bigoplus_{n \geqq 0} H_n(Q\mathscr{S})$. We call the triple (Z, Z', Z'') an *integral structure* of $\mathscr{S}$.

Theorem 5.11. *Let $\mathscr{S}$ be a set-free system of* DG *Lie algebras with integral structure (Z, Z', Z'') and let M be a positive integer. Then there are only finitely many G-simple homotopy types $[X]_s$ of finite G-complexes of torsion bound M with simple model $\mathscr{S}$ such that the map $\beta: \mathscr{S} \to L(X)$ induces isomorphisms between Z, Z', and Z'' on the one hand and the corresponding natural lattices of $L(X)$ on the other.*

Proof of Theorem 5.7. The construction of the geometric model is inductive on the skeleta of X. As before we identify X with a G-simplicial set in $\underline{\underline{SS}}$ so that $L(X)$ makes sense. Suppose we have constructed

$$\mathscr{L}^{(n-1)} \xrightarrow{\alpha} L(X^n)$$

which is a geometric model for the n^{th} skeleton X^n. In each orbit of $(n+1)$-simplices of X we choose exactly one simplex. Let σ be one such simplex and let G_σ be its isotropy group. First, we will adjoin a new generator x of degree n to $\mathscr{L}^{(n-1)}(G/G_\sigma)$ by choosing its differential properly.

Let $S^n = \Delta(n)/\Delta(n)^{\cdot}$ (standard n-simplex with boundary collapsed to a point) and let $D^{n+1} = \Delta(n+1)/V(n+1, 0)$ (standard $n+1$ simplex with all faces but the last collapsed to a point). Let

$$f: (D^{n+1}, S^n) \to ((X^{n+1})^{G_\sigma}, (X^n)^{G_\sigma})$$

be the map induced by the characteristic map of σ. We have the following commutative diagram

$$\begin{array}{ccccc} L(S^n) & \xrightarrow{p} & QL(S^n) & \longrightarrow & s^{-1}\bar{C}(S^n) \\ \cap & & \cap & & \cap \\ L(D^{n+1}) & \underset{p}{\longrightarrow} & QL(D^{n+1}) & \underset{q}{\longrightarrow} & s^{-1}\bar{C}(D^{n+1}). \end{array}$$

Let c and e be the canonical generators of $s^{-1}\bar{C}(S^n)$ and $s^{-1}\bar{C}(D^{n+1})$ respectively such that $de = c$, and let $\bar{c} \in L(S^n)$ and $\bar{e} \in L(D^{n+1})$ be elements in their preimages via $q \circ p$ respectively such that $d\bar{e} = \bar{c}$. The elements $\bar{c}$ and $\bar{e}$ exist because $q \circ p$ is surjective and a homology isomorphism in dimension $n-1$. Let $\bar{a} \in L((X^n)^{G_\sigma})$, $\bar{b} \in L((X^{n+1})^{G_\sigma})$, $a \in s^{-1}\bar{C}((X^n)^{G_\sigma})$, and $b \in s^{-1}\bar{C}((X^{n+1})^{G_\sigma})$ be the images of $\bar{c}$, $\bar{e}$, c, and e via f respectively. Note that $d\bar{b} = \bar{a}$ and $db = a$. Consider the diagram

$$\begin{array}{ccc} \mathscr{L}^{(n-1)}(G/G_\sigma) & \xrightarrow{\alpha} & L((X^n)^{G_\sigma}) \subseteq L((X^{n+1})^{G_\sigma}) \\ \downarrow & & \downarrow p \\ s^{-1}\bar{c}((X^n)^{G_\sigma}) \cong Q(\mathscr{L}^{(n-1)}(G/G_\sigma)) & \xrightarrow{Q\alpha} & QL((X^n)^{G_\sigma}) \\ & & \downarrow q \\ & & s^{-1}\bar{C}((X^n)^{G_\sigma}). \end{array}$$

The maps α, $Q\alpha$, and q are homology isomorphisms and $q \circ Q\alpha$ is an isomorphism. We identify the two groups via this last isomorphism. The cycles $Q\alpha(a)$ and $p(\bar{a})$ represent the same homology class in $QL((X^n)^{G_\sigma})$ because $qQ\alpha(a) = qp(\bar{a})$. Since α is a homology isomorphism, a lifts to a cycle $y \in \mathscr{L}^{(n-1)}(G/G_\sigma)$ such that $\alpha(y) = \bar{a} + dz$. Observe that $qp(z) = 0$ since z is of degree n and $(s^{-1}\bar{C}((X^n)^{G^\sigma}))_n = 0$.

Now we adjoin a new generator x of degree n to $\mathscr{L}(G/G_\sigma)$ and define

$$dx = y$$

and

$$\alpha(x) = \bar{b} + z \in L((X^{n+1})^{G_\sigma}).$$

We extend this definition as follows: Let S be the set of all $n+1$ simplices of X. S is a finite G-set. Consider the free system of graded Lie algebras $L(n) \equiv F[\underline{\mathbb{Q}}(S)]$. Define $\mathscr{L}^{(n)}$ as the free product

$$\mathscr{L}^{(n)} = \mathscr{L}^{(n-1)} \vee L(n).$$

The differential on $L(n)$ is defined by choosing the differential of one element in each orbit of S (as above) and then extending it by the action of G. Similarly we define α on $L(n)$.

By construction $Q\mathscr{L}^{(n)} \xrightarrow{Q\alpha} QL(X^{n+1}) \twoheadrightarrow s^{-1}\bar{C}(X^{n+1})$ is an isomorphism. Moreover, this map is a simple homotopy equivalence because it is a simple homotopy equivalence up to degree $n-1$ (by assumption) and in degree n (last degree) it maps basis elements to basis elements bijectively. This completes the proof of the theorem.

Remark. The construction of the geometric model is similar but more subtle than the construction of the chain algebra of a loop space in [AH].

Proof of Theorem 5.9. Statement (a) follows immediately from the lifting Theorem 2.11.

Now consider the diagram

$$\begin{array}{ccccc} Q\mathscr{L} & \xrightarrow{Q\alpha} & QL(X) & \xrightarrow{q} & s^{-1}\bar{C}(X) \\ \downarrow{\scriptstyle Q\tilde{f}} & & \downarrow{\scriptstyle QL(f)} & & \downarrow{\scriptstyle f_*} \\ Q\mathscr{L}' & \xrightarrow{Q\alpha'} & QL(X') & \xrightarrow{q'} & s^{-1}\bar{C}(X') \end{array}.$$

If H is a homotopy between $L(f) \circ \alpha$ and $\alpha' \circ \tilde{f}$, then $H \circ S$ is a chain homotopy between $QL(f) \circ Q\alpha$ and $Q\alpha' \circ Q\tilde{f}$ (see Sect. 2). This implies that $f_* \circ q \circ Q\alpha$ is chain homotopic to $q' \circ Q\alpha' \circ Q\tilde{f}$. Since $q \circ Q\alpha$ and $q' \circ Q\alpha'$ are isomorphisms and simple homotopy equivalences and f_* is a simple homotopy equivalence, $Q\tilde{f}$ is a simple homotopy equivalence as well. This proves statement (*b*) of the theorem.

Theorem 5.8 is an immediate corollary of Theorem 5.9.

Proof of Theorem 5.11. First we observe that there are only finitely many G-homotopy types of torsion bound M with simple model $\mathscr{S}$ and lattices Z and Z'. This follows from Theorem 4.13, since a minimal model Λ of $\mathscr{S}$ is an equivariant minimal Lie model for the spaces X and the induced lattices on Λ are mapped onto the natural lattices of $L(X)$. Let $[X]$ be one of these G-homotopy types. We will show that only finitely many simple homotopy types of spaces in $[X]$ can have simple model $\mathscr{S}$ and system of lattices Z''.

Consider the commutative diagram

$$\begin{array}{ccccc}
\operatorname{aut}_G(X) & \xrightarrow{\ \tau\ } & Wh(G;\mathbb{Z}) & \xrightarrow{\ p\ } & \bar{W} \\
\downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \bar{h}} \\
\operatorname{aut}(\mathscr{S}) & \xrightarrow{\ \tau_{\mathbb{Q}}\ } & Wh(G;\mathbb{Q}) & \xrightarrow{\ p_{\mathbb{Q}}\ } & \bar{W}_{\mathbb{Q}}\,; \\
\downarrow{\scriptstyle q} & \nearrow{\scriptstyle \tau'} & & \searrow{\scriptstyle p'} & \downarrow{\scriptstyle h'} \\
\Gamma \subseteqq \operatorname{aut}(H(Q\mathscr{S})) & & & & W'_{\mathbb{Q}}
\end{array}$$

here $\operatorname{aut}_G(X)$ is the group of G-homotopy classes of G-self homotopy equivalences of X, $\operatorname{aut}(\mathscr{S})$ is the group of homotopy classes of self homotopy equivalences of $\mathscr{S}$, τ, and $\tau_{\mathbb{Q}}$ are the generalized Whitehead torsions over $\mathbb{Z}$ and over $\mathbb{Q}$ respectively and $\bar{W}$ and $\bar{W}_{\mathbb{Q}}$ are the cokernels of τ and $t_{\mathbb{Q}} \circ \ell$ respectively.

$\bar{W}$ measures the simple homotopy types in the G-homotopy type $[X]$ as follows: a homotopy equivalence $f: X \to Y$ induces a unique element $\bar{\tau}(f) \in \bar{W}$ which does not depend on f or Y but depends only on the simple homotopy type of Y. It is easy to see that this correspondence is bijective.

Let $\operatorname{aut}(H(Q\mathscr{S}))$ be the quotient of $\operatorname{aut}(\mathscr{S})$ divided by the following equivalence relation: $a \sim b$ if $a_* = b_*$ on $H(Q\mathscr{S})$. We will need the following statement.

Proposition 5.12. *The map* $\tau_{\mathbb{Q}}: \operatorname{aut}(\mathscr{S}) \to Wh(G;\mathbb{Q})$ *factors uniquely via* $\operatorname{aut}(H(Q\mathscr{S}))$.

Granting the proposition, we denote by $W'_{\mathbb{Q}}$ the cokernel of $\tau'|_{\Gamma}: \Gamma \to Wh(G;\mathbb{Q})$, where Γ is the subgroup of automorphisms of $H(Q(\mathscr{S}))$ which preserve the system of lattices Z''.

Let $[Y]_s$ be a G-simple homotopy type in $[X]$ with simple model $\mathscr{S}$ and lattices Z''. We'll show that $[Y]_s$ lies in the kernel of $h' \circ \bar{h}$. Let $f: X \to Y$ be a G-homotopy equivalence and let $\tilde{f}: \mathscr{S} \to \mathscr{S}$ be an induced homotopy equivalence (which obviously preserves the lattice Z'') such that

$$\begin{array}{ccccc}
\mathscr{S} & \xrightarrow{\ \beta_X\ } & L(X) & \twoheadrightarrow & s^{-1}\bar{C}(X) \\
\downarrow{\scriptstyle \tilde{f}} & \curvearrowright \simeq & \downarrow & \curvearrowright & \downarrow{\scriptstyle f_*} \\
\mathscr{S} & \xrightarrow{\ \beta_Y\ } & L(Y) & \twoheadrightarrow & s^{-1}\bar{C}(Y)
\end{array}.$$

Clearly $\tau_{\mathbb{Q}}(\tilde{f}) = \tau_{\mathbb{Q}}(f_*) = h \circ \tau(f)$ and $\tau_{\mathbb{Q}}(\tilde{f}) = \tau' \circ q(\tilde{f})$, where $q(\tilde{f})$ is an automorphism of $H(Q\mathscr{S})$ which preserves Z'' isomorphically. Hence, $h(\tau(f))$ lies in $\tau'(\Gamma)$ and therefore maps to zero in $W'_{\mathbb{Q}}$. This implies that $h'\bar{h}([Y]_s) = h'\bar{h}p(\tau(f)) = p'h\tau(f) = 0$.

Therefore, it suffices to prove that the kernel of $h'\bar{h}$ is finite. By Wall [W], the kernel of $h: Wh(G;\mathbb{Z}) \to Wh(G;\mathbb{Q})$ is finite and hence the kernel of $\bar{h}$ is finite.

In order to show that the kernel of h' is finite we use certain results involving algebraic and arithmetic groups which are proved in $[T_3]$. We know that $\operatorname{aut}(\mathscr{S})$ is an algebraic matrix $\mathbb{Q}$-group and $\ell(\operatorname{aut}_G(X))$ is an arithmetic subgroup of $\operatorname{aut}(\mathscr{S})$. Moreover, $\operatorname{aut}(H(Q\mathscr{S}))$ is also an algebraic matrix $\mathbb{Q}$-group, the projection q is a map of algebraic groups and Γ is an arithmetic subgroup of aut $(H(Q\mathcal{S}))$. It is a well known fact in the theory of arithmetic groups that if $k: K \to K'$ is a surjective map of algebraic groups and A and A' are arithmetic subgroups of K and K' respectively then $k(A)$ and A' are commensurable arithmetic subgroups of K' [BHC]. In our case, since $q\ell(\mathrm{aut}_G(X)) \subseteqq \Gamma$, $q\ell(\mathrm{aut}_G(X))$ has finite index in Γ. Therefore $\tau_{\mathbb{Q}}\ell(\mathrm{aut}_G(X)) = \tau' q\ell(\mathrm{aut}_G(X))$ has finite index in $\tau'(\Gamma)$. This implies that h' has finite kernel.

It only remains to prove the proposition above.

Proof of Proposition 5.12. We must show that for $\alpha \in \mathrm{aut}(\mathcal{S})$ which is the identity on $H(Q\mathcal{S})$, $\tau_{\mathbb{Q}}(\alpha) = 0$.

By the decomposition formula for generalized torsion [R_1], the problem can be reduced easily to the following: Suppose C is a finite free based $\mathbb{Q}(G)$-complex and $\alpha: C \to C$ induces the identity on homology. Then $\tau_{\mathbb{Q}}(\alpha) = 0$.

We consider several cases:

(i) First consider the case where $H_*(C)$ is stably free over $\mathbb{Q}(G)$ (i.e. there is a free $\mathbb{Q}(G)$-complex F such that $H_*(C) \oplus F$ is a free $\mathbb{Q}(G)$-complex). Then $\tau_{\mathbb{Q}}(C)$ can be defined as an absolute invariant and $\tau_{\mathbb{Q}}(\alpha) = \tau_{\mathbb{Q}}(C) - \tau_{\mathbb{Q}}(C) = 0$ by [Mi].

If C is of dimension n and $H_k(C) = 0$ for $k < n$ then $H_n(C)$ is stably free and we are in case (i).

(ii) Consider the case where there exists a complex C' of dimension n such that $C \subset C'$ and α extends to $\alpha': C' \to C'$. Moreover assume that α' induces the identity on homology, $H_k(C') = 0$ for $k < n$, and $\tau_{\mathbb{Q}}(\bar{\alpha}) = 0$, where $\bar{\alpha}: C'/C \to C'/C$.

Then $\tau_{\mathbb{Q}}(\alpha) = 0$, because $\tau_{\mathbb{Q}}(\alpha) = \tau_{\mathbb{Q}}(\alpha') - \tau_{\mathbb{Q}}(\bar{\alpha}) = 0 - 0 = 0$.

Case (ii) can be achieved if the following case (iii) holds.

(iii) Suppose C is a free based complex of dim n with $H_j(C) = 0$ for $j < k < n$ and let $\alpha: C \to C$ be a map which induces the identity on homology. Let $a \in Z_k(C)$ such that $0 \neq [a] \in H_k(C)$. Define C' by $C'_\ell = C_\ell$ for $\ell \neq k+1$ and

$$C'_{k+1} = C_{k+1} + \mathbb{Q}(G)(x).$$

The differential of C' restricts to the differential of C and $dx \equiv a$.

Then α can be extended to α' on C' such that α' induces the identity both on homology and on the quotient $C'/C = \mathbb{Q}(G)$.

Indeed, we have an exact sequence

$$0 \to H_{k+1}(C) \to H_{k+1}(C') \to \mathbb{Q}(G)(x) \xrightarrow[j]{} H_k(C) \to H_k(C') \to 0,$$

where $j(x) = [a]$. By the semisimplicty of $\mathbb{Q}(G)$, we can write $1 = i_0 + i_1$ in $\mathbb{Q}(G)$ with $i_1 i_0 = 0$, $i_0^2 = i_0$, and $i_1^2 = i_1$. Then $\mathbb{Q}(G)(x) = \mathbb{Q}_0(i_0 x) + \mathbb{Q}_1(i_1 x)$, where $\mathbb{Q}_1(i_1 x) = \ker j$.

We can now split the exact sequence as follows:

Since $i_1[a] = 0$, $i_1 a = da'$, $a' \in C_{k+1}$. Then $d i_1 a' = i_1(i_1 a) = i_1 a$ and hence $d(i_1(x - a')) = 0$, i.e. $i_1(x - a') \in Z_{k+1}(C')$.

The correspondence $x \to i_1(x - a')$ induces a map

$$\mathbb{Q}(G)(x) \to Z_{k+1}(C')$$

which annihilates $\mathbb{Q}_0(i_0 x)$ and maps $\mathbb{Q}_1(i_1 x)$ injectively thus splitting the exact sequence

$$0 \to H_{n+1}(C) \to H_{n+1}(C') \rightleftarrows \mathbb{Q}_1(i_1 x) \to 0.$$

Now by hypothesis $\alpha(a)-a=db$ with $b\in C_{k+1}$. Replace b by $\bar{b}=i_0b+\alpha(a')-a'$. Then $d\bar{b}=\alpha(a)-a$.

Now define $\alpha':C'\to C'$ by $\alpha'|C=\alpha$ and $\alpha'(x)=x+\bar{b}$. Then α' is a chain map and it induces the identity on both summands of $H_{n+1}(C')$. Indeed

$$\begin{aligned}\alpha'(i_1(x-a')) &= i_1\alpha'(x)-\alpha'(i_1a')=i_1\alpha'(x)-i_1\alpha(a')\\ &= i_1x+i_1\bar{b}-i_1\alpha(a')\\ &= i_1x+i_1\alpha(a')-i_1a'-i_1\alpha(a')\\ &= i_1(x-a').\end{aligned}$$

This completes the proof of the proposition.

Proof of Theorem 5.6. We shall prove the theorem only for $G=\{e\}$. The general case follows through complication of terminology by essential the same argument. We prove the theorem in three steps. (Although these steps are elementary, we give the arguments because they are so chosen as to work for $G\neq\{e\}$ as well.)

Step 1. X is simple homotopy equivalent to X', where X' has only one zero-cell.

Indeed, by a standard argument there is a maximal tree T (i.e. a maximal collapsible one-complex) in X. Let $X'=X/T$. Since $\pi_0(X)=0$, X' has only one zero-cell.

Step 2. X is simple homotopy equivalent to X', where X' has one zero-cell and no one-cells.

By Step 1, we can assume without loss of generality that X has only one zero-cell. Let

$$X_1=S_1\vee S_2\vee\,\dots\,S_j$$

be the one-skeleton, where S_i is a circle for $i=1,\dots,j$. Let

$$\bar{X}=X\cup D_1\cup\,\dots\,D_j.$$

where D_i is a two-cell attached to X along S_i. Since $\pi_1(X)=0$, the map $S_i\to X$ extends to a map

$$f_i:D_i'\to X_2,$$

where D_i' is a two-disk and X_2 is the two-skeleton of X.

Let $\tilde{X}=\bar{X}\cup\bar{D}_1\cup\bar{D}_2\cup\,\dots\,\bar{D}_j$, where $\bar{D}_i$ is a three-cell attached to $\bar{X}$ as follows: Let $\partial\bar{D}_i=D_i\cup D_i'$, where $D_i\cap D_i'=S_i$ for every i. We map $g_i:\partial\bar{D}_i\to\bar{X}$ by $g_i|D_i$ $=$inclusion and $g_i|D_i'=f_i$.

Then we collapse along $(\bar{D}_i,D_i)$ and have that $\tilde{X}$ is simple homotopy equivalent to X. On the other hand the subcomplex

$$Y=\tilde{X}_1\cup D_1\cup D_2\dots\cup D_j=D_1\vee\,\dots\,\vee D_j$$

of $\tilde{X}$ is a collapsible. Hence $\tilde{X}$ is simple homotopy equivalent to $\tilde{X}/Y=X'$, where X' is a CW-complex with one zero-cell and no one-cells.

Step 3. X is simple homotopy equivalent to K/K' as in the theorem.

Proof. By Steps 1 and 2 we can assume

$$X_2 = S_1^2 \vee \ldots \vee S_r^2,$$

i.e. X_2 is the wedge of two-spheres. The theorem is certainly true for a wedge of two-spheres. Let $X = D^n \bigcup_f Y$ with $f: S^{n-1} \to Y$ for $n \geqq 3$, and assume inductively that there is a simple homotopy equivalence $g: Y \to L/L'$, where L and L' satisfy the conditions of the theorem.

Note that L' being collapsible implies that the projection $L \to L/L'$ is also a simple homotopy equivalence. Thus we can factor g up to homotopy

$$\begin{array}{ccc} Y & \xrightarrow{\bar{g}} & L \\ & \searrow^{g} & \downarrow \\ & & L/L' \end{array}$$

and by rechoosing g we can assume that the above diagram commutes. Finally by homotopy extension we can assume that the composite

$$S^{n-1} \xrightarrow{f} Y \xrightarrow{\bar{g}} L$$

is simplicial with respect to some triangulation of S^{n-1} which we fix.

Then $X = D^n \bigcup_f Y$ is simple homotopy equivalent to $D^n \bigcup_{\bar{g}f} L$.

Therefore, it suffices to show that $K \equiv D^n \bigcup_{\bar{g}f} L$ contains a collapsible two-complex which in turn contains the one-skeleton K_1 in some triangulation.

As a simplicial complex K we take initially $C(S^{n-1}) \cup M(\bar{g}f)$, where $C(S^{n-1})$ is the cone over S^{n-1} and $M(\bar{g}f)$ is the simplicial mapping cylinder of $\bar{g}f: S^{n-1} \to L$.

Let $\tilde{M} = M(h)$, where $h = \bar{g}f|_{S_1^{n-1}}: S_1^{n-1} \to L' \subset L$. Then $\tilde{M}$ is a collapsible subcomplex of $M(\bar{g}f)$ containing its one-skeleton.

Now we use the following elementary fact: Let W be any subdivision of $\partial \Delta_n$. Then W can be extended to a subdivision $\bar{W}$ of Δ_n which contains a two-subcomplex V having the following property: V contains the one-skeleton $\bar{W}_1$ of $\bar{W}$ and it collapses to $\bar{W}_1$.

Because of this fact, $K' = V \cup M(h)$ is a collapsible two-dimensional subcomplex of K containing the one-skeleton. This completes the proof of the theorem.

6. Applications

Let $\mathbb{Z}_n$ be the cyclic group with n elements. We consider topological $\mathbb{Z}_{p^k}$-actions on the sphere S^n, where p is a prime number. We assume that the fixed point sets $(S^n)^H$ are nonempty and simple-connected for all subgroups $H \subsetneqq \mathbb{Z}_{p^k}$. By Smith theory $(S^n)^H$ is a $\mathbb{Z}_p$-homology sphere of dimension say $n(H) < n-1$ for every $H \subsetneqq \mathbb{Z}_{p^k}$. We denote the collection of dimensions of the fixed point sets of a $\mathbb{Z}_{p^k}$-action on S^n by $\{n(H)\}$.

Theorem 6.1. *There are only finite many* $\mathbb{Z}_{p^k}$-homotopy types of $\mathbb{Z}_{p^k}$*-actions on* S^n *with each* $n(H)$ *odd and with given torsion bound* M.

Proof. Let X be a $\mathbb{Z}_{p^k}$-action on S^n as in the theorem and let $r: X \to X_0$ be an equivariant rationalization of X ([T$_1$]) meaning that $r^H: X^H \to X_0^H$ is an ordinary rationalization for every $H \subseteqq \mathbb{Z}_{p^k}$. For every H we consider $H_{n(H)}(X^H)$/torsion as a subgroup of $H_{n(H)}(X_0^H) \cong \mathbb{Q}$ via r_*^H.

Let Y be any other $\mathbb{Z}_{p^k}$-action on S^n with the same collection of dimensions. It follows by using elementary equivariant obstruction theory that there exists a $\mathbb{Z}_{p^k}$-map

$$f: Y \to X_0$$

which is a $\mathbb{Z}_{p^k}$-rational homotopy equivalence. In fact, f can be chosen so that

$$f_*^H: H_{n(H)}(Y^H)/\text{torsion} \to H_{n(H)}(X^H)/\text{torsion} \subseteqq H_{n(H)}(X_0^H)$$

and the degree d_H of this map satisfies $0 < |d_H| < |\mathbb{Z}_{p^k}/H|$ for every $H \subseteqq \mathbb{Z}_{p^k}$.

Therefore all $\mathbb{Z}_{p^k}$-actions on S^n with dimensions $\{n(H)\}$ have equivariant minimal Lie models isomorphic to the model Λ of X_0. $H(\Lambda)$ has a system of lattices Z induced by the integral homotopy of X. Moreover, $H(\Lambda)$ has systems of lattices induced by the integral homotopy of the Y's via the above mentioned isomorphisms. But because of the control of the degrees d_H, there are only finitely many distinct systems of lattices on $H(\Lambda)$.

Now we have to check the lattices on $H_G^{n+2}(Q\Lambda_{(n)}; H_{n+1}(\Lambda))$ for every n. From [T$_2$] we know that in the case of $\mathbb{Z}_{p^k}$-actions the Bredon spectral sequence collapses to a short sequence rationally:

$$\begin{aligned} 0 &\to \text{Ext}(H_{n+1}(Q\Lambda_{(n)}), H_{n+1}(\Lambda)) \to H_G^{n+2}(Q\Lambda_{(n)}; H_{n+1}(\Lambda)) \\ &\to \text{Hom}(H_{n+2}(Q\Lambda_{(n)}), H_{n+1}(\Lambda)) \to 0\,. \end{aligned}$$

By construction $H_i(Q\Lambda_{(n)}) \cong H_{i+1}(X_{0,n+1})$ for every i, where $X_{0,n+1}$ is the $(n+1)st$ stage of the equivariant Postnikov tower of X_0. We have $\pi_i(X_{0,n+1}^H) = 0$ for $i > n+1$ and every H and $\pi_i(X_{0,n+1}^H) = \pi_i(X_0^H)$ for $i \leqq n+1$ and every H. Therefore $X_{0,n+1}^H$ is either an odd-dimensional Eilenberg-MacLane space of dimension less than $n+2$ or a point [recall that we assumed $n(H)$ odd]. In either case $H_i(Q\Lambda_{(n)}) = 0$ if $i \geqq n+1$. Therefore, $H_G^{n+2}(Q\Lambda_{(n)}; H_{n+1}(\Lambda)) = 0$ for every n.

Now the assumptions of Theorem 4.13 are satisfied and we get the required finiteness result for $\mathbb{Z}_{p^k}$-actions on S^n.

We consider $\mathbb{Z}_{p^k}$-actions on S^n such that each $\mathbb{Z}_{p^k}/H$ acts trivially on $H_*((S^n)^H)$. We recall that the Reidemeister torsion of such an action lies in $\bigoplus_{H \subseteq \mathbb{Z}_{pk}} \mathbb{Q}_*(\mathbb{Z}_{p^k}/H)$, where $\mathbb{Q}_*(G)$ means the units of the group ring ([B], [R$_1$]).

Theorem 6.2. *Consider $\mathbb{Z}_{p^k}$-actions on S^n with non-empty, simply connected fixed point sets $(S^n)^H$ of odd cohomological dimensions such that $\mathbb{Z}_{p^k}/H$ acts trivially on $H_*((S^n)^H)$ for every $H \subseteqq \mathbb{Z}_{p^k}$. Then there are only finitely many $\mathbb{Z}_{p^k}$ simple homotopy types of such actions having torsion bound M and given equivariant Reidemeister torsion $v \in \bigoplus_{H \subseteqq G} \mathbb{Q}_*(\mathbb{Z}_{p^k})$.*

Proof. By Theorem 6.1 there are only finitely many $\mathbb{Z}_{p^k}$-homotopy types of $\mathbb{Z}_{p^k}$-actions on S^n with the required properties. Now let X and Y be two $\mathbb{Z}_{p^k}$-actions in the same $\mathbb{Z}_{p^k}$-homotopy type, and let $f: X \to Y$ be a $\mathbb{Z}_{p^k}$-homotopy equivalence.

For G abelian, if X and Y have the same Reidemeister torsion v, f must be a rational G-simple homotopy equivalence [B] and therefore X and Y have a common simple model $\mathscr{S}$. Therefore all $\mathbb{Z}_{p^k}$-actions which satisfy the conditions of the theorem correspond to a finite number of simple models $\mathscr{S}$. On the other hand, we can argue as in the previous theorem that because of the control of the degrees of the maps f^H, the various $\mathbb{Z}_{p^k}$-actions define a finite number of lattices on each model $\mathscr{S}$. Now we apply Theorem 5.11 completing the proof.

Next we prove similar results for semilinear actions.

Definition 6.3. A semilinear G-sphere is a finite G-complex X such that for each subgroup H of G the fixed point set X^H is an $n(H)$-dimensional space which is homotopy equivalent to the sphere $S^{n(H)}$. Moreover we assume that $n(H)$ is odd for each H.

Theorem 6.4. *Given a finite group G and a positive integer n, there are only finitely many G-homotopy types of semilinear G-spheres of dimension n.*

Remark. This result for semilinear G-spheres with the additional condition that $N(H)$ acts trivially on $H_{n(H)}(X^H;\mathbb{Z})$ follows from work of tom Dieck and Petrie [DP]. Moreover in [DP], an estimate of finiteness is given in terms of the Burnside ring of G.

Proof. Let X be a semilinear G-sphere of dimension n. We take the join of X with an appropriate linear G-sphere X' such that the resulting semilinear sphere $X * X'$ has the following properties:

(i) $(X * X')^H$ is nonempty, connected and simply connected of odd dimension for every subgroup H of G.

(ii) the gaps between the dimensions of the fixed point sets are larger than the length of the group G. Here the *length of G* is defined to be the length of the largest chain of distinct subgroups

$$\{e\} \subset H_1 \subset H_2 \subset \ldots \subset G$$

from $\{e\}$ to G. It is easy to show that X' can be chosen to be a linear sphere of dimension m depending on n and G but not on X. Then the dimension of $X * X'$ is $N \equiv n+m$ for every X.

First we will show that there are only finitely many G-homotopy types of semilinear G-spheres of dimension N which have properties (i) and (ii) and then we will show that the process of taking joins with X' is a finite-to-one correspondence. For the first statement it suffices to prove that there are only finitely many G-homotopy types of such actions with given collection of dimensions $\{n(H)\}$.

Let Y be a semilinear G-sphere satisfying (i) and (ii) and having dimensions $\{n(H)\}$. For any other such G-sphere Y', there exists a G-map $f: Y \to Y'$ which is a rational G-homotopy equivalence. Moreover, the degrees d_H of $f^H: Y^H \to Y'^H$ can be chosen such that $0 < |d_H| < |NH/H|$. This follows by elementary equivariant obstruction theory. Let Λ be an equivariant minimal Lie model of Y. Then Λ is an equivariant minimal model of Y' as well. Λ has an integral structure (Z, Z') induced by Y. On the other hand, the maps f induce integral structures on Λ which correspond to the homotopy and Bredon cohomology of the Postnikov tower of the Y''s. We will show that there are only finitely many distinct induced integral structures. This is obviously true for the systems of lattices on $H(\Lambda)$ because the degrees d_H are bounded.

We claim that $H_G^{n+1}(Q\Lambda_{(n-1)}; H_n(\Lambda)) \cong H_G^{n+2}(Y_n; \underline{\pi}_{n+1}(Y)) \otimes \mathbb{Q} = 0$ for every n, where $\{Y_n\}$ is the equivariant Postnikov tower of X. In order to see this we consider the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(\underline{H}_q(Y_n); \underline{\pi}_{n+1}(Y)) \Rightarrow H_G^{p+q}(Y_n; \underline{\pi}_{n+1}(Y))$$

which converges to the Bredon cohomology [Br]. Here we assume that everything is tensored over $\mathbb{Q}$ but for brevity we omit writing it.

Let $\underline{\pi}_{n+1}(Y) \neq 0$. (Otherwise the Bredon cohomology above is zero.) Let M be the length of G. Because of the gap condition (ii) and the assumption of odd dimensions $n(H)$, $\underline{H}_i(Y_n) = 0$ for $i > n - M$, i.e. the E_2-term of the spectral sequence vanishes above $n - M$.

On the other hand we can show

Lemma 6.5. $\mathrm{Ext}^p(A, B) = 0$ *for any systems of rational vector spaces A, B for G if p is larger than the length of G.*

Granting the lemma, the E_2-term vanishes to the right of M as well. In particular, this means that $E_2^{p,q} = 0$ for $p + q = n + 2$ and that

$$E_2^{p,q} = E_\infty^{p,q} = 0 \quad \text{for} \quad p + q = n + 2 .$$

This then implies that $H_G^{n+2}(X_n; \underline{\pi}_{n+1}(X)) = 0$ rationally. Therefore, finiteness for the lattices Z' of Λ is trivially satisfied. Now we apply Theorem 4.13 and get the finiteness result for semilinear G-spheres of dimension N which satisfy conditions (i) and (ii).

Proof of Lemma 6.5. The statement follows essentially from the existence of minimal projective covers. Let

$$\ldots \to P_i \xrightarrow{p_i} P_{i-1} \to \ldots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A$$

be a projective resolution of A where P_0 is a minimal projective cover of A and each P_i is a minimal projective cover of the kernel of the map p_{i-1}. By construction of a minimal projective cover in [T$_1$], $P_0(G/G) = A(G/G)$ and therefore $P_1(G/G) = 0$. Similarly, if G' is a maximal proper subgroup of G then $P_1(G/G') = \ker P_0(G/G')$ and therefore $P_2(G/G') = 0$. We continue inductively and have the required result.

Remark. The above argument works equally well for rational semilinear spheres.

In order to complete the proof of Theorem 6.4 it suffices to prove the following

Theorem 6.6. *Let A_n be the family of G-homotopy classes of semilinear G-spheres of dimension n. Let Z be a semilinear G-sphere of dimension $k - 1$ with $Z^G \neq \emptyset$. Then the map $A_n \to A_{n+k}$ given by $X \to X * Z$ is finite to one.*

This is a corollary of Theorem 5.2 in [R$_2$]. We recall the terminology and statement of that theorem. Let $X, Y \in A_n$ and $f: X * Z \to Y * Z$ be a G-map. Assume $Z^G \neq \emptyset$ and $n(H) =$ dimension of $X^H \leqq$ dimension of Y^H. Let $k(H) =$ dimension of

Z^H+1. We have then the composite

$$\tilde{H}_{n(H)}(X^H) \overset{\lambda_1}{\cong} \tilde{H}_{n(H)+k(H)}(X^H * Z^H) = \tilde{H}_{n(H)+k(H)}((X * Z)^H)$$

$$\xrightarrow{f_*} \tilde{H}_{n(H)+k(H)}((Y * Z)^H) = \tilde{H}_{n(H)+k(H)}(Y^H * Z^H) \overset{\lambda_2}{\cong} \tilde{H}_{n(H)}(Y^H)$$

which we designate $\deg_H(f)$. The isomorphisms λ_1 and λ_2 depend on choosing a generator of $\tilde{H}_{k(H)-1}(Z^H) \cong \mathbb{Z}$, but if we choose the same generator for both λ_1 and λ_2 the composite $\deg_H(f)$ is independent of the choice. The map $\deg_H(f)$ is a stable invariant independent of Z, i.e. if one replaces Z by $Z * W$ and f by $f * id$ then $\deg_H(f) = \deg_H(f * id)$.

Theorem 5.2 *in* $[R_2]$: *Let* $Z^G = \emptyset$ *and let* $f: X * Z \to Y * Z$ *be a G-map. We assume that dimension* $X^H \leqq$ *dimension* Y^H *and that for* $H \subsetneqq H'$ *and* $X^H = X^{H'}$, $\deg_H(f) = j \deg_{H'}(f)$, *where* $j: \tilde{H}_{n(H)}(Y^{H'}) \to \tilde{H}_{n(H)}(Y^H)$. *Then there exists* $f': X \to Y$ *with* $\deg_H(f') = \deg_H(f)$ *for all* $H \subsetneqq G$.

Note. If f in the theorem is a G-homotopy equivalence so is f'. In this case $\deg_H(f)$ is an isomorphism from one copy of $\mathbb{Z}$ to another and there are only two choices. This is the basis of the following proof.

Proof of Theorem 6.6. We must show that if $X_1, X_2, \ldots, X_r, \ldots$ is an infinite family of distinct elements of A_n it is not possible that $X_1 * Z, X_2 * Z, \ldots, X_r * Z, \ldots$ are all equal in A_{n+k}. Assume on the contrary that there exists a G-homotopy equivalence $f_j: X_j * Z \to X_1 * Z$ for each j. Then dimension $X_j^H =$ dimension $(X_j * Z)^H - k(H)$ $=$ dimension $(X_1 * Z)^H - k(H) =$ dimension X_1^H. We call this common value $n(H)$. For every H and r, we choose a generator $\lambda_{H,r}$ of $\tilde{H}_{n(H)}(X_r^H)$ such that $\lambda_{H',r} = \lambda_{H,r}$ if $X_r^H = X_r^{H'}$. Note that by constancy of $n(H)$ and the fact of semilinearity, $X_r^H = X_r^{H'}$ for every r if it is true for any one r.

Let W be the set of subgroups $H \subsetneqq G$ such that $X_r^H \neq \emptyset$. Given any G-homotopy equivalence $h: X_j * Z \to X_r * Z$, $\deg_H(h)$ as a function of $H \in W$ is determined by assigning H the value $+1$ if $\deg_H(h)(\lambda_{H,j}) = \lambda_{H,r}$ and assigning H the value -1 if $\deg_H(h)(\lambda_{H,j}) = -\lambda_{H,r}$. That is $\deg_H(h)$, $H \in W$, can be interpreted as a function $d_h: W \to \mathbb{Z}_2$. It is obvious that for $h = h_1 \circ h_2$, $d_h = d_{h_1} \cdot d_{h_2}$, where the multiplication is meant to be in $\mathbb{Z}_2$. In particular if $d_{h_1} = d_{h_2}$ then $d_h = d_{id}$, the constant function with value $+1$. On the other hand if $d_h = d_{id}$ then h must satisfy the hypothesis of Theorem 5.2 in $[R_2]$. Thus is "desuspends" to a homotopy equivalence of X_j to X_r.

Now, consider the G-homotopy equivalences $f_j: X_j * Z \to X_1 * Z$. Since the number of functions $W \to \mathbb{Z}_2$ is finite we must have $d_{f_j} = d_{f_r}$ for some $j \neq r$. If h_j is a homotopy inverse to f_j then $d_{h_j} = d_{f_j}$, whence $d_{h_j f_k} = d_{id}$. Thus there exists a G-homotopy equivalence $X_k \to X_j$ contradicting the assumption. This completes the proof of Theorem 6.6.

We can have a version of Theorem 6.2 for semilinear actions as well. We consider only finite cyclic groups G.

Theorem 6.7. *Let* G *be a finite cyclic group and let* $v \in \bigoplus_{H \subseteq G} \mathbb{Q}_*(G/H)$. *There are only finitely many G-simple homotopy types of semilinear G-spheres of given dimension n* *and Reidemeister torsion v, such that each G/H acts trivially on the homology of the fixed point set under H.*

The proof is the same as in Theorem 6.2 (using Theorem 6.6 this time).

Remark. The condition that G be abelian is probably unnecessary.

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Tensor Product of Several Spaces and Nuclearity

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1. Introduction and Definitions

In his now famous memoir [2], Grothendieck posed the following problem: Let the ε- and π-topologies coincide on the tensor product $X_1 \otimes X_2$ of two locally convex spaces X_1, X_2, i.e.

$$X_1 \otimes_\varepsilon X_2 = X_1 \otimes_\pi X_2 . \tag{1}$$

Does it necessarily follow that at least one of the spaces X_1, X_2 is a nuclear space? Hard work on this problem was done in the Banach space case; Pisier found an infinite-dimensional Banach space P such that

$$P \otimes_\varepsilon P = P \otimes_\pi P . \tag{2}$$

This negatively answers the question of Grothendieck.

Now we may ask other questions, e.g.: Do there exist non-nuclear spaces X_1 and X_2 or P such that (1) or (2) is satisfied and moreover X_1, X_2 or P satisfy some additional property $\mathscr{P}$? For $\mathscr{P}$ we may take e.g. the following properties of the space: reflexivity, having or not having a basis, being a Schwartz or Fréchat space. Examples of non-nuclear reflexive Fréchet-Schwartz spaces with bases satisfying (1) was given in [6], while the existence of reflexive Banach spaces X_1, X_2, P satisfying (1) or (2) is still an open problem. The aim of this paper is to answer the following more general question:

Let $X_1, \ldots, X_r$ be locally convex spaces and let the ε- and π-topologies coincide on their tensor product, that is

$$(\mathrm{G}_r) \qquad X_1 \otimes_\varepsilon \cdots \otimes_\varepsilon X_r = X_1 \otimes_\pi \cdots \otimes_\pi X_r .$$

Does it follow that at least one of the spaces $X_1, \ldots, X_r$ is nuclear?

In the case $r=2$ this question coincides with Grothendieck's question and has a negative answer as was shown by Pisier. Our results are the following:

A) *For every r ($r \geqq 2$) the question (G_r) has negative answer.*

In contrast to (2) we have the following complement to A):

B) *If* $X=X_1=\ldots=X_r$ *and if* $r\geqq 3$ *then the question* (G_r) *has a positive answer:* Thus we have e.g.: *A locally convex space* X *is nuclear iff* $X\otimes_\varepsilon X\otimes_\varepsilon X=X\otimes_\pi X\otimes_\pi X$.

C) *If* $X_1,\ldots,X_r$ *are normed spaces and* (1) *is satisfied then at least* $r-2$ *of the spaces* $X_1,\ldots,X_r$ *must have finite dimension.*

In fact, in 3.2. and 3.3., slightly more general results are presented, e.g. a counterexample $X_1,\ldots,X_r$ to (G_r) may be given such that all X_j are Fréchet-Schwartz spaces with bases and every continuous r-linear form on $X_1\times\ldots\times X_r$ is strongly nuclear. Thus we have also negative to the following question [8]:

Let $X_1,\ldots,X_r$ *be locally convex spaces such that every* (P_r) *continuous r-linear form on* $X_1\times\ldots\times X_r$ *is nuclear.*

Must one of the spaces $X_1,\ldots,X_r$ *be nuclear?*

In the proof of A) it seems advantageous to use the theory of ideals of multilinear operators. This theory was introduced by Pietsch [11]. It is not the aim of this paper further to develop this theory. In Sect. 2 we only state some definitions and results, which we shall mostly use in Sect. 3 for the proof of A) and B). Finally, in the last section (Sect. 4) we sketch an alternative proof of A) and C). Also we give here a corrected version of [7, Lemma 2.3].

We shall use the notation from [8] and [9]. Here we recall some: If $x=(x_i)$ is a sequence of real numbers and $p>0$ then $l_p(x)=l_p(x_i)=\left(\sum_{i=1}^{\infty}|x_i|^p\right)^{1/p}$ and $l_\infty(x)=l_\infty(x_i)=\max|x_i|$. If $x=(x_i)$ is a sequence of elements of a Banach space X then $w_p(x)=w_p(x_i)=\sup\{l_p(\langle x_i, x'\rangle);\ x'\in X',\ \|x'\|\leqq 1\}$. Now let p, q, r be positive numbers, n a natural number and $A: X\to Y$ an operator between Banach spaces X and Y. Then $\mathbb{P}^{(n)}_{(p,q,r)}(A)$ is the infimum from the constants K such that

$$l_p(\langle Ax_i, y_i'\rangle_{i=1}^n)\leqq K w_q((x_i)_{i=1}^n)\cdot w_r((y_i')_{i=1}^n)$$

for all $x_1,\ldots,x_n\in X$ and all $y_1',\ldots,y_n'\in Y'$. As usual, we put $\mathbb{P}_{(p,q,r)}(A)=\sup\mathbb{P}^{(n)}_{(p,q,r)}(A)$ and we denote by $\mathfrak{P}_{(p,q,r)}$ the ideal of absolutely (p,q,r)-summing operators:

$$\mathfrak{P}_{(p,q,r)}=\{A;\ \mathbb{P}_{(p,q,r)}(A)<\infty\}.$$

Now let $(X_1,p_1),\ldots,(X_r,p_r)$ be seminormed spaces. Then we can define the ε- and π-seminorms on the tensor product $X_1\otimes\ldots\otimes X_r$. Namely, let $z=\sum_{i=1}^{n}x_{1i}\otimes\ldots\otimes x_{ri}\in X_1\otimes\ldots\otimes X_r$. Then we put

$$\varepsilon(z)=\varepsilon_{p_1,\ldots,p_r}(z)=\sup\left\{\left|\sum_{i=1}^{n}a_1(x_{1i})\ldots a_r(x_{ri})\right|;\ a_i\in X_i',\ p_i(a_i)\leqq 1\right\}$$

and

$$\pi(z)=\pi_{p_1,\ldots,p_r}(z)=\inf\left\{\sum_{i=1}^{m}p_1(x_{1i})\ldots p_r(x_{ri})\right\},$$

where the infimum is taken over all representations $z=\sum_{i=1}^{m}x_{1i}\otimes\ldots\otimes x_{ri}$.

If $X_1, \ldots, X_r$ are locally convex spaces then the ε- (respectively the π-) topology on $X_1 \otimes \ldots \otimes X_r$ is generated by the system of all seminorms $\{\varepsilon_{p_1, \ldots, p_r}\}$ (respectively $\{\pi_{p_1, \ldots, p_r}\}$), where p_i are continuous seminorms on X_i $(i = 1, \ldots, r)$.

Let $X_1 \otimes_\varepsilon \ldots \otimes_\varepsilon X_r$ denote $X_1 \otimes \ldots \otimes X_r$ with the ε-topology (respectively with the ε-norm if $X_1, \ldots, X_r$ are normed spaces); similar we define $X_1 \otimes_\pi \ldots \otimes_\pi X_r$. Finally we shall say that a locally convex space X is generated by a Banach space E if there are seminorms $\{p_\alpha\}$ on X generating the topology of the space X such that the canonical Banach spaces

$$(\widetilde{X, p_\alpha}) = \text{completion of } X/\{x;\, p_\alpha(x) = 0\}$$

are all isometric to E.

2. Ideals of Multilinear Forms on Banach Spaces

In all of what follows in this paragraph r is a fixed natural number. It seems useful to study the theory of ideals of r-linear operators. It was Pietsch [11] who started this new field. Here we only repeat the basic definitions, give some new ones and state some propositions, which will be suitable for our purposes. Also we restrict ourselves exclusively to the ideal of r-linear forms, $r \geqq 2$. We even do not formulate the notion of the ideal of r-linear forms. We will use only the fact that it is a collection $\mathfrak{A}$ of the components

$$\mathfrak{A}(Y_1, \ldots, Y_r) \subset \mathfrak{L}(Y_1, \ldots, Y_r)$$

with the properties analogous to the usual notion of the ideal of operators. $\mathfrak{L}(Y_1, \ldots, Y_r)$ denotes the vector space of all real (or complex) valued r-linear forms s on the cartesian product $Y_1 x \ldots x Y_r$ of Banach spaces $Y_1, \ldots, Y_r$. Thus, for example if $s \in \mathfrak{A}(Y_1, \ldots, Y_r)$ and $A_i \in L(X_i, Y_i)$ are linear operators $(i = 1, \ldots, r)$ then the composition

$$s \circ (A_1, \ldots, A_r) \in \mathfrak{A}(X_1, \ldots, X_r)$$

is defined by the formula

$$s \circ (A_1, \ldots, A_r)\,(x_1, \ldots, x_r) = s(A_1 x_1, \ldots, A_r x_r)\,.$$

We will consider $\mathfrak{A}$ to be equipped with a quasi-norm $\mathbb{A}$ on each vector space $\mathfrak{A}(Y_1, \ldots, Y_r)$, $\mathbb{A}$ having the properties analogous to the properties of the usual quasi-norm on an operator ideal, e.g.

$$\mathbb{A}(s \circ (A_1, \ldots, A_r)) \leqq \mathbb{A}(s) \cdot \|A_1\| \ldots \|A_r\|\,,$$

$\mathbb{A}(m) = 1$, where m is the r-form of the multiplication: $m(k_1, \ldots, k_r) = \prod_{i=1}^{r} k_i$, each k_i belonging to the scalar field K. An example of such an ideal norm is the usual norm of an r-linear form $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$:

$$\|s\| = \sup\{|s(y_1, \ldots, y_r)|;\ \|y_i\| \leqq 1\}\,. \tag{3}$$

An r-form $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$ is of finite rank n if it can be expressed in the form

$$s = \sum_{i=1}^{n} y'_{1i} \otimes \ldots \otimes y'_{ri}\,, \quad \text{i.e.} \quad s(y_1, \ldots, y_r) = \sum_{i=1}^{n} y'_{1i}(y_1) \ldots y'_{ri}(y_r)\,,$$

where $y'_{ji} \in Y'_j$ $(j=1, \ldots, r)$ and n is minimal for such an expression. Thus our notion of rank n is the "tensor rank n" in the terminology of Pietsch [11]. Let us agree to denote by $\mathfrak{F}$ the ideal of all finite rank r-forms. Evidently,

$$\mathfrak{F}(Y_1, \ldots, Y_r) = Y'_1 \otimes \ldots \otimes Y'_r .$$

Similarly we have in an obvious way the natural canonical imbedding

$$Y_1 \otimes \ldots \otimes Y_r \subset \mathfrak{F}(Y'_r, \ldots, Y'_r) . \tag{4}$$

Thus, if $\mathbb{A}$ is a quasi-norm on $\mathfrak{F}$ then (4) induces a quasi-norm on every tensor product $Y_1 \otimes \ldots \otimes Y_r$.

Every linear operator $A \in L(X, Y)$ gives rise to a 2-form $s_A \in \mathfrak{L}(X, Y')$ such that $s_A(x, y') = \langle Ax, y' \rangle$ and every *2-form* $s \in \mathfrak{L}(X, Y)$ gives rise to an operator $A_s : X \to Y'$ such that $s(x, y) = \langle A_s x, y \rangle$. Thus, the usual operator ideals correspond to the ideals of 2-linear forms.

2.1. Nuclear r-Linear Forms and Approximation Numbers

In this section s will be an element of $\mathfrak{L}(Y_1, \ldots, Y_r)$ and $A_i \in L(X_i, Y_i)$ will be linear operators for all $i = 1, \ldots, r$.

An r-linear form $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$ is called nuclear if it admists a representation

$$s(y_1, \ldots, y_r) = \sum_{i=1}^{\infty} \lambda_i a_{1i}(y_1) \ldots a_{ri}(y_r) , \tag{5}$$

where $\|a_{ji}\| \leqq 1$, $a_{ji} \in Y'_j$ for all $j = 1, \ldots, r$ and

$$\lambda = \sum_{i=1}^{\infty} |\lambda_i| < \infty .$$

The infimum of λ over all representations (5) is denoted by $\mathbb{N}(s)$. If moreover $s \in \mathfrak{F}$ and if we allow in (5) only finite representations we similarly obtain the finite nuclear norm $\mathbb{N}^0(s)$ of s.

2.1.1. Proposition. *For all* $y = \sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri} \in Y_1 \otimes \ldots \otimes Y_r$ *we have*

$$\pi\left(\sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri}\right) = \mathbb{N}^0\left(\sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri}\right) \tag{6}$$

and

$$\varepsilon\left(\sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri}\right) = \left\|\sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri}\right\| . \tag{7}$$

Formula (6) *expresses the fact that* $Y_1 \otimes_\pi \ldots \otimes_\pi Y_r$ *is a normed subspace of* $Y''_1 \otimes_\pi \ldots \otimes_\pi Y''_r$.

Proof. Evidently $\pi(y_1 \otimes \ldots \otimes y_r) = \|y_1\| \ldots \|y_r\|$. This easily implies that the dual of the space $Y_1 \otimes_\pi \ldots \otimes_\pi Y_r$ is $\mathfrak{L}(Y_1, \ldots, Y_r)$ with the usual norm (3). This yields that

$$\pi\left(\sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri}\right) = \sup\left\{\left|\sum_{i=1}^{r} s(y_1, \ldots, y_r)\right| ; \|s\| \leqq 1\right\} . \tag{8}$$

But every $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$ can be extended to $Y_1'' x \ldots x Y_r''$ with the same norm. Thus by (8) $Y_1 \otimes_\pi \ldots \otimes_\pi Y_r$ is a normed subspace of $Y_1'' \otimes_\pi \ldots \otimes_\pi Y_r''$. The definition of $\mathbb{N}^0$ immediately implies that the norm $\mathbb{N}^0$ on $Y_1 \otimes \ldots \otimes Y_r$ is equal to the induced norm from $Y_1'' \otimes_\pi \ldots \otimes_\pi Y_r''$.

(7) is immediate from the definitions.

It is well known that if $A : X \to Y$ is an operator then $\mathbb{N}(A) \leqq 12 \sum_{i=1}^{\infty} a_i(A)$, where $a_i(A)$ are the approximation numbers of A. We want to obtain similar estimates for r-linear forms. To that end we introduce the following definition:

2.1.2. Definition a) Let $n \geqq 1$ be an integer. The n^{th} approximation number $a_n(s)$ of an element $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$ is defined as usual:

$$a_n(s) = \inf\{\|s-t\|;\, t \in \mathfrak{F}(Y_1, \ldots, Y_r), \operatorname{rank} t < n\}.$$

b) If $p > 0$ then the $\mathbf{S}_p$ quasi-norm and the ideal $\mathfrak{S}_p$ of r-forms is defined as usual

$$\mathbf{S}_p(s) = \left(\sum_{n=1}^{\infty} a_n(s)^p \right)^{\frac{1}{p}} \quad \text{and} \quad \mathfrak{S}_p = \{s;\, \mathbf{S}_p(s) < \infty\}.$$

c) If $A_i \in L(X_i, Y_i)$ are linear operators for $i = 1, \ldots, r$, then we also define

$$a_n(A_1, \ldots, A_r) = \sup\{a_n(s \circ (A_1, \ldots, A_r));\ \|s\| \leqq 1, s \in \mathfrak{L}(Y_1, \ldots, Y_r)\}.$$

d) Moreover, if $\mathbb{A}$ is a quasi-norm on an ideal $\mathfrak{A}$ of r-linear forms then we put

$$\mathbb{A}(A_1, \ldots, A_r) = \sup\{\mathbb{A}(s \circ (A_1, \ldots, A_r));\, s \in \mathfrak{F}(Y_1, \ldots, Y_r), \|s\| \leqq 1\}.$$

Evidently we have for $s \in \mathfrak{F}$:

$$\mathbb{A}(s \circ (A_1, \ldots, A_r)) \leqq \|s\| \mathbb{A}(A_1, \ldots, A_r)$$
$$\mathbb{A}(s \circ (A_1, \ldots, A_r)) \leqq \mathbb{A}(s) \cdot \|(A_1, \ldots, A_r)\| = \mathbb{A}(s) \cdot \|A_1\| \ldots \|A_r\|.$$

2.1.3. Proposition. *For all integers $m \geqq 1$ and $n \geqq 1$ we have*

$$a_{m+n-1}(s \circ (A_1, \ldots, A_r)) \leqq a_m(s) \cdot a_n(A_1, \ldots, A_r)$$

and

$$a_{m+n-1}(A_1 \circ B_1, \ldots, A_r \circ B_r) = a_m(A_1, \ldots, A_r) \cdot a_n(B_1, \ldots, B_r).$$

(Here $B_i \in L(Z_i, X_i)$ are linear operators.)

Proof. Given $\varepsilon > 0$, let $s_m \in \mathfrak{F}(Y_1, \ldots, Y_r)$ be such that $\|s - s_m\| \leqq (1+\varepsilon) a_m(s)$ and $\operatorname{rank} s_m < m$. If we put $r = \|s - s_m\|^{-1}(s - s_m) \circ (A_1, \ldots, A_r)$ then we can find $r_n \in \mathfrak{F}(X_1, \ldots, X_r)$ such that $\|r - r_n\| \leqq (1+\varepsilon) a_n(r)$ and $\operatorname{rank} r_n < n$. Evidently, the rank of $v = s_m \circ (A_1, \ldots, A_r) - \|s - s_m\| r_n$ is smaller than $m + n - 1$. This in turn implies that

$$\begin{aligned} a_{m+n-1}(s \circ (A_1, \ldots, A_r)) &\leqq \|s \circ (A_1, \ldots, A_r) - v\| \\ &\leqq \|s - s_m\| \cdot \|r - r_n\| \leqq (1+\varepsilon)^2 a_m(s) \cdot a_n(r). \end{aligned}$$

From the definition it follows that $a_n(r) \leqq a_n(A_1, \ldots, A_n)$.

The second inequality is proved similarly: first, we choose an $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$, $\|s\| \leqq 1$ such that

$$a_{m+n-1}(A_1 \circ B_1, \ldots, A_r \circ B_r) \leqq (1+\varepsilon) a_{m+n-1}(s \circ (A_1 \circ B_1, \ldots, A_r \circ B_r)).$$

Then we choose $t_m \in \mathfrak{F}(X_1, \ldots, X_r)$ such that $\operatorname{rank} t_m < m$ and such that

$$\|s \circ (A_1, \ldots, A_r) - t_m\| \leqq (1+\varepsilon) a_m(s \circ (A_1, \ldots, A_r)).$$

We put $p = s \circ (A_1, \ldots, A_r) - t_m$ and $r = \|p\|^{-1} \cdot p \circ (B_1, \ldots, B_r)$. Finally, we choose $r_n \in \mathfrak{F}(Z_1, \ldots, Z_r)$, $\operatorname{rank} r_n < n$, such that $\|r - r_n\| = (1+\varepsilon) a_n(r)$. Then the rank of $t_m \circ (B_1, \ldots, B_r) + \|p\| \cdot r_n$ is smaller than $m+n+1$ and we can estimate

$$\begin{aligned} a_{m+n-1}(s \circ (A_1 \circ B_1, \ldots, A_r \circ B_r)) &\leqq \|p \circ (B_1, \ldots, B_r) - \|p\| \cdot r_n\| \\ &= \|p\| \cdot \|r - r_n\| \leqq (1+\varepsilon)^2 a_m(s \circ (A_1, \ldots, A_r)) \cdot a_n(r) \\ &= (1+\varepsilon)^2 a_m(A_1, \ldots, A_r) \cdot a_n(B_1, \ldots, B_r). \end{aligned}$$

We shall need the next lemma which generalizes the well known result for the case $r = 2$.

2.1.4. Lemma. *Let* $t = \sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri} \in Y_1 \otimes \ldots \otimes Y_r$ *be a tensor of* rank n. *Then* t *can be written as*

$$t = \sum_{i=1}^{n^{r-1}} z_{1i} \otimes \ldots \otimes z_{ri}, \tag{9}$$

where $z_{ji} \in Y_j$ *and* $\|z_{1i}\| \ldots \|z_{ri}\| \leqq \varepsilon(t) = \|t\|$ *for all* $j = 1, \ldots, r$ *and* $i = 1, \ldots, n$.

Thus we have

$$\mathbb{N}^0(t) \leqq (\operatorname{rank} t)^{r-1} \cdot \|t\| \quad \text{for all} \quad t \in \mathfrak{F}. \tag{10}$$

Proof. Let $[Y_j]$ be the linear span of $\{y_{j1}, \ldots, y_{jn}\} \subset Y_j$. We choose in the finite-dimensional space $[Y_j]$ the Auerbach's basis $\{z_{j,k_j}\}_{k_j}$ with its biorthogonal forms $\{z'_{j,k_j}\}$. According to the Hahn-Banach theorem we may assume that $z'_{j,k_j} \in Y'_j$ and $\|z'_{j,k_j}\| = 1$. Then $y_{ji} = \sum_{k_j} z'_{j,k_j}(y_{ji}) z_{j,k_j}$ and we obtain

$$t = \sum_{i=1}^{n} y_{1i} \otimes \left(\sum_{k_2} z'_{2,k_2}(y_{2i}) z_{2,k_2} \right) \otimes \ldots \otimes \left(\sum_{k_r} z'_{r,k_r}(y_{ri}) z_{r,k_r} \right).$$

If we put $y_{k_2,\ldots,k_r} = \sum_{i=1}^{n} z'_{2,k_2}(y_{2i}) \ldots z'_{r,k_r}(y_{ri}) \cdot y_{1i}$ then $y_{k_2,\ldots,k_r} \in Y_1, \|y_{k_2,\ldots,k_r}\| \leqq \|t\|$ and

$$t = \sum_{k_2,\ldots,k_r} y_{k_2,\ldots,k_r} \otimes y_{2,k_2} \otimes \ldots \otimes y_{r,k_r}$$

which implies (9) if we suitably change the notation.

2.1.5. Proposition. *Every* $s \in \mathfrak{S}_p(Y_1, \ldots, Y_r)$ *can be written*

$$s = \sum_{i=1}^{\infty} \lambda_i a_{1li} \otimes \ldots \otimes a_{ri},$$

where $a_{ji} \in Y_j'$, $\|a_{ji}\| \leqq 1$ *and* $(\lambda_i) \in l_p$ *is such that*

$$\left(\sum_{i=1} |\lambda_i|^p\right)^{\frac{1}{p}} \leqq 2 \cdot 6^{\frac{r-1}{p}} \mathbf{S}_{\frac{p}{r-1}}(s)$$

Moreover, if $s \in \mathfrak{F}$ *and* $\operatorname{rank} s < 2^k$ *then* $\lambda_i = 0$ *if* $i > 3^{r-1}(2^{k(r-1)} - 1)$.

The *proof* is quite similar to that of the well known case $r = 2$, but we repeat it here for the convenience of the reader.

We choose $\varepsilon_n > 0$. For every natural number n let us choose $s_n \in \mathfrak{F}(Y_1, \ldots, Y_r)$ with $\operatorname{rank} s_n < 2^n$ and $\|s - s_n\| < (1 + \varepsilon_n) a_{2^n}(s)$. Put $s_0 = 0$ and $u_n = s_{n+1} - s_n$; then, $\dim u_n < 3 \cdot 2^n$. From $\lim a_n(s) = 0$ we obtain that $\lim \|s - s_n\| = 0$; hence $s_n = \sum_{k=1}^{n} s_k - s_{k-1}$ implies that $s = \lim_n \sum_{k=1}^{n} u_{k-1} = \sum_{k=0}^{\infty} u_k$. On the basis of 2.1.4. we may write

$$u_n = \sum_{i=1}^{d_n} \lambda_{ni} a_{n1i} \otimes a_{n2i} \otimes \ldots \otimes a_{nri},$$

where $a_{nji} \in Y_j'$, $\|a_{nji}\| \leqq 1$, $0 \leqq \lambda_{ni} \leqq \|u_n\|$ and $d_n \leqq (3 \cdot 2^n)^{r-1} - 1$. Next, we observe that

$$\begin{aligned}\sum_{i=1}^{d_n} \lambda_{ni}^p &\leqq d_n \|u_n\|^p \leqq 3^{r-1} \cdot 2^{n(r-1)}(\|s - s_{n+1}\| + \|s - s_n\|)^p - \|u_n\|^p \\ &\leqq (1 + \varepsilon_n)^p \cdot 3^{r-1} \cdot 2^{n(r-1)+p} a_{2^n}(s)^p - \|u_n\|^p \\ &\leqq 2^p \cdot 6^{r-1} \cdot 2^{(n-1)(r-1)} a_{2^n}(s)^p.\end{aligned}$$

if ε_n are sufficiently small.
Finally we have

$$s = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \sum_{i=1}^{d_n} \lambda_{ni} a_{n1i} \otimes a_{n2i} \otimes \ldots \otimes a_{nri},$$

where

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{i=1}^{d_n} \lambda_{ni}^p &\leqq 2^p \cdot 6^{r-1} \sum_{n=0}^{\infty} 2^{(n-1)(r-1)} a_{2^n}(s)^p \\ &\leqq 2^p \cdot 6^{r-1} \left(\sum_{n=0}^{\infty} 2^{n-1} a_{2^n}(s)^{\frac{p}{r-1}}\right)^{r-1} \\ &\leqq 2^p \cdot 6^{r-1} \left(\sum_{n=1}^{\infty} a_n(s)^{\frac{p}{r-1}}\right)^{r-1}.\end{aligned}$$

Moreover, if $s \in \mathfrak{F}$, $\operatorname{rank} s < 2^k$, then $a_{2^k}(s) = 0$ and we may choose $s_k = s$, i.e. $u_k = 0$. Thus we obtain

$$s = s_k = \sum_{l=1}^{k} u_{l-1} = \sum_{l=0}^{k-1} u_l$$

and $\lambda_i = 0$ for

$$i > \sum_{l=0}^{k-1} d_l \leqq \sum_{l=0}^{k-1} (3 \cdot 2^l)^{r-1} = 3^{r-1} \sum_{l=0}^{k-1} 2^{l(r-1)} = 3^{r-1}(2^{k(r-1)} - 1).$$

2.1.6. Corollary. a) $\mathbf{N}^0(s) \leqq 2 \cdot 6^{r-1} \cdot \mathbf{S}_{\frac{1}{r-1}}(s)$ *for all* $s \in \mathfrak{F}(Y_1, \ldots, Y_r)$.

b) $\mathbf{N}^0(A_1, \ldots, A_r) \leqq 2 \cdot 6^{r-1} \mathbf{S}_{\frac{1}{r-1}}(A_1, \ldots, A_r)$ *for all linear operators* $A_1, \ldots, A_r$.

2.2. The Inequality of Horn and von Neumann

The multiplicativity of the approximation numbers of operators and the correspondence between operators and 2-forms imply that in the case $r=2$ we have the following estimate

$$\mathbf{S}_p(A_1, A_2) \leqq 2^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} a_n(A_1)^p a_n(A_2)^p \right)^{\frac{1}{p}},$$

which is a generalization of a result proved by Horn and von Neumann. We want to obtain a generalization of this inequality for arbitrary integer $r \geqq 2$.

Suppose we have fixed operators $A_1, \ldots, A_r$ and $K_1, \ldots, K_r$, $A_i \in L(X_i, Y_i)$, $K_i \in L(X_i, Y_i)$ and an r-linear form $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$. Now let $K \subset \{1, \ldots, r\}$ be a set of integers. Let us agree to put $A_i^K = B_i^K = A_i$ if $i \notin K$, $A_i^K = A_i - K_i$ if $i \in K$ and $B_i^K = K_i$ if $i \in K$ for all $i = 1, \ldots, r$. Furthermore, we put

$$S_0 = s \circ (A_1, \ldots, A_r),$$

$$S_1 = \sum_{i=1}^{r} s \circ (A_1^{\{i\}}, \ldots, A_r^{\{i\}}) = \sum_{i=1}^{r} s \circ (A_1, \ldots, A_{i-1}, A_i - K_i, A_{i+1}, \ldots, A_r),$$

and generally

$$S_k = \sum_{|K|=k} s \circ (A_1^K, \ldots, A_r^K),$$

where $|K|$ is the cardinality of the set $K \subset \{1, \ldots, r\}$. Similarly we put

$$T_k = \sum_{|K|=k} s \circ (B_1^K, \ldots, B_r^K).$$

Thus, $S_0 = T_0$ and $S_r = s \circ (A_1 - K_1, \ldots, A_r - K_r)$, $T_r = s \circ (K_1, \ldots, K_r)$. Now we claim that

$$S_k = \sum_{k=0}^{k} (-1)^j \binom{r-j}{k-j} T_j \quad \text{for all} \quad k = 0, 1, \ldots, r. \tag{11}$$

Indeed, the equality $s \circ (A_1^K, \ldots, A_r^K) = \sum_{M \subset K} (-1)^{|M|} s \circ (B_1^M, \ldots, B_r^M)$ implies that

$$\begin{aligned} S_k &= \sum_{|K|=k} s \circ (A_1^K, \ldots, A_r^K) \\ &= \sum_{j=0}^{k} (-1)^j \left(\sum_{|K|=k} \sum_{\substack{M \subset K \\ |M|=j}} \right) s \circ (B_1^M, \ldots, B_r^M) \\ &= \sum_{j=1}^{k} (-1)^j \binom{r-j}{k-j} \sum_{|M|=j} s \circ (B_1^M, \ldots, B_r^M) \\ &= \sum_{j=0}^{k} (-1)^j \binom{r-j}{k-j} T_j . \end{aligned}$$

If $k=0$ then (11) reduces to $S_0 = T_0$ and if $k=1$ then (11) gives $S_1 + T_1 = rT_0$ which is obviously true. If we had written $T_j - K_j$ instead of K_j then the roles of T_k and S_k would have changed and thus we may also write

$$T_k = \sum_{j=0}^{k} (-1)^j \binom{r-j}{k-j} S_j \quad \text{for all} \quad k = 0, 1, \ldots, r. \tag{12}$$

Thus $T_{r-1}=\sum_{j=0}^{r-1}(-1)^j(r-j)S_j=\sum_{j=0}^{r}(-1)^j(r-j)S_j$ and $T_r=\sum_{j=0}^{r}(-1)^jS_j$. This in turn yields

$$T_{r-1}-(r-1)T_r=S_0+\sum_{j=2}^{r}(-1)^j(r-j-(r-1))S_j,$$

or

$$T_0-(T_{r-1}-(r-1)T_r)=\sum_{j=2}^{r}(-1)^j(j-1)S_j\,. \tag{13}$$

The following lemma is the key to the next proposition and its proof is obvious.

2.2.1. Lemma. *Let K_j be the finite-dimensional operators,* $\dim K_j=k_j$ $(j=1,\ldots,r)$. *Then, for all $1\leqq i\leqq r$ we have*

$$\operatorname{rank} s\circ(K_1,\ldots,K_{i-1},A_i,K_{i+1},\ldots,K_r)\leqq\prod_{j\neq i}k_j\,.$$

Now we observe that

$$T_{r-1}-(r-1)T_r=\sum_{i=1}^{r}s\circ(K_1,\ldots,K_1,A_i-(r-1)r^{-1}K_i,K_{i+1},\ldots,K_r)$$

and thus

$$\operatorname{rank}(T_{r-1}-(r-1)T_r)\leqq\sum_{i=1}^{r}\prod_{j\neq i}\dim K_j\,. \tag{14}$$

Given arbitrary $\varepsilon>0$ we choose $K_j\in L(X_j,Y_j)$, such that $\dim K_j<k_j$ and $\|A_j-K_j\|<(1+\varepsilon)a_{k_j}(A_j)$. Then (14) implies that

$$\operatorname{rank}(T_{r-1}-(r-1)T_r)\leqq\sum_{i=1}^{r}\prod_{j\neq i}(k_j-1)=k\,. \tag{15}$$

The relations (13), (15) now imply that

$$a_{k+1}(s\circ(A_1,\ldots,A_r))\leqq\sum_{j=2}^{r}(j-1)\,\|S_j\|\,. \tag{16}$$

Suppose now that $\|A_j\|\leqq 1$ for all $j=1,\ldots,r$. By the definition of S_j we may estimate

$$\begin{aligned}\|S_l\|&\leqq\sum_{|K|=l}\|s\circ(A_1^K,\ldots,A_1^K)\|\leqq\|s\|\sum_{|K|=l}\prod_{i\in K}\|A_i-K_i\|\\&\leqq(1+\varepsilon)^l\cdot\|s\|\sum_{|K|=l}\prod_{i\in K}a_{k_i}(A_i)\,.\end{aligned}$$

As $a_{k_i}(A_i)\leqq a_1(A_i)=\|A_i\|\leqq 1$ we may further estimate as in the proof of (11):

$$\begin{aligned}\|S_l\|&\leqq(1+\varepsilon)^l\cdot\|s\|\cdot\sum_{|K|=l}\prod_{i\in K}a_{k_i}(A_i)\\&\leqq(1+\varepsilon)^l\cdot\|s\|\sum_{i\neq j}\Bigg(\sum_{\substack{|K|=l\\ i,j\in K}}a_{k_i}(A_i)a_{k_j}(A_j)\Bigg)\\&=(1+\varepsilon)^l\cdot\|s\|\binom{r-2}{l-2}\sum_{i\neq j}a_{k_i}(A_i)a_{k_j}(A_j)\,.\end{aligned}$$

Thus we have proved:

2.2.2. Proposition. *Let k_j be natural numbers, $k=\sum_{i=1}^{r}\prod_{j\neq i}(k_j-1)$ and let A_j be linear operators, $\|A_j\|\leqq 1$ for all $j=1,\ldots,r$. Then*

$$a_{k+1}(A_1,\ldots,A_r)\leqq \sum_{l=2}^{r}(l-1)\sum_{|K|=l}\prod_{i\in K}a_{k_i}(A_i)$$
$$\leqq \sum_{l=2}^{r}(l-1)\binom{r-2}{l-2}\sum_{i\neq j}a_{k_i}(A_i)a_{k_j}(A_j)\,.$$

(K is a subset of the set $\{1,\ldots,r\}$.)

If we choose $k_j=n$ then especially we have

$$a_{r(n-1)^{r-1}+1}(A_1,\ldots,A_r)\leqq C(r)\sum_{i\neq j}a_n(A_i)a_n(A_j)\,,$$

where $C(r)=\sum_{l=2}^{r}(l-1)\binom{r-2}{l-2}$.

2.2.3. Corollary. *Let $p>0$ and let A_j be linear operators $\|A_j\|\leqq 1$ for all $j=1,\ldots,r$. Then there are constants C_1, C_2 depending on p and r only such that we may write the following estimates:*

$$\mathbf{S}_p(A_1,\ldots,A_r)$$
$$=\sup_{\|s\|\leqq 1}\left\{\sum_{n=1}^{\infty}a_n(s\circ(A_1,\ldots,A_r))^p\right\}^{\frac{1}{p}}$$
$$\leqq \sup_{\|s\|\leqq 1}\left\{\sum_{n=1} r\cdot(r-1)\cdot n^{r-2}a_{r(n-1)^{r-1}+1}(s\circ(A_1,\ldots,A_r))^p\right\}^{\frac{1}{p}}$$
$$\leqq r^{\frac{2}{p}}\left\{\sum_n n^{r-2}\left(\sum_{l=2}^{r}(l-1)\sum_{|K|=l}\prod_{i\in K}a_n(A_i)\right)^p\right\}^{\frac{1}{p}}$$
$$\leqq C_1(p,r)\sum_{l=2}^{r}(l-1)\sum_{|K|=l}\left\{\sum_n n^{r-2}\prod_{i\in K}a_n(A_i)^p\right\}^{\frac{1}{p}}$$
$$\leqq C_1(p,r)\sum_{l=2}^{r}(l-1)\sum_{i\neq j}\sum_{\substack{|K|=l\\ i,j\in K}}\left\{\sum_n n^{r-2}a_n(A_i)^p a_n(A_j)^p\right\}^{\frac{1}{p}}$$
$$=C_1(p,r)\sum_{l=2}^{r}(l-1)\binom{r-2}{l-2}\sum_{i\neq j}\left\{\sum_n n^{r-2}a_n(A_i)^p a_n(A_j)^p\right\}^{\frac{1}{p}}$$
$$=C_2(p,r)\sum_{i\neq j}\left(\sum_n a_n(A_i)^q\cdot a_n^q(A_j)\right)^{\frac{1}{q}},$$

where $\dfrac{1}{q}-\dfrac{1}{p}=r-2.$

We have used the fact that $\sum_{i=1}^{\infty} a_i \leqq \sum_{n=1}^{\infty} (i_{n+1}-i_n)\cdot a_{i_n}$, where $i_n = r(n-1)^{r-1}+1$, so that $i_{n+1}-i_n = r[n^{r-1}-(n-1)^{r-1}] \leqq r(r-1)n^{r-2}$.

Also we have used here the fact that

$$l_{q,p}(x_i) = \sum_{i=1} i^{\frac{1}{q}-\frac{1}{p}} |x_i|^{*p}$$

defines a quasi-norm on the corresponding Lorenz sequence space and $l_{q,p}(x_i) \leqq l_{q,q}(x_i) = l_q(x_i)$ if $q \leqq p$.

2.2.4. Theorem. *There is a real function $K(r)$ such that*

$$\mathbb{N}^0(A_1, \ldots, A_r) \leqq K(r) \sum_{i \neq j} \left\{ \sum_{n=1} (a_n(A_i) a_n(A_j))^{\frac{1}{2r-3}} \right\}^{2r-3}.$$

Proof. We use 2.1.6. b) and the above result, where we put $p=(r-1)^{-1}$.

2.3. Adjoint Norms

Let $\mathbb{A}$ be a norm on an ideal of r-linear forms $\mathfrak{A}$. If $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$ and $t \in Y_1 \otimes \ldots \otimes Y_r$ then we have the natural pairing $\langle s, t\rangle$:

$$\langle s, t\rangle = \langle t, s\rangle = \sum_{i=1}^{n} s(y_{1i}, \ldots, y_{ri}),$$

where

$$t = \sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri}.$$

We can define

$$\mathbb{A}^{\triangle}(s) = \sup\{|\langle s, t\rangle|;\ \mathbb{A}(t) \leqq 1, t \in Y_1 \otimes \ldots \otimes Y_r\}.$$

Thus we have

$$|\langle s, t\rangle| \leqq \mathbb{A}^{\triangle}(s) \cdot \mathbb{A}(t) \quad \text{if} \quad t \in Y_1 \otimes \ldots \otimes Y_r.$$

More convenient for us is the adjoint "norm" $\mathbb{A}^*$, which we now define:

$$\mathbb{A}^*(s) = \sup\{|\langle s \circ (A_1, \ldots, A_r), t\rangle|\},$$

where the supremum is taken over all finite operators $A_i \in L_{W^*}(X_i', Y_i) = X_i \otimes Y_i$, $\|A_i\| \leqq 1$ for $i = 1, \ldots, r$ and $t \in \mathfrak{A}(X_1, \ldots, X_r)$, $\mathbb{A}(t) \leqq 1$.

Evidently we have for all $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$ and all $t \in \mathfrak{L}(X_1, \ldots, X_r)$

$$\begin{aligned} \langle s \circ (A_1, \ldots, A_r), t\rangle &= \langle s, t \circ (A_1', \ldots, A_r')\rangle \\ &= \sum_{i_1, \ldots, i_r} s(y_{i_1}, \ldots, y_{i_r}) \cdot t(x_{i_1}, \ldots, x_{i_r}), \end{aligned}$$

where $A_j = \sum_{i_j} x_{i_j} \otimes y_{i_j}$ are the finite representations of A_j. (Observe that $t \circ (A_1', \ldots, A_r') \in Y_1 \otimes \ldots \otimes Y_r$ and thus $\langle s, t \circ (A_1', \ldots, A_r')\rangle$ makes sense.)

Now, if $t \in \mathfrak{A}(X_1, \ldots, X_r)$ and A_i are finite operators then

$$|\langle s \circ (A_1, \ldots, A_r), t\rangle| \leqq \mathbb{A}^*(s) \cdot \|A_1\| \ldots \|A_r\| \mathbb{A}(t). \tag{17}$$

On the other hand, if $s \in \mathfrak{A}(Y_1, \ldots, Y_r)$, then

$$|\langle s \circ (A_1, \ldots, A_r), t\rangle| = |\langle t(A_1', \ldots, A_r'), s\rangle| \leqq \mathbb{A}^*(t)\, \|A_1\| \ldots \|A_r\| \cdot \mathbb{A}(s)$$

This in turn implies

$$\mathbb{A}^{**}(s) \leqq \mathbb{A}(s) \quad \text{for all} \quad s \in \mathfrak{A}. \tag{18}$$

The proof of the inequality

$$\mathbb{A}^*(s) \leqq \mathbb{A}^{\triangle}(s) \quad \text{for all} \quad s \in \mathfrak{L} \tag{19}$$

is left to the reader.

We define the integral "norm" of $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$:

$$\mathbb{I}(s) = \|s\|^{\triangle} = \sup\{|\langle s, t\rangle|;\ \|t\| \leqq 1, t \in Y_1 \otimes \ldots \otimes Y_r\}.$$

2.3.1. Proposition. $\mathbb{I}(s) = \|s\|^*$ *for all* $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$.

Proof. Let
$$t = \sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri} \subset Y_1 \otimes \ldots \otimes Y_r \subset \mathfrak{F}(Y_1', \ldots, Y_r').$$

Let $X_j = \operatorname{span}\{y_{ji}\}_{i=1}^n$ and let t_x be t considered as an element of

$$X_1 \otimes \ldots \otimes X_r \subset \mathfrak{L}(X_1', \ldots, X_r').$$

Evidently we have $t = t_x \circ (A_1', \ldots, A_r')$ where $A_j : X_j \to Y_j$ are the imbeddings. This in turn implies

$$|\langle s, t\rangle| = |\langle s, t_x \circ (A_1', \ldots, A_r')\rangle| \leqq \|s\|^* \cdot \|t\|.$$

From the definition of $\|s\|^{\triangle}$ we obtain $\|s\|^{\triangle} \leqq \|s\|^*$. The equality now follows from (19).

2.3.2. $\mathbb{N}^{0*}(s) = \mathbb{N}^{0\triangle}(s) = \|s\|$ *for all* $s \in \mathfrak{L}(Y_1, \ldots, Y_r)$.

Proof. Let $t = \sum_{i=1}^{n} y_{1i} \otimes \ldots \otimes y_{ri} \in Y_1 \otimes \ldots \otimes Y_r$. Then

$$|\langle s, t\rangle| = \left|\sum_{i=1}^{n} s(y_{1i}, \ldots, y_{ri})\right| = \|s\| \sum_{i=1}^{n} \|y_{1i}\| \ldots \|y_{ri}\|,$$

which implies that $|\langle s, t\rangle| = \|s\| \cdot \mathbb{N}^0(t)$. This means that $\mathbb{N}^{0\triangle}(s) \leqq \|s\|$.

On the other hand, given $\varepsilon > 0$, there are $y_j \in Y_j$, $\|y_j\| = 1$ such that $\|s\| \leqq |s(y_1, \ldots, y_r)| + \varepsilon$. Let X_j be the one dimensional subspace of Y_j spanned by y_j and let $A_j : X_j \to Y_j$ be the inclusions. Then

$$\begin{aligned} |s(y_1, \ldots, y_r)| &= |s \circ (A_1, \ldots, A_r)(y_1, \ldots, y_r)| \\ &= |\langle s \circ (A_1, \ldots, A_r), y_1 \otimes \ldots \otimes y_r\rangle| \\ &\leqq \mathbb{N}^{0*}(s) \cdot \mathbb{N}^0(y_1 \otimes \ldots \otimes y_r) = \mathbb{N}^{0*}(s). \end{aligned}$$

This shows that $\|s\| = \mathbb{N}^{0*}(s)$. Using also (19) we obtain 2.3.2.

The following is also well known for $r = 2$.

2.3.3. Proposition. *Let $s \in \mathfrak{L}(Y_1, \dots, Y_r)$ and let all but one spaces $Y_1, \dots, Y_r$ be finite-dimensional. Then $s \in \mathfrak{F}$ and $\mathbb{I}(s) = \mathbb{N}(s) = \mathbb{N}^0(s)$.*

Proof. We suppose that $Y_2, \dots, Y_r$ are finite-dimensional with biorthogonal bases

$$(y_{i_2}, y'_{i_2}), \dots, (y_{i_r}, y'_{i_r}) \quad \text{in} \quad Y_2, \dots, Y_r,$$

respectively. Evidently $\mathfrak{L}(Y_1, \dots, Y_r) = \mathfrak{F}(Y_1, \dots, Y_r)$ and thus the norm $\mathbb{N}^0$ is defined on $\mathfrak{L}(Y_1, \dots, Y_r)$. Applying the Hahn-Banach theorem, we can find a functional L on $\mathfrak{L}(Y_1, \dots, Y_r)$ such that $L(s) = \mathbb{N}^0(s)$ and $|L(t)| \leqq \mathbb{N}^0(t)$ for all $t \in \mathfrak{L}(Y_1, \dots, Y_r)$.

Let $y'_j \in Y'_j$ for all $j = 1, \dots, r$. Then,

$$\begin{aligned} &L(y'_1 \otimes \dots \otimes y'_r) \\ &= \sum_{i_2, \dots, i_r} y'_2(y_{i_2}) y'_3(y_{i_3}) \dots y'_r(y_{i_r}) L(y'_1 \otimes y'_{i_2} \otimes \dots \otimes y'_{i_r}). \end{aligned}$$

If we define

$$f_{i_2, \dots, i_r}(y') = L(y' \otimes y'_{i_2} \otimes y'_{i_3} \otimes \dots \otimes y'_{i_r})$$

for $y' \in Y'_1$ then $f_{i_2, \dots, i_r} \in Y''_1$ and we may write

$$\begin{aligned} L = l = \sum_{i_2, \dots, i_r} & f_{i_2, \dots, i_r} \otimes y_{i_2} \\ & \otimes \dots \otimes y_{i_r} \in Y''_1 \otimes Y_2 \otimes \dots \otimes Y_r. \end{aligned}$$

Thus $L(t) = \langle l, t \rangle$ for all $t \in \mathfrak{L}(Y_1, \dots, Y_r)$. It follows from

$$\begin{aligned} &\langle l, y'_1 \otimes \dots \otimes y'_r \rangle = L(y'_1 \otimes \dots \otimes y'_r) \\ &\leqq \mathbb{N}^0(y'_1 \otimes \dots \otimes y'_r) \leqq \|y'_1\| \dots \|y'_r\| \quad \text{that} \quad \|l\| \leqq 1. \end{aligned}$$

Since $s \in Y'_1 \otimes \dots \otimes Y'_r$ and $l \in Y''_1 \otimes \dots \otimes Y''_r$ we have

$$\mathbb{N}^0(s) = L(s) = \langle s, l \rangle \leqq \mathbb{I}(s) \, \|l\| \leqq \mathbb{I}(s).$$

As in the case $r = 2$, we easily obtain that $\mathbb{N}^0$ is the greatest norm on $\mathfrak{F}(Y_1, \dots, Y_r)$, and thus

$$\mathbb{I}(s) \leqq \mathbb{N}(s) \leqq \mathbb{N}^0(s) \quad \text{for all} \quad s \in \mathfrak{F}. \tag{20}$$

2.3.4. Proposition. (See [6, 8d)].) *Let $A: X \to Y$ be an operator. Then*

$$\mathbb{P}^r_{(r,r,r)}(A) \leqq \mathbb{I}(A_1, \dots, A_r) \leqq \mathbb{N}^0(A_1, \dots, A_r), \tag{21}$$

where $A_1 = \dots = A_r = A$.

Proof. Let (a_{ij}) be an (n, n)-matrix. Then evidently (see e.g. [5, (6)])

$$\left| \sum_{i=1}^n a_{ii} \right| \leqq \sup \left\{ \sum_{i,j=1}^n a_{ij} t_i s_j ; \, |t_i| \leqq 1, |s_j| \leqq 1 \right\}. \tag{22}$$

Suppose we have arbitrary $x_1, \dots, x_n \in X$ and $y'_1, \dots, y'_n \in Y'$. Let the numbers α_i be such that $\alpha_i \langle A x_i, y'_i \rangle = |\langle A x_i, y'_i \rangle|$ and put $a_{ij} = \alpha_i^r \langle A x_i, y'_j \rangle^r$. Let us define

$$s = \sum_{j=1}^n s_j y'_j \otimes \dots \otimes y'_j \in \mathfrak{L}(Y_1, \dots, Y_r).$$

It follows from (22) that

$$\begin{aligned}
&\sum_{i=1}^{n} |\langle Ax_i, y_i'\rangle|^r \\
&\leqq \sup\left\{\sum_{i,j=1}^{n} \alpha_i^r \langle Ax_i, y_j'\rangle^r t_i s_j;\ |t_i|\leqq 1, |s_j|\leqq 1\right\} \\
&= \sup\left\{\sum_{i=1}^{n} \alpha_i^r t_i s(A_1 x_i, \ldots, A_r x_i)\right\} \\
&= \sup\left\langle s\circ(A_1, \ldots, A_t), \sum_{i=1}^{n} \alpha_i^r t_i x_i \otimes \ldots \otimes x_i\right\rangle \\
&\leqq \sup \mathbb{I}(s\circ(A_1, \ldots, A_r)) \cdot \left\|\sum_{i=1}^{n} \alpha_i^r t_i x_i \otimes \ldots \otimes x_i\right\|.
\end{aligned}$$

Evidently,

$$\mathbb{I}(s\circ(A_1, \ldots, A_r)) \leqq \|s\|\, \mathbb{I}(A_1, \ldots, A_r), \qquad \|s\| \leqq w_r^r(y_j')$$

and

$$\left\|\sum_{i=1}^{n} \alpha_i^r t_i x_i \otimes \ldots \otimes x_i\right\| \leqq w_r^r(x_i).$$

2.3.5. Proposition. *For every integer $r \geqq 2$ and every operators $A_1, \ldots, A_r$ we have*

$$\prod_{k=1}^{r} \mathbb{P}^{(n)}_{(1, p_k, q_k)}(A_k) \leqq n^{r-1} \mathbb{I}(A_1, \ldots, A_r) \leqq n^{r-1} \mathbb{N}^0(A_1, \ldots, A_r), \tag{23}$$

where $\sum_{k=1}^{r} \frac{1}{p_k} = \sum_{k=1}^{r} \frac{1}{q_k} = 1$ with all p_k and q_k positive.

Proof. Let us denote by $*$ the sum of integers modulo n. Then we have for all numbers $a_{j,i}$

$$\prod_{j=1}^{r} \sum_{i=1}^{n} a_{j,i} = \sum_{i=1}^{n} \sum_{i_2, \ldots, i_r = 1}^{n} a_{1,i} \cdot a_{2,i*i_2} \cdot a_{3,i*i_3} \ldots a_{r,i*i_r}. \tag{24}$$

Now, let $x_{k,i} \in X_k$, $y_{k,i}' \in Y_k'$ for $k=1, \ldots, r$ and $i=1, \ldots, n$, and let the numbers $\alpha_{k,i}$ be such that $\alpha_{k,i}\langle A_k x_{k,i}, y_{k,i}'\rangle = |\langle A_k x_{k,i}, y_{k,i}'\rangle|$. If $a=(s_1, \ldots, s_n)$ is an n-tuple of numbers s_j, we define the r-linear forms $t_{a,i_2,\ldots,i_r} \in \mathfrak{L}(Y_1, \ldots, Y_r)$. Namely we put

$$t_{a,i_2,\ldots,i_r} = \sum_{j=1}^{n} s_j y_{1,j}' \otimes y_{2,j*i_2}' \otimes \ldots \otimes y_{r,j*i_r}',$$

and

$$\mathrm{a}_{ij} = \sum_{i_2, \ldots, i_r = 1}^{n} \alpha_{1,i} \langle A_1 x_{1,i}, y_{1,j}'\rangle \cdot \prod_{k=2}^{n} \alpha_{k,i*i_k} \langle A_k x_{k,i*i_k}, y_{k,j*i_k}'\rangle.$$

Using (24) and (22) we have

$$\begin{aligned}
&\prod_{k=1}^{r}\sum_{i=1}^{n}|\langle A_k x_{k,i}, y'_{k,i}\rangle| \\
&= \sum_{i=1}^{n}\sum_{i_2,\dots,i_r=1}^{n}\alpha_{1,i}\langle A_1 x_{1,i}, y'_{1,i}\rangle\prod_{k=2}^{r}\alpha_{k,i*i_k} \\
&\quad\cdot\langle A_k x_{k,i*i_k}, y'_{k,i*i_k}\rangle \\
&= \sum_{i=1}^{n} a_{ii} \leqq \sup\left\{\sum_{i,j=1}^{n} a_{ij}t_i s_j \,|t_i|\leqq 1, |s_j|\leqq 1\right\} \\
&\leqq \sup\Big\{\sum_{i,i_2,\dots,i_r} t_{a,i_2,\dots,i_r}(t_i\alpha_{1,i}A_1 x_{1,i}, \alpha_{2,i*i_2} \\
&\quad\cdot A_2 x_{2,i*i_2}, \dots, \alpha_{r,i*i_r}A_r x_{r,i*i_r})\Big\} \\
&= \sup\Big\{\sum_{i_2,\dots,i_r}\langle t_{a,i_2,\dots,i_r}\circ(A_1,\dots,A_r), s_{b,i_2,\dots,i_r}\rangle; \\
&\quad |t_i|\leqq 1, |s_i|\leqq 1\Big\},
\end{aligned}$$

where we have put $b=(t_1,\dots,t_n)$ and

$$\begin{aligned}
s_{b,i_2,\dots,i_r} = \sum_{i=1}^{n} t_i\alpha_{1,i}x_{1,i}\otimes\alpha_{2,i*i_2}x_{2,i*i_2} \\
\otimes\dots\otimes\alpha_{r,i*i_r}x_{r,i*i_r}\in X_1\otimes\dots\otimes X_r.
\end{aligned}$$

But

$$\begin{aligned}
&\langle t_{a,i_2,\dots,i_r}\circ(A_1,\dots,A_r), s_{b,i_2,\dots,i_r}\rangle \\
&\quad\leqq \mathbb{I}(t_{a,i_2,\dots,i_r}\circ(A_1,\dots,A_r)\cdot\|s_{b,i_2,\dots,i_r}\| \\
&\quad\leqq \mathbb{I}(A_1,\dots,A_r)\,\|t_{a,i_2,\dots,i_r}\|\cdot\|s_{b,i_2,\dots,i_r}\|.
\end{aligned}$$

Evidently,

$$\|s_{b,i_2,\dots,i_r}\|\leqq\prod_{k=1}^{r} w_{p_k}((x_{k,i})_i)$$

and

$$\|t_{a,i_2,\dots,i_r}\|\leqq\prod_{k=1}^{r} w_{q_k}((y'_{k,i})_i).$$

Thus, we have obtained

$$\prod_{k=1}^{r}\sum_{i=1}^{n}|\langle A_k x_{k,i}, y'_{k,i}\rangle| \leqq n^{r-1}\mathbb{I}(A_1,\dots,A_r)\prod_{k=1}^{r} w_{p_k}((x_{k,i})_i)\cdot w_{q_k}((y_{k,i})_i. \tag{25}$$

From the definition of $\mathbb{P}^{(n)}_{(1,p_k,q_k)}(A_k)$ it follows that, given $\varepsilon>0$, there are $x_{k,i}\in X_k$ and $y'_{k,i}\in Y'_k$ ($i=1,\dots,n$) such that

$$\prod_{k=1}^{r} \mathbb{P}^{(n)}_{(1,p_k,q_k)}(A_k)\cdot w_{p_k}((x_{k,i})_i)\cdot w_{q_k}((y_{k,i})_i) < (1+\varepsilon)\prod_{k=1}^{r}\sum_{i=1}^{n}|\langle A_k x_{k,i}, y'_{k,i}\rangle| .$$

This together with (25) gives (23).

Remark. Probably, it is possible (using similar techniques as in the proof of [5, 2.1]) to deduce a more general result, namely

$$\prod_{k=1}^{r} \mathbb{P}^{(n)}_{(p_k,q_k,r_k)}(A_k)\leqq n^{r-1}\mathbb{I}(A_1,\dots,A_r),$$

where

$$\sum_{k=1}^{r}\left(\frac{1}{q_k}+\frac{1}{r_k}-\frac{1}{p_k}\right)=2-r,\qquad p_k\geqq 1,\qquad q_k,\qquad r_k>0 .$$

3. Tensor Products and Nuclearity

The following observation will be basic to this paragraph.

3.1. Proposition. *Let $A^i: X_i\to Y_i$ be linear operators ($i=1,\dots,r$). Then*

a) $$\mathbb{N}^0(A_1,\dots,A_r)=\|A'_1\otimes\dots\otimes A'_r: Y'_1\otimes_\varepsilon\dots\otimes_\varepsilon Y'_r \to X'_1\otimes_\pi\dots\otimes_\pi X'_r\|$$

b) $$\begin{aligned}\mathbb{I}(A_1,\dots,A_r)&\leqq\|A_1\otimes\dots\otimes A_r: X_1\otimes_\varepsilon\dots\otimes_\varepsilon X_r\to Y_1\otimes_\pi\dots\otimes_\pi Y_r\|\\ &\leqq\|A''_1\otimes\dots\otimes A''_r: X''_1\otimes_\varepsilon\dots\otimes_\varepsilon X''_r\to Y''_1\otimes_\pi\dots\otimes_\pi Y_r\|\\ &=\mathbb{N}^0(A'_1,\dots,A'_r)\end{aligned}$$

and the equality in b) *holds if e.g. all spaces Y''_i ($i=1,\dots,r$) have the metric approximation property.*

Proof. a) is just the definition of $\mathbb{N}^0(A_1,\dots,A_r)$ and 2.1.1. b) follows from a) and from the fact that $X_1\otimes_\varepsilon\dots\otimes_\varepsilon X_r$ is isometrically imbedded into $X''_1\otimes_\varepsilon\dots\otimes_\varepsilon X''_r$ and $Y_1\otimes_\pi\dots\otimes_\pi Y_r$ is isometrically imbedded into $Y''_1\otimes_\pi\dots\otimes_\pi Y''_r$.

Suppose now that Y''_i ($i=1,\dots,r$) have the metric approximation property. Then (the completion of) $Y''_1\otimes_\pi\dots\otimes_\pi Y''_{r-1}$ also has the metric approximation property. Repeated use of Remark 4.4. yields

$$\|A''_1\otimes\dots\otimes A''_r\|=\|(A''_1\otimes\dots\otimes A''_{r-1})\otimes A_r\|=\dots=\|A_1\otimes\dots\otimes A_r\| .$$

The main results of this paper are the following two theorems.

3.2. Theorem. *Let $E_1,\dots,E_r$ be Banach spaces with Schauder bases. Then there are non-nuclear Fréchet-Schwartz spaces $X_1,\dots,X_r$ with bases such that X_i is generated* *by E_i $(i=1,\ldots,r)$, and*

$$\bigotimes_{i=1}^{r}{}_{\varepsilon} X_i = \bigotimes_{i=1}^{r}{}_{\pi} X_i \tag{28}$$

3.3. Theorem. a) *Let $r \geqq 3$. Then a locally convex space $X = X_1 = \ldots = X_r$ is nuclear iff* (28) *holds, i.e. iff*

$$\bigotimes_{i=1}^{r}{}_{\varepsilon} X = \bigotimes_{i=1}^{r}{}_{\pi} X$$

b) *Let $r \geqq 3$ and let $X_1, \ldots, X_r$ be locally convex spaces, at least two of them are infinite-dimensional normed spaces, and let $X_1, \ldots, X_r$ satisfy* (28). *Then all the other spaces $X_1, \ldots, X_r$ are nuclear spaces.*

In particular, we have: If $X_1, \ldots, X_r$ are Banach spaces satisfying (28) *then all but two of the spaces $X_1, \ldots, X_r$ are of finite dimension.*

Proof of 3.3. a) is contained in [6, 8e)] and it in fact follows from 2.3.4. because (21) gives

$$\mathbb{P}^r_{(r,r,r)}(A) = \mathbb{P}^r_{(r,r,r)}(A') \leqq \|A \otimes \ldots \otimes A\|.$$

For the proof of 3.3. b) we observe: Let X be a Banach space of infinite dimension. Then $\mathbb{P}^{(n)}_{(1,2,\infty)}(\mathrm{Id}_X) \geqq n$ for all $n = 1, 2, \ldots$ Indeed, let $\varepsilon > 0$ be given. Then the Dvoretzky's theorem allows us to find for each n a subspace $H_n \subset X$ and an isomorphism L_n of l_2^n onto H_n such that $\|L_n\| = 1$, $\|L_n^{-1}\| = 1 + \varepsilon$. Let $x_i = L_n e_i$ and $h_i = (L_n^{-1})' f_i$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of l_2^n with its biorthogonals $\{f_1, \ldots, f_n\}$. Then $w_2^2(x_i) \leqq w_2^2(e_i) = 1$. An application of the Hahn-Banach theorem yields $y_1, \ldots, y_n \in X'$, such that $\|y_i\| = \|h_i\| \leqq 1 + \varepsilon$ and $\langle x_i, y_j \rangle = \delta_{ij}$. Finally, we obtain

$$\sum_{i=1}^{n} \langle x_i, y_i \rangle = n \geqq n(1+\varepsilon)^{-1} w_2(x_i) w_\infty(y_i),$$

which verifies our claim.

Lemma. *If $A: X \to Y$ is a linear operator and if X_1 and X_2 are Banach spaces of infinite dimension, then we have e.g.*

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}_{(1,1,\infty)}(A) \\ &\leqq \|\mathrm{Id}_{X_1} \otimes \mathrm{Id}_{X_2} \otimes A : X_1 \otimes_\varepsilon X_2 \otimes_\varepsilon X \to Y_1 \otimes_\pi Y_2 \otimes_\pi Y\| \end{aligned}$$

Indeed, 2.3.5. and 3.1. b) imply

$$\begin{aligned} &\mathbb{P}^{(n)}_{(1,1,\infty)}(A) \cdot \mathbb{P}^{(n)}_{(1,2,\infty)}(\mathrm{Id}_{X_1}) \cdot \mathbb{P}^{(n)}_{(1,2,\infty)}(\mathrm{Id}_{X_2}) \\ &\leqq n^2 \mathbb{I}^0(\mathrm{Id}_{X_1}, \mathrm{Id}_{X_2}, A) = n^2 \|\mathrm{Id}_{X_1} \otimes \mathrm{Id}_{X_2} \otimes A\|. \end{aligned}$$

The above claim now yields the lemma.

Now we can finish the proof of 3.3. b): Suppose X_1 and X_2 are infinite-dimensional Banach spaces. Then evidently $X_1 \otimes_\varepsilon X_2 \otimes_\varepsilon X_i$ is a subspace of $X_1 \otimes_\varepsilon \ldots \otimes_\varepsilon X_r$ and the same holds for π-products. The Lemma now easily implies that X_i is nuclear, because the product of two absolutely summing operators is nuclear.

To prove Theorem 3.2, we will need:

3.4. Lemma. *Let $\{\alpha_n\}$ be an arbitrary non-increasing sequence of positive numbers, $a_1 \leqq 1$ and $\lim_{\alpha_n} = 0$. Let r be a natural number. Then there are sequences*

$$a_1 = \{a_{1,n}\}_n, \quad a_2 = \{a_{2,n}\}_n, \ldots, a_r = \{a_{r,n}\}_n \tag{29}$$

of positive numbers such that

(i) $a_{i,n} \cdot a_{j,n} \leqq \alpha_n$ *for all n and all $i \neq j$, $1 \leqq i, j \leqq r$*

(ii) $\{a_{i,n}\}_n$ *is non-increasing (in n) and* $\lim_{n\to\infty} a_{i,n} = 0$ *for all $i = 1, \ldots, r$*

(iii) $\{a_{i,n}\}_n = a_i \notin l_p$ *for all $p > 0$ and all $i = 1, \ldots, n$.*

Proof. We choose $\{a_{i,n}\}$ inductively together with an increasing sequence $\{n_k\}$ of natural numbers. We put $n_1 = a_{1,1} = a_{2,1} = \ldots = a_{r,1} = 1$. Now let us suppose that n_k has been chosen together with $a_{i,n}$ for all $i = 1, \ldots, r$ and all $n \leqq n_k$, and such that $a_{i,n}$ satisfy (i) for all $n \leqq n_k$, are non-increasing in $n \leqq n_k$ and satisfy (30)–(33) below, where we write l instead of k and $l \leqq k$.

Suppose that $k \equiv i$ modulo r for some natural number $i \leqq r$, i.e. $k = m \cdot r + i$, where m is an integer $\geqq 0$.

Then, we can choose n_{k+1} such that

$$k \leqq a_{i,n_{k+1}} \cdot n_{k+1}^{\frac{1}{k}} \quad \text{and} \quad k \cdot \alpha_{n^{k+1}} \leqq a_{i,n_k} \tag{30}$$

and

$$\alpha_{n_{k+1}} \leqq a_{i,n_k} \cdot a_{j,n_k} \quad \text{for all} \quad j \leqq r\,. \tag{31}$$

Now, we put

$$a_{i,n} = a_{i,n_k} \quad \text{for} \quad n_k < n \leqq n_{k+1} \tag{32}$$

and

$$a_{j,n} = \alpha_{n_{k+1}} \cdot a_{i,n}^{-1} = \alpha_{n_{k+1}} \cdot a_{i,n_k}^{-1} \quad \text{for} \quad n_k < n \leqq n_{k+1} \tag{33}$$

and for all $j \leqq r$, $j \neq i$.

This inductively defines the sequences (29) and $\{n_k\}$. If $j \neq i$ and n is in the interval $n_k < n \leqq n_{k+1}$ then from (33) we see that $a_{i,n} \cdot a_{j,n} = \alpha_{n_{k+1}} \leqq \alpha_n$ and from (33) and (31) we obtain

$$a_{j',n} \cdot a_{j,n} = \alpha_{n_{k+1}} \cdot (\alpha_{n_{k+1}} \cdot a_{i,n_k}^{-2}) \leqq \alpha_{n_{k+1}} \leqq \alpha_n\,.$$

This shows (i).

The sequences (29) are constant on every interval $k_n < n \leqq k_{n+1}$. The sequence a_i is non-increasing for all $n \leqq n_{k+1}$, which follows from (32) and from the induction hypothesis. Similarly (33) and (31) imply that for n in the interval $n_k < n \leqq n_{k+1}$ we have $a_{j,n} = \alpha_{n_{k+1}} \cdot a_{i,n_k}^{-1} \leqq a_{j,n_k}$. Thus, all a_j $(j \neq i)$ are non-increasing.

From (33) and (30) we can see that

$$a_{j,n_{k+1}} = \alpha_{n_{k+1}} \cdot a_{i,n_k}^{-1} \leqq k^{-1} \quad \text{if} \quad j \neq i\,.$$

This together with the monotonicity of the sequences (29) implies (ii).

To show (iii) suppose that p is an arbitrary positive number. Now (30) and (33) imply that for any $k \geqq p$ we have

$$k \leqq a_{i,n_{k+1}} \cdot n_{k+1}^{\frac{1}{k}} \leqq a_{i,n_{k+1}} \cdot n_{k+1}^{\frac{1}{p}}. \tag{34}$$

As $a_i = \{a_{i,n}\}_n$ is non-increasing we observe that

$$n_{k+1} a_{i,n_{k+1}}^p \leqq \sum_{n=1}^{n_{k+1}} a_{i,n}^p. \tag{35}$$

Evidently (34) and (35) yield

$$k^p = \sum_{i=1}^{\infty} a_{i,n}^p. \tag{36}$$

Finally, we observe that i is dependent on k and thus, choosing suitably $k \geqq p$, we can obtain (36) for all $i = 1, \ldots, r$ and all k arbitrarily large. This accomplishes the proof of (iii).

Proof of 3.2. Theorem. We may suppose that $\{e_{i,k}\}$ is the monotone Schauder basis in the space $(E_i, \|\cdot\|_i)$ for all $i = 1, \ldots, r$. Let (29) be the sequences given by Lemma 3.4. and suppose that $\sum \alpha_n < \infty$. Let $X_{i,n}$ be the Banach space of all sequences of real numbers such that $p_{i,n}(\{x_k\}) < \infty$, where

$$p_{i,n}(\{x_k\}) = \left\| \sum_{k=1} a_{i,k}^n x_k e_{i,k} \right\|_i.$$

Then, evidently, $\{e_{i,n,k}\}_k = \{a_{i,k}^n e_{i,k}\}_k$ is a Schauder basis of the Banach space $X_{i,n}$, isometric to the basis $\{e_{i,k}\}$ of E_i. Let $X_i = \bigcap_n X_{i,n}$. Then the countable set of norms $\{p_{i,n}\}_n$ induces on X_i the topology of a Fréchet-Schwartz space with the basis $(e_k) = (\{\delta_{k,l}\}_l)$.

Let m, n be the natural number. Then the identity map of X_i is an element of $L(X_{m+n}, X_n)$ (see [7, (22)] and thus Id_{X_i} extends to the canonical continuous operator

$$X_{i,m+n,n} : X_{i,m+n} \to X_{i,n}$$

between Banach spaces $X_{i,m+n}$ and $X_{i,n}$. Evidently $e_{i,m+n,k} = a_{i,k}^m \cdot e_{i,n,k}$ and thus $X_{i,m+n,n}(e_{i,m+n,k}) = a_{i,k}^m \cdot e_{i,n,k}$. Now [7, Lemma 3.3] and [9, 11.7.4] implies that

$$a_k(X_{i,m+n,n}) = a_{i,k}^m = a_k(X'_{i,m+n,n}). \tag{37}$$

The property (iii) of the sequence $a_i = \{a_{i,k}\}_k$ and (37) imply that

$$\sum_{k=1}^{\infty} a_k(X_{i,m+n,n})^p = \sum_{k=1}^{\infty} a_{i,k}^{mp} = \infty \quad \text{for all} \quad p > 0.$$

This implies that the spaces X_i are not nuclear [3, 8]. Property (ii) implies that the mapping $X_{i,n+1,n}$ are compact, which shows that X_i is a Schwartz space.

It remains to show (28). To this end, let $n_1, \ldots, n_r$ be arbitrary natural numbers. Similarly as in the proof of [7, Theorem 3.2] the ε-topology on $\bigotimes_{i=1}^{r} X_i$ will be finer than the π-topology if the operator

$$X = X_{i,n_1+1,n_1} \otimes X_{i,n_2+1,n_2} \otimes \ldots \otimes X_{i,n_r+1,n_r} : \bigotimes_{l=1}^{r}{}_{\varepsilon} X_{i,n_l+1} \to \bigotimes_{l=1}^{r}{}_{\pi} X_{i,n_l}$$

is continuous. From 3.1. b, 2.2.4, (37) and (i) follows that

$$\|X\| \leqq K(r) \sum_{i \neq j}^{r} \left\{ \sum_{n=1}^{\infty} (a_{i,n} \cdot a_{j,n})^{\frac{1}{2r-3}} \right\}^{2r-3} < \infty .$$

This finishes the proof of 3.2.

4. Remarks

4.1. Let $X_1, \ldots, X_r$ be Banach spaces. We can call an r-linear form $s \in \mathfrak{L}(X_1, \ldots, X_r)$ strongly nuclear if, for example, $l_p(a_n(s)) < \infty$ for all $p > 0$. If $X_1, \ldots, X_r$ are locally convex spaces then a continuous linear r-form on $X_1 \times \ldots \times X_r$ is called strongly nuclear if there are continuous seminorms p_i on X_i ($i = 1, \ldots, r$) such that s is strongly nuclear if considered as a continuous r-linear form on $(\widetilde{X_1 p_1}) \times \ldots \times (\widetilde{X_r, p_r})$. Now we can formulate

Proposition. *The spaces $X_1., \ldots, X_r$ in 3.2. Theorem may be chosen so that every continuous r-linear form on $X_1 \times \ldots \times X_r$ is strongly nuclear.*

To show this, it is sufficient to choose $\{\alpha_n\}$ in the proof of 3.2 such that $\{\alpha_n\} \in l_p$ for every $p > 0$. Then if s is any continuous r-linear form on $X_1 \times \ldots \times X_r$ then there are $n_1, \ldots, n_r$ such that $s \in \mathfrak{L}(X_{1,n_1}, \ldots, X_{r,n_r})$; (we keep the notation of the proof of 3.2.). Let $A_i = X_{i,n_i+1,n_i}$ be the canonical imbeddings. Then $a_n(A_i) = a_{i,n}$ and thus (2.1.2. c), 2.2.3. and (i) gives

$$\mathbf{S}_p(s \circ (A_1, \ldots, A_r)) \leqq \|s\| \cdot \mathbf{S}_p(A_1, \ldots, A_r) \leqq C_2(p, r) \left(\sum_{i \neq j}^{r} (a_{i,n} \cdot a_{j,n})^q \right)^{\frac{1}{q}} < \infty .$$

4.2. Alternatively we can obtain Theorem 3.2. (at least for $E_1 = E_2 = E_3 = l_2$) as follows:

a) *Let $A_i : l_2 \to l_2$ be two compact operators, $i = 1, 2$. Let us consider the operator $A_1 \otimes A_2$*

$$A_1 \otimes A_2 : l_2 \otimes_\varepsilon l_2 \to l_2 \otimes_\pi l_2 .$$

If it is continuous we have

$$\begin{aligned} &a_{m \cdot n+1}(A_1 \otimes A_2) \\ &\leqq \sum_{i=1}^{\infty} a_{i+m}(A_1) a_i(A_2) + \sum_{i=1}^{m} a_i(A_1) a_{i+n}(A_2) \\ &\leqq a_{m+1}(A_1) \sum_{i=1}^{\infty} a_i(A_2) + a_{n+1}(A_2) \sum_{i=1}^{m} a_i(A_1) . \end{aligned} \tag{38}$$

Indeed the Schmidt representation gives us the expression

$$A_1 x = \sum_{i=1}^{\infty} a_i(A_1)(e_i, x) f_i,$$

where $\{e_i\}$ and $\{f_i\}$ are orthonormal sequences in l_2. Let K_1 be the operator of rank m:

$$K_1 x = \sum_{i=1}^{m} a_i(A_1)(e_i, x) f_i.$$

Then $a_i(T_1 - K_1) = a_{m+i}(T_1)$ for $i = 1, 2, \ldots$, and $a_i(K_1) = a_i(A_1)$ for $i = 1, \ldots, m$, $a_{m+1}(K_1) = 0$ (cf. [9, 11.3.3]). Similarly, we choose K_2 such that, $\dim K_2 = n$, $a_i(A_2 - K_2) = a_{i+n}(A_2)$, $i = 1, 2, \ldots$.

Now we have

$$\begin{aligned} a_{m \cdot n+1}(A_1 \otimes A_2) &\leqq \|A_1 \otimes A_2 - K_1 \otimes K_2\| \\ &= \|(A_1 - K_1) \otimes A_2 + K_1 \otimes (A_2 - K_2)\| \\ &\leqq \|(A_1 - K_1) \otimes A_2\| + \|K_1 \otimes (A_2 - K_2)\| \\ &\leqq \sum_{i=1}^{\infty} a_i(A_1 - K_1) a_i(A_2) + \sum_{i=1}^{\infty} a_i(K_1) \cdot a_i(A_2 - K_2), \end{aligned}$$

which gives (38).

In particular, from (38) we obtain

$$a_{n^2+1}(A_1 \otimes A_2) \leqq \sum_{i=1}^{\infty} a_{i+n}(A_1) a_i(A_2) + \sum_{i=1}^{n} a_i(A_1) a_{i+n}(A_2). \tag{39}$$

b) *Let $X_1, \ldots, X_r$ be Banach spaces. Then we have the natural isometric imbedding*

$$I: X_1 \otimes_\varepsilon \ldots \otimes_\varepsilon X_r \to L(X_1', X_2 \otimes_\varepsilon \ldots \otimes_\varepsilon X_r),$$

$$I\left(\sum_{i=1}^{n} x_{1i} \otimes \ldots \otimes x_{ri}\right)(x') = \sum_{i=1}^{n} x'(x_{1i}) \cdot x_{2i} \otimes \ldots \otimes x_{ri}, \qquad x' \in X_1',$$

It is not difficult to see that the same formula gives also the canonical imbedding

$$J: X_1 \otimes_\pi \ldots \otimes_\pi X_r \to \mathfrak{N}^0(X_1', X_2 \otimes_\pi \ldots \otimes_\pi X_r),$$

i.e. $\mathbb{N}^0(Jx) = \pi(x)$.

Evidently

$$J(A_1 \otimes \ldots \otimes A_r)x = (A_2 \otimes \ldots \otimes A_r) \circ Ix \circ A_1'$$

for $x \in X_1 \otimes_\varepsilon \ldots \otimes_\varepsilon X_r$.

c) From a) and b) we see that

$$\begin{aligned} &\|A_1 \otimes A_2 \otimes A_3\| \\ &\quad = \sup\{\pi(A_1 \otimes A_2 \otimes A_3)(x));\ x \in X_1 \otimes_\varepsilon X_2 \otimes_\varepsilon X_3,\ \varepsilon(x) \leqq 1\} \\ &\quad = \sup\{\mathbb{N}^0((A_2 \otimes A_3) \circ Ix \circ A_1');\ \varepsilon(x) \leqq 1\}. \end{aligned}$$

The multiplicativity of the approximation numbers of operators and (39) give

$$\begin{aligned}
&\mathbb{N}^0((A_2 \otimes A_3) \circ Ix \circ A_1') \\
&\leqq 12 \sum_{n=1}^{\infty} a_n((A_2 \otimes A_3) \circ Ix \circ A_1') \\
&\leqq 12 \sum_{n=1}^{\infty} 2(n+1) a_{n^2+n-1}((A_2 \otimes A_3) \circ Ix \circ A_1') \\
&\leqq 24 \sum_{n=1}^{\infty} (n+1) a_n(Ix \circ A_1') \cdot a_{n^2}(A_2 \otimes A_3) \\
&\leqq 24 \|Ix\| \sum_{n=1}^{\infty} (n+1) a_n(A_1') \cdot a_{(n-1)^2+1}(A_2 \otimes A_3) \\
&\leqq 24 \|x\| \sum_{n=1}^{\infty} (n+1) a_n(A_1') \\
&\quad \cdot \left\{ \sum_{i=1}^{\infty} a_{i+n-1}(A_2) a_i(A_3) + \sum_{i=1}^{n} a_i(A_2) a_{i+n-1}(A_3) \right\}.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} a_i(A_3) \sum_{n=1}^{\infty} (n+1) a_{i+n-1}(A_2) a_n(A_1') \\
&\quad \leqq 2 \sum_{i=1}^{\infty} a_i(A_3) \left\{ \sum_{n=1}^{\infty} (a_n(A_1') a_{n+i-1}(A_2))^{1/2} \right\}^2.
\end{aligned}$$

Here we used the fact that $\sum_n n a_n \leqq \left(\sum_n \sqrt{a_n} \right)^2$.

Putting all this together we obtain

$$\begin{aligned}
&\|A_1 \otimes A_2 \otimes A_3\| \\
&\leqq 48 \sum_{i=1}^{\infty} a_i(A_3) \left\{ \sum_{n=1}^{\infty} (a_n(A_1') a_{i+n-1}(A_2))^{1/2} \right\}^2 \\
&\quad + 48 \sum_{i=1}^{\infty} a_i(A_2) \left\{ \sum_{n=1}^{\infty} (a_n(A_1') a_{i+n-1}(A_3))^{1/2} \right\}^2 \\
&\leqq 48 \left\{ \sum_{i,n=1} (a_i(A_3) a_n(A_1') a_{i+n-1}(A_2))^{1/2} + (a_i(A_2) a_n(A_1') a_{i+n-1}(A_3))^{1/2} \right\}^2.
\end{aligned}$$

The proof of Theorem 3.2 in our special case now proceeds similarly as in 3.2, using (40) instead of 2.2.4. Of course, a more elaborate version of Lemma 3.4 is needed.

4.3. Let $\mathfrak{Ten}_r(X, Y)$ denote the class of operators $A : X \to Y$ such that

$$\left\| \bigotimes_{i=1}^{r} A : \bigotimes_{i=1}^{r}{}_{\varepsilon} X \to \bigotimes_{i=1}^{r}{}_{\pi} Y \right\| = \mathbb{I}(A_1, \ldots, A)$$

is finite. We do not know whether $\mathfrak{Ten}_r(X, Y)$ is closed under addition of operators or not, or if $\left\| \bigotimes_{i=1}^{r} A \right\|^{\frac{1}{r}}$ is a quasi-norm (cf. [6]). It follows from (21) that $\mathfrak{Ten}_r(X, Y)$

$\subset \mathfrak{P}_{(r,r,r)}(X, Y)$. Evidently, $\mathfrak{Ten}_{r+1} \subset \mathfrak{Ten}_r$. This leads to the following question: It is true that $\mathfrak{P}_{(a,a,a)} \subset \mathfrak{P}_{(b,b,b)}$ if $b \leqq a$?

4.4. A. Defant noticed that the proof of [7, Lemma 2.3.] is not correct because the operator $\tilde{U}$ (in the notation of [7]) depends on the subspace L and we cannot conclude that $\mathbb{N}^0(A\tilde{U}B') = \inf_L \mathbb{N}^0({}_L A\tilde{U}B')$. The number $\mathbb{N}_0(A, B)$ was introduced in [7] and we have by the definition

$$\begin{aligned}\mathbb{N}_0(A, B') &= \sup\{\mathbb{N}^0(AUB');\ \|U\| \leqq 1, U \in \mathfrak{F}(Y_1', X_1)\}\\ &= \|A \otimes B'' : X_1 \otimes_\varepsilon Y_1' \to X_2 \otimes_\pi Y_2''\|\,.\end{aligned}$$

We are able to prove the lemma under additional assumptions, namely we have the following

Corrected Version of [7, Lemma 2.3]:

Let $A: X_1 \to X_2$, $B: Y_1 \to Y_2$ be two operators between Banach spaces and let the space X_2 have the metric approximation property. Then

$$\mathbb{N}_0(A, B') = \|A \otimes B : X_1 \otimes_\varepsilon Y_1 \to X_2 \otimes_\pi Y_2\|\,.$$

Proof. Let $\varepsilon > 0$ be given. Then there is $U = \sum_{i=1}^{n} x_{1i} \otimes y_{1i}'' \in \mathfrak{F}(Y_1', X_1)$ such that $\|U\| \leqq 1$ and

$$\mathbb{N}_0(A, B') - \varepsilon \leqq \mathbb{N}^0(AUB')\,. \tag{40}$$

The metric approximation property of X_2 implies ([9, 10.2.5]) the existence of an operator $Q: X_2 \to X_2$ such that $QAUB' = AUB'$, and $\|Q\| \leqq 1 + \varepsilon$. Let $L = Q(X_2)$. Then $AUB'(Y_2') \subset L$ and we have

$$\mathbb{N}^0(AUB') \leqq \mathbb{N}^0({}_L AUB') = \mathbb{N}^0({}_L QAUB') = \mathbb{I}({}_L QAUB')\,. \tag{41}$$

(If $Z: X \to Y$ is an operator, $Z(X) \subset L \subset Y$, then by ${}_L Z$ we denote the astriction of the range space Y to L, i.e. ${}_L Z: X \to L$.) Now by the definition of the integral norm $\mathbb{I}$ we can choose $V \in \mathfrak{F}(L, Y_2')$ such that

$$\mathbb{I}({}_L QAUB') - \varepsilon \leqq \operatorname{tr}(QAUB'V) \quad \text{and} \quad \|V\| \leqq 1\,. \tag{42}$$

The principle of local reflexivity now gives an operator T: $\operatorname{sp}\{y_{1i}''\}_{i=1}^n \to Y_1$ such that $\|T\| \leqq 1 + \varepsilon$ and $y_{1i}(f) = y_i(f)$ for all $f \in B'V(L)$, where we have put $y_i = Ty_{1i}$. Now, if we define

$$\tilde{U} = \sum x_{1i} \otimes y_i \in \mathfrak{F}_{w*}(Y_1', X_1) = X_1 \otimes Y_1\,,$$

we see that $\tilde{U} = (TU')'$, and thus $\varepsilon(\tilde{U}) = \|\tilde{U}\| \leqq 1 + \varepsilon$. We also have $\operatorname{tr}(QA\tilde{U}B'V) = \operatorname{tr}(QAUB'V)$. Putting this together with (41) and (42) we obtain

$$\begin{aligned}\mathbb{N}_0(A, B') - 2\varepsilon &\leqq \operatorname{tr}(QA\tilde{U}B'V) \leqq \mathbb{I}({}_L QA\tilde{U}B') = \mathbb{N}^0(QA\tilde{U}B')\\ &\leqq \|Q\|\, \mathbb{N}^0(A\tilde{U}B)\,.\end{aligned} \tag{43}$$

Using the identification of $\tilde{U} = \sum x_{1i} \otimes y_i \in X_1 \otimes Y_1$ with the operator $\tilde{U}: Y_1' \to X_1$ we have

$$(A \otimes B)\tilde{U} = \sum Ax_{1i} \otimes By_i = A\tilde{U}B'\,. \tag{44}$$

Finally (43) and (44) yield

$$\mathbb{N}_0(A, B') \leqq \sup\{\pi((A \otimes B)\tilde{U});\ \tilde{U} \in X_1 \otimes Y_1, \varepsilon(\tilde{U}) \leqq 1\} = \|A \otimes B\|.$$

4.5. The last remark is due to Kwapień. Theorem 3.3. b) can be deduced easily as follows. Let X_1, X_2 be infinite-dimensional normed spaces, X_3 a locally convex space and let

$$X_1 \otimes_\varepsilon X_2 \otimes_\varepsilon X_3 = X_1 \otimes_\pi X_2 \otimes_\pi X_3. \tag{45}$$

Then, $Y = X_1 \otimes_\varepsilon X_2$ is isomorphic to $X_1 \otimes_\pi X_2$. On the basis of Dvoretzky's theorem Y contains uniformly $l_2^n \otimes_\varepsilon l_2^n$, which in turn contains isometrically l_∞^n. From (45) we see that

$$Y \otimes_\varepsilon X_3 = Y \otimes_\pi X_3,$$

where Y contains uniformly (and uniformly complemented l_∞^n). The main theorem in [4] now implies that X_3 is nuclear.

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The Smoothing Components of a Triangle Singularity. II

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Introduction

In this paper we begin a detailed study of the universal deformation of a triangle singularity. Our principal purpose is to prove most of the statements announced in Part I [12] or in certain cases generalizations thereof. Nevertheless, the present paper can be read independently of *I*.

Virtually all our knowledge of the miniversal deformation of a triangle singularity comes through a moduli space interpretation of its "negatively weighted" part, an idea due to Pinkham. Actually, this moduli space interpretation exists for any reduced isolated singularity with $\mathbb{C}^*$-action (see the appendix), but in the case of a triangle singularity it can be phrased in much more geometric terms. This is in fact true for a larger class of singularities – a class we have baptized Fuchsian singularities**. (We call a singularity X with good $\mathbb{C}^*$-action *Fuchsian* if there is a Fuchsian group Γ acting on the Poincaré upper half plane such that $X \cong \operatorname{Spec} \bigoplus_{n=0}^{\infty} A_n^{\Gamma}$, where A_n denotes the space of Γ-automorphic forms of degree $2n$.) In the triangle case little information is lost by restricting ourselves to the negatively weighted part of the universal deformation, as this is "topologically versal" (although we prove only a slightly weaker result). This enables us to conclude (in Sect. 3) that a triangle singularity is smoothable if and only if there exists a $K3$ surface with a certain configuration of curves on it. (For Fuchsian singularities the "if" part of this statement holds.) The existence of such surfaces with curve configurations turns out to be a purely lattice theoretic affair: in Sect. 4 we phrase this in terms of the seemingly elusive notion of a good embedding, but after some work (in Sect. 5) we get a good hold on this, making it for instance a relatively simple matter to compute the number of smoothing components of a given triangle singularity. The nature of these results is such that, if we combine

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** Some authors use the term *canonical* singularities them with the Torelli theorem for $K3$ surfaces, we obtain a precise description of the "rational double point part" in the negatively weighted part of a universal deformation (Sect. 6). The paper concludes with a proof that Wahl's list of singularities adjacent to a triangle singularity is complete. Surprisingly, it turns out that all these adjacencies can be described in terms of a good embeddings.

1. Fuchsian Singularities and Associated Fine Moduli Spaces

(1.1) Let X be a normal affine surface with good $\mathbb{C}^*$-action. So its coordinate ring R has the structure of a graded $\mathbb{C}$-algebra: $R=\bigoplus_{l=0}^{\infty} R_l$, $R_0=\mathbb{C}$, and X is smooth outside its *vertex* x $\left(\text{defined by the maximal ideal } R_+:=\bigoplus_{l=1}^{\infty} R_l\right)$. The $\mathbb{C}^*$-action and the grading are related by $f(\lambda\cdot y)=\lambda^{-k}f(y)$ if $y\in X$, $\lambda\in\mathbb{C}^*$, $f\in R_l$, so $\lambda\cdot y\to x$ if $|\lambda|\to\infty$. The standard projectivization $X\subset\bar{X}$ is defined by setting $\bar{R}_l=R_0\oplus\ldots\oplus R_l$ and taking $\bar{X}:=\operatorname{Proj}\bigoplus_{l=0}^{\infty}\bar{R}_l$ [see (A.1) for more discussion]. This is a normal projective variety which inherits the $\mathbb{C}^*$-action. The curve $\bar{X}_\infty:=\bar{X}-X$ added to X is isomorphic to $\operatorname{Proj}\bigoplus_{l=0}^{\infty}R_l$ and can be identified with the $\mathbb{C}^*$-orbit space of $\bar{X}-\{x\}$. According to Orlik and Wagreich [16], the closure of an irregular $\mathbb{C}^*$-orbit in $\bar{X}-\{x\}$ meets $\bar{X}_\infty$ in a singular point of $\bar{X}$ (in fact, a cyclic quotient singularity) and this yields a bijection between such orbits and the singular points of $X-\{x\}$.

Following Dolgachev [7], there exist a simply-connected Riemann surface $\mathscr{D}$, a discrete cocompact subgroup Γ of $\operatorname{Aut}(\mathscr{D})$ and a line bundle ℓ over $\mathscr{D}$ to which the action of Γ lifts (all essentially unique) such that $R\cong\bigoplus_{k=0}^{\infty}H^0(\mathscr{D},\ell^k)^\Gamma$ (as graded $\mathbb{C}$-algebras). In geometric terms, this means that X is obtained from the orbit space $\Gamma\backslash\operatorname{Tot}(\ell^{-1})$ by collapsing the image of the zero-section to a point. It is then not hard to see that $\bar{X}$ is isomorphic to the Thom space of ℓ (the isomorphism takes $\bar{X}_\infty$ to the image of the zero section).

Recall that the dualizing sheaf of a normal surface Z is simply $\omega_Z:=i_*\Omega^2_{Z_{\mathrm{reg}}}$, where $i: Z_{\mathrm{reg}}\subset Z$ is the inclusion of the smooth part of Z.

(1.2) **Proposition.** *The dualizing sheaf of $\bar{X}$ is trivial if and only if we can take for $\mathscr{D}$ the Poincaré upper half plane $\mathscr{H}$ and for ℓ its canonical bundle.*

Proof. If $\omega_{\bar{X}}$ is trivial, then let ω be a generating section. Since $\Omega^2_{\operatorname{Tot}(\ell)}\otimes\mathcal{O}_{\mathscr{D}}$ (we identify $\mathscr{D}$ with the zero-section of ℓ) is naturally isomorphic to $\ell^{-1}\otimes\Omega_{\mathscr{D}}$, ω determines a Γ-equivariant isomorphism between ℓ and $\Omega_{\mathscr{D}}$. We cannot have $\mathscr{D}\cong\mathbb{P}^1$ because $\Omega^{\otimes k}_{\mathbb{P}^1}$ has no nonzero sections for $k>0$. Nor can we have $\mathscr{D}\cong\mathbb{C}$, for any section of $\Omega^{\otimes k}_{\mathbb{C}}$ invariant under a cocompact group is a scalar multiple of $dz^{\otimes k}$. So $\mathscr{D}\cong\mathscr{H}$.

Conversely, if $\mathscr{D}\cong\mathscr{H}$ and $\ell\cong\Omega_{\mathscr{H}}$ (Γ-equivariantly), then the isomorphism $\ell\cong\Omega_{\mathscr{D}}$ defines a nowhere zero section of $\ell^{-1}\otimes\Omega_{\mathscr{D}}\cong\Omega^2_{\operatorname{Tot}(\ell)}|\mathscr{D}$. This section extends uniquely to a $\mathbb{C}^*$-invariant section of $\Omega^2_{\operatorname{Tot}(\ell)}$.

(1.3) If (X,x) satisfies one of the equivalent conditions of (1.2), we call (X,x) a *Fuchsian singularity*. From now on we suppose we are dealing with such a singularity. We also fix a generating section ω_* of ω_X.

Let $(p_-:\mathscr{X}_-\to S_-, \iota: X\cong X_*)$ be the negatively weighted part of a miniversal deformation of X and let $(\bar{p}_-:(\bar{\mathscr{X}}_-,\bar{\mathscr{X}}_\infty)\to S_-,\ \bar{\iota}:(\bar{X},\bar{X}_\infty)\cong(\bar{X}_*,\bar{X}_{\infty,*}),\ \bar{X}_\infty\times S_-\cong\bar{\mathscr{X}}_\infty)$ be its standard projectivization (usually we will not mention these two isomorphisms). In (A.5) we characterize $\bar{p}_-$ as the fine moduli space of R-polarized surfaces. We will show [Theorem (1.8)] that in the case at hand, the notion of an R-polarized surface admits a more geometric description. The following two lemmas pave the way.

(1.4) **Lemma.** *Each fibre $\bar{X}_t$ of $\bar{p}_-$ is a normal regular surface.*

Proof. By construction, $\bar{X}$ is normal; as normality is preserved under a small deformation, nearby fibres will be also normal. According to Pinkham [19, (6.4)], $\bar{X}_*$ is a regular surface. Since $t\in S_-\mapsto \dim_{\mathbb{C}} H^1(\mathcal{O}_{\bar{X}_t})$ is semi-continuous, nearby fibres are also regular. The fact that the $\mathbb{C}^*$-action on S_- is good now gives the result for all fibres.

The dualizing sheaf $\omega_{\bar{\mathscr{X}}_-/S_-}$ of $\bar{p}_-$ is given by $j_*j^*\Omega^2_{\bar{\mathscr{X}}/S_-}$, where j denotes the inclusion of the set of $y\in\bar{\mathscr{X}}_-$ where $\bar{p}_-$ is smooth in $\bar{\mathscr{X}}_-$. Since $H^1(\omega_{\bar{X}_*})\cong H^1(\mathcal{O}_{\bar{X}_*})=0$, a standard argument shows that $\bar{p}_{-*}\omega_{\bar{X}/S_-}$ is in $*\in S$ a free $\mathcal{O}_{S_-,*}$-module of rank $\dim_{\mathbb{C}} H^0(\bar{\omega}_{\bar{X}})=1$, whose fibre over $*$ is $H^0(\bar{\omega}_X)$

(1.5) **Lemma.** *There exists a generating section ω of $\omega_{\bar{\mathscr{X}}_-/S_-}$ which is homogeneous of degree -1 with respect to the $\mathbb{C}^*$-action and reduces to ω_* on $\bar{X}_*$. Furthermore ω is unique.*

Proof. By homogeneity, $\bar{p}_{-*}\omega_{\bar{\mathscr{X}}_-/S_-}$ is a free $\mathcal{O}_{S_-,*}$-module of rank 1, so we only have to show that $\mathbb{C}^*$ acts on $\omega_{\bar{X}}$ with weight -1. Choose a smooth point z of $\bar{X}_\infty$ (which is then also smooth in $\bar{X}$). Since $\bar{X}_\infty$ is pointwise fixed under the $\mathbb{C}^*$-action, $\mathbb{C}^*$ acts trivially on the tangent space $T_z\bar{X}_\infty$. As $\mathbb{C}^*$ acts on the normal space $T_z\bar{X}/T_z\bar{X}_\infty$ with weight 1, it acts on $\Lambda^2 T_z^*\bar{X}$ with weight -1.

(1.6) As mentioned earlier, each singular point of $\bar{X}-\{x\}$ lies on $\bar{X}_\infty$ and is a cyclic quotient singularity. Since $\bar{X}$ has trivial dualizing sheaf, such a singularity is Gorenstein and is hence of type A_l for some $l\geqq 1$. Let $\tilde{X}\to X$ minimally resolve these singularities. The exceptional locus of an A_ℓ singularity is a chain of ℓ smooth rational curves of self-intersection -2 (briefly: *nodal curves*). So if we have m singular points $a_1,\ldots,a_m$ of type $A_{p_1-1},\ldots,A_{p_m-1}$, respectively, then the pre-image $\tilde{X}_\infty$ of $\bar{X}_\infty$ will look like this

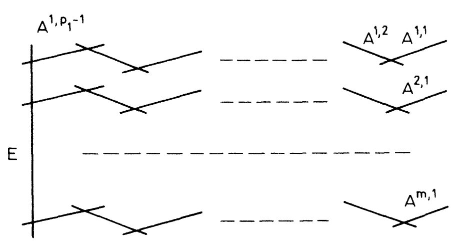


Fig. 1

Here E denotes the strict transform of $\bar{X}_\infty$ and $A^{\mu,1}\cup\ldots\cup A^{\mu,p_\mu-1}$ is the exceptional locus over a_μ. By the adjunction formula, the genus g of E is given by $1+\frac{1}{2}E\cdot E$.

We perform this resolution simultaneously in the family $\bar{p}_-$ and thus find $\tilde{p}_-:(\tilde{\mathscr{X}}_-,\tilde{\mathscr{X}}_\infty)\to(\bar{\mathscr{X}}_-,\bar{\mathscr{X}}_\infty)\to S_-$. Since we only resolved rational double points, ω lifts to a generating section $\tilde{\omega}$ of $\omega_{\tilde{\mathscr{X}}_-/S_-}$. By the same token, each fibre $\tilde{X}_t$ is regular.

Occasionally, a Fuchsian singularity with the numerical data $(p_1,\ldots,p_m;g)$ is denoted $D_{p_1,\ldots,p_m}(g)$ (or $D_{p_1,\ldots,p_m}$ if $g=0$). In general this doesn't refer to a single isomorphism type because we have moduli if $g>0$ or $m>3$. For $g=0$, $m=3$ there are no moduli and D_{p_1,p_2,p_3} represents a unique isomorphism type. These are the *triangle singularities*. Those with $g=0$, $m=4$ are called the *quadrilateral singularities*.

From work of Orlik and Wagreich [16] it follows that on the minimal resolution $\hat{X}\to\tilde{X}$ (of the unique singularity $\tilde{x}$ of $\tilde{X}$) the above configuration of curves entends to

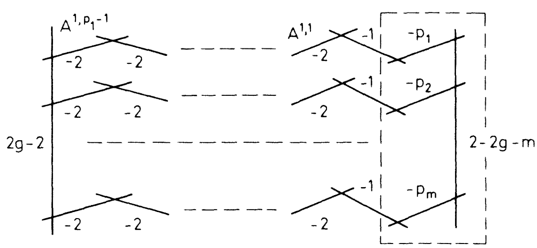


Fig. 2

in which the slant lines are smooth rational curves and the two vertical curves are copies of E, self-intersections being as indicated. The exceptional locus of $\hat{X}\to\tilde{X}$ is the union of curves in the box. By successively contracting the (-1)-curves we end up with a ruled surface isomorphic to $\mathbb{P}_E(\Omega_E\oplus\mathcal{O}_E)$. Conversely, we can start out from this surface and blow up points until we reach the above configuration.

A necessary and sufficient condition that the configuration in the box is contractible is that its intersection matrix is negative definite. The latter condition is equivalent to $\sum_{\mu=1}^{m} 1/p_\mu<m+2g-2$ and $p_\mu\geqq 2$ for all μ. So:

(1.7) A $D_{p_1,\ldots,p_m}(g)$-singularity exists if and only if $p_\mu\geqq 2$ for all μ and $\sum_{\mu=1}^{m} 1/p_\mu<m+2g-2$, the second condition being a consequence of the first if $g\geqq 2$ or $m\geqq 5$.

(1.8) **Theorem.** *Let $\tilde{Y}$ be a complete normal surface with trivial dualizing sheaf ω_Y, $\tilde{Y}_\infty\subset\tilde{Y}$ a reduced curve on $\tilde{Y}_{\text{reg}}$ such that $\tilde{Y}-\tilde{Y}_\infty$ is affine and $\phi:\tilde{Y}_\infty\to\tilde{X}_\infty$ an isomorphism. Given a generator ζ of $H^0(\omega_{\tilde{Y}})$, there exists a unique $t\in S_-$ and a unique isomorphism of $(\tilde{Y},\tilde{Y}_\infty,\zeta)$ onto $(\tilde{X}_t,\tilde{X}_{t,\infty}\cong\tilde{X}_\infty,\omega_t)$ compatible with ϕ.*

Proof. The uniqueness assertions easily follow from Proposition (A.5) and the fact that ω is homogeneous of degree -1. The existence part is to a large extent implicit in a result of Pinkham [19, (6.13)] (and indeed, can be derived from it). As a consequence our proof follows his arguments rather closely.

Let us (abusively) denote the irreducible components of $\tilde{Y}_\infty$ by the same symbols as those of $\tilde{X}_\infty: E, A_{ij}$ (via the isomorphism ϕ). If $(\tilde{Y},\tilde{Y}_\infty)\to(\bar{Y},\bar{Y}_\infty)$ is obtained by contracting each of the connected components of $\tilde{Y}_\infty - E$ (analytically), then following Pinkham the total transform of the Weil divisor $l\bar{Y}_\infty$ is given by

$$E^{(l)} := lE + \sum_{\nu=1}^{n} \sum_{j=1}^{p_n-1} \left[\frac{lj}{p_\nu}\right] A^{\nu j},$$

where [ ] stands for "integral part". In particular $H^0(\mathcal{O}_{\bar{Y}}(l\bar{Y}_\infty)) \cong H^0(\mathcal{O}_{\tilde{Y}}(l\tilde{Y}_\infty))$. Denoting the latter by $\mathcal{R}_l$, then $\mathcal{R} := \bigoplus_{l=0}^{\infty} \mathcal{R}_l$ is in a natural way a $\mathbb{C}$-algebra, for $E^{(l)} + E^{(m)} \leqq E^{(l+m)}$. Since $\mathcal{E}^{(l)} := \mathcal{O}_{\tilde{Y}}(E^{(l)}) \otimes \mathcal{O}_E$ only depends on $\tilde{Y}_\infty$, ϕ maps it isomorphically onto $\mathcal{O}_{\tilde{X}}(E^{(l)}) \otimes \mathcal{O}_E$ [in Pinkham's notation, this is $\mathcal{O}_E(D^{(l)})$] and it is shown in [19, (5.1)] that $H^0(\mathcal{O}_{\tilde{X}}(E^{(l)} \otimes \mathcal{O}_E) \cong \mathcal{R}_l$. We therefore identify $H^0(\mathcal{E}^{(l)})$ with R_l. Let $t \in \mathcal{R}^1$ denote the element 1 and let $\mathcal{S}_l$ be the complex

$$\mathcal{S}_l : 0 \to \mathcal{R}_{l-1} \xrightarrow{\cdot t} \mathcal{R}_l \to R_l \to 0.$$

Notice that in case $\tilde{Y} = \tilde{X}$, we have $\mathcal{R}^l = R_0 \oplus \ldots \oplus R_l$, so that then $\mathcal{S}_l$ is exact. In that case, we also have that $H^1(\mathcal{O}_{\tilde{X}}(E^{(l)})) = 0$, $l = 0, 1, 2, \ldots$ [19, (6.5)]. We use these facts to show:

(1.9) **Lemma.** *$\mathcal{S}_l$ is exact* ($l = 0, 1, 2, \ldots$).

Proof. The cohomology sequence of

$$0 \to \mathcal{O}_{\tilde{Y}}(E^{(l)} - E) \to \mathcal{O}_{\tilde{Y}}(E^{(l)}) \to \mathcal{E}^{(l)} \to 0$$

is

$$0 \to H^0(\mathcal{O}_{\tilde{Y}}(E^{(l)} - E)) \xrightarrow{\cdot t} \mathcal{R}_l \xrightarrow{\beta_0} R_l \to H^1(\mathcal{O}_{\tilde{Y}}(E^{(l)} - E)) \xrightarrow{\alpha_1} H^1(\mathcal{O}_{\tilde{Y}}(E^{(l)}))$$

$$\to H^1(\mathcal{E}^{(l)}) \to H^2(\mathcal{O}_{\tilde{Y}}(E^{(l)} - E)) \to H^2(\mathcal{O}_{\tilde{Y}}(E^{(l)})) \to 0.$$

Our aim is to identify the second term with $\mathcal{R}_{l-1}$ and to show that β_0 is surjective. We do this via two intermediate results

Assertion 1. $H^2(\mathcal{O}_{\tilde{Y}}(E^{(l)})) = 0$ ($l \geqq 1$), $H^2(\mathcal{O}_{\tilde{Y}}(E^{(l)} - E)) = 0$ ($l \geqq 2$) *and α_1 is surjective*

Proof. Since $\tilde{Y}$ has trivial dualizing sheaf, Serre duality gives

$$h^2(\mathcal{O}_{\tilde{Y}}(E^{(l)})) = h^0(\mathcal{O}_{\tilde{Y}}(-E^{(l)})) = 0 \quad (l \geqq 1).$$

Similarly, $h^2(\mathcal{O}_{\tilde{Y}}(E^{(l)} - E)) = 0$ resp. 1 for $l \geqq 2$ resp. $l = 1$. Hence the surjectivity of α_1 is equivalent to $h^1(\mathcal{E}^{(l)}) = 0$ ($l \geqq 2$) or $h^1(\mathcal{E}^{(1)}) = 1$ ($l = 1$). So we only need to verify the surjectivity of α_1 in case $\tilde{Y} = \tilde{X}$. But this is immediate from the vanishing of $H^1(\mathcal{O}_{\tilde{X}}(E^{(l)})$ ($l = 0, 1, \ldots$), [19, (6.5)].

Assertion 2. *The inclusion* $i_l : \mathcal{O}_{\tilde{Y}}(E^{(l-1)}) \subset \mathcal{O}_{\tilde{Y}}(E^{(l)} - E)$ *(defined, because* $E^{(l-1)} \leqq E^{(l)} - E$*) induces an isomorphism on cohomology* ($l = 1, 2, \ldots$).

Proof. One easily checks that $E^{(l)} - E - E^{(l-1)}$ is a reduced effective divisor supported by $\tilde{Y}_\infty$.

So the cokernel of i_l only depends on $\tilde{Y}_\infty = \tilde{X}_\infty$. Hence the assertion needs only be verified for $\tilde{Y} = \tilde{X}$. In that case β_0 is surjective and $\mathrm{Ker}(\beta_0) \cong \mathscr{R}_{l-1}$, so that i_l induces an isomorphism on H^0. Moreover, the surjectivity of β_0 gives that α_1 is injective. This implies that $h^1(\mathcal{O}_{\tilde{Y}}(E^{(l)} - E)) = h^1(\mathcal{O}_{\tilde{Y}}(E^{(l)})) = 0$ $(l = 1, 2, \dots)$, so i_l is also an isomorphism on H^1. Finally, H^2 of either side is trivial for $l \geqq 2$ (by Assertion 1). This completes the proof since i_1 is an equality.

We can now finish the proof of the lemma. By Assertion 2, $\mathrm{Ker}(\beta_0)$ may be identified with $\mathscr{R}_{l-1}$. The surjectivity of α_1 and Assertion 2 yield a chain of surjections

$$H^1(\mathcal{O}_{\tilde{Y}}) \twoheadrightarrow H^1(\mathcal{O}_{\tilde{Y}}(E^{(1)})) \cong H^1(\mathcal{O}_{\tilde{Y}}(E^{(2)} - E^{(1)})) \twoheadrightarrow H^1(\mathcal{O}_{\tilde{X}}(E^{(2)})) \cong \dots \cong H^1(\mathcal{O}_{\tilde{Y}}(E^{(l)} - E)).$$

Since $H^1(\mathcal{O}_{\tilde{Y}}) = 0$ by assumption, all groups are trivial and hence β_0 is surjective.

We complete the proof of the theorem by showing that $(\bar{Y}, \bar{Y}_\infty)$ is in a natural way R-polarized [in the sense of (A.5)], so that we may conclude via (A.6). By assumption $\bar{Y} - \bar{Y}_\infty \cong \tilde{Y} - \tilde{Y}_\infty$ is affine; since $\bar{Y}_\infty$ is irreducible, it follows that $\bar{Y}_\infty$ is ample. Since $E^{(l)}$ is the total transform of $l\bar{Y}_\infty$, we have a natural isomorphism $\mathscr{R}_l \cong H^0(\mathcal{O}_{\bar{Y}}(l\bar{Y}_\infty))$. By the lemma, $(\bar{Y}, \bar{Y}_\infty)$ is then R-polarized.

2. Rough Classification of the Fibres

(2.1) We next address the question of where the fibres of $\tilde{p}_-$ fit in Kodaira's classification. Here we could invoke the classification of Mérindol [15] and Umezu [21] of normal surfaces with trivial dualizing sheaf, but in this specific situation the result we want is easily obtained directly.

Let $(\tilde{Y}, \tilde{Y}_\infty)$ be a pair as in the hypotheses of Theorem (1.7): $\tilde{Y}$ is a complete regular normal surface with trivial dualizing sheaf, $\tilde{Y} - \tilde{Y}_\infty$ is affine and $\tilde{Y}_\infty = E \bigcup \cup_{\mu j} A^{\mu j}$ is as in (1.6). Let $\pi: \hat{Y} \to \tilde{Y}$ be a minimal resolution of the singularities of $\tilde{Y}$. Then $\pi_* \mathcal{O}_{\hat{Y}} = \mathcal{O}_{\tilde{Y}}$ ($\tilde{Y}$ is normal) and hence the Leray spectral sequence yields the exact sequence

$$\begin{array}{l} \mathscr{S}: 0 \to H^1(\mathcal{O}_{\tilde{Y}}) \to H^1(\mathcal{O}_{\hat{Y}}) \to H^0(R^1\pi_* \mathcal{O}_{\hat{Y}}) \to H^2(\mathcal{O}_{\tilde{Y}}) \to H^2(\mathcal{O}_{\hat{Y}}) \to 0 \\ \qquad\qquad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \wr\| \\ \qquad\qquad 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{C} \end{array}$$

[we have $h^2(\mathcal{O}_{\tilde{Y}}) = h^0(\omega_{\tilde{Y}}) = 1$, by Serre duality].

Let us first consider the case when $H^0(R^1\pi_* \mathcal{O}_{\hat{Y}}) = \bigoplus_{z \in \tilde{Y}_{\mathrm{sing}}} (R^1\pi_* \mathcal{O}_{\hat{Y}})_z$ is trivial. Then the singularities of $\tilde{Y}$ are rational (by definition); as they are also Gorenstein, each of them is a rational double point (abbreviated as RDP). A generating section of $\omega_{\tilde{Y}}$ lifts to a generating section of $\omega_{\hat{Y}}$ (so that $\hat{Y}$ has trivial canonical sheaf) and $H^1(\mathcal{O}_{\hat{Y}}) = 0$. These two properties say that $\hat{Y}$ is a minimal $K3$ surface.

If the term in the middle is nonzero, then $\tilde{Y}$ contains an irrational singularity y. A generating section of $\omega_{\tilde{Y}}$ lifts to a meromorphic section of $\omega_{\hat{Y}}$ whose divisor is strictly negative. This implies that all the plurigenera $P_m(\hat{Y}) := h^0(\Omega_{\hat{Y}}^{\otimes m})$, $m \geqq 1$, vanish. Following Castelnuovo, $\hat{Y}$ is then birationally ruled over a curve C of genus $g(C) = h^1(\mathcal{O}_{\hat{Y}})$. If $g(C) > 0$, then the Albanese map provides a morphism $\psi: \hat{Y} \to C$ whose general fibre is a $\mathbb{P}^1$.

We cannot have that $\hat{Y}_\infty$ is contained in a fibre for $H^0(\mathcal{O}_{\hat{Y}-\hat{Y}_\infty}) \cong H^0(_{\tilde{Y}-\tilde{Y}_\infty})$ has transcendence degree 2 while that of $H^0(\mathcal{O}_{\hat{Y}\text{-fibre}}) \cong H^0(\mathcal{O}_{C-pt})$ equals 1. On the other hand, ψ maps each curve of genus $< g(C) = h^1(\mathcal{O}_{\hat{Y}})$ to a point, so that $g(E) \geqq h^1(\mathcal{O}_{\hat{Y}})$. Following Mérindol and Umezu there are actually two possibilities:

(i) y is the unique irrational singularity of $\hat{Y}$; its genus $(:= \dim_{\mathbb{C}}(R^1\pi_*\mathcal{O}_{\hat{Y}})_y)$ equals $g(C)+1$ or

(ii) C is an elliptic curve and the only irrational singularities of $\tilde{Y}$ are two simply-elliptic singularities.

For instance, if we take $(\tilde{Y}, \tilde{Y}_\infty) = (\tilde{X}, \tilde{X}_\infty)$, then $\tilde{X}$ is birationally equivalent to $E \times \mathbb{P}^1$, so that the genus of (X, x) is equal to $g(E)+1$.

We will mostly be concerned with the case when E is rational. We are then in case (i): $\tilde{Y}$ has a unique irrational singularity of genus 1 (this is also immediate from the exact sequence $\mathscr{S}$). As $\tilde{Y}$ is Gorenstein, the singularity is minimally elliptic in the sense of Laufer [10]. Hence:

(2.2) If (X, x) is minimally elliptic (or equivalently, if $\bar{X}_\infty$ is rational), then for any fibre $\tilde{X}_t$ of $\tilde{p}_-$ either $\tilde{X}_t$ is a $K3$ surface with only RDP's or $\tilde{X}_t$ is a rational surface whose singular locus consists of a minimally-elliptic singularity and (possibly) some RDP's.

3. Versality in the Non-Positive Weight Part

(3.1) Instead of considering the negative weight of a universal deformation of X, we could as well investigate the non-positive weight part. This too, is representable by an affine morphism $p_{\leqq} : \mathscr{X}_{\leqq} \to S_{\leqq}$. If $S_0 \subset S_{\leqq}$ denotes the weight zero part, then there is a natural retraction $r: S_{\leqq} \overset{\leftarrow}{\to} S_0$ such that $S_- = r^{-1}(*)$. We regard S_0 as a germ at $*$. The morphism $p_{\leqq}$ is projectivized in the same way as $p_{\leqq}$:

$$\bar{p}_{\leqq} : \bar{\mathscr{X}}_{\leqq} \to S_{\leqq},$$

the main difference with $\bar{p}_-$ now being that the added part $\bar{\mathscr{X}}_{\leqq,\infty}$ need not be trivial over $S_{\leqq}$. The same arguments show that the fibres of $\bar{p}_{\leqq}$ are regular normal surfaces with trivial dualizing sheaf. In particular, each fibre of $p_0 : \mathscr{X}_0 \to S_0$ (the restriction of $p_{\leqq}$ over S_0) defines a Fuchsian singularity. It is well-known that the numerical data $(g(E); p_1, \ldots, p_m)$ are constant in this family, so that it is possible resolve the singular sections of p_0 lying on $\mathscr{X}_{0,\infty}$ in a uniform way. This gives a morphism

$$\tilde{p}_{\leqq} : (\tilde{\mathscr{X}}_{\leqq}, \tilde{\mathscr{X}}_{\leqq,\infty}) \to S_{\leqq}.$$

The following proposition will imply that in certain cases not too much information about the full universal deformation is lost by restricting ourselves to $p_{\leqq}$.

(3.2) **Proposition.** *If $H^0(\theta_{X_t}) = 0$ $(t \in S_{\leqq})$, then $p_{\leqq}$ is versal over t.*

Proof. Write Y, $\tilde{Y}$, $\tilde{Y}_\infty$ for X_t, $\tilde{X}_t$, $X_{t,\infty}$ and let $\Omega_{\tilde{Y}}\langle \tilde{Y}_\infty \rangle$ denote the sheaf of differentials on X having at most a logarithmic pole along $\tilde{Y}_\infty$. Its $\mathcal{O}_{\tilde{Y}}$-dual is the sheaf $\theta_{\tilde{Y}}\langle \tilde{Y}_\infty \rangle$ of derivations on $\tilde{Y}$ which preserve $\tilde{Y}_\infty$. The proposition will be a formal consequence of the following

(3.3) **Lemma.** $h^2(\theta_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle)=h^0(\theta_{\tilde{Y}})$.

Proof. The $\mathcal{O}_{\tilde{Y}}$-dual of $\theta_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle$ is $j_*j^*\Omega_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle$, where $j:\tilde{Y}_{\mathrm{reg}}\subset \tilde{Y}$ denotes the inclusion of the smooth part. By Serre duality, $h^2(\theta_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle)=h^0(\Omega_{\tilde{Y}_{\mathrm{reg}}}\langle \tilde{Y}_\infty\rangle)$. The cohomology sequence of

$$0\to\Omega_{\tilde{Y}_{\mathrm{reg}}}\to\Omega_{\tilde{Y}_{\mathrm{reg}}}\langle \hat{Y}_\infty\rangle\to\mathcal{O}_E\oplus\bigoplus_{\mu j}\mathcal{O}_{A^{\mu j}}\to 0$$

begins with

$$0\to H^0(\Omega_{\tilde{Y}_{\mathrm{reg}}})\to H^0(\Omega_{\tilde{Y}_{\mathrm{reg}}}\langle \tilde{Y}_\infty\rangle)\to\mathbb{C}_E\oplus\bigoplus_{\mu,j}\mathbb{C}_{A^{\mu j}}\xrightarrow{\delta} H^1(\Omega_{\tilde{Y}_{\mathrm{reg}}}).$$

The differential δ composed with the natural map

$$H^1(\Omega_{\tilde{Y}_{\mathrm{reg}}})\to H^1(\Omega_D)\cong\mathbb{C}_E\oplus\bigoplus_{\mu,j}\mathbb{C}_{A^{\mu j}}$$

is represented by the intersection matrix of $\{E, A^{\mu j}\}$. As the latter is nondegenerate ($\tilde{Y}-\tilde{Y}_\infty$ is affine), it follows that δ is injective. Hence $H^0(\Omega_{\tilde{Y}_{\mathrm{reg}}})\cong H^0(\Omega_{\tilde{Y}_{\mathrm{reg}}}\langle \tilde{Y}_\infty\rangle)$. A generating section of $\omega_{\tilde{Y}}$ determines an isomorphism $H^0(\Omega_{\tilde{Y}_{\mathrm{reg}}})\cong H^0(\theta_{\tilde{Y}_{\mathrm{reg}}})$. Since $\theta_{\tilde{Y}}$ is the dual of a coherent sheaf and $\tilde{Y}$ is normal, the last group is equal to $H^0(\theta_{\tilde{Y}})$.

Proof of (3.2). The spectral sequence

$$E_2^{pq}=H^p(\tilde{Y},\mathcal{E}xt^q(\Omega_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle,\mathcal{O}_{\tilde{Y}}))\ \Rightarrow\ \mathrm{Ext}^{p+q}(\Omega_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle,\mathcal{O}_{\tilde{Y}})$$

yields the exact sequence

$$0\to H^1(\theta_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle)\to\mathrm{Ext}^1(\Omega_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle,\mathcal{O}_{\tilde{Y}})\xrightarrow{\varrho_1}\bigoplus_{y\in \tilde{Y}_{\mathrm{sing}}}\mathrm{Ext}^1(\Omega_{\tilde{Y},y},\mathcal{O}_{\tilde{Y},y})$$

$$\to H^2(\theta_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle)\to\mathrm{Ext}^2(\Omega_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle,\mathcal{O}_{\tilde{Y}})\xrightarrow{\varrho_2}\bigoplus_{y\in \tilde{Y}_{\mathrm{sing}}}\mathrm{Ext}^2(\Omega_{\tilde{Y},y},\mathcal{O}_{\tilde{Y},y})\to 0.$$

Since $H^2(\theta_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle)=0$ by Lemma (3.3), ϱ_1 is surjective and ϱ_2 is an isomorphism. Interpreting ϱ_1 resp. ϱ_2 as a map between infinitesimal deformation spaces resp. obstruction spaces, we find that any deformation of the multi-germ $(\tilde{Y},\tilde{Y}_{\mathrm{sing}})$ is induced by a deformation of $\tilde{Y}$ preserving $\tilde{Y}_\infty$.

(3.4) **Corollary.** *If (X,x) is minimally-elliptic (or equivalently, if $\bar{X}_\infty$ is rational), then $p_{\leqq}$ is versal (in the global sense) off the weight zero part. In particular, any configuration of singularities on a fibre (multi-germ) of p occurs over $S_{\leqq}$, except the equisingular deformations of (X,x).*

Proof. Let $(\tilde{Y},\tilde{Y}_\infty):=(\tilde{X}_t,\tilde{X}_{\infty,t})$ be a fibre of $p_{\leqq}$. The assumption means that E is rational. Hence $E\cdot E=-2$ (by the adjunction formula). So any global vectorfield η on $\tilde{Y}$ will preserve E. For the same reason η will preserve the $A^{\mu j}$'s and thus defines a section of $\theta_{\tilde{Y}}\langle Y_\infty\rangle$. Since $m\geqq 3$, $\deg\theta_{\tilde{Y}}\langle \tilde{Y}_\infty\rangle\otimes\mathcal{O}_E=2-m<0$, so that in fact η lies in $H^0(\theta_{\tilde{Y}}(-\tilde{Y}_\infty))$. Combining this with Proposition 1.2 of [5] enables us to identify $H^0(\theta_{\tilde{Y}})$ with $H^0(\theta_{\tilde{Y}}(-\bar{Y}_\infty))$. The latter is just the Lie algebra of the group G_Y of automorphisms of $\bar{Y}$ acting as the identity on $\bar{Y}_\infty$. Following Theorem (A.2), G_Y is a subgroup of the group G of automorphisms of $(\bar{X}_{r(t)},\bar{X}_{\infty,r(t)})$. Without loss of generality we may assume that $(\bar{X}_{r(t)}, \bar{X}_{\infty,r(t)}) = (\bar{X}, \bar{X}_{\infty})$. Since G_Y acts trivially on $\bar{X}_{\prime}$, it follows that $G_Y \subset \mathbb{C}^*$. So $H^0(\theta_{\tilde{Y}}) \neq 0$ implies that $\tilde{Y}$ is $\mathbb{C}^*$-invariant and hence that $t \in S_0$. This proves the first part of the corollary. The rest follows from the observation that any $\mathbb{C}^*$-invariant neighbourhood of $S_{\leqq} - S_0$ in S is in fact a neighbourhood of S_0 in S from which $S_{\geqq}$ has been deleted.

(3.5) *Remarks*. 1. Wahl proved in [24] that the miniversal deformation of $\bar{X}$ with X as in (3.4) (so without preserving $\bar{X}_{\infty}$) induces a versal deformation of the germ (X, x). For (X, x) a complete intersection, Corollary (3.4) follows from work of Wirthmüller [25] and Damon [6].

2. With some more effort it can be shown that p is topologically trivial along S_0, compare the work of Wirthmüller and Damon, cited above.

3. It can be shown that in the situation of (3.4), $S_{\leqq}, *)$ is smooth and is not an irreducible component of $(S, *)$. It follows that the irreducible components of the affine variety $S_{\leqq}$ bijectively correspond to those of the germ $(S, *)$. This correspondence preserves the configuration of the singularities of the generic fibre, in particular smoothing components correspond to smoothing components.

4. Corollary (3.4) is false if we replace $S_{\leqq}$ by S_-. This was already observed by Pham [17] some time ago.

4. *K*3 Surfaces with Certain Curve Configurations

In Sect. 2 we found that the fibres of $\tilde{p}_-$ with at most RDP's were in fact $K3$ surfaces equipped with a curve isomorphic to $\tilde{X}_{\infty}$. Theorem (1.7) provided a converse to this: any pair $(\tilde{Y}, \tilde{Y}_{\infty})$ in which $\tilde{Y}$ is a $K3$ surface with only RDP's and $\tilde{Y}_{\infty}$ a curve in the smooth part of $\tilde{Y}$ isomorphic to $\tilde{X}_{\infty}$ such that $\tilde{Y} - \tilde{Y}_{\infty}$ is affine, occurs in fact as a fibre of $\tilde{p}_-$. The main goal of this section is to characterize such pairs $(\tilde{Y}, \tilde{Y}_{\infty})$, in terms of the structure of $\mathrm{Pic}(\tilde{Y})$ and the classes of the irreducible components of $\tilde{Y}_{\infty}$ in $\mathrm{Pic}(\tilde{Y})$. Our approach fails in two aspects: (i) the discrete nature of Pic forces us to neglect the moduli $\tilde{X}_{\infty}$ might have, so our results only concern the generic point of the weight zero part S_0 of S, but more seriously (ii): we cannot treat the cases where E is rational and the number n of nodal strings exceeds 4. So our results are optimal for the triangle singularities since S_0 is then a singleton. The elimination of (i) should involve the explicit construction of certain automorphic forms on the period domain of certain $K3$ surfaces, see the remarks (4.11).

(4.1) We begin some remarks and definitions pertaining to root systems. Suppose we are given a lattice Q (i.e. a free $\mathbb{Z}$-module of finite type equipped with a symmetric bilinear form). For the purpose of this paper we call a subset B of Q a *root basis* if $\alpha \cdot \alpha = -2$ for all $\alpha \in B$ and $\alpha \cdot \beta \geqq 0$ for all distinct $\alpha, \beta \in B$. Then each $\alpha \in B$ determines an orthogonal reflection s_α in $Q: x \mapsto x + (x \cdot \alpha)\alpha$. The group W_B generated by these reflections is the *Weyl group* of B and the W_B-orbit R of B is the *root system* generated B (occasionally we also use this terminology if B is an arbitrary set of elements in Q of square norm -2). The *intersection graph* of B is simply the graph on B obtained by connecting distinct $\alpha, \beta \in B$ with $\alpha \cdot \beta$ bonds. If this graph is connected we call B *indecomposable*. We mention some facts (see [2] and [22] for more discussion), in case B spans a negative semi-definite sublattice of Q.

(i) R is finite if and only if B spans a negative definite sublattice.

(ii) If Q is negative definite, then $\{\alpha \in Q : \alpha \cdot \alpha = -2\}$ is a root system (with respect to some root basis B).

(iii) If B spans a negative definite lattice, and is indecomposable, then B is of type A_l $(l \geqq 1)$, D_l $(l \geqq 4)$ or E_l $(l = 6, 7, 8)$. Any other root basis for R is in the W_B-orbit of B.

(iv) If B is indecomposable and spans a degenerate negative semi-definite lattice, then B is of type $\hat{A}_l$ $(l \geqq 1)$, $\hat{D}_l$ $(l \geqq 4)$ or $\hat{E}_l$ $(l = 6, 7, 8)$. The radical of $\mathbb{Z} \cdot B$ is spanned by a unique element of the form $n = \sum_{\alpha \in B} n_\alpha \alpha$ with all $n_\alpha > 0$, called the *fundamental isotropic element*. Any root bases for R having the same fundamental isotropic element as B is in the W_B-orbit of B. We call R an *affine root system* in this case.

(v) If Q is a negative semi-definite lattice, whose radical is of rank one, then $\{\alpha \in Q : \alpha \cdot \alpha = -2\}$ decomposes orthogonally (and uniquely) into root systems of affine type only.

(vi) Let B be an affine root basis with fundamental isotropic element n and let A be the affine space $\{\phi \in \mathrm{Hom}(\mathbb{R}\,.\,B, \mathbb{R}) : \phi(n) = 1\}$. Then W_B acts as a discrete group of affine-linear transformations on A, whose subgroup of translations may be identified with the lattice $\mathbb{Z}\,.\,B/\mathbb{Z}n$ under the correspondence $q \in \mathbb{Z}\,.\,B \mapsto t_q$ where $t_q(\phi)(n) = \phi(x) + q \cdot x$. A strict fundamental domain for this action is the closed simplex $\bar{\Delta} := \{\phi \in A : \phi(\alpha) \geqq 0 \text{ for all } \alpha \in B\}$.

(4.2) Consider a $D_{p_1, \ldots, p_m}(g)$ singularity (X, x). The irreducible components $E, A^{1,1}, \ldots, A^{m, p_m - 1}$ of $\tilde{X}_\infty$ in $\mathrm{Pic}(\tilde{X})$ generate a lattice with a distinguished basis B which we denote by $Q_{p_1, \ldots, p_m}(g)$ (or $Q_{p_1, \ldots, p_m}$ when $g = 0$). So if we denote the elements of B, $e, a^{1,1}, \ldots, a^{m, p_m - 1}$, then $e^2 = 2 - 2g$, $(a^{i,j})^2 = -2$, etc. Recall from (1.7) that $p_\mu \geqq 2$ for all μ and $\sum_{i=1}^{m} 1/p_\mu < 2g - 2 + m$ (this last inequality is automatic if $g \geqq 2$ or $m \geqq 5$). The lattice is nondegenerate and has signature $(1, \#(B) - 1)$.

Suppose $g = 0$. Then for $m = 3, 4$ there is a (unique) nonzero element $n = \Sigma_{\alpha \in B} n_\alpha \alpha$ with $n_\alpha \in \mathbb{Z}_+$ and $n \cdot \alpha \in \{0, 1\}$ for all $\alpha \in B$ (if $m \geqq 5$ such an element does not exist). The coefficients n_α are given in the diagrams below.

$Q_{2,3,p_3}$, $p_3 \geqq 7$

Q_{2,p_2,p_3}, $p_2 p_3 \geqq 4$

Q_{p_1,p_2,p_3}, $p_i \geqq 3$

Q_{p_1,p_2,p_3,p_4}, $p_i \geqq 2$

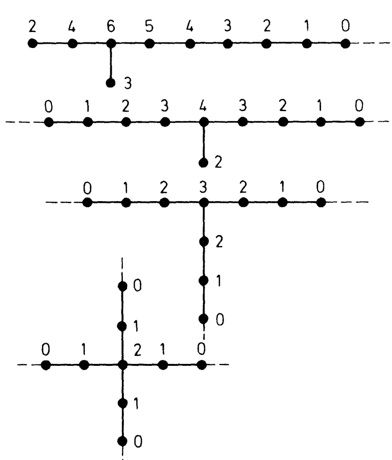


Fig. 3

We denote $\operatorname{supp}(n)=\{\alpha\in B: n_\alpha \neq 0\}$ by B_0. This is an affine root basis (of type $\hat{E}_8$, $\hat{E}_7$, $\hat{E}_6$, $\hat{D}_4$, respectively) whose fundamental isotropic element is just n. We often refer to n as the *fundamental isotropic element of* B (or of $Q_{p_1,\ldots,p_m}$). An $\alpha\in B_0$ with $n_\alpha=1$ is called *special* (in accordance with a similar terminology for affine Weyl groups).

Now suppose we are given an embedding j of $Q_{p_1,\ldots,p_m}$ $(m=3,4)$ in some lattice M. We say that j is *critical* if there is an affine root system in M which *strictly* contains the affine root system generated by $j(B_0)$. (This is equivalent to the existence of an $x\in M$ with $x\cdot x=-2$ such that $j(B_0)$ and x generate a negative semi-definite lattice and x is not orthogonal to B_0.) Since an affine system of type $\hat{E}_8$ is not contained in any larger affine system, embeddings of $Q_{2,3,6+t}$ $(t>0)$ are never critical. We say that the embedding j is *bad* if the induced embedding of $Q_{p_1,\ldots,p_m}$ in the primitive saturation of the image of j [i.e. $j(Q_{p_1,\ldots,p_m})_{\mathbb{Q}}\cap M$] is critical or equivalently, if j factors over a critical embedding $Q_{p_1,\ldots,p_m}\to M'$ with finite cokernel. Otherwise j is called a *good* embedding.

(4.3) In order to relate the preceding with the fibres of $\tilde{p}_-$, we have to recall a few facts on algebraic $K3$ surfaces. For proofs and more discussion we refer to [14] and [13, Sect. 1]. For a (minimal, nonsingular) $K3$ surface Z, $H_2(Z,\mathbb{Z})$ is free of rank 22. Equipped with the intersection pairing, $H_2(Z,\mathbb{Z})$ is isomorphic to the lattice $(-E_8)^{\oplus 2}\oplus H^{\oplus 3}$. The latter is an even unimodular lattice of signature (3, 19); we shall denote it by L. If ω is nonzero holomorphic 2-form on Z, then $\omega\wedge\omega=0$, $\omega\wedge\bar{\omega}>0$, so that its cohomology class $[\omega]\in H^2(Z,\mathbb{C})$ satisfies similar (in)equalities: $[\omega]\cdot[\omega]=0$, $[\omega]\cdot[\bar{\omega}]>0$. If we regard ω as a homomorphism $H_2(Z,\mathbb{Z})\to\mathbb{C}$, then its kernel may be identified with the Picard group $\operatorname{Pic}(Z)$ of Z. This is a nondegenerate sublattice of $H^2(Z,\mathbb{Z})$ of signature $(1,\varrho-1)$, where $\varrho=rk\operatorname{Pic}(Z)$. So $\{x\in\operatorname{Pic}(Z)_{\mathbb{R}}: x\cdot x>0\}$ has two connected components (an open cone and its antipode). Only one of these contains ample classes; we call this the *positive cone.*

(1) If $x\in\operatorname{Pic}(Z)$ is such that $x\cdot x\geqq -2$, then $\pm x$ is representable by an effective divisor.

(2) If C is an irreducible curve on Z with $C\cdot C=-2$, then C is smooth and rational (such C are called *nodal curves*).

We denote by R_Z the set of $\alpha\in\operatorname{Pic}(Z)$ with $\alpha\cdot\alpha=-2$ and let $B_Z\subset R_Z$ be the set of classes of nodal curves. Notice that the group W_Z generated by the reflections s_α, $\alpha\in R_Z$, leaves $\operatorname{Pic}(Z)$ and the positive cone invariant. It can be shown that B_Z is a root basis for R_Z (so that W_Z is its Weyl group), but we won't need this.

(3) If $d\in\operatorname{Pic}(Z)$ is in the closure of the positive cone, then there is a unique element d' in its W_Z-orbit such that $d'\cdot\alpha\geqq 0$ for all $\alpha\in B_Z$.

(4) If $d\in\operatorname{Pic}(Z)$ is in the positive cone and $d\cdot\alpha\geqq 0$ for all $\alpha\in B_Z$, then the linear system $|d|$ has no fixed points and contains a smooth irreducible curve of genus $\frac{1}{2}d\cdot d+1$. Furthermore, $R^d=\{\alpha\in R_Z:\alpha\cdot d=0\}$ is a finite root system of which $B_Z\cap R^d$ is a root basis. The union of the nodal curves corresponding to the elements of $B_Z\cap R^d$ is exceptional locus of the morphism $\|d\|: Z\to\operatorname{Proj}\bigoplus_{n=0}^{\infty} H^0(nd)$ yielding a bijection between the set of indecomposable components of R^d and the singularities (RDP's) of the latter variety.

(5) If $d \in \mathrm{Pic}(Z) - \{0\}$ is a primitive (= indivisible) element with $d \cdot d = 0$ such d is on the boundary of the positive cone and $d \cdot \alpha \geqq 0$ for all $\alpha \in B_Z$, then $|d|$ defines an elliptic fibration $Z \to \mathbb{P}^1$. The types of the reducible fibres of this fibration can be read off from the position of d in $\mathrm{Pic}(z)$: $R^d := \{\alpha \in R_Z : \alpha \cdot d = 0\}$ decomposes into mutually orthogonal affine root systems $R_1^d, \ldots, R_k^d$, $B_Z \cap R_i^d$ is a root basis of R_i^d and the nodal curves corresponding of the elements of $B_Z \cap R_i^d$ make up a reducible fibre of $|d|$. This yields a bijection between the set of indecomposable summands of R^d and the set of reducible fibres of $|d|$.

We are now ready to state and prove the main result of this section. It generalizes *I*, Proposition 3.

(4.4) **Theorem.** *Let Z be a $K3$ surface endowed with an embedding $j: Q_{p_1, \ldots, p_m} \to \mathrm{Pic}(Z)$ with $m = 3$ or 4. Then the following are equivalent*

(i) *There exists a $w \in W_Z$ such that $\pm w \circ j$ maps every $\alpha \in B$ on the class of a nodal curve.*

(ii) *The embedding j is good.*

Almost simultaneously we prove:

(4.5) **Theorem.** *Let Z be a $K3$ surface endowed with an embedding $j: Q_{p_1, \ldots, p_m}(g) \to \mathrm{Pic}(Z)$, $g > 0$. Then there exists a $w \in W_Z$ such that $\pm w \circ j$ maps every $\alpha \in B$ on the class of a smooth irreducible curve.*

Proof of (4.4). Let us first show (i) $\Rightarrow$ (ii). Since (ii) is a property invariant under automorphisms in $\mathrm{Pic}(Z)$, we may assume that each $j(\alpha)$, $\alpha \in B_0$ represents a nodal curve C_α. Then $N := \Sigma_{\alpha \in B_0} n_\alpha C_\alpha$ is a fibre of an elliptic fibration $|N|: Z \to \mathbb{P}^1$. Let $\beta \in \mathrm{Pic}(Z)$ be such that $\beta \cdot \beta = -2$, $\beta \cdot n = 0$ [we identify $Q_{p_1, \ldots, p_m}$ with its image in $\mathrm{Pic}(Z)$]. Represent $\pm\beta$ by an effective divisor D. Since $D \cdot N = 0$, D must be contained in a union of fibres of $|N|$. Then D is linearly equivalent to $\lambda N + D_0$ for some $\lambda \geqq 0$ where D_0 is connected and effective. If $\mathrm{supp}(D_0) \subset \mathrm{supp}(N)$, then it is clear that $\beta \in \mathbb{Z}.B_0$. Otherwise $\mathrm{supp}(D_0)$ and $\mathrm{supp}(N)$ are disjoint and so β is orthogonal to $\mathbb{Z}.B_0$.

We begin the proof (ii) $\Rightarrow$ (i). By composing j with some element of $\{\pm 1\}.W_Z$ we may assume that n is the class of an elliptic fibration. The hypothesis (ii) just means that B_0 is a root basis of a component R_0 of the disjoint union of affine root systems $\{\alpha \in R_Z : (\alpha \cdot n) = 0\}$. Since $B_Z \cap R_0$ is also a root basis of R_0, there exists a $w \in W(R_0)$ which maps B_0 onto $B_Z \cap R_0$. So we may in fact assume that each $\alpha \in B_0$ is represented by a nodal curve C_α. Now n is represented by the Kodaira cycle $N := \Sigma_{\alpha \in B_0} n_\alpha C_\alpha$. From here we proceed with induction on the cardinal of $B - B_0$: let $\alpha \in B$ be an "end" of B (i.e. α is connected with a unique $\alpha' \in B - \{\alpha\}$) such that $\alpha \notin B_0$. Put $B' := B - \{\alpha\}$. The inductive step is now provided by Lemma (4.7) below.

Proof of (4.5). Since $e \cdot e \geqq 0$, there exists a $w \in \{\pm 1\} \cdot W_Z$ such that $w(e)$ can be represented by a smooth irreducible curve E. If $B \neq \{e\}$, then let $\alpha \in B$ be an end of B and assume inductively that each element β of $B' := B - \{\alpha\}$ can be represented by a smooth irreducible curve C_β. If we write n for e, N for E and B_0 for $\{e\}$, then again the inductive step is provided by Lemma (4.7).

(4.6) **Lemma.** *We can represent α by an effective divisor D which is relatively prime to $C:=\Sigma_{\beta\in B'}C_\beta$.*

Proof. Since $\alpha\cdot\alpha=-2$, $\pm\alpha$ is representable by an effective divisor D. In fact, only $+\alpha$ is: choose an effective divisor C' without fixed components and $\mathrm{supp}(C')=\mathrm{supp}(C)$: then $\alpha\cdot[C']>0$ and $D\cdot C'\geqq 0$, so $\alpha=[D]$. Now write $D=D'+\Sigma_{\beta\in B'}m_\beta C_\beta$ with D' relatively prime to C. By changing D in its linear equivalence class, we may assume that $\Sigma_{\beta\in B'}m_\beta C_\beta$ is a fixed part of $|D|$. By the Riemann-Roch inequality, $\Sigma_\beta m_\beta C_\beta=0$ or $\dim H^0(\mathcal{O}_Z(\Sigma_\beta m_\beta C_\beta))\geqq 2+\frac{1}{2}(\Sigma_\beta m_\beta C_\beta)^2$, so in the last case $(\Sigma_\beta m_\beta C_\beta)^2<0$. This implies that D' meets C: otherwise we would have $(\Sigma_\beta m_\beta C_\beta)^2=D\cdot(\Sigma_\beta m_\beta C_\beta)=m_{\alpha'}\geqq 0$. But then $\Sigma_\beta m_\beta C_\beta=0$ and so $D'\cdot C=D\cdot C=1$ contradicts $D'\cdot C=0$.

Let us first consider the case when $B'\neq B_0\cup\{\alpha'\}$ (recall that α' is the unique element of B' which is connected with α). Then $D\cdot N=\alpha\cdot n=0$, for α' does not occur in n. Since a multiple of N has no fixed components, it follows that $D'\cdot N=0$ and $\Sigma m_\beta C_\beta\cdot N=0$. Choose $\alpha''\in B-B_0-\{\alpha'\}$ connected with B_0. It follows from $0=N\cdot\Sigma m_\beta C_\beta\geqq m_{\alpha''}\geqq 0$ that $m_{\alpha''}=0$. Since $D\cdot C_{\alpha''}=\alpha\cdot\alpha''=0$, this implies that $m_\beta=0$ for all $\beta\in B'$ adjacent to α''. Proceeding with induction we find that $D=D'$ (the induction works since B' is connected).

If $B'=B_0\cup\{\alpha'\}$, then the preceding argument shows that $m_{\alpha'}=0$ so that $\Sigma m_\beta C_\beta$ has the same support as N. Since $(\Sigma m_\beta C_\beta)\cdot N=0$, this can only be if $\Sigma m_\beta C_\beta$ is a multiple of N. But now $(\Sigma m_\beta C_\beta)^2=0$ implies that $D=D'$.

If $B'=B_0$, then we only need to do the case when B_0 is an affine root basis. In that case we apply fact (vi) of (4.1) to the linear forms d resp. d' on $V:=\mathbb{R}.B_0$ defined by D resp. D'. Since $d,d'\in\bar{\Delta}$ and $d-d'$ differ by a translation t_q $(q=\Sigma m_\beta\beta+\mathbb{Z}n)$, it follows that $q\in\mathbb{Z}n$. In other words $\Sigma m_\beta\beta$ is a multiple of n. Since n is isotropic, this implies that $\Sigma m_\beta\beta=0$.

(4.7) **Lemma.** *There exists a $w\in W_Z$ which leaves B' pointwise fixed such that $w(\alpha)$ can be represented by a nodal curve.*

Proof. Fix an ample class $\eta\in\mathrm{Pic}(Z)$. By the preceding lemma, α may be represented by an effective divisor D relatively prime to C. Since $D\cdot C_\beta=0$ if $\beta\neq\alpha'$, $D\cdot C_{\alpha'}=1$, it follows that D is of the form C_0+D' with $C_0\cdot C_\beta=0$ if $\beta\neq\alpha'$, $C_0\cdot C_{\alpha'}=1$ and $\mathrm{supp}(D')\cap\mathrm{supp}(C)=\emptyset$. Write $D'=D''+\sum_{i=1}^{k}m_iD_i$, where $D_1,\ldots,D_k$ are nodal, $m_i\geqq 0$ and each irreducible component of D'' has self-intersection $\geqq 0$. If $D\cdot D_i<0$ for some i, then replace α by $\tilde{\alpha}:=s_{[D_i]}(\alpha)$. Since $s_{[D_i]}$ leaves B' invariant, $\tilde{\alpha}$ can be represented by an effective divisor (by the above lemma) so that $\tilde{\alpha}\cdot\eta>0$. On the other hand, $\tilde{\alpha}\cdot\eta=(\alpha\cdot\eta)+(D_i\cdot D)<(\alpha\cdot\eta)-1$, which shows that we cannot repeat this process indefinitely. So we may assume that $D\cdot D_i\geqq 0$, $i=1,\ldots,k$. We show that this implies $D=C_0$. Then $C_0^2=D^2=-2$ and we are done.

From the chain of (in)equalities

$$\begin{aligned}-2=D^2=D(C_0+D''+\Sigma m_iD_i)&\geqq D(C_0+D')\\ &\geqq C_0^2+(D'')^2+2(C_0\cdot D'')+\Sigma m_iD_i(C_0+D'')\geqq -2\end{aligned}$$

it follows that C_0,D'' and Σm_iD_i are orthogonal and that $C_0^2=-2$, $(D'')^2=(\Sigma m_iD_i)^2=0$. In particular, $C_0\cdot D'=0$ and $(D')^2=0$.

Suppose $D' \neq 0$. Since $C \cdot D' = 0$, $C^2 \geqq 0$, this can only be if C and D' a proportional in $\mathrm{Pic}(Z)$: $D' \sim \lambda C$ for some $\lambda \in \mathbb{Q} - \{0\}$. But then $C_0 \cdot D' = C_0 \cdot \lambda C = \lambda \neq 0$ yields a contradiction. So $D' = 0$ and hence $D = C_0$.

Combining Theorems (4.4), (4.5) with result of the earlier sections yields the following smoothability criteria for Fuchsian singularities.

(4.8) **Corollary.** *Some $D_{p_1,\ldots,p_m}(g)$-singularity with $g > 0$ is smoothable if there exists an embedding of the lattice $Q_{p_1,\ldots,p_m}(g)$ in the K3 lattice $L = (-E_8)^{\oplus} \oplus H^{\oplus 3}$.*

(4.9) **Corollary.** *The triangle singularity or some quadrilateral singularity $D_{p_1,\ldots,p_m}$ $(m = 3, 4)$ is smoothable if and only if there exists a good embedding of the lattice $Q_{p_1,\ldots,p_m}$ in the K3 lattice L.*

(4.10) Since $Q_{p_1,\ldots,p_m}$ resp. L has signature $(1, p_1 + \ldots + p_m - m)$ resp. $(3, 19)$ it follows that in the case $g = 0$ a necessary condition for smoothability is $p_1 + \ldots + p_m \geqq 19 + m$, a result earlier obtained by Wahl [24]. Pinkham, in an appendix [20] to I, has shown that in the triangle case $(m = 3, g = 0)$, the condition $p_1 + p_2 + p_3 \leqq 22$ implies the existence of a good, usually primitive, embedding of Q_{p_1,p_2,p_3} in L except when $(p_1, p_2, p_3) = (2, 10, 10)$. He also shows the non-smoothability of this singularity by means of geometric methods. (An alternative proof consists in showing that there is no embedding of $Q_{2,10,10}$ in L.) The fine structure of the smoothing components of triangle singularities will be discussed in the following sections and in greater detail in Part III.

(4.11) When one tries to eliminate the adjective "some" in Corollary (4.9) one is confronted with the interesting problem of expressing the analytic invariants of nodal curve configurations on a $K3$ surface in terms of the periods of that surface. For instance, if X is a $K3$ surface on which we have a $\hat{D}_4$ configuration of nodal curves, then the four crossing points have a cross ratio on the central curve and the problem is to express this cross ratio in terms of the periods.

5. Structure of Critical Embeddings

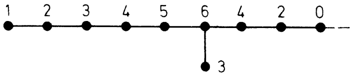
In the remainder of this paper, we shall limit ourselves to triangle singularities. For a detailed study of the negatively weighted part of a miniversal deformation of such a singularity it is necessary to have a simple characterization for the notions "good" and "critical" introduced in the last section. This is provided by Theorem (5.1) below and its corollaries.

(5.1) **Theorem.** *An embedding of $Q := Q_{p,q,r}$ in a lattice M is critical if and only if*

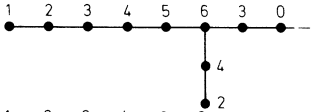
(1) *there exist two ends $\alpha, \beta \in B$ of B not in B_0 and an $y \in M$ such that $y \cdot y = -2$, $y \cdot \alpha = 1$, $y \cdot \beta = -1$, and $y \cdot \gamma = 0$ for all $\gamma \in B - \{\alpha, \beta\}$. If (after some permutation of p, q, r) α resp. β is the end of the p resp. q branch, then there is an isomorphism of $Q_{p,q,r} + \mathbb{Z}y$ onto $Q_{2,r+1,p+q-2}$ which maps the fundamental isotropic of $Q_{p,q,r}$ onto the one of $Q_{2,r+1,p+q-2}$ or*

(2) *there exist an isotropic $m \in M$ and an integer $k \geqq 2$ such that the orthogonal projection $\bar{m}$ of m in $Q_{\mathbb{Q}}$ is of the form $\frac{1}{k} n_1$, where $n_1 \in \mathbb{Z}_+$. B is a nonzero isotropic*

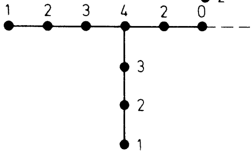
element. After a possible permutation of p, q, r we are in one of the following three cases

(2i) $(p, q, r) = (2, 6, 3+t)$, $n_1 =$ 

, $k=2$.

(2ii) $(p, q, r) = (3, 6, 2+t)$, $n_1 =$ 

, $k=3$.

(2iii) $(p, q, r) = (4, 4, 2+t)$, $n_1 =$ 

, $k=2$.

Fig. 4

Moreover, there is an isomorphism of $Q + \mathbb{Z}\frac{1}{k}n_1$ onto $Q' := Q_{2,3,6+t}, Q_{2,3,6+t}, Q_{2,4,4+t}$ respectively which sends the fundamental isotropic element of Q to the one of Q'.

(5.2) **Corollary.** *An embedding of $Q_{p,q,r}$ is bad if and only if $\{p, q, r\}$ belongs to one of the triples listed under* (2) *of* (5.1) *and $\frac{1}{k}n_1 \in M$, in other words, the embedding factors over one the bad embeddings*

$$Q_{2,6,3+t} \hookrightarrow Q_{2,3,6+t}, \qquad Q_{3,6,2+t} \hookrightarrow Q_{2,3,6+t}, \qquad Q_{4,4,2+t} \hookrightarrow Q_{2,4,4+t}.$$

In particular, any primitive embedding is good.

(5.3) **Corollary.** *A good embedding $Q_{p,q,r} \hookrightarrow M$ is critical if and only if after a permutation of p, q, r, it factors over one of the embeddings j listed below.*

(1) *The embedding $j: Q_{p,q,r} \hookrightarrow Q_{2,p+1,q+r-2}$ if* (5.1), *case* (1) *or*
(2) *The triple (p, q, r) is listed under* (2) *of* (5.1) *and j is the composite*

$$Q_{p,q,r} \subset Q_{p,q,r} \oplus (0) \subset Q'_{p,q,r}.$$

Here (0) *denotes the* rank 1 *isotropic lattice (with generator u), the first inclusion is the obvious one and*

$$Q'_{p,q,r} := Q_{p,q,r} \oplus (0) + \frac{1}{k}\mathbb{Z}(n_1, u).$$

So we are in one of the following cases

(2i) $j: Q_{2,6,3+t} \hookrightarrow Q_{2,3,6+t} \oplus (0)$
(2ii) $j: Q_{3,6,2+t} \hookrightarrow Q_{2,3,6+t} \oplus (0)$
(2iii) $j: Q_{4,4,2+t} \hookrightarrow Q_{2,4,4+t} \oplus (0)$.

Proof of (5.1). We begin with the "if" part. First case (1). Let y, α, and β be as in the theorem. Without loss of generality we may assume that $\alpha = a_1$ and $\beta = b_1$. Consider the element $y' \in y + Q$ with the coefficients v_γ of $y' - y = \Sigma_{\gamma \in B} v_\gamma \gamma$ given by

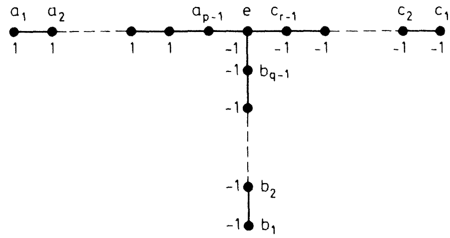


Fig. 5

It is easily seen that $y' \in W_B y$, so that $y' \cdot y' = -2$. Then the intersection diagram on

$$B' := \{a_{p-1}, y', c_1, \ldots, c_{r-1}, -a_{p-3}, \ldots, -a_1, -y, b_1, \ldots, b_{q-1}, e\}$$

is of type $T_{2,r+1,p+q-2}$. Now is B' a basis of $Q + \mathbb{Z}y$ and in this basis n is given by

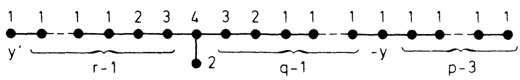


Fig. 6

which is clearly $W_{B'}$-equivalent to the fundamental isotropic element of B'.

In the cases 2(i)–(iii), observe that $n \cdot n_1 = k$. So $\delta := n - \frac{1}{k} n_1$ is an element of $Q + \mathbb{Z}\frac{1}{k} n_1$ with $\delta \cdot \delta = -2$. Let α be the unique element of $\mathrm{supp}(n_1)$ connected with B-$\mathrm{supp}(n_1)$ and let β be any special vertex of $\mathrm{supp}(n)$. If B_1 is obtained from B by replacing α by $\alpha - \frac{1}{k} n_1$ and β by $\beta + \frac{1}{k} n_1$, then B_1 generates $Q + \frac{1}{k} n_1$, the intersection diagram of B_1 is of the asserted type and its fundamental isotropic element equals $\frac{1}{k} n_1$. Since $-s_\delta$ maps n to n_1 and $\mathrm{supp}(n)$ into the integral span of $\mathrm{supp}(n_1)$, it follows that the embedding $Q \hookrightarrow Q + \mathbb{Z}\frac{1}{k} n_1$ is bad. Since $n_1 - km$ is in the radical of $Q + \mathbb{Z}m$ this implies that the embedding $Q \hookrightarrow Q + \mathbb{Z}m$ is critical.

Next we prove the "only if" part. Assume that $j: Q \hookrightarrow M$ is critical embedding. Let R be a maximal affine root system in M containing B_0. By definition, R strictly contains the affine system generated by B_0. Without loss of generality, we may assume that Q and R span M (otherwise, replace M by this span). If $I \subset M$ is a primitive isotropic sublattice then the contents of the statement we want to prove is not affected if we replace M by M/I. We therefore also assume that $Q^\perp$ is negative definite. This implies that R is *the* maximal affine root system in M containing B_0: if $\alpha \in R$ and $\beta \in M$ is such that $\beta \cdot n = 0$, $\beta \cdot \beta = -2$, $\beta \cdot \alpha \neq 0$, then $\beta \in R$.

After these preparations we make one further assumption: we suppose that B_0 is of type $\hat{E}_6$. The case when B_0 is of type $\hat{E}_7$ is dealt with in the same way (and slightly easier). From here the proof is completed in four steps. Straightforward, but possibly tedious calculations are usually left to the reader.

Step 1. Let $B_1 \subset B$ denote the set of $\alpha \in B$ connected with B_0. If we are not in case (2ii) of the theorem, then B_0 is contained in a root system R' of type $\hat{E}_7$ which is orthogonal to $B - B_1$.

Proof. Since B_0 is an $\hat{E}_6$ root basis, R is of type $\hat{E}_7$ or $\hat{E}_8$. If R is orthogonal to $B-B_1$, then we can always find an $\hat{E}_7$ system $R' \subset R$ which contains B_0.

If some $\delta \in B_1$ is not orthogonal to R, then $\delta \in R$ (for $\delta \cdot n = 0$) and by the same token, R contains the connected component B_2 of δ in $B-B_1$. Now is $R \cap (B-B_1)$ in the orthogonal complement of an $\hat{E}_6$ root basis in R. Since the orthogonal complement of an E_6 subsystem of an E_8 system is of type A_2, $R \cap (B-B_1)$ is of type A_1 or A_2. In particular, $R \cap (B-B_1) = B_2$. If $B_2 = \{\delta\}$, then R is of type $\hat{E}_8$ (for an E_7 system doesn't contain a $E_6 \times A_1$ system) and so the orthogonal complement R' of δ in R will be of type $\hat{E}_7$. Clearly, this R' is as required.

We finish the proof by showing that if B_2 is of type A_2 we are in case (2ii). We label the first four elements of the branch of B containing $B_2 : \alpha_3, \alpha_2, \alpha_1, \alpha_0$. We begin at the end, so $B_2 = \{\alpha_3, \alpha_2\}$ and α_0 is special vertex of B_0. Since R is in the $\mathbb{Q}$-span of $B_0 \cup B_2$, each $x \in R$ is uniquely written $\bar{x} + \lambda n$ with $\mathrm{supp}(\bar{x}) \subset (B_0 - \{\alpha_0\}) \cup B_2$ and $\lambda \in \mathbb{Q}$. We denote the elements $\bar{x}$ so obtained by $\bar{R}$. Note that $\bar{R}$ is a root system of type E_8 containing the $E_6 \times A_2$ root basis $(B_0 - \{\alpha_0\}) \cup B_2$. There are just two $W(E_8)$-equivalence classes of embeddings of an $E_6 \oplus A_2$ root basis in an E_8 root system; these two are interchanged by applying the involution of the E_6 factor. Using the standard embeddings as representative, it follows that there is a root $\bar{x} \in \bar{R}$ with $\bar{x} \cdot \alpha_3 = 0$, $\bar{x} \cdot \alpha_2 = -1$ and $\bar{x}$ orthogonal to all elements of $B_0 - \{\alpha_0\}$ except for a special $\beta_0 \in B_0$ different from α_0 with which $\bar{x}$ has inner product -1. A straightforward calculation shows that $\bar{x}$ is given by:

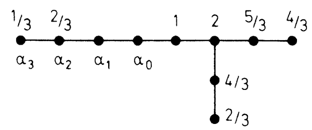


Fig. 7

As $x = \bar{x} + \lambda n$ must have integral inner product with α_1, we have $\lambda \equiv -2/3 \bmod \mathbb{Z}$. Since the coefficient of x on β_0 is then $2/3 \bmod \mathbb{Z}$, it follows that β_0 is an end of B. It is clear that $x \equiv \frac{1}{3} n_1 \bmod Q$ where n_1 is as in (2ii) of the theorem.

From now on we assume that we are not in case (2ii) of the theorem. We let R' be as in step 1.

Step 2. Let α_0 and β_0 be the two special elements of B_0 on the branches of B of shortest length. Then there exists an $x \in R'$ with $x \cdot \alpha_0 = 1$, $x \cdot \beta_0 = -1$ and orthogonal to all other elements of B_0.

Proof. Consider the E_6 root basis $B_0 - \{\beta_0\}$ and regard it as a subset of the $\hat{E}_7$ system R'. The embeddings of a E_6 root basis in an $\hat{E}_7$ root system fall into two $W(\hat{E}_7)$-equivalence classes; representatives are the standard inclusion and its composite with the involutive symmetry of the $\hat{E}_7$ root basis. It follows that there is an $x \in R'$ with $x \cdot \alpha_0 = 1$ which is orthogonal to $B_0 - \{\alpha_0, \beta_0\}$. Since $x \cdot n = 0$, this implies that $x \cdot \beta_0 = -1$.

Step 3. Let α_1 and β_1 be the elements of $B_1 - B_0$ adjacent to α_0 and β_0 respectively (if they exist). Then there exists an $x \in R'$ orthogonal to $B - \{\alpha_0, \beta_0, \alpha_1, \beta_1\}$ with $(x \cdot \alpha_0) = 1$, $(x \cdot \beta_0) = -1$, the ordered pair $((x \cdot \alpha_1), (x \cdot \beta_1))$ belongs to $(-1, 1)$, $(-1, 0)$, $(0, 1)$, $(0, 0)$. In the last case we are in the situation of (2iii) of the theorem.

Proof. Let x be as in step 2. By adding a suitable multiple of n to x we may assume that x is orthogonal to the unique element γ_0 of $B_1 - B_0$ distinct from α_0 and β_0. (This element does exist, since it is on the longest branch of B.) We let $\bar{x}$ denote the orthogonal projection of x in $Q \cdot B_1$. Since the orthogonal complement of B_1 in $\mathbb{Q} . B_1 + \mathbb{Q}x$ is negative definite we have $\bar{x} \cdot \bar{x} \geqq x \cdot x = -2$ and equality holds if and only if $x = \bar{x} \in \mathbb{Q} . B_1$. A straightforward calculation shows that the conditions $\bar{x} \cdot \bar{x} \geqq -2$, $\bar{x} \cdot \alpha_0 = 1$, $\bar{x} \cdot \beta_0 = -1$, $\bar{x}$ orthogonal to the other elements of B_0, are only satisfied in the four cases mentioned. In the last case we find that $\bar{x} \cdot \bar{x} = -2$ (so $x = \bar{x}$) and that $\bar{x}$ is given by

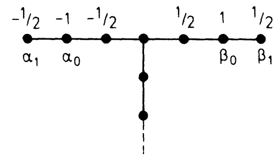


Fig. 8

This implies that α_1 and β_1 are ends of B. Clearly, $\bar{x} \equiv 1/2n_1 \bmod Q$, where n_1 is as in case (2iii) of the theorem.

Step 4. Completion of the proofs of (5.1) and (5.2).

We now suppose that we are not in the cases (ii) and (iii) of (5.1). Let $x \in R'$ be as in Step 3. Then following Step 3, there are three possibilities for the pair $((x \cdot \alpha_1), (x \cdot \beta_1))$. We define in each case a new element $y \in x + Q$ by letting $y = x \in Q$ be

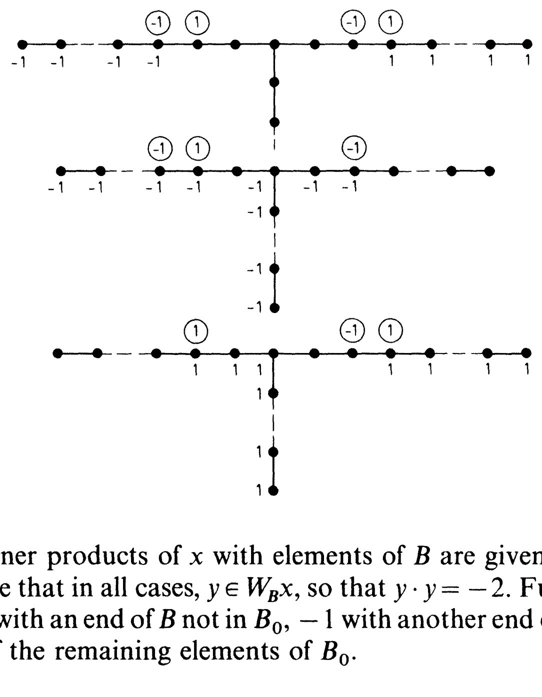


Fig. 9

(The nonzero inner products of x with elements of B are given by the encircled numbers.) Notice that in all cases, $y \in W_B x$, so that $y \cdot y = -2$. Furthermore, y has inner product 1 with an end of B not in B_0, -1 with another end of B not in B_0 and is orthogonal of the remaining elements of B_0.

6. The Rational Double Point Part

Let X be a $D_{p,q,r}$-singularity and S_- the negatively weighted part of its miniversal deformation. The *rational double point part* $S_{-,f}$ is defined as the set of $t \in S_-$ for which $\tilde{X}_s$ has only RDP's as singularities. With the help of the results of Sect. 4 and two fundamental (and deep) theorems on $K3$ surfaces (the Torelli theorem and the surjectivity of the period mapping) we shall show that each smoothing component of $S_{-,f}$ has in a natural way the structure of an orbit space.

We abbreviate $Q_{p,q,r}$ by Q.

(6.1) Let j be a good embedding of $Q=Q_{p,q,r}$ in the $K3$ lattice L. Then its orthogonal complement M has signature $(2, 22-p-q-r)$ (so $p+q+r \leqq 22$). We put

$$\Omega_j = \{\omega \in \mathrm{Hom}(L, \mathbb{C}) : j^*\omega = 0, \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0\},$$

where the nondegenerate inner product on L has been transfered to $\mathrm{Hom}(L, \mathbb{C})$ in the usual way. The set Ω_j is open in a quadric cone and has two connected components which are interchanged by complex conjugation. Let Ω be one of its connected components. The identity component of the real Lie group

$$G_{\mathbb{R}} = \{g \in \mathrm{Aut}(L_{\mathbb{R}}) : g \circ j = j\} = \mathrm{Aut}(M_{\mathbb{R}})$$

acts properly and transitively on $\Omega/\mathbb{C}^*$ as the symmetric domain associated to $G_{\mathbb{R}}$. It is of type IV for $p+q+r<20$, isomorphic to $\mathscr{H} \times \mathscr{H}$ for $p+q+r=20$, isomorphic to $\mathscr{H}$ for $p+q+r=21$ and a singleton for $p+q+r=22$. Put

$$\Gamma = \{g \in \mathrm{Aut}(L) : g \circ j = j, g(\Omega) = \Omega\}.$$

This is a subgroup of finite index of $\mathrm{Aut}(M)$ and hence an arithmetic subgroup of $G_{\mathbb{R}}$.

Let $\mathscr{E} \subset L$ denote the set of elements $y \in L$ which are as in (5.1)–(1). Clearly, $\mathscr{E}$ is Γ-invariant. For $y \in \mathscr{E}$, we set $\Omega(y) = \{\omega \in \Omega : \omega(y) = 0\}$.

(6.2) **Lemma.** *$\mathscr{E}$ is a finite union of Γ-orbits and $\{\Omega(y) : y \in \mathscr{E}\}$ is locally finite on Ω.*

Proof. If $y_1, y_2 \in L$ have the same component in $Q^\#$ under the inclusion $L \subset Q^\# \oplus M^\#$, then clearly their $M^\#$ components have the same norm (Here is

$$M^\# := \{x \in M_{\mathbb{Q}} : x \cdot u \in \mathbb{Z} \text{ all } u \in M\};$$

it contains M as a submodule of finite index.) Now it is well-known that the automorphism group of any lattice has only finitely many orbits in the set of primitive elements of given norm [1, 9.11 Theorem]. From this it easily follows that Γ has finitely many orbits in the set of $y \in L$ with given $Q^\#$-component. This proves that $\mathscr{E}$ consists of a finite number of Γ-orbits.

For the second part of the lemma we must show that a compact neighbourhood K of $\omega_0 \in \Omega$ meets only finitely many $\Omega(y)$, $y \in \mathscr{E}$. Let V_1 denote the orthogonal complement of $Q_{\mathbb{R}}$ in $\mathrm{Ker}(\omega_0)$ and let V_2 be the orthogonal complement of V_1 in $M_{\mathbb{R}}$. From $\omega_0 \cdot \omega_0 = 0$, $\omega_0 \cdot \bar{\omega}_0 > 0$ it follows that V_2 is a positive definite plane and that V_1 is negative definite. Write $y = y_Q + y_1 + y_2$ according to the decomposition $L_{\mathbb{R}} = Q_{\mathbb{R}} \oplus V_1 \oplus V_2$. If $y^0 \in \mathscr{E}$, then for any $y \in \Gamma y^0$

we have $y_Q = y_Q^0$. If moreover $\Omega(y) \cap K \neq \emptyset$, then y_1 must be contained in compact neighbourhood K_1 of 0 in V_1 (for V_1 is positive definite). Since $y \cdot y = y^0 \cdot y^0$ and V_2 is negative definite, y_2 is then in a compact $K_2 \subset V_2$. Since $L \cap (\{y_0\} \times K_1 \times K_2)$ is finite, the local finiteness follows.

We put $\Omega' := \Omega - \cup\{\Omega(y) : y \in \mathscr{E}\}$. By the previous lemma, Ω' is a connected open subset of Ω. From the definitions and (5.1), the following lemma is immediate.

(6.3) **Lemma.** *Let $\omega \in \Omega$. Then $\omega \in \Omega'$ if and only if the embedding $Q_{p,q,r} \hookrightarrow \mathrm{Ker}(\omega)$ induced by j is not critical.*

We return to the $D_{p,q,r}$-singularity X. We let $S(j, \Omega)$ denote the set of $t \in S_-$ with the property that (i) the fibre $\tilde{X}_t$ of $\tilde{p}_-$ has RDP's as singularities and (ii) if $\hat{X}_t \to \tilde{X}_t$ resolves these singularities minimally, then there exists an isomorphism $\phi: L \to H_2(\hat{X}, \mathbb{Z})$ of lattices such that $\phi \circ j$ is the natural embedding and $\phi^* \hat{\omega}_t \in \Omega$. (Here $\hat{\omega}_t$ denotes the lift of $\tilde{\omega}_t$.) If there is another such ϕ, then the two choices differ by an element of Γ, so that the assignment $t \mapsto \phi^* \hat{\omega}_t$ yields a well-defined mapping $\Phi: S(j, \Omega) \to \Omega/\Gamma$.

(6.4) **Theorem** (= Proposition 4 of I). *The mapping Φ maps $S(j, \Omega)$ isomorphically and $\mathbb{C}^*$-anti-equivariantly onto Ω'/Γ. Moreover, this sets up a bijective correspondence between the smoothing components of the $D_{p,q,r}$-singularity (X, x) and the* $\mathrm{Aut}(L_{\mathbb{R}})$*-equivalence classes of pairs (j, Ω) where $j: Q_{p,q,r} \to L$ is a good embedding and Ω is a component of Ω_j.*

Proof. The anti-equivariance is immediate from (1.5). The local Torelli theorem for $K3$ surfaces implies that Φ is a local isomorphism. To prove that Φ is injective, we use the global Torelli theorem for polarized $K3$ surfaces. If $\Phi(s) = \Phi(t)$, then by definition, there exists an isomorphism $\hat{\varphi}: H_2(\hat{X}_s, \mathbb{Z}) \to H_2(\hat{X}_t, \mathbb{Z})$ compatible with the natural embedding of $Q_{p,q,r}$ and such that $\hat{\varphi}^* \hat{\omega}_t = \hat{\omega}_s$. In particular, $\hat{\varphi}$ maps $\mathrm{Pic}(\hat{X}_s)$ onto $\mathrm{Pic}(\hat{X}_t)$. Now choose $d \in \mathbb{Z}_+ . B$ such that $d \cdot \alpha > 0$ for all $\alpha \in B$. Then d induces an ample class in $\mathrm{Pic}(\tilde{X}_s)$. If $x \in \mathrm{Pic}(\hat{X}_s)$ is such that $x \cdot x \geqq -2$, $x \notin R_s$, then $x \cdot d \neq 0$ and we have, $x \cdot d > 0$ if and only if x is effective. It follows that the effective classes in $\mathrm{Pic}(\hat{X}_s)$ are generated by B_s and $\{x \in \mathrm{Pic}(\hat{X}_s) : x \cdot x \geqq -2, x \cdot d > 0\}$.

Composing $\hat{\varphi}$ (on the left) with an element of the Weyl group of R_s enables us to assume that $\hat{\varphi}$ takes B_s to B_t. By the preceding, $\hat{\varphi}$ then takes effective classes to effective classes. The Torelli theorem mentioned above, asserts that $\hat{\varphi}$ is induced by an isomorphism $\hat{\Psi}: \hat{X}_s \to \hat{X}_t$. Clearly $\hat{\Psi}$ drops to an isomorphism $\tilde{\Psi}: \tilde{X}_s \to \tilde{X}_t$, compatible with the injections $\tilde{X}_\infty \hookrightarrow \tilde{X}_s$, $\tilde{X}_\infty \hookrightarrow \tilde{X}_t$. This proves the injectivity of Φ. The same argument shows that $S(j, \Omega) \cap S(j', \Omega') \neq \emptyset$ implies that (j, Ω) is Γ-equivalent to (j', Ω'). On the other hand, if $\tilde{X}_t$ has only RDP's as singularities, then t is in some $S(j, \Omega)$.

From Lemmas (6.3) and (5.3) it is immediate that Φ maps into Ω'/Γ. In order to prove that this is in fact the image of Φ, we use the surjectivity of the period map for $K3$ surfaces, which implies that for any $\omega \in \Omega'$ there exist a $K3$ surface $\hat{Y}$ and an isomorphism $\phi: H_2(\hat{Y}, \mathbb{Z}) \to L$ such that $\phi^* \omega$ is the class of a holomorphic 2-form ω_Y on $\hat{Y}$. Since $\omega \in \Omega'$, the embedding $\phi^{-1} \circ j: Q_{p,q,r} \subset \mathrm{Ker} \phi^* \omega$ is not critical. Altering ϕ by an element of the Weyl group of $\hat{Y}$ we may assume by (4.5) that $\phi^{-1} \circ j$ maps each element of B to the class of a nodal curve. Let $\hat{Y}_\infty$ denote the union of these nodal curves and let $(\hat{Y}, \hat{Y}_\infty) \to (\tilde{Y}, \tilde{Y}_\infty)$ contract all nodal curves disjoint from

$\hat{Y}_\infty$. Then $\hat{\omega}_{\hat{Y}}$ is the pull-back of a unique $\omega_{\tilde{Y}}$ on $\tilde{Y}$ and by (1.8), $(\tilde{Y}, \tilde{Y}_\infty, \omega_{\tilde{Y}})$ is isomorphic to a fibre of $\tilde{p}_-$.

(6.5) **Corollary.** *A configuration V of* RDP*'s occurs as the singular locus over some fibre of $\tilde{p}_-$ over $S(j, \Omega)$ if and only if the good embedding j factors as $Q \hookrightarrow P \hookrightarrow L$ where $Q \to P$ is non-critical, $P \hookrightarrow L$ is primitive, and*

$$R := \{x \in P \cap Q^\perp : x \cdot x = -2\}$$

is a finite root system whose indecomposable components are in bijective and type-preserving correspondence with the elements of V. (We may, of course, assume that $P_{\mathbb{Q}}$ is spanned by Q and R.)

Thus, determining the possible RDP configurations on fibres of a versal deformation of (X, x) has become a problem of lattice-theoretic nature.

7. Remaining Adjacencies

We continue our investigation of the miniversal deformation of a $D_{p,q,r}$ singularity (X, x). In (2.2) we found that each fibre of $\tilde{p}_-$ is either a $K3$ surface with RDP's or a rational surface with a minimally elliptic singularity. Conversely, it was found in (1.8), that a rational surface $\tilde{Y}$ with trivial dualizing sheaf and endowed with a $T_{p,q,r}$-curve $\tilde{Y}_\infty$ on its smooth part such that $\tilde{Y} - \tilde{Y}_\infty$ is affine, occurs as a fibre of $\tilde{p}_-$. The purpose of this section is to classify all such pairs $(\tilde{Y}, \tilde{Y}_\infty)$. In view of the preceding this will give us a complete description of all configurations of singularities on fibres of a versal deformation of (X, x). One of our findings will be that Wahl's list of adjacencies [23, (5.4), (5.6)] is complete in so far as a triangle singularity is involved (as he conjectured). This was already known for the 14 triangle singularities which embed in $\mathbb{C}^3$, for Brieskorn [3] had listed these earlier.

(7.1) Let $(\tilde{Y}, \tilde{Y}_\infty)$ be as above. We know that the singular locus of $\tilde{Y}$ consists of a unique minimally-elliptic singularity z_0 and some RDP's. So the exceptional locus Z of the minimal resolution $\pi : (\hat{Y}, \hat{Y}_\infty) \to (\tilde{Y}, \tilde{Y}_\infty)$ consists of the exceptional locus Z_0 of z_0 and some RDP configurations. Since $\tilde{Y}$ has trivial dualizing sheaf, Z_0 is the support of a canonical divisor $K_{\hat{Y}}$ for $\hat{Y}$. It is well-known, that $-K_{\hat{Y}}$ is effective. Our aim is to describe $\hat{Y}$ explicitly as a blown up projective plane (on which we are given two distinguished curves corresponding to $\hat{Y}_\infty$ and Z_0). Let C_α denote the irreducible component of $\hat{Y}_\infty$ corresponding to $\alpha \in B$. Then $N := \Sigma_{\alpha \in B_0} n_\alpha C_\alpha$ is the fibre of an elliptic fibration $\varepsilon : \hat{Y} \to \mathbb{P}^1$. Since Z_0 is connected and disjoint from N, Z_0 is contained in a fibre F_0 of ε. The same is true, of course, for any other connected component of Z (necessarily a RDP configuration). We want to locate the exceptional curves of the first kind (briefly, *exceptional curves*) on $\hat{Y}$ contained in a fibre. Notice that these will be all disjoint: if C and C' are exceptional curves in a fibre with $C \cdot C' > 0$, then blowing down C' maps C into a curve $\bar{C}$ with $\bar{C} \cdot \bar{C} \geqq 0$ ($=0$ if and only if $\bar{C}$ is smooth). But this can only be if $\bar{C} \cdot \bar{C} = 0$ and (some multiple of) $\bar{C}$ is a fibre. Since ε is an elliptic fibration, we must have $p_a(\bar{C}) = 1$, contradicting the smoothness of $\bar{C}$.

Let C be an exceptional curve in a fibre. Since $C \cdot K_{\hat{Y}} = -1$, we have in fact $C \subset F_0$. The resolution $\pi : \hat{Y} \to \tilde{Y}$ is minimal, so C is not contained in Z. Since $\tilde{Y} - \tilde{Y}_\infty$

is affine, it follows that C meets $\hat{Y}_\infty$. Let C_α be an irreducible component of $\hat{Y}_\infty$ with $C \cdot C_\alpha > 0$. Since N and C are in distinct fibres, $\alpha \notin B_0$. We distinguish two cases.

Case 1. $C_\alpha \cdot N = 0$. We show that then α is an end of B, that $C \cdot C_\alpha = 1$ and that C is the unique exceptional curve in a fibre meeting any of the curves C_β, β in the connected component P of α in $B - B_0$.

Since $C \cup C_\alpha$ is contained in an elliptic fibre, the intersection matrix of $\{C, C_\alpha\}$ is negative semi-definite, so $C \cdot C_\alpha = 1$. If α is not an end of B, then let α_1 and α_2 be the adjacent elements of B with $\alpha_2 \in B_0$. Successive contraction of C, C_α, C_{α_2} preserves the fibration and maps C_{α_1} on a smooth rational curve $\bar{C}_{\alpha_1}$ of self-intersection zero. This is impossible for both a fibre component and a section on an elliptic rational surface. But clearly there are no other possibilities for $\bar{C}_{\alpha_1}$. A similar argument shows that there are no other exceptional curves in a fibre meeting C_α.

Case 2. $C_\alpha \cdot N = 1$ (so C_α is a section of ε). We claim that then $C \cdot C_\alpha = 1$ and that the fibre F_α of ε which contains the connected component of $\hat{Y}_\infty - C_\alpha$ not containing N is distinct from F_0.

Since C_α is a section and C is in a fibre, we must have $C \cdot C_\alpha = 1$. By the preceding case, C is disjoint from the curve C_β, $\beta \in B - B_0$, which is connected with C_α. Since $F_\alpha \cap C_\alpha = C_\beta \cap C_\alpha$ and $F_0 \cap C_\alpha = C \cap C_\alpha$, it follows that $F_\alpha \neq F_0$.

We conclude that there is at most one exceptional curve in F_0 meeting a given connected component of $\hat{Y}_\infty - N_{\text{red}}$. Now start contracting: in Case 1 blow down successively C and the curves in the corresponding connected component of $\hat{Y}_\infty - N_{\text{red}}$, except for the last one (the section). This preserves the fibration and the section has become exceptional. In Case 2, we only blow down C: the corresponding section is now also exceptional. We denote the resulting surface with its elliptic fibration $\varepsilon': Y' \to \mathbb{P}^1$. More generally, a prime indicates the image of a subset of $\hat{Y}$ in Y'. From the contraction process it is clear that any newly created exceptional curve C' on Y' must meet one of the exceptional sections. This forbids C' to be in a fibre: otherwise contraction of C' preserves the fibration and yields a section with zero self-intersection. So $\varepsilon': Y' \to \mathbb{P}^1$ is minimal as an elliptic surface. Since Y' is rational, each fibre of ε' is then an anti-canonical divisor. In particular, $Z'_0 = F'_0$. Notice that we only contracted in F_0, so that in Case 2 the fibre F_α is not affected.

Now it is not hard to show that the elliptic fibration $\varepsilon': Y' \to \mathbb{P}^1$ with its distinguished fibres N' and Z'_0 is obtained as follows. If N is of type $\hat{E}_{9-k}$ $(k = 1, 2, 3)$ start out with a line v and a reduced cubic ζ in $\mathbb{P}^2$ such that v meets ζ in k nonsingular points of ζ and blow up $\mathbb{P}^2$ 9 times over these points to make the pencil through ζ and $3v$ free of base points. The result is an elliptic fibration $\tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ with two distinguished fibres F_v and F_ζ. With some patience it is possible to determine the configurations of singular fibres and sections that occur. For each other configuration we must select for each special $\alpha_0 \in B_0$ at most one section C'_{α_1} meeting C'_{α_0} so that sections $C'_{\alpha_1}, C'_{\beta_1}$ chosen for distinct special $\alpha_0, \beta_0 \in B_0$ do not meet or meet with multiplicity one on F_ζ (so that blowing up the intersection point separates the sections). We have tabulated all possible cases below. Some comments on the notation are in order: each special curve of N' (i.e. curve C_α with $n_\alpha = 1$) is represented by a slant line, whereas the rest of N' (which is disjoint from any section) is indicated by a dotted line. The points of Z'_0 over which we must blow up in order to recover $\hat{Y}$ are fattened and the number of blow ups is mentioned. Lines which are (almost) horizontal represent sections. The strict transform Z of Z' is always one of the following three types:

(i) a smooth elliptic curve with self-intersection $-d$; it is the exceptional locus of an elliptic singularity $El(d)$,

(ii) a cycle of rational curves (at most 3) of self-intersection $-b_1, -b_2, \ldots$ which is the exceptional locus of the cusp singularity $C(b_1, b_2, \ldots)$

(iii) the exceptional locus of a triangle singularity (or its non-quasi-homogeneous companion):

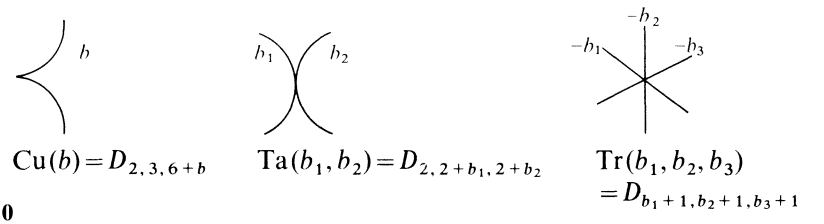


Fig. 10

(We have used Laufer's notation [10].) Hopefully, the rest of our notation is self-explanatory. Singular fibres are only denoted when relevant.

Case 1. N' of type $\hat{E}_8$, so $(p, q, r) = (2, 3, 6+t)$.

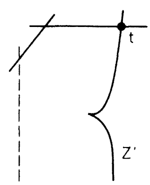


Fig. 11

or with the cuspidal cubic replaced by a less degenerate fibre: a nodal cubic or a smooth elliptic curve. Then $Z = Z_0$ is of type $\mathrm{Cu}(t)$, $C(t)$ or $\mathrm{El}(t)$, respectively.

Case 2. N' of type $\hat{E}_7$, so $(p, q, r) = (2, 4+t_1, 4+t_2)$.

2a.

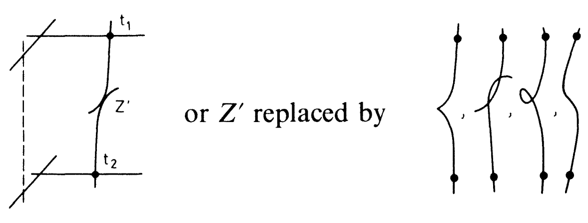


Fig. 12

Then $Z = Z_0$ is of type $\mathrm{Ta}(2+t_1, 2+t_2)$, $\mathrm{Cu}(t_1+t_2)$, $C(2+t_1, 2+t_2)$, $C(t_1+t_2)$, $\mathrm{El}(t_1+t_2)$, respectively.

2b. $(p,q,r)=(2,4,4+t)$ 2c. $(p,q,r)=(2,6,3+t)$

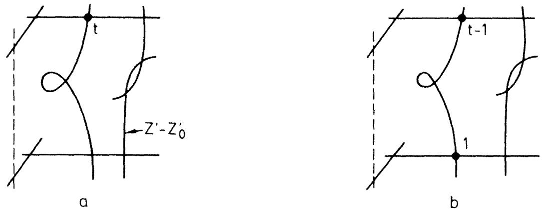


Fig. 13a and b

In either case, the nodal cubic may be replaced by a smooth cubic. In Case 2b, Z_0 is of type $C(t)$ or $\mathrm{El}(t)$ and $Z-Z_0$ is a nodal curve (corresponding to a A_1-singularity). In Case 2c, $Z=Z_0$ is of type $C(t)$.

Case 3. N' of type $\hat{E}_6$, so $(p,q,r)=(3+t_1, 2+t_2, 3+t_3)$.

3a.

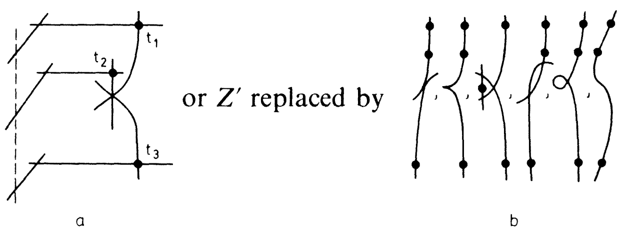
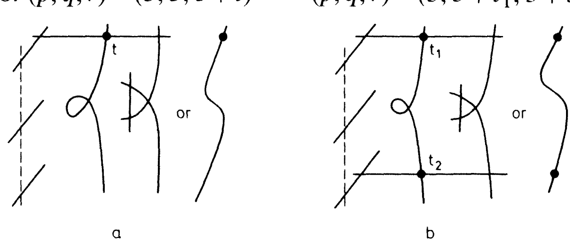


Fig. 14

yielding a $Z=Z_0$ of type $\mathrm{Tr}(2+t_1, 2+t_2, 2+t_3)$, $\mathrm{Ta}(2+t_3, 2+t_1+t_2)$, $\mathrm{Cu}(t_1+t_2+t_3)$, $C(2+t_1, 2+t_2, 2+t_3)$, $C(2+t_3, 2+t_1+t_2)$, $C(t_1+t_2+t_3)$, $\mathrm{El}(t_1+t_2+t_3)$, respectively.

3b. $(p,q,r)=(3,3,3+t)$ $(p,q,r)=(3,3+t_1,3+t_2)$



$(p,q,r)=(3+t_1, 3+t_2, 3+t_3)$

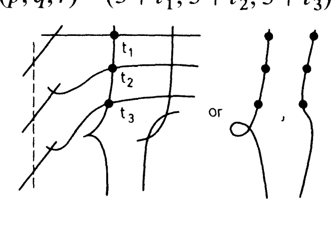
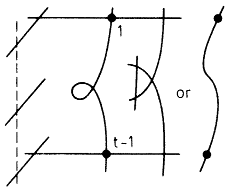


Fig. 15a–c

In these three cases, Z is of type $C(t)+A_2$ or $\mathrm{El}(t)+A_2$, $C(t_1+t_2)+A_1$ or $\mathrm{El}(t_1+t_2)+A_1$, $\mathrm{Cu}(t_1+t_2+t_3)$ [or $C(t_1+t_2+t_3)$ or $\mathrm{El}(t_1+t_2+t_3)+A_1$], respectively.

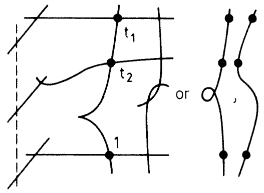
3c. $(p,q,r)=(3,6,2+t)$



a

Z of type $C(t)$ or $\mathrm{El}(t)$.

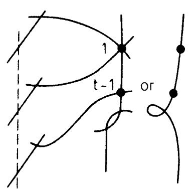
3d. $(p,q,r)=(5,3+t_1,3+t_2)$



b

Z of type $\mathrm{Cu}(t_1+t_2+1)$, $C(t_1+t_2+1)$ or $\mathrm{El}(t_1+t_2+1)$.

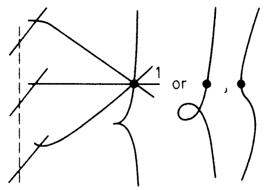
3e. $(p,q,r)=(4,4,2+t)$



c

Z of type $C(2,2+t)$, $C(4+t)$ or $\mathrm{El}(4+t)$.

3f. $(p,q,r)=(4,4,4)$



d

Z of type $\mathrm{Cu}(1)$, $C(1)$ or $\mathrm{El}(1)$.

Fig. 16a–d

(7.2) This completes the description of the singularity configurations that occur on fibres of the versal deformation of a $D_{p,q,r}$ singularity. In the third part of this series we shall determine which of these singularities occur over a given smoothing component. The result may be described as follows. (Compare also Brieskorn [4a] and Friedman and Pinkham [8].)

Let $j: Q_{p,q,r} \to L$ be a good embedding and let $S(j,\Omega)$ be one of (the at most two) associated smoothing components. Then after a permutation of p,q,r:

I. A factorization $j: Q_{p,q,r} \hookrightarrow Q_{2,p+1,q+r-2} \hookrightarrow L$ with the first embedding as in (5.1)–(1) and the second embedding good determines a $D_{2,p+1,q+r-2}$ singularity over $\bar{S}(j,\Omega)$.

I′. Assume $p=5$, $q,r \geqq 4$. A factorization $j: Q_{5,q,r} \hookrightarrow Q_{2,6,q+r-2} \hookrightarrow Q_{2,3,q+r+1} \hookrightarrow L$ with the first embedding as in (5.1)–(1) and the second as in (5.2) determines a $D_{2,3,q+r+1}$-singularity over $\bar{S}(j,\Omega)$.

II. A factorization $j: Q_{p,q,r} \hookrightarrow Q_{p,q,r} \oplus (0) \hookrightarrow L$ such that the induced embedding of $Q_{p,q,r}$ in the primitive saturation of $Q_{p,q,r} \oplus (0)$ in L is good determines a $C(p-1,q-1,r-1)$-cusp singularity over $\bar{S}(j,\Omega)$.

II′. Assume (p,q,r) is one of the (three) triples of (5.3)–(2). A factorization of j over one of the embeddings described there determines resp. a cusp singularity of type $C(t)\,(=C(1,2,5+t))$, $C(t)$, $C(2,2+t)$ over $\bar{S}(j,\Omega)$.

III. A factorization $j: Q_{p,q,r} \oplus (0) \oplus (0) \to L$ in which the first map is good determines a simply-elliptic singularity over $\bar{S}(j,\Omega)$ of degree $r-6$ (if $p=2$, $q=3$), $q+r-8$ (if $p=2$, $q,r \geqq 4$), $p+q+r-9$ (if $p,q,r \geqq 3$), respectively.

IV. Any cusp or triangle singularity over $\bar{S}(j,\Omega)$ not listed under I–II′ is a deformation of one of the triangle singularities over $\bar{S}(j,\Omega)$ listed under I′ and corresponds to a factorization of j by an embedding as in I′ and one of the embeddings in I–II′ (with different p,q,r).

IV′. A simply-elliptic singularity over $\bar{S}(j,\Omega)$ not listed under III is a deformation of a cusp singularity listed under II′ or IV and corresponds to a factorization

$$j: Q_{p,q,r} \to Q_{p',q',r'} \oplus (0) \to Q_{p',q',r'} \oplus (0) \oplus (0) \to L,$$

where the first map is as in II′ or obtained from IV and the second map induces a good embedding of $Q_{p',q',r'}$ in $Q_{p',q',r'} \oplus (0) \oplus (0)$.

Thus the determination of the singularities over $\bar{S}(j,\Omega)$ is a purely arithmetic problem. The above result will be best understood with the help of theory for Stein completions of orbit spaces of the form Ω'/Γ which we will discuss in Part III. In the hypersurface case (where S is smooth and hence irreducible) this should be compared with Brieskorn [4a].

Appendix: Deformations with Good $\mathbb{G}_m$-Action

(A.1) All schemes in this appendix will be of finite type over a fixed algebraically closed field K of char 0. Let $X = \operatorname{Spec} R$, where $R = \bigoplus_{l=0}^{\infty} R_l$ is a reduced $\mathbb{Z}_+$-graded K-algebra, with $R_0 = K$. The graded structure on R corresponds to a $\mathbb{G}_m$-action on X (defined by letting the contragradient action on R be $\lambda \cdot \phi = \lambda^l \phi$ if $\lambda \in \mathbb{G}_m$, $\phi \in R_l$); the condition that $R_0 = K$, $R_l = 0$ for $l < 0$ says that this action is *good*: its unique fixed point (vertex) called x is defined by the maximal ideal $R_+ = \bigoplus_{l=1}^{\infty} R_l$ and $\mathbb{G}_m \times X \to X$, $(\lambda, y) \to \lambda^{-1} \cdot y$ entends to $K \times X \to X$ with $K \times \{x\}$ mapping onto $\{x\}$. We suppose that x is the unique singular point of X. The *standard projectivization* $X \subset \bar{X}$ is defined as follows: let $\bar{R}_l := R_0 \oplus \ldots \oplus R_l$ so that $\bar{R} := \bigoplus_{l=0}^{\infty} \bar{R}_l$ becomes in a natural way a graded $\mathbb{Z}_+$-algebra and put $\bar{X} := \operatorname{Proj} \bar{R}$. If $t = (1,0) \in \bar{R}_1 = R_0 \oplus R_1$, then $\bar{R} = R[t]$. Note that X becomes a subscheme of $\bar{X}$ by making t invertible. The complement $\bar{X}_\infty := \bar{X} - X$ is the divisor defined by t; we have $\bar{X}_\infty = \operatorname{Proj} R$.

This construction works also in a relative situation. Let us say that a deformation $(f: (\mathscr{X}, X_{s_0}) \to (S, s_0), \iota: X \cong X_{s_0})$ of X has (good) $\mathbb{G}_m$-action of both $(\mathscr{X}, X_{s_0})$ and (S, s_0) are equipped with (good) $\mathbb{G}_m$-actions making ι and f $\mathbb{G}_m$-equivariant. In case of a good $\mathbb{G}_m$-action, the (formal) schemes $(\mathscr{X}, X_{s_0})$ and (S, s_0) may be replaced be affine schemes, which we denote by $\mathscr{X} = \operatorname{Spec}(\mathscr{R})$ and $S = \operatorname{Spec}(A)$, so that $f^*: A \to \mathscr{R}$ is a homomorphism of $\mathbb{Z}_+$-graded algebras. Put $\bar{\mathscr{R}}_l := A\mathscr{R}_0 + \ldots + A\mathscr{R}_l$. Since $\mathscr{R}$ is flat over A, each $\bar{\mathscr{R}}_l$ is flat (hence free) over A.

Now $\bar{\mathscr{R}}:=\bigoplus_{l=0}^{\infty}\bar{\mathscr{R}}_l$ is graded A-algebra so that we can define $\bar{\mathscr{X}}:=\operatorname{Proj}_A\bar{\mathscr{R}}$. The projection $\bar{f}:\bar{\mathscr{X}}\to S$ is a flat morphism. If $t\in\bar{\mathscr{R}}_1=\mathscr{R}_0A+\mathscr{R}_1A$ corresponds to 1, then one easily verifies that $\bar{\mathscr{R}}/t\bar{\mathscr{R}}$ is naturally isomorphic (as a graded A-algebra) to $A\bigotimes_K R$. So the divisor $\bar{\mathscr{X}}_\infty\subset\bar{\mathscr{X}}$ defined by (t) is naturally isomorphic to $\bar{X}_\infty\times S$. Notice that $\bar{\mathscr{R}}_l\cong H^0(\mathcal{O}_{\bar{\mathscr{X}}}(l\bar{\mathscr{X}}_\infty))$. We further may identify $\mathscr{X}$ with $\bar{\mathscr{X}}-\bar{\mathscr{X}}_\infty$.

The deformations of X with (good) $\mathbb{G}_m$-action are the objects of a category whose morphisms are defined in an obvious ($\mathbb{G}_m$-equivariant) way. Following Pinkham [19, Theorem (2.3)], there exists a miniversal object (ι,p) for the deformations of X with $\mathbb{G}_m$-action which (by forgetting the actions) induces also a miniversal deformation of the isolated singularity (X,x). If p_- denotes the part of p of negative weight (i.e. the maximal submorphism of p on which we have a good $\mathbb{G}_m$-action), then it is clear that p_- is miniversal for deformations of X with good $\mathbb{G}_m$-action. The main purpose of this section is to show that p_- is actually *universal* for this property:

(A.2) *Theorem.* Let X be a reduced affine scheme with good $\mathbb{G}_m$-action such that its vertex is the unique singular point. Then the part p_- of negative weight of a miniversal deformation of X is a final object in the category of deformations of X with good $\mathbb{G}_m$-action. The group G of automorphisms of X commuting with the $\mathbb{G}_m$-action acts in a natural manner on p_- and its projectivization $\bar{p}_-$ [as defined in (A.1)]. Any isomorphism between two fibres of $\bar{p}_-$ that preserves the pieces at infinity is induced by a unique element of G.

Choose $k\in\mathbb{N}$ such that $V_\infty=\bigoplus_{j=1}^{k}R_j$ generates R as a K-algebra and put

$$\mathbb{P}_\infty:=\operatorname{Proj}\bigoplus_{l=0}^{\infty}S_l(V_\infty),$$

where the grading on $S_l(V_\infty)$ is induced by that of V(so $\mathbb{P}_\infty$ is the twisted projective space associated to V_∞^*). This enables us to regard $\bar{X}_\infty$ as a closed subscheme of $\mathbb{P}_\infty$. If we identify V_∞ with a K-vectorspace in $\bar{R}$ spanned by $R_j\subset\bar{R}_j$ $(j=1,\ldots,k)$, then $V:=V_\infty\oplus Kt$ generates $\bar{R}$ as a K-algebra. We put $\mathbb{P}:=\operatorname{Proj}\bigoplus_{l=0}^{\infty}S_l(V)$ and regard both $\bar{X}$ and $\mathbb{P}_\infty$ as closed subschemes of $\mathbb{P}$. Note that then $X_\infty=\bar{X}\cap\mathbb{P}_\infty$. The embedding of $\bar{X}$ in $\mathbb{P}$ is natural to the extent that the action of G on X naturally extends to $\mathbb{P}$ in a way as to preserve $\mathbb{P}_\infty$ and X. This enables us to regard G as a subgroup of $\operatorname{Aut}(\mathbb{P})$.

Let us denote by B the subgroup of $\operatorname{Aut}(\mathbb{P})$ acting as the identity on $\mathbb{P}_\infty$ and let $U\subset B$ be its unipotent radical.

(A.3) **Lemma.** *The B-stabilizer of $\bar{X}$ is $\mathbb{G}_m$ and $B=\mathbb{G}_m\times U$ (semi-direct product).*

Proof. If $b\in B$ leaves $\bar{X}$ invariant, then b fixes the unique singular point x of X. It is clear than then $b\in\mathbb{G}_m$. The action of B on $R_0\oplus\ldots\oplus R_k$ leaves invariant the filtration $R_0\subset R_0\oplus R_1\subset\ldots$ and acts on its successive quotients by homotheties. This last action factors over the action of $\mathbb{G}_m$ and thus determines a retraction-homomorphism $B\to\mathbb{G}_m$. Its kernel is clearly unipotent. As $\mathbb{G}_m$ is reductive, it is in fact the unipotent radical of B.

We consider deformations of $\bar{X}$ in $\mathbb{P}$ which preserve $\bar{X}_\infty$. Let F be the functor from schemes to sets which associates to a scheme S the set of families

$$\begin{array}{c} \bar{\mathscr{Z}} \subset S \times \mathbb{P} \\ \big\downarrow \bar{\pi} \\ S \end{array}$$

such that $\bar{\mathscr{Z}}_\infty (:= \bar{\mathscr{Z}} \cap (S \times \mathbb{P}_\infty)) = S \times \bar{X}_\infty$ and $\bar{\pi}_* \mathcal{O}_{\bar{\mathscr{Z}}}(l \bar{\mathscr{Z}}_\infty)$ is a flat $\mathcal{O}_S$-module for all $l \geqq 0$. By a minor modification of Grothendieck's construction [9] one finds this functor to be representable by a family $S \times H_1 \supset \bar{\mathscr{Z}} \xrightarrow{\bar{\pi}} H_1$. Notice that $\mathbb{G}_m$ acts on this family. The element $* \in H_1$ representing $\bar{X} \subset \mathbb{P}$ is a fixed point of $\mathbb{G}_m$. Let $H \subset H_1$ denote the maximal affine subscheme through $*$ on which the action of $\mathbb{G}_m$ is good. Both B and G will act on $\bar{\pi}_H : \bar{\mathscr{Z}}_H \to H$. By Lemma (A.3) the B-stabilizer of $*$ is just $\mathbb{G}_m$. Since $\mathbb{G}_m$ is reductive, there exists a $\mathbb{G}_m$-equivariant (formal) slice $(T, *) \subset (H, *)$ to the U-orbit of $*$ in H. Since $\mathbb{G}_m$ acts well on $(T, *)$, there is a unique $\mathbb{G}_m$-invariant closed subscheme $T \subset H$ lifting $(T, *)$.

Let $\mathbb{G}_m$ act on U by conjugation.

(A.4) **Lemma** (i) *The evaluation map $e: U \times T \to H$ is a $\mathbb{G}_m$-equivariant isomorphism and determines a G_m-equivariant cartesian retraction:*

$$\begin{array}{ccc} \bar{\mathscr{Z}}_H & \xrightarrow{\tilde{r}} & \bar{\mathscr{Z}}_T \\ \big\downarrow \bar{\pi}_H & & \big\downarrow \bar{\pi}_T \\ H & \xrightarrow{r} & T \end{array}$$

in which $\tilde{r}$ induces the projection $\bar{\mathscr{Z}}_{H,\infty} = \bar{X}_\infty \times H \to \bar{X}_\infty \times T = \bar{\mathscr{Z}}_{T,\infty}$.

(ii) *G can be made to act on $\bar{\pi}_T$ such that $(\tilde{r}, r)$ is G-equivariant. (N.B. We are not claiming that T is G-invariant.)*

(iii) *Any isomorphism $(\bar{\sigma}, \bar{\sigma}_\infty): (\bar{Z}_{t_1}, \bar{Z}_{t_1,\infty}) \to (\bar{Z}_{t_2}, \bar{Z}_{t_2,\infty})$ between two fibres of $\bar{\pi}_T$ is induced by some $g \in G$. If $\bar{\sigma}_\infty = 1$, then $g \in \mathbb{G}_m$.*

Proof. (i) It is clear that e is $\mathbb{G}_m$-equivariant. Since the action on source and target is good and e induces a formal isomorphism in the vertex, e itself must be an isomorphism. Define r by $r(h) = pr_T \circ e^{-1}(h)$ and $\tilde{r}$ by $\tilde{r}(z) = (pr_U \circ e^{-1}(h))^{-1} \cdot z$ where $h = \bar{\pi}_H(z)$.

(ii) Since G normalizes U, G leaves the U-orbit in $\bar{\mathscr{Z}}_H$ invariant. So if we regard $\bar{\mathscr{Z}}_T$ as the U-orbit space of $\bar{\mathscr{Z}}_H$, then G will act on it.

(iii) For any fibre $(\bar{Z}, \bar{Z}_\infty)$ of $\bar{\pi}_H$, we may identify $\mathbb{P}$ with the twisted projective space associated to the dual of $\bigoplus_{l=0}^{k} H^0(\mathcal{O}_{\bar{Z}}(l\bar{Z}_\infty)) \oplus Kt$ where $t \in H^0(\mathcal{O}_{\bar{Z}}(\bar{Z}_\infty))$ corresponds to the element 1. Hence the isomorphism $\bar{\sigma}$ naturally extends to an automorphism of $\mathbb{P}$. Since $\bar{\sigma}$ leaves $\bar{X}_\infty$ invariant, it also leaves $\mathbb{P}_\infty$ invariant. The automorphism $\bar{\sigma}_\infty$ of $\bar{X}_\infty$ is induced by some $g' \in G$, so that we can write $\bar{\sigma} = g' \cdot b$ with $b \in B$. Write $b = u \cdot \lambda$ with $u \in U$, $\lambda \in \mathbb{G}_m$. Then $g := g' \cdot \lambda$ will be as desired. If $\bar{\sigma}_\infty = 1$, then we may take $g' = 1$, so that $g \in \mathbb{G}_m$.

In view of the preceding lemma, the proof of the theorem will be complete if we prove that $\pi_T : \mathscr{Z}_T \to T$ is a universal as a deformation of X with good $\mathbb{G}_m$-action. This is what we will do.

Let $p : \mathscr{X} = \operatorname{Spec}(\mathscr{R}) \to S = \operatorname{Spec}(A)$, $\iota : X \cong X_{s_0}$, be a deformation with good $\mathbb{G}_m$-action (so $\mathscr{R}$ and A are $\mathbb{Z}_+$ graded). Lift a K-basis of R_l to $\mathscr{R}_l$ $(l=0, \ldots, k)$. This defines a section of the projection $\mathscr{R}_1 \oplus \ldots \oplus \mathscr{R}_k \to R_1 \oplus \ldots \oplus R_k = V$ [where $\mathscr{R}$ is as in (A.1)]. The image of this section plus the element "1" of $\bar{\mathscr{R}}_1$ generate $\bar{\mathscr{R}}$ as an A-algebra over a neighbourhood S' of s_0. For reasons of homogeneity (or equivalently, because of the good $\mathbb{G}_m$-action around), we may take $S' = S$. The ensuing epimorphism $\bigoplus_{l=0}^{\infty} S_l(V) \bigotimes_K A \to \bar{\mathscr{R}}$ determines an embedding $\bar{\mathscr{X}} \to \mathbb{P} \times S$ with $\bar{\mathscr{X}}_\infty \cong \bar{X}_\infty \times S \to \mathbb{P} \times S$ being the obvious map. By the definition of $\bar{\pi}_H$, $\bar{\mathscr{X}} \times S$ is the pull-back of $\bar{\mathscr{Z}}_H \to H$ over a unique morphism $S \times H$. Compose this with r to get a morphism

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\tilde{\phi}} & \mathscr{Z}_T \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_T} \\ S & \xrightarrow{\phi} & \end{array}$$

of deformations of X with good $\mathbb{G}_m$-action. It remains to show that $(\tilde{\phi}, \phi)$ is unique. Suppose that $(\tilde{\phi}', \phi')$ is another morphism from p to π_T. For any $s \in S$, $\tilde{\phi}'_s \circ \tilde{\phi}_s^{-1} : Z_{\phi(s)} \to Z_{\phi'(s)}$ is an isomorphism inducing the identity at infinity, so by (A.4)-iii is induced by some $\lambda \in \mathbb{G}_m$. Since $\tilde{\phi}$ and $\tilde{\phi}'$ are $\mathbb{G}_m$-equivariant, it follows that $\tilde{\phi}' = \lambda \cdot \tilde{\phi}$ on the $\mathbb{G}_m$-orbit of X_s. This must then also be the case on X_{s_0} (which is in the closure of this orbit). As $\tilde{\phi}'$ and $\tilde{\phi}$ are equal on X_{s_0}, it follows that $\lambda = 1$, so that $\tilde{\phi}'_s = \phi_s$.

(A.5) There is also an interpretation of $\bar{p}_- : (\bar{\mathscr{X}}_-, \bar{\mathscr{X}}_\infty) \to S_-$ as a fine moduli space. Let us define a *R-polarized scheme* as a triple $(\bar{Z}, \bar{Z}_\infty, \phi^*)$ where $\bar{Z}$ is a projective scheme, $\bar{Z}_\infty$ an ample reduced Weil divisor on $\bar{Z}$ and $\phi^* : R_{\bar{Z}}/tR_{\bar{Z}} \to R$ an isomorphism of graded K-algebras, where

$$R_{\bar{Z}} := \bigoplus_{l=0}^{\infty} H^0(\mathcal{O}_{\bar{Z}}(l\bar{Z}_\infty)) \quad \text{and} \quad t \in H^0(\mathcal{O}_{\bar{Z}}(\bar{Z}_\infty))$$

the element corresponding to 1. Notice that G acts on such triples, simply by composing the isomorphism ϕ with the action of G on $\bar{R}$.

(*A*.6) **Proposition.** *The morphism $\bar{p}_- : (\bar{\mathscr{X}}, \bar{X}_\infty \times S_-) \to S_-$ is a fine moduli space for R-polarized schemes.*

Proof. Let $\mathscr{Z} := \operatorname{Spec} R_{\bar{Z}}$. The element $t \in R_{\bar{Z}}$ is not a zero divisor, hence defines a flat morphism to the affine line, $f : \mathscr{Z} \to \mathbb{A}$. Since t is homogeneous (of degree 1), f has good $\mathbb{G}_m$-action. The map ϕ^* corresponds to a G-equivariant isomorphism $\phi : X \cong Z_0$, so the pair (f, ι) is a deformation of X with good $\mathbb{G}$-action. According to (A.2) there is a unique morphism of such deformations from (f, ι) to p_-. If we projectivize as in (A.1) we find a cartesian diagram

$$\begin{array}{ccc} (\bar{\mathscr{Z}}, \bar{\mathscr{Z}}_\infty) & \longrightarrow & (\bar{\mathscr{X}}_-, \bar{X}_\infty \times S_-) \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle \bar{p}_-} \\ \mathbb{A} & \longrightarrow & S_- \end{array}$$

The pair $(\bar{Z}, \bar{Z}_\infty)$ is in a natural way isomorphic to the fibre $(\bar{Z}_1, \bar{Z}_{1,\infty})$ of $\bar{f}$. Composing with the above morphism realizes it as a fibre of $\bar{p}_-$. The uniqueness of this fibre follows from (A.2).

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Anticanonical Models of Rational Surfaces

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Let X be a non-singular rational surface over an algebraically closed field k. If K is a canonical divisor of X, $-K$ is called an *anticanonical divisor* of X. For a positive integer m, let $\Phi_m\ (=\Phi_{-mK})$ denote the m-th anticanonical map defined by the linear system $|-mK|$. The *anti-Kodaira* dimension (or the *anticanonical dimension*) of X, written by $\kappa^{-1}(X)$, is defined as follows:

$$\kappa^{-1}(X)=\begin{cases}\max\limits_{m>0}\dim\Phi_m(X), & \\ -\infty & \text{if}\quad P_{-m}(X)=0\quad\text{for all}\quad m>0,\end{cases}$$

where $P_{-m}(X)=\dim H^0(X,\mathcal{O}(-mK))$, the m-th *antigenus*. Note that $\kappa^{-1}(X)$ can take one of the values $-\infty$,0, 1, 2. It turns out that the *anticanonical ring*

$$R^{-1}(X)=\bigoplus_{m\geqq 0} H^0(X,\mathcal{O}(-mK))$$

is finitely generated over k (see Sect. 4). The *anticanonical model* Y of X is defined to be $\operatorname{Proj} R^{-1}(X)$.

The purpose of this paper is to study rational surfaces with $\kappa^{-1}(X)=2$. Our results are based on the Zariski decomposition of divisors established by Zariski [26]. In Sect. 2 we recall and extend somewhat properties of Zariski decompositions. In case $\kappa^{-1}(X)\geqq 0$, we have the Zariski decomposition of the anticanonical divisor: $-K=P+N$, where the P is a numerically effective $\mathbb{Q}$-divisor. We define $d(X)=P^2$ and call it the *degree* of X. We remark that $\kappa^{-1}(X)=2 \Leftrightarrow d(X)>0$. In Sect. 3 we discuss the structure of rational surfaces with $\kappa^{-1}(X)=0$ and 1. In Sect. 4 we prove as a main result that if $\kappa^{-1}(X)=2$, then the anticanonical model Y satisfies the following properties: (i) Y has only isolated rational singularities, (ii) some multiple of $-K_Y$ is an ample Cartier divisor, where the K_Y denotes a canonical divisor of Y (as a Weil divisor). We also show that if conversely a normal projective surface Y satisfies the above two properties, then the minimal resolution X of Y is a rational surface with $\kappa^{-1}(X)=2$. In Sect. 5 we give a dimension formula for pluri-antigenera [mainly for the case $\operatorname{char}(k)=0$]. Let r be the least integer such that rP is integral. If $m\equiv 0$ or $r-1 \bmod r$, then we have

$$P_{-m}(X)=\tfrac{1}{2}m(m+1)d(X)+1\,.$$

For general m, there exists a supplementary term depending on the singularities of Y. In Sect. 6 we discuss the effect of blowing ups on the degrees. We show for instance that if a non-singular rational surface X with $\kappa^{-1}(X) \geqq 0$ is obtained from X' by blowing up n points, then we have the inequality: $d(X) \leqq d(X') \leqq d(X)+n$. In Sect. 7 we give several examples of rational surfaces with $\kappa^{-1}(X)=2$ such as (i) rational ruled surfaces $\mathbb{F}_e$, (ii) Del Pezzo surfaces, (iii) blown up $\mathbb{P}^2$, (iv) certain blown up Del Pezzo surfaces, (v) quotient surfaces, (vi) rational surfaces obtained by torus embeddings, (vii) minimal normal compactifications of $\mathbb{C}^2$.

In Sect. 1 local properties of singularities are collected. Given a normal surface singularity y, we define a $\mathbb{Q}$-divisor Δ, which reflects many important properties of y. In Appendix we prove a local version of the Miyaoka-Ramanujam vanishing theorem, which is used in Sect. 1.

Notation

X: a non-singular rational surface
$-K$: an anticanonical divisor of X
$\kappa^{-1}(X)$: the anti-Kodaira dimension of X
$P_{-m}(X)$: the m-th antigenus of X $[=\dim H^0(X, \mathcal{O}(-mK))]$
$\mathcal{O}$: the structure sheaf of X $(=\mathcal{O}_X)$
$d(X)$: the degree of X
$\varrho(X)$: the Picard number of X
$\sim$: the linear equivalence of divisors
$\approx$: the numerical equivalence of divisors.

Sometimes X', X_0 also denote non-singular rational surfaces and $-K'$, $-K_0$ will mean anticanonical divisors, respectively.

A $\mathbb{Q}$-divisor is a linear combination of divisors with $\mathbb{Q}$-coefficients. Given a $\mathbb{Q}$-divisor $D=\sum \alpha_i E_i$ with irreducible components E_i, we set

$$[D]=\sum[\alpha_i]E_i, \qquad \{D\}=\sum\{\alpha_i\}E_i,$$

where the $[\alpha]$ (resp. $\{\alpha\}$) denotes the greatest integer smaller than or equal to α (resp. the least integer greater than or equal to α). Since $[-\alpha]=-\{\alpha\}$, we get $[-D]=-\{D\}$. If $D \sim D'$ (i.e., $D-D'$ is a principal divisor of a non-zero rational function), then $[D] \sim [D']$. For $\mathbb{Q}$-divisors, we refer to [6, 22].

For a point x, the *multiplicity* of D at x is given by

$$\operatorname{mult}_x(D)=\sum \alpha_i \operatorname{mult}_x(E_i),$$

where $\operatorname{mult}_x(E_i)$ denotes the usual multiplicity of E_i at x.

For a curve C on a surface, we denote by $\mathcal{N}_C$ the normal sheaf $\mathcal{O}(C) \otimes \mathcal{O}_C$.

1. Preliminaries

We collect some fundamental facts concerning surface singularities. Let V be an affine surface having only one normal singularity y. Let $\pi: U \to V$ be the minimal resolution in the sense that the exceptional set $A=\pi^{-1}(y)$ contains no exceptional curves of the first kind. Let $A=E_1 \cup \ldots \cup E_n$ be the decomposition into irreducible components. We denote by K $(=K_U)$ a canonical divisor of U. We write $\mathcal{O}$ in place of $\mathcal{O}_U$.

$\mathbb{Q}$-divisor Δ. 1.1. Now we define a $\mathbb{Q}$-divisor $\Delta=\sum\delta_i E_i$ by the equations:

(1.1.1) $$\sum\delta_i E_i E_j = -KE_j \quad \text{for} \quad j=1,\dots,n.$$

Since the intersection matrix (E_iE_j) is negative definite, the solution is unique. The minimality of π implies that $KE_j\geqq 0$ for all j. It follows that Δ is an effective $\mathbb{Q}$-divisor (cf. Lemma 2.1). We note that $\Delta=0$ if and only if y is a rational double point and otherwise $\mathrm{Supp}(\Delta)=A$. In order to indicate y, we sometimes write as

(1.1.2) $$\Delta=\Delta_y.$$

Definition 1.2. Let

$$d(y)=-\Delta^2$$

denote the *degree* of the singularity y.

Remark 1.3. We have the following characterizations:

(1.3.1) y is a Gorenstein singularity $\Leftrightarrow \Delta$ is integral, $\omega_\Delta\cong\mathcal{O}_\Delta$ [3, 21],

(1.3.2) y is a quotient singularity $\Leftrightarrow [\Delta]=0$ ($k=\mathbb{C}$, [25]).

In Appendix we give an interpretation of Δ in the Zariski decomposition of $-K$ (local version).

$\pi_*\mathcal{O}(-mK)$. 1.4. For a Weil divisor D on V, let $\mathcal{O}_V(D)$ denote the corresponding reflexive sheaf of rank one. If $i: V\backslash y\to V$ is the inclusion, then $\mathcal{O}_V(D)=i_*\mathcal{O}(D_{|V\backslash y})$. Let K_V be a canonical divisor of V (as a Weil divisor).

Lemma 1.5 ([3], see also Appendix).
(i) $R^1\pi_*\mathcal{O}(mK)=0$ *for* $m>0$,
(ii) $H^1_A(U,\mathcal{O}(-mK))=0$ *for* $m\geqq 0$.

Lemma 1.6. *We have*

$$\pi_*\mathcal{O}(-mK)\cong\mathcal{O}_V(-mK_V) \quad \textit{for} \quad m\geqq 0.$$

Proof. We have the exact sequence of local cohomologies

$$0\to H^0(U,\mathcal{O}(-mK))\to H^0(U\backslash A,\mathcal{O}(-mK))\to H^1_A(U,\mathcal{O}(-mK)).$$

By Lemma 1.5, $H^1_A(U,\mathcal{O}(-mK))=0$. On the other hand,

$$H^0(U\backslash A,\mathcal{O}(-mK))\cong H^0(V\backslash y,\mathcal{O}(-mK_{V\backslash y}))\cong H^0(V,\mathcal{O}_V(-mK_V)).$$

Thus

$$H^0(U,\mathcal{O}(-mK))\cong H^0(V,\mathcal{O}_V(-mK_V)),$$

which yields the required result. Q.E.D.

$R^1\pi_*\mathcal{O}(-mK)$. 1.7. We set

$$l_m(y)=\dim R^1\pi_*\mathcal{O}(-mK)_y \quad \text{for} \quad m\geqq 0.$$

The term $l(y)=l_1(y)$ has been introduced by Wahl in a different context [24, Theorem (1.12)].

Lemma 1.8. $R^1\pi_*\mathcal{O}(-mK-[(m+1)\Delta])=0$ *for* $m\geqq 0$.

Proof. We have only to apply Theorem A.2 in Appendix to the divisor $-(m+1)K$. Q.E.D.

As a corollary, we obtain

Lemma 1.9. $l_m(y)=\dim H^1([(m+1)\Delta],\mathcal{O}_{[(m+1)\Delta]}(-mK))$ *for* $m\geqq 0$.

Corollary 1.9.1. *In particular, for* $m=0$, *we have*

$$\dim R^1\pi_*\mathcal{O}_U=\dim H^1([\Delta],\mathcal{O}_{[\Delta]}).$$

Rational singularity. 1.10. The singularity y is *rational* if

$$R^1\pi_*\mathcal{O}_U=0.$$

Lemma 1.11. *We have the following equivalent conditions:*
(i) y *is a rational singularity,*
(ii) $H^1(Z\cdot\mathcal{O}_Z)=0$ *for any effective divisor* Z *supported in* A,
(iii) $H^1([\Delta],\mathcal{O}_{[\Delta]})=0$.

Proof. (i)$\Leftrightarrow$(ii) is a general result [2, p. 130]. (i)$\Leftrightarrow$(iii) follows from Corollary 1.9.1. Q.E.D.

As a consequence, if y is rational, then the exceptional set A consists of a tree of non-singular rational curves.

Lemma 1.12. *Suppose that* y *is a rational singularity. Let* r *be the least integer such that* $\tilde{\Delta}=r\Delta$ *is integral. Then we have the following properties:*
(i) $H^1(\tilde{\Delta},\mathcal{O}_{\tilde{\Delta}})=0$,
(ii) $\mathcal{O}(rK+\tilde{\Delta})\otimes\mathcal{O}_{\tilde{\Delta}}\cong_{\tilde{\Delta}}$,
(iii) rK_V *is a Cartier divisor,*
(iv) $\pi^*(rK_V)\sim rK+\tilde{\Delta}$.

Proof. These facts are more or less well known [1–3]. (i) follows from Lemma 1.11. For $\tilde{\Delta}$ with (i), numerical equivalence of Cartier divisor implies linear equivalence [1], hence we get (ii). Thus

$$0\to\mathcal{O}(rK)\to\mathcal{O}(rK+\tilde{\Delta})\to\mathcal{O}_{\tilde{\Delta}}\to 0$$

is an exact sequence. Using the vanishing $H^1(U,\mathcal{O}(rK))=0$, we see the exact cohomology sequence

$$0\to H^0(U,\mathcal{O}(rK))\to H^0(U,\mathcal{O}(rK+\tilde{\Delta}))\to H^0(\tilde{\Delta},\mathcal{O}_{\tilde{\Delta}})\to 0.$$

By taking V small, we find that $\mathcal{O}(rK+\tilde{\Delta})\cong\mathcal{O}$. Hence

$$\mathcal{O}_V(rK_V)\cong i_*\mathcal{O}(rK_{V\setminus y})\cong i_*\mathcal{O}_{V\setminus y}\cong\mathcal{O}_V,$$

from which follows (iii). Now (iv) is clear. Q.E.D.

Lemma 1.13. *With the same hypothesis as in Lemma 1.12, if* $m\equiv s \bmod r$ *with* $0\leqq s<r$, *then*

$$l_m(y)=\tfrac{1}{2}m(m+1)d(y)+l_s(y)-\tfrac{1}{2}s(s+1)d(y).$$

Corollary 1.13.1. *If $m \equiv 0$ or $r-1 \bmod r$, then*

$$l_m(y)=\tfrac{1}{2}m(m+1)d(y).$$

In particular, we have

(1.13.2) $$l_{r-1}(y)=\tfrac{1}{2}r(r-1)d(y),$$

(1.13.3) $$l_r(y)=\tfrac{1}{2}r(r+1)d(y).$$

Proof of Lemma 1.13. We have $[(m+1)\Delta]=(m-s)\Delta+[(s+1)\Delta]$. Put $C=(m-s)\Delta$, $D=[(s+1)\Delta]$. Note that $(m-s)(K+\Delta)\sim 0$ on $C+D$. The dual form of Lemma 1.9 gives

$$l_m(y)=\dim H^0(C+D, \mathcal{O}_{C+D}((s+1)K+D)).$$

There is an exact sequence

$$0\to\mathcal{O}_C((s+1)K)\to\mathcal{O}_{C+D}((s+1)K+D)\to\mathcal{O}_D((s+1)K+D)\to 0.$$

On the other hand, $H^1(C, \mathcal{O}_C((s+1)K))=0$, because $H^1(U, \mathcal{O}((s+1)K))=0$. Therefore, the cohomology sequence yields

$$\begin{aligned} l_m(y)&=\chi(\mathcal{O}_C((s+1)K))+l_s(y)\\ &=\tfrac{1}{2}m(m+1)d(y)-\tfrac{1}{2}s(s+1)d(y)+l_s(y). \end{aligned}$$ Q.E.D.

Proof of Corollary 1.13.1. In case $m\equiv 0 \bmod r$, the assertion is clear. The above argument also works for the case $s=-1$ and we get (1.13.2), which proves the case $m\equiv r-1 \bmod r$. Q.E.D.

Remark 1.14. We have a characterization:

(1.14.1) y is a rational double point $\Leftrightarrow$ $l_m(y)=0$ for all $m\geqq 0$.

Singular surfaces. 1.15. Given a normal projective surface Y, we have a canonical divisor K_Y as a Weil divisor. We extend the definition of the anticanonical ring to the singular case:

(1.15.1) $$R^{-1}(Y)=\bigoplus_{m\geqq 0} H^0(Y, \mathcal{O}_Y(-mK_Y)).$$

Let $\pi: X\to Y$ be the minimal resolution. By Lemma 1.6, we find that

$$\pi_*\mathcal{O}(-mK)\cong\mathcal{O}_Y(-mK_Y) \quad \text{for} \quad m\geqq 0.$$

Therefore, we get natural isomorphisms

$$H^0(X, \mathcal{O}(-mK))\cong H^0(Y, \mathcal{O}_Y(-mK_Y)) \quad \text{for} \quad m\geqq 0.$$

Lemma 1.16. *With X, Y as above, we have the isomorphism*

$$R^{-1}(X)\cong R^{-1}(Y).$$

Proposition 1.17. *Let Y be a normal projective surface such that some multiple of $-K_Y$ is an ample Cartier divisor. Let X be the minimal resolution of Y. Then $R^{-1}(X)$ is finitely generated and we have*

$$Y\cong\operatorname{Proj} R^{-1}(X).$$

Proof. By a standard reasoning (we refer to [6] and [26]), the graded ring $R^{-1}(Y)$ is finitely generated and $Y \cong \operatorname{Proj} R^{-1}(Y)$. Since $R^{-1}(X) \cong R^{-1}(Y)$, we obtain the above result. Q.E.D.

The Picard number $\varrho(Y)$ of Y can be defined to be the rank of the vector space $\{\operatorname{Pic}(Y)/\approx\} \otimes \mathbb{Q}$. For each singular point $y_i \in \operatorname{Sing}(Y)$, let

(1.18) $\varrho(y_i) =$ the number of irreducible curves in $\pi^{-1}(y_i)$.

Lemma 1.19. *If Y has only rational singularities, then*

$$\varrho(X) = \varrho(Y) + \sum_{y_i \in \operatorname{Sing}(Y)} \varrho(y_i).$$

Proof. Let D be a $\mathbb{Q}$-divisor on X which is orthogonal to every irreducible component of $\pi^{-1}(\operatorname{Sing}(Y))$. Since Y has only rational singularities, some multiple of D is trivial near $\pi^{-1}(\operatorname{Sing}(Y))$ and hence induces a Cartier divisor on Y. Q.E.D.

Finally, we recall a general result.

Lemma 1.20 (Kleiman [9]). *Let $f: Y \to Y'$ be a surjective morphism between projective surfaces Y and Y'. Then $\varrho(Y) \geqq \varrho(Y')$.*

2. Zariski Decomposition

In this section we recall generalities on the Zariski decomposition of divisors (Zariski [26], Fujita [7], see also [14, 22]). Let X be a non-singular projective surface. A $\mathbb{Q}$-divisor D is called *pseudo effective* if $DH \geqq 0$ for all ample divisors H on X. A $\mathbb{Q}$-divisor P is called *numerically effective* if $PC \geqq 0$ for all curves C on X.

Given a pseudo effective $\mathbb{Q}$-divisor D, there exists a unique *Zariski decomposition*:

$$D = P + N,$$

where the P is a numerically effective $\mathbb{Q}$-divisor and the N is an effective $\mathbb{Q}$-divisor such that either $N = 0$ or if we write $N = \sum \alpha_i E_i$,

(i) the intersection matrix $(E_i E_j)$ is negative definite,

(ii) $PE_j = 0$ for each j.

If D is effective, then so is P. In the proof, one of the key facts was the following

Lemma 2.1 [26]. *Let $E_1, \ldots, E_n$ be irreducible curves on X such that the intersection matrix $(E_i E_j)$ is negative definite. Let D be an effective $\mathbb{Q}$-divisor. Suppose that*

$$(D - M)E_j \leqq 0 \quad \textit{for} \quad j = 1, \ldots, n,$$

where the M is a $\mathbb{Q}$-divisor supported in $E_1 \cup \ldots \cup E_n$. Then $D - M$ is effective.

The numerically effective part P can be regarded as a maximal element in the following sense.

Lemma 2.2. *Let D be a pseudo effective $\mathbb{Q}$-divisor on X with the Zariski decomposition: $D = P + N$. If there is a decomposition: $D = \Gamma + Z$, where the Γ is a numerically effective $\mathbb{Q}$-divisor and the Z is an effective $\mathbb{Q}$-divisor, then $P \geqq \Gamma$, $Z \geqq N$.*

Proof. In case $N=0$, there is nothing to prove. Assume that $N>0$. We write as $N=\sum \alpha_i E_i$. The intersection matrix $(E_i E_j)$ is negative definite. We see that

$$(Z-N)E_j=(P-\Gamma)E_j \leqq 0 \quad \text{for all } j.$$

By Lemma 2.1, $Z \geqq N$ and hence $P \geqq \Gamma$. Q.E.D.

Corollary 2.2.1. *With the same hypothesis, we have* $P^2 \geqq \Gamma^2$.

Proof. Let $Z=P_Z+N_Z$ denote the Zariski decomposition. By the above lemma, we find that $N_Z \geqq N$. We have $P-(N_Z-N)=\Gamma+P_Z$. Hence

$$P^2 \geqq (P-(N_Z-N))^2=(\Gamma+P_Z)^2 \geqq \Gamma^2. \quad \text{Q.E.D.}$$

The uniqueness of the Zariski decomposition is a consequence of Lemma 2.2. We can also formulate as follows.

Lemma 2.3. *Let D, D' be two pseudo effective $\mathbb{Q}$-divisors on X. Let $D=P+N$, $D'=P'+N'$ be the Zariski decompositions, respectively. If $D \approx D'$, then $P \approx P'$, $N=N'$. In particular, $P^2=P'^2$.*

Corollary 2.3.1. *If furthermore $D \sim D'$, then $P \sim P'$, $N=N'$.*

Proof. By definition, $D-D'=(f)$ with a non-zero rational function f. Since $N=N'$, we see that $P-P'=(f)$. Q.E.D.

Lemma 2.4 [22]. *Let D be a pseudo effective divisor (integral) on X. Let $D=P+N$ be the Zariski decomposition. Then $\{N\}$ is in the fixed part of the complete linear system $|D|$. In other words,*

$$|D|=|[P]|+\{N\}.$$

Proof. Take an effective divisor $D' \in |D|$. Let $D'=P'+N'$ denote the Zariski decomposition. By Lemma 2.3, $N=N'$. So $D' \geqq N$, because P' is effective. Since D' is integral, we must have $D' \geqq \{N\}$ and hence $D'-\{N\} \in |[P]|$. Q.E.D.

Corollary 2.4.1. *With D, P as above, we have*

$$\dim H^0(X, \mathcal{O}(D))=\dim H^0(X, \mathcal{O}([P])).$$

Lemma 2.5. *Let $\varphi: X \to X'$ be a surjective morphism between non-singular projective surfaces X and X'. Given a pseudo effective $\mathbb{Q}$-divisor D' and X', let $D'=P'+N'$ be the Zariski decomposition. Then φ^*P' is numerically effective and the Zariski decomposition of φ^*D' is given by*

$$\varphi^*D'=(\varphi^*P')+(\varphi^*N').$$

3. Anticanonical Divisor

In what follows we consider rational surfaces. Let X be a non-singular rational surface and $-K$ an anticanonical divisor of X. We study the Zariski decomposition of $-K$. First, we determine when $-K$ is pseudo effective.

Lemma 3.1. *The following conditions are equivalent:*
(i) $\kappa^{-1}(X) \geqq 0$,
(ii) $-K$ *is pseudo effective.*

Proof. (i) ⇒ (ii) is immediate. To prove (ii) ⇒ (i), let $-K = P + N$ be the Zariski decomposition. By virtue of the properties of P and N, $-KP = P^2$. So if $P^2 = 0$, we have $KP = 0$. In view of Table I in [22], we find that $\kappa^{-1}(X) \geqq 0$. Q.E.D.

Definition 3.2. In case $\kappa^{-1}(X) \geqq 0$, let

$$-K = P + N$$

be the Zariski decomposition. The *degree* of X, written by $d(X)$, is defined by

$$d(X) = P^2 .$$

As a convention, we put $d(X) = -\infty$ in case $\kappa^{-1}(X) = -\infty$.
We have the following relation between $d(X)$ and $\kappa^{-1}(X)$.

Table 3.3

$d(X)$	$\kappa^{-1}(X)$
$-\infty$	$-\infty$
0	$\left\{\begin{array}{l}0\\1\end{array}\right.$
+	2

Now we give a brief description of rational surfaces with $\kappa^{-1}(X) = 0$ and 1, although our main objective is the case $\kappa^{-1}(X) = 2$.

Theorem 3.4. *Let X be a non-singular rational surface with $d(X) = 0$. We have two types according as P is numerically equivalent to zero or not.*

Type (b). $P \approx 0$. *X is of this type if and only if there is an effective divisor $Z \in |-mK|$ for some $m > 0$ such that the intersection matrix of irreducible components of Z is negative definite. In particular, we have $\kappa^{-1}(X) = 0$.*

Type (c). $P \not\approx 0$. *There exists a birational morphism $\varphi: X \to X_0$ to a non-singular rational surface X_0 such that $-K_0 = P_0$ in the Zariski decomposition and that P and $\varphi^* P_0$ are proportional, i.e., $nP \sim n_0 \varphi^* P_0$ for some positive integers n and n_0. So $\kappa^{-1}(X) = \kappa^{-1}(X_0)$. We have two subcases.*

Table 3.4.1

$\kappa^{-1}(X)$	Structure of X_0
0	$\exists$ an indecomposable curve of canonical type $C_0 \sim -K_0$ such that the normal sheaf $\mathcal{N}_{C_0}$ is not a torsion element of $\mathrm{Pic}(C_0)$
1	$\exists$ a minimal elliptic fibration $p: X_0 \to \mathbb{P}^1$

In either case, X_0 is obtained from $\mathbb{P}^2$ by blowing up 9 points.

Proof. The assertions for the type (b) are clear. We consider the type (c). Take a positive integer n so that nP is integral and $|nP| \neq \emptyset$. Fix an effective divisor $F \in |nP|$. According to Proposition 2 in [22], there is a birational morphism $\varphi: X \to X_0$ onto a non-singular rational surface X_0 such that there exists an effective divisor F_0 on X_0 with $F = \varphi^* F_0$ and that there are no exceptional curves of the first kind E having the property: $F_0 E = 0$. Note that F_0 is numerically effective. Also we have $K_0 F_0 = 0$, because

$$K_0 F_0 = (\varphi^* K_0)(\varphi^* F_0) = nKP = 0 \quad \text{(cf. Lemma 3.1)}.$$

In view of Theorem 1 in [22], we have two possibilities.

Case (i). $\kappa^{-1}(X_0) = 0$, $F_0 = n_0 C_0$, where the C_0 is an indecomposable curve of canonical type $\in |-K_0|$. In particular, $-K_0$ is numerically effective, so $-K_0 = P_0$ in the Zariski decomposition. Consequently, we have $nP \sim n_0 \varphi^* P_0$.

Case (ii). $\kappa^{-1}(X_0) = 1$, X_0 has a minimal elliptic fibration $p: X_0 \to \mathbb{P}^1$ and $F_0 = \sum \alpha_i f_i$, where each f_i is a fibre and each α_i is a positive rational number. Since the base curve is rational, $f_i \sim f_j$ for all i and j. We know that p has at most one multiple fibre. Let n_0 be the multiplicity ($n_0 = 1$ if there is no multiple fibre). Then $-n_0 K_0 \sim f$ for a fibre f, hence $-K_0 = P_0$ in the Zariski decomposition. Combining these together, we conclude that P and $\varphi^* P_0$ are proportional.

The remaining assertions can be found in [22]. Q.E.D.

Remark 3.5. In case $\kappa^{-1}(X) = 1$, the anticanonical model $Y = \operatorname{Proj} R^{-1}(X)$ is a non-singular rational curve. For details on rational elliptic surfaces, we refer to [8, 13].

4. Structure of Anticanonical Models

We proceed to study a non-singular rational surface X with $\kappa^{-1}(X) = 2$. As before, let $-K = P + N$ be the Zariski decomposition.

Definition 4.1. An exceptional curve of the first kind E is said to be *redundant* if $PE = 0$.

We get rid of redundant exceptional curves by successive contractions.

Proposition 4.2. *Let X be a non-singular rational surface with $\kappa^{-1}(X) = 2$. Then X birationally dominates a non-singular rational surface X_0 with $\kappa^{-1}(X_0) = 2$, without redundant exceptional curves. Furthermore, the following conditions are satisfied:*
(i) $R^{-1}(X) \cong R^{-1}(X_0)$,
(ii) $d(X) = d(X_0)$.

Proof. Suppose that there exists a redundant exceptional curve E on X. Let $\mu: X \to X'$ be the contraction of E. Then $\kappa^{-1}(X') = 2$. Let $-K' = P' + N'$ be the Zariski decomposition. By the Hodge index theorem, the intersection matrix of irreducible components of $N + E$ is negative definite. We see easily that (see Corollary 6.7)

$$(4.2.1) \qquad P \sim \mu^* P', \qquad N + E = \mu^* N'.$$

For $m>0$, we have the isomorphism:

$$[mP] \sim \mu^*[mP'] + n_m E, \tag{4.2.2}$$

where the n_m is a non-negative integer. We prove this. Clearly,

$$mP = [mP] + \{mN\} - mN,$$

$$mP' = [mP'] + \{mN'\} - mN'.$$

Hence

$$[mP] \sim \mu^*[mP'] + G,$$

where we have put $G = \mu^*\{mN'\} - m\mu^* N' - \{mN\} + mN$. Using (4.2.1), we get

$$G = \mu^*\{mN'\} - \{m\mu^* N'\}.$$

If we write $N' = \sum \alpha_i E_i'$, letting $\mu^* E_i' = E_i + v_i E$ with E_i the strict transform of E_i', then

$$\mu^*\{mN'\} = \sum \{m\alpha_i\} E_i + (\sum \{m\alpha_i\} v_i) E,$$

$$\{m\mu^* N'\} = \sum \{m\alpha_i\} E_i + (\sum \{m\alpha_i v_i\}) E.$$

So if we put $n_m = \sum(\{m\alpha_i\} v_i - \{m\alpha_i v_i\})$, then $G = n_m E$.

Via (4.2.2), we obtain the isomorphisms

$$H^0(X, \mathcal{O}(-mK)) \cong H^0(X', \mathcal{O}(-mK')) \quad \text{for} \quad m \geqq 0.$$

It follows then that $R^{-1}(X) \cong R^{-1}(X')$. Of course, $d(X) = d(X')$.

Thus by successive contractions of redundant exceptional curves, we can arrive at the required surface X_0. Q.E.D.

In the rest of this paragraph we prove the following structure theorem of the anticanonical models.

Theorem 4.3. *Let X be a non-singular rational surface with $\kappa^{-1}(X) = 2$. Then the anticanonical ring $R^{-1}(X)$ is finitely generated and the anticanonical model $Y = \operatorname{Proj} R^{-1}(X)$ satisfies the following properties:*

$$(*) \qquad \begin{cases} \text{(i) } Y \text{ has only isolated rational singularities}, \\ \text{(ii) some multiple of } -K_Y \text{ is an ample Cartier divisor}. \end{cases}$$

Furthermore, there is a birational morphism $\pi: X \to Y$. If X contains no redundant exceptional curves, then X coincides with the minimal resolution of Y.

Conversely, if Y is a normal projective surface having the properties $(*)$, then the minimal resolution X of Y is a rational surface with $\kappa^{-1}(X) = 2$ and Y is isomorphic to the anticanonical model of X.

Proof. In view of Proposition 4.2, we may assume that X contains no redundant exceptional curves. We set

$$A = \{\text{irreducible curves } E \mid PE = 0\}.$$

By the Hodge index theorem, this is a finite set. Also the intersection matrix of irreducible curves in A is negative definite. We decompose A into connected components: $A = \cup A_i$.

Now let Z be any effective divisor supported in A_i. Since $K+Z=-P-N+Z$, we get $(K+Z)P=-P^2<0$, hence $H^0(X,\mathcal{O}(K+Z))=0$. We have the exact sequence

$$0\to\mathcal{O}(K)\to\mathcal{O}(K+Z)\to\omega_Z\to 0.$$

Noting that $H^1(X,\mathcal{O}(K))=0$, the cohomology sequence shows that $H^0(Z,\omega_Z)=0$. The duality theorem says that $H^1(Z,\mathcal{O}_Z)=0$. Thus, according to Lemma 1.11, each A_i can be contracted to a rational singularity. Furthermore, by contracting A, we have a birational morphism $\pi: X\to Y$ to a normal projective surface Y (cf. [1]). Let y_i be the rational singularity of Y corresponding to A_i. By the assumption, A contains no exceptional curves of the first kind, so π is nothing but the minimal resolution of Y.

Let $\Delta_i=\Delta_{y_i}$ be the $\mathbb{Q}$-divisor for each y_i defined as in Sect. 1. We claim that

(4.3.1) $$N=\sum\Delta_i.$$

In fact, since $\mathrm{Supp}(N)\subset A$, our assertion follows from the equality: $NE=-KE$ for every irreducible curve E in A. Let r_i be the least integer such that $r_i\Delta_i$ is integral, and let $r=\mathrm{l.m.c.}(r_i)$. By Lemma 1.12, rK_Y is a Cartier divisor. Also $\pi^*(rK_Y)\sim rK+rN$. In other words $\pi^*(-rK_Y)\sim rP$. Next, we see that the divisor $-rK_Y$ is ample. For an irreducible curve C on Y, let $\tilde{C}$ denote the strict transform of C by π. Then we have

$$(-rK_Y)C=rP\tilde{C}>0,$$

because $\tilde{C}$ is not contained in A. By Nakai's criterion, we assert that $-rK_Y$ is ample. Thus we have seen that Y has the property (*). We deduce from Proposition 1.17 that $R^{-1}(X)$ is finitely generated and that $Y\cong\mathrm{Proj}\,R^{-1}(X)$. So Y is the anticanonical model of X.

Conversely, let Y be a normal projective surface having the property (*). Assume that $-rK_Y$ is an ample Cartier divisor for a positive integer r. Let $\pi: X\to Y$ be the minimal resolution. As before, let $\Delta_i=\Delta_{y_i}$ for $y_i\in\mathrm{Sing}(Y)$, and let $\Delta=\sum\Delta_i$. Then $\tilde{\Delta}=r\Delta$ must be an integral divisor and we have

$$-rK\sim\pi^*(-rK_Y)+\tilde{\Delta}.$$

We infer that $\kappa^{-1}(X)=2$ (this is also a consequence of Lemma 1.16). It remains to show that X is rational. Since $\kappa^{-1}(X)=2$, X is birationally a ruled surface. Assume that X is a ruled surface of genus $g\geqq 1$. Let S be a relatively minimal model of X. Again $\kappa^{-1}(S)=2$. An easy calculation [20] shows that $\kappa^{-1}(S)=2$ if and only if S has the form $\mathbb{P}\mathcal{O}\oplus\mathcal{O}(\mathfrak{e}))$ with $e>2g-2$, where we have put $e=-\deg\mathfrak{e}$. So in case $g\geqq 1$, $e>0$. Therefore, there exists a section b with $b^2=-e<0$ for the ruled fibration. Let C be the strict transform of b. Clearly, $C^2<0$, also genus $(C)=g\geqq 1$. It follows that $-KC<0$. On the other hand, since Y has only rational singularities, $\pi^{-1}(\mathrm{Sing}(Y))$ consists of non-singular rational curves. So C cannot be a component of $\tilde{\Delta}$. Consequently,

$$(-rK)C=\pi^*(-rK_Y)C+\tilde{\Delta}C\geqq 0,$$

which is a contradiction. We know that Y is isomorphic to the anticanonical model of X (Proposition 1.17). Q.E.D.

Now we study irreducible curves with negative self-intersection.

Proposition 4.4. *Let X be a non-singular rational surface with $\kappa^{-1}(X)=2$. Then the number of irreducible curves with negative self-intersection is finite.*

Proof. Let $-K=P+N$ be the Zariski decomposition. Let B denote the set of those irreducible curves C such that $C^2<0$. We write as $B=\mathrm{Supp}(N)\cup B'$. If $C\subset B'$, then $-KC=PC+NC\geqq 0$. Noting that $2p_a(C)-2=KC+C^2$, we must have $p_a(C)=0$, hence C is a non-singular rational curve. It follows that $PC=C^2+2-NC\leqq 1$. It is now standard to see that B' is a finite set (cf. [10, p. 190], [4, p. 176]). Q.E.D.

Remark 4.5. Some particular cases have been considered by Nagata [17, Theorem 4a, Lemma 6.1].

5. Pluri-Antigenera

In this section we further study anticanonical models of rational surfaces. We shall obtain a dimension formula for pluri-antigenera. Let X be a non-singular rational surface with $\kappa^{-1}(X)=2$. Let Y be the anticanonical model. To see the structure of Y, we assume that X contains no redundant exceptional curves. By Theorem 4.3, it follows that $\pi: X\to Y$ is the minimal resolution. We fix the notation as in the proof of Theorem 4.3. Let $\Delta_i=\Delta_{y_i}$ for $y_i\in\mathrm{Sing}(Y)$. Let r_i be the least integer such that $r_i\Delta_i$ is integral, and let $r=\mathrm{l.c.m.}(r_i)$. We know that $-rK_Y$ is an ample Cartier divisor. If $-K=P+N$ is the Zariski decomposition, then we have

$$\mathcal{O}(rP)\cong\pi^*\mathcal{O}_Y(-rK_Y),\qquad N=\sum\Delta_i.$$

We have the following alternative definition of the degree:

$$d(X)=(-rK_Y)^2/r^2, \tag{5.1}$$

which can be regarded as the degree of Y. We note the following formula:

$$K^2=d(X)-\sum d(y_i). \tag{5.2}$$

First, we deal with the Picard number.

Lemma 5.3. $\varrho(X)+K^2=10$.

Proof. This is a well known fact. Since the sum $\varrho(X)+K^2$ is preserved by blowing ups, we have only to check the relatively minimal case. Q.E.D.

Proposition 5.4. *We have*

$$d(X)+\varrho(Y)=10-\sum_{y_i\in\mathrm{Sing}(Y)}(\varrho(y_i)-d(y_i)).$$

Proof. This follows from Lemma 5.3, using (5.2) and Lemma 1.19. Q.E.D.

Corollary 5.4.1. *We have the inequality:*

$$\sum\varrho(y_i)<9+\sum d(y_i).$$

Next, we consider cohomological invariants of Y.

Lemma 5.5. $H^i(Y,\mathcal{O}_Y)=0$ *for* $i>0$.

Proof. Since Y has only rational singularities, we have $R^1\pi_*\mathcal{O}=0$. Hence we get isomorphisms

$$H^i(Y,\mathcal{O}_Y)\cong H^i(X,\mathcal{O}) \quad \text{for all } i. \quad \text{Q.E.D.}$$

Now we restrict to the case in which $\mathrm{char}(k)=0$. We show the following vanishing results.

Theorem 5.6 [$\mathrm{char}(k)=0$].

$$H^i(Y,\mathcal{O}_Y(-mK_Y))=0 \quad \textit{for} \quad i>0, \quad m\geqq 0.$$

Proof. We need the following Miyaoka-Ramanujam vanishing theorem [15].

Theorem 5.6.1 [$\mathrm{char}(k)=0$]. *Let D be a divisor on a non-singular projective surface X. Suppose that $\kappa(D,X)=2$. Let $D=P+N$ be the Zariski decomposition. Then*

$$H^i(X,\mathcal{O}(K+D-[N]))=0 \quad \textit{for} \quad i>0.$$

Applying this to the divisor $-(m+1)K$, we get

(5.6.2) $$H^i(X,\mathcal{O}(-mK-[(m+1)\Delta]))=0 \quad \text{for} \quad i>0, \quad m\geqq 0.$$

We put $G=[(m+1)\Delta]$. Since $R^1\pi_*\mathcal{O}(-mK-G)=0$ (Lemma 1.8), we get

$$H^i(X,\mathcal{O}(-mK-G))\cong H^i(Y,\pi_*\mathcal{O}(-mK-G)) \quad \text{for all } i.$$

Therefore, it follows then from (5.6.2) that

(5.6.3) $$H^i(Y,\pi_*\mathcal{O}(-mK-G))=0 \quad \text{for} \quad i>0, \quad m\geqq 0.$$

By taking the direct image sheaves of the exact sequence

$$0\to\mathcal{O}(-mK-G)\to\mathcal{O}(-mK)\to\mathcal{O}_G(-mK)\to 0,$$

we obtain the exact sequence

$$0\to\pi_*\mathcal{O}(-mK-G)\to\pi_*\mathcal{O}(-mK)\to\mathcal{T}\to 0,$$

where $\mathcal{T}=\pi_*\mathcal{O}_G(-mK)$. Since $\mathcal{T}$ is supported on $\mathrm{Sing}(Y)$, we find that $H^i(Y,\mathcal{T})=0$ for $i>0$. Note that $\pi_*\mathcal{O}(-mK)\cong\mathcal{O}_Y(-mK_Y)$ (Lemma 1.6).

We have, therefore,

$$0=H^i(Y,\pi_*\mathcal{O}(-mK-G))\to H^i(Y,\mathcal{O}_Y(-mK_Y))\to H^i(Y,\mathcal{T})=0$$

for $i>0$. So we obtain the desired results. Q.E.D.

Corollary 5.7. *We have*

$$\dim H^1(X,\mathcal{O}(-mK))=\sum l_m(y_i).$$

Proof. The Leray spectral sequence

$$H^i(Y,R^j\pi_*\mathcal{O}(-mK)) \Rightarrow H^{i+j}(X,\mathcal{O}(-mK))$$

yields the exact sequence

$$H^1(Y,\mathcal{O}_Y(-mK_Y))\to H^1(X,\mathcal{O}(-mK))\to H^0(Y,R^1\pi_*\mathcal{O}(-mK))$$
$$\to H^2(Y,\mathcal{O}_Y(-mK_Y)).$$

Using the vanishings in Theorem 5.6, we get

$$H^1(X, \mathcal{O}(-mK)) \cong H^0(Y, R^1\pi_*\mathcal{O}(-mK)).$$

In view of the definition 1.7, we deduce the required formula. Q.E.D.

As a consequence, we obtain a dimension formula for pluri-antigenera. Obviously, $H^2(X, \mathcal{O}(-mK))=0$ for $m \geqq 0$. The Riemann-Roch formula together with Corollary 5.7 yields

(5.8) $$P_{-m}(X)=\tfrac{1}{2}m(m+1)K^2+1+\sum l_m(y_i) \quad \text{for} \quad m \geqq 0.$$

Theorem 5.9 [char$(k)=0$]. *Let X be a non-singular rational surface with $\kappa^{-1}(X)=2$, and let Y be the anticanonical model of X. Then the pluri-antigenera $P_{-m}(X)$ are given by the formula for $m \geqq 0$:*

$$P_{-m}(X)=\tfrac{1}{2}m(m+1)d(X)+1+\sum_{y_i \in \mathrm{Sing}(Y)} (l_m(y_i)-\tfrac{1}{2}m(m+1)d(y_i)).$$

Proof. In view of Proposition 4.2, we may assume that X contains no redundant exceptional curves. Using (5.2), the required formula follows from (5.8). Q.E.D.

Corollary 5.10. *If $m \equiv s \bmod r$, $0 \leqq s < r$, then*

(5.10.1) $$P_{-m}(X)=\tfrac{1}{2}m(m+1)d(X)+P_{-s}(X)-\tfrac{1}{2}s(s+1)d(X).$$

In particular, if $m \equiv 0$ or $r-1 \bmod r$, then

(5.10.2) $$P_{-m}(X)=\tfrac{1}{2}m(m+1)d(X)+1.$$

Proof. In view of the periodicity of l_m in Lemma 1.13, by the above formula, we get (5.10.1). Also by Corollary 1.13.1, we get (5.10.2). Q.E.D.

Now we observe the general case in which char$(k) \geqq 0$. Though we do not know whether Theorem 5.6.1 holds for rational surfaces in case char$(k)>0$, we can show that $H^2(Y, \mathcal{O}_Y(-mK_Y))$ vanishes in general. Indeed, $H^2(X, \mathcal{O}(-mK-G))$ is dual to $H^0(X, \mathcal{O}((m+1)K+G))$, which vanishes, because $((m+1)K+G)P = -(m+1)P^2<0$. As in the proof of Theorem 6.6, we see that $H^2(Y, \mathcal{O}_Y(-mK_Y))=0$. So at least we can say that

$$\dim H^1(X, \mathcal{O}(-mK)) \geqq \sum l_m(y_i).$$

Hence we obtain the inequality:

(5.11) $$P_{-m}(X) \geqq \tfrac{1}{2}m(m+1)K^2+1+\sum l_m(y_i) \quad \text{for} \quad m \geqq 0.$$

In particular, we have the inequality:

(5.11.1) $$P_{-m}(X) \geqq \tfrac{1}{2}m(m+1)d(X)+1 \quad \text{for} \quad m \geqq 0, \quad m \equiv 0 \bmod r.$$

In case $m \equiv r-1 \bmod r$, Theorem 5.6.1 is reduced to the usual Ramanujam vanishing theorem, which holds in all characteristics when X is regular. Hence

(5.11.2) $$P_{-m}(X)=\tfrac{1}{2}m(m+1)d(X)+1 \quad \text{for} \quad m \geqq 0, \quad m \equiv r-1 \bmod r.$$

Remark 5.12. The inequality (5.11.1) can be shown by the following simple argument. First, we observe the inequality:

(5.12.1) $$P_{-m}(X) \geqq \tfrac{1}{2}([mP]-K)[mP]+1 .$$

This follows from the Riemann-Roch formula for the divisor $[mP]$. So if $m \equiv 0 \bmod r$, we deduce (5.11.1).

6. Degrees and Blowing Ups

In this section we consider the effect of blowing ups on the degrees of rational surfaces. Let X' be a non-singular rational surface. Let $\mu: X \to X'$ be the blowing up at a point $x' \in X'$. Let E be the exceptional curve and let $-K$, $-K'$ be anticanonical divisors of X, X', respectively. It is well known that $-K+E \sim \mu^*(-K')$. It follows that $\kappa^{-1}(X) \leqq \kappa^{-1}(X')$.

First, we consider the case in which $\kappa^{-1}(X) \geqq 0$. So we have also $\kappa^{-1}(X') \geqq 0$. Let
$$-K=P+N, \qquad -K'=P'+N'$$
denote the Zariski decompositions, respectively. Then

(6.1) $$P+N+E \sim \mu^*P' + \mu^*N' .$$

We set

(6.2) $$\begin{cases} \alpha = PE , \\ \beta = 1 - \mathrm{mult}_{x'}(N') . \end{cases}$$

Since $KE=-1$, we have

(6.2.1) $$\alpha = 1 - NE .$$

If $\bar{N}'$ denotes the strict transform of N', then $\mu^*N' = \bar{N}' + (1-\beta)E$. Hence we have

(6.2.2) $$\beta = 1 - \bar{N}'E .$$

Lemma 6.3. *Possible values of* α *and* β *are as follows:*
(i) $\alpha=0$, $\beta<0$ *if* $E \subset \mathrm{Supp}(N)$,
(ii) $0 \leqq \alpha \leqq \beta \leqq 1$ *if* $E \not\subset \mathrm{Supp}(N)$.

Proof. If $E \subset \mathrm{Supp}(N)$, clearly,

(6.3.1) $$P \sim \mu^*P' , \qquad N+E = \mu^*N' ,$$

hence $\alpha=0$, $\beta<0$.

If $E \not\subset \mathrm{Supp}(N)$, then by (6.2.1), we have $0 \leqq \alpha \leqq 1$, because $NE \geqq 0$. We obtain the decomposition:

(6.3.2.) $$P+N+E = (P+\alpha E) + (N + (1-\alpha)E) ,$$

with $N+(1-\alpha)E$ being effective. It is easy to see that $P+\alpha E$ is numerically effective. It follows from Lemma 2.2 together with Lemma 2.3 that

(6.3.3) $$N+(1-\alpha)E \geqq \mu^*N' = \bar{N}' + (1-\beta)E .$$

So $(1-\alpha)E \geqq (1-\beta)E$, hence $\alpha \leqq \beta$. Of course, $\beta \leqq 1$, by definition. Q.E.D.

Lemma 6.4. *We have*

(i) $d(X') = d(X)$ *if* $E \subset \mathrm{Supp}(N)$,

(ii) $\alpha^2 \leqq d(X') - d(X) \leqq \beta^2$ *if* $E \not\subset \mathrm{Supp}(N)$.

Proof. (i) follows from (6.3.1). We show (ii). Using (6.3.2), by Corollary 2.2.1, we have $P'^2 \geqq (P+\alpha E)^2 = P^2 + \alpha^2$. By (6.3.3), $N \geqq \bar{N}'$, because neither N nor $\bar{N}'$ contains E. If we put $Z = N - \bar{N}'$, then $P + Z + \beta E \sim \mu^* P'$. Thus

(6.4.1) $$P'^2 = P^2 + Z^2 + \beta^2,$$

since $(P+Z)E = (-K - \bar{N}')E = \beta$. Noting that $Z^2 \leqq 0$, we get the inequality: $P'^2 \leqq P^2 + \beta^2$. Q.E.D.

Corollary 6.5. *In particular, we have*

$$d(X) \leqq d(X') \leqq d(X) + 1 .$$

Remark 6.5.1. When $\kappa^{-1}(X) = 2$, there is another proof of the above inequality. Since $-mK \sim \mu^*(-mK') - mE$, we easily see that

(6.5.2) $$P_{-m}(X) \leqq P_{-m}(X') \leqq P_{-m}(X) + \tfrac{1}{2} m(m+1) .$$

Let r, r' be the least integers such that rP, $r'P'$ are integral, respectively. Using the dimension formula (5.10.2) [see (5.11.2) for the case $\mathrm{char}(k) > 0$] for $m \equiv rr' - 1 \bmod rr'$, we have

$$P_{-m}(X) = \tfrac{1}{2} m(m+1) d(X) + 1 , \qquad P_{-m}(X') = \tfrac{1}{2} m(m+1) d(X') + 1 .$$

Combining these with (6.5.2), we arrive at the required inequality.

Corollary 6.6. *The following conditions are equivalent:*

(i) $d(X') = d(X) + 1$,

(ii) $P + E \sim \mu^* P'$, $N = \mu^* N'$,

(iii) $PE = 1$.

Proof. (ii) ⇒ (iii) is trivial. (iii) ⇒ (i) follows from Lemma 6.4. It remains to see (i) ⇒ (ii). In view of Lemma 6.4, we must have $\beta = 1$. We infer from (6.4.1) that $Z = 0$, or equivalently, $P + E \sim \mu^* P'$. Q.E.D.

Corollary 6.7. *The following conditions are equivalent:*

(i) $P \sim \mu^* P'$, $N + E = \mu^* N'$,

(ii) $PE = 0$ *and the intersection matrix of the irreducible components of* $N + E$ *is negative definite.*

(iii) $\mathrm{mult}_{x'}(N') \geqq 1$.

Remark 6.7.1. If $\kappa^{-1}(X) = 2$, (ii) can be replaced by (ii)′ $PE = 0$.

Using Corollary 6.5 successively, we obtain

Theorem 6.8. *Let X be a non-singular rational surface with $\kappa^{-1}(X) \geqq 0$. If X is obtained from a non-singular rational surface X' by blowing up n points, then we have*

$$0 \leqq d(X') - d(X) \leqq n .$$

As before, let $\mu : X \to X'$ be the blowing up at a point x'. We turn our attention to the conditions on X' which imply that $\kappa^{-1}(X) \geqq 0$.

Lemma 6.9. *Suppose that there is an effective divisor $D' \in |-mK'|$ for some positive integer m. Then*

(i) *if* $\mathrm{mult}_{x'}(D') \geqq m$, *then* $\kappa^{-1}(X) \geqq 0$,

(ii) *if* $\mathrm{mult}_{x'}(D') > m$, *then* $\kappa^{-1}(X) = \kappa^{-1}(X')$.

Proof. Let D be the strict transform of D'. If $v = \mathrm{mult}_{x'}(D')$, then $\mu^* D' = D + vE$. On the other hand, $-mK \sim D + (v-m)E$. So if $v \geqq m$, we get $\kappa^{-1}(X) \geqq 0$. In case $v > m$, we have

$$\kappa(D + (v-m)E, X) = \kappa(D + vE, X) = \kappa(\mu^* D', X) = \kappa(D', X').$$

We conclude therefore that $\kappa^{-1}(X) = \kappa^{-1}(X')$. Q.E.D.

Lemma 6.10. (i) *If* $d(X') = 1$, *then* $\kappa^{-1}(X) \geqq 0$,

(ii) *if* $d(X') > 1$, *then* $\kappa^{-1}(X) = 2$.

Proof. Assume that $d(X') \geqq 1$. Let $-K' = P' + N'$ be the Zariski decomposition, let r' be the least integer such that $r'P'$ is integral. We see from (5.10.2) [cf. (5.11.1)] that $\dim |r'P'| \geqq r'(r'+1)/2$. Therefore, we can find an effective divisor $D' \in |r'P'|$ such that $\mathrm{mult}_{x'}(D') \geqq r'$. The divisor $D = \mu^* D' - r'E$ is then effective. Since $-r'K \sim D + r'\mu^* N'$, we infer that $\kappa^{-1}(X) \geqq 0$. Furthermore,

$$D^2 = D'^2 - r'^2 = r'^2(d(X') - 1).$$

So if $d(X') > 1$, then $\kappa(D, X) = 2$, hence $\kappa^{-1}(X) = 2$. Q.E.D.

By induction, Lemma 6.10 together with Corollary 6.5 proves

Theorem 6.11. *Let X' be a non-singular rational surface with $\kappa^{-1}(X') = 2$. Let X be any rational surface obtained from X' by blowing up n points. If $d(X') = n$, then $\kappa^{-1}(X) \geqq 0$. If $d(X') > n$, then $\kappa^{-1}(X) = 2$.*

7. Examples

We consider examples of rational surfaces with $\kappa^{-1}(X) = 2$.

Rational Ruled Surface. 7.1. We denote by $\mathbb{F}_e$ the rational ruled surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$ over $\mathbb{P}^1$ with $e \geqq 0$. There is a section b with $b^2 = -e$, which is unique if $e > 0$. Let f denote a fibre in the ruled fibration. Then $-K \sim 2b + (e+2)f$. The Zariski decomposition is given by

$$P = -K, \qquad N = 0 \qquad \text{in case} \quad e = 0, 1, 2,$$

$$P = (1 + 2/e)(b + ef), \qquad N = (1 - 2/e)b \quad \text{in case} \quad e \geqq 3.$$

Let Y_e be the anticanonical model of $\mathbb{F}_e$. For $e = 0, 1$, $Y_e = \mathbb{F}_e$, and for $e \geqq 2$, Y_e has only one singularity. We list the invariants.

Table 7.1.1

e	$d(\mathbb{F}_e)$	$P_{-1}(\mathbb{F}_e)$	$\varrho(Y_e)$
0, 1	8	9	2
2	8	9	1
$\geqq 3$	$e + 4(1 + 1/e)$	$e + 6$	1

By Theorem 6.11, we have the following result:

(7.1.2) let X be obtained from $\mathbb{F}_e$ by blowing up n points. If

$$n \leqq \begin{cases} 7 & \text{in case } e \leqq 3, \\ e+4 & \text{in case } e \geqq 4, \end{cases}$$

then always $\kappa^{-1}(X)=2$.

Del Pezzo Surfaces. 7.2. If $-K$ is ample, then X is usually called a *Del Pezzo surface*. Del Pezzo surfaces have been classified as follows. We denote by S_n a surface obtained from $\mathbb{P}^2$ by blowing up n points.

Table 7.2.1

$d(X)$	Structure
9	$\mathbb{P}^2$
8	$\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1$
$1 \leqq d \leqq 7$	S_{9-d}

These $9-d$ points should be in "general position". For details, see for instance [5].

If $-K$ is not ample but numerically effective [with $\kappa^{-1}(X)=2$], X is called a *degenerate* Del Pezzo surface. Such surfaces (except $\mathbb{F}_2$) can be constructed from $\mathbb{P}^2$ by blowing up at most 8 points in "almost general position" but not in general position.

Table 7.2.2

$d(X)$	Structure
8	$\mathbb{F}_2$
$1 \leqq d \leqq 7$	S_{9-d}

The anticanonical model Y has a finite number of rational double points. We must have $\sum \varrho(y_i) \leqq 8$.

Blown up $\mathbb{P}^2$. *7.3.* By repeating the process in Lemma 6.9(ii), we obtain an infinite series. For example, we start with $\mathbb{P}^2$. Take a nodal cubic curve C. Let $\varrho: X \to \mathbb{P}^2$ be the blowing up of the node n times. Let E_0 be the strict transform of C, and let $E_1, \ldots, E_n$ be the strict transform of exceptional curves. Then we have

$$E_0^2 = -(6-n), \quad E_1^2 = \ldots = E_{n-1}^2 = -2, \quad E_n^2 = -1.$$

These curves $E_0, \ldots, E_n$ form a cycle of $\mathbb{P}^1$'s, and we have $\sum E_i \in |-K|$. For the case in which $n \geqq 9$, an easy calculation shows

$$d(X) = (n^2 - 7n + 1)^{-1} 9, \quad P_{-1}(X) = 1, \quad \varrho(Y) = 1.$$

Blown up Del Pezzo Surfaces. 7.4. Let S be a (degenerate) Del Pezzo surface with $d(S)=1$. As we have seen in 7.2, such a surface can be constructed from $\mathbb{P}^2$ by blowing up 8 points. Take a point s in S, and let $\mu: X \to S$ be the blowing up at s. Let E denote the exceptional curve. We are interested in the value $\kappa^{-1}(X)$. We know that $P_{-1}(S)=2$ and that $|-K_S|$ has a unique base point s_0 [5]. Let $\mu_0: X_0 \to S$ be the blowing up at s_0, and let E_0 be the exceptional curve. Then $\kappa^{-1}(X_0)=1$ and hence X_0 has a minimal elliptic fibration $p: X_0 \to \mathbb{P}^1$, which has no multiple fibre. Note that E_0 is a section in this fibration. If f is a fibre of p, $f \sim -K_0$, hence $f+E_0 \sim \mu_0^*(-K_S)$. Therefore, $f+E_0=\mu_0^* D_f$, where $D_f \in |-K_S|$. If f is a regular fibre, D_f is a non-singular elliptic curve. If f is a singular fibre, we divide into two cases: (i) D_f is an irreducible rational curve with a node or a cusp, whose self-intersection is -1, (ii) D_f consists of non-singular rational curves with self-intersection -2 except one with self-intersection -1.

Now we assume that $s \neq s_0$. There is a unique anticanonical divisor $D=D_f \in |-K_S|$ which contains s. In case s is a singular point of D, we see that $\kappa^{-1}(X)=2$ (Lemma 6.9). In case s is a non-singular point of D, let us write as $\mu^* D = C+E$ with C being the strict transform of D. Then $C \sim -K$. If s lies on a -2 curve, C contains a -1 curve, from which we infer that $\kappa^{-1}(X)=2$. Otherwise, C becomes an indecomposable curve of canonical type. In view of Proposition 5 in [22], we get

$$\kappa^{-1}(X)=\begin{cases} 0 & \text{if } \mathcal{N}_C \text{ is not a torsion element of } \operatorname{Pic}(C), \\ 1 & \text{if } \mathcal{N}_C \text{ is a torsion element of } \operatorname{Pic}(C). \end{cases}$$

To summarize, let $f_1, \ldots, f_n$ be the singular fibres of p, and let $D_i = D_{f_i}$ for each i. We set

$$\sum = \cup \operatorname{Sing}(D_i) \cup (-2 \text{ curves in } D_i).$$

Then we have seen the equivalence:

(7.4.1) $$\kappa^{-1}(X)=2 \Leftrightarrow s \in \sum.$$

In this case, the anticanonical model Y has one or two rational triple points and also $d(X)=\sum d(y_i)$. It is not difficult to give all possible types of singularities.

Quotient Surfaces ($k=\mathbb{C}$). *7.5.* Consider an action of $G=\mathbb{Z}/5\mathbb{Z}$ on $\mathbb{P}^2$ by

$$(X_0, X_1, X_2) \to (X_0, \zeta X_1, \zeta^2 X_2),$$

where ζ is a primitive 5-th root of the unity. Then G has three fixed points on $\mathbb{P}^2$. Correspondingly, the quotient surface $\mathbb{P}^2/G$ has two rational triple points and a rational double point. In this case, the minimal resolution X is a rational surface with $\kappa^{-1}(X)=2$ with the following invariants:

$$d(X)=\tfrac{9}{5}, \quad P_{-1}(X)=1, \quad \varrho(Y)=1.$$

In general, if a finite group G acts with only isolated fixed points on a non-singular rational surface X' with $\kappa^{-1}(X')=2$, then the minimal resolution X of the quotient X'/G is again a rational surface with $\kappa^{-1}(X)=2$. Furthermore, $d(X) = d(X')/|G|$, where $|G|$ denotes the order of G.

Torus Embeddings. 7.6. Let X be a non-singular rational surface obtained by a torus embedding. Let T denote the torus $k^* \times k^*$. The boundary $X \backslash T$ consists of a cycle of non-singular rational curves $E_{-1}, E_0, \ldots, E_\varrho$, where $\varrho = \varrho(X)$. It turns out that $\sum E_i \in |-K|$. We infer from the explicit construction in [18] that $\kappa^{-1}(X)=2$ (cf. Lemma 6.9). The anticanonical model Y has only quotient singularities.

Compactifications of $\mathbb{C}^2$. *7.7.* We remark that any minimal normal compactification X of $\mathbb{C}^2$ is an example of a rational surface with $\kappa^{-1}(X)=2$ [16].

Appendix. Vanishing Theorem

Let V, y, U, π, $A = E_1 \cup \ldots \cup E_n$ have the same meaning as in Sect. 1. But we do not necessarily assume that π is minimal. First, we show a local version of the Zariski decomposition of divisors.

Theorem A.1. *Let D be a divisor on U. Then there exists a unique decomposition:*

$$D = P + N$$

satisfying the following properties:

(i) *the P is a $\mathbb{Q}$-divisor such that $PE_j \geqq 0$ for all j,*
(ii) *the N is an effective $\mathbb{Q}$-divisor supported in A,*
(iii) *if E_j is a component of N, then $PE_j = 0$.*

Proof. The following proof is just an easier version of that of the global Zariski decomposition [7]. Set

$$A^{(1)} = \{E_j | DE_j < 0\}.$$

If $A^{(1)} = \emptyset$, we have only to put $P = D$, $N = 0$. If not, define a $\mathbb{Q}$-divisor N_1 supported in $A^{(1)}$ by the equations:

$$N_1 E_j = DE_j \quad \text{for all } E_j \text{ in } A^{(1)}.$$

We infer that N_1 is effective (Lemma 2.1). Letting $P_1 = D - N_1$, we set

$$A^{(2)} = \{E_j | P_1 E_j < 0\}.$$

If $A^{(2)} = \emptyset$, we would put $P = P_1$, $N = N_1$. Otherwise, define an effective $\mathbb{Q}$-divisor N_2 supported in $A^{(1)} \cup A^{(2)}$ by

$$N_2 E_j = P_1 E_j \quad \text{for all } E_j \text{ in } \quad A^{(1)} \cup A^{(2)}.$$

Letting $P_2 = P_1 - N_2$, we set again

$$A^{(3)} = \{E_j | P_2 E_j < 0\}.$$

If $A^{(3)} = \emptyset$, we would put $P = P_2$, $N = N_1 + N_2$. If not, we repeat the above process. The proof of the uniqueness part has been omitted (cf. Lemma 2.2).

We now prove the following local version of the Miyaoka-Ramanujam vanishing theorem (Theorem 5.6.1).

Theorem A.2. *Let D be a divisor on U, and let $D = P + N$ denote the Zariski decomposition. Then*

$$R^1 \pi_* \mathcal{O}(K + D - [N]) = 0.$$

Proof.[1] In view of the Grothendieck's formal function theorem, it suffices to prove the vanishing of the cohomology group $H^1(Z, \mathcal{O}_Z(K+D-[N]))$ for any effective divisor Z supported in A. By duality, we have to show that $H^0(Z, \mathcal{O}_Z(Z-D+[N]))=0$. We have $Z-D+[N]=Z-(N-[N])-P$. If we write $Z-(N-[N])$ as $\Gamma'-\Gamma''$, where both Γ' and Γ'' are effective and have no common components. We can find an irreducible component E of Γ' such that $\Gamma'E<0$. Since each coefficient of $N-[N]$ is less than one, $\mathrm{Supp}(Z)=\mathrm{Supp}(\Gamma')$. So E is also a component of Z. Now we put $Z'=Z-E$. We have

$$\deg_E(\mathcal{O}_E(Z-D+[N]))=\Gamma'E-\Gamma''E-PE<0,$$

hence $H^0(E, \mathcal{O}_E(Z-D+[N]))=0$. By the long exact sequence of

$$0\to\mathcal{O}_{Z'}(Z'-D+[N])\to\mathcal{O}_Z(Z-D+[N])\to\mathcal{O}_E(Z-D+[N])\to 0,$$

we obtain the isomorphism:

$$H^0(Z', \mathcal{O}_{Z'}(Z'-D+[N]))\cong H^0(Z, \mathcal{O}_Z(Z-D+[N])).$$

Thus by induction on the sum of the coefficients of Z, we get the required vanishing. Q.E.D.

Corollary A.3 (dual form). *With the same hypothesis,*

$$H^1_A(U, \mathcal{O}(-D+[N]))=0.$$

Proof. By the duality theory (Bădescu [3], Lipman [12]), we have the isomorphism

$$H^1_A(U, \mathcal{O}(-D+[N]))\cong H^1(U, \mathcal{O}(K+D-[N]))^{\vee}. \quad \text{Q.E.D.}$$

Remark A.4. When π is minimal, the Zariski decompositions of K and $-K$ are as follows.

Table A.4.1

	P	N
K	K	0
$-K$	$-K-\Delta$	Δ

Correspondingly, we obtain the following vanishings (cf. Lemmata 1.5 and 1.8):

$$R^1\pi_*\mathcal{O}(mK)=0 \quad [H^1_A(U, \mathcal{O}(-(m-1)K))=0] \quad \text{for} \quad m>0,$$

$$R^1\pi_*\mathcal{O}(-mK-[(m+1)\Delta])=0 \quad [H^1_A(U, \mathcal{O}((m+1)K+[(m+1)\Delta]))=0] \quad \text{for} \quad m\geqq 0.$$

1 For proofs when $N=0$, we refer to Laufer [11] (Bădescu [3]), Ramanujam [19], and Wahl [23] (dual form). Our first proof was along the Laufer's argument. I would like to thank T. Fujita for correcting a mistake there

Note added in proof. For a further systematic study of Weil divisors on normal surfaces, we refer to the forthcoming paper: Weil divisors on normal surfaces, to appear in Duke Math. J.

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A Remark on Two Hilbert Space Scattering Theory

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In the mathematical analysis of scattering processes it is of particular interest to compare the long time behaviour of a given system with a more easily understood reference configuration. For applications it is important to have an abstract theory mimicking the naturally given assumptions of leading examples. This requirement is – with respect to an important class of problems related to selfadjoint realizations of partial differential operators – satisfied by the Birman-Belopolskii approach to two Hilbert space scattering theory, [Be]. The Birman-Belopolskii theory has experienced extension and considerable simplification by the work of Pearson, [Pe]. The occurence of infinite dimensional eigenspaces in examples of particular interest like Maxwell's equations and the first order system of linearized elasticity theory led to difficulties.

To meet these difficulties Lyford proposed a more sophisticated setting of the assumptions [Ly 1], but his proof contained a gap, which was pointed out in [Ly 2]. Nevertheless, such a result is desirable.

Fortunately we are able to introduce an additional assumption to Lyford's conditions, that seems to be quite naturally satisfied in the applications and that corrects the argument in Lyford's proof. We are concerned with the following situation.

Let A_i be a selfadjoint operator with domain $D(A_i)$ in the separable Hilbert space H_i, $i=1,2$. In order to compare these operators we need an "identification operator"

$$J: H_1 \to H_2 \quad \text{linear and bounded}, \tag{1}$$

such that

$$\begin{aligned} &J(D(A_1)) \subset D(A_2), \\ &J^*(D(A_2)) \subset D(A_1). \end{aligned} \tag{2}$$

Let $(\Pi_i(\lambda))_{\lambda\in\mathbb{R}}$ denote the spectral family of A_i and P_i be the projection on the corresponding absolute continuous subspace of H_i, $i=1,2$. The generalized wave operators $W_\pm$ are the strong limits, when they exist,

$$W_\pm = \underset{t\to\pm\infty}{s\text{-lim}}\, W(t)P_1$$

with

$$W(t)=e^{itA_2}Je^{-itA_1}.$$

For a proper comparison between A_1 and A_2 it is desirable to have $W_\pm$ as unitary operators from the absolute continuous subspace $H_{1,ac}\equiv P_1H_1$ onto the absolute continuous subspace $H_{2,ac}=P_2H_2$. In this case the wave operators are called complete.

We shall now formulate the main result about existence and completeness of the wave operators.

Theorem 1. *Let $(I_n)_{n\in\mathbb{N}}$ ($\mathbb{N}$ set of positive integers) be a family of open, disjoint, bounded intervals on the real line $\mathbb{R}$ such that $N=\mathbb{R}\setminus\bigcup_{n=1}^{\infty} I_n$ is a Lebesgue null set and*

(a) $(A_2J-JA_1)\Pi_1(L)$ *trace class,*
(b) $(J^*J-1)\Pi_1(L)$ *compact,*
(c) $(JJ^*-1)\Pi_2(L)$ *compact,*
(d) $(J^*A_2-A_1J^*)\Pi_2(L)$ *compact,*

for all $L\equiv\bigcup_{n=1}^{\infty} I_n\cap I$, where I is an arbitrary bounded open interval of $\mathbb{R}$. Under these assumptions the strong limits

$$W_\pm=\underset{t\to\pm\infty}{s\text{-lim}}\ W(t)P_1,$$

$$V_\pm=\underset{t\to\pm\infty}{s\text{-lim}}\ W(t)^*P_2,$$

exist. The restrictions $W_\pm: H_{1,ac}\to H_{2,ac}$, $V_\pm: H_{2,ac}\to H_{1,ac}$ are unitary operators and

$$W_\pm^*=V_\pm.$$

Remark. We have chosen to formulate the theorem not in the most general, but in the most convenient way as far as applications are concerned that we have in mind. Following the details of the proof the reader will be able to find the technically necessary assumptions, (compare [Ly 1]).

The idea of the proof is to localize the question with respect to the partition $(I_n)_n$. The arguments will be based on a result by Pearson, [Pe].

Theorem 2 (D. Pearson). *Let $L: H_1\to H_2$ be a bounded linear operator such that*

$$L(D(A_1))\subset D(A_2),$$

and the closure of

$$A_2L-LA_1,$$

is trace class. Then the strong limit

$$\underset{t\to\pm\infty}{s\text{-lim}}\ e^{itA_2}Le^{-itA_1}P_2$$

exists.

Proof of Theorem 1. Pearson's theorem yields the existence of

$$W_{\pm}(I_n) := \underset{t\to\pm\infty}{s\text{-lim}}\, W(t)\Pi_1(I_n)P_1\,, \qquad n\in\mathbb{N}\,. \tag{3}$$

Without loss of generality we may restrict our attention to the limit $t\to+\infty$. For $W_+(I_n)$, $n\in\mathbb{N}$, we obtain the following properties:

$$\Pi_2(\lambda)W_+(I_n) = W_+(I_n)\Pi_1(\lambda)\,, \qquad \lambda\in\mathbb{R}\,, \tag{4.1}$$

$$P_2W_+(I_n) = W_+(I_n)\,, \tag{4.2}$$

$$W_+(I_n)H_1 \subset \Pi_2(I_n)P_2H_2\,. \tag{4.3}$$

Furthermore, it is allowed to add up all the local wave operators:

$$W_+ := \underset{N\to\infty}{s\text{-lim}} \sum_{n=1}^{N} W_+(I_n) \quad \text{exists}\,. \tag{5}$$

W_+ is in fact the wave operator

$$W_+ = \underset{t\to\infty}{s\text{-lim}}\, W(t)P_1\,,$$

and by (4.2)

$$P_2W_+ = W_+\,. \tag{6}$$

Using assumption (b) of the theorem we obtain by standard arguments

$$\|W_+u\| = \|P_1u\|\,, \qquad u\in H_1\,. \tag{7}$$

Applying Pearson's theorem again we obtain the existence of

$$V_+(I_n) := \underset{t\to\infty}{s\text{-lim}}\, \Pi_1(I_n)W(t)^*P_2\,.$$

$V_+(I_n)$ has similar properties as $W_+(I_n)$,

$$\Pi_1(\lambda)V_+(I_n) = V_+(I_n)\Pi_2(\lambda)\,, \qquad \lambda\in\mathbb{R}\,, \tag{8.1}$$

$$P_1V_+(I_n) = V_+(I_n)\,, \tag{8.2}$$

$$V_+(I_n)H_2 = \Pi_1(I_n)P_1H_1\,. \tag{8.3}$$

Furthermore, we have

$$V_+(I_n)^* = W_+\Pi_1(I_n)\,. \tag{9}$$

The existence of

$$V_+ = \underset{N\to\infty}{s\text{-lim}} \sum_{n=1}^{N} V_+(I_n) \tag{10}$$

follows in the same way as equality (5); for details see [PS].

The proof of the isometry of V_+ is the actually difficult part of the proof of the main result. So, we state this separately as a lemma.

Lemma. *We have*

$$\|V_+v\| = \|P_2v\| \quad \textit{for all} \quad v\in H_2\,.$$

From the lemma we can conclude that

$$W_+ H_1 = P_2 H_2 .$$

in fact

$$W_+ V_+ \quad \text{is the identity on} \quad H_{2,ac} = P_2 H_2 . \tag{11}$$

From (11) we can see – using assumption (c) – that V_+ defined in (10) is indeed the strong limit of $W(t)^* P_2$ as $t \to \infty$:

$$V_+ = \operatorname*{s\text{-lim}}_{t\to\infty} W(t)^* P_2 . \tag{12}$$

Assuming that the lemma is valid this completes the proof of our main result. □

Proof of the Lemma. Let $K \Subset I_n$, and for $r>0$ let $\Delta_r := (-r, r)$ and $R_r := (-\infty, -r] \cup [r, \infty)$. Then for large $r>0$ we obtain

$$\begin{aligned}&(J\Pi_1(I_n)J^* - 1)\Pi_2(K)\\ &\quad = (JJ^* - 1)\Pi_2(K) - J\Pi_1(R_r)J^*\Pi_2(K) - J\Pi_1(\Delta_r \setminus I_n)J^*\Pi_2(K) .\end{aligned} \tag{13}$$

The first term on the r.h.s. of (13) is compact by assumption (c), and the second can be estimated

$$\|J\Pi_1(R_r)J^*\Pi_2(K)\| = 0(r^{-1}) \quad \text{as} \quad r \to \infty . \tag{14}$$

For the third term we have

$$\begin{aligned}&\Pi_1(\Delta_r \setminus I_n)J^*\Pi_2(K)\\ &\quad = \int_{\lambda\in\Delta_r\setminus I_n} \int_{\mu\in K} (\lambda-\mu)^{-1} d\Pi_1(\lambda)\Pi_1(\Delta_r)(A_1 J^* - J^* A_2)\Pi_2(K) d\Pi_2(\mu) ,\end{aligned} \tag{15}$$

and, since dist $(\Delta_r \setminus I_n, K) > 0$, the double-integral transforms compact operators into compact operators (see [Bi], Theorem 9). From assumption (d) now follows that $J\Pi_1(\Delta_r \setminus I)J^*\Pi_2(K)$ is compact. Hence $(J\Pi_1(I_n)J^* - 1)\Pi_2(K)$ is compact, and we conclude

$$\|V_+ \Pi_2(K) v\| = \|\Pi_2(K) P_2 v\| \quad \text{for all} \quad v \in H_2 , \ \text{all} \ K \Subset I_n . \tag{16}$$

From (16) the assertion is easily deduced. □

Remark. Under the given assumptions the "invariance principle" holds: Let $L_1, \ldots, L_n$ be a finite collection of intervals with $\mathbb{R} = \bigcup_{n=1}^{N} L_n$ and let ϕ be a real-valued function on $\mathbb{R}$, that is continuously differentiable on each open L_n with ϕ' positive and locally of bounded variation. Then the wave operators

$$\operatorname*{s\text{-lim}}_{t\to\pm\infty} e^{i\phi(A_2)t} J e^{-i\phi(A_1)t} P_1$$

exist and are equal to $W_\pm$. To prove this, we observe first, that the invariance principle holds in the situation of Pearson's theorem. The idea for the proof of this step may be found in the book of Kato ([Ka], Chap. X). Then, by multiplying from the left with $\Pi_1(I_n)$, we obtain invariance on $\Pi_1(I_n)H_1$. The invariance principle now follows again by adding up.

Remark. The significance of Theorem 1 can be seen from initial boundary value problems of classical physics, e.g. the equations of linearized elasticity theory or Maxwell-type equations, where infinite-dimensional eigenspaces occur (compare [Pi]). In such cases Lyford's condition, i.e.

$$\Pi_2(\Delta)(A_2J-JA_1)\Pi_1(\Delta) \quad \text{trace-class for every bounded real interval } \Delta,$$

is clearly not satisfied, but conditions (a)–(d) of Theorem 1 are. As (a)–(c) are concerned, this is shown in [Pi]. Validity of (d) can be proved in the same manner as (c), for after suitable choice of the identification operator, $(J^*A_2-A_1J^*)$ appears to be a bounded, localizing operator, and so (d) follows from the local compactness property for the operator A_2.

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Submanifolds with Geodesic Normal Sections

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Dedicated to Professor Kentaro Yano on his 70th birthday

1. Introduction

Let M be a connected n-dimensional ($n \geqq 2$) submanifold of an m-dimensional Euclidean space $\mathbb{E}^m$. For a point p in M and a unit vector t tangent to M at p, the vector t and the normal space $T_p^\perp M$ to M at p determine an $(m-n+1)$-dimensional affine subspace $E(p,t)$ in $\mathbb{E}^m$ through p. The intersection of M and $E(p,t)$ gives rise to a curve γ in a neighborhood of p which is called the *normal section* of M at p in the direction t [4]. In [4, 5, 9] submanifolds in $\mathbb{E}^m$ with (pointwise) planar normal sections were investigated. In this paper we ask the following geometric question:

"*Which submanifolds of* $\mathbb{E}^m$ *have geodesic normal sections?*"

In Sect. 2 we observe that "standard" immersions of a strongly harmonic space in $\mathbb{E}^m$ have geodesic normal sections (Remark 1). Moreover, we claim that every submanifold in $\mathbb{E}^m$ with geodesic normal sections is always constant isotropic (Theorem 1). Moreover in this paragraph, two characterizations and some basic results for submanifolds with geodesic normal sections are also obtained.

In Sect. 3, we will concentrate on surfaces with geodesic normal sections. Several classification theorems are obtained. In particular, we will show that if M is a surface in $\mathbb{E}^5$ with geodesic normal sections, then M is contained in a 2-plane, or in a 2-sphere or in a Veronese surface as an open submanifold.

The results of this paper have been announced in [6]. The authors would like to take the opportunity to express their many thanks to Professor T. Nagano for his valuable suggestions and to Professor P. Dombrowski and the referee for their useful comments.

2. Submanifolds with Geodesic Normal Sections

In this paragraph, we assume that M is an n-dimensional submanifold in $\mathbb{E}^m$. Let ∇ and $\tilde{\nabla}$ be the covariant derivatives of M and $\mathbb{E}^m$, respectively. For any two vector

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fields X and Y tangent to M, the second fundamental form h is given by $h(X, Y) = \tilde{\nabla}_{XY} - \nabla_X Y$. For any vector field ξ normal to M, we put $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ and $D_X \xi$ denote the tangential and normal components of $\tilde{\nabla}_X \xi$, respectively. D is called the normal connection of the normal bundle $T^\perp M$.

For the second fundamental form h its covariant derivative, denoted by $\bar{\nabla} h$, is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{1}$$

for vector fields X, Y, and Z tangent to M.

We say that a submanifold M of $\mathbb{E}^m$ has *geodesic normal sections* when each normal section of M in $\mathbb{E}^m$ is a geodesic of M (or, equivalently, each geodesic of M is a normal section of M in $\mathbb{E}^m$). The uniqueness theorem of geodesics implies the following.

Lemma 1. *If a submanifold M of $\mathbb{E}^m$ has geodesic normal sections, then any curve γ in M, which is a normal section of M at $\gamma(0)$ in the direction $\gamma'(0)$ remains for all s in the domain of γ a normal section of M at $\gamma(s)$ in the direction $\gamma'(s)$, i.e. for all $s \in \operatorname{dom}\gamma$, γ is contained in the affine subspace $E(\gamma(s), \gamma'(s))$ of $\mathbb{E}^m$; in particular for all $s \in \operatorname{dom}\gamma$, all derivatives $\gamma^{(k)}$ $(k \in \mathbb{N}_0)$ are contained in* Span$\{\gamma'(s), T^\perp_{\gamma(s)}M\}$. □

We recall that a submanifold M of $\mathbb{E}^m$ is said to be *isotropic* (in the sense of O'Neill [12]) if for each point p in M and each unit vector t tangent to M at p, the length of $h(t, t)$, $\|h(t, t)\|$, depends only on p and not on t at p. In other words, each geodesic of M emanating from p has – considered as a curve in $\mathbb{E}^m$ – the same first curvature κ_1 at p. In particular, when $\|h(t, t)\|$ is also independent of the point p in M, then M is said to be *constant isotropic*.

Theorem 1 (Constant isotropy). *Each submanifold M in $\mathbb{E}^m$ with geodesic normal sections is constant isotropic.*

Proof. Assume that M has geodesic normal sections. Let p be a point in M and γ a normal section at p parametrized by arc length with $p = \gamma(0)$. Denote γ' by T. Because γ is a geodesic of M, we have

$$\gamma'' = \gamma^{(2)} = h(T, T), \tag{2}$$

$$\gamma^{(3)} = -A_{h(T,T)}T + D_T(h(T, T)), \tag{3}$$

$$\gamma^{(4)} = -\nabla_T(A_{h(T,T)}T) - A_{D_T(h(T,T))}T - h(A_{h(T,T)}T, T) + D_T(D_T(h(T, T))). \tag{4}$$

From Lemma 1, we see that for any fixed s, $\gamma^{(3)}(s)$ is a linear combination of $T(s)$ and vectors of $T^\perp_{\gamma(s)}M$. Thus we get

$$A_{h(T,T)}T \wedge T = 0.$$

In particular, with $t = T(0)$ at $p = \gamma(0)$, we have

$$\langle h(t, t), h(t, z)\rangle = 0 \tag{5}$$

for each $z \in T_pM$ such that $\langle z, t\rangle = 0$. Because (5) holds for each $t \in TM$, M is isotropic [12, p. 207], i.e.

$$\|h(t,t)\| = \lambda(p) \tag{6}$$

for all unit vectors $t \in T_pM$, $p \in M$, where λ is a function on M.

From (4) and Lemma 1 we obtain

$$\langle \nabla_t(A_{h(T,T)}T) + A_{D_t(h(T,T))}t, z\rangle = 0 \tag{7}$$

for $z \in T_pM$ with $\langle z, t\rangle = 0$. Extend z to a vector field Z tangent to M which is parallel along γ and extend t on a neighborhood of $p = \gamma(0)$ such that T stays a unit vector orthogonal to Z. From $\langle A_{h(T,T)}T, Z\rangle = 0$ and (7) we get

$$\begin{aligned}
0 &= \langle A_{h(T,T)}T, \nabla_t Z\rangle - \langle D_t(h(T,T)), h(t,z)\rangle \\
&= \langle h(t,t), D_t(h(T,Z))\rangle \quad [\text{using (5) and } \nabla_t Z = 0] \\
&= \langle h(t,t), (\bar\nabla_z h)(t,t)\rangle \quad (\text{using } \nabla_t T = \nabla_t Z = 0 \text{ and the Codazzi equation}) \\
&= \langle h(t,t), D_z(h(T,T))\rangle - 2\langle h(t,t), h(t, \nabla_z T)\rangle \\
&= \tfrac{1}{2} z\langle h(T,T), h(T,T)\rangle \quad [\text{using } \nabla_z T \perp t \text{ and (5)}] \\
&= \tfrac{1}{2} z(\lambda^2).
\end{aligned}$$

Since $\dim M \geqq 2$, this implies that M is constant isotropic. □

Now, we give the following.

Theorem 2 (Characterization). *Let M be a submanifold in $\mathbb{E}^m$. Then the following three statements are equivalent:*

(a) *M has geodesic normal sections;*

(b) *all normal sections of M – considered as curves in $\mathbb{E}^m$ – have the same constant first curvature;*

(c) *every curve γ of M which is a normal section of M at $\gamma(0)$ in the direction $\gamma'(0)$ remains for all $s \in \operatorname{dom}\gamma$ a normal section of M at $\gamma(s)$ in the direction $\gamma'(s)$.*

Proof. We first show the equivalence of (a) and (b). Let γ be a normal section of M at $\gamma(0) = p$ in the direction $\gamma'(0)$. Then we have

$$\gamma'' = \tilde\nabla_T T = \nabla_T T + h(T,T), \tag{8}$$

where $T = \gamma'$. Since $\nabla_T T$ is perpendicular to T and γ is a normal section at p, (8) implies that $\nabla_t T = 0$ and, therefore, that the first curvature $\kappa_1 = \|\gamma''\|$ of γ satisfies

$$\kappa_1(0) = \|h(t,t)\|. \tag{9}$$

If M has geodesic normal sections, Theorem 1 and (9) show that all normal sections have the same constant first curvature. Conversely, assume that all normal sections of M have the same constant first curvature, say λ. Let p and q be two points of M and t and u two unit vectors with $t \in T_pM$ and $u \in T_qM$. Let γ and $\bar\gamma$ be the normal sections determined by (p,t) and (q,u), respectively. Then by (9) and the hypothesis we have $\|h(t,t)\| = \|h(u,u)\|$. So M is constant isotropic. Now, from (8) and (9) we have $\lambda^2 = \kappa_1^2 = \|\nabla_T T\|^2 + \|h(T,T)\|^2 = \|\nabla_T T\|^2 + \lambda^2$, so $\nabla_T T = 0$. This proves (a) $\Leftrightarrow$ (b).

The implication (a) $\Rightarrow$ (c) is just Lemma 1. So we only need to prove (c) $\Rightarrow$ (a). Assume statement (c) holds. Let γ be a normal section in M. For any fixed $s_0 \in \operatorname{dom}\gamma$, $\gamma(s)$ lies for all $s \in \operatorname{dom}\gamma$ in $E(\gamma(s_0), \gamma'(s_0))$. Thus, from $\gamma''(s_0) = \nabla_{\gamma'(s_0)}\gamma' + h(\gamma'(s_0), \gamma'(s_0))$, we find $\nabla_{\gamma'(s_0)}\gamma' = 0$ because $\nabla_{\gamma'(s_0)}\gamma'$ is perpendicular to $E(\gamma(s_0), \gamma'(s_0))$. Since this is true for all $s_0 \in \operatorname{dom}\gamma$, the normal section γ is a geodesic. □

From Theorem 2 we obtain the following.

Corollary 1. *If M is a compact submanifold of $\mathbb{E}^m$ with geodesic normal sections, then all geodesics of M are closed curves and – considered as curves in $\mathbb{E}^m$ – they have the same constant first curvature. In particular, if M is a compact symmetric space, then M is of rank one.*

The first part of this corollary follows from Theorem 3 [especially from the fact that a normal section, which is a geodesic of M, is defined on the entire real line and it is given globally by $M \cap E(p, t)$ for some point p in M and a unit vector $t \in T_pM$]. The second part follows from the fact that every flat torus of dimension $\geqq 2$ contains non-closed geodesics.

Combining Theorem 2 and Theorem 3 of [4] we obtain the following.

Corollary 2. *Let M be a submanifold of $\mathbb{E}^m$. If all geodesics of M considered as curves in $\mathbb{E}^m$ are 2-planar, then M has geodesic normal sections.* □

Here a curve in $\mathbb{E}^m$ is said to be *q-planar* if it is contained in a q-plane of $\mathbb{E}^m$. A q-planar curve α such that no arc of α is r-planar for some $r < q$ is said to be *proper q-planar*.

As a converse of Theorem 1 we give the following.

Proposition 1. *Let M be a submanifold of $\mathbb{E}^m$ such that all its geodesics, considered as curves in $\mathbb{E}^m$, are 3-planar. Then M has geodesic normal sections if and only if M is isotropic.*

Proof. Assume that M is isotropic and all geodesics of M are 3-planar. Let β be a geodesic in M. Consider as a curve in $\mathbb{E}^m$, β lies in $\beta(s) + \operatorname{Span}\{\beta'(s), \beta^{(2)}(s), \beta^{(3)}(s)\}$ for all s. Put $\beta' = T$. Since $\beta^{(2)} = h(T, T)$, $\beta^{(3)} = -A_{h(T,T)}T + D_T(h(T, T))$ and M is isotropic, we have $A_{h(T,T)}T \in \operatorname{Span}\{T\}$. Hence β lies in $\operatorname{Span}\{T(s), h(T(s), T(s)), D_{T(s)}(h(T, T))\}$ for all $s \in \operatorname{dom}\gamma$. In particular, this shows that for a fixed $s \in \operatorname{dom}\beta$, β lies in the affine subspace $E(\beta(s), \beta'(s))$ which is spanned by $\beta'(s)$ and $T^{\perp}_{\beta(s)}M$. Thus β coincides with the normal section of M at $\beta(s)$ in the direction $\beta'(s)$. Therefore, M has geodesic normal sections. The converse of this is given in Theorem 1. □

Proposition 2. *Let M be a submanifold of $\mathbb{E}^m$ all of whose geodesics (considered as curves in $\mathbb{E}^m$) are 4-planar. Then M has geodesic normal sections if and only if M is constant isotropic.*

Proof. Assume that M is constant isotropic and has geodesics which are all 4-planar. Let β be a geodesic in M. Considering β as a space curve and using the same notations as in the proof of Proposition 1 we have

$$\beta^{(4)} = -\lambda^2 h(T, T) - A_{D_T(h(T,T))}T + D_T(D_T(h(T, T))),$$

where $\lambda^2 = \|h(T, T)\|^2$. Let z be any vector in $T_{\beta(s)}M$ with $\langle z, \beta'(s)\rangle = 0$ for a fixed s in dom β. Extend z to a vector field Z tangent to M such that $\nabla_T Z = 0$. Then we have

$$\begin{aligned}\langle A_{D_T(h(T,T))}T, Z\rangle &= \langle D_T(h(T,T)), h(T,Z)\rangle \\ &= -\langle h(T,T), D_T(h(T,Z))\rangle \\ &= -\langle h(T,T), (\bar{\nabla}_T h)(T,Z)\rangle \\ &= -\langle h(T,T), (\bar{\nabla}_Z h)(T,T)\rangle .\end{aligned}$$

Extend T to a unit tangent vector field on M (which will also be denoted by T). The isotropy of M implies $\langle h(T,T), h(\nabla_Z T, T)\rangle = 0$ such that

$$\langle A_{D_T(h(T,T))}T, Z\rangle = -\tfrac{1}{2}Z(\lambda^2) = 0$$

because M is constant isotropic. From this we conclude that β lies in $\beta(s) + \mathrm{Span}\{T(s), h(T(s), T(s)), D_{T(s)}(h(T,T)), D_{T(s)}(D_T(h(T,T)))\}$. Hence M has geodesic normal sections. The converse follows again from Theorem 1. □

Proposition 3. *Let M be an n-dimensional submanifold in $\mathbb{E}^m$ with geodesic normal sections ($n<m$). Then M admits no geodesic arc which is proper $(m-n+1)$-planar unless $m=n+1$ and M is contained in a hypersphere of $\mathbb{E}^{n+1}$ as an open manifold.*

Proof. Let M be a submanifold in $\mathbb{E}^m$ with geodesic normal sections. Suppose γ to be a geodesic arc in M which is proper $(m-n+1)$-planar. Then without loss of generality we may assume that

$$\gamma' \wedge \gamma^{(2)} \wedge \ldots \wedge \gamma^{(m-n+1)} \neq 0$$

along γ [i.e. $\gamma'(s), \gamma^{(2)}(s), \ldots, \gamma^{(m-n+1)}(s)$ are linearly independent for each s]. Thus, by continuity, for any fixed s, a geodesic $\tilde{\gamma}$ through $\gamma(s)$ which is sufficiently close to γ is also proper $(m-n+1)$-planar in a small neighborhood of $\gamma(s)$.

Now let $\{e_1, \ldots, e_n, \xi_{n+1}, \ldots, \xi_m\}$ be an orthonormal frame such that $e_1, \ldots, e_n$ are tangent to M and e_1 restricted to γ equals γ'. Because γ lies in an $(m-n+1)$-plane $\mathbb{E}^{m-n+1}$ and normal sections are geodesics, $\mathbb{E}^{m-n+1}$ is spanned by e_1, $\xi_{n+1}, \ldots, \xi_m$ along γ. Thus we have

$$\tilde{\nabla}_{e_1}(e_2 \wedge e_3 \wedge \ldots \wedge e_n) = 0 \quad \text{along } \gamma .$$

Since $\nabla_{e_1} e_1 = 0$ along γ, this implies

$$\begin{aligned}0 = {}& h(e_1, e_2) \wedge e_3 \wedge \ldots \wedge e_n + e_2 \wedge h(e_1, e_3) \wedge e_4 \wedge \ldots \wedge e_n + \ldots \\ & \ldots + e_2 \wedge \ldots \wedge e_{n-1} \wedge h(e_1, e_n) \quad \text{along } \gamma .\end{aligned}$$

Consequently, $h(e_1, u) = 0$ for each u orthogonal to e_1 along γ.

Let u be a unit vector in T_pM, $p = \gamma(0)$, orthogonal to $e_1(p)$. Consider a rotation in T_pM given by

$$\begin{cases}\bar{e}_1 = \cos\theta e_1(p) + \sin\theta u , \\ \bar{u} \;= -\sin\theta e_1(p) + \cos\theta u .\end{cases}$$

Since γ is proper $(m-n+1)$-planar, for sufficiently small θ, $\bar{e}_1$ gives a new geodesic normal section which is proper $(m-n+1)$-planar. Thus

$$0 = h(\bar{e}_1, \bar{u}) = \cos\theta \sin\theta [h(u,u) - h(e_1(p), e_1(p))]$$

for sufficiently small θ. Therefore, we have

$$h(e_1(p), e_1(p)) = h(u, u)$$

and

$$h(e_1(p), u) = 0$$

for any unit vector u in T_pM with $\langle u, e_1(p)\rangle = 0$, from which we conclude that p is an umbilical point. Since p can be any point on the geodesic γ, each point of γ is umbilical. Now, because geodesic arcs which are sufficiently close to γ are also proper $(m-n+1)$-planar, we may conclude that there exists a sufficiently small open subset U of M such that U is totally umbilical. Hence U is either contained in an n-plane or an n-dimensional hypersphere in an $(n+1)$-dimensional Euclidean subspace of $\mathbb{E}^m$. Since γ is assumed to be proper $(m+n-1)$-planar, the first case cannot occur and the second case occurs only when $m = n+1$. □

A submanifold M of $\mathbb{E}^m$ is said to be *spherical* if M is contained in a hypersphere of $\mathbb{E}^m$.

Proposition 4. *Let M be a spherical submanifold in $\mathbb{E}^m$. If M has geodesic normal sections, then M admits no geodesic arcs which are proper 3-planar.*

Proof. Let M be a submanifold of a hypersphere S_a^{m-1} of radius $a \in \mathbb{R}_0^+$ in $\mathbb{E}^m$ centered at the origin. Denote by h and h' the second fundamental forms of M in $\mathbb{E}^m$ and S_a^{m-1}, respectively. Because the second fundamental form $\tilde{h}$ of S_a^{m-1} in $\mathbb{E}^m$ is given by $\frac{1}{a}\langle\ ,\ \rangle\eta$, where η is the inner unit normal of S_a^{m-1} in $\mathbb{E}^m$, and $h = h' + \tilde{h}$, we have

$$\bar{\nabla} h = \bar{\nabla}' h'$$

and

$$\bar{\nabla}\bar{\nabla} h = \bar{\nabla}'\bar{\nabla}' h',$$

where $\bar{\nabla}'$ is defined in the same way as $\bar{\nabla}$ in (1) but using the normal connection D' of M in S_a^{m-1}.

Suppose that γ is a geodesic in M which is proper 3-planar. Put $T = \gamma'$ (where γ is considered as a curve in $\mathbb{E}^m$). Then we have $\gamma' \wedge \gamma^{(2)} \wedge \gamma^{(3)} \wedge \gamma^{(4)} = 0$, i.e.

$$(\bar{\nabla}_T\bar{\nabla}_T h)(T, T) \wedge (\bar{\nabla}_T h)(T, T) \wedge h(T, T) = 0.$$

Since $h(T, T)$ has a nontrivial component in the direction of η, this gives

$$(\bar{\nabla}'_T\bar{\nabla}'_T h')(T, T) \wedge (\bar{\nabla}'_T h')(T, T) = 0. \tag{10}$$

Now, because M is constant isotropic in both $\mathbb{E}^m$ and S_a^{m-1}, we have

$$\|(\bar{\nabla}'_T h')(T, T)\|^2 = -\langle(\bar{\nabla}'_T\bar{\nabla}'_T h')(T, T), h'(T, T)\rangle \tag{11}$$

and

$$\langle(\bar{\nabla}'_T h')(T, T), h'(T, T)\rangle = 0. \tag{12}$$

From (10), (11), and (12) we find $(\bar{V}'_T h')(T,T)=0$. Consequently, $(\bar{V}_T h)(T,T)=0$. This implies that $\gamma' \wedge \gamma^{(2)} \wedge \gamma^{(3)}=0$. Thus γ is not proper 3-planar which gives the desired contradiction. □

Let M be a submanifold in $\mathbb{E}^m$. Then M is called a *helical submanifold* of order d if each geodesic β of M – considered as a curve in $\mathbb{E}^m$ – is proper d-planar in $\mathbb{E}^m$ and has constant curvatures $\kappa_1, \ldots, \kappa_{d-1}$ which do not depend upon β. The Frenet frame $\{T, N_1, \ldots, N_{d-1}\}$ of β is given by (see, e.g., Theorem 3.1 of [14])

$$T=\beta',$$
$$N_j=\sum_i b_{ji}(\bar{V}_T^{i-2}h)(T,T)$$

for $j=1,\ldots,d-1$, where i runs over the range $\{2,4,\ldots,j+1\}$ if j is odd and $\{3,5,\ldots,j+1\}$ if j is even. The coefficients b_{ji} are certain functions of $\kappa_1,\ldots,\kappa_j$. This shows that the geodesic β lies in $\beta(s)+\operatorname{Span}\{T(s), N_1(s), \ldots, N_{d-1}(s)\}$ which is an affine subspace of $E(\beta(s), T(s))$ for any fixed s in the domain of β. Consequently, β is the normal section of M at $\beta(s)$ in the direction $\beta'(s)$ for any fixed s. Because this is true for any geodesic β, M has geodesic normal sections. This proves the following.

Proposition 5. *If M is a helical submanifold of $\mathbb{E}^m$, then M has geodesic normal sections.* □

Remark 1. Let M be a compact Riemannian manifold. Then M has a unique kernel of the heat equation [1, p. 155]:

$$K: M\times M\times \mathbb{R}_0^+ \to \mathbb{R}.$$

Let δ denote the distance function on M. Then M is called a strongly harmonic manifold if there exists a function $\Psi:\mathbb{R}^+\times\mathbb{R}_0^+\to\mathbb{R}$ such that $K(x,y,t)=\Psi(\delta(x,y),t)$ for $x,y\in M$ and $t\in\mathbb{R}_0^+$. Compact symmetric spaces of rank one are known examples of strongly harmonic manifolds [1, p. 158].

Let λ_k be the k-th nonzero eigenvalue of the Laplace operator Δ. Let $V_k=\{f\in C^\infty(M)|\Delta f=\lambda_k f\}$ be the eigenspace of Δ with eigenvalue λ_k. On V_k we can define an inner product by $\langle\langle f,g\rangle\rangle=\int_M fg*1$ for $f,g\in V_k$. V_k together with $\langle\langle\ ,\ \rangle\rangle$ becomes a finite dimensional Euclidean space. Let $\varphi_k^1,\ldots,\varphi_k^m$ be an orthonormal basis of V_k w.r.t. $\langle\langle\ ,\ \rangle\rangle$. Then the mapping

$$\varphi_k: M\to\mathbb{E}^m: x\mapsto c_k(\varphi_k^1(x),\ldots,\varphi_k^m(x))$$

defines a helical isometric immersion for some suitable constant c_k [1, p. 178]. From Proposition 5 we see that each such immersion gives rise to a submanifold with geodesic normal sections.

Remark 2. Let M be a compact symmetric space of rank one and let φ_1 be the isometric immersion given above. Then M has 2-planar geodesic normal sections. In particular, for a real projective plane we get the Veronese surface, an imbedding of $\mathbb{R}P^2$ into $\mathbb{E}^5$. In the following by a Veronese surface we mean an imbedding of a real projective plane into $\mathbb{E}^5$ which is obtained from the imbedding mentioned above up to isometries and homotheties of $\mathbb{E}^5$.

Remark 3. Consider the third standard immersion of an n-sphere

$$\varphi_3 : S^n(c(3)) \to \mathbb{E}^{N(3)+1}$$

given in [7], where $S^n(c(3))$ is the n-sphere of constant sectional curvature $n/3(n+2)$ and $N(3)+1 = n(n+1)(n+5)/6$. It can be verified that this immersion gives rise to a submanifold with geodesic normal sections whose geodesics are proper 4-planar. In particular, if $n=2$, this gives us an example of a surface of positive constant Gauss curvature in $\mathbb{E}^7$ with geodesic normal sections whose geodesics are all proper 4-planar.

3. Surfaces with Geodesic Normal Section

In this paragraph we study surfaces in $\mathbb{E}^m$ with geodesic normal sections. First, we give the following.

Lemma 2. *If M is an isotropic surface in $\mathbb{E}^m$, then M is pseudo-umbilical, i.e. the Weingarten map A_H in the direction of the mean curvature vector is proportional to the identity map.*

Proof. Let M be a λ-isotropic surface in $\mathbb{E}^m$, where λ is a function on M. Then, for any orthonormal vectors X and Y tangent to M, we have

$$\langle h(X,X), h(X,Y)\rangle = 0 . \tag{13}$$

Let $\{e_1, e_2\}$ be an orthonormal frame tangent to M. Then $H = \frac{1}{2}[h(e_1, e_2) + h(e_2, e_2)]$. Put

$$a = \langle h(e_1, e_1), h(e_2, e_2)\rangle . \tag{14}$$

Since M is isotropic, (13) and (14) give

$$2A_H = \begin{pmatrix} \lambda^2 + a & 0 \\ 0 & \lambda^2 + a \end{pmatrix} . \tag{15}$$

This proves Lemma 2. □

Lemma 3. *If M is an isotropic surface in $\mathbb{E}^m$, then, with respect to a suitable orthonormal frame $\{e_1, e_2, \xi_3, \ldots, \xi_m\}$, where e_1, e_2 are tangent to M and $\xi_3, \ldots, \xi_m$ normal to M, we have*

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_4 = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}, A_6 = \ldots = A_m = 0 \tag{16}$$

for some functions α and β on M.

Proof. Let $\alpha = \|H\|$ and $M_1 = \{p \in M | \alpha(p) = 0\}$. On M_1, on chooses ξ_4 in the direction of $h(e_1, e_1)$ and ξ_5 in the direction of $h(e_1, e_2)$, and put $\beta = \|h(e_1, e_2)\|$, then

$$A_3 = A_6 = \ldots = A_m , \quad A_4 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix},$$

for a suitable choice of $\xi_3, \xi_6, \dots, \xi_m$ and where M is λ-isotropic. From the isotropy, it follows that for all $\theta \in [0, 2\pi[$,

$$\begin{aligned}\lambda^2 &= \|h(\cos\theta e_1 + \sin\theta e_2, \cos\theta e_1 + \sin\theta e_2)\|^2 \\ &= \lambda^2 \cos^2 2\theta + \beta^2 \sin^2 2\theta,\end{aligned}$$

which implies $\beta = \lambda$.

On $M \backslash M_1$, one chooses ξ_3 in the direction of H, ξ_4 in the direction of $h(e_1, e_1) - h(e_2, e_2)$, and ξ_5 in the direction of $h(e_1, e_2)$. Then we have (16) with $\beta = \|h(e_1, e_2)\|$. □

Combining Theorem 1, Lemma 3, and Lemma 4 we obtain the following.

Proposition 6. *Let M be a surface in $\mathbb{E}^m$. If M has geodesic normal sections, then M is pseudo-umbilical and there is an adapted orthonormal frame $\{e_1, e_2, \xi_3, \dots, \xi_m\}$ (e_1, e_2 tangent to M and $\xi_3, \dots, \xi_m$ normal to M) such that with respect to this frame we have*

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_4 = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \quad A_5 = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}, \quad A_6 = \dots = A_m = 0$$

for some functions α and β on M, where $\alpha^2 + \beta^2$ is constant. □

From Proposition 6 we obtain the next results.

Theorem 3. *Let M be a minimal surface in $\mathbb{E}^m$. If M has geodesic normal sections, then M is totally geodesic in $\mathbb{E}^m$.*

Proof. Under the hypothesis, we obtain from Theorem 1 and Proposition 6 that M has constant Gauss curvature. Since M is minimal in $\mathbb{E}^m$, M is totally geodesic in $\mathbb{E}^m$ (see Remark 2 of [3]). □

Theorem 4. *Let M be a surface in $\mathbb{E}^m$ with geodesic normal sections. If either M has flat normal connection or $m = 4$, then M is totally umbilical in $\mathbb{E}^m$ and hence M is contained either in a 2-plane in $\mathbb{E}^m$ or in 2-sphere in a 3-dimensional Euclidean subspace $\mathbb{E}^3$ of $\mathbb{E}^m$ as an open submanifold.*

This theorem follows immediately from Proposition 6. □

Proposition 7. *Let M be a surface in $\mathbb{E}^m$ with geodesic normal sections* $(2 < m)$. *If M admits a geodesic arc which is proper $(m-2)$-planar, then either*

(a) *$m = 3$ and M is totally geodesic in $\mathbb{E}^3$, or*

(b) *$m = 4$ and M is contained in a 2-sphere of a 3-plane $\mathbb{E}^3$ of $\mathbb{E}^4$ as an open submanifold.*

Proof. Under the hypothesis, $M = M_1 \cup M_2 \cup M_3$, where $M_1 = \{p \in M | \alpha(p) = 0\}$, $M_2 = \{p \in M | \alpha(p) \neq 0,\ \beta(p) = 0\}$, and $M_3 = \{p \in M | \alpha(p)\beta(p) \neq 0\}$, where we use the notations of Proposition 6. From Theorem 4 we know that $\mathrm{int}(M_1 \cup M_2)$ is totally umbilical, i.e. each component of $\mathrm{int}(M_1 \cup M_2)$ is either contained in a 2-plane $\mathbb{E}^2$ of $\mathbb{E}^m$ or contained in a 2-sphere of a 3-plane $\mathbb{E}^3$ of $\mathbb{E}^m$. In either case, if there is a geodesic arc in $\mathrm{int}(M_1 \cup M_2)$ which is proper $(m-2)$-planar, m is either 3 or 4 and this proposition follows from Theorem 3.

Now, assume that $M_3 \neq \emptyset$ and there is a geodesic arc γ in M_3 which is proper $(m-2)$-planar. Choose an adapted orthonormal frame $\{e_1, e_2, \xi_3, \ldots, \xi_m\}$ as given in Proposition 6 such that e_1 is equal to γ' along γ. Since γ is assumed to be proper $(m-2)$-planar and it is a normal section, there is a unit normal vector field η such that

$$0 = \tilde{\nabla}_{e_1}(e_2 \wedge \eta) \quad \text{along } \gamma, \tag{17}$$

from which we find

$$h(e_1, e_2) \wedge \eta = 0 \quad \text{along } \gamma \tag{18}$$

and

$$D_{e_1}\eta = 0 \quad \text{along } \gamma. \tag{19}$$

These imply that η must be equal to ξ_5 or $-\xi_5$ and $D_{e_1}\xi_5 = 0$. Thus we find

$$D_{e_1(\gamma(s))}\left(\frac{h(e_1, e_2)}{\beta}\right) = 0 \tag{20}$$

for each s. From Proposition 3 we know that there is no geodesic arc which is proper $(m-1)$-planar in M_3. Thus by considering a sufficiently small rotation as in the proof of Proposition 3 we also obtain

$$D_{\bar{e}_1(\gamma(s))}\left(\frac{h(\bar{e}_1, \bar{e}_2)}{\beta}\right) = 0 \tag{21}$$

for each s and all small θ, where

$$\bar{e}_1 = \cos\theta e_1 + \sin\theta e_2$$

and

$$\bar{e}_2 = -\sin\theta e_1 + \cos\theta e_2.$$

Hence from (21) we get

$$\begin{aligned} 0 = &-2\cos^2\theta D_{e_1(\gamma(s))}\xi_5 - 2\cos\theta\sin\theta D_{e_2(\gamma(s))}\xi_4 \\ &+ (\cos^2\theta - \sin^2\theta) D_{e_2(\gamma(s))}\xi_5. \end{aligned}$$

Since this is true for all sufficiently small θ, we have

$$0 = -2D_{e_1(\gamma(s))}\xi_4 + D_{e_2(\gamma(s))}\xi_5$$

and

$$0 = -2\cos\theta D_{e_2(\gamma(s))}\xi_4 - \sin\theta D_{e_2(\gamma(s))}\xi_5$$

for each s, from which we find

$$D\xi_4 = D\xi_5 = 0 \quad \text{along } \gamma. \tag{22}$$

Using the same argument as in the proof of Proposition 3 we get

$$D\xi_4 = D\xi_5 = 0 \tag{23}$$

on an open subset U of M_3. On U the equation of Ricci implies

$$[A_{\xi_4}, A_{\xi_5}] = 0. \tag{24}$$

Thus $\beta = 0$ on U which gives a contradiction.

Consequently, $M_3 = \emptyset$ and the proposition is proved. □

Combining Propositions 3 and 7 we find that, if M is a surface in $\mathbb{E}^5$ with geodesic normal sections, then geodesics of M are 2-planar. By applying results of [10, p. 265] or [13, p. 52], we obtain the following.

Theorem 5. *Let M be a surface in $\mathbb{E}^5$. Then M has geodesic normal sections if and only if M is contained in one of the following surfaces as an open submanifold:*

(i) *a 2-plane $\mathbb{E}^2$ in $\mathbb{E}^5$;*

(ii) *an ordinary 2-sphere in a 3-plane of $\mathbb{E}^5$;*

(iii) *a Veronese surface in $\mathbb{E}^5$.* □

From Propositions 3 and 7 we also obtain the following.

Lemma 4. *Let M be a surface in $\mathbb{E}^6$. If M has geodesic normal sections, then all geodesics of M, considered as curves in $\mathbb{E}^6$, are 3-planar.* □

A combination of Proposition 4 and Lemma 4 yields the following result.

Theorem 6. *Let M be a spherical surface in $\mathbb{E}^6$. If M has geodesic normal sections, then M has 2-planar geodesics and hence M is contained in one of the following surfaces as an open submanifold:*

(i) *an ordinary 2-sphere in a $\mathbb{E}^3$ in $\mathbb{E}^6$;*

(ii) *a Veronese surface in a hyperplane of $\mathbb{E}^6$.* □

Finally, for surfaces in $\mathbb{E}^6$ with constant Gauss curvature, we give the following.

Theorem 7. *Let M be a surface in $\mathbb{E}^6$ with constant Gauss or mean curvature. Then M has geodesic normal sections if and only if M lies in a 5-dimensional Euclidean subspace $\mathbb{E}^5$ of $\mathbb{E}^6$ and M is a surface mentioned in Theorem 5.*

Proof. In view of Lemma 4 we only need to show that there are no geodesic arcs in M which are proper 3-planar when considered as curves in $\mathbb{E}^6$. So suppose γ to be a geodesic arc through $p = \gamma(0)$ which is proper 3-planar. Then we have $\gamma' \wedge \gamma^{(2)} \wedge \gamma^{(3)} \neq 0$ everywhere. Choose an orthonormal frame $\{e_1, e_2, \xi_3, \xi_4, \xi_5, \xi_6\}$ as in Proposition 6 such that $e_1 \circ \gamma = \gamma'$ and $\nabla_{e_1} e_2 = 0$ along γ. Because the Gauss curvature $K = \alpha^2 - 2\beta^2$ is constant or α is constant and M is $(\alpha^2 + \beta^2)$-isotropic, α and β are both constant. Again we assume $\alpha\beta \neq 0$. For $i, j \in \{1, 2\}$ and $x, y \in \{3, 4, 5, 6\}$ we put

$$\tilde{\nabla} e_i = \sum_j \omega_i^j \otimes e_j + \sum_x \omega_i^x \otimes \xi_x$$

and

$$\tilde{\nabla} \xi_x = \sum_i \omega_x^i \otimes e_i + \sum_y \omega_x^y \otimes \xi_y,$$

where ω_A^B ($A, B \in \{1,2,3,4,5,6\}$) are 1-forms on M satisfying $\omega_A^B + \omega_B^A = 0$. From the equation of Codazzi we have $(\bar{\nabla}_{e_1} h)(e_1, e_2) = (\bar{\nabla}_{e_2} h)(e_1, e_1)$. Thus we find

$$\omega_5^3(e_1) = \omega_4^3(e_2), \tag{25}$$

$$\beta\omega_5^4(e_1) + 2\beta\omega_1^2(e_1) = \alpha\omega_3^4(e_2), \tag{26}$$

$$0 = \alpha\omega_3^5(e_2) + \beta\omega_4^5(e_2) - 2\beta\omega_1^2(e_2), \tag{27}$$

$$\beta\omega_5^6(e_1) = \alpha\omega_3^6(e_2) + \beta\omega_4^6(e_2). \tag{28}$$

Similarly, by $(\bar{\nabla}_{e_2} h)(e_1, e_2) = (\bar{\nabla}_{e_1} h)(e_2, e_2)$, we obtain

$$\omega_5^3(e_2) = -\omega_4^3(e_1), \tag{29}$$

$$\beta\omega_5^4(e_2) + 2\beta\omega_1^2(e_2) = \alpha\omega_3^4(e_1), \tag{30}$$

$$0 = \alpha\omega_3^5(e_1) - \beta\omega_4^5(e_1) + 2\beta\omega_1^2(e_1), \tag{31}$$

$$\beta\omega_5^6(e_2) = \alpha\omega_3^6(e_1) - \beta\omega_4^6(e_1). \tag{32}$$

From (26) and (31) we derive $\omega_3^4(e_2) = -\omega_3^5(e_1)$ and from (27) and (30) we get $\omega_3^4(e_1) = \omega_3^5(e_2)$. Combining these with (25) and (26) we obtain

$$\omega_3^4 = \omega_3^5 = 0. \tag{33}$$

Therefore, by (26), (30), and (33) we have

$$\omega_4^5 = 2\omega_1^2. \tag{34}$$

Consequently, we have $(\bar{\nabla}_{e_1} h)(e_1, e_1) = [\alpha\omega_3^6(e_1) + \beta\omega_4^6(e_1)]\xi_6$. In particular, we find

$$(\bar{\nabla}_{e_1(\gamma(s))} h)(e_1(\gamma(s)), e_1(\gamma(s))) \wedge \xi_6(\gamma(s)) = 0 \tag{35}$$

for each s. Since γ is 3-planar, γ is contained in $\gamma(s) + \text{Span}\{e_1(\gamma(s)), h(e_1(\gamma(s)), e_1(\gamma(s))), \xi_6(\gamma(s))\}$ for each s. Because this 3-plane is parallel along γ with respect to $\tilde{\nabla}$, we have

$$D_{e_1(p)}\xi_6 \wedge h(e_1(p), e_1(p)) = 0. \tag{36}$$

Now, since ξ_6 is independent of the choice of e_1, by applying the same argument to the geodesic emanating from $p = \gamma(0)$ and tangent to $\bar{e}_1(p) = \cos\theta e_1(p) + \sin\theta e_2(p)$ at p for sufficiently small θ, we find $D_{\bar{e}_1(p)}\xi_6 \wedge h(\bar{e}_1(p), \bar{e}_1(p)) = 0$. Using (36), this means

$$\begin{aligned} 0 = D_{e_1(p)}\xi_6 &\wedge [2\cos^2\theta h(e_1(p), e_2(p)) + \sin\theta\cos\theta h(e_2(p), e_2(p))] \\ &+ D_{e_2(p)}\xi_6 \wedge [\cos^2\theta h(e_1(p), e_1(p)) + \sin^2\theta h(e_2(p), e_2(p)) \\ &+ 2\sin\theta\cos\theta h(e_1(p), e_2(p))]. \end{aligned} \tag{37}$$

Because (37) holds for any sufficiently small θ, we have

$$D_{e_2(p)}\xi_6 \wedge h(e_2(p), e_2(p)) = 0. \tag{38}$$

Combining (36) and (38) we conclude

$$\omega_6^5(p) = 0. \tag{39}$$

From Proposition 6, (33) and the structure equation, we obtain

$$\omega_3^6 \wedge \omega_6^5 = -2\alpha\beta\omega^1 \wedge \omega^2 , \tag{40}$$

where (ω^1, ω^2) are the 1-forms on M dual to (e_1, e_2). Since α and β are constant, (39) and (40) imply that $\alpha\beta = 0$. This is a contradiction. □

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Über das Synthese-Problem für nilpotente Liesche Gruppen

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In der Theorie der kommutativen Banachschen Algebren versteht man unter einer Synthese-Menge (oder auch einer Wienerschen Menge) eine Teilmenge $\mathscr{X}$ des Strukturraumes einer Banachschen Algebra $\mathscr{A}$ mit der Eigenschaft, daß es zu $\mathscr{X}$ nur ein abgeschlossenes (zweiseitiges) Ideal I in $\mathscr{A}$, nämlich den Kern $k(\mathscr{X})$, gibt, dessen Hülle gerade $\mathscr{X}$ ist. Insbesondere für L^1-Faltungsalgebren lokalkompakter abelscher Gruppen ist die Frage ausgiebig, wenn auch keineswegs erschöpfend, untersucht worden, wleche Teilmengen Synthese-Mengen ist. Allgemeiner kann man versuchen, zu gegebenem $\mathscr{X}$ sämtliche abgeschlossenen zweiseitigen Ideale I mit Hülle $\mathscr{X}$ zu bestimmen. In dieser Allgemeinheit erscheint eine befriedigende Antwort einigermaßen aussichtslos. Doch für manche Zwecke ist es sehr hilfreich, wenigstens das folgende qualitative Resultat zur Verfügung zu haben: Es gibt eine nur von $\mathscr{X}$ abhängige natürliche Zahl m mit $(k(\mathscr{X})/I)^m = 0$ für jedes I mit Hülle $\mathscr{X}$. Im Falle $\mathscr{A} = L^1(\mathbb{R}^n)$ hat Müller, [25], ein solches Ergebnis für weite Klassen von Untermannigfaltigkeiten $\mathscr{X}$ von $\mathbb{R}^n$ erhalten.

Die obigen Begriffsbildungen und Fragestellungen lassen sich ohne Mühe auf zweiseitige abgeschlossene Ideale in nicht-kommutativen Banachschen Algebren $\mathscr{A}$ (insbesondere L^1-Faltungsalgebren nicht-kommutativer lokalkompakter Gruppen) übertragen, wenn man sich einigt, was man hier unter dem Strukturraum verstehen will. Der nächstliegende Kandidat ist der Raum $\mathrm{Priv}(\mathscr{A})$ aller primitiven Ideale, d. h. der Annullatoren algebraisch irreduzibler Darstellungen von $\mathscr{A}$. Als ein anderer Kandidat bietet sich im Falle involutiver Banachscher Algebren, also beispielsweise L^1-Algebren, der Raum $\mathrm{Priv}_*(\mathscr{A})$ aller Kerne involutiver irreduzibler Darstellungen von $\mathscr{A}$ in Hilbertschen Räumen an, und zwar weniger aus Analogie-Betrachtungen als aus den pragmatischen Erwägungen, daß über diesen Raum viel mehr Information zur Verfügung steht und daß es vielleicht nützlicher ist, L^∞-Funktionen durch Matrix-Koeffizienten unitärer Darstellungen approximieren zu können als durch Koeffizienten von weniger leicht handhabbaren Darstellungen (auf den „Approximations-Aspekt" der Synthese-Eigenschaft möchte ich nicht näher eingehen, da dieser im folgenden keine Rolle spielt). Im Falle von L^1-Algebren nilpotenter Liescher Gruppen N (und nur für diese werden hier Ergebnisse vorgelegt) gibt es zum Glück eine solche

Alternative gar nicht, da dann $\mathrm{Priv}(L^1(N))$ mit $\mathrm{Priv}_*(L^1(N))$ und auch mit dem Raum der maximalen Ideale in $L^1(N)$ übereinstimmt, siehe [26].

Ich möchte nun bekennen, daß ich den Titel dieses Artikels aus einem recht oberflächlichen Grunde gewählt habe, nämlich um den Leser mit ihm vertrauten Begriffen anzusprechen. Rechtfertigen kann ich den Titel damit, daß die hier erhaltenen Ergebnisse in der Tat Konsequenzen für das Synthese-Problem haben (siehe Abschn. 4, Satz 7). Meine ursprüngliche Motivation war und meine Sicht des Problems ist hingegen eine andere, und von daher hätte ich besser den Titel „Infinitesimal bestimmte Ideale in L^1-Algebren nilpotenter Liescher Gruppen" genommen. Mir geht es eigentlich darum, die inzwischen gut ausgebaute Idealtheorie in universellen Einhüllenden Liescher Algebren für die Darstellungstheorie Liescher Gruppen zu nutzen. So weiß man beispielsweise, siehe [10], daß für irgendeine stark stetige, vollständig topologisch irreduzible Darstellung π einer zusammenhängenden Lieschen Gruppe G in einem Banachschen Raum E der Annullator $\mathfrak{q}$ von E_∞, dem Raum der unendlich oft differenzierbaren Vektoren in E, in $\mathscr{U}\mathfrak{g}$ – $\mathscr{U}\mathfrak{g}$ bezeichnet, wie stets in diesem Artikel, die komplexe universelle Einhüllende der reellen Lieschen Algebra $\mathfrak{g}$ (von G) – ein primitives Ideal ist. Ist weiter N ein zusammenhängender (nilpotenter) Normalteiler in G mit Liescher Algebra $\mathfrak{n}$, so ist $\mathfrak{p} := \mathfrak{q} \cap \mathscr{U}\mathfrak{n}$ ein Primideal in $\mathscr{U}\mathfrak{n}$, dessen Hülle im Raum $\mathrm{Priv}\,\mathscr{U}\mathfrak{n}$ der primitiven Ideale von $\mathscr{U}\mathfrak{n}$ der (Jacobson-) Abschluß einer Γ-Bahn ist, wobei mit Γ der komplexe Zariski-Abschluß der adjungierten Gruppe von G bezeichnet ist. Dann wird E von $\mathfrak{p} * \mathscr{D}(N)$ annulliert; dabei ist $\mathscr{D}(N)$ der Raum der Testfunktionen auf N, und $\mathscr{D}(N)$, ist, in wohlbekannter Manier, siehe unten, ein $\mathscr{U}\mathfrak{n}$-Bimodul. Nehmen wir nun der Einfachheit halber für diese Diskussion an, daß π gleichmäßig beschränkt (auf N) ist (im allgemeinen Fall hat man zusätzlich ein Gewicht auf N einzuführen). Dann wird E auch von dem Abschluß von $\mathfrak{p} * \mathscr{D}(N)$ in $L^1(N)$ annulliert, und man interessiert sich für dieses annullierende Ideal. Genauer gesagt, man möchte beweisen, daß dieses Ideal so groß als eben möglich ist: Man kann nämlich $\mathrm{Priv}\,L^1(N) = \mathrm{Priv}_*\,L^1(N)$ in $\mathrm{Priv}\,\mathscr{U}\mathfrak{n}$ einbetten, siehe [11], und daher den Durchschnitt $\mathscr{X} = h(\mathfrak{p}) \cap \mathrm{Priv}(L^1(N))$ bilden, und man hätte gerne, daß der Abschluß von $\mathfrak{p} * \mathscr{D}(N)$ in $L^1(N)$ gleich dem Kern von $\mathscr{X}$ in $L^1(N)$ ist.

Diese Fragestellung läßt sich allgemeiner formulieren, und man kommt zu dem Begriff des „infinitesimal bestimmten Ideals". Ist nämlich G irgendeine zusammenhängende Liesche Gruppe und I ein abgeschlossenes zweiseitiges Ideal in $L^1(G)$ – oder in einer Beurlingschen Algebra auf G –, so kann man dem Ideal I ein zweiseitiges Ideal $\mathfrak{p} = \mathfrak{p}_I$ in $\mathscr{U}\mathfrak{g}$ assoziieren durch die Setzung $\mathfrak{p} = \{u \in \mathscr{U}\mathfrak{g};\ u * \mathscr{D}(G) \subseteqq I\}$ oder, da $\mathscr{D}(G)$ approximierende Einsen für $L^1(G)$ enthält,

$$\mathfrak{p} = \{u \in \mathscr{U}\mathfrak{g};\ \mathscr{D}(G) * u * \mathscr{D}(G) \subseteqq I\} = \{u \in \mathscr{U}\mathfrak{g};\ \mathscr{D}(G) * u \subseteqq I\}.$$

Und man kann sich fragen, ob I mit dem Abschluß (der linearen Hülle) von $\mathfrak{p} * \mathscr{D}(G)$ übereinstimmt, also aus $\mathfrak{p}$ zurückgewonnen werden kann. In diesem Falle wollen wir das Ideal I infinitesimal bestimmt nennen. Ich möchte die Gelegenheit nutzen, um für dieses Problem etwas Reklame zu machen, auch wenn ich derzeit nur unter sehr einschränkenden Bedingungen, siehe Satz 6, eine positive Lösung vorweisen kann. Nebenbei bemerkt, es scheint mir, daß selbst im Falle $G = \mathbb{R}^n$ und $I = k(\mathscr{X})$ mit einer abgeschlossenen Menge $\mathscr{X}$ in $\hat{G} \cong \mathbb{R}^n$ das obige Problem in dieser Formulierung bisher nicht systematisch behandelt wurde. In diesem Falle läuft durch Dualisieren die obige Dichtigkeits-Aussage darauf hinaus, daß man die *beschränkten* Lösungen eines gewissen Systems homogener partieller Differentialgleichungen mit konstanten Koeffizienten durch Linearkombinationen von Charakteren in $\mathscr{X}$ schwach approximieren kann. Man hat hingegen sehr wohl das Problem behandelt, beliebige Lösungen durch sogenannte Exponentialpolynome, in die dann auch unbeschränkte Charaktere eingehen, zu approximieren, siehe etwa [29, 30, 23] und [12].

Nach diesen allgemeinen Bemerkungen möchte ich nun den Inhalt dieses Artikels etwas detaillierter vorstellen. Ab jetzt sei mit N stets eine einfachzusammenhängende, zusammenhängende nilpotente Liesche Gruppe N mit Liescher Algebra $\mathfrak{n}$ bezeichnet. Die Voraussetzung des einfachen Zusammenhangs ist für alle folgenden Ergebnisse unerheblich; bei gewissen Konstruktionen erweist sie sich gelegentlich als nützlich. Für solche Gruppen kann man neben $\mathscr{D}(N)$ auch den Raum $\mathscr{S}(N)$ der Schwartzschen Funktionen bilden, $\mathscr{S}(N)$ besteht aus allen Funktionen $f: N \to \mathbb{C}$ mit der Eigenschaft, daß $f \circ \exp$ eine Schwartzsche Funktion auf dem Vektorraum $\mathfrak{n}$ ist. Da die Gruppenoperationen in den (hier globalen) Koordinaten erster Art durch Polynome gegeben sind, ist $\mathscr{S}(N)$ eine Faltungsalgebra, in der Tat eine topologische Algebra, siehe auch [14]. Weiter operiert $\mathscr{U}\mathfrak{n}$ von links und von rechts auf allen unendlich oft differenzierbaren Funktionen auf N, wobei für $X \in \mathfrak{n}$ die Wirkung durch $(X * f)(y) = \left.\frac{d}{dt}\right|_{t=0} f(\exp(-tX)y)$ bzw. $(f * X)(y) = \left.\frac{d}{dt}\right|_{t=0} f(y\exp(-tX))$ gegeben ist, und $\mathscr{S}(N)$ ist invariant unter dieser Wirkung, also ein $\mathscr{U}\mathfrak{n}$-Bimodul. Oben hatten wir jedem zweiseitigen abgeschlossenen Ideal I in $L^1(N)$ mittels der Testfunktionen ein Ideal $\mathfrak{p} = \mathfrak{p}_I$ in $\mathscr{U}\mathfrak{n}$ zugeordnet. Zur Definition von $\mathfrak{p}$ kann man ebenso gut die Schwartzschen Funktionen verwenden; denn mit Hilfe von approximierenden Einsen in $\mathscr{D}(N)$ sieht man leicht, daß

$$\begin{aligned}\mathfrak{p} &= \{u \in \mathscr{U}\mathfrak{n};\ u * \mathscr{S}(N) \subseteqq I\} = \{u \in \mathscr{U}\mathfrak{n};\ \mathscr{S}(N) * u \subseteqq I\} \\ &= \{u \in \mathscr{U}\mathfrak{n};\ \mathscr{S}(N) * u * \mathscr{S}(N) \subseteqq I\}.\end{aligned}$$

Aus demselben Grunde stimmt auch der Abschluß $\langle \mathfrak{p} * \mathscr{D}(N)\rangle^-$ des von $\mathfrak{p} * \mathscr{D}(N)$ erzeugten Ideals mit $\langle \mathfrak{p} * \mathscr{S}(N)\rangle^-$, mit $\langle \mathscr{D}(N) * \mathfrak{p} * \mathscr{D}(N)\rangle^-$ und mit $\langle \mathscr{S}(N) * \mathfrak{p} * \mathscr{S}(N)\rangle^-$ überein. Der Vorteil der Schwartzschen Funktionen liegt darin, daß $\mathscr{S}(\mathfrak{n})$ unter der Fouriertransformation auf $\mathscr{S}(\mathfrak{n}^*)$ übergeht, der Vorteil der Testfunktionen darin, daß man sich durch die Faltung von $\mathfrak{p}$ mit $\mathscr{D}(N)$ Funktionen mit kompaktem Träger in I verschaffen kann. Sei nun speziell I der Kern einer abgeschlossenen Menge $\mathscr{X}$ in Priv $(L^1(N)) = \mathrm{Priv}_*(L^1(N))$, abgeschlossen bezüglich der Jacobson-Topologie, die laut [1] unter der Identifikation von $\mathrm{Priv}_*(L^1(N))$ mit $\mathrm{Priv}_*(C^*(N))$ mit der Jacobson-Topologie auf $\mathrm{Priv}_*(C^*(N))$, also mit der Topologie auf dem unitären Dual $\hat{N}$ übereinstimmt. Dann kann man $\mathfrak{p} = \mathfrak{p}_I$ leicht „ausrechnen“, $\mathfrak{p}$ ist nämlich nichts anderes als der Kern von $\mathscr{X}$, aufgefaßt als Teil von Priv $\mathscr{U}\mathfrak{n}$, also auch der Kern von $\bar{\mathscr{X}}$, wobei der Abschluß $\bar{\mathscr{X}}$ in der Jacobson-Topologie von Priv $\mathscr{U}\mathfrak{n}$ zu bilden ist. Folglich ist das abgeschlossene Ideal $\langle \mathfrak{p} * \mathscr{D}(N)\rangle^-$ von $L^1(N)$ jedenfalls in $k(\bar{\mathscr{X}} \cap \mathrm{Priv}(L^1(N))$ enthalten. Für die gewünschte Gleichung $\langle \mathfrak{p} * \mathscr{D}(N)\rangle^- = I = k(\mathscr{X})$ ist also notwendig, daß $\mathscr{X} = \bar{\mathscr{X}} \cap \mathrm{Priv}(L^1(N))$ ist. Interpretiert man $\mathscr{X}$ im Kirillowschen Bild als eine N-invariante Teilmenge von $\mathfrak{n}^*$, so muß $\mathscr{X}$ notwendigerweise eine algebraische Menge sein.

Die Ergebnisse dieses Artikels beziehen sich ausschließlich auf den Fall, daß $\mathscr{X}$ eine Bahn unter einer Automorphismengruppe G von N ist, wobei noch vorausgesetzt wird, daß G ein semidirektes Produkt aus einer kompakten abelschen Lieschen Gruppe T (die nicht zusammenhängend zu sein braucht) und einer (auf $\mathfrak{n}$) unipotent wirkenden, zusammenhängenden nilpotenten Lieschen Gruppe M ist. Im Sinne der obigen notwendigen Bedingung wird im Abschn. 1 ein (etwas allgemeinerer) Bahnensatz bewiesen, welcher zeigt, daß für Bahnen $\mathscr{X}$ unter solchen Gruppen die Gleichung $\mathscr{X}=\bar{\mathscr{X}}\cap \mathrm{Priv}(L^1(N))$ allemal richtig ist. Danach behandeln wir erst einmal den Fall eines trivialen M und erhalten Ergebnisse, die auch von unabhängigem Interesse sind. So wird im Abschn. 2 die topologische Algebra $\mathscr{S}(N)/\mathscr{S}(N)\cap k(\mathscr{X})$ bestimmt. Für einen einzelnen Punkt $\mathscr{X}=\{\Omega\}$ wurde dies bereits von Howe getan, [14], und der hier gegebene Beweis verwendet viele seiner Methoden. Im Abschn. 3 wird gezeigt, daß $\mathfrak{p} * \mathscr{S}(N)$ total in $k(\mathscr{X})\cap\mathscr{S}(N)$ ist. Auch dieser Satz ist für einzelne Punkte bereits bekannt, er wurde in [22] von Ludwig bewiesen. Mit ähnlichen Methoden wie in [21] kann man aus den Ergebnissen der Abschnitte 2 und 3 herleiten, daß $\mathfrak{p}*\mathscr{S}(N)$ total in $I=k(\mathscr{X})\lhd L^1(N)$ ist. Weiter wird im Abschn. 4 ausgeführt, daß eine solche Aussage dann auch für nicht-triviales M wahr ist, und zwar indem man sie mit Hilfe eines einfachen, bereits in [27] verwendeten Tricks auf den Fall eines trivialen M zurückführt. Als eine Anwendung erhält man daraus, daß es eine nur von N abhängige natürliche Zahl d mit der Eigenschaft gibt, daß $(k(\mathscr{X})/J)^d=0$ ist für jedes abgeschlossene zweiseitige Ideal J in $L^1(N)$ mit Hülle $\mathscr{X}$.

1. Ein Bahnensatz

Hier wird neben dem in der Einleitung angekündigten Bahnensatz ein weiterer, im Abschn. 3 benötigter (Hilfs)Satz 2 bewiesen. Ein ganz wesentliches Hilfsmittel für Satz 2 ist ein Satz von Borho aus [3]. Die in diesem Paragraphen verwendeten Grundbegriffe aus der Theorie der (komplexen) algebraischen Gruppen findet man etwa in [2] oder [15].

Satz 1. *Es seien T eine kompakte Liesche Gruppe und N eine einfachzusammenhängende nilpotente Liesche Gruppe. T operiere stetig und homomorph durch Automorphismen auf N, so daß man das semidirekte Produkt $G=T\ltimes N$ bilden kann. Weiter seien V ein endlich-dimensionaler reeller Vektorraum und $\pi: G\to \mathrm{Aut}(V)$ eine stetige Darstellung von G in V mit der Eigenschaft, daß $\pi(N)$ aus unipotenten Transformationen besteht. Mit K sei der (komplexe) Zariski-Abschluß von $\pi(G)$ in $\mathrm{Aut}(V_{\mathbb{C}})$ bezeichnet; dabei ist $V_{\mathbb{C}}=V+iV$ die Komplexifizierung von V. Dann gelten für jedes $x\in V$:*

(i) *Die Zusammenhangskomponente $(G_x)_0$ der Standuntergruppe G_x ist von endlichem Index in G_x.*

(ii) *Es sei $T':=\{t\in T;\ \pi(t)x\in\pi(N)x\}=G_xN\cap T$. Dann gibt es $y\in\pi(N)x$ mit $T_y=T'$ und $G_y=T_yN_y$.*

(iii) *Der Zariski-Abschluß von $\pi(G)x$ in $V_{\mathbb{C}}$ ist gleich Kx.*

(iv) $Kx\cap V=\pi(G)x$.

Bemerkung 1. Kx ist natürlich stets im Abschluß von $\pi(G)x$ enthalten. (iii) bedeutet also nichts anderes, als daß Kx Zariski-abgeschlossen ist.

Bemerkung 2. Es ist wesentlich, daß $x \in V$ vorausgesetzt wird. Für beliebige $x \in V_{\mathbb{C}}$ ist (iii) offenbar im allgemeinen nicht richtig. Das sieht man bereits an dem Beispiel $N = \{1\}$, $T = \mathbb{T}$, $V = \mathbb{C}$ (aufgefaßt als zweidimensionaler reeller Vektorraum) und $\pi(t)v = tv$. Bezeichnet man nämlich mit $e_1 = 1$, $e_2 = i$ die kanonische Basis des reellen Vektorraumes V, so ist $f_1 = e_1 + ie_2$, $f_2 = e_1 - ie_2$ eine Basis des komplexen Vektorraumes $V_{\mathbb{C}} = V \oplus iV$. Eine einfache Rechnung zeigt, daß $\pi(t)f_1 = t^{-1}f_1$ und $\pi(t)f_2 = tf_2$ für alle $t \in \mathbb{T}$ gilt. Erklärt man die lineare Transformation $A_z: V_{\mathbb{C}} \to V_{\mathbb{C}}$ für $z \in \mathbb{C}^x$ durch $A_z f_1 = z^{-1}f_1$ und $A_z f_2 = zf_2$, so ist also $K = \{A_z;\, z \in \mathbb{C}^x\}$, und Kf_1 ist ersichtlich nicht Zariski-abgeschlossen.

Beweis. (i) und (ii): Bekanntlich sind Bahnen unipotenter Gruppen abgeschlossen, also ist $\pi(N)x$ abgeschlossen, woraus man erhält, daß T' und $G_x N = T'N$ abgeschlossene Untergruppen von G sind. Insbesondere sind G_x/N_x, G_xN/N und T' isomorphe topologische Gruppen. Da natürlich T'_0 von endlichem Index in T' ist, ist $(G_x/N_x)_0$ von endlichem Index in G_x/N_x. Nun ist aber N_x bekanntlich zusammenhängend und daher $(G_x/N_x)_0 = (G_x)_0/N_x$, womit (i) bewiesen ist. – Wegen (i) gibt es eine maximale kompakte Untergruppe H in G_x, und da N_x zusammenhängend ist, wird H unter der Quotientenabbildung $G_x \to G_x/N_x \cong T'$ auf eine maximale kompakte Untergruppe von T', d. h. auf T' abgebildet. Es gilt also $G_x = N_x H$. Weiter ist H in einer zu T konjugierten Untergruppe enthalten, es gibt $a \in G$ mit $H \subsetneqq aTa^{-1}$, und man kann natürlich a in N wählen. Dann sei $y = \pi(a)^{-1}x$. Damit ist $G_y = a^{-1}G_x a = a^{-1}HaN_y = T_y N_y$ und dann auch $T' = T_y$.

(iii) und (iv): Für diese beiden Aussagen kann man ohne Beschränkung der Allgemeinheit annehmen, daß T zusammenhängend ist.

Weiter ist π unendlich oft differenzierbar, und $\pi(G)$ ist eine abgeschlossene Liesche Untergruppe von $\mathrm{Aut}(V)$ mit Liescher Algebra $d\pi(\mathfrak{g})$. Man kann dann die komplexe Liesche Unteralgebra $d\pi(\mathfrak{g}) + id\pi(\mathfrak{g})$ von $\mathrm{End}(V_{\mathbb{C}})$ bilden. Aus den Voraussetzungen und der Charakterisierung „algebraischer" Liescher Algebren (d. h. solcher, welche Liesche Algebren von algebraischen Gruppen sind, vgl. [24] sowie [5] und [6]) folgt, daß $d\pi(\mathfrak{g}) + id\pi(\mathfrak{g})$ eine algebraische Liesche Algebra ist. Daher stimmt die Liesche Algebra $\mathfrak{k}$ von K mit $d\pi(\mathfrak{g}) + id\pi(\mathfrak{g})$ überein. Mit M wird die Zariski-Hülle von $\pi(N)$ bezeichnet; ihre Liesche Algebra $\mathfrak{m}$ ist gleich $d\pi(\mathfrak{n}) + id\pi(\mathfrak{n})$. Ferner gilt $K = \pi(T)\exp(id\pi(\mathfrak{t}))M$, und die entsprechende Zerlegung eines Elementes aus K ist eindeutig.

Der folgende Beweis für (iii) ist eine Erweiterung des Argumentes für Operationen unipotenter Gruppen. Sei $B = Kx$ und B' der Zariski-Abschluß von B. Bekanntlich sind Bahnen algebraischer Gruppen Zariski-lokal abgeschlossen, also ist $B' \setminus B$ Zariski-abgeschlossen. Nehmen wir im Gegensatz zur Behauptung an, daß $B' \setminus B$ nicht leer ist. Es seien I bzw. J die Ideale zu B' bzw. $B' \setminus B$ in der Algebra $\mathbb{C}[V_{\mathbb{C}}]$ der komplexen Polynome auf $V_{\mathbb{C}}$. Offenbar sind I und J invariant unter K, und I ist ein echter Teil von J. Da die unipotente Gruppe M in J/I nicht-triviale Fixpunkte besitzt, gibt es $f \in J \setminus I$ mit $f(my) = f(y)$ für alle $m \in M$ und $y \in B'$. Anders ausgedrückt: Setzt man

$$A = \{g \in J;\, g(my) = g(y) \text{ für alle } m \in M \text{ und } y \in B'\},$$

so ist A nicht in I enthalten. Des weiteren sind I, J sowie A invariant unter komplexer Konjugation, d. h. unter $g \to g^*$, $g^*(v) = \overline{g(\bar{v})}$. Daher gibt es ein reelles Polynom $f \in \mathbb{R}[V]$, $f \in A$, $f \notin I$. Wir fixieren nun ein solches f. Die reellen

Polynome in A, deren Grad den von f nicht übersteigt, bilden einen endlichdimensionalen reellen Unterraum W von $A \cap \mathbb{R}[V]$ mit $f \in W$. Mit A ist auch W invariant unter $\pi(G)$, also insbesondere unter $\pi(T)$. Da $\pi(T)$ eine kompakte Gruppe ist, gibt es eine Basis $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ von W derart, daß die Wirkung von $\pi(T)$ auf W in dieser Basis durch orthogonale Matrizen gegeben ist. Man verifiziert ohne Mühe, daß dann $\varphi := f_1^2 + \ldots + f_n^2$ invariant unter $\pi(T)$ und dann auch unter der Komplexifizierung $\pi(T)_{\mathbb{C}} = \pi(T)\exp(id\pi(\mathfrak{t}))$ ist. Da φ außerdem in A liegt (A ist eine Algebra), ist φ konstant auf B'. Wegen $\varphi \in J$ und $B' \backslash B \neq \emptyset$ ist diese Konstante gleich Null. Für jedes $y \in \pi(G)x$ ist also $0 = \varphi(y) = f_1(y)^2 + \ldots + f_n(y)^2$, daher $f_j(y) = 0$ und mithin $f(y) = 0$. Folglich ist auch $f(B') = 0$ im Widerspruch zu $f \notin I$.

Nun wird (iv) bewiesen, zunächst für triviales T, der allgemeine Fall wird danach auf den Spezialfall zurückgeführt. Bei unipotenten Gruppen kann man leicht mit Induktion argumentieren. Sei dazu W ein eindimensionaler $\pi(G) = \pi(N)$-invarianter Unterraum von V. Ist $y \in Kx \cap V = Mx \cap V$, so gibt es $a \in N$ mit $y \equiv \pi(a)x \bmod W$, also $y = \pi(a)x + w$ mit $w \in W$. Ist $w = 0$, so ist nichts mehr zu zeigen. Wir können daher $w \neq 0$ annehmen. Nach Voraussetzung gibt es $b \in M$ mit $bx = y = \pi(a)x + w$. Dann liegt $b^{-1}\pi(a)$ in $M' := \{c \in M; cx \equiv x \bmod W_{\mathbb{C}}\}$. M' ist eine zusammenhängende, unter komplexer Konjugation invariante Untergruppe von M. Daher ist ihre Liesche Algebra $\mathfrak{m}'$ von der Form $\mathfrak{m}' = d\pi(\mathfrak{n}') + id\pi(\mathfrak{n}')$ mit einer passenden Unteralgebra $\mathfrak{n}'$ von $\mathfrak{n}$. Definieren wir $\varphi: M' \to W_{\mathbb{C}}$ durch $\varphi(c) = cx - x$, so ist φ ein Homomorphismus. Wegen $-w \in \operatorname{Bild} \varphi$ ist φ nicht-trivial und folglich surjektiv. Des weiteren ist $\varphi(\pi(N')) = W$, insbesondere gibt es $c \in N'$ mit $\pi(c)x - x = w$. Dann ist aber $y = \pi(a)\pi(c)x \in \pi(N)x$.

Im allgemeinen Fall können wir wegen (ii) annehmen, daß $G_x = T_x N_x$. Die Liesche Algebra $\mathfrak{k}_x$ von K_x ist die Komplexifizierung von $d\pi(\mathfrak{g}_x)$, also $\mathfrak{k}_x = d\pi(\mathfrak{t}_x) + id\pi(\mathfrak{t}_x) + d\pi(\mathfrak{n}_x) + id\pi(\mathfrak{n}_x)$, und es gilt $(K_x)_0 = \pi(T_x)_0 \exp(id\pi(\mathfrak{t}_x)) M_x$. Sei nun $y \in Kx \cap V$, also $y = \pi(a)tmx \in V$ mit $a \in T$, $t \in \exp(id\pi(\mathfrak{t}))$ und $m \in M$. Ohne Beschränkung der Allgemeinheit können wir $a = 1$ annehmen. Nun ist $tmx = y = \bar{y} = \bar{t}\bar{m}x = t^{-1}\bar{m}x$, folglich $x = \bar{m}^{-1}t^2mx$ oder $\bar{m}^{-1}t^2m \in K_x$. Da K_x eine lineare algebraische Gruppe ist, ist $(K_x)_0$ von endlichem Index in K_x. Insbesondere gibt es eine natürliche Zahl j mit $(\bar{m}^{-1}t^2m)^j \in (K_x)_0$. Nun ist $(\bar{m}^{-1}t^2m)^j = t^{2j}n$ mit $n \in M$, und wegen der oben hingeschriebenen Zerlegung von $(K_x)_0$ gilt $t^{2j} \in K_x$, d.h. $t^{2j}x = x$. Da t eine halbeinfache lineare Transformation mit positiven Eigenwerten ist, gilt dann auch $tx = x$. Man erhält $y = tmx = tmt^{-1}x \in V \cap Mx$, und folglich $y \in \pi(N)x$ auf Grund des Spezialfalles.

Als eine Anwendung von Satz 1 stellen wir den im Abschn. 3 benötigten Satz 2 bereit. Die Formulierung ist ganz den dortigen Bedürfnissen angepaßt und dem Inhalt des Satzes nicht ganz angemessen, denn der Satz hat zunächst mit der Lieschen Gruppe N und ihrer Faltungsalgebra nichts zu tun, sondern ist eigentlich eine Aussage über die Idealtheorie in der universellen Einhüllenden einer reellen nilpotenten Lieschen Algebra.

Satz 2. *Seien N eine einfachzusammenhängende, zusammenhängende nilpotente Liesche Gruppe mit Liescher Algebra $\mathfrak{n}$ und T eine kompakte abelsche Liesche Gruppe. T operiere stetig und homomorph durch Automorphismen auf N, $(t, x) \to x^t$, und damit auch auf $\mathfrak{n}$ und $\mathscr{U}\mathfrak{n}$. Ω sei ein primitives Ideal in $L^1(N)$, mit Ω_∞ sei das entsprechende (selbstadjungierte) primitive Ideal in $\mathscr{U}\mathfrak{n}$ bezeichnet. Weiter sei $\mathfrak{w}$ ein* *zweidimensionales zentrales Ideal in* $\mathfrak{n}$, $\mathfrak{w}$ *sei T-invariant und ein irreduzibler T-Modul. Ist* T' *der Stabilisator von* $\Omega_\infty \cap \mathscr{U}\mathfrak{w}$ *im Raum der maximalen Ideale von* $\mathscr{U}\mathfrak{w}$, *d.h. in* $\mathfrak{w}_{\mathbb{C}}^*$, *so gilt*

$$\bigcap_{t\in T} \Omega_\infty^t + (\mathscr{U}\mathfrak{w} \cap \Omega_\infty)\mathscr{U}\mathfrak{n} = \bigcap_{t\in T'} \Omega_\infty^t .$$

Bemerkung. Ist $\mathscr{U}\mathfrak{w}\cap\Omega_\infty$ gerade das von $\mathfrak{w}$ erzeugte Ideal in $\mathscr{U}\mathfrak{w}$, so ist $T'=T$, und der Satz is trivialerweise richtig. Wir werden also im folgenden stets annehmen, daß $\mathfrak{w}$ nicht in $\mathscr{U}\mathfrak{w}\cap\Omega_\infty$ enthalten ist. Dann ist, wegen der Irreduzibilität von $\mathfrak{w}$, $T'=\{t\in T;\ w^t=w \text{ für alle } w\in\mathfrak{w}\}$.

Beweis. Es ist klar, daß das auf der linken Seite der Gleichung stehende Ideal in dem auf der rechten Seite stehenden enthalten ist. Man braucht also nur die andere Inklusion zu beweisen.

Abkürzend setzen wir $\mathfrak{p}=\bigcap_{t\in T}\Omega_\infty^t = k(T\Omega_\infty)$. Die Hülle $h(\mathfrak{p})$ von $\mathfrak{p}$ in $\operatorname{Priv}\mathscr{U}\mathfrak{n}$ $=\operatorname{Max}\mathscr{U}\mathfrak{n}$ stimmt natürlich im allgemeinen nicht mit $\mathscr{X}:=\{\Omega_\infty^t;\ t\in T\}$ überein, sondern ist gleich dem Abschluß $\bar{\mathscr{X}}$ von $\mathscr{X}$ in der Jacobson-Topologie von $\operatorname{Priv}\mathscr{U}\mathfrak{n}$. Nun induziert die Dixmier-Abbildung $\operatorname{Dix}: \mathfrak{n}_{\mathbb{C}}^* \to \operatorname{Priv}\mathscr{U}\mathfrak{n}$ eine Bijektion vom Raum $\mathfrak{n}_{\mathbb{C}}^*/N_{\mathbb{C}}$ der Bahnen in $\mathfrak{n}_{\mathbb{C}}^*$ unter der coadjungierten Darstellung auf $\operatorname{Priv}\mathscr{U}\mathfrak{n}$, und nach [7] ist diese Abbildung sogar ein Homöomorphismus (für die Zariski- bzw. die Jacobson-Topologie). Zu Ω_∞ existiert ein reelles $g\in\mathfrak{n}_{\mathbb{C}}^*$ mit $\operatorname{Dix}(g)=\Omega_\infty$; man kann nämlich für g irgendeinen Punkt in derjenigen N-Bahn in $\mathfrak{n}^*$ nehmen, die unter der Kirillowschen Korrespondenz dem Ideal Ω entspricht, vgl. [11]. Bildet man in $\mathfrak{n}_{\mathbb{C}}^*$ den Zariski-Abschluß $\mathscr{Y}$ der $T\ltimes N$-Bahn von g, so ist wegen der oben genannten Sätze $\bar{\mathscr{X}}$ gerade gleich $\operatorname{Dix}(\mathscr{Y})$. Wir wählen nun ein Repräsentantensystem $s_1,\dots,s_m$ für die T_0-Nebenklassen in T, mit K bezeichnen wir die zu $T_0\ltimes N$ gehörige lineare algebraische Gruppe auf $\mathfrak{n}_{\mathbb{C}}^*$. Dann ist $\mathscr{Y}$ wegen Satz 1 gerade gleich $\bigcup_{j=1}^m Ks_jg$. Läßt man die übereinstimmenden K-Bahnen fort, so erhält man eine disjunkte Zerlegung $\mathscr{Y}=\bigcup_{j=1}^n Kt_jg$. Setzt man weiter $\bar{\mathscr{X}}_j=\operatorname{Dix}(Kt_jg)$, so ist auch $\bar{\mathscr{X}}=\bigcup_{j=1}^n\bar{\mathscr{X}}_j$ eine disjunkte Zerlegung in abgeschlossene irreduzible Mengen. $\bar{\mathscr{X}}_j$ kann man auch beschreiben als den Abschluß von

$$\mathscr{X}_j := \{\Omega_\infty^{t_j^{-1}t};\ t\in T_0\} \subseteqq \operatorname{Priv}\mathscr{U}\mathfrak{n}.$$

Wegen der Irreduzibilität von $\bar{\mathscr{X}}_j$ sind die Ideale $\mathfrak{p}_j := k(\bar{\mathscr{X}}_j)=k(\mathscr{X}_j)$prim. Es ist $\mathfrak{p}=\bigcap_{j=1}^n\mathfrak{p}_j$, und für $1\leqq j<r\leqq n$ gilt $\mathfrak{p}_j+\mathfrak{p}_r=\mathscr{U}\mathfrak{n}$ wegen der Disjunktheit der $\bar{\mathscr{X}}_i$. Aus dem Chinesischen Restsatz folgt dann, daß $\mathscr{U}\mathfrak{n}/\mathfrak{p}$ isomorph zur direkten Summe $\bigoplus_{j=1}^n\mathscr{U}\mathfrak{n}/\mathfrak{p}_j$ ist. Unter Verwendung eines Satzes von Borho, siehe [3, 6.8], wollen wir nun die Struktur der Algebren $B_j := \mathscr{U}\mathfrak{n}/\mathfrak{p}_j$ bestimmen. Dazu bilden wir die komplexe auflösbare Liesche Algebra $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}\ltimes\mathfrak{n}_{\mathbb{C}}$. $\mathfrak{p}_j$ ist natürlich $\mathfrak{h}$-invariant, und auch alle übrigen Voraussetzungen des oben zitierten Satzes sind erfüllt. Es gibt daher einen zentralen $\mathfrak{h}$-Eigenvektor e in B_j mit $(B_j)_e\cong Z(B_j)_e\otimes A_m$, wobei mit $(B_j)_e$ die Lokalisierung von B_j nach e, mit $Z(B_j)_e$, die Lokalisierung des Zentrums $Z(B_j)$

nach e und mit A_m die Weylsche Algebra in m Erzeugenden bezeichnet ist. e ist dann auch ein Eigenvektor für die Wirkung von T_0 auf B_j, es gibt also ein $\chi \in \hat{T}_0$ mit $e^t = \chi(t)e$ für $t \in T_0$. Nun ist $\mathfrak{p}_j = k(\mathscr{X}_j)$ ein selbstadjungiertes Ideal in $\mathscr{U}\mathfrak{n}$, also $\mathscr{U}\mathfrak{n}/\mathfrak{p}_j$ eine involutive Algebra, und für $u := e^*e \neq 0$ gilt

$$u^t = e^{*t}e^t = e^{t*}e^t = (\chi(t)e)^*\chi(t)e = \bar{\chi}(t)e^*\chi(t)e = u \quad \text{für alle} \quad t \in T_0 .$$

Bezeichnet man mit $q_j: B_j \to \mathscr{U}\mathfrak{n}/\Omega_\infty^{t_j^{-1}}$ die Quotientenabbildung, so ist $q_j(u)$ ein Vielfaches des Einselementes, sagen wir $q_j(u) = \lambda$. Folglich liegt $u - \lambda$ in $\operatorname{Kern} q_j$, und wegen der obigen Beziehung liegt auch $u^t - \lambda = (u - \lambda)^t$ in $\operatorname{Kern} q_j$, also $u - \lambda$ in $\bigcap_{t \in T_0} (\operatorname{Kern} q_j)^t$, woraus man wegen $\mathfrak{p}_j = k(\mathscr{X}_j)$ erhält, daß $u - \lambda = 0$ oder $u = \lambda$. Mithin ist e invertierbar und $Z(B_j)_e = Z(B_j)$ sowie $(B_j)_e = B_j$, also $B_j \cong Z(B_j) \otimes A_m$. Im übrigen ist m unabhängig von j, denn A_m ist isomorph zu $\mathscr{U}\mathfrak{n}/\Omega_\infty^{t_j^{-1}} \cong \mathscr{U}\mathfrak{n}/\Omega_\infty$.

Zusammen mit der oben festgestellten Isomorphie zwischen $\mathscr{U}\mathfrak{n}/\mathfrak{p}$ und $\bigoplus_{j=1}^{n} \mathscr{U}\mathfrak{n}/\mathfrak{p}_j$ ergibt sich nun, daß $B := \mathscr{U}\mathfrak{n}/\mathfrak{p}$ isomorph zu $Z(B) \otimes A_m$ ist. Wir haben zu zeigen, daß $I := \bigcap_{t \in T'} \Omega_\infty^t/\mathfrak{p}$ in $J := \mathfrak{p} + (\mathscr{U}\mathfrak{w} \cap \Omega_\infty)\mathscr{U}(\mathfrak{n})/\mathfrak{p}$ enthalten ist. Bei der obigen Isomorphie entspricht jedes maximale (primitive) Ideal Λ in B dem Ideal $\Lambda \cap Z(B) \otimes A_m$. Daraus folgt, daß I von $I \cap Z(B)$ erzeugt wird. Es genügt also zu zeigen, daß $I \cap Z(B)$ in J enthalten ist. Weiter zerfällt $Z(B)$ in T-Eigenräume, für $\gamma \in \hat{T}$ sei $Z(B)_\gamma = \{v \in Z(B);\ v^t = \gamma(t)v\}$. Wie oben kann man sich überlegen, daß der Raum $Z(B)^T$ der T-Fixpunkte nur aus den Vielfachen von Eins besteht. Daraus folgt nun leicht, daß jedes $Z(B)_\gamma$ höchstens eindimensional ist (in der Tat ist $Z(B)_\gamma$ genau dann eindimensional, wenn $\gamma = 1$ auf dem Stabilisator T_Ω von Ω ist, und $Z(B)$ ist der $\mathbb{C}$-Gruppenring der Gruppe $(T/T_\Omega)\hat{}$; aber das benötigen wir im folgenden nicht). Ist nämlich $u \in Z(B)_\gamma$, $u \neq 0$, so ist $u^*u \in Z(B)^T$, siehe oben. Also ist $u^*u = \lambda 1$ mit passendem $\lambda \in \mathbb{C}$. Nun ist $u^*u \neq 0$, denn zu u existiert wenigstens ein $t \in T$ mit $q_t(u) \neq 0$, wobei mit $q_t: B \to \mathscr{U}\mathfrak{n}/\Omega_\infty^t$ die Quotientenabbildung bezeichnet ist; $q_t(u)$ ist ein von Null verschiedener Skalar μ, und es gilt $q_t(u^*u) = q_t(u)^*q_t(u) = \bar{\mu}\mu \neq 0$. Ist nun v ein beliebiges weiteres Element in $Z(B)_\gamma$, so liegt u^*v in $Z(B)^T$, also $u^*v = \nu 1$ mit $\nu \in \mathbb{C}$ und folglich $uu^*v = \nu u$ oder $v = \lambda^{-1}\nu u$.

Sei nun $u \in I \cap Z(B)$. Wir wollen zeigen, daß u in J liegt. O.B.d.A. können wir annehmen, daß u ein T'-Eigenvektor ist, also $u^t = \alpha(t)u$ für $t \in T'$ mit einem $\alpha \in (T')\hat{}$. u läßt sich eindeutig in T-Eigenvektoren entwickeln, $u = \sum_{\gamma \in \hat{T}} u_\gamma$ mit $u_\gamma \in Z(B)_\gamma$, wobei natürlich nur endlich viele u_γ von Null verschieden sind. Da u ein α-Eigenvektor ist, ist $u_\gamma = 0$, falls $\gamma|_{T'} \neq \alpha$. Wir können annehmen, daß wenigstens ein u_γ von Null verschieden ist. Indem man für ein solches u_γ das invertierbare Element u_γ^* heranmultipliziert, sieht man, daß man zusätzlich o.B.d.A. $\alpha = 1$ annehmen darf, also $u = \sum_{\gamma \in (T/T')\hat{}} u_\gamma \in I \cap Z(B)$.

Wie wir oben gesehen haben, brauchen wir nur den Fall $g(\mathfrak{w}) \neq 0$ zu behandeln. Dann gibt es eine Basis e_1, e_2 von $\mathfrak{w}$ und einen Homomorphismus $\delta: T \to \mathbb{T}$ mit den folgenden Eigenschaften:

(1) $\operatorname{Kern} \delta = T'$.

(2) $g(e_1) = 0$, $g(e_2) = 1$.

(3) Für $f_1 := e_1 + ie_2$ und $f_2 := e_1 - ie_2$ gelten $f_1^t = \delta(t)^{-1} f_1$ und $f_2^t = \delta(t) f_2$.

Das Ideal $\Omega_\infty \cap \mathcal{U}\mathfrak{w}$ wird erzeugt von $w - ig(w)$, $w \in \mathfrak{w}$ oder $w \in \mathfrak{w}_{\mathbb{C}}$; insbesondere liegt $1 - \mathrm{f}_2$ in diesem Ideal. Bezeichnet man mit f das Bild von f_2 unter $\mathcal{U}\mathfrak{n} \to B$, so liegt also $1 - f$ in J, sogar in $J \cap Z(B)$. Nach Konstruktion ist f ein (von Null verschiedenes) Element in $Z(B)_\delta$. Die Charaktergruppe $(T/T')\hat{}$ wird von δ erzeugt, daher ist jedes u_γ von der Form $\lambda_k f^k$ mit passenden $k \in \mathbb{Z}$ und $\lambda_k \in \mathbb{C}$, also $u = \sum_{k=-\infty}^{\infty} \lambda_k f^k$. Eine solche Darstellung ist i.a. nicht eindeutig, nämlich dann, wenn T/T' endlich ist. Offenbar ist $f \equiv 1$ modulo I und damit $u \equiv \sum_{k=-\infty}^{\infty} \lambda_k$ modulo I, also $\sum_{k=-\infty}^{\infty} \lambda_k = 0$ wegen $u \in I$. Wir wollen zeigen, daß u in der Form $u = (1-f)v$ mit $v \in B$ geschrieben werden kann. Damit ist dann der Satz bewiesen, denn $(1-f)v$ liegt in J. Dazu bildet man, zunächst formal,

$$v = (1-f)^{-1} u = \sum_{j \geqq 0} f^j u = \sum_{j \geqq 0} \sum_{k=-\infty}^{\infty} \lambda_k f^{j+k}$$

$$= \sum_{i=-\infty}^{\infty} \Big(\sum_{k \leqq i} \lambda_k \Big) f^i .$$

Durch $v := \sum_{i=-\infty}^{\infty} \Big(\sum_{k \leqq i} \lambda_k \Big) f^i$ ist aber tatsächlich ein Element in B definiert; denn für genügend große i ist $\sum_{k \leqq i} \lambda_k = \sum_{k=-\infty}^{\infty} \lambda_k = 0$, und für genügend kleine i ist $\lambda_k = 0$ für alle $k \leqq i$. Mit diesem v verifiziert man mühelos, daß $u = (1-f)v$ gilt.

Zur späteren Verwendung im Abschn. 4 notieren wir die folgende Bemerkung, die sich leicht aus Satz 1 (sogar nur auf triviales T angewendet) und den im Laufe des Beweises von Satz 2 zitierten Sätzen über die Idealtheorie in universellen Einhüllenden nilpotenter Liescher Algebren ergibt. Wir benutzten dabei die oben eingeführten Bezeichnungen.

Bemerkung. Seien $\mathfrak{n}$, Ω und Ω_∞ wie oben. Weiter sei L eine einfachzusammenhängende, zusammenhängende nilpotente Liesche Gruppe, welche stetig und homomorph durch unipotente Automorphismen auf $\mathfrak{n}$ operieren möge, $\gamma: L \to \mathrm{Aut}(\mathfrak{n})$. Dann ist die Hülle $h(\mathfrak{q})$ von $\mathfrak{q} := \bigcap_{a \in L} \gamma(a)\Omega_\infty$ in $\mathrm{Priv}\,\mathcal{U}\mathfrak{n}$ gerade die $\gamma(L)_{\mathbb{C}}$-Bahn von Ω_∞.

Beweis. Es sei g ein reelles lineares Funktional auf $\mathfrak{n}$ mit $\mathrm{Dix}(g) = \Omega_\infty$. Dann ist $\mathrm{h}(\mathfrak{q})$ nichts anderes als das Bild des Zariski-Abschlusses von $\gamma(L)^*_{\mathbb{C}}\mathrm{Ad}(N)^*_{\mathbb{C}} g$ in $\mathfrak{n}^*_{\mathbb{C}}$ unter Dix. Nun ist aber $\gamma(L)^*_{\mathbb{C}}\mathrm{Ad}(N)^*_{\mathbb{C}} g$ nach Satz 1 Zariski-abgeschlossen, also $h(\mathfrak{q}) = \{a\Omega_\infty;\ a \in \gamma(L)_{\mathbb{C}}\}$ wie behauptet.

2. Die topologische Algebra $\mathcal{S}(N)/\mathcal{S}(N) \cap k(T\Omega)$

Wir übernehmen hier die folgende Notation aus [14]. Seien H eine einfachzusammenhängende, zusammenhängende nilpotente Liesche Gruppe, U eine zusammenhängende Untergruppe von H und γ ein unitärer Charakter von U. Weiter sei $x_1, \ldots, x_r$ eine Basis in einem (passenden) linearen Komplement von $\mathfrak{u}$ in $\mathfrak{h}$ derart, daß $\mathfrak{u}_j := \mathfrak{u} + \mathbb{R}x_1 + \ldots + \mathbb{R}x_j$ eine Liesche Unteralgebra von $\mathfrak{h}$ und ein Ideal in $\mathfrak{u}_{j+1}$ ist. Dann läßt sich jedes $h \in H$ eindeutig in der Form

$h=\exp(t_r x_s)\cdot\ldots\cdot\exp(t_1 x_1)u$ mit $t_j\in\mathbb{R}$ und $u\in U$ schreiben. Mit $\mathcal{S}(H,U,\gamma)$ bezeichnen wir den Raum aller unendlich oft differenzierbaren Funktionen $\varphi: H\to\mathbb{C}$ mit $\varphi(hu)=\gamma(u)^{-1}\varphi(h)$ für $u\in U$ und $h\in H$ und mit der Eigenschaft, daß φ, mittels obiger Zerlegung von H aufgefaßt als eine Funktion in den Variablen $t_1,\ldots,t_s$, eine Schwartzsche Funktion auf $\mathbb{R}^r$ ist. Durch Strukturübertragung können wir $\mathcal{S}(H,U,\gamma)$ auch zu einem topologischen Vektorraum machen. Es ist leicht zu sehen, vgl. [14], daß dieser topologische Vektorraum von der Auswahl der Basis $x_1,\ldots,x_s$ unabhängig ist. Ist zusätzlich T eine kompakte abelsche Liesche Gruppe, so kann man analog den Raum $\mathcal{S}(T\times H,U,\gamma)$ bilden; $\mathcal{S}(T\times H,U,\gamma)$ ist als topologischer Vektorraum isomorph zu $\mathcal{S}(T\times\mathbb{R}^r)$.

Satz 3. *Seien N eine einfachzusammenhängende, zusammenhängende nilpotente Liesche Gruppe mit Liescher Algebra $\mathfrak{n}$ und T eine kompakte abelsche Liesche Gruppe. T operiere stetig und homomorph durch Automorphismen auf N, $(t,x)\to x^t$ für $x\in N$, $t\in T$. Weiter sei Ω ein primitives (=maximales) Ideal in $L^1(N)$, der Stabilisator T_Ω von Ω in T sei endlich. Dann gibt es Unteralgebren $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ von $\mathfrak{n}$ und ein $g\in\mathfrak{n}^*$ mit den folgenden Eigenschaften:*

(i) $T_\Omega g=g$.

(ii) $\mathfrak{a}$, $\mathfrak{b}$ *und* $\mathfrak{c}$ *sind invariant unter* T_Ω.

(iii) $\mathfrak{a}\subset\mathfrak{c}\subset\mathfrak{b}$, $\mathfrak{a}$ *und* $\mathfrak{c}$ *sind Ideale in* $\mathfrak{b}$, $\mathfrak{a}=\mathfrak{c}\cap\operatorname{Kern} g$, $\mathfrak{c}=\{X\in\mathfrak{n};\, g([X,\mathfrak{b}])=0\}$, $\mathfrak{b}/\mathfrak{a}$ *ist eine Heisenbergsche Algebra mit Zentrum* $\mathfrak{c}/\mathfrak{a}$.

Mit A, B, C seien die $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ entsprechenden Untergruppen von N bezeichnet. $\chi_g: C\to\mathbb{T}$ sei definiert durch $\chi_g(\exp X)=e^{ig(X)}$. Weiter sei mit U die Untergruppe $U:=\{(b,bc);\, b\in B,\, c\in C\}$ von $N\times N$ bezeichnet; $\gamma: U\to\mathbb{T}$ sei definiert durch $\gamma(b,bc)=\chi_g(c)^{-1}$.

(iv) $\Omega=\operatorname{Kern}\operatorname{ind}_C^N\chi_g$.

(v) *Für $\varphi\in\mathcal{S}(N)$ sei $R\varphi: T\times N\times N\to\mathbb{C}$ definiert durch*

$$(R\varphi)(t,x,y)=\int_C\varphi^t(xcy^{-1})\chi_g(c)dc=\int_C\varphi((xcy^{-1})^{t^{-1}})\chi_g(c)dc.$$

Für jedes $\varphi\in\mathcal{S}(N)$ liegt $R\varphi$ in $\mathcal{S}(T\times N\times N,U,\gamma)$, und R ist eine stetige Surjektion von $\mathcal{S}(N)$ auf die T_Ω-Fixpunkte in $\mathcal{S}(T\times N\times N,U,\gamma)$, d.h. auf den Raum derjenigen $\psi\in\mathcal{S}(T\times N\times N,U,\gamma)$, für welche $\psi(tt_0,x,y)=\psi(t,x^{t_0^{-1}},y^{t_0^{-1}})$ gilt, wobei $t\in T$, $t_0\in T_\Omega$ und $x,y\in N$.

(vi) *Zu der surjektiven stetigen linearen Abbildung R gibt es eine stetige lineare Umkehrabbildung, d.h. R ist ein Retrakt.*

Bemerkung 1. Die Existenz von $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ zu gegebenem g wurde, sogar viel allgemeiner, bereits von Howe bewiesen, [13].

Bemerkung 2. Im allgemeinen ist $\operatorname{ind}_C^N\chi_g$ nicht irreduzibel, aber $\operatorname{ind}_C^N\chi_g$ ist ein Faktor, und zwar ein Vielfaches der im Kirillowschen Bild zu g gehörigen irreduziblen Darstellung.

Bemerkung 3. Die Funktion $(x,y)\to(R\varphi)(t,x,y)$ ist natürlich nichts anderes als der Kern zum Operator $\left(\operatorname{ind}_C^N\chi_g\right)(\varphi^t)$. Daher ist $\operatorname{Kern} R=\mathcal{S}(N)\cap\bigcap_{t\in T}\Omega^t$, und aus (v) und (vi) folgt, daß $\mathcal{S}(N)/\mathcal{S}(N)\cap\bigcap_{t\in T}\Omega^t$ zu $\mathcal{S}(T\times N\times N,U,\gamma)^{T_\Omega}$ isomorph ist.

Beweis. Unabhängig von allem Übrigen ist es klar, daß die Funktion $R\varphi$ die angegebenen Transformationseigenschaften besitzt. Man nutzt dabei aus, daß C/A zentral in B/A ist. Die anderen Eigenschaften von R (also z. B., daß $R\varphi$ überhaupt eine Schwartzsche Funktion ist) ergeben sich aus einem Induktionsbeweis mit den üblichen Fallunterscheidungen.

Es sei $z(Z)$ das Zentrum von $\mathfrak{n}(N)$. Zu dem Ideal Ω gehört ein eindeutig bestimmter Charakter $\eta: Z \to \mathbb{T}$ mit

$$\{f \in L^1(Z); f * L^1(N) \subseteqq \Omega\} = \left\{f \in L^1(Z); \int_Z f(z)\eta(z)dz = 0\right\}.$$

1. Fall. Es gibt eine nicht-triviale, zusammenhängende T-invariante Untergruppe W von Z mit $\eta(W) = 1$.

Die Existenz von g, $\mathfrak{a}$, $\mathfrak{b}$ und $\mathfrak{c}$ mit den Eigenschaften (i)–(v) folgt sofort aus der Existenz der entsprechenden Größen für N/W mit der induzierten T-Wirkung; als maximales Ideal in $L^1(N/W)$ nimmt man natürlich das Bild von Ω unter der kanonischen Surjektion $L^1(N) \to L^1(N/W)$. Um (vi) zu beweisen, hat man sich noch zu überlegen, daß $\mathscr{S}(N) \to \mathscr{S}(N/W)$ eine stetige lineare Umkehrabbildung zuläßt. Dazu wähle man ein Komplement $\mathfrak{v}$ zu $\mathfrak{w}$ (= Liesche Algebra von W) in $\mathfrak{n}$. Dann definiert die Multiplikation in N einen Diffeomorphismus von $\exp(\mathfrak{v}) \times W$ auf N, und die Einschränkung der Quotientenabbildung $N \to N/W$ auf $\exp(\mathfrak{v})$ etabliert einen Diffeomorphismus $\exp(\mathfrak{v}) \to N/W$; mit $s: N/W \to \exp(\mathfrak{v})$ sei die Umkehrabbildung bezeichnet. Wählt man noch eine Funktion f_0 in $\mathscr{S}(W)$ mit $\int_W f_0(x)dx = 1$, so ist durch $\varphi \to \tilde{\varphi}$, $\tilde{\varphi}(s(y)w) = \varphi(y)f_0(w)$ für $\varphi \in \mathscr{S}(N/W)$, $y \in N/W$, $w \in W$ eine Umkehrabbildung zu $\mathscr{S}(N) \to \mathscr{S}(N/W)$ angegeben.

2. Fall. Es gibt eine zweidimensionale zusammenhängende Untergruppe W von Z mit $\eta(W) = \mathbb{T}$, auf welcher T als eindimensionale Automorphismengruppe operiert.

Man kann also W mit $\mathbb{C}$ identifizieren, so daß sich die Wirkung von T auf W in der Form $w^t = \delta(t)w$ mit einem surjektiven Charakter $\delta: T \to \mathbb{T}$ schreiben läßt. Offenbar liegt T_Ω in $T' := \operatorname{Kern}\delta$. Weiter gibt es eine zu $\mathbb{T}$ isomorphe Untergruppe T_c von T mit $T = T'T_c$; und der Durchschnitt $F := T' \cap T_c$ ist endlich.

Für $\varphi \in \mathscr{S}(N)$ definieren wir $J\varphi: T_c \times N \to \mathbb{C}$ durch $(J\varphi)(t, x) = \int_W \varphi^t(xw)\eta(w)dw$. Man überzeugt sich, daß $J\varphi$ in $\mathscr{S}(T_c \times N, W, \eta)$ liegt, in der Tat liegt $J\varphi$ in $\mathscr{S}(T_c \times N, W, \eta)^F := \{\psi \in \mathscr{S}(T_c \times N, W, \eta); \psi(tt_0, x) = \psi(t, x^{t_0^{-1}})$ für alle $t_0 \in F$, $t \in T_c$, $x \in N\}$. Wir behaupten, daß J ein Retrakt von $\mathscr{S}(N)$ auf $\mathscr{S}(T_c \times N, W, \eta)^F$ ist. Um eine Umkehrabbildung zu konstruieren (und damit gleichzeitig die Surjektivität zu beweisen), gehe man wie folgt vor. Man wähle ein T-invariantes Komplement $\mathfrak{v}$ zu $\mathfrak{w}$ in $\mathfrak{n}$ und definiere $\mathscr{S}(T_c \times \mathfrak{v}) \to \mathscr{S}(T_c \times N, W, \eta)$, $f \to f'$, durch $f'(t, \exp(X)w) = \bar{\eta}(w)f(t, X^{t^{-1}})$ für $t \in T_c$, $w \in W$ und $X \in \mathfrak{v}$. Diese Abbildung ist ein Isomorphismus (da die Variablentransformation $(t, X) \to (t, X^{t^{-1}})$ von $T_c \times \mathfrak{v}$ auf sich selbst ein Diffeomorphismus ist, dessen sämtliche Ableitungen nur polynomial wachsen) mit Umkehrabbildung $f \to \tilde{f}$, $\tilde{f}(t, X) = f(t, \exp(X^t))$. Der Isomorphismus $f \to \tilde{f}$ induziert einen Isomorphismus von $\mathscr{S}(T_c \times N, W, \eta)^F$ auf $\mathscr{S}(T_c/F \times \mathfrak{v})$. Es genügt daher zu zeigen, daß es zu der Abbildung

$I: \mathscr{S}(N) \to \mathscr{S}(T_c/F \times \mathfrak{v})$, definiert durch $I\varphi = (J\varphi)^{\check{}}$, d.h. $(I\varphi)(t, X) = \int\limits_W \varphi(\exp(X) w^{t^{-1}}) \eta(w) dw$, eine stetige lineare Umkehrabbildung gibt. Dies folgt aber, mittels der Fourierschen Transformation auf der Gruppe W, leicht aus der Tatsache, daß die Restriktion $\mathscr{S}(\mathbb{C}) \to \mathscr{S}(\mathbb{T}) = \mathscr{D}(\mathbb{T})$ eine stetige lineare Umkehrung besitzt.

Wir wenden nun die Induktionsvoraussetzung auf die Gruppe $N' = N/(\mathrm{Kern}\,\eta)_0$ mit operierender Gruppe $T' = \mathrm{Kern}\,\delta$ und auf das maximale Ideal Ω' in $L^1(N')$ an, welches gegeben ist als Bild von Ω unter der kanonischen Surjektion $L^1(N) \to L^1(N')$. Man findet also Unteralgebren $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$ von $\mathfrak{n}'$ und ein lineares Funktional g' auf $\mathfrak{n}'$ mit den Eigenschaften (i)–(vi). Insbesondere ist die Abbildung $R': \mathscr{S}(N') \to \mathscr{S}(T' \times N' \times N', U', \gamma')^{T_{\Omega'}}$, gegeben durch

$$R'\varphi(t, x, y) = \int\limits_{C'} \varphi^t(xcy^{-1}) \chi_{g'}(c) dc,$$

ein Retrakt. Den Größen $\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$ und g' entsprechen durch Zurückziehen die Unteralgebren $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ von $\mathfrak{n}$ und das Funktional g. Es ist leicht zu sehen, daß mit dieser Setzung die Eigenschaften (i)–(iv) erfüllt sind; man beachte, daß $T'_{\Omega'} = T_\Omega$. $\mathscr{S}(T' \times N' \times N', U', \gamma')$ ist kanonisch isomorph zu $\mathscr{S}(T' \times N \times N, U, \gamma)$, und die T_Ω-Fixpunkte entsprechen bei diesem Isomorphismus einander. Man kann also R' auffassen als Surjektion von $\mathscr{S}(N')$ auf $\mathscr{S}(T' \times N \times N, U, \gamma)^{T_\Omega}$. Weiter hat man eine kanonische Surjektion $\mathscr{S}(N') \to \mathscr{S}(N, W, \eta)$, gegeben durch

$$(L\varphi)(x) = \int\limits_{W/(\mathrm{Kern}\,\eta)_0} \varphi(xw) \eta(w) dw$$

für $\varphi \in \mathscr{S}(N')$, und R' faktorisiert durch L, d.h. es gibt $R'': \mathscr{S}(N, W, \eta) \to \mathscr{S}(T' \times N \times N, U, \gamma)^{T_\Omega}$ mit $R''L = R'$, nämlich

$$(R''\psi)(t, x, y) = \int\limits_{C/W} \psi^t(xcy^{-1}) \chi_g(c) dc$$

für $\psi \in \mathscr{S}(N, W, \eta)$, $x, y \in N$ und $t \in T'$; man beachte, daß der Integrand auf Nebenklassen modulo W konstant ist, da $\chi_g|_W = \eta$ und T' auf W trivial operiert. Mit R' besitzt offenbar auch R'' eine stetige lineare Umkehrung. Dann ist aber auch $\mathrm{id} \otimes R'': \mathscr{S}(T_c \times N, W, \eta) \to \mathscr{S}(T_c \times T' \times N \times N, U, \gamma)^{T_\Omega}$ ein Retrakt. Schließlich definieren wir noch $M: \mathscr{S}(T \times N \times N, U, \gamma)^{T_\Omega} \to \mathscr{S}(T_c \times T' \times N \times N, U, \gamma)^{T_\Omega}$ durch $(M\psi)(t, t', x, y) = \psi(tt', x, y)$. M ist ein stetiger Isomorphismus von $\mathscr{S}(T \times N \times N, U, \gamma)^{T_\Omega}$ auf den Raum $\mathscr{S}(T_c \times T' \times N \times N, U, \gamma)^{T_\Omega, F}$ derjenigen $\psi \in \mathscr{S}(T_c \times T' \times N \times N, U, \gamma)^{T_\Omega}$ mit $\psi(tt_0, t't_0^{-1}, x, y) = \psi(t, t', x, y)$ für $x, y \in N$, $t \in T_c$, $t' \in T'$, $t_0 \in F$. Man überzeugt sich nun, daß $R = M^{-1}(id \otimes \mathbb{R}'')J$ gilt, insbesondere besteht das Bild von R aus Schwartzschen Funktionen. Indem man einen beliebigen linearen stetigen Schnitt $\mathscr{S}(T_c \times T' \times N \times N, U, \gamma)^{T_\Omega} \to \mathscr{S}(T_c \times N, W, \eta)$ zu $id \otimes R''$ über die endliche Gruppe F mittelt, erhält man einen F-verkettenden Schnitt, insbesondere einen solchen, welcher $\mathscr{S}(T_c \times T' \times N \times N, U, \gamma)^{T_\Omega, F}$ in $\mathscr{S}(T_c \times N, W, \eta)^F$ überführt. Zusammen mit der Tatsache, daß J eine stetige lineare Umkehrung zuläßt, folgt nun, daß R surjektiv ist und eine stetige lineare Umkehrung besitzt.

Wenn weder der erste noch der zweite Fall vorliegt, so operiert T als endliche Automorphismengruppe auf z und Z. Im nächsten Schritt wollen wir zeigen, daß man dann auch ohne Beschränkung der Allgemeinheit annehmen kann, daß T

trivial auf z operiert. Sei dazu $\mathfrak{w}$ ein T-irreduzibler Teilraum von z mit zugehöriger Gruppe W, und T' sei der gemeinsame Stabilisator in T der Elemente von $\mathfrak{w}$. Wir können annehmen, daß $\eta(W)=\mathbb{T}$. Es soll gezeigt werden, daß aus der Gültigkeit des Satzes für T', N, Ω die Richtigkeit für T, N, Ω folgt. Da T_Ω in T' liegt, haben die zu T', N, Ω existierenden $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, g natürlich auch die Eigenschaften (i)–(iv) bezüglich T, N, Ω. Es sind noch (v) und (vi) zu zeigen. Sei dazu $T=\bigcup_{j=1}^{n} t_j T'$ eine disjunkte Zerlegung von T in T'-Nebenklassen, $t_1=1$. Die Charaktere η_j, $1\leqq j\leqq n$, von W seien definiert durch $\eta_j(w)=\eta(w^{t_j})$. Aus der Gültigkeit des Satzes für T', N, Ω folgt, daß die Abbildung $P_1:\mathscr{S}(N,W,\eta)\to\mathscr{S}(T'\times N\times N,U,\gamma)^{T_\Omega}$, definiert durch

$$(P_1\varphi)(t,x,y)=\int_{C/W}\varphi^t(xcy^{-1})\chi_g(c)d\dot{c},$$

ein Retrakt ist. Wir definieren nun Isomorphismen $V_j:\mathscr{S}(N,W,\eta_j)\to\mathscr{S}(N,W,\eta)$ durch $(V_j\psi)(x)=\psi(x^{t_j^{-1}})$ und setzen $P_j=P_1V_j$ für $1\leqq j\leqq n$. Definiert man noch den Isomorphismus

$$D:\mathscr{S}(T\times N\times N,U,\gamma)^{T_\Omega}\to\bigoplus_{j=1}^{n}\mathscr{S}(T'\times N\times N,U,\gamma)^{T_\Omega}$$

durch $Df=(f|_{t_jT'\times N\times N})_{1\leqq j\leqq n}$ sowie $Q:\mathscr{S}(N)\to\bigoplus_{j=1}^{n}\mathscr{S}(N,W,\eta_j)$ durch $(Q\varphi)_j(x)=\int_W\varphi(xw)\eta_j(w)dw$, so ist das Diagramm

$$\begin{array}{ccc}\mathscr{S}(N) & \xrightarrow{Q} & \bigoplus_{j=1}^{n}\mathscr{S}(N,W,\eta_j)\\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \oplus P_j}\\ \mathscr{S}(T\times N\times N,U,\gamma)^{T_\Omega} & \xrightarrow{D} & \bigoplus_{j=1}^{n}\mathscr{S}(T'\times N\times N,U,\gamma)^{T_\Omega}\end{array}$$

kommutativ. Aus der Tatsache, daß die Charaktere $\eta_1,\dots,\eta_n$ paarweise verschieden sind, folgt leicht, daß Q surjektiv ist und eine stetige Umkehrung zuläßt. Damit ist klar, daß R die entsprechenden Eigenschaften besitzt.

Also können wir nunmehr annehmen, daß $\dim\mathfrak{z}=1$, $\eta(Z)=\mathbb{T}$ und daß T trivial auf $\mathfrak{z}$ operiert.

3. Fall. $\dim\mathfrak{z}=1$, $\eta(Z)=\mathbb{T}$, T operiert trivial auf $\mathfrak{z}$. Es gibt ein abelsches T-invariantes Ideal $\mathfrak{w}$, welches $\mathfrak{z}$ echt umfaßt.

Dann gibt es auch ein abelsches Ideal $\mathfrak{w}$, für welches zusätzlich gilt, daß $\mathfrak{w}/\mathfrak{z}$ zentral in $\mathfrak{n}/\mathfrak{z}$ und $\mathfrak{w}/\mathfrak{z}$ ein irreduzibler T-Modul ist. Insbesondere ist $\mathfrak{w}$ zwei- oder dreidimensional. Mit $\mathfrak{h}$ sei der Zentralisator von $\mathfrak{w}$ bezeichnet. Wir wählen T-invariante Komplemente $\mathfrak{v}$ zu $\mathfrak{h}$ in $\mathfrak{n}$ und $\mathfrak{y}$ zu $\mathfrak{z}$ in $\mathfrak{w}$, also $\mathfrak{n}=\mathfrak{v}\oplus\mathfrak{h}$ und $\mathfrak{w}=\mathfrak{y}\oplus\mathfrak{z}$. Es ist leicht zu sehen, daß die Liesche Klammer eine nicht-ausgeartete Bilinearform $\mathfrak{v}\times\mathfrak{y}\to\mathfrak{z}$ induziert. Schließlich sei noch $\mathfrak{h}'=\mathfrak{h}/\mathfrak{y}$ gesetzt. Nach Konstruktion operiert T auf $\mathfrak{h}'$.

Wie man aus der wohlbekannten Darstellungstheorie nilpotenter Liescher Gruppen, vgl. etwa [17] oder [28], weiß, gibt es zu dem maximalen Ideal Ω (und ebenso zu jedem Ω^t) genau ein maximales Ideal Λ' in $L^1(H')$ mit der folgenden Eigenschaft: Ist Λ das volle Urbild von Λ' unter der kanonischen Abbildung $L^1(H)$

$\to L^1(H')$, so gilt

$$\Omega = \left\{ \left(\bigcap_{x \in N} \Lambda^x \right) * L^1(N) \right\}^- .$$

Daraus ergibt sich insbesondere, daß der Stabilisator von Λ' in T mit T_Ω übereinstimmt.

Auf Grund der Induktionsvoraussetzung existieren zu Λ' Unteralgebren $\mathfrak{a}'$, $\mathfrak{b}'$ und $\mathfrak{c}'$ von $\mathfrak{h}'$ sowie ein reelles Funktional g' auf $\mathfrak{h}'$ mit (i)–(vi). Mit $\mathfrak{a}$, $\mathfrak{b}$, und $\mathfrak{c}$ seien die Urbilder von $\mathfrak{a}'$, $\mathfrak{b}'$, und $\mathfrak{c}'$ unter $\mathfrak{h} \to \mathfrak{h}'$ bezeichnet. Das Funktional g auf $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{h}$ ist, nach Definition, auf $\mathfrak{v}$ identisch Null und auf $\mathfrak{h}$ die Zusammensetzung aus g' und $\mathfrak{h} \to \mathfrak{h}'$.

Es ist leicht zu sehen, daß das Quadrupel $(g, \mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ die Eigenschaften (i)–(iv) besitzt.

Zum Nachweis von (v) und (vi) zeigen wir zunächst, daß durch

$$\begin{aligned}(\mathscr{F}\varphi)(a,b,h) &= \int_Y dr \varphi(\exp(a) h r \exp(-b)) \\ &= \int_{\mathfrak{y}} dr \varphi(\exp(a) h \exp(r) \exp(-b))\end{aligned}$$

ein Isomorphismus $\mathscr{F} : \mathscr{S}(N, Z, \eta) \to \mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta)$ gegeben ist; Z können wir ja ebensogut als Teil von H' auffassen. Zur Konstruktion der Umkehrabbildung wählen wir ein Komplement $\mathfrak{k}$ zu $\mathfrak{w} = \mathfrak{y} \oplus \mathfrak{z}$ in $\mathfrak{h}$, $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{y} \oplus \mathfrak{z}$. Es existieren dann eindeutig bestimmte Funktionen K, β und ξ auf $\mathfrak{v} \times \mathfrak{v} \times \mathfrak{k}$ mit Werten in $\mathfrak{k}$, $\mathfrak{y}$ bzw. $\mathfrak{z}$ derart, daß

$$\begin{aligned}&\exp a \exp k \exp b \\ &\quad = \exp(a+b) \exp K(a,b,k) \exp \beta(a,b,k) \exp \xi(a,b,k)\end{aligned}$$

für alle $a, b \in \mathfrak{v}$ und $k \in \mathfrak{k}$ gilt. Nun wird

$$\mathscr{G} : \mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta) \to \mathscr{S}(N, Z, \eta)$$

definiert durch

$$\begin{aligned}&(\mathscr{G}\psi)(\exp a \exp \mu \exp s) \\ &\quad = \int_{\mathfrak{v}} db \, e^{ig[b,s]} e^{-ig\xi(a,b,\mu)} \psi(a+b, b, \exp K(a,b,\mu))\end{aligned}$$

für $a \in \mathfrak{v}$, $\mu \in \mathfrak{k}$ und $s \in \mathfrak{y}$. Wir behaupten, daß $\mathscr{G}$ ein zu $\mathscr{F}$ inverser Isomorphismus ist.

Um einzusehen, daß $\mathscr{G}$ tatsächlich ein Isomorphismus ist, geht man folgendermaßen vor. Da die (teilweise) Fourier-Transformation ja bekanntlich ein Isomorphismus der Schwartzschen Räume ist, hat man lediglich zu zeigen, daß durch

$$\varphi \to \tilde{\varphi}, \qquad \tilde{\varphi}(a,b,\mu) = e^{-ig\xi(a,b,\mu)} \varphi(a+b, b, K(a,b,\mu)),$$

ein Automorphismus von $\mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times \mathfrak{k})$ definiert ist. Nun sind ξ und K polynomiale Funktionen, und für feste a, b ist $K_{a,b} : \mathfrak{k} \to \mathfrak{k}$, definiert durch $K_{a,b}(\mu) = K(a,b,\mu)$ eine bireguläre Abbildung im Sinne der algebraischen Geometrie. Es gilt nämlich, wie man leicht bestätigt (vgl. auch die weiter unten stehende Rechnung), $K_{a,b}^{-1}(\mu) = K(a+b, -b, \mu)$; insbesondere hängt auch die Umkehrung $K_{a,b}^{-1}$ nicht nur von μ, sondern auch von a und b in polynomialer Weise ab. Daher ist die Variablen-Transformation $(\mathrm{a}, \mathrm{b}, \mu) \to (\mathrm{a}+\mathrm{b}, \mathrm{b}, \mathrm{K}(\mathrm{a}, \mathrm{b}, \mu))$ insgesamt biregulär, woraus man schließen kann, daß $\varphi \to \tilde{\varphi}$ in der Tat ein Automorphismus ist.

Der Beweis für die Behauptung, daß $\mathscr{F}$ ein Isomorphismus ist (insbesondere dafür, daß die Werte von $\mathscr{F}$ überhaupt in $\mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta)$ liegen), ist erbracht, wenn wir zeigen können, daß $\mathscr{F} \circ \mathscr{G} = id$ ist, jedenfalls bis auf eine positive Konstante. Sei also $\psi \in \mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta)$, $\varphi = \mathscr{G}\psi$ und $f = \mathscr{F}\varphi$. Für $a, b \in \mathfrak{v}$ und $k \in \mathfrak{k}$ ist

$$\begin{aligned} f(a, b, \exp k) &= (\mathscr{F}\varphi)(a, b, \exp k) \\ &= \int_{\mathfrak{y}} dr \varphi(\exp a \exp k \exp r \exp(-b)) . \end{aligned}$$

Nun ist

$$\begin{aligned} &\exp a \exp k \exp r \exp(-b) \\ &\quad = \exp a \exp k \exp(-b) \exp b \exp r \exp(-b) \\ &\quad = \exp(a-b) \exp K(a, -b, k) \exp \beta(a, -b, k) \\ &\qquad \cdot \exp \xi(a, -b, k) \exp r \exp[b, r] , \end{aligned}$$

also

$$\begin{aligned} &\varphi(\exp a \exp k \exp r \exp(-b)) \\ &\quad = \mathrm{e}^{-ig\{\xi(a, -b, k) + [b, r]\}} \varphi(\exp(a-b) \exp K(a, -b, k) \\ &\qquad \cdot \exp(r + \beta(a, -b, k))) . \end{aligned}$$

Setzen wir abkürzend $\xi_0 = \xi(a, -b, k)$, $\beta_0 = \beta(a, -b, k)$ und $K_0 = K(a, -b, k)$, so ergibt die Variablentransformation $r' = r + \beta_0$, daß

$$\begin{aligned} &f(a, b, \exp k) \\ &\quad = \int_{\mathfrak{y}} dr\, \mathrm{e}^{-ig\{\xi_0 + [b, r-\beta_0]\}} \varphi(\exp(a-b) \exp K_0 \exp r) \\ &\quad = \int_{\mathfrak{y}} dr\, \mathrm{e}^{-ig\{\xi_0 + [b, r-\beta_0]\}} \int_{\mathfrak{v}} dc\, \mathrm{e}^{ig\{[c, r] - \xi(a-b, c, K_0)\}} \\ &\qquad \cdot \psi(a-b+c, c, \exp K(a-b, c, K_0)) . \end{aligned}$$

Da die Zusammensetzung aus $[\,,\,] : \mathfrak{v} \times \mathfrak{y} \to \mathfrak{z}$ und g eine Dualität zwischen $\mathfrak{v}$ und $\mathfrak{y}$ etabliert, liefert die Fouriersche Inversions-Formel, angewandt auf die Funktion

$$\begin{aligned} h(c) &= \mathrm{e}^{-ig\xi(a-b, c, K_0)} \psi(a-b+c, c, \exp K(a-b, c, K_0)) , \; f(a, b, \exp k) \\ &= E\, \mathrm{e}^{-ig\{\xi_0 - [b, \beta_0]\}} \mathrm{e}^{-ig\xi(a-b, b, K_0)} \psi(a, b, \exp K(a-b, b, K_0)) \end{aligned}$$

mit einer gewissen positiven Konstanten E.

$\xi(a-b, b, K_0)$ und $K(a-b, b, K_0)$ sind definiert durch die Gleichung

$$\begin{aligned} &\exp(a-b) \exp K_0 \exp(b) \\ &\quad = \exp a \exp K(a-b, b, K_0) \exp \beta(a-b, b, K_0) \exp \xi(a-b, b, K_0) . \end{aligned}$$

Aus der Definition von K_0, β_0 und ξ_0 ergibt sich

$$\exp a \exp k \exp(-b) = \exp(a-b) \exp K_0 \exp \beta_0 \exp \xi_0 ,$$

also

$$\begin{aligned}&\exp(a-b)\exp K_0\\&\quad=\exp a\exp k\exp(-b)\exp(-\beta_0)\exp(-\xi_0).\end{aligned}$$

Durch Multiplikation mit $\exp b$ und Vergleich mit der obigen Beziehung erhält man

$$\begin{aligned}&\exp(a)\exp k\exp(-b)\exp(-\beta_0)\exp(-\xi_0)\exp b\\&\quad=\exp a\exp K(a-b,b,K_0)\exp\beta(a-b,b,K_0)\\&\quad\cdot\exp\xi(a-b,b,K_0),\end{aligned}$$

also

$$\begin{aligned}&\exp k\exp(-\beta_0)\exp(-\xi_0+[b,\beta_0])\\&\quad=\exp K(a-b,b,K_0)\exp\beta(a-b,b,K_0)\exp\xi(a-b,b,K_0)\end{aligned}$$

und damit $k=K(a-b,b,K_0)$ und $\xi(a-b,b,K_0)=-\xi_0+[b,\beta_0]$. Trägt man das in den gewonnenen Ausdruck für $f(a,b,\exp k)$ ein, so folgt $f(a,b,\exp k)=E\psi(a,b,\exp k)$, wie behauptet.

Nach Induktionsvoraussetzung ist die Abbildung

$$R':\mathscr{S}(H',Z,\eta)\to\mathscr{S}(T\times H'\times H',U',\gamma')^{T_\Omega},$$

gegeben durch

$$(R'f)(t,h',k')=\int_{C'/Z}d\dot{c}\chi_{g'}(c)f((h'ck'^{-1}),$$

ein Retrakt. Dabei ist natürlich $U'=\{(b',b'c');\ b'\in B',\ c'\in C'\}$, und $\gamma':U'\to\mathbb{T}$ ist gegeben durch $\gamma'(b',b'c')=\chi_{g'}(c')^{-1}$. Dann hat auch

$$id\otimes R':\mathscr{S}(\mathfrak{v}\times\mathfrak{v}\times H',Z,\eta)\to\mathscr{S}(\mathfrak{v}\times\mathfrak{v}\times T\times H'\times H',U',\gamma')^{T_\Omega}$$

die entsprechenden Eigenschaften.

Der Raum $\mathscr{S}(T\times N\times N,U,\gamma)$ ist wegen der Zerlegung $N=\exp(\mathfrak{v})H$ isomorph zu $\mathscr{S}(T\times\mathfrak{v}\times H\times\mathfrak{v}\times H,U,\gamma)$. Die Funktionen in $\mathscr{S}(T\times\mathfrak{v}\times H\times\mathfrak{v}\times H,U,\gamma)$ sind aber konstant auf den Nebenklassen modulo $Y\times Y$ und lassen sich daher als Elemente von $\mathscr{S}(T\times\mathfrak{v}\times H'\times\mathfrak{v}\times H',U',\gamma')$ auffassen. Mit anderen Worten, durch $\psi\to\tilde{\psi}$, $\tilde{\psi}(t,a,h',b,k')=\psi(t,\ \exp(a)h,\ \exp(b)k)$ – wobei h und k irgendwelche Urbilder von h' bzw. k' unter $H\to H'$ sind – ist ein Isomorphismus von $\mathscr{S}(T\times N\times N,U,\gamma)$ auf $\mathscr{S}(T\times\mathfrak{v}\times H'\times\mathfrak{v}\times H',U',\gamma')$ definiert. Unter diesem Isomorphismus wird $\mathscr{S}(T\times N\times N,U,\gamma)^{T_\Omega}$ auf den Teilraum

$$\mathscr{S}(T\times\mathfrak{v}\times H'\times\mathfrak{v}\times H',U',\gamma')^{T_\Omega},$$

bestehend aus allen f mit

$$f(tt_0^{-1},a,h',b,k')=f(t,a^{t_0},h'^{t_0},b^{t_0},k'^{t_0})$$

für $a,\ b\in\mathfrak{v}$, $t\in T$, $t_0\in T_\Omega$ und $h',\ k'\in H'$, abgebildet. Weiter ist

$$\mathscr{K}:\mathscr{S}(T\times\mathfrak{v}\times H'\times\mathfrak{v}\times H',U',\gamma')\to\mathscr{S}(\mathfrak{v}\times\mathfrak{v}\times T\times H'\times H',U',\gamma'),$$

gegeben durch

$$(\mathscr{K} f)(a, b, t, h', k') = f(t, a^t, h', b^t, k'),$$

ein Isomorphismus, da die zugrundeliegende Variablen-Transformation wiederum ein Diffeomorphismus mit höchstens polynomial wachsenden Ableitungen ist. Unter $\mathscr{K}$ geht

$$\mathscr{S}(T \times \mathfrak{v} \times H' \times \mathfrak{v} \times H', U', \gamma')^{T_\Omega}$$

in

$$\mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times T \times H' \times H', U', \gamma')^{T_\Omega}$$

über. Also ist durch $\mathscr{L}\psi := \mathscr{K}\tilde{\psi}$ ein Isomorphismus $\mathscr{L}$ von $\mathscr{S}(T \times N \times N, U, \gamma)^{T_\Omega}$ auf $\mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times T \times H' \times H', U', \gamma')^{T_\Omega}$ definiert.

Die Definitionen sind gerade so eingerichtet, daß das Diagramm

$$\begin{array}{cccc} & \mathscr{S}N, Z, \eta) & \xrightarrow{\mathscr{F}} & \mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta) \\ \mathscr{S}(N) \nearrow & \downarrow \hat{R} & & \downarrow id \otimes R' \\ \searrow^{R} & \mathscr{S}(T \times N \times N, U, \gamma)^{T_\Omega} & \xrightarrow{\mathscr{L}} & \mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times T \times H' \times H', U', \gamma')^{T_\Omega} \end{array}$$

kommutiert. Dabei ist mit $\mathscr{S}(N) \to \mathscr{S}(N, Z, \eta)$ natürlich die kanonische Abbildung $\varphi \to \hat{\varphi}$, $\hat{\varphi}(x) = \int_Z \varphi(xz)\eta(z)dz$, gemeint, und $\hat{R}$ ist gegeben durch $\hat{R}\hat{\varphi} = R\varphi$. Aus der Kommutativität des Diagramms folgt erst einmal, daß das Bild von R aus Schwartzschen Funktionen besteht (genauer müßte man eigentlich die Kommutativität des Diagramms als $\hat{R} = \mathscr{L}^{-1}(id \otimes R')\mathscr{F}$ formulieren). Weiter ist mit R' auch $\hat{R}$ ein Retrakt. Da $\mathscr{S}(N) \to \mathscr{S}(N, Z, \eta)$ eine stetige lineare Umkehrung zuläßt, ist dann R ebenfalls ein Retrakt.

4. Fall. $\mathfrak{n}$ ist isomorph zu einer Heisenbergschen Algebra, $\eta(Z) = \mathbb{T}$, und T operiert trivial auf $\mathfrak{z}$.

Dann ist $T = T_\Omega$ endlich. Man wähle $\mathfrak{a} = 0$, $\mathfrak{b} = \mathfrak{n}$ und $\mathfrak{c} = \mathfrak{z}$. Ist ferner $\mathfrak{v}$ ein T-invariantes Komplement zu $\mathfrak{z}$ in $\mathfrak{n}$, so setze man $g = 0$ auf $\mathfrak{v}$ und definiere g auf $\mathfrak{z}$ durch $\eta(\exp X) = e^{ig(x)}$ für $X \in \mathfrak{z}$. Es ist leicht zu sehen, daß damit (i)–(vi) erfüllt sind.

3. $\left(\bigcap_{t \in T} \Omega^t_\infty\right) * \mathscr{S}(N)$ ist total in $\mathscr{S}(N) \cap \bigcap_{t \in T} \Omega^t$.

Wir behalten die Bezeichnungen des vorigen Paragraphen weitestgehend bei, auch bei den analog zu dort zu diskutierenden möglichen Fällen. Insbesondere seien g und R wie im Abschn. 2 konstruiert. $\mathscr{S}(N) \cap \bigcap_{t \in T} \Omega^t$ ist nichts anderes als der Kern von R.

Satz 4. *Seien N eine einfachzusammenhängende zusammenhängende nilpotente Liesche Gruppe mit Liescher Algebra $\mathfrak{n}$ und T eine kompakte abelsche Liesche Gruppe. T operiere stetig und homomorph durch Automorphismen auf N, $(t, x) \to x^t$. Ω sei ein primitives Ideal in $L^1(N)$, der Stabilisator T_Ω von Ω in T sei endlich. Weiter sei Ω_∞ das Ω entsprechende primitive Ideal in $\mathscr{U}\mathfrak{n}$ und $\mathfrak{p} = \bigcap_{t \in T} \Omega^t_\infty$. Dann liegt $\mathfrak{p} * \mathscr{S}(N)$ total in $\mathscr{S}(N) \cap \bigcap_{t \in T} \Omega^t$.*

Bemerkung. Die Voraussetzung der Endlichkeit von T_Ω ist offenbar überflüssig: Man kann den Fall eines beliebigen T_Ω sofort auf endliches T_Ω reduzieren (durch Verkleinern von T). Sie ist nur gemacht worden, um die Ergebnisse vom Abschn. 2 unmittelbar anwenden zu können.

Beweis. Der Beweis erfolgt natürlich wieder durch vollständige Induktion über die Dimension von N. Wiederum sei η der zu Ω gehörige Charakter auf dem Zentrum Z von N.

1. Fall. Es gibt eine nicht-triviale, zusammenhängende, T-invariante Untergruppe W von Z mit $\eta(W)=1$.

Es sei $\mathfrak{n}'=\mathfrak{n}/\mathfrak{w}$, Ω' sei das Bild von Ω unter der kanonischen Surjektion $L^1(N)\to L^1(N/W)$ und $\mathfrak{p}'$ sei das Bild von $\mathfrak{p}$ unter der Surjektion $\mathscr{U}\mathfrak{n}\to\mathscr{U}(\mathfrak{n}')$. Dann ist offensichtlich Ω'_∞ das Bild von Ω_∞ unter $\mathscr{U}\mathfrak{n}\to\mathscr{U}(\mathfrak{n}')$ und $\mathfrak{p}'$ ist gleich $\bigcap_{t\in T}\Omega'^t_\infty$. Die Behauptung des Satzes folgt aus der entsprechenden Aussage für $\mathfrak{p}'$, $\mathfrak{n}'$ und Ω', wenn wir zeigen, daß der Kern der Surjektion $Q:\mathscr{S}(N)\to\mathscr{S}(N/W)$ im Abschluß des linearen Erzeugnisses von $\mathfrak{p}*\mathscr{S}(N)$ liegt. Nun ist $\mathfrak{p}\cap\mathscr{U}\mathfrak{w}=\mathfrak{w}\mathscr{U}\mathfrak{w}$, und in der Tat liegt $\operatorname{Kern}Q$ in dem vom $\mathfrak{w}*\mathscr{S}(N)$ aufgespannten abgeschlossenen Unterraum von $\mathscr{S}(N)$. Um dieses einzusehen, wähle man ein lineares Komplement $\mathfrak{v}$ zu $\mathfrak{w}$ in $\mathfrak{n}$. Identifiziert man $\mathscr{S}(N)$ mit $\mathscr{S}(\mathfrak{n})=\mathscr{S}(\mathfrak{v}\oplus\mathfrak{w})$, so geht $\operatorname{Kern}Q$ in

$$E:=\left\{f\in\mathscr{S}(\mathfrak{v}\oplus\mathfrak{w});\ \int_{\mathfrak{w}} f(X,Y)\,dY=0 \text{ für alle } X\in\mathfrak{v}\right\}$$

über. Und $\mathfrak{w}*\mathscr{S}(N)$ geht in $\mathfrak{w}*\mathscr{S}(\mathfrak{v}\oplus\mathfrak{w})$ über, wobei die Operatoren aus $\mathfrak{w}$ natürlich nur auf das zweite Argument wirken. Es ist leicht zu sehen, daß E als abgeschlossener Unterraum von $\mathscr{S}(\mathfrak{v}\oplus\mathfrak{w})$ von Tensoren $f=f_1\otimes f_2\in\mathscr{S}(\mathfrak{v})\otimes\mathscr{S}(\mathfrak{w})$ mit $\int_{\mathfrak{w}} f_2(Y)\,dY=0$ erzeugt wird. Nun läßt sich jede Schwartzsche Funktion φ auf $\mathbb{R}^n$ mit $\int_{\mathbb{R}^n}\varphi(x)\,dx=0$ in der Form $\varphi=\frac{\partial}{\partial x_1}\varphi_1+\ldots+\frac{\partial}{\partial x_n}\varphi_n$ mit passenden $\varphi_j\in\mathscr{S}(\mathbb{R}^n)$ schreiben. Daraus folgt sofort, daß E in dem Abschluß des linearen Erzeugnisses von $\mathfrak{w}*\mathscr{S}(\mathfrak{v}\oplus\mathfrak{w})$ enthalten ist.

2. Fall. Es gibt eine zweidimensionale zusammenhängende Untergruppe W von Z mit $\eta(W)=\mathbb{T}$, auf welcher T als eindimensionale Automorphismengruppe wirkt.

Wir verwenden dieselben Bezeichnungen wie im entsprechenden Fall beim Beweis von Satz 3 im Abschn. 2, insbesondere zerlegen wir T in $T=T'T_c$ mit endlichem Durchschnitt $F=T'\cap T_c$. Zur Abkürzung sei $\mathscr{S}=\mathscr{S}(T_c\times N, W,\eta)$, und man hat die Abbildung $J:\mathscr{S}(N)\to\mathscr{S}$ mit Bild $J=\mathscr{S}^F$. In einem ersten Schritt zeigen wir:

(1) Kern J ist enthalten im Abschluß des von $\mathfrak{p}*\mathscr{S}(N)$ erzeugten abgeschlossenen Unterraumes von $\mathscr{S}(N)$.

Genauer zeigen wir, daß $\operatorname{Kern}J$ im Abschluß des von $(\mathscr{U}\mathfrak{w}\cap\mathfrak{p})*\mathscr{S}(N)$ erzeugten abgeschlossenen Unterraumes enthalten ist. $\mathfrak{w}$ besitzt eine Basis W_1, W_2 derart, daß mit $D=W_1+iW_2\in\mathscr{U}\mathfrak{w}$ gilt: $D^t=\delta(t)D$, $\bar{D}^t=\overline{\delta(t)}\bar{D}$ für $t\in T$. Das Ideal $\Omega_\infty\cap\mathscr{U}\mathfrak{w}$ in $\mathscr{U}\mathfrak{w}$ wird von $\{X-ig(X);\ X\in\mathfrak{w}\}$ erzeugt. Setzt man $\lambda=g(W_1)^2$

$+g(W_2)^2$, so wird $\mathfrak{p}\cap\mathcal{U}\mathfrak{w}=\bigcap_{t\in T}\{\Omega_\infty\cap\mathcal{U}\mathfrak{w}\}^t$ von $D\bar{D}+\lambda$ erzeugt. Nun ist Kern J nichts anderes als Kern I, I wie im Abschn. 2, d.h.

$$I:\mathcal{S}(N)\to\mathcal{S}(T_c/F\times\mathfrak{v}),\qquad (I\varphi)(t,X)=\int_W\varphi(\exp(X)w^{t^{-1}})\eta(w)dw,$$

mit einem T-invarianten Komplement $\mathfrak{v}$ zu $\mathfrak{w}$ in $\mathfrak{n}$. Identifiziert man $\mathcal{S}(N)$ mit $\mathcal{S}(\mathfrak{v}\oplus\mathfrak{w})$, so geht $(\mathfrak{p}\cap\mathcal{U}\mathfrak{w})*\mathcal{S}(N)$ in $(\mathfrak{p}\cap\mathcal{U}\mathfrak{w})*\mathcal{S}(\mathfrak{v}\oplus\mathfrak{w})$ über, wobei die Operatoren in $\mathfrak{p}\cap\mathcal{U}\mathfrak{w}$ natürlich nur auf die zweite Variable wirken; explizit: für $P\in\mathfrak{w}$ und $f\in\mathcal{S}(\mathfrak{v}\oplus\mathfrak{w})$ ist

$$(P*f)(X,Y)=-\partial_P f(X,Y)=\left.\frac{d}{ds}\right|_{s=0} f(X,Y-sP).$$

Bei dieser Identifikation geht Kern $I=$ Kern J in

$$E:=\left\{f\in\mathcal{S}(\mathfrak{v}\oplus\mathfrak{w});\ \int_{\mathfrak{w}} f(X+Y^{t^{-1}})e^{ig(Y)}dY=0 \text{ für } X\in\mathfrak{v},\ t\in T_c\right\}$$

über. Da I ein Retrakt ist, wird E als topologischer Vektorraum erzeugt von Tensoren $f_1\otimes f_2$ mit $f_1\in\mathcal{S}(\mathfrak{v})$, $f_2\in\mathcal{S}(\mathfrak{w})$ und

$$\int_{\mathfrak{w}} f_2(Y^{t^{-1}})e^{ig(Y)}dY=0\quad \text{für alle}\quad t\in T_c.$$

Es genügt daher zu zeigen, daß

$$\left\{f\in\mathcal{S}(\mathfrak{w});\ \int_{\mathfrak{w}} f(Y^{t^{-1}})e^{ig(Y)}dY=0 \text{ für alle } t\in T_c\right\}$$

in $(D\bar{D}+\lambda)*\mathcal{S}(\mathfrak{w})$ enthalten ist. Diese Inklusion beweist man leicht mittels Fourierscher Transformation auf $\mathfrak{w}$; in der Tat stimmen die beiden Räume überein.

Wegen (1) und der im Abschn. 2 hergestellten Beziehung zwischen $R:\mathcal{S}(N)\to\mathcal{S}(T\times N\times N,U,\gamma)$ und $R'':\mathcal{S}\to\mathcal{S}(T'\times N\times N,U,\gamma)$ genügt es, das Folgende zu beweisen:

(2) $J(\mathfrak{p}*\mathcal{S}(N))$ ist total in $\mathcal{S}^F\cap\mathrm{Kern}(id\otimes R'')$.

Dazu definieren wir zunächst eine $\mathcal{U}\mathfrak{n}$-Wirkung auf $\mathcal{S}$ derart, daß $J:\mathcal{S}(N)\to\mathcal{S}$ eine $\mathcal{U}\mathfrak{n}$-lineare Abbildung wird. Natürlich operiert $\mathcal{U}\mathfrak{n}$ auf dem Quotienten $\mathcal{S}(N,W,\eta)$ von $\mathcal{S}(N)$; diese Wirkung ist nicht treu, sondern hat gerade das von $\Omega_\infty\cap\mathcal{U}\mathfrak{w}$ erzeugte Ideal als Kern. Für $u\in\mathcal{U}\mathfrak{n}$ und $\psi\in\mathcal{S}$ erklären wir nun

$$(u\psi)(t,x)=[u^t*\psi(t,-)](x).$$

Es ist leicht zu sehen, daß J damit $\mathcal{U}\mathfrak{n}$-linear wird. Daher ist die Behauptung (2) äquivalent zu: $\mathfrak{p}(\mathcal{S}^F)$ ist total in $\mathcal{S}^F\cap\mathrm{Kern}(id\otimes R'')$. Dazu beweisen wir erst einmal

(3) $\mathfrak{p}\mathcal{S}$ ist total in $\mathrm{Kern}(id\otimes R'')$.

Zu (3): Da R'' ein Retrakt ist, wird $\mathrm{Kern}(id\otimes R'')$ als topologischer Vektorraum von Tensoren $\psi_1\otimes\psi_2$ mit $\psi_1\in\mathcal{S}(T_c)$ und $\psi_2\in\mathrm{Kern}\,R''\subset\mathcal{S}(N,W,\eta)$ erzeugt. Nun ist $\mathcal{S}(N,W,\eta)$ ein Quotient von $\mathcal{S}(N/(\mathrm{Kern}\,\eta)_0)$. Setzt man $\mathfrak{p}':=\bigcap_{t\in T'}\Omega_\infty^t$, so ist laut

Induktionsannahme $\mathfrak{p}' * \mathscr{S}(N, W, \eta)$ total in Kern R''. Es reicht also zu zeigen, daß $\psi \otimes v * f$ für jedes $\psi \in \mathscr{S}(T_c)$, jedes $v \in \mathfrak{p}'$ und jedes $f \in \mathscr{S}(N, W, \eta)$ im Abschluß des von $\mathfrak{p}\mathscr{S}$ erzeugten Unterraumes liegt.

Nach Satz 2 läßt sich v in der Form $v = u + q$ mit $u \in \mathfrak{p}$ und $q \in (\mathscr{U}\mathfrak{w} \cap \Omega_\infty)\mathscr{U}\mathfrak{n}$ schreiben. Da $\mathfrak{p}$ in T_c-Eigenräume zerfällt, gibt es eine Darstellung $u = \sum_{j=1}^{n} u_j$ mit $u_j \in \mathfrak{p}$ und $u_j^t = \chi_j(t) u_j$ für gewisse Charaktere χ_j von T_c. Für jedes j liegt $u_j(x_j^{-1}\psi \otimes f)$ in $\mathfrak{p}\mathscr{S}$, und es gilt

$$\{u_j(\chi_j^{-1}\psi \otimes f)\}(t, x) = \psi(t)(u_j * f)(x),$$

also

$$\sum_{j=1}^{n} \{u_j(\chi_j^{-1}\psi \otimes f)\} = \psi \otimes u * f = \psi \otimes v * f.$$

(3)→(2): Für $\psi \in \mathscr{S}$ sei $\psi^{\#} \in \mathscr{S}$ durch $\psi^{\#}(t, x) = \frac{1}{|F|} \sum_{s \in F} \psi(ts, x^s)$ definiert; $\psi \to \psi^{\#}$ ist eine stetige Projektion von $\mathscr{S}$ auf $\mathscr{S}^F$. Man rechnet leicht nach, daß $\#$ mit der $\mathscr{U}\mathfrak{n}$-Modulstruktur auf $\mathscr{S}$ verträglich ist: $(u\psi)^{\#} = u(\psi^{\#})$. Wir haben zu zeigen, daß jedes f aus $\mathscr{S}^F \cap \mathrm{Kern}(id \otimes R'')$ durch Linearkombinationen von Elementen aus $\mathfrak{p}(\mathscr{S}^F)$ approximiert werden kann. Nun läßt sich f nach (3) durch $\sum_{j=1}^{n} u_j \psi_j$ mit $u_j \in \mathfrak{p}$, $\psi_j \in \mathscr{S}$ approximieren. Dann wird $f = f^{\#}$ durch

$$\left(\sum_{j=1}^{n} u_j \psi_j\right)^{\#} = \sum_{j=1}^{n} u_j(\psi_j^{\#}) \in \mathfrak{p}(\mathscr{S}^F)$$

approximiert.

Wie im Abschn. 2 können wir nun annehmen, daß T als endliche Automorphismengruppe auf $\mathfrak{z}$ und Z operiert. Auch hier wollen wir uns als nächstes überlegen, daß man dann ohne Beschränkung der Allgemeinheit annehmen darf, daß T trivial auf Z operiert. Dazu sei wieder $\mathfrak{w}$ ein T-irreduzibler Teilraum von $\mathfrak{z}$ mit $\eta(W) = \mathbb{T}$ und T' der gemeinsame Stabilisator in T der Elemente von $\mathfrak{w}$. Es sei $T = \bigcup_{j=1}^{n} t_j T'$ eine Zerlegung in Nebenklassen, $\eta_j(w) = \eta(w^{t_j})$ für $w \in W$. Auch $Q: \mathscr{S}(N) \to \bigoplus_{j=1}^{n} \mathscr{S}(N, W, \eta_j)$ und $V_j: \mathscr{S}(N, W, \eta_j) \to \mathscr{S}(N, W, \eta)$ seien wie im Abschn. 2. Des weiteren sei $\mathfrak{p}' = \bigcap_{t \in T'} \Omega_\infty^t$. Man überlegt sich leicht, daß Kern Q im Abschluß der linearen Hülle von $\mathfrak{p} * \mathscr{S}(N)$ enthalten ist, offenbar sogar im Abschluß der linearen Hülle von $(\mathscr{U}\mathfrak{w} \cap \mathfrak{p}) * \mathscr{S}(N)$. Es ist daher der von

$$\mathfrak{p} * Q(\mathscr{S}(N)) = \mathfrak{p} * \bigoplus_{j=1}^{n} \mathscr{S}(N, , W, \eta_j)$$

aufgespannte abgeschlossene Unterraum zu bestimmen. Transformiert man $\mathfrak{p} * \bigoplus_{j=1}^{n} \mathscr{S}(N, W, \eta_j)$ mittels $\bigoplus_{j=1}^{n} V_j$ in $\bigoplus_{j=1}^{n} \mathscr{S}(N, W, \eta)$, so erhält man

$$E := \{(u^{t_1} * f_1, \ldots, u^{t_n} * f_n);\ u \in \mathfrak{p},\ f_j \in \mathscr{S}(N, W, \eta)\}.$$

Nun wird aber $\mathscr{S}(N, W, \eta)$ von $(\mathscr{U}\mathfrak{w} \cap \Omega_\infty)\mathscr{U}\mathfrak{n}$ annulliert. Da die Charaktere $\eta_1, \ldots, \eta_n$ paarweise verschieden sind, folgt aus dem chinesischen Restsatz, daß die Abbildung

$$\mathscr{U}\mathfrak{w} \to \bigoplus_{j=1}^{n} \mathscr{U}\mathfrak{w}/\mathscr{U}\mathfrak{w} \cap \Omega_\infty (\cong \mathbb{C}^n), \qquad u \to (u^{t_1}, \ldots, u^{t_n})$$

surjektiv ist. Das gilt dann auch für die entsprechende Abbildung $\mathscr{U}\mathfrak{n} \to \bigoplus_{j=1}^{n} \mathscr{U}\mathfrak{n}/(\mathscr{U}\mathfrak{w} \cap \Omega_\infty)\mathscr{U}\mathfrak{n}$. Man weist leicht nach, daß das Bild von $\mathfrak{p}$ unter dieser Abbildung gerade $\bigoplus_{j=1}^{n} \mathfrak{p}'/(\mathscr{U}\mathfrak{w} \cap \Omega_\infty)\mathscr{U}\mathfrak{n}$ ist. Also ist

$$E = \{(u_1 * f_1, \ldots, u_n * f_n);\ u_j \in \mathfrak{p}',\ f_j \in \mathscr{S}(N, W, \eta)\}.$$

Nimmt man an, daß $\mathfrak{p}' * \mathscr{S}(N)$ total in $\Omega' = \bigcap_{t \in T'} \Omega^t$ ist, so folgt leicht, daß $\mathfrak{p} * \mathscr{S}(N)$ in Ω total ist.

3. Fall. $\dim \mathfrak{z} = 1$, $\eta(Z) = \mathbb{T}$, T operiert trivial auf $\mathfrak{z}$. Es gibt ein abelsches T-invariantes Ideal $\mathfrak{w}$, welches $\mathfrak{z}$ echt umfaßt.

Genauso wie im entsprechenden Fall im Abschn. 2 gibt es dann ein abelsches Ideal $\mathfrak{w}$, für welches zusätzlich gilt, daß $\mathfrak{w}/\mathfrak{z}$ zentral in $\mathfrak{n}/\mathfrak{z}$ und $\mathfrak{w}/\mathfrak{z}$ ein irreduzibler T-Modul ist. Wie dort sei $\mathfrak{h}$ der Zentralisator von $\mathfrak{w}$ in $\mathfrak{n}$, $\mathfrak{w} = \mathfrak{y} \oplus \mathfrak{z}$, $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{h}$ und $\mathfrak{h}' = \mathfrak{h}/\mathfrak{y}$.

Wir wollen im folgenden annehmen, daß $\mathfrak{y}$ (und dann auch $\mathfrak{v}$) zweidimensional ist, da in diesem Falle gewisse Komplikationen auftreten. Wenn man den zweidimensionalen Fall beherrscht, ist klar, wie der eindimensionale zu behandeln ist.

Die Komplikationen rühren daher, daß $\mathfrak{v}$ im allgemeinen keine (abelsche) Algebra zu sein braucht, was dazu führt, daß der Transport der $\mathscr{U}\mathfrak{n}$-Modulstruktur von $\mathscr{S}(N, Z, \eta)$ auf $\mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta)$ längs $\mathscr{F}$ zu unnötig komplizierten Formeln führt. Wir betrachten stattdessen zwei entsprechende „eindimensionale Transformationen" und zeigen, daß sich deren Kompositum in genau angebbarer Weise von $\mathscr{F}$ unterscheidet.

Dazu wählen wir irgendeine Zerlegung $\mathfrak{y} = \mathfrak{y}_1 \oplus \mathfrak{y}_2$ von $\mathfrak{y}$ in eindimensionale Unterräume. Dieser entspricht einer Zerlegung $\mathfrak{v} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ mit $[\mathfrak{v}_i, \mathfrak{y}_j] = \delta_{ij}\mathfrak{z}$. Dann sei $\mathfrak{h}_2 = \mathfrak{h} + \mathfrak{v}_2$ und $\mathfrak{h}_2' = \mathfrak{h}_2/\mathfrak{y}_1$. Wie im Abschn. 2 ist durch

$$(\mathscr{F}_1 \varphi)(a_1, b_1, h_2') = \int_{\mathfrak{y}_1} dr\, \varphi(\exp(a_1) h_2 \exp r \exp(-b_1))$$

ein Isomorphismus $\mathscr{F}_1 : \mathscr{S}(N, Z, \eta) \to \mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times H_2', Z, \eta)$ definiert. Entsprechend ist $\mathscr{S}(H_2', Z, \eta)$ zu $\mathscr{S}(\mathfrak{v}_2 \times \mathfrak{v}_2 \times H', Z, \eta)$ isomorph und dann, durch Tensorieren mit der Identität, $\mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times H_2', Z, \eta)$ zu $\mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times \mathfrak{v}_2 \times \mathfrak{v}_2 \times H', Z, \eta)$ isomorph. Explizit ist ein solcher Isomorphismus $\mathscr{F}_2$ durch

$$(\mathscr{F}_2 \psi)(a_1, b_1, a_2, b_2, h') = \int_{\mathfrak{y}_2} dr\, \psi(a_1, b_1, \exp(a_2) h \exp r \exp(-b_2))$$

für $\psi \in \mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times H_2', Z, \eta)$ gegeben (dabei ist natürlich h irgendein Urbild von h' unter $H/Y_1 \to H' = H/Y$).

Für $a_1 \in \mathfrak{v}_1$ und $a_2 \in \mathfrak{v}_2$ sei $s(a_1, a_2) \in H$ durch

$$\exp(a_1)\exp(a_2) = \exp(a_1 + a_2) s(a_1, a_2)$$

definiert; mit $s'(a_1, a_2)$ bezeichnen wir das Bild von $s(a_1, a_2)$ in H'. Damit erklären wir

$$S: \mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta) \to \mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times \mathfrak{v}_2 \times \mathfrak{v}_2 \times H', Z, \eta)$$

durch

$$(Sf)(a_1, b_1, a_2, b_2, h') = f(a_1 + a_2, b_1 + b_2, s'(a_1, a_2) h' s'(b_1, b_2)^{-1})$$

und behaupten, daß das Diagramm

$$\begin{array}{ccccc} \mathscr{S}(N, Z, \eta) & \xrightarrow{\mathscr{F}_1} & \mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times H'_2, Z, \eta) & \xrightarrow{\mathscr{F}_2} & \mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times \mathfrak{v}_2 \times \mathfrak{v}_2 \times H', Z, \eta) \\ & {\scriptstyle\mathscr{F}} \searrow & & \nearrow {\scriptstyle S} & \\ & & \mathscr{S}(\mathfrak{v} \times \mathfrak{v} \times H', Z, \eta) & & \end{array}$$

kommutiert.

Für $\varphi \in \mathscr{S}(N, Z, \eta)$ ist

$$\begin{aligned} &((\mathscr{F}_2 \circ \mathscr{F}_1)\varphi)(a_1, b_1, a_2, b_2, h') \\ &= \int_{\mathfrak{y}_2} dr_2 (\mathscr{F}_1 \varphi)(a_1, b_1, \exp(a_2) h \exp r_2 \exp(-b_2)) \\ &= \int_{\mathfrak{y}_2} dr_2 \int_{\mathfrak{y}_1} dr_1 \, \varphi(\exp(a_1) \exp(a_2) h \exp r_2 \\ &\quad \cdot \exp(-b_2) \exp r_1 \exp(-b_1)) \\ &= \int_{\mathfrak{y}_2} \int_{\mathfrak{y}_1} dr_1 \, dr_2 \, \varphi(\exp(a_1) \exp(a_2) h \exp(r_1 + r_2) \\ &\quad \cdot \exp(-b_2) \exp(-b_1)), \end{aligned}$$

da $\exp(-b_2)$ und $\exp r_1$ kommutieren, und andererseits

$$\begin{aligned} &\{(\mathscr{S} \circ \mathscr{F})\varphi\}(a_1, b_1, a_2, b_2, h') \\ &= (\mathscr{F}\varphi)(a_1 + a_2, b_1 + b_2, s'(a_1, a_2) h' s'(b_1, b_2)^{-1}) \\ &= \int_{\mathfrak{y}_1} \int_{\mathfrak{y}_2} dr_1 \, dr_2 \, \varphi(\exp(a_1 + a_2) s(a_1, a_2) h s(b_1, b_2)^{-1} \\ &\quad \cdot \exp(r_1 + r_2) \exp(-b_1 - b_2)). \end{aligned}$$

Da $\exp(r_1 + r_2)$ mit $s(b_1, b_2) \in H$ vertauscht und

$$\exp(a_1 + a_2) s(a_1, a_2) = \exp(a_1) \exp(a_2)$$

sowie

$$s(b_1, b_2)^{-1} \exp(-b_1 - b_2) = \exp(-b_2) \exp(-b_1)$$

gelten, folgt $\mathscr{F}_2 \circ \mathscr{F}_1 = S \circ \mathscr{F}$ wie behauptet.

Zu dem Tripel (N, T, Ω) gehören Unteralgebren $\mathfrak{a}$, $\mathfrak{b}$ und $\mathfrak{c}$ von $\mathfrak{n}$ sowie ein Funktional g auf $\mathfrak{n}$, wie im Abschn. 2 konstruiert. Insbesondere können wir annehmen, daß $\mathfrak{y}$ in $\mathfrak{a}$ gelegen ist, so daß g auf $\mathfrak{y}$ verschwindet und daher ein Funktional g' auf $\mathfrak{h}'$ definiert. Zu dem Paar (g, C) bilden wir die Abbildung R, ebenso R' zu (g', C') mit $C' = C/Y$. Kern R ist nichts anderes als $\mathscr{S}(N) \cap \bigcap_{t \in T} \Omega^t$, und Kern R' ist gleich $\mathscr{S}(H') \cap \bigcap_{t \in T} \Lambda'^t$ wobei Λ' das zu g' gehörige maximale Ideal in $L^1(H')$ bezeichnet. Wir fassen nun R und R' als Abbildungen auf den Quotienten $\mathscr{S}(N, Z, \eta)$ und $\mathscr{S}(H', Z, \eta)$ von $\mathscr{S}(N)$ bzw. $\mathscr{S}(H')$ auf. Auf Grund der Untersuchungen im Abschn. 2 ist $\mathscr{F}(\operatorname{Kern} R)=\operatorname{Kern}(id\otimes R')$. Da R' ein Retrakt ist, ist $\operatorname{Kern}(id\otimes R')$ der Abschluß von $\mathscr{S}(\mathfrak{v}\times\mathfrak{v})\otimes\operatorname{Kern} R'$. Nun ist $\operatorname{Kern}(id\otimes R')$ invariant unter dem Automorphismus $f\to\tilde{f}$ von $\mathscr{S}(\mathfrak{v}\times\mathfrak{v}\times H', Z, \eta)$, welcher durch

$$\tilde{f}(a_1+a_2, b_1+b_2, h')=f(a_1+a_2, b_1+b_2, s'(a_1,a_2)^{-1}h's'(b_1,b_2)^{-1})$$

mit $a_1, b_1\in\mathfrak{v}_1$, $a_2, b_2\in\mathfrak{v}_2$ und $h'\in H'$ gegeben ist; denn $\operatorname{Kern} R'$ ist invariant unter Links- und Rechtstranslationen. Also ist $\operatorname{Kern}(id\otimes R')$ auch der Abschluß von

$$\{\tilde{f};\ f\in\mathscr{S}(\mathfrak{v}\times\mathfrak{v})\otimes\operatorname{Kern} R'\}.$$

Dann ist aber

$$S\mathscr{F}(\operatorname{Kern} R) = S(\operatorname{Kern} id\otimes R')$$

der Abschluß von

$$\{S\tilde{f};\ f\in\mathscr{S}(\mathfrak{v}\times\mathfrak{v})\otimes\operatorname{Kern} R'\}=\mathscr{S}(\mathfrak{v}_1\times\mathfrak{v}_1\times\mathfrak{v}_2\times\mathfrak{v}_2)\otimes\operatorname{Kern} R'.$$

Um zu zeigen, daß $\mathfrak{p}*\mathscr{S}(N)$ total in $\bigcap_{t\in T}\Omega^t\cap\mathscr{S}(N)$ ist, reicht es also zu beweisen, daß die Elemente aus dem dichten Teil $\mathscr{S}(\mathfrak{v}_1\times\mathfrak{v}_1\times\mathfrak{v}_2\times\mathfrak{v}_2)\otimes\operatorname{Kern} R'$ von $S\,\mathscr{F}(\operatorname{Kern} R)=\mathscr{F}_2\mathscr{F}_1(\operatorname{Kern} R)$ durch Linearkombinationen von Elementen aus $\mathscr{F}_2\mathscr{F}_1(\mathfrak{p}*\mathscr{S}(N, Z, \tilde{\eta}))$ approximiert werden können. Da auf Grund der Induktionsannahme $\mathfrak{p}'*\mathscr{S}(H', Z, \eta)$ total in $\operatorname{Kern} R'$ ist $\left(\text{mit } \mathfrak{p}'=\bigcap_{t\in T}\Lambda_\infty^{\prime t}\right)$, braucht man sich nur davon zu überzeugen, daß $\mathscr{S}(\mathfrak{v}_1\times\mathfrak{v}_1\times\mathfrak{v}_2\times\mathfrak{v}_2)\otimes\mathfrak{p}'*\mathscr{S}(H', Z, \eta)$ in $\mathscr{F}_2\mathscr{F}_1(\mathfrak{p}*\mathscr{S}(N, Z, \eta))$ enthalten ist.

Dazu zeigen wir zunächst, daß zu jedem $u'\in\mathscr{U}\mathfrak{h}'$ ein $u\in\mathscr{U}\mathfrak{h}$ existiert mit $(\mathscr{F}_1\mathscr{F}_2)(u*\varphi)=u'*(\mathscr{F}_2\mathscr{F}_1)\varphi$ für $\varphi\in\mathscr{S}(N, Z, \tilde{\eta})$, wobei $u'*\psi$ für $\psi\in\mathscr{S}(\mathfrak{v}_1\times\mathfrak{v}_1\times\mathfrak{v}_2\times\mathfrak{v}_2\times H', Z, \eta)$ natürlich die Anwendung von u' auf die letzte Variable bedeutet. Im übrigen ist u modulo dem von $\{z-ig(z);\ z\in\mathfrak{z}\}$ erzeugten Ideal eindeutig bestimmt. All das wird leicht aus den Eigenschaften der „reduzierenden Quadrupel", siehe [9, 4.7.7], folgen. Die Konstruktion von u erfolgt in zwei Schritten. Man zeigt zunächst, daß zu u' ein $u_2\in\mathscr{U}(\mathfrak{h}/\mathfrak{y}_1)$ mit $\mathscr{F}_2(u_2*\psi)=u'*\mathscr{F}_2\psi$ für alle $\psi\in\mathscr{S}(\mathfrak{v}_1\times\mathfrak{v}_1\times H_2', Z, \eta)$ existiert und dann, daß zu (jedem) $u_2\in\mathscr{U}(\mathfrak{h}/\mathfrak{y}_1)$ ein $u\in\mathscr{U}\mathfrak{h}$ mit $\mathscr{F}_1(u*\varphi)=u_2*\mathscr{F}_1\varphi$ für alle $\varphi\in\mathscr{S}(N, Z, \eta)$ existiert.

Es wird nur die Konstruktion von u_2 vorgeführt; die Konstruktion von u zu gegebenem u_2 verläuft völlig analog. Dazu sei $z\in\mathfrak{z}$ mit $g(z)=1$ gewählt, weiter seien $x_2\in\mathfrak{v}_2\subset\mathfrak{h}+\mathfrak{v}_2/\mathfrak{y}_1=\mathfrak{h}_2'$ und $y_2\in\mathfrak{y}/\mathfrak{y}_1\cong\mathfrak{y}_2\subset\mathfrak{h}_2'$ mit $[x_2, y_2]=z$ gewählt. Der Zentralisator von y_2 in $\mathfrak{h}_2'$ ist gerade $\mathfrak{h}/\mathfrak{y}_1$, und $(x_2, y_2, z, \mathfrak{h}/\mathfrak{y}_1)$ ist ein reduzierendes Quadrupel für $\mathfrak{h}_2'$. Die Ergebnisse aus [9] werden später auf das reduzierende Quadrupel $(-x_2, iy_2, -iz, \mathfrak{h}/\mathfrak{y}_1\otimes\mathbb{C})$ von $\mathfrak{h}_2'\otimes\mathbb{C}$ angewendet. Für $v\in\mathfrak{h}_2'$ läßt sich $\mathscr{F}_2(v*\psi)$ leicht direkt durch v und $\mathscr{F}_2\psi$ ausdrücken. Ist $v=x_2$, so gilt:

$$\begin{aligned}
&\mathscr{F}_2(x_2*\psi)(a_1, b_1, a_2, b_2, h')\\
&\quad=\int_{\mathfrak{y}_2} dr_2(x_2*\psi)(a_1, b_1, \exp(a_2)h\exp r_2\exp(-b_2))\\
&\quad=\frac{d}{dt}\bigg|_{t=0}\int_{\mathfrak{y}_2} dr_2\psi(a_1, b_1, \exp(-tx_2)\exp(a_2)h\exp r_2\exp(-b_2))\\
&\quad=\frac{d}{dt}\bigg|_{t=0}\mathscr{F}_2\psi(a_1, b_1, a_2-tx_2, b_2, h'),
\end{aligned}$$

d. h.

$$\mathcal{F}_2(x_2 * \psi) = -\frac{\partial}{\partial x_2}\mathcal{F}_2\psi .$$

Ist $v \in \mathfrak{h}/\mathfrak{y}_1 \subseteqq \mathfrak{h}'_2$, so ist

$$\begin{aligned}&\mathcal{F}_2(v * \psi)(a_1, b_1, a_2, b_2, h') \\ &= \int_{\mathfrak{y}_2} dr_2 \frac{d}{dt}\Big|_{t=0} \psi(a_1, b_1, \exp(-tv)\exp(a_2)h\exp r_2 \exp(-b_2)) .\end{aligned}$$

Nun ist

$$\exp(-tv)\exp(a_2) = \exp(a_2)\exp\left(-t\sum_{n=0}^{\infty}\frac{1}{n!}ad(-a_2)^n(v)\right).$$

Mit den Setzungen $a_2 = \alpha x_2$ und

$$v_\alpha = \sum_{n=0}^{\infty}\frac{1}{n!}ad(-a_2)^n(v) = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}ad(-x_2)^n(v) \in \mathfrak{h}/\mathfrak{y}_1$$

erhält man

$$\begin{aligned}&\mathcal{F}_2(v * \psi)(a_1, b_1, a_2, b_2, h') \\ &= \frac{d}{dt}\Big|_{t=0} \int_{\mathfrak{y}_2} dr_2 \psi(a_1, b_1, \exp(a_2)\exp(-tv_\alpha)h\exp r_2 \exp(-b_2)) \\ &= \frac{d}{dt}\Big|_{t=0} (\mathcal{F}_2\psi)(a_1, b_1, a_2, b_2, \exp(-tv_\alpha)'h') \\ &= \{v'_\alpha * \mathcal{F}_2\psi(a_1, b_1, a_2, b_2, -)\}(h'),\end{aligned}$$

d. h. v_α wirkt nur auf die letzte Variable, die Wirkung hängt aber von der dritten Variablen ab; mit v'_α ist natürlich das Bild von v_α unter $\mathfrak{h}/\mathfrak{y}_1 \to h/\mathfrak{y}$ bezeichnet. Wegen $g(z) = 1$ ist $z * \psi = i\psi$. Durch Spezialisieren auf $v = y_2$ ergibt die obige Formel (wegen $y'_2 = 0$)

$$\mathcal{F}_2(y_2 * \psi)(a_1, b_1, a_2, b_2, h') = -i\alpha\mathcal{F}_2\psi(a_1, b_1, a_2, b_2, h') .$$

Bezeichnet man, analog zu [9, 4.7.8], die durch $-x_2$ induzierte Derivation auf $\mathcal{U}(\mathfrak{h}/\mathfrak{y}_1)$ mit δ, so lehren die dortigen Rechnungen zusammen mit den obigen Formeln: Ist $v \in \mathcal{U}(\mathfrak{h}/\mathfrak{y}_1)$ irgendein Urbild von u' unter der kanonischen Abbildung $\mathcal{U}(\mathfrak{h}/\mathfrak{y}_1) \to \mathcal{U}(\mathfrak{h}/\mathfrak{y})$, so ist durch

$$u_2 := \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\delta^m v(iy_2)^m$$

ein Element in $\mathcal{U}(\mathfrak{h}/\mathfrak{y}_1)$ mit $\mathcal{F}_2(u_2 * \psi) = u' * \mathcal{F}_2\psi$ definiert.

Folglich gibt es zu jedem $u' \in \mathfrak{p}' = \bigcap_{t \in T} \Lambda^{\prime t}_\infty \lhd \mathcal{U}(\mathfrak{h}/\mathfrak{y})$ ein $u \in \mathcal{U}\mathfrak{h}$ mit $(\mathcal{F}_2\mathcal{F}_1)(u * \varphi) = u' * (\mathcal{F}_2\mathcal{F}_1)\varphi$ für alle $\varphi \in \mathcal{S}(N, Z, \eta)$. Ein solches u liegt dann notwendigerweise in $\mathfrak{p}$; denn $u \in \mathfrak{p}$ ist äquivalent zu

$$u * \mathcal{S}(N, Z, \eta) \subseteqq \operatorname{Kern} R$$

und zu

$$(\mathscr{F}_2\mathscr{F}_1)(u * \mathscr{S}(N, Z, \eta))$$
$$= u' * \mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times \mathfrak{v}_2 \times \mathfrak{v}_2, H', Z, \eta) \subseteqq \mathscr{F}_2\mathscr{F}_1(\mathrm{Kern}\, R),$$

was wegen $u' \in \mathfrak{p}'$ wahr ist. Daher ist

$$\mathscr{S}(\mathfrak{v}_1 \times \mathfrak{v}_1 \times \mathfrak{v}_2 \times \mathfrak{v}_2) \otimes \mathfrak{p}' * \mathscr{S}(H', Z, \eta)$$

in

$$\mathscr{F}_2\mathscr{F}_1(\mathfrak{p} * \mathscr{S}(N, Z, \eta))$$

enthalten. Wie wir oben gesehen haben, folgt daraus die Behauptung des Satzes auch in diesem Fall.

4. Fall. $\mathfrak{n}$ ist isomorph zu einer Heisenbergschen Algebra, $\eta(Z) = \mathbb{T}$, und T operiert trivial auf $\mathfrak{z}$.

Dann ist $\mathfrak{p} = \Omega_\infty$ und $\mathscr{S}(N) \cap \bigcap_{t \in T} \Omega^t = \Omega \cap \mathscr{S}(N)$, d. h., die Wirkung der Gruppe T ist für die Aussage des Satzes irrelevant. Man kann also $T = \{1\}$ annehmen. Dann befindet man sich aber wieder im 3. Fall, wenn man von dem trivialen Fall eines eindimensionalen $\mathfrak{n}$ absieht.

4. Kerne von $T \ltimes M$-Bahnen sind infinitesimal bestimmt

Als erstes wird gezeigt, daß jedenfalls die Kerne von T-Bahnen infinitesimal bestimmt sind. Der folgende Beweis basiert auf derselben Idee wie der Beweis für den entsprechenden Satz in [21].

Satz 5. *Seien N eine einfachzusammenhängende zusammenhängende nilpotente Liesche Gruppe mit Liescher Algebra $\mathfrak{n}$ und T eine kompakte abelsche Liesche Gruppe. T operiere stetig und homomorph durch Automorphismen auf N, $(t, x) \to x^t$. Ω sei ein maximales Ideal in $L^1(N)$, Ω_∞ sei das entsprechende Ideal in $\mathscr{U}\mathfrak{n}$. Weiter sei $\mathfrak{p} = \bigcap_{t \in T} \Omega_\infty^t$. Dann ist $\mathfrak{p} * \mathscr{D}(N)$ total in $\bigcap_{t \in T} \Omega^t$ (in der L^1-Norm).*

Bemerkung. Der Beweis wird zeigen, daß ein entsprechender Satz auch für Beurlingsche Algebren $L^1_w(N)$ richtig ist, d. h. $\mathfrak{p} * \mathscr{D}(N)$ liegt dicht in $L^1_w(N) \cap \bigcap_{t \in T} \Omega^t$, sofern man voraussetzt, daß das Gewicht w höchstens polynomial wächst. Diese Voraussetzung stellt sicher, daß $\mathscr{S}(N)$ in $L^1_w(N)$ enthalten ist.

Beweis. Durch Übergang zu einer Untergruppe von T, falls erforderlich, können wir ohne Beschränkung der Allgemeinheit annehmen, daß der Stabilisator T_Ω von Ω in T endlich ist. Da $\mathscr{D}(N)$ approximierende Einsen für $L^1(N)$ enthält und da $\mathfrak{p}$ unter $Ad(N)$, ausgedehnt auf $\mathscr{U}\mathfrak{n}$, invariant ist, ist

$$\langle \mathfrak{p} * \mathscr{D}(N) \rangle^- = \langle \mathscr{D}(N) * \mathfrak{p} \rangle^- = \langle \mathscr{D}(N) * \mathfrak{p} * \mathscr{D}(N) \rangle^- = \langle \mathfrak{p} * \mathscr{S}(N) \rangle^-$$
$$= \langle \mathscr{S}(N) * \mathfrak{p} \rangle^- = \langle \mathscr{S}(N) * \mathfrak{p} * \mathscr{S}(N) \rangle^- .$$

Es genügt also zu zeigen, daß $\mathfrak{p} * \mathscr{S}(N)$ total in $\bigcap_{t \in T} \Omega^t$ ist.

Zu den Daten T, N, Ω seien $g \in \mathfrak{n}^*$ sowie Unteralgebren $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ von $\mathfrak{n}$ gemäß Satz 3 gewählt. Dann haben wir den Retrakt $R: \mathscr{S}(N) \to \mathscr{S}(T \times N \times N, U, \gamma)^{T_\Omega}$ mit $\operatorname{Kern} R = \mathscr{S}(N) \cap \bigcap_{t \in T} \Omega^t$. Sei nun ξ ein beschränktes lineares Funktional auf $L^1(N)$ mit $\xi(\mathfrak{p} * \mathscr{S}(N)) = 0$. Wir haben zu zeigen, daß auch $\xi\left(\bigcap_{t \in T} \Omega^t\right) = 0$ ist. Für jeden Charakter μ von T sei

$$L^1(N)^\mu = \{f \in L^1(N);\ f^t = \mu(t) f \text{ für alle } t \in T\}$$

und $\mathscr{S}(N)^\mu = \mathscr{S}(N) \cap L^1(N)^\mu$; $\mathscr{S}(N)^\mu$ liegt offensichtlich dicht in $L^1(N)^\mu$. Wir fixieren nun erst einmal $\mu \in \hat{T}$ sowie $p, q \in \mathscr{S}(N)^T = \mathscr{S}(N)^1$ und zeigen, daß das regularisierte Funktional $\varphi \to \xi(p * \varphi * q)$ sogar C^*-stetig auf $L^1(N)^\mu$ ist. Durch Einschränkung definiert ξ ein stetiges lineares Funktional auf $\mathscr{S}(N)$, und nach Satz 4 faktorisiert ξ durch R, es gibt also ein stetiges lineares Funktional $\tilde{\xi}$ auf $\mathscr{S}(T \times N \times N, U, \gamma)^{T_\Omega}$ mit $\xi(\varphi) = \tilde{\xi}(R\varphi)$. Insbesondere interessieren uns die Werte von $\tilde{\xi}$ auf $R(\mathscr{S}(N)^\mu)$; man beachte, daß mit φ auch $p * \varphi * q$ in $\mathscr{S}(N)^\mu$ liegt. $R(\mathscr{S}(N)^\mu)$ ist offenbar nichts anderes als

$$\begin{aligned}\mathscr{S}(T \times N \times N, U, \gamma)^\mu := \{\psi \in \mathscr{S}(T \times N \times N, U, \gamma);\ & \psi(t, x, y) = \mu(t)\psi(1, x, y) \\ & \text{sowie } \psi(1, x^{t_0}, y^{t_0}) = \mu(t_0)^{-1}\psi(1, x, y) \\ & \text{für alle } x, y \in N,\ t \in T,\ t_0 \in T_\Omega\}.\end{aligned}$$

Definieren wir, für späteren Gebrauch gleich etwas allgemeiner,

$$\mathscr{S}(T \times N \times N, U, \gamma) \to \mathscr{S}(N \times N, U, \gamma), \qquad \psi \to \psi_1,$$

durch $\psi_1(x, y) = \psi(1, x, y)$, so liefert diese Abbildung einen Isomorphismus von $\mathscr{S}(T \times N \times N, U, \gamma)^\mu$ auf einen abgeschlossenen Teilraum von $\mathscr{S}(N \times N, U, \gamma)$.

Nun wählen wir eine Polarisierung $\mathfrak{h}$ für g mit $\mathfrak{c} \subset \mathfrak{h} \subset \mathfrak{b}$; es gilt natürlich $\dim \mathfrak{h}/\mathfrak{c} = \dim \mathfrak{b}/\mathfrak{h}$, $\mathfrak{h}$ ist im wesentlichen eine Polarisierung für g auf der Heisenbergschen Algebra $\mathfrak{b}/\mathfrak{a}$. Mit H sei die entsprechende Untergruppe von N bezeichnet, $\chi_g: H \to \mathbb{T}$ sei definiert durch $\chi_g(\exp X) = e^{ig(X)}$ für $X \in \mathfrak{h}$ (dies ist eine Fortsetzung des ursprünglichen $\chi_g: C \to \mathbb{T}$). $\mathscr{S}(N \times N, U, \gamma)$ ist isomorph zu $\mathscr{S}(N \times N, H \times H, \chi_g \times \bar{\chi}_g)$ via

$$(\mathscr{F}\psi)(x, y) := \int_{H/C} d\dot{a}\, \chi_g(a)\psi(xa, y)$$

für $\psi \in \mathscr{S}(N \times N, U, \gamma)$. Die Umkehrabbildung ist, bis auf eine Konstante (die von der Normalisierung der invarianten Maße abhängt), gegeben durch

$$(\mathscr{G}\varphi)(x, y) = \int_{B/H} d\dot{b}\, \varphi(xb, yb).$$

Diese Isomorphismen stellen, nebenbei bemerkt, den Zusammenhang zwischen der hier behandelten Beschreibung von $\mathscr{S}(N)/\mathscr{S}(N) \cap \Omega$ und der früher von Howe in [14] gegebenen her.

Wir beweisen exemplarisch, daß $(\mathscr{F} \circ \mathscr{G})\varphi = \varphi$ für $\varphi \in \mathscr{S}(N \times N, H \times H, \chi_g \times \bar{\chi}_g)$ ist. Der Beweis für $\mathscr{G} \circ \mathscr{F} = id$ verläuft ähnlich (in der Tat ist er sogar etwas einfacher) und beruht ebenso auf der Fourierschen Inversionsformel für Vektorgruppen.

Nach Definition ist

$$\{(\mathscr{F} \circ \mathscr{G})\varphi\}(x, y) = \int_{H/C} d\dot{a}\, \chi_g(a) \int_{B/H} d\dot{b}\, \varphi(xab, yb).$$

Nun ist

$$\varphi(xab, yb) = \varphi(xbb^{-1}ab, yb) = \chi_g(b^{-1}a^{-1}b)\varphi(xb, yb).$$

Definieren wir bei festen x, y die Funktion f auf $\dot{B} := B/H$ durch $f(\dot{b}) = \varphi(xb, yb)$, so ist also

$$\{(\mathscr{F} \circ \mathscr{G})\varphi\}(x, y) = \int_{H/C} d\dot{a} \int_{\dot{B}} d\dot{b}\, \chi_g(a)\chi_g(\dot{b}^{-1}a^{-1}\dot{b}) f(\dot{b}).$$

Erklärt man für $\dot{a} \in H/C$ die Funktion $\tau_{\dot{a}} : \dot{B} \to \mathbb{T}$ durch $\tau_{\dot{a}}(\dot{b}) = \chi_g(a)\chi_g(\dot{b}^{-1}a^{-1}\dot{b})$, so ist $\tau_{\dot{a}}$ ein Charakter von $\dot{B}$ und $\dot{a} \to \tau_{\dot{a}}$ ist ein Isomorphismus von H/C auf die Charaktergruppe von $\dot{B}$. Die Homomorphie von $\tau_a : \dot{B} \to \mathbb{T}$ ist gleichbedeutend mit

$$\chi_g(a)\chi_g(b^{-1}a^{-1}b)\chi_g(a)\chi_g(d^{-1}a^{-1}d) = \chi_g(a)\chi_g(d^{-1}b^{-1}a^{-1}bd)$$

für alle $b, d \in B$ und $a \in H$ und mit

$$\chi_g(b^{-1}a^{-1}b)\chi_g(a) = \chi_g(d^{-1}b^{-1}a^{-1}bd)\chi_g(d^{-1}ad).$$

Nun ist

$$\chi_g(d^{-1}b^{-1}a^{-1}bd)\chi_g(d^{-1}ad) = \chi_g(d^{-1}b^{-1}a^{-1}bad).$$

Aber $b^{-1}a^{-1}ba$ ist zentral in B/A. Also gilt

$$\chi_g(d^{-1}b^{-1}a^{-1}bad) = \chi_g(b^{-1}a^{-1}ba) = \chi_g(b^{-1}a^{-1}b)\chi_g(a),$$

wie behauptet. Es ist nicht schwer zu sehen, daß $\dot{a} \to \tau_{\dot{a}}$ homomorph in $\dot{a}$ und daß $\dot{a} \to \tau_{\dot{a}}$ injektiv ist. Aus Dimensionsgründen ist dann $\dot{a} \to \tau_{\dot{a}}$ ein Isomorphismus. Die Fouriersche Inversionsformel für die Vektorgruppe $\dot{B} = B/H$ liefert

$$\{(\mathscr{F} \circ \mathscr{G})\varphi\}(x, y) = \int_{H/C} d\dot{a} \int_{\dot{B}} d\dot{b}\, \tau_{\dot{a}}(\dot{b}) = f(\dot{e}) = \varphi(x, y).$$

Weiter ist $\mathscr{S}(N \times N, H \times H, \chi_g \times \bar{\chi}_g)$ isomorph zu $\mathscr{S}(\mathbb{R}^{2n})$ mit $n = \dim N/H$. Einen solchen, sehr willkürlichen Isomorphismus kann man wie folgt konstruieren, vgl. [14] und den Anfang vom Abschn. 2. Für eine passende Basis $X_1, \ldots, X_n$ in einem passenden linearen Komplement von $\mathfrak{h}$ in $\mathfrak{n}$ ist die Abbildung

$$\mathbb{R}^n \times H \to N, \qquad (s_1, \ldots, s_n, h) \to \exp(s_n X_n) \cdot \ldots \cdot \exp(s_1 X_1) h$$

ein Diffeomorphismus und für integrierbare Funktionen ψ auf N/H gilt

$$\int_{N/H} \psi(\dot{\alpha}) d\dot{\alpha} = \int_{\mathbb{R}^n} \psi(\exp(s_n X_n) \cdot \ldots \cdot \exp(s_1 X_1)) ds_1 \cdot \ldots \cdot ds_n$$

bis auf eine feste positive Konstante. Man definiert dann

$$\mathscr{K} : \mathscr{S}(N \times N, H \times H, \chi_q \times \bar{\chi}_q) \to \mathscr{S}(\mathbb{R}^{2n})$$

durch

$$(\mathscr{K}\varphi)(r_1, \ldots, r_n, s_1, \ldots, s_n)$$
$$= \varphi(\exp(r_n X_n) \cdot \ldots \cdot \exp(r_1 X_1), \exp(s_n X_n) \cdot \ldots \cdot \exp(s_1 X_1)).$$

Durch Zusammensetzen erhält man schließlich eine Einbettung von $S(T \times N \times N, U, \gamma)^\mu$ auf einen abgeschlossenen Unterraum von $\mathscr{S}(\mathbb{R}^{2n})$

$$\mathscr{S}(T \times N \times N, U, \gamma)^\mu$$
$$\longrightarrow \mathscr{S}(N \times N, U, \gamma) \xrightarrow{\mathscr{F}} \mathscr{S}(N \times N, H \times H, \chi_g \times \bar{\chi}_g) \xrightarrow{\mathscr{K}} \mathscr{S}(\mathbb{R}^{2n}).$$

Das gegebene stetige lineare Funktional $\tilde{\xi}$ auf $\mathscr{S}(T\times N\times N, U, \gamma)^\mu$ kann man mittels dieser Einbettung als ein stetiges Funktional auf einem abgeschlossenen Teilraum von $\mathscr{S}(\mathbb{R}^{2n})$ auffassen. Dort erlaubt es eine stetige lineare Fortsetzung, es gibt also eine temperierte Distribution ξ' auf $\mathbb{R}^{2n}$ mit

$$\tilde{\xi}(\psi)=(\xi'\circ\mathscr{K}\circ\mathscr{F})(\psi_1)$$

für $\psi\in\mathscr{S}(T\times N\times N, U, \gamma)^\mu$.

Zu ξ' existieren, vgl. [31], p. 239, ein stetiges $\zeta\in L^2(\mathbb{R}^{2n})$ und ein Differentialoperator D auf $\mathbb{R}^{2n}$ mit polynomialen Koeffizienten derart, daß

$$\xi'(v)=\int_{\mathbb{R}^{2n}}\zeta(a)(Dv)(a)da$$

für alle $v\in\mathscr{S}(\mathbb{R}^{2n})$ gilt.

Wir drücken nun das Funktional $\varphi\to\xi(p*\varphi*q)$, $\varphi\in\mathscr{S}(N)^\mu$, durch ξ' und damit durch ζ und D aus. Sei $f=p*\varphi*q$ mit einem $\varphi\in\mathscr{S}(N)^\mu$. Es ist

$$\xi(f)=\tilde{\xi}(Rf)=(\xi'\circ\mathscr{K}\circ\mathscr{F})((Rf)_1).$$

Und für $(Rf)_1$ findet man durch einige Variablen-Transformationen (hinter denen natürlich nichts anderes als die Tatsache steckt, daß die $(Rf)_1$, $(R\varphi)_1$, ... die Kerne zu gewissen Operatoren im Darstellungsraum von $\operatorname{ind}_C^N\chi_g$ sind):

$$(Rf)_1(x,y)=\int_{N/C}d\dot{\alpha}\int_{N/C}d\dot{\beta}(Rp)_1(x,\beta)(R\varphi)_1(\beta,\alpha)(Rq)_1(\alpha,y).$$

Aus dem gleichen Grunde gilt

$$[\mathscr{F}((Rf)_1)](x,y)=\int_{N/H}d\dot{\alpha}\int_{N/H}d\dot{\beta}[\mathscr{F}((Rp)_1)](x,y)[\mathscr{F}((R\varphi)_1)](\beta,\alpha)\cdot[\mathscr{F}((Rq)_1)](\alpha,y)$$

oder, indem man $\tilde{f}:=\mathscr{F}((Rf)_1)$ setzt und entsprechend $\tilde{\varphi}$, $\tilde{p}$ und $\tilde{q}$ erklärt,

$$\tilde{f}(x,y)=\int_{N/H}d\dot{\alpha}\int_{N/H}d\dot{\beta}\,\tilde{p}(x,\beta)\tilde{\varphi}(\beta,\alpha)\tilde{q}(\alpha,y).$$

Setzt man weiter $f'=\mathscr{K}\tilde{f}$ und erklärt φ', p' und q' entsprechend, so erhält man durch Ausnutzen der Beziehung zwischen dem invarianten Maß auf N/H und dem Lebesgueschen Maß auf $\mathbb{R}^n$

$$\begin{aligned}&f'(r_1,\dots,r_n,s_1,\dots,s_n)\\&=\int_{\mathbb{R}^n}d\alpha_1,\dots,d\alpha_n\int_{\mathbb{R}^n}d\beta_1,\dots,d\beta_n p'(r_1,\dots,r_n,\beta_1,\dots,\beta_n)\\&\quad\cdot\varphi'(\beta_1,\dots,\beta_n,\alpha_1,\dots,\alpha_n)q'(\alpha_1,\dots,\alpha_n,s_1,\dots,s_n),\end{aligned}$$

bis auf eine feste positive Konstante, welche allein von der Normalisierung der jeweiligen invarianten Maße abhängt und für die folgende Argumentation unerheblich ist.

Also ist

$$\begin{aligned}\xi(p*\varphi*q)&=\xi(f)=(\xi'\circ\mathscr{K}\circ\mathscr{F})((Rf)_1)=\xi'(f')\\&=\int_{\mathbb{R}^n}dr\int_{\mathbb{R}^n}ds\,\zeta(r,s)(Df')(r,s)\end{aligned}$$

mit

$$r=(r_1,\dots,r_n), \qquad s=(s_1,\dots,s_n)$$

und daher

$$|\xi(p*\varphi*q)|\leqq\|\zeta\|_2\,\|Df'\|_2 .$$

Nun läßt sich Df' in der Form

$$(Df')(r,s)=\sum_{j=1}^{k}\int_{\mathbb{R}^n}d\alpha\int_{\mathbb{R}^n}d\beta\,P_j(r,\beta)\varphi'(\beta,\alpha)Q_j(\alpha,s)$$

mit gewissen Schwartzschen Funktionen $P_1,\dots,P_k$, $Q_1,\dots,Q_k$ schreiben, die sich durch Anwendung von D auf p' und q' ergeben. Deutet man P_j und φ' als Kerne von Integraloperatoren K_j bzw. Φ auf $L^2(\mathbb{R}^n)$,

$$(\Phi\psi)(\beta)=\int_{\mathbb{R}^n}\varphi'(\beta,\alpha)\psi(a)d\alpha$$

und

$$(K_j\psi)(\beta)=\int_{\mathbb{R}^n}P_j(\beta,\alpha)\psi(a)d\alpha ,$$

so erhält man

$$(Df')(r,s)=\sum_{j=1}^{k}[(K_j\circ\Phi)(Q_j(-,s))](r)$$

und damit

$$\|Df'\|_2\leqq\sum_{j=1}^{k}\|K_j\|\,\|\Phi\|\,\|Q_j\|_2 .$$

Also gilt

$$|\xi(p*\varphi*q)|\leqq\|\zeta\|_2\sum_{j=1}^{k}\|K_j\|\,\|\Phi\|\,\|Q_j\|_2=E\|\Phi\|$$

mit einer Konstanten E, die natürlich von p und q abhängt. Bildet man die induzierte Darstellung $\pi=\operatorname{ind}_H^N\chi_g$ und realisiert diese in $L^2(\mathbb{R}^n)$ mittels des oben angegebenen Diffeomorphismus $\mathbb{R}^n\times H\to N$, so ist Φ nichts anderes als $\pi(\varphi)$. Damit ergibt sich

$$|\xi(p*\varphi*q)|\leqq E\|\pi(\varphi)\|\quad \text{für alle}\quad \varphi\in\mathcal{S}(N)^\mu .$$

Diese Ungleichung gilt dann auch für alle $\varphi\in L^1(N)^\mu$. Liegt nun φ in $L^1(N)^\mu\cap\bigcap_{t\in T}\Omega_t$ und damit insbesondere in $\Omega=\operatorname{Kern}\pi$, so ist $\xi(p*\varphi*q)=0$ für alle $p,q\in\mathcal{S}(N)^T$. Läßt man p und q eine approximierende Eins von $L^1(N)$ durchlaufen, so ergibt sich $\xi\left(L^1(N)^\mu\cap\bigcap_{t\in T}\Omega^t\right)=0$ für jedes $\mu\in\hat{T}$. Da aber $\sum_{\mu\in T}L^1(N)^\mu\cap\bigcap_{t\in T}\Omega^t$ dicht in $\bigcap_{t\in T}\Omega^t$ liegt, folgt $\xi\left(\bigcap_{t\in T}\Omega^t\right)=0$ wie behauptet.

Mit einem ähnlichen Trick wie in [27] kann man Satz 5 wesentlich verschärfen.

Satz 6. *Seien N einfachzusammenhängende, zusammenhängende nilpotente Liesche Gruppe mit Liescher Algebra $\mathfrak{n}$ und Λ ein primitives Ideal in $L^1(N)$, Λ_γ sei das entsprechende Ideal in $\mathcal{U}\mathfrak{n}$. G sei ein semidirektes Produkt $T \ltimes M$ aus einer kompakten abelschen Lieschen Gruppe T und einer einfachzusammenhängenden, zusammenhängenden nilpotenten Lieschen Gruppe M. G operiere stetig und homomorph durch Automorphismen auf N und dann auch auf $\mathfrak{n}$, dabei operiere M durch unipotente Automorphismen auf $\mathfrak{n}$. Es sei $\mathcal{X} = \{\Lambda^x;\ x \in G\}$, $k(\mathcal{X}) = \bigcap_{x\in G} \Lambda^x \lhd L^1(N)$ und $\mathfrak{q} = \bigcap_{x \in G} \Lambda^x_\gamma \lhd \mathcal{U}\mathfrak{n}$. Dann ist $\mathfrak{q} * \mathcal{S}(N)$ total in $k(\mathcal{X})$.*

Bemerkung. Vermutlich ist ein entsprechender Satz auch für Beurlingsche Algebren $L^1_w(N)$ mit einem polynomial wachsenden Gewicht w richtig, vgl. auch die Bemerkung im Anschluß an Satz 5. Allerdings läßt sich der unten stehende Beweis nicht ohne weiteres auf die allgemeinere Situation übertragen, da die dort konstruierte Darstellung ϱ von H in $L^1(N)$ den Unterraum $L^1_w(N)$ nicht immer invariant läßt.

Beweis. G ist als Menge das kartesische Produkt $T \times M$, und die Multiplikation in G ist gegeben durch

$$(t, a)(s, b) = (ts, \alpha(s)^{-1}(a)b)$$

mit einem stetigen Homomorphismus $\alpha: T \to \mathrm{Aut}(M)$. Die Tatsache, daß G auf N operiert, bedeutet, daß man zwei stetige Homomorphismen $\beta: T \to \mathrm{Aut}(N)$ und $\gamma: M \to \mathrm{Aut}(N)$ mit

$$\gamma(\alpha(s)(a)) = \beta(s)\gamma(a)\beta(s)^{-1}$$

für alle $a \in M$, $s \in T$ gegeben hat.

Wir bilden die Hilfsgruppe H, H ist als Menge gleich $M \times N \times N$, und die Multiplikation ist definiert durch:

$$(a', x', y')(a, x, y) = (a'a, \gamma(a)^{-1}(x')x, x^{-1}\gamma(a)^{-1}(y')xy).$$

Auf Grund der Voraussetzungen ist H eine einfachzusammenhängende, zusammenhängende nilpotente Liesche Gruppe. Wir werden bei den folgenden Rechnungen H vornehmlich als das semidirekte Produkt aus der Untergruppe $K = M \times N \times \{e\}$ und dem Normalteiler $\{e\} \times \{e\} \times N \cong N$ ansehen, $H = K \ltimes N$. Es ist leicht nachzurechnen, daß durch

$$\delta(t)(a, x, y) = (\alpha(t)(a), \beta(t)(x), \beta(t)(y)) \quad \text{für} \quad t \in T, \quad (a, x, y) \in H,$$

ein stetiger Homomorphismus $\delta: T \to \mathrm{Aut}(H)$ definiert ist.

H wirkt auf Funktionen $f: N \to \mathbb{C}$ durch

$$\{\varrho(a, x, y)f\}(z) = f(y^{-1}x^{-1}\gamma(a)^{-1}(z)x) \quad \text{für} \quad z \in N, \quad (a, x, y) \in H.$$

Im besonderen ist hierdurch eine stark stetige Darstellung von H in dem Banachschen Raum $L^1(N)$ erklärt. Jedes $\varrho(a, x, y)$ ist eine lineare Isometrie auf $L^1(N)$; daher kann ϱ zu einer Darstellung von $L^1(H)$ integriert werden, die ebenfalls mit ϱ bezeichnet wird. Es sei I der Abschluß der linearen Hülle von $\mathfrak{q} * \mathscr{D}(N)$ in $L^1(N)$. Da I und $k(\mathscr{X})$ unter der Darstellung ϱ invariant sind, liefert diese auch Darstellungen ϱ_I und ϱ_k von H (und von $L^1(H)$) in $E := L^1(N)/I$ bzw. in $L^1(N)/k(\mathscr{X})$. Mit Hilfe von Satz 5 werden wir beweisen, daß $\mathrm{Kern}_{L^1(H)}\varrho_I = \mathrm{Kern}_{L^1(H)}\varrho_k$ ist. Daraus folgt Satz 6, denn dann stimmen auch $\mathrm{Kern}_{L^1(N)}\varrho_I = I$ und $\mathrm{Kern}_{L^1(N)}\varrho_k = k(\mathscr{X})$ überein. Mit $\mathrm{Ann}_{\mathscr{U}\mathfrak{h}}(E)$ bezeichnen wir den infinitesimalen Annullator von E in der universellen Einhüllenden $\mathscr{U}\mathfrak{h}$ der Lieschen Algebra $\mathfrak{h}$ von H, d.h.

$$\begin{aligned}\mathrm{Ann}_{\mathscr{U}\mathfrak{h}}(E) &= \{w \in \mathscr{U}\mathfrak{h}; w * \mathscr{D}(H) \subseteqq \mathrm{Ann}_{L^1(H)}(E)\} \\ &= \{w \in \mathscr{U}\mathfrak{h};\ \varrho(w * \mathscr{D}(H)) \subseteqq I\}.\end{aligned}$$

Korrekterweise sollte man wohl $\mathrm{Ann}_{\mathscr{U}\mathfrak{h}}(E_{\mathscr{X}})$ schreiben, doch die Bezeichnung $\mathrm{Ann}_{\mathscr{U}\mathfrak{h}}(E)$ ist unmißverständlich.

Die nächsten Betrachtungen werden zeigen, daß man das Ideal $\mathrm{Ann}_{\mathscr{U}\mathfrak{h}}(E)$ auch direkt aus $\mathfrak{q}$ gewinnen kann. Für die folgende Konstruktion vergleiche man auch [9, 2.5.3]. $\mathscr{U}\mathfrak{n}$ operiert durch Linksmultiplikation auf $\mathscr{U}\mathfrak{n}/\mathfrak{q}$, $\sigma_\mathfrak{q} : \mathscr{U}\mathfrak{n} \to \mathrm{End}(\mathscr{U}\mathfrak{n}/\mathfrak{q})$. Die Liesche Algebra $\mathfrak{k}$ von K operiert auf $\mathfrak{n}$ durch Derivationen, die sich eindeutig zu Derivationen von $\mathscr{U}\mathfrak{n}$ fortsetzen lassen. Diese Derivationen lassen $\mathfrak{q}$ invariant (da $\mathfrak{q}$ invariant unter Konjugation mit Elementen aus K ist), und man erhält eine lineare Abbildung $\sigma_\mathfrak{q} : \mathfrak{k} \to \mathrm{End}(\mathscr{U}\mathfrak{n}/\mathfrak{q})$. Zusammen ergibt sich eine lineare Abbildung $\sigma_\mathfrak{q}$ von $\mathfrak{h} = \mathfrak{k} + \mathfrak{n}$ in $\mathrm{End}(\mathscr{U}\mathfrak{n}/\mathfrak{q})$, und man rechnet leicht nach, daß $\sigma_\mathfrak{q}$ eine Darstellung der Lieschen Algebra $\mathfrak{h}$ und damit auch von $\mathfrak{U}\mathfrak{h}$ ist. Es wird sich erweisen, daß $\mathrm{Ann}_{\mathscr{U}\mathfrak{h}}(E)$ gerade gleich $\mathrm{Kern}\,\sigma_\mathfrak{q} = \mathrm{Ann}_{\mathscr{U}\mathfrak{h}}(\mathscr{U}\mathfrak{n}/\mathfrak{q})$ ist.

Zur Bestimmung von $\mathrm{Kern}\,\sigma_\mathfrak{q}$ berechnen wir den Annullator der Nebenklasse $[1] \in \mathscr{U}\mathfrak{n}/\mathfrak{q}$. Jedes Element in $\mathscr{U}\mathfrak{h}$ läßt sich in der Form $\sum_{j=1}^m u_j v_j$ mit $u_j \in \mathscr{U}\mathfrak{n}$, $v_j \in \mathscr{U}\mathfrak{k}$ schreiben, [9, 2.2.10]. Für $v \in \mathscr{U}\mathfrak{k}$ gilt $v[1] = \chi(v)[1]$, wobei mit $\chi(v)$ der konstante Term in v bezeichnet ist, oder auch: χ ist derjenige Algebrenhomomorphismus $\mathscr{U}\mathfrak{k} \to \mathbb{C}$ mit $\mathrm{Kern}\,\chi = \mathfrak{k}\mathscr{U}\mathfrak{k}$. Also ist

$$\begin{aligned}\sigma_\mathfrak{q}\left(\sum_{j=1}^m u_j v_j\right)[1] &= \sum_{j=1}^m \sigma_\mathfrak{q}(u_j)(\chi(v_j)[1]) \\ &= \sum_{j=1}^m \chi(v_j)[u_j] = \left[\sum_{j=1}^m \chi(v_j) u_j\right].\end{aligned}$$

Daher gilt

$$\mathrm{Ann}_{\mathscr{U}\mathfrak{h}}([1]) = \Delta_\mathfrak{q} \quad \text{mit}$$

$$\Delta_\mathfrak{q} := \left\{\sum_{j=1}^m u_j v_j;\ u_j \in \mathscr{U}\mathfrak{n},\ v_j \in \mathscr{U}\mathfrak{k},\ \sum_{j=1}^m \chi(v_j) u_j \in \mathfrak{q}\right\}. \tag{1}$$

Da $\mathscr{U}\mathfrak{n}/\mathfrak{q}$ offensichtlich ein zyklischer $\mathscr{U}\mathfrak{h}$-Modul ist, ist $\mathrm{Kern}\,\sigma_\mathfrak{q}$ gleich dem größten in $\Delta_\mathfrak{q}$ enthaltenen zweiseitigen Ideal. Die obige Konstruktion läßt sich natürlich für jedes K-invariante Linksideal in $\mathscr{U}\mathfrak{n}$ oder, was dasselbe ist, jedes M-invariante zweiseitige Ideal in $\mathscr{U}\mathfrak{n}$ durchführen. Nun ist $\mathfrak{c} = \bigcap_{a \in M} \gamma(a)(\Lambda_\infty)$ ein solches Ideal, und man findet eine Darstellung $\sigma_\mathfrak{c}$ von $\mathscr{U}\mathfrak{h}$ in $\mathscr{U}\mathfrak{n}/\mathfrak{c}$. $\sigma_\mathfrak{c}$ ist sogar eine irreduzible Darstellung von $\mathscr{U}\mathfrak{h}$. Denn ein $\sigma_\mathfrak{c}(\mathscr{U}\mathfrak{h})$-invarianter Unterraum von $\mathscr{U}\mathfrak{n}/\mathfrak{c}$ entspricht einem M-invarianten, zweiseitigen Ideal $\mathfrak{d}$ in $\mathcal{U}\mathfrak{n}$ mit $\mathfrak{c}\subset\mathfrak{d}\subset\mathcal{U}\mathfrak{n}$. Ist $\mathfrak{d}$ ein echtes Ideal, so gibt es ein maximales Ideal $\mathfrak{b}$ mit $\mathfrak{d}\subset\mathfrak{b}$. Insbesondere liegt $\mathfrak{b}$ in der Hülle $h(\mathfrak{c})$ von $\mathfrak{c}$. Diese ist aber nach der Bemerkung am Ende des Abschn. 1 nichts anderes als die $M_{\mathbb{C}}$-Bahn von Λ_{∞}. Da $\mathfrak{d}$ nicht nur unter M, sondern auch unter $M_{\mathbb{C}}$ invariant ist, ergibt sich daraus, daß $\mathfrak{d}$ in $\mathfrak{c}$ enthalten ist. Also ist $\operatorname{Kern}\sigma_{\mathfrak{c}}$ ein primitives Ideal in $\mathcal{U}\mathfrak{h}$ und folglich sogar maximal.

Der Dualraum von $L^1(N)$ läßt sich mit $L^\infty(N)$ via $\langle\varphi,f\rangle=\int_N\varphi(x)f(x)dx$ für $\varphi\in L^\infty(N)$, $f\in L^1(N)$ identifizieren, und der Dualraum von $E=L^1(N)/I$ ist identifizierbar mit

$$I^\perp=\left\{\varphi\in L^\infty(N);\ \int_N\varphi(x)f(x)dx=0 \text{ für alle } f\in I\right\}.$$

Sei $E'\subseteqq I^\perp$ der Raum der Gårdingschen Vektoren für die zu ϱ_I kontragrediente Darstellung von H. Es gibt dann, vgl. [32, p. 256], eine Anti-Darstellung ϱ_I' von $\mathcal{U}\mathfrak{h}$ in E' mit $\langle\varrho_I'(w)\varphi,f\rangle=\langle\varphi,\varrho_{I,\infty}(w)f\rangle$ für $\varphi\in E'$, $f\in E_\infty$ und $w\in\mathcal{U}\mathfrak{h}$. Da E' schwach dicht in $I^\perp$ liegt, ist $\operatorname{Ann}_{\mathcal{U}\mathfrak{h}}(E)=\operatorname{Kern}\varrho_I'$.

Setzt man wie üblich $\check\varphi(x)=\varphi(x^{-1})$ für Funktionen φ auf N und bezeichnet mit $u\to\check u$, $u\in\mathcal{U}\mathfrak{n}$, den entsprechenden Anti-Isomorphismus von $\mathcal{U}\mathfrak{n}$, so rechnet man ohne Mühe nach, daß

$$\varrho_I'(u)\varphi=(\check\varphi*u)\check{}=\check u*\varphi \quad \text{für} \quad u\in\mathcal{U}\mathfrak{n} \quad \text{und} \quad \varphi\in E' \quad \text{gilt.} \tag{2}$$

Insbesondere ist $\check{\mathfrak{q}}*E'=0$. (2) folgt übrigens auch aus der allgemeineren Formel

$$\{\varrho_I'(X)\varphi\}(y)=\frac{d}{ds}\bigg|_{s=0}\{\varrho(\exp(-sX))\varphi\}(y) \quad \text{für} \quad X\in\mathfrak{h}, y\in N$$

und $\varphi\in E'$. (3)

Zu (3): Sei $f\in\mathcal{D}(N)$, und $f'\in\mathcal{S}(N)$ sei definiert durch

$$f'(y)=\frac{d}{ds}\bigg|_{s=0}[\varrho(\exp(sX))f\}(y).$$

Mit $[f]$ bzw. $[f']$ bezeichnen wir die entsprechenden Nebenklassen in $E=L^1(N)/I$. Damit ist $\varrho_{I,\infty}([f])=[f']$. Da die $[f]$, $f\in\mathcal{S}(N)$, in E dicht liegen, ist die Funktion $\psi:=\varrho_I'(X)\varphi\in E'$ durch die Gleichung $\langle\psi,[f]\rangle=\langle\varphi,[f']\rangle$ oder $\langle\psi,f\rangle=\langle\varphi,f'\rangle$ eindeutig festgelegt. Es ist daher lediglich nachzuprüfen, daß die Funktion $\tilde\varphi$,

$$\tilde\varphi(y)=\frac{d}{ds}\bigg|_{s=0}\{\varrho(\exp(-sX))\varphi\}(y),$$

der Beziehung $\langle\tilde\varphi,f\rangle=\langle\varphi,f'\rangle$ genügt. Letzteres folgt aber leicht aus der Identität $\langle\varrho(h^{-1})\varphi,f\rangle=\langle\varphi,\varrho(h)f\rangle$ für alle $h\in H$, $f\in\mathcal{D}(N)$ und $\varphi\in E'$.

Aus (3) zieht man die für das Folgende wichtige Konsequenz, daß $\{\varrho_I'(X)\varphi\}(e)=0$ für $X\in\mathfrak{k}$ und $\varphi\in E'$ und mithin $\{\varrho_I'(v)\varphi\}(e)=\chi(v)\varphi(e)$ für $v\in\mathcal{U}\mathfrak{k}$. Ist weiter w ein beliebiges Element in $\mathcal{U}\mathfrak{h}$, $w=\sum_{j=1}^m u_jv_j$ mit $u_j\in\mathcal{U}\mathfrak{n}$ und $v_j\in\mathcal{U}\mathfrak{k}$, so gilt

$$\varrho_I'(w)\varphi=\sum_{j=1}^m\varrho_I'(v_j)(\check u_j*\varphi)$$

und also

$$(\varrho_I'(w)\varphi)(e) = \sum_{j=1}^{m} \chi(v_j)(\check{u}_j * \varphi)(e).$$

Nun können wir zeigen, daß

$$\operatorname{Kern}\sigma_{\mathfrak{q}} \subseteqq \operatorname{Kern}\varrho_I' = \operatorname{Kern}\varrho_{I,\mathfrak{X}} \subseteqq \operatorname{Kern}\varrho_{k,\mathfrak{X}}. \tag{4}$$

Die mittlere Gleichung und die letzte Inklusion sind klar. Es bleibt also die erste Inklusion zu zeigen. Ist $w \in \operatorname{Kern}\sigma_{\mathfrak{q}}$, so ist für $\varphi \in E'$ und $y \in N$:

$$\begin{aligned}(\varrho_I'(w)\varphi)(y) &= \{\varrho(e,e,y^{-1})\varrho'(w)\varphi\}(e)\\ &= \{\varrho(e,e,y^{-1})\varrho_I'(w)\varrho(e,e,y)\varrho(e,e,y^{-1})\varphi\}(e)\\ &= (\varrho_I'(\tilde{w})\psi)(e) \quad \text{mit} \quad \psi = \varrho(e,e,y^{-1})\varphi \in E'\\ &\quad \text{und} \quad \tilde{w} = (e,e,y^{-1})(w),\end{aligned}$$

womit natürlich die Anwendung (= Konjugation) von $(e,e,y^{-1}) \in H$ auf $w \in \mathscr{U}\mathfrak{h}$ gemeint ist. Mit w liegt auch $\tilde{w}$ in $\operatorname{Kern}\sigma_{\mathfrak{q}}$, also insbesondere in $\Delta_{\mathfrak{q}}$. Es gibt folglich eine Darstellung $\tilde{w} = \sum_{j=1}^{m} u_j v_j$ mit $u_j \in \mathscr{U}\mathfrak{n}$, $v_j \in \mathscr{U}\mathfrak{k}$ und $u := \sum_{j=1}^{m} \chi(v_j)u_j \in \mathfrak{q}$. Man erhält

$$\begin{aligned}(\varrho_I'(w)\varphi)(y) &= (\varrho_I'(\tilde{w})\psi)(e)\\ &= \sum_{j=1}^{m} \chi(v_j)(\check{u} * \psi)(e) = (\check{u} * \psi)(e) = 0,\end{aligned}$$

da $u \in \mathfrak{q}$ und $\psi \in E'$. Damit ist der Beweis für (4) erbracht.

Wir wollen Satz 5 auf ein passendes maximales Ideal Ω in $L^1(H)$ anwenden. Dieses wird wie folgt konstruiert. Man bilde $F := L^1(N) \bigcap_{a \in M} \gamma(a)\Lambda$. Die Wirkung ϱ von H auf $L^1(N)$ macht auch F zu einem H- und $L^1(H)$-Modul, und man nimmt als Ω den Annullator von F in $L^1(H)$.

Man kann auf verschiedene Weisen einsehen, daß Ω maximal ist, z. B. kann man, von einer irreduziblen involutiven Darstellung τ von $L^1(N)$ mit $\operatorname{Kern}\tau = \Lambda$ ausgehend, ohne Schwierigkeiten eine irreduzible involutive Darstellung π von $L^1(H)$ mit $\operatorname{Kern}\pi = \Omega$ angeben (dies wurde in [27] getan). Eine andere Möglichkeit ist die folgende. Wendet man die obigen Betrachtungen auf den Fall $T = \{1\}$ an $\left(\text{dann geht } k(\mathfrak{X}) \text{ in } \bigcap_{a \in M} \gamma(a)\Lambda \text{ und } \mathfrak{q} \text{ in } \mathfrak{c} = \bigcap_{a \in M} \gamma(a)\Lambda_{\infty} \text{ über}\right)$, so liefert (4) speziell, daß $\operatorname{Kern}\sigma_{\mathfrak{c}}$ in $\Omega_{\infty} := \operatorname{Ann}_{\mathscr{U}\mathfrak{h}}(F)$ enthalten ist. Nun ist aber $\operatorname{Kern}\sigma_{\mathfrak{c}}$ maximal, also gilt

$$\operatorname{Kern}\sigma_{\mathfrak{c}} = \Omega_{\infty}. \tag{5}$$

Nach [19] („Wienersche Eigenschaft" nilpotenter Liescher Gruppen) gibt es zu Ω eine irreduzible involutive Darstellung π von $L^1(H)$ mit $\Omega \subseteqq \operatorname{Kern}\pi$. Dann ist aber auch Ω_{∞} in $\operatorname{Kern}\pi_{\infty}$ enthalten. Wegen der Maximalität von $\operatorname{Kern}\sigma_{\mathfrak{c}}$ ist $\operatorname{Kern}\pi_{\infty} = \Omega_{\infty}$. Also gilt $\operatorname{Kern}\pi_{\infty} * \mathscr{D}(H) \subseteqq \Omega$ und dann auch $\operatorname{Kern}\pi \subseteqq \Omega$ nach Satz 5; folglich ist $\Omega = \operatorname{Kern}\pi$ maximal.

Wie im Satz 5 setzen wir $\mathfrak{p} := \bigcap_{t \in T} \delta(t)(\Omega_\infty)$. Es gilt zunächst

$$\operatorname{Kern} \varrho_{k,\infty} = \operatorname{Ann}_{\mathscr{U}\mathfrak{h}}(L^1(N)/k(\mathscr{X})) \subseteqq \mathfrak{p}\,. \tag{6}$$

Da $\operatorname{Kern} \varrho_{k,\infty}$ invariant unter T ist, genügt es zu zeigen, daß $\operatorname{Kern} \varrho_{k,\infty}$ in Ω_∞ gelegen ist. Das ist aber klar, da $k(\mathscr{X})$ in $\bigcap_{a \in M} \gamma(a)\Lambda$ enthalten ist.

Weiter gilt

$$\mathfrak{p} \subseteqq \operatorname{Ann}_{\mathscr{U}\mathfrak{h}}(\mathscr{U}\mathfrak{n}/\mathfrak{q}) = \operatorname{Kern} \sigma_{\mathfrak{q}}\,. \tag{7}$$

Da $\operatorname{Kern} \sigma_{\mathfrak{q}}$ das größte in $\Delta_{\mathfrak{q}}$ gelegene zweiseitige Ideal ist, genügt es zu zeigen, daß $\mathfrak{p}$ in $\Delta_{\mathfrak{q}}$ enthalten ist. Nun liegt aber $\bigcap_{t \in T} \delta(t)(\Delta_{\mathfrak{c}})$ in $\Delta_{\mathfrak{q}}$ denn: Ist

$$\sum_{j=1}^{m} u_j v_j \in \bigcap_{t \in T} \delta(t)(\Delta_{\mathfrak{c}})$$

mit $u_j \in \mathscr{U}\mathfrak{n}$ und $v_j \in \mathscr{U}\mathfrak{n}$ und $v_j \in \mathscr{U}\mathfrak{k}$, so ist

$$\delta(t)\left(\sum_{j=1}^{m} u_j v_j\right) = \sum_{j=1}^{m} \delta(t)(u_j)\delta(t)(v_j) \in \Delta_{\mathfrak{c}} \quad \text{für alle} \quad t \in T,$$

also

$$\sum_{j=1}^{m} \chi(\delta(t)v_j)\delta(t)(u_j)$$
$$= \sum_{j=1}^{m} \chi(v_j)\delta(t)(u_j) = \sum_{j=1}^{m} \chi(v_j)\beta(t)(u_j) \in \mathfrak{c}$$

und folglich

$$\sum_{j=1}^{m} \chi(v_j)u_j \in \beta(t^{-1})\mathfrak{c} \quad \text{für alle} \quad t \in T.$$

Mithin liegt $\sum_{j=1}^{m} \chi(v_j)u_j$ in $\bigcap_{t \in T} \beta(t)\mathfrak{c} = \mathfrak{q}$, also $\sum_{j=1}^{m} u_j v_j$ in $\Delta_{\mathfrak{q}}$. Es reicht daher zu zeigen, daß $\mathfrak{p} = \bigcap_{t \in T} \delta(t)(\Omega_\infty)$ in $\bigcap_{t \in T} \delta(t)(\Delta_{\mathfrak{c}})$ enthalten ist. Das ist aber klar wegen (5).

(4), (6) und (7) ergeben zusammen

$$\mathfrak{p} = \operatorname{Kern} \sigma_{\mathfrak{q}} = \operatorname{Kern} \varrho_{I,\infty} = \operatorname{Kern} \varrho_{k,\infty}\,. \tag{8}$$

Mithin gilt

$$\{\mathfrak{p} * \mathscr{D}(H)\}^{-} \subseteqq \operatorname{Ann}_{L^1(H)}(E) \subseteqq \operatorname{Ann}_{L^1(H)}(L^1(N)/k(\mathscr{X})) \subseteqq \bigcap_{t \in T} \delta(t)\Omega\,,$$

wobei die beiden letzteren Inklusionen offensichtlich sind. Nach Satz 5 ist aber $\{\mathfrak{p} * \mathscr{D}(H)\}^{-} = \bigcap_{t \in T} \delta(t)\Omega$ und folglich

$$\operatorname{Ann}_{L^1(H)}(E) = \operatorname{Ann}_{L^1(H)}(L^1(N)/k(\mathscr{X}))\,,$$

woraus sich, wie oben festgestellt,

$$I = \{\mathfrak{q} * \mathscr{D}(N)\}^{-} = k(\mathscr{X})$$

ergibt.

Sei nun zunächst $\mathscr{X}$ eine beliebige abgeschlossene Menge im unitären Dual $\hat{N}$ von N. $\hat{N}$ läßt sich mit dem Raum der maximalen Ideale in $L^1(N)$ identifizieren. Zu $\mathscr{X}$ existiert allemal ein größtes Ideal in $L^1(N)$, nämlich der Kern $k(\mathscr{X})$, dessen Hülle gerade wieder $\mathscr{X}$ ist. Aber es gibt auch, wie im Falle abelscher Gruppen, ein kleinstes abgeschlossenes Ideal mit $\mathscr{X}$ als Hülle. Die Existenz eines solchen Ideals wurde in [20] bewiesen; dieses Ideal bezeichnen wir wie dort mit $j(\mathscr{X})$. Mit ähnlichen Mitteln wie in [22], vgl. auch [27], kann man nun aus Satz 6 herleiten, daß für gewisse $\mathscr{X}$ der Quotient $k(\mathscr{X})/j(\mathscr{X})$ nicht nur eine Radikalalgebra ist – das ist er stets –, sondern sogar eine nilpotente Algebra ist. Bekanntlich, vgl. etwa [8], ist N eine Gruppe mit polynomial wachsendem Haarschen Maß: es gibt eine natürliche Zahl d derart, daß zu jeder kompakten Menge K in N eine Konstante $C = C_K$ existiert mit Maß $(K^n) \leqq Cn^d$ für alle natürlichen Zahlen n. Mit $d(N)$ sei die kleinste derartige Zahl bezeichnet.

Satz 7. *Die Voraussetzungen seien dieselben wie im Satz 6, insbesondere sei also N eine zusammenhängende nilpotente Liesche Gruppe und $\mathscr{X} \subseteqq \hat{N}$ sei eine Bahn unter einer Gruppe der Form $T \ltimes M$. Dann ist $\{k(\mathscr{X})/j(\mathscr{X})\}^m = 0$ mit $m = 2^{d(N)+4} - 1$.*

Beweis. Der Vollständigkeit halber wollen wir Satz 7 hier beweisen, obwohl man sämtliche Argumente in [27] und [22] nachlesen kann. Aus Satz 6 folgt, daß $k(\mathscr{X}) \cap \mathscr{D}(N)$ dicht in $k(\mathscr{X})$ liegt – und das ist in der Tat die einzige Eigenschaft von $\mathscr{X}$, die man für die folgende Argumentation benötigt. Grundlegend für die Konstruktion von $j(\mathscr{X})$ (und viele andere idealtheoretische Eigenschaften von $L^1(N)$) ist der Dixmiersche Funktionalkalkül, siehe [8]:

Ist $f = f^* \in \mathscr{D}(N)$ und $\varphi : \mathbb{R} \to \mathbb{C}$ eine r-mal, wobei $r := d(N) + 4$, stetig differenzierbare Funktion mit kompaktem Träger und $\varphi(0) = 0$, so gibt es (eindeutig) $\varphi\{f\} \in L^1(N)$ mit $\pi(\varphi\{f\}) = \varphi(\pi(f))$ für alle involutiven Darstellungen π; dabei ist $\varphi(\pi(f))$ im Sinne des gewöhnlichen Funktionalkalküls in C^*-Algebren zu verstehen. Weiter wurde dort gezeigt, daß zu $f = f^* \in \mathscr{D}(N)$ eine Familie von r-mal stetig differenzierbaren Funktionen $\varphi : \mathbb{R} \to \mathbb{C}$ mit kompaktem Träger existiert, so daß jedes φ sogar in einer ganzen Umgebung von 0 verschwindet, und daß man die r-te Faltungspotenz f^r beliebig gut durch diese $\varphi\{f\}$ approximieren kann.

Ist nun speziell $f = f^* \in k(\mathscr{X}) \cap \mathscr{D}(N)$, so liegen die mit solchen φ gebildeten $\varphi\{f\}$ in $j(\mathscr{X})$ – das folgt aus der Konstruktion von $j(\mathscr{X})$, [20]. Nebenbei bemerkt, man kann sehr schnell einsehen (unter Verwendung der Symmetrie der involutiven Banachschen Algebra $L^1(N)$), daß solche $\varphi\{f\}$ in jedem abgeschlossenen Ideal liegen müssen, welches $\mathscr{X}$ als Hülle hat, vgl. das entsprechende Argument in [20]. Also liegt f^r in $j(\mathscr{X})$ für jedes $f = f^* \in k(\mathscr{X}) \cap \mathscr{D}(N)$. Das ist dann aber auch richtig für jedes $f \in k(\mathscr{X}) \cap \mathscr{D}(N)$; denn: Zerlegt man f in $f = f_1 + i f_2$ mit $f_j = f_j^*$, so liegen f_1 und f_2 in $k(\mathscr{X}) \cap \mathscr{D}(N)$. Bildet man das Polynom $Q(z) = (f_1 + z f_2)^r$, $z \in \mathbb{C}$, so ist $Q(z)$ kongruent zu Null modulo $j(\mathscr{X})$ für alle reellen z, folglich für alle z. Insbesondere liegt $Q(i) = f^r$ in $j(\mathscr{X})$. Da nun $k(\mathscr{X}) \cap \mathscr{D}(N)$ dicht in $k(\mathscr{X})$ ist, gilt $f^r \in j(\mathscr{X})$ für jedes $f \in k(\mathscr{X})$.

Wendet man den Satz von Nagata–Higman, vgl. etwa [16], Appendix C, auf die Algebra $k(\mathscr{X})/j(\mathscr{X})$ an, so sieht man, daß $\{k(\mathscr{X})/j(\mathscr{X})\}^m = 0$ mit $m = 2^r - 1$, d. h. $f_1 * \ldots * f_m \in j(\mathscr{X})$ für $f_1, \ldots, f_m \in k(\mathscr{X})$.

Schlußbemerkung

Es besteht eine gewisse Aussicht, daß man die Ergebnisse dieses Artikels auf nichtabelsche kompakte Liesche Gruppen T übertragen kann, die zusammen mit einer unipotenten Gruppe M auf einer nilpotenten Lieschen Gruppe N operieren – das ist jedenfalls der sich anbietende nächste Schritt im Sinne des in der Einleitung vorgestellten Problemkreises. Noch etwas allgemeiner kann man die Kerne von Bahnen solcher Gruppen, $T \ltimes M$ im unitären Dual von Gruppen desselben Typs auf ihre infinitesimale Bestimmtheit untersuchen. – Wie in der Einleitung angedeutet, kann man Ergebnisse dieser Art beispielsweise zur Bestimmung von Annullatoren topologisch irreduzibler Darstellungen verwenden. In einer folgenden Arbeit werde ich eine Beschreibung aller primitiven Idelae in der L^1-Algebra einer auflösbaren Lieschen Gruppe geben. Dabei werden die Ergebnisse dieses Artikels eine wesentliche Rolle spielen.

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Tauberian L^1-Convergence Classes of Fourier Series. II

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1. Introduction

In part I of this paper [1] we extended and generalized results in [2] concerning the problem of L^1-convergence of Fourier series,

$$S(f) \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}, \qquad t \in T = \mathbb{R}/2\pi\mathbb{Z},$$

where $f \in L^1(T)$. Specifically, two general Tauberian L^1-convergence classes were studied: the first is due to Stanojević [2] and is defined by

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta \hat{f}(k)|^p = 0, \tag{1.1}$$

where $1 < p \leqq 2$; the second is defined by

$$\overline{\lim_{n\to\infty}}\, l(n)^{-1/q} \left(\sum_{k=n}^{n+[n/l(n)]} k^{p-1} |\Delta \hat{f}(k)|^p \right)^{1/p} = 0, \tag{1.2}$$

where $1 < p \leqq 2$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\{l_{(n)}\}_{|n|<\infty} \in G(f)$, the class of all even sequences satisfying the following conditions

$$l(n) > 0, \qquad l(n) \to +\infty, \qquad l(n) = o(n) \quad (n \to \infty) \tag{1.3}$$

$$\|\sigma_{n+[n/l(n)]}(f) - \sigma_n(f)\| \, l(n) = o(n) \quad (n \to \infty), \tag{1.4}$$

$\sigma_n(f) = \sigma_n(f, t)$ being the $(C, 1)$ means of the partial sums $S_n(f) = S_n(f, t)$ of $S(f)$ and $\|\cdot\| = \|\cdot\|_{L^1(T)}$ (a denotation to be used throughout the rest of the paper). It was shown in [1] that $G(f)$ is nonempty for all $f \in L^1(T)$; in fact $G(f) \supset L(f)$, where this latter class is that of all sequences satisfying (1.3) and

$$\|\sigma_n(f) - f\| \, l(n)^{1/2} = o(1) \quad (n \to \infty).$$

In the case of (1.1) we assumed that the Fourier coefficients $\{\hat{f}(n)\}$ were asymptotically even, i.e.

$$\frac{1}{n}\sum_{k=1}^{n} |\hat{f}(k)-f(-k)|\lg k = o(1) \quad (n\to\infty) \tag{1.5}$$

$$\lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]} |\Delta(\hat{f}(k)-\hat{f}(-k))|\lg k = 0. \tag{1.6}$$

Analogous conditions arose in the case of (1.2): $\{\hat{f}(n)\}$ is assumed to be asymptotically even with respect to $G(f)$, i.e., for some $\{l(n)\}\in G(f)$,

$$\frac{1}{[n/l(n)]}\sum_{k=n}^{n+[n/l(n)]} |\hat{f}(k)-\hat{f}(-k)|\lg k = o(1) \quad (n\to\infty) \tag{1.7}$$

$$\sum_{k=n}^{n+[n/l(n)]} |\Delta(\hat{f}(k)-\hat{f}(-k))|\lg k = o(1) \quad (n\to\infty). \tag{1.8}$$

In either case the Stanojević [2] necessary and sufficient condition

$$\|\hat{f}(n)E_n+\hat{f}(-n)E_{-n}\| = o(1) \quad (n\to\infty),$$

where

$$E_m = E_m(t) = \sum_{k=0}^{m} e^{ikt}, \qquad m\in\mathbb{Z},$$

was reduced to the classical form of a speed condition on the rate that $\{\hat{f}(n)\}$ tends to zero, i.e., $\hat{f}(n)1g|n| = o(1)$ $(|n|\to\infty)$.

These results take the following precise form.

Theorem A. *Let $f\in L^1(T)$ and for some $1<p\leqq 2$, let* (1.1), (1.5), *and* (1.6) *hold. Then*

$$\|S_n(f)-f\| = o(1) \quad (n\to\infty)$$

if and only if $\hat{f}(n)1g|n| = o(1)$ $(|n|\to\infty)$.

Theorem B. *Let $f\in L^1(T)$ and for some $\{l(n)\}\in G(f)$ and some $1<p\leqq 2$, let* (1.2), (1.7), *and* (1.8) *hold. Then* $\|S_n(f)-f\| = o(1)$ $(n\to\infty)$ *if and only if* $\hat{f}(n)\lg|n|/l(n) = o(1)$ $(|n|\to\infty)$.

In [1] numerous corollaries to both Theorem A and Theorem B were found. These provide an intrinsic view of the nature of difficulties in L^1-convergence theory and the relation between the smoothness of f and the smoothness of $\{\hat{f}(n)\}$.

Part II of this paper has a three-fold purpose centering on the method of proof of Theorem A and Theorem B. The focus of Sect. 3 is on developing Theorem A and Theorem B beyond the constraint of asymptotic evenness. This is a significant improvement both in form and content of the previous result.

The key idea in the proof of all these results is a more refined estimation of $\|S_n(f)-\sigma_n(f)\|$, a situation analogous to the classical Hardy Tauberian theorem ([3], p. 52). In Sect. 4 this idea will be used to give results on the order of magnitude of $\|S_n(\mu)\|$ for certain classes of Borel measures μ, on T. To be more specific, let $M(T)$ be the class of all Borel measures on T, then it is well known that $\mu\in M(T)$ is equivalent to

$$\|\sigma_n(\mu)\| = 0(1) \quad (n\to\infty),$$

where $\sigma_n(\mu)$ are the $(C, 1)$ means of the partial sums $S_n(\mu)$ of the Fourier Stieljes series

$$S(\mu) \sim \sum_{|n|<\infty} \hat{\mu}(n) e^{int}.$$

Our methods will be used to give a subclass of $M(T)$ for which

$$\|S_n(\mu)\| = K_n |\hat{\mu}(n)| \lg n + 0(1) \quad (|n| \to \infty), \tag{1.8}$$

where K_n is a bounded sequence of real numbers. Due to the nature of the technicalities (Lemma 2.1), this result will involve another notion of asymptotic evenness. Results of Edwards [4] and Stanojević [5] will be given as corollaries.

Also note from (1.8) that this result gives a sharper version of Helson [6] result that $\|S_n(\mu)\| = 0(1)$ $(n \to \infty)$ implies $\hat{\mu}(n) = o(1)$ $(|n| \to \infty)$.

The third purpose of this paper is based on the following observation: almost all classical sufficient conditions for a cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

to be the Fourier series of some $f \in L^1(0, \pi)$ are also sufficient for $\|g_n - f\| = o(1)$ $(n \to \infty)$, where g_n is the modified cosine sum of Garrett and Stanojević [7], i.e.,

$$g_n(x) = \tfrac{1}{2} \sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n} \left(\sum_{l=k}^{n} \Delta a_l \right) \cos kx = S_n(x) - a_{n+1} D_n(x),$$

where

$$D_n(x) = \frac{\sin(n+\frac{1}{2})x}{2 \sin \frac{x}{2}}$$

is the Dirichlét kernel. Consequently, Sect. 4 is devoted to finding classes of sequences $\{c(n)\}_{|n|<\infty}$ for which the L^1-convergence of the modified sums is both sufficient and necessary for trigonometric series $\sum_{|n|<\infty} c(n) e^{int}$ to be the Fourier series of some $f \in L^1(T)$. The analogous situation for measures is also discussed. Since these considerations are based on the observation on cosine series, this provides another significant application of asymptotic evenness conditions.

Section 6 concludes with some related comments.

2. Lemmas

The following lemmas are convenient for the results of Sects. 3 and 4. To state these in a succinct manner a modification of E_m is introduced: for $m \in \mathbb{Z}$, let

$$\begin{aligned} A_m(t) &= E_m(t) + \operatorname{sgn}(m) \frac{e^{-i \operatorname{sgn}(m) t/2}}{2i \sin t/2} \\ &= \operatorname{sgn}(m) \frac{e^{i \operatorname{sgn}(m)(|m| + \frac{1}{2})t}}{2i \sin t/2}. \end{aligned}$$

For an increasing sequence ϕ_n of natural numbers the set $(-\pi, -\pi/\phi_n)\cup(\pi/\phi_n, \pi)$ is denoted by T_n^ϕ; in the case that $\phi_n = n$, the set is denoted by T_n.

Lemma 2.1. *Let $\{c(n)\}_{|n|<\infty} \subset C$ and let $\{\phi_n\}_{|n|<\infty}$ be an increasing even ($\phi_{-n} = \phi_n$) sequence of natural numbers such that $\phi_n = o(n)$ $(n \to \infty)$. If $(c(n) - c(-n)) \lg \phi_n = 0(1)$ $(n \to \infty)$, then*

$$K|c(n)| \lg \phi_n + 0(1) \leqq \int_{T_n^\phi} |c(n)A_n(t) + c(-n)A_{-n}(t)| dt$$
$$\leqq \pi |c(n)| \lg \phi_n + 0(1) \quad (n \to \infty),$$

where K is an absolute constant.

Proof. Denote by I_n the integral above. Then,

$$I_n \leqq |c(n)| \int_{T_n^\phi} |A_n(t) + A_{-n}(t)| dt + |c(n) - c(-n)| \int_{T_n^\phi} |A_{-n}(t)| dt$$
$$= |c(n)| \int_{T_n^\phi} |D_n(t)| dt + |c(n) - c(-n)| \int_{T_n^\phi} |A_{-n}(t)| dt.$$

Since, $\int_{T_n^\phi} |D_n(t)| dt \leqq \pi \lg \phi_n$ and $\int_{T_n^\phi} |A_m(t)| dt \leqq \pi \lg \phi_n$, we have

$$I_n \leqq \pi |c(n)| \lg \phi_n + 0(1) \quad (n \to \infty),$$

which is the right-hand inequality. For the left-hand inequality, we have

$$c(n)D_n(t) = c(n)(A_n(t) + A_{-n}(t))$$
$$= (c(n)A_n(t) + c(-n)A_{-n}(t)) + (c(n) - c(-n))A_{-n}(t).$$

Thus,

$$|c(n)| \int_{T_n^\phi} |D_n(t)| dt \leqq I_n + |c(n) - c(-n)| \int_{T_n^\phi} |A_{-n}(t)| dt.$$

Since

$$\int_{T_n^\phi} |D_n(t)| dt \geqq K \lg \phi_n, \tag{2.2}$$

for an appropriate constant K, the desired conclusion follows.

It should be noted that the condition $\phi_n = o(n)$ $(n \to \infty)$ is used in obtaining estimate (2.2). In the case (useful in proof of Theorem 3.1) $\phi_n = n$ a similar estimate holds, i.e.,

$$\int_{T_n} |D_n(t)| dt \geqq K \lg n,$$

whose proof is simpler than that of (2.2). For latter use, this case is summarized in the following.

Lemma 2.2. *Let $\{c(n)\}_{|n|<\infty} \subset C$. If $(c(n) - c(-n)) \lg n = 0(1)$ $(n \to \infty)$, then*

$$K|c(n)| \lg n + 0(1) \leqq \int_{T_n} |c(n)A_n(t) + c(-n)A_{-n}(t)| dt$$
$$\leqq \pi |c(n)| \lg n + 0(1) \quad (n \to \infty) \tag{2.3}$$

where K is an absolute constant.

In regards to our latter results, note that the conclusion of the above lemmas may be put in the following form,

$$I_n = K_n|c(n)|\lg\phi_n + O(1) \quad (|n| \to +\infty),$$

where K_n is a bounded sequence of real numbers.

As a further remark, both lemmas hold with $O(1)$ replaced by $o(1)$ throughout; this is useful as follows.

Lemma 2.3. *Let $\{c(n)\}$ and $\{\phi_n\}$ be as in Lemma 2.1. Then $I_n = o(1)$ $(n \to \infty)$ if and only if $c(n)\lg\phi_n = o(1)$ $(|n| \to \infty)$.*

Lemma 2.4. *Let $\{c(n)\} \subset C$. Then $I_n = o(1)$ $(n \to \infty)$ if and only if $c(n)\lg|n| = o(1)$ $(|n| \to \infty)$.*

Both of these follow from the fact that $I_n = o(1)$ $(n \to \infty)$ implies $(c(n) - c(-n))\lg\phi_n = o(1)$ $(n \to \infty)$ in the first case and $(c(n) - c(-n))\lg n = o(1)$ $(n \to \infty)$ in the second.

The proof of the next lemma is similar to the proof of Lemma 2.2 in [1]. It is an application of Hölder's inequality and the Riesz extension of the Hausdorff-Young inequality [8].

Lemma 2.5. *Let $\{c(n)\}_{|n| < \infty} \subset C$, let $\{\phi_n\}_0^\infty$ be an increasing sequence of natural numbers and let $1 < p \leqq 2$. Then, for $m > n$,*

$$\int_{T_n^\phi} \left| \sum_{|k|=n}^{m} c(k)A_k(t) \right| dt \leqq C_p \phi_n^{1/q} \left(\sum_{|k|=n}^{m} |c(k)|^p \right)^{1/p},$$

where C_p is an absolute constant.

3. L^1-convergence Classes

The results of this section are concerned with the problem of L^1-convergence of the partial sums of Fourier Series. These results take two forms corresponding to two choices for m in the following identity: for $m > n$,

$$S_n(t) - \sigma_n(t) = \frac{m+1}{m-n}(\sigma_m(t) - \sigma_n(t)) - Q_n^m(t), \tag{3.1}$$

where,

$$Q_n^m(t) = \sum_{|k|=n+1}^{m} \frac{m-|k|+1}{m-n} c(k)e^{ikt}.$$

Here,

$$S_n(t) = \sum_{|k| \leqq n} c(k)e^{ikt},$$

and $\sigma_n(t)$ are the corresponding Fejer sums. The two choices are $m = [\lambda n]$, where $\lambda > 1$, and $m = n + [n/l(n)]$, where $\{l(n)\}$ is appropriately chosen. The former has the advantage of significant generalization of classical results whereas the latter indicates certain trade-offs which can be made.

The following is the main result of the former type.

Theorem 3.1. *Let $f \in L^1(T)$ and for some $1 < p \leqq 2$, let*

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta \hat{f}(k)|^p = 0 . \tag{3.2}$$

Then $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ if and only if $\hat{f}(n) \lg |n| = o(1)$ $(|n| \to \infty)$.

Proof. For $\lambda > 1$, identity (3.1) takes the form

$$S_n(f,t) - \sigma_n(f,t) = \frac{[\lambda n]+1}{[\lambda n]-n} (\sigma_{[\lambda n]}(f,t) - \sigma_n(f,t)) - Q_n^{[\lambda n]}(f,t)$$

where

$$Q_n^{[\lambda n]}(f,t) = \sum_{|k|=n+1}^{[\lambda n]} \frac{[\lambda n]-|k|+1}{[\lambda n]-n} \hat{f}(k) e^{ikt} .$$

Since

$$\|\sigma_{[\lambda n]}(f) - \sigma_n(f)\| = o(1) \quad (n \to \infty) \tag{3.3}$$

and

$$\int_{-\pi/n}^{\pi/n} |Q_n^{[\lambda n]}(f,t)| dt \leqq \frac{2\pi}{n} \sum_{|k|=n+1}^{[\lambda n]} |\hat{f}(k)| = o(1) \quad (n \to \infty) , \tag{3.4}$$

it follows that

$$\|S_n(f) - \sigma_n(f)\| = o(1) \quad (n \to \infty) \tag{3.5}$$

if and only if

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \int_{T_n} |Q_n^{[\lambda n]}(f,t)| dt = 0 . \tag{3.6}$$

Consequently, we have to show that (3.6) is equivalent with $f(n) \lg |n| = o(1)$ $(|n| \to \infty)$. Applying summation by parts to $Q_n^{[\lambda n]}(f,t)$ gives:

$$\begin{aligned} Q_n^{[\lambda n]}(f,t) = {} & \sum_{|k|=n}^{[\lambda n]-1} \Delta \left(\frac{[\lambda n]-|k|+1}{[\lambda n]-n} f(k) \right) E_k(t) \\ & + \frac{\hat{f}([\lambda n]) E_{[\lambda n]}(t) + f(-[\lambda n]) E_{-[\lambda n]}(t)}{[\lambda n]-n} \\ & - \frac{[\lambda n]-n+1}{[\lambda n]-n} (f(n) E_n(t) + f(-n) E_{-n}(t)) . \end{aligned}$$

Rewriting as two sums over positive and negative integers k, introducing $A_n(t)$ and eliminating the resulting collapsing sums yields

$$\begin{aligned} Q_n^{[\lambda n]}(f,t) = {} & \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n]-|k|}{[\lambda n]-n} \Delta \hat{f}(k) A_k(t) + \frac{1}{[\lambda n]-n} \sum_{|k|=n+1}^{[\lambda n]} f(k) A_k(t) \\ & - (\hat{f}(n) A_n(t) + \hat{f}(-n) A_{-n}(t)) . \end{aligned} \tag{3.7}$$

Integrating the absolute value of the first two terms on the right-hand side of (3.7) and applying Lemma 2.5 provides the following estimates:

$$I_n^1 = \int_{T_n} \left| \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n]-|k|}{[\lambda n]-n} \Delta \hat{f}(k) A_k(t) \right| dt$$

$$\leqq C_p n^{1/q} \left(\sum_{|k|=n}^{[\lambda n]-1} \left(\frac{[\lambda n]-|k|}{[\lambda n]-n} \right)^p |\Delta \hat{f}(k)|^p \right)^{1/p}$$

$$\leqq C_p \left(\sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta \hat{f}(k)|^p \right)^{1/p}, \tag{3.8}$$

and,

$$I_n^2 = \int_{T_n} \left| \frac{1}{[\lambda n]-n} \sum_{|k|=n+1}^{[\lambda n]} \hat{f}(k) A_k(t) \right| dt$$

$$\leqq B_p \left(\frac{n}{[\lambda n]-n} \right)^{1/q} \left(\frac{1}{[\lambda n]-n} \int_{|k|=n+1}^{[\lambda n]} |\hat{f}(k)|^p \right)^{1/p}, \tag{3.9}$$

where C_p and B_p are absolute constants. Thus, (3.2) implies $\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} I_n^1 = 0$, and the fact that $\hat{f}(n) = o(1)$ $(|n| \to \infty)$ implies $\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} I_n^2 = 0$. Returning to (3.7), it follows that (3.5) holds if and only if

$$\int_{T_n} |\hat{f}(n) A_n(t) + \hat{f}(-n) A_{-n}(t)| dt = o(1) \quad (n \to \infty),$$

which by Lemma 2.4 is equivalent with $\hat{f}(n) \lg |n| = o(1)$ $(|n| \to \infty)$. This concludes the proof.

To shortcut the writing of various corollaries, the statement, $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ if and only if $\hat{f}(n) \lg |n| = o(1)$ $(|n| - > \infty)$, will be denoted (Y).

The following corollaries to Theorem 3.1 serve to illustrate two aspects of conditions (3.2). Notice that (3.2) implies

$$\text{(BOX)} \qquad |n|^{1/q} |\Delta \hat{f}(k)| = o(1) \quad (|n| \to \infty), \quad \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$$

as do almost all known L^1-convergence class conditions. A natural question is: can (3.2) be weakened to (BOX)? Our conjecture is that the answer is negative and that sequences satisfying (1.1) are dense in the class (BOX). It will also be seen that (3.2) is a way of controlling the nature of possible gaps in the sequences $\{\hat{f}(n)\}$. A simple case is the following.

Corollary 3.1.1. *Let $f \in L^1(T)$ and for some $1 < p \leqq 2$ $\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$ and some $\lambda > 1$, let*

$$\left(\sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta \hat{f}(k)|^p \right)^{1/p} = 0(|n|^{1/q} |\Delta \hat{f}(n)|) \quad (|n| \to \infty). \tag{3.10}$$

If (BOX), then (Y) holds.

Further emphasis on the gap aspect of (3.2) is found by considering for $\lambda > 1$, $\tau_n(f, \lambda)$, the number of non-zero $\Delta\hat{f}(k)$ where $n \leqq |k| \leqq [\lambda n]$.

Corollary 3.1.2. *Let $f \in L^1(T)$ and for some $1 < p \leqq 2$, let*

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} [\tau_n(f, \lambda)]^{1/p} \max_{n \leqq |k| \leqq [\lambda n]} |k|^{1/q} |\Delta\hat{f}(k)| = 0 .$$

Then Y holds.

Observe that this preceeding result has the following corollary, in its original form due to Stanojević [2].

Corollary 3.1.3. *Let $f \in L^1(T)$ and let $n\Delta\hat{f}(n) = 0(1)$ ($|n| \to \infty$). Then (Y) holds.*

That this follows from Corollary 3.1,2 is seen in the inequality

$$[\tau_n(f, \lambda)]^{1/p} \max_{n \leqq |k| \leqq [\lambda n]} |k|^{1/q} |\Delta\hat{f}(k)| \leqq (2\lambda)^{1/p} (\lambda - 1)^{1/p} \max_{n \leqq |k| \leqq [\lambda n]} |k\Delta\hat{f}(k)| .$$

As a further striving for (BOX) we have the following result [1].

Corollary 3.1.4. *Let $f \in L^1(T)$ and for some $1 < p \leqq 2$, let*

$$\overline{\lim_{n\to\infty}} \left(\sum_{|k|=n}^{[\lambda n]} |k|^{(p-1)/2} |\Delta\hat{f}(k)|^{p/2} \right)^{1/p} = 0(1) \quad (\lambda \downarrow 1) .$$

If (BOX), then (Y) holds.

An intrinsic refinement of this result can be made as follows.

Corollary 3.1.5. *Let $f \in L^1(T)$ and for some $1 < p \geqq 2$, let*

$$\overline{\lim_{n\to\infty}} \left(\sum_{|k|=n}^{[\lambda n]} |k|^{(p-1)/\lambda} |\Delta\hat{f}(k)|^{p/\lambda} \right)^{1/p} = 0(1) \quad (\lambda \downarrow 1). \tag{3.11}$$

If (BOX), then (Y) holds.

In particular, setting $\lambda = q - 1$ in (3.11) gives a curious result.

Corollary 3.1.6. *Let $f \in L^1(T)$ and let*

$$\overline{\lim_{n\to\infty}} \left(\sum_{|k|=n}^{[(q-1)n]} |k|^{(p-1)^2} |\Delta\hat{f}(k)|^{p(p-1)} \right)^{1/p} = 0(1) \quad (q \downarrow 2)$$

$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$. *If $|n|^{1/2} |\Delta\hat{f}(n)| = o(1)$ ($|n| \to \infty$), then (Y) holds.*

To further exemplify the nature of (3.2) consider the following.

Example. If $f \in L^1(T)$ and $f(t) \sin t/2$ is of bounded variation on T, then $n\Delta\hat{f}(n) = 0(1)$ ($|n| \to \infty$)).

We calculate for $n>0$,

$$\begin{aligned}\Delta\hat{f}(n)&\frac{1}{2\pi}\int_T f(t)\,[e^{-int}-e^{-i(n+1)t}]dt\\ &=\frac{1}{2\pi}\int_T f(t)e^{-i(n+\frac{1}{2})t}(e^{it/2}-e^{-it/2})dt\\ &=\frac{i}{\pi}\int_T [f(t)\sin t/2]e^{-i(n+\frac{1}{2})t}dt\,.\end{aligned}$$

Similar calculation holds for $n<0$. Now, mimicing the proof of a classical theorem (see [9]) on the order of magnitude of the Fourier coefficients of functions of bounded variation gives the result.

Another corollary to Theorem 3.1 are stated below; it is a generalization of a result of Stanojević [5].

Corollary 3.1.7. *Let $f\in L^1(T)$ and for some $1<p\leqq 2$, let*

$$\frac{1}{n}\sum_{1\leqq|k|\leqq n}|k|^p|\Delta\hat{f}(k)|^p=o(1)\quad(n\to\infty)\tag{3.12}$$

Then (Y) holds.

The alternative approach to the preceding results has the following as the main result.

Theorem 3.2. *Let $f\in L^1(T)$ and for some $\{l(n)\}_0^\infty\in G(f)$ and some $1<p\leqq 2$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, let*

$$\overline{\lim_{n\to\infty}}\, l(n)^{-1/q}\left(\sum_{|k|=n}^{n+[n/l(n)]}|k|^{p-1}|\Delta\hat{f}(k)|^p\right)^{1/p}=0\,.\tag{3.13}$$

Then $\|S_n(f)-f\|=o(1)$ $(n\to\infty)$ if and only if $\hat{f}(n)\lg\frac{|n|}{l(n)}=o(1)$ $(n\to\infty)$.

Proof. The proof is a modification of the proof of Theorem 3.1. The hypothesis $\{l(n)\}\in G(f)$ shows that (3.5) holds if and only if

$$\overline{\lim_{n\to\infty}}\,\|Q_n^{n+[n/l(n)]}(f)\|=0\,;$$

where $Q_n^{n+[n/l(n)]}(f,t)$ is $Q_n^m(t)$ with $m=n+[n/l(n)]$. However,

$$\int_{-\pi/[n/l(n)]}^{\pi/[n/l(n)]}|Q_n^{n+[n/l(n)]}(f,t)|dt\leqq\frac{2}{[n/l(n)]}\sum_{|k|=n+1}^{n+[n/l(n)]}|\hat{f}(k)|=o(1)\quad(n\to\infty)\,,$$

and consequently, (3.5) holds if and only if

$$\overline{\lim_{n\to\infty}}\int_{T_n^\phi}|Q_n^{n+[n/l(n)]}(f,t)|dt=0\,,$$

where $\phi=\{\phi_n\}=\{[n/l(n)]\}$.

Now, following the proof of Theorem 3.1, in particular until application of Lemma 2.5 with T_n replaced by T_n^Φ, the estimates (3.8) and (3.9) become:

$$I_n^1 \leqq A_p \left(\frac{n}{l(n)}\right)^{1/q} \left(\sum_{|k|=n}^{n+[n/l(n)]} |\Delta \hat{f}(k)|^p\right)^{1/p}$$
$$\leqq A_p l(n)^{-1/q} \left(\sum_{|k|=n}^{n+[n/l(n)]} |k|^{p-1} |\Delta \hat{f}(k)|^p\right)^{1/p}$$

and

$$I_n^2 \leqq A_p \left(\frac{1}{[n/l(n)]} \sum_{|k|=n}^{n+[n/l(n)]} |\hat{f}(k)|^p\right)^{1/p}.$$

Thus, (3.5) holds if and only if

$$\int_{T_n^\Phi} |\hat{f}(n)A_n(t) + \hat{f}(-n)A_{-n}(t)| dt = o(1) \quad (n \to \infty).$$

Appealing to Lemma 2.3 concludes the proof.

The cumbersome appearance of this result becomes more transparent in the case of Fourier series when we choose $l(n) = [1/\|\sigma_n(f) - f\|] \in G(f)$. In this case, the condition $l(n) = o(n)$ $(n \to \infty)$ forces the condition $n\|\sigma_n(f) - f\| \to \infty (n \to \infty)$. However, this is a mild requirement as $n\|\sigma_n(f) - f\| = o(1)$ $(n \to \infty)$ forces f to be a constant function (see [10] for this and related results). The condition $n\|\sigma_n(f) - f\| \to \infty$ will be tacitly assumed in the discussions below.

Corollary 3.2.1. *Let* $f \in L^1(T)$ *and let*

$$\|\sigma_n(f) - f\| \max_{n \leqq |k| \leqq n(1 + \|\sigma_n(f) - f\|)} |k \Delta \hat{f}(k)| = o(1) \quad (n \to \infty).$$

Then $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ *if and only if*

$$\hat{f}(n) \lg(|n| \, \|\sigma_{|n|}(f) - f\|) = o(1) \quad (|n| \to \infty).$$

The proof of this following from the inequality:

$$l(n)^{-1/q} \left(\sum_{|k|=n}^{n+[n/l(n)]} |k|^{p-1} |\Delta \hat{f}(k)|^p\right)^{1/p} \leqq l(n)^{-1} \max_{n \leqq |k| \leqq n + [n/l(n)]} |k| \, |\hat{f}(k)|.$$

Notice that functions in $\mathrm{Lip}_\alpha^*(T)$ $(0 < \alpha \leqq 1, w_1(f, \delta) = 0(|\delta|^\alpha)$ $(\delta \to 0)$, $w_1(f, \delta)$ being the integral modulus of continuity) trivially satisfy all conditions in the above corollary.

A final corollary of Theorem 3.2 is analogous to Corollary 3.1.3 and is indicative of the limitations of the trade-offs which are made in using the alternative approach.

Corollary 3.2.2. *Let* $f \in L^1(T)$ *and let* $n \Delta \hat{f}(n) = 0(1)$ $(n \to \infty)$. *Then* $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ *if and only if* $\hat{f}(n) \lg(|n| \, \|\sigma_{|n|}(f) - f\|) = o(1)$ $(|n| \to \infty)$.

The case of real Fourier sine series was considered in [11].

In the case of complex quasi-monotone coefficients in the sense of *V.* Stanojević [12], all above results take a sharper form.

4. Order of Magnitude of $\|S_n(\mu)\|$

Our purpose in this section is to relate the order of magnitude of $\|S_n(\mu)\|$ with the order of magnitude of $\hat{\mu}(n)\lg|n|$ for certain subclasses of $M(T)$. The main results is the following.

Theorem 4.1. *Let* $\mu\in M(T)$, *let* $(\hat{\mu}(n)-\hat{\mu}(-n))\lg n=0(1)$ $(n\to\infty)$ *and for some* $1<p\leqq 2$ *and some* $\lambda>1$, *let*

$$\sum_{|k|=n}^{[\lambda n]}|k|^{p-1}|\Delta\hat{\mu}(k)|^p=0(1)\quad(n\to\infty).\tag{4.1}$$

Then

$$\|S_n(\mu)\|=K_n|\hat{\mu}(n)|\lg n+0(1)\quad(n\to\infty)\tag{4.2}$$

where $\{K_n\}$ *is a bounded sequence of real numbers.*

Proof. Recall the identity

$$S_n(\mu,t)-\sigma_n(\mu,t)=\frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda n]}(\mu,t)-\sigma_n(\mu,t))-Q_n^\lambda(\mu,t),$$

where

$$Q_n^\lambda(\mu,t)=\sum_{|k|=n+1}^{[\lambda n]}\frac{[\lambda n]-|k|+1}{[\lambda n]-n}\hat{\mu}(k)e^{ikt}.$$

Then, from the fact that $\|\sigma_n(\mu)\|=0(1)$ $(n\to\infty)$, it follows that

$$\|S_n(\mu)\|=\|Q_n^\lambda(\mu)\|+0(1)\quad(n\to\infty).$$

Following the proof of Theorem 3.1 [in particular estimates (3.8) and (3.9)] we see that,

$$\|S_n(\mu)\|=\int_{T_n}|\hat{\mu}(n)A_n(t)+\hat{\mu}(-n)A_{-n}(t)|dt$$
$$+0\left(\left(\sum_{|k|=n}^{[\lambda n]}|k|^{p-1}|\Delta\hat{\mu}(k)|^p\right)^{1/p}\right)+0\left(\left(\frac{1}{[\lambda n]-n}\sum_{|k|=n+1}^{[\lambda n]}|\hat{\mu}(k)|^p\right)^{1/p}\right).$$

Applying (4.1), and the fact that $\hat{\mu}(n)=0(1)$ $(|n|\to\infty)$, this estimate becomes

$$\|S_n(\mu)\|=\int_{T_n}|\hat{\mu}(n)A_n(t)+\hat{\mu}(-n)A_{-n}(t)|dt+0(1)\quad(n\to\infty).$$

An application of Lemma 2.2 completes the proof.

One fine point concerning the result is that it is unnecessary to take the limit as λ tends to one. This is so because

$$\lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{|k|=n}^{[\lambda n]}|k|^{p-1}|\Delta\hat{\mu}(k)|^p=A<+\infty,$$

immediately implies (4.1).

Numerous corollaries to this result follow in the same manner as those to Theorem 3.1. For the sake of brevity we leave the bulk of these to the reader to ascertain being content with two representative examples.

Corollary 4.1.1. *Let $\mu \in M(T)$, let $(\hat{\mu}(n)-\hat{\mu}(-n))\lg n = 0(1)$ $(n\to\infty)$ and for some $1<p\leqq 2$ and some $\lambda>1$, let*

$$[\tau_n(\mu,\lambda)]^{1/p} \max_{n\leqq|k|\leqq[\lambda n]} |k|^{1/q}|\Delta\hat{\mu}(k)| = 0(1) \quad (n\to\infty)$$

$\left(\frac{1}{p}+\frac{1}{q}=1\right)$. *Then* (4.2) *holds.*

Corollary 4.1.2. *Let $\mu \in M(T)$, let $(\hat{\mu}(n)-\hat{\mu}(-n))\lg n = 0(1)$ $(n\to\infty)$ and let $n\Delta\hat{\mu}(n)=0(1)$ $(|n|\to\infty)$. Then* (4.2) *holds.*

To further exemplify Theorem 4.1, consider a result of Edwards [4] for Fourier-Stieljes cosine series (C) that if the sequence $\{a_n\}_0^\infty$ is quasi-convex, i.e.

$$\sum_{n=0}^{\infty}(n+1)|\Delta^2 a_n| < \infty\,,$$

then $\|S_n(\mu)\| = 0(1)$ $(n\to\infty)$ if and only if $a_n \lg n = 0(1)$ $(n\to\infty)$. This result is a special case of Corollary 4.1.2 since quasi-convexity and the condition $a_n = 0(1)$ $(n\to\infty)$ implies $n\Delta a_n = 0(1)$ $(n\to\infty)$.

Theorem 4.1 and its corollaries also generalize a result of Stanojević [5] concerning Fourier-Stieljes series.

One could attempt to prove an analogue of Theorem 3.2 for the problem considered here. However, a glance at the proof of Theorem 3.2 will show that the further hypothesis

$$\|\sigma_{n+[n/l(n)]}(\mu) - \sigma_n(\mu)\|\, l(n) = 0(1) \quad (n\to\infty)$$

should be required. In the context of $M(T)$ this latter hypothesis seems somewhat unnatural; a natural choice for $\{l(n)\}$ apparently does not exist for $M(T)$ as it does for $L^1(T)$. This comment also applies to the next section.

5. Fourier and Fourier-Stieljes Character of Trigonometric Series

The modified trigonometric sums corresponding to a trigonometric series $\sum_{|n|<\infty} c(n)e^{int}$ to be considered here are defined by

$$g_n(t) = \sum_{k=0}^{n} \Delta c(k) E_k(t) + \sum_{k=1}^{n} \Delta c(-k)(E_{-k}(t)-1)\,. \tag{5.1}$$

These are analogues to the sums considered by Garrett and Stanojević [7]. The primary purpose of this section is to relate the L^1-convergence and L^1-boundness of the modified sums to that of the $(C,1)$ means $\sigma_n(t)$ via conditions on the coefficients. This will allow characterization of trigonometric series as a Fourier or Fourier-Stieljes series in terms of the behavior in L^1-norm of the modified sums.

In all that follows, $\{c(n)\}_{|n|<\infty}$ is a sequence of complex numbers and $\sigma_n^g = \sigma_n^g(t)$ denotes the $(C, 1)$ means of the sequence $\{g_n(t)\}$. The desired characterizations will be obtained via the following two lemmas.

Lemma 5.1. *Let $\lambda > 1$ and $1 < p \leqq 2$. Then*

$$\|g_n - \sigma_n^g\| \leqq \frac{[\lambda n]+1}{[\lambda n]-1} \|\sigma_{[\lambda n]}^g - \sigma_n^g\| + A_p \left(\sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta c(k)|^p \right)^{1/p} + A \sum_{k=n}^{[\lambda n]} |\Delta(c(k)-c(-k))| \lg k \tag{5.2}$$

where A and A_p are absolute constants.

Proof. Using the identify $D_n(t) = E_n(t) + E_{-n}(t) - 1$, (5.1) is rewritten as

$$g_n(t) = \sum_{k=0}^{n} \Delta c(k) D_k(t) + \sum_{k=0}^{n} \Delta(c(-k)-c(k))(E_{-k}(t)-1). \tag{5.3}$$

A useful identity is given by

$$\begin{aligned} g_n(t) - \sigma_n^g(t) = {} & \frac{[\lambda n]+1}{[\lambda n]-n} (\sigma_{[\lambda n]}^g(t) - \sigma_n^g(t)) \\ & - \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta c(k) D_k(t) \\ & - \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta(c(-k)-c(k))(E_{-k}(t)-1). \end{aligned}$$

Consequently,

$$\begin{aligned} \|g_n - \sigma_n^g\| \leqq {} & \frac{[\lambda n]+1}{[\lambda n]-n} \|\sigma_{[\lambda n]}^g - \sigma_n^g\| + \left\| \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta c(k) D_k(t) \right\| \\ & + \left\| \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta(c(-k)-c(k))(E_{-k}(t)-1) \right\| \\ = {} & I_1 + I_2 + I_3. \end{aligned}$$

Since $\|E_m\| \leqq A \lg |m| + 0(1)$ $(|m| \to \infty)$, where A is an absolute constant, we have

$$\begin{aligned} I_3 & \leqq A \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} |\Delta(c(k)-c(-k))| \lg k \\ & \leqq A \sum_{k=n}^{[\lambda n]} |\Delta(c(k)-c(-k))| \lg k. \end{aligned}$$

I_2 is handled as follows (analogous to the proof of Lemma 2.5):

$$I_2 = \frac{1}{\pi} \left\{ \int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi} \right\} \left| \sum_{k=n}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta c(k) D_k(t) \right| dt = I_2' + I_2''.$$

Since $|D_k(t)| \leqq (k+\frac{1}{2})$, we have,

$$I_1' \leqq \frac{1}{n+1} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} |\Delta c(k)| \left(k+\frac{1}{2}\right) \leqq \frac{[\lambda n]+1}{n+1} \sum_{k=n}^{[\lambda n]} |\Delta c(k)|$$
$$= \frac{[\lambda n]+1}{n+1} ([\lambda n]-n) \left(\frac{1}{[\lambda n]-n} \sum_{k=n+1}^{[\lambda n]} |\Delta c(k)| \right)$$
$$\leqq \frac{[\lambda n]+1}{n+1} ([\lambda n]-n)^{1-1/p} \left(\sum_{k=n+1}^{[\lambda n]} |\Delta c(k)|^p \right)^{1/p}$$
$$\leqq (\lambda^2-1) \left(\sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta c(k)|^p \right)^{1/p}.$$

Applying the inequalities of Hölder and Riesz to I_2'' one obtains $\left(\|\cdot\|_q \text{ denotes } L^q(T)\text{-norm}, \frac{1}{p}+\frac{1}{q}=1 \right)$:

$$I_2'' \leqq 2 \left(\left\{ \int_{\pi/n+1}^{\pi} \frac{dt}{\sin^p t/2} \right\} \right)^{1/p} \left\| \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n]-k+1}{[\lambda n]-n} \Delta c(k) \sin\left(k+\frac{1}{2}\right) t \right\|_q$$
$$\leqq A_p' \left(\sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta c(k)|^p \right)^{1/p}.$$

Combining the estimates for I_2,

$$I_2 \leqq A_p \left(\sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta c(k)|^p \right)^{1/p}.$$

This completes the proof of the lemma.

A second lemma concerning the almost everywhere convergence of $g_n(t)$ is convenient; it has independent interest.

Lemma 5.2. *Let $c(n)=o(1)$ $(|n|\to\infty)$, and let*

$$\sum_{k=1}^{\infty} |\Delta(c(k)-c(-k))| \lg k < \infty. \tag{5.4}$$

If for some $1<p\leqq 2$,

$$\sum_{k=1}^{\infty} |\Delta_0(k)|^p < \infty, \tag{5.5}$$

then, $\lim_{n\to\infty} S_n(t) = \lim_{n\to\infty} g_n(t) = f(t)$ exists a.e. in $T-\{0\}$.

Proof. The case $p=1$ is classical so we assume $1<p\leqq 2$. Denote the two terms on the right hand side of (5.3) by $g_{1n}(t)$ and $g_{2n}(t)$, respectively. For $0<\delta\leqq|t|\leqq\pi$, the estimate $|E_n(t)|\leqq\pi/\delta$ gives,

$$|g_{2n}(t)| \leqq \left(\frac{\pi}{\delta}+1\right) \sum_{k=0}^{n} |\Delta(c(-k)-c(k))| \lg k.$$

Consequently, (5.4) implies $\lim_{n\to\infty} g_{2n}(t)=f_2(t)$ exists for $t\in T-\{0\}$. For $g_{1n}(t)$, by (5.5), the theorem of Riesz [8] gives that $\sum_{n=0}^{\infty} \Delta c(n)\sin(n+\frac{1}{2})t$ is the Fourier series of some $f_1\in L^q(T)\cdot\left(\frac{1}{p}+\frac{1}{q}=1\right)$. Thus, since $f_1\in L^2(T)$, by the Carleson result [13],

$$\lim_{n\to\infty}\sum_{k=0}^{n} \Delta c(k)\sin(k+\tfrac{1}{2})t=f_2(t)$$

a.e. in T. This proves the lemma with $f(t)=f_1(t)/\sin t/2+f_2(t)$.

Note that we could have replaced (5.4) and (5.5) with the single condition, $\sum_{|n|<\infty} |\Delta c(n)|^p<\infty$. However, the ideas of this part of the paper depend heavily on the notion of asymptotic evenness, a notion to which (5.4) belongs.

For convenient statements of our results three classes of coefficients are introduced. The null sequence $\{c(n)\}$ is said to belong to the class AE (asymptotically even) if

$$\frac{1}{n}\sum_{k=1}^{n} |c(k)-c(-k)|\lg k=o(1)\quad (n\to\infty) \tag{5.6}$$

and

$$\lim_{\lambda\downarrow 1}\overline{\lim_{n\to\infty}}\sum_{k=n}^{[\lambda n]} |\Delta(c(k)-c(-k))|\lg k=0. \tag{5.7}$$

The bounded sequence $c(n)$ is said to belong to the class BAE if $0(1)$ replaces $o(1)$ in (5.6) and (5.7) is replaced by: for some $\lambda>1$,

$$\sum_{k=n}^{[\lambda n]} |\Delta(c(k)-c(-k))|\lg k=0(1)\quad (n\to\infty).$$

With these in mind, our main results now are formulated.

Theorem 5.1. (i) *Let* $\{c(n)\}\in\mathrm{AE}$, *and let* (5.4) *hold. If for* $1<p\leqq 2$,

$$\frac{1}{n}\sum_{k=1}^{n} k^p|\Delta c(k)|^p=o(1)\quad (n\to\infty), \tag{5.8}$$

then

$$\lim_{n\to\infty} g_n(t)=f(t) \text{ exists a.e. in } T-\{0\}, \tag{5.9}$$

and, the trigonometric series $\sum_{|n|<\infty} c(n)c^{int}$ *is the Fourier series of* f *if and only if* $\|g_n-f\|=o(1)$ $(n\to\infty)$.

(ii) *Let* $c(n)\in\mathrm{BAE}$. *If for* $1<p\leqq 2$,

$$\frac{1}{n}\sum_{k=1}^{n} k^p|\Delta c(k)|^p=0(1)\quad (n\to\infty),$$

then trigonometric series $\sum_{|n|<\infty} c(n)c^{int}$ *is the Fourier-Stieljes series of some* $\mu\in M(T)$ *if and only if* $\|g_n\|=0(1)$ $(n\to\infty)$.

Proof. The details for (i) are given, (ii) parallels it. Notice that (5.8) implies (5.5). Indeed,

$$\begin{aligned}\sum_{k=1}^{n} |\Delta c(k)|^p &= \sum_{k=1}^{n-1} \Delta\left(\frac{1}{k^p}\right) \sum_{l=1}^{k} l^p |\Delta c(l)|^p + \frac{1}{n^p} \sum_{k=1}^{n} k^p |\Delta c(k)|^p \\ &\leqq \sum_{k=1}^{n} \frac{1}{k^p}\left(\frac{1}{k}\sum_{l=1}^{k} l^p |\Delta c(l)|^p\right) + o(1) \\ &= 0\left(\sum_{k=1}^{n} \frac{1}{k^p}\right), \quad (n\to\infty).\end{aligned}$$

Thus, by Lemma 5.2, we have (5.9). Now suppose $\|g_n - f\| = o(1)$ $(n\to\infty)$, then $\|\sigma_n^g - f\| = o(1)$ $(n\to\infty)$.

Since

$$\begin{aligned}g_n(t) &= S_n(t) - [c(n+1)E_n(t) + c(-(n+1))E_{-n}(t) - 1)] \\ &= S_n(t) - c(n+1)D_n(t) + [c(n+1) - c(-(n+1))](E_n(t) - 1),\end{aligned}$$

it follows that,

$$\begin{aligned}\sigma_n(t) - f(t) = (\sigma_n^g(t) - f(t)) &+ \frac{1}{n+1}\sum_{k=1}^{n} c(k+1)D_k(t) \\ &- \frac{1}{n+1}\sum_{k=1}^{n} [c(k+1) - c(-(k+1))](E_{-k}(t) - 1).\end{aligned}$$

Denote the L^1-norm of second and third terms on the right hand side by I_1 and I_2, respectively. Then as in the proof of Lemma 5.1,

$$I_1 \leqq A_p\left(\frac{1}{n+1}\sum_{k=0}^{n} |c(k)|^p\right)^{1/p} = o(1) \quad (n\to\infty)$$

and

$$I_2 \leqq \frac{A}{n+1}\sum_{k=1}^{n} |c(k+1) - c(-(k+1))| \lg k = o(1) \quad (n\to\infty),$$

where A_p and A are absolute constants. Consequently, $\|\sigma_n - f\| = o(1)$ $(n\to\infty)$, and trigonometric series is the Fourier series of f. To obtain the converse, recall (5.2). Then $\|g_{[\lambda n]} - g_n\| = o(1)$ $(n\to\infty)$ by considerations like those immediately above. Also,

$$\lim_{\lambda\downarrow 1}\lim_{n\to\infty}\sum_{k=n}^{[\lambda n]} k^{p-1}|\Delta c(k)|^p = 0$$

by (5.8) and finally (5.7) takes care of the rest. This completes the proof.

This result has several corollaries.

Corollary 5.1.1. (i) *Let* $\{c(n)\} \in \mathrm{AE}$, *and let* (5.4) *hold.*

If

$$\frac{1}{n}\sum_{k=1}^{n} k|\Delta c(k)| = o(1) \quad (n\to\infty), \tag{5.10}$$

and

$$n\Delta c(n) = 0(1) \quad (n \to \infty), \tag{5.11}$$

then (5.9) *holds and trigonometric series is the Fourier series of f if and only if* $\|g_n - f\| = o(1)$ $(n \to \infty)$.

(ii) *Let* $\{c(n)\} \in \mathrm{BAE}$ *and let* (5.11) *hold.*

Then trigonometric series is the Fourier-Stieljes series of some $\mu \in M(T)$ if and only if $\|g_n\| = 0(1)$ $(n \to \infty)$.

Part (i) follows from the fact that (5.10) and (5.11) imply (5.8).

Corollary 5.1.2. *Let* $\{c(n)\} \in \mathrm{AE}$, *let* (5.4) *hold with* $p = 1$. *If* (5.11) *hold, then* (5.9) *holds and trigonometric series is the Fourier series of f if and only if* $\|g_n - f\| = o(1)$ $(n \to \infty)$.

The proof follows from the fact that (5.5) with $p = 1$ implies (5.10).

Corollary 5.1.3. *Let* $\{c(n)\} \in \mathrm{AE}$, *and let* (5.4) *hold. If for* $1 < p \leqq 2$,

$$\sum_{n=1}^{\infty} n^{p-1} |\Delta c(n)|^p < \infty, \tag{5.12}$$

then (5.9) *holds and trigonometric series is the Fourier series of if and only if* $\|g_n - f\| = o(1)$ $(n \to \infty)$.

Notice that the usefulness of the Tauberian nature of the inequality such as (5.2) has not been used yet (as in Sect. 3). For this one should devise a result like those above whose sole condition is

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta c(k)|^p = 0. \tag{5.13}$$

However (5.13) alone evidently does not imply (5.5), i.e. the a.e. limit of the sequence $g_n(t)$ is not given directly. More will be said in this direction in Sect. 6. The following result making use of (5.13) generalizes Corollary 5.1.1 significantly.

Theorem 5.2. *Let* $\{c(n)\} \in \mathrm{AE}$, *and let* (5.4) *and* (5.11) *hold. Then* (5.9) *holds and trigonometric series is the Fourier series of f if and only if* $\|g_n - f\| = o(1)$ $(n \to \infty)$.

The proof of this is done by noting that (5.11) implies both (5.5) and (5.12), then use Lemma 5.1 in the same was as in the proof of Theorem 5.1.

6. Further Comments

The results of Sect. 3 seem (in particular) to indicate that L^1-convergence classes are obtained through tight local regularity of the sequence $\{\Delta \hat{f}(n)\}$.

Returning to the remarks preceding Theorem 5.2, the following result is apparent.

Proposition. *Let* $f \in L^1(T)$. *If* $\{f(n)\} \in \mathrm{AE}$ *and let* (5.13) *hold, then* $\|g_n - f\| = o(1)$ $(n \to \infty)$.

Numerous corollaries to this in the vein of Sect. 3 could easily be developed. Also note that Theorem A is a corollary to this. These considerations show that with respect to representation of f in L^1-norm, the sequence of modified sums is better behaved than partial sums. This in combination with earlier observations concerning cosine series further modivates the results of Sect. 5. Also note that mapping $L^p \to L^p$ given by $f \to g_n(f)$ defines a linear operator which is unbounded on L^p for all $p \geqq 1$. A study of this operator on L^p, $p > 1$ will be the topic in further investigation in theory of characterization of Fourier series and Fourier-Stieljes series.

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Cohomology of Induced Representations for Algebraic Groups

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Introduction

Let G denote a reductive algebraic group over an algebraically closed field K of characteristic $p>0$. This paper is concerned with the computation of the cohomology for the (first) Frobenius kernel G_1 of G with coefficients in a G-module induced from a Borel subgroup B.

Fix a maximal torus T in B and let U be the unipotent radical of B. If W denotes the Weyl group for G with respect to T, if $w \in W$ has length $\ell(w)$ and if λ is a dominant character of T, then our main result (cf. 3.7 and 5.5) can be formulated as follows: If $w \cdot 0 + p\lambda$ is dominant, then

$$H^i(G_1, H^0(G/B, w \cdot 0 + p\lambda))^{(-1)} \simeq H^0(G/B, S^{(i-\ell(w))/2}(\mathfrak{u}^*) \otimes \lambda), \tag{1}$$

where $S(\mathfrak{u}^*)$ is the symmetric algebra on $\mathfrak{u}^* = (\operatorname{Lie} U)^*$. [We prove this under some very mild (not completely unnecessary) restrictions on p and for groups of type E and F under some (hopefully unnecessary) restrictions on λ.] Observe that $H^i(G_1, H^0(G/B, \mu)) = 0$ for any μ not considered in (1) and that we get an explicit character formula for the right hand side in (1) considered as a G-module, as we prove that the higher cohomology vanishes. For $\lambda = 0$ and $w = 1$ we obtain in particular Friedlander and Parshall's results in [10] and [11] on $H^\cdot(G_1, K)$. In fact, our work was started by trying to understand and generalize the first of these two papers and benefited from informations about some results and methods of the second one.

Let E be a B-module. Our idea is to examine the relation between the G_1-cohomology of the induced G-module $H^0(G/B, E)$ and the cohomology $H^\cdot(G/B, H^\cdot(B_1, E)^{(-1)})$. Here B_1 denotes the Frobenius kernel of B. This relation is described by two spectral sequences having the same abutment. We show that for a one dimensional B-module given by a dominant weight λ both these spectral sequences degenerate. In the first case this follows from Kempf's vanishing theorem, whereas in the second case it requires first an explicit determination of the B_1-cohomology and then a vanishing theorem for the higher G/B-cohomology of $S(\mathfrak{u}^*) \otimes \lambda$.

To compute $H^{\cdot}(B_1, \lambda)$ we first use the method from [10] to find $H^{\cdot}(B_1, K)$ and then we show that the rest may be obtained from this via translation. The vanishing of $H^i(G/B, S(\mathfrak{u}^*)\otimes\lambda)$ for $i>0$ is proved by considering a Koszul resolution ending $S(\mathfrak{g}^*)\to S(\mathfrak{u}^*)\to 0$ where $\mathfrak{g}=\operatorname{Lie} G$. There are some technical difficulties when λ is close to one of the walls of the dominant chamber, and, in fact, for such λ we prove the result only for classical groups. The case $\lambda=0$ is easy, however.

Our description of $H^{\cdot}(G_1, H^0(G/B, \lambda))^{(-1)}$ also allows us to prove (for $\lambda=0$ or far from the walls) that it has a "good" filtration. The key to the proof is again the Koszul resolution which reduces the problem to show that $S(\mathfrak{g}^*)$ has a good filtration. For the classical groups we prove that this is so for all odd primes, whereas for the exceptional groups we have to exclude also $p=3$ and (for E_8) $p=5$. (In [11] the same statement is proved, but with considerably stronger restrictions on p.)

Part of our method for computing G_1-cohomology works also for the higher Frobenius kernels $G_r, r>1$. As we point out, however, already for SL_2 the G_r-cohomology is much more complicated for $r=2$ than for $r=1$. Whenever possible we have worked with general r.

The paper is organized as follows. In Sect. 1 we consider arbitrary algebraic groups and extend the results of [10] to such groups. In particular, we describe explicitly the cohomology ring $H^{\cdot}(V_r, K)$, where V_r is the r^{th} Frobenius kernel of a vector space V over K. For a general reduced algebraic group M this gives us the terms in the spectral sequence relating $H^{\cdot}(M_r, K)$ to $H^{\cdot}(\mathfrak{m}_r, K)$ where $\mathfrak{m}$ is the Lie algebra of M. The computation of B_1-cohomology is done in Sect. 2 and then used in Sect. 3 to obtain the above mentioned results on G_1-cohomology. In Sect. 4 we prove that certain G-modules have a good filtration, in particular, we examine $S(\mathfrak{g}^*)$. We then discuss some applications, mainly to cohomology groups, but also to invariant theory. The Sect. 5 contains the more technical proof for the case of a weight close to the wall of the dominant chamber. Here we carry out a case by case analysis for the classical groups. In Sect. 6 we show that we get different results for small p in some cases.

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1. The Spectral Sequence for the Cohomology of Frobenius Kernels

In this section we rederive and extend to their proper generality the results of [10], 5.1, working with arbitrary groups and arbitrary Frobenius kernels.

Throughout this paper let K be an algebraically closed field.

1.1. Let M be an affine group scheme over K with algebra of regular functions $K[M]$ and augmentation ideal $I\subset K[M]$. Then $K[M]$ has a natural filtration by the powers of I, and we can form the associated graded algebra $grK[M]=\bigoplus_{n\geq 0} I^n/I^{n+1}$. As the comultiplication Δ and the coinverse σ satisfy $\Delta(I)\subset I\otimes K[M]+K[M]\otimes I$ and $\sigma(I)=I$ they induce a comultiplication and a coinverse on $\operatorname{gr} K[M]$ so that we may regard $\operatorname{gr} K[M]$ as the ring of regular functions $K[\operatorname{gr} M]$ on an affine group scheme $\operatorname{gr} M$ over K.

Let $K[X]$ be the polynomial ring over K in one variable, i.e. the ring of regular functions on K. As Δ satisfies more precisely $\Delta(x) \in 1 \otimes x + x \otimes 1 + I \otimes I$ for all $x \in I$, we see that the map $K[\operatorname{gr} M] \to K[X] \otimes K[\operatorname{gr} M]$ sending any homogeneous element f of degree n to $X^n \otimes f$ or the corresponding map $K \times \operatorname{gr} M \to \operatorname{gr} M$ have all the formal properties of a vector space structure on $\operatorname{gr} M$. Thus we may regard $\operatorname{gr} M$ not only as a group scheme, but even as a vector space scheme over K, i.e. an $\mathfrak{O}_K$-module in the sense of [7].

Obviously, $K[\operatorname{gr} M]$ is a homomorphic image of the symmetric algebra $S(I/I^2)$, which we may consider as the ring of regular functions on the Lie algebra $\mathfrak{m} \simeq (I/I^2)^*$ of M. It is now clear that the surjection of $K[\mathfrak{m}] \simeq S(I/I^2)$ onto $K[\operatorname{gr} M]$ is compatible with the vector space structures. We can therefore identify $\operatorname{gr} M$ with a closed subspace scheme of $\mathfrak{m}$.

1.2. Lemma. *Let M be a reduced algebraic group over K with Lie algebra $\mathfrak{m}$. Then $\operatorname{gr} M$ is naturally isomorphic to $\mathfrak{m}$. Suppose $\operatorname{char} K = p \neq 0$ and denote the r^{th} Frobenius kernel of M (respectively $\mathfrak{m}$) by M_r (respectively $\mathfrak{m}_r$). Then $\operatorname{gr}(M_r)$ is naturally isomorphic to $\mathfrak{m}_r$ for any $r > 0$.*

Proof. In order to prove the first statement we have to show that the natural surjection from $S(I/I^2)$ onto $\operatorname{gr} K[M]$ is an isomorphism. This however follows from [7], II, Sect. 5, 2.1 or more explicitly from [26], Lemma 11.4 and Remark on p. 108.

In the second case observe that $K[M_r] = K[M]/K[M] I^{(q)}$ where $q = p^r$ and $I^{(q)} = \{x^q | x \in I\}$. Similarly $K[\mathfrak{m}_r] \simeq S(I/I^2)/(I/I^2)^{(q)}$. It is now easily checked that the isomorphism $K[\mathfrak{m}] \to \operatorname{gr} K[M]$ already established induces an isomorphism $K[\mathfrak{m}_r] \to \operatorname{gr} K[M_r]$, proving our claim.

1.3. Let M be as in 1.1 and let E be an M-module. We define a filtration of the Hochschild complex $C^\cdot(M, E)$, where

$$C^n(M, E) = E \otimes K[M] \otimes K[M] \otimes \ldots \otimes K[M]$$

(n factors $K[M]$), by

$$C^n(M, E)_{(m)} = \sum E \otimes I^{r(1)} \otimes I^{r(2)} \otimes \ldots \otimes I^{r(n)},$$

where we sum over all n-tuples $(r(i))_{1 \leqq i \leqq n}$ with $\sum_{i=1}^{n} r(i) \geqq m$. Again because of $\Delta(I) \subset K[M] \otimes I + I \otimes K[M]$ we see that the coboundary operators map (cf. [7, II, Sect. 3, 3.1]) each $C^n(M, E)_{(m)}$ into $C^{n+1}(M, E)_{(m)}$. By general results about filtered complexes (cf. [13, I, 4.2]) there is a spectral sequence converging (for connected M) to $H^\cdot(M, E)$ with E_1-term

$$E_1^{s,t} = H^{s+t}(\operatorname{gr}_s C^\cdot(M, E)). \tag{1}$$

Let us on the other hand consider E as a trivial $(\operatorname{gr} M)$-module, i.e., with comodule map $E \to E \otimes K[\operatorname{gr} M]$ sending each $e \in E$ to $e \otimes 1$. The grading on $K[\operatorname{gr} M]$ induces a grading on $C^\cdot(\operatorname{gr} M, E)$. We may obviously identify $\operatorname{gr}_s C^\cdot(M, E)$ with $C^\cdot(\operatorname{gr} M, E)_s$, the homogeneous part of degree s of $C^\cdot(\operatorname{gr} M, E)$. This identification is compatible with the coboundary operators, as the given comodule map $E \to E \otimes K[M]$ sends each $e \in E$ to an element of $e \otimes 1 + E \otimes I$. We can therefore deduce from (1):

$$E_1^{s,t} \simeq E \otimes H^{s+t}(\mathrm{gr}\, M, K)_s. \tag{2}$$

(Observe that the differentials in the spectral sequence involve E.)

We may apply all this to the trivial module K. As we have in general

$$C^n(M,E)_{(m)} \otimes C^{n'}(M,K)_{(m')} \subset C^{n+n'}(M,E)_{(m+m')},$$

we see that the spectral sequence is compatible with the $H^{\cdot}(\mathrm{gr}\, M, K)$-module structure on $H^{\cdot}(\mathrm{gr}\, M, E)$ and the $H^{\cdot}(M,K)$-module structure on $H^{\cdot}(M,E)$. In the case of $E = K$ this means a compatibility with the ring structure.

1.4. Combining 1.2 and 1.3 we get:

Proposition. *Let M be a reduced algebraic group over K, where* $\mathrm{char}\, K = p \neq 0$. *Let* $\mathfrak{m} = \mathrm{Lie}\, M$ *and denote by $M_r, \mathfrak{m}_r$ the r^{th} Frobenius kernels. Then there is for each M_r-module E a spectral sequence*

$$E_1^{s,t} = H^{s+t}(\mathfrak{m}_r, K)_s \otimes E \ \Rightarrow\ H^{s+t}(M_r, E).$$

Remark. Note that we regard $\mathfrak{m}_r$ not just as an affine group scheme, but as a vector space scheme in order to have a fixed grading on $K[\mathfrak{m}_r]$. (Note that the choice of an operation of $K^{\times}$ on an affine scheme X amounts to the choice of a $\mathbf{Z}$-grading on $K[X]$.)

1.5. Suppose from now on $\mathrm{char}\, K = p \neq 0$.

In order to apply 1.4 later on we are now going to describe $H^{\cdot}(V_r, K)$, where V is a finite dimensional vector space over K. We may assume V to be defined over $\mathbf{F}_p$ with Frobenius endomorphism F. If $n = \dim V$, we identify $K[V]$ with the polynomial ring $K[X_1, \ldots, X_n]$, mapping the dual space V^* onto $\sum K X_i$. We choose the X_i defined over $\mathbf{F}_p$, i.e. with $F^* X_i = X_i^p$ where F^* is the corresponding geometric Frobenius endomorphism on $K[V]$.

The restriction of functions induces now isomorphisms from the space $K[V,r]$ generated by all monomials $X_1^{m(1)} X_2^{m(2)} \ldots X_n^{m(n)}$ with $m(i) < p^r$ for all i onto $K[V_r]$ as well as onto $K[V(p^r)]$, where $V(p^r)$ is the finite group of fixed points of F^r, isomorphic to the elementary abelian p-group $(\mathbf{Z}/\mathbf{Z}p)^{rn}$.

As observed in [4], p. 151 for $\dim V = 1$ the

$$C^m(V,K,r) = K[V,r] \otimes K[V,r] \otimes \ldots \otimes K[V,r]$$

for r fixed form a subcomplex (stable under the cup product) of the Hochschild complex $C^{\cdot}(V,K)$. It is (under restriction of functions) isomorphic to $C^{\cdot}(V_r, K)$ and to $C^{\cdot}(V(p^r), K)$. Therefore both complexes have the same cohomology, proving

Proposition. *For each r there is an isomorphism of graded algebras*

$$H^{\cdot}(V(p^r), K) \simeq H^{\cdot}(V_r, K).$$

1.6. The cohomology of cyclic groups being well known (cf. e.g. [22]) we can now describe $H^{\cdot}(V_r, K)$ explicitly. For any vector space V' we denote by $S(V')$

respectively $\Lambda(V')$ its symmetric respectively exterior algebra given its natural grading. If we change the grading in $S(V')$ by giving all elements of V' degree 2, we denote the graded algebra thus constructed by $S'(V')$.

Suppose we are given a Frobenius endomorphism F on V'. We denote by $V'^{(i)}$ the space V', on which we let any $g \in GL(V')$ act as $F^i(g) = F^i \circ g \circ F^{-i}$.

Proposition. *Set* $V' = V^* \oplus V^{*(1)} \oplus \ldots \oplus V^{*(r-1)}$. *Then*

$$H^\cdot(V_r, K) \simeq \begin{cases} S'(V'^{(1)}) \otimes \Lambda(V') & \text{if } p \neq 2 \\ S(V') & \text{if } p = 2. \end{cases}$$

Proof. By 1.5 and e.g. the description in [4], 4.1 we have $H^\cdot(V_r, K) \simeq S(H^1(V_r, K))$ if $p=2$ and $H^\cdot(V_r, K) \simeq S'(\beta H^1(V_r, K)) \otimes \Lambda H^1(V_r, K)$ if $p \neq 2$, where $\beta: H^1(V_r, K) \to H^2(V_r, K)$ is the Bockstein operator. Furthermore, $H^1(V_r, K)$ has a basis consisting of the classes of all $X_i^{p^j} = F^j(X_i)$ with $1 \leq i \leq n$ and $0 \leq j < r$. The classes of the $F^j(X_i)$ for fixed j form a basis of a vector space which can be identified with $V^{*(j)}$ as a $GL(V)$-module. This proves $H^1(V_r, K) \simeq V'$. For $p \neq 2$ we have furthermore that β is injective and maps the class of any x into that of $\sum_{i=1}^{p-1} \left\{ {p \atop i} \right\} x^i \otimes x^{p-i}$, where $\left\{ {p \atop i} \right\} = p^{-1} \binom{p}{i} \in \mathbf{N}$. Therefore, β commutes with the action of $GL(V)$ and is F-linear. This implies $\beta(V') \simeq V'^{(1)}$, hence the proposition.

Remark. We regard V_r here not just as an algebraic group scheme, but as a vector space scheme. Otherwise we had no grading on $K[V_r]$ and we would have to replace V' in the proposition by a vector space with a filtration, where the factors are isomorphic to $V^{*(r-1)}, V^{*(r-2)}, \ldots, V^*$ from bottom to top.

1.7. The grading of $K[V] \simeq S(V^*)$ induces one on each $C^n(V, K, r)$ and hence also one on $H^n(V_r, K)$. As $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in V^*$ we see that Δ respects this grading, and hence we have also a grading on $H^n(V_r, K)$. The explicit description of the generators of $H^\cdot(V_r, K)$ in the proof of 1.6 shows:

The elements of $V^{*(i)} \subset V'$ *and of* $V^{*(i-1)(1)} \subset V'^{(1)}$ *are homogeneous of degree* p^i.

1.8. Let M be a reduced algebraic group over K with Lie algebra $\mathfrak{m}$. We denote the r^{th} Frobenius kernels of M and $\mathfrak{m}$ by M_r and $\mathfrak{m}_r$. Let us for the sake of simplicity assume M to be defined over $\mathbf{F}_p$. As M operates by conjugation linearly on $\mathfrak{m}_r$ we get combining 1.6/7 with 1.4:

Proposition. *There is a spectral sequence* $E_n^{s,t}$ *of* M*-modules converging to* $H^\cdot(M_r, K)$ *where* $E_1^{s,t}$ *is the direct sum of all*

$$(S^{m(1)}\mathfrak{m}^*)^{(1)} \otimes (S^{m(2)}\mathfrak{m}^*)^{(2)} \otimes \ldots \otimes (S^{m(r)}\mathfrak{m}^*)^{(r)} \otimes \Lambda^{n(1)}\mathfrak{m}^*$$
$$\otimes (\Lambda^{n(2)}\mathfrak{m}^*)^{(1)} \otimes \ldots \otimes (\Lambda^{n(r)}\mathfrak{m}^*)^{(r-1)}$$

with $\sum_{i=1}^{r}(2m(i)+n(i))=s+t$ *and* $\sum_{i=1}^{r}(m(i)p^i+n(i)p^{i-1})=s$ *if* $p\neq 2$, *respectively of all*

$$S^{m(1)}\mathfrak{m}^*\otimes(S^{m(2)}\mathfrak{m}^*)^{(1)}\otimes\ldots\otimes(S^{m(r)}\mathfrak{m}^*)^{(r-1)}$$

with $\sum_{i=1}^{r}m(i)=s+t$ *and* $\sum_{i=1}^{r}m(i)p^{i-1}=s$ *if* $p=2$.

Remark. For $r=1$ and $p\neq 2$ we may as in [10], 5.1 observe that the differentials $d_m^{s,t}$ in the spectral sequence are zero if $m\not\equiv 1 \bmod (p-2)$. Therefore, we may re-index the $E_n^{s,t}$, calling now $E_n^{s,t}$ the old $E_{n(p-2)+1}^{(p-1)s+t,\,-(p-2)s}$ in order to get a spectral sequence with

$$E_0^{s,t}=S^s(\mathfrak{m}^*)^{(1)}\otimes\Lambda^{t-s}\mathfrak{m}^*. \tag{1}$$

2. The B_1-Cohomology

From now on we assume $\operatorname{char}K=p\neq 0$, though some of our results later on will make sense also if $\operatorname{char}K=0$.

In this section we compute under some mild restriction on p the B_1-cohomology of one dimensional B-modules, where B is a Borel subgroup of a reductive algebraic group over K.

2.1. We shall now introduce some notations which will be used in all remaining parts of this paper.

Let G be a connected, simply connected reductive algebraic group over K. Let $B=TU$ be a Borel subgroup of G with its unipotent radical U and a maximal torus T. We denote the corresponding Lie algebras by $\mathfrak{g}=\operatorname{Lie}G$, $\mathfrak{b}=\operatorname{Lie}B$, $\mathfrak{u}=\operatorname{Lie}U$, but $\mathfrak{h}=\operatorname{Lie}T$.

Let $X(T)$ be the group of characters of T and $R\subset X(T)$ be the set of roots. We choose an ordering of the roots, so that the roots in B are negative. We write R_+ for the set of positive roots (i.e. the weights of T on $\mathfrak{g}/\mathfrak{b}$) and $X(T)^+$ for the set of dominant weights. Let $\leq$ be the usual order relation on $X(T)$ defined by R_+ so that $\lambda\leq\mu$ if and only if $\mu-\lambda$ is a positive linear combination of positive roots.

We denote by ϱ half the sum of the positive roots and by W the Weyl group of T. We write $w\cdot\lambda=w(\lambda+\varrho)-\varrho$ for $w\in W$ and $\lambda\in X(T)$. For each $\alpha\in R$ there is a reflection $s_\alpha\in W$ with respect to α. The s_α with α simple generate W; for $w\in W$ let $\ell(w)$ be the length of w with respect to this set of generators.

For each T-module V and each $\lambda\in X(T)$ let V^λ denote the λ-weight space of V and let $\operatorname{ch}V$ denote (for $\dim V<\infty$) the formal character of V. For all $\lambda\in X(T)$ set $\chi(\lambda)=\sum_i(-1)^i\operatorname{ch}H^i(G/B,\lambda)$. Then we have $\chi(w\cdot\lambda)=(-1)^{\ell(w)}\chi(\lambda)$ for all $w\in W$ and $\chi(\lambda)=\operatorname{ch}H^0(G/B,\lambda)$ for $\lambda\in X(T)^+$ by Kempf's vanishing theorem.

If R is indecomposable, we write α_0 for its largest short root and $h=\langle\varrho,\alpha_0^\vee\rangle+1$ for its Coxeter number. In general we denote by h the maximum of the Coxeter numbers of the indecomposable components of R.

For the sake of simplicity we assume G, B, T, U to be defined over $\mathbf{F}_p$ with Frobenius endomorphism F and T to be split over $\mathbf{F}_p$. We denote r^{th} Frobenius kernels again by $G_r, B_r, \ldots$. If we twist a G-module V by F^r we denote the twisted module by $V^{(r)}$. If a representation of G on some space V factors through G/G_r, then there is a unique G-module denoted by $V^{(-r)}$ such that $V \simeq (V^{(-r)})^{(r)}$, because G/G_r is isomorphic to G via F^r. We use similar conventions for the other groups (B, T, U).

2.2. We shall now prove some properties of the weights of $\Lambda\mathfrak{g}^*$, some of which are needed immediately, some others later on. We shall assume in our proofs R to be indecomposable and G semisimple. It is then easy to treat the general case.

(1) *Suppose $p > h$. If λ is a weight of Λg^* vanishing on T_1, then $\lambda = 0$.*

Proof. We may assume λ to be dominant. As it vanishes on T_1 there is $\mu \in X(T)^+$ with $\lambda = p\mu$. Now λ is a sum of different roots, hence $\lambda \leqq 2\varrho$. This implies $0 \leqq p\langle \mu, \alpha_0^\vee \rangle \leqq 2\langle \varrho, \alpha_0^\vee \rangle = 2(h-1) < 2p$, hence $\langle \mu, \alpha_0^\vee \rangle \in \{0, 1\}$. If $\langle \mu, \alpha_0^\vee \rangle = 1$, then μ is a fundamental weight not in $\mathbf{Z}R$. Indeed the μ with this property form a system of representatives for the non-zero classes in $X(T)/\mathbf{Z}R$ (cf. [3, Chap. VI, Sect. 1, Exercise 24c, Sect. 2, Exercise 5a]). As p does not divide the order of $X(T)/\mathbf{Z}R$ (for $p > h$) we see that also $\lambda = p\mu \notin \mathbf{Z}R$ contradicting the fact that λ is a weight of $\Lambda\mathfrak{g}^*$. Hence $\langle \mu, \alpha_0^\vee \rangle = 0$ and then $\mu = \lambda = 0$. (For type A_n we have $h = n+1$, and there is a weight of $\Lambda\mathfrak{g}^*$ divisible by $n+1$. Thus the bound cannot be improved in this case.)

(2) *Let v be a weight of $\Lambda(\mathfrak{g}/\mathfrak{b})^*$. Then $w(v+\varrho) \leqq \varrho$ for all $w \in W$ and*

$$|\langle v+\varrho, \alpha^\vee \rangle| \leqq h-1$$

for all $\alpha \in R$.

Proof. Obviously v is a sum of different negative roots. Therefore $v + \varrho$ has the form

$$\tfrac{1}{2} \sum_{\alpha \in Q} \alpha - \tfrac{1}{2} \sum_{\alpha \in Q'} \alpha$$

for some disjoint decomposition $R_+ = Q \cup Q'$. Then every $w(v+\varrho)$ has also this form, which implies $w(\varrho + v) \leqq \varrho$. Choose now $w \in W$ with $w(\varrho + v) \in X(T)^+$. Then

$$\max_{\alpha \in R} |\langle v+\varrho|, \alpha^\vee \rangle| = \max_{\alpha \in R} |\langle w(v+\varrho), \alpha^\vee \rangle| = \langle w(v+\varrho), \alpha_0^\vee \rangle \leqq \langle \varrho, \alpha_0^\vee \rangle = h-1 .$$

[If there are two root lengths and if α is long, we may replace α_0 in the last argument by the longest root $\tilde{\alpha}$. As $\langle \varrho, \tilde{\alpha}^\vee \rangle = 2(n-2)$ (respectively n, 8, 3) for type B_n (respectively C_n, F_4, G_2) we get a slightly better bound for long roots in these cases.]

2.3. **Proposition.** *Suppose $p > h$. Then there is an isomorphism of graded B-algebras*

$$H^\cdot(B_1, K) \simeq S^\cdot(\mathfrak{u}^*)^{(1)}.$$

Proof. By 1.8 there is a spectral sequence of B-modules converging to $H^\cdot(U_1, K)$ with $E_0^{m, m+n} \simeq S^m(\mathfrak{u}^*)^{(1)} \otimes \Lambda^n(\mathfrak{u}^*)$. All T_1-modules being semisimple we can identify $H^\cdot(B_1, K)$ with the T_1-fixed points of $H^\cdot(U_1, K)$:

$$H^\cdot(B_1, K) \simeq H^0(T_1, H^\cdot(U_1, K)) = H^0(U_1, K)^{T_1}.$$

The spectral sequence $(E_n^{s,t})^{T_1}$ converges to this cohomology.

The weights of T on $\Lambda^n\mathfrak{u}^*$ are sums of n different positive roots, hence non-zero for $n>0$. By 2.2 (1) this implies that they do not vanish on T_1, hence $(\Lambda^n\mathfrak{u}^*)^{T_1}=0$ for $n>0$ and $(E_0^{m,n})^{T_1}=0$ for $m\neq n$. Any differential d_i having bidegree $(i, 1-i)$ this spectral sequence degenerates. Therefore, $H^{2m}(B_1, K)$, is isomorphic to $S^m(\mathfrak{u}^*)^{(1)}$, whereas $H^{2m+1}(B_1, K)=0$. The linear isomorphism $H^{\cdot}(B_1, K)\simeq S^{\cdot}(\mathfrak{u}^*)^{(1)}$ obtained so far is compatible with the multiplication and with the B-action, as the spectral sequence is. This proves our claim.

2.4. For $r>1$ the spectral sequence from 1.8 will not degenerate after taking T_r-fixed points, as it does for $r=1$. Let us, however, consider now the very special case where G has semi-simple rank 1. There we can get some results on $H^{\cdot}(B_r, K)$ also for $r>1$.

Let us denote the only positive root by α and suppose at first $p\neq 2$. If $x_1, \dots, x_n$ is a basis of some vector space V', let us write $S(x_1, \dots, x_n)$ and $\Lambda(x_1, \dots, x_n)$ instead of $S(V')$ and $\Lambda(V')$; in this notation we shall leave the choice of the grading open. As U is isomorphic to the additive group, we have by 1.6

$$H^{\cdot}(U_r, K)\simeq S(x_1, \dots, x_r)\otimes\Lambda(y_1, \dots, y_r), \tag{1}$$

where each y_i (respectively x_i) is a vector of weight $p^{i-1}\alpha$ (respectively $p^i\alpha$) living in degree 1 (respectively 2). This isomorphism is compatible with the B-action, as B operates linearly on U under conjugation.

From (1) we deduce immediately

$$H^{\cdot}(U_r, K)^{T_1}\simeq S(x_1, \dots, x_r)\otimes\Lambda(y_2, \dots, y_r)$$

and

$$H^{\cdot}(U_r, K)^{T_2}=(H^{\cdot}(U_r, K)^{T_1})^{T_2}\simeq S(x_2, \dots, x_r)\otimes\Lambda(y_3, \dots, y_r)\otimes(S(x_1)\otimes\Lambda(y_2))^{T_2}.$$

It is easy to check that $(S(x_1)\otimes\Lambda(y_2))^{T_2}$ is spanned by the $x_1^{pm}\otimes 1$ and the $x_1^{pm-1}\otimes y_2$. We can, therefore, identify this algebra with $S(x_1^p)\otimes\Lambda(x_1^{p-1}\otimes y_2)$. Thus we get

$$H^{\cdot}(B_2, K)\simeq H^{\cdot}(U_2, K)^{T_2}\simeq(S(x, x')\otimes\Lambda(y))^{(2)}, \tag{2}$$

where all three generators have weight α and where we assign to x the degree 2, to x' degree $2p$ and to y degree $2p-1$. For $r>2$ the results get more and more complicated.

Let us now look at $p=2$, the only case excluded for our G in 2.3. As $\alpha=2\varrho$ vanishes on T_1 we get

$$H^{\cdot}(B_1, K)\simeq H(U_1, K)\simeq S(\mathfrak{u}^*), \quad (p=2) \tag{3}$$

and

$$H^{\cdot}(B_2, K)\simeq S(x, x')^{(1)}, \quad (p=2) \tag{4}$$

with x, x' both of weight α and living in degree 1 and 2.

2.5. From now on G will again be as in 2.1. For each $\lambda\in X(T)$ let K_λ be the one dimensional B-module, on which T operates via λ and U trivially. For $\lambda\in p^rX(T)$

we may regard K_λ also as a G_rB-module. We shall often write simply λ instead of K_λ [e.g. $H^i(B_r,\lambda)=H^i(B_r,K_\lambda)$ or $V\otimes\lambda=V\otimes K_\lambda$], whenever no confusion is possible.

The induced module $H^0(G_r/B_r,\lambda)$ is in a natural way (because of $G_rB/B\simeq G_r/B_r$) a G_rB-module, which we shall denote by $\hat{Z}_r(\lambda)$. [This convention differs from that in [17], [18]. The $\hat{Z}(r,\lambda)$ constructed there using the Borel subgroup $B'\supset T$ corresponding to the positive roots give the $\hat{Z}_r(\lambda)$ here after conjugation by some $g\in N(T)$ with $gB'g^{-1}=B$ and after relabelling.]

Induction from B_r to G_r being an exact functor we get isomorphisms of B-modules

$$H^\cdot(G_r,\hat{Z}_r(\lambda))\simeq H^\cdot(B_r,\lambda)\quad\text{for each}\quad \lambda\in X(T). \tag{1}$$

Using the spectral sequence $H^i(G_rT/G_r,H^j(G_r,E)) \Rightarrow H^{i+j}(G_rT,E)$ and the identification $G_rT/G_r\simeq T/T_r$ we may identify $H^i(G_rT,E\otimes p^r\nu)$ for each G_rT-module E and each $\nu\in X(T)$ with the $(-p^r\nu)$-weight space of $H^i(G_r,E)$ or, equivalently, the $(-\nu)$-weight space of $H^i(G_r,E)^{(-r)}$. This implies especially

$$H^\cdot(G_r,E)=\bigoplus_{\nu\in X(T)} H^\cdot(G_rT,E\otimes p^r\nu). \tag{2}$$

For each residue class $Y\in X(T)/\mathbf{Z}R$ the sum of all $p^r\nu$-weight spaces in $H^\cdot(G_r,E)$ with $\nu\in Y$ is a G-submodule $H^\cdot(G_r,E)_Y$ of $H^\cdot(G_r,E)$, and we have

$$H^\cdot(G_r,E)=\bigoplus_{Y\in X(T)/\mathbf{Z}R} H^\cdot(G_r,E)_Y \tag{3}$$

and

$$H^\cdot(G_r,E)_Y=\bigoplus_{\nu\in Y} H^\cdot(G_rT,E\otimes(-p^r\nu)). \tag{4}$$

Suppose E has all its weights in $\mathbf{Z}R$. Then all weights of $H^i(G_r,E)$ belong to $\mathbf{Z}R\cap p^rX(T)$. If $p\nmid(X(T):\mathbf{Z}R)$ this intersection is equal to $p^r\mathbf{Z}R$, hence in this case all weights of $H^i(G_r,E)$ belong to $p^r\mathbf{Z}R$, i.e., we get:

$$H^\cdot(G_r,E)=H^\cdot(G_r,E)_{\mathbf{Z}R}\quad (E,p\text{ as above}). \tag{5}$$

Similar results hold also for B_r-cohomology.

2.6. For $\lambda\in X(T)^+$ let $L(\lambda)$ denote the simple G-module with highest weight λ. Any $\lambda\in X(T)$ can be written uniquely in the form $\lambda=\lambda_0+p^r\lambda_1$ where λ_0 satisfies $0\leqq\langle\lambda_0,\alpha^\vee\rangle<p^r$ for all simple roots α. Then $\hat{L}_r(\lambda)=L(\lambda_0)\otimes p^r\lambda_1$ is a simple G_rB-module and any simple G_rB-module is isomorphic to a unique $L_r(\lambda)$.

We claim for all $\lambda,\mu\in X(T)$:

$$\text{If } \operatorname{Ext}^1_{G_rB}(\hat{L}_r(\lambda),\hat{L}_r(\mu))\neq 0,\quad\text{then}\quad \lambda\in W\cdot\mu+p\mathbf{Z}R. \tag{1}$$

Let us decompose $\lambda=\lambda_0+p^r\lambda_1$, and $\mu=\mu_0+p^r\mu_1$ as above. The Hochschild-Serre spectral sequence for $G_r\lhd G_rB$ gives the exact sequence

$$\begin{aligned}0&\to\operatorname{Ext}^1_B(\lambda_1,\operatorname{Hom}_{G_r}(L(\lambda_0),L(\mu_0))^{(-r)}\otimes\mu_1)\to\operatorname{Ext}^1_{G_rB}(\hat{L}_r(\lambda),\hat{L}_r(\mu))\\&\to(\operatorname{Ext}^1_{G_r}(L(\lambda_0),L(\mu_0))^{(-r)}\otimes(\mu_1-\lambda_1))^B.\end{aligned}$$

If the first term is non-zero, then $\lambda_0=\mu_0$ and $\lambda_1-\mu_1\in\mathbf{Z}R$, hence $\lambda\in\mu+p^r\mathbf{Z}R$. Let us suppose the last term to be non-zero. Now we may embed the B-invariants into the T-invariants which are equal to $\mathrm{Ext}^1_{G_rT}(L_r(\lambda), L_r(\mu))$ and may then apply the known result for G_rT-modules (cf. [18], p. 168). We may also identify the last term with

$$(\mathrm{Ext}^1_{G_r}(L(\lambda_0), L(\mu_0))^{(-r)}\otimes H^0(G/B, \mu_1-\lambda_1))^G.$$

Using the Hochschild-Serre spectral sequence for $G_r\lhd G$ we may embed the last group into

$$\mathrm{Ext}^1_G(L(\lambda_0), L(\mu_0)\otimes H^0(G/B, \mu_1-\lambda_1)^{(r)}).$$

Using the strong linkage principle for G we see that this term is zero, unless there is a weight ν of $H^0(G/B, \mu_1-\lambda_1)$ such that

$$\lambda_0\in W\cdot(\mu_0+p^r\nu)+p\mathbf{Z}R.$$

As $\nu\in\mu_1-\lambda_1+\mathbf{Z}R$ the claim follows. (This argument was first used in [10, 3.8].)

Formula (1) implies of course a statement about blocks. If there is $\alpha\in R$ such that $\langle\lambda+\varrho, \alpha^\vee\rangle$ is not divisible by p the proofs of [17], 5.5(2), (3) show that all $\mu\in W\cdot\lambda+p\mathbf{Z}R$ belong to the same block as λ. [The maps between the $\hat{Z}_r(\lambda)$ used there are homomorphisms of G_rB-modules].

Set now

$$\bar{C}_1=\{\lambda\in X(T)|0\leqq\langle\lambda+\varrho, \alpha^\vee\rangle\leqq p \text{ for all } \alpha\in R_+\}.$$

For each $\lambda, \mu\in\bar{C}_1$ there is a translation functor T_λ^μ on the category of G-modules. By (1) we can extend it to G_rB-modules, as it was done for G_rT-modules in [18, 5.2].

Suppose $p\geqq h$. Then 0 belongs to the interior of $\bar{C}_1$ and for each $\mu\in\bar{C}_1$ with $\langle\mu+\varrho, \alpha^\vee\rangle=0$ for some $\alpha\in R$ we have, e.g. by [18, 5.2(6)]

$$T_0^\mu K=0.$$

The functors T_λ^μ and T_μ^λ on the categories of G_rT-modules are adjoint to each other. This implies for μ as above and any G_rT-module E

$$H^\cdot(G_rT, T_\mu^0 E)\simeq\mathrm{Ext}^\cdot_{G_rT}(K, T_\mu^0 E)=\mathrm{Ext}^\cdot_{G_rT}(T_0^\mu K, E)=0.$$

For $\nu\in\mathbf{Z}R$ we have $(T_\mu^0 E)\otimes p^r\nu\simeq T_\mu^0(E\otimes p^r\nu)$ as adding $p^r\nu$ does not change the sets $W\cdot\lambda+p\mathbf{Z}R$. Hence 2.5(3) implies

$$H^\cdot(G_r, T_\mu^0 E)_{\mathbf{Z}R}=0. \tag{2}$$

2.7. Lemma. *Suppose $p\geqq h$. For each $w\in W$, each simple root α with $\ell(ws_\alpha)=\ell(w)+1$ and each $i\in\mathbf{Z}$ there is an isomorphism of B-modules*

$$H^i(G_r, \hat{Z}_r(ws_\alpha\cdot 0))_{\mathbf{Z}R}\simeq H^{i-1}(G_r, \hat{Z}_r(w\cdot 0))_{\mathbf{Z}R}.$$

Proof. As $p\geqq h$, we can find $\mu\in\bar{C}_1$ such that $\langle\mu+\varrho, \alpha^\vee\rangle=0$, but $0<\langle\mu+\varrho, \beta^\vee\rangle<p$ for every other positive root β (cf. [16, Satz 10]). The arguments in [18, 5.1/2] show that there is an exact sequence of G_rB-modules

$$0\to\hat{Z}_r(ws_\alpha\cdot 0)\to T_\mu^0\hat{Z}_r(w\cdot\mu)\to\hat{Z}_r(w\cdot 0)\to 0.$$

The long exact sequence of cohomology being compatible with the B-action, the desired isomorphisms follow from 2.6(2).

2.8. Using induction on $\ell(w)$ we get from Lemma 2.7 and from 2.5(1) immediately:

Proposition. *Suppose $p \geqq h$. For each $w \in W$ and each $i \in \mathbf{Z}$ there are isomorphisms of B-modules*

$$H^i(G_r, \hat{Z}_r(w \cdot 0))_{\mathbf{Z}R} \simeq H^i(B_r, w \cdot 0)_{\mathbf{Z}R} \simeq H^{i-\ell(w)}(B_r, K)_{\mathbf{Z}R}.$$

Remark. For $p > h$ we may drop the index $\mathbf{Z}R$. This follows from 2.5(5) as all modules have weights in $\mathbf{Z}R$.

2.9. For each $\lambda, \nu \in X(T)$ we have isomorphisms $\hat{Z}_r(\lambda + p^r\nu) \simeq \hat{Z}_r(\lambda) \otimes p^r\nu$ and

$$H^{\cdot}(B_r, \lambda + p^r\nu) \simeq H^{\cdot}(B_r, \lambda) \otimes p^r\nu.$$

We see especially (if $p > h$) for each $w \in W$ and $\nu \in X(T)$

$$H^i(B_r, w \cdot 0 + p^r\nu) \simeq H^{i-\ell(w)}(B_r, K) \otimes p^r\nu \quad \text{for all} \quad i \in \mathbf{Z}. \tag{1}$$

Suppose $r = 1$. Then every $\lambda \in X(T)$ not of the form $w \cdot 0 + p\nu$ with $w \in W$ and $\nu \in X(T)$ does not belong to the same block as 0 (cf. [17], 5.5). This implies $H^{\cdot}(G_1, \hat{Z}_1(\lambda)) = 0$ and also $H^{\cdot}(B_1, \lambda) = 0$ by 2.5(1). Therefore we have for $p > h$ a description of all $H^{\cdot}(G_1, \hat{Z}_1(\lambda))$ or $H^{\cdot}(B_1, \lambda)$ combining (1) and Proposition 2.3. More explicitly we have for $w \in W$ and $\nu \in X(T)$:

$$H^i(B_1, w \cdot 0 + p\nu) \simeq \begin{cases} S^{(i-\ell(w))/2}(u^*)^{(1)} \otimes p\nu & \text{if } i \equiv \ell(w) \bmod 2, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

3. The G_1-Cohomology

In this section we study the groups $H^{\cdot}(G_1, H^0(G/B, \lambda))$ for $\lambda \in X(T)^+$. From the linkage principle it is clear that this cohomology can be non-zero only if λ has the form $\lambda = w \cdot 0 + p\lambda'$ for some $\lambda' \in X(T)^+$ and $w \in W$. Suppose $p > h$. We compute in this section the cohomology in the case $\lambda = 0$ (proving results from [11] in a different way) and in the case where λ' is "strongly dominant", i.e. where its coordinates in terms of the fundamental weights are at least $h-1$. We shall be able to deal with arbitrary λ in Sect. 5 provided G is not of exceptional type.

3.1. The induction from B to G is a left exact functor, carrying injective B-modules into injective G-modules, which are also injective as G_r-modules. Therefore the composite functor $\mathscr{F}$, sending each B-module E to the (untwisted) G_r-fixed points $\mathscr{F}(E) = (H^0(G/B, E)^{G_r})^{(-r)}$ of $H^0(G/B, E)$ is again left exact, admitting derived functors $R^q\mathscr{F}$ and a spectral sequence

$$E_2^{s,t} = H^s(G_r, H^t(G/B, E))^{(-r)} \Rightarrow (R^{s+t}\mathscr{F})(E) \tag{1}$$

converging to the derived functors of $\mathscr{F}$.

Let us consider on the other hand the functor $\mathscr{F}'$ from B-modules to G-modules, which is the composite of at first sending E to $(E^{B_r})^{(-r)}$ and then sending a B-module E' to $H^0(G/B, E')$, i.e. $\mathscr{F}'(E) = H^0(G/B, (E^{B_r})^{(-r)})$. We claim that $E \mapsto (E^{B_r})^{(-r)}$ sends injective modules to B-modules which are acyclic for the induction functor $H^0(G/B, ?)$. But it is enough to consider $E = K[B]$, and since $K[B]^{B_r} \simeq K[B]^{(r)}$ the statement is clear in this case. Therefore the spectral sequence associated to the composition of these functors converges to the derived functors $R^q\mathscr{F}'$:

$$E_2'^{s,t} = H^s(G/B, H^t(B_r, E)^{(-r)}) \Rightarrow (R^{s+t}\mathscr{F}')(E). \tag{2}$$

Proposition. *The functors $\mathscr{F}$ and $\mathscr{F}'$ are isomorphic.*

Proof. Let E be a B-module. The canonical B-homomorphism $\varphi: H^0(G/B, E) \to E$ maps $H^0(G/B, E)^{G_r}$ to E^{B_r}. The universal property of induction gives for each G-module V that the map $\alpha: \psi \mapsto \varphi \circ \psi$ is an isomorphism

$$\alpha: \mathrm{Hom}_G(V^{(r)}, H^0(G/B, E)) \xrightarrow{\sim} \mathrm{Hom}_B(V^{(r)}, E).$$

As G_r and B_r operate trivially on $V^{(r)}$, we may rewrite this isomorphism as

$$\alpha: \mathrm{Hom}_G(V^{(r)}, H^0(G/B, E)^{G_r}) \xrightarrow{\sim} \mathrm{Hom}_B(V^{(r)}, E^{B_r}),$$

or also by untwisting the Frobenius twist

$$\mathrm{Hom}_G(V, H^0(G/B, E)^{G_r})^{(-r)}) \xrightarrow{\sim} \mathrm{Hom}_B(V, (E^{B_r})^{(-r)}).$$

This means that $\mathscr{F}(E) = (H^0(G/B, E)^{G_r})^{(-r)}$ together with φ satisfies the universal property of $H^0(G/B, (E^{B_r})^{(-r)} = \mathscr{F}'(E)$. Therefore $\mathscr{F}(E)$ and $\mathscr{F}'(E)$ are canonically isomorphic, hence $\mathscr{F}$ and $\mathscr{F}'$ isomorphic.

3.2. The following corollary is immediate from the proposition:

Corollary. *Let E be a B-module.*

a) *If $H^j(G/B, E) = 0$ for all $j > 0$, then the spectral sequence $H^s(G/B, H^t(B_r, E)^{(-r)})$ converges to $H^{s+t}(G_r, H^0(G/B, E))^{(-r)}$*

(b) *If E in addition has the property that $H^s(G/B, H^t(B_r, E)^{(-r)}) = 0$ for all $s > 0$, then we have isomorphisms of G- and $H^\cdot(G_r, K)$-modules*

$$H^i(G_r, H^0(G/B, E))^{(-r)} \simeq H^0(G/B, H^i(B_r, E)^{(-r)}).$$

Remark. Note that if E is a B-module then a G_r-invariant element in $H^0(G/B, E)$ takes values in E^{B_r}, i.e. may be considered as an element in $H^0(G/B, E^{B_r})$. The Proposition 3.1 says that this correspondence gives an isomorphism $\mathscr{F}(E) \xrightarrow{\sim} \mathscr{F}'(E)$.

If E' is another B-module, then we also see that the isomorphism between $\mathscr{F}$ and $\mathscr{F}'$ respects the cup-products, i.e. we have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}(E) \times \mathscr{F}(E') & \longrightarrow & \mathscr{F}(E \otimes E') \\ \wr\downarrow & & \wr\downarrow \\ \mathscr{F}'(E) \times \mathscr{F}'(E') & \longrightarrow & \mathscr{F}'(E \otimes E') \end{array}.$$

This implies that the isomorphisms between the derived functors of $\mathscr{F}$ and $\mathscr{F}'$ respect cup-products. Furthermore, the diagram above gives rise to (by taking injective resolutions of E and E') the following commutative diagram

$$\begin{array}{ccc} H^i(G_r, H^0(G/B, E))^{(-r)} \times H^j(G_r, H^0(G/B, E'))^{(-r)} & \to & H^{i+j}(G_r, H^0(G/B, E \otimes E'))^{(-r)} \\ \downarrow & & \downarrow \\ H^0(G/B, H^i(B_r, E)^{(-r)}) \times H^0(G/B, H^j(B_r, E')^{(-r)}) & \to & H^0(G/B, H^{i+j}(B_r, E \otimes E')^{(-r)}) \end{array}.$$

In particular this shows that the map $H^\cdot(G_r, K)^{(-r)} \to H^0(G/B, H^\cdot(B_r, K)^{(-r)})$ is a ring homomorphism and that for any B-module E the map

$$H^\cdot(G_r, H^0(G/B, E)^{(-r)}) \to H^0(G/B, H^\cdot(B_r, E)^{(-r)})$$

is a $H^\cdot(G_r, K)$-module homomorphism.

3.3. Let $\lambda \in X(T)^+$. By Kempf's vanishing theorem we have that $H^i(G/B, \lambda) = 0$ for $i > 0$. Hence the above corollary shows that there is a spectral sequence

$$H^s(G/B, H^t(B_1, \lambda)^{(-1)}) \Rightarrow H^{s+t}(G_1, H^0(G/B, \lambda))^{(-1)}. \tag{1}$$

Now everything here vanishes, unless λ is linked to 0, i.e. $\lambda = w \cdot 0 + pv$ for some $w \in W$ and $v \in X(T)^+$. For such λ we have a description of $H^\cdot(B_1, \lambda)$ in 2.9(2) if $p > h$. That result shows that (1) has the form

$$H^s(G/B, S^t(\mathfrak{u}^*) \otimes v) \Rightarrow H^{s+\ell(w)+2t}(G_1, H^0(G/B, \lambda))^{(-1)}. \tag{2}$$

If $H^s(G/B, S^t(\mathfrak{u}^*) \otimes v) = 0$ for all $s > 0$ (for a given v) then we get isomorphisms

$$H^0(G/B, S^t(\mathfrak{u}^*) \otimes v) \simeq H^{\ell(w)+2t}(G_1, H^0(G/B, \lambda))^{(-1)}, \tag{3}$$

whereas $H^i(G_1, H^0(G/B, \lambda)) = 0$ if $i \not\equiv \ell(w) \bmod 2$. In most cases we shall be able to show that this assumption is satisfied (3.6, 5.4).

3.4. Let us call $\lambda \in X(T)^+$ *strongly dominant, if* $\langle \lambda, \alpha^\vee \rangle \geqq h - 1$ for all simple roots α. (Observe that this condition occurred already in [18], Lemma 3.4).

Lemma. *Let* $i \in \mathbf{N}$.

a) *We have* $H^j(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{b})^*) = 0$ *for all* $j \neq i$, *and* $H^i(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{b})^*)$ *is the trivial G-module of dimension* $\# \{w \in W | \ell(w) = i\}$.

b) *If* $\lambda \in X(T)^+$ *is strongly dominant, then*

$$H^j(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{b})^* \otimes \lambda) = H^j(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{u})^* \otimes \lambda) = 0$$

for all $j > 0$.

Proof. a) Any weight v of $\Lambda^i(\mathfrak{g}/\mathfrak{b})^*$ is the sum of i different negative roots. Choose $w \in W$ such that $w(v + \varrho) \in X(T)^+$. By 2.2(2) we have $w \cdot v \leqq 0$. Proposition 2.5 and Theorem 2.6 in [35] imply that $H^j(G/B, v) = 0$ for all j, if $w \cdot v \neq 0$, and for all $j \neq \ell(w)$, if $w \cdot v = 0$, whereas $H^{\ell(w)}(G/B, v) \simeq K$ in this case. In the second case $-v = \varrho - w^{-1}\varrho$ is the sum of the $\ell(w)$ different positive roots α with $w(\alpha) < 0$, and this is the only way of writing $-v = \varrho - w^{-1}\varrho$ as a sum of different positive roots ([19], 5.10.2). Hence $i = \ell(w)$. As the trivial one dimensional G-module does not extend with itself but trivially, we get a).

b) In this case $\lambda + v + \varrho \in X(T)^+$ for every weight v of $\Lambda(\mathfrak{g}/\mathfrak{b})^*$ or of $\Lambda(\mathfrak{g}/\mathfrak{u})^*$ by 2.2 (2), hence $H^j(G/B, \lambda + v) = 0$ for all $j > 0$ by Kempf's vanishing theorem [for

$\lambda+\nu\in X(T)^+$] or by [1], Lemma 1.1 [for $\lambda+\nu\notin X(T)^+$]. This implies the result immediately.

Remark. This lemma makes sense and is true also if char $K=0$. The first part says that the Hodge cohomology $H^{\cdot}(G/B,\Omega^{\cdot}_{G/B})$ is "diagonal", a fact which is well known at least in characteristic zero, see e.g. [20]. It was pointed out in the first author's review of [20] [see Math. Rev. 58 (1979), #22094] that this proof (which more or less is the same as the proof given above) carries over to arbitrary characteristic.

3.5. Let $\varphi: V\to V'$ be a surjective homomorphism of vector spaces over K. Then φ induces for each n a surjective homomorphism: $S^nV\to S^nV'$ which can be continued to an exact sequence (where $E=\ker\varphi$):

$$\dots\to S^{n-i}V\otimes\Lambda^iE\xrightarrow{\varphi_i}S^{n-i+1}V\otimes\Lambda^{i-1}E\to\dots\to S^{n-1}V\otimes E\to S^nV\to S^nV'\to 0, \quad (1)$$

which we are going to call the Koszul resolution for (V,V'). The map φ_i above is given by

$$\varphi_i(v\otimes e_1\wedge e_2\wedge\dots\wedge e_i)=\sum_{j=1}^{i}(-1)^j ve_j\otimes e_1\wedge\dots\wedge\hat{e}_j\wedge\dots\wedge e_i.$$

If a group H is operating on V and V' and if φ is equivariant, then the whole resolution is compatible with the H-action.

Another useful fact about symmetric and exterior powers is the following. For V,V',φ as above we have for each n filtrations

$$S^nV=F_n\supset F_{n-1}\supset\dots \quad\text{and}\quad \Lambda^nV=F'_n\supset F'_{n-1}\supset\dots$$

such that

$$F_i/F_{i-1}\simeq S^{n-i}E\otimes S^iV' \quad (2)$$

and

$$F'_i/F'_{i-1}\simeq\Lambda^{n-i}E\otimes\Lambda^iV'. \quad (3)$$

If again a group is operating and φ equivariant, then the filtrations and the isomorphisms are compatible with the group action.

3.6. Theorem. *Let $n\in\mathbb{N}$.*

a) *We have $H^i(G/B,S^n(\mathfrak{u}^*))=0$ for all $i>0$.*

b) *If $\lambda\in X(T)^+$ is strongly dominant, then $H^i(G/B,S^n(\mathfrak{u}^*)\otimes\lambda)=0$ for all $i>0$.*

Proof. a) Induction on n together with the filtration 3.5 (2) for $V=\mathfrak{b}^*$, $V'=\mathfrak{u}^*$, and $E\simeq\mathfrak{h}^*$ shows that it is enough to prove $H^i(G/B,S^n(\mathfrak{b}^*))=0$ for $i>0$. Using the Koszul resolution for $(\mathfrak{g}^*,\mathfrak{b}^*)$ in 3.5 (1) we see that it is even enough to prove

$$0=H^j(G/B,S^{n-i}(\mathfrak{g}^*)\otimes\Lambda^i(\mathfrak{g}/\mathfrak{b})^*)\cong S^{n-i}(\mathfrak{g}^*)\otimes H^j(G/B,\Lambda^i(\mathfrak{g}/\mathfrak{b})^*)$$

for $j>i$. But this is clear from Lemma 3.4.a.

b) Here we may use a similar line of argument applying Lemma 3.4.b.

Remark. In case b) the stronger result noted in 3.4.b proves that there is an exact sequence of G-modules (for strongly dominant λ):

$$\dots \to S^{n-i}(\mathfrak{g}^*)\otimes H^0(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{u})^*\otimes\lambda)\to \dots \to S^{n-1}(\mathfrak{g}^*)\otimes H^0(G/B, \mathfrak{u}^*\otimes\lambda) \to S^n(\mathfrak{g}^*)\otimes H^0(G/B,\lambda)\to H^0(G/B, S^n(\mathfrak{u}^*)\otimes\lambda)\to 0. \quad (1)$$

3.7. Combining Theorem 3.6 with 3.3(3) we obtain

Corollary. *Suppose* $p>h$.

a)
$$H^i(G_1,K)\simeq \begin{cases} H^0(G/B, S^{i/2}(\mathfrak{u}^*))^{(1)} & \text{if } i \text{ is even},\\ 0 & \text{otherwise}.\end{cases}$$

b) *Let* $\lambda\in X(T)^+$ *be strongly dominant. Then for all* $w\in W$

$$H^i(G_1, H^0(G/B, w\cdot 0+p\lambda))\simeq \begin{cases} H^0(G/B, S^{(i-\ell(w))/2}(\mathfrak{u}^*)\otimes\lambda)^{(1)} & \text{if } i\equiv \ell(w) \bmod 2,\\ 0 & \text{otherwise}.\end{cases}$$

3.8. Let $\lambda\in X(T)^+$. We want to compute $\operatorname{ch} H^0(G/B, S^n(\mathfrak{u}^*)\otimes\lambda)$ under the assumption that $H^i(G/B, S^n(\mathfrak{u}^*)\otimes\lambda)=0$ for all $i>0$. In that case

$$\begin{aligned}\operatorname{ch} H^0(G/B, S^n(\mathfrak{u}^*)\otimes\lambda) &= \sum_{i\ge 0} (-1)^i \operatorname{ch} H^i(G/B, S^n(\mathfrak{u}^*)\otimes\lambda)\\ &= \sum_{\nu\in X(T)} \dim S^n(\mathfrak{u}^*)^\nu \sum_{i\ge 0}(-1)^i \operatorname{ch} H^i(G/B,\lambda+\nu)\\ &= \sum_{\nu\in X(T)} P_n(\nu)\chi(\lambda+\nu)\\ &= \sum_{\mu\in X(T)^+}\sum_{w\in W} (-1)^{\ell(w)} P_n(w\cdot\mu-\lambda)\chi(\mu),\end{aligned}$$

where $P_n(\nu)=\dim S^n(\mathfrak{u}^*)^\nu$ is the number of R_+-tuples $(n(\alpha))_\alpha$ with $\nu=\sum_{\alpha\in R_+} n(\alpha)\alpha$ and $n=\sum_{\alpha\in R_+} n(\alpha)$.

For $\lambda=0$ or for λ strongly dominant we get in this way (if $p>h$) also the formal character of all $H^i(G_1,K)$ or $H^i(G_1, H^0(G/B, w\cdot 0+p\lambda))$. Let us remark that especially

$$\operatorname{ch} H^{2n}(G_1,K)=\sum_{\mu\in X(T)^+}\sum_{w\in W}(-1)^{\ell(w)}P_n(w\cdot\mu)\chi(\mu). \quad (1)$$

Therefore in this case we have given another proof for the results in [11], 1.4 and 2.5 (proved there for $p\ge 3h-1$).

3.9. Let $\mathcal{N}=\{x\in\mathfrak{g}\mid x \text{ nilpotent}\}=G\mathfrak{u}$ be the nilpotent variety of $\mathfrak{g}$.

The restriction of functions $S(\mathfrak{g}^*)\to S(\mathfrak{u}^*)$ induces a G-homomorphism

$$\varphi: S(\mathfrak{g}^*)\to H^0(G/B, S(\mathfrak{u}^*))$$

mapping each $f\in S(\mathfrak{g}^*)$ to the function $g\mapsto (g^{-1}f)|_{\mathfrak{u}}$. The kernel of φ is

$$\begin{aligned}\ker\varphi &= \{f\in S(\mathfrak{g}^*)\mid (g^{-1}f)|_{\mathfrak{u}}=0 \text{ for all } g\in G\}\\ &=\{f\in S(\mathfrak{g}^*)\mid f(\mathcal{N})=0\}.\end{aligned}$$

Therefore φ induces an injective homomorphism $\bar\varphi: K[\mathcal{N}]\to H^0(G/B, S(\mathfrak{u}^*))$.

Lemma. *Suppose* $(p,|W|)=1$. *This map* $\bar{\varphi}$ *is an isomorphism of graded G-algebras*

$$K[\mathcal{N}] \xrightarrow{\sim} H^0(G/B, S(\mathfrak{u}^*)).$$

Proof. We just have to check that the image of each $S^n(\mathfrak{g}^*)$ in $K[\mathcal{N}]$ has the same formal character as $H^0(G/B, S^n(\mathfrak{u}^*))$. The second one has been computed in 3.8 (because of 3.6.a). In order to show that the first one is the same, one has to apply the work of Veldkamp [24] and use the calculations in [14], cf. the arguments in [11], 2.6.

Remarks. 1) Combining this Lemma with Corollary 3.7.a we regain the results in [11], 1.4 and 2.6.

2) Observe that this Lemma as well as the computations in 3.8, Theorem 3.6, Lemma 3.4 and the constructions in 3.5 make sense also for char $K=0$ and follow from our arguments also in that case.

3.10. Let us for the moment suppose that G has semi-simple rank 1. As the only positive root α is a dominant weight, all weights of $H^\cdot(U_r, K)$ are dominant, and hence so are also all weights of $H^\cdot(B_r, \lambda) = (H^\cdot(U_r, K) \otimes \lambda)^{T_r}$ for $\lambda \in X(T)^+$. By 3.2 we have therefore for all i an isomorphism

$$H^i(G_r, H^0(G/B, \lambda))^{(-r)} \simeq H^0(G/B, H^i(B_r, \lambda)^{(-r)}). \tag{1}$$

Furthermore each $H^i(G_r, H^0(G/B, \lambda))^{(-r)}$ has a filtration by modules of the form $H^0(G/B, \mu)$ with $\mu \in X(T)^+$, each $H^0(G/B, \mu)$ occurring $\dim H^i(B_r, \lambda)^{p^r\mu}$ times. This implies especially

$$\operatorname{ch} H^i(G_r, H^0(G/B, \lambda))^{(-r)} = \sum_{\mu \in X(T)^+} \dim H^i(B_r, \lambda)^{p^r\mu} \chi(\mu). \tag{2}$$

Suppose now $p \neq 2$. Using the notations from 2.4 we have inside $H^\cdot(B_r, K) = H^\cdot(U_r, K)^{T_r}$ a subalgebra $S(x_r)$ with x_r of weight $p^r\alpha$ in degree 2. Then $H^0(G/B, S(x_r)^{(-r)})^{(r)}$ will be a subalgebra in $H^\cdot(G_r, K)$ isomorphic to $H^\cdot(G_1, K)^{(r-1)}$. (The existence of such a subalgebra for arbitrary G was observed in [11], 1.8). The explicit description of $H^\cdot(B_2, G)$ in 2.4(2) shows however that $H^\cdot(G_r, K)$ will be strictly greater than $H^\cdot(G_1, K)^{(r-1)}$ and that the cohomology may not completely vanish in odd degree.

Suppose $p=2$. Here 2.4(3) implies for all $i \in \mathbf{N}$

$$H^i(G_1, K)^{(-1)} \simeq H^0(G/B, i\varrho), \quad (p=2) \tag{3}$$

As we have $H^0(G/B, i\alpha) \simeq H^{2i}(G_1, K)^{(-1)}$ for $p \neq 2$ and $H^{2i+1}(G_1, K)=0$ in that case, we see already now that some restriction on p in our theorems is necessary. We shall return to that in Sect. 6.

4. Good Filtrations

In those cases in which we have determined the G-modules $H^j(G_1, H^0(G/B, \lambda))^{(-1)}$ in the last section we prove now, that these modules have a "good filtration". This enables us then to compute the

$$\operatorname{Ext}^j_G(H^0(G/B, \mu)^{(1)*}, H^0(G/B, \lambda))$$

for these λ and all dominant weights μ.

4.1. A filtration of a module for an arbitrary connected algebraic group H is called *good*, if the factors in the filtration are isomorphic to modules induced from a one dimensional representation of a Borel subgroup of H. We shall now formulate some properties of this notion working with G only, though the results generalize to arbitrary H.

The first property was already stated in [18], 5.2(3). We have for any two G-modules V, V':

(1) *$V \oplus V'$ has a good filtration if and only if V and V' have a good filtration.*

By [8], 1.3 a finite dimensional G-module V has a good filtration if and only if $H^1(G, V \otimes H^0(G/B, \lambda)) = 0$ for all $\lambda \in X(T)^+$ [or, equivalently, if and only if $H^n(G, V \otimes H^0(G/B, \lambda)) = 0$ for all $n > 0$ and all $\lambda \in X(T)^+$]. As noted also in [11], 2.1 this implies for any exact sequence $0 \to V_n \to V_{n-1} \to \dots \to V_1 \to V \to 0$ of finite dimensional G-modules

(2) *If all V_i have a good filtration, so has V.*

For two G-modules V and V' we have by [9] (cf. [25]):

(3) *If V and V' have good filtrations, so has $V \otimes V'$* provided that we exclude all those cases, where G has factors of type E_7 or E_8 and where at the same time $p = 2$. We shall in this section assume that G satisfies (3), without mentioning these (hopefully unnecessary) restrictions explicitly. If $V' = H^0(G/B, \lambda)$ for some $\lambda \in X(T)^+$, then the factors of a good filtration of $V \otimes H^0(G/B, \lambda)$ will have the form $H^0(G/B, \lambda + \nu)$, where ν is a weight of V with $\lambda + \nu \in X(T)^+$.

Using the isomorphisms $S^n(V \oplus V') \simeq \bigoplus_{i=0}^{n} S^{n-i}(V) \otimes S^i(V')$ and (1), (3) we see for any two G-modules V and V':

(4) *$S(V \oplus V')$ has a good filtration if and only if $S(V)$ and $S(V')$ have a good filtration.*

If $p > i$, then $S^i(V)$ and $\Lambda^i V$ are direct summands of the i^{th} tensor power of V. Thus (1) and (3) imply:

(5) *If V has a good filtration and if $p > i$, then $\Lambda^i V$ and $S^i V$ have a good filtration.*

Another simple property was also noted in the proof of [11], 2.4:

(6) *If V is a finite dimensional G-module, such that $\langle \nu + \varrho, \alpha^\vee \rangle \leqq p$ for all weights ν of V and all $\alpha \in R_+$, then V has a good filtration.*

4.2. Let V be a G-module with $\dim V = n < \infty$. Choose a maximal torus $T(V)$ and a Borel subgroup $B(V) \supset T(V)$ of $GL(V)$. There are $\omega_i \in X(T(V))$ such that $\Lambda^i V \simeq H^0(GL(V)/B(V), \omega_i)$ for $1 \leqq i \leqq n$. Then $X(T(V))^+ = \sum_{i=1}^{n-1} \mathbf{N}\omega_i + \mathbf{Z}\omega_n$. For each $\pi \in X(T(V))^+$ let us compose the given representation $G \to GL(V)$ with the operation of $GL(V)$ on $H^0(GL(V)/B(V), \pi)$. In this way we get a G-module which we shall denote by $S_\pi V$. For $\pi = r\omega_1$ we get for example the symmetric powers $S_\pi V = S^r V$. [To each partition $m_1 \geqq m_2 \geqq \dots \geqq m_n \geqq 0$ into at most n parts there corresponds

$$\pi = \sum_{i=0}^{n-1} (m_i - m_{i+1})\omega_i + m_n\omega_n \in X(T(V))^+ .$$

Then S_π is often called the Schur functor associated to this partition.]

Lemma. *If $\Lambda(V)$ has a good filtration, so have all $S_\pi V$ with $\pi \in X(T(V))^+$.*

Proof. For all $r \in \mathbf{Z}$ the module $S_{r\omega_n}V$ is one dimensional, every $g \in GL(V)$ operating as multiplication with $\det(g)^r$. For an arbitrary π we then have $S_\pi V \otimes S_{r\omega_n} V \simeq S_{\pi - r\omega_n} V$ and for any $\lambda \in X(T)^+$ we have

$$H^0(G/B, \lambda) \otimes S_{r\omega_n} V \simeq H^0(G/B, \lambda \otimes S_{r\omega_n} V) .$$

Therefore we may restrict ourselves to the case $\pi \in \sum_{i=1}^{n-1} \mathbf{N}\omega_i$.

Now we use induction on π. For $\pi = \omega_i$ the claim holds by assumption. If $\pi \neq \omega_i$ for all i, then we can write $\pi = \pi_1 + \pi_2$ with π_1 and π_2 both strictly smaller than π. By induction $S_{\pi_1}V$ and $S_{\pi_2}V$ have a good filtration, by 4.1 (3) so has $S_{\pi_1}V \otimes S_{\pi_2}V$. Applying 4.1 (3) to $GL(V)$ we see that $S_{\pi_1}V \otimes S_{\pi_2}V$ has another filtration with top factor $S_\pi V$ and all other factors of the form $S_{\pi'}V$ with $\pi' < \pi$. We may assume that all these $S_{\pi'}V$ have a good filtration, hence so has $S_\pi V$ by 4.1 (2).

4.3. The following application of Lemma 4.2 was at first proved in [11], 2.2:

(1) *If V is a finite dimensional G-module, such that $\Lambda(V)$ has a good filtration, then $S(V)$ has a good filtration.*

This result is a special case ($V' = K$) of the following more general result:

(2) *If V and V' are finite dimensional G-modules such that $\Lambda(V)$ and $\Lambda(V')$ have good filtrations, then $S(V \otimes V')$ has a good filtration.*

One may interprete Theorem 3.2 in [5] as saying that each $S^r(V \otimes V')$ has a good filtration as a $GL(V) \times GL(V')$-module. The factors in this filtration have the form $S_\pi V \otimes S_{\pi'} V'$ with $\pi \in X(T(V))^+$ and $\pi' \in X(T(V'))^+$. Now (2) follows from 4.1 (3) and 4.2.

Let V be a vector space of dimension m and suppose $G = SL(V)$. Then

$$\Lambda^i V \simeq H^0(G/B, \omega_i) \quad \text{and} \quad \Lambda^i V^* \simeq H^0(G/B, \omega_{m+1-i}),$$

where the ω_i are the fundamental weights with their usual numbering. It follows from (2) that $S(V \otimes V^*) \otimes S(\mathrm{End}\, V) \simeq S(\mathrm{End}\, V)^*$ has a good filtration. The natural injection $\mathfrak{g} \to \mathrm{End}(V)$ induces a surjection $\mathrm{End}(V)^* \to \mathfrak{g}^*$ with a one dimensional kernel which is trivial as a G-Module. Therefore we have for all n an exact sequence of G-modules

$$0 \to S^{n-1}(\mathrm{End}\, V)^* \to S^n(\mathrm{End}\, V)^* \to S^n \mathfrak{g}^* \to 0 ;$$

this is a (trivial) special case of 3.5 (1). Now 4.1 (2) implies:

(3) *If $G = SL(V)$, then $S\mathfrak{g}^*$ has a good filtration.*

4.4. It was noted already in [11], 2.4 that 4.3 (3) generalizes to arbitrary G if $p \geqq 3h-3$. (This follows from 4.1 (6) and 4.3 (1) as 2ϱ is the highest weight of $\Lambda\mathfrak{g}^*$). This bound on p is certainly too large. Let us use the notion of a good prime as in [23], p. 106, i.e. if R is indecomposable, then p is good if:

type A_n:	p arbitrary,
type B_n, C_n, D_n:	$p \neq 2$,
type G_2, F_4, E_6, E_7:	$p \neq 2, 3$,
type E_8:	$p \neq 2, 3, 5$.

In general p is good if it is good for all simple components of R.

Proposition. *Suppose G is semisimple and simply connected. If p is good, then $S\mathfrak{g}^*$ has a good filtration.*

Proof. We may restrict ourselves to the case that R is indecomposable. For R of type A_n the proposition is contained in 4.3 (3). Let us now exclude this case. From our assumption on p it now follows that $\mathfrak{g}^* \simeq \mathfrak{g}$ is a simple G-module isomorphic to $L(\tilde{\alpha})$, where $\tilde{\alpha}$ is the largest root (cf. the survey in [15], p. 17–22). Also, we may (and shall for the types B_n, D_n) replace G by any isogenous image without changing $\mathfrak{g}$.

Let us consider $K[G]$ as a G-module under the conjugation action. Then $K = K1$, the augmentation ideal I and I^2 are submodules with $K[G] = K \oplus I$ and $\mathfrak{g}^* \simeq I/I^2 \simeq K[G]/(K \oplus I^2)$. Let us prove: If

$$0 \to K \oplus I^2 \to K[G] \to \mathfrak{g}^* \to 0 \tag{1}$$

splits, then $S\mathfrak{g}^*$ has a good filtration.

Let us consider a G-submodule $M \simeq \mathfrak{g}^*$ of $K[G]$ which is a complement to $K \oplus I^2$. Then $M \subset I$ as otherwise $M \cap I = 0$ which is absurd. This implies $I = M \oplus I^2$ and by induction $I^m = M^m + I^{m+1}$ for all $m \in \mathbb{N}$. Therefore the natural map from $S^m M$ onto M^m induces a surjection of G-modules $S^m\mathfrak{g}^* \to I^m/I^{m+1}$. The smoothness of G implies (as in 1.2) however $\dim I^m/I^{m+1} = \dim S^m\mathfrak{g}$ for all m, hence $I^m \simeq S^m\mathfrak{g}^* \oplus I^{m+1}$ and $K[G] \simeq \bigoplus_{i=0}^{m} S^i\mathfrak{g}^* \oplus I^{m+1}$. Now, Donkin and Koppinen ([31]) proved independently that $K[G]$ as a (G, G)-bimodule has a good filtration with factors of the form $H^0(G/B, \lambda) \otimes H^0(G/B, \mu)$. This implies by 4.1 (3) that $K[G]$ as a G-module under conjugation has a good filtration, hence so have all $S^i\mathfrak{g}^*$. (The theorem in [31] is stated only for G simply connected, but the general case follows as then $K[G]$ is a direct summand of the same object for a simply connected covering.)

Consider a non-trivial G-module V. The map $c: V \otimes V^* \to K[G]$ with $c(v \otimes \varphi)(g) = \varphi(g^{-1}v)$ for $v \in V$, $\varphi \in V$, and $g \in G$ is a homomorphism of G-modules. Without changing $\mathfrak{g}$ we may replace G by its image in $SL(V)$, hence may assume that $K[G]$ is generated by $c(V \otimes V^*)$ as a K-algebra. This implies $K[G] = K + c(V \otimes V^*) + I^2$. Suppose now that we have a decomposition $V \otimes V^* = V_1 \oplus V_2$, where $V_1 \simeq L(\tilde{\alpha})$, whereas $L(\tilde{\alpha})$ is no composition factor of V_2. Then $c(V_2)$ is mapped to 0 under the natural projection $K[G] \to K[G]/(K + I^2) \simeq L(\tilde{\alpha})$. This implies $K[G] = c(V_1) + (K + I^2)$, and by dimension comparison even $K[G] = c(V_1) \oplus (K + I^2)$. Hence (1) splits, and $S\mathfrak{g}^*$ has a good filtration.

We therefore have to find V with a decomposition $V \otimes V^* = V_1 \oplus V_2$ as above. For the classical groups we take the natural representation as a special orthogonal or a symplectic group. For the exceptional groups we choose V of minimal dimension. In all cases there is (for good p) a fundamental weight λ such that $V \simeq H^0(G/B, \lambda) = L(\lambda)$. Now $V \otimes V^*$ has a good filtration, and one factor is always $H^0(G/B, \tilde{\alpha}) = L(\tilde{\alpha}) \simeq \mathfrak{g}$. [This occurrence of $L(\tilde{\alpha})$ is clear by comparing with characteristic 0 as $\mathfrak{g}$ is embedded in $\mathrm{End}(V)$.] Suppose we can show for all dominant weights μ with $\tilde{\alpha} < \mu \leqq \lambda' = \lambda - w_0 \lambda$ that $L(\tilde{\alpha})$ is no composition factor of $H^0(G/B, \mu)$. As λ' is the highest weight of $V \otimes V^*$ this implies that $L(\tilde{\alpha})$ occurs with multiplicity 1 in $V \otimes V^*$ and that it does not extend with any other composition factor of $V \otimes V^*$ (cf. [4], 3.10). Therefore $L(\tilde{\alpha})$ is a direct summand of $V \otimes V^*$, and we get the decomposition of $V \otimes V^*$ as desired.

Let us use the notations of [3], planches II–IX. In type C_n we have $\tilde{\alpha} = 2\omega_1 = \lambda'$ and nothing is to be shown. In type B_n, $n > 2$ and D_n, $n > 3$ we have $\tilde{\alpha} = \omega_2$ and $\lambda' = 2\omega_1 = \omega_2 + \alpha_1$. This immediately implies that $L(\tilde{\alpha})$ is a composition factor of $H^0(G/B, \lambda')$ only for $p = 2$ and that only $\mu = \lambda'$ has to be considered. Let us now look at the exceptional types. The W-invariant scalar product $(\,|\,)$ as in [3] satisfies $(\mathbf{Z}R | \mathbf{Z}R) \subset \mathbf{Z}$. If $L(\tilde{\alpha})$ is a composition factor of $H^0(G/B, \mu)$ then $\tilde{\alpha} + \varrho$ and $\mu + \varrho$ have (by [2]) to be conjugate under the affine Weyl group. This implies easily $(\tilde{\alpha} + \varrho | \tilde{\alpha} + \varrho) \equiv (\mu + \varrho | \mu + \varrho) \bmod p$. Now for all μ with $\tilde{\alpha} < \mu \leqq \lambda'$ one checks without difficulty $(\tilde{\alpha} + \varrho | \tilde{\alpha} + \varrho) \not\equiv (\mu + \varrho | \mu + \varrho) \bmod p$ except for the case $\mu = \omega_6$ for type E_7 and $p = 5$. But here $\mu - \tilde{\alpha}$ is a linear combination of those simple roots forming a basis of the subsystem of type D_6. It then follows from [30] (or already from [15], p. 15) that the multiplicity of $L(\tilde{\alpha})$ in $H^0(G/B, \mu)$ is equal to the multiplicity of the trivial representation in the adjoint representation for a group of type D_6 hence equal to 0 for $p \neq 2$.

Remark. The main ideas of this proof are due to Donkin. By our own arguments we got a worse bound on p for the exceptional cases.

4.5. We can now reformulate one of our remarks in 3.10:

(1) *If G has semisimple* rank 1, *then each $H^i(G_r, H^0(G/B, \lambda))^{(-r)}$ with $\lambda \in X(T)^+$ has a good filtration.*

Donkin conjectured in talks in Turku (1980) that (1) should hold for arbitrary G with $H^0(G/B, \lambda)$ replaced by a G-module having a good filtration. For λ strongly dominant and any $w \in W$ we get a good filtration for $H^i(G_1, H^0(G/B, w \cdot 0 + p\lambda))^{(-1)}$ if p is large enough, using Corollary 3.7.b, the discussion in 4.4 and the following lemma:

Lemma. *Let $\lambda \in X(T)^+$ be strongly dominant. If $S(\mathfrak{g}^*)$ has a good filtration, so has each $H^0(G/B, S^n(\mathfrak{u}^*) \otimes \lambda)$ with $n \in \mathbf{N}$.*

Proof. For each weight ν of $\Lambda(\mathfrak{g}/\mathfrak{u})^*$ we have $H^j(G/B, \lambda + \nu) = 0$ for all $j > 0$ (cf. proof of 3.4.b). Therefore $H^0(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{u})^* \otimes \lambda)$ has a good filtration for all i. We apply now 4.1 (3), (2) to the exact sequence 3.6 (1) in order to get our claim.

4.6. Suppose $S(\mathfrak{g}^*)$ has a good filtration. It was observed in [11], 2.4 that each $H^j(G_1, K)^{(-1)}$ has a good filtration. Let us briefly indicate how this follows from our description of these groups as $H^0(G/B, S^n(\mathfrak{u}^*))$.

Using the filtration in 3.5(2) for the surjection $\mathfrak{b}^* \to \mathfrak{u}^*$, Theorem 3.6.a, induction on n, and (4.1 (2) we see that it is enough to prove that all $H^0(G/B, S^n\mathfrak{b}^*)$ have a good filtration. Let us now use the Koszul resolution 3.5(1) for $(\mathfrak{g}^*, \mathfrak{b}^*)$ and break it into a series of short exact sequences

$$0 \to L_{i+1} \to S^{n-i}(\mathfrak{g}^*) \otimes \Lambda^i(\mathfrak{g}/\mathfrak{b})^* \to L_i \to 0$$

where $L_0 = S^n(\mathfrak{b}^*)$ and $L_N = 0$ for $N > \dim(\mathfrak{g}/\mathfrak{b})$. By 3.4.a we know that

$$H^j(G/B, S^{n-i}(\mathfrak{g}^*) \otimes \Lambda^i(\mathfrak{g}/\mathfrak{b})^*) \simeq S^{n-i}(\mathfrak{g}^*) \otimes H^j(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{b})^*)$$

vanishes for $j \neq i$ and has a good filtration for $j = i$. Induction from above using 4.1(2) proves that $H^j(G/B, L_j)$ vanishes for $j \neq i$ and has a good filtration for $j = i$. Taking $i = 0$ we get the desired result for $H^0(G/B, S^n(\mathfrak{b}^*))$.

4.7. Let V be a finite dimensional G-module with a good filtration. For all $\lambda, \mu \in X(T)^+$ we have $\mathrm{Hom}_G(H^0(G/B, \lambda)^*, H^0(G/B, \mu)) = 0$ if $\lambda^* \neq \mu$, whereas this space is one dimensional for $\lambda^* = \mu$. Here $\lambda^* = -w_0(\lambda)$, w_0 the longest word in W. This together with [4], 3.3 implies for each $\lambda \in X(T)^+$, that the number of factors isomorphic to $H^0(G/B, \lambda^*)$ in a good filtration of V is equal to the dimension of

$$\mathrm{Hom}_G(H^0(G/B, \lambda)^*, V) \cong (H^0(G/B, \lambda) \otimes V)^G = H^0(G, H^0(G/B, \lambda) \otimes V). \quad (1)$$

Lemma. *Let $\lambda \in X(T)^+$. Suppose all $H^j(G_1, H^0(G/B, \lambda))^{(-1)}$ have a good filtration. Then for each $\mu \in X(T)^+$ and for all j the number of factors isomorphic to $H^0(G/B, \mu^*)$ in a good filtration of $H^j(G_1, H^0(G/B, \lambda))^{(-1)}$ is equal to*

$$\dim \mathrm{Ext}^j_G(H^0(G/B, \mu)^{(1)*}, H^0(G/B, \lambda)).$$

Proof. Let us write $H^0(\lambda) = H^0(G/B, \lambda)$ for the moment. By (1) the number we consider equals the dimension of

$$\begin{aligned} &H^0(G, H^0(\mu) \otimes H^j(G_1, H^0(\lambda))^{(-1)}) \\ &\simeq H^0(G/G_1, H^0(\mu)^{(1)} \otimes H^j(G_1, H^0(\lambda))) \\ &\simeq H^0(G/G_1, H^j(G_1, H^0(\lambda) \otimes H^0(\mu)^{(1)})). \end{aligned}$$

By 4.1(3) the G-module $H^j(G_1, H^0(\lambda))^{(-1)} \otimes H^0(\mu)$ has a good filtration, hence

$$H^i(G/G_1, H^j(G_1, H^0(\mu)^{(1)} \otimes H^0(\lambda))) = 0$$

for $i > 0$. Therefore the Lyndon spectral sequence for G/G_1 degenerates, and the space above is also isomorphic to

$$H^j(G, H^0(\mu)^{(1)} \otimes H^0(\lambda)) \simeq \mathrm{Ext}^j_G(H^0(\mu)^{(1)*}, H^0(\lambda)),$$

as claimed.

Remark. The calculation above is more or less the same as in the proof of [11], 3.2, which is the special case $\lambda = 0$ of 4.8 below.

4.8. Combining 4.7 and 4.5 with 3.7/8 we get now

Proposition. *Suppose $p > h$. Consider $\lambda \in X(T)^+$ of the form $\lambda = w' \cdot 0 + p\lambda'$ with $w' \in W$ and $\lambda' \in X(T)^+$ strongly dominant or $w' = 1$ and $\lambda' = 0$. Then for all $\mu \in X(T)^+$ and all*

$j \in \mathbb{N}$ we have

$$\mathrm{Ext}^j_G(H^0(G/B,\mu)^{(1)*}, H^0(G/B,\lambda)) = 0 \quad \textit{for} \quad j-\ell(w') \textit{ odd},$$

whereas this space has dimension

$$\sum_{w \in W} (-1)^{\ell(w)} P_{(j-\ell(w'))/2}(w \cdot \mu^* - \lambda')$$

for $j-\ell(w')$ even.

4.9. Let V be a finite dimensional G-module such that $S(V)$ has a good filtration. By 4.7(1) we see for each n that $\dim S^n(V)^G$ is equal to the number of factors isomorphic to $K \simeq H^0(G/B, 0)$ in a good filtration of $S^n(V)$. Therefore $\dim S^n(V)^G$ can be computed from the formal character $\mathrm{ch}\, S^n(V)$, hence from $\mathrm{ch}\, V$. Thus the invariant theory of V looks very much the same as for a representation with the same formal character of the corresponding complex Lie group of the same type as G.

We can make this more precise. There is a group scheme $\mathscr{G}$ over $\mathbf{Z}$ such that $G \simeq \mathscr{G} \underset{\mathbf{Z}}{\times} \mathrm{Spec}\, K$ and such that $G(\mathbf{C}) = \mathscr{G} \underset{\mathbf{Z}}{\times} \mathrm{Spec}\, \mathbf{C}$ is a reductive algebraic group over $\mathbf{C}$ of the same type. Suppose that there is a $\mathscr{G}$-module $V_{\mathbf{Z}}$, free over $\mathbf{Z}$ with $V \simeq V_{\mathbf{Z}} \otimes_{\mathbf{Z}} K$. Then $S(V)^G \simeq S(V_{\mathbf{Z}})^{\mathscr{G}} \otimes_{\mathbf{Z}} K$, and $S(V_{\mathbf{Z}})^{\mathscr{G}}$ is a lattice in $S(V_{\mathbf{C}})^{G(\mathbf{C})}$, where $V_{\mathbf{C}} = V_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$.

Let us mention some examples where this explains some old results. Let V_n be the space of homogeneous binary forms of degree n, considered as a G-module for $G = SL_2(K)$. Then $V_n = H^0(G/B, n\varrho)$. For $p \geqq \dim V_n = n+1$ the exterior algebra $\Lambda(V_n)$ has a good filtration by 4.1 (4), hence so has $S(V_n)$ by 4.3 (1). Thus $S(V_n)^G$ has the same form as over $\mathbf{C}$ if $p > n$, as proved in [12].

Let G be now one of the groups $GL(V)$, $SL(V)$, $SO(V)$ or $Sp(V)$. It is not difficult to prove that each $S^n V$ has a good filtration.

Then by 4.1 (4) also each $S^n E$ has a good filtration, where E is one of the following spaces $V \oplus V \oplus \ldots \oplus V$ or [for $GL(V)$ only] $V \oplus V \oplus \ldots \oplus V \oplus V^* \oplus V^* \oplus \ldots \oplus V^*$. Hence the invariant theory of E has the same form for every p as over $\mathbf{C}$, as observed in [6].

For $GL(V)$ and $SL(V)$ we get the good filtrations on $S(V)$ and $S(V^*)$ from good filtrations on $\Lambda(V)$ and $\Lambda(V^*)$ by 4.3 (1). That the latter exists was observed already before 4.3 (3) for $SL(V)$. The argument for $GL(V)$ is similar.

For $G = Sp(V)$, too, we can prove that each $\Lambda^i V$ has a good filtration. Because of the duality between $\Lambda^i V$ and $\Lambda^{(\dim V - i)}(V^*) \simeq \Lambda^{(\dim V - i)} V$ we may restrict ourselves to the case $i \leqq (\dim V)/2$. The cases $i = 0, 1$ are trivial, and for $2 \leqq i \leqq (\dim V)/2$ the invariant symplectic form induces an embedding $\Lambda^{i-2} V \hookrightarrow \Lambda^i V$. We have to prove $\Lambda^i V / \Lambda^{i-2} V \simeq H^0(G/B, \omega_i)$, where ω_i is the i^{th} fundamental weight. It will be enough to construct a surjection $\varphi: \Lambda^i V \to H^0(G/B, \omega_i)$ because then necessarily $\varphi(\Lambda^{i-2} V) = 0$ as ω_i is no weight of $\Lambda^{i-2} V$, hence $\ker \varphi = \Lambda^{i-2} V$ by dimension considerations. It certainly amounts to the same, to construct an embedding $\psi: V(\omega_i) \to \Lambda^i V \simeq (\Lambda^i V)^*$, where $V(\omega_i) \simeq H^0(G/B, \omega_i)^*$ is the Weyl module with highest weight ω_i. Now V and $V(\omega_i)$ come from modules $V_{\mathbf{Z}}$ and $V(\omega_i)_{\mathbf{Z}}$ over $\mathbf{Z}$ (for an appropriate group scheme over $\mathbf{Z}$), and $V(\omega_i)_{\mathbf{Z}}$ is constructed as a submodule of $\Lambda^i V_{\mathbf{Z}}$. Reduction mod p gives a map $V(\omega_i) \to \Lambda^i V$; in order to prove its injectivity we have to show that a basis of $V(\omega_i)_{\mathbf{Z}}$ as a $\mathbf{Z}$-module remains linearly independent in $\Lambda^i V$ after reduction mod p. Now there is a basis given in [15], Lemma I 10 for the zero weight space, which remains linearly independent in $\Lambda^i V$ by the same arguments proving its linear independence in $\Lambda^i V_{\mathbf{Z}}$. From this the result follows for all weight spaces by the same arguments as used in [15], p. 16. [The same basis as in [15] is also used in [21] and there (Proposition 1) also the embedding $V(\omega_i) \to \Lambda^i V$ is constructed.]

For $G = SO(V)$ and $p \neq 2$ the $\Lambda^i V$ are of the form $H^0(G/B, \lambda_i)$ for some $\lambda_i \in X(T)^+$. [It is known from characteristic 0 that the formal characters coincide, and one has to check the irreducibility of the $H^0(G/B, \lambda_i)$, which was observed in [27], p. 65–67, cf. the correction in [15], Satz I 13]. For $p=2$ this statement is no longer true, as the $\Lambda^i V$ are self-dual whereas the $H^0(G/B, \lambda_i)$ are not (in general). But for the $S^i V$ we can give a direct proof for the existence of good filtration. The cases $i=0,1$ are trivial. Suppose $i \geqq 2$. The invariant quadratic form defines an embedding $S^{i-2} V \hookrightarrow S^i V$. We have to show $S^i V / S^{i-2} V \simeq H^0(G/B, i\omega)$, where ω_1 is the highest weight of V. As $i\omega_1$ is no weight of $S^{i-2} V$ we have $\mathrm{Hom}_G(S^{i-2} V, H^0(G/B, i\omega_1)) = 0$, hence it will suffice to find a surjection $S^i V \to H^0(G/B, i\omega_1)$. It will be even enough to find a surjection $\otimes^i V \to H^0(G/B, i\omega_1)$ as this has to factor through $S^i V$ by the same reasoning. But $\otimes^i V$ has a good filtration by 4.1 (3) and $i\omega_1$ is its highest weight, hence there is a surjection from $\otimes^i V$ onto $H^0(G/B, i\omega_1)$ as claimed.

5. More Vanishing

In Sect. 3 we could prove our main results (e.g. the vanishing theorem 3.6) only for λ strongly dominant or for $\lambda = 0$. We want to show now that at least for classical groups this restriction is unnecessary.

In this section we can also admit the case $\mathrm{char}\, K = 0$ (except for Corollary 5.5).

5.1. Let $P \supset B$ be a parabolic subgroup of G and $\mathfrak{p} = \mathrm{Lie} P$. We shall write S_P for the set of simple roots in the Levi factor $M \supset T$ of P and $R^*(P)$ for the set of positive roots not in P, i.e. the weights of $\mathfrak{g}/\mathfrak{p}$. Set

$$X(T)_P^+ = \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geqq 0 \text{ for all } \alpha \in S_P\}.$$

Suppose we are given $\lambda \in X(T)$ and a weight ν of $\Lambda(\mathfrak{p}/\mathfrak{b})^*$. There is a w in the Weyl group of M such that $w(\lambda + \varrho + \nu) \in X(T)_P^+$. We claim:

$$\textit{If } \lambda + \varrho \in X(T)^+, \quad \textit{then} \quad w(\lambda + \varrho + \nu) \in X(T)^+. \tag{1}$$

Proof. We have to show $\langle w(\lambda + \nu + \varrho), \beta^\vee \rangle \geqq 0$ for all simple roots β not in S_P. But

$$w(\lambda + \nu + \varrho) \leqq \lambda + w(\nu + \varrho) \leqq \lambda + \varrho$$

by 2.2 (2). On the other hand the difference $\lambda + \varrho - w(\lambda + \varrho + \nu)$ is certainly a linear combination of roots in M, hence a positive linear combination of the $\alpha \in S_P$. As they satisfy $\langle \alpha, \beta^\vee \rangle \leqq 0$, we get $\langle w(\lambda + \varrho + \nu), \beta^\vee \rangle \geqq \langle \lambda + \varrho, \beta^\vee \rangle \geqq 0$ as needed.

5.2. Let us call a parabolic subgroup $P \supset B$ of G *nice*, if for every dominant weight $\lambda \in X(T)^+$ and every weight ν of some $\Lambda^k(\mathfrak{g}/\mathfrak{p})^*$ with $\lambda + \nu \in X(T)_P^+$ and $\langle \lambda + \varrho + \nu, \alpha^\vee \rangle \neq 0$ for all $\alpha \in R$ we have

$$\# \{\alpha \in R_+ \mid \langle \lambda + \varrho + \nu, \alpha^\vee \rangle < 0\} \leqq k. \tag{1}$$

Lemma. *Suppose $p \geqq h-1$. A parabolic subgroup $P \supset B$ of G is nice if and only if for each $\lambda \in X(T)^+$ and each weight ν of any $\Lambda^k(\mathfrak{g}/\mathfrak{p})^*$ with $\lambda+\nu \in X(T)_P^+$ we have either*

$$H^{\cdot}(G/B, \lambda+\nu)=0 \tag{2}$$

or there is a $k' \leqq k$ and a $\lambda' \in W.(\lambda+\nu) \cap X(T)^+$ with

$$H^j(G/B, \lambda+\nu)=\begin{cases} H^0(G/B, \lambda') & \text{if } j=k', \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Proof. For each λ and ν as above and each root $\alpha \in R_+$ we have

$$\langle \lambda+\varrho+\nu, \alpha^\vee \rangle > -(h-1) \geqq -p \tag{4}$$

by 2.2 (2). Choose $w \in W$ such that $w(\lambda+\nu+\varrho) \in X(T)^+$. Because of (4) Corollary 2.4 (ii) in [1] yields isomorphisms

$$H^i(G/B, \lambda+\nu) \simeq H^{i-\ell(w)}(G/B, w.(\lambda+\nu)) \tag{5}$$

for all i. If $\langle \lambda+\varrho+\nu, \alpha^\vee \rangle=0$ for some $\alpha \in R$ then (e.g. by Lemma 1.1 in [1]) condition (2) is satisfied. If not, then $H^i(G/B, \lambda+\nu)=0$ for $i \neq \ell(w)$ by Kempf's vanishing theorem, whereas $H^{\ell(w)}(G/B, \lambda+\nu) \neq 0$. Therefore the condition in the lemma is satisfied if and only if $\ell(w) \leqq k$ in all these cases. But $\ell(w)$ is just the number on the left-hand side of (1).

5.3. Let $P \supset P' \supset B$ be two parabolic subgroups of G, and let $M \supset T$ be the standard Levi factor of P. Then $M \cap B$ is a Borel subgroup of M with $M/(M \cap B) \simeq P/B$, and $M \cap P'$ is a parabolic subgroup of M with $M/(M \cap P') \simeq P/P'$. We shall write $\mathfrak{p}=\operatorname{Lie} P$, $\mathfrak{p}'=\operatorname{Lie} P'$ and $\mathfrak{m}=\operatorname{Lie} M$.

Lemma. *Suppose $p \geqq h-1$ and suppose $M \cap P'$ is nice in M. If for all $\lambda \in -\varrho+X(T)^+$ and all $i>0$ we have*

$$H^i(G/B, S^n(\mathfrak{p}^*) \otimes \lambda)=0, \tag{1}$$

then we have for all $\lambda \in -\varrho+X(T)^+$ and $i>0$

$$H^i(G/B, S^n(\mathfrak{p}'^*) \otimes \lambda)=0. \tag{2}$$

Proof. Using the Koszul resolution 3.5 (1) for $(\mathfrak{p}^*, \mathfrak{p}'^*)$ we see that it is enough to show

$$H^i(G/B, S^{n-j}(\mathfrak{p}^*) \otimes \Lambda^j(\mathfrak{p}/\mathfrak{p}')^* \otimes \lambda)=0 \quad \text{for} \quad i>j. \tag{3}$$

As $S^{n-j}(\mathfrak{p}^*)$ and $\Lambda^j(\mathfrak{p}/\mathfrak{p}')^*$ are P'-modules the left hand side is (by Kempf's vanishing theorem) equal to

$$H^i(G/P', S^{n-j}(\mathfrak{p}^*) \otimes \Lambda^j(\mathfrak{p}/\mathfrak{p}')^* \otimes H^0(P'/B, \lambda)). \tag{4}$$

For each weight ν of $\Lambda^j(\mathfrak{p}/\mathfrak{p}')^*$ and each root α we have $\langle \nu+\varrho, \alpha^\vee \rangle \leqq h-1 \leqq p$ by 2.2 (2). Therefore the P'-module $\Lambda^j(\mathfrak{p}/\mathfrak{p}'^*)$ has a good filtration by 4.1 (3) hence so has $\Lambda^j(\mathfrak{p}/\mathfrak{p}')^* \otimes H^0(P'/B, \lambda)$ by 4.1 (2). The factors in this filtration have the form $H^0(P'/B, \lambda+\nu)$, where ν is a weight of $\Lambda^j(\mathfrak{p}/\mathfrak{p}')^*$ with $\lambda+\nu \in X(T)_{P'}^+$. Therefore (3) will follow, if we can show for all such ν:

$$H^i(G/P', S^{n-j}(\mathfrak{p}^*) \otimes H^0(P'/B, \lambda+\nu))=0 \quad \text{for} \quad i>j. \tag{5}$$

Again by Kempf's vanishing theorem the left-hand side is isomorphic to

$$H^i(G/B, S^{n-j}(\mathfrak{p}^*)\otimes(\lambda+\nu)). \tag{6}$$

and there is a spectral sequence converging to it with E_2-term

$$E_2^{s,t} = H^s(G/P, S^{n-j}(\mathfrak{p}^*)\otimes H^t(P/B, \lambda+\nu)). \tag{7}$$

As $P'\cap M$ is nice in M, we see using the identifications $M/(M\cap B)\simeq P/B$ and $P/P'\simeq M/(M\cap P')$: If some $H^t(P/B, \lambda+\nu)$ is non-zero, then all the others vanish, we have $t\leqq j$ and an isomorphism $H^t(P/B, \lambda+\nu)\simeq H^0(P/B, \lambda')$ for some $\lambda'\in X(T)_P^+$ conjugate to $\lambda+\nu$ under the Weyl group of M. Therefore $\lambda'\in -\varrho+X(T)^+$ by 5.1 (1). Now the spectral sequence degenerates (if not zero at all) and yields isomorphisms

$$\begin{aligned} H^i(G/B, S^{n-j}(\mathfrak{p}^*)\otimes(\lambda+\nu)) &\simeq H^{i-t}(G/P, S^{n-j}(\mathfrak{p}^*)\otimes H^0(P/B, \lambda')) \\ &\simeq H^{i-t}(G/B, S^{n-j}(\mathfrak{p}^*)\otimes\lambda'). \end{aligned}$$

Now we apply (1) and see that these groups are 0 for $i>t$, hence for $i>j$, as we had to show.

5.4. We call a chain

$$G = P_0 \supset P_1 \supset \dots \supset P_{m-1} \supset P_m = B$$

of parabolic subgroups *nice*, if for each i the intersection $M_i\cap P_{i+1}$ is a nice parabolic subgroup of M_i, where $M_i \supset T$ denotes the standard Levi factor of P_i.

Proposition. *Suppose $p\geqq h-1$ and suppose there is a nice chain of parabolic subgroups from G to B. Then we have for all $\lambda\in -\varrho+X(T)^+$ all $n\in\mathbb{N}$ and $j>0$:*

$$H^j(G/B, S^n(\mathfrak{b}^*)\otimes\lambda) = H^j(G/B, S^n(\mathfrak{u}^*)\otimes\lambda) = 0.$$

Proof. By Kempf's vanishing theorem we have

$$H^j(G/B, S^n(\mathfrak{g}^*)\otimes\lambda)\simeq S^n(\mathfrak{g}^*)\otimes H^j(G/B, \lambda) = 0$$

for all $j>0$. From this we get the vanishing of $H^j(G/B, S^n(\mathfrak{b}^*)\otimes\lambda)$ applying Lemma 5.3 several times. The other result follows using the Koszul resolution for $(\mathfrak{b}^*, \mathfrak{u}^*)$ or the filtration in 3.5 (2).

5.5. Corollary. *Suppose $p>h$ and suppose there is a nice chain of parabolic subgroups from G to B. Then we have for all $w\in W$ and $\lambda\in X(T)^+$ with $w\cdot 0 + p\lambda\in X(T)^+$:*

$$H^i(G_1, H^0(G/B, w\cdot 0+p\lambda)) \simeq \begin{cases} H^0(G/B, S^{(i-\ell(w))/2}(\mathfrak{u}^*)\otimes\lambda)^{(1)} & \textit{if } i\equiv\ell(w) \bmod 2, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. This follows from 5.4 in the same way as Corollary 3.7 follows from 3.6.

Remark. The assumption of this corollary is satisfied, if all indecomposable components of R have type A, B, C or D. This follows by induction from the following proposition.

5.6. Proposition. *If R is indecomposable of type A_n (respectively B_n, C_n, D_n) and if $P \supset B$ is a parabolic subgroup with Levi factor of type A_{n-1} (respectively B_{n-1}, C_{n-1}, D_{n-1}), then P is nice.*

Proof. Let us number the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ as in [3], Chap. VI, planches I–IV and let us also use the notations ε_i as there. We assume that $\alpha_2, \ldots, \alpha_n$ are the simple roots in the Levi factor of P. Then $R^* = \{\alpha \in R_+ | \alpha_1 \leqq \alpha\}$ is the set of positive roots not in P. We have to show: If $\lambda \in X(T)^+$ and if v is a sum of k different elements of R^* such that $\lambda - v \in X(T)_P^+$ and $\langle \lambda + \varrho - v, \alpha^\vee \rangle \neq 0$ for all $\alpha \in R$, then

$$\{\alpha \in R^* | \langle \lambda + \varrho - v, \alpha^\vee \rangle < 0\} \leqq k. \tag{1}$$

Case 1. *Type A_n.* In this case $\langle \beta, \alpha^\vee \rangle = 1$ for all $\alpha, \beta \in R^*$ with $\alpha \neq \beta$. This implies $\langle \lambda + \varrho - v, \alpha^\vee \rangle \geqq 1 - \langle v, \alpha^\vee \rangle \geqq 1 - (k - 1 + 2) = -k$ for all λ, v as above and all $\alpha \in R^*$. For $\alpha, \beta \in R^*$ with $\alpha \neq \beta$ the difference $\alpha^\vee - \beta^\vee$ is again some coroot $\gamma^\vee$ with $\gamma \in R$. As $\langle \lambda + \varrho - v, \gamma^\vee \rangle \neq 0$ we get $\langle \lambda + \varrho - v, \alpha^\vee \rangle \neq \langle \lambda + \varrho - v, \beta^\vee \rangle$. Thus the elements $\langle \lambda + \varrho - v, \alpha^\vee \rangle$ with $\alpha \in R^*$ are pairwise different and all $\geqq -k$. Therefore, there can be at most k roots $\alpha \in R^*$ with $\langle \lambda + \varrho - v, \alpha^\vee \rangle < 0$, as we have to show. [In this case we do not need the assumption $\lambda - v \in X(T)_P^+$ which, however, will be necessary in other cases.]

Case 2. *Type B_n.* We have

$$R^* = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \ldots, \varepsilon_1 - \varepsilon_n, \varepsilon_1, \varepsilon_1 + \varepsilon_n, \ldots, \varepsilon_1 + \varepsilon_2\}.$$

Let us denote these roots in this order by $\beta_1 = \alpha_1, \beta_2, \beta_3, \ldots$. As in case 1 we see $\langle v, \alpha_1^\vee \rangle \leqq k+1$, hence $\langle \lambda + \varrho - v, \alpha_1^\vee \rangle \geqq -k$. For each i with $\varepsilon_1 \neq \beta_i, \beta_{i+1}$ the difference $\beta_i^\vee - \beta_{i+1}^\vee$ is of the form $\gamma^\vee$ for some $\gamma \in R_+$, $\gamma \notin R^*$. As $\lambda - v \in X(T)_P^+$ this implies $\langle \lambda + \varrho - v, \gamma^\vee \rangle > 0$, hence $\langle \lambda + \varrho - v, \beta_i^\vee \rangle < \langle \lambda + \varrho - v, \beta_{i+1}^\vee \rangle$. The sequence of integers $\langle \lambda + \varrho - v, \beta_i^\vee \rangle$ is hence strictly increasing (by steps of at least 1 each time) except possibly for the two steps involving $\beta_i = \varepsilon_1$. At those places we replace $\beta_i^\vee$ by $\frac{1}{2}\beta_i^\vee$ and get then (by the same argument as before) still a strict increase, though possibly only by $\frac{1}{2}$ at these two steps. Therefore there can be at most $k+1$ roots $\alpha \in R^*$ with $\langle \lambda + \varrho - v, \alpha^\vee \rangle < 0$. We have to show that this maximum is not reached. If it were we had necessarily $\langle \lambda + \varrho - v, \alpha_1^\vee \rangle = -k$ and $\langle \lambda + \varrho - v, \alpha_i^\vee \rangle = 1$ for $2 \leqq i \leqq n$, i.e.

$$\lambda + \varrho - v = -\frac{2(k-n)+3}{2}\varepsilon_1 + \sum_{i=2}^{n} \frac{2(n-i)+1}{2}\varepsilon_i.$$

Because of $\langle \lambda + \varrho - v, \gamma^\vee \rangle \neq 0$ for all $\gamma \in R$ we had $2(k-n) + 3 \neq \pm 1, \pm 3, \ldots, \pm(2n-3)$, hence $k \geqq 2n-1$. As there are only $2n-1$ roots in R^* there cannot be $k+1 \geqq 2n$ of them negative on $\lambda + \varrho - v$. Thus we have got a contradiction.

Case 3. *Type C_n.* We have

$$R^* = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \ldots, \varepsilon_1 - \varepsilon_n, 2\varepsilon_1, \varepsilon_1 + \varepsilon_n, \ldots, \varepsilon_1 + \varepsilon_2\}$$

and denote these roots in this order by $\beta_1 = \alpha_1, \beta_2, \beta_3, \ldots$. In this case $\beta_{i+1}^\vee - \beta_i^\vee$ is without exception of the form $\gamma^\vee$ with $\gamma \in R_+ \backslash R^*$. Therefore the sequence of integers $\langle \lambda + \varrho - v, \beta_i^\vee \rangle$ is strictly increasing. As $\langle 2\varepsilon_1, \alpha_1^\vee \rangle = 2$ we now have $\langle v, \alpha_1^\vee \rangle \leqq k+2$, where $\langle \nu, \alpha_1^\vee \rangle = k+2$ can occur only if both α_1 and $2\varepsilon_1$ are among the k roots adding up to ν, whereas $\varepsilon_1 + \varepsilon_2$ is not. We see now that $\langle \lambda + \varrho - \nu, \alpha_1^\vee \rangle \geqq -(k+1)$, hence at most $k+1$ roots in R^* can be negative on $\lambda + \varrho - \nu$.

Let us suppose that this maximum is achieved. Then necessarily $\langle \lambda + \varrho - \nu, \beta_i^\vee \rangle = -(k+2-i)$ for $1 \leqq i \leqq k+1$, hence

$$\langle \lambda + \varrho - \nu, \alpha_i^\vee \rangle = 1 \quad \text{for} \quad 2 \leqq i \leqq \min(n, k+1). \tag{2}$$

Furthermore, we have $\langle \lambda, \alpha_1^\vee \rangle = 0$ and $\langle \nu, \alpha_1^\vee \rangle = k+2$ so that α_1 and $2\varepsilon_1$ occur in ν, but $\varepsilon_1 + \varepsilon_2$ does not. Let us write $\lambda + \varrho = \sum_{i=1}^{n} m_i \varepsilon_i$ with integers $m_1 > m_2 > \ldots > m_n > 0$; because of $\langle \lambda, \alpha_1^\vee \rangle = 0$ we have $m_1 = m_2 + 1$. As ν starts $\nu = (k+1)\varepsilon_1 - \varepsilon_2 + \ldots$, we get $\lambda + \varrho - \nu = (m_2 - k)\varepsilon_1 + (m_2+1)\varepsilon_2 + \ldots$. By (2) the coefficient of ε_i in $\lambda + \varrho - \nu$ is $m_2 - i + 3$ for $2 \leqq i \leqq \min(n, k+2)$. On the other hand it can differ at most by 1 from m_i. As $m_i \leqq m_2 - i + 2$, we get $m_i = m_2 - i + 2$ and that $\varepsilon_1 - \varepsilon_i$ occurs in ν, whereas $\varepsilon_1 + \varepsilon_i$ does not, for all i with $2 \leqq i \leqq \min(n, k+2)$. If $k+2 \leqq n$ then we had now already $k+1$ roots occurring in ν which is impossible. Thus $n < k+2$ and $\nu = 2\varepsilon_1 + \sum_{i=2}^{n} (\varepsilon_1 - \varepsilon_i)$, hence $k = n$. Furthermore we have

$$\lambda + \varrho - \nu = (m_2 - n)\varepsilon_1 + \sum_{i=2}^{n} (m_2 - i + 3)\varepsilon_i.$$

As $\beta_{k+1} = \varepsilon_1 + \varepsilon_n$ we must have $m_2 - n + m_2 - n + 3 = \langle \lambda + \varrho - \nu, (\varepsilon_1 + \varepsilon_n)^\vee \rangle < 0$, i.e. $m_2 \leqq n-2$. Now $m_n = m_2 - n + 2 > 0$ yields the desired contradiction.

Case 4. *Type D_n.* We have

$$R^* = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \ldots, \varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_n, \ldots, \varepsilon_1 + \varepsilon_2\}$$

and denote these roots in this order by $\beta_1 = \alpha_1, \beta_2, \ldots, \beta_{n-1} = \varepsilon_1 - \varepsilon_n, \beta'_{n-1} = \varepsilon_1 + \varepsilon_n, \beta_n = \varepsilon_1 + \varepsilon_{n-1}, \ldots$. Then again each $\beta_{i+1}^\vee - \beta_i^\vee$ is of the form $\gamma^\vee$ for some $\gamma \in R^+, \gamma \notin R^*$ implying $\langle \lambda + \varrho - \nu, \beta_i^\vee \rangle < \langle \lambda + \varrho - \nu, \beta_{i+1}^\vee \rangle$. The same is true, if we replace $\beta_{n-1}^\vee$ by $\beta_{n-1}'^\vee$. Furthermore we have $\langle \nu, \alpha_1^\vee \rangle \leqq k+1$, and this maximum can be reached only if $\varepsilon_1 - \varepsilon_2$ occurs in ν, whereas $\varepsilon_1 + \varepsilon_2$ does not. (Hence $k \leqq 2n-3$ in this case.) We deduce from this $\langle \lambda + \varrho - \nu, \alpha_1^\vee \rangle \geqq -k$; therefore at most $k+1$ roots in R^* can be negative on $\lambda + \varrho - \nu$.

Let us suppose that this maximum is achieved. Then $\langle \lambda, \alpha_1^\vee \rangle = 0$ and $\langle \nu, \alpha_1^\vee \rangle = k+1$, furthermore both β_{n-1} and β'_{n-1} are among those $k+1$ roots negative on $\lambda + \varrho - \nu$. The increase from $\langle \lambda + \varrho - \nu, \beta_i^\vee \rangle$ to $\langle \lambda + \varrho - \nu, \beta_{i+1}^\vee \rangle$ has to be exactly 1 in the region of those $k+1$ roots. This implies

$$\lambda + \varrho - \nu = (n - k + 2)\varepsilon_1 + \sum_{i=2}^{n-2} (n-i)\varepsilon_i.$$

As $\langle \lambda + \varrho - \nu, \gamma^\vee \rangle \neq 0$ for all $\gamma \in R$, we have $n - k - 2 \neq \pm 1, \pm 2, \ldots, \pm(n-2)$, hence $n - k - 2 < -(n-2)$ and $k > 2n - 4$. As $k \leqq 2n-3$ by an observation above, we get $k = 2n-3$ and $\nu = k\varepsilon_1 - \varepsilon_2$. Then $\lambda + \varrho = (n-2)\varepsilon_1 + (n-3)\varepsilon_2 + (n-3)\varepsilon_3 + \ldots$ and $\langle \lambda, \alpha_2^\vee \rangle = -1$ contradicting $\lambda \in X(T)^+$.

5.7. We leave it to the reader to check that $P \subset G$ is nice, if R is of type G_2 and if S_P consists of the unique long simple root β. If α is the other simple root one ought to order $R^* = \{\alpha, 3\alpha+\beta, 2\alpha+\beta, 3\alpha+2\beta, \alpha+\beta\}$.

6. Small Primes

In this section we discuss some cases where $p \leqq h$. It turns out that for such p the B_1-cohomology has a somewhat different description. The difficulty is that the spectral sequence from 1.8 does not degenerate. Still, in several cases we are able to compute its abutment.

For simplicity we assume R to be indecomposable. We let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the simple roots and $\omega_1, \ldots, \omega_n$ the fundamental weights. (We use always the numbering of [3, Planches I–IX] with ℓ replaced by n). As before α_0 is the highest short root.

A. *The case* $p = h > 2$.

6.1. In this case R has to be of type A_n with $p = n+1$. The proof of 2.2 (1) shows: If λ is a non-zero weight of $\Lambda\mathfrak{g}^*$ vanishing on T_1, then there is a fundamental weight ω_i such that $\lambda \in pW\omega_i$ and hence $\lambda \notin \mathbf{Z}R$. This implies

$$\text{If } \lambda \in p\mathbf{Z}R \text{ is a weight of } \Lambda\mathfrak{g}^*, \text{ then } \lambda = 0. \tag{1}$$

Let us now consider the spectral sequence $E_n^{s,t}$ from 1.8 converging to $H^\cdot(U_1, K)$. Everything is compatible with the T-action, thus we get a spectral sequence for each weight space of $H^\cdot(U_1, K)$. Take a weight of the form $p\lambda$ with $\lambda \in \mathbf{Z}R$. Now $E_0^{m,m+r} \simeq S^m(\mathfrak{u}^*)^{(1)} \otimes \Lambda^r(\mathfrak{u}^*)$, and all weights of $S^m(\mathfrak{u}^*)^{(1)}$ belong to $p\mathbf{Z}R$. As the weights of $\Lambda^r\mathfrak{u}^*$ are non-zero for $r > 0$ we see for the $p\lambda$-weight space $(E_0^{m,m})^{p\lambda} \simeq (S^m(\mathfrak{u}^*)^\lambda)^{(1)}$ and $(E_0^{m,m'})^{p\lambda} = 0$ for $m \neq m'$. Therefore for these weight spaces the spectral sequence degenerates, and we get $H^\cdot(U_1, K)^{p\lambda} \simeq S^\cdot(\mathfrak{u}^*)^{\lambda(1)}$. Obviously, $H^\cdot(B_1, K) = H^\cdot(U_1, K)^{T_1} = \bigoplus_{\nu \in X(T)} H^\cdot(U_1, K)^{p\nu}$. If we extend the notation of 2.5 (3) to more general T-modules we get

$$H^\cdot(B_1, K)_{\mathbf{Z}R}^{(-1)} = \bigoplus_{\nu \in \mathbf{Z}R} H^\cdot(U_1, K)^{p\nu(-1)} \simeq S^\cdot(\mathfrak{u}^*). \tag{2}$$

Furthermore, Proposition 2.8 implies for all $w \in W$ and $i \in \mathbf{Z}$

$$H^i(B_1, w \cdot 0)_{\mathbf{Z}R}^{(-1)} \simeq H^{i-\ell(w)}(B_1, K)_{\mathbf{Z}R}^{(-1)}. \tag{3}$$

6.2. Let us identify W with the symmetric group in $n+1$ letters (as in [3], Pl. I) and let $\sigma \in W$ be the cyclic permutation $(1, 2, \ldots, n+1)$. Then elementary computations show $\sigma(\omega_i) = \omega_{i+1} - \omega_1$ for $1 \leqq i < n$ and $\sigma(\omega_n) = -\omega_1$, hence $\sigma(\varrho) = \varrho - (n+1)\omega_1 = \varrho - p\omega_1$ and

$$\sigma^i \cdot 0 = -p\omega_i \quad \text{for} \quad 0 \leqq i \leqq n. \tag{1}$$

We use here and later on the convention

$$\omega_0 = 0. \tag{2}$$

Proposition. *For all $w \in W$ and $j \in \mathbf{Z}$ we have*

$$H^j(B_1, w\cdot 0)^{(-1)} \simeq \begin{cases} \bigoplus_{i=0}^{n} S^{(j-\ell(w\sigma^i))/2}(\mathfrak{u}^*)\otimes w(\omega_i) & \text{if } j \equiv \ell(w) \bmod 2, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. As $w\cdot 0 = w\cdot(\sigma^i\cdot 0 + p\omega_i) = w\sigma^i\cdot 0 + pw(\omega_i)$, we have for all i

$$H^j(B_1, w\cdot 0)^{(-1)} \simeq H^j(B_1, w\sigma^i\cdot 0)^{(-1)}\otimes w(\omega_i),$$

hence for all $Y \in X(T)/\mathbf{Z}R$

$$H^j(B_1, w\cdot 0)^{(-1)}_Y \simeq H^j(B_1, w\sigma^i\cdot 0)^{(-1)}_{-\omega_i+Y}\otimes w(\omega_i),$$

especially

$$H^j(B_1, w\cdot 0)^{(-1)}_{\omega_i+\mathbf{Z}R} \simeq H^j(B_1, w\sigma^i\cdot 0)^{(-1)}_{\mathbf{Z}R}\otimes w(\omega_i).$$

The proposition follows now from 6.1 (2), (3) as the ω_i are representatives for $X(T)/\mathbf{Z}R$ and as $\ell(\sigma)$ is even, hence $\ell(w\sigma^i) \equiv \ell(w) \bmod 2$. In fact one easily checks

$$\ell(\sigma^i) = i(n+1-i) \equiv 0 \bmod 2. \tag{3}$$

6.3. Corollary. *For all $w \in W$, $\lambda \in X(T)^+$ and $j \in \mathbf{Z}$ with $\mu = w\cdot 0 + p\lambda \in X(T)^+$ we have*

$$H^j(G_1, H^0(G/B,\mu))^{(-1)} \simeq \begin{cases} \bigoplus_{i=0}^{n} H^0(G/B, S^{(j-\ell(w\sigma^i))/2}(\mathfrak{u}^*)\otimes(\lambda + w\omega_i)) & \text{if } j \equiv \ell(w) \bmod 2, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Each $\omega_i \neq 0$ is a minuscule weight, therefore all $w(\omega_i)$ belong to $-\varrho + X(T)^+$, hence so do the $\lambda + w(\omega_i)$. The corollary now follows from 5.4, the proposition and 3.2.b.

6.4. Let us remember that G was assumed to be simply connected (since 2.1). Let $\bar{G}$ be the adjoint group, i.e. $\bar{G} = G/Z$, where Z is the schematic centre of G. Then Z is isomorphic to the first Frobenius kernel of the multiplicative group, hence a diagonalizable group scheme and thus has the property $H^i(Z, E) = 0$ for all $i > 0$ and all Z-modules E. This implies easily $H^i(G, E) \simeq H^i(\bar{G}, E)$ for all $\bar{G}$-modules E. The situation is different for G_1 and $\bar{G}_1$, however.

In our case Z is contained in G_1 and there are exact sequences $1 \to G_1/Z \to \bar{G}_1 \to Z \to 1$, and $1 \to T_1/Z \to \bar{T}_1 \to Z \to 1$ where $\bar{T} = T/Z$. The character group $X(\bar{T}_1)$ can be identified with $\mathbf{Z}R/p\mathbf{Z}R$ and T_1/Z is the subgroup scheme defined by the subgroup $pX(T)/p\mathbf{Z}R$. Hence $X(Z)$ is here identified with $pX(T)/p\mathbf{Z}R$. Furthermore, we have an isomorphism $(G_1T)/Z \simeq \bar{G}_1T$. We may therefore consider every $\bar{G}_1T$-module E as a G_1T-module. It follows then from the Lyndon-Hochschild-Serre spectral sequence that $H^{\cdot}(G_1, E) \simeq H^{\cdot}(G_1/Z, E)$ and $H^{\cdot}(\bar{G}_1, E) \simeq H(G_1/Z, E)^Z$, hence

$$H^{\cdot}(\bar{G}_1, E)^{(-1)} \simeq H^{\cdot}(G_1, E)^{(-1)}_{\mathbf{Z}R}. \tag{1}$$

A similar statement holds for B instead of G (taking $\bar{B}=B/Z$). Proposition 6.2 therefore implies for all $j\in\mathbf{Z}$ and $w\in W$

$$H^j(\bar{B}_1, w\cdot 0)^{(-1)}\simeq\begin{cases} S^{(j-\ell(w))/2}(\mathfrak{u}^*) & \text{if } j\equiv\ell(w) \bmod 2, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

If we have in addition $\lambda\in X(T)^+$ such that $\mu=w\cdot 0+p\lambda\in X(T)^+$, then $\mu\in\mathbf{Z}R$ and $H^0(G/B,\mu)$ is a $\bar{G}$-module. There is exactly one i with $0\leqq i\leqq n$ and $\lambda+\omega_i\in\mathbf{Z}R$. For this i Corollary 6.3 implies

$$H^j(\bar{G}_1, H^0(G/B,\mu))\simeq\begin{cases} H^0(G/B, S^{(j-\ell(w\sigma^i))/2}(\mathfrak{u}^*)\otimes(\lambda+w\omega_i)) & \textit{if } j\equiv\ell(w) \bmod 2, \\ 0 & \textit{otherwise}. \end{cases} \tag{3}$$

B. *The Case* $p=h-1>2$.

In this case p does not divide the order of $X(T)/\mathbf{Z}R$.

6.5. Lemma.

$$(\Lambda^i(\mathfrak{u}^*))^{T_1}=\begin{cases} K & \textit{for} \quad i=0, \\ p\alpha_0 & \textit{for} \quad i=\ell(s_{\alpha_0}), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Consider $Y\subset R_+$ non-empty and set $\lambda=\sum_{\alpha\in Y}\alpha$. Suppose $\lambda\in pX(T)$ or (equivalently) $\lambda\in p\mathbf{Z}R$.

We claim at first that λ is dominant. In fact by 2.2 (2) (or rather its analogue for positive roots) we have $|\langle\lambda-\varrho, \beta^\vee\rangle|\leqq h-1=p$ for all $\beta\in R$. In particular, for β simple we get $\langle\lambda,\beta^\vee\rangle-1\geqq -p$. As $\langle\lambda,\beta^\vee\rangle$ is a multiple of p it has to be at least 0.

Set now $S_0=\{\beta\in S|\langle\beta,\alpha_0^\vee\rangle=0\}$ and $R_0=R\cap\mathbf{Z}S_0=\{\alpha\in R|\langle\alpha,\alpha_0^\vee\rangle=0\}$. If $\langle\lambda,\alpha_0^\vee\rangle=0$ then $Y\subset R_0$ and $\lambda\in p\mathbf{Z}R_0$. But this is impossible by 2.2 (1) since $p=h-1$ is strictly larger than the Coxeter number of R_0.

On the other hand, $\langle\lambda,\alpha_0^\vee\rangle$ has to be divisible by p, it is at most $\langle 2\varrho,\alpha_0^\vee\rangle=2p$ and cannot be equal to p by the argument used in the proof of 2.2 (1). Therefore, $\langle\lambda,\alpha_0^\vee\rangle=2p=\langle 2\varrho,\alpha_0^\vee\rangle$. This implies $Y\supset Y_1=\{\alpha\in R_+|\langle\alpha,\alpha_0^\vee\rangle<0\}$. Obviously, $s_\beta(Y_1)=Y_1$ for all $\beta\in S_0$ and hence $s_\beta(\lambda_1)=\lambda_1$, where $\lambda_1=\sum_{\alpha\in Y_1}\alpha$. It is now easy to see that $\lambda_1\in\mathbf{Q}\alpha_0$ and since $\langle\lambda_1,\alpha_0^\vee\rangle=2p$ we conclude $\lambda_1=p\alpha_0$. Therefore, $\lambda-\lambda_1\in p\mathbf{Z}R$. As $\lambda-\lambda_1$ is the sum of all $\alpha\in Y\backslash Y_1$ and as $\langle\lambda-\lambda_1,\alpha_0^\vee\rangle=0$ the arguments above prove $\lambda-\lambda_1=0$. Hence $\lambda=p\alpha_0$ and $Y=Y_1$.

To finish the proof of the lemma we only need to verify that $|Y_1|=\ell(s_{\alpha_0})$. Now for the longest root $\tilde{\alpha}$ it is clear that $\langle\alpha,\tilde{\alpha}^\vee\rangle>0$ for $\alpha\in R_+$ is equivalent to $s_{\tilde{\alpha}}(\alpha)<0$, so that $\ell(s_{\tilde{\alpha}})=|\{\alpha\in R_+|\langle\alpha,\tilde{\alpha}^\vee\rangle<0\}|$. We get the same result for α_0 going over to the dual root system $R^\vee$ using that $\alpha_0^\vee$ is the largest root of $R^\vee$.

6.6. For $\lambda\in\mathbf{Z}R$ we can write $\lambda=\sum_{i=1}^n m_i\alpha_i$ with $n_i\in\mathbf{Z}$ and we set

$$ht(\lambda)=\sum_{i=1}^n m_i.$$

Lemma. $ht(\alpha_0)=(\ell(s_{\alpha_0})+1)/2$.

Proof. If $\tilde{\alpha}$ denotes the largest root in R then $\{\alpha\in R_+|s_{\tilde{\alpha}}(\alpha)<0\}=\{\tilde{\alpha}\}\cup\{\alpha\in R_+|\ \langle\alpha,\tilde{\alpha}^\vee\rangle=1\}$ and thus $\ell(s_{\tilde{\alpha}})=\sum_{\alpha\in R_+}\langle\alpha,\tilde{\alpha}^\vee\rangle-1=\langle 2\varrho,\tilde{\alpha}^\vee\rangle-1$. Now $\alpha_0^\vee$ is the largest root in $R^\vee$ and hence $\ell(s_{\alpha_0})+1=\langle 2\sigma,\alpha_0^{\vee\vee}\rangle$, where 2σ is the sum of the positive roots in $R^\vee$. By [3, Chap. VI, Sect. 1, No. 10, cor. de la prop 29] we have that $ht(\alpha_0)=\langle\alpha_0,\sigma\rangle$ and our lemma follows.

6.7. For the sake of brevity let us write

$$r=ht(\alpha_0).$$

As $s_{\alpha_0}\cdot 0=-\langle\varrho,\alpha_0^\vee\rangle\alpha_0=-(h-1)\alpha_0=-p\alpha_0$ Proposition 2.2 in [28] and 6.6 above imply

$$H^i(B,-p\alpha_0)=\begin{cases}K & \text{for}\quad i=2r-1,\\ 0 & \text{otherwise}.\end{cases}\tag{1}$$

Lemma. *We have*

$$H^i(B,S^m(\mathfrak{u}^*)\otimes(-\alpha_0))=0\quad\textit{for}\quad i+m>r\tag{2}$$

and

$$H^1(B,S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0))=K.\tag{3}$$

Proof. Let λ be a weight of $S^m(\mathfrak{u}^*)\otimes(-\alpha_0)$ and assume $\lambda<0$. Write $\lambda=-\sum_{i=1}^n a_i\alpha_i$. As $\lambda>-\alpha_0$ we see that $a_i<p$ for all i. It follows now from [32] that $H^i(B,\lambda)=0$ unless $\lambda=-\sum_{j=1}^i\beta_j$ for some $\beta_j\in R_+$. If this is the case we get

$$-\sum_{j=1}^i\beta_j=\sum_{j=1}^m\gamma_j-\alpha_0$$

for some $\gamma_j\in R_+$ and hence $r=ht(\alpha_0)\geqq i+m$. This implies (2). Write now $\alpha_0=\sum_{i=1}^n c_i\alpha_i$. Then $\sum_{i=1}^n c_i=r$ and we see that the minimal weights in $S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0)$ are $\sum_{i=1}^n b_i\alpha_i-\sum_{i=1}^n a_i\alpha_i$ for some non-negative b_i with $\sum_{i=1}^n b_i=r-1$. In particular, the only negative weights are $-\alpha_1,\dots,-\alpha_n$ each occurring with multiplicity one. Hence $S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0)$ contains a B-submodule $E\simeq\bigoplus_{i=1}^n(-\alpha_i)$. The 0-weight space in $(S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0))/E$ is B-invariant and has dimension $n-1$, as there are exactly $n-1$ roots of the form $\alpha_i+\alpha_j$. This implies $H^0(B,(S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0))/E)\simeq K^{n-1}$, whereas $H^1(B,(S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0))/E)=0$ as this module has no negative weights.

By [28, 2.4] we have $H^1(B,-\alpha_i)=K$ for all i. Therefore, we get an exact sequence

$$H^0(B,S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0))\to K^{n-1}\to K^n\to H^1(B,S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0))\to 0.$$

If the first term vanishes the formula (3) follows. Hence we have to prove that α_0 is no weight of $S^{r-1}(\mathfrak{u}^*)^U$. It is, however, an easy consequence of the existence of regular nilpotent elements in good characteristic and of their description in [33] that $S(\mathfrak{u}^*)^U$ is generated by elements $x_1, \ldots, x_n$ in $\mathfrak{u}^*$ of weight $\alpha_1, \ldots, \alpha_n$. Any weight λ of any $S^m(\mathfrak{u}^*)^U$ satisfies therefore $ht(\lambda)=m$. Thus α_0 is no weight for $r-1 = ht(\alpha_0)-1$.

6.8. Let r be as in 6.7. Consider the spectral sequence (cf. 1.8 and 2.3)

$$E_0^{s,t}=S^s(\mathfrak{u}^*)^{(1)}\otimes(\Lambda^{t-s}\mathfrak{u}^*)^{T_1} \Rightarrow H^{s+t}(B_1,K).$$

Lemma. *The only non-zero differentials in the spectral sequence are $d_r^{s,s+2r-1}$, and these are all injective.*

Proof. It is clear from 6.5 that only the $d_r^{s,s+2r-1}$ can be non-zero. To prove injectivity it is by the derivation property of d_r enough to consider the case $s=0$. But if $d_r^{0,2r-1}$ is not injective on $(\Lambda^{2r-1}\mathfrak{u}^*)^{T_1}=p\alpha_0$ it is the zero map in which case (1) gives

$$H^{2r-1}(B_1,K)\simeq p\alpha_0.$$

Hence to prove the lemma it is enough to prove

$$H^{2r-1}(B_1,-p\alpha_0)^{B/B_1}=0. \tag{2}$$

Now this is the $E_2^{0,2r-1}$-term in the spectral sequence

$$H^i(B/B_1,H^j(B_1,-p\alpha_0)) \Rightarrow H^{i+j}(B,-p\alpha_0). \tag{3}$$

By the above we have for $j<2r-1$

$$H^j(B_1,-p\alpha_0)^{(-1)}\simeq\begin{cases}S^{j/2}(\mathfrak{u}^*)\otimes(-\alpha_0) & \text{for } j \text{ even},\\ 0 & \text{for } j \text{ odd}.\end{cases}$$

If $i+j=2r-1$ and $i>1$ we get now

$$H^i(B/B_1,H^j(B_1,-p\alpha_0))=0$$

by 6.7 (2). The spectral sequence (3), therefore, yields a short exact sequence

$$0\to H^1(B,S^{r-1}(\mathfrak{u}^*)\otimes(-\alpha_0))\to H^{2r-1}(B,-p\alpha_0)\to H^0(B/B_1,H^{2r-1}(B_1,-p\alpha_0))\to 0.$$

The first two terms are equal to K by 6.7 (3), (1), hence the last one has to be 0, i.e. (2) has to hold.

6.9. Proposition ($p=h-1$). *Set $r=ht(\alpha_0)$. Then:*

$$H^j(B_1,K)^{(-1)}\simeq\begin{cases}S^{j/2}(\mathfrak{u}^*)/S^{(j/2)-r}(\mathfrak{u}^*)\otimes\alpha_0 & \text{for } j \text{ even},\\ 0 & \text{for } j \text{ odd}.\end{cases}$$

This is now clear.

6.10. Assume that $H^i(G/B,S(\mathfrak{u}^*)\otimes\alpha_0)=0$ for all $i>0$. This condition is certainly satisfied if G is of classical type by the results in Sect. 5. Proposition 6.9 implies at first $H^i(G/B,H^j(B_1,K)^{(-1)})=0$ for all $i>0$ and then via 3.2:

Corollary ($p=h-1$). *With the above assumption we have*

a) *if j is odd, then $H^j(G_1, K)=0$;*

b) *if $j=2i$ is even, then we have a short exact sequence*

$$0 \to H^0(G/B, S^{i-r}(\mathfrak{u}^*) \otimes \alpha_0) \to H^0(G/B, S^i(\mathfrak{u}^*)) \to H^{2i}(G_1, K)^{(-1)} \to 0 .$$

C. *The Case G of Type G_2 and $p=3$.*

This is the only case, where G has rank 2 and where $p>2$, which has not been dealt with so far. It is also an example of what may happen in general for $2<p<h-1$.

6.11. Let thus R be of type G_2. In order to avoid double indices let us write $\alpha=\alpha_1$ and $\beta=\alpha_2$, where we choose α to be short. For all $\gamma \in R_+$ let X_γ denote a generator for the γ weight space in $\mathfrak{u}^*$. It is now easy to check that the following elements form a basis of $(\Lambda \mathfrak{u}^*)^{T_1}$:

$$v_0 = 1 ,$$

$$v_1 = X_\beta \wedge X_{3\alpha+2\beta}, \qquad v_2 = X_{3\alpha+\beta} \wedge X_{3\alpha+2\beta}, \qquad v_3 = X_\beta \wedge X_{\alpha+\beta} \wedge X_{2\alpha+\beta},$$

$$v_4' = X_\alpha \wedge X_{2\alpha+\beta} \wedge X_{3\alpha+2\beta}, \qquad v_4'' = X_{\alpha+\beta} \wedge X_{2\alpha+\beta} \wedge X_{3\alpha+\beta},$$

$$v_5 = X_\alpha \wedge X_\beta \wedge X_{2\alpha+\beta} \wedge X_{3\alpha+\beta},$$

$$v_6 = X_\beta \wedge X_{\alpha+\beta} \wedge X_{2\alpha+\beta} \wedge X_{3\alpha+\beta} \wedge X_{3\alpha+2\beta} = v_3 \wedge v_2 .$$

The weights of these elements are 0 for v_0, $3(\omega_2-\omega_1)$ for v_1, v_3, $3\omega_1$ for v_2, v_4', v_4'', v_5, and $3\omega_2$ for v_6. It is now easy to compute the T_1 fixed points of the usual Lie algebra cohomology $H^\cdot(\mathfrak{u}, K)$. The differentials $\Lambda^i \mathfrak{u}^* \to \Lambda^{i+1}\mathfrak{u}^*$ map all v_i to 0 and map v_4', v_4'' to a non-zero multiple of v_5. Let v_4 be a linear combination of v_4' and v_4'' mapped to zero. We may sum up the computations as follows [identifying any v_i with its image in $H^\cdot(\mathfrak{u}, K)$]:

$$H^0(\mathfrak{u}, K)^{T_1} = K v_0 \quad \text{(of weight 0)}, \tag{1}$$

$$H^i(\mathfrak{u}, K)^{T_1} = 0 \quad \text{for} \quad i=1,4,6, \tag{2}$$

$$H^2(\mathfrak{u}, K)^{T_1} = K v_1 \oplus K v_2 \quad [\text{of weight } 3(\omega_2-\omega_1), 3\omega_1], \tag{3}$$

$$H^3(\mathfrak{u}, K)^{T_1} = K v_3 \oplus K v_4 \quad [\text{of weight } 3(\omega_2-\omega_1), 3\omega_1], \tag{4}$$

$$H^5(\mathfrak{u}, K)^{T_1} = K v_6 \quad (\text{of weight } 3\omega_2). \tag{5}$$

6.12. We want to use the information from 6.11 to compute B_1-cohomology using the spectral sequence (cf. [10, 5.2])

$$E_1^{i,j} = S^i(\mathfrak{u}^*)^{(1)} \otimes H^{j-i}(\mathfrak{u}, K)^{T_1} \Rightarrow H^{i+j}(B_1, K). \tag{1}$$

We get the first two cohomology groups immediately as all differentials involved have to be zero by 6.11 (1)–6.11 (3):

$$H^1(B_1, K) = 0. \tag{2}$$

There is an exact sequence of B-modules

$$0 \to \mathfrak{u}^* \to H^2(B_1, K)^{(-1)} \to E \to 0, \tag{3}$$

where $E = ((\Lambda^2 \mathfrak{u}^*)^{T_1})^{(-1)}$ is a non-split extension of ω_1 by $\omega_2 - \omega_1$.

In order to go on we need information about B-cohomology in a special case. Take $\lambda=-\omega_1$ or $\lambda=\omega_1-\omega_2$. As $\lambda+\varrho\in X(T)^+$, but $\lambda\notin X(T)^+$ we have $H^\cdot(G/B,\lambda)=0$. There is $w\in W$ with $w(3\lambda+\varrho)\in X(T)^+$. An explicit computation gives $w\cdot 3\lambda=-\omega_2=-(3\alpha+2\beta)$, respectively $w\cdot 3\lambda=\omega_1-\omega_2=-(\alpha+\beta)$. So no dominant weight is less or equal to $w\cdot 3\lambda$, hence $H^\cdot(G/B,3\lambda)=0$ by the strong linkage principle. Using the spectral sequence in [28, 1.1] we get

$$H^\cdot(B,\lambda)=H^\cdot(B,3\lambda)=0 \quad \text{for} \quad \lambda\in\{-\omega_1,\omega_1-\omega_2\}. \tag{4}$$

Thus the abutment of the spectral sequence

$$H^i(B/B_1,H^j(B_1,3\lambda)) \Rightarrow H^{i+j}(B,3\lambda) \tag{5}$$

is zero. As $H^i(B/B_1,H^j(B_1,3\lambda))\simeq H^i(B,H^j(B_1,K)^{(-1)}\otimes\lambda)$ we get from (4) for $j=0$ and from (2) for $j=1$:

$$H^\cdot(B/B_1,H^j(B_1,3\lambda))=0 \quad \text{for} \quad j=0,1. \tag{6}$$

We claim that also

$$H^2(B/B_1,H^2(B_1,3\lambda))=0. \tag{7}$$

In order to prove this one checks for all weights v of $\mathfrak{u}^*$ and of E from (3) that $H^2(B,v+\lambda)=0$ using [28, Proposition 2.9.i]. Then (7) follows from (3). By (6) and (7) all differentials on the $E_2^{0,3}$ terms in (5) vanish. As the abutment is zero we get

$$H^0(B/B_1,H^3(B_1,3\lambda))=0,$$

i.e., that -3λ is no weight of $H^3(B_1,K)$. The spectral sequence (1) shows that we can identify $H^3(B_1,K)$ with the $E_r^{0,3}$ term there, hence can have as weights only $3(\omega_2-\omega_1)$ and $3\omega_1$. As they do not occur we get $H^3(B_1,K)=0$ and

$$E_r^{0,3}=0 \quad \text{for all} \quad r>2. \tag{8}$$

6.13. By 6.11 the only non-zero E_1 terms in the spectral sequence 6.12(1) are the $E_1^{i,i}$, $E_1^{i,i+2}$, $E_1^{i,i+3}$, $E_1^{i,i+5}$. Hence all $d_1^{i,j}$ vanish except (possibly)

$$d_1^{i,i+3}: S^i(\mathfrak{u}^*)^{(1)}\otimes H^3(\mathfrak{u},K)^{T_1}\to S^{i+1}(\mathfrak{u}^*)^{(1)}\otimes H^2(\mathfrak{u},K)^{T_1}.$$

The derivation property of d_1 shows

$$d_1^{i,i+3}(f\otimes v_j)=(f\otimes 1)d_1^{0,3}(1\otimes v_j).$$

We have, therefore, only to compute the images of v_3 and v_4. The image of v_3 is 0, as $3(\omega_2-\omega_1)$ is no weight of $\mathfrak{u}^{*(1)}\otimes H^2(\mathfrak{u},K)^{T_1}$. We claim that v_4 (after replacing it by a multiple, if necessary) is mapped to $X_\alpha\otimes v_1$. If not, its image is 0 by weight considerations, hence $d_1=0$. Then by 6.12(8) the differential $d_2^{0,3}$ is injective. Looking at weight spaces we see that $d_2^{0,3}(v_4)$ is a non-zero multiple of $X_\alpha X_{\alpha+\beta}$. This element is not invariant under the root subgroup $U_{-\beta}$, whereas v_4 is. Thus we get a contradiction, and d_1 is given by

$$d_1^{i,i+3}(f\otimes v_j)=\begin{cases} fX_\alpha\otimes v_1 & \text{for} \quad j=4, \\ 0 & \text{for} \quad j=3 \end{cases} \tag{1}$$

for all $f\in S^i(\mathfrak{u}^*)$.

6.14. Lemma. *The spectral sequence 6.12 (1) has the following E_2-terms:*

$$E_2^{i,i} = S^i(\mathfrak{u}^*)^{(1)},$$
$$E_2^{i,i+2} = S^i(\mathfrak{u}^*)^{(1)} \otimes (\Lambda^2 \mathfrak{u}^*)^{T_1} / (S^{i-1}(\mathfrak{u}^*) X_\alpha)^{(1)} \otimes v_1,$$
$$E_2^{i,i+3} = S^i(\mathfrak{u}^*)^{(1)} \otimes 3(\omega_2 - \omega_1),$$
$$E_2^{i,i+5} = S^i(\mathfrak{u}^*)^{(1)} \otimes 3\omega_2.$$

All other terms vanish. The differentials $d_2^{i,i+3}$ and $d_2^{i,i+5}$ are injective. All other differentials vanish.

Proof. The description of the E_2-terms and the vanishing of the $d_2^{i,j}$ for $j \neq i+3, i+5$ is clear from the above. From 6.12 (8) it follows that $d_2^{0,3}$ is injective. Weight space considerations show that it maps v_3 to some $cX_\alpha X_\beta$ with $c \in K$, $c \neq 0$. Using the derivation property of d_2 we get $d_2^{i,i+3}(f \otimes v_3) = cfX_\alpha X_\beta$ for all $f \in S^i(\mathfrak{u}^*)$ and $d_2^{i,i+5}(v_6) = d_2^{i,i+5}(v_3 \wedge v_2) = d_2^{i,i+3}(v_3) \otimes v_2$ hence $d_2^{i,i+5}(f \otimes v_6) = cfX_\alpha X_\beta \otimes v_2$ for all $f \in S^i(\mathfrak{u}^*)$. This proves the injectivity of $d_2^{i,i+3}$ and $d_2^{i,i+5}$.

6.15. Proposition. *Let $j \in \mathbb{N}$.*

a) $H^{2j+1}(B_1, K) = 0$.

b) *There is an exact sequence of B-modules*

$$0 \to S^j(\mathfrak{u}^*)/S^{j-2}(\mathfrak{u}^*) \otimes (\omega_2 - \omega_1) \to H^{2j}(B_1, K)^{(-1)}$$
$$\to S^{j-1}(\mathfrak{u}^*) \otimes E / (S^{j-2}(\mathfrak{u}^*) \otimes \omega_1 \oplus S^{j-3}(\mathfrak{u}^*) \otimes \omega_2) \to 0$$

with E as in 6.12 (3).

Proof. It follows from 6.14 that $E_\infty^{i,j} = 0$ for $j \neq i, i+2$ in 6.12 (1), that $E_\infty^{i,j} = E_3^{i,j}$ and that there are exact sequences

$$0 \to E_2^{i-2,i+1} \to E_2^{i,i} \to E_3^{i,i} \to 0$$

and

$$0 \to E_2^{i-2,i+3} \to E_2^{i,i+2} \to E_3^{i,i+2} \to 0.$$

The explicit description of the E_2 terms in 6.14 yields the proposition.

6.16. Corollary. *Let $j \in \mathbb{N}$.*

a) $H^{2j+1}(G_1, K) = 0$.

b) *There is an exact sequence of G-modules*

$$0 \to H^0(G/B, S^j(\mathfrak{u}^*))/H^0(G/B, S^{j-2}(\mathfrak{u}^*) \otimes (\omega_2 - \omega_1)) \to H^{2j}(G_1 K)^{(-1)}$$
$$\to H^0(G/B, S^{j-1}(\mathfrak{u}^*) \otimes E)/H^0(G/B, S^{j-2}(\mathfrak{u}^*) \otimes \omega_1 \oplus S^{j-3}(\mathfrak{u}^*) \otimes \omega_2) \to 0$$

with E as in 6.12(3).

Proof. By checking all weights of $\Lambda(\mathfrak{g}/\mathfrak{b})^*$ one verifies for $\lambda \in \{0, \omega_1, \omega_2, \omega_2 - \omega_1\}$ that

$$H^i(G/B, \Lambda(\mathfrak{g}/\mathfrak{b})^* \otimes \lambda) = 0 \quad \text{for} \quad i > 1.$$

Using the Koszul resolution of $(\mathfrak{g}^*, \mathfrak{b}^*)$ as in 3.4 we get for these values of λ that $H^i(G/B, S(\mathfrak{u}^*) \otimes \lambda) = 0$ for $i > 0$. The corollary follows now from 6.15 via the spectral sequence 3.3 (1).

D. *The Case* $p=2$.

6.17. For $p=2$ and $r=1$ the spectral sequence from 1.8 converging to $H^{\cdot}(U_1, K)$ has E_1 terms $E_1^{i,0}=S^i(\mathfrak{u}^*)$ and $E_1^{i,j}=0$ for $j\neq 0$. Therefore, $H^{\cdot}(U_1, K)$ is the cohomology of the complex

$$0\to K\to \mathfrak{u}^*\to S^2\mathfrak{u}^*\to S^3\mathfrak{u}^*\to \dots .$$

The differential d_1 is a derivation on $S\mathfrak{u}^*$, it is hence determined by its restriction to $\mathfrak{u}^*$. As d_1 is T-equivariant and as 2γ is no root for any $\gamma\in R$ the image of $\mathfrak{u}^*$ is contained in $\bigoplus_{\lambda\notin 2R}(S^2\mathfrak{u}^*)^{\lambda}$. This space is mapped injectively to $\Lambda^2\mathfrak{u}^*$ under the natural (in characteristic 2) surjection $S^2\mathfrak{u}^*\to\Lambda^2\mathfrak{u}^*$ with kernel $\{f^2 | f\in\mathfrak{u}^*\}$. Therefore, it suffices to compute the composition $\mathfrak{u}^*\to\Lambda^2\mathfrak{u}^*$ of d_1 with this surjection. The argument in [10, 5.2] shows, however, that this map is just the coboundary operator for the usual Lie algebra cohomology, i.e. dual to $\Lambda^2\mathfrak{u}\to\mathfrak{u}$ with $a\wedge b\to[a,b]$. Thus for $p=2$ we can write down explicitly the differentials for the spectral sequence.

6.18. Let us use 6.17 in order to compute the cohomology in few cases. Let us choose for each $\gamma\in R_+$ a generator X_γ for the γ-weight space in $\mathfrak{u}^*$. We can identify $S(\mathfrak{u}^*)$ with the polynomials in the X_γ. The polynomials in the X_γ^2 then form a subalgebra, which we may identify with $S(\mathfrak{u}^*)^{(1)}$, on which d_1 is zero. Then $S(\mathfrak{u}^*)$ and $S(\mathfrak{u}^*)^{T_1}$ are free modules over $S(\mathfrak{u}^*)^{(1)}$ having $\prod_{\gamma\in I} X_\gamma$ for all $I\subset R_+$, respectively, just the T_1 invariant elements among these as a basis. We have to compute d_1 only on these basis elements.

Having dealt with A_1 in 2.4(3) and 3.10(3) we now turn to the rank 2 cases. In order to avoid double indices we write $\alpha=\alpha_1$ and $\beta=\alpha_1$ with α short in the case of two root lengths. We always have $d_1X_\alpha=0=d_1X_\beta$ and $d_1X_{\alpha+\beta}=X_\alpha X_\beta$. For type A_2 the only T_1 invariant basis element besides 1 is $X_\alpha X_\beta X_{\alpha+\beta}$ which is mapped to $X_\alpha^2X_\beta^2$ by d_1. This implies easily for all $i\in\mathbb{N}$ (and type A_2):

$$H^{2i+1}(B_1, K)=0. \tag{1}$$

There is an isomorphism of B-modules

$$H^{2i}(B_1, K)^{(-1)}\simeq S^i(\mathfrak{u}^*)/S^{i-2}(\mathfrak{u}^*)X_\alpha X_\beta. \tag{2}$$

By Proposition 5.4 we have $H^j(G/B, S^{i-2}(\mathfrak{u}^*)X_\alpha X_\beta)=0$ for all $j>0$ and hence (together with 3.6.a and 3.2):

$$H^{2i+1}(G_1, K)=0. \tag{3}$$

There is an exact sequence of G-modules

$$0\to H^0(G/B, S^{i-2}(\mathfrak{u}^*)\otimes\varrho)\to H^0(G/B, S^i(\mathfrak{u}^*))\to H^{2i}(G_1, K)^{(-1)}\to 0. \tag{4}$$

For type C_2 we have $d_1X_{2\alpha+\beta}=2X_\alpha X_{\alpha+\beta}=0$. We may regard $S(\mathfrak{u}^*)^{T_1}$ as $K[X_\alpha^2, X_\beta, X_{\alpha+\beta}^2, X_{2\alpha+\beta}]\otimes(K\oplus KX_\alpha X_{\alpha+\beta})$. As $d_1(X_\alpha X_{\alpha+\beta})=X_\alpha^2X_\beta$, we have $H^{\cdot}(B_1, K)\simeq K[X_\alpha^2, X_\beta, X_{\alpha+\beta}^2, X_{2\alpha+\beta}]/(X_\alpha^2X_\beta)$. We may state this result as follows:

There are exact sequences of B-modules for all $i \in \mathbf{N}$ (and type C_2):

$$0 \to S^i(\mathfrak{u}^*)/S^{i-2}(\mathfrak{u}^*) \otimes \omega_2 \to H^{2i}(B_1, K)^{(-1)} \to S^{i-1}(\mathfrak{u}^*) \otimes \omega_2 / S^{i-2}(\mathfrak{u}^*) \otimes 2\omega_1 \to 0, \tag{5}$$

$$0 \to S^i(\mathfrak{u}^*) \otimes (\omega_2 - \omega_1)/S^{i-1}(\mathfrak{u}^*) \otimes \omega_1 \to H^{2i+1}(B_1, K)^{(-1)} \to S^i(\mathfrak{u}^*) \otimes \omega_1 / S^{i-2}(\mathfrak{u}^*) \otimes \varrho \to 0. \tag{6}$$

In this case $p=2$ divides $(X(T):\mathbf{Z}R)$ and we see that $H^{2i}(B_1, K)_{\mathbf{Z}R} = H^{2i}(B_1, K)$, whereas $H^{2i+1}(B_1, K)_{\mathbf{Z}R} = 0$.

One may check that $H^j(G/B, \Lambda^i(\mathfrak{g}/\mathfrak{b})^* \otimes \nu) = 0$ for $j > i$ and $\nu \in \{\omega_1, \omega_2, 2\omega_1, \omega_2 - \omega_1, \varrho\}$. As before this implies that we may compute $H^\cdot(G_1, K)$ by applying $H^0(G/B, ?)$ to (5) and (6).

For type G_2 we have $d_1 X_{2\alpha+\beta} = 0$, $d_1 X_{3\alpha+\beta} = X_\alpha X_{2\alpha+\beta}$ and $d_1 X_{3\alpha+2\beta} = X_\beta X_{3\alpha+\beta} + X_{\alpha+\beta} X_{2\alpha+\beta}$. There are 16 "square-free" T_1-invariant monomials in the X_γ, eight in degree 3, three each in degree 2 and 4, and one each in degree 0 and 6. We leave it to the reader to check that for all $i \in \mathbf{N}$ (and type G_2)

$$H^{2i+1}(B_1, K) = 0 \tag{7}$$

and that there is a filtration of B-modules $0 \subset F_{i0} \subset F_{i1} \subset F_{i2} = H^{2i}(B_1, K)^{(-1)}$, where

$$F_{i0} \simeq S^i(\mathfrak{u}^*)/S^{i-2}(\mathfrak{u}^*)(K X_\alpha X_\beta + K X_\alpha X_{2\alpha+\beta} + K Y) \tag{8}$$

with $Y = X_{\alpha+\beta} X_{2\alpha+\beta} + X_\beta X_{3\alpha+\beta}$,

$$F_{i1}/F_{i0} \simeq (S^{i-1}(\mathfrak{u}^*)/(S^{i-2}(\mathfrak{u}^*) X_\alpha + S^{i-3}(\mathfrak{u}^*) Y)) \otimes (\omega_2 - \omega_1), \tag{9}$$

$$F_{i2}/F_{i1} \simeq (S^{i-1}(\mathfrak{u}^*)/S^{i-2}(\mathfrak{u}^*)(K X_\beta + K X_{2\alpha+\beta})) \otimes \omega_1. \tag{10}$$

6.19. Let us consider the case C_n ($n \geqq 1$). This (and $p=2$) is the only case (for any p), where there are roots in $pX(T)$. Using the notations of [3, planche III] for $n \geqq 2$ these are the roots $\alpha = 2\varepsilon_i$ and for these one easily checks $d_1 X_\alpha = 0$. Hence $H^1(B_1, K)^{(-1)}$ has dimension equal to n with weights ε_i ($1 \leqq i \leqq n$). As $\varepsilon_i + \varrho \in X(T)_+$ but $\varepsilon_i \notin X(T)_+$ for $i > 1$, whereas $\varepsilon_1 = \omega_1$ we see $H^j(G/B, H^1(B_1, K)^{(-1)}) = 0$ for $j > 0$ and $H^0(G/B, H^1(B_1, K)^{(-1)}) = H^0(G/B, \omega_1)$. Thus 3.2 implies

$$H^1(G_1, K)^{(-1)} = H^0(G/B, \omega_1). \tag{1}$$

This is, of course, well-known (cf. e.g. [29, 6.10]) and is the only case, where $H^1(G_1, K) \neq 0$. Observe that $H^1(G, K)_{\mathbf{Z}R} = 0$.

Let us look at $S^2(\mathfrak{u}^*)$. The T_1 invariants are spanned by the X^2 with $X \in \mathfrak{u}^*$ (forming a subspace isomorphic to $\mathfrak{u}^{*(1)}$ and annihilated by d_1), the $X_{2\varepsilon_i} X_{2\varepsilon_j}$ (mapped to zero by d_1) and the $X_{\varepsilon_i - \varepsilon_j} X_{\varepsilon_i + \varepsilon_j}$ on which d_1 turns out to be injective. Hence we have an exact sequence

$$0 \to \mathfrak{u}^* \to H^2(B_1, K)^{(-1)} \to E \to 0,$$

where E is a B-module having weights $\varepsilon_i + \varepsilon_j$ ($1 \leqq i < j \leqq n$). Except for $\varepsilon_1 + \varepsilon_2 = \omega_2$ we have $\varepsilon_i + \varepsilon_j \notin X(T)_+$ but $\varepsilon_i + \varepsilon_j + \varrho \in X(T)_+$. Thus 3.2 gives an exact sequence of G-modules

$$0 \to \mathfrak{g}^* \to H^2(G_1, K)^{(-1)} \to H^0(G/B, \omega_2) \to 0. \tag{2}$$

Note that in all other cases $H^2(G_1, K)$ is known:

If R is indecomposable not of type C_n then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, hence a result by Hochschild gives a short exact sequence

$$0 \to \mathfrak{g}^* \to H^2(G_1, K)^{(-1)} \to H^2(\mathfrak{g}, K)^{(-1)} \to 0$$

(cf. [10, 5.2, footnote]), and $H^2(\mathfrak{g}, K)$ has been computed by W. van der Kallen. It is equal to the G-module $\ker \pi$ in [34, 5.2]. (The same statement holds for $p \neq 2$ and arbitrary R.)

6.20. Let us, finally, look at the type A_5. For any positive root α we have $\alpha \notin 2X(T)$, for any different positive roots α, β we have $\alpha + \beta \notin 2X(T)$. This yields

$$H^0(B_1, K) = K, \quad H^1(B_1, K) = 0, \quad H^2(B_1, K)^{(-1)} \simeq \mathfrak{u}^* \tag{1}$$

and

$$H^0(G_1, K) = K, \quad H^1(G_1, K) = 0, \quad H^2(G_1, K)^{(-1)} \simeq \mathfrak{g}^*. \tag{2}$$

Let us consider $S^3(\mathfrak{u}^*)$. For $\alpha, \beta \in R^+$ with $\alpha + \beta \in R$ the element $X_\alpha X_\beta X_{\alpha+\beta}$ is T_1 invariant and mapped to $X_\alpha^2 X_\beta^2$ under d_1. Thus d_1 is injective on this part of $S^3(\mathfrak{u}^*)$. Suppose $\alpha, \beta, \gamma \in R_+$ satisfy $\alpha + \beta + \gamma \in 2X(T)$. Then $\langle \alpha + \beta + \gamma, \alpha_i^\vee \rangle \geqq -3$ for all i and as $\alpha + \beta + \gamma \in 2X(T)$ even $\langle \alpha + \beta + \gamma, \alpha_i^\vee \rangle \geqq -2$. Hence each weight λ of $H^3(B_1, K)^{(-1)}$ satisfies $\lambda + \varrho \in X(T)^+$. One checks easily that the only dominant weight is ω_3 (using the notations of [3, planche I]). The only other dominant weight of $S^3(\mathfrak{u}^*)$ is $2(\omega_1 + \omega_5)$, but d_1 is injective on the corresponding weight space. For each permutation σ of $\{4, 5, 6\}$ we have $X_\sigma = X_{\varepsilon_1 - \varepsilon_{\sigma(4)}} X_{\varepsilon_2 - \varepsilon_{\sigma(5)}} X_{\varepsilon_3 - \varepsilon_{\sigma(6)}}$ in $S^3(\mathfrak{u}^*)$ of weight $2\omega_3$ and these six elements form a basis of this weight space. Using the formula $d_1(X_{\varepsilon_i - \varepsilon_j}) = \sum_{k=i+1}^{j-1} X_{\varepsilon_i - \varepsilon_k} X_{\varepsilon_k - \varepsilon_j}$ for $i<j$ one checks $d_1\left(\sum_\sigma X_\sigma\right) = 0$ and that the kernel of d_1 on this weight space is spanned by this element. Thus $H^3(B_1, K)^{(-1)}$ has a one dimensional ω_3-weight space. The description of the other weights above, (1) and 3.2 then imply

$$H^3(G_1, K)^{(-1)} \simeq H^0(G/B, \omega_3). \tag{3}$$

This shows that there are cases besides the type C_n, where the cohomology does not vanish in all odd degrees. Observe that $H^3(G_1, K)_{\mathbf{Z}R} = 0$ in this case.

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Hölder and L^p-Estimates for the ∂ Equation in Some Convex Domains with Real-Analytic Boundary* **

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1. Introduction

Using the so-called Henkin-Ramirez reproducing kernel introduced in [2, 7], Grauert-Lieb and Henkin were able to define for a strictly pseudoconvex domain D an integral operator T satisfying $\bar{\partial}Tf=f$ for $\bar{\partial}$-closed forms f and for which a Hölder 1/2-estimate holds [1, 3, 5]. The problem arises whether something similar is true for weakly pseudoconvex domains.

Reproducing kernels can (trivially) be constructed in convex domains and so the operator T is also defined in this case (see Sect. 2). In [8], Range considered the domain

$$D=\left\{z:\sum_{i=1}^{n}|z_i|^{m_i}\leqq 1\right\},$$

where $m_1, \ldots, m_n$ are positive even integers, and proved that T satisfies a Hölder $1/m$-estimate, where $m=\max\{m_1, \ldots, m_n\}$. It is natural to ask whether this result holds for all convex domains with real analytic boundary.

In this paper we show that this is the case for domains $D=\{z:r(z)<0\}$ where r is a real-analytic, convex function of the type

$$r(z)=\sum_{i=1}^{n} s_i(|z_i|^2)-1,$$

and obtain as well L^p-estimates for the $\bar{\partial}$-equation. As in the strictly pseudoconvex case, the results depend on having a good control of the support function $F(\xi, z)$ $\xi\in bD$, $z\in D$, which appears in the denominator of the reproducing kernel (in the convex case is simply

$$F(\xi,z)=\sum_{i=1}^{n}\frac{\partial r}{\partial \xi_i}(\xi)(\xi_i-z_i)\Big).$$

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In the strictly pseudoconvex case this control is

$$2\operatorname{Re} F(\xi, z) \geqq -r(z) + c|\xi - z|^2, \qquad \xi \in bD, z \in D, |\xi - z| \text{ small}.$$

As in [8], the crucial point is to replace this by

$$2\operatorname{Re} F(\xi, z) \geqq -r(z) + c\,\mathrm{Lr}(\xi)(\xi - z)^2 + c|\xi - z|^m \tag{$*$}$$

where $\mathrm{Lr}(\xi)$ denotes the complex Hessian of r at ξ and m is some positive integer. The motivation for $(*)$ comes from the fact, also pointed out in [4], that the eigenvalues of $\mathrm{Lr}(\xi)$ appear in the numerator of the kernels. Once $(*)$ is established, the proof of [8] essentially works and we slightly modify it so as to give as well the L^p-estimates.

The paper is organized as follows. We consider first a real version of $(*)$

$$2\operatorname{Re} F(\xi, z) \geqq -r(z) + c\,\mathrm{Hr}(\xi)(\xi - z)^2 + c|\xi - z|^m, \quad \xi \in bD \tag{$**$}$$

where $\mathrm{Hr}(\xi)$ denotes the real hessian of r at ξ. Note that $(**)$ amounts to say that $\mathrm{Hr}(\xi)$ absorbs, even if it degenerates along certain directions, the remainder terms of the Taylor series of r at ξ. We are able to prove $(**)$ only if, all ξ_i, z_i are small (Sect. 2). Then in Sect. 3 we use it to obtain $(*)$ if $s_i(|z_i|^2)$ is strictly convex away from zero and in this case we show that the results hold for T (Theorems 3.3 and 3.4). In the general case (Sect. 4) we must construct another support function ϕ and consider a modification of T introduced in [9].

One final comment is in order: we feel that $(**)$ is true for all convex (bounded) domains with real analytic boundary. If this were the case, surely it could be applied to treat more general types of domains[1].

Finally, concerning notation, we use c to denote most of the constants and also employ $\simeq$ to mean that two quantities are of the same order of growth.

2. The Canonical Support Function and Its Estimate

2.1

From now on D will denote a bounded domain of the type $D = \{z : r(z) < 0\}$ where

$$r(z) = \sum_{i=1}^{n} s_i(|z_i|^2) - 1$$

is convex and real-analytic. We will denote by r_i the function of one complex variable defined by $r_i(w) = s_i(|w|^2)$. To be precise, the s_i are assumed to be real-analytic functions in an interval $[0, a_i]$ such that

(i) $s_i'(t) \geqq 0$, $s_i'(t) + 2t\, s_i''(t) \geqq 0$ for $0 \leqq t < a_i$ (i.e. r_i is convex)

(ii) $s_i(0) = 0$ and $s_i(a_i) > 1$ (i.e. D is bounded).

If $s_i(t) = b_k t^k + \dots$ is the development near 0, note that this implies $b_k > 0$, hence $s_i'(t) > 0$ for small $t \neq 0$, hence for all $0 < t < a_i$ (because $r_i(t)$ is convex and $r_i'(t) = 2t\, s_i'(t^2)$ increases). In particular, $s_i'(t) + t\, s_i''(t) > 0$ if $0 < t < a$. Since the complex Hessian of r_i at w is $s_i'(|w|^2) + |w|^2 s_i''(|w|^2)$ this shows that r_i is strictly subharmonic if $w \neq 0$. To exclude the strictly pseudoconvex case we assume of course

(iii) $s_i'(0) = 0$.

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1 See the note at the end of the paper

Thus $\mathrm{Lr}_i(w)$ degenerates exactly at $w=0$, but $\mathrm{Hr}_i(w)$ may degenerate at some other points.

2.2

We define

$$F_i(w,v)=\frac{\partial r_i}{\partial w}(w)(w-v).$$

Using Taylor's development and the hypothesis on s_i it is easy to see that

$$\begin{aligned} g_i(w,v) &\overset{\text{def}}{=} r_i(v)-r_i(w)+2\,\mathrm{Re}\,F_i(w,v)\geqq 0 \\ &\text{and is} \quad =0 \quad \text{only if} \quad w=v\,. \end{aligned} \tag{1}$$

The function $F(\xi,z)=\sum_i F_i(\xi_i,z_i)$ is the canonical support function for the domain D.

2.3

We shall obtain two different estimates of the function g_i in (1). In this subsection we drop out the index i and write r for r_i, g for g_i, F for F_i.

The first estimate is well known and is based on the following result [11]:

Theorem (Lojasiewicz). *Let g be a real analytic function in an open set U in $\mathbb{R}^n$ and let $z(g)$ denote the set of zeroes of g. Given a function $h\in C^\infty(U)$ vanishing on $z(g)$ and a compact set $K\subset U$, there exist a positive constant c and a positive integer m such that $|g(x)|\geqq c|h(x)|^m$, $x\in K$.*

By (1), we can apply the above to $g(w,v)$ in (1) with $h(w,v)=|w-v|^2$. Hence we get

$$g(w,v)\geqq c|w-v|^m \tag{2}$$

(this is the observation that all domains with real-analytic boundary are of finite type).

The second estimate we will need is the one contained in the following theorem. We use the notation

$$\mathrm{Hr}(w)(w-v)^2=(w-v,\bar{w}-\bar{v})\begin{pmatrix} r_{w\bar{w}} & r_{ww} \\ r_{\bar{w}\bar{w}} & r_{w\bar{w}}\end{pmatrix}\begin{pmatrix}\bar{w}-\bar{v} \\ w-v\end{pmatrix}$$

Theorem. *There exists $\varepsilon>0$ and $c>0$ such that*

$$g(w,v)\geqq c\,\mathrm{Hr}(w)(w-v)^2 \quad \textit{for} \quad |w|,|v|\leqq\varepsilon\,. \tag{3}$$

(Note that the theorem simply says that if r is a radial, convex and analytic function in a neighbourhood of 0 in the complex plane, then $\mathrm{Hr}(w)(w-v)^2$ controls all the other terms of higher order in the Taylor's development of $r(v)$ at w, for small v, w).

Proof. We regard both terms in (3) as functions of (w,v) in a neighbourhood of the origin in $\mathbb{C}^2$. First we prove the theorem for $r(z)=|z|^{2k}$. In this case by homogeneity we may suppose $(w, v) \in S^3$, the unit sphere in $\mathbb{C}^2$ and, because of (1), we may suppose as well that (w, v) is near the diagonal of S^3. But then w is far from zero and the result follows, for $\mathrm{Hr}(w)$ does not degenerate there.

The general case will be essentially reduced to this one. The main tool is the so-called "curve selection lemma", whose proof can be found in [6]:

Curve Selection Lemma. *Let U be an open set in $\mathbb{R}^m$ which is defined by inequalities between real-analytic functions. Then, if 0 is an accumulation point of U, there exists an analytic curve $\gamma: [0, \alpha) \to \mathbb{R}^m$ such that $\gamma(0)=0$ and $\gamma(t) \in U$ for $0<t<\alpha$.*

We use it in the following form:

Corollary. *Let G_1, G_2 be two analytic functions in a neighbourhood of 0 in $\mathbb{R}^m$. Then to prove that $G_1 \geqq G_2$ in some neighbourhood of 0 it is enough to prove that for each analytic curve $\gamma(t)$, with $\gamma(0)=0$, there exists $\alpha>0$ (which may depend on γ!) such that $G_1(\gamma(t)) \geqq G_2(\gamma(t))$ for $0 \leqq t < \alpha$.*

Hence it is enough to find $c>0$ such that for any analytic curve $\gamma(t) =(w(t), v(t))$, $\gamma(0)=0$, there exists $\alpha=\alpha(\gamma)$ such that

$$g(w(t), v(t)) \geqq c\,\mathrm{Hr}(w(t))\,(w(t)-v(t))^2$$

for $0<t<\alpha$. Lets consider the developments of w, v

$$w(t)=w_0 t^p + o(t^p), \qquad v(t)=v_0 t^p + o(t^p),$$
$$w_0 \text{ or } v_0 \neq 0, \qquad p \geqq 1$$

and that of r, $r(z)=\sum_{m=k}^{\infty} b_m |z|^{2m}$ with $b_k>0$. Here we write $r_k(z)=|z|^{2k}$, g_k for the corresponding function defined by (1) and let $c_k>0$ be some constant corresponding to this case. Then, clearly,

$$g(w(t), v(t)) = b_k g_k(w_0, v_0) t^{2pk} + o(t^{2pk})$$
$$\mathrm{Hr}(w(t))\,(w(t)-v(t))^2 = b_k \,\mathrm{Hr}_k(w_0)\,(w_0-v_0)^2 t^{2pk} + o(t^{2pk}).$$

Hence, if $w_0 \neq v_0$, t^{2pk} is the lower order term and we can choose any $c<c_k$.

If $w_0=v_0$ we must find out which is the lower order term in $g(w(t), v(t))$. To do so we will make a change of parameter. Note that, due to the rotation invariance of (3) we may assume that w is real and take, for small values of t, w as a new parameter (assuming $w_0>0$). Let $x(t)$, $y(t)$ denote the real and imaginary parts of $v(t)$. Since now we are assuming that $v_0=w_0$ is real

$$\begin{aligned} x(w) &= w + \varphi w^{\alpha} + o(|w|^{\alpha}), \\ y(w) &= \sigma w^{\beta} + o(|w|^{\beta}), \end{aligned} \tag{4}$$

with α, β rationals greater than 1. Lets denote as before $r_m(z)=|z|^{2m}$ and g_m for the corresponding function. A computation shows that

$$g_m(x(w), y(w), w) = \sum_{k_1+k_2+k_3=m}{}' \frac{m!}{k_1!k_2!k_3!} 2^{k_2} w^{2k_1+k_2}(x-w)^{k_2}((x-w)^2+y^2)^{k_3}, \quad k_i \geqq 0 \tag{5}$$

where by the tilda we mean that the combinations $k_1 = m$, $k_2 = k_3 = 0$ and $k_1 = m - 1$, $k_2 = 1$, $k_3 = 0$ are omitted. When we use (4) we find that the k-th term of (5) is, if $\varrho = \min(\alpha, \beta)$

$$O(w^{2k_1 + k_2 + k_2\alpha + 2k_3\varrho}) + \text{higher order terms}.$$

Since $2k_1 + k_2 + k_2\alpha + k_3\varrho = 2m + (\alpha - 1)k_2 + 2(\varrho - 1)k_3$, it turns out that

$$g_m = \text{const}\, w^{2m + 2(\varrho - 1)} + \text{higher order terms}$$

(in fact this is the contribution of the term $k_2 = 0$ $k_3 = 1$ and also that of $k_2 = 2$ and $k_3 = 0$ if $\alpha = \varrho$). This implies

$$g(x(w), y(w), w) = A w^{2k + 2(\varrho - 1)} + o(|w|^{2k + 2(\varrho - 1)})$$

with $A > 0$, i.e., what this analysis reveals is that the lower order term of $g(x(w), y(w), w)$ comes *only* from the first term $b_k|z|^{2k}$ of r (and A depends on b_k, k, φ^2, and σ^2). The same will happen with $\mathrm{Hr}(w)\,(r - w)^2$, which consists in the terms $k_2 = 0$, $k_3 = 1$ (with its corresponding coefficients) of (5). In fact we will have, of course,

$$\mathrm{Hr}(w)\,(v - w)^2 = 2A w^{2k + 2(\varrho - 1)} + o(|w|^{2k + 2(\varrho - 1)}).$$

Therefore it suffices to take c also smaller than 1/2. This ends the proof of the theorem. □

Remark. We have been unable to obtain a more satisfactory proof of the theorem, in particular one not so heavily based on the radial character of r. It would be very interesting to find it, and for other types of functions. As said in the introduction we feel that the theorem holds for all real-analytic and convex functions[1].

For w real we already noticed that $\mathrm{Hr}(w)$ is a diagonal matrix with entries $2s'(w^2) + 4w^2 s''(w^2)$ and $2s'(w^2)$, and $\mathrm{Lr}(w) = s'(w^2) + w^2 s''(w^2)$. Now it is clear that these three quantities are of the same order if w is near 0. Hence $\mathrm{Hr}(w)\,(w - v)^2 \simeq \mathrm{Lr}(w)|w - v|^2$ if $|w|$ is small. Thus we can replace Hr by Lr in (3). Using then (2), we get

$$\begin{aligned} & r(v) - r(w) + 2\,\mathrm{Re}\,F(w, v) \\ & \geqq c\,\mathrm{Lr}(w)|w - v|^2 + c|w - v|^m \quad \text{for} \quad |w|, |v| \leqq \varepsilon. \end{aligned} \tag{6}$$

3. The Results in a Particular Case

3.1

In this section we will prove our results with the auxiliary hypothesis

(iv) $$s_i'(t) + 2t s_i''(t) > \text{for} \quad 0 < t < a_i,$$

which means that $\mathrm{Hr}_i(w)$ only degenerates at $w = 0$. As said in the introduction, (iv) is not necessary but will allow us to obtain the results for the canonical kernels associated with the canonical support function $F(\xi, z)$ in 2.2. To make more clear the exposition we prefer to treat first this case and then deal in the next section with the general case.

1 See the note at the end of the paper

Since now Hr_i and Lr_i only degenerate at $w=0$ it is clear that g_i will satisfy (3) with Hr_i replaced by Lr_i for all $|w| \geqq \varepsilon$ if $|w-v|$ is small. Thus (6) holds for r_i if $|w-v|$ is small, hence for all w, v because of (1), and adding on i we obtain the fundamental inequality

$$r(\xi)-r(z)+2\operatorname{Re}F(\xi,z) \geqq c\,\mathrm{Lr}(\xi)(\xi-z)^2+c|\xi-z|^m, \qquad \xi, z \in \bar{D} \tag{7}$$

which is the basis of all estimates.

3.2

Now we briefly recall the definition of the reproducing kernel $H(\xi,z)$ and the integral solution operator T. Set

$$w'(\xi)=\sum_{i=1}^{n}(-1)^{i-1}\xi_i \bigwedge_{j\neq i} d\xi_j, \qquad w(\xi)=d\xi_1\wedge\ldots\wedge s\xi_n$$

$$P_i(\xi)=\frac{\partial r}{\partial \xi_i}(\xi), \qquad P(\xi)=(P_1(\xi),\ldots,P_n(\xi)).$$

For $u\in C^1(\bar{D})$ one has the decomposition formula

$$u(z)=c_n\int_{bD} u(\xi)H(\xi,z)+c_n\int_{bDx[0,1]} \bar{\partial}u(\xi)\wedge w'(\eta)\wedge w(\xi)$$
$$-c_n\int_D \bar{\partial}u(\xi)\wedge B(\xi,z)$$

where

$$\eta=(1-\mathrm{t})\frac{\bar{\xi}-\bar{\mathrm{z}}}{|\xi-\mathrm{z}|^2}+\mathrm{t}\frac{\mathrm{P}(\xi)}{\mathrm{F}(\xi,\mathrm{z})}, \qquad \mathrm{c}_n=(-1)^{n(n-1)/2}(\mathrm{n}-1)!\,(2\pi i)^{-n}$$

and

$$H(\xi,z)=w'\left(\frac{P}{F}\right)\wedge w(\xi),$$

$$B(\xi,z)=w'\left(\frac{\bar{\xi}-\bar{z}}{|\xi-z|^2}\right)\wedge w(\xi)$$

are respectively the reproducing kernel and the Bochner-Martinelli kernel. The integral solution operator T is then, for a $(0,1)$ form f regular to the boundary,

$$Tf(z)=T_1f(z)+T_2f(z),$$

where

$$T_1f(z)=c_n\int_{bDx[0,1]} f(\xi)\wedge w'(\eta)\wedge w(\xi),$$

$$T_2f(z)=-c_n\int_D f(\xi)\wedge B(\xi,z).$$

Using the relation

$$(n-1)!\ w'(\eta)\wedge w(\xi) = (-1)^{n(n-1)/2}\left(\sum_{i=1}^{n}\eta_i d\xi_i\right)\wedge\left(\sum_{i=1}^{n} d\eta_i\wedge d\xi_i\right)^{n-1}$$

and Newton's binomial formula for two forms, it is easily seen that the component in dt of $w'(\eta)\wedge w(\xi)$ to be considered is

$$\frac{(-1)^{n(n-1)/2}}{(n-2)!}\gamma_1\wedge\gamma_2\wedge(t\bar{\partial}_\xi\gamma_2+(1-t)\bar{\partial}_\xi\gamma_1)^{n-2}\wedge dt$$

where

$$\gamma_1=\frac{\partial_\xi|\xi-z|^2}{|\xi-z|^2}\quad\text{and}\quad\gamma_2=\frac{\partial r(\xi)}{F(\xi,z)}$$

Now, since

$$\bar{\partial}_\xi\gamma_1=\frac{\bar{\partial}_\xi\partial_\xi|\xi-z|^2}{|\xi-z|^2}+\gamma_1\wedge\ldots,\ \bar{\partial}_\xi\gamma_2=\frac{\bar{\partial}\partial r(\xi)}{F(\xi,z)}+\gamma_2\wedge\ldots$$

we finally obtain, performing the integration in t, that $T_1 f$ can be written

$$T_1=\sum_{k=0}^{n-2}c(n,k)H_k$$

where the $c(n,k)$ are constants and

$$H_k f(z)=\int_{bD} f(\xi)\wedge H_k(\xi,z)\,,\qquad z\in D$$

$$H_k(\xi,z)=|\xi-z|^{2(k+1-n)}F(\xi,z)^{-k-1}\partial_\xi|\xi-z|^2 \wedge\partial r(\xi)\wedge(\bar{\partial}\partial r(\xi))^k\wedge(\bar{\partial}\partial|\xi-z|^2)^{n-k-2}\,. \tag{8}$$

In the same manner the reproducing kernel $H(\xi,z)$ can be written

$$H(\xi,z)=\frac{(-1)^{n(n-1)/2}}{(n-1)!}\,\frac{\partial r(\xi)\wedge(\bar{\partial}\partial r(\xi))^{n-1}}{F(\xi,z)^n}\,. \tag{9}$$

3.3. Theorem. *If m is as in (7), for each $\alpha<1/m$ there exists $c(\alpha)>0$ such that*

$$|Tf(z)-Tf(w)|\leq c(\alpha)\|f\|_\infty|z-w|^\alpha\,,\qquad z,w\in D\,. \tag{10}$$

Proof. Once one has (7) the proof of [8] can be applied. We include it here presented in a somewhat different way, because some aspects of the computation will be needed as well to obtain the L^p-estimates.

It is well known that $T_2 f$ has modulus of continuity $\delta|\log\delta|$. So, taking into account the expression of $T_1 f$ obtained above it is enough to prove

$$\left|\int_{bD} v_z H_k(\xi,z)\wedge f(\xi)\right|\leq c(\alpha)|r(z)|^{\alpha-1}\quad v=\text{normal field} \tag{11}$$

for it is also well known that this implies that $H_k f$ satisfies (10). Now, there exists r_0 and δ_0 such that if $|r(z)|<r_0$ and $\frac{\partial r}{\partial z_j}(z)\neq 0$, say $j=1$, then

$$t_1=r(\xi)-r(z), \quad t_2=\operatorname{Im}F(\xi,z),$$
$$t_{2j-1}=\operatorname{Re}(\xi_j-z_j), \quad t_{2_j}=\operatorname{Im}(\xi_j-z_j) \quad j=2,\ldots,n$$

is a real coordinate system in the ball $B(z,\delta_0)$ such that $t(z)=0$, $|t|^2\simeq|\xi-z|^2$ and $d\sigma(\xi)\simeq dt_2 dt_3\ldots dt_{2n}$ in $bD\cap B(z,\delta_0)$. Of course, in proving (11) we may assume that $|r(z)|<r_0$ and estimate only the contribution of $bD\cap B(z,\delta_0)$ in the integral. Hence we proceed to estimate $v_z H_k(\xi,z)\wedge f(\xi)$ as a multiple of $dt_2\wedge\ldots\wedge dt_{2n}$ in $B(z,\delta_0)$. From (8),

$$v_z H_k(\xi,z)=\Bigg\{(k+1-n)\frac{v_z|\xi-z|^2}{|\xi-z|^{2(n-k)}}\frac{\partial_\xi|\xi-z|^2}{F(\xi,z)^{k+1}}$$
$$-(k+1)\frac{v_zF(\xi,z)}{F(\xi,z)^{k+2}}\frac{\partial_\xi|\xi-z|^2}{|\xi-z|^{2(n-k-1)}}$$
$$+\frac{v_z\partial_\xi|\xi-z|^2}{|\xi-z|^{2(n-1-k)}F(\xi,z)^{k+1}}\Bigg\}$$
$$\wedge\,\partial r(\xi)\wedge(\bar\partial\partial r(\xi))^k\wedge(\bar\partial\partial|\xi-z|^2)^{n-k-2}.$$

We use the notations

$$\alpha_i(\xi)=\frac{\partial^2 r}{\partial\bar\xi_i,\partial\xi_i}(\xi), \qquad P_i(\xi)=\frac{\partial r}{\partial\xi_i}(\xi).$$

Using $\partial r, d\xi_2,\ldots,d\xi_n$, $\bar\partial r, d\bar\xi_2,\ldots,d\bar\xi$ as basis of 1-forms it is easy to see that $\partial r(\xi)\wedge(\bar\partial\partial r(\xi))^k$ equals the exterior product of $\partial r(\xi)$ with

$$\left(\sum_{i=2}^n\alpha_i d\bar\xi_i\wedge d\xi_i\right)^k$$
$$+k\alpha_1|P_1|^{-2}\left(\sum_{i=2}^n\alpha_i d\bar\xi_i\wedge d\xi_i\right)^{k-1}\wedge\sum_{i=2}^n\bar P_i d\bar\xi_i\wedge\sum_{i=2}^n P_i d\xi_i.$$

This last expression is in turn a sum of forms of type (k,k) in $d\xi_2,\ldots,d\xi_n$ whose coefficients are products of k different α's between the $\alpha_2,\ldots,\alpha_n$ or products of $k-1$ different α's, $\alpha_{i_1},\ldots,\alpha_{i_{k-1}}$ between the $\alpha_2,\ldots,\alpha_n$, with $k\alpha_1|P_1|^{-2}\bar P_j P_k$, j,k different from $1, i_1,\ldots,i_{k-1}$.

Now observe that the hypothesis on r imply that

$$|P_j|\leqq c\alpha_j$$

To summarize, $v_z H_k(\xi,z)\wedge f(\xi)$ is in $bD\cap B(z,\delta_0)\ d\sigma(\xi)$ times a function which is a finite linear combination of functions of type

$$h^1_{i,i_1\ldots i_k}(\xi)\frac{f_i(\xi)\alpha_{i_1}(\xi)\ldots\alpha_{i_k}(\xi)}{|\xi-z|^{2n-2k-2}F(\xi,z)^{k+1}} \tag{12}$$

or

$$h^2_{i,i_1\ldots i_k}(\xi)\frac{f_i(\xi)\alpha_{i_1}(\xi)\ldots\alpha_{i_k}(\xi)}{|\xi-z|^{2n-2k-3}F(\xi,z)^{k+2}} \tag{12}$$

where $f=\sum f_i d\bar{\xi}_i$, the $i_1, \dots, i_k$ are different indexes between $2, \dots, n$ and the h's are bounded functions (with bounds independent of z).

From (7) follows that

$$\frac{\alpha_j(\xi)}{|F(\xi,z)|} \leqq c \frac{1}{|r(z)|+|\xi_j - z_j|^2}, \qquad \xi \in bD, \qquad z \in D \tag{13}$$

Writing t' for $(t_3, \dots, t_{2n})$, observing that $t_1 = |r(z)|$ is the equation of bD and using (7) and (13) one is lead to the estimates (assuming $i_1 = 2, \dots, i_k = k+1$)

$$\begin{aligned} I_1(k,t_1) = & \int\limits_{\substack{|t_2| \leqq \delta_0 \\ |t'| \leqq \delta_0}} \\ & \cdot \frac{dt_2 dt'}{(t_1 + |t_2| + |t'|)^{2n-2k-2}(t_1 + |t_2| + |t'|^m) \prod\limits_{j=2}^{k+1} (t_1 + t_{2-1}^2 + t_2^2)} \leqq c(\alpha) t_1^{\alpha-1} \\ I_2(k,t_1) = & \int\limits_{\substack{|t_2| \leqq \delta_0 \\ |t'| \leqq \delta_0}} \\ & \cdot \frac{dt_2 dt'}{(t_1 + |t_2| + |t'|)^{2n-2k-3}(t_1 + |t_2| + |t'|^m)^2 \prod\limits_{j=2}^{k+1} (t_1 + t_{2j-1}^1 + t_{2j}^2)} \leqq c(\alpha) t_1^{\alpha-1} \end{aligned} \tag{14}$$

which can be consulted in [8]. □

3.4. Theorem. *For each p, $1 \leqq p \leqq \infty$, there exists $c(p) > 0$ such that $\|Tf\|_{L^p} \leqq c(p) \|f\|_{L^p}$.*

Proof. Again it is enough to consider T_1, i.e., to prove the estimate for each H_k, $k = 0, \dots, n-2$. It is then convenient to have an expression of $H_k f$ as a volume integral. To do this we employ

$$A(\xi, z) = -r(\xi) + F(\xi, z)$$

as a continuation of $F(\xi, z)$ inside D. Note that $\operatorname{Im} A = \operatorname{Im} F$ and that (7) gives

$$2 \operatorname{Re} A(\xi, z) \geqq -r(\xi) - r(z) + c \operatorname{Lr}(\xi)(\xi - z)^2 + c|\xi - z|^m \tag{15}$$

and so $A(\xi, z)$ never vanishes for $\xi, z \in D$. By the explicit expression (8) we see that

$$|H_k(\xi, z)| = 0(|\xi - z|^{2k+3-2n}),$$
$$|\bar{\partial}_\xi H_k(\xi, z)| = 0(|\xi - z|^{2k+2-2n}).$$

So, applying Stokes theorem in $D \backslash B(z, \varepsilon)$ and making $\varepsilon \to 0$ we obtain

$$H_k f(z) = \int\limits_D f(\xi) \wedge \bar{\partial}_\xi H_k(\xi, z)$$

with H_k now given by (8) with F replaced by A. Clearly it is enough to show (denoting by dm the Lebesgue measure in D)

$$\int\limits_D |\bar{\partial}_\xi H_k(\xi, z)| \mathrm{dm}(\xi) \leqq c, \qquad \int\limits_D |\bar{\partial}_\xi H_k(\xi, z)| \mathrm{dm}(z) \leqq c.$$

To prove the first, by (15) we may assume, with the notations used in the proof of Theorem 3.3, that $|r(z)|<r_0$ and just estimate the integral over $D\cap B(z,\delta_0)$ where $\mathrm{dm}(\xi)\simeq dt_1\ldots dt_{2n}$. In a similar way as in Theorem 3.3, it turns out that the coefficients of $\bar\partial_\xi H_k(\xi,z)$ are in $D\cap B(z,\delta_0)$ of the type (12) with F replaced by A (and without the f_i's). Using then (15), one is lead as before to the estimates

$$\int_0^{\delta_0} I(k,t_1)dt_1 \leqq c\,, \qquad \int_0^{\delta_0} I(k,t_1)dt_1 \leqq c\,, \qquad k=0,\ldots,n-2$$

which follow from the ones in (14). The second integral is evaluated exactly in the same way. □

Remark. As it can be easily seen from the proofs, Theorems 3.3 and 3.4 hold for all convex domains with real analytic boundary if $n=2$. This is so because in this case the estimate $r(z)-r(\xi)+2\operatorname{Re}F(\xi,z)\geqq c|\xi-z|^m$ is already sufficient (this is noticed in [8] for Theorem 3.3).

3.5

Finally we point out that in a similar way, with (9) and the basic inequalities (7) and (13) one can prove the following.

Theorem. *The reproducing kernel $H(\xi,z)$ satisfies*

$$\int_{bD} |H(\xi,z)|\,|\xi-z|d\sigma(\xi)\leqq c\,, \qquad z\in D\,.$$

Using this, one can extend to D a series of well known results for the strictly pseudoconvex domains as for instance, Cole-Range's Theorem on the structure of Henkin measures and its corollaries concerning the equivalence of the notions of zero set, peak set and peak-interpolation set for the algebra of holomorphic functions in D, continuous on $\bar D$ (see [10, p. 198]).

4. The Results in the General Case

4.1

In the absence of condition (iv) of 3.1. we cannot prove that the canonical support function satisfies (7) (because we are not able to prove (3) for all w, v near a point of degeneracy of Hr_i different from zero). Thus our first taks in the general case is to replace F by another support function ϕ for which (7) holds. This can easily be done using the fact that the r_i are strictly subharmonic away from zero.

We define

$$G_i(w,v)=F_i(w,v)+\frac{1}{2}\frac{\partial^2 r_i}{\partial w^2}(w)\,(w-v)^2\,.$$

Let $\varepsilon>0$ be as in (3). By strict subharmonicity in $\varepsilon/2<|w|<a_i^{1/2}$, one has

$$r_i(v)-r_i(w)+2\operatorname{Re}G_i(w,v)\geqq c\,\mathrm{Lr}_i(w)|w-v|^2\simeq|w-v|^2 \tag{16}$$

if $|w|\geqq\varepsilon/2$ and $|w-v|$ is small, say $|w-v|<\delta$.

Let now $\varphi(w)$ be a C^∞-function in the complex plane, $0 \leqq \varphi \leqq 1$ equal to 1 in $|w| < \varepsilon/2$ and zero outside $|w| < 2\varepsilon/3$. We define now

$$F_i'(w, v) = \varphi(w) F_i(w, v) + (1 - \varphi(w)) G_i(w, v) = P_i'(w, v)(w - v)$$

with

$$P_i'(w, v) = P_i(w) + (1 - \varphi(w)) \frac{1}{2} \frac{\partial^2 r_i}{\partial w^2}(w)(w - v).$$

From (6) (for r_i and F_i) and (16) it follows, reducing δ if necessary, that

$$r_i(v) - r_i(w) + 2\,\mathrm{Re}\, F_i'(w, v) \geqq c\,\mathrm{L}r_i(w)|w - v|^2 + c|w - v|^m$$

for $|w - v| < \delta$. Now we define for $\xi, z \in \bar{D}$

$$F'(\xi, z) = \sum_{i=1}^{n} F_i'(\xi_i, z_i).$$

Then $F' \in C^\infty(\bar{D} \times \bar{D})$, is a polynomial in z and

$$r(\xi) - r(z) + 2\,\mathrm{Re}\, F'(\xi, z) \geqq c\,\mathrm{L}r(\xi)(\xi - z)^2 + c|\xi - z|^m$$

holds for $|\xi - z| < \delta$.

4.2

To avoid the Henkin-Ramirez procedure and the division problem involved, we will use the simpler method of [4] and [9], which we briefly recall. Choose $\chi \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ such that $0 \leqq \chi \leqq 1$, $\chi \leqq 1$ for $|\xi - z| \leqq \delta/2$, and $\chi = 0$ for $|\xi - z| \geqq \delta$. For $i = 1, \ldots, n$ define

$$P_i(\xi, z) = \chi P_i'(\xi_i, z_i) + (1 - \chi)(\bar{\xi}_i - \bar{z}_i)$$

$$\phi(\xi, z) = \sum_{i=1}^{n} P_i(\xi, z)(\xi_i - z_i) = \chi F'(\xi, z) + (1 - \chi)|\xi - z|^2.$$

Then there exists $\eta > 0$ such that $|\phi(\xi, z)| \geqq c > 0$ for $\xi \in bD$, $r(z) < \eta$ if $|\xi - z| \geqq \delta/2$, $\phi(\xi, z)$ and $P_i(\xi, z)$ are holomorphic in z in $|\xi - z| < \delta/2$ and the fundamental inequality

$$r(\xi) - r(z) + 2\,\mathrm{Re}\, \phi(\xi, z) \geqq c\,\mathrm{L}r(\xi)(\xi - z)^2 + c|\xi - z|^m \tag{17}$$

remains for $|\xi - z| < \delta/2$. For $t \in [0, 1]$ and $\xi \in bD$, one sets

$$w_i(\xi, z, t) = t \frac{P_i(\xi, z)}{\phi(\xi, z)} + (1 - t) \frac{\bar{\xi}_i - \bar{z}_i}{|\xi - z|^2}$$

which is well defined for $z \in D \cup \{z : r(z) \leqq \eta,\ |z - \xi| \geqq \delta/2\}$, and $W = \sum_{i=1}^{n} w_j d\xi_i$.

Finally, the Cauchy-Fantappié form $\Omega_q(w)$ is defined, for $q = 0, 1$, by

$$\Omega_q(w) = (2\pi i)^{-n} \binom{n-1}{q} W \wedge (\bar{\partial}_{\xi,\lambda} W)^{n-q-1} \wedge (\bar{\partial}_z W)^q$$

Set now, for a (0, 1) form f

$$T_0 f = \int_{bD\times[0,1]} f \wedge \Omega_0 - \int_{D\times\{0\}} f \wedge \Omega_0$$

$$Ef = \int_{bD} f \wedge \mu * \Omega_1, \qquad \mu(\xi) = (\xi, 1).$$

In [9] it is proved the following

Lemma. *If $f \in C^1_{(0,1)}(\bar{D})$, Ef is defined and of class C^∞ on $\bar{D}_\eta = \{z : r(z) \leqq \eta\}$. If $\bar\partial f = 0$, then $\bar\partial Ef = 0$ in D and $f = Ef + \bar\partial T_0 f$. Further*

$$\|Ef\|_{L^\infty(D^\eta)} \leqq c\|f\|_{L^\infty(D)}.$$

Thus the problem of solving $\bar\partial u = f$ in D is reduced to that of solving $\bar\partial u = Ef$. But now $Ef \in C^\infty_{(0,1)}(\bar{D})$ and if T_η denotes the canonical operator T of 3.2. for the (convex) domain D_η, it follows that the operator

$$S = T_\eta E + T_0 \quad \text{satisfies} \quad \bar\partial Sf = f$$

4.3. Theorem. *The operator S satisfies the estimates*

$$|Sf(z) - Sf(w)| \leqq c(\alpha)\|f\|_\infty |z-w|^\alpha, \qquad \alpha < 1/m$$

$$\|Sf\|_{L^p(D)} \leqq c(p)\|f\|_{L^p(D)}.$$

Proof. The estimates that an integral operator like E or T_0 satisfy only depend on the singularity of its kernel near the diagonal. The kernel $\mu * \Omega_1$ of E is zero for $|\xi - z| < \delta/2$ and so there is no problem with E. Clearly for T_η we have

$$\|T_\eta Ef\|_{L^p(D)} \leqq c(p)\|Ef\|_{L^p(D_\eta)}$$

$$\|T_\eta Ef\|_{L_{ip}\,D} \leqq c(\alpha)\|Ef\|_{L^\infty(D_\eta)}.$$

Hence everything is reduced to T_0. But, because of (17), we can argue with T_0 in the same manner we did for T in Theorems 3.3 and 3.4. The only change is that in (8) one must replace $\partial r(\xi) = \sum_{i=1}^n P_i(\xi)d\xi_i$ by $\sum_{i=1}^n P_i(\xi, z)d\xi_i$. Near the diagonal $P_i = P_i'$ and it is easily seen that also $|P_i'| \leqq c\alpha_i$. The rest of the proof is identical. □

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Note added in proof: Recently Prof. A. Nagel has sent us a nice proof of inequality (**) for all convex functions r for which there exists an integer m such that for all directions v and all points x, $\sum_{j=2}^{m} |D_v^j r(x)| > 0$. This can be used to extend our results to convex domains of finite type satisfying certain geometric conditions, the most relevant being the following one:

$$Hr(w)(v,v) \simeq Hr(w)(Jv,Jv),$$

for $w \in bD$ and $v \in T_w^c(bD)$, the complex tangent space at w. Here J denotes the operator of multiplication by i and it can be easily seen that this condition does depend on the defining function r.

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The Gauss Map for Surfaces in 4-Space

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Given a connected oriented manifold M^n of dimension n and an immersion $X : M^n \to \mathbb{R}^{n+k}$, there is associated to it a Gauss map $g : M^n \to G(n, k)$, where $G(n, k)$ is the Grassmann manifold of oriented n-planes in $(n+k)$-space. That is, there is associated to X a map g which assigns to each $m \in M$ the oriented n-plane tangent to $X(M)$ at $X(m)$. This is one way in which one may generalize the classical Gauss map for surfaces in 3-space. Suppose on the other hand, we are given a map $g : M^n \to G(n, k)$. Does there exist an immersion $X : M^n \to \mathbb{R}^{n+k}$ with g as its Gauss map? Is even g locally a Gauss map and also to what extent does g uniquely determine X? In this paper, we study these questions for the case $n=k=2$, however only under the additional restriction that g is supposed to be an immersion of M^2 into $G(2, 2)$.

The main results obtained are the following ones: In Theorem 1 we state a purely algebraic necessary condition on g for the existence of X with this g as its Gauss map. If this algebraic condition on g is fulfilled, the existence of X for g reduces – under certain "regularity" assumptions (a rank condition) – to solving a system of partial differential equations. We shall study this system if it is either of hyperbolic or of elliptic type. For the hyperbolic case we prove in Theorem 2 resp. 3 the existence resp. uniqueness of an immersion X with the given g as its Gauss map, if X is prescribed along a nowhere-characteristic curve in M (Cauchy problem). For the elliptic case we prove in Theorem 4 the existence of X with the given g as its Gauss map, if M is a closed disc, and in Theorem 5 we get for arbitrary M the uniqueness of such an X, if the values of X are prescribed along any arc in M. Some of these results are in a recent paper [1] by Aminov but his point of view and methods are very different from ours. (As a by-product of our investigations we prove in the Appendix a slight sharpening of a result of Chern and Spanier.)

If ξ is a vector bundle over M and $m \in M$ we let ξ_m denote the fibre over m. In the case of the tangent bundle of M, TM, we write M_m for $(TM)_m$. If S is a set and $f : \xi \to S$ is a map, we denote the restriction of f to ξ_m by f_m. Also $\Lambda^k(\xi)$ denotes the

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bundle of k-forms on M with values in the bundle ξ. That is, $\phi \in \Lambda^k(\xi)$ is a k-form on M_m with values in ξ_m.

We use $(,)$ and $\| \ \|$ to denote an inner product and a norm in a variety of settings. If $f: M \to R$ is an immersion of a surface M into a Riemannian manifold R, we use dA_f to denote the element of area induced on M by f. Throughout this paper M denotes a connected oriented C^∞ surface. Also all maps are C^∞ unless noted otherwise.

§1. The Bundle of Admissable Maps

Let $G = G(2,2)$ denote the Grassmann manifold of oriented 2-planes in $\mathbb{R}^4$ through the origin; let $d\bar{\sigma}^2$ denote its standard riemannian metric. This metric can be represented as follows (see e.g. [6]): Let (e_1, e_2, e_3, e_4) be a field of positively oriented orthonormal frames of $\mathbb{R}^4$ defined on some open subset U of G, which is "π-adapted" for each $\pi \in U$, i.e. $(e_1(\pi), e_2(\pi))$ is a positively oriented basis of the oriented 2-plane π, for every $\pi \in U$. Then, if we define – as usual – the Pfaffian forms $\omega_i^\alpha := (de_i, e_\alpha)$ for $i \in \{1, 2\}$ and $\alpha \in \{3, 4\}$ we have on U:

$$d\bar{\sigma}^2 = \sum_{i,\alpha} (\omega_i^\alpha)^2 = (\omega_1^3)^2 + (\omega_1^4)^2 + (\omega_2^3)^2 + (\omega_2^4)^2 . \tag{1}$$

Suppose $g: M \to G$ is an immersion. Then we can pull back $d\bar{\sigma}^2$ by g to get a riemannian metric $d\sigma^2 = g^* d\bar{\sigma}^2$ for M.

Assume $X: M \to \mathbb{R}^4$ to be an immersion with the given g as its Gauss map, i.e. for all $m \in M$ the differential of X at m, $d_m X$, maps the oriented 2-plane M_m onto the oriented 2-plane $g(m)$, preserving the orientation. If we consider dX as a differential 1-form with values in the trivial bundle $\mathbb{R}_M^4$ over M with fibre $\mathbb{R}^4$, i.e. dX is a section of $\Lambda^1(\mathbb{R}_M^4)$, we have as usual $d(dX) = 0$. Therefore a necessary and (by Poincaré's Lemma even) sufficient condition for any section Φ in $\Lambda^1(\mathbb{R}_M^4)$ defined over any simply connected open subset U of M to be the differential dX of an immersion $X: U \to \mathbb{R}^4$ is given by:

$$\Phi_m(M_m) = g(m), \det(\Phi_m) > 0, \quad \text{and} \quad d_m \Phi = 0 \quad \text{for all} \quad m \in U, \tag{C}$$

where $\det(\Phi_m)$ is computed with respect to the metric $d\sigma^2$ on M_m and the metric induced from $\mathbb{R}^4$ on $g(m)$, i.e. with $\det(\Phi_m) := \det((e_i, \Phi_m(a_k)))_{i,k=1,2}$ where (a_1, a_2) resp. (e_1, e_2) is any positively oriented orthonormal 2-frame of M_m (w.r.t. $d\sigma^2$) resp. of $g(m)$.

We shall study the conditions of [C] now more closely; without causing confusion we shall consider the map $g: M \to G$ in the following as an oriented riemannian 2-plane bundle over M, a vector subbundle of $\mathbb{R}_M^4$ with $g(m) \subset \mathbb{R}^4$ as its fibre over $m \in M$, the metric being induced from $\mathbb{R}^4$. Correspondingly we obtain the oriented normal bundle $g^\perp$ of g in $\mathbb{R}_M^4$ characterized as follows: If $m \in M$ and (e_1, e_2, e_3, e_4) is a $g(m)$-adapted positively oriented orthonormal frame of $\mathbb{R}^4$, then (e_3, e_4) is a positively oriented basis of $g^\perp(m)$. Thus we get the orthogonal splitting $\mathbb{R}_M^4 = g \oplus g^\perp$ with the corresponding orthogonal projections $(\ldots)^T: \mathbb{R}_M^4 \to g$ and $(\ldots)^\perp: \mathbb{R}_M^4 \to g^\perp$.

In view of the first condition of [C] we introduce now the following rank 4 vector subbundle β of the bundle $\Lambda^1(\mathbb{R}_M^4)$, the fibre of which is given by

$\beta_m := \{\Phi : M_m \to \mathbb{R}^4$ linear map$|\Phi(M_m) \subset g(m)\}$ for all $m \in M$. Any section Φ in β can be viewed therefore as a differential 1-form with values in $\mathbb{R}^4$, hence the vanishing on the differential 2-form $d\Phi$ with values in $\mathbb{R}^4$ is (because of $\mathbb{R}^4_M = g \oplus g^\perp$) equivalent to:

$$(d\Phi)^\perp = 0\ [C'] \quad \text{and} \quad (d\Phi)^T = 0\ [C''].$$

But $\Phi \mapsto (d\Phi)^\perp$ is a tensorial operation, i.e. for all $m \in M$ the value of $(d\Phi)^\perp$ at $m \in M$ does only depend on the value of Φ at m [because $\Phi \mapsto (d\Phi)^\perp$ is additive in Φ and for every C^∞ function φ on M we have $(d(\varphi\Phi))^\perp = (d\varphi \wedge \Phi)^\perp + (\varphi d\Phi)^\perp = \varphi(d\Phi)^\perp$, since $\Phi^\perp = 0$ by definition of β.] Therefore we have obtained the

Lemma 1. *There exists a vector bundle homomorphism*

$$C : \beta \to \Lambda^2(g^\perp)$$

characterized by

$$C(\Phi) := (d\Phi)^\perp \quad \textit{for every section } \Phi \textit{ in } \beta,$$

and the condition $[C']$ *reduces to the purely algebraic condition:*

$$C(\Phi) = 0. \qquad [C']$$

Now, $C_m : \beta_m \to \Lambda^2(g^\perp(m))$ is a linear map of a 4-dimensional vector space into a 2-dimensional vector space; therefore if we let $\alpha_m = \ker(C_m)$ then

$$\dim(\alpha_m) \geqq 2 \quad \text{for all} \quad m \in M.$$

Definition 1. We say a point $m \in M$ is a *regular point* of g if $\dim(\alpha_m) = 2$; otherwise m is said to be a *singular point* of g (and as we will see in Lemma 3 we have $\dim(\alpha_m) = 3$). Also M is called *regular* (for g) if all points of M are regular points of g. Note that the set of regular points of g is always open in M. It follows easily (see the Remark after Lemma 3) that m is a singular point of g if and only if there exists a frame field (e_1, e_2, e_3, e_4) of $\mathbb{R}^4$ defined on a neighborhood U of m which is "g-adapted", i.e. $(e_1(m'), \ldots, e_4(m'))$ is $g(m')$-adapted for all $m' \in U$, with respect to which $\omega_1^4 = \omega_2^4 = 0$ at m. Here $\omega_i^\alpha = (de_i, e_\alpha)$ for $i \in \{1, 2\}$ and $\alpha \in \{3, 4\}$.

Now, if m is singular for g and if the frame field is chosen in the way just described, near m the 2-vector $e_1 \wedge e_2$, representing the 2-planes given by g, is given up to first order by

$$e_1 \wedge e_2 = (e_1(m) + \omega_1^3 e_3(m)) \wedge (e_2(m) + \omega_2^3 e_3(m)).$$

So up to first order all values of g near m are contained in the hyperplane of $\mathbb{R}^4$ orthogonal to $e_4(m)$. Thus "infinitesimally near m" our problem essentially would by one of trying to find $X : M \to \mathbb{R}^3$ (and not $X : M \to \mathbb{R}^4$). By a straightforward application of the Cartan Calculus it can be shown, that if all points of M are singular for g, then there exists a 3-dim. vector space W of $\mathbb{R}^4$ such that $g(m) \subset W$ for all $m \in M$.

Definition 2. Let $\beta = (B, p, M)$, where B is the total space of β and $p : B \to M$ the projection of B onto M. If $A = \bigcup_{m \in M} \ker(C_m)$, then

$\alpha = (A, (p|A), M)$ is called the *space of admissable maps*.

If M is regular for g, then $\alpha = \ker(C)$ is a vector subbundle of β, the bundle of admissable maps. (This terminology is motivated by Lemma 1.)

The second condition in [C], that $\det(\Phi_m) > 0$ for all $m \in M$, motivates the introduction of a function $Q: \alpha \to \mathbb{R}$ as follows: For every $m \in M$ the determinant function $\det_m: \beta_m \to \mathbb{R}$ (as defined in [C]) is a non-degenerate indefinite quadratic form of index 2 on the 4-dimensional vector space β_m (of all linear maps of the 2-dim. vector space M_m into the 2-dim. vector space $g(m)$). Therefore the restriction $Q_m := \det_m | \alpha_m$ of $\det_m$ to the vector subspace α_m of β_m is a quadratic form on α_m, which – according to the position of α_m in β_m – can be either definite, indefinite, or degenerate. It is precisely this behavior that determines whether or not there exists any $\Phi_m \in \alpha_m$ for which $\det(\Phi_m) > 0$.

Finally we shall discuss the remaining condition [C''] for a section Φ in α to be differential of an immersion $X: M \to \mathbb{R}^4$ with g as its Gauss map. In order to do so, we suppose now that M is regular for g, and since the following conditions are local, we may assume, that M is homeomorphic to $\mathbb{R}^2$ (so that all vector bundles over M are trivial). Then there exist on M sections σ_1, σ_2 of α, that are linearly independent at each point of M, and there exists on M a g-adapted field (e_1, e_2, e_3, e_4) of frames of $\mathbb{R}^4$. Then any section Φ of α may be written as

$$\Phi = \sum_{i=1} \varphi^i \sigma_i \text{ with real-valued functions } \varphi^1, \varphi^2 \text{ on } M.$$

Then

$$d\Phi = \sum_{i=1}^{2} (d\varphi^i \wedge \sigma_i + \varphi^i d\sigma_i)$$

and therefore

$$(d\Phi)^T = \sum_{i,j=1}^{2} (d\varphi^i \wedge (\sigma_i, e_j) + \varphi^i (d\sigma_i, e_j)) e_j .$$

If we introduce therefore the functions resp. Pfaffian forms

$$\sigma_i^j = (\sigma_i, e_j) \quad \text{resp. } \kappa_i^j = -(d\sigma_i, e_j),$$

then the condition [C''], i.e. $(d\Phi)^T = 0$, amounts to

$$\sum_{i=1}^{2} \sigma_i^j d\varphi^i + \varphi^i \kappa_i^j = 0 \quad \text{for} \quad j \in \{1, 2\}. \tag{2}$$

(2) is a homogeneous system of two first order linear partial differential equations for the real-valued coefficient functions φ^1, φ^2 of Φ, defined on the surface M.

§2. The Weingarten Map of g

For $\pi \in G$ it is well-known that $G_\pi \cong \mathrm{GL}(\pi, \pi^\perp)$, where $\pi^\perp \in G$ is the orthogonal complement of π with orientation determined by requiring that if (e_1, e_2, e_3, e_4) is a positively oriented orthonormal frame of $\mathbb{R}^4$ with (e_1, e_2) a positively oriented frame of π, then (e_3, e_4) is a positively oriented frame of $\pi^\perp$. In fact the isomorphism $\iota: G_\pi \to \mathrm{GL}(\pi, \pi^\perp)$ is defined as follows: Let $l \in G_\pi$ and $v \in \pi$, then

$$\iota(l)(v) := (l\bar{v})^\perp,$$

where $\bar{v}$ is an $\mathbb{R}^4$-valued function defined near π that extends v (i.e. $\bar{v}(\pi)=v$) with $\bar{v}(\lambda)\in\lambda$ for $\lambda\in G$ near π, $l\bar{v}$ denotes the directional derivative of $\bar{v}$ with respect to l, and $(\ldots)^\perp$ denotes orthogonal projection onto $\pi^\perp$. In the future we will suppress the isomorphism ι and simply write

$$l(v)=(l\bar{v})^\perp. \tag{3}$$

If $g: M\to G$ is an immersion and $a\in M_m$, then $g_*(a)\in G_{g(m)}=\mathrm{GL}(g(m), g^\perp(m))$. For $\Phi\in\beta_m$ and $b\in M_m$, $\Phi(b)\in g(m)$. Set $v=\Phi(b)$ and $l=g_*(a)$ in (3) to get

$$g_*(a)(\Phi(b))=g_*(a)(v)=(g_*(a)\bar{v})^\perp=(a(\bar{v}\circ g))^\perp=(a\Phi(b))^\perp; \tag{4}$$

the last equality reflects the fact that $(a(\bar{v}\circ g))^\perp$ depends only on the value of $\bar{v}\circ g$ at m which is $v=\Phi(b)$. If Φ is the differential of an immersion X, then the second fundamental tensor field h of X is defined by

$$h(a,b):=(a\Phi(b))^\perp \quad \text{for all} \quad (a,b)\in TM\oplus TM.$$

By (4) it follows that

$$h(a,b)=g_*(a)(\Phi(b)) \quad \text{for all} \quad (a,b)\in TM\oplus TM.$$

Therefore if $\Phi\in\beta_m$ the tensor $h_\Phi: M_m\times M_m\to g^\perp(m)$ defined by

$$h_\Phi(a,b):=g_*(a)(\Phi(b)) \quad \text{for all} \quad a,b\in M_m, \tag{5}$$

will be called the *second fundamental tensor* of Φ.

Now if Φ is a section in β near m and a, b are sections in TM near m then (4) implies

$$\begin{aligned} C(\Phi)(a,b)&=(d\Phi)^\perp(a,b)=(a\Phi(b)-b\Phi(a)-\Phi([a,b]))^\perp\\ &=g_*(a)(\Phi(b))-g_*(b)(\Phi(a))=h_\Phi(a,b)-h_\Phi(b,a).\end{aligned}$$

In particular $C(\Phi)$, for $\Phi\in\beta$, is the anti-symmetric part of h_Φ and therefore the kernel α of C is precisely the set of all Φ for which the second fundamental tensor h_Φ is symmetric (as it should be).

We may also view $g_{*|m}: M_m\to G_{g(m)}=\mathrm{GL}(g(m), g^\perp(m))$ as the linear map

$$g_{*|m}: M_m\otimes g(m)\to g^\perp(m)$$

where $g_{*|m}(a\otimes v)=g_{*|m}(a)(v)$ for all $a\in M_m$ and all $v\in g(m)$. Since both $g(m)$ and $g^\perp(m)$ can be endowed with an inner product (because both spaces lie in $\mathbb{R}^4$) we may identify the dual spaces $g(m)^*$ resp. $g^\perp(m)^*$ with $g(m)$ resp. $g^\perp(m)$. We denote the transpose of $g_{*|m}: M_m\otimes g(m)\to g^\perp(m)$ by A_m and note

$$A_m: g^\perp(m)\to M_m^*\otimes g(m)=\beta_m.$$

Thus we have a vector bundle homomorphism $A: g^\perp\to\beta$. We denote the value of A at $z\in g^\perp$ by A^z. If $\Phi\in\beta_m$, then (5) implies

$$(h_\Phi(a,b),z)=(A^z(a),\Phi(b)).$$

Thus if Φ is the differential of an immersion $X: M\to\mathbb{R}^4$, A is the Weingarten map of X (see pp. 10–15 of [4]). Apparently the Weingarten map depends only on the Gauss map of X; hence

Definition 3. For any map $g: M\to G$ we call $A: g^\perp\to\beta$ the *Weingarten map* of g.

The condition that h_Φ is symmetric in order that $\Phi \in \alpha$ may be represented using A in a way that will be particularly useful. We define a non-degenerate skew-symmetric tensor field $P: \beta \oplus \beta \to \mathbb{R}$ as follows: For Φ, $\Psi \in \beta_m$, let

$$P(\Phi, \Psi) := (\Phi(a_1), \Psi(a_2)) - (\Phi(a_2), \Psi(a_1)),$$

where (a_1, a_2) is a positively oriented orthonormal frame of M_m (with respect to $d\sigma^2$). For $\Phi \in \beta_m$, h_Φ is symmetric if and only if

$$\begin{aligned}(h_\Phi(a_1, a_2) - h_\Phi(a_2, a_1), z) &= (A^z(a_1), \Phi(a_2)) - (A^z(a_2), \Phi(a_1)) \\ &= P(A^z, \Phi) = 0 \quad \text{for all} \quad z \in g^\perp(m).\end{aligned}$$

This yields

Lemma 2. $\alpha_m = \{\Phi \in \beta_m | P(A^z, \Phi) = 0$ *for all* $z \in g^\perp(m)\}$.

Since g is an immersion, $g_{*|m}: \beta_m \to g^\perp(m)$ has non-trivial range. So, the range of $g_{*|m}$, $\mathrm{ran}(g_{*|m})$, has dimension 1 or 2. Since the range of A_m, $\mathrm{ran}(A_m)$, has the same dimension as the range of $g_{*|m}$, its adjoint, $\mathrm{ran}(A_m)$ also has dimension 1 or 2; since P is non-degenerate Lemma 2 implies $\dim(\alpha_m) = 3$ or 2, resp. This proves

Lemma 3. *The* $\dim(\alpha_m) = 2$ *resp. 3 if and only if* $\dim(\mathrm{ran}(g_{*|m})) = 2$ *resp.* 1.

Remark. Let (e_1, e_2, e_3, e_4) be a g-adapted frame field defined near $m \in M$. If we define Pfaffian forms ω_i^α on M near m by $\omega_i^\alpha := (de_i, e_\alpha)$ for $i \in \{1, 2\}$ and $\alpha \in \{3, 4\}$, then one may show $\omega_i^\alpha(a) = (g_*(a \otimes e_i), e_\alpha)$ for all vector fields a on M near m. Thus, if m is a singular point, i.e. $\dim(\alpha_m) \neq 2$, then $\mathrm{ran}(g_{*|m})$ has dimension 1 and we may choose $e_4(m) \perp \mathrm{ran}(g_{*|m})$; consequently $\omega_1^4 = \omega_2^4 = 0$ at m.

§3. The Behavior of Q_m

We turn our attention to the behavior of Q_m, i.e. determining whether Q_m is degenerate, indefinite, positive definite, etc. Two invariants of Q_m that determine this behavior, when m is a regular point of g, are the sign of $\det(Q_m)$ – the determinant taken with respect to some inner product on α_m – and the signature of Q_m, $\sigma(Q_m)$, which equals the number of positive eigenvalues of Q_m minus the number of negative eigenvalues of Q_m. This is so since $\dim(\alpha_m) = 2$ when m is a regular point.

Beside the quadratic form Q_m we will introduce other quadratic forms that arise by restricting $\det_m: \beta_m \to \mathbb{R}$ to other subspaces of β_m. First let $\alpha_m^\perp$ denote the $\det_m$-orthogonal complement of α_m in β_m. Then the restriction

$$Q_m^\perp := \det_m | \alpha_m^\perp \text{ resp. } Q_m' = \det_m | \mathrm{ran}(A_m)$$

is a quadratic form on $\alpha_m^\perp$ resp. $\mathrm{ran}(A_m)$. We want to compute the determinants of these quadratic forms in order to be able to study their behavior. In order to do this we introduce any arbitrary riemannian metric on β and use the symbolism Det when taking the determinant of any quadratic form defined on any subspace of β_m with respect to the restriction of that riemannian metric to the subspace.

First observe that Q_m and $Q_m^\perp$ are either simultaneously degenerate or simultaneously non-degenerate, and if they are non-degenerate then

$$\mathrm{sign}\,\mathrm{Det}(Q_m) = \mathrm{sign}\,\mathrm{Det}(Q_m^\perp).$$

This follows since both are degenerate or not according as $\alpha_m = \alpha_m^\perp$ or $\alpha_m \neq \alpha_m^\perp$. When $\alpha_m \cap \alpha_m^\perp = \{0\}$ it is clear that $\operatorname{sign}\operatorname{Det}(Q_m) = \operatorname{sign}\operatorname{Det}(Q_m^\perp)$ since $\operatorname{Det}(\det_m) > 0$ (because $\det_m$ is non-degenerate of index 2 on a 4-dimensional space). Finally note that if $\alpha_m \neq \alpha_m^\perp$ but $\alpha_m \cap \alpha_m^\perp \neq \{0\}$ then $\operatorname{Det}(Q_m) = \operatorname{Det}(Q_m^\perp) = 0$.

Both P_m and the polarization of $\det_m$, also denoted $\det_m$, are non-degenerate and so they determine in a standard fashion isomorphisms $\iota_P, \iota_{\det}: \beta \to \beta^*$, resp. Define an isomorphism $I: \beta \to \beta$ by setting $I := \iota_{\det}^{-1} \circ \iota_P$. The isomorphism I has two important properties:

$$1)\ I(\operatorname{ran}(A_m)) = \alpha_m^\perp; \qquad 2)\ I^* Q_m^\perp = Q_m' \tag{6}$$

Property 1) is a consequence of Lemma 2 and the observation that

$$\det_m(I(A^z), \Phi) = P(A^z, \Phi) \quad \text{for all} \quad \Phi \in \beta_m \quad \text{and all} \quad z \in g^\perp(m).$$

Property 2) is easily checked by introducing coordinates: If $\Phi \in \beta_m$ corresponds to $\begin{pmatrix} \varphi_1^1 & \varphi_2^1 \\ \varphi_1^2 & \varphi_2^2 \end{pmatrix}$ with respect to a choice of bases for M_m and $g(m)$, then $I(\Phi)$ corresponds to $\begin{pmatrix} -\varphi_1^2 & -\varphi_2^2 \\ \varphi_1^1 & \varphi_2^1 \end{pmatrix}$ with respect to the same bases. Hence, we arrive at

Lemma 4. *For all $m \in M$, (regular or not)*

$$\operatorname{sign}\operatorname{Det}(Q_m) = \operatorname{sign}\operatorname{Det}(Q_m').$$

Moreover Q_m is degenerate if and only if Q_m' is degenerate.

Now let's turn our attention to $\sigma(Q_m)$. Before we proceed we need the following lemma.

Lemma 5. *If B is a non-degenerate quadratic form on a vector space V and W is a subspace of V, then*

$$\sigma(B) = \sigma(B|W) + \sigma(B|W^\perp),$$

where $W^\perp$ is the B-orthogonal complement of W.

Proof. First we make the following observation: Let U be a $2k$-dimensional vector space with non-degenerate quadratic form B. If U contains an isotropy subspace I – i.e. $B|I = 0$ – of dimension k, then $\sigma(B) = 0$. For, say, $\sigma(B) > 0$; then there exists a positive subspace P – i.e. $B|P$ is positive definite – of dimension greater than k. Hence $P \cap I \neq \{0\}$; this is a contradiction.

Now suppose $B|W$ has a null space I – i.e. its polarization has I for its nullspace – of dimension k. Then I is also the nullspace of $B|W^\perp$. Now define subspaces S and T of W and $W^\perp$, resp., by $W = S \oplus I$ and $W^\perp = T \oplus I$. Clearly $\sigma(B|W) = \sigma(B|S)$ and $\sigma(B|W^\perp) = \sigma(B|T)$. Now define another subspace C of V by $(S \oplus T)^\perp = I \oplus C$. Clearly $V = (I \oplus C) \oplus S \oplus T$ is a decomposition of V by B-orthogonal subspaces. Hence

$$\sigma(B) = \sigma(B|I \oplus C) + \sigma(B|W) + \sigma(B|W^\perp).$$

But $V = C \oplus (W + W^\perp)$ and $\dim(W \cap W^\perp) = k$ implies that $\dim(C) = k$. Thus $I \oplus C$ is of dimension $2k$ and has an isotropy subspace I of dimension k. Thus, from our initial observation $\sigma(B|I \oplus C) = 0$, and the result follows.

By Lemma 5, $\sigma(Q_m) = -\sigma(Q_m^\perp)$ since $\sigma(\det_m) = 0$. But (6) implies $\sigma(Q_m^\perp) = \sigma(Q'_m)$. This yields

Lemma 6. *For all $m \in M$*

$$\sigma(Q_m) = -\sigma(Q'_m).$$

Lemmas 4 and 6 allows us to determine the behavior of Q_m by examining the behavior of $Q'_m = \det_m | \mathrm{ran}(A_m)$, which is more intimately connected to g since A_m is the adjoint of $g_{*|m}$. We have finally come to the point at which we will introduce the invariants of g by which we describe the behavior of Q_m.

Let $X: M \to \mathbb{R}^4$ be an immersion with Gauss map g. The Gauss curvature equation for the Gaussian curvature of M (in the metric induced by X) in terms of the Weingarten map is

$$K = \det(A^{e_3}) + \det(A^{e_4}),$$

where (e_3, e_4) is any orthonormal frame field of $g^\perp$. These determinants are computed by identifying TM with g by means of dX. By analogy we are led to

Definition 4. Let $g: M \to G$ be an immersion. We define the *precurvature* k of g by

$$k := \det(A^{e_3}) + \det(A^{e_4}),$$

where (e_3, e_4) is any orthonormal frame field of $g^\perp$. But, in this case, $\det(A^{e_\alpha})$, $\alpha \in \{3, 4\}$, is computed with respect to $d\sigma^2$ on M and the metric induced on g from $\mathbb{R}^4$. It will become clear in §5 that k is a first order invariant of g.

Now $g^\perp: M \to G$ is also an immersion; for if $p: G \to G$ is defined by $p(\pi) := \pi^\perp$, for all $\pi \in G$, it turns out that p is an isometry (e.g. see [2]) and clearly $g^\perp = p \circ g$. Let $\bar{A}$ be the adjoint of $g_*^\perp: \bar{\beta} \to g$, where $\bar{\beta} = TM^* \otimes g^\perp$. Then define the *pre-normal curvature* n of g to be the precurvature of $g^\perp$, i.e.

$$n := \det(\bar{A}^{e_1}) + \det(\bar{A}^{e_2}),$$

where (e_1, e_2) is any orthonormal frame field of g. If g is the Gauss map of an immersion X and $\det(\bar{A}^{e_i})$, $i \in \{1, 2\}$, is computed using the metric induced on M by X rather than g (as is done in the definition of n) then one gets $\det(\bar{A}^{e_1}) + \det(\bar{A}^{e_2}) = N$, the normal curvature of X.

Define scalars l_{ij}^α as follows:

$$l_{ij}^\alpha := (g_*(a_j \otimes e_i), e_\alpha),\ i, j \in \{1, 2\}, \alpha \in \{3, 4\}, \tag{7}$$

for orthonormal frames (a_1, a_2), (e_1, e_2), and (e_3, e_4) of M_m, $g(m)$, and $g^\perp(m)$, resp. Note that l_{ij}^α, for $i \in \{1, 2\}$ and $\alpha \in \{3, 4\}$, are the components of $g_*(a_j)$ with respect to an orthonormal basis of $G_{g(m)}$ for $j \in \{1, 2\}$; hence

$$\sum_{i,\alpha} l_{ij}^\alpha l_{ik}^\alpha = \delta_{jk}, \quad \text{for} \quad j, k \in \{1, 2\}. \tag{8}$$

Also $(l_{ij}^\alpha)_{1 \leq i, j \leq 2}$ represents A^{e_α} with respect to orthonormal bases of M_m and $g(m)$, for $\alpha \in \{3, 4\}$. Hence

$$k(m) = |l_{ij}^3| + |l_{ij}^4|,$$

where $|l_{ij}^{\alpha}|$ denotes the determinant of (l_{ij}^{α}). Then using the Cauchy-Schwarz inequality we find

$$\begin{aligned}|k(m)| &= |l_{11}^3 l_{22}^3 - l_{21}^3 l_{12}^3 + l_{11}^4 l_{22}^4 - l_{21}^4 l_{12}^4| \\ &\leqq \sqrt{\sum_{i,\alpha}(l_{i1}^{\alpha})^2}\sqrt{\sum_{i,\alpha}(l_{i2}^{\alpha})^2} = 1 .\end{aligned}$$

The last equality follows from (8). Since $n(m)$ is the precurvature of $g^{\perp}$ at m, we also obtain $|n(m)| \leqq 1$. Hence, we have

Lemma 7. $-1 \leqq k, n \leqq 1$.

The transformation A_m can be used to pull Q'_m back to a quadratic form on $g^{\perp}(m)$ which is called F_m, i.e. $F_m := A_m^* Q'_m$ for all $m \in M$. Thus we obtain a quadratic form F on $g^{\perp}$. The form F is an analogue on the quadratic form $\mathscr{F}$ defined on the normal bundle of an immersion X by Little [7, p. 267]; this analogy is pursued further in §5. When the determinant or trace of F_m is computed, it is to be understood that this is done with respect to the inner product on $g^{\perp}(m)$ induced from $\mathbb{R}^4$. Clearly $\operatorname{sign}\det(F_m) = \operatorname{sign}\operatorname{Det}(Q'_m)$ and $\sigma(F_m) = \sigma(Q'_m)$ if m is a regular point of g since $A_m : g^{\perp}(m) \to \operatorname{ran}(A_m)$ is an isomorphism when m is regular. Using Lemmas 4 and 6 we obtain

Lemma 8. *If m is a regular point of g, then*

$$\operatorname{sign}\det(F_m) = \operatorname{sign}\operatorname{Det}(Q_m) \quad \textit{and} \quad \sigma(F_m) = -\sigma(Q_m).$$

We only need to know $\sigma(F_m)$ when $\det(F_m) \geqq 0$ to determine the behavior of Q_m. This is so because Q_m is indefinite when $\det(F_m) < 0$. But when $\det(F_m) \geqq 0$, $\operatorname{sign}\operatorname{tr}(F_m) = \operatorname{sign}\sigma(F_m)$. So now we proceed to compute $\det(F_m)$ and $\operatorname{tr}(F_m)$ in terms of $k(m)$ and $n(m)$. We also characterize the singular points of M in terms of k and n.

Lemma 9. *For all $m \in M$*

$$\operatorname{tr}(F_m) = k(m) \quad \textit{and} \quad \det(F_m) = (1/4)(k^2 + n^2 - 1)(m).$$

Proof. Since $F_m = A_m^*(\det_m)$, it follows immediately from the definition of $k(m)$ that $\operatorname{tr}(F_m) = k(m)$. Define scalars l_{ij}^{α} as we did in (7) and note that

$$l_{ij}^{\alpha} = (A^{e_\alpha}(a_j), e_i), \quad \text{for} \quad i,j \in \{1,2\} \quad \text{and} \quad \alpha \in \{3,4\}.$$

Also, for $\bar{A}$, the adjoint of $g_*^{\perp}$,

$$\begin{aligned}(\bar{A}^{e_i}(a_j), e_\alpha) &= -(de_i(a_j), e_\alpha) = (de_\alpha(a_j), e_i) \\ &= -(A^{e_\alpha}(a_j), e_i) = -l_{ij}^{\alpha}\end{aligned} \tag{9}$$

for $i,j \in \{1,2\}$ and $\alpha \in \{3,4\}$. Let

$$l := \begin{pmatrix} l_{11}^3 & l_{21}^3 & l_{11}^4 & l_{21}^4 \\ l_{12}^3 & l_{22}^3 & l_{12}^4 & l_{22}^4 \end{pmatrix}.$$

Also define scalars p^{rs} for $r, s \in \{1,2,3,4\}$ to be the determinant of the 2×2 matrix formed from the r^{th} and s^{th} columns of l in that order. In order to compute the determinant of F_m we need to find $F_m(e_\alpha, e_\beta)$ for $\alpha, \beta \in \{3, 4\}$. Here $F_m(\cdot, \cdot)$ denotes the polarization of $F_m(\cdot)$. We obtain:

$$F_m(e_\alpha, e_\alpha) = \det(A^{e_\alpha}) = \begin{cases} p^{12} & \text{if } \alpha = 3 \\ p^{34} & \text{if } \alpha = 4 \end{cases},$$

$$F_m(e_3, e_4) = \det_m(A^{e_3}, A^{e_4}) = 1/2(p^{14} - p^{23}).$$

Also note that $k(m) = p^{12} + p^{34}$, and (9) implies that $n(m) = p^{13} + p^{24}$. Hence, at m,

$$\begin{aligned} k^2 + n^2 - 1 &= (p^{12} + p^{34})^2 + (p^{13} + p^{24})^2 - 1 \\ &= (p^{12})^2 + (p^{34})^2 + (p^{13})^2 + (p^{24})^2 - 1 + 2(p^{12}p^{34} + p^{13}p^{24}). \end{aligned}$$

However, (8) implies $\sum_{r<s} (p^{rs})^2 = 1$, and there is the following identity due to Plücker

$$p^{13}p^{24} = p^{12}p^{34} + p^{14}p^{23}.$$

Hence, at m,

$$\begin{aligned} k^2 + n^2 - 1 &= -(p^{14})^2 - (p^{23})^2 + 4p^{12}p^{34} + 2p^{14}p^{23} \\ &= 4p^{12}p^{34} - (p^{14} - p^{23})^2 = 4\det(F_m). \end{aligned}$$

Lemma 10. *The point $m \in M$ is a singular point of g if and only if $k(m) = \pm 1$ and $n(m) = 0$.*

Proof. Suppose m is a singular point of g. Then there is a unit vector, say e_4 of the orthonormal frame (e_3, e_4) of $g^\perp(m)$, such that $A^{e_4} = 0$. Hence $F(e_4, z) = 0$ for all $z \in g^\perp(m)$. Thus $\det(F_m) = 0$. Also the rows of $(l_{ij}^3)_{1 \leqq i,j \leqq 2}$ form an orthonormal set by (8) since $l_{ij}^4 = 0$ for $i, j \in \{1, 2\}$; hence $k(m) = \pm 1$. Since $k^2(m) + n^2(m) - 1 = 0$ by Lemma 9 it follows that $n(m) = 0$.

Now suppose $k(m) = \pm 1$ and $n(m) = 0$. Lemma 9 implies $\det(F_m) = 0$. Hence, there is a vector, say e_4 of the orthonormal basis (e_3, e_4) of $g^\perp(m)$, such that $F_m(e_4, z) = 0$ for all $z \in g^\perp(m)$. In particular $\det(A^{e_4}) = 0$. Since $\operatorname{tr}(F_m) = k(m) = \pm 1$, we obtain $|\det(A^{e_3})| = 1$. But, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |\det(A^{e_3})| &= |l_{11}^3 l_{22}^3 - l_{21}^3 l_{12}^3| \leqq \sqrt{\sum_i (l_{i1}^3)^2} \sqrt{\sum_i (l_{i2}^3)^2} \\ &\leqq \sqrt{\sum_{i,\alpha} (l_{i1}^\alpha)^2} \sqrt{\sum_{i,\alpha} (l_{i2}^\alpha)^2} = 1. \end{aligned}$$

Hence $l_{i,j}^4 = 0$ for $i, j \in \{1, 2\}$. Therefore $e_4 \in \ker(A_m)$ and thus m is a singular point of g.

Combining Lemmas 8, 9, and 10 we obtain the following proposition in which we let $\delta := 1/4(k^2 + n^2 - 1)$.

Proposition 1. *Let $g: M \to G$ be an immersion.*

a) *A point $m \in M$ is a singular point of g if and only if $k(m) = \pm 1$ and $n(m) = 0$*

b) *If m is a regular point of g then*

$$\operatorname{sign} \operatorname{Det}(Q_m) = \operatorname{sign} \delta(m)$$

and if $\operatorname{sign} \operatorname{Det}(Q_m) \geqq 0$, *then*

$$\operatorname{sign} \sigma(Q_m) = -\operatorname{sign} k(m).$$

§4. The Type of (2)

When $Q_m \neq 0$ we will show that the type of the system (2) at $m \in M$ is determined by the sign of $\delta(m)$, i.e. (2) is elliptic, parabolic, or hyperbolic according as $\delta(m)$ is positive, zero, or negative. For the definition of the type of a system like system (2) as well as description of the normal forms for each type, see for example [3]. When $Q_m = 0$ at m, then either m is a singular point for g, i.e. $k = \pm 1$ and $n = 0$ at m, or $k = 0$ and $n = \pm 1$ at m; when $k = 0$ and $n = \pm 1$ we won't require a knowledge of the type of (2).

We will establish the type of (2) at $m \in M$ by showing the system (2) has the appropriate normal form at m. Suppose, for example, that $\delta(m) < 0$ so that Q_m is indefinite by Proposition 1. Then there exist two linearly independent elements σ_1, $\sigma_2 \in \alpha_m$ such that $\det_m(\sigma_i) = Q_m(\sigma_i) = 0$, for $i \in \{1, 2\}$. Necessarily σ_1 and σ_2 are rank 1 elements of $\mathrm{GL}(M_m, g(m))$. Hence there exists Pfaffian forms Ω^i at m and vectors $f_i \in g(m)$, for $i \in \{1, 2\}$, such that

$$\sigma_i = f_i \otimes \Omega^i \quad \text{for} \quad i \in \{1, 2\}.$$

Since $\{f_1 \otimes \Omega^1, f_2 \otimes \Omega^2\}$ is a basis for α_m it follows that $\{f_1, f_2\}$ and $\{\Omega^1, \Omega^2\}$ are each linearly independent; if not, every element in α_m would have rank 1 which implies $Q_m = 0$ and thus $\delta(m) = 0$. Now extend σ_1 and σ_2 to sections in α near m and call the extensions σ_1 and σ_2 too. Then any section Φ of α near m may be written

$$\Phi = \varphi^1 \sigma_1 + \varphi^2 \sigma_2$$

for real-valued functions φ^i, $i \in \{1, 2\}$. Hence

$$(d\Phi)^T = d\varphi^1 \wedge \sigma_1 + d\varphi^2 \wedge \sigma_2 + (\text{terms free of } d\varphi^i, i \in \{1, 2\}).$$

The f_1 and f_2 components of $(d\Phi)^T = 0$ – and hence a system equivalent to system (2) – at m looks like:

$$d\varphi^1 \wedge \Omega^1 + (\text{terms free of } d\varphi^i, i \in \{1, 2\}) = 0$$
$$d\varphi^2 \wedge \Omega^2 + (\text{terms free of } d\varphi^i, i \in \{1, 2\}) = 0.$$

This is the normal form of a hyperbolic system. Thus (2) is hyperbolic at m. Similarly, we can show that when $\delta(m) = 0$ and $Q_m \neq 0$ then the system (2) is parabolic at m. In this case choose linearly independent $\sigma_1, \sigma_2 \in \alpha_m$ so that – for the polarization $Q_m(\cdot, \cdot)$ of $Q_m(\cdot)$ – $Q_m(\sigma_1, \Phi) = 0$ for all $\Phi \in \alpha_m$. When $\delta(m) > 0$, complexify α_m, Q_m, etc. so that we can choose $\sigma_2 = \bar{\sigma}_1 \neq 0$ and $Q_m(\sigma_i) = 0$, for $i \in \{1, 2\}$, in order to show the system (2) is elliptic at m.

Proposition 2. *Suppose $Q_m \neq 0$, then the type of the system (2) – i.e. $(d\Phi)^T = 0$ – at m is elliptic, parabolic, or hyperbolic according as $\delta(m)$ is positive, zero, or negative.*

Definition 5. Let $g: M \to G$ be an immersion. A point $m \in M$ at which $Q_m \neq 0$ is called *elliptic*, *parabolic*, or *hyperbolic* according as $\delta(m)$ is positive, zero, or negative. If each point of M is a hyperbolic resp. an elliptic point of g, then we say g is *hyperbolic* resp. *elliptic*.

§5. Interpretations of k, n and δ

In this section we will show that $k(m)$, $n(m)$, and hence $\delta(m)$ are completely determined by the position of the tangent plane $g_*(M_m)$ in $G_{g(m)}$, and the sign of $\delta(m)$ is a convexity condition at m on any immersion with Gauss map g.

It is well-known that $G = S^2 \times S^2$, where each factor of this riemannian product, S^2, is a 2-sphere of radius $1/\sqrt{2}$ and thus has constant sectional curvature 2 [2, 6]. Let $p_i: G \to S^2$ be the projection onto the ith factor, for $i \in \{1, 2\}$. Let dA_i denote the pullback under p_i of the element of area of S^2, for $i \in \{1, 2\}$. If $\pi \in G$, then the sectional curvature K_G of the 2-plane $P \subset G_\pi$ is given by

$$K_G(P) = 2[dA_1(P)^2 + dA_2(P)^2], \tag{10}$$

where P is represented in the arguments of dA_i, $i \in \{1, 2\}$, by a unit 2-vector. If we write $G_\pi = S_1 \oplus S_2$, where S_i denotes the plane in G_π tangent to the ith factor of G, for $i \in \{1, 2\}$, then $K_G(P)$ equals twice the sum of the squares of the cosines of the angle between P and S_1 and the angle between P and S_2.

Let (e_1, e_2, e_3, e_4) be a field of positively oriented orthonormal frames of $\mathbb{R}^4$ defined on an open set $U \subset G$ which is π-adapted for all $\pi \in U$. Define Pfaffian forms ω_r^s by $\omega_r^s := (de_r, e_s)$, for $r, s \in \{1, 2, 3, 4\}$. Then by (1), $(\omega_1^3, \omega_2^3, \omega_1^4, \omega_2^4)$ is a field of orthonormal frames of T^*G on U. The ω_r^s also satisfy the Cartan structural equations:

$$d\omega_r^s = \sum_{t=1}^{4} \omega_r^t \wedge \omega_t^s. \tag{11}$$

In terms of ω_r^s define Pfaffian forms ψ^r as follows:

$$\psi^1 := (\omega_1^4 + \omega_2^3)/\sqrt{2},\ \psi^2 := (\omega_2^4 - \omega_1^3)/\sqrt{2}$$
$$\psi^3 := (\omega_1^4 - \omega_2^3)/\sqrt{2},\ \psi^4 := (\omega_1^3 + \omega_2^4)/\sqrt{2}.$$

Then $(\psi^1, \psi^2, \psi^3, \psi^4)$ is another field of orthonormal frames of T^*G on U. Associated to this field are the connection forms ψ_r^s, for $r, s \in \{1, 2, 3, 4\}$, defined by

$$d\psi^s := \sum_{r=1}^{4} \psi^r \wedge \psi_r^s,\ \psi_r^s + \psi_s^r := 0. \tag{12}$$

Using (11) one may show

$$\left\{\begin{aligned} &\psi_1^2 = -\psi_2^1 = \omega_1^2 + \omega_3^4 \\ &\psi_3^4 = -\psi_4^3 = \omega_1^2 - \omega_3^4 \\ &\psi_r^s = 0 \quad \text{for} \quad \{r, s\} \neq \{1,2\} \text{ or } \{3,4\}. \end{aligned}\right. \tag{13}$$

For the curvature forms $\Psi_r^s := d\psi_r^s - \sum_{t=1}^{4} \psi_r^t \wedge \psi_t^s$, we obtain

$$\left\{\begin{aligned} &\Psi_1^2 = -\Psi_2^1 = -2\psi^1 \wedge \psi^2 \\ &\Psi_3^4 = -\Psi_4^3 = -2\psi^3 \wedge \psi^4 \\ &\Psi_r^s = 0 \quad \text{for} \quad \{r, s\} \neq \{1,2\} \text{ or } \{3,4\}. \end{aligned}\right. \tag{14}$$

Therefore by (12) and (13) the differential systems $\psi^1=\psi^2=0$ and $\psi^3=\psi^4=0$ are both completely integrable and by (14) the sectional curvatures in G of the tangent planes to their integral manifolds are always 2. Thus the integral manifolds of these differential systems are the "factors" of $G=S^2\times S^2$. In particular, at any point $\pi\in U$ (and hence G) we may suppose $\psi^1\wedge\psi^2=dA_1$ and $\psi^3\wedge\psi^4=dA_2$. Also note:

$$\begin{cases}\omega_1^3\wedge\omega_2^3+\omega_1^4\wedge\omega_2^4=\psi^1\wedge\psi^2+\psi^3\wedge\psi^4=dA_1+dA_2\\ \omega_1^3\wedge\omega_1^4+\omega_2^3\wedge\omega_2^4=\psi^1\wedge\psi^2-\psi^3\wedge\psi^4=dA_1-dA_2\,.\end{cases}\tag{15}$$

Let $g:M\to G$ be an immersion. It is a straightforward computation to check that

$$kdA_g=g^*(\omega_1^3\wedge\omega_2^3+\omega_1^4\wedge\omega_2^4)\tag{16}$$

and

$$ndA_g=g^*(\omega_1^3\wedge\omega_1^4+\omega_2^3\wedge\omega_2^4),\tag{17}$$

where dA_g is the element of area on M induced by g. This should be no surprise to anyone used to working with moving frames. Hence using Eqs. (10), (15), (16), and (17) we obtain the next proposition in which $K_g:M\to\mathbb{R}$ is defined by

$$K_g(m):=K_G(g_*(M_m)),\quad\text{for all}\quad m\in M\,.$$

Proposition 3. *For each $m\in M$, the following holds:*

$$\begin{aligned}k(m)&=dA_1(g_*(M_m))+dA_2(g_*(M_m))\\ n(m)&=dA_1(g_*(M_m))-dA_2(g_*(M_m))\\ \delta(m)&=1/4(K_g(m)-1)\,.\end{aligned}$$

In particular, $k(m)$, $n(m)$, and $\delta(m)$ are completely determined by the angles between $g_*(M_m)$ and the factors of $G=S^2\times S^2$.

Let $X:M\to\mathbb{R}^4$ be an immersion with Gauss map g which need not be an immersion. Define $\varrho:M\to\mathbb{R}$ by

$$dA_g:=\varrho dA_X\,.$$

We call ϱ the *Jacobian of the Gauss map*; clearly $\varrho=1/Q(dX)$. Let K and N denote the Gaussian curvature and the normal curvature of X, resp. From (16) and (17) it is easy to see that

$$K=\varrho k\quad\text{and}\quad N=\varrho n\tag{18}$$

on M where g is an immersion.

In [7] Little defines a quadratic form $\mathscr{F}$ on $g^\perp$ as follows: For $z\in g^\perp(m)$, $\mathscr{F}(z)=\det(A^z)$, where the determinant of $A^z:M_m\to g(m)$ is taken using the metric induced by X on M_m – rather than the metric induced by g on M_m – and the metric induced from $\mathbb{R}^4$ on $g(m)$. Hence, if g is an immersion at m, it follows that

$$\mathscr{F}=\varrho F\,.$$

Little then defines $\Delta:=\det(\mathscr{F})$, where the determinant is taken with respect to the metric on $g^\perp$ induced from $\mathbb{R}^4$, i.e. with respect to the same metric as was used to define δ. Hence

$$\Delta = \varrho^2 \delta . \tag{19}$$

At each point $m \in M$ there is an ellipse $\mathscr{E}_m \subset g^{\perp}(m)$ which is the image of the unit circle in M_m (with respect to the metric induced by X) under the second fundamental form h. The ellipse may be a line or a point in degenerate cases. A point is called an inflection point if $\mathscr{E}_m$ lies in a 1-dimensional linear subspace of $g^{\perp}(m)$. According to Little [7] the following results hold:

Lemma 11. *Let $X: M \to \mathbb{R}^4$ be an immersion.*

a) *A point m is an inflection point of X if and only if $\Delta(m) = N(m) = 0$.*

b) *If m is not an inflection point of X, then the origin 0 of $g^{\perp}(m)$ is inside, on, or outside the ellipse $\mathscr{E}_m$ according as $\Delta(m)$ is positive, zero, or negative.*

Note that when the origin 0 of $g^{\perp}(m)$ is outside $\mathscr{E}_m$ there exists a unit vector $z \in g^{\perp}(m)$ such that (h, z) is positive definite. Hence there is a neighborhood U of m in M such that $X(U)$ is on one side of the hyperplane L through $X(m)$ with normal z. I.e. L is a local supporting hyperplane of $X(M)$ at $X(m)$. When 0 is outside $\mathscr{E}_m$ we say that X is *convex* at m. When the origin 0 is inside $\mathscr{E}_m$ there are clearly no local supporting hyperplanes of $X(M)$ at $X(m)$. In this case we say X is *aconvex* at m. Using Lemma 11 and (19) we obtain:

Proposition 4. *Let X be an immersion with Gauss map g. If $\delta(m) > 0$ resp. $\delta(m) < 0$ – so that g is necessarily an immersion at m – then the immersion X is convex resp. aconvex at m.*

Thus if $g: M \to G$ is an immersion, then sign $\delta(m)$ is a convexity condition at m on any immersion $X: M \to \mathbb{R}^4$ that may exist and have g as its Gauss map near m.

Using the ideas of this section we can prove a slight sharpening of a result of Chern and Spanier [2]. This appears in the Appendix.

§6. Proper and Improper Points

Let $m \in M$ be a regular point of an immersion $g: M \to G$. In addition assume $\delta(m) \geqq 0$ and $k(m) \geqq 0$. Proposition 1 implies $\mathrm{Det}(Q_m) \geqq 0$ and $\sigma(Q_m) \leqq 0$. Hence Q_m is negative semi-definite, i.e. for all $\Phi \in \alpha_m$, $Q_m(\Phi) \leqq 0$. Thus there exists no immersion $X: M \to \mathbb{R}^4$ with g as its Gauss map by condition $[C]$ of §1.

Definition 6. We call any regular point m of g for which Q_m is negative semi-definite an *improper point* of g. Otherwise a regular point is a *proper point* of g. From now on, we agree to call a point elliptic or parabolic only if it is a proper point of g.

It is worth noting that if an immersion X had existed with Gauss map g for which m was an improper point, then m would not have been an inflection point of X (because of Lemma 11 and Eq. (7), since $n(m) \neq 0$ by Lemma 7), $\Delta(m) \geqq 0$, and $K(m) \geqq 0$. But if m had not been an inflection point and $\Delta(m) \geqq 0$, then the origin 0 of $g^{\perp}(m)$ would not have been on or inside $\mathscr{E}_m$; moreover, $\mathscr{E}_m$ would not have been degenerate. This configuration of 0 and $\mathscr{E}_m$ implies $K(m) < 0$; we would have a contradiction.

The preceding paragraphs give two proofs of the next theorem.

Theorem 1. *Let $g: M \to G$ be an immersion. If there exists an improper point of g on M then there exists no immersion $X: M \to \mathbb{R}^4$ with Gauss map g.*

Remark. If the orientation of M changes then the sign of k changes. Hence by changing the orientation, improper points may become elliptic or parabolic, and vica versa. A point m at which $k(m)=0$, $n(m)=\pm 1$ is improper under either orientation of M. Thus if M has elliptic or parabolic points in addition to improper points, or points at which $k=0$, $n=\pm 1$, then there exists no immersion X on M with either orientation that has g as its Gauss map.

§7. Hyperbolic Maps

Suppose M is an open disk and g is hyperbolic. By the results of §4 there exist globally defined sections σ_1, σ_2 in α with $\sigma_i = E_i \otimes \Omega^i$, for $i \in \{1, 2\}$, where E_1, E_2 resp. Ω^1, Ω^2 are pointwise linearly independent sections in g resp. T^*M. Moreover we may suppose that (E_1, E_2) is positively oriented and $\|E_i\| = 1$, for $i \in \{1, 2\}$. Also we may suppose that $\Omega^1 = dx$ and $\Omega^2 = dy$ for smooth functions $x, y: M \to \mathbb{R}$, by introducing integrating factors if necessary, and that (Ω^1, Ω^2) is positively oriented. Then x, y is a positively oriented coordinate system when restricted to a suitable neighborhood of any point in M. We call x, y *characteristic coordinates*. Hence $\sigma_1 = E_1 \otimes dx$ and $\sigma_2 = E_2 \otimes dy$.

If Φ is any section of α on M, then

$$\Phi = U E_1 \otimes dx + V E_2 \otimes dy,$$

where U, V: $M \to \mathbb{R}$ are smooth. If we introduce Pfaffian forms W_i^j, for $i, j \in \{1, 2\}$, by $(dE_i)^T = \sum_{j=1}^{2} W_i^j E_j$, then (2) becomes

$$\left\{ \begin{aligned} &\frac{\partial U}{\partial x} + W_2^2(\partial/\partial x) U - W_1^2(\partial/\partial y) V = 0, \\ &\frac{\partial V}{\partial y} - W_2^1(\partial/\partial x) U + W_1^1(\partial/\partial y) V = 0. \end{aligned} \right. \tag{20}$$

The system (20) is, of course, the normal form of (2), when (2) is hyperbolic on M. The directions of $\partial/\partial x$ and $\partial/\partial y$ define the characteristic directions and the curves $x = \text{const.}$, $y = \text{const.}$ are the characteristic curves of (2).

Let I be an interval; a smooth curve $\gamma: I \to M$ is said to be *regular* if it is an embedding of I into M. A regular curve γ is *nowhere characteristic* if it is nowhere tangent to a characteristic curve. Let t be a coordinate on I. In terms of the functions x, y defined above, which locally are coordinates, we introduce $\gamma^1, \gamma^2: I \to \mathbb{R}$ by $\gamma_*(\partial/\partial t) = \gamma' = \gamma^1 \partial/\partial x + \gamma^2 \partial/\partial y$. Then γ is a regular nowhere characteristic curve if and only if the product $\gamma^1\gamma^2$ never vanishes on I. Let $\Gamma: I \to \mathbb{R}^4$ be a smooth curve. We describe what we mean when we say that *Γ is compatible with $g \circ \gamma$*. First $\Gamma' = \Gamma_*(\partial/\partial t)$ never vanishes and $\Gamma'(t) \in g(\gamma(t))$ for all $t \in I$. In terms of the vector fields E_1, E_2 defined above, let $\Gamma^1, \Gamma^2: I \to \mathbb{R}$ be defined by $\Gamma' = \Gamma^1 E_1 \circ \gamma + \Gamma^2 E_2 \circ \gamma$; then second we require that $\operatorname{sign}(\Gamma^1\Gamma^2) = \operatorname{sign}(\gamma^1\gamma^2)$. The fact that the orientations of E_1, E_2 and dx, dy are positive insures that this definition is well-defined.

Theorem 2. *Let $g: M \to G$ be hyperbolic, $\gamma: I \to M$ be a regular nowhere characteristic curve in M, and $\Gamma: I \to \mathbb{R}^4$ be a curve compatible with $g \circ \gamma$. Then there exists an immersion $X: N \to \mathbb{R}^4$, where N is a neighborhood of $\gamma(I)$, such that the Gauss map of X is $g|N$ and $X \circ \gamma = \Gamma$.*

Proof. On a simply connected neighborhood N'' of $\gamma(I)$ we introduce E_1, E_2 and x, y as above. Define functions U_0 and V_0 along γ by $U_0 = \Gamma^2/\gamma^2$ and $V_0 = \Gamma^1/\gamma^1$. Since the coefficients in (20) are C^∞ functions and the initial data is C^∞, the standard existence theorems for hyperbolic systems with initial data along curves which are nowhere characteristic [3] insures the existence of a neighborhood N' of $\gamma(I)$ on which there is a solution U, V of (20) satisfying the initial data $U \circ \gamma = U_0$ and $V \circ \gamma = V_0$. On N' set $\Phi = VE_1 \otimes dx + UE_2 \otimes dy$. Clearly $Q(\Phi) = (\text{positive})\, UV$. But UV on γ is $U_0 V_0 = \Gamma^1 \Gamma^2/\gamma^1 \gamma^2 > 0$. Thus there is a simply connected neighborhood N of $\gamma(I)$ on which $UV > 0$ and hence $Q(\Phi) > 0$. On N, Φ is a section of α that satisfies (20); thus $d\Phi = 0$. But since N is simply connected there exists a mapping X such that $\Phi = dX$. Clearly X is an immersion having Gauss map g. Also $(X \circ \gamma)' = \Phi \circ \gamma' = V_0 \gamma^1 E_1 \circ \gamma + U_0 \gamma^2 E_2 \circ \gamma = \Gamma^1 E_1 \circ \gamma + \Gamma^2 E_2 \circ \gamma = \Gamma'$. Hence by adding a suitable fixed vector to the immersion X, $X \circ \gamma = \Gamma$.

Remark. Suppose $\text{sign}(\Gamma^1 \Gamma^2) = -\text{sign}(\gamma^1 \gamma^2)$. If the orientation of M is changed, then there would exist an immersion X in a neighborhood of $\gamma(I)$ with Gauss map g and $X \circ \gamma = \Gamma$.

Since the set of hyperbolic points of $g: M \to G$ is open we obtain the following obvious

Corollary. *Let $g: M \to G$ be an immersion and suppose $m \in M$ is hyperbolic point of g. Then g is the Gauss map of some immersion in a neighborhood N of m.*

Definition 7. Assume again that g is hyperbolic and $\gamma: I \to M$ is a regular curve. A point $m \in M$ is said to be in the *domain of determinancy* of γ is the following holds: Either $m \in \gamma(I)$ or there are characteristic curves c_1 and c_2 running from m to $\gamma(I)$ which are tangent to different characteristic directions at m; moreover, if m_i is the point where c_i terminates along $\gamma(I)$ for $i \in \{1, 2\}$, and γ^* is the arc of γ connecting m_1 and m_2, then the union of the configurations c_1, c_2, and γ^* bounds a disk; see Fig. 1. Note that m_i need not be the first point on $\gamma(I)$ met by c_i as we run along c_i from m to $\gamma(I)$. We let $D\gamma$ denote the domain of determinancy of γ.

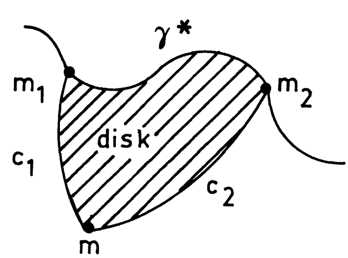

Fig. 1

Lemma 12. *Let m_0 and m_1 be distinct points of M. Let c_1 and c_2 be characteristic curves which run from m_0 to m_1. Suppose that at m_0 the curves are tangent to different characteristic directions. Then the union of the configurations of c_1 and c_2 does not bound a disk.*

Proof. Suppose the union of the configurations of c_1 and c_2 bounds a disk B. Then we may introduce characteristic coordinates x, y on a neighborhood of B. Then, say, x is constant on c_2 and y is constant on c_1. Therefore $x(m_0)=x(m_1)$ implies $dx=0$ somewhere along c_1 which is a contradiction.

A consequence of Lemma 12 is that if γ is a characteristic curve then $D\gamma=\gamma$. However, when γ is not a characteristic curve then $D\gamma$ has non-empty interior.

In the next theorem, two immersions $X_i: M \to \mathbb{R}^4$, $i=1,2$, are considered. Associated with each immersion are various invariants defined in M. We use the subscripts 1 and 2 to distinguish between the invariants induce on M by X_1 and X_2, respectively.

Theorem 3. *Let $X_i: M \to \mathbb{R}^4$, $i \in \{1,2\}$, be immersions with the same Gauss map. Suppose $\Delta_1<0$ (and hence $\Delta_2<0$) on M. Let $\gamma: I \to M$ be a regular curve. If $X_1 \circ \gamma = X_2 \circ \gamma$, then $X_1 = X_2$ on $D\gamma$.*

Proof. The result follows trivially if γ is a characteristic for then $D\gamma=\gamma$. Therefore suppose γ is not a characteristic curve and $m \in D\gamma - \gamma(I)$. Then m lies in a closed disk B bounded by the characteristic curves from m to γ and an arc γ^* of γ. We introduce sections $E_1 \otimes dx$ and $E_2 \otimes dy$ in $\alpha|B$ as above. Then

$$dX_i = V_i E_1 \otimes dx + U_i E_2 \otimes dy, \quad \text{for} \quad i \in \{1,2\}$$

on B. Also on B, $\gamma' = \gamma^1 \partial/\partial x + \gamma^2 \partial/\partial y$. Since $X_1 \circ \gamma = X_2 \circ \gamma$ it follows that $dX_1(\gamma') = dX_2(\gamma')$, i.e.,

$$V_1 \gamma^1 E_1 + U_1 \gamma^2 E_2 = V_2 \gamma^1 E_1 + U_2 \gamma^2 E_2$$

along γ^*. But E_1, E_2 are linearly independent; therefore $V_1\gamma^1 = V_2\gamma^1$ and $U_1\gamma^2 = U_2\gamma^2$. Set $U = U_1 - U_2$ and $V = V_1 - V_2$; then $V\gamma^1 = 0$ and $U\gamma^2 = 0$ along γ^*. Let $Z = \{p \in \gamma^* | \gamma^1(p) = 0\}$. Note that $Z \neq \gamma^*$ for otherwise $m \in D\gamma^* = \gamma^* \subset \gamma$. Therefore $V=0$ on $\gamma^* - Z \neq \phi$. If Z has no interior in γ^*, then $V=0$ on γ^*. So suppose Z has non-empty interior in γ^*. Let J be a component of Z with non-empty interior. On J, γ^2 is nowhere 0 so that $U=0$ on J. Hence on J, from (20),

$$\frac{\partial V}{\partial y} = -W_1^1(\partial/\partial y)V.$$

Since $V=0$ at an endpoint of J it follows that $V=0$ on J. Consequently $V=0$ on γ^*. Similarly $U=0$ on γ^*. The standard uniqueness result for hyperbolic systems with Cauchy data on a nowhere characteristic curve can be extended to show that if $U=V=0$ on the regular curve γ^*, then $U=V=0$ on B. Hence $U_1=U_2$ and $V_1=V_2$ on B; thus $dX_1 = dX_2$ on B. Since $X_1 \circ \gamma = X_2 \circ \gamma$ it follows that $X_1 = X_2$ on B and, in particular, $X_1(m) = X_2(m)$.

Corollary. *Let M be a closed disk with boundary, ∂M, and $X_i: M \to \mathbb{R}^4$, $i \in \{1,2\}$, be immersions with the same Gauss map. If $\Delta_1<0$ on M and $X_1|\partial M = X_2|\partial M$, then $X_1 = X_2$.*

Proof. We need only show that $D(\partial M) = M$. This follows immediately from the existence of the functions x, y on M with $dx=0$, $dy=0$ defining the characteristic directions. For if $m \in M$, two characteristic curves with distinct characteristic directions originating at m must terminate on ∂M. This is most easily seen by looking at the image of M under the continuous $(x, y): M \to \mathbb{R}^2$. Also by Lemma 12 these two characteristic curves will not meet again. Hence $m \in D(\partial M)$.

Remark. When g is the Gauss map of an immersion X and TM is identified with g by dX, then E_1 and E_2 determine the characteristic directions. Also, one may show that E_1, E_2 are characterized up to scalar multiples as the only linear independent pair of vectors such that $h(E_1, E_2) = 0$.

§8. Elliptic Maps

Suppose for the moment that M is a closed disk and g is elliptic. Then Q is positive definite at each point of M. Let α^c denote the complexification of α so that we may view each $\Phi \in \alpha^c_m$ as a complex linear map from the complexified tangent space M^c_m into $\mathbb{C}^4$. For each $m \in M$, Q_m extends to a quadratic form, also denoted by Q_m, on α^c_m by setting $Q_m(\Phi) = \det_m(\Phi)$ for all $\Phi \in \alpha^c_m$. Then there exists a globally defined unit vector field $E: M \to \mathbb{C}^4$ and a nowhere vanishing globally defined complex 1-form Ω such that $\sigma = E \otimes \Omega$ is a section in α^c. Of course, $\bar{\sigma} = \bar{E} \otimes \Omega$ is also a section in α. By introducing a nowhere zero complex-valued integrating factor, if necessary, we may write $\Omega = dz$ where z is a complex coordinate on M. To see this note that $\Omega\bar{\Omega}$ is a Riemannian metric on M. Introduce isothermal coordinates $z = x + iy$ on M such that $\Omega\bar{\Omega} = |\lambda|^2 dz\, d\bar{z}$. Then $\Omega = \lambda\, dz$, or if $\Omega = \lambda\, d\bar{z}$ relabel the isothermal coordinates x, y to get $\Omega = \lambda\, dz$. Hence, we may let $\sigma = E \otimes dz$. We call z the normal coordinate.

Let Φ be a section in α and hence in α^c on M. Then

$$\Phi = UE \otimes dz + \bar{U}\bar{E} \otimes d\bar{z},$$

where U is a complex-valued function on M.

Now introduce complex 1-forms W and W' by $(dE)^T = WE + W'\bar{E}$. Then (2) becomes

$$\frac{\partial U}{\partial \bar{z}} + W(\partial/\partial \bar{z})U - \bar{W}'(\partial/\partial z)\bar{U} = 0, \tag{21}$$

where $\bar{W}'$ is the complex conjugate of W'. Equation (21) is the normal form of (2), when (2) is elliptic on M.

Theorem 4. *Let M be a closed disk and g be elliptic. There exists an immersion $X: M \to \mathbb{R}^4$ with Gauss map g.*

Proof. Consider the homogeneous boundary value problem of the first kind consisting of (21) and the boundary condition $\operatorname{Re} U|\partial M = 0$. This problem has a solution U that has no zeros in M; see [3, p. 323]. Let $\Phi = UE \otimes dz + \bar{U}\bar{E} \otimes d\bar{z}$. Clearly $Q(\Phi) > 0$ on M since Φ never vanishes on M. Also since Φ is a section in α and satisfies (21), i.e. (2), $d\Phi = 0$. Since M is simply connected there is a mapping X such that $dX = \Phi$. Clearly X is an immersion having Gauss map g.

Corollary. *Let $g: M \to G$ be an immersion and suppose g is elliptic at $m \in M$. Then g is the Gauss map of some immersion defined in a neighborhood of m.*

In the following theorem M is no longer restricted to be a closed disk. Also by an arc A in M we mean the image of a regular curve.

Theorem 5. *Let $X_i: M \to \mathbb{R}^4$ be two immersions with the same Gauss map. If $\Delta_1 > 0$ (and hence $\Delta_2 > 0$) on M and $X_1|A = X_2|A$, where A is some arc in M, then $X_1 = X_2$.*

Proof. Let $m \in M$. Then there is a closed disk B in M which contains m and meets A in a subarc A^*. Let $E \otimes dz$ be the section in $\alpha^c|B$ described above. Then $dX_i = U_i E \otimes dz + \bar{U}_i \bar{E} \otimes d\bar{z}$ where U_i is a complex-valued function defined on B, $i = 1, 2$. Of course, U_1 and U_2 satisfy (21) on B.

Now let $\gamma: I \to B$ be a regular curve with $\gamma(I) = A^*$. Then $X_1 \circ \gamma = X_2 \circ \gamma$ implies $dX_1(\gamma') = dX_2(\gamma')$. Consequently $U_1 = U_2$ along A^*. Let $U = U_1 - U_2$; then U satisfies (21) and $U = 0$ on A^*. But $U = f(z)\exp[s(z, \bar{z})]$ on B, where f is analytic; see [3, p. 261]. However $f = 0$ along A^* implies f is identically zero on B. Hence $U = 0$ or $U_1 = U_2$ on B. Consequently $dX_1 = dX_2$ on B. Since $X_1 = X_2$ on A^*, then $X_1 = X_2$ on B. In particular $X_1(m) = X_2(m)$. Thus $X_1 = X_2$ on M.

Appendix

Let $X: M \to \mathbb{R}^4$ be an immersion with Gauss map g and set $g_i = p_i \circ g$, where $p_i: G \to S^2$ is projection onto the ith factor of $G = S^2 \times S^2$, for $i \in \{1, 2\}$. Also let A_i equal the algebraic area of the image of M under g_i, $i \in \{1, 2\}$; A_i is, in fact, equal to 2π times the degree of the map g_i.

Proposition 5. *If $X: M \to \mathbb{R}^4$ is an immersion with M closed then*

$$A_1 + A_2 = \int_M K\, dA_X,$$

and

$$A_1 - A_2 = \int_M N\, dA_X.$$

Proof. Let $M' = \{m \in M | \varrho(m) \neq 0\}$. Then, for $i \in \{1, 2\}$,

$$A_i = \int_M g^*(dA_i) = \int_{M'} g^*(dA_i).$$

By Proposition 3, which is valid on M', we have $A_1 + A_2 = \int_{M'} k\, dA_g$. But the definition of the Jacobian, ϱ, and (18) implies $\int_{M'} k\, dA_g = \int_{M'} K\, dA_X$. Therefore $A_1 + A_2 = \int_{M'} K\, dA_X = \int_M K\, dA_X$. Similarly one may show $A_1 - A_2 = \int_M N\, dA_X$.

A corollary of this is the theorem of Chern and Spanier.

Corollary. *If X is an embedding of a closed surface M, then*

$$A_1 = A_2 = 1/2 \int_M K\, dA_X.$$

Proof. If X is an embedding then the algebraic number of self-intersections of X is zero. A theorem of Whitney [5, 7 – Appendix A] says that $\frac{1}{2\pi}\int_M N dA_X$ equals twice the algebraic number of self-intersections of X. Thus $A_1 - A_2 = 0$. The result follows immediately from Proposition 5.

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Bilipschitz Extensions of Maps Having Quasiconformal Extensions

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1. Introduction

This paper deals with the curious phenomenon that sometimes a quasiconformal property implies the corresponding bilipschitz property. This was first discovered by Ahlfors, who proved [A, Lemma 3, p. 80] that if a planar curve through ∞ admits a quasiconformal reflection, it also admits a bilipschitz reflection. This result was extended to dimensions $n \neq 4$ by the authors in [TV3, 7.15]. Furthermore, Gehring [G2, Theorems 6,7] gave generalizations of the Ahlfors theorem in the plane.

The main result of the present paper states that if $X \subset R^n$, $n \neq 4$, and if a bilipschitz map $f: X \to R^n$ has a quasiconformal extension to R^n, then it also has a bilipschitz extension to R^n. This situation occurs, for example, if X and fX are images of the p-sphere S^p or the p-disc $\bar{B}^p$ under quasiconformal maps of R^n, $1 \leqq p \leqq n-1$. We apply the result to the flattening theory of lipschitz manifolds.

2. Extension

The main results of this section are Theorems 2.12 and 2.19. The first of these states that a quasiconformal map of R^n, $n \neq 4$, can be replaced by a bilipschitz map without changing it on a set X at which it already is bilipschitz. The proof depends on the approximation of quasiconformal maps by quasihyperbolic ones (Definition in 2.1), proved by the authors in [TV3]. We also need an auxiliary result (Lemma 2.3) whose lengthy proof could be substantially simplified in the case where X is connected and unbounded. In Theorem 2.19 we apply 2.12 to prove that a bilipschitz homeomorphism between subsets X, Y of R^n can be extended to a bilipschitz homeomorphism of R^n, provided that X and Y satisfy certain conditions given in terms of quasisymmetry.

2.1. Notation and Terminology. We let $(e_1, \ldots, e_n)$ denote the standard basis of the euclidean n-space R^n. Open balls in R^n are written as $B^n(x,r)$ and spheres as $S^{n-1}(x,r)$; the superscripts may be dropped. We also set

$$B^n = B^n(0,1), \qquad S^{n-1} = S^{n-1}(0,1), \qquad R^n_+ = \{x \in R^n : x_n \geqq 0\}, \qquad B^n_+ = B^n \cap R^n_+ .$$

Let $G \subset R^n$ and let $f: G \to R^n$. For $x \in G$ we write

$$L(x,f) = \limsup_{y \to x} \frac{|f(y)-f(x)|}{|y-x|}, \qquad l(x,f) = \liminf_{y \to x} \frac{|f(y)-f(x)|}{|y-x|}.$$

We use the abbreviations QC = quasiconformal, QS = quasisymmetric, LQS = locally quasisymmetric, LIP = locally lipschitz. The basic theory of QS maps is given in [TV 1]. A map $f: X \to Y$ between metric spaces is L-bilipschitz if

$$|x-y|/L \leqq |f(x)-f(y)| \leqq L|x-y|$$

for all $x, y \in X$. An embedding $f: X \to Y$ is η-QS if $\eta: [0,\infty) \to [0,\infty)$ is a homeomorphism and if $|f(a)-f(x)| \leqq \eta(t)|f(b)-f(x)|$ whenever $|a-x| \leqq t|b-x|$. Recall that a QC map $f: R^n \to R^n$, and hence a bilipschitz map, is always a homeomorphism onto R^n.

If D is a proper subdomain of R^n, the quasihyperbolic (abbreviated QH) metric k_D of D is defined by

$$k_D(x, y) = \inf_\alpha \int_\alpha \frac{|dx|}{d(x, \partial D)},$$

where the infimum is taken over all rectifiable paths α joining x and y in D. We shall make use of the fact that each pair of points in D can be joined by a QH geodesic, [GO, p. 53], although this could easily be avoided.

An embedding $f: D \to R^n$ is called *H-quasihyperbolic* if $fD \neq R^n$ and if f is H-bilipschitz in the QH metrics of D and fD. If U is an open proper subset of R^n, an embedding $f: U \to R^n$ is called H-QH if $f|D$ is H-QH for every component D of U.

In what follows, we shall assume $n \geqq 2$. However, the results are easily seen to be also true for $n=1$. Indeed, if $X \subset R^1$, every monotone L-bilipschitz map $f: X \to R^1$ can be extended to an L-bilipschitz homeomorphism $F: R^1 \to R^1$ which is affine on each component of $R^1 \backslash X$.

2.2. Basic Radial Maps. For $y \in R^n$ and $0 \leqq a < b \leqq \infty$, the set

$$A(y,a,b) = \{x \in R^n : a < |x-y| < b\}$$

is called an *annulus*. If $a > 0$ and $b < \infty$, the annulus is *nondegenerate*. Then $A(y,a,b) = B(y,b) \backslash \bar{B}(y,a)$.

Suppose that $A_1 = A(y_1, a_1, b_1)$ and $A_2 = A(y_2, a_2, b_2)$ are nondegenerate annuli. Then there is a unique homeomorphism $h: \bar{A}_1 \to \bar{A}_2$ of the form

$$h(y_1 + re) = y_2 + \beta r^\alpha e,$$

where $e \in S^{n-1}$, $a_1 \leqq r \leqq b_1$, $\beta > 0$, $\alpha > 0$. Indeed,

$$\alpha = \frac{\log(b_2/a_2)}{\log(b_1/a_1)}, \qquad \beta = a_1^{-\alpha} a_2 = b_1^{-\alpha} b_2.$$

We say that $h: \bar{A}_1 \to \bar{A}_2$ is the *basic radial map*. It is K-QC with $K = \max(\alpha^{n-1}, \alpha^{1-n})$. Moreover,

$$L(x,h) = \beta|x-y_1|^{\alpha-1} \max(\alpha, 1),$$
$$l(x,h) = \beta|x-y_1|^{\alpha-1} \min(\alpha, 1),$$

cf. [V 1, 16.2].

2.3. Lemma. *Suppose that* $X \neq \emptyset$ *is a closed set in* R^n, *and that* $f: R^n \to R^n$ *is a homeomorphism such that*

(1) f *is* $K-$QC,
(2) $f|X$ *is* L*-bilipschitz*,
(3) $f|R^n \setminus X$ *is* $H-$QH.

Then there is a homeomorphism $g: R^n \to R^n$ *such that*

(a) $g|X = f|X$,
(b) $gD = fD$ *for every component* D *of* $R^n \setminus X$,
(c) g *is* L_1*-bilipschitz for some* L_1 *depending only on* H, K, L *and* n.

Proof. There is a homeomorphism $\eta: [0, \infty) \to [0, \infty)$, depending only on K and n, such that every $K-$QC map $f: R^n \to R^n$ is $\eta-$QS, see [V3, 2.4]. Choose numbers $q \geqq 3$ and $M \geqq 4$ such that

(2.4) $$\eta(1/q) \leqq 1/2, \quad \eta(1/M) \leqq 1/4, \quad \eta(L/qM) \leqq 1/4q.$$

Then q and M depend only on K, L, n.

We let $\mathscr{A}$ denote the family of all annuli $A = A(y, a, b)$ in R^n such that:

(i) $y \in X$.
(ii) $A \cap X = \emptyset$.
(iii) Each boundary component of A meets X.
(iv) $b/a \geqq q^2 M^2$ or $b/a = \infty$.

Note that (iv) implies $b/a \geqq 144$.

The family $\mathscr{A}$ can be empty. This happens, for example, if X is connected and unbounded. In this case we choose $g = f$. Otherwise, f will be altered in each of the annuli $A \in \mathscr{A}$.

For each $A \in \mathscr{A}$, we let B_A denote the bounded component of $R^n \setminus A$. By (i), $B_A \cap X \neq \emptyset$. We define an equivalence relation in $\mathscr{A}$ by setting

$$A_1 \sim A_2 \quad \text{whenever} \quad B_{A_1} \cap X = B_{A_2} \cap X.$$

Furthermore, for each $A = A(y, a, b) \in \mathscr{A}$ we set

$$A^* = A(y, qa, b/q).$$

Here $qa < b/q$ by (iv).

Fact 1. *If* $A_1 \not\sim A_2$, *then* $d(A_1^*, A_2^*) \geqq \min(d(A_1^*), d(A_2^*))/2$.

To prove this, let $A_j = A(y_j, a_j, b_j)$, and assume first that $B_{A_1} \cap B_{A_2} \cap X = \emptyset$. Setting $C_{A_j} = R^n \setminus B^n(y_j, b_j)$ we then have $y_1 \in C_{A_2}$ and $y_2 \in C_{A_1}$. We may assume $b_1 \geqq b_2$. Let $x_1 \in A_1^*$ and $x_2 \in A_2^*$. Since $q \geqq 3$, we have

$$|x_1 - y_2| \geqq b_1(1 - 1/q) \geqq 2b_2/3,$$
$$|x_2 - y_2| \leqq b_2/q \leqq b_2/3,$$

and therefore

$$|x_1 - x_2| \geqq b_2/3 \geqq b_2/q = \min(d(A_1^*), d(A_2^*))/2.$$

Next assume that $B_{A_1} \cap B_{A_2} \cap X$ contains a point z. Since $A_1 \not\sim A_2$, we may assume that there is $z' \in X \cap (B_{A_1} \setminus B_{A_2})$. Then $y_2 \in B_{A_1}$, since the case $y_2 \in C_{A_1}$ is clearly impossible. Now $b_2 \leqq |y_2 - z'| \leqq 2a_1$. If $x_j \in A_j^*$, we have

$$|x_1 - y_1| \geqq qa_1 \geqq 3a_1 ,$$
$$|x_2 - y_1| \leqq |x_2 - y_2| + |y_2 - y_1| \leqq b_2/q + a_1 < 2a_1 ,$$

and thus

$$|x_1 - x_2| \geqq a_1 > b_2/q .$$

Fact 1 is proved.

We form a family $\mathscr{A}_0 \subset \mathscr{A}$ by choosing exactly one annulus from each equivalence class. Set $\mathscr{A}^* = \{A^* : A \in \mathscr{A}_0\}$. It follows from Fact 1 that the members of $\mathscr{A}^*$ are disjoint. The family $\mathscr{A}^*$ need not be locally finite, but the following result follows easily from Fact 1:

Fact 2. *Every point in $R^n \setminus X$ has a neighborhood which meets at most one member of $\mathscr{A}^*$.*

We set

$$E = R^n \setminus \cup \mathscr{A}^* .$$

Clearly $X \subset E$.

Fact 3. *If $x \in E$ and $y \in X$ with $|x - y| = r > 0$, then the annulus $A = A(y, r/Mq, Mqr)$ meets X.*

To prove this, assume that $A \cap X = \emptyset$. Then A is contained in a unique annulus $A_1 = A(y, a_1, b_1) \in \mathscr{A}$ such that $a_1 \leqq r/Mq$, $b_1 \geqq Mqr$. If $a_1 = 0$, then $A_1 \in \mathscr{A}_0$, which implies $x \notin E$, a contradiction. Suppose $a_1 > 0$. Choose $A_2 = A(y_2, a_2, b_2) \in \mathscr{A}_0$ with $A_1 \sim A_2$. Then $|y - y_2| \leqq \min(a_1, a_2)$, and

$$a_2 \leqq d(B_{A_2} \cap X) = d(B_{A_1} \cap X) \leqq 2a_1 .$$

Since $M \geqq 4$, and $q \geqq 3$, we obtain

$$|x - y_2| \geqq r - a_1 \geqq (Mq - 1)a_1 > 2qa_1 \geqq qa_2 .$$

If $b_1 = \infty$, then also $b_2 = \infty$, and hence $x \in A_2^*$, a contradiction. If $b_1 < \infty$, we have

$$|x - y_2| \leqq r + a_1 \leqq b_1/Mq + b_1/M^2q^2 < b_1/2q .$$

Here

$$b_1 = d(y, X \setminus B_{A_1}) = d(y, X \setminus B_{A_2}) \leqq a_2 + b_2 < 2b_2 .$$

Thus $|x - y_2| < b_2/q$, which implies $x \in A_2^*$ and hence $x \notin E$. This contradiction proves Fact 3.

We next define a homeomorphism $h: R^n \to R^n$. We let $h|E = \mathrm{id}$. For $A \in \mathscr{A}_0$, $h|\bar{A}^*$ will be a radial homeomorphism with $h|\partial A^* = \mathrm{id}$. To simplify notation, we assume that $A = A(0, a, b)$ and that $f(0) = 0$. The general case reduces to this by auxiliary translations.

Suppose first that A is nondegenerate. There is a unique integer k such that

$$aq^2M^k \leqq b < aq^2M^{k+1} .$$

Since $b \geqq aq^2M^2$, $k \geqq 2$. We set $s_j = aqM^j$ for $0 \leqq j \leqq k-1$ and $s_k = b/q$. Then

(2.5) $$M \leqq s_j/s_{j-1} \leqq M^2$$

for $1 \leqq j \leqq k$. For $1 \leqq j \leqq k-1$ we set $t_j = |f^{-1}(s_j e_1)|$. We also define $t_0 = aq$, $t_k = b/q$. Since f^{-1} is η-QS, we have

$$t_j/t_{j-1} \leqq \eta(M), \qquad t_{j-1}/t_j \leqq \eta(1/M) \leqq 1/4$$

for $2 \leqq j \leqq k-1$; here we also made use of (2.4). We next estimate this ratio for $j=1$ and $j=k$. Choose $y_1 \in X$ with $|y_1| = a$. Since $f|X$ is L-bilipschitz, $a/L \leqq |f(y_1)| \leqq La$. Hence

$$\frac{t_1}{t_0} = \frac{|f^{-1}(s_1 e_1)|}{q|y_1|} \leqq \eta\left(\frac{s_1}{|f(y_1)|}\right) \leqq \eta(qML),$$

$$\frac{t_0}{t_1} \leqq q\eta\left(\frac{|f(y_1)|}{aqM}\right) \leqq q\eta\left(\frac{L}{qM}\right) \leqq 1/4.$$

Similarly, choosing $y_2 \in X$ with $|y_2| = b$ we have $b/L = |f(y_2)| \leqq Lb$, and obtain

$$\frac{t_k}{t_{k-1}} < \frac{b}{|f^{-1}(s_{k-1}e_1)|} \leqq \eta\left(\frac{|f(y_2)|}{s_{k-1}}\right) \leqq \eta(qLM^2),$$

$$\frac{t_{k-1}}{t_k} = \frac{q|f^{-1}(s_{k-1}e_1)|}{b} \leqq q\eta\left(\frac{s_{k-1}}{|f(y_2)|}\right) \leqq q\eta(L/qM) \leqq 1/4.$$

Summing up, we have

(2.6) $$4 \leqq t_j/t_{j-1} \leqq M_1$$

for all $j = 1, \ldots, k$, where $M_1 = \eta(qLM^2)$.

We now define $h|\bar{A}^*: \bar{A}^* \to \bar{A}^*$ so that h maps each $\bar{A}_j = \bar{A}(0, s_{j-1}, s_j)$ onto $\bar{A}(0, t_{j-1}, t_j)$ by a basic radial map, see 2.2.

Next assume that A is degenerate. We may assume that $a > 0$ or $b < \infty$, since otherwise $\operatorname{card} X = 1$ and the lemma is trivial.

Suppose that $a > 0$, $b = \infty$. We now divide A^* into an infinite number of subannuli $A_j = A(0, s_{j-1}, s_j)$ with $s_j = aqM^j$, $j \geqq 0$. Define $t_0 = aq$ and $t_j = |f^{-1}(s_j e_1)|$ for $j \geqq 1$. Then (2.5) and (2.6) are true for all $j \geqq 1$, and we define $h|\bar{A}^*$ as above.

In the case $a = 0$, $b < \infty$, we set $s_0 = t_0 = b/q$ and $s_j = bM^j/q$, $t_j = |f^{-1}(s_j e_1)|$ for $j = -1, -2, \ldots$. Again (2.5) and (2.6) are true for all $j \leqq 0$, and we define $h|\bar{A}^*$ as above.

We have thus defined a bijective map $h: R^n \to R^n$ with $h|X = \mathrm{id}$. From Fact 1 it easily follows that h is a homeomorphism.

Let $A^* \in \mathscr{A}^*$, and assume again $A = A(0, a, b)$, $f(0) = 0$. Let $A_j = A(0, s_{j-1}, s_j)$ be one of the subannuli of A^* defined above. Thus $hA_j = A(0, t_{j-1}, t_j)$, and

$$h(x) = \beta_j |x|^{\alpha_j - 1} x, \qquad \alpha_j = \frac{\log(t_j/t_{j-1})}{\log(s_j/s_{j-1})},$$

for $x \in \bar{A}_j$. By (2.5) and (2.6), we have

$$\frac{\log 4}{2 \log M} \leqq \alpha_j \leqq \frac{\log M_1}{\log M};$$

hence $M_2^{-1} \leqq \alpha_j \leqq M_2$ for some M_2 depending only on K, L and n. By 2.2 and [V1, 35.1], $h|A^*$ is K_1-QC with $K_1 = M_2^{n-1}$. In fact, the map $h: R^n \to R^n$ is $K_1 - QC$. This follows at once from [V2, Theorem 2], but it can be proved more elementarily as follows: We define a sequence of homeomorphisms $h_j: R^n \to R^n$ by setting $h_j|A^* = h|A^*$ for all $A^* \in \mathscr{A}^*$ with $d(A^*) \geqq 1/j$, and $h_j = \mathrm{id}$ outside these annuli. By Fact 1, the family $\{A^* \in \mathscr{A}^*: d(A^*) \geqq 1/j\}$ is locally finite. As above, this implies that h_j is K_1-QC. Since $h_j \to h$ uniformly, h is K_1-QC.

Set $U = R^n \setminus X$. For $x \in U$ we write

$$L_Q(x, h) = \frac{L(x, h) d(x, X)}{d(h(x), X)}, \qquad l_Q(x, h) = \frac{l(x, h) d(x, X)}{d(h(x), X)}.$$

We next show that

$$H_1^{-1} \leqq l_Q(x, h) \leqq L_Q(x, h) \leqq H_1 \tag{2.7}$$

for all $x \in U$, where the constant H_1 depends only on K, L and n. Incidentally, (2.7) means that $h|U$ is H_1-QH, but we shall not make use of this fact.

Let $x \in A_j$ and suppose, for example, that $\alpha_j \geqq 1$. Since

$$|y| - a \leqq d(y, X) \leqq |y|$$

for every $y \in A^*$, the formulas in 2.2 yield

$$L_Q(x, h) \leqq \frac{\alpha_j |h(x)|}{|h(x)| - a} \leqq \frac{\alpha_j}{1 - 1/q} \leqq 2M_2,$$

$$l_Q(x, h) \geqq \frac{|x| - a}{|x|} \geqq 1 - 1/q \geqq 1/2.$$

If $\alpha_j \leqq 1$, we similarly obtain

$$1/2M_2 \leqq l_Q(x, h) \leqq L_Q(x, h) \leqq 2.$$

These inequalities imply (2.7) with $H_1 = 2M_2$.

We shall show that the homeomorphism $g = fh: R^n \to R^n$ is the desired map. The conditions (a) and (b) are clearly satisfied. It remains to show that g is L_1-bilipschitz for some L_1 depending only on the quadruple

$$v = (H, K, L, n).$$

Since $g|X = f|X$ is L-bilipschitz, it suffices to prove that $g|U$ is locally L_1-bilipschitz. From (3) and (2.7) we obtain

$$(HH_1)^{-1} \leqq l_Q(x, g) \leqq L_Q(x, g) \leqq HH_1,$$

cf. [TV3, 6.5]. Hence it suffices to prove that there is $C = C(v)$ such that

$$1/C \leqq \frac{d(g(x), fX)}{d(x, X)} \leqq C \tag{2.8}$$

for all $x \in U$, cf. [F, 2.2.7].

Case 1. $x \in E$. Pick $y \in X$ such that $|y-x| = d(x, X) = r$. By Fact 3, we can choose $z \in X$ with $r/Mq \leqq |z-y| \leqq Mqr$. Since f is η − QS,

$$\frac{|f(x)-f(y)|}{|f(z)-f(y)|} \leqq \eta(Mq).$$

Since $f(x) = g(x)$, this implies

$$d(g(x), fX) \leqq \eta(Mq)|f(z)-f(y)| \leqq \eta(Mq)LMqr,$$

which proves the second inequality of (2.8) with $C = \eta(Mq)LMq$.

To prove the first inequality, we choose $y', z' \in X$ such that $|f(x)-f(y')| = d(f(x), fX) = r'$ and $|x-y'|/Mq \leqq |z'-y'| \leqq Mq|x-y'|$. Then $r'\eta(Mq) \geqq |f(z') - f(y')| \geqq |x-y'|/LMq \geqq d(x, X)/LMq$.

Case 2. $x \notin E$. Now there is $A = A(y, a, b) \in \mathscr{A}_0$ with $x \in A^*$. Assume again $y = 0$, $f(0) = 0$. We first show that there is $M_3 = M_3(v)$ such that

$$\frac{1}{M_3} \leqq \frac{|g(x)|}{|x|} \leqq M_3. \tag{2.9}$$

Since g is KK_1 − QC, it is η_1 − QS with η_1 depending only on v. Choose j such that the closure of the annulus $A_j = A(0, s_{j-1}, s_j)$ contains x. If $s_j < b/q$, we set $x' = g^{-1}(s_j e_1)$ and have $|x'| = s_j$. Since $|x'|/M \leqq |x| \leqq |x'|$, we obtain

$$|g(x)| \leqq \eta_1(1)s_j \leqq \eta_1(1)M|x|,$$
$$|g(x)| \geqq s_j/\eta_1(M) \geqq |x|/\eta_1(M).$$

If $s_j = b/q$, we set $x' = g^{-1}(s_{j-1}e_1)$. Now $|x'| = s_{j-1}$ and $|x'| \leqq |x| \leqq M^2|x'|$, and we obtain

$$|g(x)| \leqq \eta_1(M^2)|x|, \quad |g(x)| \geqq |x|/\eta_1(1)M^2.$$

Hence (2.9) is true with $M_3 = M^2\eta_1(M^2)$.

Since

$$d(g(x), fX) \leqq |g(x)|, \quad d(x, X) \geqq |x| - a \geqq |x|/2,$$

(2.9) gives the second inequality of (2.8) with $C = 2M_3$.

To prove the first inequality, we first show that

$$d(g(x), fX) = d(g(x), f[X \cap B_A]). \tag{2.10}$$

If $y_1 \in X \backslash B_A$, then $|y_1| \geqq b \geqq q|x|$. By (2.4), this implies

$$|f(x)| \leqq \eta(1/q)|f(y_1)| \leqq |f(y_1)|/2.$$

Since $fA^* = gA^*$ and since $0 \in fX$, this proves (2.10).

If $z \in X \cap B_A$, $|x| \geqq 2|z|$, and hence $|x| \leqq 2|z-x|$. Since g is η_1 − QS, this implies $|g(x)| \leqq \eta_1(2)|g(z)-g(x)|$. Hence

$$\frac{d(x, X)}{d(g(x), fX)} \leqq \frac{\eta_1(2)|x|}{|g(x)|} \leqq \eta_1(2)M_3.$$

This completes the proof of Lemma 2.3. □

2.11. Lemma. *Let D be a proper subdomain of R^n, and let $x, y \in D$ be such that $k_D(x, y) \leqq 1/3$. Then $|x-y| \leqq d(x, \partial D)/2$.*

Proof. Assume $|x-y| > d(x, \partial D)/2 = r/2$. Let α be a QH geodesic joining x to y. It has a proper subpath β joining x to $S(x, r/2)$ in $B(x, r/2)$. For $z \in |\beta|$ we have $d(z, \partial D) \leqq 3r/2$. Hence

$$k_D(x, y) > \int_\beta \frac{|dz|}{d(z, \partial D)} \geqq \frac{2}{3r} \int_\beta |dz| \geqq 1/3,$$

which is a contradiction. □

2.12. Theorem. *Suppose that X is a closed set in R^n, $n \neq 4$, and that $f: R^n \to R^n$ is a $K-$QC map such that $f|X$ is L-bilipschitz. Then there is an L_1-bilipschitz map $g: R^n \to R^n$ such that*

(1) $g|X = f|X$,

(2) $gD = fD$ *for each component D of $R^n \setminus X$,*

(3) L_1 *depends only on K, L and n.*

Proof. We may assume $X \neq \emptyset$. For every component D of $U = R^n \setminus X$ we apply [TV3, 7.12] and find a homeomorphism h_D: $D \to fD$ such that

(i) $k_{fD}(h_D, f|D) \leqq 1/3$,

(ii) h_D is $H-$QH with $H = H(K, n)$,

(iii) h_D is $H^{2n-2}-$QC.

Define $h: R^n \to R^n$ by $h|X = f|X$ and by $h|D = h_D$ for every component D of U. Clearly h is bijective. Moreover, $h|X$ and $h|U$ are continuous. If $x \in X$ and $y \in D$, (i) and 2.11 give

$$|h(y) - h(x)| \leqq |h_D(y) - f(y)| + |f(y) - f(x)| \leqq 3|f(y) - f(x)|/2.$$

Consequently, h is continuous and hence a homeomorphism. Similarly we obtain

$$|h(y) - h(x)| \geqq |f(y) - f(x)|/2.$$

Hence the linear dilatations of f and h satisfy the inequality $H(x, h) \leqq 3H(x, f)$ for all $x \in X$. By (iii), this implies that h is K_1-QC with K_1 depending on K and n. The theorem follows now from Lemma 2.3 with the substitution $f \mapsto h$. □

2.13. Remark. As a special case, Theorem 2.12 gives the results on QC and bilipschitz reflections, mentioned in the introduction.

2.14. We want to apply Theorem 2.12 to extension problems. For that purpose, we introduce the following notion: A closed set $X \subset R^n$ is said to be QS *extendable* if every $\eta-$QS homeomorphism f: $X \to X$ has a $K-$QC extension to R^n with K depending only on η, X and n. The correspondence $(\eta, n) \mapsto K$ is the *extendability function* of X.

The following result is obvious:

2.15. Lemma. *If $X \subset R^n$ is* QS *extendable and if $f: R^n \to R^n$ is* QC, *then fX is* QS *extendable.* □

2.16. Lemma. *Let $u: S^p \to \bar{R}^p$ be the stereographic projection with $u(e_{p+1}) = \infty$, and let $f: S^p \to S^p$ be $\eta-$QS with $f(e_{p+1}) = e_{p+1}$. Then the map $f_1: R^p \to R^p$, defined by ufu^{-1}, is $\theta-$QS with θ depending only on η.*

Proof. The most natural proof is based on *quasimöbius* maps, whose theory will appear elsewhere [V4]. An embedding $g: X \to Y$ is called θ-quasimöbius if $\theta: R^1_+ \to R^1_+$ is a homeomorphism and if for each quadruple a, b, c, d of distinct points in X, their cross ratio $\tau = |a, b, c, d|$ and the image cross ratio $\tau' = |f(a), f(b), f(c), f(d)|$ satisfy the inequality $\tau' \leqq \theta(\tau)$. Each $\eta-$QS map is θ-quasimöbius with $\theta = \theta_\eta$. Since u is möbius, ufu^{-1}: $\bar{R}^p \to \bar{R}^p$ is θ-quasimöbius. Since it fixes ∞, f_1 is $\theta-$QS.

For $p \geqq 2$, an alternative proof for a slightly weaker result can be based on the fact that u is conformal. Indeed, this implies that f_1 is locally $K-$QC and hence $K-$QC with $K = \eta(1)^{p-1}$. This gives the result with θ depending on η and p, which would be sufficient to our purposes. However, this argument is not valid for $p = 1$. A direct proof seems rather awkward, cf. [TV4, 5.22]. □

2.17. Lemma. *Suppose that $f: \bar{B}^n \to \bar{B}^n$ is a homeomorphism such that $f|B^n$ is $K-$QC and $f|S^{n-1}$ is $\eta-$QS. Then $|f(0)| \leqq \varphi(\eta, K, n) < 1$.*

Proof. Suppose that the lemma is false. Then there is a sequence of homeomorphisms $f_j: \bar{B}^n \to \bar{B}^n$ such that $|f_j(0)| \to 1$ and such that $f_j|B^n$ is $K-$QC and $f_j|S^{n-1}$ is $\eta-$QS for every j. Applying [TV1, 3.5 and 3.6] and passing to a subsequence we may assume that $f_j|S^{n-1}$ converge uniformly to an $\eta-$QS homeomorphism $h: S^{n-1} \to S^{n-1}$. Extend each f_j by reflection to a $K-$QC map $g_j: \bar{R}^n \to \bar{R}^n$. If a_1, a_2, a_3 are distinct points of S^{n-1}, the distances $|f_j(a_i) - f_j(a_k)|$, $i \neq k$, are bounded away from zero. Applying [V1, 19.4 and 21.1] and passing again to a subsequence, we may assume that g_j converge to a homeomorphism $g: \bar{R}^n \to \bar{R}^n$. Since $|f_j(0)| \to 1$ and since $f_j|S^{n-1} \to h$, this leads to a contradiction. □

2.18. Theorem. *The sets R^p and S^p are* QS *extendable in R^n for $p \leqq n-1$. The sets R^p_+ and $\bar{B}^p$ are* QS *extendable in R^n for $p \leqq n$.*

Proof. (a) $X = R^p$. This was proved in [TV2]; the cases $n = 2, 3, 4$ had been proved earlier by Beurling-Ahlfors, Ahlfors, and Carleson, respectively.

(b) $X = S^p$. This case was sketched in [TV2, 3.15.4]. Indeed, with the aid of 2.16 and 2.17 it easily reduces to the first case.

(c) $X = R^p_+$. By [TV2], we may assume $p = n$. Let $f: R^n_+ \to R^n_+$ be $\eta-$QS. Then f is $K-$QC with $K = \eta(1)^{n-1}$. By reflection, we can extend f to a $K-$QC map $F: R^n \to R^n$.

(d) $X = \bar{B}^p$. We may again assume $p = n$. Let $f: \bar{B}^n \to \bar{B}^n$ be $\eta-$QS. Then

$$\frac{1 + |f(0)|}{1 - |f(0)|} \leqq \eta(1),$$

which implies $|f(0)| \leqq 1 - 1/\eta(1)$. Hence there is a K_1-QC map $g: R^n \to R^n$ such that $g(f(0)) = 0$, $g|R^n \setminus B^n = \mathrm{id}$, and $K_1 = K_1(\eta)$. Since f is $K-$QC with $K = \eta(1)^{n-1}$, we can extend gf by reflection to a KK_1-QC map $F: R^n \to R^n$. The desired extension of f is then the KK_1^2-QC map $g^{-1}F$. □

2.19. Theorem. *Suppose that X is QS extendable in R^n, that $\varphi_1, \varphi_2: R^n \to R^n$ are $K-$QC maps, and that $f: \varphi_1 X \to \varphi_2 X$ is a homeomorphism.*

(1) *If f is $\eta-$QS, then f can be extended to a K_1-QC map $F: R^n \to R^n$ with K_1 depending only on K, η, n and on the extendability function of X.*

(2) *If $n \neq 4$ and if is L-bilipschitz, then f can be extended to an L_1-bilipschitz map $F: R^n \to R^n$ with L_1 depending only on K, L, n and on the extendability function of X.*

Proof. (1) The first part is an almost direct consequence of the definitions. Since φ_1, φ_2 are $K-$QC, they are η_1-QS for η_1 depending only on K and n. Hence $g = \varphi_2^{-1} f \varphi_1 | X$ is an η_2-QS homeomorphism onto X with η_2 depending only on K, η and n. Choose a K_1-QC extension $G: R^n \to R^n$ of g given by the definition of QS extendability. Then the $K^2 K_1-$QC map $F = \varphi_2 G \varphi_1^{-1}$ is the desired extension of f.

(2) Since a bilipschitz map is QS, part (1) gives a K_1-QC extension $F_1: R^n \to R^n$ of f. The desired L_1-bilipschitz extension is given by Theorem 2.12. □

2.20. Remarks. 1. Suppose that X and φ_1 are as in 2.19 and that $f: \varphi_1 X \to R^n$ is bilipschitz, $n \neq 4$. Then we see by the second part of Theorem 2.19 that f extends to a bilipschitz map of R^n if and only if $f \varphi_1 X$ is the image of X under some QC map of R^n.

In some cases this condition is automatically satisfied. For example, let $n=2$ and let X be R^1 or S^1. If $g: X \to R^2$ is QS, then gX is easily seen to be of bounded turning [LV, II.8.7], and hence gX is the image of X under a QC map of R^2 [LV, Theorem II.8.6]. Theorem 2.19 gives then Corollary 2 of [G2, p. 218].

If $n=3$ and $X=S^1$, this condition cannot be dropped, since $f \varphi_1 X$ could be knotted. Or, if $X = S^2$, $f \varphi_1 X$ could be the Fox-Artin wild sphere [G1, Theorem 3].

2. A result related to the case $X = R^{n-1}$ of 2.19. (2) has recently been given by Latfullin [L].

3. Flattening

3.1. We shall apply the theory of the preceding section to the flattening theory of lipschitz manifolds. We thank Jouni Luukkainen for pointing out this application and for the permission to publish it in this paper.

A *lipschitz* (or LIP) *n-manifold* is a separable metric space M such that each point $x \in M$ has a neighborhood bilipschitz homeomorphic to B^n or B^n_+. A LIP submanifold of M is a subset which is a LIP manifold in the induced metric.

Let C be one of the categories TOP, LQS or LIP. Let M be a LIP manifold, and let $Q \subset \operatorname{int} M$ be a LIP p-submanifold. Then Q is said to be locally C-*flat* at $x \in Q$ if x has a neighborhood U such that $(U, U \cap Q)$ is C-homeomorphic to (B^n, B^p) or to (B^n, B^p_+).

The Fox-Artin arc and the Fox-Artin sphere [R, p. 63 and p. 67], if constructed in a regular manner, give examples of LIP 1-manifolds and LIP 2-manifolds in R^3

which are not even locally TOP flat. There are also LIP arcs in R^3 which are locally TOP flat but not locally LIP flat [T, p. 68]. Our next result shows that such an arc cannot be locally LQS flat.

Incidentally, we know of no examples of LIP submanifolds which are not locally LIP flat in codimensions greater than two.

3.2. Theorem. *Suppose that* $n \neq 4$, *that* M *is a* LIP *n-manifold, and that* $Q \subset \operatorname{int} M$ *is a* LIP *submanifold. If* Q *is locally* LQS *flat at* $x \in Q$, Q *is locally* LIP *flat at* x.

Proof. We prove the case where x is a boundary point of Q; the case $x \in \operatorname{int} Q$ is similar but slightly easier.

Since the question is local, we may assume that $M = R^n$. Choose a neighborhood E of x in Q and a bilipschitz homeomorphism $f: E \to \bar{B}^p_+$ with $f(x) = 0$. Then choose a neighborhood V of x in R^n and a QS homeomorphism $h: (V, V \cap Q) \to (2B^n, 2B^p_+)$ such that $h(x) = 0$ and $V \cap Q \subset E$. By the QC Schoenflies theorem (see [V1, 41.3]), we can extend $h^{-1}|\bar{B}^n$ to a QC map $\varphi_1: R^n \to R^n$. The map fh^{-1} defines a QS embedding $\psi: 2B^p_+ \to B^p_+$ with $\psi[2B^{p-1}] \subset B^{p-1}$. We extend ψ by reflection to a QC embedding $\psi_1: 2B^p \to B^p$, and then $\psi_1|\bar{B}^p$ by the QC Schoenflies theorem and by [TV2] to a QC map $\varphi_2: R^n \to R^n$. By 2.15 and 2.18, $\bar{B}^p_+$ is QS extendable in R^n. Since f defines a bilipschitz homeomorphism $u: \varphi_1 \bar{B}^p_+ \to \varphi_2 \bar{B}^p_+$, it follows from 2.19 that u has a bilipschitz extension $F: R^n \to R^n$.

The set $W = F\varphi_1 B^n$ is a neighborhood of 0 in R^n. Choose a ball $rB^n \subset W$ such that $rB^p_+ \subset F\varphi_1 B^p_+$. Then F^{-1} defines a bilipschitz embedding $w: rB^n \to R^n$ with $w(0) = x$ and $w^{-1}Q = rB^p_+$. □

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