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A Brun-Titchmarsh theorem for multiplicative functions

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1. Introduction

Let $d(n)$ be the divisor function, and let a and k be integers satisfying

$$(1.1) \quad 0 < a < k, \quad (a, k) = 1.$$

In 1957 Linnik and Vinogradov [7] proved that if $0 < \alpha < \frac{1}{2}$, then, as $x \rightarrow \infty$,

$$(1.2) \quad \sum_{\substack{n \leq x \\ n \equiv a \pmod{k}}} d(n) \ll \frac{\phi(k)}{k^2} x \log x$$

uniformly in a and k , provided only that $k < x^{1-\alpha}$. Here $\phi(k)$ is Euler's function and the implied constant depends only on α . Their proof reduces to estimating the number $\Psi(x, y)$ of positive integers not exceeding x having no prime factor greater than y . Unfortunately they made use of a uniform upper estimate of $\Psi(x, y)$ by A. I. Vinogradov which is incorrect (see Norton [9]). Later Linnik [6] stated the generalisation of the problem to the functions

$$d_r^l(n) = (d_r(n))^l, \quad r = 2, 3, \dots; \quad l = 1, 2, \dots$$

where $d_r(n)$ is the number of ways of writing n as a product of r factors, taking account of ordering. The expected result is that, as $x \rightarrow \infty$,

$$(1.3) \quad \sum_{\substack{n \leq x \\ n \equiv a \pmod{k}}} d_r^l(n) \ll \frac{x}{k} \left(\frac{\phi(k)}{k} \log x \right)^{r^{l-1}}$$

uniformly in $k < x^{1-\alpha}$ with the implied constant depending on r, l and α . This result has been used by others (see, for example, [8]) to tackle a variety of problems.

The merit of these results centers on the range of uniformity for the modulus k being large. Indeed, in the shorter range $k < x^{2/3-\alpha}$, Selberg (unpublished manuscript) has obtained even an asymptotic formula for the sum in (1.2) (see also Hooley [4])

and Heath-Brown [3]). The situation is similar to that in estimating $\pi(x; k, a)$, the number of primes $p \leq x$ such that $p \equiv a \pmod{k}$; the Brun-Titchmarsh inequality

$$\pi(x; k, a) \ll \frac{x}{\phi(k) \log \frac{x}{k}},$$

valid uniformly in $k < x$, is often used to supplement the asymptotic formula for $\pi(x; k, a)$ given by the Siegel-Walfisz theorem, which is known to be valid only for a much shorter range of k . Indeed, the Brun-Titchmarsh inequality is sometimes used to supplement even Bombieri's theorem; for example, in the solution to the Titchmarsh-Linnik divisor problem (see [2]).

We remark however that, unlike the Brun-Titchmarsh inequality, (1.2) cannot be extended all the way to $k < x$. To see this, we choose k to be the largest prime less than x ; then the right hand side of (1.2) is of order $\log x$, whereas we can choose $a < k$ such that $d(a) > \log^2 x$.

In 1971 Wolke [12] applied the method of Erdős [1]¹⁾ to study the sum $\sum_{n \leq x} f(a_n)$ where f is a non-negative multiplicative function satisfying the condition that there exist positive constants C_1 and C_2 such that for all primes p and all $l \geq 1$,

$$(1.4) \quad f(p^l) \leq C_1 l^{C_2};$$

and (a_n) is a strictly increasing sequence of positive integers. However, the results they obtained are not uniform with respect to any given class of sequences (a_n) , such as the class of arithmetic progressions. Nevertheless, using their method, we can now give the generalisation of the Brun-Titchmarsh problem for a class of non-negative multiplicative functions f which satisfy conditions weaker than (1.4), and in a short interval $x - y < n \leq x$.

I would like to thank Professor H. Halberstam who suggested the problem to me in the first place and who has given me much valuable advice and criticism.

2. The main theorem

We shall consider the class M of arithmetic functions f which are non-negative, multiplicative and satisfy the following two conditions:

- (i) There exists a positive constant A_1 such that

$$f(p^l) \leq A_1^l, \quad p \text{ prime}, \quad l \geq 1.$$

- (ii) For every $\varepsilon > 0$, there exists a positive constant $A_2 = A_2(\varepsilon)$ such that

$$f(n) \leq A_2 n^\varepsilon, \quad n \geq 1.$$

¹⁾ Actually [1] antedates [7] and contains a more delicate argument.

The following is our main theorem.

Theorem 1. Let $f \in M$, $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{1}{2}$ and let a, k be integers satisfying (1. 1). Then, as $x \rightarrow \infty$,

$$(2. 1) \quad \sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k}}} f(n) \ll \frac{y}{\phi(k)} \frac{1}{\log x} \exp \left(\sum_{\substack{p \leq x \\ p \nmid k}} \frac{f(p)}{p} \right),$$

uniformly in a, k and y provided that

$$(2. 2) \quad k < y^{1-\alpha}, \quad x^\beta < y \leq x.$$

We make the following remarks on our conditions (i) and (ii). First, the condition (1. 4) clearly implies both (i) and (ii), and it is easy to construct a function $f \in M$ which does not satisfy (1. 4). Secondly, if $f \in M$ and $\delta > 0$, then there exists $A_3 = A_3(\delta)$ such that

$$(2. 3) \quad \sum_p \sum_{l=2}^{\infty} \frac{f(p^l)}{p^{l\delta}} \leq A_3.$$

Finally we point out that the rather strong condition (ii) is actually vital to the truth of the theorem, and, in particular, we cannot replace it by (2. 3). For suppose that (ii) does not hold; then there exist $\varepsilon_0 > 0$ and a strictly increasing sequence of positive integers (n_r) such that $f(n_r) > n_r^{\varepsilon_0}$. We can now show that (2. 1) does not hold by setting $\alpha = \frac{1}{10} \min(\varepsilon_0, 1)$. We define a, k, x and y as follows: For large r , we put

$$a = n_r, \quad k = \text{the least prime in the interval } (a, 2a), \quad x = y = k^{\frac{1}{1-\alpha}} + 1.$$

Then the left hand side of (2. 1) is at least $f(a) > a^{\varepsilon_0}$, whereas the right hand side is at most

$$\frac{y}{\phi(k)} y^\alpha = \frac{y^{1+\alpha}}{k-1} \ll k^{\frac{1+\alpha}{1-\alpha}-1} \ll k^{3\alpha} \ll a^{3\alpha} < a^{3\varepsilon_0/10}.$$

We note, in particular, that Theorem 1 cannot be extended to cover the case when $f(n) = 2^{\Omega(n)}$. We shall give some examples in which Theorem 1 is applicable in section 6.

3. Notations

The letters $a, b, d, h, i, j, k, l, m, n, r$ and s are used to denote positive integers and we shall assume that (1. 1) always holds. The letters p and q are reserved for prime numbers. We let $p(n)$ and $q(n)$ denote the greatest and least prime factors of n respectively, while $\Omega(n)$ is the number of prime factors of n , taking account of multiplicity, and $p(1) = q(1) = \Omega(1) = 1$.

The letters $x, y, z, \alpha, \beta, \delta$ and ε denote positive real numbers and λ a real number. We put

$$\Psi(x, z) = \sum_{\substack{n \leq x \\ p(n) \leq z}} 1, \quad \text{and} \quad \Phi(x, y, z; k, a) = \sum_{\substack{x-y < n < x \\ n \equiv a \pmod{k} \\ q(n) > z}} 1.$$

All the constants implied by the $\ll$ and O -notations depend at most on α, β and f .

4. Preliminary lemmas

We shall require the following lemmas.

Lemma 1. *For all sufficiently large x , we have*

$$\Psi(x, \log x \log \log x) \leq \exp \left\{ \frac{3 \log x}{(\log \log x)^{\frac{1}{2}}} \right\}.$$

Proof. As a consequence of the prime number theorem, we have, for all large y , that

$$(4.1) \quad \sum_{p \leq y} \frac{1}{\log p} \leq \frac{2y}{\log^2 y}.$$

We now use Rankin's method (see [10]). Let $\delta > 0$. We have, for large y ,

$$\begin{aligned} \Psi(x, y) &= \sum_{\substack{n \leq x \\ p(n) \leq y}} 1 \leq x^\delta \sum_{\substack{n \leq x \\ p(n) \leq y}} \frac{1}{n^\delta} \leq x^\delta \sum_{\substack{n \geq 1 \\ p(n) \leq y}} \frac{1}{n^\delta} = x^\delta \prod_{p \leq y} \left(1 + \frac{1}{p^\delta} + \frac{1}{p^{2\delta}} + \cdots \right) \\ &= x^\delta \prod_{p \leq y} \left(1 + \frac{1}{p^\delta - 1} \right) \leq \exp \left(\delta \log x + \sum_{p \leq y} \frac{1}{p^\delta - 1} \right) \\ &\leq \exp \left(\delta \log x + \frac{1}{\delta} \sum_{p \leq y} \frac{1}{\log p} \right) \leq \exp \left(\delta \log x + \frac{2y}{\delta \log^2 y} \right) \end{aligned}$$

by (4.1). Now put $y = \log x \log \log x$ and set $\delta = (\log \log x)^{-\frac{1}{2}}$ and we see that the result follows.

Lemma 2. *Let a and k satisfy (1.1) and suppose that $k < y \leq x$ and $z \geq 2$. Then*

$$\Phi(x, y, z; k, a) \leq \frac{y}{\phi(k) \log z} + z^2.$$

This can be proved using the simplest Selberg upper bound sieve method; see [2], p. 104.

Lemma 3. *Let $f \in M$. Then as $x \rightarrow \infty$,*

$$\sum_{\substack{n \leq x \\ (n, k) = 1}} \frac{f(n)}{n} \ll \exp \left(\sum_{\substack{p \leq x \\ p \nmid k}} \frac{f(p)}{p} \right).$$

uniformly in k .

Proof. Since $f \in M$, we have, by (2.3), that

$$\begin{aligned} \sum_{\substack{n \leq x \\ (n, k) = 1}} \frac{f(n)}{n} &\leq \prod_{\substack{p \leq x \\ p \nmid k}} \left(1 + \frac{f(p)}{p} + \sum_{l=2}^{\infty} \frac{f(p^l)}{p^l} \right) \leq \exp \left\{ \sum_{\substack{p \leq x \\ p \nmid k}} \left(\frac{f(p)}{p} + \sum_{l=2}^{\infty} \frac{f(p^l)}{p^l} \right) \right\} \\ &= \exp \left\{ \sum_{\substack{p \leq x \\ p \nmid k}} \frac{f(p)}{p} + O(1) \right\} \ll \exp \left(\sum_{\substack{p \leq x \\ p \nmid k}} \frac{f(p)}{p} \right). \end{aligned}$$

Lemma 4. Let $f \in M$. Then, as $z \rightarrow \infty$,

$$\sum_{\substack{n \geq z^{\frac{1}{2}} \\ p(n) \leq z^{\frac{1}{r}} \\ (n, k) = 1}} \frac{f(n)}{n} \ll \exp \left(\sum_{\substack{p \leq z \\ p \nmid k}} \frac{f(p)}{p} - \frac{1}{10} r \log r \right)$$

uniformly in k and r , provided that $1 \leq r \leq \frac{\log z}{\log \log z}$.

This lemma corresponds to Lemma 3 in [12] except that we generalise it to incorporate the condition $(n, k) = 1$, and replace the implicit constant for the coefficient of $r \log r$ by an absolute constant. Since it is crucial for our application that the constants involved are independent of k , and our class M is larger than Wolke's, we shall give the detailed proof here.

Proof. Again we use Rankin's method. Let $\frac{3}{4} \leq \delta \leq 1$. We have, by (2. 3), that

$$\begin{aligned} \sum_{\substack{n \geq x \\ p(n) \leq y \\ (n, k) = 1}} \frac{f(n)}{n} &\leq x^{\delta-1} \sum_{\substack{n \geq x \\ p(n) \leq y \\ (n, k) = 1}} \frac{f(n)}{n^{\delta}} \leq x^{\delta-1} \sum_{\substack{n \geq 1 \\ p(n) \leq y \\ (n, k) = 1}} \frac{f(n)}{n^{\delta}} \\ &= x^{\delta-1} \prod_{\substack{p \leq y \\ p \nmid k}} \left(1 + \frac{f(p)}{p^{\delta}} + \sum_{l=2}^{\infty} \frac{f(p^l)}{p^{l\delta}} \right) \leq \exp \left\{ (\delta-1) \log x + \sum_{\substack{p \leq y \\ p \nmid k}} \frac{f(p)}{p^{\delta}} + O(1) \right\}. \end{aligned}$$

Now $\frac{f(p)}{p^{\delta}} = \frac{f(p)}{p} + \frac{f(p)}{p} (p^{1-\delta} - 1)$, and, by condition (i),

$$\begin{aligned} \sum_{p \leq y} \frac{f(p)}{p} (p^{1-\delta} - 1) &\leq \sum_{p \leq y} \frac{A_1}{p} \sum_{n=1}^{\infty} \frac{((1-\delta) \log p)^n}{n!} \leq A_1 \sum_{n=1}^{\infty} \frac{(1-\delta)^n \log^{n-1} y}{n!} \sum_{p \leq y} \frac{\log p}{p} \\ &\leq 2 A_1 \sum_{n=1}^{\infty} \frac{((1-\delta) \log y)^n}{n!} \leq 2 A_1 \exp \{ (1-\delta) \log y \} = 2 A_1 y^{1-\delta}. \end{aligned}$$

Therefore we have

$$\sum_{\substack{n \geq x \\ p(n) \leq y \\ (n, k) = 1}} \frac{f(n)}{n} \ll \exp \left\{ (\delta-1) \log x + \sum_{\substack{p \leq y \\ p \nmid k}} \frac{f(p)}{p} + 2 A_1 y^{1-\delta} \right\}.$$

Now put $x = z^{\frac{1}{2}}$, $y = z^{\frac{1}{r}}$ and set $\delta = 1 - \frac{r \log r}{4 \log z}$. Note that if

$$1 \leq r \leq \log z / \log \log z, \text{ then } r \log r < \log z \text{ and so } \frac{3}{4} < \delta \leq 1.$$

Moreover, we now have

$$(1-\delta) \log x = \frac{r \log r}{4 \log z} \cdot \frac{1}{2} \log z = \frac{1}{8} r \log r$$

and

$$y^{1-\delta} = z^{\frac{1-\delta}{r}} = z^{\frac{\log r}{4 \log z}} = r^{\frac{1}{4}}.$$

We now have

$$\sum_{\substack{n \geq z^{\frac{1}{2}} \\ p(n) \leq z^{\frac{1}{r}} \\ (n, k) = 1}} \frac{f(n)}{n} \ll \exp \left(\sum_{\substack{p \leq z^{\frac{1}{r}} \\ p \nmid k}} \frac{f(p)}{p} - \frac{1}{8} r \log r + 2 A_1 r^{\frac{1}{4}} \right)$$

and the required result follows at once.

5. Proof of the main theorem

Let k and y satisfy (2. 2) and put

$$(5. 1) \quad z = y^{\frac{\alpha}{10}}.$$

For each n satisfying $x - y < n \leq x$, $n \equiv a \pmod{k}$ we express n in the form

$$n = p_1^{s_1} \cdots p_j^{s_j} p_{j+1}^{s_{j+1}} \cdots p_l^{s_l} = b_n d_n, \quad (p_1 < p_2 < \cdots < p_l),$$

where b_n is chosen so that

$$(5. 2) \quad b_n \leq z < b_n p_{j+1}^{s_{j+1}}.$$

We divide the set of such integers n into the following classes:

$$\text{I: } q(d_n) > z^{\frac{1}{2}};$$

$$\text{II: } q(d_n) \leq z^{\frac{1}{2}}, \quad b_n \leq z^{\frac{1}{2}};$$

$$\text{III: } q(d_n) \leq \log x \log \log x, \quad b_n > z^{\frac{1}{2}};$$

$$\text{IV: } \log x \log \log x < q(d_n) \leq z^{\frac{1}{2}}, \quad b_n > z^{\frac{1}{2}}.$$

First we have

$$\sum_{n \in \mathbf{I}} f(n) = \sum_{n \in \mathbf{I}} f(b_n) f(d_n) \leq \sum_{b \leq z} f(b) \sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k} \\ n \equiv 0 \pmod{b} \\ q\left(\frac{n}{b}\right) > z^{\frac{1}{2}}}} f\left(\frac{n}{b}\right) = \sum_{\substack{b \leq z \\ (b, k) = 1}} f(b) \sum_{\substack{\frac{x}{b} - \frac{y}{b} < d \leq \frac{x}{b} \\ d \equiv a' \pmod{k} \\ q(d) > z^{\frac{1}{2}}}} f(d),$$

where $a' \equiv a\bar{b}$, $b\bar{b} \equiv 1 \pmod{k}$. From (5. 1) we have $q(d) > z^{\frac{1}{2}} = y^{\frac{\alpha}{20}} > x^{\frac{\alpha\beta}{20}}$, so that $\Omega(d) \leq \frac{\log x}{\log q(d)} < \frac{20}{\alpha\beta}$, and hence, by condition (i) $f(d) \leq A_1^{\Omega(d)} \leq A_1^{\frac{20}{\alpha\beta}}$. We have therefore

$$\sum_{n \in \mathbf{I}} f(n) \ll \sum_{\substack{b \leq z \\ (b, k) = 1}} f(b) \Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{2}}; k, a'\right).$$

Since $k < y^{1-\alpha}$ and $b \leq z < y^\alpha$, so that $kb < y$, it follows from Lemma 2 that $\Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{2}}; k, a'\right) \leq \frac{2y}{\phi(k)b \log z} + z$, and therefore

$$\sum_{n \in \mathbf{I}} f(n) \ll \left\{ \frac{y}{\phi(k) \log z} + z^2 \right\} \sum_{\substack{b \leq z \\ (b, k) = 1}} \frac{f(b)}{b}.$$

From Lemma 3 we arrive at

$$(5.3) \quad \sum_{n \in \mathbf{I}} f(n) \ll \left\{ \frac{y}{\phi(k) \log z} + z^2 \right\} \exp \left(\sum_{\substack{p \leq z \\ p \nmid k}} \frac{f(p)}{p} \right).$$

Next, to each $n \in \mathbf{II}$, there correspond p and s such that $p^s \parallel n$, $p \leq z^{\frac{1}{2}}$ and by (5.2) $p^s > z^{\frac{1}{2}}$. Let s_p denote the least positive integer s satisfying $p^s > z^{\frac{1}{2}}$ so that $s_p \geq 2$ and hence $p^{-s_p} \leq \min(z^{-\frac{1}{2}}, p^{-2})$; thus $\sum_{p \leq z^{\frac{1}{2}}} \frac{1}{p^{s_p}} \leq \sum_{p \leq z^{\frac{1}{4}}} z^{-\frac{1}{2}} + \sum_{p > z^{\frac{1}{4}}} p^{-2} \ll z^{-\frac{1}{4}}$.

It now follows that

$$(5.4) \quad \sum_{n \in \mathbf{II}} 1 \leq \sum_{p \leq z^{\frac{1}{2}}} \sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k} \\ n \equiv 0 \pmod{p^{s_p}}}} 1 = \sum_{\substack{p \leq z^{\frac{1}{2}} \\ p \nmid k}} \left\{ \frac{y}{kp^{s_p}} + O(1) \right\} \ll \frac{y}{k} z^{-\frac{1}{4}} + z^{\frac{1}{2}}.$$

Suppose next that $n \in \mathbf{III}$. Then there exists b such that $b|n$, $z^{\frac{1}{2}} < b \leq z$, and $p(b) < \log x \log \log x$. Consequently we have

$$(5.5) \quad \begin{aligned} \sum_{n \in \mathbf{III}} 1 &\leq \sum_{\substack{z^{\frac{1}{2}} < b \leq z \\ p(b) < \log x \log \log x}} \sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k} \\ n \equiv 0 \pmod{b}}} 1 = \sum_{\substack{z^{\frac{1}{2}} < b \leq z \\ p(b) < \log x \log \log x \\ (b, k) = 1}} \left\{ \frac{y}{kb} + O(1) \right\} \\ &\leq \frac{y}{k} z^{-\frac{1}{2}} \Psi(z, \log x \log \log x) + O(z) \ll \frac{y}{k} z^{-\frac{1}{4}} + z \end{aligned}$$

by Lemma 1. Since $k < y^{1-\alpha}$ we have, by (5.1), $z < y^\alpha z^{-\frac{1}{4}} < \frac{y}{k} z^{-\frac{1}{4}}$, and so, from (5.4) and (5.5), we have

$$\sum_{n \in \mathbf{II}} 1 + \sum_{n \in \mathbf{III}} 1 \ll \frac{y}{k} z^{-\frac{1}{4}}.$$

For $y > x^\beta$ we have, by condition (ii) and (5.1), that

$$f(n) \ll n^{\frac{\alpha\beta}{80}} \leq x^{\frac{\alpha\beta}{80}} < y^{\frac{\alpha}{80}} = z^{\frac{1}{8}}$$

so that we arrive at

$$(5.6) \quad \sum_{n \in \mathbf{II}} f(n) + \sum_{n \in \mathbf{III}} f(n) \ll \frac{y}{k} z^{-\frac{1}{8}}.$$

Lastly we deal with the class IV. We have

$$\sum_{n \in \text{IV}} f(n) = \sum_{n \in \text{IV}} f(b_n) f(d_n) \leq \sum_{z^{\frac{1}{2}} < b \leq z} f(b) \sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k} \\ b_n = b, q(d_n) > p(b) \\ \log x \log \log x < q(d_n) \leq z^{\frac{1}{2}}}} f(d_n).$$

Let us put

$$(5.7) \quad r_0 = \left\lceil \frac{\log z}{\log(\log x \log \log x)} \right\rceil,$$

so that $\log x \log \log x > z^{\frac{1}{r_0+1}}$. Let $2 \leq r \leq r_0$ and consider those n for which $z^{\frac{1}{r+1}} < q(d_n) \leq z^{\frac{1}{r}}$.

For such n , we have $p(b_n) = p(b) < q(d_n) < z^{\frac{1}{r}}$ and moreover, as before,

$$\Omega(d_n) \leq \frac{\log x}{\log q(d_n)} \leq \frac{(r+1) \log x}{\log z} < \frac{10(r+1)}{\alpha\beta} < \frac{20r}{\alpha\beta}$$

so that $f(d_n) \leq A_1^{\Omega(d_n)} \leq A_5^r$, $A_5 = A_1^{\frac{20}{\alpha\beta}}$.

It follows that

$$\begin{aligned} \sum_{n \in \text{IV}} f(n) &\leq \sum_{2 \leq r \leq r_0} A_5^r \sum_{\substack{z^{\frac{1}{2}} < b \leq z \\ p(b) < z^{\frac{1}{r}}}} f(b) \sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k} \\ n \equiv 0 \pmod{b} \\ z^{\frac{1}{r+1}} < q\left(\frac{n}{b}\right) < z^{\frac{1}{r}}}} 1 \\ &\leq \sum_{2 \leq r \leq r_0} A_5^r \sum_{\substack{z^{\frac{1}{2}} < b \leq z \\ p(b) < z^{\frac{1}{r}} \\ (b, k) = 1}} f(b) \Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{r+1}}; k, a'\right) \end{aligned}$$

where $a' \equiv a\bar{b}$, $b\bar{b} \equiv 1 \pmod{k}$. From Lemma 2 we have

$$\Phi\left(\frac{x}{b}, \frac{y}{b}, z^{\frac{1}{r+1}}; k, a'\right) \leq \frac{y(r+1)}{\phi(k) b \log z} + z^{\frac{2}{r+1}},$$

and therefore

$$\sum_{n \in \text{IV}} f(n) \leq \left(\frac{y}{\phi(k) \log z} + z^2 \right) \sum_{2 \leq r \leq r_0} (r+1) A_5^r \sum_{\substack{z^{\frac{1}{2}} < b \leq z \\ p(b) < z^{\frac{1}{r}} \\ (b, k) = 1}} \frac{f(b)}{b}.$$

From (5.7) we see that we can apply Lemma 4 to the inner sum here giving

$$\begin{aligned} (5.8) \quad \sum_{n \in \text{IV}} f(n) &\ll \left(\frac{y}{\phi(k) \log z} + z^2 \right) \exp\left(\sum_{\substack{p \leq z \\ p \nmid k}} \frac{f(p)}{p} \right) \sum_{2 \leq r \leq r_0} r A_5^r \exp\left(-\frac{1}{10} r \log r \right) \\ &\ll \left(\frac{y}{\phi(k) \log z} + z^2 \right) \exp\left(\sum_{\substack{p \leq z \\ p \nmid k}} \frac{f(p)}{p} \right). \end{aligned}$$

For $k < y^{1-\alpha}$ we have, by (5. 1),

$$z^2 < \frac{k}{\phi(k)} \frac{z^3}{\log z} < \frac{y^{1-\alpha+3\alpha/10}}{\phi(k) \log z} < \frac{y}{\phi(k) \log z}.$$

From (5. 3) and (5. 8) we now have

$$\sum_{n \in \mathbf{I}} f(n) + \sum_{n \in \mathbf{IV}} f(n) \ll \frac{y}{\phi(k) \log z} \exp \left(\sum_{\substack{p \leq z \\ p \nmid k}} \frac{f(p)}{p} \right)$$

and this, together with (5. 6), gives the desired result (2. 1).

6. Applications of the main theorem

It is easy to see that the expected result (1. 3) is an immediate consequence of our main theorem; indeed we have:

Theorem 2. Let α, β, λ be real numbers and let a, k, r be integers. Suppose that

$$0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad r \geq 2, \quad 0 < a < k, \quad (a, k) = 1.$$

We have, as $x \rightarrow \infty$,

$$\sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k}}} d_r^\lambda(n) \ll \frac{y}{k} \left(\frac{\phi(k)}{k} \log x \right)^{r^\lambda - 1}$$

uniformly in a, k and y provided that

$$k < y^{1-\alpha}, \quad x^\beta < y \leq x.$$

Next, let $r(n)$ be the number of ways of writing n as a sum of two squares so that if $f(n) = r(n)/4$, then $f \in M$. Smith [11] has obtained an asymptotic formula for the sum $\sum_{\substack{n \leq x \\ n \equiv a \pmod{k}}} r(n)$ valid for $k < x^{2/3-\alpha}$. Here we have

Theorem 3. Let α, β, λ be real numbers and let a, k be integers. Suppose that

$$0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad 0 < a < k, \quad (a, k) = 1.$$

We have, as $x \rightarrow \infty$,

$$(6.1) \quad \sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k} \\ r(n) > 0}} r^\lambda(n) \ll \frac{y}{\phi(k)} \prod_{\substack{p|k \\ p \equiv 1 \pmod{4} \\ p > 2^\lambda}} \left(1 - \frac{2^\lambda}{p} \right) (\log x)^{2^\lambda - 1 - 1}$$

uniformly in a, k and y provided that

$$k < y^{1-\alpha}, \quad x^\beta < y \leq x.$$

Note that if we put $\lambda = 0$, then (6. 1) becomes

$$\sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k} \\ r(n) > 0}} 1 \ll \frac{1}{\phi(k)} \prod_{\substack{p|k \\ p \equiv 1 \pmod{4}}} \left(1 - \frac{1}{p} \right) \cdot \frac{y}{\sqrt{\log x}}.$$

We call a positive integer s a square full integer if s is a product of squares and cubes, and let $\delta(n)$ be the number of square full divisors of n . Knopfmacher [5] has obtained an asymptotic formula for the sum $\sum_{\substack{n \leq x \\ n \equiv a \pmod{k}}} \delta(n)$, valid for $k < x^{7/36-\alpha}$. As our

last application of the main theorem, we have

Theorem 4. *Let α, β, λ be real numbers and a, k be integers. Suppose that*

$$0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad 0 < a < k, \quad (a, k) = 1.$$

We have, as $x \rightarrow \infty$,

$$\sum_{\substack{x-y < n \leq x \\ n \equiv a \pmod{k}}} \delta^\lambda(n) \ll \frac{y}{k}$$

uniformly in a, k and y provided that

$$k < y^{1-\alpha}, \quad x^\beta < y \leq x.$$

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