Notes on random functions.

By

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1. The introduction of the notion of the random into analysis is in the first instance the work of Borel[1]). His theory of *probabilités dénombrables* concerns itself with quantities depending upon an infinite sequence of choices and with their average values. In the simplest case, the choices in question are between two alternatives, which may be taken as the signs + and —. Thus questions as to the probability of convergence of the series

$$(1.01) \qquad \sum_{n=0}^{\infty} \pm C_n$$

and of the distribution of its sums belong to this order of ideas. Such questions have been much discussed in the pages of Fundamenta Mathematicae and Studia Mathematica, and are associated with the names of Steinhaus and Rademacher, among others. To Steinhaus in particular is due the reduction of such questions to questions concerning the Lebesgue integral.

The next step forward in introducing the notion of probability into analysis consists in replacing numbers dependent on an infinite sequence of choices by functions dependent on an infinite sequence of choices. This step was taken by Paley and Zygmund[2]), who have developed the theory of the analytic properties shared by "almost all" functions of the form

$$(1.02) \qquad \sum_{n=0}^{\infty} \pm C_n f_n(x),$$

and the closely related problem of the properties shared by "almost all" functions of the form

$$(1.03) \qquad \sum_{n=0}^{\infty} C_n e^{2\pi i \vartheta_n} f_n(x),$$

[1]) Borel.

[2]) Paley and Zygmund (1), (2), (3) hereinafter referred to as PZ 1, 2, 3.

where the numbers ϑ_n vary independently in the interval (0, 1). In this they were systematically developing a method of Littlewood for the production of *Gegenbeispiele*. From this point of view, random functions present themselves as a somewhat sophisticated analytical tool, and may seem to belong rather to the pathology than to the natural history of functions of the real variable.

There is however a way in which functions with an element of randomness present themselves to the physicist. The Brownian movement, the motion of a suspended particle in a fluid in response to the irregularities of the molecular collisions to which it is subjected, is of so strikingly random a character that it does not even seem to possess a velocity. The physicist Perrin[3]) has commented on the striking way in which this motion suggests the non-differentiable continuous functions of Weierstrass, and Borel[4]) has commented on this in connection with his theory of denumerable probabilities. The components of the displacement of such a particle, regarded as functions of the time, are random functions in a sense differing considerably from that of Zygmund and Paley. Wiener[5]) has reduced the theory of such random functions to a mathematically definite form, also depending upon a Lebesgue integration. For the purposes of the present paper, we shall represent one of the components of the displacement of the moving particle in the Wiener theory by $\chi(\alpha, t)$, where t is the time and α the parameter on which Lebesgue integration is performd for the purpose of averaging over all functions and determining probabilities. Wiener's $\chi(\alpha, t)$ is then, as he shows, a continuous function of t for almost all values of α, defined for $0 \leqq \alpha \leqq 1$, $0 \leqq t \leqq 1$, and vanishing for $t = 0$.

The purpose of this paper is to bridge the gap between the PZ and the W theories, by proving for the W random functions theorems analogous to those proved in the PZ papers. The analogy in question takes two different forms, which are in a manner of speaking Fourier duals of one another. The simplest and most obvious correspondence between the two theories is that $\chi(\alpha, t)$ determines what is really the limit of a large number of choices of sign of small quantities, each choice being whether over a small period of time a particle subject to the Brownian motion will wander to the right or to the left. Thus formally

$$(1.04) \qquad \int_0^1 f(x, t)\, c(t)\, d_t \chi(\alpha, t)$$

3) E. Perrin (1)

4) E. Borel (1), (2).

5) N. Wiener (1).

is an analogue to (1.02). On the other hand, it may be established that if $\{\gamma_n(t)\}$ is a set of normal and orthogonal functions $(0, 1)$, the quantities

$$(1.05) \qquad \int_0^1 \gamma_n(t)\, d_t \chi(\alpha, t)$$

appropriately defined all have a Gaussian distribution with the same modulus, and are completely independent of one another. Thus

$$(1.06) \qquad \sum_{n=0}^{\infty} c_n f_n(x) \int_0^1 \gamma_n(t)\, d_t \chi(\alpha, t)$$

is an analogue of (1.02), and like it depends on an infinite sequence of like independent probabilities, with the difference that the probabilities of (1.02) are all-or-none probabilities such as arise in the tossing of a coin, whereas the probabilities of (1.06) are continuously distributed like those of the errors in shooting at a target.

It will turn out that by far the greater part of the PZ theory has a precise W analogue in both possible senses. The W theory is a little less adapted to the construction of Gegenbeispiele than the PZ theory, in that the absolute magnitude of the coefficients in (1.02) is determined in advance, while that in (1.06) itself depends on probability considerations. On the other hand, many averages concerning which the PZ theory only yields inequalities are expressible in closed form in the *W* theory.

If the finite range of t in $\chi(\alpha, t)$ is replaced by an infinite range, as is easily possible, the theory undergoes but little change. On the other hand, the Fourier series is replaced by the Fourier integral, and the symmetry of the latter makes the two analogues of the PZ theory reduce essentially to one. Furthermore, a new type of functional arises over the infinite range — the mean as contrasted with the integral, and the evaluation of the distribution of such means introduces new considerations. Under certain circumstances, such means will almost always assume a certain fixed value.

2. By W 1, if $t_0 = 0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq 1$,

$$(2.01) \qquad \int_0^1 F(\chi(\alpha, t_1), \chi(\alpha, t_2) - \chi(\alpha, t_1), \ldots, \chi(\alpha, t_n) - \chi(\alpha, t_{n-1}))\, d\alpha$$

$$= \pi^{-n/2} \prod_1^n (t_k - t_{k-1})^{-1/2} \int_{-\infty}^{\infty} du_1 \ldots \int_{-\infty}^{\infty} du_n$$

$$\exp. \left[- \sum_1^n \frac{u_k^2}{t_k - t_{k-1}} \right] F(u_1, \ldots, u_n).$$

From this it is easy to show that if $t_2 \geqq t_1$,

$$(2.02)\qquad \int_0^1 [\chi(\alpha, t_2) - \chi(\alpha, t_1)]^{2n}\, d\alpha = \frac{1\cdot 3\cdot 5 \ldots 2n-1}{2^n}(t_2 - t_1)^n;$$

$$\int_0^1 [\chi(\alpha, t_2) - \chi(\alpha, t_1)]^{2n+1}\, d\alpha = 0;$$

and that if none of the intervals $(t_{1,p}, t_{2,p})$ overlap,

$$(2.03)\qquad \int_0^1 \prod_{p=1}^{\nu} [\chi(\alpha, t_{2,p}) - \chi(\alpha, t_{1,p})]^{n_p}\, d\alpha$$

$$= \prod_{p=1}^{\nu} \int_0^1 [\chi(\alpha, t_{2,p}) - \chi(\alpha, t_{1,p})]^{n_p}\, d\alpha.$$

Since $1\cdot 3\cdot 5 \ldots (2n-1)$ represents the number of distinct ways of parcelling $2n$ objects into pairs, it follows from (2.02) and (2.03) that the integral in (2.03) may be evaluated by parcelling the product in the integrand into pairs in all possible ways, integrating the product of each pair, multiplying the integrals for all the pairs in each parcelling, and adding these products over all parcellings. From this it results immediately that

$$(2.04)\qquad \int_0^1 \chi(\alpha, t_1)\chi(\alpha, t_2) \ldots \chi(\alpha, t_{2n})\, d\alpha$$

$$= \int_0^1 \chi(\alpha, t_1)\chi(\alpha, t_2)\, d\alpha \int_0^1 \chi(\alpha, t_3)\chi(\alpha, t_4)\, d\alpha \ldots \int_0^1 \chi(\alpha_1, t_{2n-1})\chi(\alpha, t_{2n})\, d\alpha$$

+ all similar sums for all other parcellings of $t_1, \ldots, t_{2n}$ into pairs;

$$\int_0^1 \chi(\alpha, t_1)\chi(\alpha, t_2) \ldots \chi(\alpha, t_{2n+1})\, d\alpha = 0.$$

This enables us to establish:

Lemma 1. *Let $\varrho_1(t), \varrho_2(t), \ldots \varrho_{2n}(t)$ be functions of limited total variation over* (0. 1). *Then*

$$(2.05)\qquad \int_0^1 d\alpha \int_0^1 \chi(\alpha, t_1)\, d\varrho_1(t_1) \int_0^1 \chi(\alpha, t_2)\, d\varrho_2(t_2) \ldots \int_0^1 \chi(\alpha, t_{2n})\, d\varrho_{2n}(t_{2n})$$

$$= \int_0^1 d\varrho_1(t_1) \ldots \int_0^1 d\varrho_{2n}(t_{2n}) \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_n \{\chi(\alpha_1, t_1)\chi(\alpha_1 t_2)\chi(\alpha_2, t_3)\chi(\alpha_2, t_4) \ldots$$
$$\chi(\alpha_n, t_{2n-1})\chi(\alpha_n, t_{2n}) + \text{etc.}\},$$

where the "etc." stands for all other distributions of the $2n$ t_k's over the n α_k's, two to each, permutations of the α_k's among themselves not being counted as distinct.

Another very easy result to establish is that if ϱ_1 and ϱ_2 are of limited total variation and $\varphi_1(1) = \varphi_2(1) = 0$,

$$(2.06)\qquad \int_0^1 d\alpha \int_0^1 \chi(\alpha, t_1)\, d\varrho_1(t_1) \int_0^1 \chi(\alpha, t_2)\, d\varrho_2(t_2)$$

$$= \tfrac{1}{2} \int_0^1 t_1\, d\varrho_1(t_1) \int_{t_1}^1 d\varrho_2(t_2) + \tfrac{1}{2} \int_0^1 t_2\, d\varrho_2(t_2) \int_{t_2}^1 d\varrho_1(t_1)$$

$$= -\tfrac{1}{2} \int_0^1 t\, \{\varrho_2(t)\, d\varrho_1(t) + \varrho_1(t)\, d\varrho_2(t)\} = \tfrac{1}{2} \int_0^1 \varrho_1(t)\, \varrho_2(t)\, dt.$$

Thus we may give to Lemma 1 the somewhat compacter form:

Lemma 1′. *Let* $\varrho_1(t), \ldots \varrho_{2n}(t)$ *be functions of limited total variation over* $(0, 1)$. *Then*

$$(2.07)\qquad \int_0^1 d\alpha \int_0^1 \chi(\alpha, t_1)\, d\varrho_1(t_1) \ldots \int_0^1 \chi(\alpha, t_{2n})\, d\varrho_{2n}(t_{2n})$$

$$= 2^{-n} \int_0^1 \varrho_1(t)\, \varrho_2(t)\, dt \int_0^1 \varrho_3(t)\, \varrho_4(t)\, dt \ldots \int_0^1 \varrho_{2n-1}(t)\, \varrho_{2n}(t)\, dt + \text{etc.}$$

In particular, if $\int_0^1 [\varrho(t)]^2 = 1$, and $\varrho(t)$ is of limited total variation,

$$(2.08)\qquad \int_0^1 d\alpha \left[\int_0^1 \chi(\alpha, t)\, d\varrho(t)\right]^{2n} = \frac{1 \cdot 3 \cdot 5 \ldots 2n-1}{2^n}.$$

From this we may readily prove that if $P(w)$ is any polynomial in w,

$$(2.09)\qquad \int_0^1 d\alpha\, P\left[\int_0^1 \chi(\alpha, t)\, d\varrho(t)\right] = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} P(u)\, e^{-u^2}\, du.$$

We may readily extend this result to:

Lemma 2. *Let* $\varrho(t)$ *be a function of limited total variation, and let* $\int_0^1 [\varrho(t)]^2\, dt = 1$. *Let*

$$(2.10)\qquad I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} F(u)\, e^{-u^2}\, du$$

exist as a Lebesgue integral. Then

$$(2.11)\qquad \int_0^1 d\alpha\, F\left[\int_0^1 \chi(d, t)\, d\varrho(t)\right] = I.$$

This may be expressed in words by saying that $\int_0^1 \chi(\alpha, t)\, d\varrho(t)$ has a Gaussian distribution with modulus dependent only on $\int_0^1 [\varrho(t)]^2\, dt$.

A reasoning of the same sort will establish:

Lemma 3. *Let $\varrho_1(t), \ldots, \varrho_n(t)$ be functions of limited total variation for which*

$$(2.12) \qquad \int_0^1 \varrho_k(t)\,\varrho_2(t)\,dt = \begin{cases} 0; & [k \neq l] \\ 1; & [k = l] \end{cases}$$

and let

$$(2.13) \qquad \pi^{-n\,2} \int_{-\infty}^{\infty} d\,u_1 \ldots \int_{-\infty}^{\infty} d\,u_n F(u_1, \ldots, u_n)\, e^{-\sum_1^n u_k^2}$$

exist as a Lebesgue integral. Then

$$(2.14) \qquad \int_0^1 d\alpha\, F\left[\int_0^1 \chi(\alpha, t)\, d\,\varrho_1(t), \ldots, \int_0^1 \chi(\alpha, t)\, d\,\varrho_n(t)\right]$$

exists and has the same value.

Let it be noted that we have hereby established that the set of functions

$$(2.15) \qquad g_{\lambda_1, \lambda_2, \ldots, \lambda_n}(\alpha) = \int_0^1 \chi(\alpha, t)\, d\,\varrho_{\lambda_1}(t) \ldots \int_0^1 \chi(\alpha, t)\, d\,\varrho_{\lambda_n}(t) \qquad [\lambda_j \neq \lambda_k \text{ if } j \neq k]$$

is normal and orthogonal. They are analogous to a set of normal and orthogonal functions given by Walsh, and assuming only the values ± 1. Unlike the Walsh functions, they are continuous.

We may put Lemma 3 more strikingly by saying that the quantities $\int_0^1 \chi(\alpha, t)\, d\,\varrho_u(t)$ are completely independent — not merely linearly independent — of one another, and that they all have a Gaussian distribution with the same modulus.

3. Let us now introduce:

Definition 1. *If $\varrho(t)$ is a function of limited total variation over $(0,1)$, we shall write*

$$(3.01) \qquad \int_0^1 \varrho(t)\, d\,\chi(\alpha, t) = \varrho(1)\,\chi(\alpha, 1) - \int_0^1 \chi(\alpha, t)\, d\,\varrho(t)$$

As before, let $\{\varrho_n(t)\}$ be a complete set of normal and orthogonal functions of limited total variation. Let $\sum_1^\infty a_n^2$ converge. We wish to consider the behaviour of

$$(3.02) \qquad \int_0^1 \left[\sum_1^n a_k\,\varrho_k(t)\right] d\,\chi(\alpha, t)$$

as $n \to \infty$. By Lemma 3, this question may be answered in terms of the distribution of $\sum_1^n a_k u_k$ when the individual variables u_k have independent Gaussian distributions with the same modulus. This again is formally the same question as that of the distribution of $\chi(\alpha, a_1^2 + a_2^2 + \ldots + a_n^2)$. As has been said, Wiener has shown that for almost all values of α, $\chi(\alpha, t)$ is continuous in t. Thus almost always

$$\lim_{n \to \infty} \chi\left(\alpha, \sum_1^n a_k^2\right) = \chi\left(\alpha, \sum_1^\infty a_k^2\right) \tag{3.03}$$

from which it follows that $\sum_1^\infty a_k u_k$ almost always converges, and that

$$\lim_{n \to \infty} \int_0^1 \left[\sum_1^n a_k \varrho_k(t)\right] d\chi(\alpha, t) \tag{3.04}$$

almost always exists. We now introduce:

Definition 2. *Let $\varrho(t)$ belong to L_2, and let*

$$\varrho(t) \sim \sum_1^\infty a_k \varrho_k(t). \tag{3.05}$$

Then we shall put

$$\int_0^1 \varrho(t)\, d\chi(\alpha, t) = \lim_{n \to \infty} \int_0^1 \left[\sum_1^n a_k \varrho_k(t)\right] d\chi(\alpha, t). \tag{3.06}$$

We now wish to show that Definitions 1 and 2 are consistent, in the sense that they almost always lead to the same value of $\int_0^1 \varrho(t)\, d\chi(\alpha, t)$ when they are both applicable. Let us then adhere to Definition 1, and let us suppose that

$$\int_0^1 [\varrho(t)]^2 dt - \sum_1^n a_k^2 < \varepsilon^2. \tag{3.07}$$

It will follow from Lemma 2 that

$$\left| \int_0^1 \varrho(t)\, d\chi(\alpha, t) - \int_0^1 \left[\sum_1^n a_k \varrho_k(t)\right] d\chi(\alpha, t) \right| < \varepsilon \tag{3.08}$$

except over a set of values of α of measure not exceeding

$$\frac{1}{\varepsilon\sqrt{\pi}} \int_{-\infty}^{\infty} (\varepsilon u)^2 e^{-u^2} du = \varepsilon/2. \tag{3.09}$$

Thus (3.08) cannot tend almost everywhere to any limit other than 0, and we have established the consistency of the two definitions. A precisely similar argument will show that Definition 2 is independent of the particular choice of the set $\{\varrho_k(t)\}$.

Let it be noted that by (2.08)

$$\int_0^1 d\alpha \left[\sum_1^n a_k \int_0^1 \varrho_k(t)\, d\chi(\alpha, t)\right]^{2m} \leqq \frac{1 \cdot 3 \cdot 5 \ldots 2m-1}{2^m} \left[\int_0^1 [\varrho(t)]^2 dt\right]^m. \tag{3.10}$$

Let it also be noted that

$$(3.11)\qquad \lim_{n\to\infty}\int_0^1 d\alpha\left\{\int_0^1\left[\sum_1^n a_k\varrho_k(t)\,d\chi(\alpha,t)-\int_0^1\varrho(t)\,d\chi(\alpha,t)\right\}^2=0.$$

This results from (3.06) and the fact that by (2.08)

$$(3.111)\qquad \lim_{m,n\to\infty}\int_0^1 d\alpha\left\{\int_0^1\sum_m^n a_k\varrho_k(t)\right]d\chi(\alpha,t)\right\}^2=0.$$

From (3.11) and (3.10) it readily follows that

$$(3.12)\qquad \int_0^1 d\alpha\left[\int_0^1\varrho(t)\,d\chi(\alpha,t)\right]^{2m}\leqq\frac{1\cdot3\cdot5\ldots2m-1}{2^m}\left[\int_0^1[\varrho(t)]^2\,dt\right]^m.$$

Applying this to

$$(3.13)\qquad \varrho_*(t)=\varrho(t)-\sum_1^n a_k\varrho_k(t),$$

we get

$$(3.14)\qquad \lim_{n\to\infty}\int_0^1 d\alpha\left[\int_0^1\varrho_*(t)\,d\chi(\alpha,t)\right]^{2m}=0.$$

This together with (2.08) yields

$$(3.15)\qquad \int_0^1 d\alpha\left[\int_0^1\varrho(t)\,d\chi(\alpha,t)\right]^{2m}=\frac{1\cdot3\cdot5\ldots2m-1}{2^m}\left[\int_0^1[\varrho(t)]^2\,dt\right]^m.$$

We have thus established:

Lemma 2′. *Let*

$$(2.10)\qquad I=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}F(u)\,e^{-u^2}\,du$$

exist as a Lebesgue integral, and let $\int_0^1[\varrho(t)]^2\,dt=1$. *Then*

$$(3.16)\qquad \int_0^1 d\alpha\,F\left[\int_0^1\varrho(t)\,d\chi(\alpha,t)\right]=I.$$

Lemma 3 also possesses an extension of the same sort to functions $\varrho_k(t)$ not of limited total variation. Once these lemmas are established, it follows that Definition 2 will yield the same result even if the ϱ_k's are not functions of limited total variation. Of course, in this form it would be circular and of no value as a definition.

4. Let us now consider a function of t and the parameter τ of the form

$$(4.01)\qquad \Phi_n(\tau,t)=\sum_1^n\sigma_k(\tau)\,\varrho_k(t)$$

where the functions $\varrho_k(t)$ are normal and orthogonal over $(0,1)$ and

$$(4.02)\qquad \sum_1^n\int_0^1[\sigma_k(\tau)]^2\,d\tau=S<\infty.$$

Then except for a set of values of τ of zero measure,

$$(4.03)\qquad \int_0^1 d\alpha\left[\int_0^1 \Phi_n(\tau,t)\,d\chi(\alpha,t)\right]^{2m} = \frac{1\cdot 3\cdot 5\ldots 2m-1}{2^m}\left[\sum_1^n[\sigma_k(\tau)]^2\right]^m$$

$$= \frac{1\cdot 3\cdot 5\ldots 2m-1}{2^m}\left[\int_0^1[\Phi_n(\tau,t)]\,dt\right]^m.$$

It follows that

$$(4.04)\qquad \int_0^1 d\alpha\int_0^1 d\tau\left[\int_0^1 \Phi_n(\tau,t)\,d\chi(\alpha,t)\right]^{2m}$$

$$= \frac{1\cdot 3\cdot 5\ldots 2m-1}{2^m}\int_0^1\left[\sum_1^n[\sigma(\tau)]^2\right]^m d\tau.$$

Thus if $\Phi(\tau,t)$ is any function belonging to L_2 as a function of t and τ simultaneously, and

$$(4.05)\qquad \Phi_n(\tau,t) = \sum_1^n \varrho_k(t)\int_0^1 \Phi(\tau,s)\,\varrho_k(s)\,ds,$$

an argument precisely similar to that of the last paragraph will show that for almost all values of α

$$(4.06)\qquad \underset{n\to\infty}{\mathrm{l.\,i.\,m}}\int_0^1 \Phi_n(\tau,t)\,d\chi(\alpha,t)$$

exists, and that as before we may *define* $\int_0^1 \Phi(\tau,t)\,d\chi(\alpha,t)$ by

$$(4.07)\qquad \int_0^1 \Phi(\tau,t)\,d\chi(\alpha,t) = \underset{n\to\infty}{\mathrm{l.\,i.\,m.}}\int_0^1 \Phi_0(\tau,t)\,d\chi(\alpha,t).$$

The consistency of this definition with previous definitions and its independence of the choice of ϱ_n's involve no new discussion. If we reflect that the finiteness of

$$(4.08)\qquad \int_0^1 d\tau\left[\int_0^1[\Phi(\tau,t)]^2\,dt\right]^m$$

carries with it the boundedness of

$$(4.09)\qquad \int_0^1 d\tau\left[\int_0^1[\Phi_n(\tau,t)]^2\,dt\right]^m$$

for all n, and the fact that

$$(4.10)\qquad \lim_{m,n\to\infty}\int_0^1 d\tau\left[\int_0^1[\Phi_m(\tau,t)-\Phi_n(\tau,t)]^2\,dt\right]^m = 0,$$

then the methods by which we have established Lemma 3′ are adequate to establish:

Theorem I. *If*

$$\int_0^1 d\tau \left[\int_0^1 [\Phi(\tau, t)]^2 dt\right]^m < \infty, \tag{4.11}$$

then

$$\int_0^1 d\alpha \int_0^1 d\tau \left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right]^{2m} = \frac{1\cdot 3\cdot 5 \ldots 2m-1}{2^m} \int_0^1 d\tau \left[\int_0^1 [\Phi(\tau, t)]^2 dt\right]^m, \tag{4.12}$$

and

$$\int_0^1 d\tau \left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right]^{2m} \tag{4.13}$$

exists for almost all values of α.

Particular cases for which (4, 11) is valid in case $q = 2m$ are: (a)

$$\Phi(\tau, t) = \beta(t) f(\tau, t), \tag{4.14}$$

where $\beta(t)$ belongs to L_2 and

$$\int_0^1 |f(\tau, t)|^q d\tau < B \tag{4.15}$$

and B is an absolute constant; (b)

$$\Phi(\tau, t) \sim \sum_1^\infty b_n f_n(\tau)\, \varrho_n(t), \tag{4.16}$$

where

$$\sum_1^\infty b_n^2 < \infty; \tag{4.17}$$

$$\int_0^1 \varrho_k(t)\, \varrho_l(t)\, dt = \begin{Bmatrix} 1; [k = l] \\ 0; [k \neq l] \end{Bmatrix}, \tag{4.18}$$

$$\int_0^1 | f_n(\tau)|^q d\tau < B; \tag{4.19}$$

and B is an absolute constant. These two cases form the two different analogues by the two methods we have mentioned, of Theorem II of PZ 1. A more precise analogue of this is:

Theorem II. *If hypothesis* (a) *or hypothesis* (b) *holds, and* $q \geqq 2$, *without necessarily being an integer,*

$$\int_0^1 d\tau \left|\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right|^q \tag{4.20}$$

exists for almost all values of α.

Here the proof is exactly alike in the two cases. We have in case (a) for $2m - 2 < q \leqq 2m$

$$(4.21)\quad \left\{\int_0^1 d\alpha \left|\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right|^q\right\}^{1/q} \leqq \left\{\int_0^1 d\alpha \left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right]^{2m}\right\}^{1/2m}$$

$$= \left(\frac{1\cdot 3\cdot 5 \ldots 2m-1}{2^m}\right)^{1/2m} \left[\int_0^1 [\beta(t)]^2 [f(\tau, t)]^2\, dt\right]^{1/2}$$

$$\leqq C_q \left[\int_0^1 [\beta(t)]^2 |f(\tau, t)|^q\, dt\right]^{1/q} \left[\int_0^1 [\beta(t)]^2\, dt\right]^{q-2/2q}$$

where C_q depends only on q.

Raising both sides to the qth power, and integrating with respect to τ, we get:

$$(4.22)\quad \int_0^1 d\alpha \int_0^1 d\tau \left|\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right|^q$$

$$\leqq C_q^q \left[\int_0^1 [\beta(t)]^2\, dt\right]^{q-2/2} \int_0^1 d\tau \int_0^1 [\beta(t)]^2 |f(\tau, t)|^q\, dt$$

$$\leqq C_q^q \left[\int_0^1 [\beta(t)]^2\, dt\right]^{q/2} B,$$

from which Theorem II follows at once.

It follows at once from Theorem I that:

Theorem III. *Let*

$$(4.23)\quad \int_0^1 [\Phi(\tau, t)]^2\, dt \leqq w^2$$

where w is independent of τ. Then if

$$(4.24)\quad G(u) = \sum_0^\infty a_n u^n$$

has only positive coefficients in its Maclaurin series, and

$$(4.25)\quad \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} G(uw)\, e^{-u^2}\, du < \infty$$

then for almost all α

$$(4.26)\quad \int_0^1 d\tau\, G\left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right] < \infty.$$

In particular, almost always

$$(4.27)\quad \int_0^1 d\tau \exp \lambda \left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t)\right]^2 < \infty$$

where

$$(428)\quad \lambda < 1/w^2.$$

In the particular case where $\Phi(\tau, t)$ is of form (4.16), where (4.17) and (4.18) hold, and where (4.19) is replaced by the condition that $f_n(\tau)$ is bounded for all values of n and τ, the methods of Theorem III of PZ 1 will enable us to show that condition (4.28) is unnecessary.

5. Now that we have obtained the formulae for the mean values of expressions of the form

$$\int_0^1 d\tau \left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t) \right]^{2m} \tag{5.01}$$

let us investigate the mean squares of the departures from these means. We have formally:

$$\begin{aligned} &\int_0^1 d\alpha \left\{ \int_0^1 d\tau \left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t) \right]^{2m} - \frac{1 \cdot 3 \cdot 5 \ldots 2m-1}{2^m} \int_0^1 d\tau \left[\int_0^1 [\Phi(\tau, t)]^2\, dt \right]^m \right\}^2 \\ &= \int_0^1 d\alpha \int_0^1 d\sigma \int_0^1 d\tau \left[\int_0^1 \Phi(\tau, t)\, d\chi(\alpha, t) \right]^{2m} \left[\int_0^1 \Phi(\sigma, t)\, d\chi(\alpha, t) \right]^{2m} \\ &\quad - \left\{ \frac{1 \cdot 3 \cdot 5 \ldots 2m-1}{2^m} \int_0^1 d\tau \left[\int_0^1 [\Phi(\tau, t)]^2\, dt \right]^m \right\}^2. \end{aligned} \tag{5.02}$$

If we evaluate (5.02) in accordance with Lemma 1′, we get for the case $m = 1$

$$\frac{1}{2} \int_0^1 d\sigma \int_0^1 d\tau \left[\int_0^1 \Phi(\sigma, t)\, \Phi(\tau, t)\, dt \right]^2; \tag{5.03}$$

for the case $m = 2$,

$$\begin{aligned} &\frac{9}{8} \int_0^1 d\sigma \int_0^1 d\tau \int_0^1 \Phi(\sigma, t)\, \Phi(\tau, t)\, dt \int_0^1 [\Phi(\sigma, t)]^2\, dt \int_0^1 [\Phi(\tau, t)]^2\, dt \\ &\quad + \frac{3}{4} \int_0^1 d\sigma \int_0^1 d\tau \left[\int_0^1 \Phi(\sigma, t)\, \Phi(\tau, t)\, dt \right]^3; \end{aligned} \tag{5.04}$$

and for the case $m = 3$,

$$\begin{aligned} &\frac{9}{2} \int_0^1 d\sigma \int_0^1 d\tau \left[\int_0^1 \Phi(\sigma, t)\, \Phi(\tau, t)\, dt \right]^2 \int_0^1 [\Phi(\sigma, t)]^2\, dt \int_0^1 [\Phi(\tau, t)]^2\, dt \\ &\quad + \frac{3}{2} \int_0^1 d\sigma \int_0^1 d\tau \left[\int_0^1 \Phi(\sigma, t)\, \Phi(\tau, t) \right]^4. \end{aligned} \tag{5.05}$$

By the Schwarz inequality, each of these expressions will be finite with the appropriate

$$\int_0^1 d\tau \left[\int_0^1 [\Phi(\tau, t)]^2\, dt \right]^m. \tag{5.06}$$

In the first instance, we have only established (5.03), (5.04), and (5.05) in the case of functions $\Phi(\sigma, t)$ of limited total variation in t, but there is no difficulty in extending them by processes now familiar to the general case for which (5.06) is finite.

If the mean square departure of a quantity from the value A is B^2, the quantity cannot differ from A by an amount exceeding B except over a set of values of measure not exceeding B^2/C_u^2. We thus obtain:

Theorem IV. *Let*

$$\frac{1}{2}\int_0^1 d\tau \int_0^1 dt[\Phi_n(\tau, t)]^2 = A \qquad [n = 1, 2, \ldots] \tag{5.07}$$

and let

$$\sum_{n=1}^{\infty}\left\{\frac{1}{2}\int_0^1 d\sigma \int_0^1 d\tau\left[\int_0^1 \Phi_n(\sigma, t)\,\Phi_n(\tau, t)\,dt\right]^2\right\}^{1/2} \tag{5.08}$$

converge. Then except for a set of values of α of zero measure,

$$\lim_{n\to\infty}\int_0^1 d\tau\left[\int_0^1 \Phi_n(\tau, t)\,d\chi(\alpha, t)\right]^2 = A. \tag{5.09}$$

Similar theorems hold for cases (5.04), (5.05), etc.

In particular, let

$$\Phi_n(\tau, t) = n^2 \text{ if } \tau - \frac{1}{n^4} < t < \tau,\ a + \frac{1}{n^4} < \tau < t; \tag{5.10}$$
$$\Phi_n(\tau, t) = 0 \text{ otherwise.}$$

Then

$$\int_0^1 \Phi_n(\sigma, t)\,\Phi_n(\tau, t)\,dt = O(1) \text{ if } |\sigma - \tau| < n^{-4}; \tag{5.11}$$
$$\int_0^1 \Phi_n(\sigma, t)\,\Phi_n(\tau, t)\,dt = O \text{ otherwise.}$$

Thus the series of (5.08) reduces to

$$\sum_{n=1}^{\infty} O(n^{-2}) < \infty, \tag{5.12}$$

Formula (5.09) becomes

$$\lim_{n\to\infty} n^4 \int_{a+\frac{1}{n^4}}^{b}\left[\chi(\alpha, \tau) - \chi\left(\alpha, \tau - \frac{1}{n^4}\right)\right]^2 d\tau = \frac{1}{2}(b - a). \tag{5.13}$$

We have already proved, however (W 1) that except for a set of values of α of zero measure, there is an A such that

$$[\chi(\alpha, \tau) - \chi(\alpha, \sigma)]^2 < A|\tau - \sigma|^{1/2} \tag{5.14}$$

for all τ and σ. Without substantial change, the argument will show that there is almost always an A such that

$$[\chi(\alpha, \tau) - \chi(\alpha, \sigma)]^2 < A|\tau - \sigma|^{9/10} \tag{5.15}$$

for all τ and σ. Thus if $(n+1)^{-4} \leqq \varepsilon \leqq n^{-4}$,

$$(5.16) \qquad n^4 \int_{a+n^{-4}}^{b} [\chi(\alpha, \tau-\varepsilon) - \chi(\alpha, \tau - n^{-4})]^2 \, d\tau = O(n^{-2})$$

uniformly in ε, so that by (5.13) we have almost always

$$(5.17) \qquad \lim_{n \to \infty} n^4 \int_{a+n^{-4}}^{b} [\chi(\alpha, \tau) - \chi(\alpha, \tau-\varepsilon)]^2 \, d\tau = \frac{1}{2}(b-a)$$

and hence

$$(5.18) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{a+\varepsilon}^{b} [\chi(\alpha, \tau) - \chi(\alpha, \tau-\varepsilon)]^2 \, d\tau = \frac{1}{2}(b-a)$$

as $\varepsilon \to 0$ in any way whatever. This is almost always, true for every rational a and b, and it may consequently be proved to hold almost always for *every* a and b in $(0, 1)$

6. We now come to the consideration of random functions over an infinite range. If we put:

$$(6.01) \qquad \xi(\alpha, t) = \sqrt{\pi} \int_{1/2}^{\frac{1}{\pi}\cot^{-1}(-t)} \chi(\alpha, u) \, d \csc \pi u - \chi\left(\alpha \frac{1}{\pi} \cot^{-1}(-t)\right) \sqrt{1+t^2} + \chi(\alpha, \tfrac{1}{2})$$

we shall find that $\xi(\alpha, 0) = 0$, and that if $-\infty \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_n$

$$(6.02) \qquad \int_0^1 F(\xi(\alpha, t_2) - \xi(\alpha, t_1), \; \xi(\alpha, t_3) - \xi(\alpha, t_2), \; \ldots, \; \xi(\alpha, t_n) - \xi(\alpha, t_{n-1})) \, d\alpha$$

$$= \pi^{-\frac{n-1}{2}} \prod_2^n (t_k - t_{k-1}) \int_{-\infty}^{\infty} du_1 \ldots \int_{-\infty}^{\infty} du_n \exp\left[-\sum_2^n \frac{u_k^2}{t_k - t_{k-1}}\right] F(u_2, \ldots, u_n).$$

Formally,

$$(6.03) \qquad \int_{-\infty}^{\infty} \xi(\alpha, t) \, d\varrho(t) = \sqrt{\pi} \int_0^1 \chi(\alpha, t) \, d[\csc \pi t \, \varrho(-\cot \pi t)]$$

so that if $\varrho(t)$ belongs to L_2, and $\varrho(t)\sqrt{1+t^2}$ is of limited total variation over $(-\infty, \infty)$ we may define

$$(6.04) \qquad \int_{-\infty}^{\infty} \varrho(t) \, d\xi(\alpha, t) = -\int_{-\infty}^{\infty} \xi(\alpha, t) \, d\varrho(t).$$

Up to Definition 2, the theory of random functions over the infinite interval is precisely parallel to that over the finite interval. In Definition 2, we may now generalize $\int_{-\infty}^{\infty} \varrho(t) \, d\xi(\alpha, t)$ so as to eliminate not merely the condition that $\varrho(t)$ should be of limited total variation, but all conditions concerning its order of magnitude, and as before, we now merely require that $\varrho(t)$ shall belong to L_2, this time over $(-\infty, \infty)$. Up to

Theorem III, no change is made by introducing an infinite range of both t and τ, and even in Theorem III the changes that are introduced over a finite range of τ are trivial and obvious. In particular, the reasoning of Theorem IV is still valid, and all that part of W 1 which applies to the proof that the spectral density of $\int_{-\infty}^{\infty} \varrho(t)\, d\,\xi(d, t)$ is almost always

$$\frac{1}{4\pi}\left| \underset{M\to\infty}{\text{l. i. m.}} \int_{-M}^{M} \varrho(t)\, e^{-iut}\, dt \right|^2 \tag{6.05}$$

will remain valid when $\varrho(t)$ merely belongs to L_2, and

$$\int_{-\infty}^{\infty} du \left[\int_{-\infty}^{\infty} \varrho(t)\, \varrho(t+u)\, dt\right]^2 < \infty \tag{6.06}$$

If we put

$$\varrho(t) = \frac{1}{\sqrt{2\pi}}\, \underset{M\to\infty}{\text{l. i. m.}} \int_{-M}^{M} \psi(u)\, e^{iut}\, du, \tag{6.07}$$

these conditions reduce themselves to the condition that $\psi(u)$ should belong both to L_2 and L_4.

7. We now restrict ourselves to the case (4. 14) and prove a theorem analogous to Theorem VIII of PZ 1. We have first to generalize Lemma 6 of that paper. This may be done in the following way. Let

$$0 = t_0 < t_1 < t_2 \ldots < t_N = 1 \tag{7.01}$$

divide the interval (0, 1) into N portions. We write

$$F_x(\alpha) = \sum_{n=0}^{N-1} \varphi_n(x) \int_{t_n}^{t_{n+1}} \beta(t)\, \alpha\, \chi(\alpha, t) \tag{7.02}$$

where $\varphi_n(x)$ denotes Rademacher's function, defined in PZ 1. We denote by $S_x(\alpha)$ the maximum

$$S_x(\alpha) = \underset{0\leqq m\leqq N-1}{\text{Max}} \left| \sum_{n=0}^{m} \varphi_n(x) \int_{t_n}^{t_{n+1}} \beta(t)\, d\chi(\alpha, t) \right|. \tag{7.03}$$

We wish to investigate the behaviour of the average

$$\int_0^1 S_0^q(\alpha)\, d\alpha. \tag{7.04}$$

We observe first that, if we write,

$$\nu(t) = n \quad (t_n \leqq t < t_{n+1},\ 0 \leqq n \leqq N-1) \tag{7.05}$$

then, for a given value of x, we may write

$$\varphi_{\nu(t)}(x)\, d\chi(\alpha, t) = d\chi(\alpha'\, t), \tag{7.06}$$

since the left hand side of (7.06) clearly represents an arbitrary Brownian motion. The correspondance between α and α' is $1-1$ except in a set of measure zero, and measure (in space α) is preseired. It follows that

$$S_x(\alpha) = S_0(\alpha'), \tag{7.07}$$

and that

$$\int_0^1 S_x^q(\alpha)\,d\alpha = \int_0^1 S_0^q(\alpha)\,d\alpha. \tag{7.08}$$

Applying Lemma 6 of PZ 1, we have

$$\begin{aligned}\int_0^1 S_0^q(\alpha)\,d\alpha &= \int_0^1 dx \int_0^1 S_x^q(\alpha)\,d\alpha \\ &= \int_0^1 d\alpha \int_0^1 S_x^q(\alpha)\,dx \\ &\leqq C_q \int_0^1 d\alpha \int_0^1 |F_x(\alpha)|^q\,dx \\ &= C_q \int_0^1 dx \int_0^1 |F_x(\alpha)|^q\,d\alpha \\ &\leqq C_q \left(\int_0^1 \beta^2(t)\,dt\right)^{\frac{1}{2}q},\end{aligned} \tag{7.09}$$

where the constants C_q depend only on q. Now denote by $S^*(\alpha)$ the upper bound

$$S^*(\alpha) = \sup_{0 \leqq t' \leqq 1} \left| \int_0^{t'} \beta(t)\,d\chi(\alpha, t) \right|. \tag{7.10}$$

By proceeding to the limit in (7.09) we obtain

$$\int_0^1 S^{*q}(\alpha)\,d\alpha \leqq C_q \left(\int_0^1 \beta^2(t)\,dt\right)^{\frac{1}{2}q}. \tag{7.11}$$

The above result is analogous to Lemma 6. We may now prove the following result, analogous to Theorem VIII.

Theorem V. *Let $S^*(\tau, \alpha)$ denote*

$$\sup_{0 \leqq t' \leqq 1} \left| \int_0^{t'} \beta(t) f(\tau, t)\,d\chi(\alpha, t) \right|, \tag{7.12}$$

where

$$\int_0^1 \beta^2(t)\,dt < \infty, \tag{7.13}$$

$$\int_0^1 |f(\tau, t)|^q\,d\tau < B, \tag{7.14}$$

and B is an absolute constant. Then, for almost all α, we have

$$\int_0^1 S^{*q}(\tau, \alpha)\, d\tau < \infty. \tag{7.15}$$

Suppose that $\beta(t)$ is a function of bounded variation for which

$$\int_0^1 \beta^2(t)\, dt = \infty. \tag{7.16}$$

It may be shown*) that, for almost all α, the integral

$$\int_0^1 \beta(t)\, d\chi(\alpha, t) \tag{7.17}$$

does not exist is the sense (3.01). We may now prove the following theorem analogous to Theorem IX of PZ 1.

Theorem VI. *Suppose that, corresponding to every set E of positive measure in the interval (0.1), we can find a set $G - G(E)$, such that*

$$\int_G \beta^2(t)\, dt = \infty, \tag{7.18}$$

$$\int_E f^2(\tau, t)\, d\tau > \delta_E \qquad (t < G) \tag{7.19}$$

where δ_E is a positive constant, which may depend on E. Then, for almost all α, the integral

$$\int_0^1 \beta(t)\, f(\tau, t)\, d\chi(\alpha, t) \tag{7.20}$$

exists in the sense (3.01) for almost no τ.

8. It was proved in PZ 1 that almost all the functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \varphi_n(t) \qquad (z = r e^{i\theta}), \tag{8.01}$$

where Σa_n^2 converges, satisfy the inequality

$$\operatorname*{Max}_{0 \leqq \theta \leqq 2\pi} |f(r e^{i\theta})| = O\left(\sqrt{\log \frac{1}{1-r}}\right). \tag{8.02}$$

A *gegenbeispiel* was given to show that (8.02) was best possible. This *gegenbeispiel* was a lacunary trigonometric series, and therefore, being of a somewhat special type, may not be regarded as altogether sufficient evidence that almost all functions are unbounded. Leater, in PZ. 3, it was shown that the function (8.01) was unbounded for almost all t subject only to the divergence of the series

$$|a_0| + \sum_{m=0}^{\infty} \left(\sum_{n=2^m}^{2^{m+1}} a_n^2 \right)^{1/2}. \tag{8.03}$$

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*) The proof is almost identical with that of Theorem B in the paper Zygmund (1).

Here we give a very simple *gegenbeispiel* with smooth coefficients, which illustrates the phenomenon in a rather different way. We write

$$(8.04)\qquad F_y^{(\nu)}(r,\Theta) = \sqrt{\nu}\sum_{n=1}^{[k\nu]} \frac{2\sin\frac{n}{\nu}}{n}\,\varrho^{-n} r^n \cos n\Theta\, y_x,$$

where ν is an even integer, k is a number which remains to be chosen, and $y_1, y_2, \ldots, y_n$ represent independent Gaussian distributions of modulus 1, and $\varrho = \varrho_\nu$ is defined by the equation

$$(8.05)\qquad \varrho^{-\nu} = e.$$

Clearly the average value of

$$(8.05_1)\qquad \int_0^{2\pi} (F_y^{(\nu)}(1,\Theta))^2\, d\Theta$$

is less than on absolute constant independent of ν (but depending on k). Now consider

$$(8.06)\qquad F_y^{(\nu)}(\varrho,\Theta) = \sqrt{\nu}\sum_{n=1}^{(k\nu)} \frac{2\sin\frac{n}{\nu}}{n}\cos n\Theta\, y_n$$

$$= \sqrt{\nu}\left(\sum_{n=1}^{\infty}\frac{\sin n\left(\Theta+\frac{1}{\nu}\right)}{n}y_n - \sum_{n=1}^{\infty}\frac{\sin n\left(\Theta-\frac{1}{\nu}\right)}{n}y_n\right)$$

$$-\sqrt{\nu}\left(\sum_{n=[k\nu]+1}^{\infty}\frac{\sin n\left(\Theta+\frac{1}{\nu}\right)}{n}y_n - \sum_{n=[k\nu]+1}^{\infty}\frac{\sin n\left(\Theta-\frac{1}{\nu}\right)}{n}y_n\right)$$

$$= F_y^*(\varrho,\Theta) + R_y(\Theta).$$

Wiener has shown*) that

$$(8.07)\qquad \sqrt{\left(\frac{2}{\pi}\right)}\sum_{n=1}^{\infty}\frac{\sin n\,\Theta}{n}y_n$$

represents an arbitrary Brownian motion $\chi(\alpha,\Theta)$, and thus

$$(8.08)\qquad F_y^*\left(\varrho,\frac{2\lambda}{\nu}\right) \qquad (\lambda = 0, 1, \ldots, \tfrac{1}{2}\nu - 1)$$

represent independent Gaussian distributions

$$(8.09)\qquad \sqrt{\nu}\left(\chi\left(\alpha,\frac{2\lambda+1}{\nu}\right)-\chi\left(\alpha,\frac{2\lambda-1}{\nu}\right)\right),$$

whose modulus is independent of ν. Also

$$(8.10)\qquad \text{Average}\max_{0\le\Theta\le 2\pi}|F_y^*(\varrho,\Theta)| \ge \text{Average}\max_{0\le\lambda\le 1/2\,\nu-1}\left|F_y^*\left(\varrho\,\frac{2\lambda}{\nu}\right)\right|$$

$$= \int_0^1 \sqrt{\nu}\max_{0\le\lambda\le 1/2\,\nu-1}\left|\chi\left(\alpha,\frac{2\lambda+1}{\nu}\right)-\chi\left(\alpha,\frac{2\lambda-1}{\nu}\right)\right| d\alpha.$$

*) Wiener (2).

The last integral may be evaluated by means of (2.01), and exceeds a fixed multiple of $\sqrt{\log \nu}$. It follows that

$$\text{(8.11)} \qquad \text{Average} \max_{0 \leqq \Theta \leqq 2\pi} |F_y^*(\varrho, \Theta)| \geqq B \sqrt{\log\left(\frac{1}{1-\varrho}\right)}.$$

Insuring now to $R_y(\Theta)$, we observe that, since

$$\text{(8.12)} \qquad \text{Average} \int_0^{2\pi} R_y^2(\Theta)\, d\Theta = O\left(\frac{1}{k}\right),$$

we have

$$\text{(8.13)} \qquad \text{Average} \max_{0 \leqq \Theta \leqq 2\pi} \left| \sqrt{\nu} \sum_{[k\nu]+1}^{\infty} \frac{2 \sin \frac{n}{\nu} \cos n\Theta}{n} \bar{\varrho}^n y_n \right| = O\left\{k^{-1/2} \sqrt{\log\left(\frac{1}{1-\bar{\varrho}}\right)}\right\},$$

$\bar{\varrho}$ being a number which remains to be chosen. Also since

$$\text{(8.14)} \quad \text{Average} \int_0^{2\pi} \left[\frac{\partial}{\partial r}\left\{\sqrt{\nu} \sum_{[k\nu]+1}^{\infty} \frac{2 \sin \frac{n}{\nu} \cos u\Theta}{n} r^{1/2\, n} y_n\right\}\right]^2 d\Theta = O\left(\frac{\nu}{(1-r)}\right),$$

it fellows that

$$\text{(8,15)} \qquad \text{Average} \max_{0 \leqq \Theta \leqq 2\pi} \left| \frac{\partial}{\partial r}\left\{\sqrt{\nu} \sum_{[k\nu]+1}^{\infty} \frac{2 \sin \frac{n}{\nu} \cos n\Theta}{n} r^n y_n\right\}\right| = O\left(\frac{\nu \log\left(\frac{1}{1-r}\right)}{1-r}\right)^{1/2}.$$

Integrating the result (8.15) from $r = \bar{\varrho}$ to $r = 1$, we get

$$\text{(8.16)} \qquad \text{Average} \max_{0 \leqq \Theta \leqq 2\pi} \left| \sqrt{\nu} \sum_{[k\nu]+1}^{\infty} \frac{2 \sin \frac{n}{\nu} \cos n\Theta}{n} (1-\bar{\varrho}^n)\, y_n \right| = O\left\{\nu(1-\bar{\varrho}) \log\left(\frac{1}{1-\bar{\varrho}}\right)\right\}^{1/2}$$

If we now choose $\bar{\varrho}$ so that $k\nu(1-\bar{\varrho}) = 1$, then, combining (8.13) and (8.16), we have

$$\text{(8.17)} \quad \text{Average} \max_{0 \leqq \Theta \leqq 2\pi} \left| \sqrt{\nu} \sum_{[k\nu]+1}^{\infty} \frac{2 \sin \frac{n}{\nu} \cos n\Theta}{n} y_n \right| = O\left(\frac{\log(k\nu)}{k}\right)^{1/2}.$$

Finally by choosing k sufficiently large, and combining (8.11) with (8.17), we obtain the desired result

$$\text{(8.18)} \qquad \text{Average} \max_{0 \leqq \Theta \leqq 2\pi} |F_y^{(\nu)}(\varrho, \Theta)| \geqq \tfrac{1}{2} B \sqrt{\log\left(\frac{1}{1-\bar{\varrho}}\right)}.$$

It is not difficult to combine functions of the form $F_y^{(\nu)}$ so as to form a function $F_y(r, \Theta)$ for which

$$\text{(8.19)} \qquad \max_{0 \leqq \Theta \leqq 2\pi} |F_y(r, \Theta)| = \Omega\left\{\varepsilon(1-r) \sqrt{\log\left(\frac{1}{1-r}\right)}\right\},$$

for almost all y, $\varepsilon(1-r)$ being an arbitrary function which tends to zero with $1-r$.

Suppose that $\vartheta_n(m)$ denote a set of functions for which

$$\sum_{m=1}^{N} \vartheta_n^2(m) = 1; \tag{8.20}$$

$$\sum_{m=1}^{N} \vartheta_n(m)\, \vartheta_{n'}(m) = 0. \tag{8.21}$$

We may show, by means of the argument already given, that, if y_1, $y_2, \ldots, y_N$ denote independent Gaussian distributions of modulus 1, then

$$\text{Average} \max_{1 \leqq m \leqq N} \left| \sum_{n=1}^{N} \vartheta_n(m)\, y_n \right| \geqq B \sqrt{(N \log N)}, \tag{8.22}$$

where B is an absolute constant.

A further property of $\chi(\alpha, t)$, which has attracted the attention of Perrin as an experimentally observable fact, is that for almost all values of α, it nowhere possesses a derivative with respect to t. We shall in fact prove the more comprehensive.

Theorem VII. *The values of* α *for which there exists at such that*

$$\lim_{\varepsilon \to 0} (\chi(\alpha, t+\varepsilon) - \chi(\alpha, t))/\varepsilon^{\lambda} < \infty \qquad [\lambda > \tfrac{1}{2}] \tag{9.01}$$

form a set of zero measure.

To prove this, let us subdivide the interval $(0, 1)$ binarily. If $(m\,2^{-q}, (m+1)\,2^{-q})$ is a subdivision of the q^{th} order, its middle point will be $(2m+1)\,2^{-q-1}$, and the differential determining the distribution of the pair of quantities $\chi(\alpha, (2m+1)\,2^{-q-1}) - \chi(\alpha, m\,2^{-q}) = u_1$, and $\chi(\alpha, (m+1)\,2^{-q}) - \chi(\alpha, (2m+1)\,2^{-q-1}) = u_2$, will be

$$\frac{1}{\pi\, 2^{-q-1}} \exp.\left(-\frac{u_1^2 + u_2^2}{2^{-q-1}}\right) d u_1\, d u_2. \tag{9.02}$$

If we put $u_1 + u_2 = u$, this becomes

$$\frac{1}{\pi\, 2^{-q-1}} \exp.\left(-\frac{u_1^2 + (u-u_1)^2}{2^{-q-1}}\right) d u, d u$$
$$= \frac{1}{\pi\, 2^{-q-1}} \exp. \frac{-\left(u_1 - \frac{u}{2}\right)^2 - \frac{u^2}{4}}{2^{-q-1}} d u_1\, d u. \tag{9.03}$$

From this we may readily conclude that the probability that either u_1 or u_2 is less than A_q in modulus does not exceed

$$\frac{4}{\sqrt{\pi\, 2^{-q-2}}} \int_{-A}^{A} e^{-\frac{u^2}{2^{-q-2}}} d u \leqq \frac{16}{\sqrt{\pi}} A\, 2^{q/2}. \tag{9.04}$$

The probability that of the 2^{q+1} quantities $|\chi(\alpha, (\mu+1)\,2^{-q-1}) - \chi(\alpha, \mu\,2^{-q-1})|$, at least N_q do not exceed A_q, whatever the distribution of the 2^q analogous quantities of the proceeding series, thus does not exceed

$$(9.05)\qquad \sum_{\nu=N}^{2^{q+1}} \frac{(2^{q+1})!}{\nu!\,(2^{q+1}-\nu)!}\left(1-\frac{16}{\sqrt{\pi}}A\,2^{q/2}\right)^{2^{q+1}-\nu}\left(\frac{16}{\sqrt{\pi}}A\,2^{q/2}\right)^{\nu} < \int\limits_{\frac{N-\frac{16}{\sqrt{\pi}}A\,2^{\frac{3q}{2}+1}}{\sqrt{\frac{16}{\sqrt{\pi}}A\,2^{\frac{3q}{2}+1}}}}^{\infty} \frac{e^{-u^2}\,du}{\sqrt{\pi}}.$$

Let us now put

$$(9.06)\qquad A_q = 2^{-\lambda q}a,\; N_q = \frac{16}{\sqrt{\pi}}\,2^{q\left(\frac{3}{2}-\mu\right)} \qquad [0\leqq \mu < \lambda]$$

Then the dominant expression in (9.05) becomes:

$$(9,07)\qquad \int\limits_{\left(2^{(\lambda-\mu)q}-2a\right)2^{q\left(\frac{3}{4}-\frac{\lambda}{2}\right)}}^{\infty} \frac{e^{-u^2}\,du}{\sqrt{\pi}} = T_q$$

and we have

$$(9.08)\qquad \sum_1^{\infty} T_q < \infty.$$

Thus it is infinitely improbable that more than a finite number of q's can be found for which at least N_q of the 2^{q+1} quantities $|\chi(\alpha,(\mu+1)2^{-q-1}) - \chi(\alpha,\mu 2^{-q-1})|$ do not exceed A_q, N_q and A_q being defined as in (9.06). On the other hand, the probability that the N_q intervals of the q^{th} stage have any point in common for all values of q from Q_1+1 to Q_2, does not exceed

$$(9.09)\qquad 2^{Q_2+1}\prod_{Q_1+1}^{Q_2}\frac{16}{\sqrt{\pi}}\,\frac{2^{q\left(\frac{3}{2}-\mu\right)}}{2^{q+1}} \leqq C_1\,C_2^{Q_2}\,2^{\left(\frac{1}{2}-\mu\right)Q_2^2}$$

where C_1 and C_2 are constants. If $\mu>\frac{1}{2}$, this tends to 0 as $Q_2\to\infty$. Thus it is infinitely improbable that there is any point common to S_q for all q's from some stage on, S_q consisting of all the points in the N_q intervals of the q^{th} stage for which

$$(9.10)\qquad |\chi(\alpha,(\mu+1)2^{-q-1})-\chi(\alpha,\mu 2^{-q-1})| \leqq 2^{-\lambda q}a.$$

It is thus, infinitely improbable that there exist any point common to an infinite sequence of intervals satisfying (9.10), and *a fortiori* infinitely improbable that any point lie in intervals of the binary subdivision, all of which from some stage on satisfy (9.10). This is in the first instance true for one value of a, but since the sum of a denumerable number of sets of zero measure is itself of zero measure, we may extend it to the case in which (9.10) is supposed to hold for *any* value of a. From this, Theorem VII follows at once.

We have already noted that if $\lambda < \frac{1}{2}$, then except for a set of cases of zero probability,

$$\lim_{\varepsilon \to 0} (\chi(\alpha, t+\varepsilon) - \chi(\alpha, t))/\varepsilon^{\lambda} = 0 \tag{9.11}$$

uniformly for all values of t.

This was proved in W_1 for the case $\lambda = 1, 4$, and the methods are substantially unchanged. The study of precisely what happens in the case $\lambda = \frac{1}{2}$ demands the use of more refined methods, and is not treated here.

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