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On a lemma of Littlewood and Offord on the distribution of certain sums *

By

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I. Introduction

Let $a_1, \dots, a_n$ be complex numbers greater than one in magnitude. LITTLEWOOD and OFFORD [1] showed that of the 2^n sums of the form

$$\sum_{j=1}^n \varepsilon_j a_j$$

in which each ε_j is ± 1 , the number of sums lying in any circle of unit radius could be no more than $c 2^n n^{-\frac{1}{2}} \log n$. ERDŐS [2] improved this upper bound to $C 2^n n^{-\frac{1}{2}} (1)$.

In this paper we show that binomial coefficient $C_{n[n/2]}$ is an upper bound, and the best one. ERDŐS conjectured that the same result might hold if the $a_1, \dots, a_n$ were vectors in any Hilbert space. We show below that the estimate (1) holds in any finite dimensional space; and we present a proof that $C_{n[n/2]}$ is an upper bound in any finite dimensional space for sufficiently large n .

II. Two Dimensions

In this section we prove the following theorem.

Theorem I. *If $\{a_j\}$ are n complex numbers of magnitude greater than unity, the number of sums of the form*

$$\sum_{j=1}^n \varepsilon_j a_j \quad (\varepsilon_j = \pm 1)$$

which lie within any circle of unit radius in the complex plane is less than or equal to the binomial coefficient $C_{n[n/2]}$ (where $[k]$ represents the integral part of k).

This result is a direct consequence of the following theorem.

Theorem II. *Let S be an n element (finite) set, X a class of subsets of S , and T an m element subset of S . If there are no distinct A and B in X such that*

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$A \supset B$ and either $A - B \subset T$ or $A - B \subset S - T$ then the number of elements X is less than or equal to $C_{n[n/2]}$.

To obtain Theorem I from this result we let S be the indices from 1 to n ; to each sum

$$\sum_{j=1}^n \varepsilon_j a_j$$

we associate the set of indices j for which $\varepsilon_j a_j$ is in the upper half plane, and we let X represent the class of sets of indices for which the corresponding sums lies in any particular unit radius circle. We let T be the set of indices for which a_j lies in the first or third quadrants. If $A - B \subset T$ then the sums corresponding to A and B differ by the sign of one or more terms each of which is of magnitude greater than one and which lie in the first quadrant. Since the sum of numbers in the first quadrant of magnitude greater than one is likewise such a number, the numbers represented by A and B must differ by twice such a number, and hence cannot both be in X . As similar remarks hold for the case $A - B \subset S - T$, application of Theorem II yields Theorem I.

Theorem II may be proven by use of the following two lemmas, the second of which is due to ERDŐS [2].

Consider a finite, n -element set S . Define a *chain* to be an $(n+1)$ element class of subsets of S that is totally ordered by inclusion. Each chain contains exactly one subset of S having j elements in it for each j less than n . A subset of a chain obtained by omitting the chain's k smallest and k largest elements for $0 \leq k \leq n/2$ will be called a *subchain*. Note that a subchain so formed will have $n+1-2k$ elements, the smallest of which will contain k elements of S while the largest contains $n-k$ elements of S . If n is even subchains will have an odd number of elements, and vice versa. If the subchain contains j elements its largest element will contain $\frac{n+j-1}{2}$ elements of S .

Lemma I. *The class of all subsets of a finite set can be expressed as the union of a collection of disjoint subchains.*

Proof. The proof will proceed by induction. For a one element set the set of all subsets is a subchain. Suppose S contains n elements, and the theorem holds for all sets having $n-1$ or fewer elements. Let a be an element of S . $S - \{a\}$ has $n-1$ elements and therefore the set of subsets of $S - \{a\}$ can be written as

$$\bigcup_{j=1}^{\alpha} X_j$$

with each X_j (for $1 \leq j \leq \alpha$) a subchain of $S - \{a\}$. Let the largest element of each X_j be denoted by L_j . Consider the following subchains of S .

$$Y_j = X_j \cup \{L_j \cup \{a\}\},$$

$$Y_{\alpha+j} = \{A \cup \{a\} \mid A \in (X_j - \{L_j\})\}.$$

Since every subset of S is either a subset of $S - \{a\}$ or the addition of a to a subset of $S - \{a\}$, all subsets of S are in the subchains given above. They are clearly disjoint. If there are $n - 2k$ elements in X_j the smallest element of X_j has k elements, the largest has $n - k - 1$ elements. The smallest element of Y_j has therefore k elements and the largest has $n - k$ elements, while the smallest element of Y_{a+j} has $k + 1$ elements and the largest has $n - 1 - k$ elements. Thus Y_j and Y_{a+j} are indeed subchains of S if X_j is a subchain of $S - \{a\}$.

Remark. The number of subchains containing q or more elements in a decomposition of the set of subsets of S according to Lemma I is $C_{n(n+q-1)/2}$ if $n - 1 + q$ is an even integer. In general, this number is $C_{n[(n+q)/2]}$.

Proof. The number of distinct subsets of S containing $\frac{n+q-1}{2}$ elements is

$$C_{n \frac{(n+q-1)}{2}} \quad \text{if} \quad \frac{n+q-1}{2}$$

is an integer. Each such subset lies in one and only one subchain of length greater than q in a decomposition of the set of subsets of S into disjoint subchains. For, as noted above, a subchain of length q contains as largest element a subset of S having $\frac{n+q-1}{2}$ elements if $n - 1 + q$ is an even integer. If $n - 1 + q$ is not an integer $n + q$ is and the number of subchains of length greater than q is equal to the number of length greater than $q + 1$.

Lemma II. (ERDÖS [2]) Let $X_1, \dots, X_q$ be disjoint sets of subsets of S with the property that no element of X_j contains any other element of X_j for $j = 1, \dots, q$. The number of elements in

$$\bigcup_{j=1}^q X_j$$

is then less than

$$\sum_{j=1}^q C_{n[(n+j)/2]}.$$

Proof. Let $M_1, \dots, M_\alpha$ be a collection of disjoint subchains whose union is S . The number of elements in the union of the X 's that lie in any subchain M_i is less than q . Thus, with

$$\bigcup_{j=1}^q X_j = Y$$

we have

$$[Y] = \sum_{i=1}^{\alpha} [Y \cap M_i] \leq \sum_{i=1}^{\alpha} \min \cdot (q, [M_i])$$

where $[Y]$ indicates the number of elements in Y . From the remark above we have

$$\begin{aligned}
 [Y] &\leq \sum_{r=1}^{n+1} \sum_{l=1}^{\alpha} \min(q, r) \delta_{[M_l], r} \\
 &\leq \sum_{r=1}^{n+1} \min(q, r) [C_{n[(n+r)/2]} - C_{n[(n+r+1)/2]}] \\
 &\leq \sum_{r=1}^{n+1} C_{n[(n+r)/2]} (\min(q, r) - \min(q, r-1)) \\
 &\leq \sum_{j=1}^q C_{n[(n+r)/2]}.
 \end{aligned}$$

The proof of Theorem II then proceeds as follows:

Let the subsets of T be expressed as a union of disjoint subchains according to Lemma I. For each $D \subset T$, let X_D denote the set of subsets B of $S-T$ such that $D \cup B \in X$. No element of X_D can contain any other, and the sets X_D are disjoint for all D that lie in a subchain of T by the hypotheses of the theorem. By Lemma II above the total number of elements of X whose intersection with T lie in a k element subchain in the decomposition of T , must be less than

$$\sum_{r=1}^k C_{n-m[(n-m+r)/2]}.$$

The total number of elements in X , by the remark above, must be less than

$$\begin{aligned}
 &\sum_{k=1}^{m+1} \sum_{r=1}^k C_{n-m[(n-m+r)/2]} (C_{m[(m+k)/2]} - C_{m[(m+k+1)/2]}) \\
 &= \sum_{r=1}^{m+1} \sum_{k=r}^{m+1} C_{n-m[(n-m+r)/2]} (C_{m[(m+k)/2]} - C_{m[(m+k+1)/2]}) \\
 &= \sum_{r=1}^{m+1} C_{n-m[(n-m+r)/2]} C_{m[(m+r)/2]} \\
 &= \sum_{r=-m}^m C_{m[m/2]+r} C_{n-m[(n-m)/2]+r} \\
 &= C_{n[n/2]}.
 \end{aligned}$$

This is the desired result.

III. p Dimensions

The arguments of the last section may be used in any number of dimensions; the set T can correspond to the indices of vectors in a real p dimensional space lying in any 2^p -ant and in the 2^p -ant obtained from it by reflection through the origin. The set $S-T$ may be chosen likewise; S will then consist of those indices for which the corresponding vectors lie in 4 particular 2^p -ants. Clearly the 2^p -ants can be oriented so that the number k of elements of S is

at least $(4/2^p)n$, for $p \geq 2$. For each choice of epsilons for those indices which lie outside of S , (of which there are 2^{n-k} possible choices), at most $C_{k[k/2]}$ can lie within a p -sphere of unit radius, by Theorem I above. The total number of sums which can lie with a unit p -sphere is therefore less than $2^{n-k} C_{k[k/2]}$, asymptotic value of which for large k is

$$2^n (\pi k/2)^{-\frac{1}{2}}, \quad \text{or} \quad (2^{p-2})^{\frac{1}{2}} C_{n[n/2]},$$

or

$$c_p 2^n n^{-\frac{1}{2}},$$

with

$$c_p = (2\pi)^{-\frac{1}{2}} 2^{p/2}.$$

This evaluation of the coefficient c_p is definitely not a best result. The 2^p -ants above may be replaced by cones of 60° half aperture; moreover, increasing the number of dimensions produces a spreading among the sums which is not at all taken into account in the argument above.

We deduce below that in a real p -dimensional vector space, the bound on number of sums in a unit p -sphere is $C_{n[n/2]}$ for sufficiently large n . If most of the vectors a_j point in essentially the same direction, the situation is essentially one dimensional and arguments of the type above prove the result. If, on the other hand, the vectors are spread widely, a different type of argument is required. We make use of Lemma III to obtain such:

Lemma III. *Let S be an n element set, X, Y, Z disjoint classes of subsets of S with the property that if $A \in X$, $B \supset A$ then $B \in Y$, and if $A \in X$, $B \subset A$ the $B \in Z$; then*

$$[X] \leq (C_{n[n/2]}/2^n) [X \cup Y \cup Z]$$

where $[X]$ represents the number of elements of X .

Proof. Let $n(j, X)$, $n(j, Y)$ and $n(j, Z)$ represent the number of elements of X, Y and Z respectively which contain exactly j elements of S . These symbols will satisfy the relations

$$(1) \quad \begin{cases} n(k+1, Y) \geq \frac{n-k}{k+1} (n(k, Y) + n(k, X)) \\ \geq \sum_{j=0}^k n(j, X) \cdot \frac{(n-j)! j!}{(n-(k+1))! (k+1)!} \end{cases}$$

and

$$(2) \quad \begin{cases} n(k-1, Z) \geq \frac{k}{n-(k-1)} (n(k, Z) + n(k, X)) \\ \geq \sum_{j=k}^{n+1} n(j, X) \frac{(n-j)! j!}{(k-1)! (n-k+1)!} \end{cases}$$

where we have assumed that $B \in Y$ if $B \supset C$, $C \in Y$ and $B \in Z$ if $B \subset C$, $C \in Z$. The smallest classes Y and Z for a given X will satisfy these latter relations.

The numerator $n-k$ in (1) represents the number of different $k+1$ element sets that can be formed which include each k element set in X and Y , while the denominator $k+1$ represents the maximum number of times a $k+1$ element set can be formed in this manner. Similar remarks apply to relation (2). Summing these relations over k yields

$$\begin{aligned}[X \cup Y \cup Z] &= \sum_{k=0}^{n+1} (n(k, X) + n(k, Y) + n(k, Z)) \\ &\geq \sum_{j=0}^{n+1} n(j, X) \frac{j!(n-j)!}{n!} \cdot \sum_{k=0}^{n+1} \frac{n!}{k!(n-k)!} \\ &\geq 2^n \sum_{j=0}^{n+1} n(j, X)/C_{n,j},\end{aligned}$$

from which we obtain

$$\begin{aligned}[X] &= \sum_{j=0}^{n+1} n(j, X) = C_{n[n/2]} \sum_{j=0}^{n+1} n(j, X)/C_{n[n/2]} \\ &\leq C_{n[n/2]} \sum_{j=0}^{n+1} n(j, X)/C_{n,j} \\ &\leq (C_{n[n/2]}/2^n) [X \cup Y \cup Z],\end{aligned}$$

as desired.

The result if most vectors lie essentially in one single direction is Lemma IV. Here we assume that all binomial coefficient arguments are large and write $2^{-\alpha n} C_{\alpha n[\alpha n/2]}$ as $\alpha^{\frac{1}{2}} 2^{-n} C_{n[n/2]}$ for convenience; no result is critical to this approximation.

Lemma IV. *If more than αN of the N vectors a_j lie in a cone C of half aperture ϑ_0 for $1 > \alpha > \frac{2}{15}$ and ϑ_0 determined by condition (I) below, then at most $C_{n[n/2]}$ of the sums*

$$\sum_{j=1}^n \varepsilon_j a_j$$

can lie in a sphere of radius 1 where each ε_j is ± 1 and the a_j are vectors of magnitude greater than 1 in any Hilbert space.

Proof. Let the cone center be in the direction n_0 . At least one vector a_k must lie outside a cone of half angle 60° centered about n_0 , or the conjectured property may be proven by the methods above. This vector may be written as $a_k = c(x_0 n_0 + X_1 n_1)$ with $|c| \geq 1$, $|x_0|^2 + |x_1|^2 = 1$, $(n_j, n_k) = \delta_{jk}$, $|x_0| < \frac{1}{2}$. Moreover at least one vector, a_j , must lie outside the cone of 60° half angle centered about n_0 and a similar one centered about n_1 or the conjectured property may be proven by the same methods. Let this vector be written as $a_j = d(y_0 n_0 + y_1 n_1 + y_2 n_2)$ with

$$\sum_{j=0}^2 |y_j|^2 = 1, \quad (n_m, n_n) = \delta_{mn}, \quad |d| > 1, \quad |y_0| < \frac{1}{2}, \quad |y_1| < \frac{1}{2}.$$

Any of the vectors a_n lying in the cone C can be written in the same notation as

$$a_p = r_p (n_0 \cos \vartheta_p + n_1 \sin \vartheta_p \varepsilon_{1p} + n_2 \sin \vartheta_p \varepsilon_{2p} + n_3 \sin \vartheta_p \varepsilon_{3p})$$

with

$$\sum_{j=1}^3 |\varepsilon_{jp}|^2 = 1, \quad |r_p| \geq 1, \quad 0 < \vartheta_p < \vartheta_0.$$

Consider the hyperplane whose normal n is in the direction of

$$x_1 y_2 n_0 - x_0 y_2 n_1 + (x_0 y_1 - x_1 y_0) n_2.$$

This hyperplane contains the vectors a_j and a_k . Furthermore the magnitude of the scalar product of its normal with any vector in C satisfies

$$\begin{aligned} |(a_p, n)| &\geq |x_1 y_2 \cos \vartheta_p| - |\sin \vartheta_p \varepsilon_{1p} x_0 y_2| \\ &\quad - |\sin \vartheta_p (1 - \varepsilon_{1p})^{\frac{1}{2}} (x_0 y_1 - x_1 y_0)| \end{aligned}$$

so that we can choose a real angle ϑ_0 such that this magnitude will always exceed one half.

Let S be the sphere of radius 1 in question and let e be its center. To each sum lying in S we can correspond three others lying in spheres centered about three of $e \pm 2a_j$, $e \pm 2a_k$, $e \pm 2a_j \pm 2a_k$ by inverting the signs of a_j , a_k and both. By the conditions above

$$\frac{(a_j, a_k)}{\|a_j\| \cdot \|a_k\|} \leq (1 + \sqrt{3})/4.$$

Suppose

$$\frac{(a_j, a_k)}{\|a_j\| \cdot \|a_k\|} \leq \frac{1}{2}.$$

These spheres cannot then overlap; consequently only one fourth of the sums that lie in a hyperplane sheet containing S normal to n of thickness 2 can lie in S itself.

By hypothesis, at least αN of the vectors a_n lie within C , so that they have components greater than $\frac{1}{2}$ normal to this sheet. The number of sums lying in this sheet must therefore be less than $2 C_{\alpha n \lfloor \alpha n/2 \rfloor} 2^{n - \lfloor \alpha n \rfloor}$ which is approximately $2\alpha^{-\frac{1}{2}} C_{n \lfloor n/2 \rfloor}$. The number that can lie in S is therefore (approximately) less than $\alpha^{-\frac{1}{2}}/2 C_{n \lfloor n/2 \rfloor}$.

If

$$(1 + \sqrt{3})/4 \geq \frac{(a_j, a_k)}{\|a_j\| \cdot \|a_k\|} \geq \frac{1}{2}$$

the spheres can overlap. We divide the sphere S into an overlap region S_1 and a non-overlap region S_2 . The dimension of the overlap region normal to

the hyperplane sheet can be shown to be less than

$$2 \left(\frac{\sqrt{3}-1}{2} \right)^{\frac{1}{2}}.$$

If ϑ_0 is chosen to satisfy the following relation (I):

$$\left(\frac{3}{8} \right)^{\frac{1}{2}} \cos \vartheta_0 - \left(\frac{\sqrt{3}+1}{4} \right) \sin \vartheta_0 \geq \left(\frac{\sqrt{3}-1}{2} \right)^{\frac{1}{2}},$$

then by the same argument as above the number of sums lying in the hyperplane sheet containing all overlap regions must be less than $\alpha^{-\frac{1}{2}} C_{n[n/2]}$. The number $[S_1]$ in S_1 must be less than half of this or $\frac{1}{2} \alpha^{-\frac{1}{2}} C_{n[n/2]}$. The previous argument tells us that

$$4[S_2] + 2[S_1] < 2\alpha^{-\frac{1}{2}} C_{n[n/2]}.$$

We may therefore conclude that the number of sums lying in S is less than $\frac{3}{4} \alpha^{-\frac{1}{2}} C_{n[n/2]}$, or, if $\alpha > \frac{9}{16}$, then $C_{n[n/2]}$.

Theorem III. *If $a_1, a_2, \dots, a_n$ are vectors of length greater than 1 in a real vector space of finite dimension d , there is a number N_d such that if n is greater than N_d the number of sums of the form*

$$\sum_{j=1}^n a_j \varepsilon_j$$

with each $\varepsilon_j = \pm 1$, that can lie within a d -sphere of radius 1 is less than $C_{n[n/2]}$.

Proof. Let the d dimensional space in question be divided into regions in a manner symmetric under inversion through the origin such that the sum of any two vectors of length greater than s that lie in one region is also a vector of length greater than s lying in the same region, for all positive real s .

Given a sphere S of unit radius, application of Lemma III shows that

$$[(S+R_\alpha) \cup S \cup (S+(-R_\alpha))] \geq (2^{n_\alpha}/C_{n_\alpha[n_\alpha/2]}) [S]$$

where $(-R_\alpha)$ represents the region obtained by inverting region R_α through the origin; the brackets here denote the number of sums lying in the contained region; n_α is the number of a_j in R_α and $(-R_\alpha)$.

If this relation is summed over all the regions R_α it is seen that

$$2^n + R > \left[\sum_{\alpha} 2^{n_\alpha}/C_{n_\alpha[n_\alpha/2]} \right] [S]$$

where R represents the overcounting of the number of sums in all space that takes place as a result of the overlap of the various regions $S \cup (S+R_\alpha) \cup (S+(-R_\alpha))$ for different α .

The theorem is proven if for n sufficiently large a division into regions R_α can be found for which we have

$$2^n + R \leq \sum_{\alpha} (2^{n_\alpha}/C_{n_\alpha[n_\alpha/2]}) C_{n[n/2]}$$

hence $[S] \leq C_{n \lfloor n/2 \rfloor}$. Upon making use of the asymptotic properties of binomial coefficients the desired relation can be reduced to

$$1 + R 2^{-n} \leq \sum_{\alpha} (n_{\alpha}/n)^{\frac{1}{2}}$$

which will follow from

$$1 + R 2^{-n} \leq (n_1/n)^{\frac{1}{2}} + (1 - (n_1/n))^{\frac{1}{2}}.$$

If the region R_1 is contained within a cone of half aperture ϑ_0 , by Lemma IV we need only consider here the case $1 - n_1/n > \frac{1}{16}$; by suitable orientation of the regions, n_1/n may always be chosen greater than some m which depends only upon geometry and not upon n . If, in addition $R 2^{-n}$ can be shown to approach zero as n increases, the theorem is proven.

The following construction shows that indeed $R 2^{-n}$ can be made to approach zero. Let the d dimensional space be divided into the analogue of quadrants (i.e., 2^d -ants); let a cone of half aperture ϑ_0 be constructed such that one sheet lies in the center of one quadrant. Within it construct a convex non empty region R_1 with hyperplane boundaries all of which intersect at the origin. This entire structure should then be reflected through the origin. Choose as the regions R_{α} all 2^d -ants except those containing the cone; and, the various regions into which the hyperplane boundaries of R_1 divide the 2^d -ants containing R_1 or its image under inversion in the origin, when they are extended to meet the boundaries of these 2^d -ants.

The overcounting that contributes to R is confined to hyperplane sheets of thickness 2 containing the boundaries of each region. Let the region within R_1 such that the scalar product of every vector within it with each of the normals to each of the boundaries of R_1 is greater than ε be denoted by A_{ε} . Since R_1 is nonempty there must exist a nonempty region A_{ε} for some positive ε . Choose ε_0 such that A_{ε_0} is nonempty. Let the whole system of regions be oriented such that the maximum number of vectors α lie within A_{ε_0} . As n increases the number of vectors lying within A_{ε_0} can be made arbitrarily large. The number of sums lying with any hyperplane sheets of finite thickness containing boundaries can therefore be made arbitrarily small compared with 2^n , by application of Lemma II. This proves the theorem.

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