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The Existence of Wave Operators in Scattering Theory

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1. Introduction

We first recall the definition of “wave operators” for the Schrödinger equation.

Let H_0 be the self-adjoint operator in $L^2(\mathbb{R}^n)$ defined by $-\Delta = -\sum_{j=1}^n \partial^2 / \partial x_j^2$. If V is a real valued measurable function such that

$$\int (1 + |x|^2)^{-M} |V(x)|^2 dx < \infty \quad (1.1)$$

for some M , then the operator $-\Delta + V$ with domain $\mathcal{S}(\mathbb{R}^n)$ has self-adjoint extensions because the defect indices are equal. Let H denote such an extension.

The solutions of the corresponding time dependent Schrödinger equation

$$i \partial v / \partial t = H v$$

are given by $t \rightarrow e^{-itH} v$ where $v \in L^2(\mathbb{R}^n)$ denotes the state at $t=0$. (Recall that e^{-itH} is a unitary operator defined by means of the spectral theorem.) It is natural to ask if there is a solution which as $t \rightarrow +\infty$ or $t \rightarrow -\infty$ is close to a given solution of the unperturbed equation, that is, close to $t \rightarrow e^{-itH_0} u$ for a given $u \in L^2$. Since e^{itH} is unitary, this means that v should be close to $e^{itH} e^{-itH_0} u$. We are therefore led to ask if the limits

$$W_{\pm} u = \lim_{t \rightarrow \pm \infty} e^{itH} e^{-itH_0} u \quad (1.2)$$

exist for every u . Since the exponentials are unitary, this is true if the limits exist for all u in a dense subset of L^2 , and W_+ , W_- are then isometric. Moreover, the intertwining property

$$e^{isH} W_{\pm} = W_{\pm} e^{isH_0}, \quad s \in \mathbb{R}, \quad (1.3)$$

follows immediately. It implies that H is the direct sum of an operator unitarily equivalent with H_0 in the range of $W_{\pm}$ and a self-adjoint operator in the orthogonal complement. When W_+ and W_- both exist and have the same range then $S = W_+^{-1} W_- = W_+^* W_-$ is a unitary operator commuting with H_0 , called the *scattering operator*. The equation $S u_- = u_+$ means that there is a solution of the perturbed Schrödinger equation which is asymptotic to $e^{-itH_0} u_-$ at $-\infty$ and to $e^{-itH_0} u_+$ at $+\infty$, so S describes the asymptotic effects of the perturbation.

The existence of wave operators has long been known (see Cook [4], Hack [7], Jauch-Zinnes [9], Kuroda [11]) for potentials which roughly speaking are $O(|x|^{-1-\varepsilon})$ at infinity for some $\varepsilon > 0$. A large class of such perturbations, called short range perturbations, is discussed in Section 2. We allow perturbations by higher order operators and replace $-\Delta$ by any self-adjoint differential operator $P(D)$ with constant coefficients. (See also Jörgens-Weidmann [10], Veselić-Weidmann [12, 13] and the references given in these papers.)

For the Coulomb potential $V(x) = -1/|x|$ it was proved by Dollard [5, 6] that the wave operators do not exist unless the definition is modified by substitution of another convolution operator $e^{-iW_t(D)}$ for e^{-itH_0} . Further work in this direction is due to Buslaev-Matveev [3], Alsholm-Kato [2] and Alsholm [1]. In Section 3 we shall improve their results and show that modified wave operators exist for any $V = V_S + V_L$ where V_S is a short range potential and

$$|D^\alpha V_L(x)| \leq C(1 + |x|)^{-\varepsilon - |\alpha|}, \quad |\alpha| \leq k, \quad (1.4)$$

where $\varepsilon > \frac{1}{2}$ if $k = 1$, $\varepsilon > \frac{1}{3}$ if $k = 2$ or $\varepsilon > 0$ if $k \geq 3$. More precisely we show that it suffices to assume that

$$|D^\alpha V_L(x)| \leq C(1 + |x|)^{-m(|\alpha|)}, \quad |\alpha| \leq 3, \quad (1.5)$$

where $m(0), \dots, m(3)$ are positive and $m(1) + m(3) > 4$. (Buslaev and Matveev assumed (1.4) with $k = [n/2] + 2 + [1/\varepsilon]$; Alsholm and Kato assumed (1.4) with $k = 1$, $\varepsilon > \frac{1}{2}$ together with a condition depending on ε on the second order derivatives; Alsholm [1] made similar assumptions on a sequence of derivatives which are too complicated to reproduce here.)

Our proof is close to that of Buslaev and Matveev in that we apply the method of stationary phase rather than the formula

$$e^{i|p|^2 t/2} F(x) e^{-i|p|^2 t/2} = e^{-i|x|^2/2t} F(pt) e^{i|x|^2/2t}$$

of Alsholm and Kato which can only be used to study perturbations of the Laplacean. We differ from all these authors by using exact solutions of the Hamilton-Jacobi equation rather than approximate solutions obtained by iteration.

2. Short Range Perturbations

Let $P(D)$, $D = -i\partial/\partial x$, be a partial differential operator with constant real coefficients in $\mathbb{R}^n$. It is well known and elementary that the closure in $L^2(\mathbb{R}^n)$ of $P(D)$, defined originally on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, is a self-adjoint operator H_0 . If F denotes the Fourier transformation,

$$(Fu)(\xi) = \hat{u}(\xi) = \int e^{-i\langle x, \xi \rangle} u(x) dx,$$

then FH_0F^{-1} is multiplication by $P(\xi)$ on the L^2 functions such that the product is also in L^2 . Let

$$V(x, D) = \sum_{|\alpha| \leq \mu} V_\alpha(x) D^\alpha$$

be a differential operator such that for some M and all α

$$\int |V_\alpha(x)|^2 (1 + |x|^2)^{-M} dx < \infty. \quad (2.1)$$

Then V maps $\mathcal{S}$ into L^2 . We assume that V is not only symmetric on $\mathcal{S}$ but that $P(D) + V(x, D)$ with domain $\mathcal{S}$ has a self-adjoint extension H . Our purpose is to examine the existence of the wave operators (1.2). In doing so we exclude the trivial case where $P(D)$ is independent of D .

Take $u \in \mathcal{S}$ and set $u_t = e^{-itH_0}u$. Then $t \rightarrow u_t$ is a C^∞ function with values in $\mathcal{S}$, which proves the existence of the following derivative

$$\frac{d}{dt}(e^{itH} e^{-itH_0}u) = e^{itH}(iH - iH_0)u_t = ie^{itH}Vu_t.$$

The limits (1.2) exist if the norm is integrable, that is, if

$$\int_{-\infty}^{\infty} \|Vu_t\| dt < \infty. \quad (2.2)$$

We shall determine completely when there is a dense set $M \subset L^2$ with $FM \subset C_0^\infty$ for which (2.2) is valid. However, it is known (Alsholm [1]) that the wave operators may exist although (2.2) is never valid when $\hat{u} \neq 0$ is in C_0^∞ . To examine (2.2) we note that

$$u_t(x) = (2\pi)^{-n} \int e^{i(\langle x, \xi \rangle - tP(\xi))} \hat{u}(\xi) d\xi.$$

The exponential has a critical point when $x = tP'(\xi)$. To make x/t bounded away from 0 then we shall take $\hat{u} \in C_0^\infty(\Omega)$ where

$$\Omega = \{\xi; P'(\xi) \neq 0\}. \quad (2.3)$$

The complement of Ω is of measure 0 since P is not a constant, so such functions are dense in L^2 . Choose positive numbers r and R so that

$$2r \leq |P'(\xi)| \leq R/2 \quad \text{when } \xi \in \text{supp } \hat{u}.$$

If $|x/t| \leq r$ or $|x/t| \geq R$, the normalized exponent

$$\xi \rightarrow (\langle x, \xi \rangle - tP(\xi)) / (|x| + |t|)$$

belongs to a compact set of functions of ξ having no critical point in $\text{supp } \hat{u}$. Hence Lemma A.1 shows that for every N

$$|u_t(x)| \leq C_N (1 + |x| + |t|)^{-N} \quad \text{if } |x/t| \leq r \text{ or } |x/t| \geq R.$$

The same estimate is valid for $D^\alpha u_t = e^{-itH_0} D^\alpha u$, so (2.1) gives for any N

$$\left(\int_{|x/t| \notin (r, R)} |Vu_t|^2 dx \right)^{\frac{1}{2}} \leq C'_N (1 + |t|)^{-N}. \quad (2.4)$$

Thus the corresponding contribution to the integral in (2.2) is always convergent.

Without paying attention to the properties of P one obtains a crude estimate of the remaining part of (2.2) from Parseval's formula, which gives

$$\int |D^\alpha u_t|^2 dx = \int |D^\alpha u|^2 dx < \infty \quad (2.5)$$

for every α . In view of Sobolev's inequalities we have

$$\int_{B_x} |Vu_t|^2 dy \leq C \sum_{|\alpha| \leq n} \sum_{|\beta| \leq \mu} \int_{B_x} |V_\beta|^2 dy \int_{B_x} |D^{\alpha+\beta} u_t|^2 dy$$

where $B_x = \{y; |x-y| \leq 1\}$. If

$$\sum_{\alpha} \int_{B_x} |V_\alpha(y)|^2 dy \leq I(t)^2 \quad \text{when } r < |x/t| < R,$$

it follows that

$$\int_{r < |x/t| < R} |Vu_t|^2 dy \leq C I(t)^2,$$

which gives (2.2) if I is integrable. Hence we have proved

Theorem 2.1. *Let $P(D)$ be a non-constant differential operator with constant real coefficients, and let $V(x, D) = \sum_{|\alpha| \leq \mu} V_\alpha(x) D^\alpha$ be a differential operator with coefficients in L^2_{loc} such that*

$$\sum_{\alpha} \int_1^\infty \left(\sup_{t < |x| < 2t} \int_{|x-y| < 1} |V_\alpha(y)|^2 dy \right)^{\frac{1}{2}} dt < \infty. \quad (2.6)$$

If H is a self-adjoint extension of $P(D) + V(x, D)$ with domain $\mathcal{S}$ and H_0 is the closure of $P(D)$ with domain $\mathcal{S}$, then the limits (1.2) exist.

Example. If $V_\alpha \in L^2_{\text{loc}}$ and $|V_\alpha(x)| \leq C|x|^{-1-\varepsilon}$, $\varepsilon > 0$, then (2.6) is fulfilled so the wave operators exist.

The weak link in the preceding argument is that we did not examine how the contributions to the L^2 norm of u_t are distributed over the annulus $r < |x/t| < R$. This can be done by means of the method of stationary phase. For the sake of simplicity we shall first assume that $\det P''(\xi) \neq 0$.

With the notations above we set $y = x/t$, and obtain if $r \leq |y| \leq R$

$$u_t(x) = u_t(y) = (2\pi)^{-n} \int e^{it(\langle y, \xi \rangle - P(\xi))} \hat{u}(\xi) d\xi.$$

Since we assume that $\det P'' \neq 0$, the complement of

$$\Omega_0 = \{\xi \in \Omega; \det P''(\xi) \neq 0\}$$

is of measure 0, so $C_0^\infty(\Omega_0)$ is dense in L^2 . We may therefore assume that $\hat{u} \in C_0^\infty(\Omega_0)$, and obtain using Lemma A.4

$$|u_t(x)| \leq C|t|^{-n/2}, \quad r < |x/t| < R, \quad (2.7)$$

Similar estimates are valid for $D^\alpha u_t$, so it follows that

$$\int_{r < |x/t| < R} |Vu_t|^2 dx \leq C \sum_{\alpha} \int_{r < |x/t| < R} |V_\alpha(x)|^2 dx / |t|^n.$$

Hence (2.2) is fulfilled if for every α

$$\int_1^\infty \left(\int_{r < |x| < R} |V_\alpha(\pm tx)|^2 dx \right)^{\frac{1}{2}} dt < \infty. \quad (2.8)$$

It suffices of course to have (2.8) for example when $r = 1$, $R = 2$.

It was pointed out to the author by S. Agmon that (2.8) can be further relaxed in a way which is useful for the application to the many-body problem. The point is that by Lemma A.1 we know that $u_t(ty)$ is bounded by any power of $1/t$ if $r \leq |y| \leq R$ and y is outside a fixed neighborhood of $\{P'(\xi); \xi \in \text{supp } \hat{u}\}$. It is therefore sufficient to know (2.8) when the integral is taken over compact subsets of the open set

$$\{P'(\xi); \xi \in \Omega_0\}.$$

Moreover, since P' is a local diffeomorphism in Ω_0 , the inverse image $P'^{-1}(\Sigma)$ of any closed set Σ of measure 0 is closed and of measure 0. Removing it from Ω_0 we obtain an open set Ω_1 such that $C_0^\infty(\Omega_1)$ is still dense in L^2 . Thus we are led to the following definition.

Definition 2.2. The operator $V = \sum_{|\alpha| \leq \mu} V_\alpha(x) D^\alpha$ is said to be a short range perturbation with respect to P if for every α the coefficient V_α satisfies (2.1) and almost every $x_0 \in P'(\mathbb{R}^n)$ has a neighborhood ω such that

$$\sum_{\alpha} \int_1^\infty \left(\int_\omega |V_\alpha(\pm tx)|^2 dx \right)^{\frac{1}{2}} dt < \infty. \quad (2.9)$$

Summing up the discussion above, we have proved

Theorem 2.3. Let $P(D)$ be a differential operator with constant real coefficients such that $\det P''(\xi)$ is not identically 0. Denote the self-adjoint realization of $P(D)$ in $L^2(\mathbb{R}^n)$ by H_0 . If V is a short range perturbation with respect to P and $P(D) + V$ with domain $\mathcal{S}$ has a self-adjoint extension H , then the limits (1.2) exist and define isometric operators intertwining H and H_0 .

Note that short range perturbations form a linear set for fixed P . Moreover, if the variables x in $\mathbb{R}^n$ are split into two groups $x' \in \mathbb{R}^{n'}$ and $x'' \in \mathbb{R}^{n''}$, $n' + n'' = n$, and if $V_\alpha(x')$ is a short range perturbation in $\mathbb{R}^{n'}$ with respect to the Laplacean, then $V_\alpha(x') D^\alpha$ is a short range perturbation in $\mathbb{R}^n$ with respect to the Laplacean there. The only exceptional directions (x', x'') are those where $x' = 0$ or x' is one of the old exceptional directions. Thus Theorem 2.3 is applicable to the many body problem if the two body interactions are of short range.

We shall now study the case where P'' has lower rank. Let the maximal rank of P'' be k . We can label the coordinates so that $\det P''_{\xi' \xi'}$ is not identically 0 if $\xi' = (\xi_1, \dots, \xi_k)$. The definition of Ω_0 above is changed to

$$\Omega_0 = \{\xi \in \Omega; \det P''_{\xi' \xi'}(\xi) \neq 0\}. \quad (2.10)$$

With $\hat{u} \in C_0^\infty(\Omega_0)$ having support near a point $\xi_0 \in \Omega_0$ we shall study the properties of

$$u_t(ty) = (2\pi)^{-n} \int e^{it(\langle y, \xi \rangle - P(\xi))} \hat{u}(\xi) d\xi. \quad (2.11)$$

If V is a sufficiently small neighborhood of ξ_0 , then $\{y = P'(\xi); \xi \in V\}$ is a k -dimensional manifold parametrized by $y' = (y_1, \dots, y_k)$, thus of the form $y'' = g(y')$, $g \in C^\infty$. Now we apply the method of stationary phase to the integral with respect to ξ' in (2.11). The critical point of the phase is then determined by $y' = \partial P / \partial \xi'$,

which by the implicit function theorem can be solved as $\xi' = \varphi(y', \xi'')$. At this point the phase is

$$f(y, \xi'') = \langle y', \varphi \rangle + \langle y'', \xi'' \rangle - P(\varphi, \xi'')$$

with differential

$$df = \langle \varphi, dy' \rangle + d\langle y'', \xi'' \rangle - P_{\xi''}(\varphi, \xi'') d\xi''.$$

In particular,

$$f'_{\xi''} = y'' - P_{\xi''}(\varphi, \xi'') = y'' - g(y')$$

since $f'_{\xi''} = 0$ when $y'' = g(y')$ and φ is independent of y'' . Thus

$$f(y, \xi'') = f_0(y) + \langle y'' - g(y'), \xi'' \rangle.$$

For each term in the asymptotic expansion of the integral with respect to ξ' in (2.11), the following integration with respect to ξ'' is therefore a Fourier transform. It follows that for any integer N

$$|u_t(y)| \leq C_N |t|^{-k/2} ((1 + |t| |y'' - g(y')|)^{-N} + |t|^{-N}), \quad r < |y| < R. \quad (2.12)$$

Hence

$$\begin{aligned} & \int_{r < |x/t| < R} |V_0(x)|^2 |u_t(x)|^2 dx \\ & \leq C (|t|^{-3} + \int_{r < |y| < R} |V_0(ty)|^2 (1 + |t| |y'' - g(y')|)^{-N} |t|^{n-k} dy). \end{aligned}$$

We have now proved

Theorem 2.4. *Let P be a non-constant differential operator with constant real coefficients and denote by k the maximal rank of $P''(\xi)$. Let Ω_0 be defined by (2.10) and denote by Ω_1 an open subset such that $m(\Omega_0 \setminus \Omega_1) = 0$. Assume that V_α satisfies (2.1) and that for every compact subset K of Ω_1*

$$\sum_{\alpha=1}^{\infty} \int dt \left(\int |V_\alpha(\pm ty)|^2 (1 + t d_K(y))^{-N} |t|^{n-k} dy \right)^{\frac{1}{2}} < \infty \quad (2.13)$$

for some N . Here $d_K(y)$ is the distance from y to the compact set $P'(K)$ in the immersed k -dimensional manifold $P'(\Omega_1)$. Then the wave operators $W_{\pm}$ can be defined by (1.2), and they are isometric operators intertwining H and H_0 .

Theorem 2.1 is of course a very special case of Theorem 2.4 but it is clearly the best result which follows from Theorem 2.4 if one ignores completely the properties of P .

Given a symmetric operator $P(x, D)$ with coefficients satisfying (2.1) there may very well be several ways of writing $P(x, D) = P(D) + V(x, D)$ so that the hypotheses in the preceding theorem are fulfilled. A well known argument in the theory of multi-channel scattering then shows that the different wave operators which can be defined have orthogonal ranges:

Theorem 2.5. Suppose that $P_1(D) + V_1(x, D) = P_2(D) + V_2(x, D)$ with domain $\mathcal{S}$ has a self-adjoint extension H , and denote by H_j the closure of $P_j(D)$ with domain $\mathcal{S}$. If say the wave operators

$$W_+^j = s\text{-}\lim_{t \rightarrow +\infty} e^{itH} e^{-itH_j}, \quad j = 1, 2,$$

exist, then their ranges are orthogonal if $P_1 \neq P_2$.

Proof. If $u_j \in \mathcal{S}(\mathbb{R}^n)$, then the unitarity of e^{itH} gives

$$\begin{aligned} (W_+^1 u_1, W_+^2 u_2) &= \lim_{t \rightarrow +\infty} (e^{-itH_1} u_1, e^{-itH_2} u_2) \\ &= \lim_{t \rightarrow +\infty} (2\pi)^{-n} \int e^{it(P_2(\xi) - P_1(\xi))} \hat{u}_1 \bar{\hat{u}}_2 d\xi. \end{aligned}$$

Taking $u_1 = u_2 \neq 0$ we conclude that $P_1 - P_2$ cannot be constant, and if u_j is chosen with support where $P_1' - P_2' \neq 0$, then Lemma A.1 shows that the limit is 0.

Example. In $\mathbb{R}^2$ let

$$P(x, D) = D_1^2 + D_2 \chi D_2$$

where $\chi \in C^\infty$ is 1 when $x_2^4 > x_1^2 + 1$ and 0 when $x_2^4 < x_1^2$. Furthermore we assume that $D^\alpha \chi$ is bounded by a polynomial when $|\alpha| \leq 1$. Since the coefficients are real a self-adjoint extension exists, and from Theorems 2.3, 2.4 it follows that the wave operators exist if $P(x, D)$ is regarded as a perturbation either of $D_1^2 + D_2^2$ or D_1^2 .

Examples such as the preceding one show clearly that the hypotheses which allow one to prove the existence of wave operators are far too general for asymptotic completeness to be expected.

3. Long Range Perturbations

We shall now study modified definitions of wave operators which are applicable to perturbations such as the Coulomb potential which are not of short range. The idea is as follows. Let $W_t(\xi) = W(\xi, t)$ be a real valued C^∞ function. Then the convolution operator $\exp -iW_t(D)$, consisting of multiplication of the Fourier transform by $\exp -iW_t(\xi)$, is a unitary operator. If the limit

$$W_\pm u = \lim_{t \rightarrow \pm\infty} e^{itH} e^{-iW_t} u \quad (3.1)$$

exists for a dense set of $u \in L^2(\mathbb{R}^n)$, it exists for every u and $W_\pm$ is an isometric operator. Moreover, if H_0 is the self-adjoint operator defined by $P(D)$, then

$$e^{isH} W_\pm e^{-isH_0} u = W_\pm \lim_{t \rightarrow \pm\infty} \exp i(W_{t+s} - W_t - sH_0) u$$

provided that the latter limit exists. By the dominated convergence theorem the limit is equal to u if for almost every ξ and every $s \in \mathbb{R}$

$$\lim_{t \rightarrow \infty} (W(\xi, t+s) - W(\xi, t) - sP(\xi)) = 0. \quad (3.2)$$

When (3.2) is valid it follows that (3.1) gives isometric operators intertwining H and H_0 .

One might object to the modified definition of wave operators that it depends on a fairly arbitrary choice. To examine this point we denote by $\tilde{W}$ another function of (ξ, t) such that the corresponding limits $\tilde{W}_\pm$ exist. We shall determine when the range of $W_\pm$ is equal to that of $\tilde{W}_\pm$ (same sign). Since $e^{itH} e^{-iW_t} \rightarrow W_+$ strongly, it follows that $e^{iW_t} e^{-itH} u \rightarrow W_+^* u$ weakly, and $W_+^* u$ has the same norm as u if $u \in \text{Im } W_+$ so the limit is in the norm then. If W_+ and $\tilde{W}_+$ have the same range, we can replace u by $\tilde{W}_+ u$ and obtain

$$W_+^* \tilde{W}_+ u = \lim e^{i(W_t - \tilde{W}_t)} u$$

with strong convergence. Hence $W_+^* \tilde{W}_+ u = Fu$ where F is the weak L^∞ limit of $\exp i(W_t(\xi) - \tilde{W}_t(\xi))$. It follows that $|F|=1$ almost everywhere and that $F^{-1} \exp i(W_t(\xi) - \tilde{W}_t(\xi)) \rightarrow 1$ weakly in L^∞ . Hence there is convergence in measure, that is, for every compact set K and every $\varepsilon > 0$

$$m\{\xi \in K; |e^{i(W(\xi, t) - \tilde{W}(\xi, t))} - F(\xi)| > \varepsilon\} \rightarrow 0, \quad t \rightarrow +\infty. \quad (3.3)$$

The argument can obviously be reversed so we have proved

Theorem 3.1. *Assume that the limits (3.1) exist as well as the corresponding limits $\tilde{W}_\pm$ when W is replaced by $\tilde{W}$. Then $W_\pm$ has the same range as $\tilde{W}_\pm$ (same sign) if and only if $\exp i(W(\xi, t) - \tilde{W}(\xi, t))$ converges in measure as $t \rightarrow \pm\infty$ to functions $F_\pm$. In that case $\tilde{W}_\pm = W_\pm F_\pm(D)$.*

We note that the unitarity of the scattering operator would be independent of the choice of W in this situation but that the scattering operator itself is not.

As stated in the introduction we shall consider perturbations which can be written as the sum of a short range perturbation and a rather smooth long range perturbation. It is then important to make the long range part as smooth as possible by means of a regularization leaving a short range error. Similar but less elaborate arguments occur in Alsholm and Kato [2] and Alsholm [1]. We first prove a preliminary lemma (cf. Alsholm [1]).

Lemma 3.2. *Let $V \in C^k(\mathbb{R}^n)$ and assume that*

$$|D^\alpha V(x)| \leq C(1 + |x|)^{-m(|\alpha|)}, \quad |\alpha| \leq k, \quad (3.4)$$

where $m(0), \dots, m(k)$ are positive numbers. Then the same estimates are valid with another C if m is replaced by the smallest concave majorant m' such that $m'(j) - j$ is decreasing.

Proof. Integrating the differential of $D^\alpha V$ from ∞ to x we conclude that

$$|D^\alpha V(x)| \leq C(1 + |x|)^{1 - m(|\alpha| + 1)}, \quad |\alpha| < k,$$

provided that the exponent is negative. By repeated use of this argument we can replace m by m^* where

$$m^*(j) = \max_{0 \leq i \leq k-j} m(j+i) - i$$

is a majorant of m and $m^*(j) - j$ is decreasing. Now consider the derivatives of $V(x + ty)$ with respect to t of orders a and b , both $\leq k$. If

$$|y|^a (1 + |x|)^{-m^*(a)} = c^{a-b} |y|^b (1 + |x|)^{-m^*(b)},$$

that is,

$$|y| = c(1 + |x|)^{(m^*(a) - m^*(b))/(a-b)} = R,$$

we have $R < |x|/2$ if c is sufficiently small. The derivatives of $V(x + t y)$ of orders a and b with respect to t have the same bounds when $|t| < 1$, so they are bounds for the intermediate derivatives when $t = 0$. If $a < j < b$ and $|\alpha| = j$, we obtain

$$R^j |D^\alpha V(x)| \leq C(1 + |x|)^{-m^*(b)} R^b,$$

and if the definition of R is introduced we see that m^* can be replaced by the smallest concave majorant.

Lemma 3.3. *Suppose that V satisfies (3.4) and that*

$$0 < \delta < \max_{0 < j \leq k} (m(j) - 1)/j, \quad \delta < 1.$$

Then one can split $V = V_1 + V_2$ so that for some $\varepsilon > 0$

$$|V_1(x)| \leq C(1 + |x|)^{-1-\varepsilon}, \quad (3.5)$$

thus V_1 is of short range, and for all α

$$|D^\alpha V_2(x)| \leq C_\alpha(1 + |x|)^{-m'(|\alpha|)}, \quad (3.6)$$

$$m'(j) = \max_{i \leq j, i \leq k} m(i) + \delta(j - i). \quad (3.7)$$

Proof. Choose a function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ when $|x| < 1$ and $\psi(x) = 0$ when $|x| > 2$, and set

$$v_0(x) = \psi(x) V(x), \quad v_v(x) = (\psi(2^{-v}x) - \psi(2^{1-v}x)) V(x), \quad v \geq 1.$$

By Lemma 3.2 we may assume that $m(j) + i \geq m(j + i)$, which gives

$$|D_\alpha v_v(x)| \leq C'(1 + |x|)^{-m(|\alpha|)}, \quad |\alpha| \leq k. \quad (3.4)'$$

In the sum $V = \sum_0^\infty v_v$ at most three supports overlap. Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\int \chi dx = 1$, $\int x^\alpha \chi(x) dx = 0$ when $0 < |\alpha| \leq k$, and set

$$\chi_v(y) = t_v^{-n} \chi(y/t_v), \quad t_v = 2^{\delta v}.$$

If $0 < j \leq k$ we have by Taylor's formula

$$|v_v * \chi_v(x) - v_v(x)| \leq C(1 + |x|)^{-m(j)} t_v^j,$$

and since t_v is comparable to $(1 + |x|)^\delta$ in the support of v_v , we obtain

$$|V_1(x)| \leq C(1 + |x|)^{\delta j - m(j)}$$

if we define $V_1 = V - V_2$,

$$V_2(x) = \sum_0^\infty v_v * \chi_v(x).$$

By hypothesis $\delta j - m(j) < -1$ for some j so (3.5) follows. To prove (3.6), (3.7) we just note that when $|\alpha| = j \geq i$ and $i \leq k$, we can calculate $D^\alpha v_v * \chi_v(x)$ by letting i

derivatives fall on v_v and $j-i$ derivatives on χ_v . The latter differentiation contributes a factor $t_v^{-(j-i)}$ comparable to $(1+|x|)^{-\delta(j-i)}$. Since only finitely many supports overlap, the lemma follows.

We shall now introduce the class of perturbations to be studied. The conditions may seem artificial at first sight but they occur quite naturally in the proof later in this section.

Definition 3.4. $V(x, D) = \sum_{|\alpha| \leq \mu} V_\alpha(x) D^\alpha$ is an *admissible long range perturbation* of P if V can be written in the form $V = V^S + V^L$ where V^S is a short range perturbation with respect to P in the sense of Definition 2.2, the coefficients V_α^L of V^L are real and

$$|D^\beta V_\alpha^L(x)| \leq C(1+|x|)^{-m(|\beta|)}, \quad |\beta| \leq 3, \quad (3.8)$$

where $m(0), \dots, m(3)$ are positive and $m(1) + m(3) > 4$.

Remark. If the derivatives of V_α^L of positive order are $O(|x|^{-1-\varepsilon})$ for some $\varepsilon > 0$ then the difference between the adjoint and the complex conjugate of V^L is of short range. To assume V^L real is therefore equivalent to assuming that V^L can be taken symmetric which is perhaps more natural but less convenient in the proof.

By Lemma 3.2 we may always assume that $m(j) - j$ is decreasing and concave, and then the condition $m(1) + m(3) > 4$ implies $m(2) > 2$ and $m(1) > 1$. As a further explanation of the definition we prove

Proposition 3.5. *If the coefficients of V satisfy (3.4) with positive $m(0), \dots, m(k)$, then V is admissible if one of the following conditions is fulfilled:*

- a) $k=1$ and $m(1) > \frac{3}{2}$;
- b) $k=2$ and $2m(1) + 3m(2) > 9$;
- c) $k=2$ and $m(2) > \frac{11}{5}$;
- d) $k=3$ and $m(3) > 3$.

Proof. In case a) we can take $\delta > \frac{1}{2}$ in Lemma 3.3 and obtain a splitting in a short range term and a long range term corresponding to exponents m' with

$$m'(3) > m'(1) + 2/2 = m(1) + 1,$$

hence $m'(3) + m'(1) > 2m(1) + 1 > 4$. In case b) we can take any $\delta < (m(2) - 1)/2$ and obtain for $m'(3)$ any number $< m(2) + (m(2) - 1)/2$. If $m(1) + m(2) + (m(2) - 1)/2 > 4$, the splitting given by Lemma 3.3 satisfies the conditions in Definition 3.4, which proves b). c) is a consequence of b) since we may always assume that $m(1) > m(2) - 1$. d) follows since we can assume $m(1) > 1$ then.

In view of Lemmas 3.2 and 3.3 we may and shall assume in what follows that V_α^L satisfies estimates of the form (3.8) for all β and that

$$m(j) \text{ is concave, } m(j) - j \text{ decreases,} \quad (3.9)$$

$m(j) - \delta j$ increases and is constant for $j \geq 3$ for some $\delta > \frac{1}{2}$.

To avoid a minor technical complication we require

$$m(1) < 2, \quad m(3) < 3 \quad (3.9)'$$

which can always be achieved by decreasing the numbers $m(j)$.

Let us now examine formally how the proofs of Section 2 change when the definition of wave operator is modified. Thus take $\hat{u} \in C_0^\infty$ and form

$$\begin{aligned} i^{-1} e^{-itH} \frac{d}{dt} (e^{itH} e^{-iW_t} u) &= (H - \partial W_t(D)/\partial t) e^{-iW_t} u \\ &= V^S(x, D) e^{-iW_t} u + V^L(x, D) e^{-iW_t} u + e^{-iW_t} (P(D) - \partial W_t(D)/\partial t) u. \end{aligned} \quad (3.10)$$

Here we have used the decomposition of Definition 3.4, and

$$\begin{aligned} e^{-iW_t} D^\alpha u(x) &= (2\pi)^{-n} \int e^{i(\langle x, \xi \rangle - W(\xi, t))} \xi^\alpha \hat{u}(\xi) d\xi, \\ e^{-iW_t} (P(D) - \partial W_t(D)/\partial t) u(x) \\ &= (2\pi)^{-n} \int e^{i(\langle x, \xi \rangle - W(\xi, t))} (P(\xi) - \partial W(\xi, t)/\partial t) \hat{u}(\xi) d\xi. \end{aligned}$$

A formal application of the stationary phase method would show that the main contributions occur when $x = \partial W/\partial \xi$. One can expect to handle the first term on the right hand side of (3.10) exactly as in Section 2, and the leading terms in the others should cancel if

$$V^L(\partial W/\partial \xi, \xi) + P(\xi) - \partial W(\xi, t)/\partial t = 0. \quad (3.11)$$

This is the Hamilton-Jacobi equation of classical mechanics, in the momentum space variables. We shall solve it in the usual way by integrating the Hamilton equations

$$d\xi/dt = -\partial V^L(x, \xi)/\partial x, \quad dx/dt = \partial P/\partial \xi + \partial V^L(x, \xi)/\partial \xi. \quad (3.12)$$

Since V^L is small at infinity we expect the classical orbits to be nearly straight lines in the x space and nearly constant in the ξ space. Before integrating (3.12) it is convenient to have a lemma on composite functions which will help us to keep track of the estimates.

Lemma 3.6. Assume that $\psi: \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$ and $\varphi: \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_3}$ are C^∞ germs at x and at $y = \psi(x)$, depending on a parameter t , which satisfy estimates of the form

$$|D^\alpha \psi(x)| \leq C_\alpha t^{a(|\alpha|)}, \quad |D^\beta \varphi(y)| \leq C'_\beta t^{b(|\beta|)}, \quad t > 1, \quad (3.13)$$

for all α, β . If a and b are convex sequences, it follows that

$$|D^\gamma (\varphi \circ \psi)(x)| \leq C''_\gamma t^{c(|\gamma|)}, \quad \gamma \neq 0, \quad t > 1, \quad (3.14)$$

where C''_γ only depends on C_α, C'_β for $|\alpha| + |\beta| \leq |\gamma|$ and

$$c(k) = \max(b(1) + a(k), b(k) + k a(1)).$$

Moreover,

$$\begin{aligned} |D^\gamma (\varphi \circ \psi)(x) - \varphi'(y) D^\gamma \psi(x)| &\leq C'' t^{d(|\gamma|)}, \quad |\gamma| > 1, \quad t > 1, \\ d(k) &= \max(b(2) + a(1) + a(k-1), b(k) + k a(1)). \end{aligned} \quad (3.15)$$

Proof. Every term in $D^\gamma (\varphi \circ \psi)(x)$, $|\gamma| = k$, consists of a derivative of φ of order j , $1 \leq j \leq k$, multiplied with j derivatives of ψ of orders $k_1, \dots, k_j$, each at least equal to 1 and with sum k . We can estimate it by a constant times t to the power

$$b(j) + a(k_1) + \dots + a(k_j).$$

In the convex set of all $k_1, \dots, k_j$ which are ≥ 1 and have sum k , the extreme points are the points where $j-1$ indices are 1 and the remaining one is $k-j+1$. The maximum over all choices of $k_1, \dots, k_j$ is therefore

$$b(j) + (j-1)a(1) + a(k-j+1)$$

which is a convex function of j . The maximum when $1 \leq j \leq k$ or $2 \leq j \leq k$ is therefore assumed at an end point, which proves (3.14) and (3.15).

Remark. If $a=b$ and $a(1)=0$, then $c(j)=a(j)$ so we have a class of estimates which is preserved under composition then. If ψ satisfies (3.13) and $\psi'(x)$ is invertible, we conclude from this that the inverse $\varphi=\psi^{-1}$ satisfies estimates of the same type with constants only depending on C_α and a bound for $\psi'(x)^{-1}$. In fact, if we differentiate the identity $\psi \circ \varphi(y)=y$ using (3.15) we obtain $|\psi' D^\gamma \varphi| \leq C_\gamma t^{a(|\gamma|)}$ if estimates for lower order derivatives of φ are already proved.

Now we integrate the Hamilton equations.

Lemma 3.7. *Let K be a compact set in $\mathbb{R}^n$ where $P'(\xi) \neq 0$ and assume that*

$$V^L(x, \xi) = \sum_{|\alpha| \leq \mu} V_\alpha^L(x) \xi^\alpha$$

is real and that

$$|D^\beta V_\alpha^L(x)| \leq C_\beta (1+|x|)^{-m(|\beta|)} \quad (3.16)$$

for all $|\beta|$ and a positive sequence $m(j)$ satisfying (3.9), (3.9)'. If $1/t_0$ and ε are sufficiently small, then the eqs. (3.12) have a solution for all $t \geq t_0$ with arbitrary Cauchy data

$$\xi = \eta, \quad x = t_0 y \quad \text{when} \quad t = t_0, \quad (3.17)$$

such that $\eta \in K$ and $|y - P'(\eta)| < \varepsilon$. The solution represents a nearly uniform motion in the sense that

$$|D_{y,\eta}^\alpha (\xi(t, y, \eta) - \eta)| \leq C t_0^{|\alpha|+1-m(|\alpha|+1)}, \quad |\alpha| \leq 1, \quad (3.18)$$

$$|x(t, y, \eta)/t - P'(\xi(t, y, \eta))| \leq C t^{1-m(1)}. \quad (3.19)$$

Moreover,

$$|D_{y,\eta}^\alpha (\xi(t, y, \eta), x(t, y, \eta)/t)| \leq C_\alpha t^{\mu(|\alpha|)} \quad (3.20)$$

for all α , where

$$\mu(k) = \max(0, k+1-m(k+1)). \quad (3.21)$$

Proof. Choose $\varepsilon > 0$ and $\delta > 0$ so that, with the notation V for V^L ,

$$|P'(\eta)| > 3\varepsilon \quad \text{if } \eta \in K, \quad |P'(\xi) - P'(\eta)| < \varepsilon \quad \text{if } \eta \in K \quad \text{and} \quad |\eta - \xi| < \delta. \quad (3.22)$$

Let t_0 be so large that

$$|\partial V(x, \xi)/\partial \xi| < \varepsilon \quad \text{if } |x| > \varepsilon t_0 \quad \text{and} \quad |\xi - \eta| < \delta, \quad \eta \in K.$$

We can solve eqs. (3.12) for an interval $t_0 \leq t < T$ with initial data $\eta \in K$ and $|y - P'(\eta)| < \varepsilon$. As long as the solution exists and $|\xi - \eta| < \delta$, $|x| \geq \varepsilon t$, the component

of dx/dt in the direction $P'(\eta)$ is at least ε , hence $|x(t)| \geq 2\varepsilon t_0 + \varepsilon(t - t_0) > \varepsilon t \geq \varepsilon t_0$ then. Furthermore,

$$|\xi - \eta| \leq \int_{t_0}^t |\partial V(x, \xi)/\partial x| dt \leq C \int_{t_0}^{\infty} t^{-m(1)} dt < \delta/2$$

if t_0 is chosen large enough. It follows that the solution exists for all $t \geq t_0$ and satisfies the conditions $|x(t)| \geq \varepsilon t$, $|\xi(t) - \eta| < \delta$. We have already proved (3.18) when $\alpha = 0$. If ξ_{∞} is the limit of ξ as $t \rightarrow \infty$, then

$$|\xi - \xi_{\infty}| \leq C \int_t^{\infty} t^{-m(1)} dt \leq C_1 t^{1-m(1)},$$

$$|dx/dt - P'(\xi_{\infty})| \leq C(t^{1-m(1)} + t^{-m(0)}) \leq C_1 t^{1-m(1)}.$$

Integration gives $|x - t P'(\xi_{\infty})| < C t^{2-m(1)}$, hence

$$|x/t - P'(\xi)| \leq C t^{1-m(1)}$$

which proves (3.19).

Now differentiate the Hamilton equations with respect to one of the variables y, η , which we denote by s . The differentiated equations are

$$d\xi'_s/dt = -\partial^2 V/\partial x^2 x'_s - \partial^2 V/\partial x \partial \xi \xi'_s$$

$$dx'_s/dt = \partial^2 P/\partial \xi^2 \xi'_s + \partial^2 V/\partial \xi^2 \xi'_s + \partial^2 V/\partial \xi \partial x x'_s.$$

It follows that $|\xi'_s|$ and $|x'_s|$ are bounded for $t > t_0$ by positive solutions of a system of differential inequalities

$$d\Xi/dt \geq C(t^{-m(2)} X + t^{-m(1)} \Xi), \quad dX/dt \geq C(\Xi + t^{-m(1)} X)$$

which are bounds initially. We can pick such majorants for large t of the form

$$X = M_1 t, \quad \Xi = M_2(1 - t^{-\gamma} M_3)$$

where $1 < \gamma + 1 \leq m(2) - 1 \leq m(1)$ and

$$M_2 M_3 \gamma > C(M_1 + M_2), \quad M_1 > C M_2.$$

This proves that x'_s/t and ξ'_s are bounded. Integration of the preceding equations gives then

$$|\xi'_s - \eta'_s| \leq C \int_{t_0}^{\infty} t^{1-m(2)} dt$$

which completes the proof of (3.18).

Let us now consider a derivative of order k of the Hamilton equations with respect to the variables (y, η) , assuming that $k > 1$ and that (3.20) has already been proved for derivatives of order $< k$. If we write $V'_x(t, y, \xi) = \Phi(t, y, \xi)$, then

$$|D_{y, \xi}^{\alpha} \Phi| \leq C t^{-m(|\alpha|+1)+|\alpha|}$$

so Lemma 3.6 shows that in the derivatives of order k of $V'_x(x, \xi) = \Phi(t, x/t, \xi)$ with respect to y, η , the terms where Φ is differentiated at least twice can be bounded by t raised to the power

$$\max(-m(3) + 2 + \mu(k-1), -m(k+1) + k) = \mu(k) - 1$$

because

$$\mu(k) - \mu(k-1) \geq \mu(2) - \mu(1) = 3 - m(3), \quad k \geq 2.$$

Similarly we can estimate the derivatives of $V'_\xi(t x/t, \xi)$. Here the preceding estimates for the derivatives of Φ are multiplied by $t^{m(j+1)-m(j)}$, hence at most by t . Thus the k -th order derivatives of ξ, x with respect to y, η are bounded by solutions of

$$d\Xi/dt \geq C(t^{-m(2)} X + t^{-m(1)} \Xi) + t^{\mu(k)-1}$$

$$dX/dt \geq C(\Xi + t^{-m(1)} X) + t^{\mu(k)}$$

which are bounds initially. Majorants of the form

$$\Xi = t^{\mu(k)} M_1, \quad X = t^{1+\mu(k)} M_2$$

can now be used. (We have assumed (3.9)' to avoid logarithms here.) The proof of the lemma is complete.

By means of the lemma solutions of the Hamilton-Jacobi equation are obtained in a standard way:

Theorem 3.8. Assume that $V^L(x, \xi) = \sum V_\alpha^L(x) \xi^\alpha$ has real coefficients satisfying (3.16) where $m(j)$ fulfills (3.9), (3.9)'. Then there is a C^∞ function $W(\xi, t)$ such that for ξ in any compact subset of $\Omega = \{\xi; P'(\xi) \neq 0\}$ we have for large $|t|$

$$\partial W / \partial t = P(\xi) + V^L(\partial W / \partial \xi, \xi) \quad (3.23)$$

$$|D_\xi^\alpha W(\xi, t)| \leq C_\alpha t^{1+\mu(|\alpha|-1)}, \quad |\alpha| \geq 1, \quad (3.24)$$

$$|D_\xi^\alpha (W(\xi, t)/t - P(\xi))| + |D_\xi^\alpha V^L(\partial W / \partial \xi, \xi)| \leq C_\alpha t^{\nu(|\alpha|)}. \quad (3.25)$$

Here μ is defined by (3.21) and

$$\nu(k) = \max(1 - m(1) + \mu(k), k - m(k)). \quad (3.26)$$

Proof. Let $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \dots$ be open sets with union Ω , each with compact closure contained in the following one. First solve the Cauchy problem for (3.23) with initial data

$$W(\xi, t) = t P(\xi) \quad \text{when } \xi \in \Omega_2, \quad t = t_1,$$

where t_1 is large. This means that we solve the Hamilton equations with initial data

$$\xi = \eta, \quad x/t = P'(\eta) \quad \text{when } t = t_1, \quad \eta \in \Omega_2.$$

If t_1 is large enough we know that the solutions exist for $t > t_1$. By (3.18), (3.20) and the remark following Lemma 3.6 the map $\eta \rightarrow \xi$ is invertible and x/t can be expressed as a function of ξ satisfying

$$|D_\xi^\alpha (x/t)| \leq C_\alpha t^{\mu(|\alpha|)}.$$

The solution of the Hamilton-Jacobi equation will then exist at least in $\Omega_1 \times [t_1, \infty)$ and be defined by $\partial W / \partial \xi = x$. Thus (3.24) is valid and

$$|t^{-1} \partial W / \partial \xi - P'(\xi)| \leq C t^{1-m(1)}. \quad (3.27)$$

This solution we denote by W_1 .

Suppose we have already a solution W_j in $\Omega_j \times [t_j, \infty)$ satisfying (3.24) and (3.27). With $\chi \in C_0^\infty(\Omega_j)$ equal to 1 in a neighborhood of Ω_{j-1} we solve the Cauchy problem for the Hamilton-Jacobi equation with initial data in Ω_{j+2}

$$W = \chi W_j + (1 - \chi) t P', \quad t = t_{j+1}.$$

If t_{j+1} is large enough it follows again from Lemma 3.7 and Lemma 3.6 that the solution exists in $\Omega_{j+1} \times [t_{j+1}, \infty)$ and is equal to W_j in $\Omega_{j-1} \times [t_{j+1}, \infty)$; moreover, it satisfies (3.24), (3.27). Continuing in this way we obtain a function satisfying the Hamilton-Jacobi equation and (3.24), (3.27) for large t over any compact subset of Ω . We continue it in a C^∞ fashion to be 0 for $|t| < 1$ say and construct W similarly for large negative t .

The Hamilton-Jacobi equation can be written

$$\partial(W - t P(\xi))/\partial t = V^L(\partial W/\partial \xi, \xi) = \Phi(t, t^{-1} \partial W/\partial \xi, \xi)$$

where $\Phi(t, y, \xi) = V^L(t y, \xi)$ has derivatives of order $k \geq 0$ with respect to y and ξ bounded by $t^{k-m(k)}$ for y and ξ in suitable compact sets. By Lemma 3.6 and (3.24) the derivatives of $V^L(\partial W/\partial \xi, \xi)$ of order k can therefore be estimated by t raised to the power

$$\max(1 - m(1) + \mu(k), k - m(k) + k \mu(1)), \quad k > 0.$$

Since $\mu(1) = 0$ this proves that

$$|D_\xi^\alpha V^L(\partial W/\partial \xi, \xi)| \leq C t^{v(|\alpha|)},$$

which is also true when $|\alpha| = 0$. Now $v(|\alpha|) > -1$ so integration with respect to t also gives the first estimate in (3.25).

We can now state and prove the main theorem in this section.

Theorem 3.9. *Let P be a differential operator with constant real coefficients such that $\det P'' \neq 0$, and let $V(x, D)$ be an admissible perturbation such that $P(D) + V(x, D)$ has a self-adjoint extension H . If W is a solution of the Hamilton-Jacobi equation (3.11) with the properties listed in Theorem 3.8, then the modified wave operators (3.1) exist, and they are isometric operators intertwining H and H_0 .*

Proof. Since (3.25) implies that $\partial W/\partial t - P \rightarrow 0$ as $t \rightarrow \infty$, the condition (3.2) is fulfilled so the last assertion is obvious.

Let Ω_0 be an open set with complement of measure 0 such that $P' \neq 0$ and $\det P'' \neq 0$ in Ω_0 . In addition we require that V^s satisfies (2.9) for every compact subset of $P'(\Omega_0)$. We have to estimate the terms on the right hand side of (3.10). To the first the arguments of Section 2 are applicable with minor modifications, but for the others we have to examine the higher order terms in the asymptotic expansion of the stationary phase method much more carefully.

The support of $\hat{u}$ can be chosen so small that $\xi \rightarrow P'(\xi)$ is a diffeomorphism in a neighborhood. The critical points of $\langle x, \xi \rangle - W(\xi, t)$ are given by

$$x/t = P'(\xi) + t^{-1}(\partial W/\partial \xi - t P'(\xi)),$$

and by (3.25) we have

$$|D_\xi^\alpha(t^{-1} \partial W/\partial \xi - P'(\xi))| \leq C t^{v(2)}, \quad |\alpha| \leq 1.$$

Since $v(2) = \max(4 - m(1) - m(3), 2 - m(2)) < 0$, we see that there can only be one critical point in or near $\text{supp } \hat{u}$ when t is large. It is close to the critical point of $\langle x, \xi \rangle - t P(\xi)$.

Let $c < \frac{1}{2}$ be the slope $\mu(j+1) - \mu(j)$ for large j . (In the notations of (3.9) we have $c = 1 - \delta$.) Then the ξ -derivatives of order ≥ 2 of

$$\xi \rightarrow t^{2c-1} W(\xi/t^c, t)$$

are uniformly bounded in view of (3.24). We shall make a change of variables to take advantage of this, but since $\hat{u}(\xi/t^c)$ has increasing support we must then decompose by means of a partition of unity

$$\sum \chi_g(\xi) = 1.$$

Here g runs over all lattice points, $\chi_0 \in C_0^\infty$ and $\chi_g(x) = \chi_0(x - g)$. Set

$$\hat{u}_g(\xi) = \hat{u}(\xi/t^c) \chi_g(\xi).$$

Then

$$e^{-iW_t} u(x) = (2\pi)^{-n} \sum_g \int \exp i t^{1-2c} (\langle x, \xi \rangle t^{c-1} - t^{2c-1} W(\xi/t^c, t)) \hat{u}_g(\xi) d\xi/t^{nc}, \quad (3.28)$$

and the number of terms which do not vanish identically is $O(t^{nc})$. We shall now consider separately contributions close to and far from the critical point.

a) Since $t^{2c-1} W(\xi/t^c, t)$ has uniformly bounded derivatives of order ≥ 2 and the support of the integrand is fixed after a translation, we can find a constant K such that if

$$|t^{c-1}(x - W'_\xi(g/t^c, t))| > K, \quad (3.29)$$

then Lemma A.1 is applicable to prove that the corresponding term in (3.28) can be bounded by any desired power of t^{-1} or indeed by any power of $(t + |x - W'_\xi|)^{-1}$. In particular, if x/t is outside a fixed neighborhood of $P'(\text{supp } \hat{u})$ and t is large, then this estimate is applicable to all terms in (3.28) and we obtain then

$$|e^{-iW_t} u(x)| \leq C_N (1 + |x| + |t|)^{-N}.$$

b) If $\hat{u}_g$ is not identically 0 then g/t^c is at distance $O(t^{-c})$ from $\text{supp } \hat{u}$, so if

$$|x/t - W'_\xi(g/t^c, t)/t| \leq K t^{-c}, \quad (3.30)$$

we conclude using (3.25) that g/t^c is at distance $O(t^{-c})$ to a critical point of $\langle x, \xi \rangle - W(\xi, t)$. Hence there is a fixed bound for the number of such terms. Using Lemma A.4 we obtain an asymptotic expansion of the corresponding terms in (3.28) with an error term after N steps, which can be bounded by $C(t^{1-2c})^{-n/2-N}$, so it is as small as any desired power of $1/t$ if N is large. Summing up: we have seen that $e^{-iW_t} u(x)$ with an error $O((1 + |x| + |t|)^{-M})$ for any desired M is given by finitely many terms in the asymptotic expansion of the stationary phase method at the critical point. To study these terms it is convenient to use a Morse lemma with a formal change of variables starting from the original integration variables ξ .

Thus we apply Lemma A.6 to

$$f(\xi) = \langle x/t, \xi \rangle - W(\xi, t)/t, \quad a(k) = \mu(k-1).$$

It follows that in the asymptotic expansion of

$$\int e^{i(\langle x, \xi \rangle - W(\xi, t))} \hat{u}(\xi) d\xi$$

the terms are products of $e^{i(\langle x, \xi \rangle - W(\xi, t))}$ by the terms in the expansion of

$$\int e^{it\langle Ay, y \rangle / 2} v_{t,x}(y) J_{t,x}(y) dy \quad (3.31)$$

where $v_{t,x}$ is the composition of $\hat{u}$ and the change of variables, and $J_{t,x}$ is the Jacobian. The symmetric matrix A may depend on t and x but A and A^{-1} have uniform bounds. From (A.10) and Lemma 3.6 we obtain

$$|D^\alpha v_{t,x}(0)| \leq C t^{\mu(|\alpha|)}, \quad |D^\alpha J_{t,x}(0)| \leq C t^{\mu(|\alpha|+1)}. \quad (3.32)$$

We shall consider separately the terms in the expansion (A.5) of (3.31) where $v_{t,x}$ is not differentiated, for these will finally cancel out as a consequence of (3.11). In the other terms we have at the j -th step, $j \geq 1$, a factor $t^{-n/2-j}$ multiplied by derivatives of $v_{t,x}$ and $J_{t,x}$ at 0 of order a resp. b such that $a+b=2j$ and $a \geq 1$. This is bounded by a constant times t raised to the power

$$-n/2 - j + \mu(a) + \mu(b+1).$$

The maximum must be taken either when $a=1$, $b=2j-1$ or when $a=2j$, $b=0$, so it is $-n/2$ plus

$$-j + \mu(1) + \mu(2j) = \mu(2j) - j \leq \mu(2) - 1 = 2 - m(3).$$

The norm of the product of the coefficient $V_0^L(x)$ and the considered terms can therefore when $0 < r < |x/t| < R$ be bounded by

$$C t^{2-m(3)-m(0)} \leq C t^{3-m(3)-m(1)}$$

which by hypothesis is an integrable function when $t > 1$. If we replace $\hat{u}$ by $\xi^\alpha \hat{u}$ and V_0^L by V_α^L , the same conclusion is obtained.

We have now estimated all terms in the asymptotic expansion of the term $V^L(x, D) e^{-iW_t} u$ in (3.10) except $V^L(\partial W / \partial \xi, \xi) \hat{u}(\xi) e^{i(\langle x, \xi \rangle - W(\xi, t))} S$, where S denotes the terms in the formal asymptotic expansion of

$$(2\pi)^{-n} \int e^{it\langle Ay, y \rangle / 2} J_{t,x}(y) dy.$$

However, these are cancelled by terms in the asymptotic expansion of

$$e^{-iW_t}(P(D) - \partial W_t(D) / \partial t) u = -(2\pi)^{-n} \int e^{i(\langle x, \xi \rangle - W(\xi, t))} V^L(\partial W / \partial \xi, \xi) \hat{u}(\xi) d\xi.$$

Let $w_{t,x}(y)$ be the composition of $V^L(\partial W / \partial \xi, \xi)$ and the change of variables above. If we use (3.25) and observe that

$$\max(v(1) + \mu(k), v(k) + k \mu(1)) = \max(1 - m(1) + \mu(k), v(k)) = v(k),$$

it follows that

$$|D^\alpha w_{t,x}(0)| \leq C_\alpha t^{v(|\alpha|)}.$$

For the terms in the j -th step of the asymptotic expansion of

$$\int e^{it\langle Ay, y \rangle / 2} w_{t,x}(y) J_{t,x}(y) dy$$

where $w_{t,x}$ and $J_{t,x}$ are differentiated a resp. b times, we obtain as estimate a constant times t raised to the power

$$-n/2 - j + v(a) + \mu(b+1).$$

Here $a+b=2j$ and we consider only terms with $a \geq 1$. The maximum of $v(a) + \mu(b+1)$ under these conditions is

$$\max(v(1) + \mu(2j), v(2j) + \mu(1)) = \max(1 - m(1) + \mu(2j), v(2j)) = v(2j).$$

Hence t is raised to a power $\leq -n/2 - j + v(2j) \leq -n/2 - 1 + v(2)$. In the beginning of the proof we saw that $v(2) < 0$, so the norm when $r < |x/t| < R$ is an integrable function of t on $(1, \infty)$.

We have now estimated the L^2 norm of all terms in (3.10) except $V^S(x, D) e^{-iW_t} u$. However, the preceding estimates show in particular that $D^\alpha e^{-iW_t} u(x)$ can be bounded by $C t^{-n/2}$, and at the beginning of the proof we obtained the bound $C_N(1 + |x| + |t|)^{-N}$ for any N when x/t is outside a fixed neighborhood of $P'(\text{supp } \hat{u})$. The same estimate of this term as in the proof of Theorem 2.3 is therefore applicable. The proof is complete.

The definition of $W_\pm$ in Theorem 3.9 depends first on the choice of splitting $V = V^S + V^L$ and then on the choice of solution W of the Hamilton-Jacobi equation with the Hamiltonian $P + V^L$. Suppose that $V = \tilde{V}^S + \tilde{V}^L$ and $\tilde{W}$ is another choice and that $W, \tilde{W}$ satisfy a part of (3.24), (3.25) namely

$$|D_\xi^\alpha W(\xi, t)| + |D_\xi^\alpha \tilde{W}(\xi, t)| \leq C |t|, \quad |\alpha| \leq 2, \quad (3.33)$$

$$|D_\xi^\alpha(W(\xi, t) - t P(\xi))| + |D_\xi^\alpha(\tilde{W}(\xi, t) - t P(\xi))| \leq C |t|^{1-\varepsilon}, \quad |\alpha| \leq 2, \quad (3.34)$$

for some $\varepsilon > 0$ when ξ is in a compact subset K of Ω and $|t|$ is large. When $\xi \in K$ it follows for large $|t|$ by subtraction of the Hamilton-Jacobi equations and application of Taylor's formula that the difference $R = W - \tilde{W}$ satisfies a differential equation

$$\partial R / \partial t = \langle a, \partial R / \partial \xi \rangle + (\tilde{V}^S - V^S)(\partial \tilde{W} / \partial \xi, \xi), \quad (3.35)$$

where

$$a(\xi, t) = \int_0^1 V_x^{L'}(\partial W / \partial \xi + s(\partial \tilde{W} / \partial \xi - \partial W / \partial \xi), \xi) ds.$$

By (3.33), (3.34) we have if $\delta < m(2) - 2 \leq m(1) - 1$

$$|D_\xi^\alpha a(\xi, t)| \leq C t^{-1-\delta}, \quad |\alpha| \leq 1.$$

If K_1 is a compact subset of the interior of K it follows that the integral curves of the equation

$$d\xi/dt = -a(\xi, t); \quad \xi = \eta \in K_1 \text{ when } t = t_0,$$

stay in K for all $t > t_0$ if t_0 is large enough. Moreover,

$$|D_\eta^\alpha(\xi(\eta, t) - \eta)| \leq C t_0^{-\delta}, \quad |\alpha| \leq 1. \quad (3.36)$$

If t_0 is sufficiently large the map $K_1 \ni \eta \rightarrow \xi(\eta, t)$ is therefore a diffeomorphism.

Now the Eq. (3.35) can be written

$$dR/dt = F \quad (3.35)'$$

where d/dt denotes differentiation along the integral curves of a and

$$F = (\tilde{V}^S - V^S)(\partial \tilde{W}/\partial \xi, \xi)$$

considered as a function of η and t . We shall prove that $|F|$ is integrable when $\eta \in K_1$ and $t > t_0$. To do so we first observe that

$$|F| \leq C \sum_{\alpha} (|\tilde{V}_{\alpha}^S(\partial \tilde{W}/\partial \xi)| + |V_{\alpha}^S(\partial \tilde{W}/\partial \xi)|).$$

It is therefore natural to introduce $x = t^{-1} \partial \tilde{W}(\xi(\eta, t))/\partial \xi$ as a new integration variable instead of η . The Jacobian is $\partial x/\partial \eta = t^{-1} \partial^2 \tilde{W}/\partial \xi^2 \partial \xi/\partial \eta$ and from (3.34), (3.36) we obtain

$$|t^{-1} \partial^2 \tilde{W}/\partial \xi^2 - P''(\xi)| < C t^{-\varepsilon}, \quad |\partial \xi/\partial \eta - I| < \frac{1}{2}.$$

Hence

$$|\det(\partial \eta/\partial x)| \leq C \quad \text{if } t > t_0, \eta \in K_1,$$

provided that t_0 is large. If K_1 is sufficiently small, the map $\eta \rightarrow x$ is a diffeomorphism and we obtain

$$\int_{K_1} \int_{t_0}^{\infty} |F| d\eta dt \leq C \int_{t_0}^{\infty} dt \int_{\omega} \sum (|\tilde{V}_{\alpha}^S(tx)| + |V_{\alpha}^S(tx)|) dx$$

for a set ω to which (2.9) is applicable. Since an L^1 mean value is bounded by the corresponding L^2 mean value, the integrability of F follows, so (3.35)' shows that $R(\xi(\cdot, t), t)$ has limits in $L^1(K_1)$ as $t \rightarrow \pm \infty$. Denote the limits by $R_{\pm}$. Since $\eta \rightarrow \xi(\eta, t)$ has an inverse φ_t with Jacobian < 2 , say, this proves that

$$R(\cdot, t) - R_{\pm} \circ \varphi_t \rightarrow 0, \quad t \rightarrow \pm \infty,$$

in L^1 norm. But φ_t has a diffeomorphic limit φ_{∞} in the C^1 topology so this implies that

$$R(\cdot, t) - R_{\pm} \circ \varphi_{\infty} \rightarrow 0$$

in L^1 . Hence we have proved, in view of Theorem 3.1,

Theorem 3.10. *Let V be an admissible potential in the sense of Definition 3.4 for an operator $P(D)$ with $\det P'' \neq 0$, and let*

$$V = V^S + V^L, \quad \tilde{V} = \tilde{V}^S + \tilde{V}^L$$

be two splittings such that V^S and $\tilde{V}^S$ have short range and $V^L, \tilde{V}^L$ are real and satisfy (3.8) for $|\beta| \leq 2$, where we assume that $m(2) > 2$. Let $W, \tilde{W}$ be solutions of the Hamilton-Jacobi equation (3.11), with V^L replaced by $\tilde{V}^L$ in the case of $\tilde{W}$, and assume that (3.33), (3.34) are fulfilled. Then $W - \tilde{W}$ has a limit in $L^1_{\text{loc}}(\Omega)$ as $t \rightarrow +\infty$ or $-\infty$ for an open set $\Omega \subset \mathbb{R}^n$ with complement of measure 0. If $W_{\pm}, \tilde{W}_{\pm}$ are modified wave operators defined by means of W and $\tilde{W}$, it follows that W_{+} and $\tilde{W}_{+}$ (resp. W_{-} and $\tilde{W}_{-}$) have the same range.

Appendix

The Method of Stationary Phase

Our purpose is to derive asymptotic expansions of oscillatory integrals of the form

$$I(\omega) = \int e^{i\omega f(x)} u(x) dx \quad (\text{A.1})$$

where f is real valued and $\omega \rightarrow \infty$ on $\mathbb{R}$. The first elementary lemma focuses attention on the critical points of f .

Lemma A.1. *Let $K \subset \mathbb{R}^n$ be a compact set, Ω a neighborhood of K and $f \in C^{k+1}(\Omega, \mathbb{R})$ a function with $f' \neq 0$ in K . Then we have for $\omega \in \mathbb{R}$*

$$\left| \int e^{i\omega f(x)} u(x) dx \right| \leq C(1+|\omega|)^{-k} |u|_k, \quad u \in C_0^k(K), \quad (\text{A.2})$$

where

$$|u|_k = \sum_{|\alpha| \leq k} \sup |D^\alpha u|. \quad (\text{A.3})$$

The same constant can be used in (A.2) for all f in a compact subset of $C^{k+1}(\Omega)$ satisfying the hypotheses.

Proof. It suffices to prove that if $x_0 \in K$ and $f'_0(x_0) \neq 0$ then there is a neighborhood U of x_0 and a neighborhood V of f_0 in $C^{k+1}(\Omega, \mathbb{R})$ such that (A.2) is valid when $u \in C_0^\infty(U)$ and $f \in V$. In fact, by a partition of unity we then obtain (A.2) for all $u \in C_0^\infty(K)$ when f is close to a function satisfying the hypotheses in the lemma, and the full statement follows by the Borel-Lebesgue lemma. — We may assume that $\partial f_0 / \partial x_1 \neq 0$ at x_0 . By the implicit function theorem there is a neighborhood U_1 of x_0 such that

$$U_1 \ni x \rightarrow (f(x), x_2, \dots, x_n) \in \mathbb{R}^n$$

is a diffeomorphism on $f(U_1)$ with uniformly bounded inverse $g \in C^{k+1}$ provided that f is close to f_0 . Let U be relatively compact in U_1 . Then

$$\int u(x) e^{i\omega f(x)} dx = \int_{f(U)} u(g(y)) e^{i\omega y_1} |\det g'(y)| dy, \quad u \in C_0^k(U),$$

so (A.2) has now become a well known estimate for the Fourier transform proved by k partial integrations.

We shall next consider the simplest type of critical point.

Lemma A.2. *Let A be a real symmetric non-degenerate matrix of signature σ . Then the following identity is valid when $\omega > 0$, $u \in \mathcal{S}$*

$$\int e^{i\omega \langle Ax, x \rangle / 2} u(x) dx = |\det 2\pi \omega A|^{-\frac{1}{2}} e^{\pi i \sigma / 4} \int \hat{u}(\xi) e^{-i \langle A^{-1} \xi, \xi \rangle / 2 \omega} d\xi. \quad (\text{A.4})$$

For every integer $k > 0$ and every $s > 2k + n/2$, we have when $u \in \mathcal{S}$

$$\left| \int e^{i\omega \langle Ax, x \rangle / 2} u(x) dx - |\det \omega A / 2\pi|^{-\frac{1}{2}} e^{\pi i \sigma / 4} \sum_0^{k-1} \langle A^{-1} D, D \rangle^j u(0) / j! (2i\omega)^j \right| \leq C_{k,s} \omega^{-n/2-k} \|u\|_{(s)}, \quad \omega > 1. \quad (\text{A.5})$$

Here $C_{k,s}$ is a continuous function of A and s , and

$$\|u\|_{(s)} = ((2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi)^{\frac{1}{2}}$$

is the Sobolev norm.

Proof. (A.4) is just an elementary computation of the Fourier transform of $x \rightarrow e^{i\langle AX, x \rangle/2}$, and (A.5) follows if the exponential on the right hand side of (A.4) is expanded by Taylor's formula. The details can be found on p.145 of Hörmander [8].

To reduce more general phase functions to quadratic forms one needs the Morse lemma:

Lemma A.3. *Let $f \in C^\infty(\Omega)$ be a real valued function in a neighborhood Ω of 0 in $\mathbb{R}^n$ and assume that $f'(0)=0$ and that $A=f''(0)$ is non-singular. Then there exist neighborhoods Ω_1 and Ω_2 of 0 in $\mathbb{R}^n$ and a diffeomorphism $X: \Omega_1 \rightarrow \Omega_2$ such that $X(x) - x = O(|x|^2)$, $x \rightarrow 0$, and*

$$f(x) = f(0) + \langle AX, X \rangle/2, \quad x \in \Omega_1.$$

Proof. By Taylor's formula

$$f(x) - f(0) = \langle B(x)x, x \rangle/2$$

where

$$B(x) = 2 \int_0^1 f''(sx)(1-s) ds.$$

Hence $B(0)=A$. Now there exists in a neighborhood of A in the set of symmetric matrices a C^∞ map $B \rightarrow R(B)$ with values in the set of real invertible matrices such that

$$R(B)^* A R(B) = B, \quad R(A) = I.$$

In fact, the differential of $R \rightarrow R^* A R$ with respect to R when $R=I$ is $R \rightarrow R^* A + AR$, and it is surjective since $R^* A + AR = C$ if $R = A^{-1} C/2$ and C is symmetric. Now we can just take $X = R(B(x))x$.

The proof is clearly applicable uniformly to all functions in a neighborhood of f in $C^\infty(\Omega)$ having a critical point at 0. By the implicit function theorem there is automatically one close to 0 so we can always make 0 a critical point by a small translation.

Lemma A.4. *Let f be as in Lemma A.3. Then there exist differential operators $L_{f,j}$ of order $2j$ such that when $u \in C_0^\infty(K)$, K compact $\subset \Omega_1$,*

$$\left| \int u(x) e^{i\omega f(x)} dx - \sum_{j=0}^{k-1} |\det f''(0)/2\pi|^{-\frac{1}{2}} e^{\pi i \sigma/4} e^{i\omega f(0)} L_{f,j} u(0) \omega^{-j-n/2} \right| \leq C \omega^{-n/2-k} \|u\|_s, \quad (\text{A.6})$$

provided that $s > 2k + n/2$ and $\omega > 1$.

Proof. (A.6) follows from (A.5) if we apply Lemma A.3 and introduce X as variable instead of x .

From the proof it follows that the remainder term in (A.6) is uniform if f is replaced by any function in a C^∞ neighborhood. This can be exploited in computing the operators $L_{f,j}$. To do so we assume $f(0)=0$ and write

$$f(x) = \langle Ax, x \rangle / 2 + g(x), \quad g(x) = O(|x|^3).$$

Then

$$\frac{d^N}{ds^N} \int e^{i\omega(\langle Ax, x \rangle / 2 + sg(x))} u(x) dx = \int e^{i\omega(\langle Ax, x \rangle / 2 + sg(x))} (i\omega g(x))^N u(x) dx.$$

If we apply (A.6) to the right hand side, the terms in the sum vanish when $2j < 3N$ so it can be estimated by $\omega^{-n/2-k+N}$ if $k \leq 3N/2$. Thus an application of Taylor's formula with remainder term gives

$$\left| \int e^{i\omega f} u dx - \int \sum_{k \leq N} e^{i\omega \langle Ax, x \rangle / 2} (i\omega g(x))^k u(x) / k! dx \right| \leq C \omega^{-n/2-N/2}.$$

In view of (A.5) and (A.6) this means that as formal power series

$$\sum_j L_{f,j} u(0) / \omega^j = \sum_{k,l} (i\omega)^{k-l} 2^{-l} \langle A^{-1} D, D \rangle^l (g^k u)(0) / k! l!. \quad (\text{A.7})$$

Note that on the right hand side we must have $2l \geq 3k$ in a non-zero term. When $l-k=j$ this implies $l \leq 3j$, $k \leq 2j$, so there is just a finite number of such terms. Altogether $3k$ derivatives are required to remove the zero of g^k at 0, so derivatives of g are of order $\leq 3+2l-3k=3+2j-k \leq 2j+2$ when $k \geq 1$. In view of the standard formula for the inverse of a matrix we have therefore proved

Lemma A.5. $(\det f''(0))^{3j} L_{f,j}$ has coefficients which are polynomials in $f''(0), \dots, f^{(2j+2)}(0)$, homogeneous of degree $3nj-j$. They can be computed from (A.7).

From the definition it is clear that $|\det f''(0)|^{-\frac{1}{2}} L_{f,j} u(0)$ is invariant if f is transformed as a function and u as a density. Note also that these quantities only depend on the formal power series expansions of f and of u at 0, so we can control them by means of a Morse lemma on the formal power series level:

Lemma A.6. Let f be a C^∞ germ at x_0 with $f'(x_0)=0$ and $A=f''(x_0)$ non-singular. Assume that for some $t > 1$

$$|D^\alpha f(x_0)| \leq C_\alpha t^{a(|\alpha|)}, \quad |\alpha| > 1, \quad (\text{A.8})$$

where $a(k)$ is a convex sequence with $a(1)=a(2)=0$. Then there is a diffeomorphism φ of a neighborhood of 0 on a neighborhood of x_0 such that

$$f \circ \varphi(y) = f(x_0) + \langle Ay, y \rangle / 2, \quad (\text{A.9})$$

$$|D^\alpha \varphi(0)| \leq C'_\alpha t^{a(|\alpha|+1)} \quad (\text{A.10})$$

where C'_α only depends on the constants in (A.8) and on a bound for A^{-1} .

Proof. We may assume that $x_0=0$. Define the symmetric matrix $B(x)$ as in the proof of Lemma A.3. Then

$$|D^\alpha B_{jk}(0)| \leq C''_\alpha t^{a(|\alpha|+2)}.$$

In view of Lemma 3.6 it follows that

$$|D^\alpha R(B)(0)| \leq C_\alpha''' t^{a(|\alpha|+2)}$$

if we note that $ka(3) \leq a(k+2)$ because $a(3) = a(3) - a(2) \leq a(4) - a(3) \leq \dots \leq a(k+2) - a(k+1)$. For the derivatives of $X(x) = R(B(x))x$ this gives

$$|D^\alpha X(0)| \leq C_\alpha^4 t^{a(|\alpha|+1)}.$$

Estimates of the same form are valid for the inverse φ by the remark following Lemma 3.6. This completes the proof.

In our applications of the preceding results the phase f and the amplitude u will often depend on the frequency ω . This is justified by the uniformity of the preceding methods of proof.

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Note Added in Proof. Dr. P.K. Alsholm has informed the author that Lemma 3.3 above allows him to reduce the conditions in [1] to

$$m(1) + c > 2, \quad c = \max(m(k) - 1)/k.$$

Writing the condition in the form $m(1) + (m(1) + 2c) > 4$ one concludes that it implies that V is admissible in the sense of Definition 3.4. On the other hand, if $m(j) = j + \varepsilon$, $j \leq k$, then the preceding condition reduces to $\varepsilon > 1/(k+1)$ whereas Proposition 3.5 shows that V is admissible if $\varepsilon > 1/2$, $k = 1$, or $\varepsilon > 1/5$, $k = 2$ or $\varepsilon > 0$, $k > 2$.

