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An effective version of Nori's theorem

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Introduction

There has been some interest, recently, in finding effective versions of vanishing theorems for Koszul cohomology over varieties of arbitrary dimension. A major advance in this direction was the work of Ein and Lazarsfeld [EL], where for the first time the general form of the expected effective versions became clear. In this paper, we provide some more evidence for the general principle established by them, that if a theorem is true for "large enough line bundles", then a similar statement should be true for any bundle of the form $K_X^{\otimes a} \otimes A^{\otimes b}$ where A is very ample and a and b are some computable constants that depend only on the dimension of the variety. We work with the following theorem of Nori:

Theorem 0.1 [N] *Let X be a smooth projective variety of dimension $n + h$. For $i = 1, \ldots, h$, let D_i be a hypersurface of degree a_i. Assume that $Y = X \cap D_1 \cap \cdots \cap D_h$ is smooth. Let ξ be a codimension $d < n$ algebraic cycle on X whose cohomology class in $H^{2d}(X, \mathbb{Q})$ is non-zero. Then the restriction of ξ to Y is not algebraically equivalent to zero, if the degrees of the hypersurfaces a_i are large enough.*

Nori reduces the proof of his theorem to that of proving a vanishing theorem for some vector bundles constructed in terms of Ω_X^j and the line bundles $O_X(a_i)$. He then proves the vanishing theorem by repeated use of Serre's vanishing theorem. In this process, he loses control on the minimum degrees of the hypersurfaces, a_i, needed for the vanishing to hold.

The main idea of this paper is to use a complex constructed in [EL], to prove the vanishing required for the vector bundles considered by Nori. This enables us to specify the positivity needed for the line bundles $O_X(a_i)$ to ensure the vanishing required in Nori's proof. As a specific example, we get the following effective application of Nori's theorem: Let X be a nonsingular quadric hypersurface in $\mathbb{P}^7$

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and let ξ be the standard primitive codimension 3 cycle on X (so ξ is the difference of two linear subvarieties). Let $Y = X \cap D_1 \cap D_2$ be smooth. If the degrees of the hypersurfaces D_1 and D_2 are at least 72, then ξ restricted to Y is not algebraically equivalent to zero.

The plan of the paper is as follows: in Sect. 1 we prove a variation of a vanishing theorem proved in [EL]. The main vanishing theorem, Theorem 2.1, is proved in Sect. 2 and we give various applications to the theory of algebraic cycles in Sect. 3.

Our methods are very similar to those in [EL] and [N]. Our intellectual debt to Lazarsfeld, Ein and Nori will be clear to anybody who is familiar with the papers [EL] and [N]. In addition, we are grateful to them for some helpful conversations and their encouragement.

1 Preliminaries

Let X be a smooth projective variety of dimension n. Let A be a very ample line bundle on X. We shall denote the canonical bundle of X by K_X.

The following proposition is well known [EL, p. 63] and shall be our starting point.

Proposition 1.1 *$\Omega_X^j \otimes A^{\otimes(j+1)}$ is globally generated for $0 \leqq j \leqq n$.*

Proof. Embed X in $\mathbb{P}(H^0(A))$ through the bundle A. On $\mathbb{P}$, the bundle $\Omega_{\mathbb{P}}^j \otimes O_{\mathbb{P}}(j+1)$ is globally generated. Hence $\Omega_X^j \otimes A^{\otimes(j+1)}$ is globally generated, since it is a quotient of $\Omega_{\mathbb{P}}^j \otimes O_{\mathbb{P}}(j+1)$. □

We make the following definitions: For $1 \leqq j \leqq n$, let $\mu_j = \mu_j(n) = \binom{n-1}{j-1} + 1$, $\sigma_j = \sigma_j(n) = \binom{n}{j}(j+1) + (j+1) + 1$ and set $\mu_0(n) = 1$ and $\sigma_0(n) = 3$.

Corollary 1.2 *Let B be a numerically effective* (nef) *line bundle on X. Then*

$$H^i(X, K_X^{\otimes \mu_j} \otimes A^{\otimes \sigma_j} \otimes B \otimes \Omega_X^j) = 0 \quad \text{for } i \geqq 1 .$$

Proof. We apply Griffiths' vanishing theorem [SS] to the globally generated bundle $\mathscr{M} = \Omega_X^j \otimes A^{\otimes(j+1)}$ and the ample line bundle $A \otimes B$. Then the claim follows from the observation that $\det \mathscr{M} = K_X^{\otimes\binom{n-1}{j-1}} \otimes A^{\otimes\binom{n}{j}(j+1)}$. □

We next prove a variation of a vanishing theorem of [EL].

Proposition 1.3 *Let $E_1, \ldots, E_r$ be globally generated vector bundles of ranks e_i on X. Let F be another globally generated vector bundle on X of rank f. Let A be an ample line bundle on X. Then*

$$H^k\left(X, A \otimes K_X \otimes F \otimes \det F \otimes \bigwedge^{e_1 - a_1} E_1 \otimes \cdots \otimes \bigwedge^{e_r - a_r} E_r\right) = 0 \quad \text{if } k > \sum_1^r a_i .$$

Proof. Let $Y = \mathbb{P}(F)$ and $\pi: \mathbb{P}(F) \to X$ be the projection map. Now $\pi^* E_i$ is a globally generated vector bundle on Y. Also $\mathscr{L} = O_Y(f+2) \otimes \pi^*(A)$ is an ample line bundle on Y. Hence, by [EL, Proposition 1.7],

$$H^k\left(Y, K_Y \otimes \mathscr{L} \otimes \pi^*\left(\bigwedge^{e_1 - a_1} E_1 \otimes \cdots \otimes \bigwedge^{e_r - a_r} E_r\right)\right) = 0 \quad \text{if } k > \sum a_i .$$

Now, by the projection formula,

$$\pi_*\left(K_Y \otimes \mathscr{L} \otimes \pi^*\left(\overset{e_1-a_1}{\bigwedge} E_1 \otimes \cdots \otimes \overset{e_r-a_r}{\bigwedge} E_r\right)\right)$$

$$= \pi_*(K_Y \otimes O_Y(f+2)) \otimes \overset{e_1-a_1}{\bigwedge} E_1 \otimes \cdots \otimes \overset{e_r-a_r}{\bigwedge} E_r \otimes A$$

$$= K_X \otimes F \otimes \det F \otimes \overset{e_1-a_1}{\bigwedge} E_1 \otimes \cdots \otimes \overset{e_r-a_r}{\bigwedge} E_r \otimes A\ .$$

Also, all higher direct image sheaves of this sheaf vanish. Hence, by Leray's spectral sequence, we get the vanishing claimed in the proposition. □

Let N be the normal bundle of X in $\mathbb{P}(H^0(A))$.

Corollary 1.4 *Let* $d = \sigma_j + \sum_1^r a_i + r(n+1)$ *where* $a_i \geqq 0$ *and* $r \geqq 1$. *Then*

$$H^k\left(X, \Omega_X^j \otimes \overset{a_1}{\bigwedge} N^* \otimes \cdots \otimes \overset{a_r}{\bigwedge} N^* \otimes K_X^{\otimes(\mu_j+r)} \otimes A^{\otimes d} \otimes B\right) = 0$$

if $k > \sum a_i$ *and* B *is any* nef *line bundle on* X.

Proof. (Following the ideas of [EL]) The bundle $N \otimes A^*$ is globally generated, since it is a quotient of $\mathbb{T}_{\mathbb{P}^n}(-1)$. Also $F = \Omega_X^j \otimes A^{\otimes j+1}$ is globally generated by Proposition 1.1. Now,

$$\overset{a_i}{\bigwedge} N^* \simeq \overset{e-a_i}{\bigwedge} (N \otimes A^*) \otimes K_X^{-1} \otimes A^{\otimes -(n+1+a_i)}$$

where e is the codimension of X in $\mathbb{P}$. We set $E_i = \bigwedge^{e-a_i}(N \otimes A^*)$ and the claim follows from Proposition 1.3.

Remark 1.5 We note that, if $(X, A) \neq (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$, and B is a nef line bundle on X, then the line bundle $K_X^{\otimes a} \otimes A^{\otimes b} \otimes B$ is very ample if $b \geqq n \cdot a + 1$ [EL, footnote in proof of Proposition 2.4]. Further, we observe that

$$n \cdot \mu_j + 1 = n\binom{n-1}{j-1} + n + 1$$

$$= \binom{n}{j} j + n + 1 < \sigma_j + n \quad \text{for all } j\ .$$

2 Main theorem

In this section we shall prove our main vanishing theorem. We shall assume throughout this section, that $(X, A) \neq (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$. Our results are straightforward for the pair $(\mathbb{P}^n, O_{\mathbb{P}^n}(1))$ and are indicated in Remark 2.3. We fix j to be an integer between 0 and n. If L is globally generated the bundle M_L is defined by the exact sequence:

$$0 \to M_L \to H^0(X, L) \otimes O_X \to L \to 0\ .$$

Theorem 2.1 *For $i = 1, \ldots, l$ let $L_i = K_X^{\otimes \mu_j} \otimes A^{\otimes d_i} \otimes B_i$, where B_i are* nef *line bundles on X, let $a_i \geqq 0$ and let $q = \sum a_i$. Assume that $d_i \geqq \sigma_j + 2n - k$ and let $M_i = M_{L_i}$. Then*

$$H^k(X, \Omega_X^j \otimes M_1^{\otimes a_1} \otimes \ldots \otimes M_l^{\otimes a_l} \otimes C \otimes K_X^{\otimes \mu_j} \otimes A^{\otimes d}) = 0 \quad \text{for } k \geqq 1 ,$$

where C is any nef *line bundle on X and $d \geqq \sigma_j + n + q - k$.*

Remark. The case $j = 0$ of this theorem was proved in [EL]. Also, the fact that a vanishing theorem of this form, holds for all "sufficiently large" line bundles L_i was shown by Nori, in [N].

Proof. The proof will be by induction on q. If q is zero, then the vanishing claimed in the theorem follows from Corollary 1.2. We now proceed with the proof of the inductive step.

We first observe that the L_i are of the form $K_X \otimes A^{\otimes n+1} \otimes D_i$ where D_i is some nef line bundle. To verify this, it suffices to ensure that $K_X^{\otimes \mu_j - 1} \otimes A^{\otimes d_i - n - 1}$ is nef. This, in turn follows by checking that $n \cdot (\mu_j - 1) < (d_i - n - 1)$ which follows from our computations in Remark 1.5.

Now, by Proposition 2.4 of [EL], for each M_i, there is a complex $\mathscr{R}^i_\cdot$ that satisfies the following conditions:

(1) $\mathscr{R}^i_0 = M_i$.
(2) For $j \geqq 1$, $\mathscr{R}^i_j$ is a direct sum of the line bundles $A^{\otimes -j}$.
(3) $\mathscr{H}_0(\mathscr{R}^i_\cdot) = 0$ and for $j \geqq 1$,

$$\mathscr{H}_j(\mathscr{R}^i_\cdot) = \bigwedge^j N^* \otimes L_i ,$$

where N is the normal bundle of X in $\mathbb{P}(H^0(A))$.

Now, for each i, we tensor the complex $\mathscr{R}^i_\cdot$ with itself a_i times to get l complexes. We further tensor these l complexes together to get a complex $\mathscr{P}'_\cdot$. Finally, we tensor the complex $\mathscr{P}'_\cdot$ by $\Omega_X^j \otimes C \otimes K_X^{\otimes \mu_j} \otimes A^{\otimes d}$ to get a complex $\mathscr{P}_\cdot$ with the following properties:

(1) $\mathscr{P}_0 = \Omega_X^j \otimes M_1^{\otimes a_1} \otimes \cdots \otimes M_l^{\otimes a_l} \otimes C \otimes K_X^{\otimes \mu_j} \otimes A^{\otimes d}$.
(2) For $m \geqq 1$, $\mathscr{P}_m$ is a direct sum of vector bundles of the form $\Omega_X^j \otimes M_1^{\otimes b_1} \otimes \cdots \otimes M_l^{\otimes b_l} \otimes C \otimes K_X^{\otimes \mu_j} \otimes A^{\otimes d-m}$ and $\sum b_i < q$.
(3) $\mathscr{H}_m(\mathscr{P}_\cdot) = 0$ if $m \leqq q - 1$.
(4) $\mathscr{H}_m(\mathscr{P}_\cdot) =$

$$\bigoplus_{c_1 + \cdots + c_p = m} \bigwedge^{c_1} N^* \otimes \cdots \otimes \bigwedge^{c_p} N^* \otimes L_{i_1} \otimes \cdots \otimes L_{i_p} \otimes \Omega_X^j \otimes C \otimes K_X^{\otimes \mu_j} \otimes A^{\otimes d} .$$

By Lemma 2.10 of [EL], to prove our theorem, we need to prove the following:

(a) $H^{k+m-1}(X, \mathscr{P}_m) = 0$ for $m = 1, \ldots, n - k + 1$ and
(b) $H^{k+m}(X, \mathscr{H}_m(\mathscr{P}_\cdot)) = 0$ for $m = 1, \ldots, n - k$.

The vanishing in (a) follows by the induction hypothesis, since for each of the summands in $\mathscr{P}_m$, $\sum b_i < q$ and $d - m \geqq \sigma_j + n + q - k \geqq \sigma_j + \sum b_p + n - (k + m - 1)$. To prove (b) we need

$$(*)\quad H^{k+m}\left(X, \bigwedge^{c_1} N^* \otimes \cdots \otimes \bigwedge^{c_p} N^* \otimes L_{i_1} \otimes \cdots \otimes L_{i_p} \otimes \Omega_X^j \otimes C \otimes K_X^{\otimes \mu_j} \otimes A^{\otimes d}\right) = 0$$

for all integers $c_1, \ldots, c_p \geqq 1$ such that $c_1 + \cdots + c_p = m \leqq n - k$. This in turn will follow from Corollary 1.4, if we show that the line bundle

$$D = L_{i_1} \otimes \cdots \otimes L_{i_p} \otimes K_X^{\otimes \mu_j} \otimes A^{\otimes d} \otimes (K_X^{\otimes \mu_j + p} \otimes A^{\otimes e})^*$$

is nef, where $e = \sigma_j + m + p(n+1)$. We first rewrite D as follows:

$$D = [K_X^{\otimes \mu_j} \otimes A^{\otimes d} \otimes (K_X^{\otimes \mu_j} \otimes A^{\otimes \sigma_j})^*] \otimes \bigotimes_{r=1}^{p} [L_{i_r} \otimes (K_X \otimes A^{\otimes (c_r + n + 1)})^*] .$$

Now $K_X^{\otimes \mu_j} \otimes A^{\otimes d} \otimes (K_X^{\otimes \mu_j} \otimes A^{\otimes \sigma_j})^*$ is very ample with our assumption on d. Further, for r between 1 and p:

$$L_{i_r} \otimes (K_X \otimes A^{\otimes (c_r + n + 1)})^* = K_X^{\otimes \binom{n-1}{j-1}} \otimes A^{\otimes (d_{i_r} - c_r - n - 1)} .$$

But $(d_{i_r} - c_r - n - 1) - n\binom{n-1}{j-1} \geqq 1$. Hence, the line bundle in question is nef (in fact, very ample). Thus the vanishing in $(*)$ follows from Corollary 1.4. □

We define:

$$\mu(n) \overset{\text{def}}{=} \begin{cases} \mu_{k+1}(n) & \text{if } n = 2k+1 \\ \mu_k(n) & \text{if } n = 2k \end{cases} \qquad \sigma(n) \overset{\text{def}}{=} \begin{cases} \sigma_{k+1}(n) + (k+1) & \text{if } n = 2k+1 \\ \sigma_k(n) + k & \text{if } n = 2k. \end{cases}$$

Proposition 2.2 *Let* $L = K_X^{\otimes \mu(n)} \otimes A^{\otimes \sigma(n)}$. *Then* $L \otimes (K_X^{\otimes \mu_j} \otimes A^{\otimes \sigma_j})^*$ *is nef for* $0 \leqq j \leqq n$.

Proof. Since $\mu_j = \mu_{n-j}(n)$ for all j and $\sigma_{n-j}(n) \geqq \sigma_j$ if $j \leqq \frac{n}{2}$ it suffices to prove the proposition for $\frac{n}{2} \leqq j \leqq n$. In this situation one verifies that $n(\mu(n) - \mu_j) < \sigma(n) - \sigma_j$. Thus our claim follows. □

Remark 2.3 If $(X, A) = (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$ then

(a) $\Omega^j_{\mathbb{P}^n}$ is $(j+1)$-regular, and
(b) M_i is 1-regular for all line bundles $O_{\mathbb{P}^n}(a_i)$ if $a_i \geqq 1$.

Thus $\Omega^j_{\mathbb{P}^n} \otimes (\bigoplus M_i)^{\otimes q}$ is $(j+q+1)$-regular. Hence, if $L_p = O_{\mathbb{P}^n}(a_p)$ where $a_p \geqq j + q$

$$H^k(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n} \otimes (\otimes M_i)^{\otimes q} \otimes L_p) = 0 \quad \text{for all } k \geqq 1 .$$

3 Applications

All our applications will be to the theory developed in Nori's paper [N]. We briefly recall his framework and notation. Let X be a smooth, projective variety of dimension $n + h$ and let $L_1, \ldots, L_h$ be very ample line bundles on X. Let $S = \bigoplus_{i=1}^h H^0(X, L_i)$ and $A = X \times S$. Let $B \subset A$ be the subvariety corresponding to the subbundle $\bigoplus M_i$ of the vector bundle $\bigoplus_{i=1}^h H^0(X, L_i) \otimes O_X$. Let $g: T \to S$ be

a smooth, not necessarily proper, morphism. Define $A_T = A \underset{X}{\times} T$ and $B_T = B \underset{X}{\times} T$. If W is a mixed Hodge structure, then $F^k W$ denotes the k^{th} member of the Hodge filtration. Further, we define $\mu = \mu(n+h)$ and $\sigma = \sigma(n+h)$.

Corollary 3.1 *With the above notation, let* $L_i = K_X^{\otimes \mu} \otimes A^{\otimes(\sigma+2n+h+c)} \otimes B_i$, *where* B_i *are nef and* c *is a positive integer. Then,*

$$F^k H^{n+k}(A_T, B_T) = 0 \quad \text{for all } k \leqq c\,.$$

Proof. It has been shown by Nori, [N, Sect. 3 and Remark 7.6], that the vanishing claimed above follows if we can show that

$$H^\alpha(X, \Omega_X^\beta \otimes ((\bigoplus M_i)^{\otimes j})^* \otimes (\bigoplus L_i^{\otimes r})^*) = 0$$

for all $\alpha \leqq n+h-1$, $0 \leqq \beta \leqq n+c$, $0 \leqq j \leqq n+c$ and $r \geqq 1$. By Serre duality, it suffices to prove that

$$H^{n+h-\alpha}(X, \Omega_X^{n+h-\beta} \otimes ((\bigoplus M_i)^{\otimes j}) \otimes (\bigoplus L_i^{\otimes r})) = 0$$

as α, β, j and r run through all possible values specified above. This follows from Theorem 2.1 and Proposition 2.2 above. □

The following corollary was the main motivation for all our results.

Corollary 3.2 *Let* X *be a smooth projective variety of dimension* $n+h$. *Let* $D_i \in |K_X^{\otimes \mu} \otimes A^{\otimes \sigma+3n+h} \otimes B_i|$, *for* $i = 1, \ldots, h$, *where* A *is very ample and* B_i *are* nef *line bundles on* X. *Assume that* $Y = X \cap D_1 \cap \cdots \cap D_h$ *is smooth. Let* ξ *be a codimension* $d < n$ *algebraic cycle on* X *whose cohomology class in* $H^{2d}(X, \mathbb{Q})$ *is non-zero. Then the restriction of* ξ *to* Y *is not algebraically equivalent to zero.*

Proof. Follows from Corollary 3.1 and [N].

Next, we prove a variation of Theorem 2.1.

Corollary 3.3 *Let* $L = K_X^{\otimes \mu} \otimes A^{\otimes(\sigma+2n+c)} \otimes B$ *where* B *is* nef. *Let* S_c *be a codimension* c *linear subspace in* $S = H^0(X, L)$, *that is base point free. Let* M_c *be the vector bundle defined to be the kernel of the surjective map* $S_c \otimes O_X \to L$. *Then*

$$H^\alpha(X, \Omega_X^{n+h-\beta} \otimes \overset{j}{\bigwedge} M_c \otimes L^{\otimes r}) = 0$$

for α, $\beta \geqq 0$ *and* $j \leqq 2n$.

Proof. The proof uses a standard trick, introduced by Green [G]. We shall prove the result by induction on c, the codimension of W in V. The result for $c = 0$, follows from Theorem 2.2 and Proposition 2.3. For the inductive step, as in [G], we take a flag of subspaces $S_c \subset S_{c-1} \subset \cdots \subset S_0 = S$ such that each successive quotient S_{i-1}/S_i is one-dimensional. Let $M_i = M_{S_i}$. Then we have exact sequences:

$$0 \to \overset{p}{\bigwedge} M_{i+1} \to \overset{p}{\bigwedge} M_i \to \overset{p-1}{\bigwedge} M_{i+1} \to 0\,.$$

Twisting the above sequence by $\Omega_X^{n+h-\beta} \otimes L^{\otimes r}$ and arguing by downward induction on p the result follows. □

Corollary 3.4 *Let* X *be a smooth projective variety of dimension* $n+1$ *and let* L, S *and* S_c *be as in Corollary* 3.3. *Let* $B \subset A = X \times S_c$ *be the subvariety corresponding*

to the subbundle M_c. Let $T \xrightarrow{g} S_c$ be a smooth morphism. Let p be the projection map $A_T \xrightarrow{p} T$. Let W be a pure, polarisable variation of Hodge structures on T of weight e. Then

$$F^{k+e}H^{n+k}(A_T, B_T; p^*W) = 0 \quad \textit{for all } k \leqq c\ .$$

Proof. Follows from Corollary 3.3 and [N, Remark D, Sect. 0.2] and work of M. Saito.

The corollary above, was motivated by a conjecture of Nori [N, Conjecture 7.4.1]. Unfortunately, our result does not address his question in the most interesting case, where one wants a statement similar to the corollary, but with the codimension of the subspace S_c, replaced by its dimension.

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Mixed problems for fully nonlinear hyperbolic equations

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Introduction

This paper deals with general hyperbolic equations of second order and nonlinear boundary conditions under geometric assumptions on the domain and on the equations. A simplified version of them is given by

$$\begin{aligned} \partial_t^2 u_j - \partial_{x_i} F_j^i(t, x, u, D_x u) &= w_j(t, x, u, Du) \quad \text{in } \Omega_T , \\ \nu_i F_j^i(t, x, u, D_x u) &= g_j(t, x, u, Du) \quad \text{in } \Gamma_T , \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) &= u_1(x) \quad \text{in } \{0\} \times \Omega . \end{aligned} \tag{1}$$

Here $1 \leqq j \leqq N$, $1 \leqq i \leqq n$, the summation convention is used, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary Γ, ν the outer normal, $\Omega_T = [0, T[\times \Omega$ and $\Gamma_T = [0, T[\times \Gamma$, $0 < T \leqq \infty$. F and g satisfy some natural conditions which assure hyperbolicity. The main result is concerned with classical solutions for small t. Assume that $u_0 \in W^{s+1}(\Omega)^N$ and $u_1 \in W^s(\Omega)^N$, where s is an integer bigger than $1 + n/2$. This implies $u_0 \in C^2(\bar{\Omega})^N$. Under suitable assumptions the following holds:

There is a unique $0 < t_0 \leqq T$ and a unique solution $u \in C^2(\Omega_{t_0} \cup \Gamma_{t_0})$. If $0 \leqq \sigma \leqq s + 1$ and $0 \leqq t < t_0$ then $\partial_t^\sigma u(t) \in W^{s+1-\sigma}(\Omega)$. $\partial_t^\sigma u(t)$ depends continuously on u_0, u_1, w, F and t. t_0 is characterized by the alternatives that either the graph of the solutions runs out of the domain where F, w and g are defined or

$$\int_0^t \|u(\tau)\|_{W^{2,\infty}(\Omega)} + \|\partial_t u(\tau)\|_{W^{1,\infty}(\Omega)} + \|\partial_t^2 u(\tau)\|_{L^\infty(\Omega)} d\tau \to \infty \quad \text{as } t \to t_0 . \tag{2}$$

Similar results hold for the fully nonlinear equation

$$\begin{aligned} F(t, x, u, Du, D^2 u) &= 0 \quad \text{in } \Omega_T , \\ u = 0 \quad \text{in } \Gamma_T, \quad u = u_0, \quad \partial_t u &= u_1 \quad \text{in } \{0\} \times \Omega , \end{aligned} \tag{3}$$

assuming natural structure conditions on the derivatives of F.

These equations describe many physical and geometrical problems. Examples are the nonlinear wave equation in a Minkowski manifold with dissipation at the

boundary and the motion of an elastic body under hydrostatic pressure, see [Ko, KS, NS]. It is also possible to reduce certain free boundary problems to such equations, see [MT]. Other examples are hyperbolic harmonic maps, Yang Mills equations and the Einstein equations if one uses a suitable fixed gauge, see [EH] and [EM].

Many papers have been written on this topic. See [D, HKM] for the problem in $\mathbb{R}^n$, [CW] and [DH] for the Dirichlet problem, [ST] for the fully nonlinear equation (2), [W] for a local existence theorem for a class of homogeneous Neumann problems, where

$$F_j^i(x, t, u, Du) = a_{jl}^{ik}(x, t, u)\partial_{x_k}u^l .$$

See [Ka1, Ka2] for a semigroup approach to similar equations but simpler boundary conditions. He also considers the continuous dependence on the initial data. [Sh1] and [NS] use a technique similar to [Ka2] and [ST] for proving existence of a local solution to Eq. (1). They need $s \geqq [n/2] + 7$. In [KS] this is weakened to $s > n/2 + 1$ using the two long papers [Sh1] and [Sh2] on linear hyperbolic equations. There is also the related problem of nonlinear boundary conditions for some first order systems of hyperbolic equations which is considered in [M1, M2].

The structure conditions used in this paper at hand are concerned with the coefficients of the highest derivatives of the linearization. They are implied by all structure conditions of the authors above. Let

$$\partial_{x_i}(a_{jl}^{ik}(t, x, u, Du)\,\partial_{x_k}v^l)$$

be the main part of the linearization, where $\partial_{x_0} = \partial_t$, and let

$$v_i a_{jl}^{ik}(t, x, u, Du)\partial_{x_k}v^l - h_{jl}^k(t, x, u, Du)\partial_{x_k}v$$

be the part of the boundary conditions containing first derivatives. For scalar equations the structure conditions hold if the matrix (a^{ik}) is symmetric, if it defines a Minkowski metric on $\mathbb{R}^{n+1}$, if $\{t\} \times \Omega$ is a space like hypersurface and if Γ_T is time like. At the boundary we require that the vector h^k is of the form $h_0(t, x, u, Du)\theta^k(t, x, u, Du)$, with $h_0 \leqq 0$, θ tangent at Γ_T, time like and $\theta^0 = 1$.

In our notation the authors [KS] and [NS] assume additionally $h^k = 0$ if $1 \leqq k \leqq n$. The even more restrictive conditions $a^{00} = 1$, $a^{0k} = 0$ if $1 \leqq k \leqq n$ and $h^k = 0$ if $0 \leqq k \leqq n$ are used by [Ka2].

The more general structure conditions are important in general relativity, in geometric problems and for free boundary problems as considered in [MT].

For the rest of the introduction we return for simplicity to equations of type (1).

The solutions are constructed as a fixed point of the following Picard type map: Differentiating Eq. (1) by t we get equations of the form

$$\partial_{tt}^2 v - \partial_{x_i}(a_{jl}^{ik}(t, x, u, Du)\partial_{x_k}v_j + f_j^i(t, x, u, Du)) \tag{4}$$

$$= w_{jl}^i(t, x, u, Du)\partial_{x_i}v^l + \bar{w}_j(t, x, u, Du)$$

where $v = \partial_t u$. The Picard type map is defined as follows. A function u is mapped to $u_0(x) + \int_0^t v(\tau, x)\,d\tau$ where v is a solution of the linear Eq. (4). The main point is the use of sharp results for the linear equations, see Theorem 2.2 and Theorem 2.4,

which are obtained by energy methods with some similarity to the methods used by [L, Sect. 3] for parabolic equations. [Ka] and [KS] use other estimates for the linear equations which force them to use a more complicated Picard type map J: $u \to \bar{u}$ which can be defined as follows. Given a function u Eq. (4) is used to get v. Now Eq. (1) is considered as an elliptic equation for $\bar{u}$ where v and $\partial_t v$ are substituted for $\partial_t u$ and $\partial_{tt}^2 u$.

It is clear from the construction that the solution can be extended as long as the graph of (u, Du) stays in a compact subset of the domain where F and w are defined and the $W^{s+1} \times W^s$ norm stays bounded, see for example [Ka] and [KS]. This result is improved by careful estimates to a characterization of blow up in terms of (2).

Continuous dependence on the initial data is considered in [Ka2] using an abstract setting. The arguments given in Sect. 2 and Sect. 3 are simpler and totally different. Furthermore they apply to more general equations.

It is possible to treat Schrödinger equations, higher order equations and mixed parabolic–hyperbolic equations similarly.

Global existence is beyond the scope of this paper, see for example the papers [C, J, K, S1]. There are also a lot of results for special equations, see [S2, St] and others. This list is by no means complete and I refer to the references in these papers for more information.

The essential contributions of this paper are: The generality of the boundary conditions, the direct method of construction of the solution, continuous dependence on the initial values, F, g and w and the characterization of blow up.

1 Main results

Let $0 \leqq i, k \leqq n$, $1 \leqq \alpha, \beta \leqq n$, $1 \leqq j, l \leqq N$ and $t = x_0$. The summation convention is used. We consider the equation

$$
\begin{aligned}
\partial_{x_i} F_j^i(t, x, u, Du) &= w_j(t, x, u, Du) \quad \text{in } \Omega_T\,, \\
\nu_i F_j^i(t, x, u, Du) &= g_j(t, x, u, Du) \quad \text{in } \Gamma_T\,, \\
u = u_0, \quad \partial_t u &= u_1 \quad \text{in } \{0\} \times \Omega\,.
\end{aligned} \tag{1.1}
$$

where $Du = (\partial_t u, \partial_{x_1} u, \ldots, \partial_{x_n} u)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary Γ of class C^{s+2}, $s > n/2 + 1$ and ν denotes the outer normal. We denote $\Omega_T = [0, T[\times \Omega$ and $\Gamma_T = [0, T[\times \Gamma$ where $0 < T \leqq \infty$.

Assumption 1 (Regularity) We assume that $u_0 \in W^{s+1}(\Omega)$, $u_1 \in W^s(\Omega)$, where $W^s = W^{s,2}$ is the usual Sobolev space and $W_0^{s,p}$ denotes the closure of C_0^∞ in that norm. Let U be an open neighborhood of $\{0\} \times \operatorname{graph}((u_0, u_1, D_x u_0))$ in $(\Omega_T \cup \Gamma_T) \times \mathbb{R}^N \times \mathbb{R}^{N \times (n+1)}$, let $F, g \in C^{s+1}(U)$, $w \in C^s(U)$ and define

$$
a_{jl}^{ik} = \frac{\partial F_j^i}{\partial (\partial_{x_k} u^l)}, \quad h_{jl}^k = h_{jl}^{uk} + h_{jl}^{sk} = \frac{\partial g_j}{\partial (\partial_{x_k} u^l)},
$$

$$
h_{jl}^k = h_{jl}^{uk} + h_{jl}^{sk},
$$

where h^s is symmetric and h^u is skew symmetric with respect to the j, l indices. We assume that $h_{jl}^{sk} = \theta^k h_{jl}$ where $\theta(t, x, u, Du)$ is a vectorfield which is tangential at Γ_T and which satisfies $\theta^0 = 1$, and $h_{jl} = h_{lj}$ is symmetric.

Assumption 2 (Symmetry) $a_{jl}^{ik} = a_{lj}^{ki}$, $v_\alpha h^\alpha = 0$,

in U. The second condition implies that g and thus also h do not depend on the normal derivative at the boundary.

Assumption 3 (Coerciveness) For any $t \in [0, T[$, $v_0 \in C^1(\bar{\Omega})$ and $v_1 \in C(\bar{\Omega})$ with $\{t\} \times \text{graph}(v_0, v_1, D_x v_0) \subset U$ there exist $\kappa_0 > 0$ and $\mu \geqq 0$ such that for all $\phi \in C^\infty(\bar{\Omega})$ the inequality

$$(P\bar{a})_{jl}^{\alpha\beta} \partial_{x_\alpha} \phi^j, \partial_{x_\alpha} \phi^l)_\Omega \geqq \kappa_0 \| \phi \|_{W^1(\Omega)}^2 - \mu \| \phi \|_{L^2}^2 ,$$

holds where $\bar{a}_{jl}^{ik} = (a_{jl}^{ik} - v^i h_{jl}^{uk} + v^k h_{jl}^{ui})\,(t, x, v_0, v_1 D_x v_1)$ and P is induced by the projection to $\mathbb{R}^n$ along θ, P: $\xi \to x - \xi^0 \theta$. Pa is defined by

$$(Pa)_{jl}^{ik} = a_{jl}^{\bar{i}\bar{k}} (\delta_{\bar{i}}^i - \delta_{\bar{i}}^0 \theta^i)(\delta_{\bar{k}}^k - \delta_{\bar{k}}^0 \theta^k) ,$$

where δ_j^i is the Kronecker delta, $(\,,\,)_\Omega$ denotes the L^2 scalar product and D_x denotes the set of first order spatial derivatives.

Remark. This condition is equivalent to open algebraic conditions on a and h at all points of $\bar{\Omega}$. It is easy to describe this condition for interior points: $(Pa)_{jl}^{\alpha\beta} \xi_\alpha \xi_\beta$ has to be uniformly positive definite for $|\xi| = 1$. The condition at the boundary is much more complicated, see [SS]. If $\theta^\alpha = 0$ then $(P\bar{a})_{jl}^{\alpha\beta} = \bar{a}_{jl}^{\alpha\beta}$.

Assumption 4 (Time components) For any $\xi \in U$ there exists a $\kappa_1 > 0$ such that for all $\eta \in \mathbb{R}^N$ the inequality

$$a_{jl}^{00}(\xi) \eta^j \eta^l \leqq -\kappa_1 |\eta|^2 ,$$

holds. Furthermore we assume $h_{jl}(\xi) \eta^j \eta^l \leqq 0$.

Assumption 5 (Compatibility conditions) We suppose that the compatibility conditions hold up to order s. We refer to the appendix for a discussion of these conditions. Under these assumptions the main theorem of this section holds:

Theorem 1.1 1. Existence, regularity: *There exists a unique* $0 < t_0 \leqq T$ *and a unique classical solution* $u \in C^2(\Omega_{t_0} \cup \Gamma_{t_0})$ *of* (1.1) *with* $D^\sigma u(t) \in L^2(\Omega)$ *if* $0 \leqq \sigma \leqq s + 1$. *Here* $D^\sigma u$ *denotes all derivatives of order* σ.
2. Continuity: *Suppose that* u^m *is a sequence of solutions to the equations*

$$\partial_{x_i} F_j^{i,m}(t, x, u^m, Du^m) = w_j^m(t, x, u^m, Du^m) \quad \text{in } \Omega_T ,$$

$$v_i F_j^{i,m}(t, x, u^m, Du^m) = g_j^m(t, x, u^m, Du^m) \quad \text{in } \Gamma_T ,$$

$$u^m = u_0^m, \quad \partial_t u^m = u_1^m \quad \text{in } \{0\} \times \Omega ,$$

where $F^m \to F$, $g^m \to g$ *in* C^{s+1}, $w^m \to w$ *in* C^s, $u_0^m \to u_0$ *in* H^{s+1} *and* $u_1^m \to u_1$ *in* H^s. *Suppose that the regularity, the structure and the compatibility conditions are satisfied for each equation. Let* $0 < t_1 < t_0$. *Then there is a* m_0 *such that the life span* $t_0^m > t_1$ *if* $m > m_0$ *and*

$$D^\sigma u^m(t) \to D^\sigma u(t) \quad \text{in } L^2(\Omega)$$

uniformly in t *if* $0 \leqq t \leqq t_1$ *and* $0 \leqq \sigma \leqq s + 1$.

3. Blow up: t_0 *is characterized by the two alternatives: either the graph of* (u, Du) *is not precompact in* U *or*

$$\int_0^t \|D^2 u(\tau)\|_{L^\infty(\Omega)} d\tau \to \infty \quad as \quad t \to t_0 .$$

Remark. The structure assumptions for scalar equations explained in the introduction imply Assumption 2–4 above. This can be seen as follows. Assumption 2 and 4 hold obviously. We notice that $\theta(t, x, u, Du)$ is a time like vector field at the boundary. We extend it to $x \in \bar{\Omega}$ as a time like vector field in the Minkowski metric $a^{ik}(t, x, u, Du)$ such that $\theta^0 = 1$. Now Assumption 3 is equivalent to the positive definiteness of $(Pa)^{\alpha\beta}$, which holds if θ is a time like vector for the metric a^{ik} and $\{t\} \times \Omega$ is spacelike.

Similar results hold for the equation

$$F_j(t, x, u, Du, D^2 u) = 0 \quad \text{in } \Omega_T ,$$

$$u = 0 \quad \text{in } \Gamma_T, \quad u = u_0, \quad \partial_t u_1 = u_1 \quad \text{in } \{0\} \times \Omega , \tag{1.2}$$

under the following assumptions.

Assumption 1 Let $s > 2 + n/2$. Let Ω, Ω_T, Γ, Γ_T be as above. We suppose that $u_2 \in C(\bar{\Omega})$ is given such that the equation above is satisfied if we substitute u_0 for $u(0)$, u_1 for $\partial_t u(0)$ and u_2 for $\partial_t^2 u(0)$. Let $U \subset (\Omega_T \cup \Gamma_T) \times \mathbb{R}^{N(1+(n+1)+(n+1)^2)}$ be an open neighborhood of $\{0\} \times \text{graph}(u_0, u_1, D_x u_0, u_2, D_x u_1, D_x^2 u_0)$ in this set and $F \in C^{s+1}(U)$. We define $a_{jl}^{ik}(t, x, u, Du, D^2 u) = \dfrac{\partial F_j}{\partial(\partial^2_{x_i x_k} u^l)}$.

Assumption 2 (Symmetry) $a_{jl}^{ik} = a_{lj}^{ki}$.

This condition can easily be relaxed to $a_{jl}^{ik} + a_{jl}^{ki} = a_{lj}^{ik} + a_{lj}^{ki}$.

Assumption 3 (Coerciveness) For $\xi \in U$ there exists $\kappa_0 > 0$ such that

$$a_{lj}^{\alpha\beta}(\xi)\eta^l \eta^j \chi_\alpha \chi_\beta \geqq \kappa_0 |\chi|^2 |\eta|^2$$

holds for $\eta \in \mathbb{R}^N$ and $\chi \in \mathbb{R}^n$.

Assumption 4 (Time Components) For $\xi \in U$ there exists $\kappa_1 > 0$ such that $a_{jl}^{00} \chi^l \chi^j \leqq -\kappa_1 |\eta|^2$ for $\eta \in \mathbb{R}^N$.

We apply the implicit function theorem to solve the equations locally for $\partial_t^2 u(0) = u_2$ and get $u_2 \in W^{s-1}$. By formal differentiation we get expressions for the higher derivatives and substitute u_σ for these derivatives.

Assumption 5 (Compatibility) $u_\sigma \in W^{s+1-\sigma} \cap W_0^1$, for $\sigma \leqq s$.

For a more detailed discussion of the compatibility conditions we refer to the Appendix. Then the following holds.

Theorem 1.2 *Assume that the Assumptions* 1–5 *are satisfied.*

1. Existence, regularity, continuity: *There exists a unique* $0 < t_0 \leqq \infty$ *and a unique solution* $u \in C^3(\Omega_{t_0} \cup \Gamma_{t_0})$ *of Eq.* (1.2) *with* $\partial_t^2 u(0) = u_2$. *Moreover* $D^\sigma u(t)$ *depends continuously on* t, *on the initial values and on the equation in the sense of Theorem* 1.1.

2. Blow up: t_0 *is characterized by two alternatives. Either the graph of* $(u, Du, D^2 u)$ *is not precompact in* U *or*

$$\int_0^t \|D^3 u(\tau)\|_{L^\infty} \to \infty \quad \text{as } t \to t_0 .$$

2 Systems of linear hyperbolic equations

The main idea in this section is the use of a convenient weak formulation for equations in divergence form, see Definition 2.1. We construct the solution by a Galerkin approximation and study some properties of it (Theorem 2.2). Higher regularity follows essentially from the energy estimate and from standard estimates for elliptic equations. The estimates for the solution allow a direct Picard iteration for the nonlinear equation with a low and essentially optimal Sobolev exponent. Equations not in divergence form will be treated similarly in the last part of this section (Theorem 2.5).

First we deal with the system of equations

$$\partial_{x_i}(a_{jl}^{ik}\partial_{x_k}u^l) = w_j + \partial_{x_i} f_j^i \quad \text{in } \Omega_T , \tag{2.1}$$

$$v_i(a_{jl}^{ik}\partial_{x_k}u^l - f_j^i) = g_j + h_{jl}^k \partial_{x_k} u^l \quad \text{in } \Gamma_T ,$$

$$u(0) = u_0 \in W^1(\Omega), \quad \partial_t u(0) = u_1 \in L^2(\Omega) .$$

We denote $(\xi^i)_{0\le i\le n}$ by ξ^i and $(\xi^\alpha)_{1\le\alpha\le n}$ by ξ^α for $\xi \in \mathbb{R}^{n+1}$.

Assumption 1 (Domain) The domain $\Omega \subset \mathbb{R}^n$ is bounded with boundary $\Gamma \in C^2$. We use the notation $\Omega_T = [0, T] \times \Omega$ and $\Gamma_T = [0, T] \times \Gamma$ in this chapter, which differs from the rest of this paper, where we work with the intervals $[0, T[$. The outer normal ν is extended to a C^1 vector field on $\bar{\Omega}$.

Assumption 2 (Regularity) $a_{jl}^{ik} \in W^{1,\infty}([0, T], L^\infty(\Omega))$, $h_{jl}^k \in W^{1,\infty}(\Omega_T)$,

$$w_j = \bar{w}_j + \hat{w}_{jl}u^l + \check{w}_{jl}^k \partial_{x_k} u^l, \quad f_j^i = \bar{f}_j^i + \hat{f}_{jl}^i u^l, \quad g_j = \bar{g}_j + \hat{g}_{jl} u^l ,$$

where

$$\bar{w}_j \in L^2(\Omega_T), \quad \hat{w}_{jl} \in L^\infty([0, T], L^n(\Omega)), \quad \check{w}_{jl}^k \in L^\infty(\Omega_T) ,$$

$$\bar{f}_j^i \in W^1([0, T], L^2(\Omega)), \quad \hat{f}_{jl}^i \in W^{1,\infty}([0, T], L^n(\Omega)) \cap L^\infty(\Omega_T) ,$$

$$\bar{g}_j \in W^1(\Omega_T), \quad \hat{g}_{jl} \in W^{1,\infty}([0, T], L^n(\Omega)) \cap L^\infty([0, T], W^{1,n}(\Omega) \cap L^\infty(\Omega)) .$$

Thus $w \in L^2(\Omega_T)$, $f \in W^1([0, T], L^2(\Omega))$ and $g \in W^1(\Omega_T)$ if $u \in W^1(\Omega_T)$.

Assumption 3 (Symmetry) $a_{jl}^{ik} = a_{lj}^{ki}$, $h_{jl}^\alpha = -h_{lj}^\alpha$, $\nu_\alpha h_{jl}^\alpha = 0$.

Thus $h^\alpha \partial_{x_\alpha} u$ depends only on tangential derivatives.

Assumption 4 (Coerciveness) There exist $\kappa > 0$ and μ such that for $t \in [0, T]$ and $v \in W^1(\Omega)$ the following holds,

$$((a_{jl}^{\alpha\beta} + h_{jl}^\alpha \nu^\beta - h_{jl}^\beta \nu^\alpha)(t)\partial_{x_\beta} v^l, \partial_{x_\alpha} v^j)_\Omega \geqq \kappa \|v\|_{W^1(\Omega)}^2 - \mu \|v\|_{L^2(\Omega)}^2 . \tag{2.2}$$

Remark. This is equivalent to a pointwise algebraic condition if the coefficients are continuous, see [SS]. Since $\eta_\alpha \eta_\beta \nu^\beta h_{jl}^\alpha \chi^j \chi^l = 0$ for all $\eta \in \mathbb{R}^n$ and $\chi \in \mathbb{R}^N$ the

coerciveness assumption implies $\eta_\alpha \eta_\beta a_{jl}^{\alpha\beta} \chi^j \chi^l \geqq \bar{\kappa} |\eta|^2 |\chi|^2$, for some $\bar{\kappa} > 0$ if the coefficients are continuous.

Assumption 5 We assume that $a_{jl}^{00} \eta^j \eta^l \leqq -\kappa |\eta|^2$ if $\eta \in \mathbb{R}^n$, where κ is taken from Assumption 4. Furthermore we assume $h_{jl}^0 \eta^j \eta^l \leqq 0$.

Definition 2.1 *A function $u \in W^1(\Omega_T)$ is called a weak solution of Eq. (2.1) if u is weakly continuous at zero as a map from $[0, T]$ to $W^1(\Omega)$ and $\partial_t u$ is weakly continuous at zero as a map from $[0, T]$ to $L^2(\Omega)$, if*

$$u(t) \rightharpoonup u_0 \quad \text{in } W^1(\Omega), \quad \partial_t u(t) \rightharpoonup u_1 \quad \text{in } L^2(\Omega) \text{ as } t \to 0,$$

and if for all $\phi \in W_0^1([0, T], L^2(\Omega)) \cap L^2([0, T], W^1(\Omega))$ the equation

$$a_{jl}^{ik} \partial_{x_k} u^l, \partial_{x_i} \phi^j)_{\Omega_T} = -(w_j, \phi^j)_{\Omega_T} + (f_j^i, \partial_{x_i} \phi^j)_{\Omega_T} + (g_j, \phi^j)_{\Gamma_T} + (h_{jl}^k \partial_{x_k} u^l, \phi^j)_{\Gamma_T} \tag{2.3}$$

holds.

Remarks. 1. We have used a formal notation for the last boundary term. This term has to be understood in the following way:

$$(h_{jl}^k \partial_{x_k} u^l, \phi^j)_{\Gamma_T} = (((\partial_\nu h_{jl}^k) - (\partial_{x_i} \nu^k h_{jl}^i) + (\operatorname{div} \nu) h_{jl}^k) \partial_{x_k} u^l, \phi^j)_{\Omega_T} + (h_{jl}^k \nu^i \partial_{x_k} u^l, \partial_{x_i} \phi^j)_{\Omega_T} - (h_{jl}^i \nu^k \partial_{x_k} u^l, \partial_{x_i} \phi^j)_{\Omega_T},$$

where $h_{jl}^k \nu^i - h_{jl}^i \nu^k$ is symmetric. Similarly $(g_j, \phi^j)_{\Gamma_T} = (\partial_\nu g_j, \phi^j)_{\Omega_T} + (g_j, \partial_\nu \phi^j)_{\Omega_T} + ((\operatorname{div} \nu) g_j, \phi^j)_{\Omega_T}$.

2. The initial condition is also a kind of regularity condition. It implies that there is a neighborhood of $t = 0$ where $u(t)$ in $W^1(\Omega)$ and $\partial_t u(t)$ in $L^2(\Omega)$ are uniformly bounded. Later we will see that the uniform boundedness and the continuity of u as a map from $[0, T]$ to $W^1(\Omega)$ and of $\partial_t u$ as a map from $[0, T]$ to $L^2(\Omega)$ are equivalent if the function u satisfies Eq. (2.3) for all testfunctions. The proof will show that the weak continuity follows from the other assumptions if $h^0 = 0$.

3. The structure conditions will be relaxed in the remark after the proof of Theorem 2.2.

We define

$$\|u\|_E(t) = (\|u(t)\|_{W^1(\Omega)}^2 + \|\partial_t u(t)\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$$

and $|\chi(t)| = \lim_{\varepsilon \to 0} \|\chi\|_{L^\infty([t-\varepsilon, t+\varepsilon])}$, for $\chi \in L^\infty(\mathbb{R})$, with the obvious modifications for Banach space valued functions defined on some interval.

Theorem 2.2 *Assume that the Assumptions 1–5 hold.*

1. Regularity: *Let u be a weak solution of (2.1). Then $u \in C([0, T], W^1(\Omega)) \cap C^1([0, T], L^2)$ and the estimate*

$$\|u\|_E(t_2) \leqq c\left(\|u\|_E(t_1) + d_1 + \int_{t_1}^{t_2} d_2(\tau) \|u\|_E(\tau) + d_3(\tau)\, d\tau\right) \leqq c\left(\|u\|_E(t_1) + d_1 + \int_{t_1}^{t_2} d_3(\tau)\, d\tau\right) e^{c \int_{t_1}^{t_2} d_2(\tau)\, d\tau}, \tag{2.4}$$

holds for $0 \leqq t_1 \leqq t_2 \leqq T$, *where*

$$d_1 = \|\bar{g}(t_1)\|_{L^2} + \|\bar{f}^\alpha(t_1)\|_{L^2} + (\|\hat{g}(t_1)\|_{L^\infty} + \|\hat{f}^\alpha(t_1)\|_{L^\infty} + 1)\|u(t_1)\|_{L^2},$$

$$d_2(t) = \|\partial_t a(t)\|_{L^\infty} + \|\hat{w}(t)\|_{L^n} + \|\check{w}^k(t)\|_{L^\infty} + \|\partial_t \hat{f}(t)\|_{L^n} + \|\hat{f}(t)\|_{L^\infty}$$
$$+ \|\hat{g}(t)\|_{W^{1,n}\cap L^\infty} + \|\partial_t \hat{g}(t)\|_{L^n} + \|h^\alpha(t)\|_{W^{1,\infty}} + \|\partial_t h^\alpha(t)\|_{L^\infty} + 1,$$

$$d_3(t) = \|\bar{w}(t)\|_{L^2} + \|\partial_t \bar{f}(t)\|_{L^2} + \|\bar{g}\|_E(t),$$

and $c = c(\Omega, \|a\|_\infty, \|h\|_\infty, \kappa, \mu)$.

2. Existence: *Given* $u_0 \in W^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *there exists a unique weak solution.*
3. Continuity: *If, in addition, we have a sequence of equations and solutions*

$$\partial_{x_i}(a_{jl}^{ik\,m}\partial_{x_k}u^{l\,m}) = w_j^m + \partial_{x_i} f_j^{i\,m} \quad \text{in } \Omega_T,$$

$$\nu_i(a_{jl}^{ik\,m}\partial_{x_k}u^{l\,m} - f_j^{i\,m}) = g_j^m + h_{jl}^{k\,m}\partial_{x_k}u^{l\,m} \quad \text{in } \Gamma_T,$$

$$u^m(0) = u_0^m \in W^1(\Omega), \quad \partial_t u^m(0) = u_1^m \in L^2(\Omega),$$

where all the data satisfy the assumptions above and also

$$\begin{array}{lll}
u_0^m \to u_0 & \text{in} & W^1(\Omega), \quad u_1^m \to u_1 \quad \text{in } L^2(\Omega), \\
a^m \to a & \text{in} & W^{1,1}([0,T], L^\infty(\Omega)), \\
h^m \to h & \text{in} & W^{1,1}([0,T], L^\infty(\Omega)) \cap L^1([0,T], W^{1,\infty}(\Omega)), \\
\bar{w}^m \to \bar{w} & \text{in} & L^1([0,T], L^2(\Omega)), \quad \hat{w}^m \to \hat{w} \quad \text{in } L^1([0,T], L^n(\Omega)), \\
\check{w}^m \to \check{w} & \text{in} & L^1([0,T], L^\infty(\Omega)), \quad \bar{f}^m \to \bar{f} \quad \text{in } W^{1,1}([0,T], L^2(\Omega)), \\
\hat{f}^m \to \hat{f} & \text{in} & W^{1,1}([0,T], L^n(\Omega)) \cap L^1([0,T], L^\infty(\Omega)), \\
\bar{g}^m \to \bar{g} & \text{in} & W^{1,1}([0,T], L^2(\Omega)) \cap L^1([0,T], W^{1,2}(\Omega)), \\
\hat{g}^m \to \hat{g} & \text{in} & W^{1,1}([0,T], L^n(\Omega)) \cap L^1([0,T], W^{1,n} \cap L^\infty(\Omega)),
\end{array}$$

holds, then $(u^m(t), \partial_t u^m(t))$ *converges to* $(u(t), \partial_t u(t))$ *in* $W^1(\Omega) \times L^2(\Omega)$ *uniformly in* t.
4. Weak continuity: *If the functions* $u_0^m, u_1^m, \bar{w}^m, \bar{f}^m$ *and* $\bar{g}^m$ *converge only weakly, then the functions* $(u^m(t), \partial_t u^m(t))$ *converge weakly in* $W^1(\Omega) \times L^2(\Omega)$.

Outline of the proof. First we establish the energy inequality for weak solutions and $0 < t_0 \leqq t_1 \leqq T$. Formally we get the inequality by testing with $\chi_{[t_1,t_2]}\partial_t u$, where χ is the characteristic function of the interval. Since $\chi\partial_t u$ is not an admissible testfunction we regularize u and $\chi_{[t_1,t_2]}\partial_t u$ first to get the energy inequality. We construct a weak solution using a Galerkin approximation. Together with the energy estimate this implies essentially all the other results in part 1 and 2.

Part 4 of the theorem is rather trivial. Only some uniformity of the weak convergence needs to be shown at $t = 0$.

For the proof of part 3 we would like to use the equation for the difference $u^m - u$. But there are terms which are too singular. Thus we consider the equation for $u^m - u_\varepsilon$ where u_ε is the regularized solution of the limit equation. If h^0 were equal to zero we could construct a solution over $[-T, T]$ and the energy inequality would imply the result. In the general case we have the energy inequality for $u^m - u_\varepsilon$ only if $t_1 > \varepsilon$ and we need some uniformity in m of the continuity at $t = 0$. But this is true by the energy inequality, the regularity of the solution and by part 4.

Proof. By the coerciveness assumption and Remark 1 to Definition 2.1 we can assume without loss of generality that $g = 0$ and $h^\alpha = 0$ since these terms can be

absorbed into the other coefficients. To simplify the notation we suppress the indices for the image space $\mathbb{R}^N$ and the index 0 of h^0. Thus it suffices to consider the equation

$$\partial_{x_i}(a^{ik}\partial_{x_k}u) = w + \partial_{x_i}f^i \quad \text{in } \Omega_T\,, \tag{2.5}$$

$$\nu_i(a^{ik}\partial_{x_k}u - f^i) = h\partial_t u \quad \text{in } \Gamma_T\,,$$

and (2.3) is replaced by

$$(a^{ik}\partial_{x_k}u, \partial_{x_i}\phi)_{\Omega_T} = -(w, \phi)_{\Omega_T} + (f^i, \partial_{x_i}\phi)_{\Omega_T} + (h\partial_t u, \phi)_{\Gamma_T}$$

for all admissible testfunctions. The coerciveness assumption simplifies to

$$(a^{\alpha\beta}\partial_{x_\beta}v, \partial_{x_\alpha}v)_\Omega \geqq \kappa\|v\|^2_{W^1} - \mu\|v\|^2_{L^2}$$

for $v \in W^1(\Omega)$. The weak continuity of a weak solution at $t = 0$ is not needed in the first part of the proof. There we call u a weak solution of (2.5) by an abuse of notation if it satisfies (2.3) for all admissible testfunctions. For $\eta \in L^1$ we define $\eta^+(t) = \eta(t+\varepsilon)$ and $\eta_\varepsilon(t) = 1/\varepsilon \int_0^\varepsilon \eta(t+\tau)\,d\tau$. If $\eta,\ \phi \in L^2(\mathbf{R})$ then $\int_{\mathbf{R}} \eta\phi_{-\varepsilon}\,dt = \int_{\mathbf{R}} \eta_\varepsilon\phi\,dt$. Using $\phi_{-\varepsilon}$ as testfunction we obtain the following equation for u_ε and testfunctions ϕ with support in $]\varepsilon, T[\times\bar{\Omega}$,

$$(a^{ik}\partial_{x_k}u_\varepsilon, \partial_{x_i}\phi)_{\Omega_T} = -(w_\varepsilon, \phi)_{\Omega_T} + (f^i_\varepsilon, \partial_{x_i}\phi)_{\Omega_T} + (h\partial_t u_\varepsilon, \phi)_{\Gamma_T} + \Delta_1 + \Delta_2\,, \tag{2.6}$$

where

$$\Delta_1 = (a^{ik}\partial_{x_k}u_\varepsilon - (a^{ik}\partial_{x_k}u)_\varepsilon, \partial_{x_i}\phi)_{\Omega_T}\,,$$

$$\begin{aligned}\Delta_2 = {} & (((\partial_\nu h)\partial_t u)_\varepsilon - (\partial_\nu h)\partial_t u_\varepsilon - ((\partial_t h)\partial_\nu u)_\varepsilon + (\partial_t h)\partial_\nu u_\varepsilon \\ & + (\operatorname{div}\nu)((h\partial_t u)_\varepsilon - h\partial_t u_\varepsilon, \phi)_{\Omega_T} \\ & + ((h\partial_t u)_\varepsilon - h\partial_t u_\varepsilon)), \partial_\nu\phi)_{\Omega_T} - ((h\partial_\nu u)_\varepsilon - h\partial_\nu u_\varepsilon, \partial_t\phi)_{\Omega_T}\,.\end{aligned}$$

We define the approximate characteristic function for $\varepsilon < t_1 < t_1 + \gamma_1 < t_2 < t_2 + \gamma_2 < T$,

$$\chi^{\gamma_1\gamma_2}_{t_1t_2}(t) = \begin{cases} 0 & 0 \leqq t < t_1\,, \\ (t - t_1)/\gamma_1 & t_1 \leqq t < t_1 + \gamma_1\,, \\ 1 & t_1 + \gamma_1 \leqq t < t_2\,, \\ 1 - (t - t_2)/\gamma_2 & t_2 \leqq t < t_2 + \gamma_2\,, \\ 0 & t_2 + \gamma_2 \leqq t \leqq T\,. \end{cases}$$

In the following we suppress the argument and the indices of χ. Setting $\phi = \chi\partial_t u_\varepsilon$ in (2.6) we obtain a lot of terms, the first of which is

$$\begin{aligned}(a^{00}\partial_t u_\varepsilon, \partial_t(\chi\partial_t u_\varepsilon))_{\Omega_T} = {} & \frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1} (a^{00}\partial_t u_\varepsilon, \partial_t u)_\Omega\,dt \\ & - \frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2} (a^{00}\partial_t u_\varepsilon, \partial_t u)_\Omega\,dt + (a^{00}\partial_t u_\varepsilon, \chi\partial_t^2 u_\varepsilon)_{\Omega_T}\,.\end{aligned}$$

Integrating the left hand side by parts we get

$$(a^{00}\partial_t u_\varepsilon, \partial_t(\chi\partial_t u_\varepsilon))_{\Omega_T} = -(a^{00}\partial_t^2 u_\varepsilon, \chi\partial_t u_\varepsilon))_{\Omega_T} - ((\partial_t a^{00})\partial_t u_\varepsilon, \chi\partial_t u_\varepsilon)_{\Omega_T}\,.$$

We sum the terms at the right hand sides and use the symmetry of a^{00},

$$(a^{00}\partial_t u_\varepsilon, \partial_t(\chi\partial_t u_\varepsilon))_{\Omega_T} = \frac{1}{2}\Big(\frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1}(a^{00}\partial_t u_\varepsilon, \partial_t u)_\Omega\, dt$$
$$-\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2}(a^{00}\partial_t u_\varepsilon^l, \partial_t u^j)_\Omega\, dt - ((\partial_t a^{00})\partial_t u_\varepsilon, \chi\partial_t u_\varepsilon)_{\Omega_T}\Big).$$

By the symmetry of the coefficients a and an integration by parts in the second term we obtain

$$(a^{\alpha 0}\partial_t u_\varepsilon, \partial_{x_\alpha}(\chi\partial_t u_\varepsilon))_{\Omega_T} + (a^{0\beta}\partial_{x_\beta} u_\varepsilon, \partial_t(\chi\partial_t u_\varepsilon))_{\Omega_T} = -((\partial_t a^{0\beta})\partial_{x_\beta} u_\varepsilon, \chi\partial_t u_\varepsilon)_{\Omega_T},$$

and similarly

$$(a^{\alpha\beta}\partial_{x_\beta} u_\varepsilon, \partial_{x_\alpha}(\chi\partial_t u_\varepsilon))_{\Omega_T} = \frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2}(a^{\alpha\beta}\partial_{x_\beta} u_\varepsilon, \partial_{x_\alpha} u_\varepsilon)_\Omega\, dt$$
$$-\frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1}(a^{\alpha\beta}\partial_{x_\beta} u_\varepsilon, \partial_{x_\alpha} u_\varepsilon)_\Omega\, dt$$
$$-(a^{\alpha\beta}\partial_{tx_\beta} u_\varepsilon, \chi\partial_{x_\alpha} u_\varepsilon)_{\Omega_T}$$
$$-((\partial_t a^{\alpha\beta})\partial_{x_\beta} u_\varepsilon, \chi\partial_{x_\alpha} u_\varepsilon)_{\Omega_T}.$$

Summing $\frac{1}{2}$ times the left and the right hand side and using again the symmetry we arrive at

$$(a^{\alpha\beta}\partial_{x_\beta} u_\varepsilon, \partial_{x_\alpha}(\chi\partial_t u_\varepsilon))_{\Omega_T} = \frac{1}{2}\Big(\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2}(a^{\alpha\beta}\partial_{x_\beta} u_\varepsilon, \partial_{x_\alpha} u_\varepsilon)_\Omega\, dt$$
$$-\frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1}(a^{\alpha\beta}\partial_{x_\beta} u_\varepsilon, \partial_{x_\alpha} u_\varepsilon)_\Omega\, dt - ((\partial_t a^{\alpha\beta})\partial_{x_\beta} u_\varepsilon, \chi\partial_{x_\alpha} u_\varepsilon)_{\Omega_T}\Big).$$

Now $w_\varepsilon \to w$ in $L^2(\Omega_T)$, $f_\varepsilon \to f$ in $W^{1,2}([0,T], L^2(\Omega))$, and therefore

$$(w_\varepsilon, \chi\partial_t u_\varepsilon)_{\Omega_T} \to (w, \chi\partial_t u)_{\Omega_T},$$
$$(f_\varepsilon^0, \partial_t\chi\partial_t u_\varepsilon)_{\Omega_T} = -(\partial_t f_\varepsilon^0, \chi\partial_t u_\varepsilon)_{\Omega_T} \to -(\partial_t f^0, \chi\partial_t u)_{\Omega_T},$$
$$(f_\varepsilon^\alpha, \partial_\alpha\chi\partial_t u_\varepsilon)_{\Omega_T} = -(\partial_t f_\varepsilon^i, \chi\partial_{x_\alpha} u_\varepsilon)_{\Omega_T}$$
$$+\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2}(f_\varepsilon^\alpha, \partial_{x_\alpha} u_\varepsilon)_\Omega\, dt - \frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1}(f_\varepsilon^\alpha, \partial_{x_\alpha} u_\varepsilon)_\Omega\, dt$$
$$\to -(\partial_t f^\alpha, \chi\partial_{x_\alpha} u)_{\Omega_T}$$
$$+\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2}(f^\alpha, \partial_{x_\alpha} u)_\Omega\, dt - \frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1}(f^\alpha, \partial_{x_\alpha} u)_\Omega\, dt,$$

as ε tends to zero. The boundary integral in (2.6) is negative by Assumption 5,

$$(h\partial_t u_\varepsilon, \chi\partial_t u_\varepsilon)_{\Omega_T} \leqq 0.$$

It remains to estimate Δ_1 and Δ_2.

$$\Delta_1 = (a^{ik}\partial_{x_k}u_\varepsilon - (a^{ik}\partial_{x_k}u)_\varepsilon, \partial_{x_\alpha}\chi\partial_t u_\varepsilon)_{\Omega_T}$$

$$= \frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2} (a^{\alpha k}\partial_{x_k}u_\varepsilon - (a^{\alpha k}\partial_{x_k}u)_\varepsilon, \partial_{x_\alpha}u_\varepsilon)_\Omega\, d\tau$$

$$- \frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1} (a^{\alpha k}\partial_{x_k}u_\varepsilon - (a^{\alpha k}\partial_{x_k}u)_\varepsilon, \partial_{x_\alpha}u_\varepsilon)_\Omega\, d\tau$$

$$- (\partial_t(a^{ik}\partial_{x_k}u_\varepsilon - (a^{ik}\partial_{x_k}u)_\varepsilon), \partial_{x_i}\chi u_\varepsilon)_{\Omega_T}\,.$$

The first two terms of the right hand side tend to zero as ε tend to zero. We will show that

$$\partial_t(a^{ik}\partial_{x_k}u_\varepsilon - (a^{ik}\partial_{x_k}u)_\varepsilon) \rightharpoonup 0 \quad \text{in } L^2(\Omega_T)\,.$$

Now

$$-\partial_t(a^{ik}\partial_{x_k}u_\varepsilon - (a^{ik}\partial_{x_k}u)_\varepsilon) = -(\partial_t a^{ik})\partial_{x_k}u_\varepsilon - a^{ik}\partial_{x_k}\frac{1}{\varepsilon}(u^+ - u)$$

$$+\frac{1}{\varepsilon}(a^{ik+}\partial_{x_k}u^+ - a^{ik}\partial_{x_k}u) = \Delta_\varepsilon a^{ik}\partial_{x_k}u^+ - (\partial_t a^{ik})\partial_{x_k}u_\varepsilon\,,$$

and $\Delta_\varepsilon a^{ik} \to \partial_t a^{ik}$ weak* in $L^\infty(\Omega_{t_2+\gamma_2})$. Thus both terms have the same limit and the difference converges to zero weakly in $L^2(\Omega_T)$. The same argument applies to Δ_2.

We define

$$J(t) = 1/2(-(a^{00}(t)\partial_t u(t), \partial_t u(t))_\Omega + (a^{\alpha\beta}(t)\partial_{x_\beta}u(t), \partial_{x_\alpha}u(t))_\Omega) - (f^\alpha(t), \partial_{x_\alpha}u(t))_\Omega\,.$$

Now we let ε tend to zero and we obtain altogether the estimate

$$\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2} J(\tau)\,d\tau \leqq \frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1} J(\tau)\,d\tau + \frac{1}{2}((\partial_t a^{ik})\partial_{x_k}u, \chi\partial_{x_i}u)_{\Omega_T}$$

$$-(w, \chi\partial_t u)_{\Omega_T} - (\partial_t f^i, \chi\partial_{x_i}u)_{\Omega_T}\,, \tag{2.7}$$

where equality holds if $h = 0$. Using the coerciveness and this inequality we get in the L^2 sense in t,

$$J(t) \geqq \frac{\kappa}{2}\|u(t)\|^2_{W^1} - \mu\|u(t)\|^2_{L^2} - c\|f^\alpha(t)\|^2_{L^2}\,.$$

Together with the inequality

$$\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2} |\rho(t)|\,dt \leqq \frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_2} |\rho(t)|\,dt + \int_{t_1}^{t_2+\gamma_2} |\chi\partial_t\rho|\,dt\,,$$

applied to $\rho = \|u(t)\|^2_{L^2(\Omega)}$ and $\rho = \|f^\alpha(t)\|^2_{L^2(\Omega)}$ we obtain

$$\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2} \|u\|^2_E(t)\,d\tau \leqq c\Bigg(\frac{1}{\gamma_1}\int_{t_1}^{t_1+\gamma_1} (J(\tau) + c\|f(\tau)\|^2_{L^2(\Omega)} + \|u(\tau)\|^2_2)\,d\tau$$
$$+ \int_{t_1}^{t_2+\gamma_2} ((1 + \|\partial_t a(\tau)\|_\infty)\|u(\tau)\|^2_E$$
$$+ (\|w(\tau)\|_{L^2(\Omega)} + \|\partial_t f(\tau)\|_{L^2(\Omega)})$$
$$\times(\|u\|_E(\tau) + \|f(\tau)\|_{L^2(\Omega)}))\,d\tau\Bigg).$$

Thus $u \in L^\infty([t_1, T], W^1(\Omega)) \cap W^{1,\infty}([t_1, T], L^2(\Omega))$. Next we prove the continuity for $t > 0$. We show that the functions u_ε converge uniformly to u. Let $v = u - u_\varepsilon$. v is a weak solution of the equation

$$\partial_{x_i}(a^{ik}\partial_{x_k} v) = w_0 + \partial_{x_i} f_0^i \quad \text{in} \quad [\varepsilon, T-\varepsilon]\times\Omega\,,$$
$$\nu_i(a^{ik}\partial_{x_k} v - f_0^i) = h\partial_t v \quad \text{in} \quad [\varepsilon, T-\varepsilon]\times\Gamma\,,$$

where

$$w_0 = w - w_\varepsilon + ((\partial_\nu h)\partial_t u)_\varepsilon - (\partial_\nu h)\partial_t u_\varepsilon - ((\partial_t h)\partial_\nu u)_\varepsilon$$
$$+ (\partial_t h)\partial_\nu u_\varepsilon + \operatorname{div} \nu((h\partial_t u)_\varepsilon - h\partial_t u_\varepsilon)\,,$$
$$f_0^i = f^i - f_\varepsilon^i + a^{ik}\partial_{x_k} u_\varepsilon - (a^{ik}\partial_{x_k} u)_\varepsilon + \nu^i((h\partial_t u)_\varepsilon - h\partial_t u_\varepsilon) + \delta^i(h\partial_\nu u_\varepsilon - (h\partial_\nu u)_\varepsilon)\,,$$

$\delta^i = 1$ if $i = 0$ and $\delta^i = 0$ otherwise. v is a weak solution in the sense that Eq. (2.3) holds for all admissible testfunctions. The right hand side satisfies the assumptions of Theorem 2.2 and we can apply the inequality above to this equation. Using the boundedness of u as a map to W^1 and of $\partial_t u$ as a map to L^2 we obtain by the arguments used to estimate A_1 that w_0 and f_0 converge to zero in the right spaces as ε tends to zero. Thus

$$\frac{1}{\gamma_2}\int_{t_2}^{t_2+\gamma_2} \|u - u_\varepsilon\|^2_E(t)\,d\tau \to 0 \quad \text{as} \quad \varepsilon \to 0\,,$$

uniformly in γ_2 and t_2 and u is continuous as a map from $]0, T]$ to $W^1(\Omega)$ and $\partial_t u$ as a map to $L^2(\Omega)$. Now we let γ_1 and γ_2 tend to zero and obtain from (2.7)

$$J(t_2) \leqq J(t_1) + \int_{t_1}^{t_2} \frac{1}{2}(((\partial_t a^{ik}(\tau))\partial_{x_k} u(\tau), \partial_{x_i} u(\tau))_\Omega \tag{2.8}$$
$$- (w(\tau), \partial_t u(\tau))_\Omega - (\partial_t f^i(\tau), \partial_{x_i} u(\tau)))_\Omega\,,$$

and

$$\|u\|_E(t_2) \leqq c\Bigg(\|u(t_1)\|_E + \|f(t_1)\|_2 + \int_{t_1}^{t_2} (\|w(\tau)\|_2$$
$$+ \|\partial_t f(\tau)\|_2 + (\|\partial_t a^{ik}(\tau)\|_\infty + 1)\|u\|_E(\tau)\,d\tau\Bigg),$$

for $0 < t_1 \leqq t_2 \leqq T$. This inequality, together with Gronwall's lemma, implies the estimate (2.4).

Next we deal with the uniqueness of the weak solution for given data. We return to Definition 2.1 and require that a weak solution satisfies $u(0) = u_0$ and $\partial_t u(0) = u_1$ in the sense of the definition. First we assume that $u_0 = 0$, $u_1 = 0$, $\bar{f}(0) = 0$. We extend $a, h, \hat{f}, \hat{w}$ and $\check{w}$ to negative t and we set $u = \bar{w} = \bar{f} = 0$ for $t \leqq 0$. The equation

$$
\begin{aligned}
(2.9) \qquad (a^{0k}\partial_{x_k}u - f^0 + h\partial_\nu u, \partial_t\phi)_{\Omega_T} = &(- a^{\alpha k}\partial_{x_k}u + f^\alpha, \partial_{x_\alpha}\phi)_{\Omega_T} \\
&- (w + (\partial_\nu h)\partial_t u - (\partial_{x_t}h)\partial_\nu u \\
&+ (\operatorname{div} \nu)h\partial_t u, \phi)_{\Omega_T} + (h\partial_t u, \partial_\nu\phi)_{\Omega_T} ,
\end{aligned}
$$

holds for all testfunctions with compact support in $]0, T[\times \bar{\Omega}$. We show that this remains true for testfunctions with compact support in $[0, T[\times \bar{\Omega}$. It suffices to do this for $\phi \in C([0, T], W^1) \cap C^{0,1}([0, T], L^2)$ with $\Phi(T) = 0$. We write $\phi(t) = \phi_0 + \bar{\phi}(t)$ with $\bar{\phi}(0) = 0$, $\phi_0 \in W^1(\Omega)$ and define

$$
\chi_\varepsilon(t) = \begin{cases} t/\varepsilon & \text{if } t \leqq \varepsilon , \\ 1 & \text{if } t > \varepsilon , \end{cases}
$$

and $\phi_\varepsilon = \chi_\varepsilon\phi_0 + \bar{\phi}$. Equation (2.9) holds for ϕ_ε. We show that both sides converge to zero as ε tends to zero if we insert $\phi_\varepsilon - \phi$ in (2.9). This is obvious for the right hand side. The left hand side

$$
\frac{1}{\varepsilon}\int_0^\varepsilon (a^{0k}\partial_{x_k}u - f^0 + h\partial_\nu u, (1 - \chi)\phi_0)_\Omega \, dt ,
$$

tends to

$$
\frac{1}{2}(a^{00}(0)u_1 + a^{0\beta}\partial_{x_\beta}u_0 - f^0(0) + h\partial_\nu u_0, \phi_0)_\Omega = 0 .
$$

Therefore Eq. (2.3) holds for all $\phi \in W_0^1([- T, T], L^2(\Omega)) \cap L^2([- T, T], W^1(\Omega))$ and the extended u. Thus the energy inequality and inequality (2.8) hold up to time zero in this case. If u and v are solutions with the same initial value then their difference is a weak solution of the homogeneous equation with initial data $u(0) = u_t(0) = 0$. Now $u - v = 0$ by the energy inequality and by Gronwall's lemma, thus the weak solution is unique.

2. Existence. The solution is constructed in a standard way by a Galerkin approximation. In this way one gets a function $u \in L^\infty([0, T[, W^1) \cap C^{0,1}([0, T[, L^2)$ which satisfies (2.3) for all admissible testfunctions and (2.4) for $t_1 = 0$ and u_0 and u_1 inserted. The only point to show is the weak continuity at $t = 0$. Now $u \in C([0, T[, L^2)$ and the uniform boundedness imply that u is weakly continuous as a map to $W^1(\Omega)$. The right hand side of (2.9) is defined for $\phi \in L^2([0, T], W^1)$. This implies

$$
a^{0k}\partial_{x_k}u - f^0 + h\partial_\nu u \in C^{1/2}([0, T], (W^1)^*) ,
$$

where $*$ denotes the dual space. The map $t \to (a^{00}(t)\,\partial_t u(t), \phi)_\Omega$ is continuous for ϕ in W^1. But this expression is bounded for almost all t by $\|\phi\|_{L^2}$. Thus the map $t \to a^{00}\partial_t u$ is weakly continuous in the L^2 norm. The multiplication with a^{00} is an automorphism of $L^2(\Omega)$ which depends continuously on t, hence $\partial_t u$ is weakly continuous and by the construction $\partial_t u(0) = u_1$.

Next we prove the continuity of the weak solution at $t = 0$. We have

$$\begin{aligned} A(t) = \frac{1}{2}\big(& -(a^{00}(0)\partial_t(u(t)-u(0)),\, \partial_t(u(t)-u(0)))_\Omega \\ & + (a^{\alpha\beta}(t)\partial_{x_\beta}(u(t)-u(0)),\, \partial_{x_\alpha}(u(t)-u(0)))_\Omega \big) \\ = J(0) + J(t) & + (f^\alpha(0)\,\partial_{x_\alpha}u(0))_\Omega + (f^\alpha(t),\, \partial_{x_\alpha}u(t))_\Omega \\ & - \frac{1}{2}((a^{00}(0)-a^{00}(t))\,\partial_t u(t), \partial_t u(t))_\Omega \\ & + \frac{1}{2}((a^{\alpha\beta}(0)-a^{\alpha\beta}(t))\,\partial_{x_\beta}u(t), \partial_{x_\alpha}u(t))_\Omega \\ & + (a^{00}(0)\partial_t u(t), \partial_t u(0))_\Omega - (a^{\alpha\beta}(0)\partial_{x_\beta}u(t), \partial_{x_\alpha}u(t))_\Omega\,. \end{aligned}$$

The terms which are linear in $u(t)$ converge by the weak continuity of u to $-2J(t)$ as $t \to 0$. We obtain, using inequality (2.7),

$$\limsup_{t\to 0} A(t) \leqq \limsup_{t\to 0} J(t) - J(0) = 0\,.$$

Since the solution u is continuous as a map to $L^2(\Omega)$ we obtain the continuity up to time zero. This argument shows the equivalence of uniform boundedness and continuity for functions satisfying (2.3) for all testfunctions.

4. Weak continuity. Under the conditions of the theorem $\|u^m\|_E(t)$ is uniformly bounded. Repeating the arguments of the proof of Part 2 implies the result.

3. Continuous dependence. We first prove that the solutions u^m are uniformly continuous in m at $t = 0$. By Part 4 the solutions u^m converge weakly to a weak solution u of the limit equation. Let $J^m(t)$ be the functional corresponding to the m^{th} equation. The estimate (2.8) implies

$$J^m(t) \leqq J^m(0) + \int_0^t |\phi^m|\,,$$

where the functions

$$\begin{aligned} \phi^m(t) = & \|\partial_t a^m(t)\|_{L^\infty(\Omega)} + \|\bar{w}^m(t)\|_{L^2(\Omega)} + \|\hat{w}^m(t)\|_{L^n(\Omega)} + \|\check{w}^m(t)\|_{L^\infty(\Omega)} \\ & + \|\partial_t \bar{f}^m(t)\|_{L^2(\Omega)} + \|\hat{f}^m(t)\|_{L^\infty(\Omega)} + \|\partial_t \hat{f}^m(t)\|_{L^n(\Omega)}\,, \end{aligned}$$

lie in $L^1([0, T])$ and $\phi^m \to \phi$ in L^1. Given $\delta > 0$ we choose m_0 such that

$$\|\phi - \phi^m\|_1 \leqq \delta/2 \quad \text{if} \quad m > m_0\,,$$

and t small to obtain $\int_0^t |\phi| \leqq \delta/2$ and $\int_0^t |\phi^m| \leqq \delta$, if $m \leqq m_0$ for some δ with $\delta \to 0$ as $t \to 0$. Thus $J^m(t) \leqq J^m(0) + \delta(t)$ for all m and a function $\delta(t)$ with $\delta \to 0$ as $t \to 0$. We define

$$
\begin{aligned}
A^m(t) &= \frac{1}{2} - (a^{00}\partial_t(u^m - u), \partial_t(u^m - u))_\Omega + \frac{1}{2}(a^{\alpha\beta}\partial_{x_\beta}(u^m - u), \partial_{x_\alpha}(u^m - u))_\Omega \\
&= J^m + J + (f^{\alpha m}, \partial_{x_\alpha} u^m)_\Omega + (f^\alpha, \partial_{x_\alpha} u)_\Omega \\
&\quad - \frac{1}{2}((a^{00} - a^{00\,m})\partial_t u^m, \partial_t u^m)_\Omega + \frac{1}{2}((a^{\alpha\beta} - a^{\alpha\beta\,m})\partial_{x_\beta} u^m, \partial_{x_\alpha} u^m)_\Omega \\
&\quad + (a^{00}\partial_t u, \partial_t u^m)_\Omega - (a^{\alpha\beta}\partial_{x_\beta} u, \partial_{x_\alpha} u^m)_\Omega ,
\end{aligned}
$$

where all terms have to be taken at time t. $J^m(t)$ contains the only terms which are quadratic in Du^m and which might not converge. We obtain:

$$
\limsup_{m\to\infty} A^m(t) \leqq \limsup_{m\to\infty} (J^m(t) - J^m(0) + J^m(0) - J(0) + J(0) - J(t)) \leqq \bar{\delta}(t) ,
$$

with $\bar{\delta}(t) \to 0$ as $t \to 0$ since the initial values converge and $J(t)$ is continuous.

The difference of $u(t)$ and $u_m(t)$ is as small as we like if we choose t small and m large. Given $\delta_0 > 0$ there are t_0, m_0 and ε_0 such that the inequalities

$$
\sup_{\substack{0 \leqq t \leqq t_0 \\ m > m_0}} \|u^m - u\|_E(t) < \delta_0, \qquad \sup_{\substack{t_0 \leqq t \leqq T \\ 0 \leqq \varepsilon \leqq \varepsilon_0}} \|u - u_\varepsilon\|_E(t) < \delta_0 ,
$$

hold. For $v_\varepsilon^m = u_\varepsilon - u^m$ we have the equation

$$
\begin{aligned}
\partial_{x_i} a^{ik\,m}\partial_{x_k} v_\varepsilon^m &= \hat{w}^m v_\varepsilon^m + \check{w}^{k\,m}\partial_{x_k} v_\varepsilon^m + w_0 + \partial_{x_i}(\bar{f}^{i\,m} v_\varepsilon^m + f_0^i) \quad \text{in} \quad [t_0, T - \varepsilon] \times \Omega , \\
\nu_i(a^{ik\,m}\partial_{x_k} v_\varepsilon^m - \hat{f}^{i\,m} v_\varepsilon^m - f_0^i) &= h^m \partial_t v_\varepsilon^m \quad \text{in} \quad [t_0, T - \varepsilon] \times \Gamma , \\
v_{\varepsilon 0}^m(t_0) = u^m(t_0) - u(t_0)_\varepsilon, &\quad v_{\varepsilon 1}^m = \partial_t u^m(t_0) - \partial_t u_\varepsilon^m(t_0) .
\end{aligned}
$$

Here

$$
\begin{aligned}
w_0 &= \bar{w}_\varepsilon - \bar{w}^m + (\hat{w} - \hat{w}^m)u_\varepsilon + (\hat{w}u)_\varepsilon - \hat{w}u_\varepsilon + (\check{w}^k - \check{w}^{k,m})\partial_{x_k} u \\
&\quad + (\check{w}^k \partial_{x_k} u)_\varepsilon - \check{w}^k \partial_{x_k} u_\varepsilon + (\partial_\nu(h^m - h))\partial_{x_k} u_\varepsilon - (\partial_t(h^m - h))\partial_\nu u_\varepsilon \\
&\quad + \operatorname{div} \nu((h^m - h)\partial_t u_\varepsilon - (h\partial_t u)_\varepsilon + h\partial_t u_\varepsilon) \\
&\quad + ((\partial_\nu h)\partial_t u)_\varepsilon - (\partial_\nu h)\partial_t u_\varepsilon - ((\partial_t h)\partial_\nu u)_\varepsilon + (\partial_t h^k)\partial_\nu u_\varepsilon , \\
f_0^i &= \bar{f}_\varepsilon^i - f^{i\,m} + (\hat{f}^i - \hat{f}^{im})u_\varepsilon + (\hat{f}^i u)_\varepsilon - \hat{f}^i u_\varepsilon \\
&\quad + (a^{ik\,m} - a^{ik})\partial_{x_k} u + a^{ik}\partial_{x_k} u_\varepsilon - (a^{ik}\partial_{x_k} u)_\varepsilon \\
&\quad + \delta^i((h - h^m)\partial_\nu u - (h\partial_\nu u)_\varepsilon + h\partial_\nu u_\varepsilon) - \nu^i((h^m - h)\partial_t u_\varepsilon - (h\partial_t u)_\varepsilon + h\partial_t u_\varepsilon) .
\end{aligned}
$$

Now we apply the energy estimate.The terms w_0, f_0, $\bar{w}_\varepsilon - \bar{w}^m$ and $\bar{f}_\varepsilon^i - \bar{f}^{i\,m}$ and the initial values converge to zero in the right spaces. Thus the estimate

$$
\|u - u^m\|_E(t) \leqq \delta(\varepsilon, m, t_0) + \int_{t_0}^t c\,\|u^m - u_\varepsilon\|_E(\tau)\,d\tau ,
$$

holds. We apply Gronwall's lemma to arrive at $\|u - u^m\|_E(t) \leqq \hat{\delta}(\varepsilon, m, t_0)$, where $\hat{\delta}$ is as small as we like if we choose first t_0 then ε and at last m. Thus u^m converges in $L^\infty([0, T], W^1) \cap C^{0,1}([0, T], L^2)$. The proof is complete.

Remark. We can also relax the structure conditions to:

Assumption 3g. $a_{jl}^{ik} = a_{lj}^{ki}$, $v_i h^i = 0$ and $h_{jl}^k = h_{jl}^{uk} + h_{jl}^{sk}$, where h^u is the antisymmetric part and h^s is the symmetric part with respect to the jl indices. As above we assume without loss of generality that $h^u = 0$. We assume that

$$h_{jl}^{si} = \theta^i h_{jl}, \quad |\theta| \leqq c_0, \quad \theta^0 = 1 ,$$

where θ is a tangential vector field on Γ_T and h is negative semidefinite.

Assumption 4g. Let P_θ be defined as in Sect. 1. We assume that

$$((P_\theta a)_{jl}^{\alpha\beta} \partial_{x_\alpha} \phi^j, \partial_{x_\beta} \phi^l)_\Omega \geqq \kappa \|\phi\|_{W^1(\Omega)}^2 - \mu \|\phi\|_{L^2(\Omega)}^2 .$$

Assumption 2g. a, h, $\theta \in C^{0,1}(\Omega)_T$, $f \in W^{1,2}(\Omega)_T$ and $g = 0$.

Changing the coordinates to (y, t), $\partial_t y(x, t) = \theta(y(x, t))$, $y(x, 0) = x$ we obtain equations of the type (2.1). Now

$$|D_x y| \leqq \bar{c} = e^{c \int_0^t \|\vartheta(\tau)\|_{W^{1,\infty}(\Omega)} d\tau} .$$

Thus Theorem 2.2 holds also under the Assumptions 1, 2, 2g, 3g, 4g and 5. Thus the estimates (2.4) change in the obvious way. The most important point is the following: The constants depend also on $\int_0^t \|\theta(\tau)\|_{W^{1,\infty}} d\tau$.

Next we deal with the Lipschitz continuity of a weak solution. We return to the assumptions 1–6 and additionally assume $g = 0$, $h^\alpha = 0$ and

Assumption 2l (Regularity) $w \in L^\infty([0, T], L^2)$,

$$f \in C([0, T], W^1(\Omega)) \cap W^{1,\infty}([0, T], L^2(\Omega)) ,$$

$$a^{ik} \in W^{1,\infty}([0, T], C(\Omega)) \cap C([0, T], W^{1,n}(\Omega)) ,$$

$$h \in C([0, T), W^{1,n}(\Omega)) \cap W^{1,\infty}([0, T], C(\Omega)) ,$$

$$u_0 \in W^2(\Omega), \quad u_1 \in W^1(\Omega) .$$

Assumption 6 (Compatibility) We say that the compatibility condition holds if

$$v_\alpha(a^{\alpha\beta}(0)\partial_{x_\beta} u_0 + a^{\alpha 0}(0) u_1 - f^\alpha(0)) + h(0) u_1 \in W_0^1(\Omega) .$$

Lemma 2.3 *Suppose that the Assumptions* 1, 2l *and* 3–6 *hold. Then* u: $[0, T] \to W^1$ *and* $\partial_t u$: $[0, t] \to L^2$ *are Lipschitz continuous at* $t = 0$.

Proof. By Theorem A.2 there is a function $\hat{u} \in C([0, T], W^2) \cap C^1([0, T], W^1) \cap C^2([0, T], L^2)$ such that $\hat{u}(0) = u_0$, $\partial_t \hat{u}(0) = u_1$, $\partial_t^2 \hat{u}(0) = 0$ and $\sup_{0 \leqq t \leqq T} (\|\hat{u}\|_E + \|\partial_t \hat{u}\|_E) \leqq c(\|u_0\|_{W^2} + \|u_1\|_{W^1})$. We define $v(t) = u(t) - \hat{u}(t)$, and

$$g^\alpha = a^{\alpha\beta}(0)\partial_{x_\beta} u_0 + a^{\alpha 0}(0) u_1 - f^\alpha(0)) + v^\alpha h(0) u_1, \quad g^0 = 0 .$$

Now v is a solution of

$$\partial_{x_i}(a^{ik}\partial_{x_k}v) = \bar{w} + \partial_{x_i}\bar{f}^i \quad \text{in } \Omega_T\,,$$
$$\nu_\alpha(a^{\alpha\beta}\partial_{x_\beta}v - \bar{f}^\alpha) = h\partial_t v \quad \text{in } \Gamma_T\,,$$
$$v(0) = 0, \quad \partial_t v(0) = 0 \quad \text{in} \quad \{0\}\times\Omega\,,$$

where

$$\bar{w} = w - \partial_v(h\partial_t\hat{u}) - (\operatorname{div}\nu)h\partial_t\hat{u} - \partial_{x_i}g^i, \quad \bar{f}^i = f^i - a^{ik}\partial_{x_k}\hat{u} - \nu^i h\partial_t\hat{u} + g^i\,,$$

and $\bar{f}^i(0) = 0$ and $\nu_i g^i \in W_0^1(\Omega)$. Thus the energy inequality implies

$$\|u(t) - u(0)\|_E \leqq c_0\Bigg(\int_0^t \|w\|_{L^2} + \|\partial_t f\|_{L^2} + (\|a\|_{W^{1,n}\cap L^\infty} + \|\partial_t a\|_{L^n}$$
$$+ \|h\|_{W^{1,n}\cap L^\infty} + \|\partial_t h\|_{L^n} + \mu)\times(\|u_0\|_{W^2} + \|u_1\|_{W^1})\,d\tau\Bigg)\,,$$

where $c_0 = c_0(\|u_0\|_{W^2}, \|u_1\|_{W^1}, \sup\|u\|_E\|_{L^2})$. This implies the Lipschitz continuity. QED

We introduce prenorms $E^{s,p}$ on $C^\infty([0,T]\times\bar{\Omega})$,

$$\|u\|_{E^{s,p}}(t) = \Bigg(\sum_{i=0}^{s}\|\partial_t^i u(t)\|^p_{W^{s+1-i,p}}\Bigg)^{\frac{1}{p}}\,,$$

$E^{s,\infty}$ with the obvious modifications and

$$\|u\|_{G^{s,p}_{t_1,t_2}} = \sup_{t_1\leqq t\leqq t_2}\|u\|_{E^{s,p}}(t), \|u\|_{F^{s,p}_{t_1,t_2}} = \int_{t_1}^{t_2}\|u\|_{E^{s,p}}(t)\,dt\,,$$
$$G^s = G^{s,2} \quad E^s = E^{s,2} \quad F^s = F^{s,2}\,.$$

We will suppress t_1 and t_2 in the notation. Let $F^{s,p}$ and $G^{s,p}$ be the Banach subspaces of $W^{s,p}([t_1,t_2]\times\Omega)$ where the norm is finite. For these spaces the Sobolev type estimates

$$\|u\|_{E^{k_1,p_1}}(t) \leqq c\|u\|_{E^{k_2,p_2}}(t), \|u\|_{G^{k_1,p_1}} \leqq c\|u\|_{G^{k_2,p_2}}, \|u\|_{F^{k_1,p_1}} \leqq c\|u\|_{F^{k_2,p_2}}\,,$$

hold if $k_2 \geqq k_1, \dfrac{1}{p_1} + \dfrac{k_2 - k_1}{n} \geqq \dfrac{1}{p_2}, p_1 \neq \infty$. These imbeddings are compact if the strict inequalities hold, even if $p_1 = \infty$. The prenorms $(t,u)\to\|u\|_{E^{s,p}}(t)$ extend to maps $G^{s,p}\to L^\infty([t_1,t_2]), F^{s,p}\to L^1([t_1,t_2])$.

Next we consider more regular solutions for more regular data. It suffices to deal with the restricted equations

(2.10)
$$\partial_{x_i}(a^{ik}\partial_{x_k}u) = w + \partial_{x_i}f^i \quad \text{in } \Omega_T\,,$$
$$\nu_i(a^{ik}\partial_{x_k}u - f^i) = h^k\partial_{x_k}u \quad \text{in } \Gamma_T\,,$$
$$u(0) = u_0, \quad \partial_t u(0) = u_1\,,$$

where $g = 0$ and $h^u = 0$. Let $s > 0$ be an integer.

Assumption 1s. In addition to Assumption 1 we assume that $\partial\Omega$ is of class C^{s+1} and $v \in C^s$.

Assumption 2s (Regularity) Let $2 \leqq p \leqq \infty$, $p > n/s$. We assume

$$a, h \in G^{s,p} \cap C^1(\Omega_T), \quad \partial_t a, \partial_t h \in F^{s,p},$$

$$w \in G^{s-1} \cap W^{s,1}([0, T], L^2(\Omega)) \cap W^{s-1,1}([0, T], W^1(\Omega)),$$

$$f^i \in G^s, \quad Df^i \in W^{s,1}([0, T], L^2(\Omega)),$$

$$u_0 \in W^{s+1}(\Omega), \quad u_1 \in W^s(\Omega).$$

Assumption 6 (Compatibility) Suppose that the Assumptions 1s, 2s, 3g, 4g and 5 hold. We solve the equation for $\partial_t^2 u$ and define $u_2 = \partial_t^2 u(0)$. u_2 depends only on u_0 and u_1. By formal differentiation we obtain expressions for $u_i = \partial_t^i u(0)$, $2 \leqq i \leqq s+1$. We require that the equations are satisfied which one obtains by formally differentiating the boundary condition i times in time direction and replacing u_i for $\partial_t^i u(0)$. We refer to the appendix for a discussion of the compatibility conditions.

Theorem 2.4 (Regularity of the higher derivatives) *Assume that the Assumptions* 1s, 2s, 3g, 4g 5 *and* 6 *hold. Then the weak solution* u *of Eq.* (2.10) *lies in* $\bigcap_{0 \leqq i \leqq s+1} C^i([0, T], W^{s+1-i}(\Omega))$ *and the estimate*

$$\|u\|_{E^{s+1}}(t_2) \leqq \bar{c}\Big(\|u(t_1)\|_{E^{s+1}} + \|w\|_{G^{s-1}} + \|f\|_{G^s} + \int_{t_1}^{t_2} \|\partial_t^s w(t)\|_{L^2(\Omega)} + \|\partial_t^{s+1} f^i(t)\|_{L^2(\Omega)}\, dt \Big) \tag{2.11}$$

holds, where

$$c = c(\kappa, \mu, \|a\|_{G^{s,p} \cap C^1(\Omega_T)}, \|\partial_t a\|_{F^{s,p}}, \|h\|_{G^{s,p} \cap C^1(\Omega_T)}, \|\partial_t h\|_{F^{s,p}}).$$

Remark. 1. The regularity assumptions imply all regularity assumptions of Theorem 2.2.
2. Assumption 4 of Theorem 2.2 is satisfied after a transformation of the coordinates.

Outline of the proof. The proof consists of three parts. First we deal with regularity. Suppose that the coefficients are smooth and that $\theta^\alpha = 0$. The general case follows by an approximation of the coefficients and a transformation of the coordinates. We use finite differences in the time direction. The Lipschitz continuity at $t = 0$ implies uniform boundedness on the left end of the interval for the finite differences. Thus the estimate of Theorem 2.2 implies the uniform boundedness of $\Delta_\varepsilon u = \dfrac{u^+ - u}{\varepsilon}$ in G^1 and $\partial_t u$ is a weak solution of the differentiated equation with the initial values u_1 and u_2 which one obtains by a formal calculation. Now Theorem A.4 implies that $u \in G^{2,2}$. By induction we obtain the regularity of u.

For the estimates we assume that $u \in G^{s+1}$ is a weak solution of (2.1). We differentiate the equation s times in time direction and apply the estimate (2.4) to this equation. By considering the equations as elliptic equations for all t and

applying Theorem A.4 we obtain uniform estimates that depend on the norms of the assumptions.

Proof. For technical reasons we assume first $a, h \in C^\infty([0, T], W^{s,p}(\Omega))$ and $h^\alpha = 0$.

Let ϕ be a testfunction for (2.10) with support in $]\varepsilon, T[\times \bar{\Omega}$. Multiplying (2.10) with $\Delta_\varepsilon^- \phi = \dfrac{\phi - \phi^-}{\varepsilon}$ and integrating by parts we see that $v = \Delta_\varepsilon u$ is a weak solution of the equations

$$\partial_{x_i}(a^{ik}\partial_{x_k} v) = w_0 + \partial_{x_i} f_0^i \quad \text{in } [0, T-\varepsilon] \times \Omega ,$$

$$\nu_i(a^{ik}\partial_{x_k} v - f^i) = h\partial_t v \quad \text{in } [0, T-\varepsilon] \times \Omega ,$$

$$v(0) = \Delta_\varepsilon u(0), \quad \partial_t v(0) = \Delta_\varepsilon \partial_t u(0) ,$$

where

$$w_0 = \Delta_\varepsilon w - (\Delta_\varepsilon \partial_\nu h)\partial_t u^+ - (\operatorname{div} \nu)(\Delta_\varepsilon h)\partial_t u^+ + (\partial_t \Delta_\varepsilon h)\partial_\nu u^+,$$

$$f_0^i = \Delta_\varepsilon f^i - (\Delta_\varepsilon a^{ik})\partial_{x_k} u^+ + \nu^i(\Delta_\varepsilon h)\partial_t u^+ - \delta^i(\Delta_\varepsilon h)\partial_\nu u .$$

The energy inequality and the Lipschitz continuity at $t = 0$ imply that $\Delta_\varepsilon u$ is bounded in $C^0[(0, T], W^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$, independent of ε. Thus $u \in C^{1,1}([0, T], L^2) \cap C^{0,1}([0, T], W^1)$. In the limit we get an equation for $v = \partial_t u$,

$$\partial_{x_i}(a^{ik}\partial_{x_k} v) = w_0 + \partial_{x_i} f_0^i \quad \text{in } [0, T] \times \Omega ,$$

$$\nu_i(a^{ik}\partial_{x_k} v - f_0^i) = h\partial_t v \quad \text{in } [0, T] \times \Gamma ,$$

where

$$w_0 = \partial_t w - \partial_\nu((\partial_t h)\partial_t u) - (\operatorname{div} \nu)(\partial_t h)\partial_t u ,$$

$$f_0^i = \partial_t f^i - (\partial_t a^{ik})\partial_{x_k} u + \nu^i(\partial_t h)\partial_t u .$$

$v \in G^1$ is a weak solution in the sense that (2.3) holds for all admissible testfunctions. This implies the weak continuity at $t = 0$, thus v is a weak solution in the sense of Definition 2.1 and $\partial_t u$ is continuous as a map from $[0, T]$ to $W^1(\Omega)$ and $\partial_t^2 u$ as a map to $L^2(\Omega)$ by the arguments of Theorem 2.2. Before identifying the initial values, we deal with space regularity. From Eq. (2.10) we obtain for $t \in [0, T]$ and $\phi \in W^1(\Omega)$,

$$(a^{\alpha\beta}(t)\partial_{x_\beta} u(t), \partial_{x_\alpha}\phi)_\Omega = (w(t) + \partial_t(a^{0k}(t)\partial_{x_k} u(t) - f^0(t)), \phi)_\Omega$$
$$+ (f^\alpha(t) - a^{0\alpha}(t)\partial_t u(t), \partial_{x_\alpha}\phi)_\Omega + (h(t)\partial_t u(t), \phi)_{\partial\Omega} ,$$

where

$$(h(t)\partial_t u(t), \phi)_{\partial\Omega} = ((\partial_\nu h(t))\partial_t u(t) + h(t)\partial_\nu \partial_t u(t) + (\operatorname{div} \nu)h(t)\partial_t u(t), \phi)_\Omega$$
$$+ (h(t)\partial_t u(t), \partial_\nu \phi)_\Omega .$$

Thus $u(t)$ is a weak solution of this elliptic equation and $u \in C([0, T], W^2(\Omega))$. Equation (2.10) is satisfied in the L^2 sense for each t and we can solve the equation for $\partial_t^2 u(0) = u_2$. Suppose that $s \geqq 2$. The compatibility conditions of order s for Eq. (2.10) imply the compatibility conditions of order $s - 1$ for the differentiated equation, and by induction on s we obtain the full regularity of u.

Next we prove the estimate (2.11). We assume that $u \in G^{s+1}$ is a weak solution of (2.10) and that the regularity assumptions of Theorem 2.4 hold. Differentiating (2.10) s times in t direction and writing the boundary integral as interior integral we obtain

$$\partial_{x_i}(a^{ik}\partial_{x_k}\partial_t^s u) = \partial_t^s w - \partial_\nu([\partial_t^s, h^k]\partial_{x_k}u) - (\operatorname{div}\nu)[\partial_t^s, h^k]\partial_{x_k}u + \partial_{x_i}(\partial_t^s f^i - [\partial_t^s, a^{ik}]\partial_{x_k} + v^i[\partial_t^s, h^k]\partial_{x_k}u) \quad \text{in } \Omega_T ,$$

$$\nu_i(a^{ik}\partial_{x_k}\partial_t^s u - \partial_t^s f^i + [\partial_t^s, a^{ik}]\partial_{x_k}u) - [\partial_t^s, h^k]\partial_{x_k}u = h^k\partial_{x_k}\partial_t^s u \quad \text{in } \Gamma_T ,$$

and

$$\partial_t^s u(0) = u_s, \quad \partial_t^{s+1}u(0) = u_{s+1} \quad \text{in } \{0\}\times\Omega ,$$

where u_s and u_{s+1} are the formal time derivatives of the initial values as in Sect. 1. Thus (2.4) implies the estimate

$$\|\partial_t^s u\|_E(t_2) \leqq c\left(\|\partial_t^s u\|_E(t_1) + d_1(t_1) + \int_{t_1}^{t_2} d_2\, d\tau\right),$$

where

$$d_1 = \|\partial_t^s f^i - [\partial_t^s, a^{ik}(t_1)\partial_{x_k}]u + v^i[\partial_t^s, h^k]\partial_{x_k}u\|_{L^2(\Omega)} ,$$

$$d_2 = \|\partial_t^s w\|_{L^2(\Omega)} + \|\partial_t^s f^i\|_E + \|[\partial_t^s, a^{ik}\partial_{x_k}]u\|_E + \|[\partial_t^s, h^k]\partial_{x_k}u\|_E + \|\partial_t^s u\|_E .$$

Similarly we obtain from (2.10) by differentiating j times in time direction the equation

$$\partial_{x_\alpha}(a^{\alpha\beta}\partial_{x_\beta}\partial_t^j u) = \partial_{x_\alpha}(-[\partial_t^j, a^{\alpha\beta}]\partial_{x_\beta}u + \partial_t^j f^\alpha + v^\alpha\partial_t^j(h^k\partial_{x_k}u)) + \partial_t^j(w - \partial_t(a^{0k}\partial_{x_k}u + f^0) - (\operatorname{div}\nu + \partial_\nu)\partial_t^j(h^k\partial_{x_k}u)) \quad \text{in } \Omega,$$

$$\nu_\alpha(a^{\alpha\beta}\partial_{x_\beta}\partial_t^j u + [\partial_t^j, a^{\alpha\beta}]\partial_{x_\beta}u - \partial_t^j f^\alpha) - \partial_t^j(h^k\partial_{x_k}u) = 0 \quad \text{in } \partial\Omega .$$

Applying Theorem A.4 and Theorem A.1 we get

$$\|\partial_t^j u\|_{W^{s+1-j}(\Omega)} \leqq c(d_3 + \|u\|_{E(t)} + \|\partial_t^{j+1}u\|_{E^{s-j}}) ,$$

where $d_3 = (\|w\|_{E^{s-1}} + \|f\|_{E^s})$ for fixed t. By induction and Theorem A.1 we obtain the estimate in the theorem. Thus u is bounded in the G^{s+1} norm by the initial data and the norms in the assumptions of Theorem 2.4. Furthermore we obtain

$$u \in \bigcap_{0\leqq j\leqq s+1} C^j([0, T], W^{s+1-j}(\Omega)) .$$

As a next step we prove the regularity of a solution under the assumptions of Theorem 2.4. We mollify a, h, $\hat{w}$, $\check{w}$ and $\hat{f}$. After a transformation of the coordinates we obtain $h^\alpha = 0$. By Theorem A.9 there are initial values $u_{0\varepsilon}$ and $u_{1\varepsilon}$ which satisfy the compatibility conditions and

$$\|u_{0\varepsilon} - u_0\|_{W^{s+1}} + \|u_{1\varepsilon} - u_1\|_{W^s} \to 0 \text{ as } \varepsilon \to 0 .$$

For these regularized equations the solutions are regular, they are uniformly bounded in G^{s+1} by the estimate above and converge in G^1. Thus the limit u is regular in the sense that

$$u \in \bigcap_{0 \leqq j \leqq s+1} C^j([0, T], W^{s+1-j}) .$$

The proof is complete.

Finally we deal with strong solutions under Dirichlet boundary conditions,

$$a_{jl}^{ik} \partial_{x_i x_k} u^l = \bar{w}_j + \hat{w}_j^k \partial_{x_k} u \quad \text{in } \Omega_T , \tag{2.12}$$

$$u = 0 \quad \text{in } \Gamma_T, \quad u(0) = u_0 \in W^{s+1}(\Omega), \quad \partial_t u(0) = u_1 \in W^s(\Omega) ,$$

where s, Ω and Ω_T are as above.

Assumption 1 (Regularity) Let $2 \leqq p \leqq \infty$ and $s \geqq 1$ satisfy the inequality $\frac{1}{p} < \frac{s-1}{n}$. Let

$$a \in G^{s-1,p} \cap C^1(\bar{\Omega}_T), \quad \partial_t a \in F^{s-1,p}, \quad w_j = \bar{w}_j + \hat{w}_{jl}^k \partial_{x_k} u^l ,$$

$$\text{where } \bar{w} \in G^{s-1} \cap W^{s,1}([0, T], L^2), \quad \hat{w} \in G^{s-1,p}, \quad \partial_t \hat{w} \in F^{s-1,p} .$$

Assumption 2 (Coerciveness) There exists $\kappa > 0$ such that the inequalities

$$a_{jl}^{\alpha\beta}(t) \eta_\alpha \eta_\beta \chi^j \chi^l \geqq \kappa |\eta|^2 |\chi|^2, \quad a_{jl}^{00} \chi^j \chi^l \leqq -\kappa |\chi|^2 ,$$

hold in Ω_T, $\eta \in \mathbb{R}^n$ and $\chi \in \mathbb{R}^N$. Together with the continuity of a this implies that there exists $\bar{\kappa} > 0$ and μ such that for all $v \in W_0^1(\Omega)$ and all t the following is true, see [SS]

$$(a^{\alpha\beta}(t) \partial_{x_\beta} v, \partial_{x_\alpha} v)_\Omega \geqq \bar{\kappa} \| u \|^2_{W^1(\Omega)} - \mu \| v \|^2_{L^2(\Omega)} .$$

Assumption 3 (Compatibility conditions) Suppose that u_0 and u_1 satisfy the compatibility conditions of order s. This means that the formal time derivatives up to order s at $t = 0$ lie in $W_0^1(\Omega)$.

We say that $u \in G^2$ is a strong solution of (2.12) if the equations are satisfied in the L^2 sense.

Under these assumptions the following theorem holds.

Theorem 2.5 (Strong solutions) 1. Existence and uniqueness: *There exists a unique strong solution* $u \in \bigcap_{0 \leqq j \leqq s+1} C^j([0, T], W^{s+1-j}(\Omega))$. *The estimate*

$$\| u \|_{E^2}(t_2) \leqq c \left(\| u \|_{E^2}(t_1) + \int_{t_1}^{t_2} d(\tau) d\tau + \| w(t_2) \|_{L^2} \right)$$

holds where c *depends on the modulus of continuity of* a, *on* κ, Ω,

$$d = \| \partial_t w \|_{L^2(\Omega)} + \| (\partial_t a^{ik}) \partial_{x_i x_k} u \|_{L^2(\Omega)}$$

$$+ \| (\partial_{x_i} a^{ik}) \partial_{x_k t} u \|_{L^2(\Omega)} + \| u \|_{E^s} + \| (\partial_t a^{ik}) \partial_{x_k t} u \|_{L^2(\Omega)} + 1 ,$$

where d *is taken at* $t = \tau$.

2. Estimate: *The estimate*

$$\|u\|_{E^{s+1}}(t_2) \leqq \bar{c}\left(\|u\|_{E^{s+1}}(t_1) + \|w\|_{G^{s-1}} + \int_{t_1}^{t_2} (\|\partial_t^s w\|_{L^2(\Omega)} + 1) d\tau \right), \tag{2.13}$$

holds where $\bar{c} = c(\|a\|_{G^{s-1,p}}, \|\partial_t a\|_{F^{s-1,p}}, \|a\|_{G^{1,\infty}})$.

3. Continuous dependence: *Let* $s = 1$. *Assume that there is a sequence of hyperbolic equations satisfying the Assumptions* 1 *and* 2:

$$a^{ikm} \partial_{x_i x_k} u^m = w^m \quad \text{in } \Omega_T,$$

$$u^m = 0 \quad \text{in } \Omega_T, \quad u^m(0) = u_0^m, \quad \partial_t u^m(0) = u_1^m$$

where the coefficients a^m *converge to* a *in* C^1 *and* w^m *converges to* w *in* $W^{1,1}([0, T], L^2)$. *Assume that the limit equation is hyperbolic and that the initial values* $(u_0^m, u_1^m) \in (W^2 \cap W_0^1) \times W_0^1$ *converge to* (u_0, u_1). *Then the solutions* u^m *converge to the solution of the limit equation in* G^2.

Proof. We assume first that a is smooth. We prove the existence of a strong solution. The equation

$$\partial_{x_i}(a^{ik} \partial_{x_k}) u = w + (\partial_{x_i} a^{ik}) \partial_{x_k} u \,,$$

$$u = 0 \quad \text{in } \Gamma_T, \quad u(0) = u_0, \quad \partial_t u(0) = u_1 \,,$$

has a unique weak solution in G^{s+1}. Next we suppose that $s = 1$, that the assumptions of the theorem hold and that $u \in G^2$ is a solution of (2.12). We differentiate (2.12) in time direction,

$$a^{ik} \partial_{x_i x_k t} u = \partial_t w - (\partial_t a^{ik}) \partial_{x_i x_k} u \,,$$

write it again in divergence form and get

$$\partial_{x_i}(a^{ik} \partial_{x_k t} u) = \partial_t w - (\partial_t a^{ik}) \partial_{x_i x_k} u + (\partial_{x_i} a^{ik}) \partial_{x_k t} u \,.$$

Applying estimate (2.4) we get

$$\|\partial_t u\|_E(t_2) \leqq c\left(\|\partial_t u\|_E(t_1) + \int_{t_1}^{t_2} d\, d\tau \right),$$

and $u \in C^2([0, T], L^2) \cap C^1([0, T], W_0^1)$. u is also a solution of the elliptic equation,

$$a^{\alpha\beta} \partial_{x_\alpha x_\beta} u = w - 2a^{0\beta} \partial_{x_\beta t} u - a^{00} \partial_t^2 u \,.$$

Theorem A.4 implies

$$\|u(t)\|_{W^2} \leqq c(\|u(t)\|_{L^2} + \|a(t)\|_{L^\infty} \|\partial_t u(t)\|_E(t) + \|w(t)\|_2) \,,$$

and altogether we get the estimate in the theorem for $s = 1$. This estimate implies the uniqueness of such solutions. As in Theorem 2.2 the apriori estimate, Theorem A.9 and the existence result for smooth coefficients imply the existence of a unique solution in the general case. Next we prove continuous dependence on the initial values and on the equation. Since the type of this equation is invariant when

reversing time we can assume that the equations and solutions of Part 2 are given in some large time domain $[-T, T]$. We obtain an equation for u_ε,

$$a^{ik}\partial_{x_i x_k} u_\varepsilon = w_\varepsilon - ((a^{ik}\partial_{x_i x_k} u)_\varepsilon - a^{ik}\partial_{x_i x_k} u_\varepsilon) ,$$

where $(a^{ik}\partial_{x_i x_k} u)_\varepsilon - a^{ik}\partial_{x_i x_k} u_\varepsilon \to 0$ in $L^\infty([0, T], L^2)$, by the continuity of u and

$$\partial_t((a^{ik}\partial_{x_i x_k} u)_\varepsilon - a^{ik}\partial_{x_i x_k} u_\varepsilon) = \Delta a^{ik}\partial_{x_i x_k} u^+ - (\partial_t a^{ik})\partial_{x_i x_k} u_\varepsilon$$
$$\to 0 \quad \text{in } L^1([0, T], L^2) ,$$

and the arguments of Theorem 2.2 apply to this simpler situation. We define $v_\varepsilon^m = u^m - u_\varepsilon$ and obtain the equation

$$a^{ik\,m}\partial_{x_i x_k} v_\varepsilon^m = w^m - w_\varepsilon + (a^{ik\,m} - a^{ik})\partial_{x_i x_k} u_\varepsilon + ((a^{ik}\partial_{x_i x_k} u)_\varepsilon - a^{ik}\partial_{x_i x_k} u_\varepsilon) ,$$

with Dirichlet boundary conditions and initial values

$$v_\varepsilon^m(0) = u_0^m - u_\varepsilon(0), \quad \partial_t v_0^m = u_1^m - u_\varepsilon(0) .$$

Now we choose first ε and then m to make the difference of $u^m - u$ as small as we like.

Part 2 of the theorem follows from part 1 as inequality (2.11) followed from Theorem 2.2. QED

3 Construction of the solution in the divergence case

We will prove Theorem 1.1 in this section. Theorem 1.2 can be proven along the same lines, essentially referring to Theorem 2.5 and Theorem A.5 in the Appendix instead of to Theorem 2.4 and Theorem A.4. First we give a detailed outline of the proof. We construct a Picard type iteration map. For this purpose we differentiate Eq. (1.1) in time direction, obtaining

$$\partial_{x_i}(a^{ik}\partial_{x_k t} u) = \partial_t w + \partial_{x_i} f^i \quad \text{in } \Omega_T ,$$
$$\nu_\alpha(a^{\alpha k}\partial_{x_k t} u - f^\alpha) = h^k \partial_{x_k t} u + \bar{g} \quad \text{in } \Gamma_T ,$$

where

$$f^i(t, x, u, Du) = a^{ik}\partial_{x_k t} u - \partial_t F^i, \quad \bar{g} = \partial_t g - h^k \partial_{x_k t} u .$$

This can be written as follows,

$$\partial_{x_i}(\bar{a}^{ik}\partial_{x_k t} u) = \bar{w} + \partial_{x_i}\bar{f}^i \quad \text{in } \Omega_T , \tag{3.1}$$
$$\nu_\alpha(\bar{a}^{\alpha k}\partial_{x_k t} u - \bar{f}^\alpha) = h^{sk}\partial_{x_k t} u \quad \text{in } \Gamma_T ,$$

where $\bar{f}^i = f^i + \nu^i \bar{g}$, $\bar{a}^{ik} = a^{ik} - \nu^i h^{uk} + \nu^k h^{ui}$ and

$$\bar{w} = \partial_t w - ((\partial_\nu + \operatorname{div}\nu)h^{uk})\partial_{x_k t} u + (\partial_{x_i}\nu^\beta h^{ui})\partial_{x_\beta t} u - \partial_\nu \bar{g} - (\operatorname{div}\nu)\bar{g} .$$

The function $\bar{a}$ and $\bar{f}$ depend on (x, t, u, Du) and $\bar{w}$ depends also linearly on $D^2 u$. For technical reasons we assume first $s > 2 + n/2$. Let $Y_{\tau, R}$ be the subset of

$$H_\tau = \bigcap_{1 \leqq i \leqq s+1} W^{i,\infty}([0, \tau], W^{s+1-i}(\Omega)) ,$$

of functions v with $\partial_t^i v(0) = u_i$, the formal time derivatives of u for $0 \leqq i \leqq s$ at $t = 0$ and satisfying $\|v\|_{H_\tau} \leqq R$. By Theorem A.2 this set is not empty if R is chosen sufficiently large.

We want to define $J \in C(Y_{\tau,R}, H_\tau)$ as the map $v \to u$ where u solves

$$\partial_{x_i}(\bar{a}^{ik}(v)\partial_{x_k t}u) = \bar{w}(v) + \partial_{x_i}\bar{f}^i(v) \quad \text{in } \Omega_T\,, \tag{3.2}$$

$$\nu_\alpha(\bar{a}^{\alpha k}(v)\partial_{x_k t}u - \bar{f}^\alpha(v)) = h(v)\partial_t^2 u \quad \text{in } \Gamma_T\,,$$

$$\partial_t u = u_1 \quad \partial_t^2 u = u_2, \quad \text{in } \{0\}\times\Omega\,,$$

where for example $a(v) = a(t, x, v, Dv)$.

Therefore we have to check the assumptions of Theorem 2.4. The Assumptions 3g and 4g of Theorem 2.4 hold for $v \in Y$. The regularity assumption will be checked later on. The compatibility condition follows from the compatibility condition of Theorem 1.1. For large R the image of J lies in Y provided that τ is chosen sufficiently small.

Now

$$Y \subset C^3([0,\tau], L^2) \cap C^2([0,\tau], W^1) \cap C^1([0,\tau], W^2)\,.$$

The set Y is compact in the topology of this second space. Let $u_0 = J(v_0)$ and $u_1 = J(v_1)$. Then $u_1 - u_0$ solves

$$\partial_{x_i}(\tilde{a}^{ik}\partial_{x_k t}(u_1 - u_0)) = \tilde{w} + \partial_{x_i}\tilde{f}^i$$

$$\nu_\alpha(\tilde{a}^{\alpha k}\partial_{x_k t}(u_1 - u_0)) = \tilde{h}^k\partial_{x_k t}(u_1 - u_0)) + \tilde{g}\,,$$

where for example

$$\tilde{a}^{ik} = \int_0^1 a^{ik}(t, x, u_0 + \lambda(u_1 - u_0), Du_0 + \lambda(Du_1 - Du_0))\,d\lambda\,,$$

and

$$\|\partial_t\tilde{w}(t)\|_{L^2} + \|\partial_t\tilde{f}^i\|_{E^1}(t) + \|\partial_t\tilde{g}\|_{E^1}(t) \leqq c\|v_0 - v_1\|_{E^2}\,.$$

Applying Theorem 2.4 with $s = 1$ gives

$$\|\partial_t(u_1 - u_0)\|_{E^2}(t) \leqq c\int_0^t \|\partial_t w(\sigma)\|_{L^2} + \|\partial_t f^i\|_{E^1}(\sigma) + \|\partial_t g(\sigma)\|_{E^1}(\sigma)\,d\sigma$$

$$\leqq ct\|v_0 - v_1\|_{E^2}\,.$$

All the constants here depend on $\|v_0\|_{H_\tau}$ and $\|v_1\|_{H_\tau}$. We have used the fact that $w(0, x) = f^i(0, x) = g(0, x) = 0$. Thus the map J is a contraction in this metric if τ is sufficiently small. The fixed point is the solution of the original problem because by Eq. (3.2)

$$\partial_t(\partial_{x_i}F^i_j(t, x, u, Du) - w_j(t, x, u, Du)) = 0 \quad \text{in } [0, T[\times\Omega\,.$$

We differentiate the original equation and apply the elliptic regularity theory to get full regularity of the solution.

The uniqueness of C^2 solutions follows by the standard energy method. Let u and v be solutions to the system of Eq. (1.1). Then

$$\partial_{x_i}(a^{ik}\partial_{x_k}(u-v)) = w^i\partial_{x_i}(u-v) + w_0(u-v) + \partial_{x_i}(f^i(u-v)) \quad \text{in } \Omega_T\,,$$

$$\nu_i(a^{ik}\partial_{x_k}(u-v)) - f^i(u-v)) = h^k\partial_{x_k}(u-v) + g(u-v) \quad \text{in } \Gamma_T\,,$$

where for example $a^{ik} = \int_0^1 (a^{ik}(t, x, u + \lambda(v-u), Du + \lambda(Du - Dv))d\lambda$. The uniqueness follows from the energy estimate of Theorem 2.2.

Next we state an apriori estimate. We assume $s > 1 + n/2$ and that $u \in G^{s+1}$ is a solution of (1.1). Since u is a solution of (3.1) we have by Theorem 2.4 $u \in \bigcap_{1 \leqq j \leqq s+1} C^j([0, T[, W^{s+1-j})$ and the elliptic theory implies even $u \in \bigcap_{0 \leqq j \leqq s+1} C^j([0, T[, W^{s+1-j})$. Suppose that the graph of (u, Du) lies in the compact set M. For such solutions we obtain an apriori estimate,

$$\|u(t)\|_{E^{s+1}} \leqq \bar{c}\left(M, \int_0^t \|D^2u(\tau)\|_{L^\infty} d\tau\right)(1 + \|u_0\|_{W^{s+1}} + \|u_1\|_{W^s})\,. \tag{3.3}$$

Using this estimate we obtain solutions if $s > 1 + n/2$: we first mollify the functions F, w and g keeping the structure conditions above. By Theorem A.9 there are initial values in $W^{s+2} \times W^{s+1}$ satisfying the compatibility conditions of order $s+1$ near u_0 and u_1 in the $W^{s+1} \times W^s$ norm. The considerations above imply the existence of a solution to the approximate problems up to a time t_ε depending on the smoothing parameter. It is possible to extend the solution until the conditions of the theorem fail. By the uniform estimate (3.3) we obtain that t_ε is bounded away from 0 and the solutions are uniformly bounded in $G^{s+1}_{0,\tau}$ for some $\tau > 0$. Thus there is a subsequence converging in $G^s_{0,\tau}$. The limit is a weak solution of (1.1), thus we obtain a solution in $G^{s+1}_{0,\tau}$. But for such solutions we have full regularity. Again we can extend this solution until either the graph leaves U or $\|u(t)\|_{E^{s+1}}$ blows up. If this norm blows up at t_0 and the graph remains precompact in U we have by (3.3), $\int_0^t \|u\|_{E^{2,\infty}}(\tau) \to \infty$ as $t \to t_0$.

It suffices to prove the continuity for small τ. We consider a sequence of equations,

$$\partial_{x_i}F^{i\,m}_j = w^m_j \quad \text{in } \Omega_T, \quad \nu_i F^{i\,m}_j = g^m_j \quad \text{in } \Gamma_T, \quad u^m(0) = u^m_0, \quad \partial_t u^m(0) = u^m_1\,.$$

By the uniform boundedness and the convergence in the low norm we have for $0 \leqq j \leqq s+1$ and $t < \tau$, $\partial^j_t u^m(t) \rightharpoonup \partial^j_t u(t)$ in $W^{s+1-j}(\Omega)$. Let D^s_P be a finite set of differential operators of order less than or equal to s which are tangential at Γ_τ with the property that they span at every point the space of all such operators. We apply D^s_P to the equations above and obtain

$$\partial_{x_i}(\bar{a}^{ik\,m}\partial_{x_k}D^s_P u^m) = \hat{w}^m + \partial_{x_i}\hat{f}^{i\,m} \quad \text{in } \Omega_T\,, \tag{3.4}$$

$$\nu_i(\bar{a}^{ik\,m}\partial_{x_k}D^s_P u^m - \hat{f}^{i\,m}) = h^{sk\,m}\partial_{x_k}D^s_P u^m \quad \text{in } \Gamma_T\,,$$

$$D^s_P u^m(0) = u^{sm}, \quad \partial_t D^s_P u^m(0) = u^{sm}\,,$$

where

$$\begin{aligned}\hat{w}^m = {} & D^s_P w^m - (\partial_\nu + \operatorname{div}(\nu))(D^s_P g^m - h^{km}\partial_{x_k}D^s_P u^m) \\ & - ((\partial_\nu + \operatorname{div}\nu)h^{uk\,m})\partial_{x_k}D^s_P u^m + (\partial_{x_i}\nu^\beta h^{ui\,m})\partial_{x_\beta}D^s_P u^m \\ \hat{f}^{i\,m} = {} & - D^s_P F^{i\,m} + a^{ik\,m}\partial_{x_k}D^s_P u^m + \nu^i(D^s_P g^m - h^{sk\,m}\partial_{x_k}D^s_P u^m)\,.\end{aligned}$$

Using compact imbeddings for the lower order terms and the coefficients we can apply Theorem 2.2 to get $D_P^s u^m \to D_P^s u$ in $C([0, T[, W^1(\Omega)) \cap C^1([0, T[, L^2(\Omega))$. The convergence of the remaining derivatives follows from the differential equation.

It remains to check the regularity condition of Theorem 2.4 for the iteration, the estimates for the continuity result and to prove estimate (3.3). For the proof of the regularity assumptions we need a lemma similar to Theorem A.1,

Lemma 3.2 *Let* $\psi \in C^\kappa(U)$ *and* $\phi \in C^\infty(\bar{\Omega}_T)$, *assume* $\max\{n/(\kappa - 1), 2\} < p$. Then

$$\|\psi(t, x, \phi, D\phi)\|_{E^{\kappa-1,p}} \leqq c_1, \quad \|\partial_t \psi(t, x, \phi, D\phi)\|_{E^{\kappa-1,p}} \leqq c_1(1 + \|\partial_t \phi\|_{E^{\kappa,p}})$$

for fixed t, *where* c_1 *depends on the* C^s *norm of* ψ *in a neighborhood of* $t \times \operatorname{graph}(\phi, D\phi)$ *and on* $\|\phi\|_{E^{\kappa,p}}$.

We refer to Theorem A.6, which is similar, for comments on the proof. Using this lemma and $\frac{1}{2} < \frac{s-2}{n}$ we obtain

$$\|\bar{a}(v)(t)\|_{E^{s-1}} + \|\partial_t \bar{a}(v)(t)\|_{E^{s-1}} + \|\bar{w}(v)(t)\|_{E^{s-2}} + \|\partial_t^{s-1} \bar{w}(v)(t)\|_2 + \|\bar{f}(v)(t)\|_{E^{s-1}} + \|\partial_t^s \bar{f}(v)(t)\|_2 \leqq c_1(1 + \|\partial_t v(t)\|_{E^s}),$$

and thus we can apply Theorem 2.4 to Eq. (3.1) and get

$$\|\partial_t u\|_{E^s}(t) + \|u(t)\|_{W^s} \leqq c\left(\int_0^t \|\partial_t v\|_{E^s}(\tau)\, d\tau, \|u(0)\|_{W^{s+1}}, \|u_1\|_{W^s}\right).$$

We choose τ small and R large. The estimate above implies that J maps $Y_{\tau,R}$ to itself.

Next we prove the estimate (3.3). We assume that $s > n/2 + 1$ and that $u \in G^{s+1}$ is a solution of (1.1). By the arguments above we have

$$u \in \bigcap_{0 \leqq j \leqq s+1} C^j([0, T], W^{s+1-j}(\Omega)).$$

Applying D_P^s to Eq. (3.1) we obtain as above

$$\partial_{x_i}(\bar{a}^{ik} \partial_{x_k} D_P^s u) = \hat{w} + \partial_{x_i} \hat{f}^i \quad \text{in } \Omega_T,$$

$$\nu_i(\bar{a}^{ik} \partial_{x_k} D_P^s u - \hat{f}^i) = h^{sk} \partial_{x_k} D_P^s u \quad \text{in } \Gamma_T,$$

$$D_P^s u(0) = u_1^s, \quad \partial_t D_P^s u = u_2^s,$$

where u_1^s and u_2^s are obtained by formal calculations from u_0 and u_1 and

$$\hat{w} = D_P^s w - (\operatorname{div} v + \partial_v)(D_P^s g - h^k \partial_{x_k} D_P^s u) - ((\operatorname{div} v + \partial_v) h^{uk}) \partial_{x_k} D_P^s u + (\partial_{x_i} v^\beta h^{ui}) \partial_{x_\beta} D_P^s u,$$

$$\hat{f}^i = -D_P^s F^i + a^{ik} \partial_{x_k} D_P^s u + v^i (D_P^s g - h^k \partial_{x_k} D_P^s u).$$

Applying Theorem 2.2 we obtain

$$\|(D_P^s u)\|_E(t) \leqq c\left(\|u(0)\|_{E^s} + \|\hat{f}(t)\|_2 + \int_0^t \|\hat{w}\|_2 + \|\partial_t \hat{f}^i\|_2\, d\tau\right)$$

where c depends on the compact neighborhood of the graph of (u, Du). Solving the differential equation for $\partial_v^2 u$ we estimate inductively the remaining derivatives and get

$$\|u\|_{E^{s+1}}(t) \leqq c\left(\|u\|_{E^{s+1}}(0) + \int_0^t \|\hat{w}\|_2 + \|\partial_t \hat{f}^i\|_2 \, d\tau \right).$$

We have to show that

$$\|\partial_t \hat{f}^i\|_2 + \|\hat{w}\|_2 \leqq \bar{c}(1 + \|D^2 u(t)\|_\infty)(1 + \|u\|_{E^{s+1}})$$

which we do for one typical term. Defining $p_i = 2s/i$ we get by Theorem A.6

$$\begin{aligned}
\|[\partial_t^s, \bar{a}^{ik}(t)]\partial_{x_k} u(t)\|_{L^2(\Omega)} &\leqq c \sum_{\substack{j+l=s+1 \\ j,l \neq 0}} \|\partial_t^j a(t)\|_{L^{p_j-1}(\Omega)} \|\partial_t^l Du\|_{L^{p_l}(\Omega)} \\
&\leqq c \sum_{j=1}^{s} (1 + \|Du(t)\|_{W^{j,p_j-1}(\Omega)})(1 + \|Du(t)\|_{W^{s+1-j,p_{s+1-j}}(\Omega)}) \\
&\leqq c \sum_{j=1}^{s} (1 + \|Du(t)\|_{W^{1,\infty}(\Omega)}^{1-(j-1)/s} \|Du(t)\|_{W^s(\Omega)}^{(j-1)/s}) \\
&\quad \times (1 + \|Du(t)\|_{W^{1,\infty}(\Omega)}^{(j-1)/s} \|Du(t)\|_{W^s(\Omega)}^{1-(j-1)/s}) \\
&\leqq c(1 + \|Du(t)\|_{W^{1,\infty}(\Omega)})(1 + \|Du(t)\|_{W^s(\Omega)}),
\end{aligned}$$

where c depends on $\|Du\|_\infty$, κ and μ.

Appendix

This appendix has the purpose of collecting some estimates and results about Sobolev spaces, elliptic equations and applications to the compatibility conditions. The first two parts are concerned with well known results which are used to study the compatibility conditions in the third part. Most of the results of the first two parts can be found in [GT] and [Fr]. We refer to [Ko] for complete proofs of the statements in this appendix.

Theorem A.1 (Results about Sobolev spaces) 1a. *We assume that* $1 \leqq p, q < \infty$ *and* $\frac{1}{p} + \frac{s}{n} = \frac{1}{q}$. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain with smooth boundary. Then*

(A.1) $$\|u\|_{L^p(\Omega)} \leqq c(p, q, n, s)\|u\|_{W^{s,q}(\Omega)},$$

for $u \in C^\infty(\bar{\Omega})$. *The imbedding* $W^{s,q} \subset L^p$ *is compact if* $\frac{1}{p} + \frac{s}{n} > \frac{1}{q}$ *and* $s \geqq 1$.

b. *Suppose that* $0 < \alpha = s - \frac{n}{q} < 1$. *Then*

(A.2) $$\|u\|_{C^\alpha(\Omega)} \leqq c(s, q, n)\|u\|_{W^{s,q}(\Omega)}.$$

2a. *If* $1 \leqq p, q, r \leqq \infty$, $0 \leqq \lambda \leqq 1$, $\dfrac{1}{r} = \dfrac{\lambda}{p} + \dfrac{1-\lambda}{q}$ *and* $u \in L^p \cap L^q$ *then*

$$\|u\|_{L^r} \leqq \|u\|_{L^p}^{\lambda} \|u\|_{L^q}^{1-\lambda} . \tag{A.3}$$

b. *Let* η *be a multiindex and* $s > |\eta|$. *Then*

$$\|D^\eta u\|_{L^r(\Omega)} \leqq c \|u\|_{L^q(\Omega)}^{\frac{s-|\eta|}{s}} \|u\|_{W^{s,p}(\Omega)}^{\frac{|\eta|}{s}} , \tag{A.4}$$

with $c = c(\Omega, r, p, q, s)$ *holds if*

$$\frac{s-|\eta|}{sp} + \frac{|\eta|}{sq} = \frac{1}{r} .$$

3a. *Let* Ω *be as above,* $1 \leqq p_1, p_2, q_1, q_2 \leqq \infty$, $\dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2} + \dfrac{1}{q_2} = \dfrac{1}{r}$, η *be a multiindex* $s = |\eta|$ *and* $D^s u$ *the set of all derivatives of order* s. *Then, for* $u, v \in C_0^\infty(\mathbb{R}^n)$,

$$\|D^\eta(uv)\|_{L^r(\Omega)} \leqq c(\|u\|_{W^{s,p_1}(\Omega)} \|v\|_{L^{q_1}(\Omega)} + \|u\|_{L^{p_2}(\Omega)} \|v\|_{W^{s,q_2}(\Omega)}) . \tag{A.5}$$

b. *Let* $f \in C^s(\bar{\Omega} \times \mathbb{R})$ *and* $u \in C_0^\infty(\bar{\Omega})$. *Then*

$$\|D^\eta f(x,u)\|_{L^p(\Omega)} \leqq c(1 + \|u\|_{L^\infty(\Omega)}^{s-1})(1 + \|u\|_{W^{s,p}(\Omega)}) . \tag{A.6}$$

4. *Let* Ω, η, s *and* f *be as above,* $1 \leqq r, p_1, p_2, q_1, q_2 \leqq \infty$,

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{r} .$$

Then, for $u, v \in C^\infty(\bar{\Omega})$,

$$\|D^\eta(uv) - uD^\eta v\|_{L^r(\Omega)} \leqq c(\|Du\|_{L^{p_1}(\Omega)} \|v\|_{W^{s-1,q_1}(\Omega)} + \|Du\|_{W^{s-1,p_2}(\Omega)} \|v\|_{L^{q_2}} , \tag{A.7}$$

$$\|D^\eta f(u) - f'(u) D^\eta u\|_{L^r(\Omega)} \leqq c(1 + \|u\|_\infty^{s-2}) \times (1 + \|u\|_{W^{1,\infty}(\Omega)})(1 + \|u\|_{W^{s-1,r}(\Omega)}) . \tag{A.8}$$

Theorem A.2 (Functions with prescribed boundary data) 1. *Let* Ω *be a bounded domain with smooth boundary,* ν *any to* $\bar{\Omega}$ *smoothly extended normal. For* $s \geqq 1$ *there exists a linear map*

$$\bigoplus_{i=0}^{s-1} W^{s-i}(\Omega) \to W^s(\Omega), \quad \bigoplus_{i=0}^{s-1} u^i \to u ,$$

with $\partial_\nu^i u - u^i \in W_0^1(\Omega) \cap W^{s-i}(\Omega)$ *for* $0 \leqq i < s$.

2. *There is a linear map*

$$\bigoplus_{i=0}^{s} W^{s-i}(\Omega) \to \bigcap_{0 \leqq i \leqq s} C^i([0,T], W^{s-i}(\Omega)), \quad \bigoplus_{i=0}^{s} u^i \to u ,$$

with the property $\partial_t^i u(0) = u^i$.

3. *Let* $g \in W^1$. *The space*

$$H \subset W^{s+1}, \quad H = \{u | u \in W_0^s \cap W^{s+1-i}, \partial_\nu^s u - g \in W_0^1\} ,$$

is dense in W_0^s.

These results are consequences of the properties of solutions of the Laplace and the wave equation, see [Ko]. Next we collect some estimates for linear elliptic equations. We deal with the equation

$$\partial_{x_i}(a_{jl}^{ik}\partial_{x_k}u^l) = w_j + \partial_{x_i}f_j^i \quad \text{in } \Omega, \tag{A.9}$$
$$\nu_i(a_{jl}^{ik}\partial_{x_k}u^l - f_j^i) = 0, \quad \text{in } \partial\Omega\,,$$

under the following assumptions.

Assumption 1 Let $1 \leqq i,k \leqq n$, $1 \leqq j,l \leqq N$, let Ω be a bounded domain with smooth boundary, $s > 0$ be an integer and $p > \max\left\{\frac{n}{s}, 2\right\}$.

Assumption 2 (Regularity) Let $a \in W^{s,p}$, $w \in W^{s-1}$, $f \in W^s$.

Assumption 3 (Coerciveness) The coefficients a_{jl}^{ik} are called elliptic with constant $\kappa > 0$ if for all $u \in C^\infty(\bar{\Omega})$,

$$(a_{jl}^{ik}\partial_{x_k}u^l, \partial_{x_i}u^j)_\Omega \geqq \kappa\|u\|_{W^1}^2 - \mu\|u\|_{L^2}^2\,.$$

If the coefficients are continuous then this condition is a pointwise algebraic condition on the coefficients and μ depends on the modulus of continuity of a, see [SS].

Definition A.3 $u \in W^1$ is called a *weak solution of* (A.9) if for all $\phi \in (W^1(\Omega))^N$,

$$(a_{jl}^{ik}\partial_{x_k}u^l, \partial_{x_i}\phi^j)_\Omega = -(w_j, \phi^j)_\Omega + (f_j^i, \partial_{x_i}\phi^j)_\Omega\,. \tag{A.10}$$

Theorem A.4. 1. Regularity: *Let u be a weak solution of Eq.* (A.9) *and assume that the assumptions above hold. Then $u \in W^{s+1}(\Omega)$ and*

$$\|u\|_{s+1} \leqq c(\kappa, \Omega, \mu, \|a\|_{W^{s,p}})(\|u\|_{W^s} + \|w\|_{W^{s-1}} + \|f\|_{W^s})\,. \tag{A.11}$$

2. Continuous dependence: *If we have a sequence of equations,*

$$\partial_{x_i}(a^{ikm}\partial_{x_k}u^m) = w^m + \partial_{x_i}f^{im} \quad \text{in } \Omega\,,$$
$$\nu_i(a^{ikm}\partial_{x_k}u^m - f^{im}) = 0 \quad \text{in } \partial\Omega\,,$$

the data w^m and g^m and the coefficients a^m converge in the space above to w, f and g, all these equations are elliptic and the solutions u^m converge to u in L^2 then they converge in W^{s+1} and u is a solution of the limit equation.

We also deal with strong solutions and Dirichlet boundary conditions,

$$a_{jl}^{ik}\partial_{x_i x_k}u^l = w_j \quad \text{in } \Omega\,, \tag{A.12}$$
$$u = 0 \quad \text{in } \partial\Omega\,.$$

This problem is said to be elliptic if $a \in C(\Omega)$, $w \in L^2(\Omega)$, and if there is a $\kappa > 0$ such that for all $\eta \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^n$ the inequality

$$a_{jl}^{ik}\eta^l\eta^j\xi_i\xi_k \geqq \kappa|\eta|^2|\xi|^2 \tag{A.13}$$

holds. This is often called strong ellipticity. Let $s \in \mathbb{N}$, $2 \leqq p \leqq \infty$ be such that $\frac{1}{p} < \frac{s-1}{n}$. We call $u \in W_0^1 \cap W^2$ a strong solution, if (A.12) holds almost everywhere. We suppress again the indices j and l.

Theorem A.5. 1. Estimate: *There exists a constant c depending on the modulus of continuity of a and on the domain such that for all strong solutions the inequality holds*:

$$\|u\|_{W^2} \leqq c/\kappa(\|w\|_{L^2} + \|u\|_{L^2}) \tag{A.14}$$

2. Higher regularity: *Assume that* $a_{jl}^{ik} \in W^{s-1,p}$, $w \in W^{s-1}$ *and* $u \in W^2 \cap W_0^1$ *solves* (A.12). *Then u lies in* W^{s+1} *and*

$$\|u\|_{W^{s+1}} \leqq c(\kappa, \Omega, s, \|a\|_{W^{s-1,p}})(\|u\|_{W^s} + \|w\|_{W^{s-1}}) \tag{A.15}$$

holds.

3. *Continuous dependence: If there is a sequence of elliptic equations,*

$$a_{jl}^{ik\,m}\partial_{x_i x_k} u^{l\,m} = w_j^m \quad \text{in } \Omega_T,$$

$$u^m = 0 \quad \text{in } \Gamma_T, \quad u^m(0) = u_0^m, \quad \partial_t u^m(0) = u_1^m$$

where $w^m \to w$ *in* $W^{s-1}(\Omega)$, $a^m \to a$ *in* $W^{s-1,p}(\Omega)$, *the limit equation is elliptic and the solutions* u^m *converge to u in* L^2, *then* $u \in W^{s+1}$, *the solutions* u^m *converge in* W^{s+1} *and u is a solution of the limit equation.*

The compatibility conditions are natural conditions for regular solutions of all types of equations considered in this paper. Since the explicit conditions require long formulas, we give a procedure how to obtain them. We deal only with the nonlinear equations in divergence form (1.1) and state the results for the other ones, since the same arguments apply also to these equations. First we solve Eq. (1.1) for $\partial_t^2 u$. This gives $u_2 = \partial_t^2 u(0)$ in terms of the initial values u_0 and u_1. Similarly we obtain from the differentiated equation formulas for the higher time derivatives $u_i = \partial_t^i u(0)$ up to $i = s + 1$. Now the compatibility conditions for the Dirichlet boundary data read as $u_i \in W^{s+1-i}(\Omega) \cap W_0^1(\Omega)$ for $0 \leqq i \leqq s$. For the Neumann boundary conditions they are given by the differentiated boundary condition

$$\partial_t^i(g_j - \nu_\alpha F_j^\alpha)(0) \in W^{s-i}(\Omega) \cap W_0^1(\Omega)\,,$$

where we substitute u_i for $\partial_t^i u$. These formal calculations are justified by the following.

For simplicity we consider only scalar equations. The arguments carry over to systems of equations. Solving for $\partial_t^2 u$ in

$$\partial_{x_i} F^i(t, x, u, Du) = w(t, x, u, Du) \quad \text{in } \Omega_T \tag{A.16}$$

we obtain

$$\partial_t^2 u = b^{\alpha\beta}(t, x, u, Du)\partial_{x_\alpha x_\beta} u + b^\beta(t, x, u, Du)\partial_{tx_\beta} u + \bar{w}(t, x, u, Du)\,. \tag{A.17}$$

We recall some of the properties of F, g and $a^{ik} = \dfrac{\partial F^i}{\partial(\partial_{x_k} u)}$ which we need in this appendix. g depends only on the tangential derivatives of u at the boundary, $a^{00} < 0$ and $\theta_\alpha \theta_\beta a^{\alpha\beta} > 0$ for $\theta \in \mathbb{R}^n \setminus \{0\}$. We assume $s > n/2 + 1$ as in Theorem 3.1, $F, g \in C^{s+1}$ and $w \in C^s$.

Let $s_0 \leqq s$ and $1 \leqq p \leqq \infty$ and assume that the solution u lies in

$$\bigcap_{0 \leqq j \leqq s_0+1} C^j([0, T], W^{s_0+1-j,p}) \cap C^1(\Omega_T)\,.$$

Theorem A.6 *Let*

$$\psi\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} \to \mathbb{R}\ ,$$

be a smooth function, $0 \leqq j \leqq s_0$. *Then* $\partial_t^j \psi(t, x, u(t), Du(t)) \in W^{s_0-j,p}(\Omega)$, *it is a function of* t, $u(t)$ *and* $Du(t)$ *and* F, g *and* w *and satisfies*

$$\|\partial_t^j \psi(t, x, u(t), Du(t))\|_{W^{s_0-j,p}} \leqq c(1 + \|u(t)\|_{W^{s_0+1,p}} + \|\partial_t u(t)\|_{W^{s_0,p}})\ ,$$

where c *depends on* $\|u(t)\|_{W^{1,\infty}}$, *on* $\|\partial_t u(t)\|_\infty$ *and on the* C^s *norm of* ψ *restricted to a neighborhood of* $\{t\} \times \operatorname{graph}(u(t), Du(t))$.

Remark. For the fully nonlinear equations the constant depends also on second derivatives. If the equation is linear, it suffices to consider linear functions ψ. The most important function for ψ is the identity which implies the estimate of the time derivatives in terms of space derivatives. This allows us to consider the energy for $t = 0$ as a function of the initial values without having constructed a solution.

This theorem is proved by induction on j. It implies that the compatibility conditions are well defined. We need to consider function spaces on the boundary to deal with these conditions. We have

$$W^{s,p}(\Omega) \cap W_0^{1,p}(\Omega) \to W^{s,p}(\Omega) \xrightarrow{tr} W^{s,p}(\Omega)/(W^{s,p}(\Omega) \cap W_0^{1,p}(\Omega))\ ,$$

where the first map is injective, the second is surjective and the image of the first map is the kernel of the second map. The space $W^{s,p}(\Omega)/(W^{s,p}(\Omega) \cap W_0^{1,p}(\Omega))$ is isomorphic to the space $W^{s-1/p,p}(\Omega)$ which is more difficult to define. With these definitions the compatibility conditions read as

$$\operatorname{tr}(\partial_t^j(v_\alpha F^\alpha - g)(0)) = 0\ .$$

The map $(u_0, u_1) \to \operatorname{tr}(\partial_t^j(v_\alpha F^\alpha - g)(0))$ clearly factors over the space $(W^{s+1}(\Omega)/W_0^{s+1}(\Omega)) \times (W^s(\Omega)/W_0^s(\Omega))$. We examine the factorisation in some more detail.

Lemma A.7 *For* $0 \leqq j + l \leqq s$ *the maps* Φ^{jl},

$$W^{s+1} \times W^s \xrightarrow{\phi^{jl}} W^{s+1-j-l}/(W^{s+1-j-l} \cap W_0^1)\ ,$$

$$u_0 \times u_1 \to \operatorname{tr}(\partial_v^l \partial_t^j u(0))\ ,$$

factor over the space $W^{s+1}/(W_0^{j+l+1} \cap W^{s+1}) \times W^s/(W_0^{j+l} \cap W^s)$.

Lemma A.8 *For any* $0 \leqq j + l \leqq s$, j *even, the map*

$$W^{s+1} \cap W_0^{j+l} \to W^{s+1-j-l}/(W_0^1 \cap W^{s+1-j-l})\ ,$$

$$v \to \phi^{jl}(u_0 + v, u_1)\ ,$$

is surjective. Similarly the map

$$W^s \cap W_0^{j+l-1} \to W^{s+1-j-l}/(W_0^1 \cap W^{s+1-j-l})\ ,$$

$$v \to \phi^{jl}(u_0, u_1 + v)\ ,$$

is surjective for j *odd.*

These results can be proved by induction on s and Theorem A.1, A.2 and A.6. Now we can state the main theorem concerning the compatibility conditions.

Theorem A.9 (Compatibility conditions) *Suppose that the following is given: Functions u_0, u_1, F, w and g under the assumptions of Theorem 1.1 and the compatibility conditions up to order s and a sequence F^m, w^m and g^m satisfying the same assumptions and converging to F, w and g. Then there are $u_0^m \in W^{s+1}$ and $u_1^m \in W^s$ for which the compatibility conditions with F^m, w^m and g^m hold and*

$$u_0^m \to u_0 \quad \text{in } W^{s+1}, \quad u_1^m \to u_1 \quad \text{in } W^s .$$

If furthermore the functions F^m, w^m and g^m are smooth then it is possible to choose the functions $u_0^m \in W^{s+2}$ and $u_1^m \in W^{s+1}$ satisfying the compatibility conditions of order $s+1$.

Proof. By the hyperbolicity assumption we can solve the equations

$$v_\alpha F^{\alpha,m}(t, x, u, Du) = g^m(t, x, u, Du) ,$$

at the boundary for $\partial_v u$,

$$\partial_v u = \bar{g}^m(t, x, u, \partial_t u, D_T u) ,$$

where we consider u and Du as variables and denote the tangential derivatives by D_T. We have $\bar{g}^m \to \bar{g}$ as $m \to \infty$. We define $u_1^{m,1} = u_1$ and choose $v_0^{m,1} \in W_0^1 \cap W^{s+1}$ such that

$$\partial_v(u_0 + v^{m,1}) - \bar{g}^m(t, x, u_0, u_1, D_T u_0) \in W_0^1 \cap W^s ,$$

and $v^{m,1} \xrightarrow{W^{s+1}} 0$ as $m \to \infty$. Set $u_0^{m,1} = u_0 + v^{m,1}$. We proceed with induction on the order j of the compatibility conditions. Suppose we have chosen $u_0^{m,j-1} \in W^{s+1}$ and $u_1^{m,j-1} \in W^s$ such that the compatibility conditions of order $j-1$ hold and

$$u_0^{m,j-1} \to u_0 \quad \text{and} \quad u_1^{m,j-1} \to u_1 \quad \text{as } m \to \infty .$$

Suppose that j is odd. We choose $u_0^{m,j} = u_0^{m,j-1}$ and want to choose $v^{m,j} \in W^s(\Omega) \cap W_0^j(\Omega)$ such that

$$\partial_t^j(v_\alpha F^{\alpha,m}(u_0^{m,j}, u_1^{m,j-1} + v^{m,j}) - g(u_0^{m,j}, u_1^{m,j-1} \to v^{m,j})) \in W_0^1 \cap W^{s-j} .$$

The trace of the second term does not depend on v^j by Lemma A.7 and the structure conditions on g. We write the first term in the form

$$\partial_t^j(v_\alpha F^{\alpha,m}) = v_\alpha a^{\alpha k,m} \partial_{x_k} \partial_t^j u^m + N^m ,$$

where the trace of the term N^m does not depend on $v^{j,m}$. Similarly we write

$$v_\alpha a^{\alpha k,m} \partial_{x_k} \partial_t^j u^m = v_\alpha v_\beta a^{\alpha\beta,m} \partial_v \partial_t^j u^m + N^m ,$$

for some other N^m, which is independent of v^j at the boundary. But by Lemma A.8 the map $v \to \phi^{j,1}(u_0^{mj}, u_1^{m,j-1} + v^{m,j})$ is surjective, the matrix $v_\alpha v_\beta a^{\alpha\beta,m}$ is regular and there is a v^m such that the compatibility conditions up to order j are satisfied. By the open mapping principle it is possible to choose $v^{m,j}$ such that

$$v^{m,j} \xrightarrow{W^s} 0 \quad \text{as } m \to \infty .$$

The first part of the theorem is proved since the argument for even j is similar. It remains to prove the density result. For this purpose we choose $u_0^{m,0}$ and $u_1^{m,0}$ in $C^\infty(\bar{\Omega})$ such that $u_0^{m,0} \xrightarrow{W^{s+1}} u_0$ and $u_1^{m,0} \xrightarrow{W^s} u_1$. The only change is the use of Part 3 of Theorem A.2.3 for the higher order compatibility conditions. QED

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Stability and ergodicity of dominated semigroups I The uniform case[1]

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1 Introduction

In the theory of positive operators the following question is very natural:

If T is a positive operator (on a Banach lattice) satisfying a certain property (P) and S is an operator such that $0 \leqq S \leqq T$, then has S also property (P)?

Properties investigated in this setting are, e.g., compactness, weak compactness or the Dunford–Pettis property (see [AlB, M-N, Zaa] for a collection of results). Only recently, the inheritance of 'asymptotic' properties has been studied, e.g., strong and uniform ergodicity, quasi-compactness, asynchronous exponential growth or certain spectral conditions [AnM, AB2, Cas, MaM, Mar, PaS].

In the present paper we investigate if uniform ergodicity and uniform stability of an operator or a strongly continuous semigroup (shortly C_0-semigroup) is preserved under domination. Our starting-point is the following inheritance result of Caselles [Cas, Corollary 4.6].

1.1 Theorem. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ satisfy $0 \leqq S \leqq T$. If T is uniformly ergodic with finite dimensional fixed space, then S is uniformly ergodic with finite dimensional fixed space.*

We will see that the dimension of the fixed space of S is even less than or equal to the dimension of the fixed space of T (Lemma 2.1). This can be used to deduce from Theorem 1.1 an inheritance result for uniform stability (Corollary 2.6). As a consequence, if $0 \leqq S \leqq T$ and T is uniformly asymptotically periodic with corresponding periodic operator of finite rank, then S has the same property and the asymptotic period of T is a multiple of the asymptotic period of S (Proposition 2.8). In Sect. 3 we show that the inheritance results on uniform ergodicity and uniform stability for single operators are also true for C_0-semigroups of positive operators (cf. Theorem 3.4, Theorem 3.6). Section 4 is devoted to the construction of an example which also provides answer to a question of Greiner et al. [GVW].

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Our notation is standard and follows mainly the books of Nagel et al. [Nag] and Schaefer [Sch]. We want to state, however, some frequently used terms and fix some notation. For a Banach space E we denote by $\mathscr{L}(E)$ the space of *bounded (linear) operators* on E. If A is a (not necessarily bounded) operator on E, then $\rho(A) := \{\lambda \in \mathbb{C}: (\lambda - A)^{-1} \text{ exists in } \mathscr{L}(E)\}$ is the *resolvent set*, $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is the *spectrum* and $r(A) := \sup\{|\lambda|: \lambda \in \sigma(A)\}$ is the *spectral radius* of A. For $\lambda \in \rho(A)$ we set $R(\lambda, A) := (\lambda - A)^{-1}$. If E is a Banach lattice, then $E_+ := \{x \in E: x \geqq 0\}$ is the *positive cone* of E and the lattice operations sup and inf are denoted by $\vee$ and $\wedge$, respectively. The *absolute value* of $x \in E$ is defined by $|x| := x \vee (-x)$. For a Banach lattice E there is a canonical order on $\mathscr{L}(E)$ given by $S \leqq T$ if and only if $Sx \leqq Tx$ for all $x \in E_+$ ($S, T \in \mathscr{L}(E)$). Then $\mathscr{L}(E)_+ := \{T \in \mathscr{L}(E): T \geqq 0\}$ denotes the cone of *positive operators* on E.

We now give the definition of uniform ergodicity and uniform stability and mention some basic facts concerning these two notions.

1.2 Definition. Let E be a Banach space and $T \in \mathscr{L}(E)$.

a) T is called *uniformly stable* with *ergodic projection* P_T if the *Cesaro means* $T_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$, $n \in \mathbb{N}$, converge to P_T in the operator norm as $n \to \infty$.

b) T is called *uniformly stable* with *ergodic projection* P_T if the powers T^n, $n \in \mathbb{N}$, converge to P_T in the operator norm as $n \to \infty$.

Clearly if T is uniformly stable then it is also uniformly ergodic and the corresponding ergodic projections coincide. It is well known that P_T is a projection onto the *fixed space* $\mathrm{Fix}(T) := \{x \in E: Tx = x\}$ of T with kernel $\overline{(\mathrm{Id} - T)E}$ and $P_T = TP_T = P_T T$ (see [Kre, 2.1.3]). Moreover, by Lin's uniform ergodic theorem [Lin] an operator $T \in \mathscr{L}(E)$ is uniformly ergodic if and only if $\lim_n \frac{1}{n}\|T^n\| = 0$ and $(\mathrm{Id} - T)E$ is closed. Hence, if T is uniformly ergodic, then $E = \mathrm{Fix}(T) \oplus (\mathrm{Id} - T)E$. From this decomposition it follows that 1 is a pole of the resolvent $R(\lambda, T)$ of order at most one (see [DuS, VII.5.22]). Under reasonable restrictions this condition is equivalent to uniform ergodicity (see [Kre, 2.2.7; DNP, Theorem W.3]). If T is a positive operator on a Banach lattice E we have the following equivalence due to Karlin [Kar, Theorem 5]: the operator T is uniformly ergodic if and only if $r(T) \leqq 1$ and 1 is a pole of the resolvent of order at most one.

We now come to the definition of uniform ergodicity and uniform stability of C_0-semigroups.

1.3 Definition. Let E be a Banach space and $\mathscr{T} = (T_t)_{t \geqq 0}$ be a C_0-semigroup of bounded operators on E.

a) $\mathscr{T}$ is called *uniformly ergodic* with *ergodic projection* $P_{\mathscr{T}}$ if the *Cesaro means* $C(t) := \frac{1}{t}\int_0^t T_s\, ds$, $t > 0$, converge to $P_{\mathscr{T}}$ in the operator norm as $t \to \infty$. (Here the integral is defined strongly.)

b) $\mathscr{T}$ is called *uniformly stable* with *ergodic projection* $P_{\mathscr{T}}$ if the T_t, $t \geqq 0$, converge to $P_{\mathscr{T}}$ in the operator norm as $t \to \infty$.

Again it is clear that uniform stability of $\mathscr{T}$ implies uniform ergodicity and the corresponding ergodic projections coincide. If the C_0-semigroup $\mathscr{T}$ is uniformly ergodic and A denotes its generator, then $P_{\mathscr{T}}$ is a projection onto

$\bigcap_{t>0} \mathrm{Fix}(T_t) = \ker A$ with kernel $\overline{\mathrm{lin}}\{\bigcup_{t>0}(\mathrm{Id} - T)E\} = \overline{A(E)}$. Moreover, $P_{\mathscr{F}} = T_t P_{\mathscr{F}} = P_{\mathscr{F}} T_t$ for all $t \geqq 0$ (see [DuS, VIII.7.2; Mas, Theorem 1.4, Theorem 1.6]).

2 Domination of discrete semigroups

If S and T are operators satisfying the assumptions of Theorem 1.1, then S is uniformly ergodic with finite dimensional ergodic projection [Cas, Corollary 4.6]. It turns out that the ergodic projections P_S and P_T of S and T, respectively, even satisfy $\mathrm{rank}\, P_S \leqq \mathrm{rank}\, P_T$. This is an immediate consequence of the following result.

2.1 Lemma. *Let E be a Banach lattice and let $P, Q \in \mathscr{L}(E)$ be projections such that $0 \leqq Q \leqq P$. Then $Q = QPQ$. In particular* $\mathrm{rank}\, Q \leqq \mathrm{rank}\, P$.

Proof. The assumption $0 \leqq Q \leqq P$ implies $0 \leqq QPQ - Q = Q(P - Q)Q \leqq P(P - Q)Q = PQ - PQ = 0$, and hence $Q = QPQ$. □

Recall that a positive operator T on a Banach lattice E is called *irreducible* if $\{0\}$ and E are the only closed T-invariant ideals in E (cf. [Sch, III.8.1]).

Remarks. a) Strict positivity of P implies $PQ = Q$. This, in particular, is the case if P is irreducible or there exists a strictly positive P-invariant linear form.

b) If P is irreducible, then either $Q = P$ or $Q = 0$. In fact, if $P \neq 0$ then $P = x' \otimes x$ for a quasi-interior point $x \in E_+$ and a strictly positive linear form $x' \in E'$ (cf. [Sch, III.8.5]). By a) we have $Q = PQ = Q'x' \otimes x$. If $Q \neq 0$ then $\langle Q'x', x\rangle = 1$. From $Q \leqq P$ it follows $Q'x' \leqq x'$, hence $0 \leqq \langle x' - Q'x', x\rangle = \langle x', x\rangle - \langle Q'x', x\rangle = 0$. Since x is a quasi-interior point this implies $x' = Q'x'$ (cf. [Sch, II.6.3]), i.e., $Q = P$.

For sake of completeness we restate Theorem 1.1 combined with Lemma 2.1.

2.2 Theorem. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ satisfy $0 \leqq S \leqq T$. If T is uniformly ergodic and the ergodic projection P_T is of finite rank, then S is uniformly ergodic and* $\mathrm{rank}\, P_S \leqq \mathrm{rank}\, P_T$.

Easy examples show that Theorem 2.2 is no longer true if P_T is not of finite rank (cf. [Cas, Remark 4.7], or Remark a) after Corollary 2.6). For irreducible and uniformly ergodic T one can improve Theorem 2.2. This is a consequence of [Cas, Theorem 4.3].

2.3 Proposition. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ satisfy $0 \leqq S \leqq T$. If T is irreducible and uniformly ergodic, then $S = T$ or (S^n) converges uniformly to zero.*

Proof. If $r(T) < 1$, then $r(S) \leqq r(T) < 1$, and hence $\lim_n \|S^n\| = 0$. If $r(T) = 1$, then, by the positivity of T, $1 \in \sigma(T)$ (cf. [Sch, V. 4.1]). Since T is uniformly ergodic 1 is a pole of the resolvent of order one (cf. [DNP, Theorem W.3; Kre, 2.2.7]). Moreover, the irreducibility of T implies that the residue P of $R(\lambda, T)$ at 1 satisfies $\mathrm{rank}\, P_T = 1$ [Sch, III.8.5]. Thus, by a result of Caselles [Cas, Theorem 4.3] we have either $S = T$ or $r(S) < r(T) = 1$. But $r(S) < 1$ implies $\lim_n \|S^n\| = 0$. □

The main purpose of this section is to show an analogue of Theorem 2.2 for uniformly stable operators with finite dimensional ergodic projection. At first we consider the special case where the ergodic projection is of rank at most one.

2.4 Theorem. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ satisfy $0 \leqq S \leqq T$. If T is uniformly stable and* $\operatorname{rank} P_T \leqq 1$, *then S is uniformly stable and* $\operatorname{rank} P_S \leqq \operatorname{rank} P_T \leqq 1$.

We need the following lemma which relates uniform ergodicity to uniform stability through a spectral condition. By Γ we denote the unit circle $\{\lambda \in \mathbb{C}: |\lambda| = 1\}$.

2.5 Lemma. *Let E be a Banach space and $S \in \mathscr{L}(E)$. Then S is uniformly stable if and only if S is uniformly ergodic and $\sigma(S) \cap \Gamma \subseteq \{1\}$.*

Proof. Clearly, if S is uniformly stable, then S is uniformly ergodic. Moreover, if $\lambda \in \sigma(S) \cap \Gamma$ then $\lambda^{n+1} - \lambda^n \in \sigma(S^{n+1} - S^n)$ for every $n \in \mathbb{N}$. Hence $|\lambda - 1| = |\lambda^{n+1} - \lambda^n| \leqq \|S^{n+1} - S^n\| \to 0$ as $n \to \infty$, i.e., $\lambda = 1$. Conversely, let S be uniformly ergodic and $\sigma(S) \cap \Gamma \subseteq \{1\}$. Then $r(S) \leqq 1$ and 1 is a pole of $R(\lambda, S)$ of order at most one (cf. [DNP, W.1, Remark 2, Theorem W.3; Kre, 2.2.7]). The residue P of $R(\lambda, S)$ at 1 is a projection onto $\operatorname{Fix}(S)$, i.e., $E = \operatorname{Fix}(S) \oplus \ker P$ (cf. [DuS, VII.3.24]). Moreover, $\ker P$ is S-invariant and $\sigma(S_{|\ker P}) = \sigma(S) \setminus \{1\}$ (cf. [DuS, VII.3.20]). Hence $r(S_{|\ker P}) < 1$ and $\lim_n \|S^n_{|\ker P}\| = 0$. From this it easily follows that $\lim_n \|S^n - P\| = 0$, i.e., S is uniformly stable. □

Proof of Theorem 2.4 We may assume $r(S) = 1$. Since S is positive $1 \in \sigma(S)$ (cf. [Sch, V.4.1]). From Theorem 2.2 it follows that S is uniformly ergodic. Then 1 is a pole of the resolvent $R(\lambda, S)$ of order one and the residue $P = P_S$ at $\lambda = 1$ has rank one (cf. [DNP, Theorem W.3]). The Niiro–Sawashima theorem (cf. [Sch, V.5.5]) implies that $\sigma(S) \cap \Gamma$ consists entirely of poles of $R(\lambda, S)$. In particular, every $\lambda \in \sigma(S) \cap \Gamma$ is an eigenvalue of S. Furthermore, since $\sigma(S) \cap \Gamma$ is cyclic (cf. [Sch, V.4.9]) it is the union of finitely many (finite) subgroups of Γ. There exists $m \in \mathbb{N}$ such that $\lambda^m = 1$ for all $\lambda \in \sigma(S) \cap \Gamma$. Thus

$$(*) \qquad \{0\} \neq \operatorname{Fix}(S) \subseteq \{x: Sx = \lambda x \text{ for some } \lambda \in \sigma(S) \cap \Gamma\} \subseteq \operatorname{Fix}(S^m)\,.$$

On the other hand $0 \leqq S^m \leqq T^m$ and $P_T = \lim_n (T^m)^n$. Then, again by Theorem 2.2, S^m is uniformly ergodic and

$$\dim \operatorname{Fix}(S^m) = \operatorname{rank} P_{S^m} \leqq \operatorname{rank} P_T \leqq 1\,.$$

Hence $\operatorname{Fix}(S^m)$ is at most one dimensional and by $(*)$ this implies that $\sigma(S) \cap \Gamma = \{1\}$. The assertion now follows from Lemma 2.5. □

In the following corollary we permit the ergodic projection P_T to have finite rank.

2.6 Corollary. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ satisfy $0 \leqq S \leqq T$. If T is uniformly stable and* $\operatorname{rank} P_T < \infty$, *then S is uniformly stable and* $\operatorname{rank} P_S \leqq \operatorname{rank} P_T$.

Proof. Theorem 2.2 implies that S is uniformly ergodic and $\operatorname{rank} P_S \leqq \operatorname{rank} P_T$. Then, by Lemma 2.5, it suffices to show $\sigma(S) \cap \Gamma \subseteq \{1\}$. We consider the space $F := P_T E$. The set $P_T(E_+)$ is a convex cone in F and with the order induced by $P_T(E_+)$ the space F becomes a vector lattice [Sch, III.11.5]. The lattice operations in F are given by $\sup(P_T x, P_T y) := P_T(P_T x \vee P_T y)$ and $\inf(P_T x, P_T y) := P_T(P_T x \wedge P_T y)$ for $x, y \in E$. Since F is finite dimensional there exist positive,

pairwise disjoint (in the order of F) vectors $y_1, \ldots, y_k \in F$ whose linear span is F. Obviously, $Ty_l = y_l$, $1 \leqq l \leqq k$, and if E_l denotes the closed ideal generated by y_l in E, then $TE_l \subseteq E_l$. Let $T_l := T_{|E_l}$, $P_l := P_{|E_l}$, and $S_l := S_{|E_l}$, $1 \leqq l \leqq k$. Then (T_l^n) tends uniformly to P_l as $n \to \infty$. Moreover $P_l E_l = P_T E_l$ is contained in the one dimensional ideal generated by y_l in F. Hence rank $P_l = 1$. By Theorem 2.4 (S_l^n) converges uniformly for all $1 \leqq l \leqq k$. Let $\alpha \in \sigma(S) \cap \Gamma$. We have to show $\alpha = 1$. Since S is uniformly ergodic with finite dimensional ergodic projection P_S we obtain, as in the proof of Theorem 2.4, that $\sigma(S) \cap \Gamma$ contains only poles of the resolvent. In particular, α is an eigenvalue of S. Let $0 \neq z \in E$ satisfy $Sz = \alpha z$. Then $|z| = |S^n z| \leqq S^n |z| \leqq T^n |z|$ for all $n \in \mathbb{N}$. Hence $|z| \leqq P_T |z|$. Thus $|z|$ is contained in the ideal generated by F in E. By the Riesz decomposition theorem (cf. [Sch, II.1.6]) there exist vectors $z_l \in E_l$, $1 \leqq l \leqq k$, such that $z = \sum_{l=1}^k z_l$. Convergence of $(S_l^n z_l)$ for every $1 \leqq l \leqq k$ implies convergence of $(S^n z)$. However, $S^n z = \alpha^n z$ converges only if $\alpha = 1$. □

Remarks. a) The corollary is no longer true if P_T has infinite rank. Take for example $E = l_p$, $1 \leqq p < \infty$, $T = \mathrm{Id}$ and $S: E \to E$: $(\xi_n) \mapsto \left(\frac{n-1}{n} \xi_n\right)$. Then $0 \leqq S \leqq T$, (S^n) tends strongly to zero but $\|S^n\| = 1$ (even $\|S_n\| = 1$) for all $n \in \mathbb{N}$.

b) If P is a positive projection of finite rank, then each operator S such that $0 \leqq S \leqq P$ is uniformly stable.

We now apply the results from above to uniformly asymptotically periodic operators. Recall that an operator P on a Banach space E is *periodic*, if there exists an $m \geqq 1$ such that $P = P^{m+1}$. Then per $P := \inf\{m \geqq 1: P = P^{m+1}\}$ is called the *period* of P.

2.7 Definition. An operator T on a Banach space E is called *uniformly asymptotically periodic* (shortly u.a.p.) if there is a periodic operator $P \in \mathscr{L}(E)$ such that $\lim_n \|T^n - P^n\| = 0$. Then aper $T :=$ per P is called the *asymptotic period* of T.

With this notion we obtain the following inheritance result.

2.8 Proposition. *Let E be a Banach lattice and let $S, T \in \mathscr{L}(E)$ satisfy $0 \leqq S \leqq T$. If T is* u.a.p. *with corresponding periodic operator P of finite rank, then S is* u.a.p. *with corresponding periodic operator Q,* rank $Q \leqq$ rank P and aper S | aper T.

Proof. Let $m :=$ aper T. Then T^m is uniformly stable with ergodic projection P^m. By Corollary 2.6 the operator S^m is uniformly stable with ergodic projection $P_{S^m} \leqq P^m$. Let $Q := SP_{S^m}$. Then $Q^{m+1} = S^{m+1} P_{S^m} = Q$, i.e., Q is periodic with per $Q \leqq m$. If $0 \leqq k < m$, then $\lim_n \|S^{mn+k} - Q^{mn+k}\| = \lim_n \|S^k (S^m)^n - S^k P_{S^m}\| = \lim_n \|S^k ((S^m)^n - P_{S^m})\| = 0$. Hence, S is u.a.p. with corresponding periodic operator Q and aper $S =$ per $Q \leqq m$. Since Q is periodic rank $Q =$ rank Q^n for all $n \in \mathbb{N}$. Then, from $0 \leqq Q^m = P_{S^m} \leqq P^m$ and Lemma 2.1, it follows rank $Q \leqq$ rank P. It remains to show that $m = n \cdot$ per Q for some $n \in \mathbb{N}$. In fact, if $l :=$ per Q, then $m = nl + k$ for $n \in \mathbb{N}$ and $0 \leqq k < l$. Then $Q = Q^{m+1} = Q^{nl} Q^{k+1} = Q^l Q^{k+1} = Q^{l+1} Q^k = Q^{k+1}$. Since $k <$ per Q this implies that $k = 0$, i.e., $m = nl$. □

Remark. Clearly, if an operator T is u.a.p. with corresponding periodic operator P of period m, then T is uniformly ergodic with ergodic projection $P_T = \frac{1}{m} \sum_{k=1}^m P^k$. Conversely, if T is positive and uniformly ergodic with ergodic

projection of finite rank, then T is u.a.p. (cf. [Lin, Theorem 1; Bar, Theorem 2, Remark 4]). Hence, in Proposition 2.8 the uniform asymptotic periodicity of S also follows from Theorem 2.2.

3 Domination of continuous semigroups

The purpose of this section is to prove continuous parameter versions of Theorem 2.2 and Theorem 2.4. The proofs rely upon a special case of Theorem 4 of Shaw [Sha], which is formulated as our next proposition. Before doing so we introduce the following notion. Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0-semigroup with infinitesimal generator A on a Banach space E. Define

$$\omega_1(A) := \inf\left\{\operatorname{Re}\lambda\colon R_\lambda := \operatorname{so-lim}_{t\to\infty} \int_0^t e^{-\lambda s} T_s\, ds \text{ exists}\right\}$$

where the integral is defined strongly and so-lim stands for the limit in the strong operator topology. Furthermore

$$\omega_0(A) := \inf\{\omega \in \mathbb{R}\colon \text{there is } M > 0 \text{ such that } \|T_t\| \leqq M\exp(\omega t) \text{ for } t \geqq 0\}$$

is the *growth bound* or the *type* of $\mathcal{T}$ and

$$s(A) := \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\}$$

is the *spectral bound* of A. Then $s(A) \leqq \omega_1(A) \leqq \omega_0(A)$ [Nag, A-IV, Corollary 1.5] and if $\operatorname{Re}\lambda > \omega_1(A)$, then $R_\lambda = R(\lambda, A)$ [Nag, A-I, Proposition 1.11]. In particular $\lambda \mapsto R_\lambda$ is holomorphic on $\{\lambda\colon \operatorname{Re}\lambda > \omega_1(A)\}$. We now come to the announced result of Shaw [Sha, Theorem 4].

3.1 Proposition. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup of operators on a Banach space E with generator A such that $\omega_1(A) \leqq 0$. Then the following assertions are equivalent.*
(i) *$\mathcal{T}$ is uniformly ergodic.*
(ii) $\lim_{t\to\infty} \frac{1}{t} \|T_t R(1, A)\| = 0$ *and* $(\mathrm{Id} - R(1, A))E$ *is closed.*

Remark. Shaw's proof (see [Sha, p. 65]) shows, under the conditions of the proposition, that assertions (i) and (ii) imply $E = \operatorname{Fix}(R(1, A)) \oplus (\mathrm{Id} - R(1, A))E$. Then 1 is a pole of $R(\lambda, R(1, A))$ of order at most one (see [DuS, VII.5.22]).

It turns out that for positive C_0-semigroups on a Banach lattice uniform boundedness of the Cesaro means $C(t) := \frac{1}{t}\int_0^t T_s\, ds$, $t > 0$, already implies $\omega_1(A) \leqq 0$.

3.2 Lemma. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup of positive operators on a Banach lattice E with generator A. If $M := \sup_{t>0} \|C(t)\| < \infty$, then $\omega_1(A) \leqq 0$.*

Proof. It suffices to show $\int_0^\infty e^{-\lambda s}\langle T_s x, x'\rangle\, ds < \infty$ for all $x \in E_+$, all $x' \in E'_+$ and every $\lambda > 0$ (see [Cle, 7.3]). From the formula

$$\int_0^t e^{-\lambda s} T_s x\, ds = e^{-\lambda t}\int_0^t T_s x\, ds + \lambda \int_0^t s e^{-\lambda s}\left(\frac{1}{s}\int_0^s T_r x\, dr\right) ds$$

(cf. [Nag, p. 103]) we obtain, for $x \in E_+$ and $x' \in E'_+$, that

$$\int_0^t e^{-\lambda s} \langle T_s x, x' \rangle \, ds \leqq \left(te^{-\lambda t} + \lambda \int_0^t s e^{-\lambda s} \, ds \right) M \| x \| \| x' \|$$

$$= \frac{1}{\lambda} (1 - e^{-\lambda t}) M \| x \| \| x' \| .$$

Hence $\int_0^\infty e^{-\lambda s} \langle T_s x, x' \rangle \, ds \leqq \dfrac{M}{\lambda} \| x \| \| x' \| < \infty$. □

For $\mathcal{T}$ a positive C_0-semigroup it is known that $s(A) = \omega_1(A)$ [Nag, C-IV, Theorem 1.3]. Hence we obtain the following characterization of uniform ergodicity for positive semigroups.

3.3 Corollary. *Let $\mathcal{T} = (T_t)_{t \geqq 0}$ be a C_0-semigroup of positive operators on a Banach lattice E with generator A. Then the following assertions are equivalent.*

(i) *$\mathcal{T}$ is uniformly ergodic.*

(ii) *$s(A) \leqq 0$, $\lim_{t \to \infty} \dfrac{1}{t} \| T_t R(1, A) \| = 0$ and $(\mathrm{Id} - R(1, A))E$ is closed.*

(iii) *$s(A) \leqq 0$, $\lim_{t \to \infty} \dfrac{1}{t} \| T_t R(1, A) \| = 0$ and $R(1, A)$ is uniformly ergodic.*

Proof. The equivalence of (i) and (ii) is an immediate consequence of Proposition 3.1 and Lemma 3.2. That (iii) implies (ii) follows from Lin's uniform ergodic theorem [Lin]. If (ii) holds, then 1 is a pole of $R(\lambda, R(1, A))$ of order at most one (cf. Proposition 3.1, remark). Moreover, by the positivity of $\mathcal{T}$ and $s(A) \leqq 0$ the operator $R(1, A)$ is positive [Nag, C-III, Theorem 1.1] and has spectral radius less than or equal 1. Then, by Theorem 5 of Karlin [Kar], $R(1, A)$ is uniformly ergodic. □

Remarks. a) There exist uniformly ergodic positive C_0-semigroups with growth bound $\omega_0(A) > 0$ (cf. [Sha, §3; GVW, §4]; see also Example 4.2).

b) If $\mathcal{T} = (T_t)_{t \geqq 0}$ is a C_0-semigroup of positive operators, then $\mathcal{T}$ is uniformly ergodic with ergodic projection $P_{\mathcal{T}} = 0$ if and only if $s(A) < 0$. In fact, if $s(A) < 0$, then the family $tC(t) = \int_0^t T_s \, ds$, $t > 0$, tends to $-A^{-1}$ uniformly as $t \to \infty$ [Nag, C-III, Theorem 1.2], and hence $(tC(t))_{t > 0}$ is bounded. This implies that $P_{\mathcal{T}} = \lim_{t \to \infty} C(t) = 0$. Conversely, if $\mathcal{T}$ is uniformly ergodic and $P_{\mathcal{T}} = 0$, then $\mathrm{Fix}\, R(1, A) = P_{\mathcal{T}} E = \{0\}$. Since $R(1, A)$ is uniformly ergodic it follows $1 \in \rho(R(1, A))$, i.e., $0 \in \rho(A)$. On the other hand $s(A) \leqq 0$ and by the positivity of $\mathcal{T}$ we have $s(A) \in \sigma(A)$ (see [Nag, C-III, Theorem 1.1]. Hence $s(A) < 0$.

After these preparations we obtain the following continuous parameter version of Theorem 2.2.

3.4 Theorem. *Let E be a Banach lattice and let $\mathcal{S} = (S_t)_{t \geqq 0}$ and $\mathcal{T} = (T_t)_{t \geqq 0}$ be C_0-semigroups of operators on E such that $0 \leqq S_t \leqq T_t$ for all t. If $\mathcal{T}$ is uniformly ergodic and the ergodic projection $P_{\mathcal{T}}$ is of finite rank, then $\mathcal{S}$ is uniformly ergodic and* rank $P_{\mathcal{S}} \leqq$ rank $P_{\mathcal{T}}$.

Proof. Let A and B be the generators of $\mathscr{T}$ and $\mathscr{S}$, respectively. By Corollary 3.3 we have to show that $s(B) \leqq 0$, $\lim_{t\to\infty} \frac{1}{t} \| S_t R(1, B)\| = 0$ and $R(1, B)$ is uniformly ergodic. The first two assertions follow from $s(B) \leqq s(A) \leqq 0$ (Corollary 3.3, [Nag, C-II, Lemma 4.10]) and $0 \leqq R(1, B) \leqq R(1, A)$ [Nag, C-III, Theorem 1.2]. It remains to show that $R(1, B)$ is uniformly ergodic. Proposition 3.1 implies that $R(1, A)$ is uniformly ergodic and Fix $R(1, A) = \ker A = P_{\mathscr{T}} E$ [Mas, Theorem 1.6]. Hence the ergodic projection of $R(1, A)$ is of finite rank. From $0 \leqq R(1, B) \leqq R(1, A)$ and Theorem 2.2 it follows that $R(1, B)$ is uniformly ergodic. □

The following corollary can be considered as an analogue of Proposition 2.3. A strong version of it can be found in [AB2, Corollary 1.4]. Recall that a C_0-semigroup $\mathscr{T} = (T_t)_{t\geqq 0}$ is called *irreducible* if $\{0\}$ and E are the only closed ideals in E which are T_t-invariant for all $t \geqq 0$ (cf. [Nag, C-III, Definition 3.1]).

3.5 Corollary. *Let E be a Banach lattice and let $\mathscr{T} = (T_t)_{t\geqq 0}$ and $\mathscr{S} = (S_t)_{t\geqq 0}$ be C_0-semigroups on E with generators A and B, respectively, such that $0 \leqq S_t \leqq T_t$ for all t. If $\mathscr{T}$ is irreducible and uniformly ergodic, then $\mathscr{S}$ is uniformly ergodic. Moreover, if $\mathscr{S} \neq \mathscr{T}$, then $s(B) < 0$ and $\lim_{t\to\infty} \| S_t B^{-1}\| = 0$.*

Proof. Suppose $\mathscr{S} \neq \mathscr{T}$. The irreducibility of $\mathscr{T}$ implies rank $P_{\mathscr{T}} \leqq 1$ (cf. [Sch, III.8.5]). Then $\mathscr{S}$ is uniformly ergodic (Theorem 3.4) and by a result of Arendt and Batty [AB2, Corollary 1.4] $\mathscr{S} \neq \mathscr{T}$ implies that $P_{\mathscr{S}} = 0$. From Corollary 3.3, Remark b) it follows that $s(B) < 0$. In particular $0 \in \rho(B)$. The inverse of B has a representation $B^{-1} = \int_0^\infty S_s\, ds = \lim_{t\to\infty} \int_0^t S_s\, ds$ (note that $\int_0^t S_s\, ds$ is defined strongly whereas the limit $t \to \infty$ exists uniformly [Nag, C-III, Theorem 1.2]). The assertion is now a consequence of the identity

$$S_t B^{-1} = S_t \int_0^\infty S_s\, ds = \int_t^\infty S_s\, ds = B^{-1} - \int_0^t S_s\, ds$$

(cf. [Nag, A-I, Proposition 1.6]) where the right-hand side converges uniformly to 0 as $t \to \infty$. □

Remark. a) In general one cannot conclude that $\lim_{t\to\infty} \| S_t\| = 0$. In fact, there exists a positive, irreducible semigroup $\mathscr{T} = (T_t)_{t\geqq 0}$ with generator B such that $s(B) < 0 < \omega_0(B)$ (cf. remark after Example 4.2). From $s(B) < 0$ it follows that $\mathscr{S}$ is uniformly ergodic with ergodic projection 0 (cf. remark after Corollary 3.3) and $\omega_0(B) > 0$ implies $\limsup_{t\to\infty} \| S_t\| = \infty$.

b) Andreu and Mazon [AnM, Lemma 2.1] obtain the same conclusion as in Corollary 3.5 if instead of the uniform ergodicity of $\mathscr{T}$ one requires that $s(A) = 0$ is a pole of the resolvent $R(\lambda, A)$.

c) For a bounded C_0-semigroup $\mathscr{T} = (T_t)_{t\geqq 0}$ on a Banach space E with generator A satisfying $\sigma(A) \cap i\mathbb{R} = \emptyset$ Arendt and Batty showed (see [AB1, proof of the stability theorem]) that $\lim_{t\to\infty} \| T_t A^{-1}\| = 0$.

Our next result is a continuous parameter version of Corollary 2.6.

3.6 Theorem. *Let E be a Banach lattice and let $\mathscr{T} = (T_t)_{t\geqq 0}$ and $\mathscr{S} = (S_t)_{t\geqq 0}$ be C_0-semigroups of operators on E such that $0 \leqq S_t \leqq T_t$ for all t. If $\mathscr{T}$ is uniformly stable and* rank $P_{\mathscr{T}} < \infty$, *then $\mathscr{S}$ is uniformly stable and* rank $P_{\mathscr{S}} \leqq$ rank $P_{\mathscr{T}}$.

Proof. Theorem 3.4 implies that $\mathscr{S}$ is uniformly ergodic. The ergodic projection $P_{\mathscr{S}}$ satisfies $P_{\mathscr{S}}E = \bigcap_{t>0} \operatorname{Fix} S_t$ and $\ker P_{\mathscr{S}} = \overline{\operatorname{lin}}\{\bigcup_{t>0}(\mathrm{Id} - S_t)E\}$ (cf. [DuS, VIII.7.2]). On the other hand S_t is uniformly stable for every $t > 0$ (Corollary 2.6). Then, for fixed $t > 0$ and $x \in (\mathrm{Id} - S_t)E$, we obtain $0 = \lim_n S_t^n x = \lim_n S_{nt}x$. This implies that $\lim_{s\to\infty} S_s x = 0$. Hence $\lim_{t\to\infty} S_t x = 0$ for all $x \in \bigcup_{t>0}(\mathrm{Id} - S_t)E$. Since $\mathscr{S}$ is uniformly bounded we even have $\lim_{t\to\infty} S_t x = 0$ for $x \in \ker P_{\mathscr{S}}$. Consequently $P_{\mathscr{S}} = \text{so-}\lim_{t\to\infty} S_t$. Since S_1 is uniformly stable (S_n) converges uniformly to $P_{\mathscr{S}}$. From this and $S_t P_{\mathscr{S}} = P_{\mathscr{S}}$ for all $t \geqq 0$ we obtain $P_{\mathscr{S}} = \lim_{t\to\infty} S_t$ in $\mathscr{L}(E)$. □

Remarks. a) The same result, differently stated, is obtained by Martinez and Mazon [MaM]. In fact, they show that a positive C_0-semigroup which is dominated by a bounded quasi-compact C_0-semigroup (see [Nag, B-IV, Definition 2.7] for this notion) is also quasi-compact (cf. [MaM, Proposition 3.3]). On the other hand a bounded positive C_0-semigroup is quasi-compact if and only if it is uniformly stable with ergodic projection of finite rank (cf. [Nag, C-IV, Theorem 2.1]).

b) If $\mathscr{T}$ and $\mathscr{S}$ are C_0-semigroups of positive operators with generator A and B, respectively, $\mathscr{S}$ is dominated by $\mathscr{T}$ and $\mathscr{T}$ is uniformly stable with non-zero ergodic projection $P_{\mathscr{T}}$ of finite rank, then Martinez and Mazon show that $s(A) = s(B) = 0$ implies uniform stability of $\mathscr{S}$ and the ergodic projection $P_{\mathscr{S}}$ is non-zero and of finite rank (cf. [MaM, Corollary 3.4]). This follows also from Theorem 3.6 and Corollary 3.3, Remark b).

c) If $(T_t)_{t\geqq 0}$ and $(S_t)_{t\geqq 0}$ are C_0-semigroups such that $0 \leqq S_t \leqq T_t$ for all t and T_t converges uniformly to an infinite dimensional projection as $t \to \infty$, then it may happen that $(S_t)_{t\geqq 0}$ is not even mean ergodic (cf. [AB2, Example 3.1]). In particular, one cannot remove the finite rank condition in Theorem 3.4 and Theorem 3.6.

4 Example

In this section we present an example of an irreducible semigroup $\mathscr{S}$ (with generator B) such that $s(B) < \omega_0(B)$. This answers a question of Greiner et al. [GVW]. We wish to thank Günther Greiner for valuable comments concerning the construction of this example.

Let A be the generator of a C_0-semigroup on a Banach space E. It is well known that $A + C$ with $C \in \mathscr{L}(E)$ is also generator of a C_0-semigroup and that $C \mapsto \omega_0(A + C)$ is upper semicontinuous on $\mathscr{L}(E)$, i.e., given $\varepsilon > 0$ and $D \in \mathscr{L}(E)$ there exists $\delta > 0$ such that $\|C - D\| < \delta$, $C \in \mathscr{L}(E)$, implies $\omega_0(A + C) < \omega_0(A + D) + \varepsilon$ (cf. [Paz, 3.1.1]). We show that the spectral bound of the generator of a positive semigroup has a similar property.

4.1 Lemma. *Let E be a Banach lattice and let A be the generator of a positive C_0-semigroup on E satisfying $s(A) > -\infty$. Then $C \mapsto s(A + C)$, $C \in \mathscr{L}(E)_+$, is upper semicontinuous at 0 where $\mathscr{L}(E)_+ := \{C \in \mathscr{L}(E)\colon C \geqq 0\}$ is equipped with the topology induced by the operator norm.*

Proof. Let $C \in \mathscr{L}(E)_+$ and denote with $\mathscr{S} = (S_t)_{t\geqq 0}$ and $\mathscr{T} = (T_t)_{t\geqq 0}$ the semigroups generated by $A + C$ and A, respectively. Then, from the Dyson–Phillips expansion of $\mathscr{S}$ (cf. [Paz, p. 78 formulas (1.10) and (1.11)]), we obtain $0 \leqq T_t \leqq S_t$ for all $t \geqq 0$. Let now $\varepsilon > 0$. Choose $\delta_0 > 0$ such that $\omega_0(A + C) \leqq \omega_0(A) + 1$ for all $C \in \mathscr{L}(E)_+$, $\|C\| < \delta_0$. A result in Kato [Kat, IV.3.1, IV.2.14] implies that there exists $0 < \delta < \delta_0$ such that $[s(A) + \varepsilon, \omega_0(A) + 1] \subseteq \rho(A + C)$ for all $C \in \mathscr{L}(E)_+$,

$\|C\| < \delta$. Since $A + C$ generates a positive semigroup we have $s(A + C) \in \sigma(A + C)$ (cf. [Nag, C-III, Theorem 1.1]), hence either $s(A + C) < s(A) + \varepsilon$ or $s(A + C) > \omega_0(A) + 1$ for all such C. The second case, however, cannot occur since $\|C\| < \delta_0$. □

Remark. One can show the following stronger result. Let A be as in Lemma 4.1 and let $\mathscr{L}(E)_+$ be equipped with the canonical order and the topology induced by the regular norm (cf. [Sch, p. 231]). Then $C \mapsto s(A + C)$, $C \in \mathscr{L}(E)_+$ is increasing and upper semicontinuous.

The following example is based on a construction of Greiner et al. [GVW] (cf. also [MaM, Example 3.5]).

4.2 Example. Let $1 < p < \infty$. The space $E := L^p(\mathbb{R}_+, dm) \cap L^1(\mathbb{R}_+, e^x dm)$ equipped with the norm $\|f\| := \|f\|_p + \int_0^\infty |f(x)| e^x \, dm$ is a Banach lattice, where m denotes the Lebesgue measure. The shift semigroup $\mathscr{T} = (T_t)_{t \geq 0}$ defined by $T_t f(x) := f(x + t)$ is a positive C_0-semigroup on E and its generator A satisfies $s(A) = -1$ and $\omega_0(A) = 0$. Let C be an irreducible operator on E (e.g., let $Cf := (\int_0^\infty f(x)\, dm(x))g$ where g is a strictly positive function in E). By Lemma 4.1 there exists $\varepsilon > 0$ such that $s(A + \varepsilon C) < -\frac{1}{2}$. Since εC is positive we have $\omega_0(A) \leqq \omega_0(A + \varepsilon C)$. Finally the irreducibility of εC implies that $A + \varepsilon C$ generates an irreducible semigroup [Nag, C-II, Proposition 3.3].

Remark. If $A + \varepsilon C$ is as in Example 4.2 and $0 < \lambda < \frac{1}{2}$, then $B := A + \varepsilon C + \lambda \mathrm{Id}$ generates an irreducible semigroup such that $s(B) < 0 < \omega_0(B)$.

Final remark. Inheritance phenomena for strongly ergodic and strongly stable operators and C_0-semigroups are discussed in [Rae].

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Automorphism groups of countable highly homogeneous partially ordered sets*

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1 Introduction

A partially ordered set (poset) $(\Omega, \leqq)$ is said to be *highly homogeneous* if for any finite subsets Ω_1 and Ω_2 of Ω, if there is an isomorphism φ between $(\Omega_1, \leqq)$ and $(\Omega_2, \leqq)$, then there is an extension of φ to an automorphism of $(\Omega, \leqq)$. By the Ryll–Nardzewski Theorem [1], all highly homogeneous countable posets are $\aleph_0$-categorical.

Throughout, $(\mathbb{Q}, \leqq)$ denotes the linearly ordered set of rational numbers, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{N}$ denotes the set of natural numbers (i.e., the non-negative integers).

In [15], Schmerl classified the countable highly homogeneous posets. He proved that any such poset is isomorphic to either

(a) $(\mathbb{N}, \leqq)$, where $\leqq$ is the trivial partial order on $\mathbb{N}$ ($m_1 \leqq m_2$ iff $m_1 = m_2$);
(b) $\mathbb{Q} \overrightarrow{\times} M$ for some non-empty unordered set M that is either finite or $\mathbb{N}$; i.e., $(q_1, \mu_1) > (q_2, \mu_2)$ iff $q_1 > q_2$;
(c) $M \overrightarrow{\times} \mathbb{Q}$ for some non-empty unordered set M that is either finite or $\mathbb{N}$; i.e., $(\mu_1, q_1) > (\mu_2, q_2)$ iff ($\mu_1 = \mu_2$ and $q_1 > q_2$); or
(d) the countable universal poset (see Sect. 2 for the definition).

We will call a poset in (b) a *rational $|M|$-highrise*. Note that $(\mathbb{Q}, \leqq)$ is a rational 1-highrise, or a rational highrise for short. We will call a poset in (c) a *city of $|M|$ rational highrises*.

The automorphism groups of all these posets have been extensively studied with the exception of (d), the countable universal poset. Our main goal is to address this gap. We will prove:

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* To Helmut Wielandt on his 81st birthday with our appreciation and respect

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Theorem 1 *If* $(\Omega, \leqq)$ *is the countable universal poset, then* $\mathrm{Aut}(\Omega, \leqq)$, *its automorphism group, is simple.*

We actually obtain sharper results (see Theorem 1′ below).

There is a natural metric on $\mathrm{Sym}(\mathbb{N}) = \mathrm{Aut}(\mathbb{N}, \leqq)$, the full symmetric group on the set of natural numbers. This is given by:

> if f and g are distinct elements of $\mathrm{Sym}(\mathbb{N})$, then
> $d(f, g) = 1/2^{n+1}$ iff f and g agree on $\{0, \ldots, n\}$ and so
> do f^{-1} and g^{-1}, but either $(n+1)f \neq (n+1)g$ or
> $(n+1)f^{-1} \neq (n+1)g^{-1}$.

With respect to this metric, $\mathrm{Sym}(\mathbb{N})$ is a complete metric space and the group operations are continuous maps [11]. If $(\Omega, \leqq)$ is a countable poset and we identify Ω with $\mathbb{N}$, then $\mathrm{Aut}(\Omega, \leqq)$ is a closed subgroup of $\mathrm{Sym}(\mathbb{N})$ and hence a complete metric space. We use Schmerl's classification to prove an extension of the work of Dixon [2]; viz, that, with respect to the product metric on $(\mathrm{Aut}(\Omega, \leqq))^m$ ($m \geqq 2$, $m \in \mathbb{N}$):

Theorem 2 *If* $(\Omega, \leqq)$ *is a countable highly homogeneous poset, then almost all finitely generated subgroups of* $\mathrm{Aut}(\Omega, \leqq)$ *are free.*

More precisely, if $m \in \mathbb{N}$ and $m \geqq 2$, then $\{(f_1, \ldots, f_m) \in (\mathrm{Aut}(\Omega, \leqq))^m : \langle f_1, \ldots, f_m \rangle$ is not free of rank $m\}$ is a meagre subset of $(\mathrm{Aut}(\Omega, \leqq))^m$, where $\langle f_1, \ldots, f_m \rangle$ is the subgroup of $\mathrm{Aut}(\Omega, \leqq)$ generated by $\{f_1, \ldots, f_m\}$.

So "almost all" in Theorem 2 is in the Baire category sense.

Again the major new result in Theorem 2 concerns the countable universal poset, the other cases being well known [2] or easy extensions thereof.

Further results on the ubiquity of free groups in these automorphism groups are given in Sect. 4.

2 Background

A highly homogeneous countable poset that contains a copy of every finite poset is called a *countable universal poset*. Such posets are $\aleph_0$-categorical; hence there is at most one up to isomorphism [4, 1]. That there is at least one follows from the Fraissé construction [15]; finite posets satisfy the amalgamation property and are inductive. Hence we may talk of *the* countable universal poset.

We shall make repeated use of the following triviality:

Lemma 2.1 *Let* $(\Omega, \leqq)$ *be the universal countable poset, and let* Γ *be a finite subposet. Then for every finite poset* $\Delta \supseteq \Gamma$, *there is* $\Delta' \subseteq \Omega$ *such that* Δ *and* Δ' *are isomorphic (as posets) by an isomorphism fixing* Γ *pointwise. In particular,* Γ *can be augmented in* Ω *by any finite number of points having any desired relations to the points of* Γ *and to each other, subject only to the constraints of satisfying transitivity, avoiding loops, and specifying* $\gamma_1 < \delta < \gamma_2$ $(\gamma_1, \gamma_2 \in \Gamma)$ *only when* $\gamma_1 < \gamma_2$.

Proof. The universality of $(\Omega, \leqq)$ guarantees that Ω contains a copy of Δ. The proof is completed by using the high homogeneity of $(\Omega, \leqq)$. □

As with most universal objects, it is difficult to pin down specific properties. Rather, one must resort to inductive definitions. By way of illustration, let $(\Omega, \leqq)$ be the countable universal poset and let $\Gamma = \{\gamma_n : n \in \mathbb{Z}\} \subseteq \Omega$. Γ is called a *coterminal $\mathbb{Z}$-chain* if both $\gamma_n < \gamma_{n+1}$ for all $n \in \mathbb{Z}$, and for each $\alpha \in \Omega$, there is $m \in \mathbb{N}$ such that $\gamma_{-m} < \alpha < \gamma_m$.

Lemma 2.2 *If $(\Omega, \leqq)$ is the countable universal poset and $\alpha \in \Omega$, then Ω contains a coterminal $\mathbb{Z}$-chain Γ with $\alpha \in \Gamma$.*

Proof. Let $\{\alpha_j : j \in \mathbb{N}\}$ be an enumeration of Ω with $\alpha_0 = \alpha$, and let $\Gamma_0 = \{\alpha_0\}$. Let $\gamma_0 = \alpha_0$. We will define $\Gamma_m = \{\gamma_i : i \in \{0, \pm 1, \ldots, \pm m\}\}$ inductively for $m \in \mathbb{N}$ so that

(i) $\gamma_i < \gamma_{i+1}$ for all i with $i \in \{0, \pm 1, \ldots, \pm m\} \setminus \{m\}$ and
(ii) $\gamma_{-m} < \alpha_j < \gamma_m$ for all $j \in \mathbb{N}$, $j < m$.

Then, by (i) and (ii), $\Gamma = \bigcup_{m \in \mathbb{N}} \Gamma_m$ will be a coterminal $\mathbb{Z}$-chain.

Suppose that Γ_m has already been defined satisfying (i) and (ii). Then $\Gamma_m \cup \{\alpha_m\}$ is a finite poset. By Lemma 2.1, Ω contains a finite poset $\Delta \cup \{\beta_1, \beta_2\}$ with $(\Delta, \leqq)$ isomorphic to $\Gamma_m \cup \{\alpha_m\}$ and $\beta_1 < \delta < \beta_2$ for all $\delta \in \Delta$. By the high homogeneity of $(\Omega, \leqq)$, there is $g \in \mathrm{Aut}(\Omega, \leqq)$ such that $\Delta g = \Gamma_m \cup \{\alpha_m\}$. Let $\gamma_{-(m+1)} = \beta_1 g$ and $\gamma_{m+1} = \beta_2 g$. Then $\Gamma_{m+1} = \Gamma_m \cup \{\gamma_{-(m+1)}, \gamma_{m+1}\}$ satisfies (i) and (ii). This completes the induction and the proof. □

3 The automorphism group of the countable universal poset

The purpose of this section is to prove

Theorem 1′ *If $(\Omega, \leqq)$ is the countable universal poset, then $\mathrm{Aut}(\Omega, \leqq)$ is simple. Indeed, if $f, g \in \mathrm{Aut}(\Omega, \leqq)$ with $g \neq 1$, then f can be written as a product of at most 16 conjugates of, alternately, g and g^{-1}; i.e., there exist $h_j \in \mathrm{Aut}(\Omega, \leqq)$ $(j = 1, \ldots, 16)$ such that $f = \prod_{j=1}^{8} (h_{2j-1}^{-1} g h_{2j-1} \cdot h_{2j}^{-1} g^{-1} h_{2j})$.*

We achieve this in a series of lemmata. Throughout this section, $(\Omega, \leqq)$ will always denote the countable universal poset and G its automorphism group. An element $g \in G$ will be said to be *embracing* if it has an orbit Γ that is a coterminal $\mathbb{Z}$-chain and $\gamma g > \gamma$ for all $\gamma \in \Gamma$; an embracing element that satisfies $\alpha g > \alpha$ for *all* $\alpha \in \Omega$ will be called *strongly embracing*. Finally, we will call an element $g \in G$ *positively encompassing* if for all $\alpha \in \Omega$ there exist $\beta_1, \beta_2 \in \Omega$ such that $\beta_1 < \alpha < \beta_2$ and $\beta_i g > \beta_i$ $(i = 1, 2)$.

Note that if g is embracing, then $\alpha g \not\leqq \alpha$ for all $\alpha \in \Omega$; if g is strongly embracing, *all* its orbits are coterminal $\mathbb{Z}$ chains.

In a series of lemmata, we will prove that

(1) any element of G can be written as $c_1^{-1} c_2$ with c_1 and c_2 strongly embracing;
(2) any two strongly embracing elements of G are conjugate;
(3) if g is embracing, then $ghgh^{-1}$ is strongly embracing for some $h \in G$;
(4) if g is positively encompassing, then $ghgh^{-1}$ is embracing for some $h \in G$;

and

(5) if $g \in G \setminus \{1\}$, then $h^{-1}ghg^{-1}$ is positively encompassing for some $h \in G$.

Theorem 1' follows immediately from (1)–(5); the proof may be viewed as the above five main steps of which (3) is the most difficult.

The proofs in this section are of three kinds. The first is a straightforward application of high homogeneity (see Lemma 3.2), whereas the second is a use of high homogeneity to permit inductive definitions of finite partial order-preserving one-to-one functions whose union is an automorphism of the poset $\langle \Omega, \leqq \rangle$ having certain desired properties (see Lemma 3.4). The third kind of proof extends the second and permits inductive definitions of infinite partial order-preserving one-to-one functions in appropriate circumstances where the infinite prescribed restrictions in building the function reduce to a finite set of restrictions for any particular point that we wish to add to the domain or range of our partial function (see Lemmas 3.3, 3.5, 3.6, 3.7 and 3.8); cf [16].

If $\beta < \gamma$, we will write $[\beta, \gamma]$ for $\{\alpha \in \Omega: \beta \leqq \alpha \leqq \gamma\}$.

Lemma 3.1 *Let* $1 \neq g \in G$ *and* $\beta, \gamma \in \Omega$ *with* $\beta < \gamma$. *Then* $\mathrm{supp}(g) \cap [\beta, \gamma] \neq \varnothing$.

Proof. Assume there are $\beta, \gamma \in \Omega$ with $\beta < \gamma$ and $g \in G \setminus \{1\}$ such that $\mathrm{supp}(g) \cap [\beta, \gamma] = \varnothing$. Let $\alpha \in \mathrm{supp}(g)$. Without loss of generality, $\alpha g \not\leqq \alpha$.

If $\beta > \alpha$, then $\beta = \beta g > \alpha g$. Since G is homogeneous and $\langle \Omega, \leqq \rangle$ contains every finite poset, we can choose $\delta \in \Omega$ so that $\alpha < \delta < \gamma$, $\delta \| \beta$ and $\delta \| \alpha g$. Then $\delta g > \alpha g$ so $\delta g \not\leqq \delta$. Also $\delta g \| \beta g = \beta$. Let $\lambda \in [\beta, \gamma]$ be such that $\lambda > \delta$ and $\lambda \| \delta g$. Then $\lambda g > \delta g$ so $\lambda g \neq \lambda$. This is a contradiction.

The case $\beta \| \alpha$ is handled similarly, and the remaining cases are duals of these. □

We will call a partial order-preserving one-to-one map g_0 *finitary* if both its domain and range are contained in the union of a finite number of coterminal $\mathbb{Z}$-chains in Ω. If each of these coterminal $\mathbb{Z}$-chains is an orbit of g_0 and $\beta g_0 > \beta$ for all $\beta \in \mathrm{dom}(g_0)$, we will say that g_0 is *strongly finitary*.

Lemma 3.2 *If* g_0 *is finitary, then there is* $g \in G$ *such that* $g \supseteq g_0$. *If* g_0 *is strongly finitary, then* g *can be chosen to be strongly embracing.*

Proof. Let $\{\alpha_j: j \in \mathbb{N}\}$ be an enumeration of Ω. We will define finitary maps $g_0 \subseteq g_1 \subseteq g_2 \subseteq \ldots$ inductively so that $\alpha_j \in \mathrm{dom}(g_j) \cap \mathrm{dom}(g_j^{-1})$ for each $j \in \mathbb{N}$. Then $g = \bigcup \{g_j: j \in \mathbb{N}\}$ will be a one-to-one order-preserving map of Ω onto Ω; so $g \in G$ and $g \supseteq g_0$.

Assume that finitary $g_0, \ldots, g_k$ have been defined so that $g_0 \subseteq g_1 \subseteq \ldots \subseteq g_k$ and $\alpha_j \in \mathrm{dom}(g_j) \cap \mathrm{dom}(g_j^{-1})$ for all j with $0 \leqq j \leqq k$. Let $\{B_i: i \in 1\}$ be a finite collection of coterminal $\mathbb{Z}$-chains whose union contains the domain and range of g_k; say, $B_i = \{\beta_{i,n}: n \in \mathbb{Z}\}$ with $\beta_{i,n} < \beta_{i,n+1}$ for all $n \in \mathbb{Z}$. If $\alpha_{k+1} \in \mathrm{dom}(g_k)$, let $g_{k,1} = g_k$. If $\alpha_{k+1} \notin \mathrm{dom}(g_k)$, then for each $i \in I$, there exist $n_i, m_i \in \mathbb{Z}$ such that $\beta_{i,n_i} \leqq \alpha_{k+1} \leqq \beta_{i,m_i}$ but $\beta_{i,r_i} \| \alpha_{k+1}$ if $n_i < r_i < m_i$. Let $M_i \in \mathbb{Z}$ be the least $m \geqq m_i$ such that $\beta_{i,m} \in \mathrm{dom}(g_k)$, if such an m exists; define $N_i \in \mathbb{Z}$ dually. Let $\lambda \in \Omega$ bear the same relation to the finite set $\{\beta_{i,r_i} g_k: i \in I,\ N_i \leqq r_i \leqq M_i,\ \beta_{i,r_i} \in \mathrm{dom}(g_k)\}$ as α_{k+1} does to $\{\beta_{i,r_i}: i \in I, N_i \leqq r_i \leqq M_i, \beta_{i,r_i} \in \mathrm{dom}(g_k)\}$. Let $g_{k,1} = g_k \cup \{(\alpha_{k+1}, \lambda)\}$. Then $g_{k,1}$ is finitary and $\alpha_{k+1} \in \mathrm{dom}(g_{k,1})$. We similarly extend $g_{k,1}$ to finitary g_{k+1} so that $\alpha_{k+1} \in \mathrm{dom}(g_{k+1}^{-1})$. This completes the induction step and proves the first part of the lemma.

For the second part we will define strongly finitary maps $g_0 \subseteq g_1 \subseteq g_2 \subseteq \ldots$ inductively so that $\alpha_j \in \mathrm{dom}(g_j) \cap \mathrm{dom}(g_j^{-1})$ for each $j \in \mathbb{N}$. Then $g = \bigcup\{g_j: j \in \mathbb{N}\}$ will be strongly embracing.

Assume that strongly finitary $g_0, \ldots, g_k$ have been defined so that $g_0 \subseteq g_1 \subseteq \ldots \subseteq g_k$ and $\alpha_j \in \mathrm{dom}(g_j) \cap \mathrm{dom}(g_j^{-1})$ for all j with $0 \leqq j \leqq k$. Let $\{B_i: i \in I\}$ be the finite set of coterminal $\mathbb{Z}$-chains whose union is the domain of g_k, each being an orbit of $\langle g_k \rangle$. Let $\beta_{i,0} \in B_i$ and $\beta_{i,n} = \beta_{i,0} g_k^n$ $(n \in \mathbb{Z})$. Note that $\alpha_{k+1} \in \mathrm{dom}(g_k)$ if and only if $\alpha_{k+1} \in \mathrm{dom}(g_k^{-1})$. If $\alpha_{k+1} \in \mathrm{dom}(g_k)$, then $g_{k+1} = g_k$ is strongly finitary and $\alpha_{k+1} \in \mathrm{dom}(g_k) \cap \mathrm{dom}(g_k^{-1})$. If $\alpha_{k+1} \notin \mathrm{dom}(g_k)$, then for each $i \in I$, there are $n_i, m_i \in \mathbb{Z}$ such that $\beta_{i,n_i} \leqq \alpha_{k+1} \leqq \beta_{i,m_i}$ and $\beta_{i,r_i} \| \alpha_{k+1}$ if $n_i < r_i < m_i$. This is a finite amount of information. Let $\alpha_{k+1,0} = \alpha_{k+1}$ and for all $t \in \mathbb{Z} \setminus \{0\}$, let $\alpha_{k+1,t} \in \Omega$ bear the same relation to $\{\beta_{i,r_i+t}: i \in I,\ n_i \leqq r_i \leqq m_i\}$ as α_{k+1} does to $\{\beta_{i,r_i}: i \in I,\ n_i \leqq r_i \leqq m_i\}$ with $\alpha_{k+1,t} < \alpha_{k+1,t+1}$ for all $t \in \mathbb{Z}$ (existence by Lemma 2.1). Define $g_{k+1} \supseteq g_k$ so that $\alpha_{k+1,t} g_{k+1} = \alpha_{k+1,t+1}$ for all $t \in \mathbb{Z}$. Then $g_{k+1} \supseteq g_k$ and is strongly finitary. □

Corollary 3.3 *G contains a strongly embracing element.*

Proof. By Lemma 2.2, there is a coterminal $\mathbb{Z}$-chain $B \subseteq \Omega$; say, $B = \{\beta_n: n \in \mathbb{Z}\}$ with $\beta_n < \beta_{n+1}$ for all $n \in \mathbb{Z}$. Let $g_0: B \to B$ be given by $\beta_n g_0 = \beta_{n+1}$. Then g_0 is strongly finitary and the corollary follows immediately from Lemma 3.2. □

We can now prove Step (1).

Lemma 3.4 *For every $g \in G$, there is a strongly embracing $f \in G$ such that fg^{-1} is strongly embracing. Hence each $g \in G$ can be written in the form $c_1^{-1} c_2$ with c_1 and c_2 strongly embracing; so the subsemigroup of all strongly embracing elements is a generating set for the group G.*

Proof. By Corollary 3.3, there is $h \in G$ such that h is strongly embracing. Let $\{\alpha_j: j \in \mathbb{N}\}$ be an enumeration of Ω. We will define order-preserving one-to-one finite maps $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$ inductively so that, for each $j \in \mathbb{N}$,

(i) $\alpha_j \in \mathrm{dom}(f_j) \cap \mathrm{dom}(f_j^{-1})$, and
(ii) $\beta f_j \geqq \beta h, \beta hg$ for all $\beta \in \mathrm{dom}(f_j)$.

Then $f = \bigcup\{f_j: j \in \mathbb{N}\}$ and fg^{-1} will be strongly embracing.

Let $\beta_0 \geqq \alpha_0 h, \alpha_0 gh$ and $\gamma_0 \leqq \alpha_0 h^{-1}, \alpha_0 g^{-1} h^{-1}$. So $\beta_0 \geqq \alpha_0 \geqq \gamma_0$. Let f_0 map γ_0 to α_0 and α_0 to β_0; then (i) and (ii) hold for $j = 0$.

Suppose that f_j has been defined and satisfies (i) and (ii). If $\alpha_{j+1} \in \mathrm{dom}(f_j) \cap \mathrm{dom}(f_j^{-1})$, let $f_{j+1} = f_j$. If $\alpha_{j+1} \notin \mathrm{dom}(f_j)$, we can choose $\beta_{j+1} \in \Omega$ so that

(a) for all $\lambda \in \mathrm{dom}(f_j)$, $\alpha_{j+1} \begin{cases} > \\ < \\ \| \end{cases} \lambda$ iff $\beta_{j+1} \begin{cases} > \\ < \\ \| \end{cases} \lambda f_j$, and

(b) $\beta_{j+1} \geqq \alpha_{j+1} h, \alpha_{j+1} hg$.

This is possible since if $\lambda \in \mathrm{dom}(f_j)$ and $\alpha_{j+1} \ngeq \lambda$, then $\alpha_{j+1} h \ngeq \lambda h \leqq \lambda f_j$; so $\alpha_{j+1} h \ngeq \lambda f_j$; similarly for hg in place of h.

If $\alpha_{j+1} \notin \mathrm{dom}(f_j^{-1})$, we proceed similarly augmenting with γ_{j+1}. Now extend f_j to f_{j+1} as necessary so that $\alpha_{j+1} f_{j+1} = \beta_{j+1}$ and $\gamma_{j+1} f_{j+1} = \alpha_{j+1}$. Then f_{j+1} satisfies (i) and (ii) and the induction is completed. □

We next complete Step (2).

Lemma 3.5 *If f, g are strongly embracing, then f and g are conjugate in G.*

Proof. Let $\{\alpha_j: j \in \mathbb{N}\}$ be an enumeration of Ω. We will define order-preserving one-to-one finitary maps $h_0 \subseteq h_1 \subseteq h_2 \subseteq \ldots$ inductively so that $\alpha_0 h_0 = \alpha_0$ and for all $j \in \mathbb{N}$,

(i) $\alpha_j \in \mathrm{dom}(h_j) \cap \mathrm{dom}(h_j^{-1})$,
(ii) $\lambda f^{\pm 1} \in \mathrm{dom}(h_j)$ and $\lambda g^{\pm 1} \in \mathrm{range}(h_j)$ for all $\lambda \in \mathrm{dom}(h_j)$,
(iii) $\exists v_0, \ldots, v_{2j} \in \mathrm{dom}(h_j)$ such that $\bigcup_{i=0}^{2j} \{v_i f^n: n \in \mathbb{Z}\} = \mathrm{dom}(h_j)$, and
(iv) $\beta h_j^{-1} f h_j = \beta g$ for all $\beta \in \mathrm{range}(h_j)$.

Then $h = \bigcup\{h_j: j \in \mathbb{N}\} \in G$ and $h^{-1} f h = g$ as desired.

Since f and g are strongly embracing, $\lambda < \lambda f, \lambda g$ for all $\lambda \in \Omega$. Hence if we define h_0 so that $\alpha_0 f^n h_0 = \alpha_0 g^n$ for all $n \in \mathbb{Z}$, then (i)–(iv) hold with $v_0 = \alpha_0$. So assume that h_j has been defined and satisfies (i)–(iv). If $\alpha_{j+1} \in \mathrm{dom}(h_j)$, let $\bar{h}_{j+1} = h_j$. If $\alpha_{j+1} \notin \mathrm{dom}(h_j)$, then for each $i \in \{0, 1, \ldots, 2j\}$, let $n_i, m_i \in \mathbb{Z}$ be such that $v_i f^{n_i} \leqq \alpha_{j+1} \leqq v_i f^{m_i}$ and $\alpha_{j+1} \| v_i f^{r_i}$ if $n_i < r_i < m_i$. Let $\beta_{j+1} \in \Omega$ be such that for all i, we have $v_i g^{n_i} \leqq \beta_{j+1} \leqq v_i g^{m_i}$ and $\beta_{j+1} \| v_i g^{r_i}$ if $n_i < r_i < m_i$ (such a β_{j+1} exists in Ω by Lemma 2.1). Let $v_{2j+1} = \alpha_{j+1}$ and let $\bar{h}_{j+1} \supseteq h_j$ be such that $\alpha_{j+1} f^n \bar{h}_{j+1} = \beta_{j+1} g^n$ for all $n \in \mathbb{Z}$. If $\alpha_{j+1} \in \mathrm{range}(\bar{h}_{j+1})$, then $h_{j+1} = \bar{h}_{j+1}$ satisfies (i)–(iv). If $\alpha_{j+1} \notin \mathrm{range}(\bar{h}_{j+1})$, repeat this step with $\bar{h}_{j+1}$ in place of h_j and $\{v_i g^n: n \in \mathbb{Z}, i = 0, \ldots, 2j+1\}$ in place of $\{v_i f^n: n \in \mathbb{Z}, i = 0, \ldots, 2j\}$ to obtain $h_{j+1} \supseteq \bar{h}_{j+1}$ satisfying (i)–(iv). □

This completes parts (1) and (2) of our proof. We next establish part (3).

Lemma 3.6 *If g is embracing, then $ghgh^{-1}$ is strongly embracing for some $h \in G$.*

Proof. Since g is embracing, there is $B = \{\beta_n: n \in \mathbb{Z}\}$ coterminal in Ω such that $\beta_n < \beta_{n+1} = \beta_n g$ for all $n \in \mathbb{Z}$. Moreover, $\alpha g \| \alpha$ or $\alpha g > \alpha$ for all $\alpha \in \Omega$. Let $\{\alpha_j: j \in \mathbb{N}\}$ be an enumeration of Ω with $\alpha_0 = \beta_0$. We will define partial order-preserving one-to-one functions (finite extensions of $h_0 = \mathrm{id}|B$) $h_0 \subseteq h_1 \subseteq h_2 \subseteq \ldots$ inductively so that

(i) $\alpha_j \in \mathrm{dom}(h_j) \cap \mathrm{dom}(h_j^{-1})$;
(ii) $\forall \lambda \in \mathrm{dom}(h_j)$, if $\lambda g^n, \lambda g^m \in \mathrm{dom}(h_j)$ for some $m, n \in \mathbb{Z}$ with $n < m$, then $\lambda g^r \in \mathrm{dom}(h_j)$ if $n \leqq r \leqq m$;
(iii) $\lambda, \lambda g \in \mathrm{dom}(h_j) \Rightarrow \lambda g h_j g > \lambda h_j$;
(iv) $\forall \lambda, \mu \in \mathrm{dom}(h_j)\ \forall m \in \mathbb{Z}^+, \lambda h_j g^m \leqq \mu h_j \Rightarrow \lambda < \mu g^m$;
(v) $\forall \lambda, \mu \in \mathrm{dom}(h_j)\ \forall m \in \mathbb{Z}^+, \lambda g^m \leqq \mu \Rightarrow \lambda h_j < \mu h_j g^m$;

Then we take $h = \bigcup_{j \in \mathbb{N}} h_j \in G$ and $\alpha ghgh^{-1} > \alpha$ for all $\alpha \in \Omega$ by (iii). Moreover, since $\beta_n ghgh^{-1} = \beta_{n+2}$, $ghgh^{-1}$ will be strongly embracing, proving the lemma.

As stated, we let h_0 be the identity function restricted to B. Assume that h_j has been defined and satisfies (i)–(v). If $\alpha_{j+1} \in \mathrm{dom}(h_j) \cap \mathrm{dom}(h_j^{-1})$, let $h_{j+1} = h_j$. Then (i)–(v) hold for $j+1$. Hence assume $\alpha_{j+1} \notin \mathrm{dom}(h_j) \cap \mathrm{dom}(h_j^{-1})$.

We define $h_{j,1}$ and $h_{j,2}$ with $h_j \subseteq h_{j,1} \subseteq h_{j,2}$ so that $\alpha_{j+1} \in \mathrm{dom}(h_{j,1})$, $\alpha_{j+1} \in \mathrm{range}(h_{j,2})$ and $h_{j+1} = h_{j,2}$ satisfies (i)–(v). If $\alpha_{j+1} \in \mathrm{dom}(h_j)$, let $h_{j,1} = h_j$. If $\alpha_{j+1} \notin \mathrm{dom}(h_j)$ but $\alpha_{j+1} g^{-m} \in \mathrm{dom}(h_j)$ for some $m \in \mathbb{Z}^+$, then we will extend h_j to $h_{j,1}$ so that $\alpha_{j+1} g^{-m+1}, \ldots, \alpha_{j+1} g^{-1}, \alpha_{j+1}$ all belong to $\mathrm{dom}(h_{j,1})$. Since we do

this one at a time, we may assume that $\alpha_{j+1}g^{-1} \in \mathrm{dom}(h_j)$. Similarly, if $\alpha_{j+1} \notin \mathrm{dom}(h_j)$ but $\alpha_{j+1}g^m \in \mathrm{dom}(h_j)$ for some $m \in \mathbb{Z}^+$, then we may reduce to the case that $m = 1$. Thus we have reduced $\alpha_{j+1} \notin \mathrm{dom}(h_j)$ to the consideration of just the cases

($\alpha_{j+1}g^{-1} \in \mathrm{dom}(h_j)$ and $\alpha_{j+1}g^m \notin \mathrm{dom}(h_j)$ for all $m \in \mathbb{N}$) or
($\alpha_{j+1}g \in \mathrm{dom}(h_j)$ and $\alpha_{j+1}g^{-m} \notin \mathrm{dom}(h_j)$ for all $m \in \mathbb{N}$) or
($\alpha_{j+1}g^n \notin \mathrm{dom}(h_j)$ for all $n \in \mathbb{Z}$).

We seek $\sigma \in \Omega \setminus \bigcup\{\gamma h_j g^n : n \in \mathbb{Z}, \gamma \in \mathrm{dom}(h_j)\}$ so that for all $\gamma \in \mathrm{dom}(h_j)$ and $m \in \mathbb{N}$

(a1) $\sigma \not< \gamma h_j g^{-m}$ whenever $\alpha_{j+1} \not< \gamma g^m$;
(a2) $\sigma \not> \gamma h_j g^m$ whenever $\alpha_{j+1} \not> \gamma g^{-m}$;
(b1) $\sigma > \gamma h_j g^{-m}$ whenever $\alpha_{j+1} \geqq \gamma g^m$;
(b2) $\sigma < \gamma h_j g^m$ whenever $\alpha_{j+1} \leqq \gamma g^{-m}$;
(c1) $\sigma > \alpha_{j+1}g^{-1}h_j g^{-1}$ if $\alpha_{j+1}g^{-1} \in \mathrm{dom}(h_j)$; and
(c2) $\sigma < \alpha_{j+1}gh_j g$ if $\alpha_{j+1}g \in \mathrm{dom}(h_j)$.

If such a σ exists, then by defining $h_{j,1} = h_j \cup \{(\alpha_{j+1}, \sigma)\}$, we obtain (i)–(v) for $h_{j,1}$ with the possible exception of $\alpha_j \in \mathrm{dom}(h_{j,1}^{-1})$, since the conditions on σ imply that $h_{j,1}$ is order-preserving, (c1) and (c2) guarantee (iii), (a1) and (a2) ensure (iv), and (b1) and (b2) give (v). To show that such a σ does indeed exist, we note that the full set of conditions (a)–(c) follows from a finite subset thereof. For consider (a1), the other cases being similar. There are only finitely many $\gamma \in \mathrm{dom}(h_j) \setminus B$, and for each such γ we have $\alpha_{j+1} \not\leqq \gamma g^m$ for only finitely many $m \in \mathbb{N}$ (since g is embracing, $\gamma g^m > \alpha_{j+1}$ for all sufficiently large m). The conditions (a1) for the various pairs γ, m with $\gamma \in B$ follow from the single condition with γ the largest $\beta \in B$ such that $\alpha \not\leqq \beta$ and $m = 0$ (since $h_j|B = \mathrm{id}$). Therefore such an appropriate σ exists providing the conditions (a)–(c) we have written down are consistent (not self-contradictory). This we now show using Lemma 2.1.

We must first show that whenever $\xi_1 < \sigma$ and $\sigma < \xi_2$ are specified, then $\xi_1 < \xi_2$.

Note that ($\xi_1 < \sigma$ and $\sigma < \xi_2$) only arises if some combination of conditions (b) and (c) simultaneously arise.

(1) *(b1) and (b2).* So for some $\gamma, \delta \in \mathrm{dom}(h_j)$ and $m, m' \in \mathbb{N}$, we have specified that $\gamma h_j g^{-m} < \sigma$ and $\sigma < \delta h_j g^{m'}$ because $\gamma g^m \leqq \alpha_{j+1}$ and $\alpha_{j+1} \leqq \delta g^{-m'}$. Hence $\gamma g^{m+m'} < \delta$. (If $\gamma g^m = \alpha_{j+1} = \delta g^{-m'}$, then $\alpha_{j+1}g^{m'}, \alpha_{j+1}g^{-m} \in \mathrm{dom}(h_j)$; by (ii), this yields $\alpha_{j+1} \in \mathrm{dom}(h_j)$, a contradiction). By (v), $\gamma h_j g^{-m} < \delta h_j g^{m'}$, as desired.

(2) *(b1) and (c2).* So for some $\gamma \in \mathrm{dom}(h_j)$ and $m \in \mathbb{N}$, we have specified $\gamma h_j g^{-m} < \sigma$ and $\sigma < \alpha_{j+1}gh_j g$ because $\lambda = \alpha_{j+1}g \in \mathrm{dom}(h_j)$ and $\gamma g^m \leqq \alpha_{j+1}$. Equality would imply $\alpha_{j+1} \in \mathrm{dom}(h_j)$ as above, so $\gamma g^m < \alpha_{j+1}$. Hence $\gamma g^{m+1} < \lambda$; so $\gamma h_j < \lambda h_j g^{m+1}$ by (v). Therefore $\gamma h_j g^{-m} < \lambda h_j g = \alpha_{j+1}gh_j g$, as desired.

(3) *(b2) and (c1).* This is dual to (2) and follows similarly.

(4) *(c1) and (c2).* This case cannot arise since $\alpha_{j+1}g^{-1}, \alpha_{j+1}g \in \mathrm{dom}(h_j)$ implies $\alpha_{j+1} \in \mathrm{dom}(h_j)$, contrary to (ii).

We next need to show that if $\sigma < \xi_1$ and $\sigma \not< \xi_2$ are specified, then $\xi_1 \not\leqq \xi_2$. Now $\sigma \not< \xi_2$ only occurs if (a1) pertains. So assume that $\sigma \not< \gamma h_j g^{-m}$ because $\alpha_{j+1} \not\leqq \gamma g^m$ for some $m \in \mathbb{N}$ and $\gamma \in \mathrm{dom}(h_j)$. Now $\sigma < \xi_1$ can arise in two ways:

(1) (*b2*). So for some $\delta \in \mathrm{dom}(h_j)$ and $m' \in \mathbb{N}$, we have specified $\sigma < \delta h_j g^{m'}$ because $\alpha_{j+1} \leqq \delta g^{-m'}$. Hence $\delta g^{-m'} \not\leqq \gamma g^m$. By (iv), $\delta h_j g^{m+m'} \not\leqq \gamma h_j$; so $\delta h_j g^{m'} \not\leqq \gamma h_j g^{-m}$, as desired.

(2) (*c2*). So we specify that $\lambda = \alpha_{j+1} g \in \mathrm{dom}(h_j)$ and $\sigma < \lambda h_j g$. If $\lambda h_j g \leqq \gamma h_j g^{-m}$, then $\lambda \leqq \gamma g^{m+1}$ by (iv). Therefore, $\alpha_{j+1} \leqq \gamma g^m$, a contradiction. Thus $\lambda h_j g \not\leqq \gamma h_j g^{-m}$ as desired.

This shows that whenever we specify that $\sigma < \xi_1$ and $\sigma \not< \xi_2$, then $\xi_1 \not\leqq \xi_2$.

Similarly, whenever we specify that $\xi_1 < \sigma$ and $\xi_2 \not< \sigma$, then $\xi_2 \not\leqq \xi_1$.

Consequently, the conditions (a)–(c) are consistent and such a σ exists.

If $\alpha_{j+1} \in \mathrm{dom}(h_{j,1}^{-1})$, we are done. If not, we seek $\tau \in \Omega \setminus \bigcup \{\gamma g^n : n \in \mathbb{Z}, \gamma \in \mathrm{dom}(h_{j,1})\}$ so that for all $\gamma \in \mathrm{dom}(h_{j,1})$ and $m \in \mathbb{N}$

(a1) $\tau \not< \gamma g^{-m}$ whenever $\alpha_{j+1} \not< \gamma h_{j,1} g^m$;
(a2) $\tau \not> \gamma g^m$ whenever $\alpha_{j+1} \not> \gamma h_{j,1} g^{-m}$;
(b1) $\tau > \gamma g^{-m}$ whenever $\alpha_{j+1} \leqq \gamma h_{j,1} g^m$;
(b2) $\tau < \gamma g^m$ whenever $\alpha_{j+1} \leqq \gamma h_{j,1} g^{-m}$;

That such a τ exists is established in the same routine way as above. Therefore $h_{j+1} = h_{j,1} \cup \{(\tau, \alpha_{j+1})\}$ satisfies (i)–(v), there being no new instances of the hypotheses in (ii) and (iii) holding. Thus the proof is complete. □

Finally we establish parts (4) and (5).

Lemma 3.7 *If g is positively encompassing, then $ghgh^{-1}$ is embracing for some strongly embracing $h \in G$.*

Proof. Since g is positively encompassing, there is a coterminal $\mathbb{Z}$-chain $B = \{\beta_i : i \in \mathbb{Z}\}$ in Ω with $\beta_i < \beta_i g < \beta_{i+1}$ for all $i \in \mathbb{Z}$ (see the proof of Lemma 2.2). If we define h_0 by: $\beta_i g h_0 = \beta_{i+1}$, $\beta_{i+1} h_0 = \beta_{i+1} g$ for all $i \in \mathbb{Z}$, then h_0 is a one-to-one order-preserving map with domain and range the coterminal $\mathbb{Z}$-chain $B \cup Bg$; indeed, h_0 is strongly finitary. By Lemma 3.2, h_0 can be extended to a strongly embracing element h of G. But, for all $i \in \mathbb{Z}$, $\beta_i h_0 g h_0^{-1} = \beta_{i+1}$ and $\beta_i h_0^2 = \beta_{i+1}$; hence $\beta_i ghgh^{-1} = \beta_{i+1}$ for all $i \in \mathbb{Z}$. Since B is coterminal, $ghgh^{-1}$ is embracing as desired. □

Lemma 3.8 *If $g \in G \setminus \{1\}$, then $h^{-1}ghg^{-1}$ is positively encompassing for some $h \in G$.*

Proof. Let $\beta_0 \in \Omega$ with $\beta_0 g \neq \beta_0$. Let $\gamma_0 = \beta_0$ and $\Gamma = \{\gamma_n : n \in \mathbb{Z}\}$ be a coterminal $\mathbb{Z}$-chain in Ω with $\gamma_n < \gamma_{n+1}$ for all $n \in \mathbb{Z}$. We define β_m ($m \in \mathbb{Z}$, $m > 0$) inductively (using Lemma 3.1) so that $\beta_m g \neq \beta_m$, $\beta_m \geqq \gamma_m$, and $\beta_{m+1} > \beta_m, \beta_m g, \beta_m g^{-1}$; also β_m ($m \in \mathbb{Z}$, $m < 0$) inductively so that $\beta_m g \neq \beta_m$, $\beta_m \leqq \gamma_m$, and $\beta_{m-1} < \beta_m, \beta_m g, \beta_m g^{-1}$. Let $\lambda_n \in \Omega$ ($n \in \mathbb{Z}$) be such that (i) $\beta_{n-1} < \lambda_n < \beta_{n+1}$, (ii) $\beta_n g < \lambda_n$, (iii) $\lambda_n \| \beta_n$ if $\beta_n g \| \beta_n$, and (iv) $\lambda_n < \beta_n$ if $\beta_n g < \beta_n$. Define h_0 by: $\beta_n h_0 = \beta_n$ and $\beta_n g h_0 = \lambda_n$ ($n \in \mathbb{Z}$). Then h_0 is finitary $\mathrm{dom}(h_0) \cup \mathrm{dom}(h_0^{-1}) = \{\beta_n : n \in \mathbb{Z}\} \cup \{\beta_n g : n \in \mathbb{Z}\} \cup \{\lambda_{2n} : n \in \mathbb{Z}\} \cup \{\lambda_{2n+1} : n \in \mathbb{Z}\}$, the union of four coterminal $\mathbb{Z}$-chains). By Lemma 3.2, h_0 can be extended to $h \in G$. Now if $\alpha \in \Omega$, there exist m, M such that $\beta_m < \alpha < \beta_M$. Since $\beta_n h^{-1} g h g^{-1} = \lambda_n g^{-1} > \beta_n$ for all $n \in \mathbb{Z}$, $h^{-1}ghg^{-1}$ is positively encompassing. □

As pointed out in the introduction to this section, Lemmas 3.1–3.8 complete the proof of the simplicity of G (Theorem 1). We should note that in our proof of Theorem 1′ we have made no attempt whatsoever to obtain the minimal number of conjugates of g and g^{-1} required to express f. An excellent survey of results of this ilk in other automorphism groups is provided in [18].

Finally in this section we recall that every element of Sym($\mathbb{N}$) can be written as a product of involutions (elements of order 2). Although we have not been able to establish this for G, as a preliminary we note the abundance of involutions. Actually,

Proposition 3.9 *For every positive integer $n \geqq 2$, there are $2^{\aleph_0}$ elements of G of order n.*

Proof. To simplify notation, we will prove the lemma for $n = 2$; the generalization of the method to other values of n should be clear.

Let $\Omega = \{\alpha_j : j \in \mathbb{N}\}$ be an enumeration of Ω. We define inductively finite order-preserving maps $g_0 \subseteq g_1 \subseteq g_2 \subseteq \ldots$ so that

(i) $\alpha_j \in \mathrm{dom}(g_j)$ and
(ii) $\beta g_j^2 = \beta \neq \beta g_j$ for all $\beta \in \mathrm{dom}(g_j)$.

Then $g = \bigcup \{g_j : j \in \mathbb{N}\} \in G$ and $g^2 = 1$.

Let $\beta_0 \in \Omega$ with $\beta_0 \parallel \alpha_0$ and let g_0 map α_0 to β_0 and β_0 to α_0. Assume that g_j has been defined appropriately. If $\alpha_{j+1} \in \mathrm{dom}(g_j)$, let $g_{j+1} = g_j$; if $\alpha_{j+1} \notin \mathrm{dom}(g_j)$, let $\beta_{j+1} \in \Omega$ be such that $\beta_{j+1} \parallel \alpha_{j+1}$ and for all $\sigma \in \mathrm{dom}(g_j)$, $\beta_{j+1} \begin{cases} < \sigma \\ > \sigma \\ \parallel \sigma \end{cases}$ iff $\alpha_{j+1} \begin{cases} < \sigma \\ > \sigma \\ \parallel \sigma \end{cases}$.

Let $g_{j+1} \supseteq g_j$ and map α_{j+1} to β_{j+1} and β_{j+1} to α_{j+1}. Then g_{j+1} has the desired properties.

Since there are two (indeed $\aleph_0$) choices for β_{j+1} above for each $j \in \mathbb{N}$ for which $\alpha_{j+1} \notin \mathrm{dom}(g_j)$, and there are infinitely many such j since each g_j is finite, the above construction gives $2^{\aleph_0}$ elements $g \in G$ with $g^2 = 1$. □

4 Free subgroups of Aut(Ω, $\leqq$)

Our first goal is to establish the existence of a highly homogeneous two generator free subgroup of $G = \mathrm{Aut}(\Omega, \leqq)$ the automorphism group of the countable universal poset. The proof is quite similar to that of Macpherson in [12].

Proposition 4.1 *For any positive integer $m \geqq 2$, there is a free highly homogeneous m generator subgroup of G.*

Proof. To simplify notation, we assume that $m = 2$.

Let h be strongly embracing (existence by Corollary 3.3), $\{\alpha_j : j \in \mathbb{N}\}$ be an enumeration of Ω, and $\{(s_j, t_j) : j \in \mathbb{N}\}$ be an enumeration of all pairs of finite tuples with s_j and t_j order-isomorphic (say by ϕ_j) partially ordered subsets of Ω. Let $\{w_j : j \in \mathbb{N}\}$ be an enumeration of all reduced group words in the free group on $\bar{g}$ and $\bar{h}$. We will define inductively finite order-preserving maps $g_0 \subseteq g_1 \subseteq g_2 \subseteq \ldots$ so that

(i) $\alpha_j \in \mathrm{dom}(g_{3j}) \cap \mathrm{dom}(g_{3j}^{-1})$,

(ii) for some $i \in \mathbb{N}$, $\{\sigma h^i : \sigma \in s_j\} \cup \{\tau h^i : \tau \in t_j) \subseteq \mathrm{dom}(g_{3j+1})$ and $\sigma h^i g_{3j+1} = \sigma \phi_j h^i$ for all $\sigma \in s_j$, and

(iii) for some $i \in \mathbb{N}$, $\alpha_0 h^{-i} w_j(g_{3j+2}, h) h^i \neq \alpha_0$, where $g_{3j+2}^{\pm 1}$ is defined at all the necessary points.

Then $g = \bigcup\{g_j: j \in \mathbb{N}\} \in G$, and the subgroup of G generated by g and h will be free (by (iii)) and highly homogeneous (by (ii)).

Let $g_{-1} = \varnothing$.

Step 3j: Extend g_{3j-1} to g_{3j} so that $\alpha_j \in \mathrm{dom}(g_{3j}) \cap \mathrm{dom}(g_{3j}^{-1})$.

Step 3j + 1: Since h is strongly embracing, there is $i \in \mathbb{N}$ such that $\rho h^i > \mathrm{dom}(g_{3j}) \cup \mathrm{dom}(g_{3j}^{-1})$ for all $\rho \in s_j \cup t_j$. Then $\mathrm{dom}(g_{3j}) \cup s_j h^i$ and $\mathrm{range}(g_{3j}) \cup t_j h^i$ are order-isomorphic by $g_{3j+1} = g_{3j} \cup h^{-i}\phi_j h^i$.

Step 3j + 2: Let $i_1 \in \mathbb{N}$ be such that $\alpha_0 h^{-i_1} < \beta$ for all $\beta \in \mathrm{dom}(g_{3j+1}) \cup \mathrm{dom}(g_{3j+1}^{-1})$. In $\mathrm{Aut}(\mathbb{Q}, \leqq)$, let $\hat{h}$ be translation by $+1$; there is $\hat{g} \in \mathrm{Aut}(\mathbb{Q}, \leqq)$ such that $0w_j(\hat{g}, \hat{h}) \neq 0$. (This can be seen in two ways. The first is to apply White's Theorem [19] that the real maps $t_1: \alpha \mapsto \alpha + 1$ and $e_3: \alpha \mapsto \alpha^3$ generate a free group of rank 2; moreover, $\zeta w \neq \zeta$ for any $w \in \langle t_1, e_3\rangle$ and ζ transcendental. The orbit of π under $\langle t_1, e_3\rangle$ is a countable dense linearly ordered set order-isomorphic to the set of rationals. An appropriate conjugate of $\langle t_1, e_3\rangle$ to obtain translation by 1 on this renamed copy of the rationals gives the result. The second way involves totally ordering the free group on two generators via the lower central series (see [13]). The resulting linearly ordered set is order-isomorphic to the set of rationals and the "larger" generator in the ordering fixes no cut. Again, an appropriate conjugate of the right regular action of the free group on this resulting copy of $\mathbb{Q}$ gives the desired result.) Choose $i_2 \in \mathbb{N}$ and a finite linearly ordered set of points Λ in Ω such that $\alpha_0 h^{-(i_1+i_2)} \in \Lambda < \alpha_0 h^{-i_1}$, and there is a one-to-one order-preserving correspondence between Λ and the set of rationals $\{0u_j(\hat{g}, \hat{h})$: $u_j(\bar{g}, \bar{h})$ an initial subword of $w_j(\bar{g}, \bar{h})\}$ so that $\alpha_0 h^{-(i_1+i_2)}$ corresponds to 0 and if λ corresponds to $0u_j(\hat{g}, \hat{h})$ and $u_j(\bar{g}, \bar{h})\bar{h}^{\pm 1}$ is a reduced initial subword of $w_j(\bar{g}, \bar{h})$, then $\lambda h^{\pm 1}$ corresponds to $0u_j(\hat{g}, \hat{h})\hat{h}^{\pm 1}$. Let $g_{3j+2} \supseteq g_{3j+1}$ be such that if $u_j(\bar{g}, \bar{h})\bar{g}^{\pm 1}$ is a reduced initial subword of $w_j(\bar{g}, \bar{h})$, and μ corresponds to $0u_j(\hat{g}, \hat{h})$, then $\mu g_{3j+2}^{\pm 1} \in \Lambda$ corresponds to $0u_j(\hat{g}, \hat{h})\hat{g}^{\pm 1}$.

Steps $3j$, $3j + 1$ and $3j + 2$ achieve (i), (ii) and (iii) and permit the inductive definition to proceed. This proves the result. □

To prove Theorem 2, we now modify Dixon's proof for $\mathrm{Sym}(\mathbb{N})$. Recall that the free group on m generators has a faithful (highly transitive) representation on $(\mathbb{Q}, \leqq)$, see [8]. Thus, to achieve the analogue of [2, Lemma 5], it is enough to define Σ appropriately. In the case of a rational $|M|$-highrise, let Σ be a convex cofinal chain (isomorphic to $(\mathbb{Q}, \leqq)$) with $\sigma > \gamma$ for all $\sigma \in \Sigma$ and $\gamma \in \Gamma$. In the case of cities, choose Σ a convex cofinal chain above some fixed $\gamma_0 \in \Gamma$ so that $\sigma > \gamma$ for all $\sigma \in \Sigma$ and $\gamma \in \Gamma$ belonging to the same rational highrise as γ_0.

The case of the countable universal poset requires more care. Again, let $G = \mathrm{Aut}(\Omega, \leqq)$. Given any $\mathbf{g} \in G^m$, we need to show that there is $\mathbf{f} \in G^m$ such that $f_i^{\pm 1}$ agrees with $g_i^{\pm 1}$ $(i = 1, \ldots, m)$ on a given prescribed finite set Γ and $\langle \mathbf{f}\rangle$ is free of rank m. Then $F = \{\mathbf{f} \in G^m$: $\langle \mathbf{f}\rangle$ is free of rank $m\}$ will be dense in G^m, establishing [2, Lemma 5(i)] and thus completing the proof of Theorem 2. To simplify notation, we restrict to the case that $m = 2$. Let $g_0^{\pm}$, $h_0^{\pm 1}$ be defined on the finite set $\Gamma \subseteq \Omega$ and let $A = \Gamma \cup \Gamma g_0 \cup \Gamma g_0^{-1} \cup \Gamma h_0 \cup \Gamma h_0^{-1}$, a finite poset. Let $B = \{\beta_n: n \in \mathbb{N}\}$ be a coinitial chain in Ω with $\beta_0 < \alpha$ for all $\alpha \in A$ and $\beta_{n+1} < \beta_n$ for all $n \in \mathbb{N}$. Extend h_0 so that $\beta_{n+1}h_0 = \beta_n$ for all $n \in \mathbb{N}$. Then h_0 is finitary and so can be extended to $h \in G = \mathrm{Aut}(\Omega, \leqq)$ by Lemma 3.2. Enumerate $\Omega = \{\alpha_j: j \in \mathbb{N}\}$ so that A is an initial segment of the enumeration, and let $\{w_j: j \in \mathbb{N}\}$ be an enumeration of all

reduced group words in the free group on $\bar{g}$ and $\bar{h}$. We will define inductively finite order-preserving maps g_j ($j \in \mathbb{N}$) so that

(i) $\alpha_j \in \mathrm{dom}(g_{2j}) \cap \mathrm{dom}(g_{2j}^{-1})$,
(ii) $g_{j+1} \supseteq g_j$, and
(iii) for some $i \in \mathbb{N}$, $\beta_0 h^{-i} w_j(g_{2j+1}, h) h^i \neq \beta_0$, where $g_{2j+1}^{\pm 1}$ is defined at all the necessary points.

This is achieved exactly as in the proof of Lemma 4.1. If $g = \bigcup\{g_j: j \in \mathbb{N}\}$, then g extends g_0 and the subgroup of G generated by g and h is free as required. □

We now mirror Dixon's proof that almost all finitely generated subgroups of $\mathrm{Sym}(\mathbb{N})$ that are highly homogeneous are free. Our context is first $A(\mathbb{Q}) = \mathrm{Aut}(\mathbb{Q}, \leqq)$. Note that any element of $A(\mathbb{Q})$ extends uniquely to an element of $A(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}, \leqq)$, where $(\mathbb{R}, \leqq)$ is the real line under the usual ordering. Let $H = \{\mathbf{h} \in A(\mathbb{Q})^m: \langle \mathbf{h} \rangle$ is highly transitive$\}$ and $U = \{\mathbf{h} \in A(\mathbb{Q})^m: \langle \mathbf{h} \rangle$ has no finite orbits$\}$; i.e. $\mathbf{h} \in U$ if and only if no rational number is fixed by every element of $\langle \mathbf{h} \rangle$. As in [2], U is a closed subset of $A(\mathbb{Q})$ and contains H.

Let $\mathscr{S}_k$ be the set of pairs of increasing k-tuples in $\mathbb{Q}$ and $\mathscr{S} = \bigcup_{k=1}^{\infty} \mathscr{S}_k$. Then $\mathscr{S}$ is countable. For each $(\mathbf{s}_1, \mathbf{s}_2) \in \mathscr{S}$, let $H(\mathbf{s}_1, \mathbf{s}_2) = \{\mathbf{x} \in A(\mathbb{Q})^m: \mathbf{s}_2 \in \mathbf{s}_1\langle \mathbf{x} \rangle\}$. Then $H(\mathbf{s}_1, \mathbf{s}_2)$ is an open subset of $A(\mathbb{Q})^m$ and $H = \bigcap\{H(\mathbf{s}_1, \mathbf{s}_2): (\mathbf{s}_1, \mathbf{s}_2) \in \mathscr{S}\}$. We now prove an analogue of [2, Lemma 6], viz.

Lemma 4.2 *$H(\mathbf{s}_1, \mathbf{s}_2) \cap U$ is dense in U for every $(\mathbf{s}_1, \mathbf{s}_2) \in \mathscr{S}$.*

Proof. Let $\mathbf{x} \in U$ and n be a positive integer. We need to find $\mathbf{z} \in U$ such that $z_i^{\pm 1}$ agrees with $x_i^{\pm 1}$ ($i = 1, \ldots, m$) at the first n points $\alpha_1, \ldots, \alpha_n$ say, enumerated in $\mathbb{Q}$, and $\mathbf{s}_1 w = \mathbf{s}_2$ for some $w \in \langle \mathbf{z} \rangle$. Without loss of generality, n is sufficiently large that if $\alpha \in \mathbf{s}_1 \cup \mathbf{s}_2$, then α is one of $\alpha_1, \ldots, \alpha_n$. Clearly we can find $y_1 \in A(\mathbb{Q})$ so that $y_1^{\pm 1}$ agrees with $x_1^{\pm 1}$ on $A = \{\alpha_1, \ldots, \alpha_n\}$ and (its extension) has only a finite number of fixed points in $\mathbb{R}$. Let $y_2 \in A(\mathbb{Q})$ be such that $y_2^{\pm 1}$ agrees with $x_2^{\pm 1}$ on A and the set of real numbers fixed by both y_1 and y_2 is just the subset of A which is fixed by both x_1 and x_2. Let $y_i = x_i$ for $i = 3, \ldots, m$ if $m \geqq 3$. By construction, $\langle \mathbf{y} \rangle$ fixes no real number. Hence for some $u(\mathbf{y}) \in \langle \mathbf{y} \rangle$, $\alpha_j u(\mathbf{y}) > \alpha_k, \alpha_k x_i, \alpha_k x_i^{-1}$ for all $i \in \{1, \ldots, m\}$ and $j, k \in \{1, \ldots, n\}$. We may further ensure that $\alpha_j u(\mathbf{y})$ exceeds all real numbers fixed by y_1 for $j = 1, \ldots, n$. Without loss of generality, $\beta\gamma_1 > \beta$ for all $\beta > \max\{\gamma \in \mathbb{R}: \gamma y_1 = \gamma\}$. For some large positive integer M, we have that $\alpha_j u(\mathbf{y}) y_1^M > \alpha_k u'(\mathbf{y}) y_i^{\pm 1}$ for all $j, k \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$ and all initial subwords $u'(\mathbf{y})$ of $u(\mathbf{y})$. Let $z_i = y_i$ ($i = 1, 3, \ldots, m$) and z_2 agree with y_2 on $(-\infty, \inf\{\alpha_k u(\mathbf{y}) y_1^M y_2^{\pm 1}: k = 1, \ldots, n\})$ and map $\mathbf{s}_1 u(\mathbf{y}) y_1^{M+1}$ to $\mathbf{s}_2 u(\mathbf{y}) y_1^{M+1}$. Then $\mathbf{z} \in U \cap H(\mathbf{s}_1, \mathbf{s}_2)$ since $\mathbf{s}_1 u(\mathbf{z}) z_1^{M+1} z_2 z_1^{-(M+1)} u(\mathbf{z})^{-1} = \mathbf{s}_2$ and $z_i^{\pm 1}$ agrees with $x_i^{\pm 1}$ on $\{\alpha_1, \ldots, \alpha_n\}$ ($i = 1, \ldots, m$). Hence $U \cap H(\mathbf{s}_1, \mathbf{s}_2)$ is dense in U as required. □

We thus obtain, as in [2], that U is the closure of H in G^m. In the complete metric space U, we have that $U \setminus (H \cap F)$ is meagre in U by the Baire Category Theorem (since $H = \bigcap\{H(\mathbf{s}_1, \mathbf{s}_2): \mathbf{s}_1, \mathbf{s}_2 \in \mathscr{S}\}$). Consequently

Theorem 4.4 *If M is either finite or $\mathbb{N}$, and $(\Omega, \leqq)$ is a rational $|M|$-highrise, then almost all highly homogeneous finitely generated subgroups of $\mathrm{Aut}(\Omega, \leqq)$ are free. In particular, almost all highly homogeneous finitely generated subgroups of $A(\mathbb{Q})$ are free.*

5 Concluding remarks

Considerable work has been done on small index subgroups of automorphism groups of countable highly homogeneous structures. In particular, see [3, 9, 17, 5]. For posets, the only oustanding case is the countable universal poset. Specifically,

> *If* $\langle \Omega, \leqq \rangle$ *is the countable universal poset, and* H *is a subgroup of* $G = \mathrm{Aut}(\Omega, \leqq)$ *with* $[G: H] < 2^{\aleph_0}$, *does* H *contain* G_Δ, *the pointwise stabiliser of a finite subset* Δ *of* Ω?

What is needed is a finitary version of Hrushovski's lemma (see [9] or [10]) for the countable universal graph which relies on "finite" not "finitary" properties.

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Note added in proof Manfred Droste has shown that, unlike $\mathrm{Sym}(\mathbb{N})$, not every element of $\mathrm{Aut}(\Omega, \leqq)$ can be written as a product of *two* involutions: For let p be strongly embracing. If $p = gh$ with $g^2 = h^2 = 1$, let $\alpha \in \Omega$ and $\beta = \alpha g$. Then $\alpha < \alpha p$ & $\beta < \beta p$. Let $\gamma = \alpha p$ & $\delta = \beta p^{-1}$. Thus $\delta < \beta$ & $\gamma h = \beta$. Hence $\gamma = \delta g < \beta g = \alpha < \alpha p = \gamma$, a contradiction.

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Pseudo-composition algebras

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1 Introduction

Let F be a field of characteristic not 2 or 3, and let A be a commutative algebra over F. We call A a *pseudo-composition algebra* if there is a symmetric bilinear form $\varphi \neq 0$ such that

$$x^3 = \varphi(x, x)x, \quad \text{for all } x \in A\,. \tag{1.1}$$

The corresponding bilinear form φ is obviously unique; we call it the *norm* of A. Algebras of complexity 1 satisfying (1.1) have been discussed by Röhrl and Walcher [4]. Pseudo-composition algebras for which the bilinear form is associative and nondegenerate have been utilized in [1]. They are closely related to Jordan algebras of degree 3 (see [5] and [1]).

In this paper we shall study pseudo-composition algebras over an algebraically closed field of characteristic not 2 or 3. Such an algebra necessarily has an idempotent e. We say that the pseudo-composition algebra A is of *quadratic type* if every element $x \in A$ satisfies an equation of the form $x^2 + \beta(x)x + \gamma(x, x)e = 0$. For pseudo-composition algebras of quadratic type, the structure is quite straightforward. Thus we are primarily concerned with classifying the pseudo-composition algebras that are not of quadratic type. With certain minimal nondegeneracy assumptions, we show that these algebras can be constructed from alternative quadratic algebras. Specifically, we establish

Theorem 1.2 *Let A be a pseudo-composition algebra with idempotent e over an algebraically closed field of characteristic not 2 or 3. Then* Rad φ *is a nilpotent ideal of A, and one of the following is true:*

(a) *A is of quadratic type.*
(b) *$A/\mathrm{Rad}\,\varphi$ is of quadratic type.*
(c) *A may be constructed from an alternative quadratic algebra.*

The construction alluded to in part (c) of Theorem 1.2 is spelled out in Sect. 7.

We assume the algebraic closure of F in this paper for two reasons – so that A contains an idempotent, and so that we can simplify our formulas by normalizing

some of the elements which are used frequently. All the results hold if instead one assumes the existence of an idempotent, and if one divides by the norms of the appropriate key elements in the right places.

2 Background and elementary results

Linearizing (1.1), we obtain

$$x^2y + 2x(xy) = \varphi(x, x)y + 2\varphi(x, y)x \tag{2.1}$$

$$x(yz) + y(zx) + z(xy) = \varphi(x, y)z + \varphi(y, z)x + \varphi(z, x)y\,. \tag{2.2}$$

As an easy consequence of these identities, we have

Proposition 2.3 *Any proper ideal of a pseudo-composition algebra is contained in the radical of its bilinear form.*

Proof. Suppose that the ideal B of the pseudo-composition algebra A contains an element $y \notin \operatorname{Rad}\varphi$. For any $x, z \in A$, (2.1) and (2.2) imply that

$$\varphi(x, y)x \in B, \quad \varphi(x, y)z + \varphi(z, y)x \in B\,.$$

If $\varphi(x, y) \neq 0$, then $x \in B$; if $\varphi(x, y) = 0$, choose z with $\varphi(z, y) \neq 0$ to obtain $x \in B$. Thus $B \not\subset \operatorname{Rad}\varphi$ implies that $B = A$. □

It is known [4] that (1.1) implies the identity

$$\varphi(x^2, x^2) = \varphi(x, x)^2\,, \tag{2.4}$$

which has the linearizations

$$\varphi(x^2, xy) = \varphi(x, x)\varphi(x, y)\,, \tag{2.5}$$

$$\varphi(x^2, yz) + 2\varphi(xy, xz) = \varphi(x, x)\varphi(y, z) + 2\varphi(x, y)\varphi(x, z)\,. \tag{2.6}$$

The following result of Röhrl and Walcher will be very helpful:

Proposition 2.7 [4] *A pseudo-composition algebra A over a field F of characteristic not 2 or 3 has a finite scalar extension which contains an idempotent. If A contains an idempotent e, and if $A_\lambda = \{z \in A \mid ez = \lambda z\}$, then*

(a) $A = Fe \oplus A_{1/2} \oplus A_{-1}$.
(b) $0 = \varphi(e, A_{1/2}) = \varphi(e, A_{-1}) = \varphi(A_{1/2}, A_{-1})$.
(c) $A_{1/2}^2 \subset A_{-1} + Fe$, $A_{1/2}A_{-1} \subset A_{1/2}$, *and* $A_{-1}^2 \subset Fe$. *In particular,* $[y_{1/2}z_{1/2}]_1 = (1/2)\varphi(y_{1/2}, z_{1/2})e$ *and* $y_{-1}z_{-1} = -\varphi(y_{-1}, z_{-1})e$.
(d) $z(zx) = (3/4)\varphi(z, z)x$ *for all* $z \in A_{-1}$, $x \in A_{1/2}$.

We note that a pseudo-composition algebra cannot have two (or more) orthogonal idempotents, and that it cannot have an identity element unless it has dimension 1. In particular, the usual composition algebras are not pseudo-composition algebras.

Lemma 2.8 *If $x \in A_{1/2}$ is such that $x^2 \neq 0$, then $[x^2]_{-1} \neq 0$.*

Proof. If $[x^2]_{-1} = 0$, then $x^2 \in Fe$. Hence, $x^2 = (1/2)\varphi(x, x)e$ by Proposition 2.7(c), and multiplying by x gives $x^3 = (1/4)\varphi(x, x)x$. This contradicts (1.1) since $x^2 \neq 0$ implies $\varphi(x, x) \neq 0$. □

3 The quadratic case

Under slightly different assumptions this case has also been discussed in [1]. We argue as follows. Let A be an F-algebra of dimension > 1 and $e \neq 0$ an idempotent in A. We say that A is of *e-quadratic type* if there are a linear form β and a symmetric bilinear form γ on A satisfying

$$x^2 + \beta(x)x + \gamma(x, x)e = 0, \quad \text{for all } x \in A . \tag{3.1}$$

Here β and γ are necessarily unique, and are called the *e-trace* and *e-norm*, respectively, of A.

Proposition 3.2 *The following are equivalent.*

(i) *A is a pseudo-composition algebra satisfying*

$$A_{-1}(e) = 0 \quad (\textit{resp. } A_{1/2}(e) = 0)$$

(ii) *A is an algebra of e-quadratic type whose e-trace β and e-norm γ satisfy*

$$\gamma = 0, \beta \neq 0 \ (\textit{resp. } \gamma(e, e) = -3, \quad \beta = -(2/3)\gamma(e, \cdot)) .$$

Furthermore, a pseudo-composition algebra A is quadratic if and only if one of the conditions of part (i) *is satisfied. In this case, the norm φ of A is given by*

$$\varphi(x, x) = \beta(x)^2 + (1/2)\beta(e)\gamma(x, x) . \tag{3.3}$$

Proof. If $A_{-1} = 0$, then $u^2 = 0$ for all $u \in A_{1/2}$ by Lemma 2.8. For any $x \in A$, we have $x = \varphi(x, e)e + u$ for some $u \in A_{1/2}$, and hence $x^2 = \varphi(x, e)^2 e + \varphi(x, e)u = \varphi(x, e)x$. Thus A is quadratic, $\gamma = 0$ and $\beta = -\varphi(e, \cdot) \neq 0$. Conversely, if $x^2 = -\beta(x)x$ for all $x \in A$ and $\beta \neq 0$, then A satisfies (1.1) with $\varphi(x, x) = \beta(x^2)$. In this case $\beta(e) = -1$ and $2xy = -\beta(x)y - \beta(y)x$, from which we see that $A_{-1} = 0$.

If $A_{1/2} = 0$, then by Proposition 2.7 any $x \in A$ is of the form $x = \varphi(x, e)e + u$, where $u \in A_{-1}$ satisfies $u^2 = -\varphi(u, u)e = [\varphi(x, e)^2 - \varphi(x, x)]e$. Thus

$$x^2 + 2\varphi(x, e)x + [\varphi(x, x) - 4\varphi(x, e)^2]e = 0 , \tag{3.4}$$

A is quadratic, $\gamma(x, x) = \varphi(x, x) - 4\varphi(x, e)^2$ and $\beta(x) = 2\varphi(x, e)$; in particular, $\gamma(x, e) = -3\varphi(x, e)$, $\gamma(e, e) = -3$, and $\beta = -(2/3)\gamma(e, \cdot)$. Conversely, if A is of quadratic type and $e \in A$ satisfies $\gamma(e, e) = -3$, $\beta(x) = -(2/3)\gamma(e, x)$, then $x^3 = \varphi(x, x)x$ with $\varphi(x, x) = (4/9)\gamma(x, e)^2 + \gamma(x, x)$. In this case $\beta(e) = 2$, and the space orthogonal to e with respect to φ and with respect to γ coincide. Whence $A = Fe + A_{-1}$ from a linearization of (3.4). In both cases the norm φ of A is given by (3.3). □

It is immediate from this characterization that Rad φ is an ideal for algebras of quadratic type.

4 Peirce decomposition in the nonquadratic case

If A is not of quadratic type, $e \in A$ an idempotent, then

$$A_{1/2} \neq 0 \quad \text{and} \quad A_{-1} \neq 0 \tag{4.1}$$

by Proposition 3.2.

It is obvious that the structure of A depends on the behaviour of φ on the different Peirce spaces. If $\varphi(A_{-1}, A_{-1}) = 0$ then $\varphi(A_{1/2}, A_{1/2}) = 0$ as noted in [4] (it also follows from Lemma 2.8). In this case $A = Fe + \operatorname{Rad}\varphi$, $\operatorname{Rad}\varphi$ is an ideal of A, and $x^3 = 0$ for all $x \in \operatorname{Rad}\varphi$.

We shall suppose hereafter that $A_{-1} \not\subset \operatorname{Rad}\varphi$. Thus there exists $c \in A_{-1}$ with $\varphi(c, c) \neq 0$, and we can normalize to obtain $\varphi(c, c) = 4/3$. By Proposition 2.7, $L(c)$: $A_{1/2} \to A_{1/2}$ and $c(cx) = x$ for all $x \in A_{1/2}$. Thus, $A_{1/2} = U \oplus V$ where

$$U = \{u \in A_{1/2} \mid cu = u\} \quad \text{and} \quad V = \{v \in A_{1/2} \mid cv = -v\} .$$

It is immediate to verify that the elements $e' = -(1/2)e + (3/4)c$ and $e'' = -(1/2)e - (3/4)c$ are idempotents. For $u \in U$ and $v \in V$, we have

$$e'u = [-(1/2)e + (3/4)c]u = -(1/4)u + (3/4)u = (1/2)u ,$$

$$e'v = [-(1/2)e + (3/4)c]v = -(1/4)v - (3/4)v = -v ,$$

which says that $u \in A_{1/2}(e')$ and $v \in A_{-1}(e')$. By symmetry we also have $u \in A_{-1}(e'')$ and $v \in A_{1/2}(e'')$. Finally we define

$$Z = \{z \in A_{-1} \mid \varphi(z, c) = 0\} ,$$

and note that $Z \subset A_{1/2}(e') \cap A_{1/2}(e'')$. The notation established in this paragraph will be used in the remainder of the paper.

Proposition 4.2 (a) *A has the orthogonal decomposition under φ given by $A = Fe \oplus Fc \oplus U \oplus V \oplus Z$.*

(b) *$UV \subset Z$, $UZ \subset V$, and $VZ \subset U$.*

(c) *For $u \in U$, $v \in V$, $z \in Z$,*

$$u(uv) = (3/4)\varphi(u, u)v, \quad v(vu) = (3/4)\varphi(v, v)u ,$$

$$u(uz) = (3/4)\varphi(u, u)z, \quad v(vz) = (3/4)\varphi(v, v)z .$$

Proof. Since $\varphi(A_{1/2}, A_{-1}) = 0$ by Proposition 2.7, we have $\varphi(U, Z) = 0 = \varphi(V, Z)$. By the last paragraph, $\varphi(U, V) \subset \varphi(A_{1/2}(e'), A_{-1}(e')) = 0$, using Proposition 2.7 with e replaced by e'. This establishes part (a). For $u \in U$ and $v \in V$, $uv \in A_{-1}(e')A_{1/2}(e') \subset A_{1/2}(e')$. Then $\varphi(uv, e') = 0$, giving $(3/4)\varphi(uv, c) = (1/2)\varphi(uv, e)$. But $u, v \in A_{1/2}(e)$, so that $uv - (1/2)\varphi(u, v)e \in A_{-1}$ by Proposition 2.8. Since $\varphi(u, v) = 0$, $uv \in A_{-1}$, and hence $\varphi(uv, e) = 0$. Thus, $\varphi(uv, c) = 0$, and $UV \subset Z$. Also, for $u \in U$, $z \in Z$ we have $uz \in A_{1/2}(e)A_{-1}(e) \subset A_{1/2}(e)$ and $c(zu) + z(cu) = 0$ by Proposition 2.7(d). Thus, $UZ \subset V$. By symmetry, $VZ \subset U$, to give part (b).

Since $u \in U \subset A_{1/2}(e')$ and $v \in V \subset A_{-1}(e')$, Proposition 2.8 gives $v(vu) = (3/4)\varphi(v, v)u$. By symmetry, $u(uv) = (3/4)\varphi(u, u)v$. For $x \in A_{1/2}$, we have $x^2 = (1/2)\varphi(x, x)e + z'$ for some $z' \in A_{-1}$. Multiplying by any element $z \in Z$, we obtain $x^2 z = -(1/2)\varphi(x, x)z - \varphi(z', z)e$. But $\varphi(e, z) = 0$, so that $\varphi(x^2, z) = \varphi(z', z)$. Thus, $x^2 z = -(1/2)\varphi(x, x)z - \varphi(x^2, z)e$. Using (2.1) we have

$$2x(xz) = \varphi(x, x)z - x^2 z = (3/2)\varphi(x, x)z + \varphi(x^2, z)e .$$

If we choose $x = u \in U$, then $\varphi(x^2, z) = 0$, giving $u(uz) = (3/4)\varphi(u, u)z$. Similarly, choosing $x = v \in V$ gives $v(vz) = (3/4)\varphi(v, v)z$, to establish part (c). □

Since U, V, Z are contained in the (-1)-spaces of e'', e', e respectively, we get from Proposition 2.7(d) the following multiplication rules:

$$uu' = \varphi(u, u')((1/2)e + (3/4)c) ,$$

$$vv' = \varphi(v, v')((1/2)e - (3/4)c) ,$$

$$zz' = -\varphi(z, z')e , \tag{4.3}$$

for $u, u' \in U$, $v, v' \in V$, $z, z' \in Z$. We will also need some properties of φ derived from the pseudo-composition formula (2.4). Setting $z = y$ in (2.6), we obtain

$$\varphi(x^2, y^2) + 2\varphi(xy, xy) = \varphi(x, x)\varphi(y, y) + 2\varphi(x, y)^2 . \tag{4.4}$$

Replacing x by $z \in Z$ and y by $u \in U \cup V$, and using $\varphi(z, u) = 0$ and the relations $z^2 = -\varphi(z, z)e$, $u^2 = \varphi(u, u)((1/2)e + (3/4)c)$ from (4.3), Eq. (4.4) reduces to

$$-\varphi(z, z)\varphi(u, u)\varphi(e, (1/2)e + (3/4)c) + 2\varphi(zu, zu) = \varphi(z, z)\varphi(u, u) .$$

Since $\varphi(e, e) = 1$ and $\varphi(e, c) = 0$, we have shown

$$\varphi(zu, zu) = (3/4)\varphi(z, z)\varphi(u, u) \quad \text{for } z \in A_{-1}, \quad u \in U \cup V . \tag{4.5}$$

On the other hand, setting $x = u \in U$ and $y = v \in V$ in (4.4) and using (4.3) gives

$$\varphi(u, u)\varphi(v, v)\varphi((1/2)e + (3/4)c, (1/2)e - (3/4)c) + 2\varphi(uv, uv)$$

$$= \varphi(u, u)\varphi(v, v) .$$

This simplifes to

$$\varphi(uv, uv) = (3/4)\varphi(u, u)\varphi(v, v) \quad \text{for } u \in U, \quad v \in V . \tag{4.6}$$

5 When φ is zero on $A_{1/2}$ or on Z

For our main results we will need elements in U and Z with nonzero norm. If there are no such elements, φ is zero on $A_{1/2}$ or on Z. In these cases we will see in this section that $N := \operatorname{Rad}\varphi$ is an ideal, and that A/N is of quadratic type.

Proposition 5.1 (a) $N = \operatorname{Rad}\varphi$ *is always a subalgebra and contains the* (1/2)- *and* (-1)-*components of all its elements.*

(b) *If* $\varphi(A_{1/2}, A_{1/2}) = 0$ *or* $\varphi(Z, Z) = 0$, *then* N *is an ideal and* A/N *is of quadratic type.*

Proof. (a) Since any $n \in N$ is orthogonal to e, we have $N \subset A_{1/2} + A_{-1}$. Writing $n = n_{1/2} + n_{-1}$ for $n_{1/2} \in A_{1/2}$ and $n_{-1} \in A_{-1}$, we obtain from Proposition 2.7(d) that

$$0 = \varphi(n, x_{1/2}) = \varphi(n_{1/2}, x_{1/2}) = \varphi(n_{1/2}, x)$$

for all $x = x_{1/2} + x_{-1} + \alpha e \in A$. Thus, $n_{1/2} \in N$ and $n_{-1} = n - n_{1/2} \in N$. Equation (2.6) with $y \in N$ reduces to $\varphi(x^2, uy) + 2\varphi(xy, xu) = 0$, which has the linearization

$$\varphi(xz, uy) + \varphi(xy, zu) + \varphi(zy, xu) = 0 , \tag{5.2}$$

for $y \in N$. Choosing $z = e$ and $u \in N$ and noting that $eN \subset N$, we see that (5.2) reduces to $\varphi(ex, uy) = 0$ for all $x \in A$. Since $eA = A$, we have shown that $uy \in N$.

(b) Suppose first that $\varphi(Z, Z) = 0$. Then (4.6) shows that either $\varphi(U, U) = 0$ or $\varphi(V, V) = 0$. Since $A_{1/2}(e') = U + Z$ and $A_{1/2}(e'') = V + Z$, we see that either $\varphi(A_{1/2}(e'), A_{1/2}(e')) = 0$ or $\varphi(A_{1/2}(e''), A_{1/2}(e'')) = 0$. Thus the case $\varphi(Z, Z) = 0$ reduces to the case $\varphi(A_{1/2}, A_{1/2}) = 0$ after changing to another idempotent. It is sufficient then to verify that N is an ideal when $\varphi(A_{1/2}, A_{1/2}) = 0$. Here we calculate that $A_{-1}N_{1/2} \subset A_{1/2} \subset N$, $A_{-1}N_{-1} \subset \varphi(A_{-1}, N_{-1})e = 0$, and $A_{1/2}N_{-1} \subset A_{1/2} \subset N$. It remains to show that $A_{1/2}N_{1/2} \subset N_{-1}$, or equivalently that $\varphi(a_{1/2}n_{1/2}, a_{-1}) = 0$ for all $a_{1/2} \in A_{1/2}$, $n_{1/2} \in N_{1/2}$, $a_{-1} \in A_{-1}$. But from (5.2),

$$\begin{aligned}\varphi(a_{1/2}n_{1/2}, a_{-1}) &= -\varphi(a_{1/2}n_{1/2}, ea_{-1}) \\ &= \varphi(a_{1/2}a_{-1}, en_{1/2}) + \varphi(ea_{1/2}, n_{1/2}a_{-1}) \subset 0 + \varphi(ea_{1/2}, N) = 0 .\end{aligned}$$

If $\varphi(A_{1/2}, A_{1/2}) = 0$, then $A_{1/2} \subset N$ and the idempotent $e + N$ of A/N has no half space. Thus, A/N is of quadratic type. □

Hence the algebras of this section fit into the second case of the conclusion of Theorem 1.2.

6 When φ is nonzero on U and Z

We are left with the case when $\varphi(Z, Z) \neq 0$ and $\varphi(A_{1/2}, A_{1/2}) \neq 0$. The latter relation implies that either $\varphi(U, U) \neq 0$ or $\varphi(V, V) \neq 0$. After possibly replacing c by $-c$, we may assume that $\varphi(U, U) \neq 0$. This implies that there exists $a \in U$ such that $\varphi(a, a) = 4/3$. Thus the maps $L(a): Z \to V$ and $L(a): V \to Z$ are both injective by Proposition 4.2(c), and hence both isomorphisms. In particular, $\dim V = \dim Z$, $Z = aV$, and $V = aZ$. Similarly, $\varphi(Z, Z) \neq 0$ implies the existence of $z \in Z$ such that $\varphi(z, z) = 4/3$, and $L(z)$ gives isomorphisms between U and V. For $u, u' \in U$, define the operation "$*$" by

$$u' * u = [u'(a(zu))]z . \tag{6.1}$$

Using Proposition 4.2(b), it is immediate that $U * U \subset U$.

A nice property of $(U, *)$ is given in

Lemma 6.2 *If* $\lambda = (3/4)\varphi$, *then* $\lambda(u' * u, u' * u) = \lambda(u', u')\lambda(u, u)$.

Proof. Using (4.5) and (4.6) we calculate that

$$\begin{aligned}\lambda(u' * u, u' * u) &= (3/4)\varphi([u'(a(zu))]z, [u'(a(zu))]z) \\ &= (3/4)\varphi(u'(a(zu)), u'(a(zu))) \\ &= (3/4)^2\varphi(u', u')\varphi(a(zu), a(zu)) \\ &= (3/4)^2\varphi(u', u')\varphi(zu, zu) \\ &= (3/4)^2\varphi(u', u')\varphi(u, u) \\ &= \lambda(u', u')\lambda(u, u) .\end{aligned}$$

□

Lemma 6.3 *$(U, *)$ is an alternative quadratic algebra with a as its unit element.*

Proof. From

$$u' * a = [u'(a(za))]z = (u'z)z = u' ,$$

$$a * u = [a(a(zu))]z = (zu)z = u ,$$

we see that a is the unit element. When $u' = u$, by applying Proposition 4.2 we obtain

$$\begin{aligned} u * u &= [u(a(zu))]z = [-a(u(zu)) + (3/2)\varphi(u, a)zu]z \\ &= [-(3/4)\varphi(u, u)az + (3/2)\varphi(u, a)zu]z \\ &= -(3/4)\varphi(u, u)a + (3/2)\varphi(u, a)u . \end{aligned} \tag{6.4}$$

Writing $\lambda = (3/4)\varphi$, we have

$$u * u - 2\lambda(u, a)u + \lambda(u, u)a = 0 , \tag{6.5}$$

which says that $(U, *)$ is a quadratic algebra.

As usual in a quadratic algebra, it is useful to have the conjugation map $u \to \bar{u} = 2\lambda(u, a)a - u$. Linearizing u' in the relation of Lemma 6.2 and setting the new variable equal to a gives

$$\lambda(u' * u, u) = \lambda(u', a)\lambda(u, u) ,$$

and doing the same thing to u yields

$$\lambda(u' * u, a) + \lambda(u', u) = 2\lambda(u', a)\lambda(u, a) .$$

Phrasing this in terms of φ instead of λ gives

$$\varphi(u' * u, a) = (3/2)\varphi(u', a)\varphi(u, a) - \varphi(u', u) . \tag{6.6}$$

We will also need the linearization of (6.4) which is

$$u * u' + u' * u = (3/2)\varphi(u, a)u' + (3/2)\varphi(u', a)u - (3/2)\varphi(u', u)a . \tag{6.7}$$

We can now prove that conjugation is an anti-automorphism using (6.6) and (6.7):

$$\begin{aligned} \overline{(u' * u)} - \bar{u} * \bar{u}' &= (3/2)\varphi(u' * u, a)a - u' * u \\ &\quad - [(3/2)\varphi(u, a)a - u] * [(3/2)\varphi(u', a)a - u'] \\ &= (3/2)[(3/2)\varphi(u, a)\varphi(u', a) - \varphi(u', u)]a - u' * u \\ &\quad - (9/4)\varphi(u, a)\varphi(u', a)a + (3/2)\varphi(u, a)u' \\ &\quad + (3/2)\varphi(u', a)u - u * u' = (3/2)\varphi(u', u)a - u' * u \\ &\quad + (3/2)\varphi(u, a)u' + (3/2)\varphi(u', a)u - u * u' = 0 . \end{aligned}$$

If we linearize the identity $u(uv) = (3/4)\varphi(u, u)v$, we get the special case $u(av) + a(uv) = (3/2)\varphi(u, a)v$. We use this and (2.6) to verify left alternativity in $(U, *)$:

$$\begin{aligned} u*(u*u') &= [u(a(z([u(a(zu'))]z)))]z = [u(a[u(a(zu'))])]z \\ &= -[u(a[a(u(zu'))])]z + (3/2)\varphi(u, a)[u(a(zu'))]z \\ &= -[u(u(zu'))]z + (3/2)\varphi(u, a)(u*u') \\ &= -(3/4)\varphi(u, u)[zu']z + (3/2)\varphi(u, a)(u*u') \\ &= -(3/4)\varphi(u, u)u' + (3/2)\varphi(u, a)(u*u') \\ &= [-(3/4)\varphi(u, u)a + (3/2)\varphi(u, a)u]*u' = (u*u)*u' . \end{aligned}$$

Since conjugation is an anti-automorphism, the right alternative identity follows from the left alternative identity, and $(U, *)$ is alternative. □

Lemma 6.8 $\varphi(a(zu), z) = \varphi(a, u)$.

Proof. In (2.6) let $y = e$ and $z \in Z$, to obtain

$$-\varphi(x^2, z) + 2\varphi(ex, zx) = 2\varphi(e, x)\varphi(z, x) .$$

Linearizing x gives

$$-\varphi(xy, z) + \varphi(ey, zx) + \varphi(ex, zy) = \varphi(e, x)\varphi(z, y) + \varphi(e, y)\varphi(z, x) ,$$

and setting $x = a$, $y = zu$ and noting that $zy = z(zu) = u$ yields

$$-\varphi(a(zu), z) + (1/2)\varphi(zu, za) + (1/2)\varphi(a, u) = 0 .$$

Since the last two terms in this equation are equal by (4.5), we have derived $\varphi(a(zu), z) = \varphi(a, u)$. □

7 The structure theorem

We begin by constructing a class of pseudo-composition algebras. Let Q be an alternative quadratic algebra with unit a, with a symmetric bilinear form λ such that $x^2 = 2\lambda(x, a)x - \lambda(x, x)a$. We do not assume that λ is nonsingular. But even without nonsingularity, λ permits composition: $\lambda(xy, xy) = \lambda(x, x)\lambda(y, y)$ (see [2]). Let $P(Q)$ be the algebra consisting of all 5-tuples $(\xi, \xi', x_1, x_2, x_3)$ where $\xi, \xi' \in F$ and where $x_i \in Q$ for $1 \leqq i \leqq 3$. We define a product on $P(Q)$ by

$$(\xi, \xi', x_1, x_2, x_3)(\eta, \eta', y_1, y_2, y_3) = (\zeta, \zeta', z_1, z_2, z_3) ,$$

where

$$\begin{aligned} \zeta &= \xi\eta - (4/3)\xi'\eta' + (2/3)\lambda(x_1, y_1) + (2/3)\lambda(x_2, y_2) - (4/3)\lambda(x_3, y_3) , \\ \zeta' &= -\xi\eta' - \xi'\eta + \lambda(x_1, y_1) - \lambda(x_2, y_2) , \\ z_1 &= (1/2)\xi y_1 + \xi' y_1 + (1/2)\eta x_1 + \eta' x_1 + x_2\bar{y}_3 + y_2\bar{x}_3 , \\ z_2 &= (1/2)\xi y_2 - \xi' y_2 + x_1 y_3 + (1/2)\eta x_2 - \eta' x_2 + y_1 x_3 , \\ z_3 &= -\xi y_3 + \bar{x}_1 y_2 + \bar{y}_1 x_2 - \xi' x_3 . \end{aligned}$$

Proposition 7.1 *The algebra $P(Q)$ is a pseudo-composition algebra.*

Proof. It is immediate from the definition of multiplication that $P(Q)$ is commutative. If $x = (\xi, \xi', x_1, x_2, x_3)$, then

$$x^2 = (\xi^2 - (4/3)\xi'^2 + (2/3)\lambda(x_1, x_1) + (2/3)\lambda(x_2, x_2) - (4/3)\lambda(x_3, x_3),$$
$$- 2\xi\xi' + \lambda(x_1, x_1) - \lambda(x_2, x_2), (\xi + 2\xi')x_1 + 2x_2\bar{x}_3,$$
$$(\xi - 2\xi')x_2 + 2x_1x_3, - 2\xi x_3 + 2\bar{x}_1x_2).$$

Using the relations $\lambda(x_1, x_2\bar{x}_3) = \lambda(x_1x_3, x_2) = \lambda(x_3, \bar{x}_1x_2)$ which hold in an algebra with a form permitting composition (see [2]), using the alternative laws and $x_1(\bar{x}_1y_1) = \lambda(x_1, x_1)y_1$ in Q, it is straightforward to verify that $x^3 = \varphi(x, x)x$ for all $x \in P(Q)$ where

$$\varphi(x, x) = \xi^2 + (4/3)\xi'^2 + (4/3)\lambda(x_1, x_1) + (4/3)\lambda(x_2, x_2) + (4/3)\lambda(x_3, x_3).$$

Thus, $P(Q)$ is a pseudo-composition algebra. □

Theorem 7.2 *Let A be a pseudo-composition algebra over F with norm φ and idempotent $e \neq 0$. Put $\varphi_i = \varphi|_{A_i \times A_i}$ for $i = -1, 1/2$ and assume $\varphi_{1/2} \neq 0$ as well as*

$$\operatorname{codim}_{A_{-1}} \operatorname{Rad} \varphi_{-1} > 1.$$

Then $A \cong P(Q)$ for some quadratic alternative algebra Q.

Proof. The assumptions are such that we have a decomposition $A = Fe \oplus Fc \oplus U \oplus V \oplus Z$ as shown above, and that we may assume $\varphi(U, U) \neq 0 \neq \varphi(Z, Z)$. We choose $z \in Z$ with $\varphi(z, z) = 4/3$. As we saw earlier, we have $V = zU$ and $Z = a(zU)$, so that $A = Fe \oplus Fc \oplus U \oplus zU \oplus a(zU)$. We shall show that

$$\Phi: \alpha e + \beta c + u_1 + zu_2 + a(zu_3) \to (\alpha, \beta, u_1, u_2, u_3)$$

is an isomorphism of A onto the algebra $P(Q)$ constructed from the quadratic algebra $(U, *, \lambda)$. Clearly Φ is a vector space isomorphism. By comparing the action of e, c, and their images, it is immediate that $\Phi((\alpha e + \beta c)x) = \Phi(\alpha e + \beta c)\Phi(x)$ for all $x \in A$. It is also easy to check using (4.5) and (4.6) that $\Phi(yy') = \Phi(y)\Phi(y')$ if y and y' are in the same subspace U, $V = zU$, or $Z = a(zU)$. It remains to show that the homomorphism property holds for mixed products.

Multiplying (6.1) by z gives $z(u' * u) = u'(a(uz))$, and from this we obtain the case where the arguments are from U and Z:

$$\Phi(u_1(a(zu_3))) = \Phi(z(u_1 * u_3)) = (0, 0, 0, u_1 * u_3, 0) = \Phi(u_1)\Phi(a(zu_3)).$$

To determine the image of the product $u_1(zu_2) \in Z$, we observe that the element $u_3 = z(a(u_1(zu_2)))$ satisfies $u_1(zu_2) = a(zu_3)$. Linearizing the first relation of Proposition 4.2(c) yields

$$a(u_1v) = - u_1(av) + (3/4)\varphi(u_1, a)v.$$

Replacing v by zu_2 in this relation, and substituting it into the expression for u_3, we have

$$u_3 = - u_1 * u_2 + 2\lambda(u_1, a)u_2 = \bar{u}_1 * u_2.$$

Thus,

$$\Phi(u_1(zu_2)) = \Phi(a(zu_3)) = (0, 0, 0, 0, -u_1 * u_2 + 2\lambda(u_1, a)u_2) = \Phi(u_1)\Phi(zu_2) .$$

Finally we deal with products of the form $VZ = (zU)[a(zU)]$. Using the linearization of Theorem 2.7(d) and Lemma 6.8, we obtain

$$\begin{aligned}(zu_2)(a(zu_3)) &= -[u_2(a(zu_3))]z + (3/2)\varphi(a(zu_3), z)u_2 \\ &= -u_2 * u_3 + 2\lambda(u_3, a)u_2 = u_2 * \bar{u}_3 .\end{aligned}$$

Then

$$\begin{aligned}\Phi((zu_2)(a(zu_3))) &= (0, 0, -u_2 * u_3 + 2\lambda(u_3, a)u_2, 0, 0) \\ &= \Phi(zu_2)\Phi(a(zu_3)) ,\end{aligned}$$

to complete the proof. □

In particular, we get a result which is essentially contained in [5], although Springer's approach is different:

Corollary 7.3 *Any nonquadratic pseudo-composition algebra with nondegenerate bilinear form over an algebraically closed field F is isomorphic to $P(Q)$ where Q is a composition algebra over F.*

If K is the radical of λ on Q, it is straightforward to verify that $N = \{(0, 0, x_1, x_2, x_3) \in P(Q) \mid x_1, x_2, x_3 \in K\}$ is an ideal, and that it is the radical of φ on $P(Q)$. Theorem 1.2 follows from the results in Sects. 3, 5, and 7, except for the fact that $\operatorname{Rad}\varphi$ is nilpotent. For this we refer the reader to [4, p. 73].

It should be noted that extra work is required in this paper because φ is only assumed to be an arbitrary symmetric bilinear form. If we assume that φ is associative and nondegenerate, the proof of Theorem 1.2 and Corollary 7.3 simplify significantly.

As a variant of a result of Springer [5], it has been observed in [1, Theorem 4.6] that for a nondegenerate Jordan algebra of degree 3 satisfying $x^3 + q_1(x)x^2 + q_2(x)x + q_3(x)e_0 = 0$ a commutative product $*$ is defined on $B := \operatorname{Ker} q_1$ by

$$x * y := -\beta(xy + (2/3)q_2(x, y)e_0)$$

(for any $\beta \in F - \{0\}$, where e_0 is the unit element of A) and that the algebra $(B, *)$ is a pseudo-composition algebra with nondegenerate norm $\varphi = -(1/3)\beta^2 q_2$. This observation, combined with Corollary 7.3, reveals immediately the (well known) structure of these algebras. We know $(B, *) \cong (P(Q), *)$ for some composition algebra by Corollary 7.3. We change the product in $P(Q)$ to $x \circ y := 2x * y$ and verify by a simple computation that

$$(\xi, \xi', x_1, x_2, x_3) \to \begin{pmatrix} \xi & x_1 & x_2 \\ \bar{x}_1 & \xi' - (1/2)\xi & x_3 \\ \bar{x}_2 & \bar{x}_3 & -\xi' - (1/2)\xi \end{pmatrix}$$

is an isomorphism of $(P(Q), \circ)$ onto the commutative algebra of 3×3 hermitian matrices of trace 0 with entries from Q, where the product is given by

$$X * X := X^2 - (1/3)(\operatorname{trace} x^2)E .$$

The final step $A \cong H_3(Q)$ is now obvious.

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The equations of Rees algebras of ideals with linear presentation

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1 Introduction

Let R be a commutative Noetherian ring, and let $f_1, \ldots, f_n$ be elements of R. We are interested in the forms

$$h(T_1 \ldots, T_n) \in R[T_1, \ldots, T_n],$$

such that

$$h(f_1, \ldots, f_n) = 0 .$$

A great many such polynomials arise from the syzygies of the ideal I generated by the f_j's, or are obtained by straightforward elimination from the syzygies. The emphasis here is on deriving estimates for the number of generators and degrees of the ideal of forms in several cases of interest. The setting for this analysis is the theory of Rees algebras. We recall that for an ideal $I = (f_1, \ldots, f_n)$ of a commutative ring R, there exists a canonical mapping

$$\alpha\colon S(I) \to \mathscr{R}(I) = R[It]$$

from the symmetric algebra of I onto its Rees algebra. The ideal I is said to be of *linear type* if this mapping is an isomorphism. In general there will be a nonvanishing kernel to α

$$\mathscr{A} = \sum_{i \geqq 2} \mathscr{A}_i , \tag{1}$$

with trivial components in degrees 0 and 1. We will say that I is of rth *relation type* if $\mathscr{A}$ is generated by its components of degree at most r.

The ways to obtain the equations of the Rees algebras of ideals of codimension 3, or higher, which are not of linear type, are markedly different from those pursued

To Professor Jürgen Herzog on his fiftieth birthday

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in [15]. Part of the reason for the difficulty lies with the lack of generic examples. To elaborate, let R be a regular local ring of dimension d, and let I be a codimension three Cohen–Macaulay ideal with a resolution

$$0 \to R^p \xrightarrow{\phi} R^m \xrightarrow{\varphi} R^n \to I \to 0 .$$

Suppose $\mathbf{x}$ is the vector of coefficients of the matrix φ, and let

$$\mathbf{T} \cdot \varphi = \mathbf{x} \cdot B(\varphi)$$

be the equations of the symmetric algebra of I. Assume that $\mathbf{x}$ defines a complete intersection of codimension ℓ. Then $I_\ell(B(\varphi))$ is contained in the ideal generated by the equations of the Rees algebra $\mathscr{R}(I)$, which is an ideal of codimension $n-1$. It follows that the ideal $(\mathbf{x} \cdot B(\varphi), I_l(B(\varphi)))$ will never have the codimension of the generic case – quite unlike what occurred in codimension two.

In general, even with the absence of a likely set of defining equations for the Rees algebra, it is still possible to predict degrees or the number of generators of $\mathscr{A}$. This will require often that we focus on ideals of low "second analytic deviation", a notion that we now define.

Let $(R, \mathbf{m})$ be a local ring, and let I be an ideal; we recall some definitions and results from [13]. An ideal $J \subset I$ is a *reduction* of I if $JI^r = I^{r+1}$ for some integer r; the least such integer is the *reduction number* of I with respect to J. A reduction J of I is *minimal* if J is a minimal ideal among the reductions of I. The *reduction number* of I is the minimum of the reduction numbers of I with respect to J, where J ranges over all minimal reductions of I. If the residue field of R is infinite, the number of generators of any minimal reduction is equal to the Krull dimension of $\mathscr{R}(I) \otimes R/\mathbf{m}$. It is the *analytic spread* of I, and is denoted $\ell(I)$. The *deviation* of I is the non-negative integer $v(I)$ – grade I. Huckaba and Huneke [9] have defined the *analytic deviation* of I as $\ell(I)$ – codimension I; for ideals of linear type $v(I) = \ell(I)$.

Definition 1.1 The difference $v(I) - \ell(I)$ will be called the *second analytic deviation* of I.

The terminology arises because the latter is the difference between the other two deviations. This number is small for distinguished classes of Cohen–Macaulay ideals.

Before the main results are summarized, we are going to recall some additional definitions.

Definition 1.2 Let I be an ideal of a local ring R of dimension d. Let $\mathbb{K}$ be the Koszul complex on the set $\mathbf{f} = \{f_1, \ldots, f_n\}$ of generators of I. Denote by $H_i(\mathbb{K})$ the homology modules of $\mathbb{K}$. I satisfies *sliding depth* if

$$\operatorname{depth} H_i(\mathbb{K}) \geqq d - n + i, \quad \forall i .$$

There is a more stringent condition on the depth of the Koszul homology modules [11]:

Definition 1.3 I is *strongly Cohen–Macaulay* if the Koszul homology modules of I with respect to one (and then to any) generating set are Cohen–Macaulay.

The difference between these conditions is best seen reflected on the depths of the modules of cycles of $\mathbb{K}$ (see [8]):

Remark 1.4 We have

$$\operatorname{depth} Z_i \geqq \begin{cases} \min\{d, d-n+i+1\}, & \text{if } I \text{ satisfies sliding depth} \\ \min\{d, d-g+2\}, & \text{if } I \text{ is strongly Cohen–Macaulay.} \end{cases}$$

We need one final notion.

Definition 1.5 The ideal I is said to satisfy condition $\mathscr{F}_0$ if for any prime ideal $I \subset \wp$, $v(I_\wp) \leqq \dim R_\wp + 1$; if $v(I_\wp) \leqq \dim R_\wp$, then I is said to satisfy condition $\mathscr{F}_1$.

The symmetric algebra of an ideal I that satisfies sliding depth and $\mathscr{F}_0$ is Cohen–Macaulay (Theorem 2.2). This creates a framework in which to examine the ideal $\mathscr{A}$ (cf. Eq. (1)), and from there the reduction number of ideals of second analytic deviation one. Several formulas make predictions about the reduction number (Theorem 2.5, Proposition 2.7, Corollary 2.9 and Theorem 3.2).

The following section considers the case of an ideal I with a relation matrix of a particularly straightforward kind:

Theorem. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field and let I be a strongly Cohen–Macaulay ideal of codimension g having a linear presentation matrix. Suppose I is minimally generated by $d+1$ elements and I is locally of linear type on the punctured spectrum of R. Then*

(a) (Theorem 3.2) *I has reduction number $d-g+1$ (in case k is infinite), $\mathscr{A}$ is generated by a single element, and $S(I)$ is reduced.*

(b) (Theorem 3.8) *If the associated graded ring of I, $\operatorname{gr}_I(R)$, is Cohen–Macaulay then $\mathscr{R}(I)$ is Cohen–Macaulay.*

(c) (Theorem 3.10) *I is generated by forms of degree δ with $\delta \leqq \dfrac{d}{g-1}$. Furthermore $\mathscr{R}(I)$ is Cohen–Macaulay if and only if $\delta = \dfrac{d}{g-1}$.*

We expect (a) and (b) above to hold without restriction. However there is a serious gap here as we have been unable to find examples of these ideals in codimension greater than three.

In the final section, we consider one criterion for an ideal to have second analytic deviation zero.

Theorem (Theorem 4.1) *Let R be a Cohen–Macaulay local ring and let I be an ideal of codimension at least 1 satisfying sliding depth. Suppose that for every $\wp \in V(I)$, either $v(I_\wp) \leqq \dim R_\wp$ or $\ell(I_\wp) \leqq \dim R_\wp - 1$. Then I is of linear type and $\mathscr{R}(I)$ is Cohen–Macaulay.*

One of its consequences says that if the ideal I is, in addition, a prime ideal whose ordinary and symbolic powers coincide then I can be generated by fewer than $\dim R$ elements.

2 Ideals of second analytic deviation one

In this section, a class of ideals of second analytic deviation 1 is identified and their Rees algebras are studied for the equations that define it and for their Cohen–Macaulayness.

2.1 Approximation complexes

There are various complexes, that have been dubbed *approximation complexes*, that often play the role of surrogates for projective resolutions of symmetric algebras [5, 6, 7]. We briefly recall some of their constructions and refer to [loc. cit] for their basic properties.

Let F and G be R-modules and suppose there are two mappings:

$$\begin{array}{ccc} F & \xrightarrow{\psi} & G \\ {\scriptstyle\varphi}\downarrow & & \\ R\,. & & \end{array}$$

The algebra $\bigwedge F \otimes S(G)$ is a double complex with differentials

$$d_\varphi = \partial : \bigwedge^r F \otimes S_t(G) \to \bigwedge^{r-1} F \otimes S_t(G)\,,$$

which is the Koszul complex associated to the mapping φ and coefficients in $S(G)$, while

$$d_\psi = \partial' : \bigwedge^r F \otimes S_t(G) \to \bigwedge^{r-1} F \otimes S_{t+1}(G)$$

defines the Koszul mapping of ψ.

It is a straightforward computation that ∂ and ∂' (skew) commute. The resulting double complex will be denoted by $\mathscr{L} = \mathscr{L}(\varphi, \psi)$. The case of interest here is φ: $F = R^n \to R$, the mapping associated to a sequence $\mathbf{x} = \{x_1, \ldots, x_n\}$ of elements of R.

Definition 2.1 The complex $\mathscr{L} = \mathscr{L}(\mathbf{x}) = \mathscr{L}(\varphi, \text{identity})$ will be called the double Koszul complex of $\mathbf{x}$.

The complex $(\mathscr{L}, \partial)$ is the Koszul complex associated to the sequence $\mathbf{x}$ in the polynomial ring $S = S(R^n) = R[T_1, \ldots, T_n]$. $(\mathscr{L}, \partial')$, on the other hand, is also an ordinary Koszul complex but constructed over the sequence $\mathbf{T} = \{T_1, \ldots, T_n\}$. Thus we have a grading of $(\mathscr{L}, \partial')$ by subcomplexes of R-modules, $\mathscr{L} = \sum \mathscr{L}_t$

$$\mathscr{L}_t = \sum_{r+s=t} \bigwedge^r R^n \otimes S_s(R^n)\,,$$

and the $\mathscr{L}_t$ are exact for $t > 0$. If we denote by $Z = Z(\mathbf{x})$ the module of cycles of $\mathbb{K}(\mathbf{x})$, and by $H = H(\mathbf{x})$ its homology, the (skew-) commutativity of ∂ and ∂' yield several new complexes:

- $\mathscr{Z}$-complex: $\mathscr{Z} = \mathscr{Z}(\mathbf{x}) = \{Z \otimes S, \partial'\}$
- $\mathscr{H}$-complex: $\mathscr{H} = \mathscr{H}(\mathbf{x}) = \{H \otimes S, \partial'\}$.

The $\mathscr{H}$-complex, as well as the $\mathscr{Z}$-complex, are graded complexes over the polynomial ring $B = R[T_1, \ldots, T_n]$. Of particular significance for us here is the fact that $H_0(\mathscr{Z}(\mathbf{x})) = S(I)$. More explicitly, the *approximation complex* $\mathscr{Z}(I)$ of I has the form:

$$0 \to Z_n \otimes B(-n) \to Z_{n-1} \otimes B(-n+1) \to \cdots \to Z_1 \otimes B(-1) \to B \to S(I) \to 0\,,$$

which indicates the role of the high depths of the Koszul homology modules. The notation is taking into account only the grading provided by the variables T_j. We shall later need to factor in the possible grading coming from the base ring R itself.

The following is a composite of [7, Theorems 6.2 and 6.5]:

Theorem 2.2 *Let R be a Cohen–Macaulay local ring and let I be an ideal of codimension $g \geqq 1$ satisfying sliding depth and $\mathscr{F}_0$. Then the complex $\mathscr{Z}(I)$ is acyclic and $S(I)$ is a Cohen–Macaulay ring. Moreover, if R is a Gorenstein ring and if I is strongly Cohen–Macaulay and $g \geqq 2$, then $S(I)$ has Cohen–Macaulay type $g - 1$.*

2.2 The analytic spread

We determine the analytic spread of the ideals in Theorem 2.2, once they satisfy $\mathscr{F}_1$ locally on the punctured spectrum. It shows that these ideals have second analytic deviation equal to 1. We shall need the connectedness lemma of Abhyankar [1, (4.9)] and Hartshorne [4, Proposition 2.1], and for completeness we reproduce a proof given by Kaplansky in a lecture several years ago.

Lemma 2.3 *Let R be a Noetherian ring with no idempotents other than 0 or 1, and let I and J be nonzero ideals of R such that $I \cdot J = 0$. Then* $\operatorname{grade}(I + J) \leqq 1$.

Proof. We may assume that $I \cap J \subset (0\colon J) \cap (0\colon I) = 0$, as otherwise for $0 \neq x \in (0\colon J) \cap (0\colon I)$, $x(I + J) = 0$. Now by our assumptions on the idempotents in R, $(0\colon I) + (0\colon J) \neq R$. Hence localizing at a prime containing this ideal, we preserve our assumptions $I \neq 0$, $J \neq 0$ and $I \cap J = 0$. Thus we may from now on assume that R is local. Suppose $\operatorname{grade}(I + J) \geqq 1$. Let $x = a + b$, $a \in I$, $b \in J$, be a regular element; it is clear that $a \neq 0$, $b \neq 0$. Moreover, $a \notin R(a + b)$ for an equation $a = r(a + b)$ yields $(1 - r)a = rb$, which is a contradiction, whether r is a unit or not. Since $(I + J)a \subset R(a + b)$ it follows that $\operatorname{grade}(I + J) = 1$. □

Proposition 2.4 *Let $(R, \mathbf{m})$ be a Cohen–Macaulay local ring of dimension d and let I be an ideal of codimension at least 1. Suppose that $v(I) = d + 1$, that I satisfies sliding depth, and that I satisfies $\mathscr{F}_1$ on the punctured spectrum of R. Then $\ell(I) = d$.*

Proof. Let

$$0 \to \mathscr{A} \to S(I) \to \mathscr{R}(I) \to 0$$

be the canonical exact sequence. By assumption $\mathscr{A}$ is annihilated by a power $\mathbf{m}^r$ of the maximal ideal. The equation $\mathbf{m}^r \cdot \mathscr{A} = 0$ implies, by the previous lemma, that $\operatorname{grade}(\mathbf{m}^r S(I), \mathscr{A}) \leqq 1$. Since $S(I)$ is a Cohen–Macaulay ring by Theorem 2.2, the codimension of this ideal will also be bounded by 1. It follows that the image of $\mathscr{A}$ in $S(I) \otimes_R (R/\mathbf{m})$ has codimension at most 1, and therefore the analytic spread of I is at least d and thus equals d. □

2.3 The content formula

The following content formula is very similar to the codimension two case (cf. [15]):

Theorem 2.5 *Let R be a Gorenstein local ring of dimension d and let I be a strongly Cohen–Macaulay ideal of codimension $g \geqq 2$. Suppose $v(I) = d + 1$ and I satisfies $\mathscr{F}_1$ on the punctured spectrum of R. Then the symmetric and Rees algebras of I first differ in degree $d - g + 2$. Moreover in the canonical homomorphism*

$$0 \to \mathscr{A}_{d-g+2} \to S_{d-g+2}(I) \to I^{d-g+2} \to 0\,,$$

$$\mathscr{A}_{d-g+2} = \operatorname{Ext}_R^d(R/I_1(\varphi), R)\,, \tag{2}$$

where φ is the matrix of a minimal presentation of I.

Proof. The hypothesis says that I satisfies the condition $\mathscr{F}_0$, so that we can use Theorem 2.2. Since I has codimension g, the modules of cycles Z_i of the Koszul complex of I have depth at least $d-g+2$, for $i=1,\ldots,d-g+2$. Consider the component $\mathscr{Z}_t$ of $\mathscr{Z}(I)$ lying over some $S_t(I)$. If $t<d-g+2$, the acyclicity of $\mathscr{Z}_t$ implies that $S_t(I)$ has depth at least 1. Since I is of linear type on the punctured spectrum of R, each $\mathscr{A}_t$ is a module of finite length. It follows that $S_{d-g+2}(I)$ is the first component that may have non-trivial torsion.

To determine $\mathscr{A}_{d-g+2}$, we make use of the obvious isomorphism

$$\operatorname{Ext}_R^d(\mathscr{A}_{d-g+2}, R) \simeq \operatorname{Ext}_R^d(S_{d-g+2}(I), R)\,.$$

Chasing the depths in the complex

$$0 \to Z_{d-g+2} \to Z_{d-g+1} \otimes B_1 \to \cdots \to Z_1 \otimes B_{d-g+1} \to B_{d-g+2} \to S_{d-g+2}(I) \to 0$$

we obtain the exact sequence

$$\operatorname{Ext}_R^{g-2}(Z_{d-g+1} \otimes B_1, R) \xrightarrow{\psi} \operatorname{Ext}_R^{g-2}(Z_{d-g+2}, R) \to \operatorname{Ext}_R^d(S_{d-g+2}(I), R) \to 0\,.$$

It is now easy to verify that

$$\operatorname{Ext}_R^{g-2}(Z_{d-g+1}, R) = H_1(I)$$

and

$$\operatorname{Ext}_R^{g-2}(Z_{d-g+2}, R) = R/I\,.$$

Furthermore, it is not difficult to see that the mapping

$$\psi\colon H_1(I) \otimes B_1^* \to R/I$$

just reads the coefficients of the various syzygies of I. □

Example 2.6 It is not clear how to extend this formula to larger classes of ideals, such as those with sliding depth. For instance, let I be the ideal generated by the minors of order 2 of the symmetric matrix

$$\begin{bmatrix} x_1 & 0 & x_2 \\ 0 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{bmatrix}.$$

I has sliding depth, is a complete intersection on the punctured spectrum but $\mathscr{A}_2 \neq 0$.

2.4 Expected reduction number

Theorem 2.5 shows that $\mathscr{A}_{d-g+2} \neq 0$. It may occur (cf. Example 2.10 below) that this component is contained in $\mathbf{m} \cdot S(I)$, that is, the reduction number of I is greater than $d-g+1$. Nevertheless, we shall take $d-g+1$ to represent the *expected* reduction number of the ideal I.

Proposition 2.7 *Let I be an ideal as in Theorem* 2.5 *and assume that the residue field of R is infinite. Further let*

$$R^m \xrightarrow{\varphi} R^{d+1} \to I \to 0$$

be a minimal presentation. If I has reduction number $d-g+1$, then $v(I_1(\varphi)) \leqq m$.

Proof. Denote by $\mathbf{x} = [x_1, \ldots, x_r]$ a minimal set of generators of the ideal $I_1(\varphi)$. We write the equations of the symmetric algebra of $S(I)$ in its dual form [14, 15]:

$$J = \mathbf{T} \cdot \varphi = \mathbf{x} \cdot B(\varphi) .$$

By assumption there exists $f \in \mathscr{A}_{d-g+2}$, $f \notin \mathbf{m}S$. By Theorem 2.5, the ideal $(\mathbf{x})$ conducts f into J, which means that we can write

$$f \cdot \mathbf{x}^t = G \cdot B(\varphi)^t \cdot \mathbf{x}^t ,$$

where G is an $r \times m$ matrix. We thus have

$$(f \cdot I_r - G \cdot B(\varphi)^t) \cdot \mathbf{x}^t = 0 ,$$

that is,

$$f \cdot I_r - G \cdot B(\varphi)^t \tag{3}$$

consists of syzygies of $\mathbf{x}$ and thus has all of its entries in $\mathbf{m}$. This implies that reducing the matrix (3) mod $\mathbf{m}$, we must have that the rank of $B(\varphi)$ mod $\mathbf{m}$ is at least r, and therefore $m \geqq r$. □

The matrix argument above can be abstracted in the following general assertion, whose proof is left as an exercise for the reader (see also [15, Theorem 3.1.3]).

Proposition 2.8 *Let J and K be ideals, $J \subset K$. Denote by L the content of K. If $J\colon K \not\subset \sqrt{L}$, then $v(J) \geqq v(K)$.*

Corollary 2.9 *Let $(R, \mathbf{m})$ be a regular local ring of dimension d, with infinite residue field and let I be a Gorenstein ideal of codimension three. Suppose $v(I) = d + 1$, I is $\mathscr{F}_1$ on the punctured spectrum of R, and has reduction number $d - 2$. Then $v(I_1(\varphi)) = d$, in particular $I_1(\varphi)$ is generated by a system of parameters.*

Proof. We use the notation of Proposition 2.7. Now $m = d + 1$, and φ is alternating. From

$$0 = \mathbf{T} \cdot \varphi \cdot \mathbf{T}^t = \mathbf{x} \cdot B(\varphi) \cdot \mathbf{T}^t ,$$

we have that $B(\varphi) \cdot \mathbf{T}^t$ is a syzygy of $\mathbf{x}$ and thus has all of its entries in $\mathbf{m}$. This implies that the matrix $B(\varphi)$ mod $\mathbf{m}$ has $\mathbf{T}$ in the kernel, so it has rank at most d. But $I_1(\varphi)$ is an $\mathbf{m}$-primary ideal so $r \geqq d$ and thus $r = d$, as asserted. □

Example 2.10 The Rees algebras of the ideals in Corollary 2.9 may exhibit certain anomalies. For an example, consider the ideal I defined by the Pfaffians of the matrix:

$$\varphi = \begin{bmatrix} 0 & -a^2 & -b^2 & -c^2 & -d^2 \\ a^2 & 0 & -d^2 & -ab & -c^2 \\ b^2 & d^2 & 0 & -a^2 & -ab \\ c^2 & ab & a^2 & 0 & -b^2 \\ d^2 & c^2 & ab & b^2 & 0 \end{bmatrix} .$$

The content ideal, $I_1(\varphi) = (a^2, b^2, c^2, d^2, ab)$, has type two so that $\mathscr{A}_3$ is generated by 2 elements by Theorem 2.5. In particular, the reduction number could not be $d - g + 1 = 2$; in fact, the reduction number is 5. A computation showed that

the Rees algebra of I is generated by the linear forms coming from the syzygies, these two 3-forms, and a sextic. The algebra $\mathscr{R}(I)$ is neither normal nor Cohen–Macaulay ($\mathrm{gr}_I(R)$ is also not Cohen–Macaulay).

Remark 2.11 We do not know of any prime ideal that exhibits the behaviour above, that is whose Rees algebra is not Cohen–Macaulay. The defining ideal of the monomial curve (t^5, t^6, t^7, t^8) satisfies the various predictions, and in addition is normal. There is one additional equation (to those defining $S(I)$); it is not clear how this cubic arose by elimination from the 1-forms.

3 Cohen–Macaulay Algebras

The following conjecture will be the focus of our attention from now on.

Conjecture 3.1 *Let R be a regular local ring of dimension d and let I be a strongly Cohen–Macaulay ideal of codimension g. Suppose $v(I) = d + 1$ and I satisfies $\mathscr{F}_1$ on the punctured spectrum of R. If the reduction number of I is $d - g + 1$ then $\mathscr{R}(I)$ is Cohen–Macaulay.*

3.1 The number of equations of the Rees algebra
We give a count for the number of generators of $\mathscr{A}$ for ideals with a linear presentation. Note that if I has linear presentation matrix and $g \geqq 2$ (which is automatic in our set-up), then I has to be minimally generated by forms, necessarily of the same degree δ.

Theorem 3.2 *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring and let I be a strongly Cohen–Macaulay ideal of codimension g having a linear presentation matrix. Suppose $v(I) = d + 1$ and I satisfies $\mathscr{F}_1$ on the punctured spectrum of R. Then I has reduction number $d - g + 1$ (in case k is infinite), $\mathscr{A}$ is generated by a single element, and $S(I)$ is reduced.*

We start with a result in the folklore of the theory of Rees algebras. Let I be generated by forms $\{f_1, \ldots, f_n\}$ of the same degree δ, and let $k[M]$ denote the k-subalgebra of $\mathscr{R}(I)$ generated by $f_j t$. Set $\mathbf{m} = (x_1, \ldots, x_d)$.

Proposition 3.3 *There is a decomposition*

$$\mathscr{R}(I) = k[M] + \mathbf{m} \cdot \mathscr{R}(I) ,$$

where

$$k[M] \cap \mathbf{m} \cdot \mathscr{R}(I) = 0 .$$

In particular, the special fiber of $\mathscr{R}(I)$, $\mathscr{R}(I) \otimes (R/\mathbf{m})$, is an integral domain.

Proof. It is clear. □

Proof of Theorem 3.2 The prime ideal $\mathscr{A}$ is a primary component of (0) in $S(I)$. We claim that $\mathbf{m}S(I)$ is the other primary component. Let

$$k[\mathbf{T}]^m \xrightarrow{B(\varphi)} k[\mathbf{T}]^d \to F \to 0$$

be the Jacobian dual of I [14]. We have that

$$S(I) = S_{k[\mathbf{T}]}(F) ,$$

so that the prime ideal which is the $k[\mathbf{T}]$-torsion submodule of $S(I)$ is also a primary component of (0). But this component contains a power of $\mathbf{m}S(I)$ so must equal this prime ideal.

The equality $0 = \mathbf{m}S(I) \cap \mathscr{A}$ yields the exact sequence

$$0 \to \mathscr{A} \to S(I) \otimes (R/\mathbf{m}) \to \mathscr{R}(I) \otimes (R/\mathbf{m}) \to 0 .$$

From Proposition 2.4 we have that $\mathscr{A}$ is a codimension 1 ideal, while Proposition 3.3 says that it is prime. This shows that $\mathscr{A}$ is generated by one single element in degree $d - g + 2$, according to Theorem 2.5. □

3.2 Characteristic polynomial and dual module

The preceding arguments raise the issues of the explicit determinations of the module F and of the polynomial f that generates $\mathscr{A}$. With a slight abuse of terminology we use the following

Definition 3.4 In the setting of Theorem 3.2, let f be a homogeneous polynomial in $k[\mathbf{T}] = k[T_1, \ldots, T_{d+1}]$ that generates $\mathscr{A}$. We call f a *characteristic polynomial* of the ideal $\mathscr{A}$.

We point out two properties of f and F, the first of which is already embedded in the previous proof.

Proposition 3.5 *Let I be an ideal as in Theorem* 3.2. *The Rees algebra $\mathscr{R}(I)$ has the following representation,*

$$\mathscr{R}(I) \simeq S_{k[\mathbf{T}]/(f)}(F) ,$$

where f is an irreducible polynomial and $F \neq 0$ is an ideal of $k[T]/(f)$, of linear type.

The last assertion implies that viewed as a $Q = k[\mathbf{T}]/(f)$-module, F satisfies the condition $\mathscr{F}_1$ – which means that in the ring $k[\mathbf{T}]$ the codimensions of the determinantal ideals of the matrix $B(\varphi)$ satisfy the estimates

$$\text{codimension } I_t(B(\varphi)) \geqq d - t + 2, \quad 1 \leqq t \leqq d - 1 .$$

Proposition 3.6 *Let I be an ideal as in Theorem 3.2. The localization*

$$\mathscr{R}(I)_{\mathbf{m}\mathscr{R}(I)}$$

is a discrete valuation domain, and $\mathscr{R}(I)$ satisfies (S_3).

Proof. By Proposition 3.3 we know that $\mathbf{m}\mathscr{R}(I)$ is a prime ideal, and by Proposition 3.5,

$$\mathscr{R}(I)_{\mathbf{m}\mathscr{R}(I)} \simeq S_Q(F)_{(\mathbf{x})} \simeq S_K(F \otimes_R K)_{(\mathbf{x})} ,$$

where K denotes the quotient field of Q. But now $S_K(F \otimes_R K)$ is a polynomial ring over K, and therefore $\mathscr{R}(I)_{\mathbf{m}\mathscr{R}(I)}$ is a regular local ring, necessarily of dimension one (cf. Proposition 2.4).

To prove that $\mathscr{R}(I)$ satisfies (S_3), let $\mathbf{q}$ be a prime ideal of $\mathscr{R}(I)$ such that $\mathscr{R}(I)_{\mathbf{q}}$ is not Cohen–Macaulay. We want to think of $\mathbf{q}$ as a prime ideal of $B = R[T_1, \ldots, T_{d+1}]$. By our assumption on I, $\mathbf{m} \subset \mathbf{q}$, hence $\mathbf{q} = (\mathbf{m}, \mathbf{p})B$ with $\mathbf{p} = \mathbf{q} \cap k[\mathbf{T}]$.

We first suppose that $\dim \mathscr{R}(I)_{\mathbf{q}} \leqq 3$, in which case $\dim k[\mathbf{T}]_{\mathbf{p}} \leqq 3$. Now $\mathbf{q}$ contains the characteristic polynomial f, and $\mathscr{R}(I)_{\mathbf{q}} \simeq (S_Q(F))_{\mathbf{q}}$ is not Cohen–Macaulay, hence $\mathbf{p} \in V(F) \subset \operatorname{Spec}(Q)$, where $\dim Q_{\mathbf{p}} \leqq 2$. Since by Proposition 3.5, F is a Q-ideal of linear type, we conclude that $F_{\mathbf{p}}$ is an almost complete intersection with $\dim Q_{\mathbf{p}}/F_{\mathbf{p}} \leqq 1$. Therefore the first Koszul homology of $F_{\mathbf{p}}$ is Cohen–Macaulay [2, Lemma]. Now $F_{\mathbf{p}}$ satisfies sliding depth and $\mathscr{F}_1$, which would force $(S_Q(F))_{\mathbf{q}}$ to be Cohen-Macaulay.

Finally assume that $\dim \mathscr{R}(I)_{\mathbf{q}} \geqq 4$. By Theorem 3.2 there is an exact sequence

$$0 \to \mathscr{A} = Sf \to S \to \mathscr{R}(I) \to 0$$

where $S = S(I)$. From the proof of Theorem 3.2, we further know that $\mathbf{m}S \cap Sf = 0$ and $f \notin \mathbf{m}S$. Therefore $0 :_S Sf = \mathbf{m}S$, and hence $Sf \simeq S/\mathbf{m}S \simeq k[\mathbf{T}]$. Since the $B_{\mathbf{q}}$-modules $k[\mathbf{T}]_{\mathbf{q}}$ and $S_{\mathbf{q}}$ both have depth at least 4, the above exact sequence implies that depth $\mathscr{R}(I)_{\mathbf{q}} \geqq 3$. □

Corollary 3.7. *Let I be an ideal as in Theorem 3.2. Then I is normal if and only if it is normal on the punctured spectrum of R.*

There remains to make f and F more explicit. It is not entirely clear how it can be accomplished. If $g = 2$, the module F has a presentation

$$k[\mathbf{T}]^d \xrightarrow{B(\varphi)} k[\mathbf{T}]^d \to F \to 0 .$$

Since f generates the annihilator of F, and $\det B(\varphi)$ annihilates F, and since both polynomials have the same degree, we conclude that

$$(f) = (\det B(\varphi)) ,$$

a somewhat curious way of obtaining irreducible polynomials! If g is arbitrary, we can only say that (f) is the divisorial closure of the $k[\mathbf{T}]$-ideal $I_d(B(\varphi))$.

3.3 Associated Graded Ring and Cohen–Macaulayness

For a given ideal I of codimension at least 1 in a Cohen–Macaulay ring R, the relationship between the Cohen–Macaulay properties of the Rees algebra $\mathscr{R}(I)$ and the associated graded ring $\operatorname{gr}_I(R)$ is very close. On one hand, there is the result of Huneke [10] asserting that the latter is Cohen–Macaulay. if $\mathscr{R}(I)$ is so. The converse, although not universally valid, holds in many interesting cases.

Theorem 3.8 *Let R be a Gorenstein local ring of dimension d and let I be a strongly Cohen–Macaulay ideal of codimension $g \geqq 2$. Suppose $v(I) = d + 1$ and I satisfies $\mathscr{F}_1$ on the punctured spectrum of R. If $\mathscr{A}$ is singly generated and $\operatorname{gr}_I(R)$ is Cohen–Macaulay, then $\mathscr{R}(I)$ is Cohen–Macaulay.*

Proof. By assumption and Theorem 2.5, we have an exact sequence

$$0 \to \mathscr{A} = Sf \to S \to \mathscr{R}(I) \to 0 ,$$

with f a homogeneous element of degree $d - g + 2$. From Theorem 2.5 it also follows that if φ is a minimal presentation matrix of I then $I_1(\varphi)$ is a Gorenstein

ideal and $I_1(\varphi)\cdot f = 0$. Since $Sf = \mathscr{A} \not\subset \mathbf{m}S$ and $\mathbf{m}S$ is the only associated prime of $I_1(\varphi)S$, it follows that $0:_S Sf = I_1(\varphi)$. Therefore

$$Sf \simeq \bar{S} = S(I) \otimes_R R/I_1(\varphi) \simeq R/I_1(\varphi)[T_1, \ldots, T_{d+1}] .$$

We denote $B = R[T_1, \ldots, T_{d+1}]$ and examine the exact sequence of graded B-modules

$$0 \to \bar{S} \to S \to \mathscr{R}(I) \to 0 .$$

It shows $\mathscr{R}(I)$ to be nearly Cohen–Macaulay as a B-module since it has depth at least d.

The cohomology sequence of canonical modules is

$$0 \to \omega_{\mathscr{R}(I)} \to \omega_S \xrightarrow{\varepsilon} \omega_{\bar{S}} \simeq \bar{S} \to \operatorname{Ext}_B^{d+1}(\mathscr{R}(I), B) \to 0 .$$

Now $\mathscr{R}(I)$ is Cohen–Macaulay if and only if $\operatorname{Ext}_B^{d+1}(\mathscr{R}(I), B)$ vanishes.

The canonical module of S, ω_S, has by Theorem 2.2, $g - 1$ generators, so that the image of ε is either the unit ideal of $\bar{S}$ or else a homogeneous ideal of $\bar{S}$ of codimension at most $g - 1$. The proof will be complete once it is shown that $\operatorname{Ext}_B^{d+1}(\mathscr{R}(I), B)$ is a module of Krull dimension zero. In particular this will occur if $\mathscr{R}(I)$ is Cohen–Macaulay on the punctured spectrum. The assertion will now follow from the following elementary observation.

Lemma 3.9 *Let R be a Noetherian local ring and let I be an ideal of positive grade. Suppose $\mathscr{R}(I)$ is Cohen–Macaulay on the punctured spectrum of R. If $\operatorname{gr}_I(R)$ is Cohen–Macaulay on the punctured spectrum then $\mathscr{R}(I)$ is also Cohen–Macaulay on the punctured spectrum (i.e., locally at every prime except for the maximal homogeneous ideal).*

Proof. We may assume that $I = (f_1, \ldots, f_n)$ with each f_i an R-regular element. Set $\mathscr{R}(I) = R[f_1 t, \ldots, f_n t]$; it suffices to prove that $\mathscr{R}(I)_{f_i t}$ is Cohen–Macaulay for each f_i. But f_i is a regular element of $\mathscr{R}(I)_{f_i t}$ and there is the canonical isomorphism

$$\mathscr{R}(I)_{f_i t}/(f_i)_{f_i t} \simeq \operatorname{gr}_I(R)_{f_i t} .$$

But the last ring is Cohen–Macaulay. □

3.4 Degree and Cohen–Macaulayness

Theorem 3.10 *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring and let I be a strongly Cohen–Macaulay ideal of codimension g having a linear presentation matrix. Suppose $v(I) = d + 1$ and I satisfies $\mathscr{F}_1$ on the punctured spectrum of R. Then I is generated by forms of degree δ with $\delta \leqq \dfrac{d}{g-1}$. Furthermore $\mathscr{R}(I)$ is Cohen–Macaulay if and only if $\delta = \dfrac{d}{g-1}$.*

Proof. We use the same exact sequence as in the proof of Proposition 3.6

$$0 \to \mathscr{A} = Sf \to S \to \mathscr{R}(I) \to 0 ,$$

where $S = S(I)$ is Cohen–Macaulay, f is a homogeneous polynomial of degree $d - g + 2$ in $k[T_1, \ldots, T_{d+1}]$, and $0:_S Sf = \mathbf{m}S$. The polynomial ring

$$B = k[x_1, \ldots, x_d, T_1, \ldots, T_{d+1}]$$

has also a natural grading given by $\deg x_i = 1$ and $\deg T_j = \delta$. In this grading, the characteristic polynomial f has degree $(d - g + 2)\delta$, and therefore

$$\mathscr{A} = Sf \simeq k[\mathbf{T}](-(d - g + 2)\delta) .$$

Thus

$$0 \to k[\mathbf{T}](-(d - g + 2)\delta) \to S \to \mathscr{R}(I) \to 0$$

is an exact sequence of graded B-modules, which induces the cohomology exact sequence

$$\operatorname{Ext}_B^d(\mathscr{R}(I), B) \to \operatorname{Ext}_B^d(S, B) \xrightarrow{\varepsilon} \operatorname{Ext}_B^d(k[\mathbf{T}](-(d - g + 2)\delta), B) \tag{4}$$

$$\simeq k[\mathbf{T}](d + (d - g + 2)\delta) \to \operatorname{Ext}_B^{d+1}(\mathscr{R}(I), B) \to 0 .$$

Notice that $\varepsilon \neq 0$ and that ε is surjective if and only if $\mathscr{R}(I)$ is Cohen–Macaulay. We are going to examine the degrees of the generators of $\operatorname{Ext}_B^d(S, B)$.

The approximation complex

$$0 \to Z_d \otimes B \to Z_{d-1} \otimes B \to \ldots \to Z_1 \otimes B \to S \to 0$$

is a homogeneous complex (with respect to the natural grading on B) that is exact. It can be used as in [7, proof of Theorem 6.5] to show that

$$\operatorname{Ext}_B^d(S, B) \otimes_B k \simeq \bigoplus_{i=d-g+2}^{d} \operatorname{Ext}_R^{d-i}(Z_i, R) \otimes_R k \simeq k((d + 1)\delta)^{g-1} ,$$

the degrees of the Z_i being given by the acyclic portion of the Koszul complex on a homogeneous minimal generating set of I.

Comparing generator degrees in (4) and keeping in mind that $\varepsilon \neq 0$ it follows that $(d + 1)\delta \leqq d + (d - g + 2)\delta$. Thus $\delta \leqq \dfrac{d}{g - 1}$, where equality holds if and only if ε is surjective. □

Once more Theorem 3.10 does not hold if I merely satisfies sliding depth, as can be seen from Example 2.6. There $\delta = 2 < \dfrac{d}{g - 1} = \frac{5}{2}$, but nevertheless, $\mathscr{R}(I)$ is Cohen–Macaulay.

Corollary 3.11 *With the assumptions of Theorem* 3.10, *the Rees algebra of I is Cohen–Macaulay if $g \leqq 4$ or if $d \leqq 2g - 2$ (in particular if* $\dim R/I \leqq 3$).

Proof. If $d \leqq 2g - 2$ then $\dfrac{d}{g - 1} \leqq 2 \leqq \delta$, and the claim follows from Theorem 3.10.

Next assume that $g \leqq 4$ and suppose that $\mathscr{R}(I)$ is not Cohen–Macaulay. We use the notation from the proof of Theorem 3.10. As $v(\operatorname{Ext}_B^d(S, B)) = g - 1 \leqq 3$, the exact sequence (4) implies that $\dim \operatorname{Ext}_B^{d+1}(\mathscr{R}(I), B) \geqq d - 2$. Hence there exists $\mathfrak{q} \in \operatorname{Spec}(\mathscr{R}(I))$ with $\dim \mathscr{R}(I)_{\mathfrak{q}} \leqq 3$ and $\mathscr{R}(I)_{\mathfrak{q}}$ not Cohen–Macaulay. Now Proposition 3.6 yields a contradiction. □

Corollary 3.12 *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring and let I be a strongly Cohen–Macaulay ideal of codimension $g \geqq 2$ having a linear presentation matrix. Write $n = v(I)$ and suppose $v(I_\wp) \leqq \dim R_\wp$ for every $\wp \in V(I)$ with $\dim R_\wp \leqq n - 2$. Then I is generated by forms of degree $\delta \leqq \dfrac{n-1}{g-1}$. In particular, $n \geqq 2g - 1$ unless R/I is regular. Furthermore, $\delta = \frac{n-1}{g-1}$ if $g \leqq 4$.*

Proof. We may assume that k is infinite and $g + 1 \leqq n \leqq d + 1$. Then, factoring out general linear forms, we reduce to the case $n = d + 1$. Now Theorem 3.10 and Corollary 3.11 imply the assertions.

Example 3.13 Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring and let I be a homogeneous perfect ideal of codimension g. Suppose that $v(I) = g + 1$ and $g \geqq 3$, or that $v(I) = g + 2, g \geqq 4$, and I is generically a complete intersection. Then I cannot have a linear presentation matrix. This assertion follows immediately from Corollary 3.12, since I is strongly Cohen–Macaulay by [3].

4 Criterion for second analytic deviation zero

We finish this paper by providing a criterion in terms of analytic spread, for when an ideal is generated by analytically independent elements.

Theorem 4.1 *Let R be a Cohen–Macaulay local ring and let I be an ideal of codimension at least 1 satisfying sliding depth. Suppose that for every $\wp \in V(I)$, either $v(I_\wp) \leqq \dim R_\wp$ or $\ell(I_\wp) \leqq \dim R_\wp - 1$. Then I is of linear type and $\mathscr{R}(I)$ is Cohen–Macaulay.*

Proof. First we may assume that the residue field of R is infinite. By [6, Theorem 6.1] it suffices to show that I satisfies $\mathscr{F}_1$. Proceeding by induction on $d = \dim R$, we may suppose that $\mathscr{F}_1$ holds locally on the punctured spectrum of R. Thus it remains to prove that $v(I) \leqq d$, under the assumption that $\ell(I) \leqq d - 1$. Let J be a minimal reduction of I, and let $K = J : I$, $\ell = \ell(I)$. If $J = I$, then $v(I) = v(J) = \ell \leqq d - 1$, and we are done. If however $J \neq I$, then codimension $K = d$ since I is of linear type locally on the punctured spectrum of R. Now I satisfies sliding depth, $J \subset I$ with $v(J) = \ell \geqq$ codimension I, and $K = J : I$ with codimension $K = d \geqq \ell$ and codimension $(I + K) = d \geqq \ell + 1$. In this situation, [8, Theorem 3.3] implies that codimension $K = \ell \leqq d - 1$, which is impossible. □

Corollary 4.2 *Let R be a Cohen–Macaulay local ring and let I be an ideal of codimension at least 1 satisfying sliding depth. Suppose that I is generically a complete intersection and satisfies one of the following conditions:*

(i) $\ell(I_\wp) \leqq \max\{\text{codimension } I_\wp, \dim R_\wp - 1\}$ *for every* $\wp \in V(I)$,
(ii) $\bar{I}$ *is normally torsionfree,*
(iii) $\bar{I^n}$, *the integral closure of* I^n, *has no embedded primes for every* n, *and* I *is primary,*
(iv) $\operatorname{gr}_I(R)_{\text{red}}$ *is a domain,*
(v) $\operatorname{gr}_I(R)$ *is reduced and* $\operatorname{projdim} I < \infty$.

Then I is of linear type and normally torsionfree, and $\mathscr{R}(I)$ is Cohen–Macaulay.

Proof. The assertions follow from Theorem 4.1 and [6, the proof of Theorem 6.1], since either of the conditions (ii) through (v) implies (i) (cf. [10, the proof of (2) $\Rightarrow$ (3) in Theorem 2.1] and [12, Corollary 1.10]). □

Theorem 4.1 also yields the following generalization of Proposition 2.4.

Corollary 4.3 *Let R be a Cohen-Macaulay local ring of dimension d and let I be an ideal of codimension at least 1 satisfying sliding depth. Suppose that $v(I) \geqq d + 1$ and that I satisfies $\mathscr{F}_1$ on the punctured spectrum of R. Then $\ell(I) = d$.*

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Crossed squares and combinatorial homotopy

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1 Introduction

A basic and generally difficult problem in algebraic topology is the computation of the set $[X, Y]$ of homotopy classes of maps from a space X to a space Y. One strategy for attacking the problem is to look for functors

$$\mathscr{F}\text{: (topological category)} \to \text{(algebraic category)}$$

that, under suitable hypotheses, reduce the computation to a feasible algebraic calculation via a bijection

$$[X, Y] \overset{\cong}{\to} [\mathscr{F}X, \mathscr{F}Y] .$$

For instance, if X and Y are pointed and connected CW-spaces and if X is 1-dimensional, then we can take $\mathscr{F}$ to be the fundamental group: $\pi_1 X$ is a free group, and the (base-point free) homotopy classes of pointed maps $X \to Y$ correspond to (conjugacy classes of) homomorphisms $\pi_1 X \to \pi_1 Y$.

In the 1940s Whitehead [19] introduced the algebraic notion of a crossed module to solve the problem when X, Y are connected CW-spaces and X is 2-dimensional. A crossed module is a group homomorphism having the general algebraic properties of the boundary homomorphism $\pi_2(W, A) \to \pi_1(A)$ arising from any pointed pair of spaces $A \subseteq W$. (See Sect. 2 for a precise definition.) Let X^1 and Y^1 denote the 1-skeleta of X and Y. Whitehead showed that the crossed module $\rho(X)$: $\pi_2(X, X^1) \to \pi_1(X^1)$ is "free" (because X is 2-dimensional), and that the homotopy classes of maps $X \to Y$ are in one-to-one correspondence with suitably defined homotopy classes of morphisms $\rho(X) \to \rho(Y)$ of crossed modules. This result reduces a number of interesting homotopy classifications to easy algebraic calculations (cf. [12]).

The notion of a crossed module admits an obvious generalisation, known as a crossed complex, that captures the general algebraic properties of the sequence of boundary homomorphisms

$$\to \pi_n(X^n, X^{n-1}) \to \cdots \to \pi_3(X^3, X^2) \to \pi_2(X^2, X^1) \to \pi_1(X^1)$$

arising from the skeleta X^n of a pointed CW-space. Unfortunately crossed complexes do not encode enough homotopy theoretic information to classify maps between, say, arbitrary 3-dimensional connected CW-spaces. Whitehead showed that they do, however, yield an algebraic classification of 3-dimensional connected CW-spaces: the homotopy types of such spaces are in one-to-one correspondence with the suitably defined homotopy types of 3-dimensional free crossed complexes.

In a recent book by Baues [2], Whitehead's methods have been extended to cover the homotopy classification of maps from a 3-dimensional connected CW-space X to some other connected CW-space Y, and also the homotopy classification of 4-dimensional connected CW-spaces. In this extended theory, crossed modules and crossed complexes are replaced by quadratic modules and quadratic complexes (see Sect. 4 for definitions).

The present paper is heavily influenced by Baues's book. We show how work of R. Brown, J.-L. Loday and others on generalisations of crossed modules leads to a variant of Baues's theory in which quadratic modules are replaced by the notion, introduced in [15], of a crossed square. This variant has several advantages:

(i) It has an obvious extension to higher dimensions in which crossed n-cubes (as defined in [14]) are the appropriate generalisation of crossed squares. The merits of such an extension are, however, unclear as the algebra involved is extremely complicated.
(ii) Crossed squares have a geometric interpretation in terms of relative homotopy groups; no such interpretation is available for quadratic modules.
(iii) Baues's theory is related via a quotient functor q_3: (free crossed squares) $\to$ (free quadratic modules).
(iv) Non-free groups of operators can be handled. One consequence of this is a new interpretation of the third homology of a group.

2 Crossed modules and crossed squares

A *crossed module* is a group homomorphism $\partial: M \to P$ with a left action of P on M satisfying $\partial({}^p m) = p(\partial m)p^{-1}$ and ${}^{(\partial m)}m' = mm'm^{-1}$ for all $m, m' \in M$, $p \in P$. A *morphism* of crossed modules, $\phi: \partial \to \partial'$, is a commutative square of group homomorphisms

$$\begin{array}{ccc} M & \xrightarrow{\phi_M} & M' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P & \xrightarrow{\phi_P} & P' \end{array}$$

satisfying $\phi_M({}^p m) = {}^{\phi_P(p)}\phi_M(m)$ for all $m \in M$, $p \in P$.

The principal topological example of a crossed module arises from a pointed pair of spaces $A \subseteq X$: the boundary homomorphism $\partial: \pi_2(X, A) \to \pi_1(A)$ is a crossed module. Of particular interest is the case when X is a 2-dimensional connected CW-space and $A = X^1$ its 1-skeleton; the associated crossed module is then, in an appropriate sense, free [18, 5, 17]. (The definition of "free" and a proof are recalled below.)

A *crossed square* is defined in [15] to be a commutative square of groups

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & N \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\nu} \\ M & \xrightarrow{\mu} & P \end{array}$$

together with left actions of P on L, M, N and a function $h\colon M \times N \to L$. Let M and N act on M, N and L via P. The structure must satisfy the following axioms for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$, $p \in P$:

1) the homomorphisms μ, ν, λ, λ' and $\mu\lambda$ are crossed modules and both λ and λ' are P-equivariant;
2) $h(mm', n) = h({}^m m', {}^m n)h(m, n)$, $h(m, nn') = h(m, n)h({}^n m, {}^n n')$;
3) $\lambda h(m, n) = m^n m^{-1}$, $\lambda' h(m, n) = {}^m n n^{-1}$;
4) $h(\lambda\ell, n) = \ell^n \ell^{-1}$, $h(m, \lambda'\ell) = {}^m \ell \ell^{-1}$;
5) $h({}^p m, {}^p n) = {}^p h(m, n)$.

A *morphism* of crossed squares, $\phi\colon \begin{pmatrix} L & N \\ M & P \end{pmatrix} \to \begin{pmatrix} L' & N' \\ M' & P' \end{pmatrix}$, consists of four group homomorphisms $\phi_L\colon L \to L'$, $\phi_M\colon M \to M'$, $\phi_N\colon N \to N'$, $\phi_P\colon P \to P'$ such that: the resulting cube of group homomorphisms is commutative; $\phi_L h(m, n) = h(\phi_M m, \phi_N n)$ for $m \in M$, $n \in N$; each of the homomorphisms ϕ_L, ϕ_M, ϕ_N is ϕ_P-equivariant.

The principal topological example of a crossed square arises from a triad of pointed spaces $A \subseteq X$, $B \subseteq X$: the square of boundary homomorphisms

$$\begin{array}{ccc} \pi_3(X; A, B) & \to & \pi_2(B, A \cap B) \\ \downarrow & & \downarrow \\ \pi_2(A, A \cap B) & \to & \pi_1(A \cap B) \end{array}$$

is a crossed square. The triad homotopy group $\pi_3(X; A, B)$ consists of homotopy classes of pointed triad maps $(E^3; E^2_+, E^2_-) \to (X; A, B)$, where E^3 denotes the solid unit 3-ball, and E^2_+, E^2_- are the surfaces of the upper and lower hemispheres of E^3. The function $h\colon \pi_2(A, A \cap B) \times \pi_2(B, A \cap B) \to \pi_3(X; A, B)$ is the triad Whitehead product. (See [1] for precise details on homotopy groups of triads).

Of particular interest is the case when the triad $(X; A, B)$ arises from a 3-dimensional connected CW-space X as follows. We shall assume that the CW-structure on X has only one 0-cell, and that this is the base-point; these two assumptions are captured by saying that X is *reduced*. (Throughout this article we shall regularly make the implicit assumption that a given space possess a reduced CW-structure.) Let X^k denote the k-skeleton of X, and let S be the set of 2-cells of X. Each 2-cell $e \in S$ has an attaching map $\alpha_e\colon S^1 \to X^1$. These attaching maps combine to produce a map $\alpha\colon \bigvee_{e \in S} S^1 \to X^1$. Here: $\bigvee_{e \in S} S^1$ denotes a wedge of circles, one circle for each 2-cell. We denote by 0: $\bigvee_{e \in S} S^1 \to X^1$ the map that sends everything to the base point. Set $W = X^1 \vee (\bigvee_{e \in S} S^1)$; thus W is a wedge of circles corresponding to the 1-cells and 2-cells of X. We have two maps

$$(1, \alpha)\colon W \to X^1,$$

$$(1, 0)\colon W \to X^1,$$

defined by requiring that both restrict to the identity on X^1, and that they restrict to α and 0 respectively on $\bigvee_{e\in S} S^1$. By use of mapping cylinders these maps can be converted into cofibrations

$$(1, \alpha)\colon W \to X^1_+ ,$$

$$(1, 0)\colon W \to X^1_- ,$$

with homotopy equivalences $X^1 \simeq X^1_+$ and $X^1 \simeq X^1_-$. The amalgamated sum $X^1_+ \cup_W X^1_-$ is homotopy equivalent to X^2.

We shall say that X is a *canonical* CW-space if: (i) X^2 equals $X^1_+ \cup_W X^1_-$ and; (ii) the characteristic map of each 3-cell is a triad map $(E^3; E^2_+, E^2) \to (X^3; X^1_+, X^1_-)$. Every reduced CW-space is clearly homotopy equivalent to a canonical one with the same number of cells; such an equivalence can be constructed functorially. In view of this we shall simplify matters by requiring henceforth that all *reduced* CW-spaces are canonical. (This slightly strengthens the previous definition of "reduced.") Note that X^1_+ and X^1_- are not generally CW-subspaces of a canonical CW-space X. (As an illustration consider the CW-decomposition of the 2-sphere $X = S^2$ with one 0-cell and one 2-cell; the subspaces X^1_+, X^1_- are the upper and lower hemispheres in this case.)

The triad $(X; X^1_+, X^1_-)$ gives rise to a crossed square, and thus we have a functor

$$\rho\colon (3\text{-dimensional reduced CW-spaces}) \to (\text{crossed squares}).$$

We shall show that the crossed square $\rho(X)$ is, in an appropriate sense, free.

It will help to have the notion of a *precrossed module*, by which we mean a group homomorphism $\delta\colon M \to P$ with a left action of P on M satisfying $\delta({}^p m) = p(\delta m)p^{-1}$ for $m\in M$, $p\in P$. Clearly any crossed module is a precrossed module. Moreover, to any precrossed module $\delta\colon M \to P$ we can associate a crossed module $\delta^{\mathrm{cr}}\colon M^{\mathrm{cr}} \to P$; we let $\langle M, M\rangle$ be the normal subgroup of M generated by the *Peiffer commutators*

$$\langle m, m'\rangle = mm'm^{-1}({}^{\delta m}m')^{-1}$$

for $m, m'\in M$, set $M^{\mathrm{cr}} = M/\langle M, M\rangle$, and let δ^{cr} be the induced homomorphism.

We say that the precrossed module $\delta\colon M \to P$ is *free* on a function $\sigma\colon S \to P$ from a set S into P if: S is a subset of M with σ the restriction of δ; and given any other precrossed module $\delta'\colon M' \to P$, any function $\tau\colon S \to M'$ satisfying $\delta'\tau = \sigma$ extends uniquely to a P-equivariant homomorphism $\phi\colon M \to M'$ satisfying $\delta'\phi = \delta$. There is clearly, up to isomorphism, only one free precrossed module on σ; we can take M to be the free group on the set $P \times S$, the action of P on M defined on generators by ${}^p(p', x) = (pp', x)$ for $p, p'\in P$, $x\in S$; the homomorphism $\delta\colon M \to P$ is defined on generators by $\delta(p, x) = p(\sigma x)p^{-1}$. The associated crossed module $\delta^{\mathrm{cr}}\colon M^{\mathrm{cr}} \to P$ is the *free* crossed module on σ.

We shall now define the *free* crossed square *on a pair of functions* (σ_2, σ_3). Let S_1, S_2 and S_3 be sets. Take F to be the free group on S_1, and suppose given a function $\sigma_2\colon S_2 \to F$. Let $\delta\colon M \to F$ be the free precrossed module on σ_2. Using the action of F on M we can form the semi-direct product $M \rtimes F$. The canonical inclusion on $M \hookrightarrow M \rtimes F$, $m \mapsto (m, 1)$ allows us to consider M as a normal subgroup of $M \rtimes F$. (Note that any normal inclusion is a crossed module with action given by conjugation.) There is a second normal subgroup of $M \rtimes F$ arising from M, namely

$$\bar{M} = \{(m, x)\in M \rtimes F\colon \delta m = x^{-1}\} .$$

For $m \in M$ we let $\bar{m}$ denote the element $(m^{-1}, \delta m)$ in $\bar{M}$. Suppose given a function $\sigma_3 \colon S_3 \to M$ whose image lies in the kernel of the homomorphism $\delta \colon M \to F$. There is then a corresponding function $\bar{\sigma}_3 \colon S_3 \to \bar{M}$, $y \mapsto (\sigma_3 y, 1)$. We define the *free* crossed square on the pair of functions (σ_2, σ_3) to be the crossed square

$$\begin{array}{ccc} L & \xrightarrow{\lambda'} & \bar{M} \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\bar{\mu}} \\ M & \xrightarrow{\mu} & M \rtimes F \end{array}$$

where μ, $\bar{\mu}$ are the normal inclusions, and where:

(i) S_3 is a subset of L with σ_3 and $\bar{\sigma}_3$ the restrictions of λ and λ' respectively;
(ii) for any other crossed square of the form

$$\begin{array}{ccc} K & \xrightarrow{\kappa'} & \bar{M} \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\bar{\mu}} \\ M & \xrightarrow{\mu} & M \rtimes F \end{array},$$

any function $\tau \colon S_3 \to K$ satisfying $\kappa\tau = \lambda$ extends uniquely to a morphism of crossed squares $\begin{pmatrix} L & \bar{M} \\ M & M \rtimes F \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} K & \bar{M} \\ M & M \rtimes F \end{pmatrix}$ over the identity on M, $\bar{M}$ and $M \rtimes F$. Clearly the free crossed square is uniquely determined, up to isomorphism, by the functions $\sigma_2 \colon S_2 \to F$ and $\sigma_3 \colon S_3 \to M$.

A neat algebraic description of the group L in the free crossed square can be derived from [9, Proposition 5.4]. To explain this description, we need to recall details on a (nonabelian) tensor product of groups and on coproducts of crossed modules.

Let $\mu \colon M \to P$ and $\nu \colon N \to P$ be crossed modules. The groups M and N act on each other, and on themselves, via the action of P. The *tensor product* $M \otimes N$ is the group generated by symbols $m \otimes n$ for $m \in M$, $n \in N$ subject to the relations

$$mm' \otimes n = ({}^{m}m' \otimes {}^{m}n)(m \otimes n) ,$$

$$m \otimes nn' = (m \otimes n)({}^{n}m \otimes {}^{n}n') ,$$

for $m, m' \in M$, $n, n' \in N$. There are homomorphisms $\lambda \colon M \otimes N \to M$, $\lambda' \colon M \otimes N \to N$ defined on generators by $\lambda(m \otimes n) = m^{n}m$ and $\lambda'(m \otimes n) = {}^{m}nn^{-1}$. The group P acts on $M \otimes N$ by ${}^{p}(m \otimes n) = ({}^{p}m \otimes {}^{p}n)$, and there is a function $h \colon M \times N \to M \otimes N$, $(m, n) \mapsto m \otimes n$. It is verified in [8] that this structure is a crossed square $\begin{pmatrix} M \otimes N & N \\ M & P \end{pmatrix}$.

The *coproduct* of $\mu \colon M \to P$ and $\nu \colon N \to P$ (in the category of crossed modules with codomain P) is a crossed module $\partial \colon M \circ N \to P$. It is the crossed module associated to the precrossed module $\delta \colon M \rtimes N \to P$, $(m, n) \mapsto (\mu m)(\nu n)$, where P acts on the semi-direct product $M \rtimes N$ by ${}^{p}(m, n) = ({}^{p}m, {}^{p}n)$. That is, $M \circ N = (M \rtimes N)^{\mathrm{cr}} = M/\langle M \rtimes N, M \rtimes N \rangle$ and $\partial = \delta^{\mathrm{cr}}$. It is shown in [4] that the subgroup $\langle M \rtimes N, M \rtimes N \rangle$ is generated by the elements $(m^{n}m^{-1}, n^{m}n^{-1})$ with $m \in M$, $n \in N$. Note that the canonical inclusions $M \hookrightarrow M \rtimes N$, $N \hookrightarrow M \rtimes N$ induce homomorphisms $i \colon M \to M \circ N$, $j \colon N \to M \circ N$.

The following easy lemma can be verified directly or obtained as a special case of [9, Proposition 5.4].

Lemma 1 *Let* $\begin{pmatrix} L & \bar{M} \\ M & M \rtimes F \end{pmatrix}$ *be the free crossed square on functions* (σ_2, σ_3) *as described above. Let* $\partial: C \to M \rtimes F$ *be the free crossed module on the function* $S_3 \to M \rtimes F, x \mapsto (\sigma_3 x, 1)$. *Note that the image of* ∂ *lies in* $M \cap \bar{M}$. *Form the crossed module* $M \otimes \bar{M} \to M \rtimes F$. *Then* L *is isomorphic to the coproduct* $(M \otimes \bar{M}) \circ C$ *factored by the relations*

(i) $\iota(\partial c \otimes \bar{m}) = \jmath(c)\jmath({}^{\bar{m}}c^{-1})$,
(ii) $\iota(m \otimes \partial c) = \jmath({}^{m}c)\jmath(c^{-1})$,

for $c \in C$, $m \in M$. *The homomorphisms* $L \to M$, $L \to \bar{M}$ *are determined by the homomorphisms* $\lambda: M \otimes \bar{M} \to M$, $\lambda': M \otimes \bar{M}$ *and* $\partial: C \to M \cap \bar{M}$. *The h-function of the crossed square is given by* $h(m, \bar{n}) = \iota(m \otimes \bar{n})$ *for* $m, n \in M$.

Note that in the situation of this lemma, the image in M of a generator $m \otimes \bar{n} = m \otimes (n^{-1}, \delta n)$ of $M \otimes \bar{M}$ is a conjugate of a Peiffer commutator:

$$\lambda\iota(m \otimes \bar{n}) = mn^{-1}({}^{\delta n}m^{-1})n = n^{-1}\langle n, m\rangle n\,.$$

Thus $M/\lambda\iota(M \otimes \bar{M}) = M^{\mathrm{cr}}$ is a free crossed module.

A 3-dimensional reduced CW-space X determines a pair of functions (σ_2, σ_3) as follows. For $i = 1, 2, 3$ let S_i be the set of i-cells of X. Take F to be the free group on S_1. The boundary of each 2-cell $e \in S_2$ determines a word $\sigma_2(e)$ in F. We thus have a function $\sigma_2: S_2 \to F$. Let $\delta: M \to F$ be the free precrossed module on σ_2. As before, let

$$W = X^1 \vee \left(\bigvee_{e \in S_2} S^1 \right)$$

be a wedge of circles, one for each 1-cell and one for each 2-cell. The fundamental group of W can be described as the semi-direct product $\pi_1 W = M \rtimes F$ constructed using the action of F on M. Each 3-cell $e^3 \in S_3$ is attached via a triad map $(S^2; E^2_+, E^2_-) \to (X^2; X^1_+, X^1_-)$ where E^2_+ and E^2_- are the upper and lower hemispheres of S^2. Thus each e^3 determines a map $S^1 = E^2_+ \cap E^2_- \to X^1_+ \cap X^1_- = W$, and hence a word $\sigma_3(e^3)$ in $M \subseteq \pi_1 W$. This defines a function $\sigma_3: S_3 \to M$.

Theorem 2 *Let* (σ_2, σ_3) *be the pair of functions determined by a 3-dimensional reduced* CW*-space* X. *Then the crossed square* $\rho(X)$ *arising from the triad* $(X; X^1_+, X^1_-)$ *is free on* (σ_2, σ_3).

Proof. Let $0: \varnothing \to M$ denote the inclusion of the empty set into M. We can show that the crossed square

$$\begin{array}{ccc} \pi_3(X^2; X^1_+, X^1_-) & \to & \pi_2(X^1_+, W) \\ \downarrow & & \downarrow \\ \pi_2(X^1_-, W) & \to & \pi_1(W) \end{array}$$

is free on the functions $(\sigma_2, 0)$; clearly $M = \pi_2(X^1_+, W)$ and $\bar{M} = \pi_2(X^1_-, W)$; freeness follows immediately from the van Kampen Theorem for crossed squares [9, Theorem 4.1] applied to the triad $(X^2; X^1_+, X^1_-)$. (Thus $\pi_3(X^2; X^1_+, X^1_-) \cong \pi_2(X^1_+, W) \otimes \pi_2(X^1_-, W)$.) The theorem now follows from [9, Proposition 6.5] applied to the triad $(Z; U, V) = (X^2; X^1_+, X^1_-)$ and $Z \cup \{e^3_k\} = X$. □

From this theorem and the exact sequence

$$\pi_3(X^2; X^1_+, X^1_-) \xrightarrow{\lambda} \pi_2(X^1_+, W) \to \pi_2(X^2, X^1) \to 0$$

we can deduce that the crossed module $\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1)$ is free: we know that $M = \pi_2(X^1_+, W)$ is a free precrossed module; the image of λ was shown above to be generated by Peiffer commutators.

Example 1 The 2-sphere $X = S^2$ has a CW-decomposition consisting of one 0-cell and one 2-cell. Thus X^1 is a point, and X^1_+, X^1_- are say the northern and southern hemispheres. Let t denote the 2-cell. We can use Lemma 1 and Theorem 2 to determine the associated crossed square $\rho(S^2)$. We find that $F = 1$ is the trivial group, and $\sigma_2\colon \{t\} \to 1$ the trivial function. It follows that $M = \bar{M} = C_\infty$ is the infinite cyclic group generated by t. Hence L is equal to the (usual abelian) tensor product $C_\infty \otimes C_\infty = C_\infty$. Thus

$$\rho(S^2) = \begin{array}{ccc} C_\infty & \xrightarrow{0} & C_\infty \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ C_\infty & \xrightarrow{\mathrm{id}} & C_\infty \end{array}$$

with $h(t^i, t^j) = t^{ij}$, and trivial action of C_∞. Here "0" denotes the trivial homomorphism, and "id" the identity homomorphism.

Example 2 A wedge $X = \bigvee_{1 \leqq i \leqq k} S^2$ of k 2-spheres has a CW-decomposition consisting of one 0-cell and k 2-cells. Let P be the free group of rank k. The preceding example generalises to give

$$\rho(X) = \begin{array}{ccc} P \otimes P & \xrightarrow{[\,,\,]} & P \\ {\scriptstyle [\,,\,]}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ P & \xrightarrow{\mathrm{id}} & P \end{array}$$

with $h(p, p') = p \otimes p'$ for $p, p' \in P$, and with P acting on itself by conjugation. The homomorphism $[\,,\,]$ sends a generator $p \otimes p'$ to $[p, p'] = pp'p^{-1}p'^{-1}$.

Example 3 The real projective space $X = \mathbb{R}P_3$ has a CW-decomposition consisting of one n-cell e^n for $0 \leqq n \leqq 3$. For convenience we write $x = e^1$, $t = e^2$ and $r = e^3$. In this case $F = C_\infty$ is the infinite cyclic group generated by x. The group M is freely generated by the symbols (x^i, t) for $i \in \mathbb{Z}$. The homomorphism $\delta\colon M \to F$ sends $(1, t) \mapsto x^2$ and more generally, because it is equivariant, $(x^i, t) \mapsto x^{2i}$. The attaching map of r is represented by a generator of the infinite cyclic group $\pi_2(X^2) = \ker(\pi_2(X^2, X^1) \to \pi_1(X^1)) = \ker(\delta^{\mathrm{cr}}\colon M^{\mathrm{cr}} \to F)$. The element $(1, t)(x, t)^{-1}$ in M is readily seen to represent such a generator. So we can take $\sigma_3\colon \{r\} \to M$ to be the function $r \mapsto (1, t)(x, t)^{-1}$. The group L is a little complicated. The following remark shows that in practice no description of L is necessary.

A morphism

$$\phi\colon \begin{pmatrix} L & \bar{M} \\ M & M \rtimes F \end{pmatrix} \to \begin{pmatrix} L' & \bar{M}' \\ M' & M' \rtimes F' \end{pmatrix}$$

from the free crossed square on (σ_2, σ_3) to another free crossed square is uniquely determined by, and uniquely determines, three functions $\tau_1\colon S_1 \to F'$, $\tau_2\colon S_2 \to M'$, $\tau_3\colon S_3 \to L'$ for which the following two squares commute:

$$\begin{array}{ccc} S_2 & \xrightarrow{\tau_2} & M' \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta'} \\ F & \xrightarrow[\tau_1^*]{} & F' \end{array} \qquad \begin{array}{ccc} S_3 & \xrightarrow{\tau_3} & L' \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda'} \\ M & \xrightarrow[\tau_2^*]{} & M' \end{array}.$$

Here τ_1^* is the unique group homomorphism extending τ_1, and τ_2^* is the unique τ_1^*-equivariant homomorphism extending τ_2 and satisfying $\delta'\tau_2^* = \tau_1^*\delta$. Thus, all of the homotopy theoretic information in the crossed square of a 3-dimensional reduced CW-space is captured by the associated functions σ_2 and σ_3. An explicit description of L is not essential.

3 Squared complexes

In [14] crossed squares are generalised to crossed n-cubes. In the present paper we only have need of a special type of crossed n-cube, which we shall call a *squared complex*; it consists of a diagram of group homomorphisms

$$\cdots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} L \begin{array}{ccc} & \overset{\lambda'}{\nearrow} & N \\ & & \searrow^{\nu} \\ & \underset{\lambda}{\searrow} & & \nearrow_{\mu} \\ & & M \end{array} P \quad (*)$$

together with a left action of P on L, M, N and C_n for $n \geqq 4$, and a function $h\colon M \times N \to N$. The following axioms need to be satisfied.

(i) The square $\begin{pmatrix} L & N \\ M & P \end{pmatrix}$ is a crossed square.
(ii) The group C_n is abelian for $n \geqq 4$.
(iii) The boundary homomorphisms satisfy $\partial_n\partial_{n+1} = 0$ for $n \geqq 4$, and $\partial_4(C_4)$ lies in the intersection $\ker\lambda \cap \ker\lambda'$.
(iv) The action of P on C_n, $n \geqq 4$, is such that μM and νN act trivially. Thus each C_n is a π_1-module, with $\pi_1 = P/(\mu M)(\nu N)$.
(v) The homomorphisms ∂_n are π_1-equivariant for $n \geqq 4$. (This does make sense for $n = 4$ since the axioms of a crossed square imply that $\ker\lambda \cap \ker\lambda'$ is a π_1-module.)

A *morphism* of squared complexes $\phi\colon \left(C_n, \begin{pmatrix} L & N \\ M & P \end{pmatrix}\right) \to \left(C'_n, \begin{pmatrix} L' & N' \\ M' & P' \end{pmatrix}\right)$ consists of a morphism of crossed squares $\phi_L, \phi_M, \phi_N, \phi_P$ together with a family of ϕ_P-equivariant homomorphisms ϕ_n, $n \geqq 4$, satisfying $\phi_L\partial_4 = \partial'_4\phi_4$ and $\phi_{n-1}\partial_n = \partial'_n\phi_n$ for $n \geqq 5$.

The principal topological example of a squared complex is the diagram of boundary homomorphisms

$$\cdots \to \pi_5(X_5, X_4) \to \pi_4(X_4, X_3) \to \pi_3(X_3; A, B) \begin{array}{ccc} & \nearrow & \pi_2(B, A \cap B) \\ & & \searrow \\ & \searrow & & \nearrow \\ & & \pi_2(A, A \cap B) \end{array} \pi_1(A \cap B)$$

arising from a triad of pointed spaces $(X_3; A, B)$ and a filtered family $X_3 \subseteq X_4 \subseteq X_5 \subseteq \ldots$ of subspaces of a space X. Of particular interest is the case when X is a reduced CW-space, $X_n = X^n$ is the n-skeleton, and $A = X^1_+$, $B = X^1_-$. We denote the resulting squared complex by $\rho(X)$. We have already shown that the crossed square part of $\rho(X)$ is free. The Hurewicz isomorphisms

$\pi_n(X^n, X^{n-1}) \cong H_n(\tilde{X}^n, \tilde{X}^{n-1})$ for $n \geqq 4$ show that the remaining groups $\pi_n(X^n, X^{n-1})$, $n \geqq 4$, are free $\pi_1(X)$-modules with bases the set of n-cells. We thus have a functor

$$\rho\text{: (reduced CW-spaces)} \to \text{(free squared complexes)}$$

where, by a *free* squared complex, we mean one in which the crossed square is free, and in which the modules C_n are free π_1-modules for $n \geqq 4$.

The *homotopy groups* $\pi_n(\rho)$, $n \geqq 1$, of the squared complex $(*)$ are defined to be the homology groups of the complex

$$\cdots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} L \xrightarrow{\partial_3} M \rtimes N \xrightarrow{\partial_2} P \to 1$$

with $\partial_3(\ell) = (\lambda\ell^{-1}, \lambda'\ell)$ and $\partial_2(m, n) = (\mu m)(\nu n)$. The axioms of a crossed square ensure that ∂_3 and ∂_2 are homomorphisms with $\partial_4(C_4)$ normal in $\ker(\partial_3)$, $\partial_3(L)$ normal in $\ker(\partial_2)$, and $\partial_2(M \rtimes N)$ normal in P. Explicitly $\pi_n(\rho) = \ker \partial_n / \operatorname{im} \partial_{n+1}$, with ∂_1 the trivial homomorphism. Note that the homotopy groups $\pi_n(\rho X)$ of the squared complex $\rho X = \left(C_n, \begin{pmatrix} L & \bar{M} \\ M & M \rtimes F \end{pmatrix}\right)$ are the homology groups of the complex

$$\cdots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} L \xrightarrow{\lambda} M \xrightarrow{\delta} F \to 1\,.$$

Theorem 3 *Let X be a reduced* CW*-space with universal cover $\tilde{X}$. Then*

$$\pi_n(\rho X) = \begin{cases} \pi_n X & \text{for } 1 \leqq n \leqq 3\,, \\ h_4(\pi_4 X) & \text{for } n = 4\,, \\ H_n(\tilde{X}) & \text{for } n \geqq 5\,, \end{cases}$$

where $h_4\colon \pi_4 X \cong \pi_4 \tilde{X} \to H_4 \tilde{X}$ is the Hurewicz homomorphism.

Proof. The homotopy groups $\pi_n(\rho X)$ are the homology groups of the complex

$$\cdots \pi_4(X^4, X^3) \xrightarrow{\partial_4} \pi_3(X^3; X^1_+, X^1_-) \xrightarrow{\lambda} \pi_2(X^1_+, W) \xrightarrow{\delta} \pi_1(X^1_-) \to 1\,.$$

The theorem is obvious for $n \geqq 5$ because of the Hurewicz isomorphisms $\pi_n(X^n, X^{n-1}) \cong H_n(\tilde{X}^n, \tilde{X}^{n-1})$. For $n = 1, 2$ it follows from the isomorphism $\pi_2(X^3) \cong \pi_2(X)$ and the following diagram of exact rows and columns:

$$\begin{array}{ccccccccc} & & & & \pi_3(X^3; X^1_+, X^1_-) & & & & \\ & & & & \downarrow{\scriptstyle \lambda} & & & & \\ & & & & \pi_2(X^1_+, W) & & & & \\ & & & & \downarrow & \searrow{\scriptstyle \delta} & & & \\ 1 & \to & \pi_2(X^3) & \to & \pi_2(X^3, X^1_-) & \to & \pi_1(X^1_-) & \to & \pi_1(X^3) \to 1\,. \\ & & & & \downarrow & & & & \\ & & & & 1 & & & & \end{array}$$

For $n = 3$ the theorem follows from the isomorphism $\pi_3(X^4) \cong \pi_3(X)$ and the following diagram of exact rows and columns:

$$\begin{array}{ccccccc} & & \pi_4(X^4, X^3) & & & & \\ & & \downarrow & \searrow^{\partial_4} & & & \\ 1 \to \pi_3(X^3) & \to & \pi_3(X^3; X^1_+, X^1_-) & & \xrightarrow{\lambda} & & \pi_2(X^1_+, W)\,. \\ & & \downarrow & & & & \\ & & \pi_3(X^4) & & & & \\ & & \downarrow & & & & \\ & & 1 & & & & \end{array}$$

For $n = 4$ the theorem follows from the isomorphism $\pi_4(X^5) \cong \pi_4(X)$, the Hurewicz isomophisms $\pi_n(X^n, X^{n-1}) \cong H_n(\tilde{X}^n, \tilde{X}^{n-1})$ for $n = 3, 4, 5$, and the following diagram of exact rows and columns:

$$\begin{array}{ccccc} \pi_5(X^5, X^4) & & & & \\ \downarrow & \searrow^{\partial_5} & & & \\ \pi_4(X^4) & \to & \pi_4(X^4, X^3) & \xrightarrow{\partial_4} & \pi_3(X^3) \\ \downarrow & & \searrow & & \downarrow \\ \pi_4(X^5) & & & & \pi_3(X^3, X^2)\,. \\ \downarrow & & & & \\ 1 & & & & \end{array}$$

□

Note that Theorems 2 and 3 provide a combinatorial discription of the homotopy groups π_n of a reduced CW-space for $n = 1, 2, 3$. For $n = 1$ this description is of course nothing new, and for $n = 2$ the description was essentially given by Whitehead in [18]. For $n = 3$ the description is a variant of the one given by Baues in [2]. Note that we can now read off immediately the isomorphism $\pi_3(S^2) \cong \mathbb{Z}$ from the crossed square of Example 1.

Example 4. The real projective space $X = \mathbb{R}P_\infty$ has a CW-decomposition consisting of one cell e^n in each dimension $n \geqq 0$. The crossed square part of the squared complex ρX is described in Example 3. For $n \geqq 4$ the group $C_n = \pi_n(X^n, X^{n-1})$ is the free $\mathbb{Z}_2$-module on e^n. The 1-cell $x = e^1$ represents the generator of $\mathbb{Z}_2$. It is well-known that the boundary homomorphism $\partial_n\colon C_n \to C_{n-1}$ is given by $\partial_n(e^n) = e^{n-1} + (-1)^n({}^x e^{n-1})$ for $n \geqq 5$. To complete the algebraic description of ρX we need to identify the boundary homomorphism $\partial_4\colon C_4 \to L$. Note that

(i) $\partial_4(C_4)$ lies in $\pi_3(X^3) = \ker \lambda \subseteq L$, and that
(ii) the following diagram commutes:

$$\begin{array}{ccccc} H_4(\tilde{X}^4, \tilde{X}^3) \cong C_4 & \xrightarrow{\partial_4} & \pi_3(X^3) & \hookrightarrow & \pi_3(X^3; X^1_+, X^1_-) = L \\ & \searrow^{\partial} & \downarrow h_3 & & \downarrow \\ & & H_3(\tilde{X}^3) & \hookrightarrow & H_3(\tilde{X}^3, \tilde{X}^2)\,. \end{array}$$

The Hurewicz homomorphism h_3 is an isomorphism since $\pi_2(X) = 0$. Hence ∂_4 is characterised by (i) and (ii). As in Example 3, let $t = e^2$ and $r = e^3$. Then $(1, t)$ is a basis element of M, and $\overline{(x^2, t)} = ((x^2, t)^{-1}, x^2)$ is an element of $\bar{M}$. Recall that

there is a function $h: M \times \bar{M} \to L$. The element $w = h((1, t), \overline{(x^2, t)})$ lies in L. Moreover the element $r^x r w^{-1}$ lies in $\ker \lambda$ since

$$\begin{aligned}\lambda(r^x r w^{-1}) &= (1,t)(x,t)^{-1}(x,t)(x^2,t)^{-1}\lambda(w^{-1}) \\ &= (1,t)(x^2,t)^{-1}\lambda(w^{-1}) \\ &= (1,t)(x^2,t^{-1})((1,t)(x^2,t^{-1})) \\ &= 1\,.\end{aligned}$$

Therefore ∂_4 can be defined by $\partial_4(e^4) = r^x r w^{-1}$.

Further descriptions of $\rho(X)$ for specific CW-species X are obtainable by mimicking the techniques of [2, Appendix IVA] (though the formulas may be more complicated since we have imposed no nilpotency condition on squared complexes). In particular, quotients and direct products of spaces can be treated.

4 The third homology of a group and quadratic modules

Theorems 2 and 3 can be generalised to CW-spaces "whose 1-skeleton is the classifying space of a group." Let G be a group and $B = B(G)$ its (reduced CW-) classifying space. Let $B \subseteq X_2 \subseteq X_3 \ldots$ be a sequence of reduced CW-space inclusions with X_2 obtained from B by attaching 2-cells, and X_n obtained from X_{n-1} by attaching n-cells for $n \geqq 3$. Denote the union of this sequence by X. If G is a free group then B is just a wedge of circles, and $B = X^1$, $X_n = X^n$ are the usual CW-skeleta of X. In this free G case we have shown above how to construct a triad $(X_3; X^1_+, X^1_-) = (X_3; B_+, B_-)$. It is readily seen that the construction of $(X_3; B_+, B_-)$ does not need the freeness of G, and is valid for an arbitrary group G. The intersection $W = B_+ \cap B_-$ is the space $W = B \vee (\bigvee S^1)$ obtained from B by attaching a wedge of circles, one circle for each 2-cell in $X_2 \backslash B$. Thus X gives rise to a squared complex $\rho_B X$:

$$\cdots \to \pi_5(X_5, X_4) \to \pi_4(X_4, X_3) \to \pi_3(X_3; B_+, B_-) \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \pi_2(B_-, W) \\ \\ \pi_2(B_+, W) \end{matrix} \begin{matrix} \searrow \\ \\ \nearrow \end{matrix} \pi_1(W)\,.$$

The previous definitions of free precrossed module, free crossed square and free squared complex all extend automatically to the case where the free group of operators F is replaced by an arbitrary group G. Theorems 2 and 3 are easily generalised to show that the squared complex $\rho_B X$ is free (in the extended sense), and that

$$\pi_n(\rho_B X) = \begin{cases} \pi_n X & \text{for } 1 \leqq n \leqq 3\,, \\ \text{image}(\pi_4 X \to H_4(\tilde{X}, \tilde{B})) & \text{for } n = 4\,, \\ H_n(\tilde{X}, \tilde{B}) & \text{for } n \geqq 5\,. \end{cases}$$

Let S be the set of 2-cells in $X_2 \backslash B$. The boundary of each 2-cell e^2 in S determines an element $\sigma(e^2)$ in G, yielding a function $\sigma\colon S \to G$. Let $\delta\colon M \to G$ be the free precrossed module on σ. As before, the two normal subgroups $M, \bar{M}$ of $M \rtimes G$ act on each other by conjugation; using these actions we can construct the crossed module $\lambda\colon M \otimes \bar{M} \to M$, $m \otimes \bar{n} \mapsto n^{-1}\langle m, n\rangle n$. Since $\rho_B X_2$ is a free crossed square (in the extended sense) we have a new combinatorial description of $\pi_3(X_2)$:

$$\pi_3(X_2) = \ker(\lambda\colon M \otimes \bar{M} \to M)\,.$$

Whitehead's Certain Exact Sequence [20] for the space X_2 is

$$\to \pi_4(X_2) \to H_4(\tilde{X}_2) \to \Gamma(\pi_2(X_2)) \xrightarrow{\alpha} \pi_3(X_2) \to H_3(\tilde{X}_2) \to 0\,,$$

where Γ denotes Whitehead's Universal Quadratic Functor. For any (additive) abelian group π, $\Gamma(\pi)$ is the (additive) abelian group generated by the symbols $\gamma(x)$ for $x \in \pi$ subject to the relations

$$\gamma(x) = \gamma(-x)\,,$$

$$\gamma(x + y + z) + \gamma(x) + \gamma(y) + \gamma(z) = \gamma(x + y) + \gamma(x + z) + \gamma(y + z)\,,$$

for $x, y, z \in \pi$.

Since $\pi_2(X_2) = \ker(\delta^{\mathrm{cr}}\colon M^{\mathrm{cr}} \to G)$, each element x of $\pi_2(X_2)$ is the image of some (not unique) element $\hat{x}$ in $\ker(\delta\colon M \to G)$. By an argument similar to the one in [8, Lemma 4.2] we see that the homomorphism α sends a generator $\gamma(x)$ of $\Gamma(\pi_2(X_2))$ to the element $\hat{x} \otimes (\hat{x}, 1)$ in $\ker(\lambda\colon M \otimes \bar{M} \to M) = \pi_3(X_2)$. The image of α is readily seen to be normal in $M \otimes \bar{M}$. The *exterior product* of M and $\bar{M}$ is defined to be

$$M \wedge \bar{M} = \operatorname{coker}(\alpha)\,.$$

Thus $M \wedge \bar{M}$ is obtained from the tensor product $M \otimes \bar{M}$ by adding the relation $\hat{x} \otimes (\hat{x}, 1) = 1$ for each $\hat{x} \in \ker(\delta\colon M \to G)$. We let $m \wedge \bar{n}$ denote the image in $M \wedge \bar{M}$ of the tensor $m \otimes \bar{n}$. There is an induced homomorphism $\lambda\colon M \wedge \bar{M} \to M$, $m \wedge \bar{n} \mapsto n^{-1}\langle m, n\rangle n$. (In fact λ is a crossed module.) Whitehead's exact sequence yields the isomorphism

$$H_3(\tilde{X}_2) = \ker(\lambda\colon M \wedge \bar{M} \to M)\,.$$

It is a routine exercise to show that the group $M \wedge \bar{M}$ is isomorphic to the "Peiffer exterior product" defined in [10] and denoted there by $M \wedge_P M$. The hypothesis of Theorem 2 in [10] is that the homomorphism $\lambda\colon M \wedge \bar{M} \to M$ be injective. The only example of such a λ given in [10] is for G a free group. The following Theorem provides all the examples of such a λ. It involves the third integral homology $H_3(Q)$ of the group $Q = \delta(M)$.

Theorem 4 *Suppose that $\delta\colon M \to G$ is a free precrossed module. Let $Q = \delta(M)$. Then there is an isomorphism*

$$H_3(Q) \cong \ker(\lambda\colon M \wedge \bar{M} \to M)\,.$$

Proof. By the precrossed module analogue of [13, Proposition 1], the restricted precrossed module $\delta: M \twoheadrightarrow Q$ is free on some function $\sigma: S \to Q$. Let $B' = B(Q)$ the classifying space of Q. Let Y_2 be the space obtained from B' by attaching one 2-cell e^2 to B' for each element $s \in S$ in such a way that the attaching map of e^2 represents $\sigma(s)$. Then $H_3(\tilde{Y}_2) = \ker(\lambda: M \wedge \bar{M} \to M)$. Since $\pi_1(Y_2)$ is trivial we have $\tilde{Y}_2 = Y_2$. The homology exact sequence arising from the cofibration $B' \to Y_2$ implies $H_3(B') = H_3(Y_2)$. The theorem follows since $H_3(Q) = H_3(B')$. □

Miller [16] has given an interpretation of the second integral homology $H_2(G)$ of a group G as the "group of non-trivial identities among the commutators of G." In our notation Miller's result is the isomorphism $H_2(G) \cong \ker(G \wedge G \to G)$, $g \wedge g' \mapsto [g, g']$, (where $G \wedge G$ is obtained from $G \otimes G$ by adding the relation $g \otimes g = 1$ for each $g \in G$; G is assumed to act on itself is by conjugation in the definition of $G \otimes G$. cf. [8, 11]). Theorem 4 yields an analogous interpretation of the third homology of G. For let S be any subset of G that generates G, and let $\delta: M \to G$ be the free precrossed module on the inclusion $\sigma: S \hookrightarrow G$. Then, given the description of $M \wedge \bar{M}$ in [10], Theorem 4 states that $H_3(G)$ is the "group of non-trivial identities among the Peiffer commutators of M."

The following corollary of Theorem 4 generalises [2, Theorem 1.8] to non-free groups of operators; the proof given in [2] uses a Peiffer commutator version of the Magnus–Witt Theorem [3]. Given a precrossed module $\delta: M \to G$ we set $\pi_2 = \ker(\delta^{\mathrm{cr}}: M^{\mathrm{cr}} \to G)$, $Q = \delta(M)$, and $A = (M^{\mathrm{cr}})^{\mathrm{ab}}$. We let $A \otimes A$ denote the usual tensor product of abelian groups. As above, $\langle M, M \rangle$ denotes the subgroup of M generated by the Peiffer commutators. We define $\langle M, M, M \rangle = \langle M, \langle M, M \rangle \rangle \langle \langle M, M \rangle, M \rangle$ to be the subgroup of M generated by all Peiffer commutators of the form $\langle m, \langle m', m'' \rangle \rangle$ or $\langle \langle m, m' \rangle, m'' \rangle$ for $m, m', m'' \in M$. It is easily verified that $\langle M, M \rangle$ and $\langle M, M, M \rangle$ are both G-invariant normal subgroups of M.

Corollary 6. *Suppose that $\delta: M \to G$ is a free precrossed module. Let $Q = \delta(M)$. If Q^{ab} is free abelian and if the third integral homology of Q is trivial, then there is an exact sequence*

$$\Gamma(H_2(Q)) \xrightarrow{\beta} \Gamma(\pi_2) \xrightarrow{\alpha} A \otimes A \to M/\langle M, M, M \rangle \to M^{\mathrm{cr}} \to 1 .$$

Proof. We need to recall two easily verified results on the tensor product $M \otimes \bar{M}$ (cf. [7, Proposition 9 and Remark 2]). First, let N be a G-invariant normal subgroup of M, and $\bar{N}$ the corresponding subgroup of $\bar{M}$. (That is, $\bar{N} = \{(n^{-1}, \delta n): n \in N\}$.) Then the evident sequence of groups

$$(M \otimes \bar{N}) \oplus (N \otimes \bar{M}) \to M \otimes \bar{M} \to (M/N) \otimes (\bar{M}/\bar{N}) \to 1$$

is exact. Second, suppose D and E are groups that act trivially on each other and by conjugation on themselves. Then the (nonabelian) tensor product $D \otimes E$ is isomorphic to the usual tensor product $D^{\mathrm{ab}} \otimes E^{\mathrm{ab}}$ of abelian groups. As an application of these two results we have the exact sequence

$$(M \otimes \overline{\langle M, M \rangle}) \oplus (\langle M, M \rangle \otimes \bar{M}) \to M \otimes \bar{M} \to M^{\mathrm{cr}} \otimes \bar{M}^{\mathrm{cr}} \to 1 ,$$

and isomorphism $M^{\mathrm{cr}} \otimes \bar{M}^{\mathrm{cr}} \cong A \otimes A$.

The following commutative diagram of exact rows and columns implies the corollary except for $\ker(\alpha) = \mathrm{im}(\beta)$.

$$\begin{array}{ccccccc}
(M \otimes \overline{\langle M, M\rangle}) \oplus (\langle M, M\rangle \otimes \bar{M}) & \to & \langle M, M, M\rangle & \to & 1 & & \\
\downarrow & & \downarrow & & & & \\
\Gamma(\pi_2) \xrightarrow{\alpha} M \otimes \bar{M} & \longrightarrow & M & \longrightarrow & M^{\mathrm{cr}} & \to & 1 \\
\downarrow & & \downarrow & \nearrow & & & \\
A \otimes A & \longrightarrow & M/\langle M, M, M\rangle & & & & \\
\downarrow & & \downarrow & & & & \\
1 & & 1\,. & & & &
\end{array}$$

Exactness at $M \otimes \bar{M}$ of the central row follows from Theorem 4. To prove $\ker(\alpha) = \mathrm{im}(\beta)$ note that A is a free abelian group, and that the homomorphism $\alpha'\colon \Gamma(A) \to A \otimes A$, $\gamma(a) \mapsto a \otimes a$ sends the generators of $\Gamma(A)$ to part of a basis of $A \otimes A$. Thus α' is injective. The theorem in [13] provides the exact sequence

$$1 \to H_2(Q) \to \pi_2 \to A\,.$$

Since Q^{ab} is free abelian a basis of $\pi_2/H_2(Q)$ extends to a basis of A. Hence α' sends $\Gamma(\pi_2/H_2(Q))$ injectively into $A \otimes A$. The inclusion of $H_2(Q)$ into π_2 induces a homomorphism $\Gamma(H_2(Q)) \xrightarrow{\beta} \Gamma(\pi_2)$ whose cokernel is $\mathrm{coker}(\beta) = \Gamma(\pi_2/H_2(Q))$. It follows that $\ker(\alpha) = \mathrm{im}(\beta)$. □

Corollary 6 enables us to replace a free squared complex ρ by a quotient $q_3(\rho)$, in certain instances, without loosing homotopy theoretic information. Suppose that

$$\rho\colon \cdots \to C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} L \begin{array}{c} \overset{\bar{M}}{\overset{\lambda'\nearrow}{\underset{\lambda\searrow}{}}\ \ \overset{\searrow\bar{\mu}}{\underset{\nearrow\mu}{}}} \\ M \end{array} M \rtimes G$$

is a free squared complex (with G an arbitrary group). We denote by $h(M, M, M)$ the subgroup of L generated by the elements $h(m, \langle m', m''\rangle)$ and $h(\langle m, m'\rangle, \bar{m}')$ for $m, m', m'' \in M$. (We are using the notation $\bar{m} = (m^{-1}, \delta m)$ for $m \in M$.) It is easily verified that $h(M, M, M)$ is a G-invariant normal subgroup. The function $h\colon M \times \bar{M} \to L$ induces a function $\bar{h}\colon A \otimes A \to L/h(M, M, M)$. We denote by $q_3(\rho)$ the induced diagram of groups

$$\begin{array}{ccccccccc}
 & & & & A \otimes A & & & & \\
 & & & & \downarrow{\scriptstyle \bar{h}} & & & & \\
q_3(\rho)\colon \cdots & \to & C_5 & \xrightarrow{\partial_5} & C_4 & \xrightarrow{\bar{\partial}_4} & L/h(M, M, M) & \xrightarrow{\bar{\lambda}} & M/\langle M, M, M\rangle \xrightarrow{\bar{\delta}} G
\end{array}$$

together with the induced action of G on each group in the diagram. The structure $q_3(\rho)$ is, in the terminology of [2], a *free quadratic complex*. The *homotopy groups* of $q_3(\rho)$ are defined to be the homology of the horizontal complex of groups. That is, $\pi_1(q_3(\rho)) = G/\mathrm{im}(\bar{\delta})$, $\pi_2(q_3(\rho)) = \ker(\bar{\delta})/\mathrm{im}(\bar{\lambda})$, and so on.

Theorem 7 *Let ρ be a free squared complex (whose operator group is not necessarily free). For $n \neq 3, 4$ we have $\pi_n(\rho) = \pi_n(q_3(\rho))$. Let $Q = \delta(M)$. If Q^{ab} is free abelian and*

$H_3Q) = 0$, *then there is an exact sequence* $H_2(Q) \to \pi_3(\rho) \twoheadrightarrow \pi_3(q_3(\rho))$ *and injection* $\pi_4(\rho) \rightarrowtail \pi_4(q_3(\rho))$. *The injection is surjective if* $H_2(Q) = 0$.

Proof. For $n \neq 3, 4$ the theorem is a tautology. To prove the theorem for $n = 3, 4$ we consider the complex

$$q_2(\rho): \cdots \to C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\bar{\partial}_4} L/h(M, M) \xrightarrow{\bar{\lambda}} M/\langle M, M\rangle \xrightarrow{\bar{\delta}} G$$

where $h(M, M)$ is the subgroup of L generated by the elements $h(m, \bar{m}')$ for $m, m' \in M$. (Note that $q_2(\rho)$ is a crossed complex [2, 6]. The homotopy groups $\pi_n(q_2\rho)$ are defined to be the homology of the complex.) We also consider the kernels

$$K = \ker(\epsilon\colon M \otimes \bar{M} \to L)\,,$$

$$K' = \ker(\epsilon'\colon A \otimes A \to L/h(M, M, M))\,,$$

where ϵ is the canonical homomorphism described in Lemma 1, and ϵ' is induced by ϵ. The quotient homomorphism $M \otimes \bar{M} \to A \otimes A$ induces a surjection $K \twoheadrightarrow K'$. On taking homotopy groups, the evident diagram of complexes

$$\begin{array}{ccc} \rho & \twoheadrightarrow & q_2\rho \\ \downarrow & & \downarrow{=} \\ q_2\rho & \twoheadrightarrow & q_2\rho \end{array}$$

yields the commutative diagram of groups

$$\begin{array}{ccccccccc} 0 \to & \pi_4(\rho) & \longrightarrow & \pi_4(q_2\rho) \to \Gamma(\pi_2\rho)/K & \longrightarrow & \pi_3(\rho) & \longrightarrow & \pi_3(q_2\rho) \to & 0 \\ & \downarrow & & \downarrow{\mathrm{id}} & \downarrow & \downarrow & & \downarrow{\mathrm{id}} & \\ 0 \to & \pi_4(q_3\rho) & \to & \pi_4(q_2\rho) \to \Gamma(\pi_2\rho)/(K\Gamma(H_2(Q)) & \to & \pi_3(q_3\rho) & \to & \pi_3(q_2\rho) \to & 0 \end{array}$$

in which the rows are exact. (Exactness of the bottom row is due to Corollary 6.) The theorem follows. □

5 Realisability

Let $\rho = \left(C_n, \begin{pmatrix} L & \bar{M} \\ M & M \rtimes F \end{pmatrix}\right)$ be a free squared complex. (From now on we shall only consider free groups of operators F. Our results can however be extended to non-free groups of operators using the techniques of the last section.) We say that a reduced CW-space X *realises* ρ if $\rho = \rho(X)$. For an integer $d \geqq 3$ we say that ρ is of *dimension* d if $C_n = 0$ for $n \geqq d + 1$, and $C_d \neq 0$ (or $L \neq M \otimes \bar{M}$ if $d = 3$). We say ρ is of *dimension* 2 if $C_n = 0$ for $n \geqq 4$, $L = M \otimes \bar{M}$ and $M \neq 1$, and of *dimension* 1 if $C_n = 0$ for $n \geqq 4$, $L = M = 1$ and $F \neq 1$.

Proposition 8 *Suppose the free squared complex* ρ *is of dimension* $d \leqq 5$. *Then it is realisable by a reduced* CW*-space of dimension* d.

Proof. Suppose the crossed square part of ρ is free on the functions $\sigma_2\colon S_2 \to F$ and $\sigma_3\colon S_3 \to M$, with F the free group on a set S_1. Let X^1 be a wedge of circles corresponding to the elements of S_1. Let $W = X^1 \vee (\vee S^1)$ be a wedge of circles obtained from X^1 by attaching one circle for each element of S_2. Let $(1, 0)\colon W \to X^1$ be the map that is the identity on X^1 and sends each circle in $W \backslash X^1$ to the base

point. Let $(1, \alpha)$: $W \to X^1$ be a map that is the identity on X^1, and sends each circle in $W \backslash X^1$ to a loop in X^1 representing the image in $\pi_1 X^1 = F$ of the corresponding element of S_2. Let X^2 be the homotopy pushout of $(1, 0)$ and $(1, \alpha)$. Note that $\pi_1(W) = M \rtimes F$. Let V be a wedge of circles corresponding to the elements of S_3. Let β: $V \to W$ be a map that sends each circle to a loop in W representing the image in $M \subseteq \pi_1(W)$ of the corresponding element of S_3. Let C_+V, C_-V be two cones on V. Since $\sigma_3(S_3) \subseteq \ker(\delta: M \to F)$, the map β extends to maps β_+: $C_+V \to X^1_+$, β_-: $C_-V \to X^1_-$. Thus β extends to a map β_s: $SV \to X^2$ where SV is a wedge of 2-spheres corresponding to elements of S_3. The space X^3 is obtained from X^2 by attaching, for each element e^3 in S_3, a 3-cell via the restriction of β_s to the 2-sphere corresponding to e^3. It follows from Theorem 2 that the crossed square $\rho(X^3)$ is the free crossed sqare ρ.

The space X^4 is obtained from X^3 by attaching, for each basis element e^4 of C_4, a 4-cell via a map $S^3 \to X^3$ representing the element $\partial_4(e^4)$ in $\pi_3 X^3 = \ker(L \to M)$. Finally, $X = X^5$ is constructed by attaching one 5-cell to X^4 for each basis element e^5 of C_5; the attachment is via a map $S^4 \to X^4$ representing $\partial_5(e^5)$ in $\mathrm{im}(h: \pi_4 X^4 \to H_4 \tilde{X}^4) \subseteq C_4$. The CW-space X realises ρ. □

Suppose that f: $X \to Y$ is a map of reduced CW-spaces. We say that the induced morphism $\phi = \rho f$: $\rho X \to \rho Y$ is *realised* by f. More generally suppose that U is a CW-subspace of X. We say that a map g: $U \to Y$ is a *partial realisation* of ϕ if the evident triangle

$$\begin{array}{ccc} \rho U & & \\ \downarrow & \overset{\rho g}{\searrow} & \\ \rho X & \xrightarrow{\phi} & \rho Y \end{array}$$

commutes.

Proposition 9 *Let g: $U \to Y$ be a partial realisation of the morphism ϕ: $\rho X \to \rho Y$. If X is of dimension $\leqq 4$ then g can be extended to a full realisation f: $X \to Y$ of ϕ.*

Proof. We have the three triads $(X^2; X^1_+, X^1_-)$, $(U^2; U^1_+, U^1_-)$, $(Y^2; Y^1_+, Y^1_-)$ with $U^2 \subseteq X^2$, $U^1_+ \subseteq X^1_+$, $U^1_- \subseteq X^1_-$. Let $W_X = X^1_+ \cap X^1_-$, $W_U = U^1_+ \cap U^1_-$, $W_Y = Y^1_+ \cap Y^1_-$, and take g^1: $W_U \to W_Y$ to be the restriction of g. The map g^1 induces a homomorphism of free groups $\pi_1 g^1$: $\pi_1 W_U \to \pi_1 W_Y$, and the morphism ϕ provides a homomorphism of free groups ϕ_1: $\pi_1 W_X \to \pi_1 W_Y$. Clearly $\pi_1 W_U$ is a subgroup of $\pi_1 W_X$, and the basis of $\pi_1 W_U$ extends to a basis of $\pi_1 W_X$. The homomorphism $\pi_1 g^1$ is the restriction of ϕ_1. We can thus extend g^1 to a map f^1: $W_X \to W_Y$ such that $\pi_1 f^1 = \phi_1$. Since $\phi_1(\ker(\pi_1 W_X \to \pi_1 X^1_+)) \subseteq \ker(\pi_1 W_Y \to \pi_1 Y^1_+)$ we can extend f^1 to a map f^1_+: $X^1_+ \to Y^1_+$. Similarly f^1 can be extended to a map f^1_-: $X^1_- \to Y^1_-$. For $n \geqq 2$ let g^n: $U^n \to Y^n$ be the restriction of g to the n-skeleta. The union of f^1_+ and f^1_- yields a map f^2: $X^2 \to Y^2$ which extends g^2.

For each 3-cell e^3 in $X^3 \backslash U^3$ we need to extend g^3 to $X^2 \cup U^3 \cup \{e^3\}$. Let χ: $(E^3; E^2_+, E^2_-) \to (X^3; X^1_+, X^1_-)$ be the characteristic map for e^3, let $[\chi]$ be the element of $\pi_3(X^3; X^1_+, X^1_-)$ represented by χ, and let $(\bar{e}^3; \bar{e}^2_+, \bar{e}^2_-)$ denote the image of χ. The morphism ϕ provides a homomorphism ϕ_3: $\pi_3(X^3; X^1_+, X^1_-) \to \pi_3(Y^3; Y^1_+, Y^1_-)$. Let ζ: $(E^3; E^2_+, E^2_-) \to (Y^3; Y^1_+, Y^1_-)$ be a map representing the

element $[\zeta] = \phi_3[\chi]$. The map $\zeta\chi^{-1}: e^3 \to Y^3$ extends to a map $\overline{\zeta\chi^{-1}}: (\bar{e}^3; \bar{e}^2_+, \bar{e}^2_-) \to (Y^3; Y^1_+, Y^1_-)$ that agrees with g^2, up to relative homotopy, on the pairs $(\bar{e}^2_+, \bar{e}^2_+ \cap \bar{e}^2_-)$ and $(\bar{e}^2_-, \bar{e}^2_+ \cap \bar{e}^2_-)$. We can thus choose ζ so that $\overline{\zeta\chi^{-1}}$ agrees exactly on these two pairs. In this way the map g^3 is extended, cell by cell, to a map $f^3: X^3 \to Y^3$.

The map g^4 can be extended in a similar way, cell by cell, to a realisation $f = f^4$ of ϕ. This extension depends on the image of $\pi_4(Y^4, Y^3) \to \pi_3(Y^3; Y^1_+, Y^1_-)$ lying in $\pi_3(X^3)$. □

We say that a map of CW-spaces, or a morphism of free squared complexes, is a *weak equivalence* if it induces isomorphisms on homotopy groups. Since a map of CW-spaces is a weak equivalence if and only if it is a homotopy equivalence, Propositions 8 and 9 imply: (i) weak equivalence is an equivalence relation on free squared complexes of dimension $\leqq 4$, and; (ii) the following theorem

Theorem 10 *The homotopy types of* CW*-spaces of dimension* $\leqq 4$ *are in one-to-one correspondence with the weak equivalence classes of free squared complexes of dimension* $\leqq 4$ *with free groups of operators.*

Theorem 7 can be used to obtain the analogue of Theorem 10 in which free squared complexes are replaced by free quadratic complexes. This analogue is proved by different means in [2].

6 Homotopy

Let I be the unit interval, and for a reduced CW-space X with base-point $*$ define $IX = X \times I/((* \times 0 = * \times 1)$ and $I_*X = X \times I/* \times I$. Then a *free homotopy* (respectively *based homotopy*) between two maps $f, g: X \to Y$ is a map $H: IX \to Y$ (respectively $H: I_*X \to Y$) such that $f(x) = H(x, 0)$, $g(x) = H(x, 1)$. We denote the set of free (respectively based) homotopy classes by $[X, Y]$ (respectively $[X, Y]_*$), and the set of based homotopy classes of based self homotopy equivalences of X by $\mathscr{E}(X)$.

Note that the squared complexes $\rho(IX)$ and $\rho(I_*X)$ each contain $\rho(X \times 0)$, $\rho(X \times 1)$ as subcomplexes. We say that two morphisms $\phi, \psi: \rho(X) \to \rho(X)$ are *free homotopic* (respectively *based homotopic*) if there is a morphism $\eta: \rho(IX) \to \rho(Y)$ (respectively $\eta: \rho(I_*X) \to \rho(Y)$) such that $\phi = \eta|_{\rho(X \times 0)}$ and $\psi = \eta|_{\rho(X \times 1)}$. We denote the set of free (respectively based) homotopy classes by $[\rho X, \rho Y]$ (respectively $[\rho X, \rho Y]_*$), and the set of based homotopy classes of based self homotopy equivalences of ρX by $\mathscr{E}(\rho X)$. Propositions 8 and 9 imply that free and based homotopy are equivalence relations on morphisms of 3-dimensional free squared complexes; moreover they imply the following theorem.

Theorem 11 *Let* X, Y *be reduced* CW*-complexes, and suppose that* X *is of dimension* $\leqq 3$. *Then there are canonical bijections* $[X, Y] \cong [\rho X, \rho Y]$, $[X, Y]_* \cong [\rho X, \rho Y]_*$ *and* $\mathscr{E}(X) \cong \mathscr{E}(\rho X)$.

Note that with some effort one could give an algebraic description of $\rho(IX)$ and $\rho(I_*X)$ in terms of ρX, and hence a purely algebraic description of free and based homotopy of morphisms of squared complexes.

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Complements of domains with respect to hulls of outside compact sets

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Introduction

It is known that, if $K \subset \mathbb{C}^2$ is a compact set and $\Omega \subset \mathbb{C}^2$ is a Stein open set, then, provided $K \cap \Omega$ is empty, if $\hat{K}$ denotes the polynomial hull of K, the open set $\Omega \setminus \hat{K}$ is Stein. The result is contained in a theorem of Słodkowski [7, Theorem 2.1, p. 365] and a short direct proof of it, based on applications of the *Kontinuitätssatz* and of the local maximum modulus principle, has been given by Rosay and Stout [6]. The result is no longer true in $\mathbb{C}^N$ for $N \geqq 3$, as simple examples show, but one can raise the question whether for general $N \geqq 2$ there might be some weaker theorem that generalizes it. Moreover it seems also natural to raise the question whether for $N = 2$ there are parallel results for the other usual kinds of hulls of K which occur in the theory of holomorphic functions, namely the rational hull and the holomorphically convex hull. We denote these hulls by $\tilde{K}$ and $\mathscr{H}(K)$, respectively, and recall that $\tilde{K}$ is the set of $z \in \mathbb{C}^N$ such that $|g(z)| \leqq \|g\|_K$ for every rational function, g, which is holomorphic around K, or equivalently the set of $z \in \mathbb{C}^N$ such that $f(z) \in f(K)$ for every polynomial f (see [8]), whereas $\mathscr{H}(K)$ is understood here to be the intersection of all Stein open sets containing K. The answers to the mentioned questions do not seem to be known and we give them here.

We shall use the notations that, if X is a Stein manifold and $K \subset X$ a compact set, $\hat{K}_X$ and $\tilde{K}_X$ denote the $\mathscr{O}(X)$-hull and the $\mathscr{M}(X)$-hull of K, respectively. The latter is the set of $x \in X$ such that $|g(x)| \leqq \|g\|_K$ for every meromorphic function on X, g, which is holomorphic around K, or equivalently the set of $x \in X$ such that $f(x) \in f(X)$ for every $f \in \mathscr{O}(X)$.

We shall prove here the following theorem:

Theorem 1 *Let M be a Stein manifold of dimension $n \geqq 2$ and $\mathfrak{S}$ a coherent analytic sheaf on M. Let $K \subset M$ be a compact set, $X \subset M$ a Stein open set containing K and $\Omega \subset M$ an open set such that $K \cap \Omega$ is empty. Then, provided $H^{n-1}(\Omega; \mathfrak{S}) = 0$, it follows that*

$$H^{n-1}(\Omega \setminus \hat{K}_X; \mathfrak{S}) = 0, \quad H^{n-1}(\Omega \setminus \tilde{K}_X; \mathfrak{S}) = 0, \quad H^{n-1}(\Omega \setminus \mathscr{H}(K); \mathfrak{S}) = 0 .$$

On account of the characterization of Stein manifolds as the complex analytic manifolds which are countable at infinity and whose one-dimensional sheaf cohomology vanishes for every coherent analytic sheaf (see [4, pp. 243–246]), the preceding theorem implies at once:

Corollary 1 *Let M be a Stein manifold of dimension two, $K \subset M$ a compact set, $X \subset M$ a Stein open set containing K and $\Omega \subset M$ a Stein open set such that $K \cap \Omega$ is empty. Then the open sets $\Omega \setminus \hat{K}_X$, $\Omega \setminus \tilde{K}_X$ and $\Omega \setminus \mathscr{H}(K)$ are Stein.*

Hence it is seen that Theorem 1 answers in the affirmative both the questions raised above.

Moreover we point out that Theorem 1 is optimal, as far as the cohomology vanishing properties of the sets $\Omega \setminus \hat{K}_X$, $\Omega \setminus \tilde{K}_X$ and $\Omega \setminus \mathscr{H}(K)$ are concerned, since for $n \geqq 3$ the cohomology spaces $H^{n-2}(\Omega \setminus \hat{K}_X; \mathfrak{S})$, $H^{n-2}(\Omega \setminus \tilde{K}_X; \mathfrak{S})$ and $H^{n-2}(\Omega \setminus \mathscr{H}(K); \mathfrak{S})$ may not vanish, even in the case that Ω be a Stein open set. We shall exhibit an example at the end of the paper.

In the first place we need to establish a basic property concerning polynomial and rational hulls, which may conceivably have already been noticed before, but to our knowledge does not appear anywhere explicitly in the published literature. We recall that, if K is a compact subset of $\mathbb{C}^N$, or of any complex manifold, the space $\mathscr{O}(K)$ of holomorphic functions around K is made into a locally convex topological vector space as the inductive limit, in the category of locally convex topological complex vector spaces, of the inductive system formed by the spaces $\mathscr{O}(U)$, as U ranges through any fundamental system of open neighbourhoods of K, the topology of $\mathscr{O}(U)$ being that of compact convergence. If K' is another compact set, with $K \subset K'$, the restriction map $\rho: \mathscr{O}(K') \to \mathscr{O}(K)$ is a continuous linear map, but in general it may not be a topological homomorphism, which means that the induced bijective linear map $\rho_*: \dfrac{\mathscr{O}(K')}{\mathrm{Ker}(\rho)} \to \mathrm{Im}(\rho)$ should be bicontinuous. However we shall prove:

Theorem 2 *If $K \subset \mathbb{C}^N$, $N \geqq 1$, is a compact set, the restriction maps $\mathscr{O}(\hat{K}) \to \mathscr{O}(K)$ and $\mathscr{O}(\tilde{K}) \to \mathscr{O}(K)$ are topological homomorphisms.*

Proof. We need to invoke the main general properties enjoyed by the topological vector spaces under consideration, for which we refer to the book of Grothendieck [3] (see in particular pp. 140, 172, 184). Such properties are also recalled, without proofs, in the book of Bănică and Stănăşilă [1, p. 16]. These spaces are separated (i.e. Hausdorff), of type $(\mathscr{LF})$ and strong duals of $(\mathscr{F})$ and $(\mathscr{S})$ spaces, in particular they are complete and bornological, and so too are their closed subspaces. The consequences which we are interested in are the following two (see [3, p. 148]):

(a) For the restriction maps under consideration to be topological homomorphisms it is necessary and sufficient that they should have closed images.

(b) Every bounded subset of $\mathscr{O}(K)$ is the image of a bounded subset of some $\mathscr{O}(U)$ by the restriction map $r: \mathscr{O}(U) \to \mathscr{O}(K)$.

That being stated, let us prove that the restriction map $\rho: \mathscr{O}(\tilde{K}) \to \mathscr{O}(K)$ is a topological homomorphism, i.e., in view of (a), that $\rho(\mathscr{O}(\tilde{K}))$ is closed in $\mathscr{O}(K)$.

We have to show that, if $f \in \mathrm{Clos}_{\mathscr{O}(K)} \rho(\mathscr{O}(\tilde{K}))$, then there is $F \in \mathscr{O}(\tilde{K})$ with $\rho(F) = f$. We divide the proof into three steps.

(i) We can obtain $\tilde{K}$ as the intersection of a sequence $\{P_n\}$ of open rational polyhedra with $P_n \supset \bar{P}_{n+1}$ for every n (see [8]), hence $\mathcal{O}(\tilde{K})$ may be considered as the locally convex inductive limit of the inductive system formed by the spaces $\mathcal{O}(P_n)$ and the restriction maps $r_{n,m}$: $\mathcal{O}(P_n) \to \mathcal{O}(P_m)$ $(n \leqq m)$. By the Oka–Weil approximation theorem for rational polyhedra (see [5]), if A_n denotes the subset of $\mathcal{O}(P_n)$ of the rational functions on $\mathbb{C}^N$ which are analytic on P_n, then A_n is dense in $\mathcal{O}(P_n)$. It follows that, if r_n: $\mathcal{O}(P_n) \to \mathcal{O}(\tilde{K})$, $n \geqq 1$, are the restriction maps, the set $A = \bigcup_n r_n(A_n)$ is dense in $\mathcal{O}(\tilde{K})$ and hence $\mathrm{Clos}_{\mathcal{O}(K)}\rho(\mathcal{O}(\tilde{K})) = \mathrm{Clos}_{\mathcal{O}(K)}\rho(A)$. Therefore we can find a sequence $\{F_n\}$ of rational functions on $\mathbb{C}^N$, each of which is analytic around $\tilde{K}$, such that $F_n \to f$ in the topology of $\mathcal{O}(K)$.

(ii) Since the sequence $\{F_n - f\}$ is convergent to 0 in $\mathcal{O}(K)$, it is a bounded subset of $\mathcal{O}(K)$, hence, in view of (b), there exists an open neighborhood U of K such that $\{F_n - f\}$ is the image, by the restriction map r: $\mathcal{O}(U) \to \mathcal{O}(K)$, of a bounded subset of $\mathcal{O}(U)$. If U' is an open neighborhood of K with $U' \subset U$, the restriction map $\mathcal{O}(U) \to \mathcal{O}(U')$ transforms the bounded subsets of $\mathcal{O}(U)$ into bounded subsets of $\mathcal{O}(U')$, therefore any sufficiently small open neighborhood of K can be taken as U. Let us take U such that f is holomorphic on U and every connected component of U meets K, so that r is injective. Then it follows that the functions $F_n - f$ are all holomorphic on U and the sequence $\{F_n - f\}$ is a bounded subset of $\mathcal{O}(U)$. By Montel's theorem, the sequence is then a relatively compact subset of $\mathcal{O}(U)$, from which we immediately infer that $F_n \to f$ in $\mathcal{O}(U)$ and hence that $\{F_n\}$ is a Cauchy sequence in $\mathcal{O}(U)$.

(iii) Let δ be a positive real number small enough so that, for every $\zeta \in K$, the polydisc $\bar{\Delta}(\zeta, \delta)$ be contained in U and let $\Delta(K, \delta) = \bigcup_{\zeta \in K} \Delta(\zeta, \delta)$. If G is a rational function on $\mathbb{C}^N$ which is holomorphic around $\tilde{K}$ and on U, then G is holomorphic on $\Delta(\tilde{K}, \delta) = \bigcup_{\zeta \in \tilde{K}} \Delta(\zeta, \delta)$. As a matter of fact, if $\alpha = (\alpha_1, \ldots, \alpha_N)$ is any multi-index, the Cauchy inequalities imply, since $\tilde{K}$ is the rational hull of K, that

$$\left\| \frac{\partial^\alpha G}{\partial z^\alpha} \right\|_{\tilde{K}} \leqq \left\| \frac{\partial^\alpha G}{\partial z^\alpha} \right\|_K \leqq \| G \|_{\bar{\Delta}(K,\delta)} \alpha! \, \delta^{-|\alpha|} .$$

Therefore, if $\eta \in \tilde{K}$ and δ' is a positive real number with $\delta' < \delta$, we have, for every $z \in \bar{\Delta}(\eta, \delta')$,

$$\sum_\alpha \frac{1}{\alpha!} \left| \frac{\partial^\alpha G}{\partial z^\alpha}(\eta) \right| |z - \eta|^\alpha \leqq \left(\frac{\delta}{\delta - \delta'} \right)^N \| G \|_{\bar{\Delta}(K,\delta)} ,$$

which implies that G is holomorphic on $\Delta(\tilde{K}, \delta)$ and in addition that

$$\| G \|_{\bar{\Delta}(\tilde{K},\delta')} \leqq \left(\frac{\delta}{\delta - \delta'} \right)^N \| G \|_{\bar{\Delta}(K,\delta)} .$$

This applies to all the rational functions F_n and to their differences $F_n - F_{m_2}$ and hence $\{F_n\}$ is a Cauchy sequence in $\mathcal{O}(\Delta(\tilde{K}, \delta))$. By the completeness of $\mathcal{O}(\Delta(\tilde{K}, \delta))$, we conclude that there is $F \in \mathcal{O}(\Delta(\tilde{K}, \delta))$ which coincides with f on $\Delta(K, \delta)$ and hence $\rho(F) = f$.

The proof that the restriction map $\mathcal{O}(\tilde{K}) \to \mathcal{O}(K)$ is a topological homomorphism is completed. It is not necessary to develop also the parallel proof for the restriction map $\mathcal{O}(\hat{K}) \to \mathcal{O}(K)$, since the latter proof can be obtained from the former by replacing the adjective "rational", where it occurs, by the adjective "polynomial" and making some obvious simplifications. qed

Now we consider the generalization of Theorem 2 to the context of Stein manifolds.

Corollary 2 *Let X be a Stein manifold of dimension $n \geqq 1$ and $K \subset X$ a compact set. Then the restriction maps $\mathcal{O}(\hat{K}_X) \to \mathcal{O}(K)$, $\mathcal{O}(\tilde{K}_X) \to \mathcal{O}(K)$ and $\mathcal{O}(\mathscr{H}(K)) \to \mathcal{O}(K)$ are topological homomorphisms.*

Proof. By Remmert's imbedding theorem, there is a biholomorphic homeomorphism of X onto a closed complex analytic submanifold of some $\mathbb{C}^N$ ($n \leqq N \leqq 2n+1$), hence it is no loss of generality to assume that X itself is such a submanifold. Then it is known that

$$\hat{K}_X = \hat{K}_{\mathbb{C}^N} = \text{polynomial hull of } K \text{ in } \mathbb{C}^N ,$$

$$\tilde{K}_X = \tilde{K}_{\mathbb{C}^N} = \text{rational hull of } K \text{ in } \mathbb{C}^N ,$$

which follows from the facts that X is the zero-set of finitely many entire functions and that every holomorphic function on X is the restriction to X of some entire function. Moreover, if $E \subset X$ is any compact set, the restriction map $\mathcal{O}_{\mathbb{C}^N}(E) \to \mathcal{O}_X(E)$ is a surjective topological homomorphism. Indeed, if U is an open neighbourhood of E in X, one can find an open neighborhood U' of E in $\mathbb{C}^N$ such that $U = X \cap U'$ and the restriction map $\mathcal{O}_{\mathbb{C}^N}(U') \to \mathcal{O}_X(U)$ is a surjective topological homomorphism. For example one can take $U' = \rho^{-1}(U)$, where $\rho\colon V \to X$ be a holomorphic retraction of an open neighborhood V of X in $\mathbb{C}^N$ onto X (see [4, p. 257]). Then the conclusion for the case of $\hat{K}_X$ follows from Theorem 2, on account of the commutative diagram of restriction maps

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{C}^N}(\hat{K}_X) & \to & \mathcal{O}_X(\hat{K}_X) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{C}^N}(K) & \to & \mathcal{O}_X(K) \end{array} ;$$

and similarly for the case of $\tilde{K}_X$.

Finally let us consider $\mathscr{H}(K)$. We can obtain $\mathscr{H}(K)$ as the intersection $\bigcap_n \hat{K}_{U_n}$, where $\{U_n\}$ be a decreasing sequence of Stein open neighborhoods of $\mathscr{H}(K)$ with $\bigcap_n U_n = \mathscr{H}(K)$. Then, if $r\colon \mathcal{O}(\mathscr{H}(K)) \to \mathcal{O}(K)$ and $r_n\colon \mathcal{O}_{U_n}(\hat{K}_{U_n}) \to \mathcal{O}_{U_n}(K)$, $n = 1, 2, \ldots$, are the restriction maps, we have

$$r(\mathcal{O}(\mathscr{H}(K)) = \bigcap_n r_n(\mathcal{O}_{U_n}(\hat{K}_{U_n})) ,$$

and the conclusion follows by resorting again to the property (a) mentioned in the proof of Theorem 2. qed

Before going into the proof of Theorem 1, we need a further preliminary result, which requires us to consider sheaf cohomology spaces with supports in a closed set. We refer to the book of Bredon [2] for basic information on sheaf cohomology with general families of supports. We recall that, given a topological space X, a sheaf $\mathscr{A}$ of Abelian groups on X and a closed set $Y \subset X$, the cohomology $H^*_Y(X; \mathscr{A})$ with supports in Y (i.e. whose family of supports is that of the closed subsets of Y) is algebraically isomorphic to the relative cohomology $H^*(X, X \setminus Y; \mathscr{A})$; moreover, if U is any open neighborhood of Y in X, there is an excision algebraic isomorphism $H^*_Y(X; \mathscr{A}) \cong H^*_Y(U; \mathscr{A})$ (see [2, pp. 59, 61]).

Corollary 3 *Let X be a Stein manifold of dimension $n \geqq 1$ and $K \subset X$ a compact set. Let Y denote any of the three sets $\hat{K}_X \backslash K$, $\tilde{K}_X \backslash K$, $\mathscr{H}(K) \backslash K$. Then*

$$H^n_Y(X \backslash K; \mathfrak{S}) = 0$$

for every coherent analytic sheaf, $\mathfrak{S}$, on X.

Proof. It is known and not difficult to check that the sets $\hat{K}_X$, $\tilde{K}_X$ and $\mathscr{H}(K)$ have the property that all the connected components of each of them meet K, hence the restriction maps $\mathscr{O}(\hat{K}_X) \to \mathscr{O}(K)$, $\mathscr{O}(\tilde{K}_X) \to \mathscr{O}(K)$ and $\mathscr{O}(\mathscr{H}(K)) \to \mathscr{O}(K)$ are injective. It follows, by Corollary 2 and the Hahn–Banach theorem (see [3, p. 55]), that the transposed maps

$$\text{Hom Cont}(\mathscr{O}(K); \mathbb{C}) \to \text{Hom Cont}(\mathscr{O}(\hat{K}_X); \mathbb{C}), \text{ etc.}$$

are surjective topological homomorphisms. We claim that, consequently, if Ω^n denotes the sheaf of germs of holomorphic n-forms on X, the maps

$$H^n_K(X; \Omega^n) \to H^n_{\hat{K}_X}(X; \Omega^n), \quad \text{etc.},$$

induced by inclusion of supports, are surjective too. This can be derived from the relative version of the Serre duality theorem (see [1, p. 287]), which implies in particular that, if $E \subset X$ is any compact set, for each integer p with $0 \leqq p \leqq n$ the space $H^p_E(X; \Omega^n)$ has a natural locally convex topology, such that the associated separated space ${}^{\sigma}H^p_E(X; \Omega^n)$ is canonically topologically isomorphic to the space $\text{Hom Cont}({}^{\sigma}H^{n-p}(E; \mathscr{O}); \mathbb{C})$ and $H^p_E(X; \Omega^n)$ is separated if and only if so is $H^{n-p+1}(E; \mathscr{O})$. Then we have that the maps

$${}^{\sigma}H^n_K(X; \Omega^n) \to {}^{\sigma}H^n_{\hat{K}_X}(X; \Omega^n), \quad \text{etc.},$$

induced by inclusion of supports, are surjective, and, since $0 = H^1(\hat{K}_X; \mathscr{O}) = H^1(\tilde{K}_X; \mathscr{O}) = H^1(\mathscr{H}(K); \mathscr{O})$, that

$${}^{\sigma}H^n_{\hat{K}_X}(X; \Omega^n) = H^n_{\hat{K}_X}(X; \Omega^n), \quad \text{etc.},$$

which proves the claim. Now we can apply the exact cohomology sequences

$$\cdots \to H^n_K(X; \Omega^n) \to H^n_{\hat{K}_X}(X; \Omega^n) \to H^n_{(\hat{K}_X \backslash K)}(X \backslash K; \Omega^n) \to 0, \text{ etc.},$$

(see [2, p. 60]), to infer that $H^n_Y(X \backslash K; \Omega^n) = 0$.

To conclude that $H^n_Y(X \backslash K; \mathfrak{S}) = 0$, we recall the fact that, as X is Stein, a positive integer r and a locally free sheaf $\mathscr{F}$ of $\mathscr{O}$-modules on X of rank $r - 1$ exist, such that the exact sequence $0 \to \mathscr{F} \to \mathscr{O}^r \to \Omega^n \to 0$ holds on X, and hence $\mathscr{O}^r \cong \mathscr{F} \oplus \Omega^n$ and $(\Omega^n)^r \cong \text{Hom}_{\mathscr{O}}(\mathscr{F}; \Omega^n) \oplus \mathscr{O}$; moreover we recall the fact that, as $\bar{Y}$ is compact, there are a Stein open neighborhood U of $\bar{Y}$ and a positive integer s such that an exact sequence of sheaves $\mathscr{O}^s \xrightarrow{\pi} \mathfrak{S} \to 0$ holds on U and $\text{Ker}(\pi)$ is a coherent analytic sheaf on U. From the former fact we derive that $H^n_Y(X \backslash K; \mathscr{O}) = 0$, and then, on account of the excision property that $H^*_Y(X \backslash K; \mathscr{A}) \cong H^*_Y(U \backslash K; \mathscr{A})$, for every sheaf $\mathscr{A}$ of Abelian groups on X, and of the vanishing of $H^{n+1}_Y(U \backslash K; \text{Ker}(\pi))$, by the latter fact we conclude that $H^n_Y(X \backslash K; \mathfrak{S}) = 0$. qed

Now we are in a position to prove Theorem 1.

Proof of Theorem 1 Consider the closed subsets of Ω $\hat{K}_X \cap \Omega$, $\tilde{K}_X \cap \Omega$ and $\mathscr{H}(K) \cap \Omega$. If we define $E = \hat{K}_X \cap (M \backslash \Omega)$, $F = \tilde{K}_X \cap (M \backslash \Omega)$ and $G =$

$\mathscr{H}(K)\cap(M\setminus\Omega)$, it is plain that $\hat{K}_X\cap\Omega=\hat{E}_X\setminus E$, $\tilde{K}_X\cap\Omega=\tilde{F}_X\setminus F$ and $\mathscr{H}(K)\cap\Omega=\mathscr{H}(G)\setminus G$; therefore Corollary 3 implies at once that $H^n_{\hat{K}_X\cap\Omega}(X\setminus E;\mathfrak{S})=0$, $H^n_{\tilde{K}_X\cap\Omega}(X\setminus F;\mathfrak{S})=0$ and $H^n_{\mathscr{H}(K)\cap\Omega}(X\setminus G;\mathfrak{S})=0$. On account of the excision property, these cohomology vanishings remain valid if we replace $X\setminus E$, $X\setminus F$, $X\setminus G$ first by $M\setminus E$, $M\setminus F$, $M\setminus G$, respectively, and then by Ω. Thus, if Y denotes any of the three sets $\hat{K}_X\cap\Omega$, $\tilde{K}_X\cap\Omega$ and $\mathscr{H}(K)\cap\Omega$, we have that $H^n_Y(\Omega;\mathfrak{S})=0$. This is true for every coherent analytic sheaf, $\mathfrak{S}$, on M. Now, if $\mathfrak{S}$ is a fixed coherent analytic sheaf on M, such that $H^{n-1}(\Omega,\mathfrak{S})=0$, as it is assumed in Theorem 1, the exact sequence

$$\cdots\to H^{n-1}(\Omega;\mathfrak{S})\to H^{n-1}(\Omega\setminus Y;\mathfrak{S})\to H^n_Y(\Omega;\mathfrak{S})\to\cdots$$

implies immediately that $H^{n-1}(\Omega\setminus Y;\mathfrak{S})=0$, which is the desired conclusion. qed

Finally we discuss an example showing that in general, for $n\geqq 3$, the stronger assumption, in Theorem 1, that Ω be a Stein open set does not entail the vanishing of any of the spaces $H^{n-2}(\Omega\setminus\hat{K}_X;\mathfrak{S})$, $H^{n-2}(\Omega\setminus\tilde{K}_X;\mathfrak{S})$ and $H^{n-2}(\Omega\setminus\mathscr{H}(K);\mathfrak{S})$. Consider the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, $n\geqq 3$, and let K denote the 2-sphere $b\mathbb{B}_n\cap\{z\in\mathbb{C}^n:\mathscr{I}m(z_2)=0,\ z_3=0,\ldots,z_n=0\}$. Then the envelope of holomorphy and the polynomial hull of K are both equal to the closed 3-disc $\overline{\Delta^3}=\bar{\mathbb{B}}_n\cap\{z\in\mathbb{C}^n:\mathscr{I}m(z_2)=0,\ z_3=0,\ldots,z_n=0\}$, hence $\hat{K}=\tilde{K}=\mathscr{H}(K)=\overline{\Delta^3}$. We claim that

$$H^{n-2}(\mathbb{B}_n\setminus\Delta^3;\Omega^n)\neq 0\,.$$

In fact we will show that the space $H^{n-1}_{\Delta^3}(\mathbb{B}_n;\Omega^n)$ is not separated, whence not null, and so the claim will follows at once from the exact sequence of relative cohomology

$$H^{n-2}(\mathbb{B}_n;\Omega^n)\to H^{n-2}(\mathbb{B}_n\setminus\Delta^3;\Omega^n)\to H^{n-1}_{\Delta^3}(\mathbb{B}_n;\Omega^n)\to H^{n-1}(\mathbb{B}_n;\Omega^n)\,,$$

since $0=H^{n-2}(\mathbb{B}_n;\Omega^n)=H^{n-1}(\mathbb{B}_n;\Omega^n)$. By the relative Serre duality theorem (see [1, p. 287]), it suffices to show that the space $H^2_c(\Delta^3;\mathscr{O})$ is not separated. After identifying the complex 2-plane $\{z\in\mathbb{C}^n:z_3=0,\ldots,z_n=0\}$ with $\mathbb{C}^2$, one is reduced to show the non-separation of the last mentioned space for $\mathscr{O}=\mathscr{O}_{\mathbb{C}^2}$. Assume, in order to reach a contradiction, that this space is separated. Then, again by the relative Serre duality theorem, but applying it in $\mathbb{B}_2$, the open unit ball of $\mathbb{C}^2$, one has that the space $H^1_{\Delta^3}(\mathbb{B}_2;\Omega^2)$ is separated too. It follows that the latter space should be null. As a matter of fact, since $\overline{\Delta^3}$ is the envelope of holomorphy of K, the exact sequence

$$\mathscr{O}(\overline{\Delta^3})\to\mathscr{O}(K)\xrightarrow{\delta}H^1_c(\Delta^3;\mathscr{O})\to 0$$

implies that $H^1_c(\Delta^3;\mathscr{O})=0$, whence ${}^{\sigma}H^1_{\Delta^3}(\mathbb{B}_2;\Omega^2)=0$. But then, from the exact sequence

$$\mathscr{O}(\mathbb{B}_2)\to\mathscr{O}(\mathbb{B}_2\setminus\Delta^3)\xrightarrow{\delta}H^1_{\Delta^3}(\mathbb{B}_2;\Omega^2)=0\,,$$

one gets that the restriction map $\mathscr{O}(\mathbb{B}_2)\to\mathscr{O}(\mathbb{B}_2\setminus\Delta^3)$ is surjective. This is not possible, since $\mathbb{B}_2\setminus\Delta^3$ is a Stein open set, and so the required contradiction has been reached.

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Some surprising Hilbert-Kunz functions

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Introduction

Let $(\mathcal{O}, m)$ be a complete local Z/p algebra and I_n the ideal generated by all x^{p^n}, $x \in m$. Let $e_n(\mathcal{O})$ be the length of $\mathcal{O}/I_n$. The function $n \to e_n(\mathcal{O})$ is the "Hilbert-Kunz function" of $\mathcal{O}$.

Suppose that the Krull dimension, a, of $\mathcal{O}$ is >0. In [3] the second author showed that $e_n(\mathcal{O}) = C p^{an} + \Delta_n$ where C is a positive real constant and $\Delta_n = O(p^{(a-1)n})$. Furthermore, when $a=1$, C is an integer and Δ_n is an eventually periodic function of n. When $a \geqq 2$, apart from Kunz's result that $C \geqq 1$, nothing more is known about C and Δ_n, and the calculation of $e_n(\mathcal{O})$ is often difficult. In [2] Kunz determined $e_n(\mathcal{O})$ when $\mathcal{O}$ is $Z/p[\![x_1, \ldots, x_s]\!] \Big/ \left(x_1^{d_1} - \prod_2^s x_i^{d_i}\right)$ and $s = a+1 \geqq 3$. In all these cases C is rational and Δ_n can be written as $\sum_{j=0}^{a-1} \Delta^{(j)}(n)\, p^{jn}$ with each $\Delta^{(j)}$ eventually periodic.

In this paper we shall determine $e_n(\mathcal{O})$ when $\mathcal{O} = Z/p[\![x_1, \ldots, x_s]\!] \Big/ \left(\sum_1^s x_i^{d_i}\right)$, where the d_i are positive integers and $s = a+1 \geqq 3$. This investigation was started by the first author in [1]; when $s=3$ or when $p=2$ she showed that C is rational and that Δ_n is an eventually periodic function of n. When $s \geqq 4$ and $p>2$ we shall see that more complicated behavior is possible. For example when $\mathcal{O} = Z/5[\![x_1, x_2, x_3, x_4]\!] \Big/ \left(\sum_1^4 x_i^4\right)$ we show that $e_n(\mathcal{O}) = \frac{168}{61} \cdot 5^{3n} - \frac{107}{61} \cdot 3^n$. The general result is stated in Theorem 5.7. It implies that, for the family of $\mathcal{O}$ that we are considering, the power series $\sum e_n(\mathcal{O})\, t^n$ represents a rational function. Whether this is true for a larger class of $\mathcal{O}$ is an open question.

The device that enables us to go beyond the results of [1] and settle the conjectures raised there is the representation ring, Γ_F, introduced in Sect. 1. In Sect. 2 and 3 we determine a piece of the multiplication table of Γ_F. This allows us, in Sect. 4, to prove certain functional equations for $D_F(k_1, \ldots, k_s)$

where $F=Z/p$ and $D_F(k_1,\ldots,k_s)$ is the F-dimension of $F[x_1,\ldots,x_s]/(x_1^{k_1},\ldots,x_s^{k_s},x_1+\ldots+x_s)$. (The existence of such functional equations was proved when $s=3$ or $p=2$, and conjectured for general s and p in [1]). In the final section we relate $e_n(\mathcal{O})$ to D_F and use the functional equations for D_F to describe $e_n(\mathcal{O})$ to describe $e_n(\mathcal{O})$ precisely. The use of Γ_F has an auxiliary benefit – it permits one to avoid the long computations with matrices of binomial and multinomial co-efficients made in [1], and gives a fairly short self-contained route to our fundamental Theorem 5.7 and most of the results of [1].

Although this paper betrays no trace of the fact, it is very much "computer-aided." For before a result can be proved it must be formulated, and we would never have come to formulate Theorems 4.5 and 4.13 without extensive computer calculation of the function D_F.

1 D_F, l_F and Γ_F

Definition 1.1 Let F be a field and $(k_1,\ldots,k_s)$ an s-tuple of non-negative integers. $D_F(k_1,\ldots,k_s)$ is the F-dimension of the vector space

$$F[x_1,\ldots,x_s]\Big/\Big(x_1^{k_1},\ldots,x_s^{k_s},\sum_1^s x_i\Big).$$

Remarks. (1) $D_F(k_1,\ldots,k_s)$ is the dimension of $F[x_1,\ldots,x_{s-1}]/(x_1^{k_1},\ldots,x_{s-1}^{k_{s-1}},(x_1+\ldots+x_{s-1})^{k_s})$.
(2) If $k_3\geqq k_1+k_2-1$ then $(x_1+x_2)^{k_3}\in(x_1^{k_1},x_2^{k_2})$ and $D_F(k_1,k_2,k_3)=k_1\,k_2$.
(3) $D_F(i,j)=\min(i,j)$
(4) If F and F' have the same characteristic, then $D_F=D_{F'}$.

To study the function D_F we introduce a related function, l_F, and a "representation ring", $\Gamma=\Gamma_F$.

Definition 1.2 $l_F(k_1,\ldots,k_s)=(-)^s\cdot(-)^{\Sigma k_i}\sum_{\varepsilon_1,\ldots,\varepsilon_s}(-)^{\Sigma\varepsilon_i}D_F(k_1+\varepsilon_1,\ldots,k_s+\varepsilon_s)$ where each ε_i takes the values 0 and 1.

From remark (3) after Definition 1.1 we see that $l_F(i,j)=\delta_{i,j}$.

Definition 1.3 An F-object is a finitely generated $F[T]$-module on which T acts nilpotently.

If M and N are F-objects so is $M\oplus N$. Furthermore $M\otimes_F N$ can be given the structure of F-object with $T(m\otimes n)=(T(m)\otimes n)+(m\otimes T(n))$. We will denote this F-object by $M\otimes N$.

Consider all formal differences $M-M'$ where M and M' are F-objects. We say that $M-M'$ and $N-N'$ are equivalent if $M\oplus N'$ and $M'\oplus N$ are isomorphic $F[T]$-modules. The theory of modules over PIDs shows that this is an equivalence relation.

Definition 1.4 $\Gamma=\Gamma_F$ consists of all equivalence classes of formal differences $M-M'$.

The following is easily proved:

Theorem 1.5 *$\oplus$ and $\otimes$ induce binary operations, $+$ and $\cdot$, $\Gamma\times\Gamma\to\Gamma$. Under these operations Γ is a commutative ring; the multiplicative identity of Γ is the class of $F[T]/T$.*

Definition 1.6 Let j be an integer $\geqq 0$.
(1) $\delta_j \in \Gamma$ is the class of $F[T]/T^j$.
(2) $\lambda_j = (-)^j(\delta_{j+1} - \delta_j)$.

Lemma 1.7 *Γ is a free abelian group under $+$, and the λ_j form a basis.*

Proof. The theory of modules over PIDs shows that $\delta_1, \delta_2, \delta_3, \ldots$ form a Z-basis of Γ. Since δ_0 is the zero element of Γ the lemma follows.

Definition 1.8 $\alpha: \Gamma \to Z$ is the Z-linear map $\sum c_i \lambda_i \to c_0$.

Remark. α takes δ_n to 1 if $n \geqq 1$. Suppose that V is an F-object. Writing V as a direct sum of copies of $F[T]/T^{n_j}$, $n_j \geqq 1$, we see that $\alpha(V)$ is the number of summands. In other words, $\alpha(V)$ is the F-dimension of V/TV.

Theorem 1.9 (1) $\alpha\left(\prod_1^s \delta_{k_i}\right) = D_F(k_1, \ldots, k_s)$

(2) $\alpha\left(\prod_1^s \lambda_{k_i}\right) = l_F(k_1, \ldots, k_s)$.

Proof. Let $M_i = F[x_i]/x_i^{k_i}$ with T acting by multiplication by x_i. Each M_i is an F-object and $M_1 \otimes \ldots \otimes M_s$ identifies with $F[x_1, \ldots, x_s]/(x_1^{k_1}, \ldots, x_s^{k_s})$ with T acting by multiplication by $\sum x_i$. The remark after Definition 1.8 now gives (1), and (2) is an immediate consequence of (1).

Theorem 1.10 (1) $\alpha(\lambda_i \lambda_j) = \delta_{i,j}$
(2) $\prod_1^{s-1} \lambda_{k_i} = \sum_n l_F(k_1, \ldots, k_{s-1}, n)\, \lambda_n$.

Proof. Since $l_F(i, j) = \delta_{i,j}$, Theorem 1.9 gives (1). We can write $\sum_1^{s-1} \lambda_{k_i}$ as $\sum c_r \lambda_r$ with $c_r \in Z$. Multiplying by λ_n, applying α and using Theorem 1.9 together with (1) we find that $c_n = l_F(k_1, \ldots, k_{s-1}, n)$.

2 Calculation of D_F for small k_i

In this section we calculate $D_F(k_1, \ldots, k_s)$ when char $F = 0$ and the k_i are arbitrary, and when char $F = p > 0$ and each $k_i \leqq p$.

Lemma 2.1 (1) *If $k_3 > k_1 + k_2$, $l_F(k_1, k_2, k_3) = 0$.*
(2) *If $k_3 = k_1 + k_2$, $l_F(k_1, k_2, k_3)$ is 0 if the binomial co-efficient $\binom{k_3}{k_1}$ is 0 in F, and 1 otherwise.*

Proof. Suppose $k_3 > k_1 + k_2$. Remark (2) after Definition 1.1 shows that $D_F(k_1 + \varepsilon_1, k_2 + \varepsilon_2, k_3 + \varepsilon_3) = (k_1 + \varepsilon_1) \cdot (k_2 + \varepsilon_2)$, giving (1). If $k_3 = k_1 + k_2$, $D_F(k_1 + \varepsilon_1, k_2 + \varepsilon_2, k_3 + \varepsilon_3)$ is again $(k_1 + \varepsilon_1)(k_2 + \varepsilon_2)$ except possibly when $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, 0)$. Furthermore $D_F(k_1 + 1, k_2 + 1, k_3) = \dim_F F[x, y]/(x^{k_1+1}, y^{k_2+1},$

$(x+y)^{k_1+k_2})=\dim_F F[x,y]\Big/\left(x^{k_1+1}, y^{k_2+1}, \binom{k_3}{k_1}x^{k_1}y^{k_2}\right)$. This is $(k_1+1)(k_2+1)$ if $\binom{k_3}{k_1}=0$ in F and $(k_1+1)(k_2+1)-1$ otherwise, giving (2).

Theorem 2.2 (1) *If* char $F=0$ *and* $j\geqq 1$, $\lambda_1\lambda_j=\lambda_{j-1}+\lambda_j+\lambda_{j+1}$.
(2) *If* char $F=p>0$ *and* $p\nmid j$ *or* $j+1$, $\lambda_1\lambda_j=\lambda_{j-1}+\lambda_j+\lambda_{j+1}$.
(3) *If* char $F=p>0$, $\lambda_1\lambda_{pi}=\lambda_{pi+1}$, $\lambda_1\lambda_{pi-1}=\lambda_{pi-2}$.

Proofs. Suppose we are in the situation of (1) or (2). Theorem 1.10 and Lemma 2.1 show that $\lambda_1\lambda_j=\lambda_{j-1}+c\lambda_j+\lambda_{j+1}$ for some c. Now there is an obvious ring homomorphism β: $\Gamma_F\to Z$ taking an F-object M to $\dim_F M$. Then $\beta(\lambda_1\lambda_j)=\beta(\lambda_1)\cdot\beta(\lambda_j)=(-)^{j+1}$ while $\beta(\lambda_{j-1}+c\lambda_j+\lambda_{j+1})=(-)^{j+1}(2-c)$. So $c=1$. Similarly $\lambda_1\lambda_{pi}=c\lambda_{pi}+\lambda_{pi+1}$ while $\lambda_1\lambda_{pi-1}=c\lambda_{pi-1}+\lambda_{pi-2}$; applying β we find that $c=0$ in both cases.

Theorem 2.3 *Suppose* char $F=0$ *and* $i\leqq j$. *Then* $\lambda_i\lambda_j=\sum_{j-i}^{j+i}\lambda_k$.

Proof. Argue by induction on i. For $i=1$ use Theorem 2.2. If $i\geqq 2$ the induction assumption shows that $(\lambda_{i-2}+\lambda_{i-1})\lambda_j=\sum_{j-i}^{j+i-4}\lambda_{k+2}+\sum_{j-i}^{j+i-2}\lambda_{k+1}=\left(\sum_{j-i}^{j+i-2}(\lambda_{k+1}+\lambda_{k+2})\right)-\lambda_{j+i}-\lambda_{j+i-1}$. On the other hand, $(\lambda_{i-2}+\lambda_{i-1}+\lambda_i)\lambda_j=\lambda_1\lambda_{i-1}\lambda_j$. By induction this is $\lambda_1\left(\sum_{j-i}^{j+i-2}\lambda_{k+1}\right)=\sum_{j-i}^{j+i-2}(\lambda_k+\lambda_{k+1}+\lambda_{k+2})$. Subtracting the previous result we find that $\lambda_i\lambda_j=\left(\sum_{j-i}^{j+i-2}\lambda_k\right)+\lambda_{j+i}+\lambda_{j+i-1}=\sum_{j-i}^{j+i}\lambda_k$.

Lemma 2.4 *Suppose* char $F=p>0$. *Then for* $i<p$, $\lambda_i\lambda_{p-1}=\lambda_{p-1-i}$.

Proof. We argue by induction on i. If $i=1$ use Theorem 2.2. If $i\geqq 2$ the induction assumption shows that $(\lambda_{i-2}+\lambda_{i-1})\cdot\lambda_{p-1}=\lambda_{p+1-i}+\lambda_{p-i}$. Also, $(\lambda_{i-2}+\lambda_{i-1}+\lambda_i)\cdot\lambda_{p-1}=\lambda_1\lambda_{i-1}\lambda_{p-1}$. By induction this is $\lambda_1\lambda_{p-i}=\lambda_{p+1-i}+\lambda_{p-i}+\lambda_{p-1-i}$. Subtracting the previous result gives the lemma.

Theorem 2.5 *Suppose* char $F=p>0$ *and* $i\leqq j\leqq p-1$. *Then* $\lambda_i\lambda_j=\sum_{j-i}^{\min(i+j,\,2p-2-i-j)}\lambda_k$.

Proof. If $i+j\leqq p-1$ we argue by induction on i precisely as in the proof of Theorem 2.3. Suppose $i+j>p-1$. Then by Lemma 2.4, $\lambda_i\lambda_j=(\lambda_{p-1}\lambda_{p-1-i})(\lambda_{p-1}\lambda_{p-1-j})=\lambda_{p-1-j}\cdot\lambda_{p-1-i}$. Since $(p-1-j)+(p-1-i)<p-1$ this is just $\sum\lambda_k$, the sum extending from $j-i$ to $(p-1-i)+(p-1-j)$, the desired result.

Theorem 2.6 (1) *Suppose* char $F=0$. *Then* $l_F(i,j,k)=1$ *if* i, j *and* k *satisfy the triangle inequality and is* 0 *otherwise.*
(2) *Suppose* char $F=p>0$ *and* i,j,k *are all* $\leqq p-1$. *Then* $l_F(i,j,k)=1$ *if* (i,j,k) *is in the closed tetrahedron with vertices at* $(0,0,0)$, $(p-1,p-1,0)$, $(p-1,0,p-1)$ *and* $(0,p-1,p-1)$, *and is* 0 *otherwise.*

Proof. Theorems 1.10 and 2.3 give (1). When the the hypotheses of (2) hold, Theorems 1.10 and 2.5 show that $l_F(i,j,k)=1$ provided i,j,k satisfy the triangle inequality and in addition $i+j+k\leqq 2p-2$, and is 0 otherwise. This gives (2).

Theorem 2.7 (1) *Suppose* char $F=0$ and $r\geqq 2$. *Then* $\prod_1^r \lambda_{k_i}=\sum c_n \lambda_n$ *with each* $c_n\geqq 0$.

(2) *Suppose* char $F=p>0$, $r\geqq 2$ *and* $k_1,\ldots,k_r$ *are* $\leqq p-1$. *Then* $\prod_1^r \lambda_{k_i}=\sum_0^{p-1} c_n\lambda_n$ *with* $0\leqq c_n\leqq p^{r-2}$.

Proof. For $r=2$ use Theorems 2.3 and 2.5. For $r>2$ use Theorems 2.3, 2.5 and induction.

Corollary 2.8 (1) *If* char $F=0$, $l_F(k_1,\ldots,k_s)\geqq 0$.
(2) *If* char $F=p>0$, $s\geqq 3$, *and* $k_1,\ldots,k_s$ *are* $\leqq p-1$, *then* $0\leqq l_F(k_1,\ldots,k_s)\leqq p^{s-3}$.

Proof. $l_F(k_1,\ldots,k_s)$ is the coefficient of λ_{k_s} in $\prod_1^{s-1}\lambda_{k_i}$. Now apply Theorem 2.7.

Lemma 2.9 *Suppose* char $F=0$ and $i+1\geqq j$. Then $\lambda_i\cdot\delta_j=(-)^{j+1}\sum_{a=1}^{j}\lambda_{i+j+1-2a}$.

Proof. $(-)^{j+1}\cdot\delta_j=(\lambda_{j-1}-\lambda_{j-2})+(\lambda_{j-3}-\lambda_{j+4})+\ldots$ with the final term being $\lambda_1-\lambda_0$ if j is even and λ_0 if j is odd. Multiplying by λ_i and using Theorem 2.3 we find that $(-)^{j+1}\cdot\lambda_i\delta_j=(\lambda_{i+j-1}+\lambda_{i-j+1})+(\lambda_{i+j-3}+\lambda_{i-j+3})+\ldots$ with the final term being $\lambda_{i+1}+\lambda_{i-1}$ if j is even and λ_i if j is odd. This gives the lemma.

Thorem 2.10 *Suppose* char $F=0$. *Then* $\delta_i\delta_j=\sum_{a=1}^{\min(i,j)}\delta_{i+j+1-2a}$.

Proof. We argue by induction on $i+j$. We may assume $i\geqq j$. By Lemma 2.9,
$$(\delta_i-\delta_{i-1})\,\delta_j=(-)^{i-1}\lambda_{i-1}\,\delta_j=(-)^{i+j}\sum_{a=1}^{j}\lambda_{i+j-2a}=\left(\sum_{a=1}^{j}\delta_{i+j+1-2a}\right)-\left(\sum_{a=1}^{j}\delta_{i+j-2a}\right).$$
By induction, $\delta_{i-1}\cdot\delta_j=\sum_{a=1}^{j}\delta_{i+j-2a}$; the result folows.

Corollary 2.11 *Suppose* char $F=0$. *Then* $D_F(i,j,k_3,\ldots,k_s)=\sum_{a=1}^{\min(i,j)} D_F(i+j+1-2a,k_3,\ldots,k_s)$.

Proof. Multiply the identity of Theorem 2.10 by $\prod_3^s\delta_{k_i}$ and apply α.

We shall use the above corollary to give a simple explicit formula for D_F when char $F=0$.

Definition 2.12 Let $k_1,\ldots,k_s$ be non-negative integers and r a (possibly negative) integer. Then $D^{(r)}(k_1,\ldots,k_s)$ is the co-efficient of $t^{r+\gamma}$ in $\prod_1^s((1-t^{k_i})/(1-t))$ where $\gamma=[\sum(k_i-1)/2]$.

(In particular, when $\sum(k_i-1)$ is even, $D^{(0)}(k_1, \ldots, k_s)$ is the "middle" co-efficient in the polynomial $\prod_1^s((1-t^{k_i})/(1-t))$, while when $\sum(k_i-1)$ is odd it is one of the two equal "middle" co-efficients of this polynomial).

Lemma 2.13 $D^{(r)}(i,j,k_3,\ldots,k_s)=\sum_{a=1}^{\min(i,j)} D^{(r)}(i+j+1-2a, k_3, \ldots, k_s)$.

Proof. Note that $(1-t^i)(1-t^j)/(1-t)^2=\sum_{a=1}^{\min(i,j)} t^{a-1}(1-t^{i+j+1-2a})/(1-t)$. Set $\gamma=[((i+j-2)/2)+(\sum(k_i-1)/2)]$ and $\gamma_a=[((i+j-2a)/2)+(\sum(k_i-1)/2)]$ so that $\gamma_a=\gamma-(a-1)$. Then the co-efficient of $t^{r+\gamma}$ in $((1-t^i)(1-t^j)/(1-t)^2)\cdot\prod((1-t^{k_i})/(1-t))$ is $D^{(r)}(i,j,k_3,\ldots,k_s)$. On the other hand the co-efficient of $t^{r+\gamma}$ in $(t^{a-1}(1-t^{i+j+1-2a})/(1-t))\cdot\prod((1-t^{k_i})/(1-t))$ is the same as the co-efficient of $t^{r+\gamma_a}$ in $((1-t^{i+j+1-2a})/(1-t))\cdot\prod((1-t^{k_i})/(1-t))$ and this is $D^{(r)}(i+j+1-2a, k_3, \ldots, k_s)$. Multiplying the identity in the first line of this paragraph by $\prod((1-t^{k_i})/(1-t))$ and equating the co-efficients of $t^{r+\gamma}$ we get the lemma.

Theorem 2.14 *If* char $F=0$, $D_F(k_1,\ldots,k_s)=D^{(0)}(k_1,\ldots,k_s)$; *the middle co-efficient of* $\prod((1-t^{k_i})/(1-t))$.

Proof. We argue by induction on s. When $s=1$, $D_F(k)=D^{(0)}(k)=0$ if $k=0$, and $D_F(k)=D^{(0)}(k)=1$ if $k\geqq 1$. When $s\geqq 2$ we use Corollary 2.11, Lemma 2.13 with $r=0$ and induction.

When char $F=p>0$ and each $k_i\leqq p$ we shall obtain an almost equally simple explicit formula for $D_F(k_1,\ldots,k_s)$; it is $\sum_{j\equiv 0(p)} D^{(j)}(k_1,\ldots,k_s)$. We need an analogue of Theorem 2.10.

Theorem 2.15 *Suppose* char $F=p>0$ *and* $i+j\leqq p$. *Then* $\delta_i\cdot\delta_j=\sum_{a=1}^{\min(i,j)}\delta_{i+j+1-2a}$. *It follows in particular that* $D_F(i,j,k_3,\ldots,k_s)=\sum_{a=1}^{\min(i,j)} D_F(i+j+1-2a, k_3, \ldots, k_s)$.

Proof. To calculate $\delta_i\delta_j$ we only need determine $\lambda_r\lambda_s$ for $r<i,s<j$. But then $r+s\leqq p-1$ and the same formula holds for $\lambda_r\lambda_s$ in characteristics p and 0. (See Theorems 2.3 and 2.5). So the conclusion of Theorem 2.10 is still valid – to get the second result we multiply by $\prod\delta_{k_i}$ and apply α.

We also need a reduction formula for $D_F(i,j,k_3,\ldots,k_s)$ when i and j are $\leqq p$ but $i+j>p$.

Lemma 2.16 *Suppose* char $F=p>0$. *If* $k_1,\ldots,k_s$ *are* $\leqq p$ *then* $D_F(k_1,\ldots,k_s,p)=\prod k_i$.

Proof. $(x_1+\ldots+x_s)^p=\sum_1^s x_i^p\in(x_1^{k_1},\ldots,x_s^{k_s})$, and we use remark (1) after Definition 1.1.

Lemma 2.17 *Suppose* char $F=p>0$. Then for $0\leqq i\leqq p$, $\lambda_{p-1}\cdot\delta_i=(-)^p(\delta_{p-i}-\delta_p)$.

Proof. By induction on i, the case $i=0$ being obvious. Suppose $i \geqq 1$. Then $\lambda_{p-1}(\delta_i - \delta_{i-1}) = (-)^{i-1}\lambda_{p-1}\lambda_{i-1} = (-)^{i-1}\lambda_{p-i} = (-)^p(\delta_{p-i} - \delta_{p+1-i})$. By induction, $\lambda_{p-1}\cdot\delta_{i-1} = (-)^p(\delta_{p+1-i} - \delta_p)$. Adding, we get the result.

Theorem 2.18 *Suppose* $\operatorname{char} F = p > 0$ *and* $i, j, k_3, \ldots, k_s$ *are all* $\leqq p$. *Then* $D_F(i, j, k_3, \ldots, k_s) = D_F(p-i, p-j, k_3, \ldots, k_s) + (i+j-p)\prod k_i$.

Proof. Lemma 2.17 shows that $\delta_i \cdot \delta_j = \lambda_{p-1}^2 \delta_i \delta_j = (\lambda_{p-1}\delta_i)(\lambda_{p-1}\delta_j) = (\delta_{p-i} - \delta_p)(\delta_{p-j} - \delta_p)$. Multiplying by $\prod \delta_{k_i}$, applying α and using Lemma 2.16 we get the desired result.

Lemma 2.19 (1) *Suppose* i *and* j *are* $< n$ *but* $i+j \geqq n$. *Then* $D^{(r)}(i, j, k_3, \ldots, k_s) = D^{(r)}(n-i, n-j, k_3, \ldots, k_s) + D^{(r)}(n, i+j-n, k_3, \ldots, k_s)$.

(2) $\sum_{r \equiv 0(n)} D^{(r)}(n, k_1, \ldots, k_s) = \prod k_i$.

Proof. Note that $(1-t^i)(1-t^j)/(1-t)^2 = (t^{i+j-n}(1-t^{n-i})(1-t^{n-j})/(1-t)^2) + ((1-t^n)(1-t^{i+j-n})/(1-t)^2)$. If we multiply this identity by $\prod((1-t^{k_i})/(1-t))$ and equate the co-efficients of $t^{r+\gamma}$ where $\gamma = [((i+j-2)/2) + (\sum(k_i-1)/2)]$ we get (1). To prove (2), set $\prod((1-t^{k_i})/(1-t)) = \sum a_j t^j$ and let $\gamma = [((n-1)/2) + (\sum(k_i-1)/2)]$. Then $D^{(r)}(n, k_1, \ldots, k_s) = \sum_{j=0}^{n-1} a_{r+\gamma-j}$. Summing over all r, (positive, negative or zero), that are $\equiv 0(n)$ we get $\sum_j a_j$, which is precisely $\prod k_i$.

Theorem 2.20 *If* $\operatorname{char} F = p > 0$ *and each* $k_i \leqq p$, *then* $D_F(k_1, \ldots, k_s) = \sum_{j \equiv 0(p)} D^{(j)}(k_1, \ldots, k_s)$.

Proof. When $s=1$, $D^{(j)}(k) = 0$ whenever $|j| \geqq k$. Since $0 \leqq k \leqq p$, $\sum_{j \equiv 0(p)} D^{(j)}(k) = D^{(0)}(k)$; as we saw in the proof of Theorem 2.14 this is $D_F(k)$. Suppose now that $s \geqq 2$. If each $k_i \leqq p$ set $D_F^*(k_1, \ldots, k_s) = \sum_{j \equiv 0(p)} D^{(j)}(k_1, \ldots, k_s)$. If $k_1 + k_2 \leqq p$, Lemma 2.13 shows that $D_F^*(k_1, k_2, \ldots, k_s) = \sum_{a=1}^{\min(k_1, k_2)} D_F^*(k_1 + k_2 + 1 - 2a, k_3, \ldots, k_s)$. Theorem 2.15 and induction on s show that $D_F(k_1, \ldots, k_s) = D_F^*(k_1, \ldots, k_s)$. If however $k_1 + k_2 \geqq p$, Lemma 2.19 shows that $D_F^*(k_1, k_2, \ldots, k_s) = D_F^*(p-k_1, p-k_2, k_3, \ldots, k_s) + (k_1 + k_2 - p)\cdot\left(\prod_3^s k_i\right)$, while Theorem 2.18 shows that D_F satisfies the same functional equation. So we can replace k_1 and k_2 by $p-k_1$ and $p-k_2$ and reduce to the case $k_1 + k_2 \leqq p$, which we've already treated.

3 More multiplication formulae in Γ_F

For the rest of this paper F has characteristic $p > 0$; q will always denote a power of p.

Lemma 3.1 *Suppose* k_1 *and* k_2 *are* $\leqq q$, *and* $k_3 \geqq q$. *Then* $D_F(k_1, k_2, k_3) = k_1 k_2$.

Proof. Since $(x+y)^q = x^q + y^q$, $(x+y)^{k_3} \in (x^{k_1}, y^{k_2})$.

Theorem 3.2 *The additive subgroup of Γ_F generated by the λ_i, $0 \leqq i \leqq q-1$, is closed under multiplication.*

Proof. Suppose k_1 and k_2 are $\leqq q-1$ and $k_3 \geqq q$. By Lemma 3.1, $D_F(k_1+\varepsilon_1, k_2+\varepsilon_2, k_3+\varepsilon_3) = (k_1+\varepsilon_1)(k_2+\varepsilon_2)$. So $l_F(k_1, k_2, k_3) = 0$; now apply Theorem 1.10 to find that the co-efficient of λ_{k_3} in $\lambda_{k_1}\lambda_{k_2}$ is 0.

Lemma 3.3 *Suppoose $1 \leqq i \leqq q$ and $j \geqq 1$. Then:*

(1) $\delta_i \delta_{qj} = i \cdot \delta_{qj}$.
(2) $\delta_i \delta_{qj+1} = (i-1)\,\delta_{qj} + \delta_{qj+i}$.
(3) $\delta_i \delta_{qj-1} = (i-1)\,\delta_{qj} + \delta_{qj-i}$.

Proof. Let M be the F-object $F[x]/x^i$ with T acting by multiplication by x. Let N, N' and N'' be the F-objects $F[y]/y^{qj}$, $F[y]/y^{qj+1}$ and $F[y]/y^{qj-1}$ with T acting by multiplication by y. The number of T-cyclic summands in $M \otimes N$ is $D_F(i, qj) = i$, and the F-dimension of $M \otimes N$ is qij. Since $(x+y)^{qj} = (x^q + y^q)^j \in (x^q, y^{qj})$, T^{qj} annihilates $M \otimes N$, giving (1). The number of T-cyclic summands in $M \otimes N'$ is also i. So $M \otimes N' \approx \bigoplus_1^i F[T]/T^{n_k}$. Since $(x+y)^{qj+1} \in (x^i, y^{qj})$, $D_F(i, qj+1, qj) = qij$ and $\dim M \otimes N'/T^{qj}(M \otimes N') = qij$. So each $n_k \geqq qj$. Also, modulo (x^i, y^{qj+1}), $(x+y)^{qj+i-1} \equiv y^{qj} \cdot (x+y)^{i-1} \equiv y^{qj} x^{i-1} \not\equiv 0$. So T^{qj+i-1} does not annihilate $M \otimes N'$ and some $n_k \geqq qj+i$. Since the F-dimension of $M \otimes N'$ is $qij+i$ we get (2). The case $i=q, j=1$ of (3) follows from (1). In all other cases, $M \otimes N''$ has $D_F(i, qj-1) = i$ T-cyclic summands. So $M \otimes N'' \approx \bigoplus_1^i F[T]/T^{n_k}$. As in (1) we see that T^{qj} annihilates $M \otimes N''$; thus each $n_k \leqq qj$. Furthermore, modulo (x^i, y^{qj-i+1}), $(x+y)^{qj-1} \equiv \pm x^{i-1} y^{qj-i} \not\equiv 0$. So $D_F(i, qj-1, qj-i+1) < i(qj-i+1)$, $\dim(M \otimes N'')/T^{qj-i+1}(M \otimes N'') < i(qj-i+1)$, and some $n_k \leqq qj-i$. Since the F-dimension of $M \otimes N''$ is $qij-i$ we get (3).

Theorem 3.4 *Suppose $i \leqq q-1$. Then:*

(1) $\lambda_i \cdot \lambda_{qj} = \lambda_{qj+i}$.
(2) *If $j \neq 0$, $\lambda_i \cdot \lambda_{qj-1} = \lambda_{qj-1-i}$.*

Proof. If $i=0$ or $j=0$ the result is clear. In all other cases (1) and (2) of Lemma 3.3 show that $\delta_i(\delta_{qj+1} - \delta_{qj}) = \delta_{qj+i} - \delta_{qj}$. Replacing i by $i+1$ we find that $(\delta_{i+1} - \delta_i)(\delta_{qj+1} - \delta_{qj}) = \delta_{qj+i+1} - \delta_{qj+i}$. Multiplying by $(-)^i \cdot (-)^{qj}$ we get (1). The proof of (2) is similar using (1) and (3) of Lemma 3.3.

Remark. A useful consequence of Theorem 3.4 is that $(\lambda_{q-1})^2$ is the unit element, λ_0, of Γ.

Corollary 3.5 *If k_1 and k_2 are $< q$, $l_F(k_1, \ldots, k_s) = l_F(q-1-k_1, q-1-k_2, k_3, \ldots, k_s)$.*

Proof. $(\lambda_{q-1-k_1}\lambda_{q-1-k_2}) = (\lambda_{k_1}\lambda_{q-1})(\lambda_{k_2}\lambda_{q-1}) = \lambda_{k_1}\lambda_{k_2}$. Now multiply by $\prod_3^s \lambda_{k_i}$ and apply α.

Theorem 3.6 *Let $I(q)$ be the ideal of Γ_F consisting of elements annihilated by each $\delta_i - i (1 \leqq i \leqq q)$. Then $u \in I(q)$ if and only if u is a Z-linear combination of δ_{qj}. (So the additive subgroup of Γ_F generated by the δ_{qj} is an ideal.)*

Proof. Lemma 3.3 shows that each $\delta_{qj} \in I(q)$. Suppose $u = \sum_0^r a_i \lambda_i$ is in $I(q)$ but not in $\sum Z \delta_{qj}$. We may assume that $a_r \neq 0$ and that r is as small as possible. Write $r = qa + b$ with $0 \leqq b \leqq q-1$. If $b = q-1$ we may reduce r by modifying u by a multiple of δ_{qa+q}. So $b \neq q-1$. Set $c = q-1-b$. Then $\binom{r+c}{c} = \binom{qa+(q-1)}{c}$ which $\neq 0$ in F. Lemma 2.1 then shows that the co-efficient of λ_{r+c} in $\lambda_c \cdot \left(\sum_0^r a_i \lambda_i\right)$ is $\neq 0$. But since $u \in I(q)$, $\lambda_c \cdot u = (-1)^c (\delta_{c+1} - \delta_c) u = (-)^c u$. Since $c > 0$ the co-efficient of λ_{r+c} in $\lambda_c \cdot u$ is 0, a contradiction.

Our next goal is to calculate $\lambda_q \cdot \lambda_{qj}$.

Theorem 3.7 *If p/j, $\lambda_q \cdot \lambda_{qj} = \lambda_{qj+q}$. If $p/j+1$, $\lambda_q \cdot \lambda_{qj} = \lambda_{qj-q}$.*

Proof. The first assertion follows from Theorem 3.4 with q replaced by pq. For the second, note that $\lambda_{q-1} \cdot \lambda_{qj} = \lambda_{qj+q-1}$. Multiplying by $\lambda_{q-1} \cdot \lambda_q$ we find that $\lambda_q \lambda_{qj} = (\lambda_q \cdot \lambda_{q(j+1)-1}) \cdot \lambda_{q-1}$. Replacing q by pq in Theorem 3.4 we find that this is $(\lambda_{qj-1}) \lambda_{q-1}$. Finally Theorem 3.4 shows this to be λ_{qj-q}.

Lemma 3.8 $D_F(q+1,\ qj+1, qj+1) - D_F(q+1, qj, qj+1) \leqq 2$.

Proof. Let $A = (x^{q+1}, y^{qj}, (x+y)^{qj+1})/(x^{q+1},\ y^{qj+1}, (x+y)^{qj+1})$. Then A is a cyclic $F[x, y]$ module, generated by y^{qj} and annihilated by y. Also, modulo (x^{q+1}, y^{qj+1}), $x(x+y)^{qj+1} \equiv x(x+y)(x^q + y^q)^j \equiv x^2 y^{qj}$. So, in A, y^{qj} is annihilated by x^2 as well, and $\dim_F(A) \leqq 2$, giving the lemma.

Lemma 3.9 *If $p \nmid j$ or $j+1$ then:*

(1) $\delta_{q+1} \cdot \delta_{qj+1} = \delta_{qj+q+1} + \delta_{qj-q+1} + \delta_{qj+q-1} + (q-2)\ \delta_{qj}$

(2) $\delta_{q+1} \cdot \delta_{qj} = \delta_{qj+q} + \delta_{qj-q} + (q-1)\ \delta_{qj}$.

Proof. Let M be the F-object $F[x]/x^{q+1}$ with T acting by multiplication by x. Let N be the F-object $F[y]/y^{qj+1}$ with T acting by multiplication by y. The number of T-cyclic summands in $M \otimes N$ is $D_F(q+1, qj+1) = q+1$. So $M \otimes N \approx \bigoplus_1^{q+1} F[T]/T^{n_k}$.

If $n \geqq qj+q+1$, $(x+y)^n \in (x^{q+1}, y^{qj+1})$. So T^{qj+q+1} annihilates $M \otimes N$. Since the co-efficient of $x^q y^{qj}$ in $(x+y)^{qj+q}$ is $j+1$ which $\neq 0$ in F, $(x+y)^{qj+q} \notin (x^{q+1}, y^{qj+1})$, T^{qj+q} does not annihilate $M \otimes N$ and $\max n_k = qj+q+1$.

Since $(x+y)^{qj+1} \in (x^{q+1}, y^{qj-q+1})$, $D_F(q+1, qj+1, qj-q+1) = (q+1)(qj-q+1)$, $\dim(M \otimes N)/T^{qj-q+1}(M \otimes N) = (q+1)(qj-q+1)$, and each $n_k \geqq qj-q+1$. Using the fact that $j \neq 0$ in F we find that $(x+y)^{qj+1} \notin (x^{q+1}, y^{qj-q+2})$, that $D_F(q+1, qj+1, qj-q+2) < (q+1)(qj-q+2)$ and consequently that $\min n_k = qj-q+1$.

Note next that $D_F(q+1), qj+1, qj) = \dim F[x, y]/(x^{q+1}, y^{qj}, (x+y)^{qj+1}) = \dim F[x, y]/(x^{q+1}, y^{qj}, jx^q y^{qj-q+1})$. Since $j \neq 0$ in F and the annihilator of $x^q y^{qj-q+1}$ in $F[x, y]/(x^{q+1}, y^{qj})$ is (x, y^{q-1}), this dimension is $(q+1)(qj) - (q-1)$. So $\dim(M \otimes N)/T^{qj}(M \otimes N)$ is $(q+1)(qj) - (q-1)$. Since $\min(n_k) = qj - (q-1)$ it

follows that q of the n_k are $\geqq qj$. Also, by Lemma 3.8, $\dim T^{qj}(M\otimes N)/T^{qj+1}(M\otimes N)\leqq 2$. So at most two n_k are $>qj$. Thus, of the $q+1$ n_k, one is $qj-q+1$, one is $qj+q+1$ and $q-2$ (at least) are qj. Since $\dim(M\otimes N)=(q+1)(qj+1)$, the last n_k must be $qj+q-1$. This gives (1); the proof of (2) is similar but simpler.

Theorem 3.10 *If $p\nmid j$ or $j+1$ then $\lambda_q\cdot\lambda_{qj}=\lambda_{qj+q}+\lambda_{qj-q}+\lambda_{qj+q-1}$.*

Proof. By Lemma 3.9, $\delta_{q+1}(\delta_{qj+1}-\delta_{qj})=(\delta_{qj+q+1}-\delta_{qj+q})+(\delta_{qj-q+1}-\delta_{qj-q})+(\delta_{qj+q-1}-\delta_{qj})$. By Lemma 3.3 with $i=q$, $\delta_q(\delta_{qj+1}-\delta_{qj})=\delta_{qj+q}-\delta_{qj}$. So $(\delta_{q+1}-\delta_q)(\delta_{qj+1}-\delta_{qj})=(\delta_{qj+q+1}-\delta_{qj+q})+(\delta_{qj-q+1}-\delta_{qj-q})-(\delta_{qj+q}-\delta_{qj+q-1})$. Multiplying by $(-)^q\cdot(-)^{qj}$ we get the result.

Remark. Note the similarity between Theorems 2.2 and 3.10.

Theorem 3.11 *Suppose $0\leqq i\leqq q-1$.*

(1) *If $p\nmid j$ or $j+1$, $\lambda_q\cdot\lambda_{qj+i}=\lambda_{qj+q+i}+\lambda_{qj-q+i}+\lambda_{qj+q-1-i}$.*
(2) *If $p\mid j$, $\lambda_q\cdot\lambda_{qj+i}=\lambda_{qj+q+i}$.*
(3) *If $p\mid j+1$, $\lambda_q\cdot\lambda_{qj+i}=\lambda_{qj-q+i}$.*

Proof. Suppose $p\nmid j$ or $j+1$. By Theorems 3.4 and 3.10, $\lambda_q\cdot\lambda_{qj+i}=\lambda_q\cdot\lambda_{qj}\cdot\lambda_i=\lambda_i(\lambda_{qj+q}+\lambda_{qj-q}+\lambda_{qj+q-1})$. Using Theorem 3.4 once again we get (1). The proofs of (2) and (3) are similar, using Theorem 3.7.

Definition 3.12 E_q is the additive subgroup of Γ generated by the λ_n, $n\equiv 0$ or $-1(2q)$. O_q is the additive subgroup of Γ generated by the λ_n, $n\equiv q$ or $q-1(2q)$.

Lemma 3.13 *Let q' be a power of p with $q'\geqq q$. If q'/q is even, then $\lambda_{q'}\cdot E_q\subset E_q$ and $\lambda_{q'}\cdot O_q\subset O_q$. If q'/q is odd, then $\lambda_{q'}\cdot E_q\subset O_q$ and $\lambda_{q'}\cdot O_q\subset E_q$.*

Proof. Suppose $n\equiv 0, q-1, q$ or $2q-1 \bmod 2q$. Write $n=q'j+i$ with $0\leqq i\leqq q'-1$. By Theorem 3.11, $\lambda_{q'}\cdot\lambda_n$ is either $\lambda_{n+q'}$, $\lambda_{n-q'}$ or $\lambda_{n+q'}+\lambda_{n-q'}+\lambda_{2jq'+(q'-n-1)}$. So if, for example, $n\equiv 0$ $(2q)$ and q'/q is odd, then $n+q'\equiv q(2q)$, $n-q'\equiv q(2q)$ and $2jq'+(q'-n-1)\equiv q-1(2q)$. Thus $\lambda_{q'}\cdot\lambda_n\in O_q$. The argument in the remaining cases is similar.

Lemma 3.14 *Let q' be a power of p with $q'\geqq q$. If q'/q is even, then $\lambda_{q'-1}\cdot E_q\subset E_q$ and $\lambda_{q'-1}\cdot O_q\subset O_q$. If q'/q is odd, then $\lambda_{q'-1}\cdot E_q\subset O_q$ and $\lambda_{q'-1}\cdot O_q\subset E_q$.*

Proof. Suppose $n\equiv 0, q-1, q$ or $2q-1 \bmod 2q$. Write $n=q'j+i$ with $0\leqq i\leqq q-1$. Then $\lambda_{q'-1}\cdot\lambda_n=\lambda_{q'-1}\cdot\lambda_{q'j}\cdot\lambda_i=\lambda_i\cdot\lambda_{q'j+q'-1}=\lambda_{q'j+q'-1-i}=\lambda_{2q'j+(q'-n-1)}$. The rest is easy.

Theorem 3.15 *If r is even, $\lambda_{qr}\cdot E_q\subset E_q$ and $\lambda_{qr}\cdot O_q\subset O_q$. If r is odd, then $\lambda_{qr}\cdot E_q\subset O_q$ and $\lambda_{qr}\cdot O_q\subset E_q$.*

Proof. By induction on r. We may assume $r>0$. Write $qr=q'j$ with q' a power of p, $q'\geqq q$ and j prime to p. If $j=1$ use Lemma 3.13. Suppose $j>1$. Since j is prime to p, $\lambda_{q'}\cdot\lambda_{q'j-q'}$ is either $\lambda_{q'j}$ or $\lambda_{q'j}+\lambda_{q'j-2q'}+(\lambda_{q'-1})\cdot(\lambda_{q'j-q'})$. Suppose now, for example, that $r=q'j/q$ is odd and that q'/q is odd. By induction $\lambda_{q'j-q'}$ stabilizes E_q and O_q while $\lambda_{q'j-2q'}$ interchanges E_q and O_q. By Lemmas 3.13

and 3.14, $\lambda_{q'}$ and $\lambda_{q'-1}$ interchange E_q and O_q. We conclude that $\lambda_{q'j}$ interchanges E_q and O_q. The argument in the remaining cases is similar.

Corollary 3.16 $E_q \cdot E_q \subset E_q$, $E_q \cdot O_q \subset O_q$ and $O_q \cdot O_q \subset E_q$.

Corollary 3.17 $\prod_1^s \lambda_{qr_i}$ *lies in* E_q *if* $\sum r_i$ *is even and in* O_q *if* $\sum r_i$ *is odd.*

4 Functional equations for l_F and D_F

Theorem 4.1 *There is an endomorphism* ψ_q *of* Γ_F *with the following properties*:
(1) $\psi_q(\delta_{qr}) = q\,\delta_r$ *for all* r.
(2) $\psi_q(\lambda_{qr}) = (-)^{(q-1)r} \cdot \lambda_r$. *If* $r \geqq 1$, $\psi_q(\lambda_{qr-1}) = (-)^{(q-1)r} \cdot \lambda_{r-1}$.

Proof. Suppose M is an F-object with T_M: $M \to M$ the multiplication by T map. Let $\psi_q(M)$ be the F-object which is M as vector space over F, but with T operating by T_M^q. There is an obvious Z-linear map $\Gamma \to \Gamma$ taking the class of M to the class of $\psi_q(M)$ for all M; this map is also denoted by ψ_q. If M and N are F-objects, $((T_M \otimes 1) + (1 \otimes T_N))^q = \sum_{i=0}^{q} \binom{q}{i} (T_M^i \otimes T_N^{q-i}) = (T_M^q \otimes 1) + (1 \otimes T_N^q)$. So $\psi_q(M \otimes N)$ identifies with $\psi_q(M) \otimes \psi_q(N)$ as $F[T]$-module, and ψ_q is a ring endomorphism of Γ. Suppose $0 \leqq k \leqq q-1$. Let M be the $F[T]$-module $F[x]/x^{qr+k}$ with T acting by multiplication by x^q. Then M represents $\psi_q(\delta_{qr+k})$ in Γ. But M is the direct sum of T-cyclic submodules generated by $1, x, \ldots, x^{q-1}$. We conclude that $\psi_q(\delta_{qr+k}) = (q-k)\,\delta_r + k \cdot \delta_{r+1}$; (1) and (2) follow immediately.

Corollary 4.2 (1) *If* u *is in the ideal* $I(q)$ *of Theorem 3.6,* $\alpha(\psi_q(u)) = q\,\alpha(u)$.
(2) *If* u *is in* E_q, *then* $\alpha(\psi_q(u)) = \alpha(u)$.

Proof. Theorem 4.1 shows that $\alpha(\psi_q(u)) = q \cdot \alpha(u)$ whenever $u = \delta_{qr}$ and that $\alpha(\psi_q(u)) = \alpha(u)$ whenever $u = \lambda_{qr}$ or λ_{qr-1} with r even. The result follows by linearity.

Lemma 4.3 *If* $k_1, \ldots, k_s$ *are* $< q$ *then* $\prod_1^s \lambda_{k_i} = \sum_0^{q-1} a_i \lambda_i$ *with* $a_0 = l_F(k_1, \ldots, k_s)$ *and* $a_{q-1} = l_F(q-1-k_1, k_2, \ldots, k_s)$.

Proof. By Theorem 3.2, $\prod_1^s \lambda_{k_i} = \sum_0^{q-1} a_i \lambda_i$. Then $a_0 = \alpha\left(\prod_1^s \lambda_{k_i}\right) = l_F(k_1, \ldots, k_s)$. Also, $a_{q-1} = \left(\alpha\left(\lambda_{q-1} \cdot \prod_1^s \lambda_{k_i}\right)\right.$. By Theorem 3.4 this is $\alpha\left(\lambda_{q-1-k_1} \cdot \prod_2^s \lambda_{k_i}\right) = l_F(q-1-k_1, k_2, \ldots, k_s)$.

Lemma 4.4 *Let* $u = \prod_1^s \lambda_{qr_i}$. *Write* $u = \sum b_i \lambda_i$. *Then*:
(1) *If* $\sum r_i$ *is even,* $b_1 = \ldots = b_{q-1} = 0$ *and* $b_0 = l_F(r_1, \ldots, r_s)$.
(2) *If* $\sum r_i$ *is odd,* $b_0 = \ldots = b_{q-2} = 0$ *and* $b_{q-1} = l_F(r_1, \ldots, r_s)$.

Proof. If $\sum r_i$ is even, Corollary 3.17 shows that $u \in E_q$. So $b_1 = \ldots = b_{q-1} = 0$ and $b_0 = \alpha(u)$; by Corollary 4.2 this is $\alpha(\psi_q(u))$. By Theorem 4.1, $\psi_q(u) = \prod_1^s \lambda_{r_i}$. So $b_0 = l_F(r_1, \ldots, r_s)$. If $\sum r_i$ is odd, Corollary 3.17 shows that $u \in O_q$. So $b_0 = \ldots = b_{q-2} = 0$ and $b_{q-1} = \alpha(\lambda_{q-1} u)$. By Lemma 3.14, $\lambda_{q-1} \cdot u \in E_q$. So $b_{q-1} = \alpha(\psi_q(\lambda_{q-1}) \cdot \psi_q(u))$. Now Theorem 4.1 shows that $\psi_q(\lambda_{q-1}) = (-)^{q-1} \lambda_0$ while $\psi_q(u) = (-)^{q-1} \prod \lambda_{r_i}$. So $b_{q-1} = \alpha(\prod \lambda_{r_i}) = l_F(r_1, \ldots, r_s)$.

Theorem 4.5 *Suppose $k_1, \ldots, k_s$ are $< q$ and $r_1, \ldots, r_s$ are $\geqq 0$.*

(1) *If $\sum r_i$ is even, $l_F(qr_1 + k_1, \ldots, qr_s + k_s) = l_F(r_1, \ldots, r_s) \cdot l_F(k_1, \ldots, k_s)$.*
(2) *If $\sum r_i$ is odd, $l_F(qr_1 + k_1, \ldots, qr_s + k_s) = l_F(r_1, \ldots, r_s) \cdot l_F(q - 1 - k_1, k_2, \ldots, k_s)$.*

Proof. $l_F(qr_1 + k_1, \ldots, qr_s + k_s) = \alpha\left(\prod_1^s \lambda_{qr_i + k_i}\right)$. By Theorem 3.4 this is $\alpha((\prod \lambda_{k_i}) \cdot (\prod \lambda_{qr_i}))$. Write $\prod \lambda_{k_i} = \sum_0^{q-1} a_i \lambda_i$ and $\prod \lambda_{qr_i} = \sum b_i \lambda_i$. Then $l_F(qr_1 + k_1, \ldots, qr_s + k_s) = \sum_0^{q-1} a_i b_i$. Now use Lemmas 4.3 and 4.4.

Theorem 4.6 (1) *Suppose $s \geqq 3$ and $k_1, \ldots, k_s$ are all $< q$. Then $0 \leqq l_F(k_1, \ldots, k_s) \leqq q^{s-3}$.*
(2) *Suppose $p = 2$. Then $l_F(k_1, \ldots, k_s) = 0$ or 1.*

Proof. When $q = p$, (1) comes from Corollary 2.8. For $q = p^j$ one argues by induction on j using Theorem 4.5. Suppose now that each $k_i < p = 2$. Since $\lambda_{k_i} = \lambda_0$ or λ_1 and $\lambda_1^2 = \lambda_0$, $\prod \lambda_{k_i}$ is either λ_1 or λ_0, and $l_F(k_1, \ldots, k_s) = \alpha(\prod \lambda_{k_i})$ is either 0 or 1. When each $k_i < 2^j$ we argue by induction on j, using Theorem 4.5.

We next develop functional equations for D_F analogous to the functional equations of Theorem 4.5 for l_F. For applications to Hilbert-Kunz functions it will be convenient to extend D_F to a function $[0, \infty)^s \to \mathbb{R}$ by multilinear interpolation, and to derive functional equations for this extension (which we shall also denote by D_F).

Definition 4.7 Γ' is the $\mathbb{R}$-algebra $\Gamma \otimes_Z \mathbb{R}$.

λ_j and δ_j may be viewed as elements of Γ'. The map $\Gamma' \to \mathbb{R}$ taking $\sum c_j \lambda_j$ to c_0 will again be denoted by α.

Definition 4.8 Suppose $0 \leqq z \leqq 1$ and j is a non-negative integer. Then δ_{j+z} is the element $(1 - z) \cdot \delta_j + z \delta_{j+1}$ of Γ'.

Definition 4.9 Let $u_1, \ldots, u_s$ be non-negative real numbers. Then $D_F(u_1, \ldots, u_s)$ is the real number $\alpha\left(\prod_1^s \delta_{u_i}\right)$.

Lemma 4.10 *Let $u_1, \ldots, u_s$ be real numbers in $[0, q]$. Then $\prod_1^s \delta_{u_i} = \sum_0^{q-1} a_i \lambda_i$ with $a_0 = D_F(u_1, \ldots, u_s)$ and $a_{q-1} = (-)^q \left(D_F(q - u_1, u_2, \ldots, u_s) - \prod_2^s u_i\right)$.*

Proof. Each $\delta_{u_i} \in \sum_0^{q-1} \mathbb{R}\lambda_i$. Since this vector space is closed under multiplication, $\prod_1^s \delta_{u_i} = \sum_0^{q-1} a_i \lambda_i$. Then $a_0 = \alpha\left(\prod_1^s \delta_{u_i}\right) = D_F(u_1, \ldots, u_s)$. Also $a_{q-1} = \alpha\left(\lambda_{q-1} \cdot \prod_1^s \delta_{u_i}\right)$. Now if j is an integer in $[0, q]$, $\lambda_{q-1} \cdot \delta_j = (-)^q \delta_j(\delta_{q-1} - \delta_q)$; by Lemma 3.3 this is $(-)^q(\delta_{q-j} - \delta_q)$. Interpolating we find that $\lambda_{q-1} \cdot \delta_{u_1} = (-)^q(\delta_{q-u_1} - \delta_q)$. So $(-)^q \cdot a_{q-1} = \alpha\left(\delta_{q-u_1} \cdot \prod_2^s \delta_{u_i}\right) - \alpha\left(\delta_q \cdot \prod_2^s \delta_{u_i}\right)$. The first term is $D_F(q-u_1, u_2, \ldots, u_s)$. Furthermore Lemma 3.3 and interpolation show that $\delta_u \cdot \delta_q = u \cdot \delta_q$ for any $u \in [0, q]$. So the second term is $\left(\prod_2^s u_i\right) \cdot \alpha(\delta_q)$ which is $\prod_2^s u_i$.

Definition 4.11 Let $r = (r_1, \ldots, r_s)$ be an s-tuple of non-negative integers and A be a subset of $\{1, \ldots, s\}$. $n_A(r)$ is the integer $\alpha((\prod_{i \in A} (-)^{r_i} \lambda_{r_i}) \cdot (\prod_{j \notin A} \delta_{r_j}))$.

Definition 4.12 If A is a subset of $\{1, \ldots, s\}$, let $|A|$ denote the cardinality of A, and x_A the element $\prod_{i \in A} x_i$ of $Z[x_1, \ldots, x_s]$. Suppose $r = (r_1, \ldots, r_s)$ is an s-tuple of non-negative integers. Then $\phi_r \in Z[x_1, \ldots, x_s]$ is defined as follows:

(1) If $\sum r_i$ is even, $\phi_r = \sum_{|A| < s} n_A(r) \cdot x_A$.

(2) If $\sum r_i$ is odd, $\phi_r = (\sum_{|A| < s} n_A(r) \cdot x_A) - l_F(r_1, \ldots, r_s) \prod_2^s x_i$.

Remark. Suppose $u_1, \ldots, u_s$ are in $[0, 1]$. Then $D_F(r_1 + u_1, \ldots, r_s + u_s) = \alpha(\prod \delta_{r_i + u_i}) = \alpha(\prod((\delta_{r_i} + (-)^{r_i} u_i \lambda_{r_i}))$. If $\sum r_i$ is even, this is $\left(\prod_1^s u_i\right) \cdot l_F(r_1, \ldots, r_s) + \sum_{|A| < s} n_A \prod_{i \in A} u_A = l_F(r_1, \ldots, r_s) \cdot D_F(u_1, \ldots, u_s) + \phi_r(u_1, \ldots, u_s)$. If $\sum r_i$ is odd one gets $-\left(\prod_1^s u_i\right) \cdot l_F(r_1, \ldots, r_s) + \sum_{|A| < s} n_A \prod_{i \in A} u_A$. This is $(l_F(r_1, \ldots, r_s) \cdot (1 - u_1)\, u_2 \ldots u_s) + (\sum_{|A| < s} n_A \prod_{i \in A} u_i) - l_F(r_1, \ldots, r_s)\, u_2 \ldots u_s = l_F(r_1, \ldots, r_s)\, D_F(1 - u_1, u_2, \ldots, u_s) + \phi_r(u_1, \ldots, u_s)$. We now generalize this:

Theorem 4.13 *Suppose that* $u_1, \ldots, u_s$ are in $[0, q]$. *Then*:

(1) *If* $\sum r_i$ *is even,* $D_F(qr_1 + u_1, \ldots, qr_s + u_s) = l_F(r_1, \ldots, r_s)\, D_F(u_1, \ldots, u_s) + q^{s-1} \phi_r(u_1/q, \ldots, u_s/q)$.

(2) *If* $\sum r_i$ *is odd,* $D_F(qr_1 + u_1, \ldots, qr_s + u_s) = l_F(r_1, \ldots, r_s)\, D_F(q - u_1, u_2, \ldots, u_s) + q^{s-1} \phi_r(u_1/q, \ldots, u_s/q)$.

Proof. If k is an integer in $[0, q]$, $\delta_k \cdot \lambda_{qr_i} = (-1)^{qr_i} \delta_k(\delta_{qr_i + 1} - \delta_{qr_i})$. By Lemma 3.3 this is $(-)^{qr_i}(\delta_{qr_i + k} - \delta_{qr_i})$. Interpolating, we find that $\delta_{u_i} \cdot \lambda_{qr_i} = (-)^{qr_i}(\delta_{qr_i + u_i} - \delta_{qr_i})$. So $\delta_{qr_i + u_i} = \delta_{qr_i} + (-)^{qr_i} \lambda_{qr_i} \delta_{u_i}$. Multiplying these relations and applying α, we find that $D_F(qr_1 + u_1, \ldots, qr_s + u_s) = \sum_A \alpha(W_A)$ where $W_A = (\prod_{i \in A} (-)^{qr_i} \lambda_{qr_i} \delta_{u_i}) \cdot (\prod_{j \notin A} \delta_{qr_j})$. Suppose first that $A = \{1, \ldots, s\}$. Then

$\alpha(W_A)=\alpha(\prod(-)^{qr_i}\lambda_{qr_i})\cdot(\prod\delta_{u_i})$. If we write $\prod\delta_{u_i}=\sum_0^{q-1} a_i\lambda_i$ and $\prod\lambda_{qr_i}=\sum b_i\lambda_i$, then $\alpha(W_A)=(-)^{q\Sigma r_i}\sum_0^{q-1} a_i b_i$. Using Lemmas 4.10 and 4.4 we find:

(1) If $\sum r_i$ is even $\alpha(W_A)=l_F(r_1,\ldots,r_s)\,D_F(u_1,\ldots,u_s)$.

(2) If $\sum r_i$ is odd, $\alpha(W_A)=l_F(r_1,\ldots,r_s)\,D_F(q-u_1,u_2,\ldots,u_s)-l_F(r_1,\ldots,r_s)\,u_2\cdot\ldots\cdot u_s$.

Suppose next that $|A|<s$. Then there is some $j\notin A$. Since $\delta_u\cdot\delta_{qr_j}=u\cdot\delta_{qr_j}$ for each u in $[0,q]$, $W_A=(\prod_{i\in A}u_i)\cdot W_A^*$ with $W_A^*=(\prod_{i\in A}(-)^{qr_i}\lambda_{qr_i})\cdot(\prod_{j\notin A}\delta_{qr_j})$. Since there is some $j\notin A$, W_A^* is in the ideal $I(q)$ of Γ defined in Theorem 3.6. Now by Theorem 4.1, $\psi_q(W_A^*)=(\prod_{i\in A}(-)^{r_i}\lambda_{r_i})\cdot(\prod_{j\notin A}q\delta_{r_j})$. Applying α and using Corollary 4.2 we find that $\alpha(W_A^*)=q^{s-1-|A|}\cdot n_A(r)$. So $\alpha(W_A)=q^{s-1}\cdot n_A(r)\cdot\prod_{i\in A}(u_i/q)$. Summing over A, and using the last paragraph to evaluate $\alpha(W_A)$ where $|A|=s$, we get the theorem.

Corollary 4.14 *Let $r=(r_1,\ldots,r_s)$ be an s-tuple of non-negative integers, $z=(z_1,\ldots,z_s)$ an element of $[0,1]^s$ and n an integer $\geqq 0$. Set $z^*=(1-z_1,z_2,\ldots,z_s)$. Then*

(1) *If $\sum r_i$ is even, $D_F(p^n(r+z))=l_F(r)\cdot D_F(p^n z)+p^{(s-1)n}\varphi_r(z)$.*

(2) *If $\sum r_i$ is odd, $D_F((p^n(r+z))=l_F(r)\,D_F(p^n z^*)+p^{(s-1)n}\varphi_r(z)$.*

Proof. For $n=0$ this is the remark preceding Theorem 4.13. For $n>0$ apply Theorem 4.13 with $q=p^n$ and $u_i=p^n z_i$.

Example. Suppose $p\geqq 3$ and $r=(1,1,1,1)$. Then $\lambda_1^2=\lambda_0+\lambda_1+\lambda_2$. So $l_F(1,1,1,1)=\alpha(\lambda_1^2\cdot\lambda_1^2)=3$. Furthermore if $|A|=j$, $n_A(r)=(-)^j\alpha(\lambda_1^j)$ which is 1, 0, 1 or -1 according as $j=0,1,2$ or 3. So $\phi_r=1+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4-(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)$. Theorem 4.13 then gives the following remarkable functional equation for D_F:

If k_1,k_2,k_3,k_4 are $\leqq q$, then $D_F(q+k_1,q+k_2,q+k_3,q+k_4)=3D_F(k_1,k_2,k_3,k_4)+q^3+(k_1k_2+k_1k_3+k_1k_4+k_2k_3+k_2k_4+k_3k_4)\,q-(k_1k_2k_3+k_1k_2k_4+k_1k_3k_4+k_2k_3k_4)$.

Besides the functional equations of Theorem 4.13 there is one further useful functional equation for D_F.

Theorem 4.15 *Suppose that $u_1,\ldots,u_s$ are in $[0,q]$. Then, $D_F(q-u_1,q-u_2,u_3,\ldots,u_s)=D_F(u_1,\ldots,u_s)+(q-u_1-u_2)\cdot\prod_3^s u_i$.*

Proof. As in the proof of Lemma 2.17 we see that $\lambda_{q-1}\delta_u=(-)^q(\delta_{q-u}-\delta_q)$ for all integers u in $[0,q]$. By interpolation this holds for all u in $[0,q]$. So $\delta_{q-u_1}\cdot\delta_{q-u_2}=\lambda_{q-1}^2\delta_{q-u_1}\delta_{q-u_2}=(\delta_{u_1}-\delta_q)(\delta_{u_2}-\delta_q)$. Multiplying by $\prod_3^s\delta_{u_i}$ and applying α gives the theorem. Taking $q=p^n$ and $u_i=p^n z_i$ in Theorem 4.15 we get:

Corollary 4.16 *Let $z=(z_i,\ldots,z_s)$ be an element of $[0,1]^s$, and n be an integer $\geqq 0$. Then $D_F(p^n\cdot(1-z_1,1-z_2,z_3,\ldots,z_s))=D_F(p^n z)+p^{(s-1)n}(1-z_1-z_2)\cdot\prod_3^s z_i$.*

5 Applications to Hilbert-Kunz functions

In this section we study the following problem. Let $(\beta_1, \ldots, \beta_s)$ be an s-tuple of non-negative rational numbers. How does $D_F(p^n\beta)$ vary with n? We introduce some notation. Fix β and a positive integer μ.

Definition 5.1 (1) Write $\beta = r + z$ and $p^\mu z = t + z'$ where $r = (r_1, \ldots, r_s)$ and $t = (t_1, \ldots, t_s)$ are s-tuples of non-negative integers, and $z = (z_1, \ldots, z_s)$, $z' = (z'_1, \ldots, z'_s)$ are in $[0, 1]^s$. We evidently may assume that each $t_i < p^\mu$.
(2) Set $R = p^\mu r + t$, so that $p^\mu \beta = R + z'$.
(3) Set $l^* = l_F(t)$ if $\sum r_i$ is even, and $l^* = l_F(p^\mu - 1 - t_1, t_2, \ldots, t_s)$ if $\sum r_i$ is odd. Then, by Theorem 4.5, $l_F(R) = l^* \cdot l_F(r)$.

We begin with a special but important case.

Theorem 5.2 *Suppose that $z' = z$ and $\sum R_i \equiv \sum r_i$ (2). Then the numbers $p^{-(s-1)n}(D_F(p^{n+\mu}\beta) - l^* D_F(p^n\beta))$ are independent of n (and rational).*

Proof. Since the β_i are rational the numbers in question are too. Suppose that $\sum R_i$ and $\sum r_i$ are even. By Corollary 4.14:
(1) $D_F(p^n\beta) = D_F(p^n(r+z)) = l_F(r)\, D_F(p^n z) + p^{(s-1)n}\phi_r(z)$
(2) $D_F(p^{n+\mu}\beta) = D_F(p^n(R+z)) = l_F(R)\, D_F(p^n z) + p^{(s-1)n}\phi_R(z)$.
Multiplying (1) by l^*, subtracting from (2) and dividing by $p^{(s-1)n}$ we get the result. If $\sum R_i$ and $\sum r_i$ are both odd, (1) and (2) still hold provided we replace $D_F(p^n z)$ by $D_F(p^n z^*)$ with $z^* = (1 - z_1, z_2, \ldots, z_s)$. The rest of the argument is the same.

Remark. The hypotheses of Theorem 5.2 can sometimes be weakened. If $z = (z_1, \ldots, z_s)$ and $z' = (z'_1, \ldots, z'_s)$ are elements of $[0, 1]^s$ we shall say that z' is equivalent to z if it may be obtained from z by replacing z_i by $1 - z_i$ for an even number of indices i. Repeated applications of Corollary 4.16 show the following – if z' is equivalent to z then $p^{-(s-1)n}(D_F(p^n z') - D_F(p^n z))$ is independent of n. We can now prove the following. Suppose we are in the situation of Definition 5.1. Suppose further that either z' is equivalent to z and $\sum R_i \equiv \sum r_i$ (2) or z' is equivalent to $z^* = (1 - z_1, z_2, \ldots, z_s)$ and $\sum R_i \not\equiv \sum r_i$ (2). Then the conclusion of Theorem 5.2 holds. To see this, suppose for example that $\sum r_i$ is odd, $\sum R_i$ is even and z' is equivalent to z^*. Then:
(1) $D_F(p^n\beta) = D_F(p^n(r+z)) = l_F(r)\, D_F(p^n z^*) + p^{(s-1)n}\phi_r(z)$
(2) $D_F(p^{n+\mu}\beta) = D_F(p^n(R+z')) = l_F(R)\, D_F(p^n z') + p^{(s-1)n}\phi_R(z')$.
Multiplying (1) by l^*, subtracting from (2), dividing by $p^{(s-1)n}$, and using the equivalence of z' and z^* we get the result. The argument in the other cases is similar.

Theorem 5.3 *Suppose that $z' = z$ and $\sum R_i \equiv \sum r_i$(2), or more generally that the hypotheses of the above remark hold. Then there is a rational C and rational numbers $\Delta_0, \Delta_1, \Delta_2, \ldots$ such that $D_F(p^n\beta) = C p^{(s-1)n} + \Delta_n$, and $\Delta_{n+\mu} = l^* \Delta_n$ for all $n \geqq 0$.*

Proof. We claim first that $l^* \neq p^{(s-1)\mu}$. When $s = 2$ this follows from the fact that $l_F(i, j) = \delta_{i,j}$. When $s \geqq 3$, Theorem 4.6 with $q = p^\mu$ shows that $l^* \leqq p^{(s-3)\mu} < p^{(s-1)\mu}$. Now by Theorem 5.2, $D_F(p^{n+\mu}\beta) - l^* D_F(p^n\beta) = A p^{(s-1)n}$ for some rational A independent of n. Set $C = A/(p^{(s-1)\mu} - l^*)$ and $\Delta_n = D_F(p^n\beta) - C p^{(s-1)n}$. Then $\Delta_{n+\mu} - l^* \Delta_n = (A p^{(s-1)n}) - C p^{(s-1)n}(p^{(s-1)\mu} - l^*) = 0$, the desired result.

Example 1 Suppose p is odd and $\beta=(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$. If we take $\mu=1$, $r=(0,0,0,0)$ and $R=t=([p/4],[p/4],[p/4],[p/4])$. Furthermore z' is either z or $3z$. So z' is equivalent to z and the conclusions of Theorem 5.3 hold with $\mu=1$. Suppose now that $p=5$ so that $t=(1,1,1,1)$ and $l^{\#}=l_F(t)=3$. So by Theorem 5.3 there are rational numbers c_0 and c_1 such that $256D_F(5^n\beta)=c_0\cdot 5^{3n}-c_1\cdot 3^n$ for all $n\geqq 0$. Now $256D_F(\beta)=1$. Furthermore $256D_F(5\beta)=\alpha((4\delta_{5/4})^4)=\alpha((4\lambda_0-\lambda_1)^4)=\alpha((17\lambda_0-7\lambda_1+\lambda_2)^2)=17^2+7^2+1=339$. So $c_0-c_1=1$ and $125c_0-3c_1=339$. Solving we find that $256D_F(5^n/4,\ 5^n/4,\ 5^n/4,\ 5^n/4)=(168/61)\cdot 5^{3n}-(107/61)\cdot 3^n$ for all $n\geqq 0$.

Example 2 Suppose p is odd, s is arbitrary and $\beta=(\frac{1}{2},\ldots,\frac{1}{2})$. If we take $\mu=1$, $z'=z=\beta$. Since z' is equivalent to both z and z^*, the conclusions of Theorem 5.3 hold with $\mu=1$. Suppose now that $p=3$ and $s=5$ so that $t=(1,1,1,1,1)$. Since $\lambda_1^2=\lambda_0+\lambda_1+\lambda_2$ and $\lambda_1^3=\lambda_0+3\lambda_1+\lambda_2$, $l^{\#}=l_F(t)=\alpha(\lambda_1^2\cdot\lambda_1^3)=5$. So by Theorem 5.3, $32D_F(3^n\beta)=c_0\cdot 3^{4n}-c_1\cdot 5^n$ for all $n\geqq 0$, with c_0 and c_1 fixed rational numbers. Now $32D_F(\beta)=1$ while $32D_F(3\beta)=\alpha((2\delta_{3/2})^5)=\alpha((2\lambda_0-\lambda_1)^5)$. Now $(2\lambda_0-\lambda_1)^2=5\lambda_0-3\lambda_1+\lambda_2$ and $(2\lambda_0-\lambda_1)^3=13\lambda_0-9\lambda_1+5\lambda_2$. So $32D_F(3\beta)=(5\cdot 13)+(3.9)+5=97$, $c_0-c_1=1$, and $81c_0-5c_1=97$. Solving we find that $32D_F(3^n/2,\ 3^n/2,\ 3^n/2,\ 3^n/2,\ 3^n/2)=(23/19)\,3^{4n}-(4/19)\,5^n$ for all $n\geqq 0$.

We now treat arbitrary β, starting with the somewhat exceptional case $p=2$.

Theorem 5.4 *Suppose* char $F=2$. *Let* $\beta=(\beta_1,\ldots,\beta_s)$ *be an s-tuple of non-negative rational numbers. Then there is a rational number* C *such that* $D_F(2^n\beta)=C\cdot 2^{(s-1)n}+$(*an eventually periodic function of* n).

Proof. Replacing β by $2^j\beta$ with j large we may assume that the β_i have odd denominators and that $\sum\beta_i$ has even numerator. Choose $\mu>0$ so that $(2^\mu-1)\cdot\beta$ has integer co-ordinates and set $\beta=r+z$ as in Definition 5.1. Then $(2^\mu-1)\cdot z$ has integer co-ordinates and we may take $t=(2^\mu-1)z$ and $z'=z$. Since $R-r=(2^\mu\beta-z)-(\beta-z)=(2^\mu-1)\beta$, our hypotheses tell us that $\sum R_i\equiv\sum r_i$ (2). By Theorem 4.6, $l^{\#}=0$ or 1; the result now follows from Theorem 5.3.

Theorem 5.5 *Suppose* char $F=p>2$. *Let* $\beta=(\beta_1,\ldots,\beta_s)$ *be an s-tuple of non-negative rational numbers. Then*:

(1) *If* $s=2$ *or* 3 *there is a rational* C *such that* $D_F(p^n\beta)=Cp^{(s-1)n}+$(*an eventually periodic function of* n).

(2) *If* $s>3$ *there is a rational* C, *rational numbers* $\Delta_0,\Delta_1,\Delta_2,\ldots$ *and integers* $l^{\#}$ *and* μ, $\mu\geqq 1$, $0\leqq l^{\#}\leqq p^{(s-3)\mu}$ *such that* $D_F(p^n\beta)=Cp^{(s-1)n}+\Delta_n$ *and* $\Delta_{n+\mu}=l^{\#}\Delta_n$ *for all sufficiently large* n. *In particular* $D_F(p^n\beta)=Cp^{(s-1)n}+O(p^{(s-3)n})$.

Proof. Replacing β by $p^j\beta$ we may assume that the β_i have denominators prime to p. Choose $\mu>0$ so that $(p^\mu-1)\beta$ has integer co-ordinates. Replacing μ by 2μ if necessary we may assume that $(p^\mu-1)\sum\beta_i$ is even. Set $\beta=r+z$ as in Definition 5.1. Then $(p^\mu-1)\cdot z$ has integer co-ordinates and we may take $t=(p^\mu-1)\cdot z$ and $z'=z$. Since $R-r=(p^\mu-1)\beta$ our hypotheses tell us that $\sum R_i\equiv\sum r_i(2)$. If $s=2$ or 3, $l^{\#}=0$ or 1 and Theorem 5.3 gives (1). If $s>3$ we use Theorem 5.3 together with Theorem 4.6.

We now translate Theorems 5.4 and 5.5 into results about Hilbert-Kunz functions. From now on $F=Z/p$.

Lemma 5.6 *Let* $d_1,\ldots,d_s$ *be positive integers and* $\mathcal{O}=F[\![x_1,\ldots,x_s]\!]/(\sum x_i^{d_i})$. *Let* $e_n(\mathcal{O})$ *be the dimension of* $\mathcal{O}/(x_1^{p^n},\ldots,x_s^{p^n})$. *Then* $e_n(\mathcal{O})=d_1\ldots d_s\,D_F(p^n\beta)$ *where* $\beta=(1/d_1,\ldots,1/d_s)$.

Proof. Set $q=p^n$ and let M_i be the F-object $F[x_i]/x_i^q$ with T operating by multiplication by $x_i^{d_i}$. Write $q=k_i d_i+a_i$ with $k_i \in Z$ and $0 \leqq a_i < d_i$. M_i is the direct sum of T-cyclic submodules generated by $1, x_i, \ldots, x_i^{d_i-1}$. It follows easily that M_i represents $(d_i-a_i)\,\delta_{k_i}+a_i\,\delta_{k_i+1}$ in Γ and $d_i\,\delta_{q/d_i}$ in Γ'. So $\alpha(M_1 \otimes \ldots \otimes M_s) = d_1 \ldots d_s\, \alpha(\prod \delta_{q/d_i}) = d_1 \ldots d_s\, D_F(p^n \beta)$. Since $M_1 \otimes \ldots \otimes M_s$ identifies with $F[x_1, \ldots, x_s]/(x_1^q, \ldots, x_s^q) = F[\![x_1, \ldots, x_s]\!]/(x_1^q, \ldots, x_s^q)$ with T operating by multiplication by $\sum x_i^{d_i}$, $\alpha(M_1 \otimes \ldots \otimes M_s) = e_n(\mathcal{O})$, giving the lemma.

Combining Lemma 5.6 with Theorems 5.4 and 5.5 we get:

Theorem 5.7 *Let $\mathcal{O}$ be as in Lemma 5.6.*

(1) *If $p=2$ or $s=3$ then $e_n(\mathcal{O}) = C\,p^{(s-1)n} + \Delta_n$ for some rational C and eventually periodic function $n \to \Delta_n$.*

(2) *If $p>2$ and $s>3$ there is a rational C and integers $l^{\#}$ and μ, $\mu \geqq 1$, $0 \leqq l^{\#} \leqq p^{(s-3)\mu}$ such that $e_n(\mathcal{O}) = C\,p^{(s-1)n} + \Delta_n$, and $\Delta_{n+\mu} = l^{\#} \cdot \Delta_n$ for all sufficiently large n. In particular $\Delta_n = O(p^{(s-3)n})$.*

Remark. (1) was proved and (2) conjectured in [1].

Examples. Suppose $F=Z/5$ and $\mathcal{O} = F[\![x_1, x_2, x_3, x_4]\!]/\sum x_i^4$. By Lemma 5.6, $e_n(\mathcal{O}) = 256\, D_F(5^n/4, 5^n/4, 5^n/4, 5^n/4)$. Example 1 following Theorem 5.3 shows this to be $(168/61)\cdot 5^{3n} - (107/61)\,3^n$ for all $n \geqq 0$. Similarly when $F=Z/3$ and $\mathcal{O} = F[\![x_1, \ldots, x_5]\!]/\sum x_i^2$, Example 2 shows that $e_n(\mathcal{O}) = (23/19)\cdot 3^{4n} - (4/19)\,5^n$.

We can easily show that the Δ_n of Theorem 5.7 is $\leqq 0$ for all n. This comes from the following more general fact.

Theorem 5.8 *Let $\mathcal{O} = F[\![x_1, \ldots, x_s]\!]/g$ for some g. Then $e_{n+1}(\mathcal{O}) \geqq p^{s-1}\, e_n(\mathcal{O})$ for all n.*

Proof. $\dim F[\![x_1, \ldots, x_s]\!]/(x_1^{p^{n+1}}, \ldots, x_s^{p^{n+1}}, g^p) \leqq p \cdot \dim F[\![x_1, \ldots, x_s]\!]/(x_1^{p^{n+1}}, \ldots, x_s^{p^{n+1}}, g)$. Since $F[\![x_1, \ldots, x_s]\!]$ is a free rank p^s module over $F[\![x_1^p, \ldots, x_s^p]\!]$ we may use the Frobenius map to see that the term on the left hand side is $p^s e_n(\mathcal{O})$. So $p^s e_n(\mathcal{O}) \leqq p\, e_{n+1}(\mathcal{O})$.

Corollary 5.9 *In the situation of Theorem 5.7, $C \geqq 1$ and $\Delta_n \leqq 0$ for all n.*

Proof. $e_n(\mathcal{O}) \geqq p^{(s-1)n} e_0(\mathcal{O}) = p^{(s-1)n}$. So $C \geqq 1$. (This is a special case of a result of Kunz.) Suppose some $\Delta_j > 0$. Then $e_j(\mathcal{O}) = C'\,p^{(s-1)j}$ with $C' > C$. By Theorem 5.8, $e_n(\mathcal{O}) \geqq C'\,p^{(s-1)n}$ for all $n \geqq j$, contradicting Theorem 5.7.

Remark. One can generalize Corollary 5.9. Suppose that $g_1, \ldots, g_{s-r}$ is a regular sequence in the maximal ideal of $F[\![x_1, \ldots, x_s]\!]$, and let $\mathcal{O}$ be the local F-algebra $F[\![x_1, \ldots, x_s]\!]/(g_1, \ldots, g_{s-r})$ of Krull dimension r. By [3], $e_n(\mathcal{O}) = C\,p^{rn} + \Delta_n$ with $\Delta_n = O(p^{(r-1)n})$. An argument like that of Theorem 5.8 shows that $e_{n+1}(\mathcal{O}) \geqq p^r e_n(\mathcal{O})$ for all n. The proof of Corollary 5.9 then shows that $C \geqq 1$ (a special case of Kunz's result) and that $\Delta_n \leqq 0$ for all n.

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On the u-invariant for quadratic forms and the linkage of cyclic algebras

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One of the more striking recent results in the algebraic theory of quadratic forms has been Merkurjev's construction of fields with arbitrary even u-invariant, and his subsequent construction of fields whose u-invariant can grow under quadratic extensions. (The u-invariant of a field F is the maximum dimension of anisotropic F-quadratic forms. For more discussion of this result see the forthcoming paper [L-M].) In this note, we study the growth of the u-invariant in characteristic 2 and some closely related questions involving the linkage number for cyclic algebras.

In characteristic 2 there are two possible u-invariants. One has $u(F) :=$the maximum dimension of anisotropic *nonsingular* quadratic forms over F, and $\hat{u}(F) :=$the maximum dimension of anisotropic quadratic forms over F. In this paper we show that both the u-invariant and the $\hat{u}$-invariant can grow under purely inseparable quadratic extensions. The growth of the u-invariant under separable extensions can be derived using the tools of Merkurjev [M2], but since his process depends heavily upon the corestriction, different techniques have to be used in the inseparable case. Aravire and Baeza [A-B] have shown that the v-invariant cannot change under purely inseparable extensions, so it is somewhat surprising that the u-invariants can grow.

The key result needed for all the recent u-invariant constructions is the following theorem. The original proof given by Merkurjev in [M1] utilized Swans computation of K_0 of quadratic hypersurfaces. Recently, a proof has been given by Tignol [T] which avoids Swan's results.

Theorem (Merkurjev) *Suppose that F is a field of characteristic different from 2. Let ϕ be an odd-dimensional quadratic form, and denote by $F(\phi)$ the function field of the quadratic hypersurface defined by ϕ. Denote by C_0 the even part of the Clifford algebra of ϕ. Let D be an arbitrary finite-dimensional F-central*

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division algebra. Then $D \otimes_F F(\phi)$ *fails to be a division algebra if and only if* $D \cong D_0 \otimes_F C_0$ *for some division algebra* D_0.

Merkurjev's theorem has recently been extended to fields of characteristic 2 (see [M-T-W, Theorem 4]). This result is crucial for this paper and is stated next.

Theorem (Mammone, Tignol, Wadsworth) *Suppose* ψ *is a nonsingular anisotropic quadratic form of dimension* $2n \geq 2$, *and let* $\Delta(\psi) \in F$ *be a representative of the Arf invariant of* ψ.
1. *If* $\Delta(\psi) \in \wp(F)$, *a necessary and sufficient condition for* $D \otimes F(\psi)$ *not to be a division algebra is that* $M_2(D)$ *contain a subalgebra isomorphic to* $C(\psi)$.
2. *If* $\Delta(\psi) \notin \wp(F)$, *a necessary and sufficient condition for* $D \otimes F(\psi)$ *not to be a division algebra is that* D *contain a subalgebra isomorphic to* $C_0(\psi)$.

The discovery that the u-invariants can grow raises the related question of determining if the linkage number of cyclic algebras can grow under algebraic extension. This questions is studied in Sect. 2 of this paper. In that section we shall require a result of Schofield and van den Bergh [S-vdB] which is reminiscent of the result of Merkurjev. This theorem determines the index of a division algebra when the scalars are extended to the function field of a Severi-Brauer variety. In fact, this result coincides (essentially) with the result of Merkurjev in the special case where the function field considered is that of a quaternion algebra (see [L, 4.11]).

Theorem (Schofield, van den Bergh) *Suppose that* A *and* B *are central simple algebras over a field* F *and that* X_B *is the Severi-Brauer variety of* B *over* F. *Then* $\operatorname{ind}(A \otimes_F F(X_B)) = \min\{\operatorname{ind}(A \otimes_F B^{\otimes i}) \mid i = 1, 2, \ldots, \exp(B)\}$.

1 Inseparable quadratic extensions and the u-invariant

The idea behind Merkurjev's construction of a field F with u-invariant $2n$ is to successively make all $(2n+1)$-dimensional quadratic forms isotropic by passing to function fields, and yet retain an anisotropic $2n$-dimensional quadratic form. To do this Merkurjev showed that if a tensor product of s quaternion algebras is an F-division algebra, then F has a $2n+2$-dimensional anisotropic quadratic form. Therefore, the main task in making these constructions is to retain division algebras under function field extensions. In order to show that the u-invariant can grow under algebraic extensions, Merkurjev applied the same technique except making sure that a certain multiquaternion division algebra over the extension field remained a division algebra. In our results we use division algebras of higher exponent since we do not have a corestriction map for inseparable extensions.

Our first lemma applies in arbitrary finite characteristic p. For $a, b \in F$ with $b \neq 0$, let $[a, b)_F$ denote the p^2-dimensional central simple F-algebra generated by elements θ, u subject to $\wp(\theta) = a$, $u^p = b$, and $u\theta u^{-1} = \theta + 1$. To describe higher degree cyclic p-algebras one needs symbols using Witt vectors. Let $a, b \in F$ and $c \in F^*$. We use the symbol $[(a, b), c)_F$ to denote the cyclic p^4-dimensional algebra generated over F by elements θ_1, θ_2, u satisfying $u^{p^2} = c$, $(\theta_1^p, \theta_2^p) - (\theta_1, \theta_2) = (a, b) \in W_2(F(\theta_1, \theta_2))$, and $(u\theta_1 u^{-1}, u\theta_2 u^{-1}) = (\theta_1, \theta_2)$

$+(1,0)\in W_2(F(\theta_1,\theta_2))$ where $W_2(F(\theta_1,\theta_2))$ denotes Witt vectors of length 2. We remark that in $\mathrm{Br}(F)$ these symbols are additive in the left slot, multiplicative in the right slot, and $[(0,a),b)_F \sim [a,b)_F$. For more details on manipulating these symbols we cite [M-M].

Lemma 1.1 *Let F be a field of characteristic p and suppose $K=F(x,y,z)$ where x, y, z are algebraically independent over F. Then the cyclic algebra $A=[(xy,z),x)_K$ is a division algebra of exponent p^2. However, denoting $L=K(\sqrt[p]{y})$ we have $A\otimes_K L\cong[(p-1)xy,\sqrt[p]{y})_L\otimes_L[z,x)_L$ is a division algebra of exponent p.*

Proof. We first check that A has exponent p^2 over K. We have $p(xy,z)=(0,(xy)^p)$ by Witt vector algebra, and consequently $A^{\otimes p}\sim[p(xy,x),x)_K\sim[(0,x^py^p),x)_K \sim[x^py^p,x)_K\sim[xy,x)_K$. The algebra $[xy,x)_K$ is a division algebra since x and y are algebraically independent over F. Next we compute in $\mathrm{Br}(L)$:

$$\begin{aligned}
[(xy,z),x)_L &= [(xy,0)+(0,z),x)_L\\
&= [(x\sqrt[p]{y^p},0),x)_L+[z,x)_L\\
&= [(x\sqrt[p]{y^p},0),x\sqrt[p]{y^{p^2}})_L+[z,x)_L\\
&= [(x\sqrt[p]{y^p},0),x\sqrt[p]{y^p})_L+[(x\sqrt[p]{y^p},0),\sqrt[p]{y^{p(p-1)}})_L+[z,x)_L\\
&= [x\sqrt[p]{y^p},\sqrt[p]{y^{(p-1)}})_L+[z,x)_L\\
&= [(p-1)x\sqrt[p]{y^p},\sqrt[p]{y})_L+[z,x)_L.
\end{aligned}$$

We must check that $A\otimes_K L\cong[(p-1)x\sqrt[p]{y^p},\sqrt[p]{y})_L\otimes_L[z,x)_L$ is a division algebra. We rewrite $A\otimes_K L\cong[(p-1)xy,\sqrt[p]{y})_L\otimes_L[z,xy)_L$ and observe that $L=K(\sqrt[p]{y})=F(xy,z,\sqrt[p]{y})$ is rational over F. We consider the valuation $v\colon K(\sqrt[p]{y})\to\mathbb{Z}\times\mathbb{Z}$ which comes from embedding $K(\sqrt[p]{y})$ in the twice iterated Laurent series field $K'=F(xy)((z^{-1}))((\sqrt[p]{y}))$. Then $A_1=[(p-1)xy,\sqrt[p]{y})$ is a semiramified division algebra with $\overline{A_1}=F(xy)(\wp^{-1}((p-1)xy))$ and value group $\Gamma_{A_1}=\Gamma_{K'}((1/p)\,v(\sqrt[p]{y}))$ and $A_2=[z,xy)$ is a semiramified division algebra with $\overline{A_2}=F(xy)(\sqrt[p]{xy})$ and value group $\Gamma_{A_2}=\Gamma_{K'}((1/p)\,v(z))$. Since $\overline{A_1}$ and $\overline{A_2}$ are linearly disjoint extensions of $F(xy)$ and as $\Gamma_{A_1}\cap\Gamma_{A_2}=\Gamma_{K'}$ we see by [Mo, Theorem 1] that $A\otimes_K K'$ is a semiramified division algebra over K'. In particular, $A\otimes_K L$ is a division algebra and the lemma is proved. □

Let $m=[3n/2]$. Merkurjev has constructed fields F together with a separable quadratic extension L of F such that $u(F)=2n$ and $u(L)=2m$ if n is even and $u(L)=2m$ or $2m+1$ if n is odd. Unfortunately, since his proof requires the corestriction, it cannot be extended to the inseparable case. The main result of this section is the following.

Theorem 1.2 *For any $n \geq 2$ there exists a field $\tilde{F}$ of characteristic 2 with $y \in \tilde{F}$ such that $u(\tilde{F}) = 2n$ and $u(\tilde{F}(\sqrt{y})) \geq 2n+2$. In case $n=2$, we can construct $\tilde{F}$ so that $u(\tilde{F}) = \hat{u}(\tilde{F}) = 4$ and $u(\tilde{F}(\sqrt{y})) = \hat{u}(\tilde{F}(\sqrt{y})) = 6$.*

Proof. In order to prove this result, we use the technique of iterating function fields developed by Merkurjev and also used in [M-T-W].

To begin we consider the field of rational functions

$$K_0 = F(x_1, y_1, \ldots, x_{n-2}, y_{n-2}, x, y, z)$$

where F has characteristic 2 and we set

$$D = [x_1, y_1)_{K_0} \otimes [x_2, y_2)_{K_0} \otimes \ldots \otimes [x_{n-2}, y_{n-2})_{K_0} \otimes [(x\,y, z), x)_{K_0}.$$

In view of Lemma 1.1 and the independence of all the variables, D is a division algebra of index 2^n and exponent 4 over K_0. Furthermore, Lemma 1.1 together with the fact that $L_0 = K_0(\sqrt{y}) = F(x_1, y_1, \ldots, x_{n-2}, y_{n-2}, x, \sqrt{y}, z)$ is rational over F also shows that

$$D \otimes_{K_0} L_0 = [x_1, y_1)_{L_0} \otimes [x_2, y_2)_{L_0} \otimes \ldots \otimes [x_{n-2}, y_{n-2})_{L_0} \otimes [x\,y, \sqrt{y})_{L_0} \otimes [z, x)_{L_0}$$

is also a division algebra over $L_0 = K_0(\sqrt{y})$.

Four our iterative construction we must establish the following.

(*) Suppose that D is a division algebra of index 2^n over a field K, $D^{\otimes 2}$ is similar to a quaternion division algebra over K, and $L = K(\sqrt{y})$ is such that $D \otimes_K L$ is a division algebra which is isomorphic to a tensor product of n quaternion algebras. Let ϕ be a nonsingular quadratic form over K of dimension $2n+2$. Then $(D \otimes_K K(\phi))^{\otimes 2}$ is similar to a quaternion division algebra over $K(\phi)$, and $(D \otimes_K L) \otimes_L L(\phi)$ is also a division algebra.

To prove (*), as $D^{\otimes 2}$ is similar to a quaternion division algebra over K and since $\dim(\phi) \geq 6$, we know by [M-T-W, Corollary 4, first part] that $(D \otimes_K K(\phi))^{\otimes 2}$ cannot be split over $K(\phi)$.

We next must check that $(D \otimes_K L) \otimes_L L(\phi)$ is a division algebra. There are two cases. First, if ϕ becomes isotropic over L, then since ϕ is nonsingular we know that $L(\phi)$ is purely transcendental over L (see [M-T-W, Corollary 2]), and the result is clear. Second, we suppose ϕ remains anisotropic over L. Assume $(D \otimes_K L) \otimes_L L(\phi)$ is not a division algebra. Then using the Mammone-Tignol-Wadsworth Theorem we have two possibilities. First, if ϕ has trivial Arf invariant, then $M_2(D \otimes_K L)$ contains a subalgebra isomorphic to the Clifford algebra $C(\phi)$. Since $\dim(\phi) = 2n+2$ we find that $C(\phi) = [a_1, b_1) \otimes \ldots \otimes [a_{n+1}, b_{n+1})$ for $a_1, b_1, \ldots, a_{n+1}, b_{n+1} \in K$. Since dimensions agree we must have $M_2(D \otimes_K L) \cong C(\phi)$ and in particular this means that D is split by the extension $L(\sqrt{b_1}, \ldots, \sqrt{b_{n+1}})$ which is a purely inseparable extension of K of exponent 2. Applying Albert's result [A, p. 109, Theorem 32] we find that D has exponent 2 in $\mathrm{Br}(K)$, a contradiction.

In the case where ϕ has nontrivial Arf invariant, again applying the Mammone-Tignol-Wadsworth Theorem we find that $D \otimes_K L$ contains a subalgebra isomorphic to the even Clifford algebra $C_0(\phi)$. But $C_0(\phi)$ contains a tensor product of n quaternion algebras defined over K, so D does also, and again

we see that D is split by a purely inseparable extension of K of exponent 2. We are done by Albert's result as before. This proves the claim (*).

The first statement of the theorem follows using Merkurjev's iterated function field argument, starting with the division algebra D defined over K_0 above. Applying (*), one iteratively makes all $2n+2$-dimensional quadratic forms isotropic to obtain the field $\tilde{F}$. It is clear that $u(\tilde{F}) \leqq 2n$. Since $D \otimes_{K_0} \tilde{F}(\sqrt{y})$ is a division algebra, we see that $u(\tilde{F}(\sqrt{y})) \geqq 2n+2$. Finally, as $([x_1, y_1)_{K_0} \otimes [x_2, y_2)_{K_0} \otimes \ldots \otimes [x_{n-2}, y_{n-2}) \otimes_{K_0} [x, y) \otimes_{K_0} \tilde{F}$ is a division algebra (again by the Mammone-Tignol-Wadsworth Theorem), we know that $u(\tilde{F}) \geqq 2n$.

We next turn to the case in the theorem where $n=2$. In this situation, in addition to considering function fields of 6-dimensional nonsingular forms, we must prove the analogue of (*) where ϕ is a 5-dimensional singular quadratic form. (This is to ensure that $\hat{u}(F)$ is at most 4. We remark that it is possible to have $u(F)=4$ yet have $\hat{u}(F)=\infty$.) Once this is done, the stated result will follow since we will have that $u(F)=\hat{u}(F)=4$, and therefore $6 \leqq u(F(\sqrt{y})) \leqq \hat{u}(F(\sqrt{y})) \leqq 6$ (the last inequality following from Leep's theorem [Le]; see the remark at the end of this section).

As $D^{\otimes 2}$ is similar to a quaternion division algebra over K and since $\dim(\phi) \geqq 5$, we know by [M-T-W, Corollary 4, second part] that $(D \otimes_K K(\phi))^{\otimes 2}$ cannot be split over $K(\phi)$. So our main task is to show that $(D \otimes_K L) \otimes_L L(\phi)$ is a division algebra. In order to prove this we need the following.

Excellence Lemma 1.3 *Let F be a field of characteristic 2 and set $L=F(\sqrt{a})$ where $a \in F-F^2$. Suppose that ψ is a totally singular anisotropic quadratic form over F such that $\psi \otimes L$ is isotropic. Then we can express $\psi \otimes L \cong \psi' \perp [0] \perp \ldots \perp [0]$ where ψ' is defined over F and is anisotropic over L.*

Proof. We proceed by induction on $n=\dim(\psi)$. Suppose that $\vec{X}$, $\vec{Y} \in F^n$ are non-zero and that $\psi(\vec{X}+\sqrt{a}\,\vec{Y})=0$. Using the anisotropicity of ψ over F we see that $\vec{X}$, $\vec{Y}$ are linearly independent. The equation $0 \neq b=\psi(\vec{X})=a\psi(\vec{Y}) \in F$ shows that $\psi=\psi_1 \perp [b] \perp [ab]$ over F, and so $\psi_L=\psi_1 \perp [b] \perp [0]$. We are done by induction, since $\dim(\psi_1 \perp [b]) < \dim(\psi)$. □

Continuing with the proof of Theorem 1.2, we suppose that $(D \otimes_K L) \otimes_L L(\phi)$ is not a division algebra. There are two cases. First we consider the case where ϕ becomes isotropic over L. We denote by $r(\phi_L)$ the dimension the totally isotropic subspace of the radical of ϕ_L. Suppose $r(\phi_L)=0$. In this situation we know that $L(\phi)$ is purely transcendental over L by [M-T-W, p. 114, Corollary 2] and the result is clear. Now suppose that $r(\phi_L)>0$. Then ϕ is $[a, b] \perp [c_1] \perp [c_2] \perp [c_3]$, in which case $r(\phi_L)=1$, or else ϕ is $[c_1] \perp [c_2] \perp [c_3] \perp [c_4] \perp [c_5]$, in which case $r(\phi_L)=1$ or 2 (see [M-T-W, p. 112, Proposition 1]). If the first occurs, using Lemma 1.3 and scaling we can express $\phi_L=[a', b'] \perp [c_1'] \perp [1] \perp [0]$ where a', b', $c_1' \in K$. In this case, denoting the F-form $[a', b'] \perp [c_1'] \perp [1]$ by ϕ_0, we know that $L(\phi)$ is purely transcendental over $L(\phi_0)$. So in fact, $(D \otimes_K L) \otimes_L L(\phi_0)$ is not a division algebra. If ϕ_0 is isotropic over L, then since $[c_1'] \perp [1]$ is anisotropic over L, by [M-T-W, p. 114, Corollary 2] $L(\phi_0)$ is rational over L and we are done as before. So now we can assume ϕ_0 is anisotropic over L. In this case [M-T-W, Theorem 3] shows that $D \otimes_K L$ contains a subalgebra isomorphic to $[a', b'] \otimes L(\sqrt{c_1'})$. Since D has

index 4, $L(\sqrt{b'}, \sqrt{c_1'})$ is a maximal subfield of $D \otimes_K L$ and we see that D has a purely inseparable splitting field of exponent 2 over K. This contradicts the fact that the exponent of D in $\mathrm{Br}(K)$ is 4. The second situation where $\phi = [c_1] \perp [c_2] \perp [c_3] \perp [c_4] \perp [c_5]$ is handled in exactly the same way.

Finally we are in the case where ϕ is anisotropic over L. There are three possibilities. First we suppose that $\phi = [a, b] \perp [c, d] \perp [1]$. In this case [M-T-W, Theorem 3] shows (by dimension count) that $D \otimes_K L \cong [b\,a, a) \otimes_L [c\,d, c)$. Hence $L(\sqrt{a}, \sqrt{b})$ is a purely inseparable splitting field of D of exponent 2 over K, a contradiction. Second, we suppose that $\phi = [a, b] \perp [c] \perp [d] \perp [1]$. In this case [M-T-W, Theorem 3] shows that $D \otimes_K L \cong [b\,a, a) \otimes_L L(\sqrt{c}, \sqrt{d})$. This is impossible, as the latter algebra is not L-central. Third, we suppose that $\phi = [a] \perp [b] \perp [c] \perp [d] \perp [1]$. In this case [M-T-W, Theorem 3] shows that $D \otimes_K L$ contains $L(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ which is a purely inseparable field extension of L of degree at least 8. This is impossible since $D \otimes_K L$ has index 4. This concludes all possibilities and the proof of Theorem 1.2. □

Remark. The proof of Leep [Le] that $u(L) \leqq (n+1)\,u(F)/2$ whenever $[L:F] = n$ applies to $\hat{u}(F)$ in characteristic 2. It would be nice to have a similar upper bound for the growth of $u(F)$ in characteristic 2 but this is not currently available.

2 Growth of the linkage number

Merkurjev's u-invariant results depend upon showing that certain tensor products of quaternion algebras are division algebras. It makes sense to consider this issue independent of quadratic form theory. In view of this we give the following definition (also see [K]).

Definition 2.1 Let F be a field and p a prime. We say that the p-linkage number of F is m, denoted $\lambda_p(F) = m$, if every class in $\mathrm{Br}_p(F)$ is similar to a tensor product of at most m cyclic algebras of index p and if there is a class in $\mathrm{Br}_p(F)$ which is not equivalent to fewer than m.

For example, using standard techniques it is easy to check that if $K = \mathbb{C}(t_1, \ldots, t_{2n})$ (iterated Laurent series), then $\lambda_p(K) = n$ for all p. An interesting question is how the linkage number behaves under algebraic extension. We have a complete answer for the behavior of the p-linkage number under purely inseparable extensions of degree p and we begin with these results.

Theorem 2.2 *Suppose that* $\mathrm{char}(F) = p$ *and* $\lambda_p(F) = m$. *Let* $y \in F - F^p$ *and set* $L = F(\sqrt[p]{y})$. *Then* $\lambda_p(L) \leqq m+1$.

Proof. First consider a cyclic algebra $[\alpha, \beta)_L$ over L. Using the identity

$$[\alpha, x+y)_L \sim \left[\alpha \frac{x}{x+y}, x\right)_L + \left[\alpha \frac{y}{x+y}, y\right)_L \in \mathrm{Br}_p(L)$$

together with the fact that every element β of $L = F(\sqrt[p]{y})$ can be expressed as $\beta = b_0 + b_1 \sqrt[p]{y} + \ldots + b_{p-1} (\sqrt[p]{y})^{p-1}$ where $b_0, b_1, \ldots, b_{p-1} \in F$, we see that any cyclic algebra $[\alpha, \beta)_L$ over L is Brauer equivalent to a tensor product of cyclic

algebras of the form $[\gamma_i, b_i)_L$ where the $b_i \in F$ and a single algebra of the form $[\gamma, \sqrt[p]{y})_L$. Replacing γ by γ^p and each γ_i by γ_i^p we can assume they all lie in F. This shows that any algebra $A \in \mathrm{Br}_p(L)$ is similar to a product $[a_1, b_1)_L \otimes \ldots \otimes [a_s, b_s)_L \otimes [c, \sqrt[p]{y})_L$ where $a_1, b_1, \ldots, a_s, b_s, c \in F$. Since $\lambda_p(F) = m$ we can assume that $s \leqq m$ and the result follows. □

This next lemma is well-known, but we need it repeatedly.

Lemma 2.3 *Suppose that D is a division algebra over a field F and K is the rational function field $K = F(x_1, y_1, \ldots, x_n, y_n)$. Let $\hat{D} = (D \otimes_F K) \otimes_K [x_1, y_1)_K \otimes_K \ldots \otimes_K [x_n, y_n)_K$. Then $\mathrm{ind}(\hat{D}) = p^n \cdot \mathrm{ind}(D)$.*

Proof. Let v denote the usual valuation on K with residue field F and value group $\mathbb{Z}^{2n}$. Then v extends to the division algebra $D \otimes_F K$ inertially. Also, v extends to the division algebra $[x_1, y_1)_K \otimes_K \ldots \otimes_K [x_n, y_n)_K$ in a totally ramified manner. It follows that $\hat{D}$ is a (valued) division algebra using, for example, [Mo, Theorem 1]. The index statement follows. □

Let F be a field of characteristic p and let D be an F-division algebra which is similar to a tensor product of $m+1$ cyclic algebras of index p, but no fewer. We consider the rational function field $K = F(x_1, y_1, \ldots, x_m, y_m)$ and the division algebra $\hat{D} = (D \otimes_F K) \otimes_K [x_1, y_1)_K \otimes_K \ldots \otimes_K [x_m, y_m)_K$. $\hat{D}$ is a division algebra in view of Lemma 2.3. Let $X_{\hat{D}}$ denote the Severi-Brauer variety of $\hat{D}$ over K. With this notation we have the following.

Lemma 2.4 *Suppose that A is an F-division algebra of exponent p^2 and index p^{m+1}. Assume that $L = F(\sqrt[p]{y})$ for $y \in F$ and that $A \otimes_F L \cong [a_1, b_1) \otimes \ldots \otimes [a_{m+1}, b_{m+1})$ and is also a division algebra. Then $A \otimes_F K(X_{\hat{D}})$ has exponent p^2 and $(A \otimes_F K(\sqrt[p]{y})) \otimes_{K(\sqrt[p]{y})} K(\sqrt[p]{y})(X_{\hat{D} \otimes_K K(\sqrt[p]{y})})$ is a division algebra.*

Proof. We first assume that $A \otimes_F K(X_{\hat{D}})$ has exponent p. This would mean that $A^p \otimes_F K$ is split over $K(X_{\hat{D}})$ and consequently $A^p \otimes_F K \sim \hat{D}^i$ for some $i = 1, 2, \ldots, p-1$ by Amitsur's theorem for generic splitting fields. However, since the variables $x_1, y_1, \ldots, x_m, y_m$ are algebraically independent over F we see for each $i = 1, 2, \ldots, p-1$ that $\mathrm{ind}(\hat{D}^i) \geqq p^{m+1}$. As $\mathrm{ind}(A^p \otimes_F K) \leqq p^m = \mathrm{ind}(A \otimes_F K)/p$ (for example, see [P, p. 281, Corollary a]), we have a contradiction.

Now we suppose that $(A \otimes_F K(\sqrt[p]{y})) \otimes_{K(\sqrt[p]{y})} K(\sqrt[p]{y})(X_{\hat{D} \otimes_K K(\sqrt[p]{y})})$ fails to be a division algebra. Then, applying the Schofield-van den Bergh Theorem we see that $\min\{\mathrm{ind}((A \otimes_F K(\sqrt[p]{y})) \otimes_{K(\sqrt[p]{y})} (\hat{D}^i \otimes_K K(\sqrt[p]{y})) \mid i = 1, 2, \ldots, p-1\} \leqq p^m$. However, as $K(\sqrt[p]{y})$ is rational over L we must have that $\mathrm{ind}((A \otimes_F K(\sqrt[p]{y})) \otimes_{K(\sqrt[p]{y})} (\hat{D}^i \otimes_K K(\sqrt[p]{y}))) = \mathrm{ind}(A \otimes_F L \otimes_F D^i) \cdot p^m$. This means that $\mathrm{ind}(A \otimes_F L \otimes_F D^i) = 1$ for some i. In particular, since D had a purely inseparable splitting field of exponent p over F, we find that A has a purely inseparable splitting field of exponent p over F. This contradicts the fact that A has exponent p^2 and proves the lemma. □

Our next result shows that the inequality obtained in Theorem 2.2 is best possible.

Theorem 2.5 *For any $m \geqq 1$ there exists a field $\tilde{F}$ of characteristic p with $\lambda_p(\tilde{F}) = m$ and $y \in \tilde{F}$ such that $\lambda_p(\tilde{F}(\sqrt[p]{y})) = m+1$.*

Proof. We consider the field of rational functions

$$F = F_0(x_1, y_1, \ldots, x_{n-2}, y_{n-2}, x, y, z)$$

where F_0 has characteristic p and we set

$$A = [x_1, y_1) \otimes [x_2, y_2) \otimes \ldots \otimes [x_{m-1}, y_{m-1}) \otimes [(x\,y, z), x).$$

In view of Lemma 1.1 and the independence of all the variables, A is a division algebra of index p^{m+1} and exponent p^2 over F. Furthermore, Lemma 1.1 together with the fact that $F(\sqrt[p]{y}) = F_0(x_1, y_1, \ldots, x_{n-2}, y_{n-2}, x, \sqrt[p]{y}, z)$ is rational over F_0 shows that

$$A \otimes_F F(\sqrt[p]{y}) = [x_1, y_1) \otimes [x_2, y_2) \otimes \ldots \otimes [x_{m-1}, y_{m-1}) \otimes [x\,y, \sqrt[p]{y}) \otimes [z, x)$$

is also a division algebra over $F(\sqrt[p]{y})$. This gives our desired starting division algebra.

We now iteratively apply Lemma 2.4 to obtain a (large) field extension of F, say $\tilde{F}$, for which every class in $\mathrm{Br}_p(\tilde{F})$ is similar to a tensor product of at most m cyclic algebras of index p. (We apply Lemma 2.4 iteratively to each class in $\mathrm{Br}_p(F)$ which is similar to a tensor product of $m+1$ symbols but no fewer, and obtain a field F_1. We next repeat this process iteratively to obtain a sequence $F_1 \subset F_2 \subset F_3 \subset \ldots$, where every class in $\mathrm{Br}_p(F_i)$ is similar to at most m symbols in $\mathrm{Br}_p(F_{i+1})$. At the end set $\tilde{F} = \bigcup_{i=1}^{\infty} F_i$. Since every class in $\mathrm{Br}_p(\tilde{F})$ is represented by an algebra of the form $C = [a_1, b_1) \otimes \ldots \otimes [a_s, b_s)$ for $a_i, b_i \in \tilde{F}$, we know that $C \in \mathrm{Br}(F_i)$ for some i, and consequently is represented by at most m symbols in $\mathrm{Br}(F_{i+1})$.) We have that $\lambda_p(\tilde{F}) \leqq m$; but as $A \otimes_F \tilde{F}(\sqrt[p]{y})$ is a division algebra we find that $\lambda_p(\tilde{F}(\sqrt[p]{y})) \geqq m+1$. From this the theorem follows. □

When considering separable extensions of degree p we have the following result which is valid in arbitrary characteristic. These observations are based entirely on the ideas of Merkurjev contained in [M2] and further discussed in [L2]. Since the method is the same as the preceding construction, we only sketch the proof. Also, when $\mathrm{char}(F) \neq p$, one must make the obvious modifications in our definition of $\hat{D}$.

Theorem 2.6 *Suppose $1 \leqq s \leqq p$. There exist fields $\tilde{F}$ of arbitrary characteristic with $\lambda_p(\tilde{F}) = pr + s$ and a cyclic extension $\tilde{L}$ of $\tilde{F}$ of degree p such that $\lambda_p(\tilde{L}) \geqq pr + s + r + 1$.*

Proof. Let $m = pr + s$. We choose a cyclic p-extension of fields L/F with division algebras A over F and B over L such that A is isomorphic to a tensor product of m cyclic algebras of index p, B is isomorphic to a tensor product of $m+r+1$

cyclic algebras of index p, and such that $\text{ind}(\text{cor}_{L/F}(B)) = p^{pr+1}$ (see [L2] for details).

We consider a F-division algebra D, a function field extension K of F, and the K-division algebra $\hat{D}$ such as described prior to Lemma 2.4. We note that $\text{ind}(\text{cor}_{LK/K}(B \otimes_L LK) \otimes_K K(X_{\hat{D}})) = p^{pr+1}$. For otherwise, applying the Schofield-van den Bergh Theorem gives that $\text{ind}(\text{cor}_{LK/K}(B \otimes_L LK) \otimes_K \hat{D}^{\otimes i}) \leqq p^{pr}$ for some $i = 1, 2, \ldots, p-1$. But since $\text{cor}_{LK/K}(B \otimes_K LK) \cong \text{cor}_{L/F}(B) \otimes_F K$ (see [D, p. 56, Example 1]), applying Lemma 2.3 we have $\text{ind}(\text{cor}_{LK/K}(B \otimes_L \hat{D}^{\otimes i})) \geqq p^m = p^{pr+1}$, a contradiction.

Next, as $\text{ind}(\hat{D}) \geqq p^{m+1}$, $A \otimes_F K(X_{\hat{D}})$ remains a division algebra. Suppose that $B \otimes_L LK(X_{\hat{D} \otimes_K LK})$ fails to be a division algebra. The Schofield-van den Bergh Theorem gives for some $i = 1, 2, \ldots, p-1$ that $\text{ind}((B \otimes_L LK) \otimes_{LK} (\hat{D}^{\otimes i} \otimes_K LK)) \leqq p^{m+r}$. However, as LK/L is rational, the definition of $\hat{D}$ together with the inequality shows that $\text{ind}(B \otimes_L (D^{\otimes i} \otimes_K L)) \leqq p^r$. Since $D^{\otimes i} \in \text{Br}_p(K)$, we know that $\text{cor}_{L/K}(D^{\otimes i} \otimes_F L) = 0$. This shows using [D, Theorem 1, p. 53] that $\text{ind}(\text{cor}_{L/K}(B \otimes_L (D^{\otimes i} \otimes_F L))) = \text{ind}(\text{cor}_{L/F}(B)) \leqq p^{pr}$. This is a contradiction. The theorem follows by iterating function fields and using arguments analogous to those discussed in the proof of Theorem 1.2. □

Remarks. (i) Since the algebra A considered in Lemma 2.4 is a *division* algebra over $K(X_{\hat{D}})$, Theorem 2.5 in fact gives an example of a field $\tilde{F}$ with $\lambda'_p(\tilde{F}) = m$ and $y \in \tilde{F}$ such that $\lambda'_p(\tilde{F}(\sqrt[p]{y})) = m+1$ where λ'_p is defined as in [K, p. 158].
(ii) When $p = 2$, the results given by Theorem 2.6 coincide with the bounds that can be obtained using Merkurjev's technique for studying the growth of the u-invariant. It would be nice to have an upper bound for the growth of $\lambda_p(F)$ under separable extensions analogous to the result of Leep for the u-invariant. At this point no general bound is known.

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Convex billiards and a theorem by E. Hopf

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1 Introduction and main results

The billiard ball model describes the free motion of a mass-particle movig in a planar convex region bounded by a smooth simple closed curve. The particle moves along a straight line and gets reflected from the boundary according to the rule "the angle of incidence equals the angle of reflection". Such a model gives rise to a "billiard ball map" by passing from one collision to the next one. The crucial property of this map is that it can be described as area-preserving twist map of the cylinder.

Let Γ be a compact convex domain in the Euclidean plane $\mathbb{R}^2$ bounded by a simple closed curve γ, $\gamma = \partial\Gamma$. We shall assume that γ is of class C^2 and has everywhere positive curvature. Fix the counterclockwise orientation on γ, and choose an arc-length parameterization of γ r: $\mathbb{R}/p\mathbb{Z} \to \gamma$, where p is the total length of γ. Throughout this paper we identify γ with the parameterizing circle $\mathbb{R}/p\mathbb{Z}$. Denote by Ω the cylinder over γ, $\Omega = \gamma \times (-1, 1)$, we refer to Ω as the phase space of the billiard associated with γ. Define the billiard ball map T: $\Omega \to \Omega$ in the following way (see [2]): we say that T: $(x, \Phi) \mapsto (y, \Psi)$, where x, $y \in \gamma$ and Φ, $\Psi \in (-1, 1)$ if the oriented line $\ell(x, y)$ passing from x to y makes the angles φ and ψ with γ so that $\cos\varphi = \Phi$ and $\cos\psi = \Psi$ (see Fig. 1). The iterates of T correspond to successive collisions of the particle with boundary.

Introduce the distance function L:

$$L\colon\ \gamma \times \gamma \to \mathbb{R}, \qquad L(x, y) = \mathrm{dist}_{\mathbb{R}^2}(x, y).$$

With this function the map is given by the formulas:

$$T\colon\ (x, \Phi) \mapsto (y, \Psi) \begin{cases} \Phi = -L_1(x, y) \\ \Psi = L_2(x, y), \end{cases} \tag{1.1}$$

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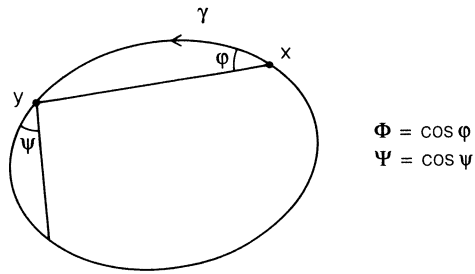


Fig. 1

here 1,2 stand for partials of L with respect to x and y respectively. It is easy to check (see also (2.1)) that $L_{12}>0$ and hence one concludes that T is a C^1-smooth mapping of Ω preserving the Lebesgue measure $d\mu=dx\,d\Phi$ of Ω. The positivity of the cross-derivative L_{12} is precisely the so-called twist condition for area-preserving map T with the generating function L (cf. [8]).

One is interested in possible dynamical behaviour of the billiard ball maps. The most deterministic behaviour corresponds to the so-called integrable billiards. The billiard is called integrable if a subset of full measure of the phase space is foliated by closed curves invariant under the billiard ball map. It was conjectured by G. Birkhoff that the only examples of integrable billiards are circular and elliptic billiards. Let me emphasize that the foliations of the phase-space by invariant curves are very different topologically in these two examples. Namely for elliptic billiards part of the phase space is foliated by null-homotopic invariant curves and for circular billiards all invariant curves are not null-homotopic.

Theorem A *If the phase space Ω of the billiard ball map corresponding to γ is foliated by continuous invariant closed curves not null-homotopic on Ω then γ is a circle.*

In what follows we use in proof of Theorems A, B (see below) a method by E. Hopf suggested for Riemannian metrics without conjugate points see [6]. Hopf's theorem states that a Riemannian metric without conjugate points on the two-dimensional torus is necessarily flat [6]. In this paper we give a discrete version of this method for convex billiards.

To formulate the analogy between these two problems let me introduce the notion of conjugate points for billiard ball problem. Pick a point $(x_0, \Phi_0)\in\Omega$ and consider the phase trajectory $(x_n, \Phi_n)=T^n(x_0, \Phi_0)$, $n\in\mathbb{Z}$. We will call $\{x_n\}$ a billiard configuration. Then $\{x_n\}$ satisfies

$$L_1(x_n, x_{n+1})+L_2(x_{n-1}, x_n)=0 \quad \text{for all } n\in\mathbb{Z} \tag{1.2}$$

by formulas (1.1).

Definition 1 A sequence $\{\xi_n\}_{n\in\mathbb{Z}}$, $\xi_n\in T_{x_n}\gamma$ is called *Jacobi field along the trajectory* $\{x_n\}$ if the following equations are satisfied for all $n\in\mathbb{Z}$

$$L_{12}(x_{n-1}, x_n)\,\xi_{n-1}+(L_{22}(x_{n-1}, x_n)+L_{11}(x_n, x_{n+1}))\,\xi_n+L_{12}(x_n, x_{n+1})\,\xi_{n+1}=0. \tag{1.3}$$

Recall that we have identified γ with $\mathbb{R}/p\mathbb{Z}$ and $T_x\gamma$ with $\mathbb{R}$ and hence each vector $\xi \in T_x\gamma$ is just a real number. It is clear that a Jacobi field is completely determined by its two successive values.

Definition 2 Two points x_M, x_N of the billiard configuration are called *conjugate* if there is a non-zero Jacobi field vanishing at x_M and x_N.

A billiard is called *to be without conjugate points* if there are no conjugate points along each billiard configuration.

Theorem B *The only billiards without conjugate points are circular billiards.*

This paper is organized as follows. In Sect. 2 we prove an inequality (Lemma 1 of § 2) for convex closed curves which replaces the Gauss-Bonnet formula in Hopf's theorem, also an intermediate result is presented which is ingredient of the proof of Theorem A, B. In Sect. 3 we construct nonvanishing Jacobi fields for billiards without conjugate points and prove Theorem B. Also we explain how Theorem A follows from Theorem B.

2 An inequality

In this section we prove a conditional statement (Theorem 1) which will be used in the proof of Theorems A, B.

We start with the following formulas for partials of the distance function L (see [11, 7] for the detailed proofs). We follow the notations of § 1.

$$L_{11}(x,y)=\frac{\hat{\Phi}^2}{L}-k(x)\,\hat{\Phi} \qquad L_{22}(x,y)=\frac{\hat{\Psi}^2}{L}-k(y)\,\hat{\Psi} \tag{2.1}$$

$$L_{12}(x,y)=\frac{\hat{\Phi}\hat{\Psi}}{L}$$

where $k(x)$ is the curvature function of γ,

$$\hat{\Phi}=\sqrt{1-\Phi^2}=\sin\varphi, \qquad \hat{\Psi}=\sqrt{1-\Psi^2}=\sin\psi \text{ (see Fig. 1).}$$

We shall also need the following easy estimates.

Proposition 1 *For all* $x, y \in \gamma$

$$\text{(a)} \qquad \frac{k_{\min}}{2}\leqq\frac{\hat{\Phi}}{L}\leqq\frac{k_{\max}}{2}, \qquad \frac{k_{\min}}{2}\leqq\frac{\hat{\Psi}}{L}\leqq\frac{k_{\max}}{2} \tag{2.2}$$

$$\text{(b)} \qquad |L_{11}|, |L_{22}|, L_{12} < K\cdot\max\{\hat{\Phi}, \hat{\Psi}\}$$

where $k_{\min}$, $k_{\max}$ *are minimal and maximal curvature and* K *is a constant depending on* $k_{\min}$, $k_{\max}$.

Proof. Estimates (2.2b) follow from (2.2a) and formulas (2.1). To prove (2.2a) apply the comparison theorem for convex domains (see [3]). This theorem implies that at each point $x \in \gamma$, the circle of radius $\frac{1}{k_{\min}}$ $\left(\text{resp. } \frac{1}{k_{\max}}\right)$ tangent to γ at x and lying on the same side of the supporting line at x as γ completely

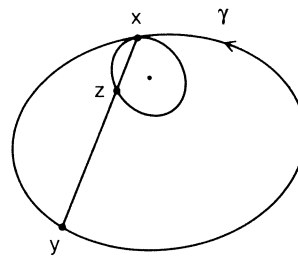


Fig. 2

contains Γ (is contained in Γ resp.). For example, to prove estimates (2.2a) from above note

$$L(x, y) \geqq |xz| = 2 \cdot \frac{1}{k_{\max}} \hat{\Phi} \quad \text{(see Fig. 2)} \qquad \text{analogously } L(x, y) \geqq \frac{2}{k_{\max}} \hat{\Psi}.$$

This completes the proof. □

Theorem 1 *Let $\omega\colon \Omega \to \mathbb{R}$ be a function of class $L_1(\Omega, \mu)$ satisfying the following inequality*

$$\omega(T(x, \Phi)) - \omega(x, \Phi) \geqq L_{11}(x, y) + 2L_{12}(x, y) + L_{22}(x, y). \tag{2.3}$$

Then γ is a circle.

Proof. Integrate both sides of the inequality (2.3) over (Ω, μ) and use that the measure μ is invariant under T

$$\int_{\Omega} [L_{11}(x, y) + 2L_{12}(x, y) + L_{22}(x, y)]\, d\mu \leqq 0.$$

Use explicit formula (2.1) in order to obtain

$$\int_{\Omega} \frac{(\hat{\Phi} + \hat{\Psi})^2}{L} d\mu \leqq \int_{\Omega} k(x)\, \hat{\Phi}\, d\mu + \int_{\Omega} k(y)\, \hat{\Psi}\, d\mu.$$

Note that both of the integrals on the right-hand side are equal to π^2, since measure μ is invariant and

$$\int_0^p k(x) \cdot dx = 2\pi, \qquad \int_{-1}^{1} \hat{\Phi} \cdot d\Phi = \frac{\pi}{2}.$$

So we obtain

$$\int_{\Omega} \frac{(\hat{\Phi} + \hat{\Psi})^2}{L} \cdot d\mu \leqq 2\pi^2.$$

Now we claim that only for circles the last inequality can be satisfied. Namely

Lemma 1 (the inequality) *For every simple closed convex curve* γ

$$\int_{\Omega} \frac{(\hat{\Phi}+\hat{\Psi})^2}{L} \cdot d\mu \geqq 2\pi^2$$

with equality only for circles.

Proof. First apply the Cauchy-Schwartz inequality

$$\int_{\Omega} \frac{(\hat{\Phi}+\hat{\Psi})^2}{L} \cdot d\mu \geqq \frac{(\int_{\Omega} (\hat{\Phi}+\hat{\Psi}) \cdot d\mu)^2}{\int_{\Omega} L \cdot d\mu} = \frac{p^2\pi^2}{\int_{\Omega} L \cdot d\mu}.$$

Recall that p is the total length of γ. Further the Santalo formula [12] gives $\int_{\Omega} L d\mu = 2\pi S$, where S is the area bounded by γ. Therefore we have

$$\int_{\Omega} \frac{(\hat{\Phi}+\hat{\Psi})^2}{L} d\mu \geqq \frac{p^2\pi^2}{2\pi S} \geqq 2\pi^2$$

the last conclusion is just the isoperimetric inequality $p^2 \geqq 4\pi S$, with equality only for circles. The proof of the inequality together with Theorem 1 is completed. □

3 Billiards without conjugate points

We are going to construct a non-vanishing Jacobi field along each billiard configuration.

Let $(x_n, \Phi_n) = T^n((x_0, \Phi_0))$ be the phase-trajectory of a point $(x_0, \Phi_0) \in \Omega$. Introduce the action function F_{MN} as follows. Fix two integers $M \leqq N$ and put

$$F_{MN}(u_M, \ldots, u_N) = L(x_{M-1}, u_M) + \sum_{i=M}^{N-1} L(u_i, u_{i+1}) + L(u_N, x_{N+1})$$

where F_{MN} is defined on the cartesian product γ^{N-M+1}. The important observation is that functional F_{MN} has the critical point at the billiard configuration $(x_M, \ldots, x_N)$. Consider the quadratic form of the second variation of F_{MN}

$$\delta^2 F_{MN}(\xi) = \sum_{i,j=M}^{N} \frac{\partial^2 F}{\partial u_i \partial u_j}(x_M \ldots x_N)\, \xi_i \xi_j$$

where $\xi = (\xi_M, \ldots, \xi_N)$, $\xi_i \in T_{x_i}\gamma$. Denote by G_{MN} the matrix of the second variation, this is the so-called Jacobi matrix (see [1]).

Lemma 2 *If the billiard has no conjugate points, then the second variation* $\delta^2_{MN} F$ *is negative definite for every* M, N *and for all billiard configurations.*

Proof. We first claim that $\delta^2 F_{MN}$ is non-degenerate along each billiard configuration without conjugate points. Indeed, otherwise $G_{MN}\xi^* = 0$ for a non-zero

$\xi^* = (\xi_M^* \dots \xi_N^*)$, but this is exactly the condition that points x_{M-1} and x_{N+1} are conjugate along $\{x_n\}$, the field $(0, \xi_M^* \dots \xi_N^*, 0)$ is Jacobi field vanishing at x_{M-1}, x_{N+1} (see § 1, (1.3)). The claim is proved. Let us mention now that given M, N the quadratic form $\delta^2 F_{MN}$ depends continuously on (x_0, Φ_0). As explained, all the quadratic forms are non-degenerate if the billiard has no conjugate points. Hence the "type" of $\delta^2 F_{MN}$ is not changed when one changes the initial point (x_0, Φ_0). Let me recall that the so-called Birkhoff's periodic orbits which were proved to exist for every convex billiard by Birkhoff [2], have the second variation negative definite. For example one can choose the following billiard configuration $\{x_n\}$; $x_{2j} = x_0$, $x_{2j+1} = x_1$ for all j, where $x_0 x_1$ is a maximal chord of Γ. The continuity argument above shows that for billiards without conjugate points $\delta^2 F_{MN}$ is negative definite for all M, N. □

Let us correspond to each Jacobi field $\{\xi_n\}$ along a billiard trajectory $\{x_n\}$ a broken line in $\mathbb{R}^2$ with the vertices (n, ξ_n), $n \in \mathbb{Z}$.

Lemma 3 *If the billiard has no conjugate points then for two different Jacobi fields $\{\xi_n\}$ and $\{\eta_n\}$ along a configuration $\{x_n\}$ the corresponding broken lines (n, ξ_n) and (n, η_n) in $\mathbb{R}^2$ intersect at most once.*

Proof. Note first that the intersection of these broken lines in two vertices contradicts immediately to no conjugate points condition. It is easy to see that by the linearity of the Jacobi Eq. (1.3) it is sufficient to prove that if $\xi_{M-1} > 0$ and $\xi_{N+1} \geqq 0$ then $\xi_i > 0$ for $i = M, \dots, N$. Indeed, $\xi = (\xi_M, \dots, \xi_N)$ satisfies the equation

$$-G_{MN}\, \xi^T = (L_{12}(x_{M-1}, x_M)\, \xi_{M-1}, 0 \dots 0, L_{12}(x_N, x_{N+1})\, \xi_{N+1})^T.$$

By Lemma 2, matrix $-G_{MN}$ is positive definite and it is easy to see that off diagonal elements of $-G_{MN}$ are negative. Then (see [1]) matrix $-G_{MN}^{-1}$ has all elements positive. This implies the result. □

Suppose that the billiard has no conjugate points. Fix a point $(x_0, \Phi_0) \in \Omega$ and its billiard configuration $\{x_n\}$. For integers $M < N$ denote by $\{\xi_n(M, N)\}$ the Jacobi field along $\{x_n\}$ which satisfies the boundary conditions

$$\xi_M(M, N) = 1 \quad \text{and} \quad \xi_N(M, N) = 0.$$

Such ξ_n exists and unique since the matrix $G_{M+1, N-1}$ is invertible by "no conjugate points" condition. Note that

$$\xi_n(M, N) = \xi_n(0, N) \cdot \xi_0(M, N). \tag{3.1}$$

This follows from the fact that both sides are Jacobi fields and coincide for $n = 0, N$. By Lemma 3 we have that for $N_1 < N_2$

$$\xi_n(0, N_1) < \xi_n(0, N_2) \quad \text{for } n > 0$$
$$\xi_n(0, N_1) > \xi_n(0, N_2) > 0 \quad \text{for } n < 0.$$

The limit

$$v_n = \lim_{N \to +\infty} \xi_n(0, N)$$

for negative n is correctly defined. Then by Jacobi Eq. (1.3) the limit exists also for positive n and the limiting sequence $\{v_n\}$ is a nonvanishing Jacobi field along $\{x_n\}$. (Note that v_n is not negative. But clearly each non-negative solution of (1.3) is positive or identically zero). The identity (3.1) shows that all this construction started from the initial point x_M instead of x_0 gives rise to a proportional Jacobi field. Therefore, function

$$\omega(x_n, \Phi_n) = -L_{11}(x_n, x_{n+1}) - L_{12}(x_n, x_{n+1}) \frac{v_{n+1}}{v_n}$$

is a well-defined function along the phase trajectory $\{(x_n, \Phi_n)\}$, $n \in \mathbb{Z}$. Therefore, we have defined a function $\omega: \Omega \to \mathbb{R}$.

Lemma 4 *The function $\omega: \Omega \to \mathbb{R}$ is measurable and belongs to $L_1(\Omega, \mu)$. In addition ω satisfies inequality* (2.3) *of the assumptions of Theorem* 1.

Proof. For any point $(x_0, \Phi_0) \in \Omega$

$$\omega(x_0, \Phi_0) = -L_{11}(x_0, x_1) - L_{12}(x_0, x_1)\, v_1$$

recall that $v_1 = \lim_{N \to +\infty} \xi_1(0, N)$. All functions $\xi_1(0, N)$ depend on (x_0, Φ_0) continuously so we get that v_1 is measurable and hence $\omega: \Omega \to \mathbb{R}$ also. To prove that $|\omega|$ is bounded, note that by the very definition of ω

$$\begin{aligned} \omega(x_n, \Phi_n) &= -L_{11}(x_n, x_{n+1}) - L_{12}(x_n, x_{n+1}) \frac{v_{n+1}}{v_n} \\ &= L_{22}(x_{n-1}, x_n) + L_{12}(x_{n-1}, x_n) \frac{v_{n-1}}{v_n} \end{aligned} \tag{3.2}$$

(the second part of the equation is just a Jacobi Eq. (1.3)).

From these identities one has (take into account $L_{12} > 0$, $v_n > 0$)

$$|\omega(x_n, \Phi_n)| \leqq \max\{|L_{22}(x_{n-1}, x_n)|, |L_{11}(x_n, x_{n+1})|\}.$$

As a result by (2.2) we obtain $|\omega(x_n, \Phi_n)| \leqq K$, and hence $\omega \in L_1(\Omega, \mu)$. On the other hand, from (3.2) one has

$$\begin{aligned} \omega(x_{n+1}, \Phi_{n+1}) - \omega(x_n, \Phi_n) &= L_{22}(x_n, x_{n+1}) + L_{11}(x_n, x_{n+1}) \\ &\quad + L_{12}(x_n, x_{n+1}) \left(\frac{v_{n+1}}{v_n} + \frac{v_n}{v_{n+1}} \right). \end{aligned}$$

The last expression clearly gives

$$\omega(x_{n+1}, \Phi_{n+1}) - \omega(x_n, \Phi_n) \geqq L_{11}(x_n, x_{n+1}) + 2L_{12}(x_n, x_{n+1}) + L_{22}(x_n, x_{n+1}).$$

This yields the proof of the Lemma. □

Proof of Theorem B. Suppose we are given a billiard without conjugate points. Apply § 3 to construct function $\omega \in L_1(\Omega, \mu)$ (Lemma 4) which meets the assumptions of Theorem 1, § 2, which was proved in § 2. This completes the proof. □

Proof of Theorem A. As we mentioned, Theorem A follows from Theorem B for the following reasons. First, it follows from G. Birkhoff's theorem that each not null-homotopic continuous curve on Ω invariant under T is a graph of a Lipshitz function from γ to $(-1, 1)$ (see [5, 4] for the proof). Then for every point (x_0, Φ_0) lying on such an invariant curve the billiard configuration $\{x_n\}$ corresponding to the phase-trajectory of (x_0, Φ_0) has no conjugate points. This follows from the following arguments.

Each segment $x_M, \ldots, x_N (M \leqq N)$ of the configuration $\{x_n\}$ is a point of local maximum of the action functional F_{MN}. (Apply the arguments of [9] in our case). Then for all $M \leqq N$ the quadratic form $\delta^2 F_{MN}$ is negative semidefinite. It can be easily checked (cf. [10]) that in this case $\delta^2 F_{MN}$ have to be negative definite. Therefore, the matrix of quadratic form $\delta^2 F_{MN}$ is non-degenerate. But this precisely means that there are no conjugate points along the configuration $\{x_n\}$. Thus if the phase space Ω is foliated by continuous not null-homotopic invariant curves then the billiard has no conjugate points. So we can now apply Theorem B. This completes the proof. □

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Continuation of positive, closed currents and analytic sets

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Introduction and results

Since the fundamental work of Lelong [7, 8] and Siu [12] it is well known, that positive, closed currents are a good generalization of analytic sets. Instead of fighting against the rigidity of analytic sets while trying to find a continuation theorem, it is possible to ask for a continuation theorem of the related positive, closed current.

1984 El Mir [2] and Sibony [11] proved independently of each other:

Let $\Omega \subset \mathbb{C}^n$ be open, $A' \subset \Omega$ a $\mathscr{C}^2$-smooth, real submanifold, $A \subset A'$ closed and let T be a positive, closed current of bidimension (p, p) on $\Omega \setminus A$.

If $p > \dim_{\mathbb{C}} \mathrm{H}_a A' + 1$ for each $a \in A'$, then the mass of T is locally finite near A, and the trivial extension $\tilde{T}$ of T to Ω is the unique determined positive, closed extension of T to Ω.

($\mathrm{H}_a A'$ is defined as the maximal complex subspace of the tangent space of A' in a.)

For totally real submanifolds $A \subset \Omega$ are the zero sets of non-negative, strictly plurisubharmonic functions, it is natural to ask now for continuation theorems of positive, closed currents across the zero sets of non-negative, strictly plurisubharmonic (or more general strictly q-convex) functions. (For $q = 1$ a (strictly) q-convex function is (strictly) plurisubharmonic.)

Theorem 1 of this article is a continuation theorem for this situation:

Theorem 1 *Let $\Omega \subset \mathbb{C}^n$ be open, $u: \Omega \to \mathbb{R}_0^+$ semicontinuous, plurisubharmonic.*

Let $A \subset \Omega$ be closed, $A \subset u^{-1}(\{0\})$, and T be a positive, closed current of bidimension (p, p) on $\Omega \setminus A$.

Suppose, that for every $a \in A$ there is an open neighbourhood U of a, $U \subset \Omega$ and that there are real constants $\lambda_o = \lambda_o(a) > 0$ and $\alpha = \alpha(a) \in]0, 1[$ and a complex coordinates' system z on U, such that

(i) $$\mathrm{dd}^c u | U \geqq \lambda_o \cdot \sum_{j=k+1}^{n} i \, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j.$$

(ii) $p > k + \frac{1}{2\alpha}$ *and* $u^{1-\alpha}|U$ *is still plurisubharmonic.*

Then there is a uniquely determined positive, closed extension of T *to* Ω. – *This uniquely determined extension is the trivial extension* $\tilde{T}$ *of* T *to* Ω.

This theorem generalizes a result of El Mir [1, Theorem 2].

Using slicing-theory Theorem 1 gives Proposition 4:

Proposition 4 *Let* $\Omega \subset \mathbb{C}^n$ *be open,* $u: \Omega \to \mathbb{R}_0^+$ *strictly q-convex (in particular two times continuous differentiable), and let* $A \subset \Omega$ *be closed,* $A \subset u^{-1}(\{0\})$.

If T *is a positive, closed current of bidimension* (p, p) *on* $\Omega \setminus A$, *and if* $p > q$, *then* T *can be continued in an unique way to a positive, closed current on* Ω.

Finally we get the results of El Mir and Sibony as a consequence of this proposition. But we only need the $\mathscr{C}^1$-smoothness of the real submanifold A'. (Corollary 1)

One of the most interesting newer continuation theorems for positive, closed currents is from El Mir [2] and Sibony [11]:

Let A *be a closed, complete pluripolar subset of an open set* $\Omega \subset \mathbb{C}^n$.

If T *is a closed, positive current on* $\Omega \setminus A$ *and if* T *has locally finite mass near* A, *then the trivial extension* $\tilde{T}$ *of* T *to* Ω *is a closed, positive current on* Ω.

($A \subset \Omega$ is called *complete pluripolar*, iff there is a plurisubharmonic function u: $\Omega \to \mathbb{R} \cup \{-\infty\}$, such that $A = u^{-1}(\{-\infty\})$.)

The next question is, in which directions generalizations of pluripolarity are possible in a manner, that there still will be continuation theorems.

If we define for example for $q \geqq 2$ a *complete q-polar set* A, such that A is the polar set of a continuous function u: $\Omega \to \mathbb{R} \cup \{-\infty\}$, which is strictly q-convex in $\Omega \setminus A$, it is easy to see, that in the general case there will be no continuation theorems for positive, closed currents.

One possible generalization of pluripolarity was defined by Sibony: He claimed, that $A \cap Y$ is pluripolar in Y for a sufficient number of $(n-q+1)$-dimensional complex submanifolds $Y \subset \Omega$, and with this definition he proved continuation theorems for positive, closed currents. (See [11].)

An other possibility according to Grauert and Riemenschneider [3] or Siu [13] is:

Definition 1 Let (Ω, ds^2) be a Kähler manifold.

(1) A continuous function u: $\Omega \to \mathbb{R} \cup \{-\infty\}$ is called *subharmonic*, iff for every open neighbourhood $U \Subset \Omega$ and every continuous function h: $\bar{U} \to \mathbb{R}$, which is harmonic in U and which satisfies $u|bU \leqq h|bU$, we have: $u \leqq h$ in U.
(2) A continuous function u: $\Omega \to \mathbb{R} \cup \{-\infty\}$ is called *strictly subharmonic*, iff for every inifinitely often differentiable function g with compact support in Ω there is an $\varepsilon > 0$, such that $u + \varepsilon g$ is still subharmonic.
(3) Let $q \in \mathbb{N}$. A continuous function u: $\Omega \to \mathbb{R} \cup \{-\infty\}$ is called *hyper-q-convex* (resp. *strictly-hyper-q-convex*), iff one has for every open set $V \subset \Omega$ and for every q-dimensional complex submanifold $X \subset V$:
$u|X$ is subharmonic (resp. strictly subharmonic) with respect to the Kähler metric, which is induced to X by ds^2.
(4) A subset A of Ω is called *complete hyper-q-polar*, $q \in \mathbb{N}$, iff there is a hyper-q-convex function u: $\Omega \to \mathbb{R} \cup \{-\infty\}$, such that $A = u^{-1}(\{-\infty\})$.

We will show, that the following theorem holds for complete hyper-q-polar sets:

Theorem 2 *Let (Ω, ds^2) be a Kähler manifold, $A \subset \Omega$ a complete hyper-q-polar subset. $(q \geqq 1.)$ If $X \subset \Omega \setminus A$ is analytic of the pure dimension q with locally finite volume near A, then $\bar{X}$ is an analytic continuation of X to Ω.*

Another continuation theorem for analytic sets, which only makes assumptions to the "interior convexity" of Reg X (that are the regular points of X), is:

Theorem 3 *Let (Ω, ds^2) be a Kähler manifold.*

Let A be a closed subset of Ω and $X \subset \Omega \setminus A$ be an analytic subset with locally finite volume near A. If there exists a continuous, subharmonic function u: Reg $X \to \mathbb{R}$ which is (locally in Ω) bounded from above and for which

$$\lim_{\substack{x \to a \\ x \in \mathrm{Reg}\, x}} u(x) = -\infty \quad \text{for every } a \in A \cap \bar{X},$$

then $\bar{X}$ is an analytic continuation of X to Ω.

1 Fundamental definitions and simple consequences

Definition 2 Let X be an hermitian complex manifold, $k, p, q \in \mathbb{N}_0$.

(1) Let $\mathscr{D}(X)$ be the space of the test-functions on X equipped with the usual topology.

$\mathscr{D}_k(X)$ is the space of the k-forms with coefficients in $\mathscr{D}(X)$.

$\mathscr{D}_{p,q}(X)$ is the subspace of the (p, q)-forms in $\mathscr{D}_{p+q}(X)$.

(2) $\mathscr{D}'_{p,q}(X) := \{T\colon \mathscr{D}_{p,q}(X) \to \mathbb{C} \mid T \text{ is linear and continuous}\}$. (We define $\mathscr{D}'_k(X)$ analogous.) An element $T \in \mathscr{D}'_{p,q}(X)$ is called *current of bidimension* (p, q).
If $n := \dim_{\mathbb{C}} X$, then $(n-p, n-q)$ is the *bidegree* of the current.

A current is said to have *order* 0, if there is a continuous continuation of the current to the space of forms with continuous coefficients and compact support.

(3) $\alpha \in \mathscr{D}_{p,p}(X)$ is *strictly positive*, iff α is a finite sum of expressions of the form

$$\left(\frac{i}{2}\right)^p \alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge \alpha_p \wedge \bar{\alpha}_p, \text{ where } \alpha_j \in \mathscr{D}_{1,0}(X).$$

(4) A current $T \in \mathscr{D}'_{p,p}(X)$ is said to be *positive*, $(T \geqq 0)$, iff $T \wedge \alpha$ is a positive distribution on X for every strictly positive form $\alpha \in \mathscr{D}_{p,p}(X)$.

E.g.: $\forall \varphi \in \mathscr{D}(X)$, $\varphi \geqq 0$ we have $(T \wedge \alpha)(\varphi) := T(\varphi \cdot \alpha) \geqq 0$.

(5) Let $T, T' \in \mathscr{D}'_{p,p}(X)$ be currents.

By $T \geqq T'$ *in distributional sense* we mean that $T - T'$ is a positive current.

Some notations. (1) If X is an oriented, $\mathscr{C}^\infty$-smooth manifold, $p \in \mathbb{N}_0$, $\varphi \in \mathscr{D}_p(X)$, $T \in \mathscr{D}'_p(X)$, we define $\langle T, \varphi \rangle := \int T \wedge \varphi := T(\varphi)$.

If $U \subset X$ is open, $T \mid U$ is defined by $T \mid U := T \mid \mathscr{D}'_p(U)$.

(2) For $I := (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ we define $|I| := k$.

$I=(i_1,\ldots,i_k)\in\mathbb{N}_0^k$ is said to be strictly (monoton) increasing, iff $i_1<i_2<\ldots<i_k$.

(3) $$\sum_{|I|=k}' := \sum_{\substack{(i_1,\ldots,i_k)\in\{1,\ldots,n\}^k \\ 1\leqq i_1<i_2<\ldots<i_k\leqq n}} \qquad \sum_{\substack{|I|=p \\ |J|=q}}' := \sum_{|I|=p}' \sum_{|J|=q}'$$

(4) For $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$, the vector dz_j is as usual a cotangential vector.

For $p,q\in\mathbb{N}_0$ with $p+q>0$ and for $I=(i_1,\ldots,i_p)\in\{1,\ldots,n\}^p$ and $J=(j_1,\ldots,j_q)\in\{1,\ldots,n\}^q$ set

$$dz_J := \begin{cases} dz_{i_1}\wedge d\bar{z}_{j_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar{z}_{j_p}\wedge d\bar{z}_{j_{p+1}}\wedge\ldots\wedge d\bar{z}_{j_q} & \text{for } 0<|I|<|J| \\ dz_{i_1}\wedge d\bar{z}_{j_1}\wedge\ldots\wedge dz_{i_q}\wedge d\bar{z}_{j_q}\wedge dz_{i_{q+1}}\wedge\ldots\wedge dz_{i_p} & \text{for } 0<|J|<|I| \\ dz_{i_1}\wedge d\bar{z}_{j_1}\wedge\ldots\wedge dz_{i_p}\wedge d\bar{z}_{j_q} & \text{for } 0<|I|=|J|. \end{cases}$$

If $|I|=0$, we set $dz_{\bar{I}}:=dz_J:=d\bar{z}_{j_1}\wedge\ldots\wedge d\bar{z}_{j_q}$, and we define analogous $dz_I:=dz_J:=dz_{i_1}\wedge\ldots\wedge dz_{i_p}$ if $|J|=0$. □

If $\Omega\subset\mathbb{C}^n$ is open, $T\in\mathscr{D}'_{p,p}(\Omega)$, then T can obviously be written in the following form:

$$T=\sum_{\substack{|I|=n-p \\ |J|=n-p}}' T_{IJ}\wedge dz_J.$$

T_{IJ} is called a *coefficient of* T.

There are coordinates' systems $w^{(\nu)}=(w_1^{(\nu)},\ldots,w_n^{(\nu)})$, $\nu=1,\ldots,n_o:=\binom{n}{p}^2$, which come from the coordinates' system z by an unitary linear transformation, and there are complex numbers $c_{IJ}^{(\nu)}$, such that we have for all strictly monoton increasing $I,J\in\{1,\ldots,n\}^{n-p}$:

$$T_{IJ}\wedge\left(\frac{i}{2}\right)^n dz_1\wedge d\bar{z}_1\wedge\ldots\wedge dz_n\wedge d\bar{z}_n=\sum_{\nu=1}^{n_o} c_{IJ}^{(\nu)}\,T_\nu,$$

with $T_\nu=T\wedge\left(\frac{i}{2}\right)^p dw_1^{(\nu)}\wedge d\bar{w}_1^{(\nu)}\wedge\ldots\wedge dw_p^{(\nu)}\wedge d\bar{w}_p^{(\nu)}$.

In particular T_ν is a positive distribution, if T is a positive current.

With standard-methods of measure theory (or of functional analysis) it is possible to extend T to an essentially greater class of forms.

This class contains the forms, which have coefficients, that are bounded, semicontinuous, real valued functions with compact support. (In particular products of the form $\varphi\cdot f$, where $\varphi\in\mathscr{D}_{p,p}(\Omega)$ and where f is a locally bounded, semicontinuous function, belong to this class.)

This extension of a current T will also be denoted by T and by dom T we will mean the space of functions to which T can be extended.

Definition 3 Let X be an n-dimensional, hermitian manifold with metric ds^2.

(1) $\beta:=\omega_{ds^2}:=-\frac{1}{2}\operatorname{Im} ds^2$ denotes the fundamental form belonging to ds^2.

(2) $dV_{ds^2}:=\dfrac{1}{n!}\,\omega_{ds^2}^n$ will denote the volume form on X belonging to ds^2.

(3) For $z \in X$ let $T_z^{*1,0} X$ be the holomorphic cotangent space of X in z and let $T_z^{*0,1} X$ be the antiholomorphic cotangent space of X in z.

For $p, q \in \mathbb{N}$ set $T_z^{*p,q} X := (\wedge^p T_z^{*1,0} X) \wedge (\wedge^q T_z^{*0,1} X)$.

(4) For $\alpha \in T_z^{*p,q} X$ define $|\alpha|_{ds^2} := \langle \alpha, \alpha \rangle_{p,q,ds^2}^{1/2}$, where $\langle \cdot, \cdot \rangle_{p,q,ds^2}$ denotes the scalar product induced to $T_z^{*p,q} X$ by ds^2.

(5) We define the following complex conjugated linear map

$$\begin{aligned} \bar{*}_{ds^2}\colon\ & T_z^{*p,q} X \to T_z^{*n-p,n-q} X \\ & \varphi \quad \mapsto \bar{*}_{ds^2} \varphi \end{aligned}, \quad \text{by:}$$

$$\langle \psi, \varphi \rangle_{p,q,ds^2} 2^n dV_{ds^2}(z) = \psi \wedge (\bar{*}_{ds^2} \varphi) \quad \text{for every } \psi \in T_z^{*p,q} X.$$

$\bar{*}_{ds^2}$ is called *conjugated Hodge-$*$-operator or Hodge-$\bar{*}$-operator.*

If a confusion is impossible, we will omit the indices p, q and ds^2.

If $\Omega \subset \mathbb{C}^n$ is open, we fix $ds^2 = \sum_{j=1}^{n} dz_j \otimes d\bar{z}_j$, if we do not denote another metric explicitely.

Definition 4 Let X be a complex manifold with hermitian metric ds^2.

(1) If $T \in \mathscr{D}'_k(X)$ is a current of order 0 and $U \subset X$ is open, we define

$$\|T\|(U) := \sup \{|T\varphi| \,|\, \varphi \in \mathscr{D}_k(X), \sup_{x \in U} |\varphi(x)| \leqq 1, \operatorname{supp} \varphi \subset U\}$$

$\|T\|$ will be extended to a Borelian measure on Borel sets $E \subset X$ as usual.
If $E \subset X$ is a Borel set, $\|T\|(E)$ is called *the mass of T on E.*

(2) Let $T \in \mathscr{D}'_{p,p}(X)$ be a positive current.

Set $\sigma_T := \sigma := T \wedge \frac{1}{p!} \beta^p$ and define for open subsets U of X:

$$\sigma_T(U) := \int_U T \wedge \frac{1}{p!} \beta^p .$$

σ_T is also extended to the Borelian sets $E \subset X$ in the canonical way.
σ_T is called *trace measure* of T.

(3) Let $A \subset X$ be closed and let $T \in \mathscr{D}'_{p,q}(X \setminus A)$ be a current of order 0.

T is said to have *locally finite mass*, iff $\|T\|(U \setminus A) < \infty$ for every relative compact subset U of X. ($U \Subset X$)

(4) Let $v\colon X \to \mathbb{R}$ be a function.

v is said to be *locally bounded from below* (*from above*), iff for every compact set $K \Subset X$ there is a constant $c = c(K) \in \mathbb{R}^+$, such that $\inf_K v \geqq -c$ (resp. $\sup_K v \leqq c$).

v is said to be *locally bounded*, if v is locally bounded from above and below.

If we denote d^c by $d^c := \frac{1}{i}(\partial - \bar{\partial})$, we obtain in a similar way as El Mir [2, Proposition I.3]:

Proposition 1 *Let $\Omega \subset \mathbb{C}^n$ be open and let $T \in \mathscr{D}'_{p,p}(\Omega)$, $1 \leqq p \leqq n$, be a positive, closed current. If $v\colon \Omega \to \mathbb{R}$ is semicontinuous, plurisubharmonic and locally bounded from below, then $dd^c(vT)$ is a well defined positive, closed current of*

bidimension $(p-1, p-1)$ *on* Ω *and, if* $v_j: \Omega \to \mathbb{R}$ *is a sequence of smooth plurisubharmonic functions, which converge pointwise monotonically decreasing to* v, *then* $\mathrm{dd^c}(vT) = \lim_{j\to\infty} \mathrm{dd^c}(v_j T)$.

Definition 5 Let X be a hermitian complex manifold, $T \in \mathscr{D}'_{p,p}(X)$ be positive and closed, $p \geqq 1$.

(1) If $f: X \to \mathbb{R} \cup \{+\infty, -\infty\}$ or $f: X \to \mathbb{C}$ is a function, such that $f \cdot \varphi$ belongs to dom T for each $\varphi \in \mathscr{D}_{p,p}(X)$, we define: $T \wedge \partial\bar{\partial} f := \partial\bar{\partial}(f \cdot T)$, $T \wedge \partial f := \partial(f \cdot T)$, $T \wedge \bar{\partial} f := \bar{\partial}(f \cdot T)$.

(2) If $v: X \to \mathbb{R}$ is a semicontinuous, plurisubharmonic function, which is locally bounded from below, we set $T \wedge \mathrm{d}v \wedge \mathrm{d^c}v := \frac{1}{2}\mathrm{dd^c}(v^2 T) - v \cdot \mathrm{dd^c}(vT)$.

(3) Let $0 \leqq k \leqq p$. For $j = 1, \ldots, k$ let $v_j: X \to \mathbb{R}$ be semicontinuous, plurisubharmonic functions, which are locally bounded from below. Inductively we define:

$$T \wedge \mathrm{dd^c}v_1 \wedge \ldots \wedge \mathrm{dd^c}v_k := \mathrm{dd^c}(v_k \cdot T \wedge \mathrm{dd^c}v_1 \wedge \ldots \wedge \mathrm{dd^c}v_{k-1}).$$

With arguments similar to the arguments of El Mir [2, Theorem I.5] it is possible to prove:

Proposition 2 *Let* X *be a hermitian complex manifold,* $u: X \to \mathbb{R}_0^+$ *a semicontinuous, plurisubharmonic function and let* $A \subset u^{-1}(\{0\})$ *be a closed subset of* Ω.

If $T \in \mathscr{D}'_{p,p}(X \setminus A)$ *is a positive, closed current with locally finite mass near* A, $p \geqq 1$, *then it is possible to continue* $T \wedge \mathrm{dd^c}u$ *to a positive closed current on* X.

In particular: $T \wedge \mathrm{dd^c}u$ *has locally finite mass near* A.

Slicing of locally flat currents. Let $\Omega \subset \mathbb{C}^n$ be open, let $T \in \mathscr{D}'_k(\Omega)$ be a locally flat current. Denote by $\pi: \Omega \to \mathbb{C}^m$ a projection, $2m \leqq k \leqq 2n$.

If $y \in G := \pi(\Omega)$, we define $\langle T, \pi, y\rangle$ as *slice of* T *on* $\pi^{-1}(\{y\})$.

$\langle T, \pi, y\rangle$ is for Lebesgue-almost-all $y \in G$ a locally flat current in $\mathscr{D}'_{k-2m}(\pi^{-1}(\{y\}))$.

2 Continuation of currents across the zero sets of non-negative strictly plurisubharmonic functions

In this section we want to prove Theorem 1, which is a generalization of a theorem of El Mir. For the comparison of the result of El Mir [1, Theorem 2] with Theorem 1 the following lemma is usefull:

Lemma 1 (Comparison lemma) *Let* $\Omega \subset \mathbb{C}^n$ *be open.*

(1) *If* $v: \Omega \to \mathbb{R}_0^+$ *is semicontinuous from above, we have:*

If there is an $\eta \in \mathbb{R}^+$, *such that*

$$\mathrm{dd^c}v \geqq v^{-\eta} \sum_{\nu=1}^{n-k} i\,\mathrm{d}z_{j_\nu} \wedge \mathrm{d}\bar{z}_{j_\nu},$$

for a strictly increasing $(j_1, \ldots, j_{n-k}) \in \{1, \ldots, n)^{n-k}$, *the following holds for* $u := v^{1+\eta}$:

(i) $\{z \in \Omega \mid u(z) = 0\} = \{z \in \Omega \mid v(z) = 0\}$.

(ii) $\mathrm{dd}^c u \geqq \lambda_o \sum_{\nu=1}^{n-k} i\,\mathrm{d}z_{j_\nu} \wedge \mathrm{d}\bar{z}_{j_\nu}$ for some $\lambda_o > 0$.

(iii) $\mathrm{dd}^c u^{1-\alpha} \geqq 0$ *for every* $\alpha \leqq \dfrac{\eta}{1+\eta}$.

(2) *If* $u: \Omega \to \mathbb{R}_0^+$ *is semicontinuous from above, then we have:*

If there is a $\lambda_o \in \mathbb{R}^+$ *and a strictly increasing* $(j_1, \ldots, j_{n-k}) \in \{1, \ldots, n)^{n-k}$, *such that*

$$\mathrm{dd}^c u \geqq \lambda_o \sum_{\nu=1}^{n-k} i\,\mathrm{d}z_{j_\nu} \wedge \mathrm{d}\bar{z}_{j_\nu}$$

and if there is an $\alpha \in]0, 1[$, *such that* $u^{1-\alpha}$ *is still plurisubharmonic, then for every* $\gamma \in]1-\alpha, 1[$ *there exists a* $C > 0$, *such that the following holds for* $v := C \cdot u^\gamma$:

(i) $\{z \in \Omega \mid v(z) = 0\} = \{z \in \Omega \mid u(z) = 0\}$.

(ii) $\mathrm{dd}^c v \geqq v^{-\eta} \sum_{\nu=1}^{n-k} i\,\mathrm{d}z_{j_\nu} \wedge \mathrm{d}\bar{z}_{j_\nu}$, *where* $\eta := \dfrac{1}{\gamma} - 1$.

Using the previous lemma it is possible to formulate the theorem of El Mir in a way, such that a direct comparison with Theorem 1 is possible:

Theorem (El Mir) *Let* $\Omega \subset \mathbb{C}^n$ *be open,* $u: \Omega \to \mathbb{R}_0^+$ *plurisubharmonic.*

Let $A \subset \Omega$ *be closed,* $A \subset u^{-1}(\{0\})$, *and suppose, that there is a neighbourhood* U *of* A, *a strictly increasing* $J = (j_1, \ldots, j_{n-k}) \in \{1, \ldots, n\}^{n-k}$, *a* $\lambda_o \in \mathbb{R}^+$ *and a real number* $\alpha > \frac{1}{3}$, *such that* $\mathrm{dd}^c u \mid U \geqq \lambda_o \sum_{\nu=1}^{n-k} i\,\mathrm{d}z_{j_\nu} \wedge \mathrm{d}\bar{z}_{j_\nu}$ *and such that* $u^{1-\alpha} \mid U$ *is one time continuous differentiable and plurisubharmonic.*

Then we have: If $T \in \mathscr{D}'_{p,p}(\Omega \setminus A)$ *is a positive, closed current,* $p > k+1$, *there is a uniquely determined positive, closed continuation of* T *to* Ω.

Proof of the Comparison Lemma. For (1) Since $v \geqq 0$, $u := v^{1+\eta}$ is non-negative and plurisubharmonic.

Approximate v by a monoton decreasing sequence of positive, smooth plurisubharmonic functions and use the theorem of monoton convergence.

For (2) For arbitrary $\gamma \in]1-\alpha, 1[$ and $\eta := \dfrac{1}{\gamma} - 1$, $v := Cu^\gamma$ is plurisubharmonic for $C \in \mathbb{R}^+$. Let $V \Subset \Omega$ be open and $(u_j^{(1-\alpha)})_{j \in \mathbb{N}}$ a sequence of smooth, plurisubharmonic functions in U, which converge pointwise monoton decreasing to $u^{1-\alpha} \mid V$.

Then $u_j := \left(u_j^{(1-\alpha)} + \dfrac{1}{j}\right)^{\frac{1}{1-\alpha}}$ is plurisubharmonic and the sequence converges monoton decreasing to $u \mid V$. In V we have for every fixed $j_o \in \mathbb{N}$:

$$u_{j_o}^{1-\gamma}\,\mathrm{dd}^c v = C \lim_{j \to \infty} \left[u_{j_o}^{1-\gamma} \gamma \frac{1}{u_j^{1-\gamma}} \left(\mathrm{dd}^c u_j - (1-\gamma) \frac{\mathrm{d}u_j \wedge \mathrm{d}^c u_j}{u_j} \right) \right].$$

For $u_j^{1-\alpha}$ is plurisubharmonic, we derive: $\dfrac{1}{u_j}\,\mathrm{d}u_j \wedge \mathrm{d}^c u_j \leqq \dfrac{1}{\alpha}\,\mathrm{dd}^c u_j$.

So we have in V:

$$u_{j_o}^{1-\gamma}\,\mathrm{dd^c}v \geqq C\cdot\gamma\lim_{j\to\infty}\left[\left(\frac{u_{j_o}}{u_j}\right)^{1-\gamma}\cdot\left(\mathrm{dd^c}u_j-\frac{(1-\gamma)}{\alpha}\,\mathrm{dd^c}u_j\right)\right]$$

$$\geqq\sum_{\nu=1}^{n-k} i\,\mathrm{d}z_{j_\nu}\wedge\mathrm{d}\bar{z}_{j_\nu}\qquad\text{for each } j_o\in\mathbb{N},\text{ if } C=\frac{\alpha}{(\gamma-(1-\alpha))\cdot\lambda_o\cdot\gamma}.$$

The statement follows for $j_o\to\infty$ from the theorem of monoton convergence. qed

For the proof of Theorem 1 we now need some lemmas:

Lemma 2 *Let $\Omega'\subset\mathbb{C}^n$ be open, S, $T\in D'_{p,p}(\Omega')$, $p\geqq 1$.*

Suppose, that S, T are positive, closed currents on Ω', such that $S\leqq T$.

Let $f\colon\Omega'\to\mathbb{R}\cup\{+\infty\}$ be semicontinuous from below.

Let v, $u\colon\Omega'\to\mathbb{R}$ be plurisubharmonic, locally bounded functions, where v is continuous.

If $\mathrm{dd^c}u\geqq f\cdot\mathrm{dd^c}v$ (in distributional sense), then $T\wedge\mathrm{dd^c}u\geqq f\cdot(S\wedge\mathrm{dd^c}v)$.

Proof. If $f\equiv c\in\mathbb{R}$, it follows from the plurisubharmonicity of $u-cv$ and the approximation of v by smooth plurisubharmonic functions, that $T\wedge\mathrm{dd^c}u\geqq T\wedge(c\cdot\mathrm{dd^c}v)\geqq c\cdot(S\wedge\mathrm{dd^c}v)$.

For an arbitrary function f the statement follows from the usual approximation methods of measure theory. qed

Lemma 3 *Let $\Omega'\subset\mathbb{C}^n$ be open, $T\in\mathscr{D}'_{p,p}(\Omega')$ be positive and closed, $p>1$.*

Let $v\colon\Omega'\to\mathbb{R}$ be continuous and plurisubharmonic and let $f\colon\Omega'\to\mathbb{R}$ be continuous.

Suppose, that $(v_j)_{j\in\mathbb{N}}$, $(f_j)_{j\in\mathbb{N}}$ are sequences of smooth functions, which converge uniformly to v resp. f on compact sets. Every v_j should be plurisubharmonic.

Then we have (in the sense of the convergence of currents):

(1) $\lim_{j\to\infty} f_j\,T\wedge\mathrm{dd^c}v_j=f(T\wedge\mathrm{dd^c}v)$.

(2) $\lim_{j\to\infty} f_j\,T\wedge\mathrm{d}v_j\wedge\mathrm{d^c}v_j=f(T\wedge\mathrm{d}v\wedge\mathrm{d^c}v)$.

Proof. (1) was proved by El Mir. (See [2, Theorem I.4].) El Mir claimed, that f and f_j should be plurisubharmonic for every j, but this is not ncessary for the proof of (1).

(2) is a consequence of the definition of $T\wedge\mathrm{d}v\wedge\mathrm{d^c}v$ and of (1).

Lemma 4 (Chern-Levine-Nirenberg estimation) *Let $\Omega\subset\mathbb{C}^n$ be open, $A\subset\Omega$ be closed. Let $T\in\mathscr{D}'_{1,1}(\Omega\setminus A)$ be positive and closed with locally finite mass near A, then we have: For each compact subset K of Ω and for each open neighbourhood U of K, $U\subset\Omega$, there is a constant $C=C(K,U)\in\mathbb{R}^+$, such that for every plurisubharmonic function $v\colon\Omega\to\mathbb{R}$, which vanishes in a neighbourhood of A, the following estimation is true:*

$$\int_{K\setminus A} T\wedge\mathrm{dd^c}v\leqq C(K,U)\cdot\|T\|(U\setminus A)\cdot\sup_U|v|.$$

If v is two times continuous differentiable and plurisubharmonic, this estimation is a result of Sibony [11, in the proof of Lemma 1.1].

Proof. Let K, U and v be as in the lemma. Withut loss of generality let $U \Subset \Omega$.

Choose an open neighborhood W_o of $A \cap \bar{U}$, such that v vanishes in an open neighbourhood of $\bar{W}_o$. Let $\psi \in \mathscr{D}(\Omega)$ with:

(i) $0 \leqq \psi \leqq 1$.
(ii) $\psi \backslash \bar{U} \backslash W_o \equiv 1$.
(iii) $\operatorname{supp} \psi \subset \Omega \backslash A$.

Let $\varphi \in \mathscr{D}(\Omega)$ with $0 \leqq \varphi \leqq 1$, $\varphi | K \equiv 1$, $\operatorname{supp} \varphi \in U$.
With this definitions we have:

$$\int_{K \backslash A} T \wedge \mathrm{dd^c} v \leqq \int T \wedge \mathrm{dd^c} v \cdot \varphi = \int T \wedge v \cdot \mathrm{dd^c}(\psi \cdot \varphi) \,(\text{for: } \operatorname{supp}(1-\psi)\,\varphi \subset \bar{W}_o \cap U)$$

$$= \int T \wedge v \cdot \mathrm{dd^c} \varphi,$$

for: $\operatorname{supp}(\mathrm{d}\psi \wedge \mathrm{d^c}\varphi + \mathrm{d}\varphi \wedge \mathrm{d^c}\psi + \varphi \mathrm{dd^c}\psi) \subset \overline{\Omega \backslash (\bar{U} \backslash W_o) \cap U} \subset \bar{W}_o$ and $v \equiv 0$ near $\bar{W}_o$.

Therefore

$$\int_{K \backslash A} T \wedge \mathrm{dd^c} v \leqq C(K, U) \cdot \| T \| (U \backslash A) \cdot \sup_U |v|,$$

where $C(K, U)$ not depends on v and T but only on $\sup |\mathrm{dd^c} \varphi|$. qed

In the case of smooth plurisubharmonic functions the following lemma was also proved by Sibony [11, Lemma 1.1].

Lemma 5 *Let $\Omega \subset \mathbb{C}^n$ be open, $A \subset \Omega$ closed. If $T \in \mathscr{D}'_{1,1}(\Omega \backslash A)$ is positive and closed with locally finite mass near A, then we have: For each compact subset K of Ω and for each open neighbourhood U of K, $U \subset \Omega$, there is a constant $C = C(K, U)$, such that for every $0 < a < b$, $a, b \in \mathbb{R}$, and for every continuous plurisubharmonic function $u: \Omega \to \mathbb{R}_0^+$ with $u | A \equiv 0$ holds:*

$$\int_{K \cap \{a \leqq u \leqq b\}} T \wedge \mathrm{d}u \wedge \mathrm{d^c} u \leqq C(K, U) \cdot \| T \| (U \backslash A) \cdot (b-a) \cdot \sup_U u.$$

Proof. Let $K \subset U \Subset \Omega$ be as in the lemma. Let $0 < a < b$ be arbitrary.

Let $(u_j)_{j \in \mathbb{N}}$ be a sequence of smooth functions on Ω, which are plurisubharmonic in U and which converge monotonically decreasing to $u | U$, such that $\sup_{\bar{U}} |u - u_j| < \dfrac{a}{4j}$.

Let $h: \mathbb{R} \to \mathbb{R}$ be a smooth, monotonically increasing, convex function, (which depends on a and b) with the following properties:

(i) $h'' \geqq \dfrac{1}{b-a} \cdot \mathbb{1}_{[a,b]}$.

(ii) $\operatorname{supp} h'' \subset \left[\dfrac{a}{2}, \dfrac{3}{2} b\right]$.

(iii) $\int_{\mathbb{R}} h''(t)\, \mathrm{d}t \leqq 2$.

(iv) $h \left| \left] -\infty, \dfrac{a}{2} \right] \right. \equiv 0$.

(v) $h(t) \leqq 2t$ for each $t \in \mathbb{R}_0^+$.

With this $h\circ u$ and $h\circ u_j$ are plurisubharmonic on U, and if $W_1 := \left\{z\in\Omega \mid u_1(z) < \frac{a}{4}\right\}$, then W_1 is an open neighbourhood of $A\cap\bar{U}$, and as well $h\circ u$ as $h\circ u_j$ vanish in an open neighbourhood of $\bar{W}_1\cap U$ (for each j).

Let $K'\subset U$ be compact, $K\subset\mathring{K}'$, and let $\varphi\in\mathscr{D}(\Omega)$, $0\leqq\varphi\leqq 1$ with supp $\varphi\subset \mathring{K}'\setminus A$ and $\varphi|K\setminus W_1\equiv 1$.

$$\begin{aligned}
&\int\limits_{K\cap\{a\leqq u\leqq b\}} T\wedge \mathrm{d}u\wedge \mathrm{d}^c u \leqq (b-a)\int h''(u)(T\wedge \mathrm{d}u\wedge \mathrm{d}^c u)\cdot\varphi\\
&=(b-a)\lim_{j\to\infty}\int h''(u_j)\, T\wedge \mathrm{d}u_j\wedge \mathrm{d}^c u_j\cdot\varphi \quad \text{(according to Lemma 3)}\\
&\leqq (b-a)\lim_{j\to\infty}\int T\wedge \mathrm{dd}^c h(u_j)\cdot\varphi \leqq (b-a)\int\limits_{K'\setminus A} T\wedge \mathrm{dd}^c(h\circ u)\\
&\leqq C'(K',U)\cdot\|T\|(U\setminus A)\cdot(b-a)\cdot 2\cdot\sup_{\bar{U}} u \quad \text{(according to Lemma 4).} \quad \text{qed}
\end{aligned}$$

The following lemma was proved by El Mir for the case of a continuous plurisubharmonic function. (See [2, Proposition III.4].) After using the comparison lemma (Lemma 1) the proof for semicontinuous plurisubharmonic functions is analogous to the proof of El Mir.

Lemma 6 *Let $\Omega'\subset\mathbb{C}^n$ be open and $u\colon \Omega'\to\mathbb{R}_0^+$ semicontinuous from above with following properties:*

(i) $\exists\lambda_o>0$, *such that* $\mathrm{dd}^c u\geqq\lambda_o\sum\limits_{j=k+1}^{n} i\,\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j$.

(ii) $\exists\alpha\in]0,1[$, *such that* $u^{1-\alpha}$ *is still plurisubharmonic.*

If $p>k$, and if $T\in\mathscr{D}'_{p,p}(\Omega')$ is a positive, closed current, then $u^{-1}(\{0\})$ is a $\|T\|$-zero set.

Notation. If $a\in\mathbb{C}^n$ and $r>0$ we denote by $\mathrm{B}(a,r)$ the ball with center a and radius r.

For deriving the finiteness of the mass of the current for all cases, which are considered here, we will prove the following lemma:

Lemma 7 *Let $a\in\mathbb{C}^n$, $r>0$, $k\in\mathbb{N}_0$, $0\leqq k<n$. (Define $\mathbb{C}^0:=\{0\}$ and $\mathrm{dV}_0\equiv 1$.)*

Let $u\colon \mathrm{B}(a,r)\to\mathbb{R}_0^+$ be semicontinuous from above with the following property: There are linear projections $\pi_o,\ldots,\pi_m\colon \mathrm{B}(a,r)\to\mathbb{C}^k$, such that $\{\pi_j^*\,\mathrm{dV}_k \mid j=0,\ldots,m\}$ generates the space of the (k,k)-forms on $\mathrm{B}(a,r)\subset\mathbb{C}^n$ and such that there is a $\lambda\in\mathbb{R}^+$, such that we have for each $\pi\in\{\pi_o,\ldots,\pi_m\}$ and every $y\in\pi(\mathrm{B}(a,r))$:

$$\mathrm{dd}^c(u|\pi^{-1}(\{y\}))\geqq\lambda i_y^*\,\mathrm{dd}^c|z|^2,$$

where $i_y\colon \pi^{-1}(\{y\})\hookrightarrow \mathrm{B}(a,r)$ is the natural inclusion.

If $A\subset u^{-1}(\{0\})$, A is closed and $T\in\mathscr{D}'_{p,p}(\mathrm{B}(a,r)\setminus A)$ is a positive, closed current, $p>k$, then $\|T\|(\mathrm{B}(a,\frac{1}{4}r)\setminus A)<\infty$.

Proof. After an affine change of coordinates we have $a=0$ and $r=1$.

For $\rho>0$ set $\mathrm{B}(\rho):=\mathrm{B}(0,\rho)$ and $\mathrm{B}:=\mathrm{B}(0,1)$.

Let $\varphi\in\mathscr{D}_{p,p}(\mathrm{B})$ be arbitrary with supp $\varphi\in\mathrm{B}(\frac{1}{4})\setminus A$ and $\sup|\varphi|\leqq 1$.

Choose a real valued function $\varphi_o\in\mathscr{D}(\mathrm{B})$, which depends on φ with the following properties:

(i) $0\leqq\varphi_o\leqq 1$,
(ii) $\varphi_0|\mathrm{supp}\,\varphi\equiv 1$,
(iii) supp $\varphi_o\subset\mathrm{B}(\frac{1}{4})\setminus A$.

With this we have: $|T(\varphi)| \leqq \|T\|(\operatorname{supp}\varphi) \leqq \text{const} \int T \wedge \varphi_o \beta^p$.

It suffices to show, that there are constants $c_o, \ldots, c_m$, such that

$$\sup\{\int T \wedge \varphi_o \beta^{p-k} \wedge \pi_j^* \, dV_k \mid \varphi_o \in \mathscr{D}(B), 0 \leqq \varphi_o \leqq 1, \operatorname{supp}\varphi_o \subset B(\tfrac{1}{4}) \setminus A\} \leqq c_j.$$

Let $\pi \in \{\pi_o, \ldots, \pi_m\}$ and let $i_y : \pi^{-1}(\{y\}) \hookrightarrow B$ be the related inclusion, if $y \in \pi(B)$.

Let $h \in \mathscr{D}(B)$ with

(i) $h | B(\frac{1}{4}) \equiv 0$,
(ii) $h | \bar{B}(\frac{3}{4}) \setminus B(\frac{1}{2}) \leqq -1$,
(iii) $h \leqq 0$.

Choose $\delta > 0$, such that $\delta \, dd^c h \geqq -\lambda \, dd^c |z|^2$. For $\varepsilon \geqq 0$ define $u_\varepsilon := \max(u + \delta h - \varepsilon, 0)$. Then we have:

(i) $u_o | B(\frac{1}{4}) \equiv u$.
(ii) $u_\varepsilon | \pi^{-1}\{(y\})$ is a plurisubharmonic for every $\varepsilon \geqq 0$ and every $y \in \pi(B)$.
(iii) $u_\varepsilon \underset{\varepsilon \searrow 0}{\nearrow} u_o$.
(iv) For each $\varepsilon > 0$ there is an open neighbourhood U_ε of A, such that $u_\varepsilon | U_\varepsilon \equiv 0$.
(v) It exists an open neighbourhood U_o of $K := (\bar{B}(\frac{3}{4}) \setminus B(\frac{1}{2})) \cap A$, such that $u_\varepsilon | U_o \equiv 0 \; \forall \varepsilon \geqq 0$.

Choose $\theta \in \mathscr{D}(B)$ with $0 \leqq \theta \leqq 1$ and

(i) $\theta | B(\frac{5}{8}) \equiv 1$,
(ii) $\operatorname{supp}\theta \subset B(\frac{3}{4})$.

Let $\varphi_o \in \mathscr{D}(B)$ be arbitrary, real-valued with $\operatorname{supp}\varphi_o \subset B(\frac{1}{4}) \setminus A$ and $0 \leqq \varphi_o \leqq 1$.

Using the slicing formula we have:

$$\int_{B \setminus A} T \wedge \varphi_o \beta^{p-k} \wedge \pi^* \, dV_k = \int_{\pi(B)} \Big(\int_{\pi^{-1}(\{y\}) \setminus A} (T, \pi, y) \wedge \varphi_o \circ i_y \, i_y^* \beta^{p-k} \Big) \, dV_k.$$

At first we want to transform the interior integral:

$$\int_{\pi^{-1}(\{y\}) \setminus A} \langle T, \pi, y) \wedge \varphi_o \circ i_y \, i_y^* \beta^{p-k}$$

$$\leqq \int_{\pi^{-1}(\{y\}) \setminus A} \langle T, \pi, y \rangle \wedge \varphi_o \circ i_y \cdot \frac{1}{\lambda} dd^c(u | \pi^{-1}(\{y\})) \wedge i_y^* \beta^{p-k-1} \qquad (p > k, \text{Lemma 2})$$

$$= \frac{1}{\lambda} \int_{\pi^{-1}(\{y\}) \setminus A} \langle T, \pi, y) \wedge i_y^* \, dd^c \varphi_o \cdot u_o \circ i_y \wedge i_y^* \beta^{p-k-1} \qquad (\text{because } u_o | B(\tfrac{1}{4}) \equiv u | B(\tfrac{1}{4}).)$$

$$\leqq \frac{1}{\lambda} \lim_{\varepsilon \searrow 0} \int_{\pi^{-1}(\{y\}) \setminus A} \langle T, \pi, y \rangle \wedge \theta \circ i_y \, dd^c(u_\varepsilon | \pi^{-1}(\{y\})) \wedge i_y^* \beta^{p-k-1}$$

(Theorem of Lebesgue)

$$= \frac{1}{\lambda} \int_{\pi^{-1}(\{y\}) \setminus A} \langle T, \pi, y \rangle \wedge i_y^* \, dd^c \theta \cdot u_o \circ i_y \wedge i_y^* \beta^{p-k-1} \qquad (\text{using Lebesgue's theorem}).$$

We have $\operatorname{supp}(u_o \cdot dd^c \theta) \subset L := B(\frac{3}{4}) \setminus (B(\frac{1}{2}) \cup U_o) \Subset B \setminus A$. Therefore we derive:

$$\int_{B \setminus A} T \wedge \varphi_o \beta^{p-k} \wedge \pi^* dV_k \leqq \frac{1}{\lambda} \|T\|(L) \cdot \sup_L |u_o \, dd^c \theta \wedge \beta^{p-k-1} \wedge \pi^* \, dV_k| < \infty.$$

So the statement is true, because θ and h were chosen independent of φ_o and φ. qed

Proof of Theorem 1 Since the problem is local, we may suppose, that $\sup_{\Omega}|u|<\infty$, that $\mathrm{dd}^c u \geqq \lambda_o \sum_{j=k+1}^{n} i\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ with a $\lambda_o>0$ and that $u^{1-\alpha}$ is plurisubharmonic for an $\alpha\in]0,1[$ with $p>k+\frac{1}{2\alpha}$.

According to the previous lemma the mass of T is locally finite near A.

The uniqueness of the continuation is a consequence of Lemma 6.

Let $0<a<b<1$, and let $\chi\colon \mathbb{R}\to\mathbb{R}$ be a smooth function with the following properties:

(i) $\chi|]-\infty, a]\equiv 0$.
(ii) $\chi|[b,\infty[\equiv 1$.
(iii) $\chi'\geqq 0$.

Then we have: $\tilde{T}=\lim_{\nu\to\infty}(\chi(\nu u)\cdot T)$.

We now want to prove, that the trivial extension $\tilde{T}$ of T to Ω is closed, if $k=0$ and $p>\frac{1}{2\alpha}$.

This is true, *because*: Let $w=(w_1,\ldots,w_n)$ be an arbitrary coordinates' system, which is deduced from the given coordinates z by an unitary change of coordinates.

Let $\omega:=\mathrm{dd}^c\sum_{j=1}^{p-1}|w_j|^2$, $\varphi_0\in\mathscr{D}(\Omega)$ and $j\in\{1,\ldots,n\}$, $\varphi=\varphi_o\,\omega^{p-1}\wedge\mathrm{d}\bar{z}_j$.

With this we have to prove: $\partial\tilde{T}(\varphi)=-\tilde{T}(\partial\varphi)=-\lim_{\nu\to\infty}\int_{\Omega} T\wedge\chi(\nu u)\,\partial\varphi=0$.

(The prove for $\varphi=\varphi_o\,\omega^{p-1}\wedge\mathrm{d}z_j$ is analogous.)

There is an open neighbourhood V of $\operatorname{supp}\varphi$, $V\Subset\Omega$, and there are smooth functions $u_\mu\colon\Omega\to\mathbb{R}_0^+$, $\mu\in\mathbb{N}$, with the following properties:

(i) $u_\mu^{1-\alpha}|V$ is plurisubharmonic. In particular $u_\mu|V$ is plurisubharmonic, too.
(ii) $u_\mu^{1-\alpha}|V$ converges monotonically decreasing to $u^{1-\alpha}|V$, if $\mu\to\infty$.
(iii) $u_\mu|\bar{V}\cap A<\frac{a}{4\mu}$. (Theorem of Dini, $u|\bar{V}\cap A\equiv 0$.)

Let $\alpha'\in[0,\alpha[$ be so near to α, that the estimation $p>\frac{1}{2\cdot\alpha'}$ is still true.

According to the Comparison Lemma we have for a suitable $C\in\mathbb{R}^+$: $C\,\mathrm{dd}^c u^{1-\alpha'}\geqq u^{-\alpha'}\mathrm{dd}^c|z|^2$. If $\tilde{\varphi}_o\in\mathscr{D}(\Omega)$ with $0\leqq\tilde{\varphi}_o\leqq 1$, $\operatorname{supp}\tilde{\varphi}_o\subset V$ and $\tilde{\varphi}_o|\operatorname{supp}\varphi_o\equiv 1$, then we have for each fixed $\nu\in\mathbb{N}$:

$$
\begin{aligned}
\Big|\int_{\Omega\setminus A} T\wedge\chi(\nu u)\,\partial\varphi\Big| &= \lim_{\mu\to\infty}\Big|\int_{\Omega\setminus A} T\wedge\chi(\nu u_\mu)\,\partial\varphi\Big| \quad\text{(Theorem of Lebesgue)}\\
&\leqq \nu\cdot\limsup_{\mu\to\infty}\Big|\int_{\Omega\setminus A} T\wedge\chi'(\nu u_\mu)\,\omega^{p-1}\wedge\mathrm{d}u_\mu\wedge\mathrm{d}^c u_\mu\cdot\tilde{\varphi}_o\Big|^{1/2}\\
&\quad\cdot\lim_{\mu\to\infty}\Big|\int_{\Omega\setminus A} T\wedge\chi'(\nu u_\mu)\,\omega^{p-1}\wedge|\varphi_o|^2\,i\,\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_j\Big|^{1/2}
\end{aligned}
$$

according to the Cauchy-Schwarz-inequality.

The second limit exists according to the theorem of Lebesgue, and the limit superior exists, because the sequence is bounded, as will be shown in the following step.

Estimation of the first factor. Choose $\mu \geqq \nu$, ν fixed. Then we have $v \cdot u_\mu | \bar{V} \cap A < \frac{a}{4}$, and:

$$\left| \int_{\Omega \setminus A} T \wedge \omega^{p-1} \cdot \chi'(v u_\mu) \wedge \mathrm{d} u_\mu \wedge \mathrm{d}^c u_\mu \cdot \tilde{\varphi}_o \right|$$

$$= \left| \int_{\Omega \setminus A} T \wedge \frac{\omega}{u_\mu^{\alpha'}} \wedge \omega^{p-2} \cdot \chi' \cdot u_\mu^{\alpha'} \wedge \mathrm{d} u_\mu \wedge \mathrm{d}^c u_\mu \cdot \tilde{\varphi}_o \right|$$

$$\leqq \text{const} \left| \int_{\Omega \setminus A} T \wedge \mathrm{dd}^c u^{1-\alpha'} \wedge \omega^{p-2} \cdot \chi' \cdot u_\mu^{\alpha'} \wedge \mathrm{d} u_\mu \wedge \mathrm{d}^c u_\mu \cdot \tilde{\varphi}_o \right|, \text{ if } p > 1 \text{ (Lemma 2).}$$

(const is a positive, real constant, which may change from one row to the next.)

Inductively we get:

$$\left| \int_{\Omega \setminus A} T \wedge \omega^{p-1} \cdot \chi'(v u_\mu) \wedge \mathrm{d} u_\mu \wedge \mathrm{d}^c u_\mu \cdot \tilde{\varphi}_o \right|$$

$$\leqq \text{const} \cdot \left(\frac{b}{v} \right)^{(p+1) \cdot \alpha'} \int_{\operatorname{supp} \tilde{\varphi}_o \cap \{z \in \Omega \mid \frac{a}{v} \leqq u_\mu(z) \leqq \frac{b}{v}\}} T \wedge (\mathrm{dd}^c u^{1-\alpha'})^{p-1} \wedge \mathrm{d} v_\mu \wedge \mathrm{d}^c v_\mu$$

$$\text{with } v_\mu := \max \left(u_\mu^{1-\alpha'} - \left(\frac{a}{2\mu} \right)^{1-\alpha'}, 0 \right)$$

because: $\operatorname{supp}(\tilde{\varphi}_o \cdot \chi'(v u_\mu)) \subset L_{\mu\nu} := \operatorname{supp} \tilde{\varphi}_o \cap \left\{ z \in \Omega \,\middle|\, \frac{a}{v} \leqq u_\mu(z) \leqq \frac{b}{v} \right\}$.

Using Lemma 5 and Proposition 2 we have for $\mu > \nu$:

$$\int_{\operatorname{supp} \tilde{\varphi}_o \cap \{z \in \Omega \mid \frac{a}{v} \leqq u_\mu \leqq \frac{b}{v}\}} T \wedge (\mathrm{dd}^c u^{1-\alpha'})^{p-1} \wedge \mathrm{d} v_\mu \wedge \mathrm{d}^c v_\mu$$

$$\leqq \text{const} \cdot \sup_{\bar{V}} u_1^{1-\alpha'} \cdot \| T \wedge (\mathrm{dd}^c u^{1-\alpha'})^{p-1} \| (V \setminus A) \cdot \left(\frac{b}{v} \right)^{1-\alpha'}.$$

Altogether we have:

$$\limsup_{\mu \to \infty} \left| \int_{\Omega \setminus A} T \wedge \chi'(v u_\mu)\, \omega^{p-1} \wedge \mathrm{d} u_\mu \wedge \mathrm{d}^c u_\mu \cdot \tilde{\varphi}_o \right| \leqq \text{const} \cdot \left(\frac{b}{v} \right)^{(p+1) \cdot \alpha' + (1-\alpha')}$$

$$= O(v^{-p \cdot \alpha' - 1}).$$

Estimation of the second factor. For ν fixed, $\mu > \nu$ we get analogous to the estimation above:

$$\left| \int_{\Omega \setminus A} T \wedge \chi'(v u_\mu)\, \omega^{p-1} \wedge |\varphi_o|^2\, i \mathrm{d} z_j \wedge \mathrm{d} \bar{z}_j \right|$$

$$\leqq \sup_V |\varphi_o|^2 \cdot \left| \int_{\Omega \setminus A} T \wedge \omega^{p-1} \cdot \chi'(v u_\mu) \cdot \tilde{\varphi}_o \wedge \mathrm{dd}^c |z|^2 \right|$$

$$\leqq \text{const} \cdot \left(\frac{b}{v} \right)^{p \cdot \alpha'} \left| \int_{\Omega \setminus A} T \wedge (\mathrm{dd}^c u^{1-\alpha'})^p \cdot \chi'(v u_\mu) \cdot \tilde{\varphi}_o \right|.$$

Using Lebesgue's theorem and Proposition 2 we get:

$$\lim_{\nu\to\infty}\Big(\lim_{\mu\to\infty}\Big|\int\limits_{\Omega\setminus A} T\wedge(\mathrm{dd}^c u^{1-\alpha'})^p\cdot\chi'(\nu u_\mu)\cdot\tilde{\varphi}_o\Big|\Big)=0.$$

So

$$\Big|\int\limits_{\Omega\setminus A} T\wedge\chi(\nu u)\,\partial\varphi\Big|^2=O(\nu^{-p\alpha'-1})\cdot o(\nu^{-p\cdot\alpha'})\xrightarrow[\nu\to\infty]{}0,\quad\text{if}\quad p\geqq\frac{1}{2\alpha'}.$$

For the cases $0<k\leqq n-1$ the statement is derived by slicing and the above shown estimates. qed

Remarks to the proof. (1) For proving the local finiteness of the mass of T near A, it is not necessary, that $u^{1-\alpha}$ is plurisubharmonic for any $\alpha\geqq 0$. It suffices, that the conditions of Lemma 7 are fulfilled locally.

(2) The structure of the proof shows, that it is sufficient, if the current and the function u fulfill the conditions of the theorem on enough "slices".

Naturally the next question is, whether there exist such numbers $\alpha\in]0,1[$, such that $u^{1-\alpha}$ is still plurisubharmonic near $u^{-1}(\{0\})$.

For two times differentiable, plurisubharmonic functions the following proposition gives an answer:

Proposition 3 *Let $\Omega\subset\mathbb{C}^n$ be open and $u\colon\Omega\to\mathbb{R}_0^+$ a two times continuous differentiable, strictly plurisubharmonic function, then we have:*

For every $a\in u^{-1}(\{0\})$ and for every $\alpha<\frac{1}{2}$ there is an open neighbourhood U_o of a, $U_o\subset\Omega$, and a two times continuous differentiable function $\rho\colon U_o\to\mathbb{R}_0^+$ with the following properties:

(i) $u^{-1}(\{0\})\cap U_o\subset\rho^{-1}(\{0\})$.
(ii) *ρ is strictly plurisubharmonic.*
(iii) *$\rho^{1-\alpha}$ is plurisubharmonic.*

Proof. Without loss of generality let $a=0\in u^{-1}(\{0\})$.

Using [6] we see, that there is an open neighbourhood U of 0, such that $u^{-1}(\{0\})\cap U\subset A$, where A is a $\mathscr{C}^1$-smooth, total real submanifold of U.

So, after shrinking U, we may suppose, that there is an one time continuous differentiable map $h:=(h_1,\ldots,h_n)$ in $V_1:=\{x\in\mathbb{R}^n\mid\exists\,y\in\mathbb{R}^n\colon x+iy\in U\}$ such that

$$h\colon V_1\to V_2:=\{y\in\mathbb{R}^n\mid\exists\,x\in\mathbb{R}^n\colon x+iy\in U\}$$

and $h(0)=0$, $\mathrm{d}h_j(0)=0\ \forall j=1,\ldots,n$ and $A\subset\{z=x+iy\in U\mid y=h(x)\}=\tilde{\rho}^{-1}(\{0\})$, where $\tilde{\rho}\colon U\to\mathbb{R}_0^+$ is defined by: $\tilde{\rho}(z):=|y-h(x)|^2$, $(z=x+iy)$.

Without loss of generality $A:=\tilde{\rho}^{-1}(\{0\})$.

For $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ we denote for $1\leqq j\leqq n$, $x_j:=\mathrm{Re}\,z_j$, $y_j:=\mathrm{Im}\,z_j$ and define $x:=\mathrm{Re}\,z:=(x_1,\ldots,x_n)$ and $y:=\mathrm{Im}\,z:=(y_1,\ldots,y_n)$. Denote $x_o=\mathrm{Re}\,z_o$ for $z_o\in\mathbb{C}^n$.

With this notation we have:

(1) $\tilde{\rho}$ is two times partial differentiable in each $z_o\in A$, so with $\nu,\mu\in\{1,\ldots,n\}$ we get:

$$\frac{\partial^2\tilde{\rho}}{\partial x_\mu\,\partial x_\nu}(z_o)=2\cdot\sum_{j=1}^{n}\frac{\partial h_j}{\partial x_\mu}(x_o)\cdot\frac{\partial h_j}{\partial x_\nu}(x_o)$$

$$\frac{\partial^2\tilde{\rho}}{\partial x_\mu\,\partial y_\nu}(z_o)=\frac{\partial^2\tilde{\rho}}{\partial y_\nu\,\partial x_\mu}(z_o)=-2\cdot\frac{\partial h_\nu}{\partial x_\mu}(x_o)$$

$$\frac{\partial^2\tilde{\rho}}{\partial y_\mu\,\partial y_\nu}(z_o)=2\delta_{\mu\nu}.$$

(2) After shrinking U again we may suppose, that

(i) $\tilde{\rho}(z) \geqq \operatorname{dist}^2(A, z)$ for every $z \in U$.
and that there is a two times continuous differentiable function $\rho\colon U \to \mathbb{R}_0^+$, which suffices the following conditions:
(ii) $\rho(z) = \tilde{\rho}(z) + o(\operatorname{dist}^2(A, z))$, where $o(\operatorname{dist}^k(A, z))$ denotes a function of z with the following property:

For every $\varepsilon > 0$ there is a $\delta > 0$, such that for each $z \in U$ with $\operatorname{dist}(A, z) \leqq \delta$ holds:

$$|o(\operatorname{dist}^k(A, z))| \leqq \varepsilon \cdot \operatorname{dist}^k(A, z).$$

(iii) The partial derivatives of ρ and $\tilde{\rho}$ of order $\leqq 2$ are equal on $A \cap U$.
(iv) $u^{-1}(\{0\}) \cap U \subset A = \tilde{\rho}^{-1}(\{0\}) = \rho^{-1}(\{0\})$.

(The existence of ρ is a consequence of the theorem of Whitney (see [9, Theorem 1.5.5]) applied to the partial derivatives of $\tilde{\rho}$ on A. The properties of ρ and $\tilde{\rho}$ follow from Taylor's formula applied to ρ resp. $\tilde{\rho}$. For details see [10].)

Since ρ is strictly plurisubharmonic in 0, we may suppose, that ρ is strictly plurisubharmonic in U.

Now we prove: For every $\alpha < \frac{1}{2}$ there is an open neighbourhood U_o of 0, $U_o \subset U$, such that $i\partial\bar{\partial}\rho^{1-\alpha} | U_o \geqq 0$. This is *because:* Choose $\alpha < \frac{1}{2}$ arbitrary, then $\varepsilon := \dfrac{1-2\alpha}{2} > 0$.

So there is an $r > 0$, such that $i\partial\bar{\partial}\rho | U' \geqq (1-\varepsilon) \cdot \dfrac{i}{2} \partial\bar{\partial}|z|^2$ in $U' := B(0, r)$ and such that $\bar{B}(0, 2r) \subset U$. In U' we have now:

$$i(\partial\bar{\partial}\rho \cdot \rho - \alpha\partial\rho \wedge \bar{\partial}\rho \geqq \left[(1-\varepsilon)\cdot\rho - \frac{\alpha}{2}\sum_{j=1}^{n}\left(\frac{\partial\rho}{\partial x_j}\right)^2 + \left(\frac{\partial\rho}{\partial y_j}\right)^2\right] \cdot \frac{i}{2}\partial\bar{\partial}|z|^2.$$

Since the partial derivatives of ρ are uniformly continuous in $B(0, 2r)$, we get for $z, z_o \in B(0, 2r)$:

$$\frac{\partial\rho}{\partial x_j}(z) = \frac{\partial\rho}{\partial x_j}(z_o) + \sum_{\nu=1}^{n} \frac{\partial^2\rho}{\partial x_j \partial x_\nu}(z_o) \cdot (x_\nu - x_{o\nu}) + \sum_{\nu=1}^{n} \frac{\partial^2\rho}{\partial x_j \partial y_\nu}(z_o) \cdot (y_\nu - y_{o\nu}) + o(|z - z_o|),$$

where $\dfrac{o(|z-z_o|)}{|z-z_o|}$ converges to 0 uniformly for all $z_o \in B(0, 2r)$, if $z \to z_o$.

A similar formula is true for $\dfrac{\partial\rho}{\partial y_j}(z)$.

If $z \in U'$, then there is a $z_o \in B(0, 2r) \cap A$, such that $\operatorname{dist}(A, z) = |z - z_o|$.

Since $d\rho(z_o)=0$, we derive:

$$\left(\frac{\partial\rho}{\partial y_j}(z)\right)^2=4\cdot\left[-\sum_{\nu=1}^{n}\frac{\partial h_j}{\partial x_\nu}(x_o)\cdot(x_\nu-x_{o\nu})+(y_j-y_{oj})\right]^2+o(\mathrm{dist}^2(A,z))$$

$$\leqq 4\cdot|z-z_o|^2\left[1+2\cdot\sum_{\nu=1}^{n}\left|\frac{\partial h_j}{\partial x_\nu}(x_o)\right|+\left(\sum_{\nu=1}^{n}\frac{\partial h_j}{\partial x_\nu}(x_o)\right)^2\right]+o(\mathrm{dist}^2(A,z)).$$

A similar estimation is true for $\left(\frac{\partial p}{\partial x_j}(z)\right)^2$.

For $\frac{\partial h_j}{\partial x_\nu}$ vanishes in 0, there is an open neighbourhood of 0, where the derivatives of h_j are small.

Now we have $\rho=\tilde{\rho}+o(\mathrm{dist}^2(A,z))$ and $\tilde{\rho}\geqq\mathrm{dist}^2(A,\cdot)$. Thus for small $|z|$ we derive:

$$(1-\varepsilon)\rho(z)-\frac{\alpha}{2}\sum_{j=1}^{n}\left(\frac{\partial\rho}{\partial x_j}(z)\right)^2+\left(\frac{\partial\rho}{\partial y_j}(z)\right)^2$$

$$\geqq\mathrm{dist}^2(A,z)\left(1-\varepsilon-2\alpha(1+\varepsilon)+\frac{o(\mathrm{dist}^2(A,z))}{\mathrm{dist}^2(A,z)}\right)\geqq 0.\quad\text{qed}$$

Slicing in directions of the surfaces, where the function u of the following theorem is strictly plurisubharmonic, we derive from Theorem 1 and Proposition 3:

Proposition 4 *Let $\Omega\subset\mathbb{C}^n$ be open, $u\colon\Omega\to\mathbb{R}_0^+$ strictly q-convex, $A\subset u^{-1}(\{0\})$ closed. Suppose, that $T\in\mathscr{D}'_{p,p}(\Omega\setminus A)$ is positive and closed, then we have:*

(1) *If $p\geqq q$, the mass of T is locally finite near A.*
(2) *If $p>q$, the trivial extension $\tilde{T}$ of T to Ω is the uniquely determined positive, closed extension of T to Ω.*

For the case of a $\mathscr{C}^2$-smooth submanifold $A'\subset\Omega$ Corollary 1 was proved by El Mir [2, Theorem III.7] and Sibony [11, Corollary 3.4]:

Corollary 1 *Let $\Omega\subset\mathbb{C}^n$ be open, $A\subset A'\subset\Omega$, A closed, and A' a $\mathscr{C}^1$-smooth, real submanifold of Ω, such that $\dim_{\mathbb{C}}\mathrm{H}_a A'<q$ for every $a\in A$.*

If $T\in\mathscr{D}'_{p,p}(\Omega\setminus A)$ is a positive, closed current, $p>q$, then there is a unique positive, closed extension of T to Ω, and this is the trivial extension $\tilde{T}$ of T.

For: For every arbitrary $a\in A$ there is an open neighbourhood U of a, such that for

$$U':=\{(\zeta',x'')\in\mathbb{C}^{q-1}\times\mathbb{R}^{n-q+1}\mid\exists z\in U:(z,\ldots,z_{q-1})=\zeta'\text{ and }(\mathrm{Re}\,z_q,\ldots,\mathrm{Re}\,z_n)=x''\}$$

there is a continuous differentiable map $h:=(h_q,\ldots,h_n)\colon U'\to\mathbb{R}^{n-q+1}$, such that

$$dh_j(a_1,\ldots,a_{q-1},\mathrm{Re}\,a_q,\ldots,\mathrm{Re}\,a_n)=0\text{ for each }j,$$

and such that $U \cap A \subset \tilde{\rho}^{-1}(\{0\})$, where $\tilde{\rho}$ is defined for $z=(z_1, \ldots, z_n) \in U$ by:

$$\tilde{\rho}(z) := \sum_{j=q}^{n} (\operatorname{Im} z_j - h_j(z_1, \ldots, z_{q-1}, \operatorname{Re} z_q, \ldots, \operatorname{Re} z_n))^2.$$

Analogous to the proof of Proposition 3 $\tilde{\rho}$ is two times continuous differentiable on A and the partial derivatives fulfill the conditions of Whitney's theorem. (See [9, Theorem 1.5.5].)

After shrinking U we may suppose, (analogous to the proof of Proposition 3) that there is a two times continuous differentiable function ρ: $U \to \mathbb{R}_0^+$ with the following properties:

(i) $\tilde{\rho}^{-1}(\{0\}) = \rho^{-1}(\{0\})$.
(ii) ρ is strictly q-convex.

So, $T | U \setminus A$ fulfills the conditions of Proposition 4 and the corollary is proved.

3 Continuation of analytic sets across polar sets

An important part in this section is played by the polar sets of hyper-q-convex functions. (See Definition 1.)

Remarks. Let $(\Omega, \mathrm{d}s^2)$ be a Kähler manifold $\dim_{\mathbb{C}} \Omega = n$.

(1) A hyper-1-convex function is plurisubharmonic. A hyper-n-convex function is subharmonic.
(2) A two times continuous differentiable function u: $\Omega \to \mathbb{R}$ is hyper-q-convex (resp. strictly hyper-q-convex), iff we have for every $z \in \Omega$ and every unitary (with respect to $\mathrm{d}s^2$) q-frame

$$\left\{ \sum_{j=1}^{n} a_{\nu j} \frac{\partial}{\partial z_j} \middle| \nu = 1, \ldots, q \right\} \quad \text{in} \quad T_z^{1,0} \Omega:$$

$$\sum_{\nu=1}^{q} \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(z)\, a_{\nu i} \bar{a}_{\nu j} \geqq 0 \qquad (\text{resp. } > 0).$$

(3) A two times continuous differentiable hyper-q-convex function is hyper-q'-convex for $q' \geqq q$.

For: (1) and (2) are trivial; for the proof of (3) see [3, Remark after Satz 5].

Some notations. Let $(\Omega, \mathrm{d}s^2)$ be a Kähler manifold, $X \subset \Omega$ analytic.

(1) $\operatorname{Reg} X$ denotes the set of the regular points of X.
(2) Let u: $\operatorname{Reg} X \to \mathbb{R}$ be a function.
u is called *locally bounded from above (from below) in* Ω, iff $\sup\{u(x) | x \in \operatorname{Reg} X \cap K\} < \infty$ (resp. $\inf\{u(x) | x \in \operatorname{Reg} X \cap K\} > -\infty$) for every compact subset $K \subset \Omega$.
u is called *locally bounded in* Ω, iff $|u|$ is locally bounded from above in Ω.
(3) Let $*_\Omega$ be the Hodge-$*$-operator, which is induced by $\mathrm{d}s^2$.
If u: $\Omega \to \mathbb{C}$ is two times continuous differentiable, we define $\Delta_\Omega u := 4 *_\Omega \bar{\partial} *_\Omega \bar{\partial} u$.
(4) Δ_X denotes the laplacian on $\operatorname{Reg} X$, which is induced by $\mathrm{d}s^2$.
$*_X$ is the by $\mathrm{d}s^2$ induced Hodge-$*$-operator on $\operatorname{Reg} X$.

By dV_x we mean the volumeform on $\operatorname{Reg} X$ induced by ds^2.
(If the situation is clear, we will omit the index X.)
(5) If $X_o \subset \Omega$ is an analytic set of dimension 0, e.g. $X_o = \{x_1, x_2, \ldots\}$, and if $u: X_o \to \mathbb{C}$ is a function, then u is locally constant on X_o, so we denote

$$\Delta_{x_o}(u|X_o) = 0 \quad \text{and} \quad d(u|X_o) = 0.$$

And further we define $\int\limits_{X_o} u\, dV_{X_o} = \sum\limits_{x \in X_o} u(x)$, if the sum is absolutely convergent. If the sum does not converge absolutely, we call u *not integrable.*

For the proof of the Theorems 2 and 3 we will need some lemmas. For the first lemmas we will consider the following situation:

Let (Ω', ds^2) be a Kähler manifold, $A' \subset \Omega'$ be closed (resp. completely hyper-q-polar).

Let $X' \subset \Omega' \setminus A'$ be an analytic set with locally finite volume near A', such that X' suffices the following condition:

For every two times continuous differentiable, bounded subharmonic function u: $\operatorname{Reg} X' \to \mathbb{R}$ and for every $\varphi \in \mathscr{D}(\Omega' \setminus A')$ holds:

$$\text{(B)} \qquad \int\limits_{\operatorname{Reg} X'} \varphi \cdot \Delta_{X'} u\, dV_{X'} = \int\limits_{\operatorname{Reg} X'} u \cdot \Delta_{X'} \varphi\, dV_{X'}.$$

At first these lemmas will be applied to $\Omega' = \Omega \setminus A$, $A' = \operatorname{Sing} X$ and $X' = \operatorname{Reg} X$, for proving, that analytic sets fulfill condition (B).

Then these lemmas will be applied again: This time for the situation of Theorem 3.

Again at first we have to prove a Chern-Levine-Nirenberg estimation:

Lemma 8 (Chern-Levine-Nirenberg estimation) *Let (Ω', ds^2) be a Kähler manifold, $A' \subset \Omega'$ be closed, and $X' \subset \Omega' \setminus A'$ be analytic with locally finite volume near A' and let condition* (B) *be fulfilled, then we have:*

For each compact set $K \subset \Omega'$ and for each open set U with $K \subset U \subset \Omega'$, there is a constant $C(K, U)$, such that

$$\int\limits_{\operatorname{Reg} X' \cap K} \Delta_{X'} v\, dV_{X'} \leqq C(K, U) \cdot \operatorname{Vol}(X' \cap U) \cdot \sup_{U \cap \operatorname{Reg} X'} |v|$$

for all functions v: $\operatorname{Reg} X' \to \mathbb{R}$, which are two times continuous differentiable, bounded and subharmonic and for which there is an open neighbourhood $W = W(v)$ of A', such that v vanishes in $W \cap \operatorname{Reg} X'$.

Using condition (B) the proof is similar to the proof of Lemma 4. Only the operator dd^c has to be substituted by $\Delta_{X'}$.

The following estimation is analoguous to Lemma 5.

Lemma 9 *Let (Ω', ds^2) be a Kähler manifold, $A' \subset \Omega'$ be closed, and $X' \subset \Omega' \setminus A'$ be analytic with locally finite volume near A' and let condition* (B) *be fulfilled, then we have:*

For each compact set $K \subset \Omega'$ and for each open set U with $K \subset U \subset \Omega'$, there is a constant $C(K, U)$, such that

$$\int\limits_{\operatorname{Reg} X' \cap K \cap \{z \in \Omega' \mid a \leqq v(z) \leqq b\}} \bar{\partial} v \wedge *_{X'} \bar{\partial} v \leqq C(K, U) \cdot \operatorname{Vol}(X' \cap U) \cdot (b - a) \cdot \sup_{\operatorname{Reg} X' \cap U} v$$

for each $0<a<b<\infty$ and for each two times continuous differentiable subharmonic function v: $\operatorname{Reg} X' \to \mathbb{R}$, which is bounded from above and for which there is an open neighbourhood $W=W(v)$ of A', such that $v|W\cap \operatorname{Reg} X' \equiv 0$.

For the proof of this lemma we use the same function h as in the proof of Lemma 5. Then we gather this statement from Lemma 8 in a similar way as we derived Lemma 5 from Lemma 4.

In both previous lemmas we supposed, that condition (B) was fulfilled by X'.

Obviously condition (B) is fulfilled, if $X'\subset\Omega'\setminus A'$ is a complex submanifold. The following lemma and the next proposition will show, that condition (B) is fulfilled by arbitrary analytic sets, too. For the proof we only use, that the singular locus of an analytic set is locally complete hyper-q-polar for every $q\in\mathbb{N}$.

Lemma 10 *Let (Ω', ds^2) be a Kähler manifold, $A'\subset\Omega'$ be completely hyper-q-polar.*

Suppose, that $X'\subset\Omega'\setminus A'$ is analytic of pure dimension $q\geqq 1$ with locally finite volume near A', and that condition (B) *is fulfilled, then we have:*

If v: $\operatorname{Reg} X' \to \mathbb{R}$ is a two times continuous differentiable, subharmonic, bounded function and if T, S: $\mathscr{D}(\Omega'\setminus A') \to \mathbb{C}$ are defined by:

$$T(\varphi) := \int\limits_{\operatorname{Reg} X'} \varphi\cdot \Delta_{X'} v \, dV_{X'} \quad \text{and} \quad S(\varphi) := \int\limits_{\operatorname{Reg} X'} \varphi\cdot \bar\partial v \wedge *_{X'} \bar\partial v,$$

then S, T are positive distributions with locally finite mass near A'.

Proof. Let g: $\Omega' \to \mathbb{R}\cup\{-\infty\}$ be hyper-q-convex with $g^{-1}(\{-\infty\}) = A'$.

Without loss of generality we may suppose, that $\Omega'\subset\mathbb{C}^n$ is open equipped with the given Kähler metric ds^2, that $g\leqq 0$ and strictly hyper-q-convex.

Using [4, Theorem 1] or [5] there is a smooth, subharmonic function f: $\operatorname{Reg} X' \to \mathbb{R}$ with $\sup\limits_{\operatorname{Reg} X'} |f-g| \leqq 1$ and $f\leqq g$. In particular:

$$\lim_{\substack{x\to a \\ x\in \operatorname{Reg} X'}} f(x) = -\infty \quad \forall a\in A'\cap \bar X'.$$

If $U\Subset\Omega'$, then $\|T\|(U\setminus A')<\infty$, *because:*

Choose a smooth, convex, monoton increasing function h: $\mathbb{R}\to\mathbb{R}$, such that $h'|\mathbb{R}_0^+ \geqq 1$ and $h|]-\infty, -1] \equiv 0$.

Let $m_o := \sup\limits_{\operatorname{Reg} X'} |v|$ and take $\varphi\in\mathscr{D}(\Omega')$ with $K := \operatorname{supp}\varphi \subset U\setminus A'$ and $\sup|\varphi|\leqq 1$.

Choose $0<\varepsilon<1$ depending on φ, such that $\min\limits_K \varepsilon g \geqq -1$, then for $v_\varepsilon := v + \varepsilon f + m_o + 2$ holds:

(i) $v_\varepsilon \geqq 0$ on $K\cap \operatorname{Reg} X'$.

(ii) $h\circ v_\varepsilon$: $\operatorname{Reg} X' \to \mathbb{R}$ is two times continuous differentiable, subharmonic and bounded. The bound is independent of ε (and in particular independent of φ).

(iii) There is an open neighbourhood W of A', such that $h\circ v_\varepsilon | \operatorname{Reg} X'\cap W \equiv 0$.

Using Lemma 8 we derive:

$$\begin{aligned} |T(\varphi)| &\leqq \int\limits_{\operatorname{Reg} X'\cap K} \Delta_{X'} v \, dV_{X'} \leqq \int\limits_{\operatorname{Reg} X'\cap \bar U} \Delta_{X'}(h\circ v_\varepsilon)\, dV_{X'} \\ &\leqq C(K, \tilde U)\cdot h(2m_o+2)\cdot \operatorname{Vol}(X'\cap \tilde U) < \infty, \end{aligned}$$

where $\tilde U\Subset\Omega'$ is an arbitrary open neighbourhood of $\bar U$.

If $U \Subset \Omega'$, then $\|S\|(U \setminus A') \leqq 2^q \int\limits_{\operatorname{Reg} X' \cap (U \setminus A')} \Delta_{X'}(v+m_o)^2 \, \mathrm{dV}_{X'} < \infty$. qed

Proposition 5 *Let $(\Omega', \mathrm{d}s^2)$ be a Kähler manifold and $X \subset \Omega'$ be analytic.*

Let u: $\operatorname{Reg} X \to \mathbb{C}$ be a two times continuous differentiable function, which is locally bounded in Ω' and suppose, that the distributions T, S: $\mathscr{D}(\Omega' \setminus \operatorname{Sing} X) \to \mathbb{C}$, which are defined by

$$T(\varphi) := \int\limits_{\operatorname{Reg} X} \varphi \cdot \Delta_X u \, \mathrm{dV}_X \quad \text{and} \quad S(\varphi) := \int\limits_{\operatorname{Reg} X} \varphi \cdot \bar{\partial} u \wedge *_X \bar{\partial} u,$$

have locally finite mass near $\operatorname{Sing} X$.

With this conditions we have for every $\varphi \in \mathscr{C}^2_o(\Omega')$:

$$\int\limits_{\operatorname{Reg} X} \varphi \cdot \Delta_X u \, \mathrm{dV}_X = \int\limits_{\operatorname{Reg} X} u \cdot \Delta_X \varphi \, \mathrm{dV}_X.$$

Proof. Without loss of generality we may suppose, that u is bounded and that there are holomorphic functions $f_1, \ldots, f_m$: $\Omega' \to \mathbb{C}$, such that

(i) $\operatorname{Sing} X = \{z \in \Omega' \mid f_1(z) = \ldots = f_m(z) = 0\}$ and

(ii) $\sup\limits_{\Omega'} \sum\limits_{j=1}^{m} |f_j|^2 \leqq 1$.

$v := \log\left(\sum\limits_{j=1}^{m} |f_j|^2\right)$ is hyper-q'-convex in Ω' for each $q' \geqq 1$ and $v|\Omega' \setminus \operatorname{Sing} X$ is smooth.

Let $a, b \in \mathbb{R}$ with $0 < a < b < 1$ and let χ: $\mathbb{R} \to \mathbb{R}$ be a smooth, monoton increasing function with

(i) $\chi|]-\infty, a] \equiv 0$ and
(ii) $\chi|[b, \infty[\equiv 1$.

Let h: $\mathbb{R} \to \mathbb{R}$ be smooth, monoton increasing, convex, such that $h|]-\infty, a] \equiv 0$ and $h(1) = 2$. For $\nu \in \mathbb{N}$ set $v_\nu := h\left(\exp\left(\frac{1}{\nu} \cdot \log\left(\sum\limits_{j=1}^{m} |f_j|^2\right)\right)\right)$.

If $\varphi \in \mathscr{C}^2_o(\Omega')$ is an arbitrary function, then we derive from Lebesgue's theorem:

$$\int\limits_{\operatorname{Reg} X} \varphi \cdot \Delta_X u \, \mathrm{dV}_X = \lim_{\nu \to \infty} \int\limits_{\operatorname{Reg} X} \chi(v_\nu) \cdot \varphi \Delta_X u \, \mathrm{dV}_X.$$

For each fixed $\nu \in \mathbb{N}$ we have $\operatorname{supp}(\chi(v_\nu) \cdot \varphi) | \operatorname{Reg} X \Subset \operatorname{Reg} X$, and so

$$\begin{aligned}
\int\limits_{\operatorname{Reg} X} \chi(v_\nu) \cdot \varphi \Delta_X u \, \mathrm{dV}_X &= \int\limits_{\operatorname{Reg} X} \Delta_X(\chi(v_\nu) \cdot \varphi) \cdot u \, \mathrm{dV}_X \\
&= \int\limits_{\operatorname{Reg} X} \Delta_X \varphi \cdot \chi(v_\nu) \cdot u \, \mathrm{dV}_X \\
&\quad + 4 \cdot \Big[\int\limits_{\operatorname{Reg} X} *(\bar{\partial}\bar{\varphi} \wedge *\bar{\partial}\chi(v_\nu)) \cdot u \, \mathrm{dV}_X \\
&\quad + \int\limits_{\operatorname{Reg} X} *(\bar{\partial}\chi(v_\nu) \wedge *\bar{\partial}\varphi) \cdot u \, \mathrm{dV}_X \Big] \\
&\quad + \int\limits_{\operatorname{Reg} X} \Delta_X(\chi(v_\nu)) \cdot \varphi \cdot u \, \mathrm{dV}_X.
\end{aligned}$$

Since $\varphi \in \mathscr{C}_o^2(\Omega')$, $|\Delta_X \varphi \cdot u|$ is bounded and using Lebesgue's theorem again, we derive:

$$\lim_{\nu \to \infty} \int_{\operatorname{Reg} X} \Delta_X \varphi \cdot \chi(v_\nu) \cdot u \, dV_X = \int_{\operatorname{Reg} X} \Delta_X \varphi \cdot u \, dV_X.$$

It only remains to prove, that the other terms converge to zero, if $\nu \to \infty$.

For the brackets [...]. Using Cauchy-Schwarz we derive for the first term: (second analogous)

$$\begin{aligned} &\Big| \int_{\operatorname{Reg} X} *(\bar\partial \bar\varphi \wedge * \bar\partial \chi(v_\nu)) \cdot u \, dV_X \Big|^2 \\ &\leqq 2^{\dim_{\mathbb{C}} \Omega'} \int_{\operatorname{Reg} X \cap \operatorname{supp} \varphi \cap \operatorname{supp} \chi'(v_\nu)} \sup \chi' \cdot \bar\partial v_\nu \wedge * \bar\partial v_\nu \\ &\quad \cdot \int_{\operatorname{Reg} X} \chi'(v_\nu) \cdot |u|^2 \cdot *(\bar\partial \bar\varphi \wedge * \bar\partial \bar\varphi) \, dV_X \\ &\xrightarrow[\nu \to \infty]{} 0, \quad \textit{because:} \end{aligned}$$

Using Lemma 9 the first factor can be estimated by a constant. The second factor converges to zero by Lebesgue's theorem.

For the remaining last term. Since $\operatorname{supp}(\chi(v_\nu)\, \varphi u) \Subset \operatorname{Reg} X$, we have:

$$\begin{aligned} &\Big| \int_{\operatorname{Reg} X} \Delta_X (\chi(v_\nu)) \cdot \varphi \cdot u \, dV_X \Big| \leqq 4 \Big| \int_{\operatorname{Reg} X} *(\bar\partial(\chi(v_\nu))) \wedge \bar\partial(u \cdot \varphi) \Big| \\ &\leqq 4 \Big| \int_{\operatorname{Reg} X} \sqrt{\chi'(v_\nu)}\, u \bar\partial \varphi \wedge * \sqrt{\chi'(v_\nu)}\, \bar\partial v_\nu \Big| + 4 \Big| \int_{\operatorname{Reg} X} \sqrt{\chi'(v_\nu)} \cdot \varphi \bar\partial u \wedge * \sqrt{\chi'(v_\nu)}\, \bar\partial v_\nu \Big|, \end{aligned}$$

where both terms converge to zero, if $\nu \to \infty$. (Cauchy-Schwarz, Lebesgue, Lemma 9.) qed

The following corollary is a consequence of Lemma 10 and Proposition 5:

Corollary 2 *Let* (Ω', ds^2) *be a Kähler manifold,* $X \subset \Omega'$ *be analytic, then we have:*
(1) *If* $u, \varphi \in \mathscr{C}^2(\Omega')$ *and if* $\operatorname{supp} \varphi \subset \Omega'$ *is compact, then*

$$\int_{\operatorname{Reg} X} \varphi \cdot \Delta_X u \, dV_X = \int_{\operatorname{Reg} X} \Delta_X \varphi \cdot u \, dV_X.$$

(2) *For each two times continuous differentiable subharmonic function* u: $\operatorname{Reg} X \to \mathbb{R}$, *which is locally bounded in* Ω' *and for every* $\varphi \in \mathscr{C}_o^2(\Omega')$ *holds:*

$$\int_{\operatorname{Reg} X} \varphi \cdot \Delta_X u \, dV_X = \int_{\operatorname{Reg} X} \Delta_X \varphi \cdot u \, dV_X.$$

Lemma 11 *Let* (Ω', ds^2) *be a Kähler manifold,* $X \subset \Omega'$ *analytic.*

Let u: $\operatorname{Reg} X \to \mathbb{R}$ *be a two times continuous differentiable, in* Ω' *locally bounded function, such that the distribution* $S \in \mathscr{D}'(\Omega' \setminus \operatorname{Sing} X)$, $S(\varphi) := \int_{\operatorname{Reg} X} \varphi \cdot \bar\partial u \wedge *_X \bar\partial u$,

in Ω' *has locally finite mass. If* $q \geqq 1$ *and if* Y *is the union of the* q*-dimensional irreducible components of* X, *then we have:*

(1) *For each* $\psi \in \mathscr{D}_{q-1,q}(\Omega')$ *holds* $\int\limits_{Y \cap \operatorname{Reg} X} i_X^* \psi \wedge \partial u = \int\limits_{Y \cap \operatorname{Reg} X} i_X^* \partial\psi \cdot u.$

(2) *For each* $\psi \in \mathscr{D}_{q,q-1}(\Omega')$ *holds* $\int\limits_{Y \cap \operatorname{Reg} X} i_X^* \psi \wedge \bar{\partial} u = \int\limits_{Y \cap \operatorname{Reg} X} i_X^* \bar{\partial}\psi \cdot u.$

Here i_X: $\operatorname{Reg} X \hookrightarrow \Omega'$ *denotes the natural inclusion.*

Proof. Without loss of generality we may suppose, that there are holomorphic functions $f_1, \ldots, f_m$: $\Omega' \to \mathbb{C}$, such that $\sum\limits_{j=1}^{m} |f_j|^2 \leqq 1$ and such that

$$\operatorname{Sing} X = \{z \in \Omega' \mid f_1(z) = \ldots = f_m(z) = 0\}.$$

Let $0 < a < b < 1$ and h: $\mathbb{R} \to \mathbb{R}$ a smooth, monotonically increasing, convex function, for which $h]-\infty, a] \equiv 0$ and $h(1) = 2$. $v_\nu := h\left(\left(\sum\limits_{j=1}^{m} |f_j|^2\right)^{1/\nu}\right)$.

Let χ: $\mathbb{R} \to \mathbb{R}$ be a smooth, monotonically increasing function $\chi]-\infty, a] \equiv 0$ and $\chi|[b, \infty[\equiv 1$, then we have:

(i) $\chi(v_\nu) \equiv 0$ in an open neighbourhood of $\operatorname{Sing} X$.
(ii) $\chi(v_\nu) \underset{\nu \to \infty}{\nearrow} \mathbb{1}_{\Omega' \setminus \operatorname{Sing} X}$.

(iii) $v_\nu | \operatorname{Reg} X$ is subharmonic and bounded. The bound is independent of ν.
(iv) v_ν vanishes in an open neighbourhood of $\operatorname{Sing} X$.

For (1). (The proof of (2) is analogous.)

$$\begin{aligned}
&\int\limits_{Y \cap \operatorname{Reg} X} i_X^* \partial\psi \cdot u = \lim_{\nu \to \infty} \int\limits_{Y \cap \operatorname{Reg} X} \chi(v_\nu) \cdot i_X^* \partial\psi \cdot u \quad \text{(Theorem of Lebesgue)} \\
&= -\lim_{\nu \to \infty} \int\limits_{Y \cap \operatorname{Reg} X} \partial(\chi(v_\nu) \cdot u) \wedge i_X^* \psi, \quad \textit{because:} \ \operatorname{supp}(\chi(v_\nu) \cdot \partial\psi) \mid Y \Subset Y \cap \operatorname{Reg} X \\
&= \lim_{\nu \to \infty} \int\limits_{Y \cap \operatorname{Reg} X} \chi(v_\nu) \cdot i_X^* \psi \wedge \partial u + \lim_{\nu \to \infty} \int\limits_{Y \cap \operatorname{Reg} X} \chi'(v_\nu) \cdot i_X^* \psi \cdot u \wedge \partial v_\nu \\
&= \int\limits_{Y \cap \operatorname{Reg} X} i_X^* \psi \wedge \partial u \quad \text{using Lebesgue's theorem and Lemma 9.} \quad \text{qed}
\end{aligned}$$

Proof of Theorem 3 Without loss of generality we may suppose, that $\Omega \subset \mathrm{B}(0, 1) \subset \mathbb{C}^n$ is open and that $\tilde{u} := u + |z|^2 \leqq 0$ in Ω.

Let Y be the union of the irreducible components of X, which have the dimension q, $q \in \{0, \ldots, n\}$. It suffices to prove, that $\bar{Y}$ is an analytic continuation of Y to Ω, where only to case $q > 0$ is interesting. (The case $q = 0$ is trivial.)

Using [4] there is a smooth, subharmonic function v: $\operatorname{Reg} X \to \mathbb{R}$ with $v \leqq \tilde{u}$ and $\sup |v - \tilde{u}| \leqq 1$.

Choose a, b, χ and h as in the proof of the previous Lemma 11.

For $\nu \in \mathbb{N}$ define $v_\nu := h\left(\exp\left(\frac{1}{\nu} \cdot v\right)\right)$.

If $T := [Y] \in \mathscr{D}'_{q,q}(\Omega \setminus A)$ is the current of integration over the regular points of Y, then the trivial extension $\tilde{T}$ of T to Ω is positive and closed,

because: The positivity of $\tilde{T}$ is trivial, so we only have to prove, that $\tilde{T}$ is closed.

For this we take an arbitrary $\varphi \in \mathscr{D}_{q-1,q}(\Omega)$. ($\varphi \in \mathscr{D}_{q,q-1}(\Omega)$ analogous.)

If i_X: $\operatorname{Reg} X \hookrightarrow \Omega$ denotes the natural inclusion, we derive from the theorem of Lebesgue:

$$(\partial \tilde{T})(\varphi) = -\int\limits_{\operatorname{Reg} Y} i_X^* \partial\varphi = -\lim_{\nu\to\infty} \int\limits_{Y\cap \operatorname{Reg} X} \chi(v_\nu)\cdot i_X^* \partial\varphi$$

$$= \lim_{\nu\to\infty} \int\limits_{Y\cap \operatorname{Reg} X} \chi'(v_\nu)\, \partial v_\nu \wedge i_X^* \varphi \quad \text{using Lemma 11.}$$

Now there is an open neighbourhood W_ν of A, such that $\chi(v_\nu)|\operatorname{Reg} X \cap W_\nu \equiv 0$. Choose an open neighbourhood W of $\operatorname{supp}\varphi \cap A$, $W \Subset W_\nu$ and a function $\psi \in \mathscr{D}(\Omega)$ with $0 \leqq \psi \leqq 1$, $\psi | W \equiv 1$ and $\operatorname{supp}\psi \subset W_\nu$.

Using Lemma 11 and ψ we are able to prove:

$$\int\limits_{Y\cap \operatorname{Reg} X} \chi(v_\nu)\cdot i_X^* \partial\varphi = -\int\limits_{Y\cap \operatorname{Reg} X} \chi'(v_\nu)\cdot \partial v_\nu \wedge i_X^* \varphi.$$

The inequality of Cauchy-Schwarz gives:

$$\Big|\int\limits_{Y\cap \operatorname{Reg} X} \chi'(v_\nu)\cdot \partial v_\nu \wedge i_X^* \varphi\Big|^2$$

$$\leqq \sup \chi' \cdot \int\limits_{\operatorname{Reg} X\cap \operatorname{supp}\varphi\cap\{z\in \operatorname{Reg} X \mid a\leqq v_\nu(z)\leqq b\}} \bar\partial v_\nu \wedge *_X \bar\partial v_\nu \cdot \int\limits_{Y\cap \operatorname{Reg} X} \chi'(v_\nu)\, i_X^* \varphi \wedge *_X i_X^* \varphi$$

$$\xrightarrow[\nu\to\infty]{} 0, \quad \textit{because:}$$

The first term is bounded (Lemma 9) and the second term converges to zero (theorem of Lebesgue). qed

Proof of Theorem 2 Theorem 2 is a direct consequence of Theorem 3.

Let $(\Omega, \mathrm{d}s^2)$ be a Kähler manifold. $A \subset \Omega$ be completely hyper-p-polar.

Let $T \in \mathscr{D}'_{p,p}(\Omega\setminus A)$ be a positive, closed current with locally finite mass near A and denote the Lelong number of T in z by $\nu(T,z)$.

For $c \in \mathbb{R}^+$ define $\mathrm{E}_c := \{z\in\Omega' \mid \nu(T,z) \geqq c\}$ and $\mathrm{E} := \bigcup\limits_{\nu\in\mathbb{N}} \mathrm{E}_{1/\nu} = \bigcup\limits_{c\in\mathbb{R}^+} \mathrm{E}_c$.

For E is a countable union of analytic sets, $T' := \mathbb{1}_{\mathrm{E}}\, T \in \mathscr{D}'_{p,p}(\Omega')$ is a well defined, positive current.

With the help of Theorem 2 and the theorem of Siu [12, main theorem] it is possible to prove, that T' is even closed in $\Omega\setminus A$. Since $T' \leqq T$, we have, that $R := R_T := T - T'$ is a positive, closed current in $\Omega\setminus A$, too.

$\mathrm{E}^{(R)} := \{z\in\Omega\setminus A \mid \nu(R,z) > 0\}$ is a set of mass zero for $\|R\|$.

This means, that the "residue-current" R is a $\|R\|$-almost-everywhere "relative smooth", positive, closed current in $\Omega\setminus A$.

For continuing T to a positive, closed current, it is "only" necessary to find a positive, closed extension of the residue-current R. Whether resp. how to extend R to a closed current, is – with the exception of special situations – a still open problem, because there are no theorems like the theorem of Siu, which would allow to apply the here used methods to the current R.

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Truncated symmetric products and configuration spaces

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0 Introduction

The spaces studied in this article are all related to the symmetric products $SP^n(Y) = Y^n/\mathcal{S}_n$ of a space Y. The symmetric group $\mathcal{S}_n$ acts on the cartesian product Y^n by permuting coordinates, and a point in $SP^n(Y)$ will be written multiplicatively $\langle y_1, \ldots, y_n \rangle = y_1 \ldots y_n$. If a basepoint $*$ in Y is given, one has inclusions $SP^n(Y) \to SP^{n+1}(Y)$ identifying $y_1 \ldots y_n$ with $y_1 \ldots y_n *$. The union over all these spaces is the infinite symmetric product $SP^\infty(Y)$. By the fundamental result about infinite symmetric products, obtained in [DT], for Y connected and of the homotopy type of a CW-complex $SP^\infty(Y)$ is a product of Eilenberg-MacLane-spaces,

$$SP^\infty(Y) = \prod_{i>0} K(H_i(Y; \mathbb{Z}), i). \tag{1}$$

In Part I we study the truncated symmetric product $TP^n(Y)$; this is the quotient of $SP^n(Y)$ by the relation $y^2 = * = 1$ for all $y \in Y$. The union of all the $TP^n(Y)$ is the infinite truncated product $TP^\infty(Y)$, the topological vector space over $\mathbb{F}_2$ generated by the points of Y modulo the subspace generated by $*$. For Y connected it is also a product of Eilenberg-MacLane-spaces,

$$TP^\infty(Y) = \prod_{i>0} K(H_i(Y; \mathbb{Z}/2\mathbb{Z}), i). \tag{2}$$

The mod(2) homology of $TP^n(Y)$ was determined in [LM] as a functor of the mod(2) homology of Y. Furthermore, there are inclusions $H_*(TP^n(Y); \mathbb{F}_2) \to H_*(TP^{n+1}(Y); \mathbb{F}_2)$ whose image we describe using a bigrading of the homology of the limit $TP^\infty(Y)$. The rational or mod(p) homology for $p > 2$ is more complicated, see 4.1 below.

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In Part II we study the configuration spaces $C^n(Y)$, the subspace of $SP^n(Y)$ consisting of all $\langle y_1, \ldots, y_n \rangle$ with $y_i \neq y_j$ when $i \neq j$. Recently, as in [Seg, MD, A-J], there has been renewed interest in these spaces and more particularly the associated deleted symmetric products

$$C^k(X) = \{\langle x_1, \ldots, x_k \rangle \mid x_i \neq x_j \text{ for } i \neq j\} \subset SP^k(X).$$

Somewhat before that [C-T1, C-T2, C-T3], there were numerous applications indicated, as well as the work of D. Anderson and P. Trauber reported on in those references which gave ample evidence, already of the importance of these spaces. For some particular manifolds the homology of these configuration spaces was computed in [A, B-C, C, V].

To amalgamate these spaces one introduces labels for the y_i in some other space, e.g. a sphere S^L; a point y with label s is cancelled from a configuration if $y = *$ or $s = \infty \in S^L$. The result is an infinite dimensional space $C(Y; S^L)$. When Y is a compact manifold of dimension m, a basic result in [MD, B1], identifies $C(Y; S^L)$ as the space of sections in a S^{m+L}-bundle over Y. Furthermore, it stably splits into a bouquet of filtration quotients $D_k(Y; S^L) = C_k(Y; S^L)/C_{k-1}(Y; S^L)$, [B2, CT1], and its homology is the homology of a product of loop spaces of spheres [BCT],

$$C(Y; S^L \simeq {}_sV D_k(Y; S^L), \tag{3}$$

$$H_*(C(Y; S^L); \mathbb{F}) \cong \bigotimes_0^m \otimes^{\beta_q} H_*(\Omega^{m-q} S^{m+L}; \mathbb{F}), \tag{4}$$

where β_q is the q^{th} $\mathbb{F}$-Betti number of Y. (Here $\text{char}(\mathbb{F}) = 2$ or $\text{char}(\mathbb{F}) \neq 2$ and $\dim(Y) + L \cong 1 \bmod (2)$.)

Our point of view in Part I is that $C^n(Y) = TP^n(Y) - TP^{n-2}(Y)$. In Part II the configuration space occurs as the base of an mL-dimensional vector bundle whose Thom space is $D_n(Y; S^L)$. Thus the quotients $TP^n(Y)$ of $SP^n(Y)$, and the subspaces $C^n(Y)$ of $SP^n(Y)$ are homologically dual to each other.

Theorem 3.1 *Let $\mathbb{F}$ be a field, then*

(a) $H^{2nk-r}(DP^k(M^{2n} - *); \mathbb{F}) \cong H_r(TP^k(M)/TP^{k-1}(M); \mathbb{F})$.

(b) $H^{2nk-r}(DP^k(M^{2n}); \mathbb{F}) \cong H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

Theorem 3.2 *Suppose M^{2n+1} is a compact, oriented, closed manifold. Then we have*

(a) $H_t^{(2n+1)k-r}(C^k(M - *); \mathbb{F}) \cong \mathcal{TP}_{r,k}(M; \mathbb{F}) = H_r(TP^k(Y), \dot{T}P^{k-1}(Y); \mathbb{Z}/p)$.

(b) $H_T^{(2n+1)k-r}(C^k(M; \mathbb{F}) = H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

(The T subscript appearing in the homology here denotes the fact that the coefficients are twisted by the $-$ action of $\mathbb{Z}/2 = \mathcal{S}_k/\mathcal{A}_k$ on the field $\mathbb{F}$.)

Part I continues with the main computational result, a spectral sequence to compute the bigraded groups $\mathcal{TP}_{*,*'}(Y) := H_*(TP^{*'}(Y), TP^{*'-1}(Y); \mathbb{Z}/p)$ which taken together form a bigraded ring.

Theorem 4.1 *There is a spectral sequence with E^2-term*

$$E^2_{*,*'} = \text{Tor}^{\mathcal{C}_{*,*'}(Y;\mathbb{F})}(\mathbb{F}, \mathbb{F})$$

converging to $\mathscr{T}\mathscr{P}_{*,*'}(\Sigma Y; \mathbb{F})$.

Here $\mathbb{F}$ is $\mathbb{Q}$, or $\mathbb{F}=\mathbb{F}_p$ with $p>2$.

We then apply this to the case $Y=M_g$, a closed, connected, oriented surface of genus g. Completing the results of [BC] is the following result which combines 5.12 and 5.13 in the body of the text

Theorem. *Let* $p>g$, *and suppose* T^2_g *is a closed Riemann surface of genus* g. *Let* f *have bidegree* (2, 1), $P^{[i]}$ *bidegree* $(2p^i+1, 2p^i)$, *the* h_j *have bidegree* (2.2), $1\leqq j\leqq 2g$, *and* $\beta P^{[i]}$ *have bidegree* $(2p^i+2, 2p^i)$, *then the bigraded algebra*

$$V_{*,*'}\cong W_g\otimes \Gamma[f, h_1, \ldots, h_j, \ldots, \beta P^{[i]}, \ldots]\otimes E[\ldots P^{[i]}\ldots]$$

satisfies $H^{2kp-j}(\mathscr{C}^k(T^2_g), \mathbb{F}_p)\cong V_{j,k}$.

Here W_g is a complicated algebra determined as the homology of a certain complex described more precisely in § 5, while $\Gamma[\,]$ denotes the divided power algebra on the stated generators and $E()$ is the exterior algebra on the given generators.

There is an interesting relation between alternating products $AP^n(Y)=Y^n/\mathscr{A}_n$ (where $\mathscr{A}_n$ is the alternating group), symmetric products, and truncated products.

Lemma 7.1 *Let* X *be a simplicial complex, then for all odd primes,* p, *there is a split exact sequence*

$$0\to H_*(TP^n(X)/TP^{n-2}(X); \mathbb{F}_p^-)\to H_*(AP^n(X); \mathbb{F}_p)\to H_*(SP^n(X); \mathbb{F}_p^+)\to 0$$

where $\mathbb{F}_p^{\pm}$ *denotes the field* $\mathbb{F}_p$ *with* $\mathbb{Z}/2$*-action* $t(a)=\pm a$.

Part II continues the methods of [BCT]. Among the results we mention a short proof of the main result of [BCT].

Corollary 8.4 [B-C-T] (i) *If* $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$ *or* m *is odd, then* $H_* C^k(M)$ *depends only on* $H_* M$ *and* m. *Furthermore there is an isomorphism of vector spaces for* $n>>0$ *and* $0<q<mk$ *given by*

$$\begin{aligned} H_q C^k(M) &\cong H_{q+2nk} C(M, S^{2n}) \\ &\cong H_{q+2nk}\left(\prod_{q=0}^{m}\prod^{\beta(q)} \Omega^{m-q} S^{m+2n}\right). \end{aligned}$$

(ii) *if* $\mathbb{F}=\mathbb{Z}/2\mathbb{Z}$ *or* m *is even, then* $H_*(C^k(M); \mathbb{F}(-1))$ *depends only on* $H_* M$ *and* m. *Furthermore there is an isomorphism of vector spaces for* $n>>0$ *and* $0<q<mk$ *given by*

$$\begin{aligned} H_q(C^k(M); \mathbb{F}(-1)) &\cong H_{q+(2n+1)k} C(M, S^{2n+1}) \\ &\cong H_{q+(2n+1)k}\left(\prod_{q=0}^{m}\prod_{\beta(q)} \Omega^{m-q} S^{m+2n+1}\right). \end{aligned}$$

Next we discuss certain product decompositions of mapping spaces of surfaces into spheres. These decompositions in turn give information about the configuration space of unordered k-tuples of points in the given surface.

Theorem 8.5 *If $n>1$, there exist principal $\Omega^2 S^{2n}$ fibrations*

$$\Omega^2 S^{2n} \to Y_{g,2n} \to (S^{2n-1})^{2g}$$

and a principal $\Omega^2 S^3$-fibration

$$\Omega^2 S^3 \to Y_{g,2} \to (S^1)^{2g}$$

such that

(1) *there is a homotopy equivalence* $\mathrm{map}_*^0(M_g, S^2) \xrightarrow{\simeq} (\Omega S^3)^{2g} \times Y_{2,g}$ *and*

(2) *there is a homotopy equivalence after inverting* 6

$$\mathrm{map}_*(M_g, S^{2n}) \xrightarrow{\simeq} (\Omega S^{4n-1})^{2g} \times Y_{g,2n}$$

for $n>1$,

(3) *there is a fibration for $n>1$*

$$Y_{g,2n} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{4n-1})^{2g}$$

which has a section after inverting 2.

As a convenience for the reader we point out that there are several distinct notations used for these spaces, both in the existing literature and in this paper: namely

(1) $\tilde{C}^n(Y) = F(Y, n)$, and
(2) $C^n(Y) = F(Y, n)/\mathscr{S}_n = B(Y, n) = TP^n(Y) - TP^{n-2}(Y)$.

Part I: Truncated symmetric products

1 Preliminaries

[M2] studied the homology and geometry of symmetric products using techniques – and largely reproducing results which had been anticipated by Steenrod [St2]. The homology results in [M2] are complete in that closed formulae are given for $H_*(SP^k(X); \mathbb{F}_p)$ for any locally finite CW-complex X.

The key new ingredient in [M2] was an explicit way of obtaining a combinatorial cell decomposition for $\coprod_k SP^k(\Sigma Y)$ if an appropriate CW-decomposition was given for $SP^\infty(Y)$.

ΣY was regarded as the identification space $I \times Y / \{0 \times Y = 1 \times Y = I \times * = *\}$ and the suspension coordinate was used to attempt to put an ordering on the unordered n-tuple $\langle (t_1, y_1), \ldots, (t_n, y_n) \rangle$. Precisely, we can partially order the points by assuming $t_1 \leqq t_2 \leqq \ldots \leqq t_n$. (Note there that $\{0 \leqq t_1 \leqq \ldots \leqq t_n \leqq 1\} = \sigma^n$ is just the ordinary n-simplex.) Where $t_i = t_{i+1}$ i.e. on the i^{th}-face of σ^n we observed that the resulting point could be identified with

$$\langle (t_1, y_1), \ldots, (t_{i-1}, y_{i-1}), (t_i, y_i \cdot y_{i+1}), \ldots, (t_n, y_n) \rangle$$

where $y_i \cdot y_{i+1}$ is the unordered pair $\langle y_i, y_{i+1} \rangle \in SP^2(Y) \subset SP^\infty(Y)$. Extending this idea the geometric bar construction was defined and explored in [M1].

For any associative H-space Y with unit $*$, set

$$B_Y = \coprod_k \sigma^k \times Y^k$$

with the compactly generated topology, where we make identifications of 3 kinds motivated by the geometry of $SP^\infty(X)$.

(i) On the face $\sigma_i^k = \{t_1, \ldots, t_i = t_{i+1}, \ldots t_k\}$, or when $y_{i+1} = *$ we identify

$$(t_1, \ldots, t_n, y_1, \ldots, y_n) \quad \text{with} \quad (t_1, \ldots, \hat{t}_{i+1}, \ldots, t_n, y_1, \ldots, y_{i-1}, y_i\, y_{i+1}, \hat{y}_{i+1}, \ldots, y_n).$$

(ii) If $t_1 = 0$ then set $(t_1, \ldots, t_n, y_1, \ldots, y_n) \sim (t_2, \ldots, t_n, y_2, \ldots, y_n)$.
(iii) If $t_n = 1$ set $(t_1, \ldots, t_n, y_1, \ldots, y_n) \sim (t_1, \ldots, t_{n-1}, y_1, \ldots, y_{n-1})$.

If $*$ is an NDR, define $E_Y = \coprod \sigma^k \times Y^k$/*relations of type* (i), (ii) *only*. One sees easily that E_Y is contractible, and there is a principal quasi-fibering

$$Y \to E_Y \to B_Y.$$

Hence B_Y is a model for the classifying space of Y. Moreover, in the case $Y = SP^\infty(X)$ then by the remarks preceeding the construction we see that

$$B_Y = SP^\infty(\Sigma X).$$

Thus, one can relate the homology of the $SP^n(\Sigma X)$ to a specific construction on pieces of $\oplus H_*(SP^n(X))$, creating an inductive situation which was relatively easy to analyze. An interesting extension of these ideas occurs in [Mc].

Recall that Cartan determined the cohomology of the Eilenberg-MacLane spaces $K(\pi, n)$ for π any finitely generated abelian group as follows. Let $\mathscr{A}_n(p)$ be the subset of the mod(p) Steenrod algebra spanned by monomials in the mod(p) Bockstein β and the P^i having excess $< n$. (For $p = 2$ we do not need the Bockstein as it is identified with Sq^1.) Then

$$H^*(K(\mathbb{Z}/p; n); \mathbb{F}_p) = S(\mathscr{A}_n(p)(\iota_n)),$$

where ι_n is the fundamental class in $H^n(K(\mathbb{Z}/p; n); \mathbb{F}_p) = \mathbb{F}_p$, and $S(W)$ is the symmetric algebra generated by W. More exactly $S(W)$ is the tensor product of the polynomial algebra on the even dimensional generators of W and the exterior algebra on the odd dimensional generators.

For $K(\mathbb{Z}, n)$ let $\mathscr{B}_n(p) = \mathscr{A}_n(p)/W$ where W is the subspace spanned by those monomials with rightmost term equal to β. Then $\mathscr{B}_n(p)$ acts on the fundamental class $\iota \in H^n(K(\mathbb{Z}, n); \mathbb{F}_p)$ and

$$H^*(K(\mathbb{Z}; n); \mathbb{F}_p) = S(\mathscr{B}_n(p)(\iota_n)).$$

The explicit structure of $H^*(SP^\infty(X); \mathbb{F}_p)$ is now given by bigrading the results of Cartan above. When X is a locally finite CW-complex:

$$H^*(SP^\infty(X); \mathbb{F}_p) = \coprod_n S_{\alpha_i}[\ldots, P^I(\alpha_i), \ldots]$$

where $P^I = \beta^{\varepsilon_1} P^{i_1} \beta^{\varepsilon_2} P^{i_2} \ldots \beta^{\varepsilon_r} P^{i_r}$ runs over all admissible monomials in the mod(p) Steenrod algebra $\mathscr{A}(p)$ of excess $< n$, while the α_i run over a selected basis for $H^n(X; \mathbb{F}_p)$. The second degree of P^I is p^r.

In the particular case where $X = S^1$ we obtain that $H^*(SP^\infty(S^1); \mathbb{Z}) = E[e_1]$, and indeed, $SP^n(S^1) \simeq S^1$ for all $n \geqq 1$. When $X = S^2$, we have $H^*(SP^\infty(S^2); \mathbb{Z}) = \mathbb{Z}[e_2]$, where e_2 has dimension 2 and second degree 1. As a consequence

$$H^*(SP^n(S^2); \mathbb{Z}) = \mathbb{Z}[e_2]/(e_2^{n+1}). \tag{1.1}$$

Indeed, it is a standard result that $SP^n(S^2) \simeq \mathbf{CP}^n$. Finally we have

$$H^*(SP^\infty(S^3); \mathbb{F}_p) = E[e_3, \ldots, P^{[i]}(e_3), \ldots] \otimes \mathbb{F}_p[\beta P^1(e_3), \ldots, \beta P^{[i]}(e_3), \ldots] \tag{1.2}$$

where $P^{[i]} = P^{p^{i-1}} P^{p^{i-2}} \ldots P^1$ runs over the admissible monomials of excess 2 in the Steenrod p-powers. Note that both $P^{[i]}$ and $\beta P^{[i]}$ have second degree p^i. Then a basis for $H^*(SP^n(S^3); \mathbb{F}_p)$ is given by all those monomials in the generators above for which the second degree is $\leqq n$.

2 The functors $TP^n(Y)$

In this section we assume all spaces are based locally finite CW-complexes.

Definition 2.1 $TP^\infty(Y) = SP^\infty(Y)/(y^2 = *, y \in Y)$.

$TP^\infty(Y)$ is a topological group – the free abelian group generated by the points of Y (with $*$ as identity) subject to $y^2 = *$ for $y \in Y$. $TP^\infty(Y)$ has been studied in [D-T], it is the free topological $\mathbb{F}_2$-vector space generated by the points of Y modulo the subspace generated by the base point. There is a natural projection $SP^\infty(Y) \to TP^\infty(Y)$ and we set

$$TP^n(Y) = \mathrm{im}(SP^n(Y)) \subset TP^\infty(Y).$$

Additionally, if $f\colon Y \to Z$ is a based map then the induced map

$$TP(f)\colon TP^\infty(Y) \to TP^\infty(Z)$$

is defined by

$$TP(f)(\langle y_1, \ldots, y_n \rangle) = \langle f(y_1), \ldots, f(y_n) \rangle.$$

Together these two operations define the TP-functor on based spaces and based, continuous maps.

Properties

(2.2) $TP^\infty(Y)$ is a homotopy functor, i.e. if $Y \simeq Z$ then $TP^\infty(Y) \simeq TP^\infty(Z)$, and $TP^n(Y)$ is a (based) homotopy functor for each n. Indeed, if

$$H_t\colon Y \to Z$$

is a based homotopy from f to g, then $TP(H_t)$ defines a homotopy from $TP(f)$ to $TP(g)$ and $TP(H_t)$ preserves the $TP^n(\,)$ functors.

(2.3) If f: $Y \to Z$ is a based map then $TP(f)$: $TP^\infty(Y) \to TP^\infty(Z)$ is a group homomorphism, and consequently is a principal fibering over its image with fiber = kernel = $TP(f^{-1})(*)$. Hence the functor TP^∞ takes cofiberings to (principal) fiberings. In particular

$$TP^\infty(Y_1 \vee Y_2) = TP^\infty(Y_1) \times TP^\infty(Y_2).$$

Additionally, if $S^{n-1} \xrightarrow{f} Y_1 \to Y_1 \cup e^n = Y$ gives Y from Y_1 by adjoining a based cell, then $TP^\infty(Y)$ is the total space of the principal fibering

$$TP^\infty(Y) = TP^\infty(Y_1) \times_{TP^\infty(S^{n-1})} TP^\infty(cS^{n-1}) \to TP^\infty(S^n)$$

where $TP^\infty(cS^{n-1})$ is the (contractible) total space of the fibering

$$TP^\infty(S^{n-1}) \to TP^\infty(cS^{n-1}) \to TP^\infty(S^n).$$

Here cY denotes the reduced cone on Y.

(2.4) More generally, $TP^\infty(\Sigma Y) = B_{TP^\infty(Y)}$ where $B_{(\ )}$ is the topological bar construction of [M1, St1] discussed in § 1.

(2.5) From (2.3) $\pi_*(TP^\infty(Y))$ is a homology theory for Y. Note that $TP^\infty(S^0) = \mathbb{Z}/2$, so by (2.3), $TP^\infty(S^1) = B_{\mathbb{Z}/2} = \mathbf{RP}^\infty$, and more generally $TP^\infty(S^n) = K(\mathbb{Z}/2, n)$. Thus we have the Dold-Thom theorem for TP^∞,

Theorem 2.6 *If Y is a based, locally finite* CW*-complex, then*

$$TP^\infty(Y) \simeq \prod_i K(H_i(Y; \mathbb{Z}/2), i).$$

Corollary 2.7 $TP(f)_*$: $\pi_*(TP^\infty(Y)) \to \pi_*(TP^\infty(Z))$ *is just the map induced in* $\mathbb{Z}/2$ *homology,*

$$f_*\colon\ H_*(Y; \mathbb{Z}/2) \to H_*(Z; \mathbb{Z}/2).$$

(2.8) The points of $TP^n(Y) - TP^{n-1}(Y)$ have the form $\langle x_1, \ldots, x_n \rangle$, $x_i \neq x_j \neq *$, if $i \neq j$. The points of $TP^{n-1}(Y)$ embed into $TP^n(Y)$ by adding one $*$, i.e. regarding them as $\langle x_1, \ldots, x_{n-1}, * \rangle$. Let

$$\mathrm{Sing}^n(Y) = \{\langle y_1, \ldots, y_n \rangle \in SP^n(Y) \mid y_i = y_j, \text{for some } i \neq j\}.$$

Then $TP^n(Y)/TP^{n-2}(Y) = SP^n(Y)/\mathrm{Sing}^n(Y)$. Similarly we have that

$$TP^n(Y)/TP^{n-1}(Y) = SP^n(Y - *)/\{\mathrm{Sing}^n(Y - *) \cup SP^{n-1}(Y)\}.$$

Theorem 2.9 $H_*(TP^n(Y); \mathbb{Z}/2) = \bigotimes_{r \leq n} H_*(TP^r(Y), TP^{r-1}(Y); \mathbb{Z}/2)$ *for Y a locally finite based* CW*-complex.*

Proof. Let

$$D_n\colon\ TP^n(Y) \to TP^\infty(TP^{n-1}(Y))$$

be defined by the formula

$$D_n(\langle y_1, \ldots, y_n\rangle) = \prod_{i=1}^{n} \langle y_1, \ldots, \hat{y}_i, \ldots, y_n\rangle. \tag{2.10}$$

D_n is continuous, and, by checking on points we have the following commutative diagram

$$\begin{array}{ccc} TP^{n-1}(Y) & \xrightarrow{i_{n-1}} & TP^n(Y) \\ \downarrow{\scriptstyle id \times D_{n-1}} & & \downarrow{\scriptstyle D_n} \\ TP^{n-1}(Y) \times TP^\infty(TP^{n-2}(Y)) & \xrightarrow{j} & TP^\infty(TP^{n-1}(Y)) \end{array} \tag{2.11}$$

where $j = \circ(i_1 \times TP(i_{n-2}))$. Now, pass to homology with $\mathbb{Z}/2$-coefficients, i.e. apply the TP^∞-functor, and use the retraction

$$r\colon\ TP^\infty(TP^\infty(Z)) \to TP^\infty(Z), \qquad r(\langle w_1, \ldots, w_s\rangle) = w_1 \cdot w_2 \ldots w_s.$$

(2.9) now follows directly (or one can apply [D, Lemma (2)] to 2.11). □

Remark 2.12 The groups $H_*(TP^n(Y), TP^{n-1}(Y); \mathbb{Z}/p)$ p an odd prime, are more complex. Of course, a class $\alpha \in H_*(TP^n(Y), TP^{n-1}(Y); \mathbb{Z}/p)$ which comes from the group $H_*(TP^n(Y); \mathbb{Z}/p)$ must, sooner or later, be in the image of the ∂-map from

$$H_*(TP^{n+r}(Y), TP^{n+r-1}(Y); \mathbb{Z}/p)$$

for some r. Nevertheless, there is an associative pairing

$$\begin{aligned} H_*(TP^n(Y), TP^{n-1}(Y); \mathbb{Z}/p) \otimes H_{*'}(TP^r(Y), TP^{r-1}(Y); \mathbb{Z}/p) \\ \to H_{*+*'}(TP^{n+r}(Y), TP^{n+r-1}(Y); \mathbb{Z}/p) \end{aligned}$$

making the groups

$$\mathcal{TP}_{*,n}(Y; \mathbb{Z}/p) = \bigoplus_{*,n} H_*(TP^n(Y), TP^{n-1}(Y); \mathbb{Z}/p)$$

into a commutative, *bigraded* ring with unit. Note that $\mathcal{TP}_{*,*'}(Y; \mathbb{Z}/p)$ is a homotopy invariant of the space Y.

Example 2.13 $S^1 = \Sigma S^0$, consequently $TP^\infty(S^1) = B_{TP^\infty(S^0)} = B_{\mathbb{Z}/2}$. But then

$$TP^n(S^1) = (B_{\mathbb{Z}/2})_n = \sigma^n \times (\mathbb{Z}/2)^n/\text{identifications}.$$

These identifications collapse σ^n to its i^{th}-face if the i^{th}-coordinate in $(\mathbb{Z}/2)^n$ is 1, or if we are on the i^{th}-face directly. Thus, there is a single n-cell in $TP^n(S^1)$, $\sigma^n \times (T)^n$, and it is a straightforward induction to show that $TP^n(S^1) \simeq \mathbf{RP}^n$ for all n.

In particular, $H_*(TP^n(S^1), TP^{n-1}(S^1); \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & *=n \\ 0 & \text{otherwise} \end{cases}$ and we have

$$\mathscr{TP}_{**'}(S^1; \mathbb{Z}/p) = E(e_{1,1}) \otimes \Gamma(b_{2,2})$$

where $E(e)$ is the exterior algebra on a one dimensional generator and $\Gamma(b)$ is the divided power algebra on a single two dimensional generator $\left(\Gamma(b)_{2i,2i} = \mathbb{Z}/p \text{ with generator } b_i \text{ and } b_i b_j = \binom{i+j}{j} b_{i+j}\right)$.

Note also that the above description of $\mathscr{TP}_{**'}(S^1; \mathbb{F})$ remains valid even when $\mathbb{F} = \mathbb{Z}$, and consequently is valid for all coefficients. In fact we have the following (formal) ring isomorphism

$$\mathscr{TP}_{*,*'}(S^1; \mathbb{F}) \cong H_*(K(\mathbb{Z}, 1); \mathbb{F}) \otimes H_*(K(\mathbb{Z}, 2); \mathbb{F}). \tag{2.14}$$

Remark 2.15 Let $Y = \Sigma Z$, then the coproduct map

$$e\colon\ \Sigma Z \to \Sigma Z \vee \Sigma Z, \qquad (t, z) \mapsto \begin{cases} 2t, z), * & t \leqq \frac{1}{2} \\ *, (2t-1, z) & t \geqq \frac{1}{2} \end{cases}$$

induces a bigraded coproduct map

$$M(e)\colon\ TP^\infty(Y) \to TP^\infty(Y \vee Y) = TP^\infty(Y) \times TP^\infty(Y)$$

which, on passing to homology makes $\mathscr{TP}_{*,*'}(Y; \mathbb{F})$ into a bigraded Hopf algebra whenever $\mathbb{F}$ is a field. Such a structure is not generally available to us when Y is not a suspension, however, since

(1) the diagonal $\Delta\colon TP^\infty(Y) \to TP^\infty(Y) \times TP^\infty(Y)$ is not filtration preserving, and
(2) $TP^\infty(Y \times Y)$ is not $TP^\infty(Y) \times TP^\infty(Y)$ so the map $TP(\Delta)$ does not help either.

Of course, the filtration of $TP^\infty(Y)$ by the $TP^n(Y)$ gives rise to a spectral sequence converging to $H_*(TP^\infty(Y); \mathbb{F})$, with E^1-term $E^1_{*,n} = \mathscr{TP}_{*,n}(Y; \mathbb{F})$. With respect to the derivations, d_r, every term $E^r_{*,*'}$ is a differential graded algebra, and when Y is a suspension is even a differential graded Hopf algebra.

Remark 2.16 Each term $E^r_{*,*'}$ in this spectral sequence is a homotopy invariant for Y.

3 Connection with deleted symmetric products

Suppose that M^{2n} is a compact, oriented manifold with base point $*$, and empty boundary. We have

Theorem 3.1 *Let $\mathbb{F}$ be a field, then*

(a) $H^{2nk-r}(C^k(M^{2n} - *); \mathbb{F}) \cong \mathscr{TP}_{r,k}(M^{2n}; \mathbb{F})$.
(b) $H^{2nk-r}(C^k(M^{2n}); \mathbb{F}) \cong H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

Proof. This is just Alexander-Poincaré duality. Indeed, if $N = N(\mathrm{Sing}^k(M^{2n}))$ is any regular neighborhood of the singular set, then ∂N is a closed manifold of dimension $2nk-1$ and $SP^k(M^{2n}) - \mathrm{int}(N)$ is an oriented manifold with bound-

ary $=\partial N$. (Unless $\mathrm{char}(\mathbb{F})=2$ the fact that this manifold is oriented is crucial when we apply duality but $C^n(M)$ is only oriented if $\dim(M)$ is even.) Thus, we have

$$\begin{aligned} H_*(SP^k(M), \mathrm{Sing}^k(M)) &= H_*(SP^k(M), N) = H_*(SP^k(M) - \mathrm{int}\, N, \partial N) \\ &= H^{2nk-*}(SP^k(M) - \bar{N}) = H^{2nk-*}(SP^k(M) - \mathrm{Sing}^k(M)) \\ &= H^{2nk-*}(C^k(M^{2k})). \end{aligned}$$

Hence, (b) follows from (2.8).

To prove (a) note that if N_1 is a regular neighborhood of

$$V_k = \{\mathrm{Sing}^k(M^{2n}) \cup SP^{k-1}(M^{2n})\},$$

thern the same argument works here since

$$SP^k(M^{2n})/V = TP^k(M^{2n})/TP^{k-1}(M^{2n}),$$

and $SP^k(M^{2n}) - V = C^k(M^{2n} - *)$.

When M is odd dimensional we must use twisted coefficients. Thus let

$$\mathbb{R} \to \xi \to C^k(M)$$

be the orientation line bundle (with $w_1(\xi) = w_1(C^k(M))$, the first Stiefel-Whitney class of $C^k(M)$). The cohomology of $C^k(M)$ with orientation twisted coefficients,

$$H^i_T(C^k(M); \mathbb{F}) \cong \tilde{H}^{i+1}(T(\xi); \mathbb{F}),$$

is defined as the ordinary reduced cohomology in one higher dimension of the Thom complex of ξ. □

Theorem 3.2 *Suppose M^{2n+1} is a compact, oriented, closed manifold. Then we have*

(a) $H_t^{(2n+1)k-r}(C^k(M - *); \mathbb{F}) \cong \mathscr{TP}_{r,k}(M; \mathbb{F}) = H_r(TP^k(Y), TP^{k-1}(Y); \mathbb{F})$.
(b) $H_t^{(2n+1)k-r}(C^k(M; \mathbb{F}) = H_r(TP^k(M)/TP^{k-2}(M); \mathbb{F})$.

(This is the way Poincaré duality works for non-oriented manifolds. Otherwise, the proof is as above.)

In [B-C-T], Cohen et al. show that if M^{2n+1} is an odd dimensional, oriented, manifold, then $H_*(C^k(M); \mathbb{F})$ depends only on $H_*(M; \mathbb{F})$. But they also show, in the same paper, that $H_*(C^k(M^{2n}); \mathbb{F}^-)$ depends only on $H_*(M^{2n}, \mathbb{F})$ when M^{2n} is even dimensional, and oriented. Hence, we now have effective methods for identifying both the twisted and untwisted homology of the deleted symmetric products of manifolds in all dimensions. Short proofs of these results are given in Part II.

4 Calculational techniques

In this section we start by introducing an important spectral sequence which allows us to understand $\mathscr{TP}_{*,*}(\Sigma Y)$. The case of more general spaces is handled in 4.6 where we identify $TP^n(Y)$ up to homology with a much larger space. This larger space, however, has a natural filtration and associated spectral

sequence which will be our main calculational tool in the remainder of Part I of this paper.

For $TP^\infty(\Sigma X)$, we have, exactly as in § 1, an identification $TP^\infty(\Sigma X) \cong B_{TP^\infty(Y)}$. In particular, this allows us to apply the Eilenberg-Moore spectral sequence with E^2-term $\mathrm{Tor}^{H_*(TP^\infty(Y))}(\mathbb{F}, \mathbb{F})$ to study $H_*(TP^\infty(\Sigma Y); \mathbb{F})$. With a little care we can even extend this to the study of $\mathscr{T}_{*,*}(\Sigma Y)$.

For a bigraded algebra $A_{*,*}$, the $\mathrm{Tor}^{A_{*,*}}(\mathbb{F}, \mathbb{F})$ groups are naturally *trigraded*. The terms of tridegree (l, m, n) come from terms of the form $|\theta_1|\ldots|\theta_l|$ with θ_i of bidegree (m_i, n_i) and $\sum m_i = m$, $\sum n_i = n$. The associated bigraded Tor groups are then given as $\mathrm{Tor}_{s,t} = \sum_{l+m=s} \mathrm{Tor}_{l,m}$. When A is a commutative, unitary algebra Tor is a bigraded commutative ring.

Theorem 4.1 *There is a spectral sequence with E^2-term*

$$E^2_{*,*'} = \mathrm{Tor}^{\mathscr{T}_{*,*'}(Y;\mathbb{F})}(\mathbb{F}, \mathbb{F})$$

converging to $\mathscr{T}_{*,*'}(\Sigma Y; \mathbb{F})$.

Proof. Filter $TP^n(\Sigma Y)$ by setting

$$TP^n(\Sigma Y)_j = TP^n(\Sigma Y) \cap (B_{TP^\infty(Y)})_j.$$

Note that $(B_{TP^\infty(Y)})_j = \mathrm{im}\{\sigma^j \times (TP^\infty(Y))^j \to B_{TP^\infty(Y)}\}$, and

$$TP^n(\Sigma Y) \cap (B_{TP^\infty(Y)})_j = \{(t_1, y_1), \ldots, (t_r, y_r) \mid \text{there are at most } j \text{ distinct values among } t_1, \ldots, t_r\}.$$

Then $TP^n(\Sigma Y)_n = TP^n(\Sigma Y)$, so the filtration is complete and the associated spectral sequence converges. Next consider the induced filtration on

$$TP^n(\Sigma Y)/TP^{n-1}(\Sigma Y).$$

The E^1_j term is obviously the direct summand of the bar construction on $C_{*,*'}(Y; \mathbb{F})$ defined as

$$\sum_{r=1}^{j} \sigma^r \times \prod_{\sum_1^r l_k = n} \mathscr{T}_{*,l_k}(Y; \mathbb{F}) = \{|\theta_1|\ldots|\theta_r| \mid \theta_r \in \mathscr{T}_{*,l_i}(Y; \mathbb{F})\}$$

and the E^2-term is the associated bigraded Tor-term as claimed. □

Corollary 4.2

$$\mathscr{T}_{*,*'}(S^n; \mathbb{F}) \cong \bigoplus_{\substack{r+s=* \\ r'+2s'=*'}} H_{r,r'}(SP^\infty(S^n); \mathbb{F}) \otimes H_{s,s'}(SP^\infty(S^{n+1}); \mathbb{F})$$

for any field $\mathbb{F}$.

Proof. We proceed by induction. The corollary amounts to the asertion that the spectral sequence of 4.1 collapses when $Y = S^n$. This is implicit in Example 2.13 (2.14) for the case S^1. For S^n we simply use the standard techniques

of [Sem] to embed the requisite *cycles* in the chain complex of $TP^m(S^n)/TP^{m-1}(S^n)$ corresponding to all the classes in the Tor-groups. □

Remark 4.3 Something like 4.2 should be expected since

$$TP^\infty(S^{n+1}) \simeq SP^\infty(S^n) \times_T SP^\infty(cS^n)$$

where T: $SP^\infty(S^n) \to SP^\infty(S^n)$ is the squaring map. Since the squaring map doubles degrees, a consistent bigrading on $SP^\infty(S^n) \times_T SP^\infty(cS^n)$ can only be realized if the bidegree in $SP^\infty(cS^n)$ is doubled.

Example 4.4 From 4.2

$$\mathscr{T}_{*,*'}(S^2; \mathbb{F}) = \bigotimes_{\substack{r'=2s'=*' \\ r+s=*}} H_{r,r'}(SP^\infty(S^2); \mathbb{F}) \otimes H_{s,s'}(SP^\infty(S^3); \mathbb{F}).$$

Also, $H_{r,r'}(SP^\infty(S^2); \mathbb{F}) = \Gamma(b_{2,1})$, while (if we let $P^{[r]} = P^{p^r} P^{p^{r-1}} \dots P^p P^1$ denote the monomial in the p^{th}-power operations in the Steenrod algebra $\mathscr{A}(p)$)

$$H^{s,s'}(SP^\infty(S^3); \mathbb{Z}/p) = E[\iota_3, P^{[0]}(\iota_3), \dots, P^{[r]}(\iota_3)\dots] \otimes \mathbb{Z}/p[\dots \beta P^{[r]}(\iota_3)\dots]$$

where the bidegree of $P^{[r]}(\iota_3)$ in $\mathscr{T}_{*,*'}(S^2; \mathbb{Z}/p)$ is $(2p^{r+1}+1, 2p^{r+1})$, and is $(2p^{r+1}+2, 2p^{r+1})$ for $\beta P^{[r]}(\iota_3)$. For the rationals $\mathbb{Q}$ the calculation reduces to $\mathscr{T}_{*,*'}(S^2; \mathbb{Q}) = \Gamma(b_{2,1}) \otimes E(\iota_{3,2})$, so

$$\mathscr{T}_{i,*}(S^2; \mathbb{Q}) = \begin{cases} \mathbb{Q} & *=2i \quad \text{generator} \quad b_{2i,i} \\ \mathbb{Q} & *=2i-1 \quad \text{generator} \quad b_{2(i-2),i-2}\,\iota_{3,2} \\ 0 & \text{otherwise} \end{cases}$$

as long as $i>1$.

Example 4.5 For $TP^\infty(S^2)$ we have that $d^1(b_{2,1}) = \iota_{3,2}$, and more generally

$$d^{p^i}(b^{p^i}) = P^{[i-1]}(\iota_{3,2}), \qquad d^{p^i}(b^{p^{i-1}(p-1)} P^{[i-2]}(\iota_{3,2})) = \beta P^{[i-1]}(\iota_{3,2})$$

for odd p, after comparing with more standard sequences, or, alternately, using the Hopf algebra structure, and the fact that for odd p the E^∞ term is 0, to force the successive differentials.

To analyze the structure of $\mathscr{C}_{*,*'}(Y)$ for more general spaces we have the following result which leads to a very useful spectral sequence.

Theorem 4.6 *Let* $\mathscr{U}_n(Y) = \bigcup_{i+2j \leqq n} SP^i(Y) \times_T SP^j(cY) \subset SP^\infty(Y) \times_T SP^\infty(cY)$, *then* $H_*(\mathscr{U}_n(Y); A) \cong H_*(TP^n(X); A)$ *for all untwisted coefficients* A.

Proof. There is a map

$$p: SP^\infty(Y) \times_T SP^\infty(xY) \to TP^\infty(Y)$$

defined on points by $p(\{a,b\}) = \{a\} \in TP^\infty(Y)$. This map is well defined since first it is a homomorphism, and second, $p(\{a,(0,x)\}) = p(\{ax^2, *\}) \in TP^\infty(Y)$. Note that the image of p when restricted to $\mathscr{U}_n(Y)$ is exactly $TP^n(Y)$, and set

$$\mathscr{U}_{n,j}(Y) = p^{-1}(TP^j(Y)) \cap \mathscr{U}_n(Y)$$

so $\mathscr{U}_{n,1}(Y) \subset \mathscr{U}_{n,2}(Y) \subset \ldots \subset \mathscr{U}_{n,n}(Y) = \mathscr{U}_n(Y)$. Then $w \in \mathscr{U}_{n,j}(Y)$ implies that w is the image of a point of the form $(\langle x_1, \ldots, x_j, \lambda_1^2, \ldots, \lambda_s^2 \rangle, (t_1, y_1), \ldots, (t_l, y_l))$ where $l \leqq [n - j/2] - s$. Moreover any such point is equivalent to the image of

$$(\langle x_1, \ldots, x_j \rangle, (0, \lambda_1), \ldots, (0, \lambda_s), \ldots, (t_l, y_l)).$$

Thus, $w \in \mathscr{U}_{n,j}(Y)$ if and only if $w \in \mathrm{im}(SP^j(Y) \times SP^{[(n-j)/2]}(cY))$.

In particular $w \in \mathscr{U}_{n,j}(Y) - \mathscr{U}_{n,j-1}(Y)$ if and only if

$$\omega \in \mathrm{im}((SP^j(Y) - (\mathrm{Sing}_j(Y) \cup SP^{j-1}(Y))) \times SP^{[(n-j)/2]}(cY)),$$

and

$$\begin{aligned} \mathscr{U}_{n,j}(Y)/\mathscr{U}_{n,j-1}(Y) &= \frac{SP^j(Y) \times SP^{[(n-j)/2]}(cY)}{SP^{j-1}(Y) \cup \mathrm{Sing}_j(Y) \times SP^{[(n-j)/2]}(cY)} \\ &\cong SP^j(Y)/\mathrm{Sing}_j(Y) \\ &= TP^j(Y)/TP^{j-1}(Y) \end{aligned}$$

since $SP^l(cY)$ is contractible for all l. Moreover, the map

$$\hat{p}: \mathscr{U}_{n,j}(Y)/\mathscr{U}_{n,j-1}(Y) \to TP^j(Y)/TP^{j-1}(Y)$$

is directly seen to be a homotopy equivalence for each j. Consider now the diagram

$$\begin{array}{ccccccc} \mathscr{U}_{n,1}(Y) & \subset & \mathscr{U}_{n,2}(Y) & \subset & \ldots & \subset & \mathscr{U}_{n,n}(Y) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_n} \\ TP^1(Y) & \subset & TP^2(Y) & \subset & \ldots & \subset & TP^n(Y). \end{array}$$

The first map p_1 is a homotopy equivalence, and so are the relative maps $\hat{p}_j$. Hence by iterate applications of the 5-lemma the homology maps p_{j*} are all isomorphisms and 4.6 follows. □

Example 4.7 $V_4(X) = \mathscr{U}_{4,1}(X)$ is given as the double mapping cylinder

$$SP^2(X) \times X \xleftarrow{\Delta \times 1} X \times X \xrightarrow{p} SP^2(X),$$

where the middle term is identified with the points of the form $\langle (t, x_1), (1, x_2) \rangle$, the first identification occurs when $t = 0$, and the second when $t = 1$. Then $\mathscr{U}_{4,2}(X)$ is the mapping cylinder of p: $X \times X \to SP^2(X)$, and thus has the homotopy type of $SP^2(X)$. Similarly, $V_4(X)/\mathscr{U}_{4,2}(X) \simeq SP^2(X) \times X/(\Delta X \times X)$ and these are the same homotopy types as in the corresponding decomposition of the singular locus of $SP^4(X)$.

Consider the Serre spectral sequence of the map

$$p: SP^\infty(Y) \times_T SP^\infty(cY) \to SP^\infty(\Sigma Y).$$

We find

Lemma 4.8 *For each prime p, we have*

(1) $E^2_{i,j}(SP^\infty(X) \times_T SP^\infty(cX))$

$$\cong \bigoplus_k H_i(SP^k(\Sigma X), SP^{k-1}(\Sigma X); \mathbb{F}_p) \otimes H_j(SP^\infty(X); \mathbb{F}_p).$$

(2) $E^2_{i,j}(\mathcal{U}_n(X))$

$$\cong \bigoplus_{k \leqq [\frac{n}{2}]} H_j(SP^k(\Sigma X), SP^{k-1}(\Sigma X); \mathbb{F}_p) \otimes \bigoplus_{k+2v \leqq n} H_j(SP^v(X), SP^{v-1}(X); \mathbb{F}_p).$$

(3) *The map of spectral sequences induced from* 4.9,

$$E^2_{i,j}(\mathcal{U}_n(X)) \to E^2_{i,j}(SP^\infty(X) \times_T SP^\infty(cX)),$$

is an injection.

(This is clear.)

The structure of differentials in the Serre spectral sequence for the quasi-fibering

$$\begin{array}{ccc} SP^\infty(Y) & \longrightarrow & SP^\infty(cY) \\ & & \downarrow \\ & & SP^\infty(\Sigma Y) \end{array}$$

is studied in [Sem, M 2]. On the other hand the quasi-fibering of 4.6 is induced from this quasi-fibering via the squaring map

$$\times 2: SP^\infty(\Sigma Y) \to SP^\infty(\Sigma Y), \qquad \times 2(\theta) = \theta^2,$$

and so we have a natural map of spectral sequences which determines the differentials in the spectral sequence for the total space $SP^\infty(Y) \times_T SP^\infty(cY)$. However, this observation does not necessarily determine the differentials when we use the spectral sequence of 4.8 on restricting the filtration of the Serre spectral sequence to the spaces $\mathcal{U}_n(Y)$ or the quotients $\mathcal{U}_n(Y)/\mathcal{U}_{n-1}(Y)$. This problem occurs typically when, for one reason or another, the first differential on an element, (which is predicted by comparing spectral sequences), is zero. Then we need a method of determining that all further differentials on that element will be zero as well. For this the following results will be quite useful.

Let M^{2n} be a closed, oriented, manifold. Suppose $* \in M^{2n}$ is chosen. Define an embedding

$$e\colon\ C^k(M^{2n} - *) \to C^{k+1}(M^{2n} - *)$$

by first deforming away from $*$, so as to assume all points of the k-tuples in $C^k(M^{2n} - *)$ lie outside a small neighborhood of $*$, then adjoin a new point in this small neighborhood.

Theorem 4.9 *The induced map in homology*

$$e_*\colon\ H_*(C^k(M^{2n} - *); \mathbb{Z}) \to H_*(C^{k+1}(M^{2n} - *); \mathbb{Z})$$

is an injection onto a direct summand.

Proof. Define a map

$$\phi_{k+1}\colon C^{k+1}(M^{2n}-*)\to SP^{\infty}(C^{k}(M^{2n}-*))$$

by $\langle x_1 \dots x_{k+1}\rangle \mapsto \prod_{i=1}^{k+1} \langle x_1 \dots \hat{x}_i \dots x_{k+1}\rangle$. Then

$$\phi_{k+1}\circ e_k(\langle x_1 \dots x_k\rangle)=\langle x_1 \dots x_k\rangle \circ (SP^{\infty}(e_k)\,\phi_k(\langle x_1 \dots x_k\rangle))$$

and the result follows by an easy induction. □

Remark 4.10 4.9 is a special case of many results already in the literature. In particular there is a stable section for e_k and the relevant spaces stably split.

Corollary 4.11 (a) *M^{2n} as above, then the E^2-term of the induced spectral sequence (from* 4.11*) for $TP^k(M^{2n})/TP^{k-1}(M^{2n})$ is given by projecting onto the terms in* 4.11(2) *of exactly degree k.*
(b) *In the restricted spectral sequence, elements of the form $\theta\circ\gamma_i([M])$ cannot be in the images of differentials on elements not of the form $\theta'\,\gamma_i([M])$.*

(The first statement is clear. The second statement follows by duality from 4.9 when we realize that the dual of the inclusion is $\cup[M^{2n}]^*$.)

5 Closed, oriented surfaces

We consider, first the case $X=M_g^2$, the closed, oriented, surface of genus g. We have

$$H_i(M_g^2;\mathbb{Z})=\begin{cases}\mathbb{Z}^{2g} & \text{generators } e_1,\dots,e_{2g},\ te_i=2e_i\\ \mathbb{Z} & \text{generator } f,\ tf=2(f+\sum_{i<g} e_i e_{g+i}).\end{cases} \tag{5.1}$$

Here t is the map in homology induced from the squaring map $M_g^2\to SP^2(M_g^2)$ sending x to x^2 for all $x\in M_g^2$.

Next, note that if we are interested in only $\mathscr{T}_{*,n}(M_g^2)$, we need only study the restriction of the differentials in the Leray-Serre spectral sequence of 4.11 to the terms of exact filtration n,

$$E^2_{j,s}=\sum_{k\leqq\left[\frac{n}{2}\right]} H_j(SP^k(\Sigma X),SP^{k-1}(\Sigma X);\mathbb{F}_p)\otimes H_s(SP^{n-2k}(X),SP^{n-2k-1}(X);\mathbb{F}_p). \tag{5.2}$$

To begin the explicit calculations we check first when $\mathbb{F}=\mathbb{Q}$, the rationals. Then $H_{i,j}(SP^{\infty}(\Sigma M_g^2);\mathbb{Q})=\mathbb{Q}[h_1,\dots,h_{2g}]\otimes E[\sigma f]$. Here, $\partial h_i=2e_i$ has lower second filtration, hence is not seen in (5.2). On the other hand, $\partial\sigma f=2(f+\sum_{i<g} e_i\circ e_{g+i})$, and this gives the reduced differential

$$\partial(\sigma f)=2\sum_{i<g} e_i\otimes e_{g+i}=\mu_g$$

for (5.2). We thus obtain

Lemma 5.3 *Let* $A_i \subset E_i[e_1, \ldots, e_{2g}]$, *be the subgroup* $\mathrm{Ann}(\mu\circ)$, *and*

$$B_i = \mathrm{im}(\mu\circ) \cap E_i[e_1, \ldots, e_{2g}],$$
$$C_{2i} = \mathbb{Q}[h_1, \ldots, h_{2g}],$$
$$D_i = E_i[e_1, \ldots, e_{2g}]/B_i.$$

Then

$$\mathscr{T}_{*,n}(M_g^2) = \sum_{s+2i+j+2=n} f^s C_{2i} A_j \sigma f \oplus \sum_{s+2i+j=n} f^s C_{2i} D_j.$$

Proof. This is the result of the d^2-differentials, i.e. the E^3-term of the spectral sequence. To see that there are no further differentials, note that we can imbed the homology of $SP^\infty\left(\bigvee_1^{2g} S^1\right)$ in its chain complex as a commutative ring. Moreover, the generators appear in the correct bidegrees, and the differential $d^2(\sigma f) = 2(f + \sum_{i<g} e_i \circ e_{g+i})$, also embeds. Thus, the elements $A_j \sigma f$, D_j are infinite cycles, being represented by actual cycles. Also, the generators h_i are non-zero in $H_2(TP^2(M_g^2), M_g^2; \mathbb{Q})$, hence their products are infinite cycles in the spectral sequence. Similarly for $f \in H_2(M_g^2; \mathbb{Q})$. But this counts all the elements occuring at E^3, and 5.3 follows. □

Example 5.4 In the case of the torus M_1^2, we have $D_* = D_1 = \langle e_1 \rangle + \langle e_2 \rangle$, while $A_* = H_*(M_1^2, \mathbb{Q})$. We thus have

$$\mathscr{T}_{*,*'}(M_1^2; \mathbb{Q}) = \mathbb{Q}[f, h_1, h_2](D_1, \tilde{H}_*(M_1^2; \mathbb{Q})\, \sigma f)$$

where the bidegree of f is (2, 1), that for h_i is (2, 2), that for the elements of D_1 is (1, 1), and that for $A_1\, \sigma f$ is (4,3), while that for $A_2\, \sigma f$ is (5, 4).

Thus determining the $E^2_{*,*}$ term above, and consequently the groups $H_*(C^n(M_g^2 - \infty); \mathbb{Q})$ reduces to analyzing the differential algebra $E[e_1, \ldots, e_{2g}, \sigma f]$ with differential $d(\sigma f) = \mu$. We write

$$E[e_1, \ldots, e_{2g}, \sigma f] = E[e_1, \ldots, e_{2g}] \oplus E[e_1, \ldots, e_{2g}]\, \sigma f$$

and let $E_k[e_1, \ldots, e_{2g}, \sigma f]$ denote the submodule of elements of degree k in the generators $e_1, \ldots, e_{2g}$. Note that E_k has dimension exactly $\binom{2g}{k}$, 1 generates E_0, while the volume form $\omega_g = e_1 e_2 \ldots e_{2g}$ generates E_{2g}.

Since μ is homogeneous of degree 2, the differential now takes the form

$$(5.5) \qquad d = \bigoplus_k d_k(g) \quad \text{with} \quad d_k(g): E_k[e_1, \ldots, e_{2g}] \to E_{k+2}[e_1, \ldots, e_{2g}]$$

and $d_k(x) = x\mu$.

We will study $d_k(g)$ by using induction on both k and g. It turns out that the (co)kernels of $d_k(g)$ are determined by the (co)kernels of $d_l(g-1)$ and $(d_l(g-1))^2$; the (co)kernels of $(d_l(g-1))^2$ are in turn determined by the (co)kernels

of $(d_m(g-2))^2$ and $(d_m(g-2))^3$; and so on. To facilitate the induction write $\mu = \sum_1^g x_i$ where $x_i = e_i \circ e_{g+i}$, so $\mu_g = \mu_{g-1} + x_g$, and

$$\begin{aligned}\mu_g^r &= \mu_{g-1}^r + r\mu_{g-1}^{r-1} x_g \\ &= r! \sum_{(l_1,\dots,l_r)} x_{l_1} \dots x_{l_r}\end{aligned}$$

where the sum runs overall all $L=(l_1, \dots, l_r)$ with $1 \leqq l_1 < \dots < l_r \leqq g$. In particular, for $r=g$ we have $\mu_g^g = g!\,\omega_g$.

For each $E_k[e_1, \dots, e_{2g}]$ the products $e_I = e_{i_1} \dots e_{i_k}$ with $I=(i_1, \dots, i_k)$ and $1 \leqq i_1 < i_2 < \dots < i_k \leqq 2g$ form the cannonical basis. There are 4 types of basis elements particularly adapted to the proposed induction

(5.6)
(1) $(i_1, \dots, i_k)$ with no $i_l = g$ or $2g$,
(2) $(i_1, \dots, i_k)$ with g but not $2g$ in the list,
(3) $(i_1, \dots, i_k)$ with $2g$ but not g in the list,
(4) $(i_1, \dots, i_k)$ with both g and $2g$ in the list,

so $E_k[g] = E_k[g-1] \oplus E_{k-1}[g-1]\ e_g \oplus E_{k-1}[g-1]\ e_{2g} \oplus E_{k-2}[g-1]\ x_g$. It is direct now to write out the operator $\wedge\mu^r$ with respect to this decomposition. In matrix form we have

$$d_k(g)^r = \begin{pmatrix} \mu_{g-1}^r & 0 & 0 & r\mu_{g-1}^{r-1} \\ 0 & \mu_{g-1}^r & 0 & 0 \\ 0 & 0 & \mu_{g-1}^r & 0 \\ 0 & 0 & 0 & \mu_{g-1}^r \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & A' \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{pmatrix}. \tag{5.7}$$

Lemma 5.8 [BC] *For $0 \leqq k \leqq 2g$ and $0 \leqq r \leqq g$ the differential $d_k(g)^r$: $E_k[g] \to E_{k+2r}[g]$ is a*

(I) *monomorphism for $k < g-r$,*
(II) *isomorphism for $k = g-r$,*
(III) *epimorphism for $k > g-r$.*

Proof. For $g=1$, the only non-trivial differential, $d_0(1)$: $E_0[1] \to E_2[1]$, is an isomorphism. For $g \geqq 2$ and $k=0$, we have $d_0(g)^r(1) = \mu_g^r$, and hence $d_0(g)^r$ is monic.

Case I. If $k < g-r$, then A, A', B as well as C in (5.7) are all monomorphisms by hypothesis. Hence, from

$$0 = d_k(g)^r(a, b_1, b_2, c) = (A(a), B(b_1), B(b_2), A'(a) + C(c))$$

we conclude that $a = b_1 = b_2 = 0$, and so $c=0$ as well. Thus the operator is a monomorphism in this case.

Case II. If $k = g-r$, then A is an epimorphism, A' and B are isomorphisms, and C is a monomorphism. Assume that $d_k(g)^r(a, b_1, b_2, c) = (A(a), B(b_1), B(b_2), A'(a) + C(c)) = 0$. First, $b_1 = b_2 = 0$. We now have $A(a) = \mu_{g-1}^r a = 0$ and $\mu_{g-1}^r c = -r\mu_{g-1}^{r-1} a$ so $\mu_{g-1} c = -ra$ since μ_{g-1}^{r-1} is an isomorphism in this degree. Hence $\mu_{g-1}^{r+1} c = \mu_{g-1}^r(-ra) = A(-rA) = 0$, but by hypothesis μ_{g-1}^{g+1} is monic here. Thus,

$c=0$. Hence, $-r\mu_{g-1}^{r-1}a=0$, and therefore $a=0$ since μ_{g-1}^{r-1} is an isomorphism here. Thus the map is a monomorphism, but the dimensions of both range and domain are equal, so it is actually an isomorphism.

Case III. It $k>g-r$, then by hypothesis A, B and C are epimorphisms, and A' is a multiple of an epimorphism. Given $(\bar{a}, \bar{b}_1, \bar{b}_2, \bar{c}) \in E_{k+2r}[g]$, we can find first a, b_1, b_2 satisfying $A(a)=\bar{a}$, $B(b_1)=\bar{b}_1$, and $B(b_2)=\bar{b}_2$. We then choose c so that $C(c)=\bar{c}-A'(a)$, and we see that we have an epimorphism in this case. □

Corollary 5.9 *The ranks of the homology groups $H_i(E[e_1, \ldots, e_2 g, \sigma f]; d)$ are given as follows,*

$$\begin{cases} \binom{2g}{i}-\binom{2g}{i-2} & \text{for } i=0,1,\ldots,g, \\ \binom{2g}{i}-\binom{2g}{i+2} & \text{for } i=g,g+1,\ldots,2g, \\ 0 & \text{in all other dimensions.} \end{cases}$$

Note the apparent duality between H_i and H_{2g-i+1}.

Next, turning to the field $\mathbb{F}_p$ for p an odd prime, we see that the differentials discussed above are still present, but the situation is more complex for the new generators, $\beta^\varepsilon P^{[i]}(\sigma f)^*$, $P^{[i]}(\sigma f)^*$. We have that $P^{[i]}(\sigma f)^*$ transgresses to the fiber, where its differential is the mod p reduction of the class $\gamma_p(f)$, a class characterized by the property that $p!\,\gamma_p(f)=f^p$ in the multiplication induced from the product pairing $SP^\infty(X)\times SP^\infty(X)\to SP^\infty(X)$. We have

$$(f+\mu)^p=\sum\binom{p}{k}f^k\mu^{p-k}=\sum\frac{p!}{k!}\gamma_k(f)\,\mu^{p-k},$$

and $\mu^{p-k}=(p-k)!\,\gamma_{p-k}(\mu)$, where

$$\gamma_{p-k}(\mu)=\sum_{i_1<i_2<\ldots<i_{p-k}\leqq g}\chi_{i_1}\otimes\chi_{i_2}\otimes\ldots\otimes\chi_{i_{p-k}},\qquad \chi_i=e_i\otimes e_{i+g}$$

so, with no indeterminacy, since $H_*(SP^\infty(M_g^2);\mathbb{Z})$ is torsion free, we have that

$$\gamma_p(f+\mu)=\sum\gamma_k(f)\,\gamma_{p-k}(\mu),$$

and factoring out by terms of lower filtration, we have that in the spectral sequence for $TP^{2p}(M_g^2)/TP^{2p-1}(M_g^2)$,

$$d_{2p+1}(P^{[1]}(\sigma f))=\gamma_p(\mu)\bmod(\text{the ideal generated by }\mu).$$

We have

Lemma 5.10 *The element $\gamma_p(\mu_g)\not\equiv 0$ mod (the ideal generated by μ), for $2p-1\leqq g$ and $p\geqq 3$, but $\gamma_p(\mu_{p+1})\in$ this ideal.*

Proof. There are $\frac{(2p-1)(2p-2)}{2}$ distinct monomials in the χ_i in $\gamma_p(\mu)$ for M^2_{2p-1}. On the other hand a basis for the part of the ideal spanned by the χ_j's in this dimension is given by all elements of the form

$$\chi_{i_1} \otimes \ldots \otimes \chi_{i_{p-1}} \mu,$$

and each of these consists of a sum of exactly p monomials. But since p and $\binom{2p-1}{2}$ are relatively prime we see that the best that could happen in terms of writing a multiple of $\gamma_p(\mu)$ as an element in the ideal, is that the multiple be divisible by p. Thus the result is true for this genus. Now, for any $g > 2p-1$, project $(S^1)^{2g} \to (S^1)^{4p-2}$, by projecting onto the first $2p-1$ coordinates and the coordinates from $g+1$ to $g+2p$ project to the next $2p-1$ coordinates. This map is a homomorphism, and, in homology, satisfies $\mu_g \mapsto \mu_{2p-1}$. Hence it preserves the ideals, and takes $\gamma_k(\mu_g)$ to $\gamma_k(\mu_{2p-1})$. This proves the first statement of 5.10.

The second statement is similar. The generators for the ideal

$$(\mu_{p+1}) \cap E_p(\chi_1, \ldots, \chi_{p+1})$$

are given by the classes $\chi_{i_1} \otimes \ldots \otimes \chi_{i_{p-1}} \otimes \mu$, and these give you monomials 2 at a time. On the other hand $\gamma_p(\mu_{p+1}) = \sum_1^{p+1} \chi_1 \otimes \ldots \hat{\chi}_i \ldots \otimes \chi_{p+1}$. Now, note that

$$\gamma_p(\mu_{p+1}) = \sum_{i=1}^{\frac{p+1}{2}} \chi_1 \otimes \ldots \hat{\chi}_{2i-1} \otimes \hat{\chi}_{2i} \ldots \otimes \chi_{p+1} \otimes \mu.$$

5.10 follows. □

We do not know what happens in the range between $p+1$ and $2p-1$, nor what happens to the elements $\gamma_{p^i}(\mu)$ relative to the ideal generated by

$$\mu, \gamma_p(\mu), \ldots, \gamma_{p^{i-1}}(\mu).$$

The next classes to study are the $\beta P^{[i]}(\sigma f)^*$. In the general spectral sequence the Kudo transgression theorem shows that

$$d_{2p^{i-1}(p-1)+1}(\beta P^{[i]}(\sigma f)^*) = (d_{2p^{i-1}+1}(P^{[i-1]}(\sigma f)^*)^{p-1} \otimes (P^{[i-1]}(\sigma f)^*).$$

The first of these is thus given by

$$\partial_{2p-1}(\beta P^1(\sigma f)^*) = \mu^{p-1} \sigma f. \tag{5.11}$$

This differential is non-zero exactly when $p-1 \leqq g$.

But we also have that the bidegree of $\beta P^{[i]}(\sigma f)^*$ is $(2p^i+2, 2p^i)$, and, among those terms of second degree exactly n, the largest dimension which can occur without the dual of a power of f^* being present is $n+2g+1$. But the differentials raise the part of the dimension involved with fiber terms while lowering dimension involved with h_j, σf, etc. on the base. And this raise can be at most $2g$

on the fiber until we involve powers of f^*. Thus, there can be no non-trivial differential of degree higher than $2g$, and we have

Lemma 5.12 *Let $p \geqq g+1$, then $E^3 = E^\infty$ for the spectral sequence converging to $H_*(TP^k(M_g^2)/TP^{k-1}(M_g^2); \mathbb{F}_p)$. Also, for $p^i \geqq g+1$ the terms $\beta P^{[i]}(\sigma f)^*$ and $P^{[i]}(\sigma f)^*$ are infinite cycles.*

Hence

$$\mathscr{T}_{*,*'}(M_g^2; \mathbb{F}_p) = V \otimes \Gamma_{i > \log_p(g)}[\ldots, P^{[i]}(\sigma f)\ldots] \otimes E_{i > \log_p(g)}[\ldots \beta P^{[i]}(\sigma f), \ldots]$$

where V is obtained from the differential $\partial \sigma f = 2\sum_1^g e_i \circ e_{i+g}$ and the higher differentials discussed above when $p < g$.

Remark 5.13 Using the results above the homology structure of the spaces $C^n(M_g^2 - \infty)$ seems not too tedious, at least for special cases. However, to obtain the groups $H_*(C^n(M_g^2); \mathbb{F}_p)$, we must consider the portion of the spectral sequence involving (5.2) for both n and $n-1$. Here, the full differential

$$\partial \sigma f = 2\left(f + \sum_1^g e_i \circ e_{i+g}\right), \qquad \partial h_i = 2e_i,$$

is required. This changes the result somewhat, (in particular all possible differentials now occur), but, once more, it is easy to handle the necessary modifications in specific cases.

6 $m-1$ connected $2m$-manifolds

The next class of spaces we will look at are $m-1$-connected, closed manifolds X of dimension $2m$, $m \geqq 2$, so for some finite $r > 0$

$$(6.1)\quad \tilde{H}_*(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}^r & * = m, \text{ generators } h_1, \ldots, h_r, \Delta h_j = h_j \otimes 1 + 1 \otimes h_j \\ \mathbb{Z} & * = 2m, \text{ generator } L, \Delta L = L \otimes 1 + 1 \otimes L + \sum a_{k,s}(h_k \otimes h_s) \\ 0 & \text{otherwise.} \end{cases}$$

We can always assume we are working over a ring containing $\frac{1}{2}$, so we have

Lemma 6.2 *In (6.1) if m is odd, then r is even, and for both even and odd m we can change variables h_i over $\mathrm{GL}_i(\mathbb{Z}(\frac{1}{2}))$ so that*

$$a_{k,s} = \begin{cases} 0 & k \neq s \text{ if } m \text{ is even} \\ 0 & (k,s) \neq (2j, 2j-1) \text{ or } (2j-1, 2j),\ 1 \leqq j \leqq \left(\frac{i}{2}\right) \text{ if } m \text{ is odd} \\ -1 & (k,s) = (2j, 2j-1) \text{ if } m \text{ is odd} \\ 1 & (k,s) = (2j-1, 2j) \text{ if } m \text{ is odd.} \end{cases}$$

Moreover, in case m is even the terms $\alpha_{j,j}$ are all units in $\mathbb{Z}(\frac{1}{2})$.

Proof. Since X is a closed Poincaré duality complex the cup product from

$$H^m(X;\mathbb{Z}(\tfrac{1}{2}))\otimes H^m(X;\mathbb{Z}(\tfrac{1}{2}))\to\mathbb{Z}(\tfrac{1}{2})$$

defined by $\langle a,b\rangle=\langle a\cup b,[X]\rangle$ is non-singular, and symmetric for m even, skew-symmetric for m odd. It is now standard reduction theory for bilinear forms to reduce to the forms in 6.2. That is, when m is even the form is diagonalizable, while when m is odd it becomes

$$\begin{pmatrix} J & 0 & 0 & \dots & 0\\ 0 & J & 0 & \dots & 0\\ . & . & . & \dots & .\\ 0 & 0 & 0 & \dots & J\end{pmatrix} \quad \text{where } J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

The fact that when m is even each term in the diagonalized form is a unit follows since the determinant for the original cup product form is ± 1. Hence, the determinant of the diagonalized form is a unit over $\mathbb{Z}(\frac{1}{2})$.

Corollary 6.3 *When m is even, the forms which occur can be chosen so that $\alpha_{j,j}=\pm 1$, $1\leqq j\leqq i$.*

Proof. First, the units of $\mathbb{Z}(\frac{1}{2})$ have the form $\pm(2^s)$, $-\infty<s<\infty$, so $U\mathbb{Z}(\frac{1}{2})=\mathbb{Z}/2\times\mathbb{Z}$, and $U/U^2=\mathbb{Z}/2\times\mathbb{Z}/2$ with generators $\pm 1, \frac{1}{2}$. Hence, we can suppose every entry $\alpha_{j,j}$ is ± 1 or $\dfrac{\pm 1}{2}$. By the change of variables $e\mapsto e+f$, $f\mapsto e-f$ the form $\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$ becomes $\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$. Similarly we can change $\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}$ to $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. This cuts us down to forms with all ± 1's along the diagonal except perhaps for a single $\dfrac{\pm 1}{2}$. But the latter situation cannot occur, since the original discriminant was ± 1. □

In the situation of 6.2, 6.3 the rational calculation becomes quite simple. We have that the E_2 terms are given by

$$E_2=\begin{cases}\mathbb{Q}[h_1,\dots,h_i,L)\otimes E[\sigma h_1,\dots,\sigma h_i,\sigma L] & m \text{ even}\\ \mathbb{Q}[L,\sigma h_1,\dots,\sigma h_i]\otimes E[\sigma L,h_1,\dots,h_i] & m \text{ odd.}\end{cases} \tag{6.4}$$

There is only one differential which does not lower bidegree since the h_i are spherical in X. It is given as

$$\partial(\sigma L)=\begin{cases}\sum\limits_k \alpha_{k,k}h_k^2 & m \text{ even},\\ 2\sum h_{2j-1}h_{2j} & 1\leqq j\leqq\left(\dfrac{i}{2}\right) \text{ if } m \text{ is odd.}\end{cases} \tag{6.5}$$

When m is even the homology of the complex given by (6.4), (6.5) is additively isomorphic to the (graded) ring

$$\mathbb{Q}[h_2,\dots,h_i]\otimes E[h_1]\otimes E[\sigma h_1,\dots,\sigma h_i] \tag{6.6}$$

with the multiplication given by $h_1^2 = \pm h_2^2 + \ldots + \pm h_i^2$. On the other hand, the complex in (6.4), (6.5) for m odd is exactly the one which has already appeared and been analyzed in § 5.

Remark 6.7 The situation at odd primes is similar to that for surfaces. But it is actually quite a bit worse from two respects. First, the ideals are much harder to manage, and second, there are now an infinite number of non-zero differentials.

7 Alternating products and deleted products of manifolds

In this section we give an elementary proof of the main results of [B-C-T], in case M is an orientable, closed manifold without boundary by using simple properties of alternating products, Poincaré duality, and some fundamental results of Dold, [D2]. The case where $\partial M \neq \emptyset$ can also be analyzed in this way, but the method does not work at all if M is not orientable.

Set $AP^n(X) = X^n/\mathscr{A}_n$, where $\mathscr{A}_n$ is the alternating group on n-letters. We also explicitly calculate the homology with coefficients $\mathbb{F}_p$, p an odd prime, of the alternating products $AP^n(X)$ for all connected spaces X having the homotopy types of CW-complexes. Let $\mathbb{F}^{\pm}$ denote the field $\mathbb{F}$ with an action of $\mathbb{Z}/2$ given by $a \mapsto \pm a$. The connection between our previous work and the $AP^n(X)$ is given by

Lemma 7.1 *Let X be a simplicial complex, then for all odd primes, p, there is a split exact sequence*

$$0 \to H_*(TP^n(X)/TP^{n-2}(X); \mathbb{F}_p^-) \to H_*(AP^n(X); \mathbb{F}_p) \to H_*(SP^n(X); \mathbb{F}_p^+) \to 0.$$

Proof. First, a map $e\colon SP^n(X) \to SP^2(AP^n(X)) \to SP^\infty(AP^n(X))$ is defined by

$$\langle x_1, \ldots, x_n \rangle \mapsto \langle [x_1, x_2, x_3, \ldots, x_n][x_2, x_1, x_3, \ldots, x_n] \rangle$$

where $[x_1, \ldots, x_n]$ denotes the equivalence class of $(x_1, \ldots, x_n)$ in $AP^n(X)$. The composition

$$AP^n(X) \xrightarrow{p} SP^n(X) \xrightarrow{e} SP^\infty(AP^n(X))$$

is $(i + Ti)$, where $T\colon AP^n(X) \to AP^n(X)$ is the evident $\mathbb{Z}/2$-action, and

$$i\colon AP^n(X) \to SP^\infty(AP^n(X))$$

is the usual inclusion. Since $T^2 = 1$ on $AP^n(X)$ we have a direct sum splitting

$$H_*(AP^n(X); \mathbb{F}_p) = H_*(AP^n(X); \mathbb{F}_p)^+ \oplus H_*(AP^n(X); \mathbb{F}_p)^-,$$

and on $H_*(AP^n(X); \mathbb{F}_p)^+$ the composite is just multiplication by 2, while the composite $SP^\infty(p) \circ e\colon H_*(SP^n(X); \mathbb{F}_p) \to H_*(SP^\infty(SP^n(X)); \mathbb{F}_p)$ is also just multiplication by 2. The identification of $H_*(AP^n(X); \mathbb{F}_p)^+$ with $H_*(SP^n(X); \mathbb{F}_p)$ follows.

It remains to study $H_*(AP^n(X); \mathbb{F}_p)^-$. First a simplicial decomposition of X^n is easily given (see e.g. [M2]), so that the simplices σ_i are either pointwise fixed by an element $\alpha \in \mathscr{S}_n$ or $\alpha\sigma_i \cap \sigma_i = \emptyset$. This gives rise to a cellular decomposi-

tion of $AP^n(X)$ so the singular set is a subcomplex. Since $T \equiv 1$ precisely on the cells of the singular set we see that $H_*(AP^n(X); \mathbb{F}_p)^-$ is given as the homology of the subcomplex $\frac{(T-1)}{2} \mathscr{C}_\#(AP^n(X)) \otimes \mathbb{F}_p$, where by $\mathscr{C}_\#$ we mean the chain complex, and this is exactly

$$\mathscr{C}_\#(SP^n(X)/\mathrm{Sing}^n(X)) \otimes \mathbb{F}_p^- .$$

7.1 follows. □

On the other hand, the main result of [D2] shows that $H_*(AP^n(X); \mathbb{F}_p)$ depends only on $H_*(X; \mathbb{F}_p)$, and we have immediately from 3.2

Corollary 7.2 [B-C-T] *Let X be a compact, connected and oriented manifold without boundary, then $H_*(C^n(X); \mathbb{F}_p)$ is a functor only of $H_*(X; \mathbb{F}_p)$ if X is odd dimensional and p is an odd prime, but also, $H_*(C^n(X); \mathbb{F}_p^-)$ is a functor only of $H_*(X; \mathbb{F}_p)$ if the dimension is even.*

It remains to describe the functors in 7.2. Here, again, [D2] reduces the problem to a much simpler one. In fact, we have

Lemma 7.3 (a) *For any odd prime p, to calculate $H_*(AP^n(X); \mathbb{F}_p)$ it suffices to calculate $H_*(AP^n(Y); \mathbb{F}_p)$ where Y is a wedge of spheres.*
(b) *If $Y = Z \vee W$ then*

$$\begin{aligned} H_*(AP^n(Y); \mathbb{F}_p) \cong & \sum_{j=0}^{n} H_*(SP^j(Z); \mathbb{F}_p) \otimes H_*(SP^{n-j}(W); \mathbb{F}_p) \\ & \oplus \sum_{j=0}^{n} H_+(TP^j(Z)/TP^{j-2}(Z); \mathbb{F}_p^-) \\ & \otimes H_*(TP^{n-j}(W)/TP^{n-j-2}(W); \mathbb{F}_p^-). \end{aligned}$$

(c) *If Y is a sphere S^n, then*

$$H_*(C^m(S^n); \mathbb{F}^\varepsilon) \cong H_*(C^m(\mathbb{R}^n); \mathbb{F}^\varepsilon) \oplus \Sigma^n H_*(C^{m-1}(\mathbb{R}^n); \mathbb{F}^\varepsilon)$$

where, $\varepsilon = +$ if n is odd $\varepsilon = -$ for n even.

Proof. The first statement is direct from [D2]. Specifically, Dold shows that any two CW-complexes, L, M, with the same $\mathbb{F}_p$ homology will satisfy $H_*(AP^n(L); \mathbb{F}_p) \cong H_*(AP^n(M); \mathbb{F}_p)$. To see (b), note that

$$\begin{aligned} AP^n(Z \vee W) = & \bigvee_{j=1}^{n-1} AP^j(Z) \times AP^{n-j}(W)/\{(x, y) \sim (Tx, Ty)\} \\ & \vee AP^n(Z) \vee AP^n(W) \end{aligned} \tag{7.4}$$

where T is the involution on $AP^j(\ )$. Next, apply 8.1 to each of the summands. Tensoring the homology groups and taking the invariant subgroup under $T \times T$ gives the result.

Finally, to show (c), we again use a result from [D2]. In [D2, §9, especially p. 76], Dold shows that if we filter $AP^n(X)$, by saying that

$\{x_1, \ldots, x_n\} \in \mathbb{F}_r(AP^n(X))$ if and only if there are at least r copies of the base point $*$ in $\{x_1, \ldots, x_n\}$, then

$$\tilde{H}_*(AP^n(X); \mathbb{F}_p) = \sum_0^{n-1} \tilde{H}_*(\mathbb{F}_r(AP^n(X))/\mathbb{F}_{r+1}(AP^n(X)); \mathbb{F}_p).$$

In particular, the filtering pieces for $AP^n(X)$ are given explicitly as

$$\begin{cases} SP^{n-r}(X)/SP^{n-r-1}(X) & r \geqq 2 \\ AP^{n-1}(X)/\mathbb{F}_1(AP^{n-1}(X)) & r = 1 \\ AP^n(X)/\mathbb{F}_1(AP^n(X)) & r = 0. \end{cases}$$

Now, applying 7.1 and our usual duality techniques, it follows that, for n odd, and p an odd prime, $H_*(AP^m(S^n)/\mathbb{F}_1(AP^m(S^n)); \mathbb{F}_p)^- = H^{nm-*}(C^m(S^n - *); \mathbb{F}$, while for n even we have $H_*(AP^m(S^n)/\mathbb{F}_1(AP^m(S^n)); \mathbb{F}_p)^- = H^{nm-*}(C^m(S^n - *); \mathbb{F}_p^-)$. This completes the proof of 7.3. □

Thus, it suffices to determine the groups $H_*(C^m(\mathbb{R}^n); \mathbb{F}_p^{\pm})$. Let $\mathbb{F}(X, k)$ denote the configuration space of ordered k-tuples of distinct points in X. Probably the easiest way to make this calculation is to use the well known calculations of the cohomology of the spaces

$$D_{n,m}(S^l) = \mathbb{F}(\mathbb{R}^n, m)_+ \wedge_{\mathscr{S}_m} (S^l)^{(m)}$$

as split summands of the homology of the loop space $\Omega^n S^{n+l}$. In [May] it was shown that a model for $\Omega^n S^n(X)$ was given as a filtered object by the construction

$$J_n(X) = \coprod_1^{\infty} \mathbb{F}(\mathbb{R}^n, k) \times_{\mathscr{S}_k} X^k/(z_1, \ldots, z_k, x_1, \ldots, x_k)$$
$$\sim (z_1, \ldots, \hat{z}_i, \ldots, z_k, x_1, \ldots, \hat{x}_i, \ldots, x_k) \quad \text{whenever } x_i = *,$$

for connected X homotopy equivalent to CW-complexes and [Sn] showed that stably $J_n(X) \simeq \bigvee_1^{\infty} D_{n,m}(X)$. $H_*(\Omega^n \Sigma^n X; \mathbb{F})$ was calculated in [M3] for any connected CW-complex X. On the other hand, $D_{n,m}(S^l)$ is just the Thom complex of the vector bundle

$$\mathbb{R}^{ml} \to \mathbb{F}(\mathbb{R}^n, m) \times_{\mathscr{S}_m} (\mathbb{R}^l)^m \to C^m(\mathbb{R}^n),$$

and as a result we have

$$H_{lm+*}(D_{n,m}(S^l); \mathbb{F}_p) \cong \begin{cases} H_*(C^m(\mathbb{R}^n); \mathbb{F}_p^+) & l \text{ even} \\ H_*(C^m(\mathbb{R}^n); \mathbb{F}_p^-) & l \text{ odd}, \end{cases} \tag{7.5}$$

so the calculation is complete.

Remark 7.6 The arguments in [D2] are quite elementary, involving, almost entirely, formal properties of semi-simplicial complexes. As the exposition given

there can hardly be improved on, we will not summarize the proofs here, but urge the interested reader to read [D2].

Remark 7.7 The splitting in 7.3(c) does not hold when n is even and ε is $+$, since, in the spectral sequence of type 4.11 converging to $H_*(TP^m(S^{2n})/TP^{m-2}(S^{2n}); \mathbb{F}_p)$, the differential $d_2(\sigma f) = 2f$ is non-trivial. This is already seen in the remarks after (5.2). When we are looking at $H_*(TP^m(S^{2n})/TP^{m-1}(S^{2n}); \mathbb{F}_p)$, we do not see this differential, and $E^2 = E^\infty$.

Remark 7.8 In some ways the calculation indicated above for $H_*(C^m(\mathbb{R}^n); \mathbb{F}_p)$ is unaesthetic. The point of view, here, is that the structure of symmetric products should be logically prior to the homology of iterated loop spaces, although the two points of view are essentially equivalent. (In hindsight we can see that the essential techniques for both approaches lie in the results of [D-T] on the criteria for a map to be a quasi-fibration.) In any case, in the next few paragraphs we give a direct method for obtaining the twisted homology groups above.

Lemma 7.9 *There is a natural injection*

$$e_n: H_*(SP^n(X)/\{\mathrm{Sing}^n(X) \cup SP^{n-1}(X)\}; \mathbb{F}_p^-) \to H_{*+n}(SP^n(\Sigma X)/SP^{n-1}(\Sigma X); \mathbb{F}_p^+)$$

for X any connected CW*-complex.*

Outline proof. One begins by replacing X by a simplicial complex, then replaces X^n by a simplicial complex for which each $\alpha \in \mathscr{S}_n$ satisfies either $\alpha\sigma \cap \sigma = \emptyset$ or $\alpha\sigma = \sigma$ and σ fixes σ pointwise for each simplex σ in X^r. In particular, the images of these simplices give cell decompositions for X^n/Γ, for all subgroups $\Gamma \subset \mathscr{S}_n$.

Next a *cellular* decomposition of $(\Sigma X)^n$ is given. Its cell are (generally) the products of the cells of X^n with I^n, but when the cells of X^n are in the singular set (necessarily a subcomplex by our assumptions above), more care must be taken. A simplicial decomposition of I^r is given, with the r-cells indexed by $\mathscr{S}_r$ as follows

$$\sigma_\alpha^r = \{(t_1, \ldots, t_r) \mid t_{\alpha(1)} \leqq t_{\alpha(2)} \leqq \cdots \leqq t_{\alpha(r)}\}.$$

Then, over the singular set, note the ordering of the suspension coordinates, so there the cells have the form $\sigma \times I^{n-r} \times \sigma_{\alpha_1}^{r_1} \times \ldots \times \sigma_{\alpha_v}^{r_v}$ with $r = r_1 + \ldots + r_v$, σ a simplex in the decomposition of X^n, and the simplexes $\sigma_{\alpha_i}^{r_i}$ occuring where the coordinates in σ are equal.

We now take the images of the cells above in $SP^n(\Sigma X)$, and study the resulting chain complex. A direct calculation shows that if $\sigma \notin \mathrm{Sing}^n(X)$, but a face $F_i(\sigma) \in \mathrm{Sing}^n(X)$ then in $\partial(\sigma \times I^n)$ the coefficient involving $F_i(\sigma) \times (?)$ is always 0. Moreover, the remainder of the boundary, is $(\partial_-(\sigma)) \times I^n$, where ∂_- denotes the boundary with twisted coefficients. It follows that the chain complex of $SP^n(X)/\mathrm{Sing}^n(X)$ with twisted coefficients, but suspended up n dimensions, is a direct summand of a chain complex for $SP^n(\Sigma X)$.

Finally, we should note the collapsing which occurs when we are at base point. This lowers dimension by at least 1 and forces us to include the basepoint conditions in 7.9. The result follows. □

As an example, we easily check that $AP^n(S^1)/AP^{n-1}(S^1) \simeq S^n$, so

$$H_*(SP^n(S^1)/\{\mathrm{Sing}^n(S^1) \cup SP^{n-1}(S^1)\}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & *=n \\ 0 & \text{otherwise,} \end{cases}$$

and this class suspends up to the generator for $H_{2n}(SP^n(S^2); \mathbb{Z}) = \mathbb{Z}$.

Next, an easy induction, based on the bar construction shows

Lemma 7.10 *When X is S^n the injection e_n of 7.9 is an isomorphism.*

Thus, we obtain a complete calculation of the twisted homology groups needed in 7.3, entirely within the context of symmetric products and Eilenberg-MacLane spaces.

Part II: Configuration spaces and mapping spaces

The purpose of this part is to continue the observations of [B-C-T] giving the homology of some configuration spaces. As at the end of § 7 we write $\mathbb{F}(M, k)$ for the subspace of M^k given by $\{(m_1, \ldots, m_k) | m_i \neq m_j \text{ if } i \neq j\}$ and as usual $C^k(M)$ for the orbit space $\mathbb{F}(M, k)/\mathcal{S}_k$ where the symmetric group $\mathcal{S}_k$ acts by permuting coordinates. Throughout Part II M is assumed to be a smooth (not necessarily closed) manifold and all homology is taken with coefficients in the field $\mathbb{F}$. With $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ and M closed, the homology of $C^k(M)$ was first given in [L-M].

Among the main results here is a short proof of the results in [B-C-T] giving the homology of $C^k(M)$ if at least one of the following is satisfied: (1) $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, (2) dimension$(M)$ is odd, or (3) dimension(M) is even and $\mathcal{S}_k$ acts on $\mathbb{F}$ by the sign representation. Also product decompositions for certain function spaces give results for surfaces which, in rational cohomology, generalize to $(n-1)$-connected $2m$-manifolds. These product decompositions give the cohomology of certain other configuration spaces.

8 Results

Let V denote a graded vector space over $\mathbb{F}$ with V concentrated in degrees at least one. Write

$$V^{\otimes k} = V \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} V$$

as a $\mathcal{S}_k$-module where $\mathcal{S}_k$ acts by permutation of coordinates with the usual sign conventions. If $v_i \in V$ write V_α for the cyclic $\mathcal{S}_k$-module generated by $\alpha = v_1 \otimes \ldots \otimes v_k$. There are 3 important examples of $\bar{V}_\alpha$ here:

(i) $V = \mathbb{F}$ concentrated in an even degree. Thus $\bar{V}_\alpha$ is a trivial $\mathcal{S}_k$-module.
(ii) $V = \mathbb{F}$ concentrated in an odd degree. Thus $\bar{V}_\alpha$ is given by the sign representation of $\mathcal{S}_k$, and we write $\bar{V}_\alpha = \mathbb{F}(-1)$ here.
(iii) The dimension of V is at least k and $v_1, \ldots, v_k$ are linearly independent. Then $\bar{V}_\alpha$ where $\alpha = v_1 \otimes \ldots \otimes v_k$ is isomorphic to $\mathbb{F}\mathcal{S}_k$ as a $\mathcal{S}_k$-module.

Next write $S_* X$ for the singular chains of a space X with a non-degenerate base point, and define

$$H_*(C^k(M); \bar{V}_\alpha) = H_*(S_* \mathbb{F}(M,k) \otimes_{\mathscr{S}_k} \bar{V}_\alpha).$$

If S_V is a bouquet of spheres with $\bar{H}_*(S_V)$ isomorphic to V as a graded vector space, then there are isomorphisms (over any field)

$$H_*(C^k(M); V^{\otimes k}) \cong H_* D_k(M, S_V)$$

where $D_k(M, S_V)$ is the quotient space

$$\mathbb{F}(M,k) \times_{\mathscr{S}_k} X^{(k)} / \mathbb{F}(M,k) \times_{\mathscr{S}_k} *$$

with $*$ the base-point in X and $X^{(k)}$ is the k-fold smash product. Furthermore, there are isomorphisms

$$H_*(C^k(M); V^{\otimes k}) \cong \bigoplus_\alpha H_*(C^k(M); V_\alpha)$$

where the right hand sum is over the obvious index set. For example if either (1) $M = \mathbb{R}^n$ or (2) if $M = M' \times \mathbb{R}^n$ and $\mathbb{F} = \mathbb{Q}$, the groups $H_*(C^k(M); V_\alpha)$ are implicit in [CT1].

We now determine $H_*(C^k(M); V^{\otimes k})$ as a direct summand of the homology of a much larger space $C(M, X)$. Namely, write

$$C(M,X) = \left\{ [S, f] \,\middle|\, \begin{matrix} (1)\ S \text{ is a finite subset of } M \\ (2)\ f : S \to X \end{matrix} \right\} / (\sim)$$

where $(\sim)$ is the equivalence relation generated by $[S, f] \sim [S - \{q\}, f|_{S-\{q\}}]$ if and only if $f(q) = *$. The space $C(M, X)$ stably splits as $\bigvee_{k \geq 1} D_k(M, X)$ for path-connected X, [B2], (the original reference here is an unpublished article of F. Cohen and L. Taylor but their proof is reproduced here). The space $C(M, X)$ is filtered with $F_j C(M, X)$ represented by $[S, f]$ where S is of cardinality at most j; similarly $H_*(C(M, X))$ has an induced filtration. Thus there are isomorphisms

$$\begin{aligned} \bar{H}_* C(M,X) &\cong \bigoplus_{k \geq 1} \bar{H}_* D_k(M,X) \\ &\cong \bigoplus_{k \geq 1} H_*(C^k(M); \bar{H}_* X^{\otimes k}). \end{aligned}$$

We shall check the following two observations.

Lemma 8.1 *There are isomorphisms of vector spaces*

$$\bar{H}_q D_k(M,X) \to \bar{H}_{q+2nk} D_k(M, \Sigma^{2n} X).$$

Lemma 8.2 *Assume that X is $(q-1)$-connected, $H_j(X; \mathbb{F}) = 0$ if $j > N$ and the dimension of M is m. Then*

(1) *$D_t(M, X)$ is $(qt-1)$-connected,*
(2) *$H_j(D_t(M, X); \mathbb{F}) = 0$ if $j > t(m+N)$,*

(3) *If ϕ is any homology isomorphism of $H_* C(M, X)$, then ϕ also preserves filtration degrees in homological degree less than $t(m+N)$ if $t(m+N-q)<q$, and*
(4) *If $n \gg N-q$ there is an isomorphism*

$$H_s D_t(M, X) \cong H_{s+2nt} C(M, \Sigma^{2n} X)$$

provided $H_s D_t(M, X)$ is non-zero.

The point of 8.1 and 8.2 is that we know $H_* D_t(M, X)$ and hence we have

$$H_*(C^t(M); V^{\otimes t})$$

provided that we know $H_* D_t(M, X)$ and thus $H_* C(M, \Sigma^{2n} X)$ as a vector space for all n. In some cases such as [B-C-T] these answers follow directly.

In particular if $D^m = [0, 1]^m$ with (1) $\bar{M} = M \cup D^m$ and (2) $M \cap D^n = [0, 1]^{n-q} \times \partial [0, 1]^q$ then we have by following work of Segal, Gromov, Dold and Thom

Proposition 8.3 [MD, B1] *If X is a connected* CW*-complex then up to homotopy equivalence there are fibrations*

$$C(M, X) \to C(\bar{M}, X) \to \Omega^{m-q} \Sigma^m X.$$

As a corollary we get the following where $\beta_q = \dim_{\mathbb{F}} H_q(M; \mathbb{F})$.

Corollary 8.4 [B-C-T] (i) *If $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ or m is odd, then $H_* C^k(M)$ depends only on $H_* M$ and m. Furthermore there is an isomorphism of vector spaces for $n>>0$ and $0<q<mk$ given by*

$$\begin{aligned} H_q C^k(M) &\cong H_{q+2nk} C(M, S^{2n}) \\ &\cong H_{q+2nk}\left(\prod_{q=0}^{m} \prod_{\beta(q)} \Omega^{m-q} S^{m+2n}\right). \end{aligned}$$

(ii) *if $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ or m is even, then $H_*(C^k(M); \mathbb{F}(-1))$ depends only on $H_* M$ and m. Furthermore, there is an isomorphism of vector spaces for $n>>0$ and $0<q<mk$ given by*

$$\begin{aligned} H_q(C^k(M); \mathbb{F}(-1)) &\cong H_{q+(2n+1)k} C(M, S^{2n+1}) \\ &\cong H_{q+(2n+1)k}\left(\prod_{q=0}^{m} \prod_{\beta(q)} \Omega^{m-q} S^{m+2n+1}\right). \end{aligned}$$

(Compare 7.3 where we must assume M is oriented.)

Next we restrict attention to a closed orientable Riemann surface of genus g, M_g. Write $\hat{M}_g = M_g - \{\text{point}\}$ and $\text{map}_*(,)$ for the pointed mapping space. If $\text{map}_*(X, Y)$ is not connected, write $\text{map}_*^0(X, Y)$ for the component of the constant map. The next theorem gives the homotopy type of the mapping space $\text{map}_*(M_g, S^{2n})$.

Theorem 8.5 *If $n>1$, there exist principal $\Omega^2 S^{2n}$ fibrations*

$$\Omega^2 S^{2n} \to Y_{g,2n} \to (S^{2n-1})^{2g}$$

and a principal $\Omega^2 S^3$-fibration

$$\Omega^2 S^3 \to Y_{g,2} \to (S^1)^{2g}$$

such that

(1) *there is a homotopy equivalence* $\mathrm{map}_*^0(M_g, S^2) \xrightarrow{\simeq} (\Omega S^3)^{2g} \times Y_{g,2}$ *and*
(2) *there is a homotopy equivalence after inverting* 6

$$\mathrm{map}_*(M_g, S^{2n}) \xrightarrow{\simeq} (\Omega S^{4n-1})^{2g} \times Y_{g,2n}$$

for $n > 1$,
(3) *there is a fibration for* $n > 1$

$$Y_{g,2n} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{4n-1})^{2g}$$

which has a section after inverting 2.

Theorem 8.6 *After localizing at p,* $p > 3$, *there are stable decompositions*

$$Y_{g,2n} \xrightarrow[\text{stable}]{\simeq} \bigvee_{k \geqq 1} D_k(Y_{g,2n}) \text{ and}$$

$$C^k(\hat{M}_g) \xrightarrow[\text{stable}]{\simeq} \bigvee_{i=0}^{k} D_i(Y_{g,2n}) \wedge D_{k-i}(\Omega S^3)^{2g}$$

where $(\Omega S^3)^{2g}$ *is given the product filtration obtained from the James filtration* [J]. *Thus there are isomorphisms*

$$\bar{H}_N C^k(\hat{M}_g) \xrightarrow{\simeq} \bigoplus_{I_j} \bigoplus_{i} \bar{H}_{N-2j} D_i(Y_{g,2})$$

where I_j *is the number of ordered partitions of j having length* $2g$, $(\alpha_1, \ldots, a_{2g})$.

Thus it suffices to understand the homology of the spaces $Y_{g,2n}$ in order to know the homology of $C^k(\hat{M}_g)$. As in (5.2) in studying $H_* Y_{g,2n}$ the exterior algebra

$$\Lambda = E[x_1, \ldots, x_{2g}, z]$$

with differential given by

$$d(z) = 2\left(\sum_{i=1}^{g} x_{2i-1} x_{2i}\right),$$

is needed but here the degrees are specified by $|x_i| = 2n-1$, $|z| = 4n-3$.

Theorem 8.7 *Assume that n is greater than one.*

(1) *In characteristic zero there are isomorphisms*

$$H^* C(\hat{M}_g, S^{2n-2}) \xrightarrow{\simeq} H_*(\Lambda \otimes \mathbb{Q}; d) \otimes H^*(\Omega S^{4n-1})^{2g} \otimes H^* \Omega S^{2n-1}.$$

(2) *In characteristic p there are isomorphisms*

$$H^* C(\hat{M}_g, S^{2n-2}) \xrightarrow{\simeq} H^* Y_{g,2n} \otimes H^*(\Omega S^{3n-1})^{2g},$$

and
(3) *In characteristic n with* $p \geqq g$, *there are isomorphisms*

$$H^* Y_{g,2n} \xrightarrow{\simeq} H_*(\Lambda \otimes \mathbb{F}; d) \otimes H^* \Omega S^{2n-1} \otimes H^* \Omega^2 S^{4n-1} // \Lambda[z].$$

Remark 8.8 8.7(1) is, of course, dual to 5.3, (2) is immediate, and (3) is dual to 5.12.

Remark 8.9 We conjecture that the stable decompositions above for $\mathrm{map}_*(M_g, S^{2n})$ and $Y_{g,2n}$ are also satisfied at $p=3$.

9 Proof of 8.1, 8.2 and 8.4

We first prove Lemma 8.1. Notice that if X_t is the fat wedge in X^t, then there is a cofibration [May]

$$\mathbf{F}(M,t) \times_{\mathscr{S}_t} X_t \to \mathbf{F}(M,t) \times_{\mathscr{S}_t} X^t \to D_t(M,X).$$

Next recall that $H_*(\mathbf{F}(M,t) \times_{\mathscr{S}_t} X^t)$ is the homology of the chain complex

$$S_* \mathbf{F}(M,t) \otimes_{\mathscr{S}_t} S_* X^{\otimes t}$$

and that over $\mathbb{F}$, there is a homology isomorphism $g\colon H_* X \to S_* X$ given by sending a class (in a choice of basis) to a representing cycle. Thus $H_*(\mathbf{F}(M,t) \times_{\mathscr{S}_t} X^t)$ is the homology of the complex $S_* \mathbf{F}(M,t) \otimes_{\mathscr{S}_t} H_* X^{\otimes t}$ and 8.1 follows as there is an isomorphism of $\mathscr{S}_t$-modules

$$(H_* X)^{\otimes t} \to (H_{*+2} \Sigma^2 X)^{\otimes t}.$$

Next we give the proof of 8.2: Notice that 8.2 (i) and 8.2 (ii) follow immediately from the cofibration given above. To check 8.2 (iii), let a_j be a non-zero element in $\bar{H}_*(D_j(M,X); \mathbb{F})$ and $|a_j|$ it's degree. Then $j(q) \leqq |a_j| \leqq j(N+m)$ if $j<t$. But since $t(m+N-q)<q$ by assumption, we have $j(m+N)<(j+1)q$ and the results in 8.2 (iii) and 8.2 (iv) follow.

Finally in this section we prove 8.4: By the above remarks it suffices to check that these are isomorphisms

$$H_* C(\bar{M}, S^{2n}) \cong H_*\left(\prod_{q=0}^{n} \prod_{\beta(q)} \Omega^{n-q} S^{m+2n}\right)$$

for m odd and

$$H_*(C(\bar{M}, S^{2n+1})) \cong H_*\left(\prod_{q=0}^{m} \prod_{\beta(q)} \Omega^{m-q} S^{m+2n+1}\right)$$

for m even.

This follows directly from naturality of the Serre spectral sequence, induction on a handle decomposition for $\bar{M}$ and that the stabilization map $\Omega^i S^N \to \Omega^\infty \Sigma^\infty S^{N-i}$ is a monomorphism in homology when $\mathbb{F}=\mathbb{F}_2$ or $N-i$ is odd: Namely consider the map of fibrations

$$\begin{array}{ccc} C(\bar{M}, S^{2n}) & \longrightarrow & \Omega^\infty \Sigma^\infty (\bar{M} \times S^q / \bar{M} \times *) \\ \downarrow & & \downarrow \\ \Omega^{m-q} S^{m+n} & \longrightarrow & \Omega^\infty \Sigma^\infty (S^{n+q}). \end{array}$$

If $m+n$ is odd, the homology of $C(\bar{M}, S^n)$ injects in the homology of $\Omega^\infty \Sigma^\infty (\bar{M} \times S^n / \bar{M} \times *)$ by induction where $\bar{M}$ is as in Proposition 8.3. The result follows.

10 Examples related to surfaces

If M is a closed, stably parallelizeable manifold then one has $C(M-\{\text{point}\}, \Sigma^l X)$ is homotopy equivalent to the pointed mapping space $\text{map}_*(M, \Sigma^l X)$ for some l, [B2]. If M is $(2m)$-dimensional and connected up to the middle dimension then M is very close to being parallelizable. If m is even, its index must be zero, and there is one further condition: M cannot contain any (connected sum) summands which are fibrations $S^m \to E \to S^m$, the associated sphere bundles of vector bundles that are stably non-trivial. In particular all the Riemann surfaces M_g^2 are stably parallelizable, $g>0$.

Hence, in these cases we have, first, a stable splitting of $\text{map}_*(M, \Sigma^m X)$ using the stable splitting of $C(M-\{\text{point}\}, X)$, and second, a good homotopy type description of $\text{map}_*(M, \Sigma^m X)$ since there is a fibration

$$\text{map}_*(M, \Sigma^m X) \to (\Omega^m \Sigma^m X)^r \xrightarrow{w^*} \Omega^{2m-1} \Sigma^m X$$

where w^* is the "hom-dual" of the attaching map $w\colon S^{2m-1} \to \bigvee_r S^m$ with cofiber M. Thus one can determine the homology of $C(M-\{\text{point}\}, X)$ by using this fibration (and either the Eilenberg-Moore spectral sequence or backing up and using the Serre spectral sequence).

In case $M=M_g$ is a closed orientable Riemann surface of genus g, there is a map

$$w_g\colon S^1 \to \bigvee_{2g} S^1$$

with cofiber M_g and w_g is given by sending a generator of $\pi_1(S^1, *)$ to

$$[x_1, x_2] \cdot [x_3, x_4] \ldots [x_{2g-1}, x_{2g}]$$

where the x_i run over a choice of generators for $\pi_1(\bigvee_{2g} S^1, *)$. Write

$$ad\colon \Omega X \times \Omega X \to \Omega X$$

for the commutator $[f,g]=fgf^{-1}g^{-1}$. Let $\mu_g\colon (\Omega X)^g \to \Omega X$ be given as loop multiplication taken in a given, fixed order. By definition we obtain the following lemma,

Lemma 10.1 *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \mathrm{map}_*(\bigvee_{2g} S^1, X) & \xrightarrow{w_g^*} & \Omega X \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \mu_g} \\ [(\Omega X)^2]^g & \xrightarrow{(a\ d)^g} & (\Omega X)^g \end{array}$$

where w_g^* is the "hom-dual" of w_g.

Thus a first step to computing the cohomology of $\mathrm{map}_*(M_g, X)$ is to understand w_g^* in cohomology. We shall work out a single example here where $X=S^{2n}$. Recall that there are isomorphisms of Hopf algebras

$$H^*(\Omega S^{2n})^{2g} \cong E[x_1, \ldots, x_{2g}] \otimes \Gamma[y_1, \ldots, y_{2g}]$$

and

$$H^*\Omega S^{2n} \cong E[u] \otimes \Gamma[v]$$

with $|x_i|=|u|=2n-1$ and $|y_i|=|v|=4n-2$. Applying Lemma 10.1 in cohomology we get

$$(w_g^*)^*(u)=0 \text{ and}$$

$$(w_g^*)^*(v)=2\sum_{i=1}^{q} x_{2i-1}x_{2i}.$$

Next we compare the morphism of fibrations in cohomology given by

$$\begin{array}{ccc} \Omega^2 S^{2n} & \xrightarrow{1} & \Omega^2 S^{2n} \\ \downarrow & & \downarrow \\ \mathrm{map}_*(M_g, S^{2n}) & \longrightarrow & * \\ \downarrow & & \downarrow \\ (\Omega S^{2n})^{2g} & \xrightarrow{w_g^*} & \Omega S^{2n}. \end{array}$$

The Serre spectral sequence in mod-2 cohomology collapses while the image of the stabilization map

$$H^*(QS^{2n-2}; \mathbb{F}) \to H^*(\Omega^2 S^{2n}; \mathbb{F})$$

consists of infinite cycles for any $\mathbb{F}$ by inspection of the map

$$\mathrm{map}_*(M_g, S^{2n}) \to \mathrm{map}_*(M_g, QS^{2n})$$

as $\mathrm{map}_*(M_g, QS^{2n})$ splits as $(QS^{2n-1})^{2g} \times QS^{2n-2}$. As a corollary, one gets the rational homology of $C(\hat{M}_g, S^{2n-2})$ and, so, again, that of $C^k(\hat{M}_g)$.

Theorem 10.2 *If $n>1$ and $\mathbb{F}=\mathbb{Q}$, there is an isomorphism*

$$H^*C(\hat{M}_g, S^{2n}) \cong H^*(\Lambda(x_1, \ldots, x_{2g}, z); d) \otimes H^*(\Omega S^{4n-1})^{2g} \otimes H^*(\Omega S^{2n-1})^{2g},$$

where $d(z)=2\sum x_{2i-1}x_{2i}$.

Proof. Recall that there is a homotopy equivalence $\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}$ after rationalization. Thus there is an equivalence $\Omega^2 S^{2n} \simeq \Omega S^{2n-1} \times \Omega^2 S^{4n-4} \cong \Omega S^{2n-1} \times S^{4n-3}$. In the Serre spectral sequence for

$$\Omega S^{2n-1} \times S^{4n-3} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{2n})^{2g},$$

the E_2-term is

$$E[x_1, \ldots, x_{2g}] \otimes \mathbb{Q}[y_1, \ldots, y_{2g}] \otimes \mathbb{Q}[w] \otimes \Lambda[z]$$

with $|w|=2n-2$ and $|z|=4n-3$. The differential $d=d_{4n-2}$ is given by $d(z)=2\sum x_{2i-1}x_{2i}$. By inspection, the algebra generators for E_{4n-1} are all infinite cycles.

11 Surfaces and product decompositions

In this section we again consider the pointed function spaces $\mathrm{map}_*(M_g, S^{2n})$, $n>1$, and $\mathrm{map}_*^0(M_g, S^2)$. Notice that there are fibrations

$$\mathrm{map}_*(M_g, S^{2n}) \xrightarrow{j} (\Omega S^{2n})^{2g} \xrightarrow{w_g^*} \Omega S^{2n}, n>1, \text{ and}$$

$$\mathrm{map}_*^0(M_g, S^2) \xrightarrow{j} (\Omega S^2)^{2g} \xrightarrow{w_g^*} \Omega S^3.$$

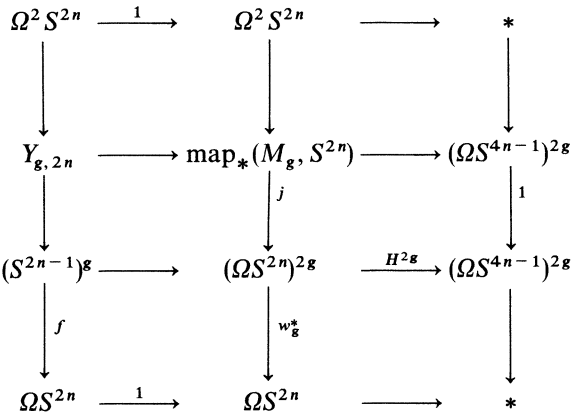
Recall that there is a fibration (over $\mathbb{Z}$) given by James [J]

$$S^{2n-1} \longrightarrow \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1}$$

and thus a commutative diagram

$$\begin{array}{ccc} \mathrm{map}_*(M_g, S^{2n}) & \xrightarrow{H^{2g}\cdot j} & (\Omega S^{4n-1})^{2g} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 1} \\ (\Omega S^{2n})^{2g} & \xrightarrow{H^{2g}} & (\Omega S^{4n-1})^{2g}. \end{array}$$

Enlarging the above to a commutative diagram giving a morphism of fibrations, we obtain spaces $Y_{g,2n}$ as follows:



(where ΩS^{2n} is replaced by ΩS^3 if $n=1$ in the bottom row).

Theorem 11.1 *There are homotopy equivalences*

(1) $\mathrm{map}_*^0(M_g, S^2) \simeq (\Omega S^3)^{2g} \times Y_{g,2}$, *and*
(2) $\mathrm{map}_*(M_g, S^{2n}) \simeq (\Omega S^{4n-1})^{2g} \times Y_{g,2n}$ *after inverting* 6.

Next, recall the bundle

$$\nu\colon \mathbf{F}(\hat{M}_g, k) \times_{\mathscr{S}_k} \mathbb{R}^k \to \mathbf{F}(\hat{M}_g, k)/\mathscr{S}_k$$

and

Proposition 11.2 [C²M²] *The bundle* ν *has order* 4 *and hence there are homotopy equivalences* $D_k(\hat{M}_g, S^{n+4}) \simeq \Sigma^{4k} D_k(\hat{M}_g, S^n)$ *and* $\hat{D}_k(\hat{M}_g, S^4) \simeq \Sigma^{4k}(C^k(\hat{M}_g)_+)$.

Filter $(\Omega S^{2n+1})^{2g}$ by the product filtration obtained from the James filtration and write $D_k(\Omega S^{2n+1})^{2g}$ for the filtration quotients.

Theorem 11.3 *After localizing at* p, $p>3$, $Y_{g,2n}$ *stably splits as* $\bigvee_{k\geq 1} D_k(Y_{g,2n})$ *and there is a stable decomposition*

$$C^k(\hat{M}_g) \underset{\simeq}{\text{stably}} \bigvee_{i=0}^{k} D_i(Y_{g,2n}) \wedge D_{k-i}(\Omega S^3)^{2g}.$$

Thus there is an isomorphism

$$\bar{H}_N(C^k(\hat{M}_g); \mathbb{F}) \cong \bigoplus_{I_j} \bar{H}_{N-2j}(D_i(Y_{g,2}); \mathbb{F})$$

where I_j *is the number of ordered partitions of* j *having length* $2g$.

Theorem 11.4 *If $p \geqq g$ and $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ for $p \geqq 3$, there are isomorphisms*

$$H^*(Y_{g,2n}) \xrightarrow{\simeq} H(\Lambda[x_1, \ldots, x_{2g}, z]; d) \otimes H^* \Omega S^{2n-1} \otimes H^* \Omega^2 S^{4n-1} // \Lambda[z],$$

and

$$H^* \mathrm{map}_*(M_g, S^{2n}) \xrightarrow{\simeq} H^* Y_{g,2n} \otimes H^*(\Omega S^{4n-1})^{2g}.$$

Proof of 11.1 Consider the unit sphere bundle in the tangent bundle of S^{2n}, $\tau(S^{2n})$. There is a bundle

$$S^{2n-1} \longrightarrow \tau(S^{2n}) \xrightarrow{\pi} S^{2n}$$

and $\tau(S^{2n})$, after localization at an odd prime is S^{4n-1}. Thus there is a p-local fibration

$$S^{2n-1} \longrightarrow S^{4n-1} \xrightarrow{\pi} S^{2n}$$

where π is the Whitehead square $[\iota_{2n}, \iota_{2n}]$. Since the triple Whitehead product

$$[\iota_{2n}, [\iota_{2n}, \iota_{2n}]] = 0$$

after localizing at primes at least 5, there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^{4n-1} \vee S^{2n} & \xrightarrow{w_{2n} \vee 1} & S^{2n} \vee S^{2n} & \xrightarrow{\text{fold}} & S^{2n} \\ \downarrow & & & & \downarrow \\ S^{4n-1} \times S^{2n} & & \xrightarrow{\mu} & & S^{2n}_{(p)}. \end{array}$$

Using this action we illustrate the proof of 11.1 by proving the case (1) for the space $\mathrm{map}^0_*(M_g, S^2)$.

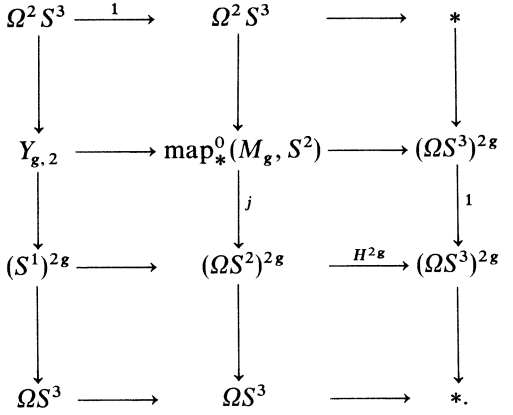
Since $\mathrm{map}_*(M_g, S^3)$ splits as $(\Omega S^3)^{2g} \times \Omega^2 S^3$, the action of S^3 on S^2 induces a homotopy commutative diagram

$$\begin{array}{ccccc} Y_{g,2} \times (\Omega S^3)^{2g} & \longrightarrow & Y_{g,2} \times \mathrm{map}_*(M_g, S^3) & \longrightarrow & \mathrm{map}^0_*(M_g, S^2) \\ \downarrow \text{project} & & & & \downarrow H^{2g} \cdot j \\ (\Omega S^3)^{2g} & & \xrightarrow{1} & & (\Omega S^3)^{2g}. \end{array}$$

But the map of $Y_{g,2} \times \{\text{point}\}$ induced by the action is the inclusion of a fibre in the space $\mathrm{map}^0_*(M_g, S^2)$ as $Y_{g,2}$ is also the fibre in

$$Y_{g,2} \to \mathrm{map}^0_*(M_g, S^2) \to (\Omega S^3)^{2g}$$

obtained from the diagram



The theorem follows.

Remark 11.5 The fibration $Y_{g,2} \to (S^1)^{2g}$ is classified by a map

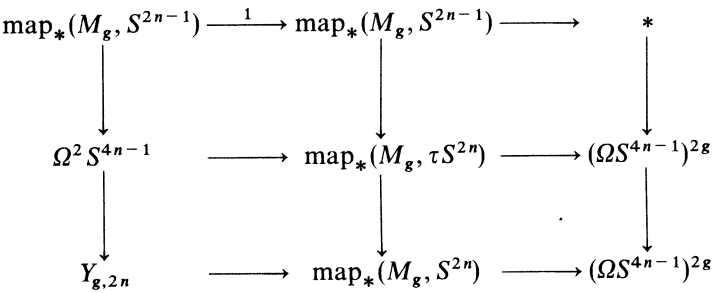
$$(S^1)^{2g} \to \Omega S^3$$

which sends the generator of $H^2(\Omega S^3, \mathbb{Z})$ to $2(\sum x_{2i-1}\, x_{2i})$.

Remark 11.6 When localized at $p=3$, there is a p-local fibration

$$Y_{g,2n} \to \mathrm{map}_*(M_g, S^{2n}) \to (\Omega S^{4n-1})^{2g}, n>1.$$

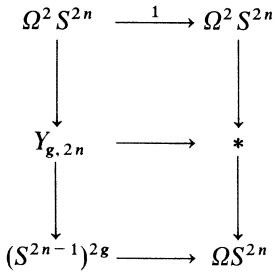
This fibration has a cross-section although we don't know that it is a trivial fibration. Nevertheless notice that there is a diagram of fibrations



Furthermore, since τS^{2n} is a 3-local H-space, S^{4n-1}, the function space $\mathrm{map}_*(M_g, \tau S^{2n})$ is homotopy equivalent to a product $(\Omega S^{4n-1})^{2g} \times \Omega^2 S^{4n-1}$.

Proof of 11.3 If $p>3$, this theorem follows from the action $S^{4n-1} \times S^{2n} \to S^{4n-1}$ given in the proof of 11.1 together with the periodicity in 8.1, 8.2, and 11.2. The proof is analogous to that given in [C²M²] for the stable decomposition of $D_k(\hat{M}_g, S^{2n-1})$ and we omit the details.

Proof of 11.4 Consider the morphisms of fibrations



and the p-local decomposition $\Omega^2 S^{2n} \simeq \Omega S^{2n-1} \times \Omega^2 S^{4n-1}$. Since a direct degree check gives that the A-module indecomposables of $H^*\Omega^2 S^{4n-1}$: degrees greater than $4n-3$, must be infinite cycles by naturality. The theorem follows for $Y_{g,2n}$.

To check the case of $\mathrm{map}_*(M_g, S^{2n})$, appeal to the product decomposition for the space $\mathrm{map}^0_*(M_g, S^2)$ and use periodicity in 11.2.

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A unitarization process for some non-locally compact topological groups

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0 Introduction

In comparison with what happens with finite dimensional Lie groups our knowledge of the unitary dual of many non locally compact groups G is rather poor; it is the case, for instance, with the infinite dimensional Lie groups of sections of smooth bundles of compact semisimple Lie groups (the so-called gauge groups). Moreover, such groups have no Haar measures, so that it seems impossible to try to expand a non commutative harmonic analysis on them.

Nevertheless, it follows from various works (see e.g. [2, 5, 6, 11, 12]) that the following situation, which seems to be similar in some sense to the surroundings of the abstract Wiener spaces, occurs for some groups G of mappings from a compact manifold into a compact Lie group: G is continuously and densely contained in a polish topological group Γ on which there exist Borel probability measures μ which are quasi invariant by the left translations by the elements of G, so that G has a natural left regular unitary representation in the space $L^2(\Gamma;\mu)$. Such a pair $(\Gamma;\mu)$ is what I have called in the present paper a polish environment of quasi invariance, (in short a PEQI), for G.

In this article it is proved that, when a non locally compact topological group G is equipped with a PEQI $(\Gamma;\mu)$, there exists a unitarization process which associates a cyclic continuous unitary representation of G with each of its continuous locally bounded representations.

1 Locally bounded continuous representations of polish groups

a) Let Γ be a topological group; with each continuous representation θ of Γ in some Hilbert space $\mathscr{H}$ let us associate the mapping ρ_θ from Γ into the set $\mathbb{R}^{*+}$ of all the strictly positive real numbers, defined for any element γ in G by:

$$\rho_\theta(\gamma)=\{\sup\|\theta(\gamma)u\|\}^{-2}, \tag{1}$$

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in which $\| \|$ denotes the hilbertian norm on $\mathscr{H}$, and sup the upper bound over the unitary vectors of $\mathscr{H}$.

Definition 1 A continuous representation θ of a topological group Γ in a Hilbert space $\mathscr{H}$ is said to be *locally bounded* if for any element γ in Γ there exist an open neighbourhood $V(\gamma)$ of γ and two strictly positive real numbers $m(\gamma)$ and $M(\gamma)$ such that for all elements γ' in $V(\gamma)$ the following inequalities: $0<m(\gamma) \leqq \rho_\theta(\gamma') \leqq M(\gamma)$ are fulfilled.

We remark that besides the bounded continuous representations of Γ in $\mathscr{H}$, and particularly the unitary representations, many continuous unbounded representations are locally bounded.

b) Let us study the locally bounded representations in the case where Γ is a polish group, that is to say a separable topological group which can be metrized in such a way that it becomes complete.

Lemma 1 *Let θ be a locally bounded continuous representation of a polish group Γ in the Hilbert space $(\mathscr{H}, \| \|)$. There exists a continuous mapping r_θ from Γ into $\mathbb{R}^{*+}$ such that for any element γ in Γ: $0<r_\theta(\gamma) \leqq \rho_\theta(\gamma)$.*

Proof. For each element γ in Γ let us select an open neighbourhood $V(\gamma)$ of γ and two strictly positive numbers $m(\gamma)$ and $M(\gamma)$ such that for all γ' in $V(\gamma)$:

$$0<m(\gamma) \leqq \rho_\theta(\gamma') \leqq M(\gamma).$$

Any polish space being paracompact (see e.g. [3, § 2]) we can assume that the family $(V(\gamma))_{\gamma \in \Gamma}$ is a locally finite open covering of Γ. One can then find a countable subset $(\gamma_p)_{p \in J}$ in Γ, where J is some subset of the set $\mathbb{N}$ of all the non negative integers, and for each index p in J an open subset V_p of Γ and two strictly positive real numbers m_p and M_p in such a way that $(V_p)_{p \in J}$ is a countable locally finite open covering of Γ and that $0<m_p \leqq \rho_\theta(\gamma) \leqq M_p$ for all γ in V_p.

Moreover, Γ being paracompact, there exists a continuous partition of the unit subordinated to $(V_p)_{p \in J}$, that is to say a family $(f_p)_{p \in J}$ of continuous mappings f_p from Γ into the closed interval $[0, 1]$ such that for each index p the support $[f_p]$ of f_p is contained in V_p, and such that the sum of all the f_p is the constant function which equals 1 on Γ.

With each index p in J let us associate the set $S(p)=\{q \in J / [f_q] \cap V_p \neq \emptyset\}$; the local finiteness of the covering $(V_p)_{p \in J}$ implies the finiteness of $S(p)$; as $p \in S(p)$, it follows that the number $s(p)$ of elements of $S(p)$ is a strictly positive integer.

Let us consider now the mapping r_θ from G into $\mathbb{R}^{*+}$ defined by:

$$r_\theta = \sum_{p \in J} (s(p))^{-1} \cdot \mu_p \cdot f_p \tag{2}$$

where μ_p denotes the smallest of the strictly positive real numbers m_q, q going over the finite set $S(p)$. We remark that for any index p in J and for any γ in V_p:

$$r_\theta(\gamma) = \mu_p \cdot \sum_{q \in S(p)} (s(q))^{-1} \cdot f_q(\gamma) \tag{3}$$

which proves that r_θ is a strictly positive continuous mapping on V_p, and then on Γ. Now let γ be any element in Γ, and let p be any index in J such that γ lies in V_p; it follows from (3) that $0<r_\theta(\gamma)\leqq\mu_p\leqq m_p\leqq\rho_\theta(\gamma)$. □

2 Polish environment of quasi invariance for non locally compact groups

a) Let G a topological group. In the case where G is locally compact, the existence of a left (or right) invariant measure h opens the way of its harmonic analysis; roughly speaking it's a matter of decomposing the associated left (or right) regular representation of G in $L^2(G; h)$ into its irreducible components.

Let us consider now, in contrast, the case where G is a non locally compact group; such a group having no Haar measure, any attempt in order to obtain its harmonic analysis seems to be, a priori, a hare-brained gamble. Nevertheless if G is a subgroup of a topological group Γ equipped with a Borel measure μ quasi invariant by the left translations by the elements of G, any Hilbert space $\mathscr{H}$ gives rise to a continuous unitary representation L of G in the Hilbert space $L^2(\Gamma, \mathscr{H}; \mu)$ by taking for any element g in G and any element Φ in $L^2(\Gamma, \mathscr{H}; \mu)$:

$$(4)\qquad (L(g)\Phi)(\gamma)=\sqrt{\frac{d\mu(g^{-1}\cdot\gamma)}{d\mu(\gamma)}}\cdot\Phi(g^{-1}\cdot\gamma),\qquad \gamma\in\Gamma.$$

The previous discussion induces to initiate the following definition:

Definition 2 Let G be a non locally compact topological group. A *polish environment of quasi invariance* (for short: a PEQI) for G is a pair $(\Gamma; \mu)$ such that:
(P1) Γ is a polish group and G is a subgroup of Γ, the topology of G being stronger than the topology induced by that of Γ;
(P2) μ is a Borel probability measure on Γ which is quasi invariant by the left translations by the elements of G.

In this context the continuous unitary representation L given by (4) will be called a left regular representation of G w.r.t. $(\Gamma; \mu)$.

Example 1 Let (i, H, B) be an abstract Wiener space (we refer to [7] for the definition and the basic properties of such objects), where B is a real separable Banach space, and H a continuously and densely embedded Hilbert space; it is well known that for any strictly positive real number t the corresponding Wiener measure μ_t on B with variance t is quasi invariant by the translations by the elements of H. Let us consider the case where H is an infinite dimensional Hilbert space; looking on H and B as non locally compact polish groups with respect to their additive group structures it follows that the space $(B; \mu_t)$ is a PEQI for H.

Example 2 Let I be the closed interval $[0, 1]$, and let K be a compact semisimple Lie group. The Banach-Lie group $C(I, K)$ of all the continuous mappings from I into K can be equipped with a Borel probability measure μ which is quasi invariant by the left translations by the elements of the dense subgroup $H^1(I, K)$ of paths of Sobolev class H^1 (cf. [1, 2, 9]). It follows that $(C(I, K); \mu)$ is a PEQI for $H^1(I, K)$.

Example 3 Let M be a smooth compact riemannian manifold, let K be a compact semisimple Lie group, and let us select two integers m and n strictly greater than the number $\frac{1}{2}\dim(M)$. It follows from [6] that it is possible to equip the Banach-Lie group $C(I, H^m(I, K))$ of all the continuous paths on the Hilbert-Lie group $H^m(I, K)$ of mappings of Sobolev class H^m from M into K with a Borel probability measure $w_{m,n}$ which is quasi invariant by the left translations by the elements of the subgroup $H_0^1(I, H^{m+n}(M, K))$ consisting of the connected component of the Hilbert-Lie group of mappings of Sobolev class H^1 from I into the Hilbert-Lie group $H^{m+n}(M, K)$. The pair $(C(I, H^m(M, K)); w_{m,n})$ is then a PEQI for the group $H_0^1(I, H^{m+n}(M, K))$; a decomposition of the associated left regular representation has been given in [11].

Example 4 The data being the same that in the previous example, the article [6] of Gaveau and Trauber (see also [5]) may be used in order to obtain a Borel probability measure $\mu_{m,n}$ on the Hilbert-Lie group $H^m(M, K)$ which is quasi invariant under the left action of the connected component $H_0^{m+n}(M, K)$ of $H^{m+n}(M, K)$, so that $(H^m(M, K); \mu_{m,n})$ is a PEQI for the group $H_0^{m+n}(M, K)$.

Remark. All these examples belong to the class often called the class of Wiener-Lie groups, which appears as the group theoretical version of the class of abstract Wiener spaces. For such groups the quasi invariant Borel probability measure arises from a diffusion process, more precisely a Brownian motion process, solution of a particular stochastic differential equation. In the next section, *I* shall give an example of PEQI for the current group $C^\infty(B_n, G)$ where B_n is a n-dimensional hollowed bowl and G a compact semisimple Lie group, and which is not of that type.

Proposition 1 *Let (Γ, μ) be a* PEQI *for a non locally compact topological group G, and let ℓ be the Lebesgue measure on $\mathbb{R}$. The action of G on $\Gamma \times \mathbb{R}$ defined for any element g of G and any element (γ, t) of $\Gamma \times \mathbb{R}$ by:*

$$g \cdot (\gamma, t) = \left(g \cdot y, \left(\frac{d\mu(g \cdot \gamma)}{d\mu(\gamma)}\right)^{-1} \cdot t\right) \tag{5}$$

leaves invariant the Borel measure $\mu \times \ell$ of $\Gamma \times \mathbb{R}$.

Proof. Let F be any element in $L^1(\Gamma \times \mathbb{R}; \mu \times \ell)$ and let g be any element in G. Then:

$$\int_{\Gamma \times \mathbb{R}} F(g^{-1} \cdot (\gamma, t)) d\mu(\gamma) d\ell(t) = \int_{\Gamma \times \mathbb{R}} F\left(g^{-1} \cdot \gamma, \left\{\frac{d\mu(g^{-1} \cdot \gamma)}{d\mu(\gamma)}\right\}^{-1} \cdot t\right) d\mu(g) d\ell(t)$$

$$= \int_{\Gamma \times \mathbb{R}} F(g^{-1} \cdot \gamma, u) \cdot \frac{d\mu(g^{-1} \cdot \gamma)}{d\mu(\gamma)} d\mu(\gamma) d\ell(u)$$

$$= \int_{\Gamma \times \mathbb{R}} F(\gamma, t) d\mu(\gamma) d\ell(t),$$

which proves the assertion. □

3 A family of PEQI for $C^\infty(B_n, G)$, where B_n is a n-dimensional hollowed bowl and G a compact semisimple Lie group

Let n be an integer $\geqq 2$, let $\mathbb{R}^n$ be equipped with its euclidean norm $\| \ \|$, let B_n be the hollow bowl:

$$B_n = \{\xi \in \mathbb{R}^n / 1 \leqq \|x\| \leqq 2\} \tag{6}$$

and let G be a connected compact semisimple Lie group.

The set $C^\infty(B_n, G)$ of all the G-valued smooth mappings on B_n equipped with the pointwise defined product and with the topology of the C^∞-uniform convergence is a non locally compact Fréchet-Lie group, and is densely contained in the polish group consisting of the Banach-Lie group $C(B_n, G)$ of all the continuous mappings from B_n into G, the underlying topology of which being the topology of the uniform convergence (see e.g. [13] or [10]).

a) Each element α of the sphere $S_{n-1} = \{\xi \in \mathbb{R}^n / \|x\| = 1\}$ defines a smooth path $\underline{\alpha}$ from $I = [0, 1]$ into B_n such that for any t in I: $\underline{\alpha}(t) = (1+t)\cdot\alpha$; moreover $\underline{\alpha}$ is a diffeomorphism from I onto the ray $I_\alpha = \{\xi = (1+t)\cdot\alpha / t \in I\}$ of B_n.

It follows that the mapping α^* from $C(B_n, G)$ into $C(I, G)$ defined by $\alpha^*(g) = g \circ \underline{\alpha}$ is a continuous homomorphism of topological groups; likewise, the mapping α^{**} from $C(I_\alpha, G)$ onto $C(I, G)$ defined for any γ in $C(I_\alpha, G)$ by $\alpha^{**}(\gamma) = \gamma \circ \underline{\alpha}$ is a continuous isomorphism of topological groups.

Lemma 2 *Let α be any element in S_{n-1}. Then:*
(i) *α^* is a strict continuous homomorphism of topological groups from $C(B_n, G)$ onto $C(I, G)$;*
(ii) *for any Borel subset A of $C(B_n, G)$ the set $\alpha^*(A)$ is a Borel subset of $C(I, G)$.*

Proof. (i) Let us recall at first, that a continuous homomorphism λ of a topological group Γ into a topological group Γ' is said to be a strict homomorphism if, e denoting the unit element in Γ', the homomorphism λ induces an isomorphism $\underline{\lambda}$ of topological groups from $\Gamma/\lambda^{-1}(\{e\})$ onto $\lambda(\Gamma)$.

Let α be an element of S_{n-1}, and let us prove first that α^* is onto; k being any non negative integer, for any element γ in the Banach-Lie group $C^k(I, G)$ let γ_α be the mapping from I_α into G such that for all t in $I = [0, 1]$: $\gamma_\alpha((1+t)\cdot\alpha) = \gamma(t)$. For all y in I_α: $\gamma_\alpha(y) = \gamma(\underline{\alpha}^{-1}(y))$, where $\underline{\alpha}$ denotes here the diffeomorphism from I onto I_α; the smoothness of $\underline{\alpha}$ implies then that γ_α lies in $C^k(I_\alpha, G)$.

Now let us consider the mapping φ from B_n into G defined by $\varphi(\xi) = \gamma_\alpha(\|\xi\| \cdot \alpha)$; from the smoothness of the mapping $\xi \mapsto \|\xi\|$ on any domain which does not contain the null vector of $\mathbb{R}^n$, it follows that φ, like γ_α, is of class C^k, and then lies in the group $C^k(B_n, G)$. Moreover for any t in I: $(\varphi \circ \underline{\alpha})(t) = \gamma_\alpha(\|\underline{\alpha}(t)\| \cdot \alpha) = \gamma_\alpha((1+t)\cdot\alpha) = \gamma(t)$; consequently $\alpha^*(\varphi) = \gamma$; in particular α^* maps $C(B_n, G)$ onto $C(I, G)$.

Let ε be the constant mapping from I into G equal to the unit element e of G; ε is the unit element of $C(I, G)$ and from the above discussion it follows that the quotient group $C(B_n, G)/(\alpha^*)^{-1}(\varepsilon)$ is isomorphic to the Banach-Lie group $C(I_\alpha, G)$; thereby one sees that the homomorphism α^* induces the isomorphism α^{**} from $C(I_\alpha, G)$ onto $C(I, G)$; we have then proved that α^* is a strict homomorphism from $C(B_n, G)$ onto $C(I, G)$.

(ii) α^* being a strict continuous homomorphism of topological groups is then an open mapping (see e.g. [4, T.G. III-16, Proposition 24]), that is to say is such that for any open (resp.: closed) subset A of $C(B_n, G)$ the set $\alpha^*(A)$ is an open (resp.: closed) subset of $C(I, G)$, from which the lemma follows. □

b) For any polish topological space P we shall denote by $\mathcal{M}(P)$ the space of all the Borel probability measures on P equipped with the weak topology; such a space can be metrized as a separable metric space; we refer to [14] for the basis properties, notions, and tools related to the space $\mathcal{M}(P)$, in particular when P is a separable metric group.

Let μ be the element of $\mathcal{M}(C(I, G))$ given in Example 2, which is quasi invariant by the translations by the elements of the dense subgroup $H^1(I, G)$. It follows from Lemma 2 that any element α in S_{n+1} gives rise to an element μ_α in $\mathcal{M}(C(B_n, G))$, defined for any Borel subset A of $C(B_n, G)$ by:

$$\mu_\alpha(A) = \mu(\alpha^*(A)). \tag{7}$$

We point out that the field $(\mu_\alpha)_{\alpha \in S_{n-1}}$ of Borel probability measures on $C(B_n, G)$ is a measurable field of Borel measures with respect to any element in $\mathcal{M}(S_{n-1})$: let $(\alpha_j)_j$ be a sequence converging to an element α in S_{n-1}; one easily sees that $(\alpha_j^*)_j$ converges to α^* in the space of continuous homomorphisms of topological groups from $C(B_n, G)$ into $C(I, G)$ (equipped with its natural topology); therefore the sequence $(\mu_{\alpha_j})_j$ converges to μ_α in $\mathcal{M}(C(B_n, G))$. It follows that for any Borel subset A of the Banach-Lie group $C(B_n, G)$ the mapping $\alpha \mapsto \mu_\alpha(A)$ is ω-integrable w.r.t. any element ω in $\mathcal{M}(S_{n-1})$, so that one obtains a new Borel probability measure Λ_ω^μ on $C(B_n, G)$ by taking for any Borel subset A:

$$\Lambda_\omega^\mu(A) = \int_{S_{n-1}} \mu_\alpha(A) \cdot d\omega(\alpha). \tag{8}$$

Proposition 2 *For any probability measure ω on S_{n-1} the pair $(C(B_n, G); \Lambda_\omega^\mu)$ is a* PEQI *for the current group $C^\infty(B_n, G)$.*

Proof. Taking into account the above discussion it is enough to prove the quasi invariance of Λ_ω^μ by the left translations by the elements of $C^\infty(B_n, G)$.

Let α be any element in S_{n-1}; for any Borel subset A of $C(B_n, G)$ and any element g in $C^\infty(B_n, G)$ we have:

$$\mu_\alpha(g^{-1} \cdot A) = \mu(\alpha^*(g^{-1} \cdot A)) = \mu(\alpha^*(g^{-1}) \cdot \alpha^*(A)) = \mu((g^{-1} \circ \underline{\alpha}) \cdot a^*(A));$$

the smoothness of g^{-1} and $\underline{\alpha}$ implies the smoothness of $\alpha^*(g^{-1})$; it follows that μ is quasi invariant by the left translation by $\alpha^*(g^{-1})$, and then μ_α is quasi invariant by the left translation by g^{-1}.

We have then proved that $(\mu_\alpha)_{\alpha \in S_{n-1}}$ is a field of quasi invariant measures under the left action of the current group $C^\infty(B_n, G)$; the quasi invariance of Λ_ω^μ follows then from the formula (8). □

4 Unitarization process for a non locally compact topological group equipped with a PEQI

Let $(\Gamma; \mu)$ be a PEQI for a non locally compact topological group G.

Let θ be locally bounded continuous representation of Γ in some Hilbert space $\mathcal{H}$. We select a strictly positive continuous mapping r_θ on Γ associated

with θ and fulfilling the property described in Lemma 1, and we denote by μ^θ the Borel measure on Γ defined by:

$$d\mu^\theta(\gamma)=r_\theta(\gamma)\cdot d\mu(\gamma). \tag{9}$$

The measure μ^θ is absolutely continuous w.r.t. μ and then, as well as μ, is quasi invariant by the left action of G. More precisely it follows from (6) that for any element g in G:

$$\frac{d\mu^\theta(g^{-1}\cdot\gamma)}{d\mu^\theta(\gamma)}=\frac{r_\theta(g^{-1}\cdot\gamma)}{r_\theta(\gamma)}\cdot\frac{d\mu(g^{-1}\gamma)}{d\mu(\gamma)}. \tag{10}$$

Thus, one obtains a continuous unitary representation L^θ realized as the left regular representation of G in the Hilbert space $L^2(\Gamma,\mathscr{H};\mu^\theta)$ of all the square μ^θ-integrable mappings from Γ into $\mathscr{H}$, by taking, for any g in G and any Φ in $L^2(\Gamma,\mathscr{H};\mu^\theta)$:

$$(L^\theta(g)\Phi)(\gamma)=\sqrt{\frac{d\mu^\theta(g^{-1}\cdot\gamma)}{d\mu^\theta(\gamma)}}\,\Phi(g^{-1}\cdot\gamma),\qquad \gamma\in\Gamma. \tag{11}$$

Let us select now a unitary vector u in $\mathscr{H}$, and let us consider the mapping $\hat{u}$ from Γ into $\mathscr{H}$ defined for any element γ in Γ by:

$$\hat{u}(\gamma)=\theta(\gamma)u. \tag{12}$$

It follows from Lemma 1 that:

$$\int_\Gamma \|\hat{u}(\gamma)\|^2\cdot d\mu^\theta(\gamma)=\int_\Gamma \|\theta(\gamma)u\|^2\cdot r_\theta(\gamma)\cdot d\mu(\gamma)\leqq\int_\Gamma d\mu(\gamma)(=1);$$

consequently $\hat{u}$ belongs to the Hilbert space $L^2(\Gamma,\mathscr{H};\mu^\theta)$. Moreover taking into account (11) and (12), one has for any g in G:

$$(L^\theta(g)\hat{u})(\gamma)=\sqrt{\frac{d\mu^\theta(g^{-1}\cdot\gamma)}{d\mu^\theta(\gamma)}}\,\theta(g^{-1})(\hat{u}(\gamma)). \tag{13}$$

Let $H(\theta;\mu;u)$ be the closed subspace of $L^2(\Gamma,\mathscr{H};\mu^\theta)$ spanned by the set:

$$S(\theta,u)=\{L^\theta(g)\hat{u}/g\in G\}. \tag{14}$$

By construction L^θ leaves invariant the Hilbert space $H(\theta;\mu;u)$; its restriction L^θ_u to $H(\theta;\mu;u)$ is then a cyclic continuous unitary representation of G for which $\hat{u}$ is a cyclic vector. The assignment $\theta\mapsto L^\theta_u$ *is then a unitarization process which associates with any locally bounded representation of* Γ a cyclic continuous unitary representation of G which is a subrepresentation of a left regular representation of G w.r.t. to the PEQI $(\Gamma;\mu)$. Consequently:

Theorem. *Let* $(\Gamma;\mu)$ *be a* PEQI *for a non locally compact group* G, *there exists a unitarization process which associates, with any locally bounded representation* θ *of* Γ *in some Hilbert space* $\mathscr{H}$ *and any unitary vector* u *in* $\mathscr{H}$, *a cyclic continuous unitary representation* L^θ_u *of* G *which is a subrepresentation of a left regular representation of* G w.r.t. $(\Gamma;\mu)$.

Remark. Let us start with the trivial representation $\mathscr{J}$ of Γ in $\mathbb{C}$, and let $\mathbb{1}$ be the constant function in $L^2(\Gamma;\mu)$ defined for any element γ in Γ by $\mathbb{1}(\gamma)=1$. Then one easily sees that $L^{\mathscr{J}}_{\mathbb{1}}$ is exactly the cyclic component of the left regular representation of G in $L^2(\Gamma;\mu)$.

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Über das quadratische Spektralmittel von Fourierkoeffizienten reell-analytischer automorpher Formen halbzahligen Gewichts

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1 Einleitung

In [11] leitete Kuznecov eine asymptotische Formel für das quadratische Spektralmittel von Fourierkoeffizienten von Maass-Formen her, siehe auch [3].

Während die von Kuznecov in derselben Arbeit hergeleitete und mittlerweile auch nach ihm benannte Summenformel und seine hierüber hinaus erzielte Abschätzung für Summen von Kloostermanschen Summen auch auf nichtverschwindende Gewichte übertragen wurde, siehe hierzu Bruggeman [2], Proskurin [18, 13] sowie Goldfeld und Sarnak [5], so ist jedoch eine asymptotische Formel der angegebenen Art für nichtverschwindende Gewichte offensichtlich bisher nicht veröffentlicht worden.

In der vorliegenden Arbeit werden wir nun ein entsprechendes Resultat (Theorem 2.3) herleiten für den Fall vektorwertiger reell-analytischer automorpher Formen zur Modulgruppe $\Gamma := \mathrm{SL}(2, \mathbf{Z})$, zum Gewicht 1/2 und einem dreidimensionalen Multiplikatorsystem, das von dem zur klassischen Thetareihe und zur Thetagruppe gehörenden Multiplikatorsystem induziert ist. Wir benutzen dies in [15], um eine verbesserte Abschätzung für das Restglied im entsprechenden geodätischen Primzahlsatz zu erhalten.

Bei dem Beweis von Theorem 2.3 werden wir uns soweit wie möglich an Kuznecovs Methode anlehnen. Ausgangspunkt ist Theorem 2.2, das als eine vorläufige Version der Kuznecovschen Summenformel angesehen werden kann. Bei der Behandlung des hierin auftretenden sogenannten Eisenstein-Terms (siehe Absatz 3.2.1) benötigen wir jedoch im Unterschied zum Fall verschwindenden Gewichts eine nichttriviale Abschätzung für das quadratische Integralmittel von Dirichletschen L-Reihen, die aus einem Ergebnis von Motohashi [16] folgt.

Außerdem benötigen wir eine Weilsche Abschätzung für verallgemeinerte Kloostermansche Summen, wie wir sie in [14] hergeleitet haben. Zusätzlich sei bemerkt, daß Theorem 2.3 auch eine Aussage über Spektralmittelwerte von Produkten von Fourierkoeffizienten zu verschiedenen Spitzenparametern einschließt.

2 Definitionen und Resultate

Wir bezeichnen mit $\mathbf{H}$ die obere Halbebene $\{z\in\mathbf{C}\,|\,\Im(z)>0\}$. Die drei klassischen Thetareihen

$$\vartheta_2(z)=\sum_{n\in\mathbf{Z}} e^{\pi i(n+\frac{1}{2})^2 z}, \quad \text{bzw.} \quad \vartheta_3(z)=\sum_{n\in\mathbf{Z}} e^{\pi i n^2 z},$$

$$\text{bzw.} \quad \vartheta_4(z)=\vartheta_3(z+1)$$

sind bekanntermaßen holomorphe Funktionen auf $\mathbf{H}$ und bilden automorphe Formen zum Gewicht 1/2 zu den Gruppen

$$\Gamma_0(2):=\left\{T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma:\ c\equiv 0(\mathrm{mod}\ 2)\right\},$$

bzw.

$$\Gamma_\vartheta:=\left\{T\in\Gamma:\ T\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(\mathrm{mod}\ 2)\right\},$$

bzw.

$$\Gamma^0(2):=\left\{T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma:\ b\equiv 0(\mathrm{mod}\ 2)\right\}$$

und gewissen Multiplikatorsystemen v_2, bzw. v_3, bzw. v_4. (Für eine genauere Beschreibung der Multiplikatorsysteme siehe z.B. Petersson [17].)

Bilden wir nun die vektorwertige Funktion

$$\Theta(z)=(\vartheta_2(z),\vartheta_3(z),\vartheta_4(z))', \qquad (' = \text{transponiert})$$

so genügt diese einem Transformationsgesetz

$$\Theta(Tz)=(cz+d)^{\frac{1}{2}}\,\mathscr{V}(T)\,\Theta(z)$$

für alle $T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$ mit einem dreidimensionalen Multiplikatorsystem $\mathscr{V}$: $\Gamma\mapsto\mathrm{Aut}(\mathbf{C}^3)$ zum Gewicht 1/2(mod 2). Dies läßt sich schreiben in der Form

$$\mathscr{V}(T)=\rho(T)\begin{pmatrix}v_2^*(T) & 0 & 0\\ 0 & v_3^*(T) & 0\\ 0 & 0 & v_4^*(T)\end{pmatrix}, \tag{1}$$

mit gewissen Erweiterungen v_i^* der v_i auf die ganze Modulgruppe (siehe z.B. [14]) und einer unitären Darstellung ρ von $\mathrm{SL}(2,\mathbf{F}_2)$ in $\mathbf{C}^3$, wobei $\mathbf{F}_2$ den zweielementigen Körper bezeichne.

$\mathscr{V}$ genügt als Multiplikatorsystem zum Gewicht 1/2(mod 2) den Relationen

(i) $\mathscr{V}(-I_2)=-iI_3$, ($I_k=k-$dim. Einheitsmatrix)
(ii) $\mathscr{V}(ST)=e^{\pi i w(S,T)}\mathscr{V}(S)\,\mathscr{V}(T)$,

mit dem 2-Kozykel

$$w(S,T)=\frac{1}{2\pi}(\arg(\gamma i+\delta)+\arg(ci+a)-\arg((\gamma i+\delta)(ci+a))).$$

Hier wie auch im weiteren ist $S=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $T=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$ und $\arg(z)\in(-\pi,\pi]$.

Wir bezeichnen ferner mit D_Γ den (uneigentlichen) Standardfundamentalbereich der Modulgruppe

$$D_\Gamma=\{z\in\mathbf{H}\colon -\tfrac{1}{2}\leqq\Re(z)\leqq\tfrac{1}{2},\ |z|\geqq 1\}$$

und mit $d\mu(z)$ das hyperbolische Flächenelement

$$d\mu(z)=\frac{dx\,dy}{y^2},$$

wobei hier und in Zukunft $x:=\Re(z)$, $y:=\Im(z)$ gesetzt sein möge.

Im folgenden sei außerdem stets $q\equiv\frac{1}{2}\bmod 2$. Dann bezeichne $A(\Gamma\backslash\mathbf{H},q,\mathscr{V})$ der Raum von Funktionen $f\colon\mathbf{H}\mapsto\mathbf{C}^3$ mit

$$f(Tz)=e^{iq\arg(cz+d)}\mathscr{V}(T)f(z)$$

sowie

$$L^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})=\{f\in A(\Gamma\backslash\mathbf{H},q,\mathscr{V})\colon f\text{ ist }\mu\text{-meßbar und }\int\limits_{D_\Gamma}|f(z)|^2\,d\mu(z)<\infty\}.$$

Mit dem Skalarprodukt

$$(f,g)=\int\limits_{D_\Gamma}f'(z)\,\bar{g}(z)\,d\mu(z)$$

wird $L^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$ in üblicher Weise zu einem Hilbertraum.

Der elliptische Differentialoperator

$$-\Delta_q=y^2\left(\frac{\partial_2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-iq\frac{\partial}{\partial x},$$

der sogenannte Laplace-Beltrami Operator, operiert als wesentlich selbstadjungierter Operator auf dem Raum $\mathscr{D}^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$ aller zweimal stetig (reell) differenzierbarer $f\in L^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$ mit $\Delta_q f\in L^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$ und läßt sich zu einem selbstadjungierten Operator $-\tilde{\Delta}_q$ mit in $L^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$ dichtem Definitionsbereich erweitern, siehe z.B. Roelcke [20].

Diejenigen Eigenfunktionen $f\in\mathscr{D}^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$ von Δ_q zum Eigenwert λ, (als Eigenfunktionen eines elliptischen Differentialoperators sind sie dann zwangsläufig reell-analytisch), die von höchstens polynomialem Wachstum in der Spitze ∞ sind, d.h. für die

$$|f(z)|\ll y^c,\ y\to\infty,\ c>0,$$

heißen reell-analytische automorphe Formen zum Gewicht q, Multiplikatorsystem $\mathscr{V}$ und Eigenwert λ. Wir bezeichnen den Raum dieser Funktionen mit $\mathscr{A}_\lambda(\Gamma\backslash\mathbf{H}, q, \mathscr{V})$.

Aus den Resultaten von Roelcke in [20, 21] folgt, daß das Spektrum des Laplace-Beltrami Operators (bzw. seiner selbstadjungierten Erweiterung) enthalten ist im Intervall $\left[\frac{|q|}{2}\left(1-\frac{|q|}{2}\right), \infty\right)$, die Eigenwerte $\lambda_0 \leqq \lambda_1 \leqq \ldots$ sich im Unendlichen häufen und daß neben dem diskreten in unserem Fall auch ein kontinuierliches Spektrum auftritt.

Es sei nun $\{\vec{u}_k\}$ ein orthonormiertes System von Eigenfunktionen, $\vec{u}_k$ gehöre zum Eigenwert λ_k. $\vec{u}_k$ läßt sich in eine Fourierreihe entwickeln. Bevor wir diese angeben können, benötigen wir jedoch noch einige Eigenschaften von $\mathscr{V}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$.

Man rechnet leicht nach, daß

$$\mathscr{V}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}e(\frac{1}{8}) & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \qquad (e(z):=e^{2\pi i z}).$$

Wir setzen $\alpha_1=0$, $\alpha_2=\frac{1}{2}$, $\alpha_3=\frac{1}{8}$. Dann hat $\mathscr{V}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ die Eigenwerte $e(\alpha_1)$, $e(\alpha_2)$, $e(\alpha_3)$ mit den entsprechenden orthonormalen Eigenvektoren

$$\vec{f}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}, \quad \vec{f}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}, \quad \vec{f}_3=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}.$$

Wir schreiben noch $\lambda_k=(1/2+it_k)(1/2-it_k)$ mit $\arg t_k \in \{0, -\pi/2\}$. Dann lautet die Fourierreihe zu $\vec{u}_k$

$$\vec{u}_k(z)=\sum_{j=1}^{3} u_{k,j}(z)\vec{f}_j,$$

wobei

$$(2) \qquad u_{k,j}(z)=\sum_{\substack{n\equiv\alpha_j \bmod 1\\ n\neq 0}} \rho_{k,j}(n)\, W_{\frac{1}{2}\operatorname{sgn}(n), it_k}(4\pi|n|\, y)\, e(nx)$$

$$+\delta_{j1}\begin{cases}\rho_{k,1}(0)\, y^{\frac{1}{2}-it_k}+\tilde{\rho}_{k,1}(0)\, y^{\frac{1}{2}+it_k}, & \text{falls } t_k\neq 0\\ \rho_{k,1}(0)\, y^{\frac{1}{2}}+\tilde{\rho}_{k,1}(0)\, y^{\frac{1}{2}}\log y, & \text{falls } t_k=0.\end{cases}$$

Hierbei bezeichnet $W_{k,\mu}(z)$ die schnell fallende Whittakerfunktion, $\rho_{k,j}(n)$, $\tilde{\rho}_{k,j}(0)$ heißen die Fourierkoeffizienten von $\vec{u}_k$.

Falls $\rho_{k,1}(0)=\tilde{\rho}_{k,1}(0)=0$, so nennt man $\vec{u}_k$ eine Spitzenform. Man sieht sofort, daß Eigenfunktionen zu einem Eigenwert $\lambda_k \geqq 1/4$ Spitzenformen sein müssen. Diejenigen Eigenwerte, die kleiner als 1/4 sind, heißen exzeptionelle Eigenwerte.

Wir betrachten nun genauer das kontinuierliche Spektrum. Aus den Resultaten in [21] folgt, daß dies das Intervall $[\frac{1}{4}, \infty)$ überdeckt und die Vielfachheit 1 besitzt. Das entsprechende Eigenpaket ist gegeben durch

$$\vec{e}_\lambda(z) = \int_0^{\sqrt{\lambda - \frac{1}{4}}} \vec{E}(z, \tfrac{1}{2} + it, \mathscr{V})\, dt,$$

wobei für $\Re(s) > 1$ die Eisensteinreihe $\vec{E}(z, s, \mathscr{V})$ durch die Reihe

$$\vec{E}(z, s, \mathscr{V}) = \tfrac{1}{2} \sum_{T \in [U] \backslash \Gamma} \mathscr{V}(T)^{-1}\, \vec{f}_1(\Im(Tz))^s\, e^{-\frac{i}{2}\arg(cz+d)},$$

$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gegeben ist. Diese Reihe besitzt eine meromorphe Fortsetzung auf die gesamte komplexe Ebene, die gegeben ist durch die folgende Fourierentwicklung.

Proposition 2.1 *Es sei*

$$\chi_n(p) = \left(\frac{D}{P}\right), \qquad \lambda_p(n) = \tfrac{1}{2} \operatorname{ord}_p(n/D)$$

wobei D die Diskriminante von $\mathbf{Q}(\sqrt{n})$ *sei und* $\left(\frac{D}{p}\right)$ *das Kroneckersymbol. Setze*

$$l_2(n, s) = 1 + \chi_n(2) - \left(\chi_n(2) - \frac{1}{2^{2s-\frac{3}{2}}}\right)\left(1 + \frac{1}{2^{2s-\frac{1}{2}}}\right) \sum_{l=0}^{\lambda_2(n)} 2^{l(2-4s)}$$

$$l_p(n, s) = p^{\lambda_p(n)(2-4s)} + \left(1 - \frac{\chi_n(p)}{p^{2s-\frac{1}{2}}}\right) \sum_{l=0}^{\lambda_p(n)-1} p^{l(2-4s)}, \quad p > 2,$$

und bezeichne mit $L_n(s)$ *die zum Charakter* χ_n *gehörende Dirichletsche Reihe.*
Definiere ferner

$$\tau_3(n, s) = \frac{|n|^{s-1}}{2^{3(2s-\frac{1}{2})}} \prod_{p \mid n} l_p(n, s)$$

$$\tau_1(n, s) = \tau_2(n, s) = \frac{|n|^{s-1}}{2^{2s+\frac{1}{2}}} \prod_{p \mid 2n} l_p(n, s).$$

Dann gilt

$$\vec{E}(z, s, \mathscr{V}) = (y^s + \beta_1(0, s)\, y^{1-s})\, \vec{f}_1 + \sum_{j=1}^{3} \sum_{\substack{n \equiv \alpha_j \bmod 1, \\ n \neq 0}} \beta_j(n, s)\, W_{\operatorname{sgn}(n)\frac{1}{4}, s-\frac{1}{2}}(4\pi |n|\, y)\, e(nx)\, \vec{f}_j$$

mit

$$\beta_1(0,s)=\frac{(2^{s-\frac{3}{4}}+2^{\frac{3}{4}-s})\,\pi^{\frac{1}{2}}\,\Gamma(2s-1)\,\zeta(4s-2)}{(2^{s-\frac{1}{4}}+2^{\frac{1}{4}-s})\,\Gamma(2s-\frac{1}{2})\,\zeta(4s-1)},$$

und für $n\neq 0$

$$\beta_1(n,s)=\frac{\sqrt{2}\pi^s L_{2n}(2s-\frac{1}{2})\,\tau_1(2n,s)}{(1+2^{2s-\frac{1}{2}})\,\zeta(4s-1)\,\Gamma(s+\operatorname{sgn}(n)\frac{1}{4})},$$

$$\beta_2(n,s)=\frac{\sqrt{2}\pi^s L_{2n+1}(2s-\frac{1}{2})\,\tau_2(2n+1,s)}{(1+2^{2s-\frac{1}{2}})\,\zeta(4s-1)\,\Gamma(s+\operatorname{sgn}(n)\frac{1}{4})},$$

$$\beta_3(n,s)=\frac{\pi^s L_{8n+1}(2s-\frac{1}{2})\,\tau_3(8n+1,s)}{(1+2^{\frac{1}{2}-2s})\,\zeta(4s-1)\,\Gamma(s+\operatorname{sgn}(n)\frac{1}{4})}.$$

Die Herleitung dieser Fourierentwicklung findet man in Siegel [22], eine etwas ausführlichere Darstellung ist gegeben in [13].

Anhand dieser Entwicklung erkennt man leicht, daß die Pole von $\vec{E}(z,s,\mathscr{V})$ (als Funktion von s) bestimmt sind durch die Nullstellen von $\zeta(4s-1)$, die Pole von $\Gamma(2s-1)$ sowie den Pol von $\zeta(4s-2)$ and der Stelle $s=3/4$.

Das Eigenpaket $\vec{e}_\lambda$ bildet nun zusammen mit $\{\vec{u}_k\}$ ein vollständiges Orthogonalsystem in $L^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$, vgl. [21, §12].

Wir beweisen noch das folgende

Theorem 2.1 (i) $\lambda_0=\frac{3}{16}$.

(ii) *Der Raum* $\mathscr{A}_{3/16}(\Gamma\backslash\mathbf{H},\frac{1}{2},\mathscr{V})$ *wird aufgespannt von* $y^{\frac{1}{4}}\Theta(z)$.

(iii) $\lambda_1>\frac{1}{4}$.

Beweis. Der Beweis für (iii) folgt ähnlich wie in [7], S. 511f. Weiter ergibt sich aus Roelcke [21, S. 302, Satz 11.2], daß zu einem Eigenwert $\lambda=(\frac{1}{2}+it)(\frac{1}{2}-it)$ genau dann eine Nichtspitzenform gehört, wenn $s_0=\frac{1}{2}+it$ ein Pol von $\vec{E}(z,s,\mathscr{V})$ (als Funktion von s) ist, der zugehörige Eigenraum ist eindimensional und wird vom Residuum der Eisensteinreihe in s_0 aufgespannt.

Der einzige Pol von $\vec{E}(z,s,\mathscr{V})$ in der Halbebene $\Re(s)\geqq\frac{1}{2}$ liegt aber, wie oben gezeigt, bei $s_0=3/4$. Die einzigen Eigenformen, die Nichtspitzenformen sind, treten also auf für $\lambda_0=3/16$, der Raum wird aufgespannt von $y^{\frac{1}{4}}\Theta(z)$, wie man leicht nachrechnen kann. □

Führen wir jetzt entsprechend [7, S. 701] zu $j=1,2,3$ und beliebigem $n\in\mathbf{Z}$ für $z\in\mathbf{H}$ und $\Re(s)>1$ die Poincaré-Reihen

$$\vec{P}_{j,n}(z,s,\mathscr{V}):=\tfrac{1}{2}\sum_{T\in\Gamma_\infty\backslash\Gamma}\mathscr{V}^{-1}(T)\,\vec{f}_j(\mathfrak{T}(Tz))^s\,j_T^{-1}(z)\,e^{-2\pi|n+\alpha_j'|\Im(Tz)+2\pi i(n+\alpha_j)\Re(Tz)}$$

ein (hierbei bezeichne Γ_∞ die von $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ erzeugte Untergruppe von Γ und $j_T(z)=e^{i2\arg(cz+d)}$, wenn $T=\begin{pmatrix}a&b\\c&d\end{pmatrix}$), so gilt für $\Re(s_1),\Re(s_2)>1$ die Parsevalsche Gleichung

$$(3)\qquad (\vec{P}_{j,n}(\cdot,s_1,\mathscr{V}),\vec{P}_{h,m}(\cdot,s_2,\mathscr{V}))=\sum_{k=0}^{\infty}c_j(k,n,s_1)\,\bar{c}_h(k,n,s_2)+\frac{1}{4\pi}\int_{-\infty}^{\infty}g_j(t,n,s_1)\,\bar{g}_h(t,m,s_2)\,dt$$

mit

$$c_j(k,n,s)=(\vec{P}_{j,n}(\cdot,s,\mathscr{V}),\vec{u}_k),$$
$$g_j(k,n,s)=(\vec{P}_{j,n}(\cdot,s,\mathscr{V}),\vec{E}(z,\tfrac{1}{2}+it,\mathscr{V})).$$

Wir haben hier benutzt, daß, wie man leicht sieht, $\vec{P}_{j,n}(\cdot,s,\mathscr{V})\in\mathscr{D}^2(\Gamma\backslash\mathbf{H},q,\mathscr{V})$. Beachte ferner, daß die Eisensteinreihe nicht quadratintegrierbar ist, die obige Schreibweise mit dem Skalarprodukt in dem Ausdruck für $g_j(k,n,s)$ jedoch dennoch Sinn macht, da das definierende Integral hier existiert.

Das Skalarprodukt der beiden Poincaré-Reihen läßt sich andererseits auch direkt auswerten, was dann im Vergleich mit der Parsevalschen Gleichung wie in Kuznecovs Beweis in [11] zu einer sog. vorläufigen Version der Kuznecovschen Summenformel führt.

Theorem 2.2 *Für $j,h\in\{1,2,3\}$ und $n,m\in\mathbf{Z}$ mit $(n+\alpha_j)(m+\alpha_h)>0$ definiere man die verallgemeinerten Kloostermanschen Summen*

$$S(n+\alpha_j,m+\alpha_h,c,\mathscr{V})=\sum_{\substack{a \bmod c\\(a,c)=1}}\vec{f}_j'\,\mathscr{V}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\vec{f}_h\cdot e\left(\left(n+\alpha_j\right)\frac{a}{c}+(m+\alpha_h)\frac{d}{c}\right),$$

wobei d derart, daß $ad\equiv 1(\mathrm{mod}\,c)$, und $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ sei aus Γ. Ferner sei $K_\nu(z)$ die Mac Donaldsche Funktion zum Index ν und

$$\Lambda_\pm(s_1,s_2,t)=\frac{\Gamma(s_1-\frac{1}{2}+it)\,\Gamma(s_1-\frac{1}{2}-it)\,\Gamma(s_2-\frac{1}{2}+it)\,\Gamma(s_2-\frac{1}{2}-it)}{\Gamma(s_1\mp\frac{1}{4})\,\Gamma(s_2\mp\frac{1}{4})}$$

$\mathscr{K}$ bezeichne den positiv orientierten Halbkreis von $-i$ bis i. Dann gilt mit der Abkürzung $\beta_{t,j}(n):=\beta_j(n,\frac{1}{2}+it)$ für alle $s_1,s_2\in\mathbf{C}$ mit $\Re(s_1)+\Re(s_2)>3/4$

$$\delta_{nm}\delta_{jh}\frac{\Gamma(s_1+s_2-1)}{(4\pi|n+\alpha_j|)^{1-s_1-s_2}}+2^{3-s_1-s_2}e^{-\frac{3\pi i}{4}}\sum_{c>0}\frac{S(n+\alpha_j,m+\alpha_h,c,\mathscr{V})}{c^{s_1+s_2}}$$
$$\cdot\int_{\mathscr{K}}K_{s_1-s_2}\left(\frac{4\pi\sqrt{(n+\alpha_j)(m+\alpha_h)}}{c}u\right)(u+u^{-1})^{s_1+s_2-2}u^{\mathrm{sgn}(n)\frac{1}{2}}\frac{du}{u}$$
$$=(4\pi\sqrt{(n+\alpha_j)(m+\alpha_h)})^{2-s_1-s_2}\left(\sum_{k\geqq 0}\bar{\rho}_{k,j}(n)\,\rho_{k,h}(m)\,\Lambda_{\mathrm{sgn}(n)}(s_1,s_2,t_k)\right.$$
$$\left.+\frac{1}{4\pi}\int_{-\infty}^{\infty}\bar{\beta}_{t,j}(n)\,\beta_{t,h}(m)\,\Lambda_{\mathrm{sgn}(n)}(s_1,s_2,t)\,dt\right).$$

Mit Hilfe dieses Theorems erhalten wir dann die angekündigte asymptotische Formel

Theorem 2.3 *Sei T derart, daß $1 \leqq T - \log T$ und $n, m \in \mathbf{Z}$ mit $nm > 0$. Dann gilt für $\varepsilon > 0$*

$$\sum_{0 \leqq t_k \leqq T} \frac{\overline{\rho_{k,j}(n)}\,\rho_{k,h}(m)}{\operatorname{ch}(\pi t_k)} = \delta_{nm}\,\delta_{jh}\,\frac{T^{2-\operatorname{sgn}(n)\frac{1}{2}}}{8\pi^2 |n+\alpha_j|(2-\operatorname{sgn}(n)\frac{1}{2})}$$
$$+ \mathcal{O}_\varepsilon((nm)^{-\frac{1}{2}}(T^{\frac{1}{2}-\operatorname{sgn}(n)\frac{1}{4}} + T^{\frac{1}{8}-\operatorname{sgn}(n)\frac{1}{8}}|n|^{\frac{1}{4}})(T^{\frac{1}{2}-\operatorname{sgn}(n)\frac{1}{4}} + T^{\frac{1}{8}-\operatorname{sgn}(n)\frac{1}{8}}|m|^{\frac{1}{4}})(Tnm)^{\varepsilon}).$$

3 Beweis der Theoreme

3.1 Beweis von Theorem 2.2

Wir benutzen im folgenden die Abkürzungen

$$n_j := n+\alpha_j, \qquad m_h := n+\alpha_h, \qquad x_T := \Re(Tz), \qquad y_T := \Im(Tz), \qquad E(z) := e^{-2\pi z}$$

und berechnen zunächst die in der Parsevalschen Formel (3) auftretenden Größen.

$$\begin{aligned} c_j(k,n,s) &= (\vec{P}_{j,n}(\cdot, s, \overline{\mathscr{V}}), u_{\lambda_k}) \\ &= \tfrac{1}{2} \int_{D_\Gamma} \sum_{T \in [U]\backslash\Gamma} \vec{f}_j' \,\overline{\mathscr{V}}(T)\, y_T^s\, E(|n_j|\, y_T)\, e(n_j x_T)\, j_T^{-1}(z)\, \bar{u}_{\lambda_k}\, d\mu(z) \\ &= \tfrac{1}{2} \sum_{T \in [U]\backslash\Gamma} \int_{T(D_r)} \vec{f}_j'\, y^s\, E(|n_j|\, y)\, e(n_j x)\, \bar{u}_{\lambda_k}\, d\mu(z) \\ &= \int_0^\infty \int_{-\frac{1}{2}}^{\frac{1}{2}} \vec{f}_j'\, y^{s-2}\, E(|n_j|\, y)\, e(n_j x)\, \bar{u}_{\lambda_k}\, dx\, dy \\ &= \bar{\rho}_{k,j}(n) \int_0^\infty y^{s-2} e^{-2\pi |n_j| y}\, W_{\operatorname{sgn}(n)\frac{1}{4}, it_k}(4\pi |n_j|\, y)\, dy \\ &= \bar{\rho}_{k,j}(n)(4\pi|n_j|)^{1-s}\, \frac{\Gamma(s-\frac{1}{2}+it_k)\,\Gamma(s-\frac{1}{2}-it_k)}{\Gamma(s-\operatorname{sgn}(n)\frac{1}{4})} \end{aligned}$$

Die letzte Gleichheit folgt aus [12, S. 316].

Ganz analog folgt

$$\begin{aligned} g_j(t,n,s) &= (\vec{P}_{j,n}(\cdot, s, \overline{\mathscr{V}}), \vec{E}_{j,n}(\cdot, \tfrac{1}{2}+it, \overline{\mathscr{V}}, \vec{f}_1)) \\ &= \bar{\beta}_{t,j}(n)\, 4\pi |n_j|^{1-s}\, \frac{\Gamma(s-\frac{1}{2}+it)\,\Gamma(s-\frac{1}{2}-it)}{\Gamma(s-\operatorname{sgn}(n)\frac{1}{4})}. \end{aligned}$$

Nun berechnen wir das Skalarprodukt der beiden Poincaré-Reihen auf direktem Wege.

$$
\begin{aligned}
(4)\quad & (\vec{P}_{j,n}(\cdot, s_1, \mathcal{V}), \vec{P}_{h,m}(\cdot, \bar{s}_2, \mathcal{V}) \\
&= \int_{D_r} \vec{P}'_{j,n}(\cdot, s_1, \mathcal{V})\, \tfrac{1}{2} \sum_{T \in [U]\backslash\Gamma} \mathcal{V}'(T)\, \vec{f}_h\, y_T^{s_2}\, E(|m_h|\, y_T)\, e(-m_h x_T)\, j_T(z)\, d\mu(z) \\
&= \int_0^\infty \int_0^1 \vec{P}'_{j,n}(\cdot, s_1, \mathcal{V})\, \vec{f}_h\, y^{s_2-2}\, E(|m_h|\, y)\, e(-m_h x)\, dx\, dy \\
&= \delta_{n_j m_h} \int_0^\infty y^{s_1+s_2-2}\, E(2|n_j|\, y)\, dy \\
&\quad + \tfrac{1}{2} \int_0^\infty \sum_{\substack{T \in [U]\backslash\Gamma \\ T \notin [U]}} \int_0^1 \vec{f}'_j\, \overline{\mathcal{V}}(T)\, \vec{f}_h\, y_T^{s_1}\, E(|n_j|\, y_T + |m_h|\, y) \\
&\quad \cdot e(n_j x_T - m_h x)\, j_T^{-1}(z)\, y^{s_2-2}\, dx\, dy \\
&= \delta_{n_j m_h} \frac{\Gamma(s_1+s_2-1)}{(4\pi|n_j|)^{s_2-s_1-1}} \\
&\quad + \frac{1}{2} \int_0^\infty \sum_{T \in [U]\backslash\Gamma/[U]} \vec{f}'_j\, \mathcal{V}(T)\, \vec{f}_h \\
&\quad \cdot \int_{-\infty}^\infty y_T^{s_1}\, E(|n_j|\, y_T + |m_h|\, y)\, e(n_j x_T - m_h x)\, j_T^{-1}(z)\, y^{s_2-2}\, dx\, dy.
\end{aligned}
$$

Wir benutzen, daß

$$y_T = \frac{y}{|cz+d|^2}, \quad x_T = \frac{a}{c} - \frac{x+d/c}{|cz+d|^2},$$

wobei wie üblich $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Außerdem beachte man, daß wegen

$$\mathcal{V}(T)\, j_T(z) = \mathcal{V}(-T)\, j_{-T}(z)$$

die Summation auf der rechten Seite von (4) eingeschränkt werden kann auf alle $T \in [U]\backslash\Gamma/[U]$ mit $c>0$, wobei dann der Faktor $\frac{1}{2}$ vor dem Integral wegfällt. Der zweite Summand auf der rechten Seite von (4) wird dann, wenn noch x durch $x+d/c$ ersetzt wird, und unter Berücksichtigung der Definition der verallgemeinerten Kloostermanschen Summen zu

$$
\begin{aligned}
(5)\quad & \int_0^\infty \sum_{c>0} S(n_j, m_h, c, \mathcal{V}) \\
&\quad \cdot \int_{-\infty}^\infty \frac{y^{s_1+s_2-2}}{|cz|^{2s_1}}\, E\left(y\left(\frac{|n_j|}{|cz|^2} + |m_h|\right)\right) e\left(-x\left(\frac{n_j}{|cz|^2} + m_h\right)\right) \left(\frac{\bar{z}}{z}\right)^{1/4} dx\, dy.
\end{aligned}
$$

Im Integral über x führt man nun zunächst die Substitution $u=x/y$ durch und anschließend im Integral über y die Substitution $v=cy\sqrt{u^2+1}$, so wird (5) zu

$$(6)\qquad \int_0^\infty \sum_{c>0} \frac{S(n_j,m_h,c,\mathscr{V})}{c^{s_1+s_2}} \int_{-\infty}^{\infty} (u^2+1)^{-\frac{s_1+s_2}{2}} v^{s_2-s_1-1} \cdot \exp\left(-\frac{2\pi}{c}\left(\frac{|n_j|}{v}+|m_h|\,v\right)\frac{\sqrt{1+\operatorname{sgn}(n)\,iu}}{\sqrt{1-\operatorname{sgn}(n)\,iu}}\right)\left(\frac{u-i}{u+i}\right)^{\frac{1}{4}} du\,dv.$$

Für die Kloostermanschen Summen gilt die Abschätzung, siehe [14],

$$(7)\qquad S(n_j,m_h,c,\mathscr{V}) \ll \sqrt{(8n_j,8m_h,c)}\,\sigma_0(c)\,c^{\frac{1}{2}},$$

und hieraus folgt mit der Argumentation wie in [11] unter Verwendung der für $\Re(z)>0$ gültige Integraldarstellung für die Mac Donaldsche Funktion

$$K_v(z)=\frac{1}{2}\int_0^\infty \exp\left(-\frac{z}{2}(v+v^{-1})\right) v^{-v-1}\,dv,$$

daß für $\Re(s_1)+\Re(s_2)>\frac{3}{4}$

$$(8)\quad (\vec{P}_{j,n}(\cdot,s_1,\mathscr{V}),\vec{P}_{h,m}(\cdot,\bar{s}_2,\mathscr{V}) = \delta_{n_jm_h}\frac{\Gamma(s_1+s_2-1)}{(4\pi|n_j|)^{s_2-s_1-1}}+2\left(\frac{n_j}{m_h}\right)^{\frac{s_2-s_1}{2}}\sum_{c>0}\frac{S(n_j,m_h,c,\mathscr{V})}{c^{s_1+s_2}} \cdot \int_{-\infty}^{\infty}(u^2+1)^{\frac{s_1+s_2}{2}}\left(\frac{u-i}{u+i}\right)^{\frac{1}{4}} K_{s_2-s_1}\left(\frac{4\pi\sqrt{|n_jm_h|}}{c}\sqrt{\frac{1+\operatorname{sgn}(n)\,iu}{1-\operatorname{sgn}(n)\,iu}}\right)du.$$

Nach der Substitution $u\mapsto\sqrt{\frac{1+\operatorname{sgn}(n)\,iu}{1-\operatorname{sgn}(n)\,iu}}$ folgt durch Vergleich der dann entstandenen rechten Seite von (8) mit der Parsevalschen Gleichung (3) die Behauptung des Theorems.

3.2 Beweis von Theorem 2.3

Wir setzen in der vorläufigen Version der Summenformel aus Theorem 2.2 $s_1=1+ir$, $s_2=1-ir$. Wir erhalten dann die Gleichung

$$(9)\quad \frac{\delta_{n_jm_h}}{4\pi|n_j|}+2e^{-\frac{3\pi i}{4}}\sum_{c>0}\frac{S(n_j,m_h,c,\mathscr{V})}{c^2}\int_{\mathscr{K}} K_{2ir}\left(\frac{4\pi\sqrt{n_jm_h}}{c}u\right)u^{\operatorname{sgn}(n)\frac{1}{2}}\frac{du}{u} = \sum_{k\geqq 0}\frac{\overline{\rho_{k,j}}(n)\,\rho_{k,h}(m)}{\operatorname{ch}(\pi t_k)}K_{\operatorname{sgn}(n)}(t_k,r)+\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\overline{\beta_{t,j}}(n)\,\beta_{t,h}(m)}{\operatorname{ch}(\pi t)}K_{\operatorname{sgn}(n)}(t,r)\,dt$$

mit

$$K_{\pm}(t,r)=\frac{\pi^2 \operatorname{ch}(\pi t)}{\operatorname{ch}(\pi(t-r))\operatorname{ch}(\pi(t+r))\,\Gamma(1\mp\frac{1}{4}+ir)\,\Gamma(1\mp\frac{1}{4}-ir)},$$

wobei wir noch die bekannte Relation

$$\Gamma(\tfrac{1}{2}+z)\,\Gamma(\tfrac{1}{2}-iz)=\frac{\pi}{\cos \pi z}$$

benutzt haben. Wir multiplizieren (9) mit

$$\operatorname{ch}(\pi r)\,\Gamma(1-\operatorname{sgn}(n)\tfrac{1}{4}+ir)\,\Gamma(1-\operatorname{sgn}(n)\tfrac{1}{4}-ir)=:g_{\operatorname{sgn}(n)}(r)$$

und integrieren die entstehende Gleichung bzgl. r über das Intervall $[Y, X]$, $1\leqq Y<X-\log X$. Für den auf der rechten Seite der Gleichung dann auftretenden Ausdruck

$$k_{Y,X}(t)=\int_Y^X K_{\pm}(t,r)\,g_{\pm}(r)\,dr$$

erhalten wir für $t>0$

$$\begin{aligned}k_{Y,X}(t)&=\frac{\pi^2}{2}\int_Y^X\left(\frac{1}{\operatorname{ch}(\pi(r+t))}+\frac{1}{\operatorname{ch}(\pi(r-t))}\right)dr\\&=\mathcal{O}(e^{-\pi(t+Y)})+\pi^2\int_{Y-t}^{X-t}\frac{e^{\pi r}}{1+e^{2\pi r}}\,dr\\&=\pi(\arctan(e^{\pi(X-t)})-\arctan(e^{\pi(Y-t)}))+\mathcal{O}(e^{-\pi Y}).\end{aligned}$$

Wir benutzen noch

$$\arctan(u)=\begin{cases}\mathcal{O}(|u|) & \text{falls}\quad u\to 0\\ \frac{\pi}{2}+\mathcal{O}(|u|^{-1}) & \text{falls}\quad u\to\infty\end{cases}$$

und erhalten für $t>0$

$$0<k_{Y,X}(t)\leqq\frac{\pi^2}{2}+\mathcal{O}(e^{-\pi Y}). \tag{10}$$

Hierüber hinaus gilt

$$0<k_{Y,X}(t)\ll X^{-\pi},\quad \text{falls}\quad Y\geqq\log X\quad\text{und}\quad 0\leqq t\leqq Y-\log X, \tag{11}$$

$$k_{Y,X}(t)=\frac{\pi^2}{2}+\mathcal{O}(X^{-1}+e^{\pi(Y-t)}),\quad \text{falls}\quad Y\leqq t\leqq X-\log X, \tag{12}$$

$$0<k_{Y,X}(t)\ll e^{\pi(X-t)},\quad \text{falls}\quad X+\log\leqq t. \tag{13}$$

Falls $0<it<\frac{1}{2}$, so sieht man leicht, daß

$$\int_Y^X K_\pm(t,r)\, g_\pm(r)\, dr \ll e^{-\pi Y}. \tag{14}$$

3.2.1 Der Eisenstein-Term. Wir benutzen dies nun zunächst für die Abschätzung des Summanden mit den Fourierkoeffizienten der Eisensteinreihe auf der rechten Seite von (9).

Außerdem benötigen wir die folgende quadratische Integralmittelabschätzung für Dirichletsche L-Reihen, die sich als direkte Folgerung eines Resultates von Motohashi [16] ergibt.

Lemma 3.1 *Es sei χ ein Kongruenzcharakter* mod q *und $L(s,\chi)$ die dazugehörige Dirichletsche L-Reihe. Dann gilt für $T>2$*

$$\int_0^T |L(\tfrac{1}{2}+it,\chi)|^2\, dt \ll (T+q^{\frac{1}{2}})(\log qT)^4.$$

Wir betrachten nun den Ausdruck

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\overline{\beta_{t,j}}(n)\,\beta_{t,h}(m)}{\operatorname{ch}(\pi t)}\, k_{Y,X}(t)\, dt \tag{15}$$

$$\ll \int_{-\infty}^{\infty} \frac{|L_{\rho_j n_j}(\frac{1}{2}+2it)||L_{\rho_h m_h}(\frac{1}{2}+2it)||\tau_j(\rho_j n_j,\frac{1}{2}+it)||\tau_h(\rho_h m_h,\frac{1}{2}+it)|}{|\zeta(1+4it)|^2|\Gamma(\frac{1}{2}+\operatorname{sgn}(n)\frac{1}{4}+it)|^2\operatorname{ch}(\pi t)}\, k_{Y,X}(t)\, dt$$

mit $\rho_1:=\rho_2:=2$, $\rho_3:=8$.

Eine elementare Rechnung zeigt, daß für $j\in\{1,2,3\}$

$$\tau_j(\rho_j n_j,\tfrac{1}{2}+it)\ll_\varepsilon |n|^{-\frac{1}{2}+\varepsilon}$$

mit einer nur von ε abhängigen impliziten Konstanten.

Außerdem benutzen wir

$$|\Gamma(\tfrac{1}{2}+\operatorname{sgn}(n)\tfrac{1}{4}+it)|^2 = 2\pi t^{\operatorname{sgn}(n)\frac{1}{2}} e^{-\pi t}(1+\mathcal{O}(t^{-1})), \qquad t>1 \tag{16}$$

sowie die bekannten Abschätzungen

$$\zeta(1+it)^{-1}\ll \log t, \qquad t\to\infty,$$

siehe z.B. [8, S. 310] und

$$L(\tfrac{1}{2}+it,\chi)\ll_\varepsilon (tq)^{\frac{1}{4}+\varepsilon},$$

wobei χ einen Kongruenzcharakter mod q bezeichne, siehe z.B. [10].

Wir schreiben $U = X + \log(X|nm|)$. Die rechte Seite von (15) ist damit

$$\ll_\varepsilon (nm)^{-\frac{1}{2}+\varepsilon} \int_1^U |L_{\rho_j n_j}(\tfrac{1}{2}+it)||L_{p_h m_h}(\tfrac{1}{2}+it)|\, t^{-\operatorname{sgn}(n)\frac{1}{2}}\, dt$$

$$+ \mathcal{O}_\varepsilon\left((nm)^{-\frac{1}{4}+\varepsilon} \int_U^\infty t^{\frac{1}{2}-\operatorname{sgn}(n)\frac{1}{2}}\, e^{\pi(X-t)}\, dt\right)$$

$$\ll_\varepsilon (nm)^{-\frac{1}{2}+\varepsilon} \sqrt{\int_1^U |L_{p_j n_j}(\tfrac{1}{2}+it)|^2\, t^{-\operatorname{sgn}(n)\frac{1}{2}}\, dt} \sqrt{\int_1^U |L_{\rho_h m_h}(\tfrac{1}{2}+it)|^2\, t^{-\operatorname{sgn}(n)\frac{1}{2}}\, dt}$$

$$+ \mathcal{O}((nm)^{-1} X^{-1}).$$

Aus Lemma 3.1 folgt

$$\int_1^U |L_{\rho_j n_j}(\tfrac{1}{2}+it)|^2\, t^{\frac{1}{2}}\, dt \ll_\varepsilon X^{\frac{1}{2}}(X+|n|^{\frac{1}{2}})(Xnm)^\varepsilon$$

und

$$\int_1^U |L_{\rho_j n_j}(\tfrac{1}{2}+it)|^2\, t^{-\frac{1}{2}} dt$$

$$\leqq \sum_{\alpha=0}^{[1+\log_2(U)]} 2^{-\frac{\alpha}{2}} \int_{2^\alpha}^{2^{\alpha+1}} |L_{\rho_j n_j}(\tfrac{1}{2}+it)|^2\, dt \ll_\varepsilon \sum_{\alpha=0}^{[1+\log_2(U)]} 2^{-\frac{\alpha}{2}}(2^{\alpha(1+\varepsilon)}+|n|^{\frac{1}{2}+\varepsilon})$$

$$\ll_\varepsilon (X^{\frac{1}{2}}+|n|^{\frac{1}{2}})(Xnm)^\varepsilon.$$

Insgesamt erhalten wir damit

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\overline{\beta_{t,j}}(n)\,\beta_{t,h}(m)}{\operatorname{ch}(\pi t)} k_{Y,X}(t)\, dt$$

$$\ll_\varepsilon (n_j m_h)^{-\frac{1}{2}}(X^{\frac{1}{2}-\operatorname{sgn}(n)\frac{1}{4}}+X^{\frac{1}{8}-\operatorname{sgn}(n)\frac{1}{8}}|n|^{\frac{1}{8}})(X^{\frac{1}{2}-\operatorname{sgn}(n)\frac{1}{4}}+X^{\frac{1}{8}-\operatorname{sgn}(n)\frac{1}{8}}|m|^{\frac{1}{8}})(Xnm)^\varepsilon.$$

3.2.2 Der Kloosterman-Term. Wir untersuchen nun den Ausdruck

$$\int_Y^X g_{\operatorname{sgn}(n)(r)} \int_{\mathcal{K}} K_{2ir}(x_{jh}u)\, u^{\operatorname{sgn}(n)\frac{1}{2}} \frac{du}{u}\, dr,$$

wobei wir zur Abkürzung $x_{jh} = \dfrac{4\pi\sqrt{n_j m_h}}{c}$ setzen. Hierbei werden wir Gebrauch machen von dem folgenden Resultat von Landau, siehe auch [8, S. 56].

Lemma 3.2 (Landau) *Sei* f: $[a, b] \mapsto \mathbf{R}$ *zweimal differenzierbar und* f' *monoton. Ferner sei* g: $[a, b] \mapsto \mathbf{R}^+$ *monoton und beschränkt mit* $G := \sup_{[a,b]}(g)$. *Dann gilt: Falls* $|f''| \geqq m > 0$ *auf* $[a, b]$

$$\left| \int_a^b g(u)\, e^{if(u)}\, du \right| \leqq 8 \frac{G}{\sqrt{m}}. \tag{18}$$

Wir benutzen zunächst die für $\Re(u) > 0$ gültige Integraldarstellung für die Mac Donaldsche Funktion, vgl. [4, S. 82]

$$K_{2ir}(x_{jh}\, u) = \int_0^\infty e^{-x_{jh} u \operatorname{ch}\xi} \cos(2r\xi)\, d\xi$$

und schreiben dies als

$$\left(\int_0^M + \int_M^\infty \right) e^{-x_{jh} u \operatorname{ch}\xi} \cos(2r\xi)\, d\xi, \qquad M > 1.$$

Falls $|\mathfrak{T}(u)| > 1/\sqrt{2}$, so läßt sich das zweite Integral mit Hilfe von (18) abschätzen durch $\mathcal{O}(x^{-\frac{1}{2}} e^{-\frac{M}{2}})$, während für $|\mathfrak{T}(u)| < 1/\sqrt{2}$, d.h. $\Re(u) > 1/\sqrt{2}$ eine direkte Rechnung zeigt, daß das zweite Integral dann $\mathcal{O}(x^{-1} e^{-M})$ ist. Insgesamt folgt also

$$K_{2ir}(x_{jh}\, u) = \int_0^M e^{-x_{jh} u \operatorname{ch}\xi} \cos(2r\xi)\, d\xi + O(x^{-\frac{1}{2}} e^{-\frac{M}{2}} + x^{-1} e^{-M}). \tag{19}$$

Es soll nun das dreifache Integral

$$I_{\operatorname{sgn}(n)} := \int_Y^X \int_{\mathscr{K}} \int_0^M g_{\operatorname{sgn}(n)}(r)\, e^{-x_{jh} u \operatorname{ch}\xi} \cos 2r\xi\, u^{\operatorname{sgn}(n)\frac{1}{2}}\, d\xi \frac{du}{u}\, dr$$

untersucht werden. Nach partieller Integration bzgl. ξ erhält man zunächst

$$\begin{aligned} I_{\operatorname{sgn}(n)} &= \frac{1}{2} \int_Y^X \frac{g_{\operatorname{sgn}(n)}(r)}{r} \sin(2r\, M)\, dr \int_{\mathscr{K}} e^{-x_{jh} u \operatorname{ch} M}\, u^{\operatorname{sgn}(n)\frac{1}{2}} \cdot \frac{du}{u} \\ &\quad + \frac{1}{2} x_{jh} \int_0^M \int_Y^X \int_{\mathscr{K}} \frac{g_{\operatorname{sgn}(n)}(r)}{r} \sin(2r\xi) \operatorname{sh}\xi\, e^{-x_{jh} u \operatorname{ch}\xi}\, u^{\operatorname{sgn}(n)\frac{1}{2}}\, du\, dr\, d\xi \\ &=: I_{1,\operatorname{sgn}(n)} + I_{2,\operatorname{sgn}(n)}. \end{aligned}$$

Wir betrachten $I_{2,\operatorname{sgn}(n)}$. Partielle Integration bzgl. u ergibt

$$I_{2,\operatorname{sgn}(n)} = \int_0^M \int_Y^X \frac{g_{\operatorname{sgn}(n)}(r)}{r} \sin(2r\xi)\, C_{\operatorname{sgn}(n)}(x_{jh}; \xi)\, dr\, d\xi \tag{20}$$

mit

$$C_{\mathrm{sgn}(n)}(x_{jh},\xi) := C_{1,\mathrm{sgn}(n)}(x_{jh},\xi)+C_{2,\mathrm{sgn}(n)}(x_{jh},\xi)$$
$$:= -\cos\left(x_{jh}\operatorname{ch}\xi-\mathrm{sgn}(n)\frac{\pi}{4}\right)\operatorname{th}\xi$$
$$+\frac{\mathrm{sgn}(n)}{4}\int_{\mathscr{K}} e^{-x_{jh}u\operatorname{ch}\xi}u^{\mathrm{sgn}(n)\frac{1}{2}}\frac{du}{u}\operatorname{th}\xi.$$

Wir benutzen (16) und erhalten

$$\frac{g_{\mathrm{sgn}(n)}(r)}{r}=\pi r^{-\mathrm{sgn}(n)\frac{1}{2}}(1+\mathcal{O}(r^{-1})),\qquad r>1,$$

und damit

$$I_{2,\mathrm{sgn}(n)}=\pi\int_0^M\int_Y^X r^{-\mathrm{sgn}(n)\frac{1}{2}}\sin(2r\xi)\,C_{\mathrm{sgn}(n)}(x_{jh},\xi)\,dr\,d\xi \tag{21}$$

$$+\mathcal{O}\left(\int_Y^X r^{-1-\mathrm{sgn}(n)\frac{1}{2}}\left|\int_0^M \sin(2r\xi)\,C_{\mathrm{sgn}(n)}(x_{jh},\xi)\,d\xi\right|dr\right)=:I_{21,\mathrm{sgn}(n)}+I_{22,\mathrm{sgn}(n)}. \tag{22}$$

Partielle Integration bzgl. r ergibt für den ersten Summanden

$$I_{21,\mathrm{sgn}(n)}=\frac{\pi}{2}\int_0^M (Y^{-\mathrm{sgn}(n)\frac{1}{2}}\cos 2Y\xi-X^{-\mathrm{sgn}(n)\frac{1}{2}}\cos 2X\xi)\,C_{\mathrm{sgn}(n)}(x_{jh},\xi)\frac{d\xi}{\xi}$$
$$-\frac{\mathrm{sgn}(n)\,\pi}{4}\int_Y^X\int_0^M r^{-1-\mathrm{sgn}(n)\frac{1}{2}}\cos(2r\xi)\,C_{\mathrm{sgn}(n)}(x_{jh},\xi)\frac{d\xi}{\xi}\,dr.$$

Wir betrachten nun die Integration bzgl. ξ und setzen zuvor $L:=\max(0,-\log x_{jh})$. Man beachte, daß $\operatorname{th}\xi/\xi\leqq 1$ auf $\mathbf{R}$. Dann ergibt sich unter Zuhilfenahme von (18), wenn man zuvor die Integration entsprechend der zwei Monotonieintervalle von $\operatorname{th}\xi/\xi$ aufspaltet und $2\cos u=e^{iu}+e^{iu}$ schreibt, mit einer geeigneten positiven Konstanten c_1

$$\left|\int_0^M \cos(2r\xi)\,C_{1,\mathrm{sgn}(n)}(x_{jh},\xi)\frac{d\xi}{\xi}\right|\leqq L+\left|\int_L^M \cos(2r\xi)\,C_{1,\mathrm{sgn}(n)}(x_{jh},\xi)\frac{d\xi}{\xi}\right| \tag{23}$$
$$\leqq L+c_1\,x_{jh}^{-\frac{1}{2}}\,e^{-L/2}.$$

Entsprechend gilt, mit geeignetem $c_2>0$

$$\left|\int_0^M \sin(2r\xi)\,C_{1,\mathrm{sgn}(n)}(x_{jh},\xi)\,d\xi\right|\leqq L+c_2\,x_{jh}^{-\frac{1}{2}}\,e^{-L/2}. \tag{24}$$

Des weiteren erhalten wir

$$\left|\int_0^M \cos(2r\xi)\, C_{2,\mathrm{sgn}(n)}(x_{jh},\xi)\,\mathrm{th}\,\xi\,\frac{d\xi}{\xi}\right|$$

$$\leqq \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\int_0^M \cos(2r\xi)\, e^{-x_{jh}(\cos t+i\sin t)\mathrm{ch}\,\xi}\,\mathrm{th}\,\xi\,\frac{d\xi}{\xi}\right| dt.$$

Falls $|t|>\pi/4$, d.h. $|\sin t|>1/\sqrt{2}$, so erhält man mit der gleichen Argumentation wie oben

$$\left|\int_0^M \cos(2r\xi)\, e^{-x_{jh}(\cos t+i\sin t)\mathrm{ch}\,\xi}\,\mathrm{th}\,\xi\,\frac{d\xi}{\xi}\right| \leqq L + c_3\, x_{jh}^{-\frac{1}{2}}\, e^{-L/2} \tag{25}$$

mit geeignetem $c_3>0$.
Falls $|t|<\pi/4$, d.h. $\cos t>1/\sqrt{2}$, so gilt für $A>0$

$$\left|\int_0^M \cos(2r\xi)\, e^{-x_{jh}(\cos t+i\sin t)\mathrm{ch}\,\xi}\,\mathrm{th}\,\xi\,\frac{d\xi}{\xi}\right| \leqq \int_0^M e^{-\frac{x_{jh}}{\sqrt{2}}\mathrm{ch}\,\xi}\, d\xi \tag{26}$$

$$\leqq A + \int_A^M e^{-\frac{x_{jh}}{\sqrt{2}}\mathrm{ch}\,\xi}\, d\xi \leqq A + \frac{1}{x_{jh}\pm \mathrm{sh}\,A}.$$

Wähle $A=\mathrm{arsh}(1/\sqrt{x_{jh}})$, falls $x_{jh}>1$ und $A=\mathrm{arsh}(1/x_{jh})$, falls $x_{jh}\leqq 1$. Damit folgt

$$\int_0^M \cos 2r\xi\, C_{2,\mathrm{sgn}(n)}(x_{jh},\xi)\,\mathrm{th}\,\xi\,\frac{d\xi}{\xi} \tag{27}$$

$$\ll \begin{cases} 1+|\log x_{jh}| & \text{falls } x_{jh}\leqq 1 \\ x_{jh}^{-\frac{1}{2}} & \text{falls } x_{jh}>1. \end{cases}.$$

Natürlich gilt (27) genauso, wenn im Integral $\cos 2r\xi$ durch $\sin 2r\xi$ ersetzt wird. Insgesamt erhalten wir so

$$I_{2,\mathrm{sgn}(n)} \ll (X^{-\mathrm{sgn}(n)\frac{1}{2}} + Y^{-\mathrm{sgn}(n)\frac{1}{2}}) \begin{cases} 1+|\log x_{jh}| & \text{falls } x_{jh}\leqq 1 \\ x_{jh}^{-\frac{1}{2}} & \text{falls } x_{jh}>1. \end{cases}$$

Führt man nun den Grenzübergang $M\to\infty$ durch, so gilt mit dem Lemma von Riemann-Lebesgue

$$\lim_{M\to\infty} I_{1,\mathrm{sgn}(n)=0},$$

da das Integral über $\mathscr{K}$ für $M \to \infty$ beschränkt bleibt. Berücksichtigt man noch (19), so folgt

$$\int_Y^X g_{\mathrm{sgn}(n)}(r) \int_{\mathscr{K}} K_{2ir}(x_{jh}u)\, u^{\mathrm{sgn}(n)\frac{1}{2}} \frac{du}{u}\, dr$$
$$\ll (X^{-\mathrm{sgn}(n)\frac{1}{2}} + Y^{-\mathrm{sgn}(n)\frac{1}{2}}) \begin{cases} 1+|\log x_{jh}| & \text{falls } x_{jh} \leqq 1 \\ x_{jh}^{-\frac{1}{2}} & \text{falls } x_{jh} > 1. \end{cases}$$

Der Term mit den Kloostermanschen Summen auf der linken Seite von (9) ist daher unter Berücksichtigung von (7) und mit den Abkürzungen $G_0 := (8n_j, 8m_h)$, $G_1 := (8n_j, 8m_h, c)$

$$(28) \quad \ll (X^{-\mathrm{sgn}(n)\frac{1}{2}} + Y^{-\mathrm{sgn}(n)\frac{1}{2}}) \Bigg((n_j m_h)^{-\frac{1}{4}} \sum_{c < 4\pi\sqrt{n_j m_h}} \frac{S(n_j, m_k, c, \mathscr{V})}{c^{\frac{3}{2}}}$$
$$+ \sum_{c \geqq 4\pi\sqrt{n_j m_h}} \frac{S(n_j, m_k, c, \mathscr{V})}{c^2} \log c \Bigg)$$
$$\ll_\varepsilon (X^{-\mathrm{sgn}(n)\frac{1}{2}} + Y^{-\mathrm{sgn}(n)\frac{1}{2}}) \Bigg((n_j m_h)^{-\frac{1}{4}} \sum_{c < 4\pi\sqrt{n_j m_h}} \frac{G_1^{\frac{1}{2}}}{c^{1-\varepsilon}} + \sum_{c < 4\pi\sqrt{n_j m_h}} \frac{G_1^{\frac{1}{2}}}{c^{\frac{3}{2}-\varepsilon}} \Bigg)$$
$$\ll_\varepsilon (X^{-\mathrm{sgn}(n)\frac{1}{2}} + Y^{-\mathrm{sgn}(n)\frac{1}{2}}) \Bigg((n_j m_h)^{-\frac{1}{4}} \sum_{k \mid G_0} k^{\frac{1}{2}} \sum_{kc' < 4\pi\sqrt{n_j m_h}} (kc')^{-1+\varepsilon}$$
$$+ \sum_{k \mid G_0} k^{\frac{1}{2}} \sum_{kc' \geqq 4\pi\sqrt{n_j m_h}} (kc')^{-\frac{3}{2}+\varepsilon} \Bigg)$$
$$\ll_\varepsilon (X^{-\mathrm{sgn}(n)\frac{1}{2}} + Y^{-\mathrm{sgn}(n)\frac{1}{2}}) (n_j m_h)^{-\frac{1}{4}+\varepsilon}.$$

3.2.3 Schluß des Beweises. Fassen wir die Ergebnisse aus den letzten Unterabschnitten zusammen, so gelangen wir zu dem Zwischenresultat

$$(29) \quad \sum_{k \geqq 0} \frac{\overline{\rho_{k,j}}(n)\, \rho_{k,h}(m)}{\mathrm{ch}(\pi t_k)} k_{Y,X}(t_k) = \frac{\delta_{n_j m_h}}{4|n_j|} \int_Y^X (r^{1-\mathrm{sgn}(n)\frac{1}{2}} + \mathscr{O}(r^{-\mathrm{sgn}(n)\frac{1}{2}}))\, dr$$
$$+ \mathscr{O}_\varepsilon((n_j m_h)^{-\frac{1}{2}} (X^{\frac{1}{2}-\mathrm{sgn}(n)\frac{1}{4}} + X^{\frac{1}{8}-\mathrm{sgn}(n)\frac{1}{8}} |n|^{\frac{1}{4}})$$
$$\cdot (X^{\frac{1}{2}-\mathrm{sgn}(n)\frac{1}{4}} + X^{\frac{1}{8}-\mathrm{sgn}(n)\frac{1}{8}} |m|^{\frac{1}{4}})(Xnm)^\varepsilon).$$

Wir setzen zur Abkürzung

$$A_{n_j m_h}(Y, X) := \sum_{Y \leqq t_k \leqq X} \frac{\overline{\rho_{k,j}}(n)\, \rho_{k,h}(m)}{\mathrm{ch}(\pi t_k)},$$
$$A^{(k)}_{n_j m_h}(Y, X) := \sum_{k \geqq 0} \frac{\overline{\rho_{k,j}}(n)\, \rho_{k,h}(m)}{\mathrm{ch}(\pi t_k)} k_{Y,X}(t_k).$$

Offensichtlich gilt für T derart, daß, $T \geqq 1 + \log T$ und $T^{**} > 1$ derart, daß $T^{**} - \log T^{**} = T$ unter Berücksichtigung von (10) und (12)

$$(30) \quad \frac{\pi^2}{2} A_{n_j n_j}(1 + \log T, T) \leqq A^{(k)}_{n_j n_j}(1, T^{**})(1 + \mathscr{O}(T^{-1})).$$

Hieraus folgt unter Zuhilfenahme von (29)

$$(31)\qquad A_{n_j n_j}(1+\log T, T) \leqq \frac{T^{*\,2-\operatorname{sgn}(n)\frac{1}{2}}}{2\pi^2 |n_j|(2-\operatorname{sgn}(n)\frac{1}{2})} + \mathcal{O}_\varepsilon((nT)^\varepsilon (T^{1-\operatorname{sgn}(n)\frac{1}{2}}|n|^{-1} + T^{\frac{1}{4}-\operatorname{sgn}(n)\frac{1}{4}}|n|^{-\frac{1}{2}})).$$

Aus (31) erhalten wir mit Hilfe partieller Summation

$$(32)\qquad \sum_{t_k > T+\log T} \frac{|\rho_{k,j}(n)|^2}{\operatorname{ch}(\pi t_k)} e^{\pi(T-t_k)} \ll_\varepsilon (nT)^\varepsilon (T^{1-\operatorname{sgn}(n)\frac{1}{2}}|n|^{-1} + |n|^{-\frac{1}{2}}).$$

Benutzen wir die Relation

$$W_{\frac{1}{4},\frac{1}{4}}\left(\frac{z^2}{2}\right) = z^{\frac{1}{2}} 2^{-\frac{1}{4}} e^{-\frac{z^2}{4}},$$

cf. [12, p. 324], so erhalten wir

$$\vartheta_3(z) = \sum_{n\neq 0} e^{\pi i n^2 z} = \sum_{n\neq 0} e^{-\pi n^2 y} e^{2\pi i n 2\frac{x}{2}} = \sum_{n\neq 0} (2n\sqrt{ny})^{-\frac{1}{2}} 2^{\frac{1}{4}} W_{\frac{1}{4},\frac{1}{4}}\left(4\pi n^2 \frac{y}{2}\right) e^{2\pi i n 2\frac{x}{2}}.$$

Analoge Rechnungen für $\vartheta_2(z)$ und $\vartheta_4(z)$ zeigen, daß für $j=1,2,3$

$$\rho_{0,j}(n) \ll n^{-\frac{1}{4}}$$

falls n ein Quadrat und $\rho_{0,j}(n)=0$ sonst.

Damit folgt wegen (14)

$$(33)\qquad \frac{|\rho_{0,j}(n)|^2}{\operatorname{ch}(\pi t_0)} k_{1+\log T, T}\left(\frac{i}{4}\right) \ll 1.$$

Hieraus folgt dann mit T^* derart, daß $T^*+\log T^* = T$

$$(34)\qquad \frac{\pi^2}{2} A_{n_j n_j}(1+\log T, T)(1+\mathcal{O}(T^{-1})) + \mathcal{O}(T^{-1} A_{n_j n_j}(0, 1+\log T)) \cdot \mathcal{O}\left(\sum_{t_k > T^*+\log T^*} \frac{|\rho_{k,j}(n)|^2}{\operatorname{ch}(\pi t_k)} e^{\pi(T^*-t_k)}\right)$$
$$\geqq A^{(k)}_{n_j n_j}(1+2\log T, T^*)$$
$$= \frac{T^{*\,2-\operatorname{sgn}(n)\frac{1}{2}}}{4|n_j|\left(2-\operatorname{sgn}(n)\frac{1}{2}\right)} + \mathcal{O}_\varepsilon((nT)^\varepsilon (T^{1-\operatorname{sgn}(n)\frac{1}{2}}|n|^{-1} + T^{\frac{1}{4}-\operatorname{sgn}(n)\frac{1}{4}}|n|^{-\frac{1}{2}})).$$

Hierbei wurde benutzt: (10) für die Summanden mit $1+\log T < t_k < T^*+\log T^*$, (11) für die Summanden mit $0 \leqq t_k < 1+\log T$, sowie (13) für die Summanden mit $t_k \geqq T^*+\log T^*$. Wir schreiben noch

$$A_{n_j n_j}(0, T_0) = A_{n_j n_j}(T_1, T_0) + \ldots A_{n_j n_j}(T_N, T_{N-1}) + A_{n_j n_j}(0, T_N)$$

mit $T_n := \log T_{n-1} + 1$, $T_0 := \log T + 1$ und N derart, daß $T_N > 2$ und $T_{N+1} \leqq 2$ und wenden (31) auf alle bis auf den letzten der $\mathcal{O}(\log\log T)$ Summanden an, der aber trivialerweise $\mathcal{O}(|n|^{-1/2+\varepsilon})$ ist. Hieraus ergibt sich also

$$A_{n_j n_j}(0, 1+\log T) \ll_\varepsilon (Tn)^\varepsilon n^{-1/2}. \tag{35}$$

Hieraus und aus (32) folgt dann zunächst

$$A_{n_j n_j}(1+\log T, T) \geqq \frac{T^{2-\operatorname{sgn}(n)\frac{1}{2}}}{2\pi^2 |n_j| \left(2-\operatorname{sgn}(n)\frac{1}{2}\right)} + \mathcal{O}_\varepsilon((nT)^\varepsilon (T^{1-\operatorname{sgn}(n)\frac{1}{2}} |n|^{-1} + T^{\frac{1}{4}-\operatorname{sgn}(n)\frac{1}{4}} |n|^{-\frac{1}{2}})). \tag{36}$$

Zusammen mit (31) ergibt dies aber unter Berücksichtigung von (35) die Behauptung von Theorem 2.3 für $n_j = m_h$.

Sei nun also $n_j \neq m_h$. Dann gilt mit T^{**} derart, daß $T^{**} - \log T^{**} = T$

$$\frac{\pi^2}{2} A_{n_j m_h}(1+\log T, T) + \mathcal{O}(A_{n_j m_h}(0, 1+\log T)) A^{(k)}_{n_j m_h}(1, T^{**})(1+\mathcal{O}(T^{-1})) - \sum_{t_k > T} \frac{\bar{\rho}_{k,j}(n)\rho_{k,h}(m)}{\operatorname{ch}(\pi t_k)} k_{1+\log T, T^{**}}(t_k)$$
$$= \mathcal{O}_\varepsilon((n_j m_h)^{-\frac{1}{2}} (X^{\frac{1}{2}-\operatorname{sgn}(n)\frac{1}{4}} + X^{\frac{1}{8}-\operatorname{sgn}(n)\frac{1}{8}} |n|^{\frac{1}{4}})(X^{\frac{1}{2}-\operatorname{sgn}(n)\frac{1}{4}} + X^{\frac{1}{8}-\operatorname{sgn}(n)\frac{1}{8}} |m|^{\frac{1}{4}})(Xnm)^\varepsilon) \tag{37}$$

$$- \sum_{t_k > T} \frac{\bar{\rho}_{k,j}(n)\rho_{k,h}(m)}{\operatorname{ch}(\pi t_k)} k_{1+\log T, T^{**}}(t_k). \tag{38}$$

Wir beachten noch, daß unter Benutzung der Schwarzschen Ungleichung, (10) und der gerade bewiesenen asymptotischen Formel für $A_{n_j n_j}$

$$\left| \sum_{T \leqq t_k \leqq T^{**} + \log T^{**}} \frac{\bar{\rho}_{k,j}(n)\rho_{k,h}(m)}{\operatorname{ch}(\pi t_k)} k_{1+\log T, T^{**}}(t_k) \right|^2$$
$$\leqq \sum_{T \leqq t_k \leqq T^{**} + \log T^{**}} \frac{|\rho_{k,j}(n)|^2}{\operatorname{ch}(\pi t_k)} k_{1+\log T, T^{**}}(t_k) \sum_{T \leqq t_k \leqq T^{**} + \log T^{**}} \frac{|\rho_{k,h}(m)|^2}{\operatorname{ch}(\pi t_k)} k_{1+\log T, T^{**}}(t_k)$$
$$\ll (A_{n_j n_j}(0, T^{**}+\log T^{**}) - A_{n_j n_j}(0, T))(A_{m_h m_h}(0, T^{**}+\log T^{**}) - A_{m_h m_h}(0, T))$$
$$\ll_\varepsilon (Tnm)^\varepsilon (T^{1-\operatorname{sgn}(n)\frac{1}{2}} |n|^{-1} + T^{\frac{1}{4}-\operatorname{sgn}(n)\frac{1}{4}} |n|^{-\frac{1}{2}})(T^{1-\operatorname{sgn}(m)\frac{1}{2}} |m|^{-1} + T^{\frac{1}{4}-\operatorname{sgn}(n)\frac{1}{4}} |m|^{-\frac{1}{2}}).$$

Die gleiche Abschätzung gilt, wie man mit (32) leicht sieht, für $|\sum_{t_k > T^{**}+\log T^{**}}|^2$. Wenden wir die Schwarzsche Ungleichung noch auf $A_{n_j m_h}(0, 1+\log T)$ an und benutzen (35), so folgt damit aus (38) die Behauptung von Theorem 2.3 auch für $n_j \neq m_h$.

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Differentiable sphere theorems for Ricci curvature

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1 Introduction

Let (M^n, g) be a compact n-dimensional Riemannian manifold satisfying $\mathrm{Ric}_{(M,g)} \geqq (n-1)g$, where $\mathrm{Ric}_{(M,g)}$ is the Ricci curvature of (M^n, g). By Myers' theorem, the diameter $\mathrm{diam}_{(M,g)}$ of M satisfies $\mathrm{diam}_{(M,g)} \leqq \pi$ and if $\mathrm{diam}_{(M,g)} = \pi$, then M is isometric to $S^n(1)$ by [Ch]. Similarly, a theorem of Lichnerowicz [L], implies that $\lambda_1(M, g) \geqq n$, while Obata [Ob] proved that $\lambda_1(M, g) = n$, implies (M, g) is isometric to $(S^n(1), g_{\mathrm{can}})$. Here $\lambda_1(M, g)$ is the first non-zero eigenvalue of (M^n, g) for the Laplace operator.

This raises the following questions: If $\mathrm{Ric}_{(M,g)} \geqq (n-1)g$ and the diameter is close to π, is M homeomorphic (diffeomorphic) to $S^n(1)$? The same question if $\lambda_1(M^n, g)$ is close to n. The answer for them is no and is due to Anderson [A2]. For $n \geqq 4$, he constructed n-manifolds (M, g_ε) satisfying $\mathrm{Ric}_{(M,g_\varepsilon)} \geqq (n-1)g_\varepsilon$, $\mathrm{diam}_{(M,g_\varepsilon)} \geqq (\pi - \varepsilon)$ for any $\varepsilon > 0$, which are not homotopy equivalent to $S^n(1)$. These manifolds (M, g_ε) also satisfy $\lambda_1(M, g_\varepsilon) \leqq n + \varepsilon$. It should be remarked that independently Otsu answered the first question for $n \geqq 5$ (see [Ot]). Thus extra hypotheses are needed to have a diameter or eigenvalue sphere theorem for Ricci curvature.

Eschenburg and Nakamura [Es, Nk] independently proved that for given $n \geqq 2$ and ρ_0, $k > 0$, there exists an $\varepsilon = \varepsilon(n, k, \rho_0) > 0$ such that if M admits a metric g satisfying $K_{(M,g)} \geqq -k^2$, $\mathrm{Ric}_{(M,g)} \geqq (n-1)$, $\mathrm{inj}_{(M,g)} \geqq \rho_0$ and $\mathrm{diam}_{(M,g)} \geqq (\pi - \varepsilon)$, then M is homeomorphic to S^n.

On the other hand Croke [Cr2] proved: If M is a compact n-dimensional Riemannian manifold with sectional curvature $K_M \geqq 1$, then there is a constant $C(n) > 1$ such that if $C(n) \cdot n > \lambda_1(M) \geqq n$ then M is homeomorphic to $S^n(1)$.

Recently Petersen [P] improved Eschenburg-Nakamura's result removing the condition on the sectional curvature. He proved the following:

Given an integer $n \geqq 2$ and $\rho_0 > 0$ there exists an $\varepsilon = \varepsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying $\mathrm{Ric}_{(M,g)} \geqq (n-1)g$, $\mathrm{inj}_{(M,g)} \geqq \rho_0$ and $\mathrm{diam}_{(M,g)} \geqq (\pi - \varepsilon)$ then M is a twisted sphere.

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The purpose of this note is to prove the following results,

Theorem 1.1 *Given an integer $n \geq 2$ and $\rho_0 > 0$ there exists an $\varepsilon = \varepsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying*

$$\text{Ric}_{(M,g)} \geq (n-1)g, \quad \text{inj}_{(M,g)} \geq \rho_0, \quad \text{diam}_{(M,g)} \geq \pi - \varepsilon \tag{1.2}$$

then M is diffeomorphic to $S^n(1)$ and the metric g is $\varepsilon' = \varepsilon'(\varepsilon)$ close in the C^α topology to the canonical metric g_{can} of curvature $+1$ on $S^n(1)$.

Theorem 1.3 *Given $n \geq 2$ and $\rho_0 > 0$ there exists an $\varepsilon = \varepsilon(n, \rho_0) > 0$ such that if M admits a metric g satisfying*

$$\text{Ric}_{(M,g)} \geq (n-1)g, \quad \text{inj}_{(M,g)} \geq \rho_0, \quad \lambda_1(M, g) \leq n + \varepsilon \tag{1.4}$$

then M is diffeomorphic to $S^n(1)$ and the metric g is $\varepsilon' = \varepsilon'(\varepsilon)$ close in the C^α topology to the canonical metric g_{can} of curvature $+1$ on $S^n(1)$.

Theorem 1.1 extends Petersen's result to the diffeomorphism case and is based on the C^α-Compactness Theorem (Theorem 2.2, see also [AC]). Theorem 1.3 is a consequence of Theorem 1.1 and Theorem 4.1, due to Croke.

Note that in Anderson's examples (as well as in Otsu's example) the volume is kept bounded below as ε goes to zero, showing that the lower bound on the injectivity radius cannot be replaced by a lower bound on the volume in dimensions bigger than or equal to 4.

2 Preliminaries

Definition 2.1 A sequence (M_i, g_i) of Riemannian manifolds is said to *converge in the C^α-topology for a fixed $\alpha < 1$ to a C^α-Riemannian manifold* (M_∞, g_∞) if M_∞ is a C^∞ manifold with a C^α metric tensor g_∞ and there is a sequence of $C^{1,\alpha}$ diffeomorphisms $f_i \colon M_\infty \to M_i$ for i sufficiently large, such that the metrics $f_i^* g_i$ converge to g_∞ in the C^α-topology on M_∞. Here the $C^{1,\alpha}$ structure is defined with respect to some fixed $C^{1,\alpha}$ atlas compatible with its C^∞ structure.

If a sequence (M_i, g_i) of Riemannian manifolds converges in the C^α-topology to a C^α-Riemannian manifold (M_∞, g_∞), we may assume that all metrics g_i are defined on a C^∞ fixed manifold M and the metrics g_i converges to g_∞ in the C^α-topology.

The following theorem (C^α-compactness, see [AC]) is the basic tool used in the proof of Theorem 1.1.

Theorem 2.2 (C^α-compactness (Anderson-Cheeger)) *The space of compact Riemannian n-manifolds (M, g) such that*

$$\text{Ric}_{(M,g)} \geq -\lambda, \quad \text{inj}_{(M,g)} \geq \rho_0 > 0, \quad \text{vol}_{(M,g)} \leq V \tag{2.3}$$

is pre-compact in the C^α topology for any $\alpha < 1$. More precisely, given any sequence of n-manifolds $\{(M_i, g_i)\}$ satisfying the bounds (2.3) and given any fixed $\alpha < 1$, there is a convergent subsequence in the C^α topology. The limit manifold (M_∞, g_∞) admits an atlas of $C^{1,\alpha}$ harmonic coordinates charts $F_\mu \colon U_\alpha \to \mathbf{R}^n$ having the following property:

(1) *The domains* U_μ *are of the form* $U_\mu = F_\mu^{-1}(B(r_h))$, *where* $B(r_h) \subset \mathbf{R}^n$, *with radius satisfying* $r_h \geqq c(n, \rho_0, \alpha, Q)$, $\alpha < 1$ *and* $Q > 1$ *fixed. The domains* $F_\mu^{-1}(B(r_h/2))$ *cover* M_∞.

3 Proof of Theorem 1.1

Consider a sequence of Riemannian manifolds $\{(M_i, g_i)\}$ satisfying

$$\text{Ric}_{(M_i, g_i)} \geqq (n-1)g_i, \quad \text{inj}_{(M_i, g_i)} \geqq \rho_0, \quad \text{diam}_{(M_i, g_i)} \geqq (\pi - \varepsilon_i) \tag{3.1}$$

where $\lim \varepsilon_i = 0$. By Theorem 2.2, $(M_i, g_i) \to (M_\infty, g_\infty)$ in the C^α topology, where (M_∞, g_∞) is a C^α Riemannian manifold. In particular for i sufficiently large all (M_i, g_i) are diffeomorphic to (M_∞, g_∞). We shall prove that (M_∞, g_∞) is isometric to (S^n, g_{can}).

From now on, we assume that all the metrics are defined on a fixed C^∞ manifold M and the metrics g_i converge to g_∞ in C^α topology.

Proposition 3.2 *Let* $\{(M, g_i)\}$ *be a sequence of Riemannian manifolds satisfying:*

$$\text{Ric}_{(M, g_i)} \geqq (n-1)g_i, \quad \text{inj}_{(M, g_i)} \geqq \rho_0, \quad \text{diam}_{(M, g_i)} \geqq \pi - \varepsilon_i \tag{3.3}$$

where $\lim_{i \to \infty} \varepsilon_i = 0$. *Then passing to a subsequence if necessary,*

$$\lambda_1(B_i(x, \pi/2), g_i) \to \lambda_1(B_{S^n}(\pi/2), g_{\text{can}}). \tag{3.4}$$

Here $\lambda_1(B_i(x, \pi/2), g_i)$ *is the first non-zero eigenvalue for the Dirichlet problem for the ball* $B_i(x, \pi/2)$ *in the metric* g_i.

To prove this proposition we need some lemmas.

Lemma 3.5 (Cheng's lemma [Ch]) *Suppose* (M, g) *a complete Riemannian n-manifold satisfying* $\text{Ric}_{(M, g)} \geqq (n-1)g$. *Then for* $x \in (M, g)$ *we have:*

$$\lambda_1(B_g(x, r), g) \leqq \lambda_1(B_{S^n}(r), g_{\text{can}})$$

and equality holds iff $(B_g(x, r), g)$ *is isometric to* $(B_{S^n}(r), g_{\text{can}})$.

Lemma 3.6 *Let* $\{(M, g_i)\}$ *be a sequence of Riemannian manifolds satisfying the bounds* (3.3), *then passing to a subsequence if necessary,*

$$\lambda_1(B_i(x, \pi/2), g_i) \to \lambda_1(B_\infty(x, \pi/2), g_\infty). \tag{3.7}$$

The proof of this is going is to be shown in 3 steps.

Step 1 Let $\{g_i\}$ be a sequence of metrics on a compact manifold M converging in C^0-topology to g_∞ and let D be a domain in M. Then,

$$\lambda_1(D, g_i) \to \lambda_1(D, g_\infty). \tag{3.8}$$

Proof. Let g and g_0 be two metrics on a closed manifold M such that $a^2 g_0 \leqq g \leqq b^2 g_0$ with $0<a^2<b^2$. Then for any k

$$\frac{a^n}{b^{n+2}} \lambda_k(M, g_0) \leqq \lambda_k(M, g) \leqq \frac{b^n}{a^{n+2}} \lambda_k(M, g_0). \tag{3.9}$$

The proof of this fact is in [GHL, p. 179], and only uses the max-min principle. The same proof also works for a compact domain $D \subset M$. Hence, if a sequence $\{g_i\}$ of metrics on M converges to g_∞ in the C^0-topology then $\lambda_1(D, g_i) \to \lambda_1(D, g_\infty)$, since we can find sequences of real numbers $\{a_i\}$ and $\{b_i\}$ satisfying $0<a_i^2<b_i^2$, $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = 1$ and $a_i^2 g_i \leqq g_\infty \leqq b_i^2 g_i$.

Step 2 Let $\{g_i\}$ be a sequence of metrics on a compact manifold M converging in C^0-topology to g_∞. Given $\delta>0$ there exists $i_0 = i_0(\delta) > 0$ such that $\forall\, i \geqq i_0$

$$B_i(x, \pi/2) \subset B_\infty(x, \pi/2+\delta) \tag{3.10}$$

$$B_i(x, \pi/2) \supset B_\infty(x, \pi/2-\delta). \tag{3.11}$$

Proof. From Theorem 2.2, part (1), we know that M has coordinate charts of the form $F_\mu^{-1}(B(r_h))$ and the domains $F_\mu^{-1}(B(r_{h/2}))$ cover M. Then $(g_{kj})_\infty$ are bounded on the closure of $F_\mu^{-1}(B(r_{h/2}))$.

Given $\varepsilon>0$ there exists $i_0>0$ such that for all $i \geqq i_0$ and for each chart

$$\sup_y |(g_{kj})_i - (g_{kj})_\infty| \leqq \varepsilon. \tag{3.12}$$

Now fix $i \geqq i_0$ and consider $y \in B_i(x, \pi/2)$. Let $\gamma\colon [0, 1] \to M$ be a minimal geodesic with respect to g_i joining x to y. Assume that $\gamma([0, 1])$ is contained in the closure of $F_\mu^{-1}(B(r_{h/2}))$ for some chart. The general case is done breaking γ in pieces such that this condition is satisfied.

In a chart $\gamma' = \sum_k v_k \partial/\partial x_k$.

$$\begin{aligned} |l_{g_i}(\gamma) - l_{g_\infty}(\gamma)| &\leqq \int_0^1 |(\sqrt{\textstyle\sum (g_{kj})_i v_k v_j} - \sqrt{\textstyle\sum (g_{kj})_\infty v_k v_j})\, dt| \\ &\leqq \int_0^1 \left| \frac{\sum ((g_{kj})_i - (g_{kj})_\infty) v_k v_j}{\sqrt{\sum ((g_{kj})_\infty v_k v_j)}} \right| dt \,. \end{aligned}$$

using twice Cauchy-Schwarz inequality we get

$$|l_{g_i}(\gamma) - l_{g_\infty}(\gamma)| \leqq \frac{\varepsilon n |v|}{\sqrt{\lambda}}, \tag{3.13}$$

where $|v|^2 = \sum v_k^2$ and λ is the smallest number such that $\sum ((g_{kj})_\infty v_k v_j) \geqq \lambda |v|^2 > 0$ in all charts $F_\mu^{-1}(B(r_{h/2}))$ covering the manifold. (Note, there are only finite number of them.) In other words

$$d_\infty(x, y) \leqq d_i(x, y) + \frac{\varepsilon n |v|}{\lambda}. \tag{3.14}$$

Now we have to show that $|v|$ does have an upper bound independent of i and y.

Since v is a tangent vector of a minimal geodesic γ: $[0, 1] \to M$ in a metric g_i, the length $\|v\|_{g_i} \leqq \mathrm{diam}_{(M, g_i)} \leqq \pi$. Then

$$\begin{aligned}\pi^2 &\geqq |\sum (g_{kj})_i v_k v_j| = |\sum ((g_{kj})_i - (g_{kj})_\infty) v_k v_j + \sum (g_{kj})_\infty v_k v_j| \\ &\geqq |\sum (g_{kj})_\infty v_k v_j| - |\sum ((g_{kj})_i - (g_{kj})_\infty) v_k v_j| \geqq \lambda |v|^2 - n\varepsilon |v|^2.\end{aligned}$$

Therefore we have that $|v| \leqq \dfrac{\pi}{\sqrt{\lambda - n\varepsilon}}$. This shows that $B_i(x, \pi/2) \subset B_\infty\left(x, \pi/2 + \dfrac{n\varepsilon\pi}{\lambda\sqrt{\lambda - n\varepsilon}}\right)$, $\forall\, i \geqq i_0$. Set $\delta = \dfrac{n\varepsilon\pi}{\lambda\sqrt{\lambda - n\varepsilon}}$ and solve for ε, this gives (by the above) $i_0 = i_0(\varepsilon(\delta)) > 0$ satisfying (3.10). The other inclusion is similarly proved.

Step 3

$$\lim_{i \to \infty} \lambda_1(B_i(x, \pi/2), g_i) = \lambda_1(B_\infty(x, \pi/2), g_\infty).$$

Proof. From (3.10) and (3.11) we have for $i \geqq i_0$:

$$\lambda_1(B_\infty(x, \pi/2 + \delta), g_i) \leqq \lambda_1(B_i(x, \pi/2), g_i) \leqq \lambda_1(B_\infty(x, \pi/2 - \delta), g_i). \tag{3.15}$$

Therefore

$$\lim_{i \to \infty} \lambda_1(B_\infty(x, \pi/2 + \delta), g_i) \leqq \lim_{i \to \infty} \lambda_1(B_i(x, \pi/2), g_i) \leqq \lim_{i \to \infty} \lambda_1(B_\infty(x, \pi/2 - \delta), g_i). \tag{3.16}$$

From (3.8) the left hand side limit is

$$\lambda_1(B_\infty(x, \pi/2 + \delta), g_\infty)$$

and the right hand side is

$$\lambda_1(B_\infty(x, \pi/2 - \delta), g_\infty).$$

Letting $\delta \to 0$ we have that

$$\lim_{i \to \infty} \lambda_1(B_i(x, \pi/2), g_i) = \lambda_1(B_\infty(x, \pi/2), g_\infty). \tag{3.17}$$

Remark. Is clear in the proof of Lemma 3.6 that given $\varepsilon > 0$ there exists $i_0 = i_0(\varepsilon) > 0$ depending only on ε such that $|\lambda_1(B_\infty(x, \pi/2)) - \lambda_1(B_i(x, \pi/2))| \leqq \varepsilon$, $\forall\, i \geqq i_0$ and any $x \in M$.

Lemma 3.18 *Let* $\{(M, g_i)\}$ *be a sequence of Riemannian manifolds satisfying:*

$$\mathrm{Ric}_{(M, g_i)} \geqq (n-1) g_i, \quad \mathrm{inj}_{(M, g_i)} \geqq \rho_0, \quad \mathrm{diam}_{(M, g_i)} \geqq \pi - \varepsilon_i \tag{3.19}$$

where $\lim_{i \to \infty} \varepsilon_i = 0$. *Then passing to a subsequence if necessary,*

$$\mathrm{Vol}(M, g_i) \to \mathrm{Vol}(S^n, g_{\mathrm{can}}). \tag{3.20}$$

Proof. By the Bishop-Gromov volume comparison theorem [G, BC], the functions

$$f_i(r)=\frac{\mathrm{Vol}(\partial B_i(x,r))}{\mathrm{Vol}(\partial B_{S^n}(r))}, \quad \hat{f}_i(r)=\frac{\mathrm{Vol}(B_i(x,r))}{\mathrm{Vol}(B_{S^n}(r))} \tag{3.21}$$

are non-increasing as function of r for any fixed x.

Here $B_i(x,r)$ is a geodesic ball in M of center x and radius r in the metric g_i.

Claim.

$$\mathrm{Vol}(\partial B_i(x,r)) \to \mathrm{Vol}(\partial B_\infty(x,r)).$$

For almost all r in $[0,\pi]$ and fixed x.

Since $g_i \to g_\infty$ in the C^α topology we have

$$\mathrm{Vol}_{g_i}(B_k(x,r)) \to \mathrm{Vol}_{g_\infty}(B_k(x,r)).$$

The proof of this fact is trivial.

Now given $\delta>0$ $\exists\, i_0=i_0(\delta)>0$ such that $\forall\, i\geqq i_0$ we have (see (3.10), (3.11)):

$$\mathrm{Vol}_{g_i}(B_\infty(x,r-\delta)) \leqq \mathrm{Vol}_{g_i}(B_i(x,r)) \leqq \mathrm{Vol}_{g_i}(B_\infty(x,r+\delta)).$$

Then

$$\lim_{i\to\infty} \mathrm{Vol}_{g_i}(B_\infty(x,r-\delta)) \leqq \lim_{i\to\infty} \mathrm{Vol}_{g_i}(B_i(x,r)) \leqq \lim_{i\to\infty} \mathrm{Vol}_{g_i}(B_\infty(x,r+\delta)).$$

Therefore

$$\mathrm{Vol}_{g_\infty}(B_\infty(x,r-\delta)) \leqq \lim_{i\to\infty} \mathrm{Vol}_{g_i}(B_i(x,r)) \leqq \mathrm{Vol}_{g_\infty}(B_\infty(x,r+\delta)).$$

Letting $\delta \to 0$ we have:

$$\lim_{i\to\infty} \mathrm{Vol}_{g_i}(B_i(x,r)) = \mathrm{Vol}_{g_\infty}(B_\infty(x,r)).$$

Thus $\hat{f}_\infty(r)=\dfrac{\mathrm{Vol}(B_\infty(x,r))}{\mathrm{Vol}(B_{S^n}(r))}$ is non-increasing as function of r and any fixed x.

The sequence of functions defined by $s \mapsto \mathrm{Vol}_{g_i}(\partial B_i(x,s))$ is uniformly bounded, just note that $f_i(r)$ is non-increasing (see (3.21)).

Then

$$\begin{aligned}\lim_{i\to\infty} \int_0^r \mathrm{Vol}_{g_i}(\partial B_i(x,s))\,ds &= \int_0^r \lim_{i\to\infty} \mathrm{Vol}_{g_i}(\partial B_i(x,s))\,ds \\ &= \lim_{i\to\infty} \mathrm{Vol}_{g_i}(B_i(x,r)) = \mathrm{Vol}_{g_\infty}(B_\infty(x,r)).\end{aligned}$$

Therefore

$$\int_0^r [\lim_{i\to\infty} \mathrm{Vol}_{g_i}(\partial B_i(x, s)) - \mathrm{Vol}_{g_\infty}(\partial B_\infty(x, s))]\, ds = 0$$

for all r in $[0, \pi]$.

This implies that $\lim_{i\to\infty} \mathrm{Vol}_{g_i}(\partial B_i(x, s)) = \mathrm{Vol}_{g_\infty}(\partial B_\infty(x, s))$ for almost all r in $[0, \pi]$. Thus

$$f_\infty(r) = \frac{\mathrm{Vol}(\partial B_\infty(x, r))}{\mathrm{Vol}(\partial B_{S^n}(r))} \tag{3.22}$$

is non-increasing as function of r almost everywhere in $[0, \pi]$ for any fixed x.

Before we proceed we need to recall a definition.

Definition 3.23 Given two points p, q in a compact Riemannian manifold (M, g) we define the *excess function* $\varepsilon_{p,q}: M \to \mathbf{R}$ *as*

$$\varepsilon_{p,q}(x) = \mathrm{dist}_g(p, x) + \mathrm{dist}_g(q, x) - \mathrm{dist}_g(p, q)$$

and the *excess of a manifold* as $\varepsilon(M, g) = \min_{p,q} \{\max_x \varepsilon_{p,q}(x)\}$.

We are going to present a proof that (M, g_∞) has excess zero due to Grove and Petersen (see [GP]).

Given two points p, q in (M, g_∞) realizing the diameter, i.e. $\mathrm{dist}_{g_\infty}(p, q) = \pi$, we just need to show that for any $\varepsilon > 0$, we have:

$$\partial B_\infty(p, \pi - \varepsilon) = \partial B_\infty(q, \varepsilon). \tag{3.24}$$

Because given any x in (M, g_∞), suppose $\mathrm{dist}_{g_\infty}(p, x) = \pi - \varepsilon$. Then $x \in \partial B_\infty(p, \pi - \varepsilon) = \partial B_\infty(q, \varepsilon)$. Thus there exists minimal geodesics γ_1, γ_2 from p to x and from x to q respectively such that the length $l(\gamma_1) = \pi - \varepsilon$ and $l(\gamma_2) = \varepsilon$. Then $\varepsilon_{p,q}(x) = \mathrm{dist}_{g_\infty}(p, x) + \mathrm{dist}_{g_\infty}(x, q) - \mathrm{dist}_{g_\infty}(p, q) = l(\gamma_1) + l(\gamma_2) - \mathrm{dist}_{g_\infty}(p, q) = 0$.

Now to prove (3.24) suppose that is not true, so there is $x \in M$ such that $\mathrm{dist}_{g_\infty}(p, x) > \pi - \varepsilon + \delta_1$ and $\mathrm{dist}_{g_\infty}(q, x) > \varepsilon + \delta_2$.

Then the balls $B_\infty(p, \pi - \varepsilon)$, $B_\infty(q, \varepsilon)$ and $B_\infty(x, \delta_3)$ are pairwise disjoint, for $\delta_3 = 1/2 \min\{\delta_1, \delta_2\}$. Since $\hat{f}_\infty(r)$ is non-increasing for every r, we have:

$$\begin{aligned} \mathrm{Vol}(M, g_\infty) &\geqq \mathrm{Vol}(B_\infty(p, \pi - \varepsilon)) + \mathrm{Vol}(B_\infty(q, \varepsilon)) + \mathrm{Vol}(B_\infty(x, \delta_3)) \\ &\geqq \frac{\mathrm{Vol}(M, g_\infty)}{\mathrm{Vol}(S^n, g_{\mathrm{can}})} (\mathrm{Vol}(B_{S^n}(\pi - \varepsilon) + \mathrm{Vol}(B_{S^n}(\varepsilon) \\ &\quad + \mathrm{Vol}(B_{S^n}(\delta_3)) > \mathrm{Vol}(M, g_\infty). \end{aligned}$$

A contradiction.

Since (M, g_∞) is a smooth manifold with a C^α metric, we have

$$\lim_{r\to 0} \frac{\mathrm{Vol}(\partial B_\infty(p, r))}{\mathrm{Vol}(\partial B_{S^n}(r))} = 1.$$

Note that the set $A=\{s\in[0,\pi]\,|\,\lim_{i\to\infty}\mathrm{Vol}_{g_i}(\partial B_i(x,s))=\mathrm{Vol}_{g_\infty}(\partial B_\infty(x,s))\}$ is dense in $[0,\pi]$ and $\mu(A)=\mu([0,\pi])$.

By (3.22) we have:

$$(3.25)\quad 1=\lim_{r\to 0}\frac{\mathrm{Vol}(\partial B_\infty(p,r))}{\mathrm{Vol}(\partial B_{S^n}(r))}\geqq\frac{\mathrm{Vol}(\partial B_\infty(p,\pi-\delta))}{\mathrm{Vol}(\partial B_{S^n}(\pi-\delta))}=\frac{\mathrm{Vol}(\partial B_\infty(q,\delta))}{\mathrm{Vol}(\partial B_{S^n}(\delta))}\to 1$$

as $\delta\to 0$.

Thus $\mathrm{Vol}(\partial B_\infty(p,r))=\mathrm{Vol}(\partial B_{S^n}(r))\ \forall r\in A$. This implies that

$$\mathrm{Vol}(M,g_\infty)=\int_0^\pi \mathrm{Vol}_{g_\infty}(\partial B_\infty(x,s))\,ds=\int_0^\pi \mathrm{Vol}(\partial B_{S^n}(s))\,ds=\mathrm{Vol}(S^n,g_{\mathrm{can}}).$$

Proof of Proposition 3.2 By Lemma 3.7 we need to show that $\forall x$, $\lambda_1(B_\infty(x,\pi/2))=\lambda_1(B_{S^n}(\pi/2))$.

Since $\mathrm{Vol}(M,g_\infty)=\mathrm{Vol}(S^n,g_{\mathrm{can}})$ (Lemma 3.20) and because $\hat{f}_\infty(r)$ is a non-increasing function of r it is easy to see that given any point x in (M,g_∞) there exists a point $y=h(x)$ such that $\mathrm{dist}_{g_\infty}(x,y)=\pi$. By the same argument showing that (M,g_∞) has excess zero is clear that the excess is realized by any two points realizing the diameter. In particular x and $h(x)$ realize the excess $e(M,g_\infty)=0$ of (M,g_∞). Moreover given x, $h(x)$ is unique, for if there exists $z\neq h(x)$ such that $\mathrm{dist}_{g_\infty}(x,z)=\pi$, we have a minimal geodesic γ from x to $h(x)$, passing through z since x and $h(x)$ realize the excess of (M,g_∞), $e(M,g_\infty)=0$. Thus $l(\gamma)=\mathrm{dist}_{g_\infty}(x,z)+\mathrm{dist}_{g_\infty}(z,h(x))=\mathrm{dist}_{g_\infty}(x,h(x))=\pi$. It contradicts the fact that $z\neq h(x)$. This defines a function $h\colon(M,g_\infty)\to(M,g_\infty)$. By the previous discussion h is well defined and injective.

Claim. h is a $C^{1,\alpha}$ isometry of (M,g_∞).

Proof. Given any two points in (M,g_∞), say x and y. There are minimal geodesics γ_1 and γ_2 from x to $h(x)$ passing through y and $h(y)$ respectively. So $\mathrm{dist}_{g_\infty}(x,h(x))=\mathrm{dist}_{g_\infty}(x,y)+\mathrm{dist}_{g_\infty}(y,h(x))=\mathrm{dist}_{g_\infty}(x,h(y))+\mathrm{dist}_{g_\infty}(h(y),h(x))$. Suppose $\mathrm{dist}_{g_\infty}(x,y)=\varepsilon$, then $\mathrm{dist}_{g_\infty}(y,h(x))=\pi-\varepsilon$. By triangle inequality we have:

$$\pi=\mathrm{dist}_{g_\infty}(y,h(y))\leqq\mathrm{dist}_{g_\infty}(y,h(x))+\mathrm{dist}_{g_\infty}(h(x),h(y))=\pi-\varepsilon+\mathrm{dist}_{g_\infty}(h(x),h(y)).$$

Then $\mathrm{dist}_{g_\infty}(h(x),h(y))\geqq\varepsilon$.

Also we have:

$$\pi=\mathrm{dist}_{g_\infty}(y,h(y))\leqq\mathrm{dist}_{g_\infty}(y,x)+\mathrm{dist}_{g_\infty}(x,h(y))=\varepsilon+\mathrm{dist}_{g_\infty}(x,h(y)).$$

Thus $\mathrm{dist}_{g_\infty}(x,h(y))\geqq\pi-\varepsilon$. Since $\mathrm{dist}_{g_\infty}(x,h(x))=\mathrm{dist}_{g_\infty}(x,h(y))+\mathrm{dist}_{g_\infty}(h(y),h(x))$, we have that $\mathrm{dist}_{g_\infty}(h(y),h(x))=\varepsilon$ i.e. h is a distance preserving function.

Theorem (Calabi and Hartman [CH]) *Let (M,g) be a connected n-dimensional C^α Riemannian manifold, $0<\alpha\leqq 1$. Then any isometry ϕ of (M,dist_g) is of $C^{1,\alpha}$ regularity. Furthermore ϕ satisfies*

$$\phi^* g=g.$$

By this result of Calabi-Hartmann, h is $C^{1,\alpha}$ isometry.

Note that $\partial B_\infty(x, \pi/2) = \partial B_\infty(h(x), \pi/2)$ see (3.24). Then if $z \in \partial B_\infty(x, \pi/2)$ by triangle inequality $h(z) \in \partial B_\infty(x, \pi/2)$. Thus $h(B_\infty(x, \pi/2)) = B_\infty(h(x), \pi/2)$.

Therefore,

$$\lambda_1(B_\infty(x, \pi/2), g_\infty) = \lambda_1(B_\infty(h(x), \pi/2), g_\infty). \tag{3.26}$$

Now we are in position to show that $\lambda_1(B_\infty(x, \pi/2), g_\infty) = \lambda_1(B_{S^n}(\pi/2), g_{\text{can}})$, $\forall x \in (M, g_\infty)$. Given $x \in (M, g_\infty)$ the balls $B_\infty(x, \pi/2)$, $B_\infty(h(x), \pi/2)$ are disjoint, $(\text{dist}_{g_\infty}(x, h(x)) = \pi)$. By Cheng's lemma [Ch, (3.5)] we have

$$\lambda_1(B_i(x, \pi/2), g_i) \leqq \lambda_1(B_{S^n}(\pi/2), g_{\text{can}}) \ \forall i. \tag{3.27}$$

Hence from Lemma 3.6

$$\lambda_1(B_\infty(x, \pi/2), g_\infty) \leqq \lambda_1(B_{S^n}(\pi/2), g_{\text{can}}). \tag{3.28}$$

By Lichnerowicz's formula (see [L, BGM]) $\lambda_1(M, g_i) \geqq n$, thus $\lambda_1(M, g_\infty) \geqq n$.

Thus

$$\begin{aligned} n \leqq \lambda_1(M, g_\infty) &\leqq \max\{\lambda_1(B_\infty(x, \pi/2), g_\infty), \lambda_1(B_\infty(h(x), \pi/2), g_\infty)\} \\ &\leqq \lambda_1(B_{S^n}(\pi/2), g_{\text{can}}) = n. \end{aligned}$$

But by (3.26), $\lambda_1(B_\infty(x, \pi/2), g_\infty) = \lambda_1(B_\infty(h(x), \pi/2), g_\infty)$. Then the Proposition 3.2 is proved.

Proposition 3.29 *The limit manifold (M, g_∞) is a real-analytic Einstein manifold.*

Proof. The proof will be in 3 steps. Let φ be the nonnegative first eigenfunction of $B_{S^n}(\pi/2)$. It is known that φ is cos. Let ρ_i be the distance function w.r.t. the point x in M and the metric g_i. Lets fix i and work on the manifold (M, g_i).

The function $\varphi \circ \rho_i$, then satisfies the boundary conditions of the Dirichlet problem. Hence:

$$\lambda_1(B_i(x, \pi/2)) \leqq \frac{\int (d\varphi \circ \rho_i, d\varphi \circ \rho_i)}{\int (\varphi \circ \rho_i)^2}. \tag{3.30}$$

Step 1 Given $\varepsilon > 0$ there is $i_0 > 0$ such that if $i > i_0$ then we have the following estimates:

$$0 \leqq \int_\xi \int_0^{a(\xi)} \varphi(t) \cdot \varphi'(t) \cdot \theta_i(t\,\xi) \cdot \ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right) dt\, d\xi \leqq \varepsilon \cdot c, \tag{3.31}$$

where $d\xi$ is the canonical measure of S^{n-1}, $\theta_i(t\,\xi)$ is $\sqrt{\det(g_i)_{k,l}} \cdot t^{n-1}$ w.r.t. normal coordinates, $a(\xi) = \min(\pi/2, \text{dist. of cut point in the direction } \xi)$ and c is a constant independent of i.

Proof. From (3.30), integrating on the tangent space of x we have:

$$\lambda_1(B_i(x, \pi/2)) \cdot \int_\xi \int_0^{a(\xi)} \varphi(t)^2 \cdot \theta_i(t\,\xi)\, dt\, d\xi \leqq \int_\xi \int_0^{a(\xi)} \varphi'(t)^2 \cdot \theta_i(t\,\xi)\, dt\, d\xi. \tag{3.32}$$

Now integrating the right hand side of (3.32) by parts we have:

$$
\begin{aligned}
(3.33)\qquad & \int_{\xi}\int_{0}^{a(\xi)} \varphi'(t)^2\,\theta_i(t\,\xi)\,dt\,d\xi \\
&= \int_{\xi} \varphi(a(\xi))\,\varphi'(a(\xi))\,\theta_i(a(\xi)\,\xi)\,d\xi \\
&\quad - \int_{\xi}\int_{0}^{a(\xi)} \varphi(t)\,\theta_i(t\,\xi)\left[\varphi''(t)+\left(\frac{\theta_i'(t\,\xi)}{\theta_i(t\,\xi)}\right.\right. \\
&\quad \left.\left. -\frac{\sin(t)'^{(n-1)}}{\sin(t)^{n-1}}+\frac{\sin(t)'^{(n-1)}}{\sin(t)^{n-1}}\,\varphi'(t)\right)\right] dt\,d\xi \\
&= \int_{\xi} \varphi(a(\xi))\,\varphi'(a(\xi))\,\theta_i(a(\xi)\,\xi)\,d\xi \\
&\quad - \int_{\xi}\int_{0}^{a(\xi)} \varphi(t)\,\varphi'(t)\,\theta_i(t\,\xi)\ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right) dt\,d\xi \\
&\quad + \int_{\xi}\int_{0}^{a(\xi)} \varphi(t)^2\,\theta_i(t\,\xi)\,\lambda_1(B_{S^n}(\pi/2)).
\end{aligned}
$$

Here we used the fact that

$$
\varphi''(t)+\frac{\sin(t)'^{(n-1)}}{\sin(t)^{n-1}}\,\varphi'(t)+\lambda_1(B_{S^n}(\pi/2))\,\varphi(t)=\Delta^{S^n}\varphi(t)+\lambda_1(B_{S^n}(\pi/2))\,\varphi(t)=0,
$$

(see [BGM, p. 56 and 164]).

Then from (3.32) and from (3.33) we have:

$$
\begin{aligned}
(3.34)\qquad & \lambda_1(B_i(x,\pi/2))\cdot\int_{\xi}\int_{0}^{a(\xi)} \varphi(t)^2\cdot\theta_i(t\,\xi)\,dt\,d\xi \leqq \int_{\xi} \varphi(a(\xi))\,\varphi'(a(\xi))\,\theta_i(a(\xi)\,\xi)\,d\xi \\
&\quad + \int_{\xi}\int_{0}^{a(\xi)} \varphi(t)(-\varphi'(t))\,\theta_i(t\,\xi)\ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right) dt\,d\xi \\
&\quad + \int_{\xi}\int_{0}^{a(\xi)} \varphi(t)^2\,\theta_i(t\,\xi)\,\lambda_1(B_{S^n}(r)).
\end{aligned}
$$

The first and the fourth term of the right hand side of (3.34) are negative.

Then

$$
\begin{aligned}
(3.35)\qquad & \lambda_1(B_i(x,\pi/2))\cdot\int_{\xi}\int_{0}^{a(\xi)} \varphi(t)^2\cdot\theta_i(t\,\xi)\,dt\,d\xi \\
&\leqq \int_{\xi}\int_{0}^{a(\xi)} \varphi(t)(-\varphi'(t))\,\theta_i(t\,\xi)\ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right) dt\,d\xi \\
&\quad + \int_{\xi}\int_{0}^{a(\xi)} \varphi(t)^2\,\theta_i(t\,\xi)\,\lambda_1(B_{S^n}(r))\,dt\,d\xi.
\end{aligned}
$$

By Proposition 3.2 and Cheng's Lemma 3.5 given ε there exists an $i_0>0$ such that if $i\geqq i_0$ then $\forall\, x$

$$0\leqq\lambda_1(B_{S^n}(\pi/2))-\lambda_1(B_i(x,\pi/2))\leqq\varepsilon. \tag{3.36}$$

From (3.35) and (3.36) it is then easy to see that

$$\begin{aligned}0\leqq\int_\xi\int_0^{a(\xi)}&\varphi(t)(-\varphi'(t))\,\theta_i(t\,\xi)\ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right)dt\,d\xi\\&+\int_\xi\int_0^{a(\xi)}\varphi(t)^2\,\theta_i(t\,\xi)[\lambda_1(B_{S^n}(r))-\lambda_1(B_i(x,\pi/2)]\,dt\,d\xi.\end{aligned} \tag{3.37}$$

From the proof of Bishop-Gromov inequality [BC], $\ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right)<0$, then the first term of the right hand side of (3.37) is negative. Therefore

$$\begin{aligned}0\leqq\int_\xi\int_0^{a(\xi)}&\varphi(t)\,\varphi'(t)\,\theta_i(t\,\xi)\ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right)dt\,d\xi\\&\leqq\int_\xi\int_0^{a(\xi)}\varphi(t)^2\,\theta_i(t\,\xi)[\lambda_1(B_{S^n}(r))-\lambda_1(B_i(x,\pi/2)]\,dt\,d\xi\leqq\varepsilon\cdot c,\end{aligned} \tag{3.38}$$

where

$$c=\int_\xi\int_0^{\pi/2}\varphi(t)^2\sin(t)^{n-1}\,dt\,d\xi. \tag{3.39}$$

Then for $i\geqq i_0$ we have

$$0\leqq\int_\xi\int_0^{a(\xi)}\varphi(t)\cdot\varphi'(t)\cdot\theta_i(t\,\xi)\cdot\ln'\left(\frac{\theta_i(t\,\xi)}{\sin(t)^{n-1}}\right)dt\,d\xi\leqq\varepsilon\cdot c. \tag{3.40}$$

Remark. Note that the integrand in (3.39) is positive hence the same estimate is true if the integral is taken from 0 to $r\leqq a(\xi)$ instead.

Step 2

$$\int_M|\mathrm{Ric}_{g_i}-(n-1)\,g_i|\,dv_{g_i}\to 0. \tag{3.41}$$

Proof. Now let's consider $r_0<\frac{\rho_0}{2}$ and fix it from now on.

Let $S_i(x,r)$ be the geodesic sphere in M_i of radius $r\leqq r_0$, and let $H_i(r,\xi)$ be the mean curvature vector at $\exp_x(r\,\xi)$ in M_i. It is well-known that $H_i(r,\xi)=\frac{\theta_i'(r\,\xi)}{\theta_i(r\,\xi)}$.

By the second variational formula,

$$H_i(r,\xi)=\sum_{k=1}^{n-1}\int_0^r |\nabla_T J_k|^2-R_i(T,J_k)\,dt \tag{3.42}$$

where J_k are Jacobi fields vanishing at x, forming an orthonormal basis at $T_{\exp_x(r\xi)}S_i(X,r)$ and T is the tangent vector of the radial geodesic $t\to\exp_x(t\xi)$.

Now by the proof of Bishop-Gromov theorem, (see [A1, BC]), we have:

$$\begin{aligned} H_i(r,\xi)&\leqq\int_0^r (n-1)f'^2-f^2\,\mathrm{Ric}_{g_i}(T,T)\,dt \\ &=(n-1)\left\{\int_0^r f'^2-f^2\right\}dt-\int_0^r[\mathrm{Ric}_{g_i}(T,T)-(n-1)]f^2\,dt \end{aligned} \tag{3.43}$$

where $f(0)=0$, $f(r)=1$. Choosing $f(s)=\dfrac{\sin(s)}{\sin(r)}$ one obtains:

$$\frac{\theta_i'(r\xi)}{\theta_i(r\xi)}=H_i(r,\xi)\leqq\ln'(\sin^{n-1}(r))-\frac{1}{\sin^2(r)}\cdot\int_0^r[\mathrm{Ric}_{g_i}(T,T)-(n-1)]\sin^2(s)\,ds. \tag{3.44}$$

Hence

$$\ln'\left(\frac{\theta_i(r\xi)}{\sin^{n-1}(r)}\right)\leqq-\frac{1}{\sin^2(r)}\cdot\int_0^r[\mathrm{Ric}_{g_i}(T,T)-(n-1)]\sin^2(s)\,ds\leqq 0. \tag{3.45}$$

Now multiplying both sides of the expression (3.45) by $\varphi(r)\cdot\varphi'(r)\cdot\theta_i(r\xi)$ we obtain:

$$\begin{aligned} &\varphi(r)\cdot\varphi'(r)\cdot\theta_i(r\xi)\cdot\ln'\left(\frac{\theta_i(r\xi)}{\sin(r)^{n-1}}\right) \\ &\geqq\frac{-\varphi(r)\cdot\varphi'(r)\cdot\theta_i(r\xi)}{\sin^2(r)}\cdot\int_0^r[\mathrm{Ric}_{g_i}(T,T)-(n-1)]\sin^2(s)\,ds\geqq 0. \end{aligned} \tag{3.46}$$

Integrating twice over S_x^{n-1}, once over M_i, (the underlying manifold M with metric g_i) and over $[0,r_0]$, we obtain:

$$\begin{aligned} &\int_{M_i}\int_{S_x^{n-1}}\int_{S_x^{n-1}}\int_0^{r_0}\varphi(r)\,\varphi'(r)\,\theta_i(r\xi)\ln'\left(\frac{\theta_i(r\xi)}{\sin(r)^{n-1}}\right)dr\,d\xi\,dT\,dx \\ &\geqq\int_{M_i}\int_{S_x^{n-1}}\int_{S_x^{n-1}}\int_0^{r_0}\frac{-\varphi(r)\cdot\varphi'(r)\cdot\theta_i(r\xi)}{\sin^2(r)} \\ &\quad\cdot\int_0^r[\mathrm{Ric}_{g_i}(T,T)-(n-1)]\sin^2(s)\,ds\,dr\,d\xi\,dT\,dx\geqq 0. \end{aligned} \tag{3.47}$$

Since $r<\frac{\rho_0}{2}$, we have by a proposition of Croke [Cr1, Proposition 14]:

$$\int_{S_x^{n-1}} \theta_i(r\xi)\, d\xi = \mathrm{vol}(S_i(x,r)) \geqq 2^{n-1} \frac{(\omega_{n-1})^n}{(\omega_n)^{n-1}} \cdot r^{n-1} = c(n,r) > 0, \tag{3.48}$$

where ω_{n-1} and ω_n are respectively the volumes of the standard $(n-1)$ and n spheres. From (3.47) and (3.48) we have the following:

$$\begin{aligned} &\int_{M_i} \int_{S_x^{n-1}} \int_0^{r_0} F_{i,x}(r)\, dr\, dT\, dx \\ &\quad \geqq \int_0^{r_0} \frac{-\varphi(r)\cdot\varphi'(r)\cdot c(n,r)}{\sin^2(r)} \\ &\qquad \cdot \int_0^r \Big[\int_{\mathrm{S}M_i} [\mathrm{Ric}_{g_i}(T,T) - (n-1) g_i]\, dT\, dx \Big] \sin^2(s)\, ds\, dr \geqq 0. \end{aligned} \tag{3.49}$$

Where $F_{i,x}\colon [0, r_0] \to \mathbf{R}$ is defined by

$$F_{i,x}(t) = \int_{S_x^{n-1}} \varphi(t)\, \varphi'(t)\, \theta_i(t\xi) \ln' \left(\frac{\theta_i(t\xi)}{\sin(t)^{n-1}} \right) d\xi.$$

From (3.31) and (3.36), $\forall x$ and $\forall i \geqq i_0$ we have that $0 \leqq \int_0^{r_0} F_{i,x}(r)\, dr \leqq \varepsilon \cdot c$.

Therefore the left hand side of (3.49) goes to zero as $i \to \infty$. The right hand side of (3.49) is the following:

$$\int_0^{r_0} \frac{-\varphi(r)\cdot\varphi'(r)\cdot c(n,r)}{\sin^2(r)} \int_0^r \int_M |s_{g_i} - (n-1)n| \sin^2(s)\, dx\, ds\, dr, \tag{3.50}$$

where s_{g_i} is the scalar curvature of M_{g_i}. This implies that $\int_M |s_{g_i} - (n-1)n| \to 0$ as $i \to \infty$.

Claim.

$$\int_M |\mathrm{Ric}_{g_i} - (n-1) g_i| \to 0. \tag{3.51}$$

Let us suppose the contrary, there then exists a unit vector field Y on an open set Ω with $\mu(\Omega) > 0$, such that $\int_\Omega |\mathrm{Ric}_{g_i}(Y,Y) - (n-1)\, g_i(Y,Y)| \geqq c > 0$. Complete Y to an orthonormal frame $\{Y, e_1, \ldots, e_{n-1}\}$. Hence

$$\begin{aligned} &\int_\Omega \sum_{k=1}^{n-1} \mathrm{Ric}_{g_i}(e_k, e_k) - (n-1)\, g_i(e_k, e_k) + \int_\Omega \mathrm{Ric}_{g_i}(Y,Y) - (n-1)\, g_i(Y,Y) \\ &\quad = \int_\Omega |s_{g_i} - (n-1)n|\,. \end{aligned}$$

Since all terms are positive it is clear that $\int_{\Omega} |s_{g_i}-(n-1)n|$ will not converge to zero. Therefore we have a contradiction.

Step 3 The limit metric g_∞ is real-analytic Einstein metric.

Proof. First we show that g_∞ is a weak $L^{1,p}$ solution of Einstein equation $\mathrm{Ric}_g = (n-1)g$.

Let $h \in L^{1,p}$ with $\|h\|_{L^{1,p}} = 1$. By Holder inequality we have:

$$\int_M < \mathrm{Ric}_{g_i} - (n-1)g_i, h > dx \leqq \|\mathrm{Ric}_{g_i} - (n-1)g_i\|_{L^1} \|h\|_{L^\infty} \tag{3.52}$$

$$\leqq K(n,p) \|\mathrm{Ric}_i - (n-1)g_i\|_{L^1} \|h\|_{L^{1,p}} = K(n,p) \|\mathrm{Ric}_i - (n-1)g_i\|_{L^1},$$

since $L^{1,p} \subset L^\infty$ for large p. We have the right hand side of (3.52) goes to zero as $i \to \infty$. This implies for all p large that g_∞ is a weak $L^{1,p}$ solution of Einstein's equation. To show that g_∞ is real analytic, recall that Einstein's equation in harmonic coordinates is given by the elliptic system (we write $g_\infty = g$ for simplicity of notation),

$$g^{kj} \frac{\partial^2 g_{rs}}{\partial x_k \partial x_j} + Q(g, \partial g) = (\mathrm{Ric}_g)_{rs} = (n-1)g_{rs}, \tag{3.53}$$

where Q is a quadratic term in g and the 1st derivatives of g, (see [DeK]).

The Eq. (3.53) is a uniformly elliptic system for which we have, locally, uniform $C^{0,\alpha}$ bounds on the coefficients g^{kj} and $L^{p/2}$ bounds on the terms Q and $(n-1)g_{rs}$, for any $p < \infty$.

Elliptic regularity [Mo, Theorem 6.2.6] gives uniform bounds on $\|g\|_{L^{2,p/2}}$, for any $p < \infty$, so that $g \in L^{2,p/2} \cap C^{1,\alpha}$. Continuing in this process, elliptic regularity implies that $g = g_\infty$ is real analytic in harmonic coordinates.

Since (M, g_∞) satisfies $\mathrm{Ric}_{(M,g_\infty)} = (n-1)g_\infty$, $\mathrm{diam}_{(M,g_\infty)} = \pi$ and g_∞ is smooth, by Cheng's maximal diameter theorem [Ch], (M, g_∞) is isometric to the n-sphere $S^n(1)$.

4 Proof of Theorem 1.3

Theorem 4.1 (Croke [Cr2]) *Let M be a compact n-dimensional Riemannian manifold satisfying $\mathrm{Ric}_M \geqq (n-1)$ and $\mathrm{diam}_M \leqq D < \pi$. Then there is a constant $C(n, D) > 1$ such that $\lambda_1(M) \geqq C(n, D) \cdot n$.*

The proof of Theorem 1.3 is as follows: choose $\tilde{\varepsilon}$ in Theorem 1.1 and choose $C(n, \pi - \tilde{\varepsilon}) > 1$ in Theorem 4.1. If $\lambda_1(M) \leqq C(n, \pi - \tilde{\varepsilon}) \cdot n$, then the diameter of M, $\mathrm{diam}_M \geqq \pi - \tilde{\varepsilon}$. Now let $\varepsilon = C(n, \pi - \tilde{\varepsilon}) \cdot n - n$.

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Fano manifolds and quadric bundles

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0 Introduction

Let X be a smooth projective variety defined over the field of complex numbers. Throughout the present paper we will assume that X is of dimension $2r$ and its first Chern class $c_1(X)$ (or, equivalently, anticanonical divisor $-K_X$) is linearly equivalent to rH where H is an ample divisor on X. If r is the largest integer dividing $c_1(X)$ then such an X will be called a Fano manifold of index r. In the present paper we prove the following (the notions of *elementary contractions, scrolls* etc. will be recalled in the subsequent section).

Theorem I *Let X be a Fano manifold of index r and dimension $2r$. Assume that $r \geqq 3$ and the second Betti number of X, $b_2(X)$, is at least 2. Then one of the following holds*:

(a) *X has a projective bundle structure over a Fano manifold of dimension $r+1$;*
(b) *X has a quadric bundle structure over a smooth variety Y;*
(c) *X has two elementary contractions: one of them makes X a non-equidimensional scroll, the other is birational and of divisorial type (see also Lemma 1.5 below for a description of the divisorial contraction);*
(d) *X admits two elementary contractions and any of them gives a structure of a nonequidimensional scroll on X.*

Remark. If X is a Fano surface of index 1 (the case $r=1$ in the above theorem) then it is called del Pezzo surface and is obtained by blowing-up $b_2(X)-1$ points on $\mathbf{P}^2$ (where $2 \leqq b_2(X) \leqq 8$). Also Fano 4-folds of index 2 are well understood, see [Mu] and [W4]. For Fano manifolds of index larger than half of their dimension we have the following

Theorem. [W2, W3] *Let X be a Fano manifold of index $\mathrm{r} > \dim X/2$. Then either $b_2(X)=1$ or X has a projective bundle structure.*

As the result of the Theorem I we obtain the following

Corollary. *Let X be Fano manifold of index r and dimension $2r$. If $r \geqq 3$ then $b_2(X) \leqq 2$ unless $X = \mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$.*

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Remark. For 6-folds this was proved by Kontani [Ko].

Fano manifolds of index r and dimension $2r$ which have projective bundle structure were studied in [PSW] where a thorough classification of them was obtained. It was proved that, except a few well understood examples which occur for $r \leqq 3$, any of them is a projectivisation of a decomposable bundle on a quadric $\mathbf{Q}^{r+1}$ or on a projective space $\mathbf{P}^{r+1}$, or it is just a product of $\mathbf{P}^{r-1}$ with a del Pezzo manifold, i.e. a Fano manifold of index r and dimension $r+1$. This settles the description of the case (a) of the above theorem.

In the present paper we discuss also the case (b) of the theorem i.e. Fano manifolds of dimension $2r$ and index r which are quadric bundles. A complete description of these is obtained:

Theorem II *Let X and Y be as in the Theorem I, case* (b). *Then one of the following holds*:

(A) $X \simeq \mathbf{Q}^r \times \mathbf{Q}^r$;

(B) $Y \simeq \mathbf{P}^r$ *and either*

(a) X *is a divisor of bidegree* (1, 2) *in the product* $\mathbf{P}^r \times \mathbf{P}^{r+1}$, *or*

(b) X *is a divisor of bidegree* (1, 1) *in the product* $\mathbf{P}^r \times \mathbf{Q}^{r+1}$, *or*

(c) X *is double covering of the product* $\mathbf{P}^r \times \mathbf{P}^r$ *branched along a divisor of bidegree* (2, 2), *or*

(d) X *is a blow-up of a smooth quadric* $\mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$ *along its linear section* $\mathbf{Q}^{r-1} = \mathbf{P}^r \cap \mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$.

As for the cases (c) and (d) from the Theorem I, we have the following examples which show that these cases can really occur

Example. Let X be a blow-up of a smooth quadric $\mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$ along a linear subspace $\mathbf{P}^{r-1} \subset \mathbf{Q}^{2r}$. Then X is the graph of a projection of $\mathbf{Q}^{2r}$ from $\mathbf{P}^{r-1}$ onto $\mathbf{P}^{r+1}$. The manifold X is Fano manifold of index $2r$ and the two maps – the projection onto $\mathbf{P}^{r+1}$ and the blow-down morphism onto $\mathbf{Q}^{2r}$ – are as in the point (c) of the theorem.

Example. Let X be a smooth complete intersection of two divisors of bidegree (1, 1) on the product $\mathbf{P}^{r+1} \times \mathbf{P}^{r+1}$. Then X has a structure described in the case (d) of the theorem: any of the two projections of X onto $\mathbf{P}^{r+1}$ is a non-equidimensional scroll.

The paper is divided into three sections: first we recall some pertinent results, then in subsequent sections we prove Theorems I and II. The proof of Theorem I involves studying deformations of rational curves on Fano manifolds and so-called Mori's *breaking-up technique*. A similar approach was used in [W2] and [W3] to deal with Fano manifolds of larger index. The proof of Theorem II depends on several structural results on *elementary contractions* which are recalled in the subsequent section.

1 Preliminaries

In this section we recall some definitions and results which will be used in the sequel. Our terminology is compatible with the one used in *minimal model theory* (cf. [KMM]) and *adjunction theory* (cf. [BSW]).

1.0 The set-up. Let X be a projective manifold of dimension n and L an ample line bundle on X. Assume that K_X is not *nef* (*nef* means *numerically effective*) and that for some integer r the divisor K_X+rL is nef but not ample. Then, according to *Contraction Theorem* (see [KMM]) a linear system $|m(K_X+rL)|$ is base-point-free for $m\gg 0$ and thus it defines a projective map $\phi: X\to Y$; we assume that ϕ is onto a normal projective variety Y and has connected fibers. The divisor K_X+rL is isomorphic to a pull-back, via ϕ, of an ample divisor from Y; we call K_X+rL a *good supporting divisor* of ϕ. Such a ϕ is called *elementary contraction* if all curves contracted by ϕ to points are numerically proportional.

We say that (X, L, ϕ) is a *scroll* (resp. a *quadric fibration*) ir $r-1=\dim X-\dim Y$ (resp. $r=\dim X-\dim Y$).

The exceptional locus of an elementary contraction ϕ, that is the set on which ϕ is not bijective (if $\dim Y<\dim X$ then the set is equal to X), will be denoted by $E(\phi)$. If $E(\phi)=X$ then ϕ is called *fiber type*, if $E(\phi)$ is of codimension 1 in X then ϕ is called *divisorial*, otherwise it is called *small*.

The first result we need is about the dimension of $E(\phi)$.

Proposition 1.1 ([I, Theorem 0.4; W, Theorem 1.1] *Assume that a map ϕ from* (1.0) *is an elementary contraction. Then for any irreducible component F of any non-trivial fiber of ϕ the following inequality holds*

$$\dim F+\dim E(\phi)\geqq n+r-1.$$

Now we recall structural results about equidimensional maps, i.e. maps whose fibers are of equal dimension.

Proposition 1.2 [F1, Lemma 2.12] *If (X, L, ϕ) is an equidimensional scroll* (i.e. *all fibers of ϕ are of dimension $r-1$) and ϕ is an extremal ray contraction then Y is smooth and $\phi: X\to Y$ is a projective bundle over Y, that is, there exists a vector bundle $\mathscr{E}$ over Y such that $X=\mathbf{P}(\mathscr{E})$ (one may assume $\mathscr{E}\simeq\phi_* L$).*

Corollary 1.3 *Let us assume that (X, L, ϕ) is as in* (1.0) *and moreover assume that ϕ is an elementary contraction. If all fibers of ϕ are of dimension $\leqq r-1$ then $X\simeq\mathbf{P}(\phi_*(L))$.*

Proposition 1.4 [ABW, Theorem C] *If (X, L, ϕ) is an equidimensional quadric bundle (i.e. all fibers of ϕ are of dimension r) and ϕ is an extremal ray contraction then Y is smooth and X embedds over Y into a projective bundle $\mathbf{P}(\mathscr{E})$ as a divisor of the relative degree 2 (as above, one may take $\mathscr{E}\simeq\phi_* L$; $\mathscr{E}$ is of rank $r+2$).*

Remark. The original statement of the Theorem C in [ABW] is about the smoothness of Y; the rest then follows easily because μ is then flat and L has a constant number of section on every fiber so that, by a theorem of Grauert, $\phi_* L$ is locally free; the embedding $X\subset\mathbf{P}(\phi_* L)$ is then given by the evaluation $\phi^*\phi_* L\to L$.

We will also need the following easy fact on divisorial elementary contractions (for its proof see e.g. [PSW, Lemma 7.1]).

Lemma 1.5 *Let (X, L, ϕ) be as at the beginning of the section. Assume that ϕ is an elementary contraction such that $E=E(\phi)$ is a divisor. Let $Z=\phi(E)$. If $\dim Z=\dim E-r$ then*

(a) *Y is smooth at a general point of Z,*
(b) *a general fiber F of* $\phi_{|E}$ *is isomorphic to* $\mathbf{P}^r$ *and* $\mathcal{O}_F(E) \simeq O_{\mathbf{P}^r}(-1)$,
(c) K_Y *is a Cartier divisor and* $K_X = \phi^* K_Y + rE$.

Next we will also need a result on exceptional fibers of a contraction morphism.

Proposition 1.6 [YZ, F1, F2] *Assume that X, L, r and* $\phi: X \to Y$ *are as at the beginning of this section. Let F be an irreducible component of a fiber of* ϕ. *If* $\dim F = r > \dim$(*general fiber of* ϕ) *then the normalization of F is isomorphic to* $\mathbf{P}^r$.

Remark. The result is proved in [YZ] (although it is not formulated there this way) and is based on a vanishing whose idea is from [F1]; to get a reference: use vanishing from Lemma 4 in [YZ] and apply Theorem 2.2 from [F2].

We will depend on the following result on maps of projective spaces and quadrics

Proposition 1.7 [L, CS, PS] *Assume that Y is a smooth variety of dimension n.*

(a) *If there exists a dominant regular map* $\psi: \mathbf{P}^n \to Y$ *then Y is isomorphic to* $\mathbf{P}^n$;

(b) *If there exists a dominant regular map* $\psi: \mathbf{Q}^n \to Y$ *then* $Y = \mathbf{P}^n$ *or* $Y = \mathbf{Q}^n$ *and in the latter case the map* ψ *is biregular.*

We will also use a result on extending sections of line bundles from a divisor to the ambient space. The following result is a version of a theorem of Sommese: its proof is the same as the proof of Proposition III from [S].

Proposition 1.8 (cf. [S, Proposition III]) *Let X be an irreducible and smooth divisor on a smooth projective variety V of dimension n. Assume that L is a line bundle on V such that* $L_X := L_{|X}$ is spanned by global sections on X. Assume also that

(i) *the evaluation map* $X \to \mathbf{P}^{\dim |L_X|}$ is onto variety of dimension $\leqq n-3$,
(ii) *the line bundle* $\mathcal{O}_X(X)$ *is ample, and*
(iii) $\mathcal{O}_V(mX) \otimes L^{-1}$ *is ample on V for* $m \gg 0$,

then every section of L_X *extends to a section of L on V (in particular* $Bs|L| \cap X = \varnothing$).

Finally, we will use a result about *uniform* vector bundles on $\mathbf{P}^r$. A vector bundles $\mathcal{E}$ of rank k on $\mathbf{P}^r$ is called *uniform* if its restriction to any line on $\mathbf{P}^r$ is isomorphic to $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_k)$ with fixed integers $a_1 \leqq \ldots \leqq a_k$; the sequence $(a_1 \ldots a_k)$ is called *splitting type* of $\mathcal{E}$. It is well-known (see e.g. [OSS, 3.2.1]) that if $\mathcal{E}$ is uniform of splitting type $(a, a, \ldots, a)$ then $\mathcal{E} \simeq \mathcal{O}(a) \oplus \ldots \mathcal{O}(a)$. We also need the following

Proposition 1.9 ([E] and [OSS, 3.4]) *If* $\mathcal{E}$ *is a uniform vector bundle on* $\mathbf{P}^r$ *of rank k with the splitting type* $(a, \ldots, a, a+1)$ (*resp.* $(a-1, a \ldots a)$) *then* $\mathcal{E}$ *is either decomposable into a sum of line bundles or (if* $k \geqq r$) *isomorphic to* $T\mathbf{P} \otimes \mathcal{O}(a-1) \oplus \mathcal{O}(a)^{\oplus(k-r)}$ (*to* $\Omega\mathbf{P} \otimes \mathcal{O}(a+1) \oplus \mathcal{O}(a)^{\oplus(k-r)}$, *respectively*), *where* $T\mathbf{P}$ (resp. $\Omega\mathbf{P}$) *denotes the tangent bundle (resp. the cotangent bundle) of* $\mathbf{P}^r$.

Proof. For $k \leqq r$ the result is known (see e.g. [OSS, 3.4]). For $r = 2$ and arbitrary k the result is due to Elencwajg [E]. The extension of these results to arbitrary

r and k is proved by induction on $k \geqq 3$. We may assume that the bundle $\mathscr{E}$ is of splitting type $(0, 0 \ldots, 0, 1)$ therefore the restriction of $\mathscr{E}$ to any hyperplane $H \simeq \mathbf{P}^{r-1}$ either splits into a sum of line bundles (in this case $\mathscr{E}$ splits because of 2.3.2 from [OSS]) or is isomorphic to $T\mathbf{P}^{r-1} \otimes \mathscr{O}(-1) \oplus \mathscr{O}^{\oplus(k-r-1)}$. Call this restriction $\mathscr{E}_H$. The bundle $\mathscr{E}_H$ is spanned and $H^i(H, \mathscr{E}_H(-1)) = 0$ for any $i \geqq 0$.

We claim that also $H^i(\mathbf{P}^r, \mathscr{E}(-1)) = 0$ for any $i \geqq 0$. Indeed, take a linear pencil of hyperplanes Λ and by Γ denote the incidence variety of Λ (the blow up of $\mathbf{P}^r$ at the base point locus of Λ) with projections $p: \Gamma \to \mathbf{P}^r$ (the blow-down map) and $q: F \to \Lambda$. Then $R^i q_*(p^* \mathscr{E}) = 0$ for $i \geqq 0$, $p_*(p^* \mathscr{E}) = \mathscr{E}$ and $R^i p_*(p^* \mathscr{E}) = 0$ for $i > 0$ and our claim follows by Leray spectral sequence.

Now using the cohomology of a sequence

$$0 \to \mathscr{E}(-1) \to \mathscr{E} \to \mathscr{E}_H \to 0$$

we find out that sections of $\mathscr{E}_H$ extend to sections of $\mathscr{E}$ so that the latter bundle is spanned by global sections. Being spanned it fits to a sequence

$$0 \to \mathscr{O}^{\oplus(k-r)} \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is a uniform vector bundle of rank r and splitting type $(0, 0 \ldots 0, 1)$. Thus $\mathscr{E}'$ is as we desire, so is $\mathscr{E}$.

2 Proof of Theorem I

Throughout the present section we assume that X is a Fano manifold of dimension $n = 2r \geqq 6$ and index $r = n/2$; let H be an ample divisor on X such that $-K_X = rH$. We will also assume that Picard number of X, $\rho(X)$, is at least two. We will examine elementary contractions of X: each elementary contraction ϕ of X fits to the description from (1.0) (for a suitable choice of L), so that its fibers should be rather large (of dimension $\geqq r-1$). On the other hand, we will frequently use the fact that no two fibers of two different elementary contractions can intersect along a curve, so that their fibers – if they meet – should not be too large. This is an easy observation:

Lemma 2.0 *Assume that F_1 and F_2 are fibers of two different elementary contractions of X. If $F_1 \cap F_2 \neq \varnothing$ then* $\dim F_1 + \dim F_2 \leqq n$ *and therefore, in view of* (1.1), $r - 1 \leqq \dim F_i \leqq r + 1$, *for* $i = 1, 2$.

First we have to show that at least one of the elementary contractions is of fiber type – so that its fibers meet fibers of other contractions.

Lemma 2.1 *There exists a rational curve C_0, $C_0 \cdot H = 1$ whose deformations sweep out the manifold X.*

Proof is very similar to this of Lemma 1 from [W3], we only sketch it here. We are to show that through every point of X there passes a curve whose intersection $\cdot$ with H is equal to 1. Assume the contrary. Then there exists a point $x \in X$ such that every rational curve pasing through this point has intersection at least 2 with H. From Mori theory, [M, Theorem 6], we know that there exists a rational curve C_1 passing through x such that $C_1 \cdot H = 2$. From deformation theory, [ibid, Sect. 1], (and our assumption that C_1 has minimal

intersection with H among curves passing through x) we know that deformation of C_1 containing x sweep out a subvariety of X of codimension at most 1. If the variety coincides with X then we conclude as the proof of Lemma 1 in [W3] or (3) in [W2]. Otherwise we note that the effective divisor swept out by deforming C_1 must have positive intersection with some extremal rational curve on X (it is clear that the curve is not a multiplicity of C_1: a divisor contracted by an elementary contraction has non-positive intersection with curves contracted by the contraction) and we conclude similarly.

2.2 Now as in [W2], or [W3] we construct a complete variety T parametrising deformations of C_0 and incidence variety V with proper maps $p: V \to X$ and $q: V \to T$, p onto X, fibers of q being rational curves. From deformation theory it follows that $\dim T \geqq n+r-3=3(r-1)$ so that fibers of p are of dimension $\geqq r-2$.

For a closed subvariety $Y \subset X$ we will denote $T_Y = q(p^{-1}(Y))$, $X_Y = p(q^{-1}(T_Y))$. The variety X_Y is closed in X.

For a closed subvariety $F \subset X$ by $NE(F) \subset NE(X)$ we will denote a cone spanned by numerical classes of curves contained in F. Similarly as Lemma (1.4.5) in [BSW] one proves the following

Lemma 2.3 *If $F \subset X$ is a fiber of a projective map of X then*

$$NE(X_F) = NE(F) + \mathbf{R}^+ [C_0].$$

Lemma 2.4 *If $F \subset X$ is a fiber of a projective map which does not contract C_0 then*

$$\dim X_F \geqq \dim F + (r-1).$$

Proof. Fibers of p are od dimension $\geqq r-2$, thus we will be done if we prove that the induced map

$$p_{|q^{-1}(T_F)}: q^{-1}(T_F) \to X_F$$

is finite-to-one outside $p^{-1}(F)$. This, however, follows from Mori's *breaking-up technique*, cf. [M, proof of Theorem 4]. Namely, otherwise we can find in T_F a curve parametrizing curves passing through a point outside F, then via normalisation we can produce a ruled surface having two non-intersecting multisections contracted by different morphisms: see e.g. (3) in [W2] for details.

Lemma 2.5 *Let ϕ_1 be an elementary contraction of X which does not contract C_0. If ϕ_1 has a fiber of dimension $\geqq r+1$ then $\rho(X)=2$ and X has a $\mathbf{P}^{r-1}$-bundle structure over a smooth Fano variety. Therefore, in view of* (1.1), *X admits no small contraction unless it has a projective bundle structure.*

Proof. Let F be a fiber in question, $\dim F \geqq r+1$. According to the above lemma $X_F = X$ so that $NE(X) = NE(F) + \mathbf{R}^+ [C_0]$. Thus $\rho(X)=2$ and C_0 is an extremal rational curve. The contraction of the ray spanned on the class of C_0 has all fibers of dimension $\leqq r-1$, (2.0), so it gives a projective bundle structure to X by (1.2).

Lemma 2.6 *X has at least one extremal ray contraction of fibre type.*

Proof. Assume the contrary. Take an extremal ray contraction with exceptional locus being a divisor, say E_1. Being effective E_1 has positive intersection with some other extremal ray and thus it must meet its locus, say E_2. Thus we can find two fibers F_1 and F_2 of two extremal ray contractions which meet, $\dim F_1 = \dim F_2 = r$ because of (1.1) and (2.0). As above, we find out that X_{F_1} is a divisor meeting F_2 so that we have a curve from F_2 whose numerical class is in $NE(X_{F_1}) = NE(F_1) + \mathbf{R}^+[C_0]$, a contradiction.

From now on we may assume that C_0 is an extremal rational curve. Let ϕ_0 denote the contraction of the ray spanned by C_0.

Lemma 2.7 *If $\rho(X) \geqq 3$ then $n=6$ and all extremal ray contractions of X have all fibers of dimension $\leqq 2$, so that X has a projective bundle structure.*

Proof. Assume $\rho(X) \geqq 3$. We will be done if we prove that no contraction of X is of divisorial type. Indeed, in such a case we take a contraction of a 2-dimensional face of $NE(X)$. The contraction is onto a variety of dimension $\geqq r-1$ (because no fiber of a contraction of the remaining extremal ray is contracted) and has a general fiber of dimension $\geqq 2r-2$ (because Picard number of a general fiber is $\geqq 2$ and we can apply the main result of [W2]) thus $n=6$ and any fiber of a contraction of a remaining extremal ray is of dimension 2.

Now we can apply an argument from the previous lemma. Assume that we have two rays not spanned by C_0 with loci E_1 and E_2, where E_1 is a divisor. Since E_1 is effective, it has positive intersection with some extremal curve; thus we can assume that either $E_1 \cdot C_0 > 0$ or, perhaps changing E_2, $E_1 \cdot C_2 > 0$, where the exeptional locus of the contraction of an extremal ray spanned on C_2 is E_2. If E_1 meets E_2 then arguing as in the previous lemma we arrive to the contradiction. Therefore we may assume that both E_1 and E_2 are divisors and E_1 has positive intersection with C_0, But then either E_2 contains a fiber of the contraction of C_0 or has a positive-dimensional intersection with a general fiber of ϕ_0, so E_1 and E_2 have to meet anyway.

Now we can improve Lemma 2.5 to get

Lemma 2.8 *If an extremal ray contraction of X has a fiber of dimension $\geqq r+1$ then X has a projective bundle structure.*

Proof. In view of (2.5)–(2.7) we can assume $\rho(X)=2$ and the contraction ϕ_0 of C_0 has a fiber of dimension $\geqq r+1$ while another contraction ϕ_1 of X is of divisorial type. But then the exceptional divisor of ϕ_1 has positive intersection with C_0 so that any two fibers of ϕ_0 and ϕ_1 meet, which yields contradiction because of (2.0) and (1.1).

Now we arrive to the conclusion of this section:

Proof of Theorem I. Let us assume that X does not have a projective bundle structure. Then, according to (2.7), $\rho(X)=2$ and X has two elementary contractions ϕ_0 and ϕ_1. According to Lemma 2.6 we may assume that ϕ_0 is of fiber type. Then, in view of (2.8), ϕ_0 has all fibers of dimension $\leqq r$, so either it is an equidimensional quadric bundle or non-equidimensional scroll. The same argument applies to ϕ_1 if it is of fiber type. If not, then, according to (1.1), it is divisorial and has all fibers of dimension r.

3 Proof of Theorem II

In this section we assume that X is a Fano manifold of index r and dimension $2r$, and has a quadric bundle structure, that is, there exists an elementary contrac-

tion $\phi: X \to Y$ which is a quadric bundle. The manifold X has two elementary contractions so let ψ denote "the other" contraction of X. From the classification of such Fano manifolds which have projective bundle structure (Main Theorem and Proposition 7.4 in [PSW]) it follows that we may assume that ψ is not a projective bundle so that it has fibers of dimension r. We will prove the following refined version of the Theorem II:

Theorem 3.0 *Let X, Y, ϕ and ψ be as above. Then one of the following holds:*

(a) $X \simeq \mathbf{Q}^r \times \mathbf{Q}^r$ *and both ϕ and ψ are projections onto factors;*

(B) $Y \simeq \mathbf{P}^r$ *and either*

(a) *X is a divisor of bidegree $(1,2)$ in the product $\mathbf{P}^r \times \mathbf{P}^{r+1}$, ϕ and ψ are projections onto $\mathbf{P}^r$ and $\mathbf{P}^{r+1}$, respectively, or*

(b) *X is a divisor of bidegree $(1,1)$ in the product $\mathbf{P}^r \times \mathbf{Q}^{r+1}$, ϕ and ψ are projections onto $\mathbf{P}^r$ and $\mathbf{Q}^{r+1}$, respectively, or*

(c) *X is double covering of the product $\mathbf{P}^r \times \mathbf{P}^r$ branched along a divisor of bidegree $(2,2)$, ϕ and ψ are projections onto each of $\mathbf{P}^r$'s (equivalently: X is an intersection of a cone over a Segre imbedding $\mathbf{P}^r \times \mathbf{P}^r \subset \mathbf{P}^{(r+1)(r+1)-1}$ with a smooth quadric in $\mathbf{P}^{(r+1)(r+1)}$), or*

(d) *X is a blow-up of a smooth quadric $\mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$ along its linear section $\mathbf{Q}^{r-1} = \mathbf{P}^r \cap \mathbf{Q}^{2r} \subset \mathbf{P}^{2r+1}$, the map ϕ is the projection from $\mathbf{P}^r \subset \mathbf{P}^{2r+1}$ and ψ is the blow-down morphism.*

The proof of the above theorem will occupy the rest of this section. First we will show that Y is either $\mathbf{P}^r$ of $\mathbf{Q}^r$ with the latter possibility only if $X \simeq \mathbf{Q}^r \times \mathbf{Q}^r$.

Lemma 3.1 *If X is as above then Y is isomomorphic to either $\mathbf{Q}^r$ (if ψ is a quadric bundle) or to $\mathbf{P}^r$.*

Proof. First note that we may assume ψ has a fiber of dimension r and thus ϕ is equidimensional, but then by a result of [ABW] Y is smooth. If ψ is equidimensional, then it is a quadric bundle and its general quadric is isomorphic to $\mathbf{Q}^r$. But then $\mathbf{Q}^r$ is mapped onto smooth Y and thus, by (1.7), Y is either $\mathbf{P}^r$ or $\mathbf{Q}^r$. The latter case is possible only if the map from the fiber onto Y is biregular.

If ψ is not equidimensional then by (1.6) an exceptional fiber of ψ has a component whose normalisation is $\mathbf{P}^r$. Then, again, we have $\mathbf{P}^r$ mapped onto smooth Y so by (1.7) $Y = \mathbf{P}^r$.

Lemma 3.2 *In the above situation, if $Y = \mathbf{Q}^r$ then $X = \mathbf{Q}^r \times \mathbf{Q}^r$.*

Proof. By the symmetry, we have a finite map $\phi \times \psi: X \to \mathbf{Q}^r \times \mathbf{Q}^r$. But, by (1.7), fibers of one of the maps (ϕ or ψ) are mapped biholomorphically onto "the other" $\mathbf{Q}^r$, so that the product map is actually biholomorphic.

In the remaining part of this section we deal with the case $Y = \mathbf{P}^r$; so that $\phi: X \to \mathbf{P}^r$. Let $\mathscr{E} := \phi_* \mathscr{O}_X(H)$. The sheaf $\mathscr{E}$ is then locally free of rank $r+2$ and X embedds into $\mathbf{P}(\mathscr{E})$ as a divisor of relative degree 2 so that $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)_{|X} \simeq \mathscr{O}(H)$. Let η denote the pullback to $\mathbf{P}(\mathscr{E})$ of a hyperplane section of $\mathbf{P}^r$ and let ξ the divisor class of $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(1)$. From the adjunction formula for $X \subset \mathbf{P}(\mathscr{E})$ we find out that $X \in |2\xi + (r+1-c)\eta|$, where $c_1(\mathscr{E}) = c_1(\mathscr{O}(c))$. The proof of the part (B) of the theorem is now divided into some lemmata.

Lemma 3.3 *In the above situation $c \leqq r+3$.*

Proof. This follows from smoothness of X. Namely, X is defined by a section of $S^2(\mathscr{E})$ whose determinant vanishes along the divisor of singular fibers (singular quadrics) of ϕ and is of degree $2c+(r+2)(r+1-c)$.

Lemma 3.4 *If ξ is ample then X is described in either* (a), *or* (b), *or* (d) *of the part* (B) *of* (3.0).

Proof. Since $\mathscr{E}$ is ample of rank $r+2$ and $c_1(\mathscr{E}) \leqq r+3$ it follows that $\mathscr{E}$ is uniform of splitting type either $(1, \ldots 1)$ or $(2, 1, \ldots 1)$. In the former case $\mathscr{E}$ is trivial twisted by $\mathscr{O}(1)$ and consequently X is as in the case (a) of the theorem. In the latter case from (1.9) it follows that $\mathscr{E}$ is either $\mathscr{O}(2)\oplus\mathscr{O}(1)\oplus\ldots\oplus\mathscr{O}(1)$ or $T\mathbf{P}^r\oplus\mathscr{O}(1)\oplus\mathscr{O}(1)$ and thus X is, respectively, as in the case (d) or (b) of the theorem.

Lemma 3.5 *If "the other" contraction ψ is of divisorial type then X is as in the case* (d) *of* (3.0).

Proof. A general description of the contraction morphism ψ in this case is provided in Lemma 1.5. Moreover, let us note that a good supporting divisor D of ψ is a multiple of either $\xi+\eta$ or $2\xi+\eta$. The proof of this is similar to an argument from [PSW]. Namely, let E be the exceptional divisor of ψ, then E has intersection -1 with an extremal rational curve contracted by ψ and thus $\xi+E$ is a good supporting divisor of ψ, and $\psi^* K_Y = r(\xi+E)$. Then, using deformation argument as in the proof of Lemma 7.2 from [PSW] we note that K_V has intersection $\leqq 2r+1$ with the image of a line contained in the fiber of ϕ. Therefore $E=\xi-a\eta$ and $D=2\xi-a\eta$, and consequently $a=1$ or 2 depending on the intersection of η with the extremal rational curve contracted by ψ.

Now we claim that the case $D=2\xi-\eta$ can not occur. For this purpose let us consider the restriction of $\mathscr{E}$ to a line $l\subset\mathbf{P}^r$ and then by X_l denote $\phi^{-1}(l)$. On $\mathbf{P}(\mathscr{E})$ we have the following intersection formulas: $\eta^2\cdot\mathbf{P}(\mathscr{E}_l)\equiv 0$, $\eta\cdot\xi^{r+1}\cdot\mathbf{P}(\mathscr{E}_l)=1$ and $\xi^{r+2}\cdot\mathbf{P}(\mathscr{E}_l)=c_1(\mathscr{E})$. Thus we can compute the intersection $E\cdot D^r$ inside X_l:

$$E\cdot D^r\cdot X_l=(\xi-\eta)\cdot(2\xi-\eta)^r\cdot(2\xi+(r+1-c)\eta)\cdot\mathbf{P}(\mathscr{E}_l)=2^r(c-1).$$

But this intersection is equal to 0 as the image of E under the contraction ψ is of dimension $r-1$. Therefore $c=1$ in this case. To complete this case note that $\mathscr{E}_l$ has a quotient bundle $\mathscr{E}'_l$ of rank $r+1$ such that $c_1(\mathscr{E}'_l)\leqq 0$. The intersection $E\cdot D^{r-1}\cdot X\cdot\mathbf{P}(\mathscr{E}'_l)$ should be then non-negative (as $\mathbf{P}(\mathscr{E}'_l)\subset\mathbf{P}(\mathscr{E}_l)$). But computing it as above we get

$$\begin{aligned}E\cdot D^{r-1}\cdot X\cdot\mathbf{P}(\mathscr{E}')&=(\xi-\eta)\cdot(2\xi-\eta)^{r-1}\cdot(2\xi+(r+1-c)\eta)\cdot\mathbf{P}(\mathscr{E}'_l)\\&=2^{r-1}(2c_1(\mathscr{E}'_l)-1)<0,\end{aligned}$$

a contradiction.

To complete the proof of the lemma we deal with the case when $E=\xi-2\eta$ and $D=\xi-\eta$. Then we compute as above that $c=r+3$. Consequently $X\in|2\xi-2\eta|=|2D|$ and thus ξ is ample on $\mathbf{P}(\mathscr{E})$ because of the following lemma and consequently we are done because of (3.3).

Lemma 3.6 *If $\mathscr{O}_X(X)$ is not ample then ξ is ample on $\mathbf{P}(\mathscr{E})$. Also, if $c>r+1$ then ξ is ample on $\mathbf{P}(\mathscr{E})$.*

Proof. The cone of effective 1-cycles on $\mathbf{P}(\mathscr{E})$ has two edges. Since $\mathscr{O}_X(X)$ is not ample X must contain curves generating the edge not contracted by $\mathbf{P}(\mathscr{E}) \to \mathbf{P}^r$. Since $\xi_{|X}$ is ample it follows that ξ has positive intersection with this edge and consequently is ample. The second part of the lemma one gets similarly.

Lemma 3.7 *Assume that $X \subset \mathbf{P}(\mathscr{E})$ is a quadric bundle such that its "other contraction" ψ is of fibre type and $\mathscr{E}$ is not ample. Then X is as in the case* (c) *of the Theorem* (3.0).

Proof. In view of (3.6) $c \leqq r+1$ and therefore $X - 2\xi = (r+1-c)\eta$ is nef on $\mathbf{P}(\mathscr{E})$. Let L_X be a pull-back – via ψ – of a very ample line bundle on the target of ψ. Since $\rho(\mathbf{P}(\mathscr{E})) = \rho(X)$ we may assume that the line bundle L_X extends to a line bundle L on $\mathbf{P}(\mathscr{E})$. Since ξ is ample on X, $X \geqq 2\xi$ on $\mathbf{P}(\mathscr{E})$ and X is effective, X is also nef on $\mathbf{P}(\mathscr{E})$. Therefore, by a similar argument as in the above lemma, it follows that for $m \gg 0$ the line bundle $\mathscr{O}_{\mathbf{P}(\mathscr{E})}(mX) \otimes L^{-1}$ is ample on $\mathbf{P}(\mathscr{E})$. Moreover, (3.6) yields that $\mathscr{O}_X(X)$ is ample on X. Thus we are in the situation of Sommese's theorem (1.8) and consequently every section of L_X extends to $\mathbf{P}(\mathscr{E})$ and base locus of L does not meet X.

Let $l \subset \mathbf{P}^r$ be a line. We have the following decomposition of $\mathscr{E}$ on l

$$\mathscr{E}_l = \mathscr{O}(v) \oplus \mathscr{E}'_l$$

where v is the smallest number in the splitting type of $\mathscr{E}$ on l. Let $C_0 \subset \mathbf{P}(\mathscr{E}_l)$ be a line such that $\xi \cdot C_0 = v$, then the bundle L is base-point-free on $\mathbf{P}(\mathscr{E}_l)$ outside C_0 (because the base-point-set of L on $\mathbf{P}(\mathscr{E}_l)$ does not meet X and thus is of dimension 1 at most). Therefore $\mathscr{E}'_l$ is ample so that $c_1(\mathscr{E}'_l) = c - v \geqq r+1$. From (3.6) it follows that $v \leqq 0$. But then C_0 is in the base locus of L. Thus X does not meet the curve C_0 and thus $X \cdot C_0 = 0$. On the other hand we find out that

$$0 = X \cdot C_0 = 2\xi \cdot C_0 + (r+1-c)\eta \cdot C_0 = 2v + r + 1 - c \leqq v$$

and therefore $v = 0$ and $c = r+1$. Consequently, the splitting type of $\mathscr{E}$ on (every line) l is $(0, 1, \ldots, 1)$. Now we use (1.9) to complete the proof. Namely, we verify easily that only $\mathscr{E} \simeq \mathscr{O} \oplus \mathscr{O}(1)^{\oplus r+1}$ satisfies our assumptions in this case, so that we are in the case (c) of (3.0) (B).

Conclusion. Note that lemmata 3.4, 3.5 and 3.7 yield the part (B) of Theorem 3.0.

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Note added in proof.

A classification of Fano manifolds of index r, dimension $2r$ and $b_2 \geqq 2$ has been completed recently by the author of the present paper and Edoardo Ballico. The result is as follows: such manifolds have either projective or quadric bundle structure (and therefore are know – see [PSW] and Theorem II of the present paper, respectively), or they are isomorphic to one of the two types of varieties discussed in the examples following the Theorem II.

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General Nörlund transforms and power series methods

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1 Introduction and main result

Let us start by recalling the following summability methods: We say a sequence (s_n) of complex numbers is summable to s by the

(i) *Cesàro-method* $s_n \to s(C_\alpha)$ for $\alpha > 0$ if

$$\frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} s_k \to s \qquad (n \to \infty).$$

(ii) *Euler-method* $s_n \to s(E_\alpha)$ for $\alpha > 0$ if

$$\frac{1}{(\alpha+1)^n} \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} s_k \to s \qquad (n \to \infty).$$

(iii) *Abel-method* $s_n \to s(A)$ if

$$(1-t) \sum_{n=0}^{\infty} s_n t^n \to s \qquad (t \to 1-).$$

(iv) *Borel-method* $s_n \to s(B)$ if

$$e^{-t} \sum_{n=0}^{\infty} \frac{s_n}{n!} t^n \to s \qquad (t \to \infty).$$

The Cesàro- and Abel-method resp. the Euler- and Borel-method are known to be closely related, see [13, 23, 25]. Recall especially the following two classical results, see [13, Theorem 92; 15] resp. [1, Theorem 0; 16].

Theorem A. (i) *For* $0 < \alpha \leqq \beta$:

$$s_n \to s(C_\alpha) \textit{ implies } s_n \to s(C_\beta) \textit{ implies } s_n \to s(A). \quad (\textit{Briefly: } (C_\alpha) \subseteq (C_\beta) \subseteq (A)).$$

(ii) *For* $s_n = \boldsymbol{O}(1)$ $s_n \to s(A)$ *implies* $s_n \to S(C_\alpha)$ *for all* $\alpha > 0$.

Theorem B. (i) *For* $0 < \alpha \leqq \beta$: $(E_\alpha) \subseteq (E_\beta) \subseteq (B)$.
(ii) *For* $s_n = \boldsymbol{O}(1)$ $s_n \to s(B)$ *implies* $s_n \to s(E_\alpha)$ *for all* $\alpha > 0$.

Statement (i) of both theorems is an implication of Abelian character. It say, that the applicability of the Cesàro- resp. Euler-methods grows wider with increasing α, in the sense that the sets of summable sequences are increasing. For $\alpha \to \infty$ the sets of summable sequences tend to the set of (A) resp. (B) summable sequences.

Statement (ii) is of Tauberian character and really says, that for bounded sequences the class of Cesàro-methods and Abels method resp. the class of Euler-methods and Borels method are equivalent, briefly $(C_\alpha) \approx (A)$ resp. $(E_\alpha) \approx (B)$.

Discussion of the properties of power series methods, of which (A) and (B) are special cases, especially discussion of Tauberian theorems, i.e. implications from convergence of the power series method to convergence of some weaker summability methods or ordinary convergence, has a long history, see e.g. the books [13, 23, 25], and has recently found new attention in the papers [12, 14, 24] and [8, 18–20].

In the latter papers general power series methods are discussed. These methods are defined in the following way.

Assume that (p_n) is a sequence of reals with

(1.1) $p_0 > 0$, $p_n \geqq 0$, $n \in \mathbb{N}$, such that the power series

$$p(t) = \sum_{n=0}^{\infty} p_n t^n \text{ has radius of convergence } R > 0.$$

For complex sequences (s_n) with increments $a_n = s_n - s_{n-1}$, $n \geqq 0$, $s_{-1} = 0$, we consider the *power series method* of summability (P), where we say

(1.2) $$s_n \to s(P), \text{ if } p_s(t) = \sum_{n=0}^{\infty} s_n p_n t^n \text{ converges for } |t| < R$$

$$\text{and if } \sigma_p(t) = \frac{p_s(t)}{p(t)} \to s \text{ as } t \to R-.$$

The power series methods contain the (J_p)- or Abel-type methods with $R = 1$ and Abel's method, $p_n = 1$, as a special case. The (B_p)- or Borel-type methods with $R = \infty$ and Borel's method, $p_n = 1/n!$, as a special case are similarly covered.

Since we can use $p_n R^n$ as weights in case $0 < R < \infty$, we only have to deal with the two cases $R = 1$ and $R = \infty$.

Furthermore we require the (P)-methods to be regular, i.e. $s_n \to s$ $(n \to \infty)$ implies $s_n \to s(P)$. By Theorem 5 in [13], we have regularity if and only if

(1.3) (A) $P_n = \sum_{k=0}^{n} p_k \to \infty$, $(n \to \infty)$, in case $R = 1$,

(B) $p(t)$ is not a polynomial, i.e.

$p_n \neq 0$ for infinitely many n in case $R = \infty$.

During the discussion of general power series methods the sequence of "maximal weights" (Δ_n) defined by

$$\Delta_n = \inf_{0<t<R} p(t)\, t^{-n} \tag{1.4}$$

played a decisive role, especially in formulating Tauberian conditions.

For a general discussion the weights p_n cannot be too irregular, so we have to impose a regularity condition.

$$p_n \sim q_n = \exp\{-g(n)\}, \quad (n\to\infty), \tag{1.5}$$

with a real function g satisfying the conditions

(C) $g\in C_2[x_0, \infty)$ for some $x_0\in\mathbb{N}$,

$g''(x)$ is positive and decreasing, with $\lim_{x\to\infty} g''(x)=0$, and

$G(x):=x^2\, g''(x)$ is increasing on $[x_0, \infty)$.

Under (1.5) all 'smooth' sequences (p_n) are included with p_n growing faster than some positive power of n. So the Abel-method is not directly included, but we treat it as belonging to the limiting case $q_n=L(n)$, with $L(\cdot)$ an increasing, slowly varying function, i.e. $L\in SV$, see [4, § 1.2] for the definition.

Under these conditions we have the relationship

$$\Delta_n = \phi(n)\, p_n \quad (n\to\infty), \tag{1.6}$$

with $\phi(t)$ a suitable, continuous function.

The aim of this note is to find matrix methods, which have similar properties related to general power series methods as our paradigms Cesàro- and Euler-methods have. We especially want to establish results similar to Theorems A and B for general power series methods. Results as in part (ii) are very useful in proving Tauberian theorems, since general Tauberian conditions always imply the boundedness of the sequence (s_n) in question, see [18, Theorem 1], and then one can work with the matrix method, which is easier to handle.

To this end we introduce the $(N, p^{*\alpha}, p)$-*method of summability* for $\alpha\in\mathbb{N}$. These methods are defined in the following way:

Let (p_n) be a sequence of reals satisfying (1.1) and

$$p_n^{*\alpha} > 0, \quad n=0, 1, 2, \ldots\ \alpha\in\mathbb{N}-\{1\}, \tag{1.7}$$

where $p_n^{*\alpha}$ is defined by

$$p_n^{*1} := p_n, \quad n=0, 1, 2, \ldots \quad \text{and} \quad p_n^{*(\alpha+1)} := \sum_{k=0}^{n} p_{n-k}^{*\alpha}\, p_k. \tag{1.8}$$

Then we say $s_n\to s$ $(N, p^{*\alpha}, p)$ if

$$\sigma_n^{\alpha} := \frac{1}{p_n^{*(\alpha+1)}} \sum_{k=0}^{n} p_{n-k}^{*\alpha}\, p_k\, s_k \to s \quad (n\to\infty). \tag{1.9}$$

Remark 1 For $\alpha>0$ a noninteger, we define $p_n^{*\alpha}$ as the n-th coefficient in the power series expansion of $p(t)^\alpha$ at 0, which is defined since $p_0 \neq 0$. We use the above definition, since we concentrate on the integer case only.

We can now state our main theorem

Theorem 1 *Let* (p_n) *be a sequence of reals satisfying* (1.1) *and* (1.5), *then*

(i) *for* $1 \leqq \alpha \leqq \beta$, $\alpha, \beta \in \mathbb{N}$, *we have*

$$(N, p^{*\alpha}, p) \subseteq (N, p^{*\beta}, p) \subseteq (P).$$

(ii) $s_n = \boldsymbol{O}(1)$ *and* $s_n \to s\,(P)$ *imply* $s_n \to s\,(N, p^{*\alpha}, p)$ *for all* $\alpha \in \mathbb{N}$.

Remark 2 (i) Since $p_n \equiv 1$ (as the limiting case) resp. $p_n = 1/n! \sim \exp\{(n+\frac{1}{2})\log n - n + \frac{1}{2}\log(2\pi)\}$ by Stirling's formula in the Abel resp. Borel case Theorems A and B are included.
(ii) Under the condition $s_n = \boldsymbol{O}(1)$ the power series methods (P), with weight sequence (p_n), and (Q), with weight sequence (q_n), are equivalent, see [19, Lemma 2], and the same is true for the generalized Nörlund means (N, p, p) and (N, q, q).
(iii) The inclusions in Theorem 1 can be obtained under weaker conditions, see § 2, Proposition 1.
(iv) The following methods are also covered, see [17–20] for further discussion and references.

Abel-type

(i) (J_p) Methods with regularly varying weights (A_{R_α}): $p_n = n^\alpha L(n)$, $\alpha > 0$, $L(\cdot) \in SV$, $\Delta_n \sim c\,n$ with $\alpha = 0$, $L(t) \equiv 1$ as the limiting case being the classical Abel method (A).
(ii) Methods with rapidly increasing weights (A_p): for example $p_n = \exp\{n^\alpha\}$, $0 < \alpha < 1$, $\Delta_n \sim c\,n^{1-\frac{\alpha}{2}} p_n$.

Borel-type

(iii) Generalized Borel-methods $(B(\alpha, \beta))$: $p_n = \dfrac{\alpha}{\Gamma(\alpha n + \beta)}$, t replaced by t^α, $\alpha, \beta > 0$, $\Delta_n \sim c\sqrt{n}\,p_n$ with $\alpha = \beta = 1$ the classical Borel-method (B).
(iv) (B_p)-Methods: for example $p_n = \exp\{-n^\beta\}$, $1 < \beta < 2$, $\Delta_n \sim c\,n^{1-\frac{\beta}{2}} p_n$.

We start out in § 2 to discuss the relation of the $(N, p^{*\alpha}, p)$-methods to other summability methods, especially the weighted mean methods and methods of moving averages. We also prove a slight generalization of Theorem 1 (i). Furthermore we give an account of the historical context of the generalized Nörlund methods.

In § 3 we concentrate on the case $\phi(t) = o(t)$ and generalize a classical theorem on the connection between Cesàro- and Euler-methods. In § 4 we complete the proof of Theorem 1 and finally in § 5 we give applications of the use of $(N, p^{*\alpha}, p)$-methods in probability theory and introduce some new families of summability methods comparable to the class of circle-methods, see [21].

2 Definitions and preliminary results

The first definition of the generalized Nörlund transform (N, a, b) was given by Borwein in [6] and is as follows:

Let (a_n) and (b_n) be sequences of real numbers such that

$$(a * b)_n = \sum_{k=0}^{n} a_{n-k} b_k \neq 0, \qquad n = 0, 1, \ldots .$$

Given a sequence (s_n) of complex numbers, its sequence (σ_n) of (N, a, b)-transforms is defined by

$$\sigma_n = \frac{1}{(a * b)_n} \sum_{k=0}^{n} a_{n-k} b_k s_k \tag{2.1}$$

and we say $s_n \to s(N, a, b)$, if $\sigma_n \to s\ (n \to \infty)$.

Using the Toeplitz theorem [13, Theorem 2] Borwein gave the necessary and sufficient conditions for the regularity of the (N, a, b)-methods,

$$\text{(i)} \quad \sum_{k=0}^{n} |a_{n-k} b_k| = \boldsymbol{O}((a * b)_n) \tag{2.2}$$

$$\text{(ii)} \quad \frac{a_{n-k} b_k}{(a * b)_n} \to 0\ (n \to \infty) \text{ for each integer } k.$$

A systematic discussion of the relation of the (N, a, b)-methods to weighted means $(N, \mathbf{1}, b)$ and Nörlund means $(N, a, \mathbf{1})$, where $\mathbf{1}$ stands for the sequence $(1, 1, \ldots)$, was given by Das in [10, 11].

Although Borwein and Das observed that Cesàro- and Euler-methods are special cases of (N, a, b)-methods, none of them mentioned the possibility of generating them as $(N, p^{*\alpha}, p)$-methods.

We now start our investigation with a slight generalization of Theorem 1(i).

Proposition 1 *Let $\alpha \leqq \beta$, $\alpha, \beta \in \mathbb{N}$ and (p_n) be a sequence of weights satisfying* (1.1) *and* (1.7).

(i) *If (p_n) generates a regular (P)-method, then*

$$(N, p^{*\alpha}, p) \subseteq (P).$$

(ii) *If (p_n) generates regular $(N, p^{*\alpha}, p)$-methods for all $\alpha \in \mathbb{N}$, then*

$$(N, p^{*\alpha}, p) \subseteq (N, p^{*\beta}, p).$$

Proof. To show the inclusion (i), we first show that each sequence $(p_n^{*\alpha})$ generates a regular (P)-method.

In case $R = 1$, $p(t)^\alpha$ still has convergence radius $R \geqq 1$ and since $(p * p)_n \geqq p_0 p_n$, we have

$$\sum_{k=0}^{n} (p * p)_k \geqq p_0 \sum_{k=0}^{n} p_k = p_0 P_n \to \infty \qquad (n \to \infty).$$

Therefore $R = 1$ and the method is regular.

In case $R=\infty$ condition (1.7) implies the regularity.

Now let $s_n \to s\,(N, p^{*\alpha}, p)$. By the regularity of all methods follows that

$$\sigma_p(t)=\frac{p_s(t)}{p(t)}=\frac{\sum_{n=0}^{\infty} p_n^{(\alpha+1)}\left(\frac{1}{p_n^{(\alpha+1)}}\sum_{k=0}^{n} p_{n-k}^{*\alpha}\,p_k\,s_k\right)t^n}{\sum_{n=0}^{\infty} p_n^{(\alpha+1)}\,t^n}\to s, \qquad (t\to R-).$$

For the second inclusion we use the Toeplitz theorem to show that

$$A=(N, p^{*\beta}, p)(N, p^{*\alpha}, p)^{-1}$$

is regular. The inverse method $(N, p^{*\alpha}, p)^{-1}$ can be obtained by using the power series $p(t)=\sum_{n=0}^{\infty} p_n t^n$. Since $p_0\neq 0$, $p(t)^{-\alpha}$ has a power series expansion in some neighbourhood of 0, say $p(t)^{-\alpha}=c(t)=\sum_{n=0}^{\infty} c_n t^n$. Then

$$(N, p^{*\alpha}, p)^{-1}_{(n,k)}=\begin{cases}\dfrac{c_{n-k}\,p_k^{*(\alpha+1)}}{p_n} & \text{for } k\leqq n\\ 0 & \text{for } n<k.\end{cases}$$

For $k\leqq n$ we hence have

$$A_{(n,k)}=a_{n,k}=\frac{1}{p_n^{*(\beta+1)}}\sum_{\nu=k}^{n} p_{n-\nu}^{*\beta}\,p_\nu\,\frac{1}{p_\nu}\,c_{\nu-k}\,p_k^{*(\alpha+1)}$$

$$=\frac{p_k^{*(\alpha+1)}}{p_n^{*(\beta+1)}}\sum_{\nu=k}^{n} p_{n-\nu}^{*\beta}\,c_{\nu-k}$$

$$=\frac{p_k^{*(\alpha+1)}\,p_{n-k}^{*(\beta-\alpha)}}{p_n^{*(\beta+1)}},$$

where we used the coefficient formula for power series.

Since the $(N, p^{*\alpha}, p)$-matrices are lower diagonal matrices so are the inverse matrices and therefore A, hence $a_{n,k}=0$, if $k>n$.

Now $a_{n,k}\geqq 0$ for all n, $k\geqq 0$ and

$$\sum_{k=0}^{n} a_{n,k}=\frac{1}{p_n^{*(\beta+1)}}\sum_{k=0}^{n} p_k^{*(\alpha+1)}\,p_{n-k}^{*(\beta-\alpha)}=\frac{p_n^{*(\beta+1)}}{p_n^{*(\beta+1)}}=1$$

by the coefficient formula. Therefore the only Toeplitz condition, which remains to be verified is that for each $k\in\mathbb{N}$

$$\frac{p_{n-k}^{*(\beta-\alpha)}}{p_n^{*(\beta+1)}}\to 0, \qquad (n\to\infty).$$

By the regularity of the $(N, p^{*\gamma}, p)$-methods, $\gamma \in \mathbb{N}$, follows

$$\frac{p_{n-k}^{*(\beta-\alpha)}}{p_n^{*(\beta+1)}} = \frac{p_{n-k}^{*(\beta-\alpha)}}{p_{n-k}^{*(\beta-\alpha+1)}} \times \frac{p_{n-k}^{*(\beta-\alpha+1)}}{p_{n-k}^{*(\beta-\alpha+2)}} \times \ldots \times \frac{p_{n-k}^{*\beta}}{p_n^{*(\beta+1)}} \to 0, \qquad (n \to \infty). \quad \square$$

Remark 3 (i) Parts of the proof can also be deduced using results of Das in [10, Theorem 1] and [11, Theorem 1].
(ii) Das also discussed in [11] the relations of (N, a, b) methods to each other. He particularly showed that if

$$\left.\begin{array}{c} p_{n+1} \geqq p_n \text{ or} \\ p_n \geqq p_{n+1} \text{ and } P_n = \boldsymbol{O}((p * p)_n) \end{array}\right\} \text{ then } (N, \mathbf{1}, p) \subseteq (N, p, p).$$

Under the condition $p_{n+1}/p_n \leqq p_{n+2}/p_{n+1}$ he also showed the reserve inclusion. In case $R = 1$ we can relax this condition.

Lemma. *Let (p_n) be a non-increasing sequence of reals satisfying* (1.1) *with $R = 1$. Then*

$$(N, p, p) \subseteq (N, \mathbf{1}, p).$$

Proof. Let $p(t)^{-1} = c(t) = \sum_{n=0}^{\infty} c_n t^n$ and $C_n = \sum_{k=0}^{n} c_k$. Then $C_0 = c_0 > 0$ and by induction $C_n \geqq 0$, since

$$0 = \sum_{k=0}^{n} c_k p_{n-k} = C_n p_0 + \sum_{k=0}^{n-1} C_k \underbrace{(p_{n-k} - p_{n-k-1})}_{\leqq 0}.$$

Now one can use the Toeplitz theorem to show that

$$(N, \mathbf{1}, p)(N, p, p)^{-1}_{n,k} = \frac{(p * p)_k C_{n-k}}{P_n}$$

is regular. $\square$

While the $(N, \mathbf{1}, p)$ and the $(N, p, \mathbf{1})$ are obviously related to the (N, p, p)-methods their use in proofs of theorems on (N, p, p) means is limited to the case $t = \boldsymbol{O}(\phi(t))$. In case $\phi(t) = \boldsymbol{o}(t)$ the methods of moving averages are more helpful. These methods are very important in probability theory, see [3, 17], and will be needed in the proof of Theorem 1(ii).

We say a sequence (s_n) of complex numbers is summable to s by the method of *moving averages* (R_ϕ), where $\phi(\cdot)$ is a self-neglecting function, $\phi \in SN$, see [3, 4 § 2.11], briefly $s_n \to s(R_\phi)$ if

$$\frac{1}{u\,\phi(n)} \sum_{n \leqq k < n + u\phi(n)} s_k \to s, \ \forall u > 0, \qquad (n \to \infty). \tag{2.3}$$

These methods are related to the Cesàro-method by the following

Proposition 2 *The following statements are equivalent*

(i) $\frac{1}{n+1}\sum_{k=0}^{n}(s_k+\varepsilon_k)=s+o\left(\frac{\phi(n)}{n}\right)(n\to\infty)$ *for some* $\varepsilon_n\to 0$.

(ii) $s_n\to s(R_\phi)$.

Proof. For $\phi(t)=t^\gamma$, $0<\gamma<1$ this is Theorem 3 in [5]. The general case follows from Theorem 2 in [2], since this theorem is applicable also for $\phi\in SN$. Compare also the proof of Theorem 1 in [3]. □

We conclude the section by computing the asymptotics of

$$p_n^{*2}=(p*p)_n=\sum_{k=0}^{n}p_{n-k}p_k$$

in two important cases.

(1.5A) $$p_n\sim n^\sigma L(n)$$

with $\sigma>-1$, $L(\cdot)$ slowly varying.

This gives an Abel-type method of polynomial growth with convergence radius $R=1$.

If we have $\lim_{x\to\infty}G(x)=\sigma>0$ we are in the general case (1.5) with $\sigma>0$, compare [8].

(1.5B) $$p_n\sim\exp\{-g(n)\}\quad\text{with}\quad\lim_{x\to\infty}G(x)=\infty.$$

These are the Abel-type methods with rapidly increasing weights and the Borel-type case. In this case the function $\phi(\cdot)$ of (1.6) is selfneglecting.

A discussion of these conditions with further implications on the behaviour of the weights and the generated power series methods is given in [8, 18–20].

Especially the asymptotic behaviour of $p_nt^n/p(t)$ has been studied in detail under these conditions. Since the methods used there are applicable for our problem, we only outline the proofs.

Proposition 3 *Let* (p_n) *be a sequence of reals satisfying* (1.1) *and* (1.5) *then*

$$(p*p)_n\sim\begin{cases}n^{2\sigma+1}L(n)^2\,B(\sigma+1,\sigma+1) & \text{in case (A)}\\ \sqrt{\dfrac{\pi}{g''(n/2)}}\exp\{-2g(n/2)\} & \text{in case (B)}\end{cases}$$

$\left(B(\sigma,\sigma)\text{ denotes the Beta-Integral }\int_0^1 t^{\sigma-1}(1-t)^{\sigma-1}\,dt.\right)$

Proof. Case (A): Compare [14].

$$\begin{aligned}(p*p)_n&\sim\int_0^n(n-x)^\sigma x^\sigma L(n-x)\,L(x)\,dx\\&=n^{2\sigma+1}L(n)^2\int_0^1(1-x)^\sigma x^\sigma\frac{L(n(1-x))\,L(x)}{L(n)^2}\,dx\\&\sim n^{2\sigma+1}L(n)^2\,B(\sigma+1,\sigma+1)\qquad(n\to\infty).\end{aligned}$$

Case (B): Compare [8, 19, 20].

We use the so-called saddle point method. Define

$$\varphi(n,k)=\varphi(k)=(-g(n-k)+g(n/2))-g(k)+g(n/2)$$
$$=-1/2(k-n/2)^2(g''(\xi_1)+g''(\xi_2))$$

by Taylor's formula with $k\leqq\xi_2\leqq n/2\leqq\xi_1\leqq n-k$.

We define now a function $\varepsilon(x)$, compare [18–20], as

$$\varepsilon(x)=x(x^2 g''(x))^{-\frac{1}{4}}.$$

Then we have

$$\frac{\varepsilon(x)}{x}\to 0 \quad \text{and} \quad \varepsilon(x)\sqrt{g''(x)}\to\infty \quad (x\to\infty).$$

Furthermore the following basic inequality holds true

$$\left|\frac{g(t)}{g(x)}-1\right|\leqq\frac{\varepsilon(x)}{x} \quad \text{for } |t-x|\leqq\varepsilon(x). \tag{2.4}$$

Then using the properties of $\varepsilon(\cdot)$, especially (2.4), and the basic inequality $|1-e^{-x}|\leqq x$, for $0\leqq x$, we obtain

$$\frac{(p*p)_n}{\exp\{-g(n/2)\}}$$
$$=2\int_{\frac{n}{2}-\varepsilon\left(\frac{n}{2}\right)}^{\frac{n}{2}}\exp\{\varphi(n,x)\}\,dx+o(1)$$
$$=\int_{\frac{n}{2}-\varepsilon\left(\frac{n}{2}\right)}^{\frac{n}{2}+\varepsilon\left(\frac{n}{2}\right)}\exp\{-g''(n/2)(x-n/2)^2\}\,dx$$
$$+2\int_{\frac{n}{2}-\varepsilon\left(\frac{n}{2}\right)}^{\frac{n}{2}}\exp\{-g''(n/2)(x-n/2)^2\}$$
$$\cdot\left\{\exp\left\{-\frac{1}{2}g''(n/2)(x-n/2)^2\left(\frac{g''(\xi_1)}{g''(n/2)}\right.\right.\right.$$
$$\left.\left.\left.-1+\frac{g''(\xi_2)}{g''(n/2)}-1\right)\right\}-1\right\}dx+o(1)$$
$$=\sqrt{\frac{\pi}{g''(n/2)}}+o(1). \quad \square$$

Remark 4 (i) The asymptotics show, that in the above cases the (N, p, p)-methods are regular, sincer $p_{n-k}/(p*p)_n\to 0$ $(n\to\infty)$ $\forall k\in\mathbb{N}$.

(ii) For higher order convolutions $p_n^{*\alpha}$ asymptotics can be obtained with the same methods. These asymptotics show the regularity of the $(N, p^{*\alpha}, p)$-methods.

(iii) If we assume p_n/p_{n+1} to be nondecreasing we have $p_{n-k}p_k \leqq p_{n-(k+1)}p_{k+1}$ if $k \leqq n/2$ and $p_{n-k}p_k \geqq p_{n-(k+1)}p_{k+1}$ if $k \geqq n/2$. Therefore the maximal weight is

$$\frac{p_{n/2}\, p_{n/2}}{(p * p)_n},$$

which is asymptotically equal to p_n/Δ_n the maximal weight in the power series method (P).

Observe that q_n/q_{n+1} is nondecreasing, where (q_n) is the exact weight sequence in (1.5).

(iv) In case (1.5A) with nonincreasing (p_n), i.e. $-1 < \sigma \leqq 0$, we have by Remark 3 and the lemma, that $(N, \mathbf{1}, p)$ summability is equivalent to (N, p, p) summability, if $c\, P_n \leqq (p * p)_n \leqq P_n$, with some constant $c > 0$.

The asymptotics of Proposition 3 and the Lemmas 3.2 and 3.4 in [14] show that this can only be possible, if we are in the classical Cesàro-case.

3 The relation of Cesàros to the $(N, p^{*\alpha}, p)$-methods

We start by recalling a classical result of Knopp, see [13, Theorem 149]

Theorem CE. *If*

$$\frac{1}{n+1}\sum_{k=0}^{n} s_k = s + o(1/\sqrt{n}), \qquad (n \to \infty)$$

then $s_n \to s(E_\alpha)$ *for every* $\alpha > 0$.

For weights (p_n) satisfying condition (1.5B) with p_n/p_{n+1} nondecreasing, we can generalize this result.

Oberve that in this cases we furthermore have $\phi(\cdot) \in SN$ (with $\phi(\cdot)$ as in (1.6)).

Theorem 2 *Let* (p_n) *be a sequence of weights satisfying* (1.5B) *and* p_n/p_{n+1} *is nondecreasing. Then*

$$\frac{1}{n+1}\sum_{k=0}^{n} s_k = s + o\left(\frac{\phi(n)}{n}\right), \qquad (n \to \infty) \tag{3.1}$$

implies $s_n \to s(N, p^{*\alpha}, p)$ *for every* $\alpha \in \mathbb{N}$.

Proof. By the inclusion theorem we only have to show the implication for the (N, p, p)-method. Because of regularity and linearity we can suppose $s = 0$. Then

$$\sum_{k=0}^{n} s_k = o(\phi(n)), \qquad (n \to \infty). \tag{3.2}$$

Furthermore from the results in [8, 19, 20] we have under (1.5B) that

$$\phi(n) \sim \sqrt{(2\pi)/g''(n)}, \qquad (n \to \infty)$$

and hence using Proposition 3

$$(p*p)_n \sim \sqrt{1/2}\,\phi(n/2)\exp\{-2g(n/2)\}, \quad (n\to\infty). \tag{3.3}$$

By Remark 4(iii) and (3.1) we have therefore for the maximal weight in the (N, p, p)-transform

$$\frac{p_{n/2}\,p_{n/2}}{(p*p)_n} \sim \frac{\sqrt{2}}{\phi(n/2)}, \quad (n\to\infty). \tag{3.4}$$

Now let $\varepsilon>0$ be given, we can find a $N\in\mathbb{N}$, such that for $n\geqq l\geqq m\geqq N$

$$\left|\sum_{k=m}^{l} s_k\right| \leqq \varepsilon\,\phi(l)\leqq\varepsilon\,\phi(n), \tag{3.5}$$

using also the monotonicity of $\phi(\cdot)$. We split the (N, p, p)-transform in three parts, namely

$$\sigma_n=\frac{1}{p_n^{*2}}\sum_{k=0}^{n} p_{n-k}\,p_k\,s_k=\frac{1}{p_n^{*2}}\left(\sum_{k=0}^{N}+\sum_{k=N+1}^{n/2-1}+\sum_{k=n/2}^{n} p_{n-k}\,p_k\,s_k\right)=\Sigma_1+\Sigma_2+\Sigma_3.$$

By Abel's Partial Summation we get

$$|\Sigma_2|\leqq\left|\sum_{k=N+1}^{n/2-1} s_k\right|\frac{p_{n/2}^2}{p_n^{*2}}+\frac{1}{p_n^{*2}}\sum_{k=N+1}^{n/2-1}\left|\sum_{\nu=N+1}^{k} s_\nu\right|\,|p_{n-k}\,p_k-p_{n-k-1}\,p_{k+1}|\,.$$

We now use (3.3) and (3.5) and the monotonicity of $p_{n-k}\,p_k$ by (3.1) to find

$$|\Sigma_2|\leqq\varepsilon+\varepsilon\,\frac{\phi(n/2)}{p_n^{*2}}\,(p_{n/2}^2-p_{n-N}\,p_N)\leqq c_2\,\varepsilon.$$

With a similar argument

$$|\Sigma_3|\leqq c_3\,\varepsilon$$

follows. Since N is fixed and $\dfrac{p_{n-k}\,p_k}{p_n^{*2}}\to 0$, $(n\to\infty)$, $\forall\,k$ it follows that

$$|\Sigma_1|\leqq c_1\,\varepsilon.$$

Since c_1, c_2, c_3 are constants the proof is complete. □

Corollary. *With (p_n) and $\phi(\cdot)$ as in Theorem 2 $s_n\to(R_\phi)$ implies $s_n\to s(N, p^{*\alpha}, p)$ for all $\alpha\in\mathbb{N}$.*

Proof. With Proposition 2 $s_n\to(R_\phi)$ implies

$$\frac{1}{n+1}\sum_{k=0}^{n}(s_k+\varepsilon_k)=s+o\left(\frac{\phi(n)}{n}\right), \quad (n\to\infty)$$

with some sequence $\varepsilon_n \to 0$, $(n \to \infty)$. Hence we obtain by the theorem $(s_n + \varepsilon_n) \to s(N, p^{*\alpha}, p)$. Since $(N, p^{*\alpha}, p)$ is a linear and regular summability method $s_n \to s(N, p^{*\alpha}, p)$ follows. □

Remark 5 (i) The result of the corollary can also be obtained by direct computation. One has to use the asymptotic properties of the (N, p, p) transform and the techniques in [17].
(ii) Also by the inclusion theorem we have $(N, p^{*\alpha}, p) \subseteq (P)$. Therefore under the conditions of Theorem 2 (P)-summability is implied.
(iii) Under the condition $s_n = \boldsymbol{O}(1)$ we can omit the assumption of monotonicity (3.1).

4 Proof of Theorem 1

Proof. The proof of part (i) follows from Proposition 1 and Remark 4(ii).

To prove part (ii) we only have to show $s_n \to s(N, p, p)$. The other implications follow by inclusion.

We distinguish two cases:

(i) If $G(x) \to \sigma > 0$, we have the case of polynomial weights. We include the limiting case $\sigma = 0$, i.e. $p_n \sim L(n)$ with $L \in SV$ and nondecreasing.

Furthermore since $s_n = \boldsymbol{O}(1)$ we can switch to the asymptotic weights $q_n = n^\sigma L(n)$, with $\sigma \geqq 0$, $L \in SV$ and nondecreasing.

We hence have $s_n \to s(Q)$, where (Q) means the power series method with weight sequence (q_n).

In this situation we use a general theorem for Laplace transforms by Stadtmüller and Trautner, see [24], to get

$$s_n \to s(N, \mathbf{1}, q).$$

Now, since (q_n) increases we have $s_n \to s(N, q, q)$ by Remark 3.

We use once again the condition $s_n = \boldsymbol{O}(1)$ to get $s_n \to s(N, p, p)$.

(ii) (p_n) as in (1.5 B), hence $G(x) \to \infty$ $(x \to \infty)$ and the asymptotic weight-sequence is $q_n = e^{-g(n)}$. Now we use Theorem 1 in Moh [22].

Define $s(u) = s_{[u]}$ if $u \geqq 0$ and $s(u) = 0$ if $u < 0$. As in Kiesel and Stadtmüller [18] we find that under $s_n = \boldsymbol{O}(1)$ $s_n \to s(P)$ implies

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} K\left(\frac{x-t}{\phi(x)}\right) s(t) \frac{dt}{\phi(x)} = s \tag{4.1}$$

with $K(x) = 1/\sqrt{2\pi} \exp\{-x^2/2\}$. The conditions of Theorem 1 of Moh, i.e. $K(x) \in L^1(-\infty, \infty)$, the Fourier transform of $K \neq 0$ for any real argument and $\phi(\cdot) \in SN$, are satisfied.

It follows that if we define

$$H(x) = \begin{cases} \dfrac{1}{\varepsilon}, & \text{if } x \in (-\varepsilon, 0), \\ 0, & \text{if } x \notin (-\varepsilon, 0) \end{cases}$$

that

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} H\left(\frac{x-t}{\phi(x)}\right)s(t)\,\frac{dt}{\phi(x)}=\lim_{x\to\infty}\frac{1}{\varepsilon\,\phi(x)}\sum_{x\leqq k<x+\varepsilon\phi(x)}s_k=s. \tag{4.2}$$

This means $s_n\to s(R_\phi)$. By the corollary to Theorem 2 and the monotonicity of q_n/q_{n+1} $s_n\to s(N,q,q)$ follows. Using again that $s_n=\boldsymbol{O}(1)$ we can conclude that $s_n\to s(N,p,p)$. □

Remark 6 We consider the case $\phi(t)=\boldsymbol{o}(t)$ only.
(i) If we use $s_n-s_{n-1}=\boldsymbol{O}(\phi(n)/n)$ as additional condition in the setting of Theorem 1 the convergence of s_n to s follows. To see that let ε tend to 0 in (4.2).
(ii) Under $s_n=\boldsymbol{O}(1)$ we have the following equivalences: $(R_\phi)\Leftrightarrow(N,p^{*\alpha},p)\Leftrightarrow(P)$, compare Bingham [1] for the Euler-Borel-case.
(iii) It is also possible to obtain a proof of part (ii) by direct computation using the asymptotic properties of the (N,p,p) and (P)-transforms.

5 Applications and complements

(i) Considering the Abelian inclusion

$$(N,p^{*\alpha},p)\subseteq(N,p^{*\beta},p)\subseteq(P)$$

for $\alpha,\beta\in\mathbb{N}$, $\alpha\leqq\beta$, the question arises whether one can get Tauberian theorems from (P) to $(N,p^{*\beta},p)$ using $(N,p^{*\alpha},p)=\boldsymbol{O}(1)$ as Tauberian condition. This so called convexity theorems are well-known for the Cesàro-Abel case, see [13, Theorem 70], and have been obtained in the Euler-Borel case quite recently, see [9].
(ii) The (N,p,p)-methods are the natural methods to be used as general matrix summability methods in the following characterisation of moment conditions by summability methods, see [17, theorem].

Let $X, X_0, X_1, \ldots$ be iid. random variables and (P) a power-series method with weight sequence (p_n) satisfying (1.5). Then the following statements are equivalent.

(i) $E(\phi^{-1}(|X|))<\infty$, $E(X)=\mu$;
(ii) $X_n\to\mu(N,p^{*\alpha},p)$ a.s. $\forall\,\alpha\in\mathbb{N}$;
(iii) $X_n\to\mu(P)$ a.s.

In the proof for $\phi(t)\approx t$ the (C_α)-methods were used and for $\phi(t)=\boldsymbol{o}(t)$ the Riesz-methods $(R,\exp\{\Phi(n)\},1)$ with $\Phi(x)=\int_1^x dt/\phi(t)$.

But $(R,\exp\{\Phi(n)\},1)\Leftrightarrow(R_\phi)$ by Lemma 3 in [17] and (R_ϕ) implies (N,p,p).
(iii) One can also use the (N,p,p)-methods to obtain new examples for Theorem 1 in Borwein [7], where weakly multiplicative sequences of random variables are studied.
(iv) In case $\phi(t)=\boldsymbol{o}(t)$ and (1.5 B) one can attempt to generalize the well-known Taylor (T) and Meyer-König (S)-methods of summability. Together with Euler-, Borel- and Valiron-method these methods build the family of circle methods, see [21].

One observes that $(T_\alpha)=(E_\alpha)^t$ and (S_α) is obtained from (T_α) by omitting the leading zeros in the (T_α) methods. Using this procedure starting with an (N, p, p)-method a (T, p, p)-method and a (S, p, p)-method can be formally obtained.

Without being explicitly stated a general Valiron method is already in long use (see the definition in (4.1) and [18]).

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An application of Banach algebra techniques to multiplicative functions

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1 Results and comments

The object of this paper is to carry out proofs of weighted versions of Wiener type inversion theorems in the theory of Banach algebras and to derive, as an application, inversion theorems for multiplicative arithmetical functions. Such theorems using general weight functions do not seem to have been known or stated explicitly, until now.

For $D=\mathbb{Z}$ or $D=\mathbb{N}\cup\{0\}$, denote by $W(D)$ the set of positive valued weight functions ω on D satisfying

$$\omega(0)=1, \quad \omega(m+n)\leqq\omega(m)\,\omega(n) \quad \text{for all } m,\ n\in D \tag{1}$$

and

$$\inf\{\sqrt[n]{\omega(bn)}:n\in\mathbb{N}\}=1 \quad \text{for each } b\in\{1,-1\}\cap D. \tag{2}$$

Consider the complex vector space $\mathscr{A}_\omega$ of the Fourier series

$$\varphi(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx} \quad (x\in\mathbb{R}) \tag{3}$$

of finite norm

$$\|\varphi\|=\sum_{n=-\infty}^{\infty}|c_n|\,\omega(n)<\infty. \tag{4}$$

Then the following weighted version of Wiener's inversion theorem holds.

Theorem 1 *For $\omega\in W(\mathbb{Z})$ let $\varphi\in\mathscr{A}_\omega$ and $\varphi(x)\neq 0$ for every $x\in\mathbb{R}$. Then $\dfrac{1}{\varphi}\in\mathscr{A}_\omega$.*

Let the complex vector space $\mathscr{A}_\omega^+$ consist of the power series

$$\varphi(z)=\sum_{n=0}^{\infty} c_n z^n \qquad (z\in\mathbb{C})$$

of finite norm

$$\|\varphi\|=\sum_{n=0}^{\infty}|c_n|\omega(n)<\infty.$$

The one-sided analogue of Theorem 1 concerning power series is then given by the following

Theorem 2 *For $\omega\in W(\mathbb{N}\cup\{0\})$ let $\varphi\in\mathscr{A}_\omega^+$ and $\varphi(z)\neq 0$ for every $z\in\mathbb{C}$, $|z|\leqq 1$. Then $\dfrac{1}{\varphi}\in\mathscr{A}_\omega^+$.*

Remark 1 If the weight function is the constant function $\omega=1$ then Theorems 1 and 2 are both well known results due to Wiener (compare Rudin [11, Theorem 18.21 and Exercise 8], or Loomis [9, 23 C]). For Lévy's extension of Wiener's result and further references see Spilker and Schwarz [15].

Remark 2 Let $\omega:\mathbb{N}\to\mathbb{R}_+$ satisfy the inequality $\omega(m+n)\leqq\omega(m)\,\omega(n)$ for all m, $n\in\mathbb{N}$. Then the sequence $(\sqrt[n]{\omega(n)})$ converges with $\lim(\sqrt[n]{\omega(n)})=\inf\{\sqrt[n]{\omega(n)}:n\in\mathbb{N}\}$. Weight functions of this type have been considered, for example, in Katznelson [7].

We shall apply Theorem 2 to multiplicative arithmetical functions. It is convenient to introduce some further notations. Denote by M the set of arithmetical functions $f:\mathbb{N}\to\mathbb{C}$ which are multiplicative, i.e. $f(1)=1$ and $f(mn)=f(m)\,f(n)$ for all relatively prime m, $n\in\mathbb{N}$. This set M together with the Dirichlet convolution $*:M^2\to M$ defined by

$$(f*g)(n)=\sum_{d\mid n}f(d)\,g\left(\frac{n}{d}\right)\qquad(n\in\mathbb{N})$$

forms an abelian group. We are interested in certain subgroups of M. Denote by L the set of real valued functions λ on $\mathbb{N}$ such that

$$\lambda(1)=1,\qquad 0\leqq\lambda(mn)\leqq\lambda(m)\,\lambda(n)\qquad\text{for all } m,\ n\in\mathbb{N},\tag{5}$$

$$\inf\{\sqrt[\nu]{\lambda(p^\nu)}:\nu\in\mathbb{N}\}=1\qquad\text{for every prime } p.\tag{6}$$

Typical examples of functions $\lambda\in L$ are given by

$$\log^\alpha(en)\quad(\alpha\geqq 0),\qquad \log^\beta(e\log(en))\quad(\beta\geqq 0),\qquad \exp(\log^\gamma n)\quad(0\leqq\gamma<1).$$

Remark 3 For every prime p and for every $\lambda\in L$, a weight function $\omega\in W(D)$ is obtained by

$$\omega(n)=\lambda(p^n)\qquad(n\in D=\mathbb{N}\cup\{0\})$$

which satisfies the conditions (1) and (2).

For $\lambda \in L$ we introduce the following sets of multiplicative functions:

$$G_\lambda = \left\{ f \in M \,\middle|\, \sum_p \frac{|f(p)|^2}{p^2} \lambda^2(p) < \infty, \sum_{p,\nu \geqq 2} \frac{|f(p^\nu)|}{p^\nu} \lambda(p^\nu) < \infty \right\},$$

$$D_\lambda = \left\{ h \in M \,\middle|\, \sum_n \frac{|h(n)|}{n} \lambda(n) < \infty \right\},$$

$$B_\lambda = \left\{ f \in G_\lambda \,\middle|\, \sum_p \frac{|f(p)|}{p} \lambda(p) < \infty \right\},$$

$$G_\lambda^* = \left\{ f \in G_\lambda \,\middle|\, \sum_{\nu=0}^{\infty} \frac{f(p^\nu)}{p^{\nu s}} \neq 0 \text{ for all primes } p \text{ and } \operatorname{Re} s \geqq 1 \right\}.$$

Properties of these sets are now given in

Theorem 3 *For $\lambda \in L$, the following statements hold:*
(i) $(G_\lambda, *)$ *is a semigroup,*
(ii) $D_\lambda = B_\lambda$,
(iii) $(G_\lambda^*, *)$ *is a group.*

As an immediate consequence of Theorem 3 we notice

Theorem 4 *For $\lambda \in L$, let $f \in G_\lambda^*$ and $g \in G_\lambda$ be λ-related,* i.e.

$$\sum_p \frac{|g(p) - f(p)|}{p} \lambda(p) < \infty. \tag{7}$$

Then there exists a function $h \in D_\lambda$ such that $g = h * f$.

Remark 4 For $f, g \in M$, the values of $h \in M$ such that $g = h * f$ are recursively determined by

$$h(p^\nu) = g(p^\nu) - \sum_{0 \leqq \rho < \nu} h(p^\rho) f(p^{\nu-\rho}) \qquad (p \text{ prime}, \nu \in \mathbb{N}).$$

Especially, we have $h(p) = g(p) - f(p)$ so that (7) is necessary for $h \in D_\lambda$. The fact that, in general, the above recursion formula cannot be made explicit causes the problem to find an additional condition, as weak as possible, which guarantees the inversion of $f \in G_\lambda$ within G_λ. It might be surprising that the condition

$$\varphi_f(p, s) := 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots \neq 0 \qquad (p \text{ prime}, \operatorname{Re} s \geqq 1)$$

is independent of $\lambda \in L$.

The significance of Theorem 4 can be described shortly as follows. Often mean properties of f can be carried over to g if the functions f, g are somehow related. The deviation of g from f is properly characterized by $h = g * f^{-1}$, and is measured in mean by $h \in D_\lambda$, with some fixed weight function $\lambda \in L$. Hitherto only the special case $\lambda = 1$ has been developed and used by many authors. Compare, for example, Delange [1], Wirsing [16, 17], Erdös and Rényi [3], Schwarz [13], Halász [4], Elliott [2]. For $\lambda = 1$ fairly general conditions have been given in [10], and in Heppner and Schwarz [6]. Theorem 4 now provides effective

relationship properties and thus solves a problem posed by Schwarz [14, Problem 8.6] in 1985.

Remark 5 The preceding theorems may be generalized to several variables. For Wiener's inversion theorem in the case $\omega=1$ see, for example, Rudin [12, Chap. 11]. The generalization of relationship properties for multiplicative functions of several variables has been worked out by Heppner [5] in the case $\lambda=1$.

By Theorem 4 we are put into position to carry over asymptotic expansions and convergence properties.

Corollary 1 *Let $k\in\mathbb{N}\cup\{0\}$, and put $\lambda(n)=\log^k(en)$. Assume that $f\in G_\lambda^*$ and $g\in G_\lambda$ are λ-related. If*

$$\sum_{n\leqq x} f(n)=xP(\log x)+o(x) \qquad (x\to\infty)$$

with a polynomial P of degree $\leqq k$ then

$$\sum_{n\leqq x} g(n)=xQ(\log x)+o(x) \qquad (x\to\infty)$$

with a polynomial Q of degree $\leqq k$. Moreover, the Dirichlet series

$$H(s)=\sum_{n=1}^{\infty}\frac{h(n)}{n^s}=\prod_p\frac{\varphi_g(p,s)}{\varphi_f(p,s)} \qquad (\operatorname{Re} s>1)$$

and its derivatives up to the order k converge absolutely at $s=1$, and

$$Q(\log x)=\sum_{\rho=0}^{k}\frac{H^{(\rho)}(1)}{\rho!}P^{(\rho)}(\log x).$$

Corollary 2 *Suppose that, under the general conditions of Corollary 1, the series*

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}\log^\rho n \qquad (\rho=0,1,\ldots,k)$$

converge with the sums S_ρ. Then the series

$$\sum_{n=1}^{\infty}\frac{g(n)}{n}\log^k n$$

converges with the sum

$$\sum_{\rho=0}^{k}\binom{k}{\rho}(-1)^\rho H^{(\rho)}(1)\,S_{k-\rho}$$

where $H(s)$ is defined in Corollary 1.

For application of Corollary 2 we may take as f the Möbius function $\mu \in M$,

$$\mu(p^\nu)=\begin{cases}-1 & \text{if } \nu=1\\ 0 & \text{if } \nu>1\end{cases} \qquad (p \text{ prime}, \nu\in\mathbb{N}).$$

It is well known (see Landau [8, §§ 157, 158]) that the series

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0, \qquad \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log n=-1$$

are conditionally convergent. As a special case Corollary 2 now yields the following

Example. If $g\in M$ satisfies

$$\sum_{p}\frac{|g(p)+1|}{p}\log p<\infty, \qquad \sum_{p,\nu\geqq 2}\frac{|g(p^\nu)|}{p^\nu}\log p^\nu<\infty$$

then

$$\sum_{n=1}^{\infty}\frac{g(n)}{n}\log n=-\prod_{p}\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\ldots\right).$$

2 Banach algebra techniques

As to the proof of Theorem 1 we notice that $\mathscr{A}_\omega$ together with the norm defined in (4) is a Banach space. The usual pointwise product $\varphi\psi$ of φ, $\psi\in\mathscr{A}_\omega$ defines a commutative multiplication on $\mathscr{A}_\omega$ since (3), (4) and

$$\psi(x)=\sum_{m=-\infty}^{\infty}d_m e^{imx}, \qquad \sum_{m=-\infty}^{\infty}|d_m|\,\omega(m)<\infty$$

imply

$$\varphi\psi(x)=\sum_{k=-\infty}^{\infty}\Big(\sum_{n+m=k}c_n d_m\Big)e^{ikx},$$

and by (1)

$$\|\varphi\psi\|=\sum_{k=-\infty}^{\infty}\Big|\sum_{n+m=k}c_n d_m\Big|\omega(k)\leqq\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}|c_n|\,|d_m|\,\omega(n+m)$$

$$\leqq\sum_{n=-\infty}^{\infty}|c_n|\omega(n)\sum_{m=-\infty}^{\infty}|d_m|\omega(m)=\|\varphi\|\,\|\psi\|<\infty.$$

The constant function 1 is the unit element in $\mathscr{A}_\omega$, and $\|1\|=\omega(0)=1$. Therefore $\mathscr{A}_\omega$ is a commutative Banach algebra with unit.

With any commutative Banach algebra $\mathscr{A}$ with unit element we associate the set $\Delta=\Delta(\mathscr{A})$ of all complex homomorphisms of $\mathscr{A}$ onto $\mathbb{C}$. From Rudin [11, Theorem 18.17], we take the following statements as

Lemma. *The inequality* $|h(\varphi)| \leqq \|\varphi\|$ *holds for every* $\varphi \in \mathscr{A}$ *and* $h \in \Delta$. *The element* $\varphi \in \mathscr{A}$ *is invertible in* $\mathscr{A}$ *if and only if* $h(\varphi) \neq 0$ *for every* $h \in \Delta$.

We put $\varphi_0(x) = e^{ix}$ and notice $\varphi_0 \in \mathscr{A}_\omega$, $\dfrac{1}{\varphi_0} \in \mathscr{A}_\omega$, and $\|\varphi_0^n\| = \omega(n)$ for $n \in \mathbb{Z}$. Let $h \in \Delta$, and assume $h(\varphi_0) = z$. By the above lemma, we obtain

$$|z^n| = |h(\varphi_0^n)| \leqq \|\varphi_0^n\| = \omega(n) \qquad (n \in \mathbb{Z})$$

so that

$$|z| \leqq \sqrt[n]{\omega(n)}, \qquad |z| \geqq \frac{1}{\sqrt[n]{\omega(-n)}} \qquad (n \in \mathbb{N}).$$

Now (2) yields $|z| = 1$. Thus, to each $h \in \Delta$ there exists a number $\alpha \in \mathbb{R}$ such that $h(\varphi_0) = e^{i\alpha}$. We obtain

$$h(\varphi_0^n) = e^{in\alpha} = \varphi_0^n(e^{i\alpha}) \qquad (n \in \mathbb{Z}),$$

and, by the continuity of h, the representation

$$\varphi = \sum_{n=-\infty}^{\infty} c_n \varphi_0^n$$

proceeds to

$$h(\varphi) = \varphi(\alpha) \qquad (\varphi \in \mathscr{A}_\omega).$$

Since, by hypothesis of Theorem 1, $\varphi(\alpha)$ does not vanish for any $\alpha \in \mathbb{R}$, φ does not belong to the kernel of any $h \in \Delta$. The lemma guarantees that φ is invertible in $\mathscr{A}_\omega$ as was stated in Theorem 1.

Turning to the proof of Theorem 2, we consider the Banach space $\mathscr{A}_\omega^+$ with a fixed weight function $\omega \in W(\mathbb{N} \cup \{0\})$. Then $\mathscr{A}_\omega^+$ is again a commutative Banach algebra. Clearly the constant function 1 and the identity function φ_0, $\varphi_0(z) = z$, belong to $\mathscr{A}_\omega^+$, but $\dfrac{1}{\varphi_0} \notin \mathscr{A}_\omega^+$. Denote by Δ^+ the set of all homomorphisms of $\mathscr{A}_\omega^+$ onto $\mathbb{C}$. The same arguments as were used in the proof of Theorem 1 show that $h(\varphi_0) = \zeta$ for $h \in \Delta^+$ implies $|\zeta| \leqq 1$ and $h(\varphi) = \varphi(\zeta)$. Since, again by hypothesis of Theorem 2, $\varphi(\zeta)$ does not vanish at any point $\zeta \in \mathbb{C}$, $|\zeta| \leqq 1$, φ does not belong to the kernel of any $h \in \Delta^+$, and the statement of Theorem 2 follows as before.

3 Multiplicative functions

We have to study the sets G_λ, D_λ, B_λ, G_λ^* $(\lambda \in L)$ of multiplicative functions defined in Sect. 1.

Let $f, g \in G_\lambda$. Then, from

$$(f*g)(p^\nu) = \sum_{\substack{\rho,\sigma \geqq 0 \\ \rho+\sigma=\nu}} f(p^\rho)\, g(p^\sigma) \qquad (p \text{ prime}, \nu \in \mathbb{N}),$$

we obtain

$$\sum_p \frac{|(f*g)(p)|^2}{p^2}\, \lambda^2(p) = \sum_p \frac{|f(p)+g(p)|^2}{p^2}\, \lambda^2(p)$$

$$\leqq 2 \sum_p \frac{|f(p)|^2 + |g(p)|^2}{p^2}\, \lambda^2(p) < \infty,$$

and by (5)

$$\sum_{p,\nu \geqq 2} \frac{|(f*g)(p^\nu)|}{p^\nu}\, \lambda(p^\nu) \leqq \sum_{\substack{p,\rho,\sigma \\ \rho+\sigma \geqq 2}} \frac{|f(p^\rho)|}{p^\rho} \frac{|g(p^\sigma)|}{p^\sigma}\, \lambda(p^\rho)\, \lambda(p^\sigma).$$

The convergence of the triple series on the right side is now easily seen by treating the four parts $\rho \in \{0,1\}, \sigma \geqq 2$; $\rho \geqq 2, \sigma \in \{0,1\}$; $\rho = \sigma = 1$; $\rho \geqq 2, \sigma \geqq 2$ separately, and by using $f(p)\, \lambda(p) \ll p$, $g(p)\, \lambda(p) \ll p$, and the Cauchy-Schwarz inequality. This shows $f*g \in G_\lambda$ and proves Theorem 3(i).

Trivially, $D_\lambda \subseteq B_\lambda$. Conversely, let $h \in B_\lambda$. Then we know

$$S := \sum_{p,\nu \geqq 1} \frac{|h(p^\nu)|}{p^\nu}\, \lambda(p^\nu) < \infty,$$

and we infer from (5) that

$$\sum_{n \leqq x} \frac{|h(n)|}{n}\, \lambda(n) \leqq \prod_{p \leqq x} \left(1 + \frac{|h(p)|}{p}\, \lambda(p) + \frac{|h(p^2)|}{p^2}\, \lambda(p^2) + \ldots\right) \leqq e^S$$

which gives $h \in D_\lambda$ and proves the statement (ii) of Theorem 3.

Obviously the identity element ε with $\varepsilon(1) = 1$, $\varepsilon(n) = 0$ for $1 < n \in \mathbb{N}$ belongs to G_λ^*. If $f, g \in G_\lambda^*$ then $\varphi_{f*g}(p,s) = \varphi_f(p,s)\, \varphi_g(p,s) \neq 0$ for $\mathrm{Re}\, s \geqq 1$ and $f*g \in G_\lambda^*$ by Theorem 3 (i). Assuming $f^{-1} \in G_\lambda$ for $f \in G_\lambda^*$ we see $1 = \varphi_\varepsilon(p,s) = \varphi_f(p,s)\, \varphi_{f^{-1}}(p,s) \neq 0$ which shows $f^{-1} \in G_\lambda^*$. In order to complete the proof of Theorem 3 (iii) it remains therefore to verify that $f \in G_\lambda^*$ implies $f^{-1} \in G_\lambda$. At this point we shall utilize Theorem 2.

Consider the convolution factor f_p of $f \in G_\lambda^*$ which singles out the values of f only at the powers p^ν of a fixed prime p,

$$f_p(n) = \begin{cases} f(n) & \text{if } n = p^\nu \quad (\nu \in \mathbb{N} \cup \{0\}) \\ 0 & \text{else.} \end{cases}$$

It is obvious that $f_p \in G_\lambda^*$ for each prime p. Applying Theorem 2 to

$$\varphi_f(p,s) = 1 + \frac{f(p)}{p}\, z + \frac{f(p^2)}{p^2}\, z^2 + \ldots \qquad (|z| \leqq 1, z = p^{1-s}),$$

$$\sum_{\nu=0}^{\infty} \frac{|f(p^\nu)|}{p^\nu}\, \omega(\nu) < \infty \qquad (\omega(\nu) = \lambda(p^\nu)),$$

according to Remark 3, we obtain $f_p^{-1} \in G_\lambda$ for each prime p.

Now we choose p_0 such that

$$\sum_{p>p_0} \frac{|f(p)|^2}{p^2}\lambda^2(p)\leqq\frac{1}{4}, \qquad \sum_{\substack{p>p_0\\ \nu\geqq 2}} \frac{|f(p^\nu)|}{p^\nu}\lambda(p^\nu)\leqq\frac{1}{6}, \tag{8}$$

and put for prime powers p^ν

$$f^*(p^\nu)=\begin{cases} f^\nu(p) & \text{if } p>p_0\\ 0 & \text{else,}\end{cases}$$

$$f_0(p^\nu)=\begin{cases} (f*\mu f)(p^\nu) & \text{if } p>p_0\\ 0 & \text{else.}\end{cases}$$

Here again μ denotes the Möbius function which is the inverse of the constant function 1, i.e. $\mu*1=\varepsilon$. By multiplicative continuation the domains of f^* and f_0 are extended to $\mathbb{N}$. Obviously f^* is totally multiplicative so that $f^{*-1}=\mu f^*$. It is easy to verify that $f^*\in G_\lambda^*$ and $f^{*-1}\in G_\lambda$. From the above definitions we obtain the representation

$$f=(\underset{p\leqq p_0}{*} f_p)*f^**f_0.$$

As we have already seen the finitely many convolution factors $f_p(p\leqq p_0)$ and f^* are invertible within G_λ. By Theorem 3 (i), the proof of Theorem 3 (iii) reduces to the proof of $f_0\in G_\lambda^*$ and $f_0^{-1}\in G_\lambda$.

By definition, we get at prime powers p^ν

$$f_0(p^\nu)=\begin{cases} f(p^\nu)-f(p)f(p^{\nu-1}) & \text{if } p>p_0, \nu\geqq 2\\ 0 & \text{else,}\end{cases}$$

and $f\in G_\lambda^*$ implies

$$\sum_{p,\nu\geqq 2} \frac{|f_0(p^\nu)|}{p^\nu}\lambda(p^\nu)\leqq \sum_{\substack{p>p_0\\ \nu\geqq 2}} \frac{|f(p^\nu)|}{p^\nu}\lambda(p^\nu)+\sum_{p>p_0}\frac{|f(p)|^2}{p^2}\lambda^2(p)$$

$$+\sum_{\substack{p>p_0\\ \nu\geqq 2}} \frac{|f(p)|}{p}\frac{|f(p^\nu)|}{p^\nu}\lambda(p)\,\dot{\lambda}(p^\nu)\leqq\frac{1}{2}$$

because of (5), (8), and $|f(p)|\,\lambda(p)\leqq\frac{1}{2}p$ for $p>p_0$. Thus $f_0\in G_\lambda$. Further we have $\varphi_{f_0}(p,s)=1\neq 0$ for $p\leqq p_0$, and

$$\varphi_{f_0}(p,s)=\varphi_f(p,s)\,\varphi_{\mu f}(p,s)=\varphi_f(p,s)\left(1-\frac{f(p)}{p^s}\right)\neq 0$$

for $p>p_0$ and $\operatorname{Re} s\geqq 1$ since $|f(p)|\leqq\frac{1}{2}p$ for $p>p_0$. This shows $f_0\in G_\lambda^*$.

By solving $f_0^{-1} * f_0 = \varepsilon$ at prime powers p^ν we obtain the recursion

$$f_0^{-1}(p^\nu) = \begin{cases} -\sum_{2 \leqq \rho \leqq \nu} f_0(p^\rho) f_0^{-1}(p^{\nu-\rho}) & \text{if } p > p_0, \nu \geqq 2 \\ 0 & \text{else.} \end{cases}$$

This gives, by considering partial sums,

$$\begin{aligned} \Sigma &:= \sum_{p,\nu \geqq 2} \frac{|f_0^{-1}(p^\nu)|}{p^\nu} \lambda(p^\nu) \\ &\leqq \sum_{p,\nu \geqq 2} \sum_{2 \leqq \rho \leqq \nu} \frac{|f_0(p^\rho)|}{p^\rho} \frac{|f_0^{-1}(p^{\nu-\rho})|}{p^{\nu-\rho}} \lambda(p^\rho) \lambda(p^{\nu-\rho}) \\ &\leqq \sum_{p,\nu \geqq 2} \frac{|f_0(p^\nu)|}{p^\nu} \lambda(p^\nu) + \sum_{\substack{p \\ \rho \geqq 2, \sigma \geqq 2}} \frac{|f_0(p^\rho)|}{p^\rho} \lambda(p^\rho) \frac{|f_0^{-1}(p^\sigma)|}{p^\sigma} \lambda(p^\sigma) \\ &\leqq (1+\Sigma) \sum_{p,\nu \geqq 2} \frac{|f_0(p^\nu)|}{p^\nu} \lambda(p^\nu) \leqq \frac{1}{2}(1+\Sigma) \end{aligned}$$

so that $\Sigma \leqq 1$. Therefore $f_0^{-1} \in G_\lambda$ which ends the proof of Theorem 3.

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Riemann-Roch for weakly-equivariant D-modules II

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This paper is a continuation of the work begun in [J2] and J3]. A Riemann-Roch theorem for coherent algebraic $\mathscr{D}$-modules that are weakly-equivariant for the action of complex diagonalizable groups (under the assumption that the actions are linear) is established in [J2]. In the first section of this paper we generalize this to allow actions by arbitrary complex linear algebraic groups. Applying [J4, (2.4)], we also obtain a generalization to higher K-theory when restricted to linear actions by diagonalizable groups.

In the later sections, we combine this Riemann-Roch theorem with the results of [J3] to provide very general character formulae for coherent algebraic $\mathscr{D}$-modules on smooth complex projective algebraic varieties that are equivariant for the action of a complex diagonalizable group. Most of the well-known character formulae for e.g. the character formulae of Weyl, Demazure or Kazdhan-Lusztig will be interpreted as special cases of our general character formula. One may regard our character formula as a natural generalization of the Atiyah-Bott-Segal-type character formula (see [A-B] and [A-S]) to $\mathscr{D}$-modules, *in the sense that when applied to extended $\mathscr{D}$-modules, our general character formula gives the familiar formulae of Atiyah-Bott-Segal.* (See (2.6.1).) When applied to holonomic $\mathscr{D}$-modules and using the Riemann-Hilbert correspondence, our character formula provides a Lefschetz-fixed point formula for equivariant perverse sheaves. One may compare this to the fact that most of the common duality theorems (e.g. Poincare, Serre or Verdier duality) may be derived from a single duality theorem at the level of $\mathscr{D}$-modules.

1

(1.0) Let X denote a complex smooth quasi-projective algebraic variety provided with the action of a complex linear algebraic group G; let $\mu: G \times X \to X (\mathrm{pr}_2 : G \times X \to X)$ denote the group action (the projection to the second factor, respectively). (We will restrict to algebraic varieties throught this paper.) The category of G-equivariant $\mathcal{O}_X$-modules will be defined as in ([T1, 1.2] and also [Mum, p. 30]). This category (the full subcategory of G-equivariant modules which are also quasi-coherent, coherent) will be denoted $Mod^G(\mathcal{O}_X)(Mod^G_{q\mathrm{coh}}(\mathcal{O}_X),$

$Mod^G_{\mathrm{coh}}(\mathcal{O}_X)$, respectively) from now on. Recall from [T1, p. 543] that these categories are *abelian.*

(1.1) Let $\mathcal{D}_X$ denote the sheaf of algebraic $\mathbb{C}$-linear differential operators on X filtered (as usual by the order of the differential operators – see [J2, (1.3)] by $\mathcal{D}^n_X = \mathrm{Diff}^n_X$. It is clear first of all that $\mathcal{D}_X$ (each $\mathcal{D}^n_X$) is a G-equivariant quasi-coherent (coherent, respectively) $\mathcal{O}_X$-module. It is shown in [J2, 1.4.2] that the map $\mu^*\mathcal{D}_X \xrightarrow{\simeq} \mathrm{pr}_2^*\,\mathcal{D}_X$ is an isomorphism of sheaves of rings (where $\mu(\mathrm{pr}_2): G \times X \to X$ is as before).

(1.2) A $\mathcal{D}_X$-module M will mean an $\mathcal{O}_X$-module which is a (sheaf of) left modules (unless we say to the contrary) over the ring $\mathcal{D}_X$. Quasi-coherent (coherent) $\mathcal{D}_X$-modules are defined in the obvious manner.

(1.3) **Definition.** A $\mathcal{D}_X$-module M will be called a *G-equivariant* (or a *weakly $-G$-equivariant) $\mathcal{D}_X$-module* provided it is a G-equivariant $\mathcal{O}_X$-module and M *comes equipped with an isomorphism* φ: $\mu^*M \xrightarrow{\simeq} \mathrm{pr}_2^*\,M$ compatible with the isomorphism $\mu^*\mathcal{D}_X \xrightarrow{\simeq} \mathrm{pr}_2^*\,\mathcal{D}_X$ and $\mu^*\mathcal{O}_X \xrightarrow{\simeq} \mathrm{pr}_2^*\,\mathcal{O}_X$. (Observe that, as M is a G-equivariant $\mathcal{O}_X$-module, the isomorphism φ satisfies the usual conditions.) The category of G-equivariant $\mathcal{D}_X$-modules (which are also quasi-coherent, coherent) and G-equivariant maps will be denoted $Mod^G(\mathcal{D}_X)$($Mod^G_{q\,\mathrm{coh}}(\mathcal{D}_X)$, $Mod^G_{\mathrm{coh}}(\mathcal{D}_X)$, respectively). The exactness of pr_2^* and μ^* shows these are again abelian categories. Consult [J2, Sect. 1] for further details on weakly-equivariant $\mathcal{D}$-modules.

(1.4) Let $f: X \to Y$ denote a G-equivariant map between smooth complex quasi-projective varieties provided with G-actions. In [J2], under the assumptions that the group G is a diagonalizable group D acting linearly on X and Y we define a functor

$$\int_f : D_b(Mod^G_{q\,\mathrm{coh}}(\mathcal{D}_X)) \to D_b(Mod^G_{q\,\mathrm{coh}}(\mathcal{D}_Y) \tag{1.4.0}$$

associated to f. (Here the D action on a smooth complex quasi-projective variety Z is said to be *linear* if there exists a linear action of D on some (sufficiently large) projective space $\mathbb{P}^n$ and Z admits a D-equivariant locally-closed immersion into $\mathbb{P}^n$. Recall from [Sum, Theorem 1] that if D is connected, we may assume without loss of generality that any action of D on a normal complex quasi-projective variety is linear.) Recall that $\int_f$ is defined usually as the composition of the functor $\mathcal{D}_{Y \leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} -$ and the derived functor of $\bar{f}_*$ where $\bar{f}$ is the induced map of ringed spaces $(X, f^{-1}(\mathcal{D}_Y)) \to (Y, \mathcal{D}_Y)$. In the equivariant case one may show there is an induced co-action of the structure sheaf $\mathcal{O}_G$ on $f^{-1}(\mathcal{D}_Y)$ as observed in [J2, (1.8.0)]. Hence one may define the category of equivariant modules $Mod^G((X, f^{-1}(\mathcal{D}_Y))$ in an obvious manner. Now it is shown in [J2, (1.8.2)(iii)] that $\bar{f}_*$ induces a map $Mod^G(X, f^{-1}(\mathcal{D}_Y)) \to Mod^G(Y, \mathcal{D}_Y)$; however it is not clear that the category $ModG((X, f^{-1}(\mathcal{D}_Y))$ has enough injectives which makes it difficult to take the right derived functor of the corresponding $\bar{f}_*$. (In [J2], as the group is restricted to be diagonalizable and the action is assumed to be linear we avoid this difficulty by making use of a Cech-resolution, which

is possible as each point has an invariant affine open neighborhood (by [Sum, pp. 8–9]). When G is an arbitrary linear algebraic group this method clearly fails.) In the place of (1.4.0) we will however succeed in defining a functor

$$\int_f^G : D_{b,\mathrm{coh}}^G(Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})) \to D_b^G(Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} Y; \mathscr{D}_{EG \underset{G}{\times} Y})) \tag{1.4.1}$$

where the derived categories are defined in (1.6.1) and (1.6.2).

(1.5.0) Assume the above situation. Let $EG \underset{G}{\times} X$ denote the simplicial scheme given by the usual bar construction (see for e.g. [Fr, p. 9]). A sheaf of rings on $EG \underset{G}{\times} X$ is a collection $R = \{R_n | n\}$, where R_n is a sheaf of algebras (with a unit) on $((EG \underset{G}{\times} X)_n, (\bigotimes^n \mathscr{O}_G) \otimes \mathscr{O}_X)$ so that for each structure map

$$\psi : (EG \underset{G}{\times} X)_n \to (EG \underset{G}{\times} X)_m,$$

there is *given* an *isomorphism* $\Psi : \psi^* R_m \to R_n$ of sheaves of algebras over

$$((EG \underset{G}{\times} X)_n, (\bigotimes^n \mathscr{O}_G) \otimes \mathscr{O}_X)$$

and the maps Ψ satisfy an obvious compatibility condition. Now modules on the above simplicial ringed space may be defined to be collections $\{M_n | n\}$ so that M_n is a sheaf of *left* R_n-modules and there exist maps $\Psi : \psi^* M_m \to M_n$ of sheaves of R_n-modules satisfying similar compatibility conditions. This category will be denoted $Mod(EG \underset{G}{\times} X; R)$. The full subcategory for which the above maps are isomorphisms will be denoted $Mod^G(EG \underset{G}{\times} X; R)$ and modules belonging to this category will be called *equivariant* modules on the simplicial ringed space $(EG \underset{G}{\times} X; R)$. (See for e.g. [J2, (1.8.2)].)

(1.5.1) One may observe that in the situation of (1.4), if $A = \mathscr{O}_X$, $\mathscr{D}_X$, $f^{-1}(\mathscr{O}_Y)$ or $f^{-1}(\mathscr{D}_Y)$ we obtain simplicial ringed spaces where $R_n = ((\bigotimes^n \mathscr{O}_G) \otimes A)$. (Using the standard simplicial identities $d_i \circ d_j = d_{j-1} \circ d_i$, $d_i s_j = s_{j-1} \circ d_i$ if $i = j$, $d_i \circ s_j$ = identity if $i = j$ or $j+1$, $d_i \circ s_j = s_j \circ d_{i-1}$ if $i > j+1$ and $s_i \circ s_j = s_j \circ s_{i-1}$ for $i > j$, one may verify that this defines a sheaf of rings on $EG \underset{G}{\times} X$ as in (1.5.0).) These will be denoted $(EG \underset{G}{\times} X; \mathscr{O}_{EG \underset{G}{\times} X})$, $(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})$, $(EG \underset{G}{\times} X; f^{-1}(\mathscr{O}_{EG \underset{G}{\times} Y}))$, $(EG \underset{G}{\times} X; f^{-1}(\mathscr{D}_{EG \underset{G}{\times} Y}))$, respectively. We will be primarily concerned with these

and may often generically denote any of these by $(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$. The abelian category of sheaves of *left* modules on

$$(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$$

will be denoted

$$Mod(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X}).$$

(1.5.2) *Remark.* Assume the above situation. First of all observe that there exists an equivalence of categories

$$Mod^G(A) \simeq Mod^G(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$$

where A is as above and the category on the left is defined as in (1.3) while the category on the right is defined as in (1.5.0). (If $M \in Mod^G(A)$, we will let $\{M_n\}$ be defined by

$$M_n = (\overset{n}{\bigotimes} \mathcal{O}_G) \otimes M;$$

one may verify as before that this defines an object of

$$Mod^G(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})).$$

$Mod^G(A)$ regarded as a subcategory of $Mod(A)$, *is not in general closed under extensions; however regarded as a full subcategory of*

$$Mod(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$$

it is. Moreover $Mod^G(A)$ *need not have enough injectives.*

(1.5.3) Now we may define

$$Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X})\,(Mod_{\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X}))$$

to be the full subcategory of $Mod(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X})$ consisting of modules $M = \{M_n | n\}$ so that M_n is a quasi-coherent (coherent, respectively) $\mathcal{O}_{(EG \underset{G}{\times} X)_n}$ module for each $n \geqq 0$. The categories

$$Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X}), \quad \text{and} \quad (Mod_{\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X}))$$

may now be defined in the obvious manner.

(1.6.1) If $\mathscr{A}$ denotes any one of the abelian categories in (1.5.1) above, we will let $C^G_+(\mathscr{A})$ denote the full-subcategory of the category $C(\mathscr{A})$ of complexes $K^\cdot$ in $\mathscr{A}$ satisfying the conditions:

(i) $\mathscr{H}^i(K^\cdot)$ is *an equivariant sheaf* i.e. belongs to the full-subcategory $Mod^G(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$ (of $Mod(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$) for each i and

(ii) there exists an integer N (depending on $K^\cdot$) so that $K^i = 0$ for all $i < N$. (i.e. the complex $K^\cdot$ is bounded below.)

The category $C^G_+(\mathscr{A})$ will be referred to as the category of bounded-below complexes in $\mathscr{A}$ *with equivariant cohomology sheaves.*

(*Remark.* The reason for considering these categories is that in view of the fact (see (1.5.2)) that the category $Mod^G(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X}$ is closed under extensions in

$$Mod(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X}),$$

one can now define the derived category associated to $C^G_+(\mathscr{A})$ – see [Hart2, p. 47]. (In general, given a K in $Mod^G(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$, one can find an injective resolution of K only by a complex in the larger category $Mod(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$.) However since the category $Mod^G(A)$ is not closed under extensions in $Mod(A)$, it is not possible to consider the derived category of complexes $K^\cdot$ in $Mod(A)$ with the cohomology sheaves $\mathscr{H}^i(K^\cdot) \in Mod^G(A)$.)

The category $C^G_+(\mathscr{A})$ ($C^G_b(\mathscr{A})$, $C^{G,b}_+(\mathscr{A})$) will similarly denote the category of bounded-below complexes in $\mathscr{A}$ with equivariant cohomology sheaves, (bounded complexes in $\mathscr{A}$ with equivariant cohomology sheaves, bounded-below complexes in $\mathscr{A}$ with bounded equivariant cohomology sheaves, respectively). $C^G_-(\mathscr{A})$, $C^{G,b}_-(\mathscr{A})$ will similarly denote the corresponding categories of bounded-above complexes.

(1.6.2) When $A = \mathscr{O}_X$ or $\mathscr{D}_X$, and $\mathscr{A}$ denotes the corresponding category in (1.5.1) we let $\mathscr{A}_{q\,\mathrm{coh}}(\mathscr{A}_{\mathrm{fl.qc}}, \mathscr{A}_{\mathrm{coh}}, \mathscr{A}_{\mathrm{lfree}})$ denote the full subcategory of $\mathscr{A}$ consisting of quasi-coherent A-modules (quasi-coherent flat A-modules, coherent modules, locally free *coherent* modules, respectively) defined as in (1.5.3). Now $C^{G,q\,\mathrm{coh}}_+(\mathscr{A})(C^{G,\mathrm{coh}}_+(\mathscr{A}))$ will denote the category of bounded-below complexes in $\mathscr{A}$ with *equivariant* cohomology sheaves in $\mathscr{A}_{q\,\mathrm{coh}}$(in $\mathscr{A}_{\mathrm{coh}}$). The categories $C^{G,q\,\mathrm{coh}}_b(\mathscr{A})$, $C^{G,\mathrm{coh}}_b(\mathscr{A})$, ($C^{G,q\,\mathrm{coh}}_-(\mathscr{A})$, $C^{G,\mathrm{coh}}_-(\mathscr{A})$, $C^{G,b\,\mathrm{coh}}_-(\mathscr{A})$) are defined similarly using bounded complexes (bounded-above complexes, respectively). The derived categories associated to the above categories of complexes will be indicated in the obvious manner.

(1.7) Assume X and Y are smooth complex quasi-projective varieties as before provided with the action of a complex linear algebraic group G. Let $f: X \to Y$ denote a G-equivariant map. *We will assume this basic situation through the rest of Sect.* 1. In this situation we proceed to define a functor

$$(1.7.^*)\quad \int_f : D^{G,\mathrm{coh}}_b(Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})) \to D^G_b(Mod(EG \underset{G}{\times} Y; \mathscr{D}_{EG \underset{G}{\times} Y}))$$

as follows.

(1.7.1) We begin by observing that the obvious inclusion functors in

$$C_{-}^{G,b,\mathrm{coh}}(Mod_{\mathrm{fl.qc}}(EG\underset{G}{\times}X;\mathscr{D}_{EG\underset{G}{\times}X}))\to C_{-}^{G,b,\mathrm{coh}}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X;\mathscr{D}_{EG\underset{G}{\times}X}))$$
$$\leftarrow C_{b}^{G,\mathrm{coh}}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X;\mathscr{D}_{EG\underset{G}{\times}X}))$$

induce equivalences of the associated derived categories. (The first (second) functor induces an equivalence of the associated derived categories by (A.1)(ii) ((A.6.7), respectively) in the appendix.) Therefore it suffices to define

$$\int_f \quad \text{on} \quad D_{-}^{G,b,\mathrm{coh}}(Mod_{\mathrm{fl.qc}}(EG\underset{G}{\times}X;\mathscr{D}_{EG\underset{G}{\times}X})).$$

Let $\mathscr{D}_{EG\underset{G}{\times}Y\leftarrow EG\underset{G}{\times}X}$ denote the left $-\mathscr{D}_{f^{-1}(EG\underset{G}{\times}Y)}$, right $\mathscr{D}_{EG\underset{G}{\times}X}$ bi-module given on $(EG\underset{G}{\times}X)_n$ by $(\bigotimes_{n}\mathscr{O}_G)\otimes\mathscr{D}_{Y\leftarrow X}$. (First of all we obtain the G-equivariant $(f^{-1}(\mathscr{D}_Y),\mathscr{D}_X)$-bi-module

$$\mathscr{D}_{Y\leftarrow X}=f^{-1}(\mathscr{D}_Y\bigotimes_{\mathscr{O}_Y}\omega_Y)\bigotimes_{f^{-1}(\mathscr{O}_Y)}\check{\omega}_X$$

where the group acts diagonally on the terms on the right. One may verify (using [Bj, p. 230]) that $\omega_Y(\omega_X)$ is an equivariant right $\mathscr{D}_Y(\mathscr{D}_X)$-module; the equivariant left $f^{-1}(\mathscr{D}_Y)$ (right $\mathscr{D}_X$)-module structure on $\mathscr{D}_{Y\leftarrow X}$ is clear).

Tensoring with $\mathscr{D}_{EG\underset{G}{\times}Y\leftarrow EG\underset{G}{\times}X}$ on the left defines an exact functor

$$(1.7.2)\quad D_{-}^{G,b,\mathrm{coh}}(Mod_{\mathrm{fl.qc}}(EG\underset{G}{\times}X;\mathscr{D}_{EG\underset{G}{\times}X}))\to D_{-}^{G,b}(Mod(EG\underset{G}{\times}X,f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y})).$$

In view of (A.6.7), the obvious functor

$$(1.7.3)\quad D_{b}^{G}(Mod(EG\underset{G}{\times}X,f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y}))\to D_{-}^{G,b}(Mod(EG\underset{G}{\times}X,f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y}))$$

is an equivalence of categories.

Let $ID_{+}^{G,b}(Mod(EG\underset{G}{\times}X,\ f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y})))$ denote the derived category of bounded-below complexes of injective modules in $Mod(EG\underset{G}{\times}X,\ f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y}))$ with bounded equivariant cohomology. The obvious functors

$$(1.7.4)\quad ID_{+}^{G,b}(Mod(EG\underset{G}{\times}X,f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y})))\to D_{+}^{G,b}(Mod(EG\underset{G}{\times}X,f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y})))$$
$$\leftarrow D_{b}^{G}(Mod(EG\underset{G}{\times}X,f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y})))$$

are clearly equivalences of categories once again by (A.6.6) and (A.6.7). Now the direct image functor

$$(1.7.5)\quad \bar{f}_*\colon ID_+^{G,b}(Mod(EG\underset{G}{\times}X, f^{-1}(\mathscr{D}_{EG\underset{G}{\times}Y}))) \to D_+(Mod(EG\underset{G}{\times}Y, \mathscr{D}_{EG\underset{G}{\times}Y}))$$

along with the functors in (1.7.1) through (1.7.4) enable us to define a functor

$$(1.7.6)\quad \int_f\colon D_b^{G,\mathrm{coh}}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X; \mathscr{D}_{EG\underset{G}{\times}X})) \to D_+(Mod(EG\underset{G}{\times}Y, \mathscr{D}_{EG\underset{G}{\times}Y}))$$

in the obvious manner.

We will now show that if $M^\cdot \in D_b^{G,\mathrm{coh}}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X;\ \mathscr{D}_{EG\underset{G}{\times}X}))$, $\int_f M^\cdot$ has bounded quasi-coherent equivariant cohomology sheaves. In view of the strongly-convergent spectral sequence

$$(1.7.7)\quad E_2^{s,t} = \mathscr{H}^s(\int_f(\mathscr{H}^t(M^\cdot))) \Rightarrow \mathscr{H}^{s+t}(\int_f M^\cdot)$$

the fact that $Mod_{\mathrm{coh}}^G(\mathscr{D}_X)$ is abelian, $\mathscr{H}^t(M^\cdot) \in Mod_{\mathrm{coh}}^G(\mathscr{D}_X)$ and (1.5.2), it suffices to prove this for a single module $N \in Mod_{\mathrm{coh}}^G(\mathscr{D}_X)$.

(1.7.8) Recall the equivariant Spencer exact sequence (see (A.4.2)) shows that any coherent weakly-equivariant algebraic $\mathscr{D}_X$-module has an equivariant resolution by a bounded complex of modules of the form $\mathscr{D}_X \otimes \mathscr{E}$ where $\mathscr{E}$ is an equivariant coherent sheaf on X. Using this and (A.4.3) one may show that if $N \in Mod_{\mathrm{coh}}^G(\mathscr{D}_X)$, $\int_f N$ has quasi-coherent equivariant cohomology. Now (A.6.7) ((A.2)(iii)) shows that the first (second, respectively) obvious inclusion in

$$(1.7.9)\quad C_b^G(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}Y, \mathscr{D}_{EG\underset{G}{\times}Y})) \to C_+^{G,b}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}Y, \mathscr{D}_{EG\underset{G}{\times}Y})) \to C_+^{G,b,q\mathrm{coh}}(Mod(EG\underset{G}{\times}Y, \mathscr{D}_{EG\underset{G}{\times}Y}))$$

induces an equivalence of the associated derived categories and completes the definition of the functor $\int_f$ as in (1.7.*).

(1.8) Similarly, under the assumptions of (1.7), we may define an exact functor

$$Rf_*\colon D_+^{G,b}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X; \mathscr{O}_{EG\underset{G}{\times}X})) \to D_+^{G,b}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}Y; \mathscr{O}_{EG\underset{G}{\times}Y})).$$

Let $ID_+^{G,b}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X; \mathscr{O}_{EG\underset{G}{\times}X}))$ denote the derived category of consisting of injective complexes in $D_+^{G,b}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X; \mathscr{O}_{EG\underset{G}{\times}X}))$. Since the obvious functor

$$ID_+^{G,b}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X; \mathscr{O}_{EG\underset{G}{\times}X})) \to D_+^{G,b}(Mod_{q\mathrm{coh}}(EG\underset{G}{\times}X; \mathscr{O}_{EG\underset{G}{\times}X}))$$

is an equivalence of categories and

$$f_*\colon Mod(EG\underset{G}{\times} X; \mathcal{O}_{EG\underset{G}{\times} X}) \to Mod(EG\underset{G}{\times} Y; \mathcal{O}_{EG\underset{G}{\times} Y})$$

preserves quasi-coherence, the functor

(1.8.*)
$$Rf_*\colon D_+^{G,b}(Mod_{q\,\mathrm{coh}}(EG\underset{G}{\times} X; \mathcal{O}_{EG\underset{G}{\times} X})) \to D_+^{G,b}(Mod_{q\,\mathrm{coh}}(EG\underset{G}{\times} Y; \mathcal{O}_{EG\underset{G}{\times} Y}))$$

may now be defined in the obvious manner.

(1.9) In case f is *proper* we may replace

$$D_b^G(Mod_{q\,\mathrm{coh}}(EG\underset{G}{\times} Y; \mathcal{D}_{EG\underset{G}{\times} Y})) \quad \text{in (1.7.*)}$$
$$(D_+^{G,b}(Mod_{q\,\mathrm{coh}}(EG\underset{G}{\times} Y; \mathcal{O}_{EG\underset{G}{\times} Y}) \quad \text{in (1.8.*))} \quad \text{by}$$
$$D_b^{G,\mathrm{coh}}(Mod_{q\,\mathrm{coh}}(EG\underset{G}{\times} Y; \mathcal{D}_{EG\underset{G}{\times} Y}))$$

($D_+^{G,b,\mathrm{coh}}(Mod_{q\,\mathrm{coh}}(EG\underset{G}{\times} Y; \mathcal{O}_{EG\underset{G}{\times} Y}))$, respectively). As in (1.7.7) and (1.7.8), once again we reduce to checking this for weakly-equivariant $\mathcal{D}_X$-modules of the form $\mathcal{D}_X \otimes \mathcal{E}$, where $\mathcal{E}$ is an equivariant coherent sheaf over $\mathcal{O}_X$. This is clear by the projection formula and the properness of f.

(1.10) Now we pause to consider the case when G is a complex diagonalizable group D acting linearly on the varieties X and Y as in (1.7). (Observe from [Sum, p. 8] that we may assume the action is linear, without any loss of generality, provided D is connected.) We proceed to define two functors:

(1.10.*)
$$Rf_*\colon C_b(Mod^D_{\mathrm{lfree}}(\mathcal{O}_X)) \to C^b_+(Mod^D_{q\,\mathrm{coh}}(\mathcal{O}_Y))$$
$$\textstyle\int_f\colon C_b(Mod^D_{\mathrm{lfree}}(\mathcal{D}_X)) \to C^b_+(Mod^D_{q\,\mathrm{coh}}(\mathcal{D}_Y)).$$

(For the higher Riemann-Roch theorem in (1.15) it is essential that we define the above functors on the category of complexes and not merely on the associated derived categories.) Recall once again from [Sum, pp. 8–9] that in this case each point of X and Y has an affine and D-invariant Zariski open neighborhood. Therefore we first consider the case when both X and Y are *affine*. In this case if $F^\cdot \in C_b(Mod^D_{\mathrm{coh}}(\mathcal{O}_X))$, ($M^\cdot \in C_b(Mod^D_{\mathrm{lfree}}(\mathcal{D}_X))$ we will let

$$Rf_*^\cdot = f_*F^\cdot\left(\textstyle\int_f M^\cdot = \vec{f}_*(D_{Y\leftarrow X} \underset{\mathcal{D}_X}{\otimes} M^\cdot), \text{ respectively}\right).$$

Under the equivalence of categories in [J2, (1.7)(iii)], observe that the complex $Rf_*F^\cdot(\int_f M^\cdot)$ of sheaves corresponds to

$$\Gamma(X, F^\cdot)\left(\Gamma(X, \mathcal{D}_{Y\leftarrow X}) \underset{\Gamma(X,\mathcal{D}_X)}{\otimes} \Gamma(X, M^\cdot), \text{ respectively}\right).$$

It follows that $Rf_* F^{\cdot}(\int_f M^{\cdot})$ is an equivariant complex of equivariant $\mathcal{O}_Y$-($\mathcal{D}_Y$-, respectively) quasi-coherent modules. Hence, in this cases, the above functors preserve quasi-isomorphisms and define the functors as required.

Next we consider the general case. Let $U=\{U_i|i\}$ be an affine Zariski open invariant cover of X. (We may assume each $f(U_i)$ is contained in an affine invariant open subset of Y.) Let $C(U)$ denote the associated simplicial scheme defined by the Cech-construction. (Recall

$$C(U)_0=\coprod_i U_i,\ C(U)_1=\coprod_{i_1<i_2}(U_{i_1}\cap U_{i_2}),\ \ldots,$$

with the obvious face maps and degeneracies.) Let $\varepsilon_n: C(U)_n\to X$ denote the structure map. If F is a D-equivariant coherent $\mathcal{O}_X$-module (M is a coherent D-equivariant and locally free $\mathcal{D}_X$-module) we let

$$C^{\cdot}(F)(C^{\cdot}(\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}M))$$

denote the Cech-complex given in dimension n by

$$\varepsilon_{n*}\,\varepsilon_n^{-1}(F)(\varepsilon_{n*}\,\varepsilon_n^{-1}(\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}M),$$

respectively. Observe this is an equivariant complex of $f^{-1}(\mathcal{D}_Y)$-modules. See [J2, (1.8.2)].) If $F^{\cdot}\in D_b(Mod^D_{\mathrm{coh}}(\mathcal{O}_X)$ ($M^{\cdot}\in D_b(Mod^D_{\mathrm{lfree}}(\mathcal{D}_X))$) we let

$$Rf_*(F^{\cdot})=\mathrm{Tot}(f_*\,C^{\cdot}(F^{\cdot}))(\textstyle\int_f M^{\cdot}=\mathrm{Tot}(f_*(C^{\cdot}(\mathcal{D}_{Y\leftarrow X}\underset{\mathcal{D}_X}{\otimes}M^{\cdot})),$$

respectively) where Tot denotes the total complex. The discussion about the affine case above shows that these define functors as in (1.10.*). If $f: X\to Y$ is also *proper*, one may readily show (use the equivariant Spencer sequence as in (A.4.2) for $\int_f$ – see for e.g. (1.7.7), (1.7.8)) that the functors in (1.10.*) induce functors

(1.10.1.*)
$$Rf_*:\ C_b(Mod^D_{\mathrm{coh}}(\mathcal{O}_X))\to C^{b,\mathrm{coh}}_+(Mod^D_{q\mathrm{coh}}(\mathcal{O}_X))$$
$$\textstyle\int_f:\ C_b(Mod^D_{\mathrm{lfree}}(\mathcal{D}_X))\to C^{b,\mathrm{coh}}_+(Mod^D_{q\mathrm{coh}}(\mathcal{D}_Y)).$$

(1.11) We will return to the general situation of (1.7) in (1.11) through (1.14). Assuming the situation of (1.7), we next define an exact functor

$$E_X:\ C^b_+(Mod^G_{q\mathrm{coh}}(X;\mathcal{O}_X))\to C^b_+(Mod^G_{q\mathrm{coh}}(X;\mathcal{D}_X))$$

by $M^{\cdot}\to\mathcal{D}_X\otimes M^{\cdot}\otimes\check{\omega}_X$. We may define exact functors E_Y similarly. Minor variations of the above functors as well as the induced maps at the level of K-theory, will all be denoted by the same symbols. Next assume for the time being that G is a complex diagonalizable group D acting lienarly. Now we obtain the isomorphisms

$$K_i(C^{b,\mathrm{coh}}_+(Mod^D_{q\mathrm{coh}}(X;\mathcal{O}_X))\cong K_i(C^b_+(Mod^D_{\mathrm{coh}}(X;\mathcal{O}_X))\cong K_i(Mod^D_{\mathrm{coh}}(X;\mathcal{O}_X))$$

and

$$K_i(C^{b,\mathrm{coh}}_+(Mod^D_{q\mathrm{coh}}(X;\mathcal{D}_X))\cong K_i(C^b_+(Mod^D_{\mathrm{coh}}(X;\mathcal{D}_X))\cong K_i(Mod^D_{\mathrm{coh}}(X;\mathcal{D}_X)).$$

The first two terms (last term) on the left in both rows are the algebraic K-groups in the sense of Waldhausen (Quillen, respectively). (See (A.6).) The first isomorphisms in both rows follow from (A.3) and (A.6.5); the last isomorphisms follow from (A.6.4). Therefore, [J4, (2.4)] shows the maps

$$E_X\colon\ K_i(C^b_+(Mod^D_{\mathrm{coh}}(X;\mathcal{O}_X)) \to K_i(C^b_+(Mod^D_{\mathrm{coh}}(X;\mathscr{D}_X)) \tag{1.11.*}$$

and

$$E_Y\colon\ K_i(C^b_+(Mod^D_{\mathrm{coh}}(X;\mathcal{O}_Y)) \to K_i(C^b_+(Mod^D_{\mathrm{coh}}(X;\mathscr{D}_Y))$$

are *isomorphisms.*

(1.12) We will now show that a map

$$d_X\colon\ K_0(Mod^G_{\mathrm{coh}}(\mathscr{D}_X)) \to K_0(Mod^G_{\mathrm{coh}}(\mathcal{O}_X))$$

may be defined just as in (2.6) of [J2] (where we restricted to actions by diagonalizable groups) and that the map E_X as in (1.11) provides a section to d_X. (Here K_0 denotes the Grothendieck group of the appropriate category.) A map d_Y may be defined similarly.

(1.13.1) We briefly recall the definition of the map d_X from [J2, (2.6)]. Let $\pi: T^*X \to X (i: X \to T^*X)$ denote the obvious projection from the cotangent bundle of X (the zero section imbedding of X in T^*X, respectively). Observe that $grd(\mathscr{D}_X) = \pi_*(\mathcal{O}_{T^*X})$. Let $\tilde{}: Mod_{q\,\mathrm{coh}}(grd(\mathscr{D}_X)) \to Mod_{q\,\mathrm{coh}}(\mathcal{O}_{T^*X})$ denote the equivalence of categories as in [EGA, 9.2.5]. One observes readily that $\tilde{}$ preserves coherence, is an exact functor and sends G-equivariant $grd(\mathscr{D}_X)$-modules to G-equivariant $\mathcal{O}_{T^*X}$-modules. Clearly $\tilde{}$ induces a functor

$$\tilde{}\ :\ K_0(Mod^G_{\mathrm{coh}}(grd(\mathscr{D}_X))) \to K_0(Mod^G_{\mathrm{coh}}(\mathcal{O}_{T^*X})).$$

(1.13.2) Let M be a G-equivariant and coherent $\mathscr{D}_X$-module provided with a good invariant filtration $\{M^k\}$. (Recall (A.4.1) shows the existence of such filtrations.) Now $grd(M)$ is a G-equivariant and coherent $grd(\mathscr{D}_X)$-module; one may show as in ([Lau, Lemma 6.1.2] and [Bj, Chap. 2, proof of 6.2] see also [B-B-Mac]) that the class of $grd(M)$ in $K_0(Mod^G_{\mathrm{coh}}(grad(\mathscr{D}_X)))$ is independent of the good G-invariant filtration chosen. We let $grd(M)^{\tilde{}}$ denote the image of $grd(M)$ in $K_0(Mod^G_{\mathrm{coh}}(\mathcal{O}_{T^*X}))$ under the map $\tilde{}$. Let

$$d_X(M) = \text{the class of } \left[\mathcal{O}_X \overset{L}{\underset{i^{-1}(\mathcal{O}_{T^*X})}{\otimes}} i^{-1}(grd(M)^{\tilde{}})\right] \underset{\mathcal{O}_X}{\otimes} \omega_X$$

using a good G-invariant filtration of M. (It is shown in [J2, (2.2)] that every coherent G-equivariant $\mathcal{O}_{T^*X}$-module of the form $grd(M)^{\tilde{}}$ has an equivariant resolution by G-equivariant locally free $\mathcal{O}_{T^*X}$-modules; this defines $\overset{L}{\otimes}$ as above.) Using (A.5.3) one may readily show that the map induced by d_X on the Grothendieck groups is a section to the map induced by the functor E_X. (A.4.2) shows d_X is also surjective and hence an isomorphism. (It is clear that in the situation of (1.7), a similar map d_Y may be defined for Y as well.)

(1.14) **Theorem** (General Rieman-Roch at the level of K_0). *Assume in addition to the hypotheses of* (1.7) *that f is* proper. *Then we obtain the commutative square*:

$$\begin{array}{ccc} K_0(Mod^G_{q\,\mathrm{coh}}(\mathscr{D}_X)) & \xrightarrow{\int_f} & K_0(Mod^G_{\mathrm{coh}}(\mathscr{D}_Y)) \\ \Big\downarrow{\scriptstyle d_X} & & \Big\downarrow{\scriptstyle d_Y} \\ K_0(Mod^G_{\mathrm{coh}}(\mathscr{O}_X)) & \xrightarrow[Rf_*]{} & K_0(Mod^G_{\mathrm{coh}}(\mathscr{O}_Y)) \end{array}$$

where the vertical maps are defined as in (1.12). *The horizontal maps are induced by the functors in* (1.7.*) *and* (1.8.*).

Proof reduces to the proof in [J2, (2.10)] in view of the fact the d_X, d_Y are isomorphisms with E_X, E_Y as their inverses. (The key observation is that it suffices to check the commutativity of the above square for $\mathscr{D}_X$-modules of the form $\mathscr{D}_X \otimes \mathscr{E}$, $\mathscr{E} \in Mod^G_{\mathrm{coh}}(\mathscr{O}_X)$. See (A.4.2).) Recall that the Riemann-Roch theorem of [J2] was proven only for linear actions by diagonalisable groups; such restrictions have now been removed.

(*Remarks.* Alternatively one may define $\int_f$ on $K_0(Mod^G_{\mathrm{coh}}(\mathscr{D}_X))$ making use of the equivariant Spencer sequence in (A.4.2). However this will have two drawbacks: (i) the map $\int_f$ on the Grothendieck-groups will not be induced by a map of derived categories and (ii) one still needs to do a bit of work to show $\int_f$ is well-defined. One may also consider defining $\int_f$ on the derived category $D_+(Mod_{q\,\mathrm{coh}}(\mathscr{D}_X))$ and try to show that if $N^\cdot \in D_+(Mod_{q\,\mathrm{coh}}(\mathscr{D}_X))$ has equivariant cohomology, then so does $\int_f N^\cdot$. However, since $Mod^G(\mathscr{D}_X)$ is not closed under extensions in $Mod(\mathscr{D}_X)$, this leads to various difficulties. Therefore, our approach seems best.)

(1.15) **Theorem** (Higher Riemann-Roch for diagonalisable group actions). *Assume in addition to the hypotheses of* (1.14) *that $G = D$ is a complex diagonalisable group acting linearly on both X and Y. Now we obtain a commutative square*

(1.15.*)
$$\begin{array}{ccc} K_i(C_b(Mod^D_{\mathrm{coh}}(X;\mathscr{D}_X))) & \xrightarrow{\int_f} & K_i(C_b(Mod^D_{\mathrm{coh}}(Y;\mathscr{D}_Y))) \\ \Big\uparrow{\scriptstyle E_X} & & \Big\uparrow{\scriptstyle E_Y} \\ K_i(C_b(Mod^D_{\mathrm{coh}}(X;\mathscr{O}_X))) & \xrightarrow[Rf_*]{} & K_i(C_b(Mod\,D_{\mathrm{coh}}(Y;\mathscr{O}_Y))) \end{array}$$

where the vertical maps are defined as in (1.11) *and are* isomorphisms. *The algebraic K-groups are the ones in the sense of Waldhausen* (see (A.6) and [Wald, (1.3)]).

Proof. First of all recall from ((A.5.1), (A.5.2)) that every $M \in Mod^D_{\mathrm{coh}}(\mathscr{D}_X)$ ($F \in Mod^D_{\mathrm{coh}}(\mathscr{O}_X)$) has an equivariant resolution by a (bounded) complex of coherent modules in $Mod^D_{\mathrm{lfree}}(\mathscr{D}_X)$ ($Mod^D_{\mathrm{lfree}}(\mathscr{O}_X)$, respectively). It follows that the obvious fully-faithful functors

$$C_b(Mod^D_{\mathrm{lfree}}(\mathscr{O}_X)) \to C_b(Mod^D_{\mathrm{coh}}(\mathscr{O}_X)),$$
$$C_b(Mod^D_{\mathrm{lfree}}(\mathscr{D}_X)) \to C_b(Mod^D_{\mathrm{coh}}(\mathscr{D}_X))$$

induce equivalences of the associated derived categories. (See (A.6.6). Recall $Mod^D_{lfree}(\mathscr{D}_X)$ denotes the category of locally-free and coherent D-equivariant $\mathscr{D}_X$-modules. $Mod^D_{lfree}(\mathscr{O}_X)$ is defined similarly.) Similarly the fully-faithful functors

$$C_b(Mod^D_{coh}(\mathscr{O}_Y)) \to C^b_+(Mod^D_{coh}(\mathscr{O}_Y)) \to C^{b,coh}_+(Mod^D_{qcoh}(\mathscr{O}_Y))$$

and

$$C_b(Mod^D_{coh}(\mathscr{D}_Y)) \to C^b_+(Mod^D_{coh}(\mathscr{D}_Y)) \to C^{b,coh}_+(Mod^D_{qcoh}(\mathscr{D}_Y))$$

induce equivalences of the associated derived categories in view of (A.3). Now it follows from (A.6.5) that the above functors induce isomorphisms of the associated Waldhausen K-theory and hence the functors in (1.10.1.*) induce the maps forming the horizontal rows of (1.15.*).

The projection formula shows there exists a natural quasi-isomorphism between the functors

$$C_b(Mod^D_{lfree}(\mathscr{O}_X)) \xrightarrow{E_X} C_b(Mod^D_{lfree}(\mathscr{D}_X)) \xrightarrow{\int_f} C^{b,coh}_+(Mod^D_{qcoh}(\mathscr{D}_Y))$$

and

$$C_b(Mod^D_{lfree}(\mathscr{O}_X)) \xrightarrow{Rf_*} C^{b,coh}_+(Mod^D_{qcoh}(\mathscr{O}_Y)) \xrightarrow{E_Y} C^{coh}_b(Mod^D_{qcoh}(\mathscr{D}_Y))$$

when $\int_f$ and Rf_* are defined as in (1.10.1.*) using the Cech-resolution. This proves the commutativity of (1.15.*). (See [Wald, Proposition 1.3.1].)

2

(2.0) Throught the rest of the paper we will assume in addition to (1.0) that the group G is a complex diagonalizable group. It will be denoted D. Let M denote the abelian group of characters of D and let $R(D)=\mathbb{Z}[M]$ denote the corresponding ring of characters. (Recall that M is finitely generated; it is a free abelian group if D is a torus and is isomorphic to $\mathbb{Z}/l^v$ if $D=$*the* l^v-th roots of unity.)

(2.1) Let D act on a smooth quasi-projective complex algebraic variety X and let $K^0_D(X)$ denote the Grothendieck group of D-equivariant locally free coherent $\mathscr{O}_X$-modules. If the action is trivial one obtains the isomorphism $K^0_D(X) \cong K^0(X) \underset{\mathbb{Z}}{\otimes} R(D)$. (See [SGA3, I, Expose 1, 4.4].) In general $K^0_D(X)$ is an algebra over $R(D)$. Observe that the tensor product makes $K_0(Mod^D_{coh}(\mathscr{O}_X))$ a $K^0_D(X)$-module and hence an $R(D)$-module. Next observe that if M is a D-equivariant coherent left $\mathscr{D}_X$-module and F is a D-equivariant locally free $\mathscr{O}_X$-module, then $M \underset{\mathscr{O}_X}{\otimes} F$ is also a coherent D-equivariant $\mathscr{D}_X$-module. (Here we let $\mathscr{D}_X$ act on the left on M and the group D diagonally on $M \underset{\mathscr{O}_X}{\otimes} F$.) It follows that $K_0(Mod^D_{coh}(\mathscr{D}_X))$ is also a $K^0_D(X)$-module and hence an $R(D)$-module. (See also (A.5.3).) If the action is trivial, using the isomorphism provided by the map d_X in (1.12), one also obtains the isomorphism $K_0(Mod^D_{coh}(\mathscr{D}_X)) \simeq K^0(X) \underset{\mathbb{Z}}{\otimes} R(D)$.

(2.2) Next assume in addition to the assumptions of (2.1) that X is also projective, $Y = \operatorname{Spec} \mathbb{C}$ and $f: X \to \operatorname{Spec} \mathbb{C}$ is the structure map of X. In this context we define a *character-map*

$$ch: K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_X)) \to R(D)$$

by letting $ch(M) =$ the class of $\int_f(M)$ in $R(D)$. We may similarly define a character map $ch: K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_X)) \to R(D)$, by letting $ch(F) =$ *the* class of $Rf_* F$. The above Riemann-Roch theorems provide the obvious relationship between these two maps.

(2.3) For each $\chi \in M$, we let e^χ denote the corresponding element of $R(D)$. Let $S' \subseteq R(D)$ denote the multiplicative subset generated by the elements of the form $1 - e^\chi$, χ a nontrivial character of D and let p be a prime ideal in $\mathbb{Z}[M]$ so that $S' \subseteq Z[M] - p$. Let $S = \mathbb{Z}[M] - p$. This will be fixed through the rest of the paper.

(2.4) Next we discuss the general form of the character formula. Assume as before that X is also a projective smooth variety. Let N denote the co-normal sheaf associated to the closed immersion $i: X^D \to X$. (Recall X^D is also smooth – see [Iver, (1.3)].) Let

$$i_*: K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_{X^D})) \to K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_X))$$

denote the Gysin map; let $\lambda_{-1}(N) =$ *the* class given by $\mathscr{O}_{X^D} - N + \lambda^2 N - \lambda^3 N + \ldots$ in $K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_{X^D}))$. Then $\lambda_{-1}(N)$ is a unit in the localized ring $S^{-1} K^0_D(X^D)$ and

$$\begin{aligned} i_*: S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_{X^D})) \cong S^{-1} K^0_D(X^D) &\xrightarrow{\simeq} \\ &\to S^{-1} K^0_D(X) \cong S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_X)) \end{aligned}$$

is an isomorphism. (Apply (A.5.3) to obtain the first and last isomorphisms.) Its inverse is the map $Li^*(\,) \cap \lambda_{-1}(N)^{-1}$. (The functor Li^* is defined using (A.5.1) and (A.5.3).) (See [Niel, p. 92].)

(2.4.1) We will let $\varphi_X: S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_X)) \to S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_{X^D}))$ denote the composition of the isomorphisms $S^{-1} d_X$, $Li^*(\,) \cap \lambda_{-1}(N)^{-1}$ and $S^{-1} E_{X^D}$ (= the inverse to $S^{-1} d_{X^D}$), where d_X, d_{X^D} are defined as in (1.12) and E_{X^D} is defined as in (1.11).

(2.5) **The general character formula** (see [J3, (1.6)]. *Assume in addition to the hypotheses of* (1.7) *that X and Y are projective. Let X^D, Y^D denote the fixed-point schemes. Let $i_X (i_Y, N)$ denote the closed immersion $X^D \to X$ ($Y^D \to Y$, the co-normal sheaf associated to the closed immersion $X^D \to X$, respectively). Let $f^D: X^D \to Y^D$ be the map induced by f. Then we obtain the commutative diagram* (*where φ_X is the map defined above*)

$$\begin{array}{ccc} S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_X)) & \xrightarrow[\varphi_X]{\simeq} & S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_{X^D}))) \\ \Big\downarrow{\scriptstyle S^{-1}\int_f} & & \Big\downarrow{\scriptstyle S^{-1}\int_{f^D}} \\ S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y)) & \xrightarrow{S^{-1}\int_{i_Y}} & S^{-1} K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_{Y^D})). \end{array} \tag{2.5.1}$$

Taking $Y = \operatorname{Spec} \mathbb{C}$, with the trivial action of D we obtain the commutative triangle

(2.5.2)
$$\begin{array}{ccc} S^{-1}K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_X)) & \xrightarrow[\simeq]{\varphi_X} & S^{-1}K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_{X\times D})) \\ \downarrow{\scriptstyle S^{-1}ch} & \swarrow{\scriptstyle S^{-1}ch} & \\ S^{-1}R(D). & & \end{array}$$

(2.5.3) *If $D = T$ is a complex torus acting trivially on Y equality of two classes in the localised Grothendieck group $S^{-1}K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y))$ implies their equality in $K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y))$. Hence if $D = T$ is a complex torus, $ch(F) = ch(\varphi_X(F))$ as classes in $R(T)$, for each $F \in Mod^D_{\mathrm{coh}}(\mathscr{D}_X)$.*

Proof. This is a generalization of [J3, (1.6)] without the assumptions of linearity on the action by D. The square (2.5.1), with the top row replaced by $S^{-1}\int_{i_X}$ clearly commutes. The Riemann-Roch theorem (1.14) (applied to the closed immersion i_X) shows φ_X is the inverse to the isomorphism $S^{-1}\int_{i_X}$. The commutativity of (2.5.1) now follows readily.

If D acts trivially on Y, we obtain $K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y)) \cong K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_Y)) \cong K^0(Y) \cong R(D) \underset{\mathbb{Z}}{\otimes} K^0(Y)$. If in addition $D = T$ is a torus, every element of S is a non-zero divisor in $K^0(Y) \cong K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_Y)) \cong K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y))$; hence two classes in $K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y))$ are equal if and only if their images in $S^{-1}K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y))$ are equal. (See [T2, p. 539].) This proves (2.5.3).

(2.6) We conclude this section by observing that the *character formulae of Atiyah-Segal, Weyl and Demazure may be derived from this characer formula.* In [T2, pp. 538–542] it is shown that these formulae can be derived from the 'coherent trace formula' (i.e. Theorem 6.4 of [T2]). For our purposes, this is merely the commutativity of the triangle

(2.6.1)
$$\begin{array}{ccc} S^{-1}K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_X)) & \xrightarrow{Li^*(\,)\cap\lambda_{-1}(N)^{-1}} & S^{-1}K_0(Mod^D_{\mathrm{coh}}(\mathscr{O}_{X\times D})) \\ \downarrow{\scriptstyle S^{-1}ch} & \swarrow{\scriptstyle S^{-1}ch} & \\ S^{-1}R(D). & & \end{array}$$

It is clear (from the definition of φ_X in (2.5.1) and (2.4.1)) that the commutativity of the above triangle is equivalent to the commutativity of the triangle in (2.5.2) for $\mathscr{D}_X$-modules of the form $\mathscr{D}_X \otimes \mathscr{E} \otimes \check{\omega}_X$, for $\mathscr{E} \in Mod^D_{\mathrm{coh}}(\mathscr{O}_X)$. (Recall also that this is precisely an algebraic version of the Atiyah-Segal character formula. See [A-B] and [A-S].)

The Weyl Character formula now follows by taking $X = a$ flag manifold, with $D = the$ maximal torus acting in the usual manner. Next take X to be a flag manifold, $i: Z \to X$ the closed immersion of a possibly singular Schubert variety in X and $\mathscr{D}$ as before. The Demazure character formula may be derived as in [T2, p. 541] by checking the commutativity of the above triangle for equivariant coherent sheaves of the form $i_* F$, where F is an equivariant vector

bundle on Z. (Making use of (A.5.1), the commutativity of the above triangle readily proves the formula in [T2, (6.17)]. Now the rest of the arguments in [T2, pp. 541–542] carry over.)

3

In this section we combine the the above Riemann-Roch theorems along with the localization theorem of [J3] to provide *general character formulae for holonomic $\mathscr{D}$-modules.* We begin by recalling the localization theorem [J3, (2.8)].

Assuming the situation of (2.0) let $Mod^D_{\mathrm{hol}}(\mathscr{D}_X)$ denote the full subcategory of $Mod^D_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of holonomic $\mathscr{D}_X$-modules. It is clearly a Serre subcategory and hence also abelian. Let X^D denote the fixed point scheme of X; as is observed in [J3, (1.3)] this is a disjoint union of its irreducible components each of which is smooth. Hence $Mod^D_{\mathrm{coh}}(\mathscr{D}_{X^D})$ and $Mod^D_{\mathrm{hol}}(\mathscr{D}_{X^D})$ may be defined in the obvious manner. Let $i_X : X^D \to X$ denote the obvious closed immersion and let $\int_{i_X}$ denotes the induced maps

$$K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_{X^D})) \to K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_X)), \qquad K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_{X^D})) \to K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_X)).$$

Recall from (2.1) and [J3, (2.6)] that the above Grothendieck-groups (maps) are modules (are maps of modules, respectively) over the representation ring $R(D)$. Let $S \subseteq R(D)$ denote the multiplicative subset as before.

(3.0) **The localization theorem** (see [J3, (2.8)] *Assume the above situation. Now $\int_i$ induces an isomorphism*

$$S^{-1}\int_{i_X} : S^{-1}K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_{X^D})) \xrightarrow{\simeq} S^{-1}K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_X)).$$

(3.1.0) Let $f : X \to Y$ denote a *proper* D-equivariant map between projective smooth complex algebraic varieties X, Y provided with an action by the complex diagonalizable group D. Let

$$\psi_X : K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_X)) \to K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_{X^D}))$$

be the map

$$M \to \sum_n (-1)^n \mathscr{H}^n(i_X^! M),$$

where $i_X : X^D \to X$ is the closed immersion. (One may define $i_X^! M$ (see [J2, (2.9′)]) by taking an equivariant resolution of M by a complex of the form $\mathscr{D}_X \otimes \mathscr{E}^\bullet$, where each $\mathscr{E}^i$ is a locally free G-equivariant $\mathscr{O}_X$-module. See (A.5.2).) Clearly ψ_X is $R(D)$-linear and is an inverse to the isomorphism $S^{-1}\int_{i_X}$ in (3.0).

(3.1.1) **Character formula for weakly-equivariant holonomic $\mathscr{D}$-modules** (see also [J3, (2.9)]. *Assume the above situation. Then we obtain the commutative diagram*

$$\begin{array}{ccc} S^{-1}K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_X)) & \xrightarrow{S^{-1}\psi_X} & S^{-1}K_0(Mod^D_{\mathrm{hol}}(\mathscr{D}_{X^D})) \\ \downarrow & & \downarrow \\ S^{-1}K_0(Mod^D_{\mathrm{coh}}(\mathscr{D}_Y)) & \xleftarrow{S^{-1}\int_{i_Y}} & S^{-1}(Mod^D_{\mathrm{coh}}(\mathscr{D}_{Y^D})). \end{array}$$

The left (right) vertical map is the composition of the obvious map $S^{-1}K_0(Mod^D_{\text{hol}}(\mathscr{D}_X)) \to S^{-1}K_0(Mod^D_{\text{coh}}(\mathscr{D}_X))$ and $S^{-1}\int_f$ (*the map* $S^{-1}K_0(Mod^D_{\text{hol}}\mathscr{D}_{X^D})) \to S^{-1}K_0(Mod^D_{\text{coh}}(\mathscr{D}_{X^D}))$ *and* $S^{-1}\int_{f^D}$, *respectively*). i_Y *is the closed immersion* $Y^D \to Y$.

Proof. This is merely the generalization of [J3, (2.8)] without the assumptions of linearity on the D-action. Since

$$\psi_X \circ \int_{i_X} \cong \textit{the identity on } K_0(Mod^D_{\text{hol}}(\mathscr{D}_{X^D}))$$

(see [J3, (2.8)], for example), the localization Theorem 3.0 shows $S^{-1}\psi_X$ is a (two sided) inverse to the isomorphism

$$S^{-1}\int_{i_X}: S^{-1}K_0(Mod^D_{\text{hol}}(\mathscr{D}_{X^D})) \to S^{-1}K_0(Mod^D_{\text{hol}}(\mathscr{D}_X)).$$

On the other hand φ_X is an inverse to the isomorphism

$$S^{-1}\int_{i_X}: S^{-1}K_0(Mod^D_{\text{coh}}(\mathscr{D}_{X^D})) \to S^{-1}K_0(Mod^D_{\text{coh}}(\mathscr{D}_X)).$$

Now one may show as in [J3, (2.9), (2.10)] that $i^!_X M \cong \psi_X(M) \cong \varphi_X(M)$ *as classes in* $S^{-1}K_0(Mod^D_{\text{coh}}(\mathscr{D}_{X^D}))$ *whenever* $M \in Mod^D_{\text{hol}}(\mathscr{D}_{X^D})$. (Since both occurrences of $S^{-1}\int_{i_X}$ above are isomorphisms, it suffices to show

$$S^{-1}\int_{i_X}(\psi_X(M)) = S^{-1}\int_{i_X}(\varphi_X(M)) \text{ as classes in } S^{-1}K_0(Mod^D_{\text{coh}}(\mathscr{D}_X));$$

this is clear in view of the fact that $\psi_X(\varphi_X)$ is an inverse to the first (second, respectively) occurrence of $S^{-1}\int_{i_X}$ above.) Now the general character formula in (2.5) proves (3.1.1).

(3.2) **Corollary** (Character formula for weakly-equivariant holonomic $\mathscr{D}$-modules). (i) Assume in addition to the hypotheses of (2.5) that $Y = \operatorname{Spec} \mathbb{C}$ (*provided with the trivial action of* D) *and* $f: X \to \operatorname{Spec} \mathbb{C}$, $f^D: X^D \to \operatorname{Spec} \mathbb{C}$ *are the structure maps. Then if* $M \in Mod^D_{\text{hol}}(\mathscr{D}_X)$, $ch(M) \cong ch(i^!_X M)$ *in* $S^{-1}K_0(R(D))$.

(ii) *Assume in addition to the hypotheses of* (2.5) *that* $D = T$ *is a complex torus acting trivially on* Y. *Now if* $M \in Mod^T_{\text{hol}}(\mathscr{D}_X)$, *both* $\int_f M$ *and* $\int_{i_Y} \circ \int_{f^T}(i^!_X M)$ *represent the same class in* $K_0(Mod^T_{\text{coh}}(\mathscr{D}_Y))$.

(iii) *Assume as in* (i) *that* $Y = \operatorname{Spec} \mathbb{C}$ *and* $D = T$ *is a complex torus as in* (ii). *Now if* $M \in Mod^T_{\text{hol}}(\mathscr{D}_X)$, $ch(M) \cong ch(i^!_X M)$ *in* $R(T)$.

Proof. (i) is clear. (ii) and (iii) follow as in the proof (2.5.3).

(3.3) *Remark.* Assume the situation of (1.0) where a complex linear algebraic group G acts on a smooth quasi-projective complex variety X. A coherent $\mathscr{D}_X$-module M is strongly G-equivariant if there exists an isomorphism $\varphi: \mu^* M \xrightarrow{\simeq} \text{pr}_2^* M$ of $D_{(G \times X)} = \mathscr{D}_G \underset{\mathbb{C}}{\otimes} \mathscr{D}_X$-modules satisfying the usual the conditions. (See [B-B, p. 16] or [Be-Ber, Sect. 3].) A strongly-equivariant $\mathscr{D}_X$-module is readily seen to be also weakly-equivariant as observed in [J3, (3.2)]. It is shown in [Bo, pp. 317–319] that if in addition G has only finitely many orbits on X, then any strongly G-equivariant coherent $\mathscr{D}_X$-module is holonomic. *Therefore, in this situation, the character formulae of* (3.2) (*with respect to a diagonalizable subgroup* D *of* G) *apply to all strongly* G*-equivariant coherent* $\mathscr{D}_X$*-modules.*

(3.4) Let X, Y denote smooth complex projective algebraic varieties acted on by a complex diagonalizable group D and let $f: X \to Y$ be a *D-equivariantly projective* map. (i.e. There exists an action of D on $Y \times \mathbb{P}^n$ for some large enough n, and the map f factors as the composition of the D-equivariant maps: $X \to Y \times \mathbb{P}^n$ (which is a closed immersion) and the projection $Y \times \mathbb{P}^n \to Y$.) Let $(X_{\text{an}}, \mathcal{O}_{X_{\text{an}}})$, $(Y_{\text{an}}, \mathcal{O}_{Y_{\text{an}}})$ denote the corresponding analytic spaces. Clearly the *analytic* group $D(\mathbb{C})$ (i.e. with structure sheaf $\mathcal{O}_{D_{\text{an}}}$) has an induced action on these analytic spaces and the map f_{an} is $D(\mathbb{C})$-equivariant. Let

$$\varepsilon_X : (X_{\text{an}}, \mathcal{O}_{X_{\text{an}}}) \to (X, \mathcal{O}_X), \qquad \varepsilon_Y : (Y_{\text{an}}, \mathcal{O}_{Y_{\text{an}}}) \to (Y, \mathcal{O}_Y)$$

denote the obvious maps of ringed spaces; let

$$\mathcal{D}_{X_{\text{an}}} = \varepsilon_X^*(\mathcal{D}_X) \quad \text{and} \quad \mathcal{D}_{Y_{\text{an}}} = \varepsilon_Y^*(\mathcal{D}_Y).$$

We will let $Mod_{\text{coh}}^{D(\mathbb{C})}(\mathcal{D}_{X_{\text{an}}})(Mod_{\text{coh}}^{D(\mathbb{C})}(\mathcal{D}_{Y_{\text{an}}}))$ denote the category of weakly-equivariant coherent $\mathcal{D}_{X_{\text{an}}}(\mathcal{D}_{Y_{\text{an}}})$, respectively) modules, which may be defined in the obvious manner as in (1.3). One observes readily that these are also abelian categories and that $\varepsilon_X(\varepsilon_Y)$ induces an exact functor

$$\varepsilon_X^* : Mod_{\text{coh}}^{D}(\mathcal{D}_X) \to Mod_{\text{coh}}^{D(\mathbb{C})}(\mathcal{D}_{X_{\text{an}}})(\varepsilon_Y^* : Mod_{\text{coh}}^{D}(\mathcal{D}_Y) \to Mod_{\text{coh}}^{D(\mathbb{C})}(D_{Y_{\text{an}}})),$$

respectively).

(3.5) **Proposition.** *Assume the above situation. Then the triangle*

$$\begin{array}{ccc} K_0(Mod_{\text{coh}}^{D}(\mathcal{D}_X)) & \xrightarrow{\int_{f_{\text{an}}} \varepsilon_X^*} & K_0(Mod_{\text{coh}}^{D(\mathbb{C})}(\mathcal{D}_{Y_{\text{an}}})) \\ \Big\downarrow \int_f & \nearrow \varepsilon_Y^* & \\ K_0(Mod_{\text{coh}}^{D}(\mathcal{D}_Y)) & & \end{array}$$

commutes. (Here $\int_{f_{\text{an}}}$ is the usual functor; one readily verifies using the equivariant Spencer-exact sequence of (A.4.2) *that $\int_{f_{\text{an}}} \circ \varepsilon_X^*(M)$ has bounded equivariant coherent cohomology, when $M \in Mod_{\text{coh}}^{D}(\mathcal{D}_X)$.)*

Proof. In view of the equivariant Spencer-exact sequence of (see (A.4.2)) it follows that we need prove this only for coherent $\mathcal{D}_X$-modules of the form $\mathcal{D}_X \otimes E \otimes \check{\omega}_X$, $E \in Mod_{\text{coh}}^{D}(\mathcal{O}_X)$. The projection formula being natural holds in the equivariant case and shows

$$\int_f (\mathcal{D}_X \otimes E \otimes \check{\omega}_X) \cong \mathcal{D}_Y \otimes Rf_* E \otimes \check{\omega}_Y \text{ as classes in } K_0(Mod_{\text{coh}}^{D}(\mathcal{D}_Y))$$

and similarly

$$\int_{f_{\text{an}}} (D_{X_{\text{an}}} \otimes (E_{\text{an}}) \otimes \check{\omega}_{X_{\text{an}}}) \cong D_{Y_{\text{an}}} \otimes Rf_{\text{an}*}(E_{\text{an}}) \otimes \check{\omega}_{Y_{\text{an}}} \quad \text{in } K_0(Mod_{\text{coh}}^{D(\mathbb{C})}(\mathcal{D}_{Y_{\text{an}}})).$$

Therefore it suffices to show that the triangle

(3.5.1)
$$\begin{array}{ccc} K_0(Mod^D_{\mathrm{coh}}(\mathcal{O}_X)) & \xrightarrow{Rf_{\mathrm{an}*}\circ\varepsilon_X^*} & K_0(Mod^{D(\mathbb{C})}_{\mathrm{coh}}(\mathcal{O}_{Y_{\mathrm{an}}})) \\ & {\scriptstyle Rf_*}\searrow \quad \nearrow{\scriptstyle \varepsilon_Y^*} & \\ & K_0(Mod^D_{\mathrm{coh}}(\mathcal{O}_Y)) & \end{array}$$

commutes. Let $F \in Mod^D_{\mathrm{coh}}(\mathcal{O}_X)$. Base change shows the existence of a natural map

(3.5.2)
$$\varphi\colon\ \varepsilon_Y^*(R^i f_* F) \to R^i f_{\mathrm{an}*} \circ \varepsilon_X^* F, \qquad \text{for each } i \geqq 0$$

which we will show is an isomorphism by a relative form of GAGA (see [GAGA]). *The naturality of the above map will show it is also equivariant for the action of* $D(\mathbb{C})$ *on* Y_{an}.

Since f is assumed to be equivariantly projective, it is the composition of an equivariant closed immersion i into $Y \times \mathbb{P}^N$ (to which the D-action extends) and the obvious projection $\mathrm{pr}_1 : Y \times \mathbb{P}^n \to Y$. Since F is an $\mathcal{O}_X$-coherent sheaf, $i_*(F)$ is a coherent sheaf on $Y \times \mathbb{P}^N$. Next observe (using flat-basechange for e.g.) that it suffices to establish the isomorphism in (3.5.2) locally on Y. Therefore we may *assume without loss of generality that* Y *is affine.* Since Y is assumed to be affine (see [Hart1, p. 121] for e.g.) F is a (non-equivariant) quotient of a vector bundle of the form $\bigotimes_n \mathcal{O}_{Y \times \mathbb{P}^N}(n)$. We will first show the map in (3.5.2) is an isomorphism when $F = \mathcal{O}_{Y \times \mathbb{P}^N}(n)$, $n \geqq 0$.

Let y denote a fixed closed-point of Y; let $\mathcal{O}_{Y,y}$, $\mathcal{O}_{Y_{\mathrm{an}},y}$ denote the corresponding local rings. Next observe that for each i, $(R^i f_* F)_Y$ is an $\mathcal{O}_{Y,y}$-module and may be computed using a Cech-resolution, flat-base-change and the Kunneth-formula to be $(\mathcal{O}_{Y,y}) \otimes H^i(\mathbb{P}^N ; \mathcal{O}_{\mathbb{P}^N}(n))$. Therefore

$$(\varepsilon_Y^* R^i f_* F)_y \cong (\mathcal{O}_{Y_{\mathrm{an}},y}) \otimes H^i(\mathbb{P}^N ; \mathcal{O}_{\mathbb{P}^N}(n)).$$

On the other hand a Kunneth-formula also shows

$$(R^i f_{\mathrm{an}*}\, \varepsilon_X^* F)_y \cong (\mathcal{O}_{Y_{\mathrm{an}},y}) \otimes H^i(\mathbb{P}^N_{\mathrm{an}} ; \mathcal{O}_{\mathbb{P}^N_{\mathrm{an}}}).$$

Hence, using [GAGA], we observe that the map in (3.5.2) is an isomorphism, when $F = \mathcal{O}_Y \otimes \mathcal{O}_{\mathbb{P}^N}(n)$. Now a snake lemma argument as in the proof of [GAGA, see top of p. 22] shows that this suffices to prove that the map in (3.5.2) is an isomorphism for any $\mathcal{O}_X$-coherent sheaf F. This completes the proof of the proposition.

(3.6) **Corollary.** *Assume in addition to the hypotheses of* (2.5) *that* D *acts linearly on* X.

(i) *Under the assumptions of* (3.2)(i)

$$\begin{aligned} R\Gamma(X_{\mathrm{an}}, \mathrm{DR}(M)) &= \textstyle\int_{f_{\mathrm{an}}} M_{\mathrm{an}},\ \int_f M,\ R\Gamma(X_{\mathrm{an}}, \mathrm{DR}(i_X^! M)) \\ &\simeq \textstyle\int_{f^T_{\mathrm{an}}} (i_X^! M)_{\mathrm{an}} \\ &\simeq \textstyle\int_{f^T_{\mathrm{an}}} i^!_{X_{\mathrm{an}}}(M_{\mathrm{an}}) \quad \text{and} \quad \int_{f^T}(i_X^! M) \end{aligned}$$

represent the same class in $S^{-1}R(D)$.

(ii) *Under the assumptions of* (3.2)(iii), they represent the same class in $R(T)$. *Similarly under the assumptions of* (3.2)(ii)

$$\int_{i_{Y_{an}}} \circ \int_{f^T_{an}} (i^!_X M)_{an} \simeq \int_{i_{Y_{an}}} \circ \int_{f^T_{an}} (i^!_{X_{an}}(M_{an}) \quad \text{and} \quad \int_{f_{an}} M_{an}$$

represent the same class in $K_0(Mod^{T(\mathbb{C})}_{coh}(\mathscr{D}_{Y_{an}}))$.

Proof. Since D is assumed to act linearly on X, observe that the map f is now equivariantly projective. Next observe that $R(D) \simeq R(D(\mathbb{C}))$. Now it suffices to show $(i^!_X M)_{an} \simeq (i^!_{X_{an}}(M_{an}))$ as classes in $K_0(Mod^{D(\mathbb{C})}_{coh}(\mathscr{D}_{X_{an}}))$. One may prove this by using the technique of (A.5.2).

4

In this section we first make use of the Riemann-Hilbert correspondence to provide character-formulae for equivariant perverse sheaves on complex projective varieties provided with actions by complex diagonalizable groups. Then we show the Kazdhan-Lusztig formula also may be interpreted as a special case of our general character formula (3.2)(iii) or (3.6).

(4.1) Let X denote a possibly-singular complex quasi-projective algebraic variety acted on by a complex diagonalizable group D. A D-equivariant perverse sheaf $F^\bullet$ on X_{an} is a perverse sheaf on X_{an} so that there exists an isomorphism

$$\psi : \mu^{-1}_{an} F^\bullet \xrightarrow{\simeq} (\mathrm{pr}_2)^{-1}_{an} F^\bullet$$

in the category of perverse sheaves on $(D \times X)_{an}$ satisfying the usual conditions (in the appropriate category of perverse sheaves). (See for e.g. [J3(3.4)].) (To see that this in-fact is *a correct definition*, see [J5, Proposition 4.2].)

Now assume that X is a smooth complex algebraic variety provided with an action by a complex diagonalizable group D. Let $Mod^{s-D}_{r.h}(\mathscr{D}_X)$, $Per^D(X_{an})$ denote the category of strongly D-equivariant regular holonomic $\mathscr{D}_X$-modules (see (3.3)) (the category of D-equivariant perverse sheaves on X_{an}, respectively.) It is shown in [J3, (3.5)] that the De-Rham functor

$$\mathrm{DR} : Mod^{s-D}_{r.h}(\mathscr{D}_X) \to Per^D(X_{an}) \tag{4.2}$$

has the following property: given any $F \in Per^D(X_{an})$, there exists an $M \in Mod^{s-D}_{r.h}(\mathscr{D}_X)$ so that $\mathrm{DR}(M) \cong F$.

(4.3) **Theorem** (see also [Kash]). (i) *Let X denote a smooth complex projective algebraic variety provided with a linear action by a complex diagonalizable group D. Let $i: X^D \to X$ denote the closed immersion of the fixed point scheme. Let $F^\bullet$ be a D-equivariant perverse sheaf on X_{an}. Then*

$$\begin{aligned} ch(R(X_{an}, F^\bullet)) &= \sum_n (-1)^n ch(H^n(X_{an}, F^\bullet)) \\ &= \sum_n (-1)^n ch(H^n((X^D)_{an}, i^!_{an} F^\bullet)) \\ &= ch\, R(X^D_{an}, i^!_{an} F^\bullet) \end{aligned}$$

in $S^{-1} R(D)$ *in general and in* $R(T)$ *if* $D = T$ *is a complex torus.*

(ii) *The same conclusions hold even if X is possibly singular.*

(*Note.* Observe from [Sum, Theorem 1] that if $D=T$ is a complex torus, and X is normal we may assume without loss of generality that the action is linear.)

Proof. Observe that by (4.2) there exists a strongly-equivariant regular holonomic $\mathscr{D}_X$-module M and quasi-isomorphisms $\eta: \mathrm{DR}(M) \to F^{\cdot}$, $\mathrm{DR}(i^! M) \simeq i_{\mathrm{an}}^! (F^{\cdot})$. Moreover one may show using (4.2) that the squares

$$\begin{array}{ccc} \mu^{-1}(\mathrm{DR}(M)) & \xrightarrow{\simeq} & \mathrm{pr}_2^{-1}(\mathrm{DR}(M)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mu^{-1}(F^{\cdot}) & \xrightarrow{\simeq} & \mathrm{pr}_2^{-1}(F^{\cdot}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mu^{-1}(\mathrm{DR}(i^! M)) & \xrightarrow{\simeq} & \mathrm{pr}_2^{-1}(\mathrm{DR}(i^!(M)) \\ \downarrow \simeq & & \downarrow \simeq \\ \mu^{-1}(i_{\mathrm{an}}^! F^{\cdot}) & \xrightarrow{\simeq} & \mathrm{pr}_2^{-1}(i_{\mathrm{an}}^! F^{\cdot}) \end{array}$$

commute in the appropriate category of perverse sheaves. Hence one may readily show

$$\sum_n (-1)^n ch H^n(X_{\mathrm{an}}, F^{\cdot}) = \sum_n (-1)^n ch H^n(X_{\mathrm{an}}, \mathrm{DR}(M))$$

and similarly

$$\sum_n (-1)^n ch H^n(X_{\mathrm{an}}^D, i_{\mathrm{an}}^! F^{\cdot}) = \sum_n (-1)^n ch H^n(X_{\mathrm{an}}^D, \mathrm{DR}(i^! M)).$$

Now (i) follows from (3.6).

(ii) Let $i: X \to \mathbb{P}^n$ denote an equivariant closed immersion. Let $i_0(i_1, i_2)$ denote the closed immersion $X \to \mathbb{P}^n((\mathbb{P}^n)^D \to \mathbb{P}^n$, $X^D \to (\mathbb{P}^n)^D$, respectively.) If $F^{\cdot}$ is an equivariant perverse sheaf on X_{an}, one observes readily that $i_{0_{\mathrm{an}}} *(F^{\cdot})$ is also an equivariant perverse sheaf on $\mathbb{P}_{\mathrm{an}}^n$. We then obtain the pull-back square

$$\begin{array}{ccc} X^D & \xrightarrow[i_2]{} & (\mathbb{P}^n)^D \\ \downarrow i & & \downarrow i_1 \\ X & \xrightarrow{i_0} & \mathbb{P}^n. \end{array}$$

Since $\mathbb{P}^n$ is smooth, we may apply (4.3)(i) along with the natural quasi-isomorphism $R i_{1\,\mathrm{an}}^! \circ i_{0\,\mathrm{an}_*} F^{\cdot} \xrightarrow{\simeq} i_{2\,\mathrm{an}_*} \circ R i_{\mathrm{an}}^! F^{\cdot}$ provided by proper-base-change to complete the proof of (ii).

(4.4) *The Kazdhan-Lusztig character formula.* Now we relate our character formula to the well-known character formula of Kazdhan and Lusztig, making use of [Bry-Kas].

Let G denote a semi-simple complex linear algebraic group with lie algebra $\mathfrak{g}$, B a fixed Borel subgroup, G/B the corresponding flag manifold and W the Weyl group. Let $K \subseteq G$ be a closed, connected subgroup with Lie algebra $\mathfrak{k}$:

assume that K has only finitely many orbits on X. Now one obtains the equivalence of categories

$$Mod_{\mathrm{coh}}^{st-K}(\mathscr{D}_X) \simeq Mod_{fg}(\mathfrak{g}, K) \tag{4.4.0}$$

where the category on the right consists of all finitely generated $(U(\mathfrak{g}), K)$-modules with trivial central character. (See [B-B, Sects. 2.2 and 2.3].)

For each $w \in W$, let $X_w(\bar{X}_w, M_w, L_w)$ denote the corresponding Schubert cell (the corresponding Schubert variety, the corresponding Verma-module, the unique irreducible sub-quotient of M_w, respectively).

Under the correspondences of (4.4.0), it is shown in [Bryl-Kas, Sect. 4] that $M_w(L_w)$ corresponds to the regular holonomic strongly K-equivariant $\mathscr{D}_X$-module $\mathscr{M}_w = \mathscr{H}^{\mathrm{codim}\, X_w}_{[X_w]}(\mathscr{O}_X)$ (a strongly$-K$-equivariant regular holonomic simple submodule $\mathscr{L}_w$ of $\mathscr{M}_w$, respectively). It is also shown in [Bry-Kas, Sects. 6 and 8] that under the Riemann-Hilbert correspondence in (4.2), $\mathscr{M}_w(\mathscr{L}_w)$ corresponds to the complex $\mathbb{C}_X[-\mathrm{codim}\, X_w]$ extended by zero outside of X_w (the shifted intersection cohomology complex $IC_{\bar{X}_w}[-\mathrm{codim}\, X_w]$ of $\bar{X}_w$, respectively). For each $y \in W$, $y \leqq w$, let $i_y : \bar{y} \to \bar{X}_w$ denote the closed immersion of the corresponding closed point of $\bar{X}_w$ fixed by the action of the maximal torus T.

Now it follows from (3.6) and (4.3) that if $y \in W$, $y \leqq w$, then

$$\begin{aligned} ch(i_y^! \mathscr{M}_w) &= ch(i_{y\,\mathrm{an}}^! \mathbb{C}_{X_w}[-\mathrm{codim}\, X_w]) \\ &= ch(\mathbb{C}_{X_w}[-\mathrm{codim}\, X_w]) \\ &= ch(\mathscr{M}_w) = ch(M_w) \quad \text{if} \quad y = w \quad \text{and} \quad = 0, \quad \text{otherwise.} \end{aligned} \tag{4.4.1}$$

Next we obtain the formula

$$ch(i_y^! \mathscr{L}_w) = (-1)^{\mathrm{codim}\, X_y} \Big(\sum_i (-1)^i \dim \mathscr{E}xt^i_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{L}_w)_y\Big)\, ch(M_y). \tag{4.4.2}$$

As both sides of this formula are additive, it may be established by writing $\mathscr{L}_w$ as an integral linear combination of various $\mathscr{M}_y$, $y \leqq w$ and then verifying the formula for each $\mathscr{M}_y$. (This follows readily from (4.4.1) and the observation

$$\mathscr{E}xt_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M}_y) = \mathrm{DR}(\mathscr{M}_y) \simeq \mathbb{C}_{X_y}[-\mathrm{codim}\, X_y].)$$

In view of (4.4.2), the Kazdhan-Lusztig formula reduces to the character formula

$$ch(\mathscr{L}_w) = \sum_{y \leqq w} ch(i_y^! \mathscr{L}_w) \tag{4.4.*}$$

which clearly is a special case of (3.6).

Appendix

Through this appendix we will assume the situation and terminology of (1.5.0)–(1.5.3).

(A.1) **Lemma.** *Assume the situation of* (1.5.0)–(1.5.3). (i) *The category* $Mod(EG \underset{G}{\times} X, A_{EG \underset{G}{\times} X})$ *when* $A = \mathcal{O}_X$, $\mathcal{D}_X$, *or* $f^{-1}(\mathcal{O}_Y)$, $f^{-1}(\mathcal{D}_Y)$ and $Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X, A_{EG \underset{G}{\times} X})$ *when* $A = \mathcal{O}_X$, $\mathcal{D}_X$ *are Grothendieck-categories and hence have injective co-generators and are complete and co-complete.*

(ii) *Let* $A = \mathcal{O}_X$ *or* $\mathcal{D}_X$ *and let* $M \in Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X})$. *Then there exists a*

$$P \in Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; A_{EG \underset{G}{\times} X}),$$

with a map $P \to M$, *so that each* P_n *is a flat* quasi-coherent $A_{(EG \underset{G}{\times} X)_n}$-module and each $P_n \to M_n$ is a surjection.

Proof makes essential use of the fact that each restriction

$$(\ )_n \colon Mod(EG \underset{G}{\times} X, A_{EG \underset{G}{\times} X}) \to Mod((EG \underset{G}{\times} X)_n, A_{(EG \underset{G}{\times} X)_n})$$

has a left adjoint L_n, given by $L_n(F)_m = \bigoplus_{\alpha:[n]\to[m]} (\alpha_m)^*(F)$. We skip the details.

(A.2) **Proposition.** *Assume the above situation.* (i) *Then the natural inclusion*

$$\varphi \colon Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X}) \to Mod(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X})$$

$$(\bar{\varphi} \colon Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{D}_{EG \underset{G}{\times} X}) \to Mod(EG \underset{G}{\times} X; \mathcal{D}_{EG \underset{G}{\times} X}))$$

has a right adjoint Q ($\bar{Q}$, *respectively*).

(ii) *If*

$$\alpha \colon Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{D}_{EG \underset{G}{\times} X}) \to Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X})$$

$$(\bar{\alpha} \colon Mod(EG \underset{G}{\times} X; \mathcal{D}_{EG \underset{G}{\times} X}) \to Mod(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X}))$$

is the obvious forgetful functor, then $\bar{\alpha} \circ \bar{\varphi} = \varphi \circ \alpha$ *and* $\alpha \circ \bar{Q} \circ \bar{\alpha}$ (i.e. *the square*

$$\begin{array}{ccc} Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{D}_{EG \underset{G}{\times} X}) & \xrightarrow{\bar{\varphi}} & Mod(EG \underset{G}{\times} X; \mathcal{D}_{EG \underset{G}{\times} X}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \bar{\alpha}} \\ Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X}) & \xrightarrow{\varphi} & Mod(EG \underset{G}{\times} X; \mathcal{O}_{EG \underset{G}{\times} X}) \end{array}$$

and the corresponding one with Q ($\bar{Q}$) *replacing* φ ($\bar{\varphi}$, *respectively*) *commutes).*

(iii) *These functors provide the equivalences*:

$$D_+^{G,b}(Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})) \simeq D_+^{G,b,q\,\mathrm{coh}}(Mod(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})) \tag{A.2.0}$$

and

$$D_+^{G,b}(Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{O}_{EG \underset{G}{\times} X})) \simeq D_+^{G,b,q\,\mathrm{coh}}(Mod(EG \underset{G}{\times} X; \mathscr{O}_{EG \underset{G}{\times} X})).$$

Proof. (i) Since φ, and $\bar{\varphi}$ preserve all small colimits, the special adjoint functor theorem shows the existence of Q and $\bar{Q}$. The commutativity of the square in (A.2) with the horizontal rows given by $\bar{\varphi}$, φ is clear; we will now prove the commutativity of the corresponding square with the horizontal rows given by $\bar{Q}$ and Q. If

$$Q_n : Mod((EG \underset{G}{\times} X)_n; \mathscr{O}_{(EG \underset{G}{\times} X)_n}) \to Mod_{q\,\mathrm{coh}}((EG \underset{G}{\times} X)_n; \mathscr{O}_{(EG \underset{G}{\times} X)_n})$$

denotes the usual quasi-coherator, one may first show that $Q = \{Q_n | n\}$.

Next let $N = \{N_n | n\} \in \mathscr{A} = Mod(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})$. Then one observes the following:

(a) $Q(\bar{\alpha}(N))$ also belongs to $\mathscr{A}_{q\,\mathrm{coh}} = Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})$; hence

$$\varphi Q(\bar{\alpha}(N)) \in \mathscr{A} = Mod(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X}).$$

(b) The map $\varphi Q(\bar{\alpha}(N)) \to \bar{\alpha}(N)$ belongs to $\mathscr{A}$.

Using (a) and (b) one may show readily that Q induces a functor

$$\tilde{Q} : Mod(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X}) \to Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})$$

right adjoint to $\bar{\varphi}$. It follows $\bar{Q} = \tilde{Q}$; now one may readily verify $\alpha \circ \bar{Q} \cong Q \circ \bar{\alpha}$. This concludes the proof of (ii).

Since the above categories all have enough injectives, one may take the right derived functors RQ, $R\bar{Q}$ of Q, $\bar{Q}$. Since α, and $\bar{\alpha}$ are exact functors we obtain: $\alpha \circ R\bar{Q} = RQ \circ \bar{\alpha}$. If

$$F^{\cdot} \in D_+^{G,b,q\,\mathrm{coh}}(Mod(EG \underset{G}{\times} X; \mathscr{O}_{EG \underset{G}{\times} X})),$$

one shows readily that the natural map $\varphi RQ(F^{\cdot}) \to F^{\cdot}$ is a quasi-isomorphism; if

$$M^{\cdot} \in D_+^{G,b,q\,\mathrm{coh}}(Mod(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X}))$$

now one may show the natural map $\bar{\varphi} R\bar{Q}(M^{\cdot}) \to M^{\cdot}$ is also a quasi-isomorphism by applying $\bar{\alpha}$ to it. It follows that RQ, $R\bar{Q}$ induce functors at the level of the equivariant derived categories in (iii) and that $\varphi(\bar{\varphi})$ is left adjoint to $RQ(R\bar{Q})$ as functors between the above derived categories. Now the proofs of the assertions in (iii) may be completed readily. (See for e.g. (A.6.6).)

(A.3) Now assume D is a complex diagonalisable group acting linearly on a smooth complex algebraic variety X. Observe in view of [J4, Proposition 1.4] that the full subcategory $Mod^D_{\mathrm{coh}}(\mathscr{D}_X)(Mod^D_{\mathrm{coh}}(\mathscr{O}_X))$ is *quasi-relevable* (as in [SGA, 6, I, (1.2)]) in $Mod^D_{q\mathrm{coh}}(\mathscr{D}_X)((Mod^D_{q\mathrm{coh}}(\mathscr{O}_X)$, respectively). It is also clearly closed under kernels of epimorphisms in $Mod^D_{q\mathrm{coh}}(\mathscr{D}_X)(Mod^D_{q\mathrm{coh}}(\mathscr{O}_X)$, respectively). Hence we obtain the equivalences

$$D_b(Mod^D_{\mathrm{coh}}(\mathscr{D}_X)) \simeq D^{\mathrm{coh}}_b(Mod^D_{q\mathrm{coh}}(\mathscr{D}_X))$$

and

$$D_b(Mod^D_{\mathrm{coh}}(\mathscr{O}_X)) \simeq D^{\mathrm{coh}}_b(Mod^D_{q\mathrm{coh}}(\mathscr{O}_X)).$$

(See [SGA 6, I 2.11.b, 3.5.b] with the trivial topology.) The use of a suitable cohomology truncation functor and (A.6.6) now provides the equivalences

$$D^{\mathrm{coh}}_b(Mod^D_{q\mathrm{coh}}(\mathscr{D}_X)) \simeq D^{b,\mathrm{coh}}_+(Mod^D_{q\mathrm{coh}}(\mathscr{D}_X)),$$
$$D^{\mathrm{coh}}_b(Mod^D_{q\mathrm{coh}}(\mathscr{D}_X)) \simeq D^{b,\mathrm{coh}}_-(Mod^D_{q\mathrm{coh}}(\mathscr{D}_X))$$

(and similarly for the corresponding derived categories obtained by replacing $\mathscr{D}_X$ with $\mathscr{O}_X$).

In (A.4.1) through (A.5.3) we recall a few techniques frequently used in this paper. (See also [J2, (2.3) and (2.4)].

(A.4.1) Assume the situation of (1.0). If M is a weakly $-G$-equivariant $\mathscr{D}_X$-module as in (1.3), then M has *a good G-invariant filtration.* i.e. M has a nondecreasing filtration by $\mathscr{O}_X$-coherent G-equivariant submodules M^k so that (i) $M^k=0$ if $k \ll 0$ (ii) $\mathscr{D}^i_X \cdot M^k \subseteq M^{k+i}$ for all k, i and (iii) there exists a k_0 so that for all $k \geqq k_0$, $\mathscr{D}^i_X \cdot M^k = M^{k+i}$ for all i and (iv) $\bigcup_i M^i = M$.

(A.4.2) Next we obtain a G-equivariant 'first-exact sequence of Spencer'. First observe that in the notation of (1.1), the tangent sheaf $T_X = (\Omega^1_X)^\vee \subseteq \mathrm{Diff}^1_X = D^1_X$. Let the dimension of X be n and let M be a G-equivariant coherent $\mathscr{D}_X$-module provided with a good G-invariant filtration $\{M^k\}$. Then the first-exact sequence of Spencer of order $k+n$ (see [Ang-Lej, p. 133]) (denoted $Sp^{k+n}(M)$) is:

(A.4.2.*)

$$\begin{aligned}
0 \to \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \wedge^n T_X \underset{\mathscr{O}_X}{\otimes} M^k \\
\to \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} \wedge^{(n-1)} T_X \underset{\mathscr{O}_X}{\otimes} M^{k+1} \\
\to \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} T_X \underset{\mathscr{O}_X}{\otimes} M^{k+n-1} \\
\to \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} M^{k+n}
\end{aligned}$$

where the differential $d : \mathscr{D}_X \otimes \wedge^p T_X \otimes M^j \to \mathscr{D}_X \otimes \wedge^{p-1} T_X \otimes M^{j+1}$ is given by

$$\begin{aligned}
d(P \otimes (t_1 \wedge \ldots \wedge t_p) \otimes m) = \sum_{i=1}^{p} (-1)^{i-1} [P t_i \otimes (t_1 \wedge \ldots \wedge \hat{t}_i \wedge \ldots \wedge t_p) \otimes m \\
- P \otimes (t_1 \wedge \ldots \wedge \hat{t}_i \wedge \ldots \wedge t_p) \otimes t_i m] \\
+ \sum_{i<j} (-1)^{i+j} P \otimes ([t_i, t_j] \wedge t_1 \\
\wedge \ldots \wedge \hat{t}_i \wedge \ldots \wedge \hat{t}_j \wedge \ldots \wedge t_p) \otimes m.
\end{aligned}$$

It is shown in [J2, (1.7) (i)] that this is a complex of weakly-equivariant $\mathscr{D}_X$-modules where the differentials are also equivariant. One observes as in [Mal, pp. 54–55] that if $k \gg 0$, the complex (A.4.2.*) is acyclic and the obvious (G-equivariant) augmentation $\varepsilon: \mathscr{D}_X \underset{\mathscr{O}_X}{\otimes} M \to M$ makes it a resolution of M. Moreover the obvious equivariant map $Sp^{k+n}(M) \to Sp^{k+n+1}(M)$ is a quasi-isomorphism again for $k \gg 0$.

(A.4.3) Let $N \in Mod^G_{q\,\mathrm{coh}}(\mathscr{O}_X)$ and let $\bar{N} = \{N_n = \overset{n}{\underset{}{\otimes}} \mathscr{O}_G \otimes \mathscr{D}_X \otimes N \otimes \check{\omega}_X\}$ denote the associated module in $Mod^G_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})$ as in (1.5.2). If

$$\int_f : D_b^{G,\,\mathrm{coh}}(Mod_{q\,\mathrm{coh}}(EG \underset{G}{\times} X; \mathscr{D}_{EG \underset{G}{\times} X})) \to D_+(Mod(EG \underset{G}{\times} Y, \mathscr{D}_{EG \underset{G}{\times} Y}))$$

is the functor in (1.7.6), observe that

$$\text{(A.4.3.*)} \qquad \int_f \bar{N} = \{R\bar{f}_{n*}(\mathscr{D}_{(EG \underset{G}{\times} Y \leftarrow EG \underset{G}{\times} X)_n} \underset{\mathscr{D}_{(EG \underset{G}{\times} X)_n}}{\otimes} \bar{N}_n \mid n\}$$

$$\cong \{R\bar{f}_{n*}(\overset{n}{\underset{}{\otimes}} \mathscr{O}_G \otimes N \underset{f_n^{-1}(\overset{n}{\underset{}{\otimes}} \mathscr{O}_G \otimes \mathscr{O}_Y)}{\otimes} f_n^{-1}(\overset{n}{\underset{}{\otimes}} \mathscr{O}_G \otimes \mathscr{D}_Y \underset{\mathscr{O}_Y}{\otimes} \check{\omega}_Y) \mid n\}$$

where

$$\bar{f}_n : ((EG \underset{G}{\times} X)_n ; f^{-1}\mathscr{D}_Y \otimes (\overset{n}{\underset{}{\otimes}} \mathscr{O}_G)) \to ((EG \underset{G}{\times} X)_n ; \mathscr{D}_Y \otimes (\overset{n}{\underset{}{\otimes}} \mathscr{O}_G))$$

is the obvious map induced by f. Since $\mathscr{D}_Y$ is locally free over $\mathscr{O}_Y$, the projection-formula applies to show the right side of (A.4.3.*) in degree n is isomorphic to $\overset{n}{\underset{}{\otimes}} \mathscr{O}_G \otimes Rf_* N \otimes \mathscr{D}_Y \otimes \check{\omega}_Y$. Clearly $Rf_* N$ has quasi-coherent equivariant cohomology. Hence so does the left-side of (A.4.3.*).

(A.5.1) Under the assumptions of (1.0) it is shown in [T1, Lemma 5.6] that each G-equivariant coherent $\mathscr{O}_X$-module is the equivariant quotient of a G-vector bundle.

(A.5.2) Similarly under the same assumptions, it is shown in [J2, (2.1)] that each G-equivariant coherent $\mathscr{D}_X$-module is the G-equivariant quotient of a locally free G-equivariant coherent $\mathscr{D}_X$-module of the form $\mathscr{D}_X \otimes \mathscr{E}$, where $\mathscr{E}$ is a G-equivariant locally-free coherent $\mathscr{O}_X$-module.

(A.5.3) It follows that we obtain the isomorphisms

$$K_0(Mod^G_{\mathrm{coh}}(\mathscr{D}_X)) \simeq K_0(Mod^G_{\mathrm{lfree}}(\mathscr{D}_X))$$

and

$$K_0(Mod^G_{\mathrm{coh}}(\mathscr{O}_X)) \simeq K_0(Mod^G_{\mathrm{lfree}}(\mathscr{O}_X)).$$

(A.6) *Waldhausen K-theory of the category of complexes in an abelian category.* (See [T-T, Sect. 1] and [Wald, pp. 328–330].)

(A.6.0) Let $\mathscr{E}$ be an exact category imbedded as a full subcategory of an abelian category $\mathscr{A}$, so that the fully-faithful functor $\mathscr{E} \to \mathscr{A}$ reflects exactness (i.e. a sequence $0 \to E' \to E \to E'' \to 0$ in $\mathscr{E}$ is exact if its image in $\mathscr{A}$ is exact) and $\mathscr{E}$ is closed under extensions in $\mathscr{A}$. Let $\mathscr{C}$ denote an abelian subcategory of $\mathscr{A}$ closed under extensions in $\mathscr{A}$. Let $\underline{\mathscr{A}}$ denote the full sub-category of complexes $M^\cdot$ in $\mathscr{A}$ with each M^k in $\mathscr{E}$ so that the cohomology objects $H^i(M^\cdot)$, computed in $\mathscr{A}$, lie in $\mathscr{C}$. We may now let $co(\underline{\mathscr{A}})(fib(\underline{\mathscr{A}}))$ denote the sub-category which has the same objects as $\underline{\mathscr{A}}$, but where a morphism $M^\cdot \to N^\cdot$ belongs to $co(\underline{\mathscr{A}})$ $(fib(\underline{\mathscr{A}}))$ precisely when $M^k \to N^k$ is an admissible monomorphism (epimorphism, respectively) for each k. The subcategory $w(\underline{\mathscr{A}})$ will have the same objects as $\underline{\mathscr{A}}$, but a morphism in $\underline{\mathscr{A}}$ will be in $w(\underline{\mathscr{A}})$ precisely when it induces an isomorphism in cohomology (computed in the abelian category $\mathscr{A}$). In this situation, $(\underline{\mathscr{A}}, co(\underline{\mathscr{A}}), fib(\underline{\mathscr{A}}), w(\underline{\mathscr{A}}))$ becomes a *complicial bi-Waldhausen* category as defined in [T-T, (1.2.11)].

(A.6.1) We may let $\mathscr{E} = \mathscr{A}$ and let $\underline{\mathscr{A}}$ denote the category of all chain complexes in $\mathscr{A}$ whose cohomology objects lie in $\mathscr{C}$. The category $co(\underline{\mathscr{A}})(fib(\underline{\mathscr{A}}))$ may now be taken to be either all injective maps or all degree-wise split injective maps (surjective maps or all degreewise surjective maps, respectively) of complexes. The category $w(\underline{\mathscr{A}})$ will be those maps that are quasi-isomorphisms. We may also consider variations of the above by considering the category of all complexes bounded-below, bounded above or bounded.

(A.6.2) The morphisms of $co(\underline{\mathscr{A}})(fib(\underline{\mathscr{A}}), w(\underline{\mathscr{A}}))$ are called cofibrations (fibrations, weak-equivalences, respectively). A functor F between two complicial bi-Waldhausen categories $\underline{\mathscr{A}}$ and $\underline{\mathscr{B}}$ (as above) is called a *complicial exaxt* functor if it is induced by an additive functor of the underlying abelian categories and it preserves cofibrations, fibrations and weak-equivalences.

(A.6.3) Let $\underline{\mathscr{A}}$ denote a complicial bi-Waldhausen category as in (A.6.0) or (A.6.1). One may now define canonical homotopy pushouts and pull-backs as follows.

(i) Let $f: M^\cdot \to N^\cdot$, $g: M^\cdot \to P^\cdot$ be maps of complexes in any of the above categories. Then the *canonical homotopy pushout* of f and g is the complex $N^\cdot \overset{h}{\coprod}_{M^\cdot} P^\cdot$ given by

$$\left(N^\cdot \overset{h}{\coprod}_{M^\cdot} P^\cdot\right)^n = N^n \oplus M^{n+1} \oplus P^n, d(n, m, p)$$

$$= (d_{N^\cdot}(n) + f(m), -d_{M^\cdot}(m), d_{P^\cdot}(p) - g(m)).$$

The canonical homotopy pull-back of $f: M^\cdot \to N^\cdot, g: P^\cdot \to N^\cdot$ may be defined similarly. (Observe that all the categories in (A.6.0) and (A.6.1) are closed under the formation of these.)

(A.6.4) *Comparison of Waldhausen and Quillen K-theories* (Gillet-Waldhausen). Assume in addition to the hypotheses of (A.6.1) that $\mathscr{A}$ is an abelian category and $\underline{\mathscr{A}}$ is the category of all bounded complexes in $\mathscr{A}$. $\mathscr{A}$ has the structure of a Waldhausen category (see [Wald, pp. 326–327]) by letting the cofibrations (weak-equivalences) be all injective maps (isomorphisms, respectively). Let $i: \mathscr{A} \to \underline{\mathscr{A}}$ be the exact functor of Waldhausen categories (i.e. a functor preserving

cofibrations and weak-equivalences) sending an object E of $\mathscr{A}$ to the complex which is E in degree 0 and 0 elsewhere. Then

(i) i induces an isomorphism $i: K_n(\mathscr{A}) \xrightarrow{\simeq} K_n(\underline{\mathscr{A}})$ for all n, where the K-theory on the left (right) is the Waldhausen K-theory of the abelian category $\mathscr{A}$ (of $\underline{\mathscr{A}}$, respectively). (See [T-T, 1.11.7].)

(ii) The Quillen K-theory of the abelian category $\mathscr{A}$ is isomorphic to the Waldhausen K-theory of $\mathscr{A}$ defined as above. (See [Wald, pp. 375–376].)

(A.6.5) *Thomason's approximation theorem for Waldhausen K-theory.* (See [T-T, 1.9.6–1.9.8].) Let $\underline{\mathscr{A}}$ denote a complicial bi-Waldhausen category as in (A.6.0) that is closed under the formation of the canonical homotopy pushouts and pull-backs. Let $\underline{\mathscr{A}}/\simeq$ *denote* the category with the same objects as $\underline{\mathscr{A}}$, but where two maps are identified if they are chain homotopic. By taking distinguished triangles to be those that are chain homotopy equivalent to mapping-cone sequences in $\underline{\mathscr{A}}$, one may show that $\underline{\mathscr{A}}/\simeq$ *becomes* a triangulated category. If $\underline{w}$ denotes the image of $w(\underline{\mathscr{A}})$ in $\underline{\mathscr{A}}/\simeq$, the existence of the canonical homotopy pushouts and pull-backs show that $\underline{w}$ is a saturated multiplicative system in the sense of [Hart2, I, Sect. 3] dand hence one may localise $\underline{\mathscr{A}}/\simeq$ at $\underline{w}$. We will denote this by $w^{-1}\underline{\mathscr{A}}$ and call it *the derived category associated to* $\underline{\mathscr{A}}$.

Theorem (see [T-T, Theorem 1.9.8]). *Let $F: \underline{\mathscr{A}} \to \underline{\mathscr{B}}$ be a complicial exact functor between two complicial bi-Waldhausen categories as in* (A.6.2). *If F induces an equivalence $w^{-1}\underline{\mathscr{A}} \to w^{-1}\underline{\mathscr{B}}$, F induces an isomorphism on Waldhausen K-theory.*

(A.6.6) We conclude by observing that if $F: \underline{\mathscr{A}} \to \underline{\mathscr{B}}$ is fully-faithful so that $F(\alpha) \in w(\underline{\mathscr{B}})$ if and only if $\alpha \in w(\underline{\mathscr{A}})$, and if for each object B of $\underline{\mathscr{B}}$, there exists an object A in $\underline{\mathscr{A}}$ and a weak-equivalence $F(A) \xrightarrow{\simeq} B$ (i.e. a map in $w(\underline{\mathscr{B}})$), then the functor F satisfies the hypotheses of the theorem. (See [T-T, 1.9.7].)

(A.6.7) Let $\mathscr{A}$ denote any one one of the abelian categories as in (A.1). Then (A.6.6) and the use of a suitable cohomology truncation functor shows the obvious functors $D_b^G(\mathscr{A}) \to D_+^{G,b}(\mathscr{A})$, $D_b^G(\mathscr{A}) \to D_-^{G,b}(\mathscr{A})$ are equivalences.

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Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations

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0 Introduction

In this paper we are concerned with the existence of global solutions to the Cauchy problem for the typical semilinear wave equations with a dissipative term:

$$u_{tt}-\Delta u+u_t-|u|^p u=0 \quad \text{in } R^N\times[0,\infty) \tag{0.1}$$

$$u(x,0)=u_0(x) \quad \text{and} \quad u_t(x,0)=u_1(x). \tag{0.2}$$

Existence of global solution to the Cauchy problem for the nonlinear wave equations related to (0.1)–(0.2) has been investigated by many authors through various approaches (cf. Segal [Se], Sather [Sa], Matsumura [Ma], Glassy [Gl], Klainerman [Kl], Kato [Ka], etc.).

Most of papers, however, treat the smooth or classical solutions and it is commonly assumed that the initial data (u_0, u_1) are sufficiently small in $H^{m+1}\times H^m$-norm ($m\geqq N/2+1$) or C^2-norm. Here, we shall prove the global existence and decay of weak solutions of (0.1)–(0.2) under the assumption that $\|u_1\|+\|\nabla u_0\|$ is small ($\|\cdot\|$ denotes L^2-norm). In some case global existence of classical solutions also follows from our result as a corollary.

Needles to say, if the nonlinear term $-|u|^p u$ is replaced by $|u|^p u$ it is well-known that there exist global solutions without any smallness conditions on (u_0, u_1) (cf. Jörgens [Jö], v. Wahl [Wa], Grillakis [Gr], Struwe [Str], etc.). But, we are interested in the term $-|u|^p u$ which is called sometimes as "blowing up" term. Indeed, if $E(0)\equiv\frac{1}{2}\{\|u_1\|^2+\|\nabla u_0\|^2\}-\frac{1}{p+2}\|u_0\|_{p+2}^{p+2}<0$ it is known that any local solution of (0.1)–(0.2) blows up in a finite time (cf. Levine [Le]).

For our purpose we employ two approaches. First, we assume that $\operatorname{supp} u_0\cup\operatorname{supp} u_1$ is compact and we utilize the finite propagation property of the solutions. For the initial-boundary value problem in a bounded domain

Dedicated to Professor J. Kajiwara on the occasion of his sixtieth birthday

similar result to ours was established by the method of "potential well" in Sattinger [Sat] and our first result will be proved by combining this method with the decay property of the energy $E(t)$.

Roughly speaking, the depth $d(t)$ of the potential well $W(t)$ associated with the domain $\Omega(t) \equiv \operatorname{supp} u(t)$ is decreasing to 0 as $t \to \infty$ and hence the decay property of the energy $E(t)$ is necessarily required for the local solution $u(t)$ to remain in $W(t)$ as $t \to \infty$ (this fact of course guarantees the global existence).

Our plan is as follows. First, by the effect of the dissipative term u_t, we derive a decay property of $E(t)$ for a local solution $u(t)$ and next, seek a condition on (u_0, u_1) which ensures $u(t)$ belongs to $W(t)$ for all $t \geqq 0$. For our plan to be successful we must assume in fact $4/(N-1) \leqq p < 4/(N-2)$.

Our first result (Theorem 2) excludes unfortunately one space dimensional equations and also the compactness of the support of $u(t)$ is essentially used. To remove these restrictions we can employ an alternative approach. That is, we introduce a concept of "modified potential well" $\tilde{W}(R^N)$ and seek directly a condition for $u(t)$ to belong to $\tilde{W}(R^N)$, $0 \leqq t < \infty$, without use of the decay property. For this, the dissipative term u_t will play an essential role again. As a consequence, we can obtain the same decay property of $E(t)$ as in the first result. In this second approach our condition on p is $4/N \leqq p < 4/(N-2)$ and the one space dimensional equations are included.

As far as the global existence is concerned, our first result is worse than the second one. However, we choose to state it for a comparison of two approaches. Any way, for our arguments the effect of the dissipative term u_t is essential. See Levine [Le2, Theorem 4.1] for a nonexistence result of global solutions to the Cauthy problem for the nondissipative semilinear wave equation.

The potential well method is also useful to investigate the life span of weak solutions for the equations without dissipation. In the final section we shall discuss on the life span for both of the equations with and without dissipative term. The life span of C^2-solutions to the Cauchy problem for semilinear equations has been investigated by many authors (cf. John [Jo], Strauss [St2] and the references cited in those), while there seems to be little result on weak solutions. Our result is very simple and hence seems to be interesting.

Finally, we note that the results of this paper remain valid even if we replace $-|u|^p u$ by $\pm |u|^{p+1}$.

1 Local solution and the potential well

We consider the problem:

$$u_{tt} - \Delta u + u_t - |u|^p u = 0 \quad \text{in } R^N \times [0, \infty) \tag{1.1}$$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x), \tag{1.2}$$

where we assume that $u_0 \in H^1$, $u_1 \in L^2$ and

$$\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset \{x \in R^N \mid |x| < L\}, \quad L > 0. \tag{1.3}$$

Then, it is well known that if $0<p\leqq 2/(N-2)$ ($0<p<\infty$ if $N=1, 2$), the problem admits a unique solution $u(t)$ on an interval $[0, T)$ such that

$$u(t)\in C([0, T); H^1)\cap C^1([0, T); L^2) \tag{1.4}$$

and

$$\operatorname{supp} u(t)\subset\{x\in R^N||x|<L+t\} \tag{1.5}$$

for $0<t<T$(cf. Strauss [St2], John [Jo]).

Thus, if $0<p\leqq 2/(N-2)$ ($0<p<\infty$ if $N=1, 2$) it suffices for the existence of a global solution to derive a priori bounds for the assumed solution $u(t)\in C([0, \infty); H^1)\cap C^1([0, \infty); L^2)$ with the property (1.5).

Motivated by above observation, let us define the potential well $W(t)$ concerning (1.1)–(1.2) and the domain $\Omega_t=\{x\in R^N||x|<L+t\}$, $t\geqq 0$.

Let $0<p<4/(N-2)$ ($0<p<\infty$ if $N=1, 2$). Then, following Sattinger [Sat] or Payne and Sattinger [PS] we define

$$W(t)\equiv\{u\in H_0^1(\Omega_t)\,|\,J(u)<d(t) \text{ and } K(u)\geqq 0\},$$

where we set

$$\begin{cases} J(u)=\dfrac{1}{2}\|\nabla u\|^2-\dfrac{1}{p+2}\|u\|_{p+2}^{p+2} \\ K(u)=\|\nabla u\|^2-\|u\|_{p+2}^{p+2} \end{cases} \tag{1.6}$$

and

$$d(t)\equiv\inf\{J(u)\,|\,u\in H_0^1(\Omega_t), K(u)=0 \text{ and } u\neq 0\}$$

(the depth of the potential well $W(t)$).

Let W and d be the potential well on Ω, $\Omega\subset R^N$, and the depth of it, respectively. Then, it is known that

$$d=J(w)=\frac{p}{2(p+2)}\|\nabla w\|^2=\frac{p}{2(p+2)}\|w\|_{p+2}^{p+2},$$

where $w(x)$ is a positive solution of the elliptic problem:

$$\Delta w+|w|^p w=0 \quad \text{in } \Omega, \quad \Omega\subset R^N, \tag{1.7}$$

$$w|_{\partial\Omega}=0 \tag{1.8}$$

(cf. [PS, Theorem 2.6 and § 3]). See also Lions [Li] for the existence of positive solutions.

When Ω is a ball $\Omega=\{x||x|<R\}$ the positive solution $w(x)$ of (1.7)–(1.8) is in fact radially symmetric and consequently unique. (Gidas et al. [GNN]. See also Fowler [Fo] and Yanagida [Ya].)

Thus, we get the following simple lemma.

Lemma 1.1 *If $0<p<4/(N-2)$ ($0<p<\infty$ if $N=1, 2$), we have*

$$d(t)=c_0(L+t)^{-(2p-Np+4)/p} \tag{1.9}$$

with a certain positive constant $c_0>0$.

Proof. Let $w_0(x)>0$ be the solution of the problem

$$\begin{cases} \Delta w_0+|w_0|^p w_0=0 & \text{in } B_1\equiv\{x\in R^N||x|\leqq 1\} \\ w_0|_{\partial B_1}=0, \end{cases}$$

and set $w(x)=k w_0(x/(L+t))$, $k>0$.

Then, w belongs to $H_0^1(\Omega_t)$ and satisfies

$$\Delta w+|w|^p w=0,$$

if we take $k=(L+t)^{-2/p}$.

Hence,

$$\begin{aligned} d(t)&=\frac{p}{2(p+2)}\int_{\Omega_t}|(L+t)^{-2/p}w_0(x/(L+t))|^{p+2}dx \\ &=\frac{p}{2(p+2)}(L+t)^{-\frac{2(p+2)}{p}+N}\|w_0\|_{p+2}^{p+2} \\ &\equiv c_0(L+t)^{-(2p+Np+4)/p}, \end{aligned}$$

where c_0 is the depth of the potential well on the unit ball. QED

Remark 1.1 When the nonlinear term $-|u|^p u$ is replaced by $\pm|u|^{p+1}$ the potential well $W(t)$ is defined in a similar way with

$$J(u)=\frac{1}{2}\|\nabla u\|^2\pm\frac{1}{p+2}\int_{\Omega_t}|u|^{p+1}u\,dx$$

and

$$K(u)=\|\nabla u\|^2\pm\int_{\Omega_t}|u|^{p+1}u\,dx.$$

In these cases we can prove also (1.9). (For the case $|u|^{p+1}$ we should consider negative solution u instead of positive one.)

We also use the following:

Lemma 1.2 *Let W be the potential well on $\Omega\subset R^N$ and let $u\in W$. Then, we have*

$$J(u)\geqq\frac{p}{2(p+2)}\|\nabla u\|^2 \tag{1.10}$$

and

$$K(u)\geqq\left\{1-\left(\frac{J(u)}{d}\right)^{p/2}\right\}\|\nabla u\|^2. \tag{1.11}$$

Proof. Since $K(u)=\|\nabla u\|^2-\|u\|_{p+2}^{p+2}\geqq 0$, it is easy to see

$$J(u)=\frac{1}{2}\|\nabla u\|^2-\frac{1}{p+2}\|u\|_{p+2}^{p+2}\geqq\frac{p}{2(p+2)}\|\nabla u\|^2.$$

The inequality (1.11) is proved in [Na3]. Here, we reproduce it for convenience of the readers. We assume $u\neq 0$ and take $\lambda>0$ such that

$$K(\lambda u)=\lambda^2\|\nabla u\|^2-\lambda^{p+2}\|u\|_{p+2}^{p+2}=0,$$

that is, $\lambda=(\|\nabla u\|^2/\|u\|_{p+2}^{p+2})^{1/p}(>1)$.

Then, by the definition of d we have

$$d\leqq J(\lambda u)=\frac{1}{2}\lambda^2\|\nabla u\|^2-\frac{1}{p+2}\lambda^{p+2}\|u\|_{p+2}^{p+2}\leqq\lambda^2 J(u)$$

and hence, we have

$$\lambda^p\geqq\left(\frac{d}{J(u)}\right)^{p/2}.$$

Therefore, we see

$$\begin{aligned}K(u)&=\|\nabla u\|^2-\lambda^{-p}\|\nabla u\|^2\\&\geqq\left\{1-\left(\frac{J(u)}{d}\right)^{p/2}\right\}\|\nabla u\|^2.\quad\text{QED}\end{aligned}$$

Motivated by the concept of the potential well we introduce here the "modified potential well"

(1.12) $$\tilde{W}(\Omega)\equiv\text{the interior of the set}\{u\in H_0^1(\Omega)\mid K(u)\geqq 0\}.$$

Note that for the definition of $\tilde{W}(\Omega)$, Ω may be unbounded and in particular we can take $\Omega=R^N$. We observe that if $4/N\leqq p\leqq 4/(N-2)$ ($4/N\leqq p<\infty$ if $N=1,2$), 0 belongs to $\tilde{W}(\Omega)$. Indeed, we see, by the Gagliardo-Nirenberg inequality,

(1.13) $$\begin{aligned}\|u\|_{p+2}^{p+2}&\leqq c\|u\|^{(p+2)(1-\theta)}\|\nabla u\|^{(p+2)\theta},\qquad\theta=Np/2(p+2),\\&\leqq c\|u\|_{H^1}^{p}\|\nabla u\|^2\end{aligned}$$

and hence $K(u)\geqq 0$ if $\|u\|_{H^1}$ is sufficiently small, which implies $0\in\tilde{W}(\Omega)$.

Further, we note that the first inequality (1.10) in Lemma 1.2 remains valid if $u\in\tilde{W}(\Omega)$.

Remark 1.2 Lemma 1.2 is valid also for the nonlinearities $\pm|u|^{p+1}$. For these nonlinearities $\tilde{W}(\Omega)$ is defined in the same manner as (1.12).

Now, we assume that $(u_0,u_1)\in H^1\times L^2$ has a compact support as (1.3) and u_0 belongs to the potential well $W(0)$. Then, by continuity there exists $T>0$

such that $u_0 \in W(T)$. Therefore, by the result in Sattinger [Sat] the initial-boundary value problem:

$$u_{tt} - \Delta u + u_t - |u|^p u = 0 \quad \text{in } \Omega_T \times [0, T) \tag{1.14}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{and} \quad u|_{\partial\Omega_T} = 0 \tag{1.15}$$

has a solution $u(t)$ in the class

$$L^\infty([0, T); H_0^1(\Omega_T)) \cap W^{1,\infty}([0, T); L^2(\Omega_T)).$$

But, we can construct such a solution $u(t)$ which meets moreover the finite propagation property:

$$\operatorname{supp} u(t) \subset \Omega_t \quad \text{for } 0 < t < T. \tag{1.16}$$

Indeed, taking suitable approximate functions $f_j(u)$ such that (cf. [St2])

$$f_j(u) = f(u) \quad \text{if } |u| \leqq j,$$
$$|f_j(u)| \leqq |f(u)| \quad \text{and} \quad |f_j(u)| \leqq c_j |u|,$$

the problem (0.1)–(0.2) with $f(u) \equiv |u|^p u$ replaced by $f_j(u)$ admits a unique solution $u_j(t) \in C([0, T]; H^1(R^N)) \cap C^1([0, T); L^2(R^N))$. Applying Appendix 1 of John [Jo], we see that $\operatorname{supp} u_j(t) \subset \Omega_t$, $0 < t < T$. Further, we can prove that $u_j(t) \in W(T)$, $0 < t < T$, for sufficiently large j, and there exists a subsequence of $\{u_j(t)\}$ which converges to a function $\tilde{u}(t)$ with $\operatorname{supp} \tilde{u}(t) \subset \Omega_t$ in certain senses. $\tilde{u}(t)$ is, in fact, a weak solution in $C([0, T); H^1(R^N)) \cap C^1([0, T); L^2(R^N))$ (cf. [St1, St2]) and such a solution is unique by Ginibre and Velo [GV] and Brenner [Br].

Next, let us assume $(u_0, u_1) \in H^1 \times L^2$ and $u_0 \in \tilde{W}(R^N)$. Then, we have again a unique local solution $u(t)$ as above for some $T > 0$. If $u(t) \in \tilde{W}(R^N)$, we see, by (1.10), $J(u(t)) \geqq c \|\nabla u(t)\|^2$ and hence the usual energy identity yields

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \leqq cE(0) < \infty.$$

This means again that the local solution $u(t)$ is continued in t as long as $u(t) \in \tilde{W}(R^N)$.

Summarizing the above we get the following local existence theorem.

Theorem 1 *Let* $0 < p < 4/(N-2)$ $(0 < p < \infty$ *if* $N = 1, 2)$ *and assume that* $(u_0, u_1) \in H^1 \times L^2$ *has a compact support as* (1.3) *and* u_0 *belongs to the potential well* $W(0)$. *Then, there exists* $T > 0$ *such that the problem* (1.12)–(1.13) *has a unique solution* $u(t)$ *on* $R^N \times [0, T)$ *in the class*

$$C([0, T); H^1) \cap C^1([0, T); L^2),$$

satisfying

$$u(t) \in W(t) \quad \textit{and} \quad \operatorname{supp} u(t) \subset \Omega_t \quad \textit{for } 0 < t < T,$$

and this solution can be continued in time t *as long as* $u(t) \in W(t)$. *The same assertion exept for the compactness of support holds for* $(u_0, u_1) \in H^1 \times L^2$ *if* $W(t)$ *is replaced by* $\tilde{W}(R^N)$.

2 Existence of global weak solution I

This section is devoted to the proof of our first result:

Theorem 2 *Let* $4/(N-1) \leqq p < 4/(N-2)$ *and let* (u_0, u_1) *have a compact support as* (1.3) *and belong to the potential well* $W(0)$, i.e.,

$$E(0) \equiv \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2) - \frac{1}{p+2}\|u_0\|_{p+2}^{p+2} < (1-\varepsilon_0)^{2/p} d(0) \tag{2.1}$$

and

$$K(u_0) = \|\nabla u_0\|^2 - \|u_0\|_{p+2}^{p+2} \geqq 0 \tag{2.2}$$

hold for some $0 < \varepsilon_0 < 1$. *Further, we assume that one of the following conditions* (a) and (b) *is satisfied*

(a) $\quad cE(0)\,\varepsilon_0^2\, I_0^{-2}(L+1) \geqq 1-\theta \quad$ *and* $\quad E(0) < c_0(1-\varepsilon_0)^{2/p}(L+1)^{-(1-\theta)}$,

(b) $\quad cE(0)\,\varepsilon_0^2\, I_0^{-2}(L+1) < 1-\theta \quad$ *and* $\quad (1-\theta)\, I_0^2 < c c_0\, \varepsilon_0^2 (1-\varepsilon_0)^{2/p}(L+1)^{\theta}$,

where c *denotes certain positive constants independent of* L *and* (u_0, u_1), *and we set* $\theta = (Np - p - 4)/p$ *and*

$$I_0^2 = 2\{\|u_0\|^2 + 2(u_1, u_0) + 4E(0)\}.$$

Then, the problem (1.1)–(1.2) *has a unique global solution* $u(t)$ *in the class*

$$C([0, \infty); H^1) \cap C^1([0, \infty); L^2),$$

which belongs to $W(t)$ *for any* $t \geqq 0$, *and meets the decay estimate*

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \leqq c_1(1+t)^{-1}, \tag{2.3}$$

where c_1 *is a constant depending on* $\|\nabla u_0\|$, $\|u_1\|$ *and other known quantities.*

Remark 2.1 Both $E(0)$ and I_0^2 are equivalent to $\|\nabla u_0\|^2 + \|u_1\|^2$ in the situation of Theorem 2, and either the condition (a) or (b) is satisfied if $\|\nabla u_0\|^2 + \|u_1\|^2$ is sufficiently small.

Proof. By the assumptions (2.1) and (2.2) we have from Lemma 1.2

$$K(u_0) > \varepsilon_0 \|\nabla u_0\|^2.$$

Let $u(t)$ be a (local) solution of (1.1)–(1.2). Then, we see by continuity that

$$u(t) \in W(t), \qquad J(u(t)) \leqq (1-\varepsilon_0)^{2/p} d(t)$$

and, by (1.11),

$$K(u(t)) \geqq \varepsilon_0 \|\nabla u(t)\|^2 \tag{2.4}$$

on some interval $[0, T)$, where we note that

$$J(u(t))=\frac{1}{2}\int_{R^N}|\nabla u(t)|^2\,dx-\frac{1}{p+2}\int_{R^N}|u(t)|^{p+2}\,dx$$
$$=\frac{1}{2}\int_{\Omega_t}|\nabla u(t)|^2\,dx-\frac{1}{p+2}\int_{\Omega_t}|u(t)|^{p+2}\,dx.$$

For our purpose it suffices to show that the estimate $J(u(t))\leqq(1-\varepsilon_0)^{2/p}d(t)$ holds for all $t>0$ as long as the solution $u(t)$ exists.

We shall estimate the energy $E(t)\equiv\frac{1}{2}\|u_t(t)\|^2+J(u(t))$. For this we shall use an argument as in [Na4]. First, we see that

$$E(t)\leqq E(0) \quad \text{for } 0\leqq t\leqq\min(1, T). \tag{2.5}$$

For a moment we consider the case $T>1$.

Then, multiplying the Eq. (1.1)–(1.2) by $u_t(x, t)$ and integrating over $R^N\times[t, t+1]$, $0<t<T-1$, we have

$$\int_t^{t+1}\|u_t(s)\|^2\,ds=E(t)-E(t+1)\equiv D(t)^2. \tag{2.6}$$

Thus, there exists $t_1\in[t, t+\frac{1}{4}]$, $t_2\in[t+\frac{3}{4}, t+1]$ such that

$$\|u_t(t_i)\|\leqq 2D(t), \qquad i=1, 2.$$

Next, multiplying the equation by $u(x, t)$ and integrating over $R^N\times[t_1, t_2]$ we get from (2.4) that

$$\begin{aligned}\varepsilon_0\int_{t_1}^{t_2}\|\nabla u(s)\|^2\,ds &\leqq\int_{t_1}^{t_2}K(u(s))\,ds\\ &=\int_{t_1}^{t_2}\|u_t(s)\|^2\,ds-(u_t(t_2), u(t_2))+(u_t(t_1), u(t_1))-\int_{t_1}^{t_2}(u_t(s), u(s))\,ds\\ &\leqq D(t)^2+5D(t)\sup_{t\leqq s\leqq t+1}\|u(s)\|.\end{aligned} \tag{2.7}$$

To estimate $\|u(t)\|$ we note the identity

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2+K(u(t))+\frac{d}{dt}(u_t(t), u(t))-\|u_t(t)\|^2=0,$$

which gives

$$\begin{aligned}\|u(t)\|^2 &\leqq\|u_0\|^2+2|(u_t(t), u(t))|+2(u_1, u_0)+2\int_0^t\|u_t(s)\|^2\,ds\\ &\leqq\|u_0\|^2+2\|u_t(t)\|^2+\frac{1}{2}\|u(t)\|^2+2(u_1, u_0)+2\int_0^t\|u_t(s)\|^2\,ds\\ &\leqq\|u_0\|^2+\frac{1}{2}\|u(t)\|^2+2(u_1, u_0)+4E(0)\end{aligned}$$

and hence, we have

$$\|u(t)\|^2 \leqq 2\{\|u_0\|^2 + 2(u_0, u_1) + 4E(0)\} \equiv I_0^2. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\varepsilon_0 \int_{t_1}^{t_2} \|\nabla u(s)\|^2 \, ds \leqq D(t)^2 + 5I_0 D(t). \tag{2.9}$$

In what follows, we denote by c generous positive constants independent of ε_0, (u_0, u_1) and T.

Now, from (2.6), (2.9) and (1.10) we have

$$\varepsilon_0 E(t_2) \leqq 2\varepsilon_0 \int_{t_1}^{t_2} E(s) \, ds \leqq c\{D(t)^2 + I_0 D(t)\}$$

and

$$\begin{aligned} E(t) &\leqq E(t_2) + \int_t^{t+1} \|u_t(s)\|^2 \, ds \\ &\leqq c\varepsilon_0^{-1}\{D(t)^2 + I_0 D(t)\} + D(t)^2 \\ &\leqq c\varepsilon_0^{-1} I_0 D(t), \end{aligned}$$

which implies by (2.6) that

$$E(t)^2 \leqq c\varepsilon_0^{-2} I_0^2 (E(t) - E(t-1)). \tag{2.10}$$

Thus, applying the lemma below to (2.10) and using (2.5) we arrive at the decay estimate

$$\begin{aligned} E(t) &\leqq \left(E(0)^{-1} + \frac{\varepsilon_0^2}{cI_0^2}[t-1]^+\right)^{-1} \\ &= E(0)\left(1 + cE(0)\,\varepsilon_0^2 I_0^{-2}[t-1]^+\right)^{-1} \end{aligned} \tag{2.11}$$

for $0 < t < T$.

Lemma 2.1 [Na1] *Let $\phi(t)$ be a nonincreasing nonnegative function on $[0, T)$, $T > 1$, such that*

$$\phi(t)^{1+r} \leqq k_0(\phi(t) - \phi(t+1)), \qquad k_0 > 0, \qquad r > 0.$$

Then, we have

$$\phi(t) \leqq (\phi(0)^{-r} + k_0^{-1} r[t-1]^+)^{-1/r} \quad \text{for } 0 \leqq t < T,$$

where $[t-1]^+ = \max\{t-1, 0\}$.

Outline of the proof of Lemma 2.1 Setting $\psi(t)\equiv\phi(t)^{-r}$, we see

$$\begin{aligned}\psi(t+1)-\psi(t)&=\int_0^1\frac{d}{d\theta}\{\theta\phi(t+1)+(1-\theta)\phi(t)\}^{-r}d\theta\\&=r\int_0^1\{\theta\phi(t+1)+(1-\theta)\phi(t)\}^{-r-1}(\phi(t)-\phi(t+1))\,d\theta\\&\geqq k_0^{-1}r.\end{aligned}$$

Hence, we get

$$\psi(t+1)\geqq\psi(0)+tk_0^{-1}r$$

and the desired estimate. QED

Now, we shall seek the condition which ensures

$$E(0)(1+cE(0)\varepsilon_0^2 I_0^{-1}[t-1]^+)^{-1}<(1-\varepsilon_0)^{2/p}d(t) \tag{2.12}$$

for all $t>0$.

Above all, by the comparison of the decay rate, we must assume $p\geqq 4/(N-1)$ and under this condition the function

$$g(t)=\{1+cE(0)\varepsilon_0^2 I_0^{-2}[t-1]^+\}(1-\varepsilon_0)^{2/p}d(t)$$

attain the minimum m_0 at $t=\alpha\equiv\dfrac{-a_0L+(1-a_0)(1-\theta)}{\theta a_0}$ with $a_0=cE(0)\,\varepsilon_0^2 I_0^{-2}$ or $t=1$, the latter being the case $\alpha\leqq 1$. Here, we set $\theta=(Np-p-4)/p$.

Observe that $\alpha=1$ is equivalent to

$$cE(0)\varepsilon_0^2 I_0^{-2}(L+1)=1-\theta.$$

It is easy to see that when $\alpha\leqq 1$, (2.12) is valid if

$$E(0)<m_0=c_0(1-\varepsilon_0)^{2/p}(L+1)^{-(1-\theta)},$$

which is the condition (a).

When $\alpha>1$, we get

$$m_0=\theta^{-\theta}(1-\theta)^{-(1-\theta)}c_0\{a_0^{-1}-(L+1)\}^{\theta}a_0(1-\varepsilon_0)^{2/p}$$

and hence (2.12) is valid if

$$\theta^{\theta}(1-\theta)^{1-\theta}E(0)<c_0\{a_0^{-1}-(L+1)\}^{\theta}a_0(1-\varepsilon_0)^{2/p}.$$

But, under the condition $\alpha>1$ this is satisfied if

$$\theta^{\theta}(1-\theta)^{1-\theta}E(0)<c_0\left\{\frac{\theta(L+1)}{1-\theta}\right\}^{\theta}a_0(1-\varepsilon_0)^{2/p},$$

or

$$(1-\theta)I_0^2<cc_0\varepsilon_0^2(1-\varepsilon_0)^{2/p}(L+1)^{\theta},$$

which is the condition (b).

Strictly speaking, the case $\theta=0$ must be treated separately, but the conclusion is unified as the statement in Theorem 2. The proof of Theorem 2 is now complete. QED

3 Existence of global weak solution II

In this section we employ another approach by use of the "modified potential well" to give our second existence result:

Theorem 3 *Assume that* $4/N\leqq p<4/(N-2)$ $(4/N\leqq p<\infty$ *if* $N=1,2)$. *Let* $(u_0, u_1)\in H^1(R^N)\times L^2(R^N)$ *and* u_0 *belong to* $\tilde{W}(R^N)$. *Then, under the additional condition*

$$\delta\equiv c I_0^{(2p-Np+4)/2} E(0)^{Np-4}<1, \tag{3.1}$$

the conclusion of Theorem 2 *except for the compactness of the support holds with* $W(t)$ *replaced by* $\tilde{W}(R^N)$, *where* I_0^2 *is the quantity given in* (2.8) *and* c *in* (3.1) *is a certain constant independent of* (u_0, u_1).

Proof. For the global existence it suffices to show that the local solution $u(t)$ belongs to $\tilde{W}\equiv\tilde{W}(R^N)$ as long as it exists. Suppose that it were not the case. Then, there exists $T>0$ such that

$$u(t)\in\tilde{W},\quad 0\leqq t<T,\quad\text{and}\quad u(T)\in\partial\tilde{W}. \tag{3.2}$$

By the definition of $\tilde{W}$ we have $K(u(T))=0$ and $u(T)\neq 0$ (note that $0\in\tilde{W}$). But, we know already by the argument in § 2 (see (2.8)) that

$$\|u(t)\|^2\leqq I_0^2,\quad 0\leqq t\leqq T, \tag{3.3}$$

if $K(u(t))\geqq 0$, $0\leqq t\leqq T$. Hence, if $4/N\leqq p<4/(N-2)$ we have (see (1.13))

$$\begin{aligned}\|u(t)\|_{p+2}^{p+2}&\leqq c I_0^{(p+2)(1-\theta)}\|\nabla u(t)\|^{(p+2)\theta},\quad \theta=Np/2(p+2),\\ &\leqq c I_0^{(p+2)(1-\theta)} J(u(t))^{(p+2)\theta/2-1}\|\nabla u(t)\|^2\quad\text{(by (1.10))}\\ &\leqq c I_0^{(2p-Np+4)/2} E(0)^{(Np-4)/4}\|\nabla u(t)\|^2\\ &\equiv\delta\|\nabla u(t)\|^2\end{aligned} \tag{3.4}$$

for $0\leqq t\leqq T$, and

$$\begin{aligned}K(u(T))&=\|\nabla u(T)\|^2-\|u(T)\|_{p+2}^{p+2}\\ &\geqq(1-\delta)\|\nabla u(T)\|^2>0\end{aligned} \tag{3.5}$$

under the assumption (3.1). Thus, we arrive at a contradiction.

In order to show the decay property of $E(t)$ we have only to repeat the argument in § 2. Indeed, setting $\varepsilon_0\equiv 1-\delta>0$ we have from (3.4)

$$K(u(t))\geqq\varepsilon_0\|\nabla u(t)\|^2,\quad 0\leqq t<\infty,$$

and we can obtain the same estimate (2.11) again for $0\leqq t<\infty$. QED

4 Global classical solution

Once the existence of global weak solution is established we can obtain the global smooth or classical solution by a standard argument (cf. [ENN, Pe, GV, Br] etc.).

Here, we shall treat the most typical and important case $N=3$ and $p=2$:

$$u_{tt}-\Delta u+u_t-u^3=0 \quad \text{in} \quad R^3\times[0,\infty), \tag{4.1}$$

$$u(x,0)=u_0(x) \quad \text{and} \quad u_t(x,0)=u_1(x). \tag{4.2}$$

Our result reads as follows:

Theorem 4 *In addition to the assumptions of Theorem 2 or Theorem 3 we suppose*

$$u_0\in H^4 \quad \text{and} \quad u_1\in H^3.$$

Then, (4.1)–(4.2) has a unique solution $u(t)$ in the class

$$\bigcap_{i=0}^{4} C^i([0,\infty);H^{4-i})$$

such that

$$\operatorname{supp} u(t)\subset\{x\in R^N \mid |x|<L+t\}$$

and it satisfies

$$\sum_{i=0}^{m}\|D_t^i D_x^{m-i}u(t)\|^2\leqq c(\|u_0\|_{H^4},\|u_1\|_{H^3})(1+t)^{-m}, \qquad m=1,2,3,4, \tag{4.3}$$

where $D_t=\dfrac{\partial}{\partial t}$ and D_x^m denotes the partial differentiations in x of order m.

At this stage the existence and uniqueness part of Theorem 4 are standard (cf. [ENN], [Pe]) and we give the proof of (4.3).

For this we prepare:

Proposition 4.1 *Let $U(t)\in C([0,\infty);H^1)\cap C^1([0,\infty);L^2)$ be the solution of the problem*

$$U_{tt}-\Delta U+U_t=F(x,t) \quad \text{in} \quad R^N\times[0,\infty)$$

with $U(x,0)=U_0(x)\in H^1$, $U_t(x,0)=U_1(x)\in L^2$ and $F\in L^2([0,\infty);L^2)$. Then, we have the difference inequality

$$\sup_{t\leqq s\leqq t+1} E_U(s)\leqq c\{D(t)^2+(D(t)+\delta(t))\sup_{t\leqq s\leqq t+1}\|U(s)\|+\delta(t)^2\}, \tag{4.4}$$

where

$$E_U(t) \equiv \frac{1}{2}(\|U_t(t)\|^2 + \|\nabla U(t)\|^2),$$

$$D(t)^2 \equiv E_U(t) - E_U(t+1) + \delta(t)^2$$

and

$$\delta(t)^2 \equiv \int_t^{t+1} \|F(s)\|^2 ds.$$

Proof. By the same argument obtaining (2.9) we can show

$$\int_{t_1}^{t_2} \|\nabla U(s)\|^2 \leqq c\{D(t)^2 + (D(t)+\delta(t)) \sup_{t \leqq s \leqq t+1} \|U(s)\|\} \equiv A(t)^2 \tag{4.5}$$

and this together with a variant of (2.6) implies

$$\int_{t_1}^{t_2} E(s)\, ds \leqq c A(t)^2.$$

Thus, there exists $t^* \in [t_1, t_2]$ such that

$$E(t^*) \leqq c A(t)^2$$

and hence

$$\begin{aligned} \sup_{t \leqq s \leqq t+1} E(s) &\leqq E(t^*) + \int_t^{t+1} \|U_t(s)\|^2 ds + \int_t^{t+1} \|F(s)\| \, \|U_t(s)\| \, ds \\ &\leqq c\{D(t)^2 + (D(t)+\delta(t)) \sup_{t \leqq s \leqq t+1} \|U(s)\| + \delta(t)^2\}. \quad \text{QED} \end{aligned}$$

Proof of Theorem 4 We proceed to the proof of (4.3). Setting $Du = U$ ($D = D_t$ or D_x), *we get*

$$U_{tt} - \Delta U + U_t = 3u^2 Du. \tag{4.6}$$

Now, applying Proposition 4.1 and using $\|Du(t)\| \leqq c_1(1+t)^{-1/2}$,

$$\sup_{t \leqq s \leqq t+1} E_U(s) \leqq c_1(1+t)^{-1/2}(D(t)+\delta(t)) + c(\delta(t)^2 + D(t)^2). \tag{4.7}$$

Here, we see

$$\delta(t)^2 \equiv \int_t^{t+1} \|u(s)^2 Du(s)\|^2 ds \tag{4.8}$$

$$\leqq \int_t^{t+1} \|u(s)\|_6^4 \|Du(s)\|_6^2 ds$$

$$\leqq c \int_t^{t+1} \|\nabla u(s)\|^4 \|\nabla Du(s)\|^2 ds$$

$$\leqq c(1+t)^{-2} \sup_{t \leqq s \leqq t+1} E_U(s).$$

It is easy to see from (4.7) and (4.8) that $E_U(t) \leqq cE_U(0) < \infty$ for some $c > 0$, and hence we can derive from (4.7) that

$$\sup_{t \leqq s \leqq t+1} E_U(s)^2 \leqq c_2(1+t)^{-1}(E_U(t) - E_U(t+1)) + c_2(1+t)^{-6} + cD(t)^4 \tag{4.9}$$

with some $c_2 \equiv c_2(\|u_0\|_{H^2}, \|u_1\|_{H^1}) < \infty$.

Noting that $D(t)^2 \leqq c_2 \leqq \infty$, (4.9) together with (4.8) implies, in particular,

$$\sup_{t \leqq s \leqq t+1} E_U(s)^2 \leqq c_2(E_U(t) - E_U(t+1))(E_U(t) - E_U(t+1)) + c_2(1+t)^{-4}. \tag{4.9.a}$$

Applying Lemma 4.1 below we have first

$$E_U(t) \leqq c_2(1+t)^{-1}.$$

Returning to the inequality (4.9) and using the estimate just above, we arrive at the inequality

$$\sup_{t \leqq s \leqq t+1} E_U(s)^2 \leqq c_2(1+t)^{-1}(E_U(t) - E_U(t+1)) + c_2(1+t)^{-6}. \tag{4.10}$$

Applying Lemma 4.1 again we conclude

$$E_U(t) \leqq c_2(1+t)^{-2},$$

which proves (4.3) for the case $m = 2$.

Lemma 4.1 [Na2] *Let $\phi(t)$ be a nonnegative function on $[0, \infty)$ satisfying*

$$\sup_{t \leqq s \leqq t+1} \phi(s)^{1+r} \leqq c(1+t)^{\theta}(\phi(t) - \phi(t+1)) + g(t)$$

with $r > 0$ and $\theta < 1$. Then, if $g(t) = o(t^{-(1-\theta)(1+1/r)})$ as $t \to \infty$, we have

$$\phi(t) \leqq c(\phi(0))(1+t)^{-(1-\theta)/r}.$$

Next, setting $U = D^2 u (D = D_t \text{ or } D_x)$ we have

$$U_{tt} - \Delta U + U_t = 3u^2 D^2 u + 6u(Du)^2.$$

(Here U should be considered as $(N+1)\times(N+1)$ matrix valued function.) Noting $\|U(t)\|\leqq c_2(t+1)^{-1}$ in this situation (see (4.10)) and using

$$\int_{R^N}(|u|^4|D^2u|^2+|u|^2|Du|^4)\,dx \leqq c(\|\nabla u\|^4\|\nabla U\|^2+\|\nabla u\|^2\|\nabla Du\|^4) \leqq c_2(t+1)^{-2}E_U(t)+c_2(t+1)^{-5}, \tag{4.11}$$

we can derive, instead of (4.10),

$$\sup_{t\leqq s\leqq t+1}E_U(s)^2\leqq c_2(1+t)^{-2}(E_U(t)-E_U(t+1))+c_2(t+1)^{-8},$$

which yields by Lemma 4.1 the estimate (4.3) for $m=3$.

The estimate for $m=4$ is also derived similarly, the details being omitted. QED

5 Lifespan of weak solutions

First we consider the equation without dissipation

$$u_{tt}-\Delta u-|u|^p u=0 \quad \text{in } R^N\times[0,T), \quad T>0, \tag{5.1}$$

$$u(x,0)=\varepsilon\phi_0(x) \quad \text{and} \quad u_t(x,0)=\varepsilon\phi_1(x) \tag{5.2}$$

with $0<\varepsilon\ll 1$, where we assume $\phi_0\in H^1$, $\phi_1\in L^2$ and

$$\operatorname{supp}\phi_0\cup\operatorname{supp}\phi_1\subset\{x\in R^N\,|\,|x|\leqq L\}. \tag{5.3}$$

As is already mentioned in Sect. 1 this problem has a local solution $u(t)\in C([0,T);H^1)\cap C^1([0,T);L^2)$ for some $T>0$ with $\operatorname{supp}u(t)\subset\Omega_t$, and this solution can be continued as long as $u(t)\in W(t)$. Here, we assume $\varepsilon>0$ is sufficiently small and we let $[0,T_\varepsilon)$ as the maximal interval of the existence of the local solution. Our purpose is to give a lower bound estimate of T_ε as $\varepsilon\to 0$.

It is easy to see that $u(t)\in W(t)$ if

$$E(t)\leqq E(0)\equiv\frac{1}{2}(\varepsilon^2\|\phi_1\|^2+\varepsilon^2\|\nabla\phi_0\|^2)-\frac{1}{p+2}\varepsilon^{p+2}\|\phi_0\|_{p+2}^{p+2} <d(t)=c_0(1+t)^{-(2p-Np+4)/p}.$$

Thus, we see for $0<\varepsilon\ll 1$,

$$T_\varepsilon>C\,E(0)^{-p/(2p-Np+4)}. \tag{5.4}$$

The following is an immediate consequence of (5.4):

Theorem 5.1 *Let $0<p<4/(N-2)$ ($0<p<\infty$ if $N=1,2$) and let T_ε be the lifespan of the local solution $u(t)$ of (5.1)–(5.2). Then, we have*

$$\liminf_{\varepsilon\to 0}\varepsilon^{2p/(2p-Np+4)}T_\varepsilon\geqq c(\|\phi_1\|^2+\|\nabla\phi_0\|^2)^{-p/(2p-Np+4)}>0. \tag{5.5}$$

The criterion on the lifespan for the C^2-solutions are investigated by many authors. But, there seems to be very little result for the weak solutions. The above one is very simple and hence seems to be interesting. We note that (5.5) is valid even if we replace $-|u|^p u$ by $\pm|u|^{p+1}$ (see Remark 1.1 in § 1).

Next, we consider again the dissipative equation:

$$u_{tt}-\Delta u+u_t-|u|^p u=0 \quad \text{in } R^N\times[0,T), \quad T>0, \tag{5.6}$$

$$u(x,0)=\varepsilon\phi_0(x) \quad \text{and} \quad u_t(x,0)=\varepsilon\phi_1(x) \tag{5.7}$$

with the condition (5.3).

As is shown in § 3 the local solution $u(t)$ exists as long as $K(u(t))\geqq 0$, i.e., $u(t)\in\tilde{W}(R^N)$. Now, we see for $0<p<4/N$,

$$\begin{aligned}\|u(t)\|_{p+2}^{p+2} &\leqq c\|u(t)\|^p\|u(t)\|^{(p+2)(1-\theta)-p}\|\nabla u(t)\|^{(p+2)\theta}, \quad \theta=Np/2(p+2),\\ &\leqq cI_0^p(L+t)^{(p+2)(1-\theta)-p}\|\nabla u(t)\|^2,\end{aligned} \tag{5.8}$$

where we have used the Poincaré inequality. Therefore $K(u(t))\geqq 0$ if

$$cI_0^p(L+t)^{(p+2)(1-\theta)-p}<1, \quad \theta=Np/2(p+2). \tag{5.9}$$

Recall that

$$\begin{aligned}I_0^2=2\Big\{&\varepsilon^2\|\phi_0\|^2+2\varepsilon^2(\phi_0,\phi_1)+2\varepsilon^2\|\nabla\phi_0\|^2+2\varepsilon^2\|\phi_1\|^2\\ &-\frac{4\varepsilon^{p+2}}{p+2}\|\phi_0\|_{p+2}^{p+2}\Big\}.\end{aligned} \tag{5.10}$$

From this we conclude:

Theorem 5.2 *Let $0<p<4/N$ and let T_ε be the life span of the solution $u(t)$ of (5.6)–(5.7). Then, we have*

$$\begin{aligned}\liminf_{\varepsilon\to 0}\varepsilon^{2p/(4-Np)}T_\varepsilon &\geqq c(\|\phi_0\|^2+(\phi_0,\phi_1)+\|\nabla\phi_0\|^2+\|\phi_1\|^2)^{-p/(4-Np)}\\ &>0.\end{aligned} \tag{5.11}$$

Remark 5.1 Let $0<p<4/(N-1)$ ($0<p<\infty$ if $N=1$) and let $[0, T_\varepsilon^*)$ be the maximal interval where $u(t)\in W(t)$. Then we can show

$$\liminf_{\varepsilon\to 0}\varepsilon^{2p/(p-Np+4)}T_\varepsilon^*>0.$$

This follows from

$$E(0)(1+cE(0)\varepsilon_0^2 I_0^{-2}[t-1]^+)^{-1}<(1-\varepsilon_0)^{2/p}d(t), \quad 0\leqq t<T_\varepsilon^*.$$

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On the point spectrum of self-adjoint extensions

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1 Introduction

Let H be a symmetric operator in a Hilbert space $\mathscr{H}$. The set of *regular points* of H

$$\begin{aligned}\Gamma(H) &= \{z \in \mathbb{C}\colon H - z \text{ is continuously invertible}\} \\ &= \{z \in \mathbb{C}\colon \exists\, v = v(z) > 0\ \forall f \in D(H)\ \|(H-z)f\| \geqq v\|f\|\}\end{aligned}$$

is an open subset of $\mathbb{C}$ containing $\mathbb{C}\setminus\mathbb{R}$. In this note we assume that H has real regular points, i.e. $\Gamma(H) \cap \mathbb{R} \neq \varnothing$. In this case the *deficiency index*

$$n(H, z) := \dim \ker(H^* - z) = \dim R(H - \bar{z})^{\perp}$$

does not depend on the special choice of the regular point z of H and consequently H has self-adjoint extensions $\tilde{H}$. Krein has posed the question what kind of spectrum can such an extension $\tilde{H}$ have within the set of real regular points of H. In this paper we present a partial solution of this old (but still not completely solved) problem. First let us recall some known results due to von Neumann [1], Stone [2], Friedrichs [3] and Krein [4]. For convenience of the reader in addition to these original sources we shall also give textbook-references.

An open interval (a, b) with $-\infty < a < b < \infty$ is called a *spectral gap* of H if

$$\left\| \left(H - \frac{a+b}{2} \right) f \right\| \geqq \frac{b-a}{2} \|f\| \qquad \forall f \in D(H).$$

If $\langle Hf, f\rangle \geqq b\|f\|^2$ we shall also say that $(-\infty, b)$ is a spectral gap of H (the latter definition is not generally used but convenient for our purposes; and it is justified since every interval (a, b) with $a < b$ is a gap of H). Let (a, b) be a spectral gap of H. Then there exists a self-adjoint extension $\tilde{H}$ of H (for instance the famous Krein or the Friedrichs extension, respectively) such that

$(a, b) \subset \rho(\tilde{H})$ (cf. [5, Theorem 5.32 and Theorem 5.34]); this is the reason why we call (a, b) a spectral gap of H.

Suppose in addition that the deficiency index of H (which is independent of the regular point z of H) equals $n \in \mathbb{N}$. Let $\tilde{H}$ be a self-adjoint extension of H. Then the sum of the multiplicities of the eigenvalues of $\tilde{H}$ within the interval (a, b) is at most equal to n, and no point of the continuous spectrum of $\tilde{H}$ lies in the interval (a, b) (cf. [5, Theorem 8.19 and Corollary 2 in §8.3]). Conversely given any finite subset $\mathscr{E}$ of (a, b) ands positive integers $p(\lambda)$, $\lambda \in \mathscr{E}$, such that $\sum_{\lambda \in \mathscr{E}} p(\lambda) \leqq n$ there exists a self-adjoint extension $\tilde{H}$ of H such that $\sigma(\tilde{H}) \cap (a, b) = \mathscr{E}$ and the multiplicity of λ as an eigenvalue of $\tilde{H}$ equals $p(\lambda)$ for each $\lambda \in \mathscr{E}$ (cf. [4]).

In the latter statement it is crucial that (a, b) is a spectral gap of H. If one only requires that (a, b) is some interval within the set of real regular points of H then the corresponding statement is false. For instance one can give a symmetric operator H such that each $\lambda \in \mathbb{R}$ is a regular point of H, the deficiency index of H equals one and each self-adjoint extension $\tilde{H}$ of H has a periodic point spectrum with period p independent of $\tilde{H}$ (cf. [5, §8.2, Example 2]).

If the deficiency index of H equals $n \in \mathbb{N}$, $\mathscr{E}$ is any finite set of real regular points of H and $\{p(\lambda)\}_{\lambda \in \mathscr{E}}$ a family of positive integers such that $\sum_{\lambda \in \mathscr{E}} p(\lambda) \leqq n$ then H has a self-adjoint extension $\tilde{H}$ such that each $\lambda \in \mathscr{E}$ is an eigenvalue of $\tilde{H}$ with multiplicity at least equal to $p(\lambda)$ (cf. [4]). However, as the above mentioned example shows, one cannot exclude that there are real regular points μ of H with $\mu \notin \mathscr{E}$ and $\mu \in \sigma_p(\tilde{H}) := \{\lambda : \lambda \text{ is an eigenvalue of } \tilde{H}\}$. One also cannot exclude that the multiplicity of λ as an eigenvalue of $\tilde{H}$ is strictly larger than $p(\lambda)$ for some $\lambda \in \mathscr{E}$.

While one has an affirmative answer to Krein's question in the case that the deficiency index of H is finite the situation is much more complicated when it is infinite. Let $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint spectral gaps of the operator H and $\{k_n\}_{n \in \mathbb{N}}$ a sequence in $\mathbb{N} \cup \{\aleph_0\}$. In a recent paper by Derkach and Malamud [6] those self-adjoint extensions of H have been characterized which have exactly k_n eigenvalues within (a_n, b_n) for each $n \in \mathbb{N}$. However nothing is said about the positions of the eigenvalues.

In the following section we shall continue this line of research using essentially different methods. Let us briefly describe our results:

Let $\mathscr{E}$ be a countable (i.e. finite or countable infinite) set of real regular points of the symmetric operator H and $p: \mathscr{E} \to \mathbb{N} \cup \{\aleph_0\}$ any mapping. Suppose that the deficiency index of H is infinite. We show that H has a self-adjoint extension $\tilde{H}$ such that each $\lambda \in \mathscr{E}$ is an eigenvalue of H with multiplicity at least $p(\lambda)$ (cf. Theorem 2.4).

Let C be a *conjugation* on $\mathscr{H}$, i.e. an antilinear bijection $C: \mathscr{H} \to \mathscr{H}$ satisfying $C^2 = 1$ and $\langle Cf, Cg \rangle = \langle g, f \rangle$ for all $f, g \in \mathscr{H}$. If H is *C-real*, i.e. if $Cf \in D(H)$ and $HCf = CHf$ for every $f \in D(H)$, then $\tilde{H}$ can be chosen such that $\tilde{H}$ is also C-real, cf. Theorem 2.3; in this case the deficiency index may also be finite, larger or equal $\sum_{\lambda \in \mathscr{E}} p(\lambda)$.

If $\mathscr{E} \subset (a, b)$ for some spectral gap (a, b) of H then, by Theorem 2.2, H has a self-adjoint extension $\tilde{H}$ with the following properties:

(i) $\sigma_p(\tilde{H}) \cap (a, b) = \mathscr{E}$,

(ii) each $\lambda \in \mathscr{E}$ is an eigenvalue of $\tilde{H}$ with multiplicity equal to $p(\lambda)$,
(iii) $\tilde{H}$ has pure point spectrum within (a, b).

It is noteworthy that, by Theorem 2.4 (Theorem 2.3), each (C-real) symmetric operator H with infinite deficiency index has a (C-real) self-adjoint extension with dense point spectrum within the set of real regular points of H. It also should be mentioned that modifying our proofs in an obvious way one gets new proofs of the above mentioned classical results.

In this paper we concentrate on the point component of the spectrum of self-adjoint extensions of H. If H has an infinite deficiency index, then a self-adjoint extension of H may also have non-empty continuous spectrum within the set of real regular points of H. We shall examine such questions in a forthcoming paper.

A sufficient condition for a symmetric operator to have self-adjoint extensions is that it commutes with some conjugation (von Neumann's theorem), cf. [5, Theorem 8.9]. It is a canonical question whether this condition is also necessary. This question is also interesting in the context of our paper, since a stronger version of Theorem 2.4 would be contained in Theorem 2.3 if the condition would be necessary. In Sect. 3 we shall present a closed symmetric operator which has a spectral gap (and therefore, in particular, self-adjoint extensions) but does not commute with any conjugation.

2 The point spectrum of self-adjoint extensions

First let H be any closed symmetric operator. A closed subspace $\mathscr{M}$ of $\mathscr{H}$ is called a *reducing subspace* of H if

$$Pf \in D(H) \quad \text{and} \quad HPf = PHf \qquad \forall f \in D(H),$$

where P denotes the orthogonal projection in $\mathscr{H}$ onto $\mathscr{M}$. Obviously along with $\mathscr{M}$ the orthogonal complement $\mathscr{M}^{\perp}$ of $\mathscr{M}$ is also a reducing subspace of H. It is easily seen that the closed span of an orthonormal system of eigenvectors of H is a reducing subspace of H. If $\mathscr{M}$ is a reducing subspace of H then the *part of* H *in* $\mathscr{M}$, i.e. the restriction of H to $D(H) \cap \mathscr{M}$, may be (and in the following will be) regarded as an operator in the Hilbert space $\mathscr{M}$. Suppose now in addition that H has a real regular point λ and the deficiency index of H is positive. We choose a one-dimensional subspace $\mathscr{L}_{\lambda}$ of $\ker(H^* - \lambda)$ and put

$$D(H') := D(H) \dotplus \mathscr{L}_{\lambda},$$

$$H'(f+g) := Hf + \lambda g \qquad \forall f \in D(H)\ \forall g \in \mathscr{L}_{\lambda}.$$

Obviously H' is a symmetric extension of H, $n(H', z) = n(H, z) - 1$ for each regular point z of H' and λ is an eigenvalue of H' and $\mathscr{L}_{\lambda}$ the corresponding eigenspace. Since the graph of H' is a one-dimensional extension of the closed graph of H the graph of H' and therefore the operator H' are also closed.

Lemma 2.1 *Assume that (a, b) is a spectral gap of H, and let H' be as above. If H' has an eigenvalue $\mu \in (a, b)$ then (a, b) is a spectral gap of the part of H' in the Hilbert space $\ker(H' - \mu)^{\perp}$.*

Proof. First we consider the case that $-\infty<a<b<\infty$. Then without loss of generality we may assume that $(a, b)=(-v, v)$. We have to show that

$$\|H'f\| \geqq v\,\|f\| \quad \text{for } f\in D(H')\cap \ker(H'-\mu)^{\perp}.$$

Assume that

$$\|H'f\| < v\,\|f\| \quad \text{(which implies } f\neq 0).$$

Let $g\in\ker(H'-\mu)$ and $g\neq 0$. We have, since $\ker(H'-\mu)$ reduces H',

$$\begin{aligned}\|H'(c_1 f+c_2 g)\|^2 &= |c_1|^2\,\|H'f\|^2+|c_2|^2\,\|H'g\|^2\\ &= |c_1|^2\,\|H'f\|^2+|c_2|^2\,\mu^2\,\|g\|^2\\ &< |c_1|^2\,v^2\,\|f\|^2+|c_2|^2\,v^2\,\|g\|^2\\ &= v^2\,\|c_1 f+c_2 g\|^2\end{aligned}$$

for all $(c_1, c_2)\in\mathbb{C}^2\setminus\{(0,0)\}$. Since f and g span a two-dimensional subspace of $D(H')$ and $\dim D(H')/D(H)=1$ it follows that $\|H'h\|<v\,\|h\|$ for some $h\in D(H)$. This is a contradiction to the hypothesis that $(-v, v)$ is a spectral gap of H. Thus we have proved the lemma in the case that $-\infty<a<b<\infty$.

The proof in the semi-bounded case is similar. One only has to replace expressions of the form $\|Ah\|$ by those of the form $\langle Ah, h\rangle$; the details are left to the reader. □

Theorem 2.2 *Let H be a closed symmetric operator with infinite deficiency index and spectral gap (a, b). Let $\mathscr{E}$ be a countable subset of (a, b) and $p\colon \mathscr{E}\to\mathbb{N}\cup\{\aleph_0\}$ be any mapping. Then there exists a self-adjoint extension $\tilde{H}$ of H with the following properties:*

(i) $\sigma_p(\tilde{H})\cap(a, b)=\mathscr{E}$,
(ii) *each $\lambda\in\mathscr{E}$ is an eigenvalue of $\tilde{H}$ with multiplicity equal to $p(\lambda)$,*
(iii) *$\tilde{H}$ has pure point spectrum within (a, b).*

Proof. The assertion of the theorem is well known in the case that $\mathscr{E}=\varnothing$, cf. [5, Theorem 5.34]. Let $\mathscr{E}\neq\varnothing$. We put $N:=\{n\in\mathbb{N}\colon n\leqq\sum_{\lambda\in\mathscr{E}}p(\lambda)\}$ and choose a sequence $\{\lambda_n\}_{n\in N}$ in $\mathscr{E}$ such that

$$\#\{n\in N\colon \lambda_n=\lambda\}=p(\lambda) \qquad \forall\,\lambda\in\mathscr{E}$$

($\#M$ denotes the cardinality of the set M); in other words $\lambda_n\in\mathscr{E}$ for all $n\in N$ and each $\lambda\in\mathscr{E}$ occurs exactly $p(\lambda)$ times in the sequence $\{\lambda_n\}_{n\in N}$. We choose a one-dimensional subspace $\mathscr{L}_{\lambda_1}$ of $\ker(H^*-\lambda_1)$ and put

$$\begin{aligned}D(H_1') &:= D(H)\dotplus\mathscr{L}_{\lambda_1},\\ H_1'(f+g) &:= Hf+\lambda_1 g \qquad \forall f\in D(H)\ \forall g\in\mathscr{L}_{\lambda_1}.\end{aligned}$$

H_1' is a closed symmetric extension of H with infinite deficiency index and λ_1 is an eigenvalue of H_1' with eigenspace $\mathscr{L}_{\lambda_1}$.

We put $\mathscr{H}_1:=\mathscr{L}_{\lambda_1}^{\perp}$ and denote by M_1 and H_1 the part of H_1' in $\mathscr{L}_{\lambda_1}$ and $\mathscr{H}_1$, respectively. Obviously M_1 is a self-adjoint operator on the one-dimensional Hilbert space $\mathscr{L}_{\lambda_1}$ with $\sigma(M_1)=\{\lambda_1\}$, H_1 is a closed symmetric operator in the

Hilbert space $\mathscr{H}_1$ with infinite deficiency index, and $H_1' = M_1 \oplus H_1$. Moreover, by the above lemma, (a, b) is a spectral gap of H_1. Thus we can replace $\mathscr{H}$ by $\mathscr{H}_1$, H by H_1 and λ_1 by λ_2 in the above conclusions.

Proceeding further in this way we get, by induction, sequences $\{\mathscr{L}_{\lambda_n}\}_{n\in N}$ and $\{\mathscr{H}_n\}_{n\in N}$ of Hilbert spaces, and sequences $\{M_n\}_{n\in N}$, $\{H_n\}_{n\in N}$ and $\{H_n'\}_{n\in N}$ of operators with the following properties:

(i) H_n' are closed symmetric extensions of H satisfying

$$H_1' \subset H_2' \subset \ldots \subset H_n' \subset \ldots,$$

(ii) M_n is a self-adjoint operator on a one-dimensional subspace $\mathscr{L}_{\lambda_n}$ of $\mathscr{H}_{n-1}$ (where $\mathscr{H}_0 := \mathscr{H}$), $\mathscr{H}_n = \mathscr{H}_{n-1} \ominus \mathscr{L}_{\lambda_n}$, with $\sigma(M_n) = \{\lambda_n\}$ for each $n \in N$,
(iii) H_n is a closed symmetric operator in the Hilbert space $\mathscr{H}_n$ with spectral gap (a, b) for each $n \in N$,
(iv) $H_n' = \left\{\bigoplus_{j=1}^{n} M_j\right\} \oplus H_n \; \forall\, n \in N$.

If $p := \sum_{\lambda\in\mathscr{E}} p(\lambda)$ is finite we choose a self-adjoint extension $\tilde{H}_p$ of H_p with spectral gap (a, b); then obviously $\tilde{H} := \left\{\bigoplus_{j=1}^{p} M_j\right\} \oplus \tilde{H}_p$ is a self-adjoint extension of H with the required properties.

Otherwise we have $N = \mathbb{N}$ and we put

$$D(H_{\infty,0}') := \bigcup_{n\in\mathbb{N}} D(H_n').$$

For each $f \in D(H_{\infty,0}')$ we choose $n \in \mathbb{N}$ such that $f \in D(H_n')$ and put

$$H_{\infty,0}' f := H_n' f$$

(since $H_n' \subset H_m'$ if $n < m$ this definition does not depend on the special choice of n). The operator $H_{\infty,0}'$ is symmetric and therefore it is closable. We denote by H_∞' the closure of $H_{\infty,0}'$. $\bigoplus_{n\in\mathbb{N}} \mathscr{L}_{\lambda_n}$ and $\bigoplus_{j=1}^{n} \mathscr{L}_{\lambda_j}$, $n \in \mathbb{N}$, are the closed spans of orthonormal systems of eigenvectors of H_∞'. Thus $\bigoplus_{j=1}^{n} \mathscr{L}_{\lambda_j}$, $n \in \mathbb{N}$, as well as $\bigoplus_{n\in\mathbb{N}} \mathscr{L}_{\lambda_n}$ and $\mathscr{H}_\infty := \mathscr{H} \ominus \{\bigoplus_{n\in\mathbb{N}} \mathscr{L}_{\lambda_n}\}$ are reducing subspaces of H_∞'. Thus we have

$$H_\infty' = \{\bigoplus_{n\in\mathbb{N}} M_n\} \oplus H_\infty,$$

where H_∞ denotes the part of H_∞' in $\mathscr{H}_\infty$. By construction, $\bigoplus_{n\in\mathbb{N}} M_n$ is a self-adjoint operator with pure point spectrum and $\sigma_p(\bigoplus_{n\in\mathbb{N}} M_n) = \mathscr{E}$ and the multiplicity of λ as an eigenvalue of $\bigoplus_{n\in\mathbb{N}} M_n$ equals $p(\lambda)$ for each $\lambda \in \mathscr{E}$. Thus it suffices to show

that H_∞ has a self-adjoint extension $\tilde{H}_\infty$ with $\sigma(\tilde{H}_\infty)\cap(a,b)=\varnothing$; then obviously $\tilde{H}:=\{\bigoplus_{n\in\mathbb{N}} M_n\}\oplus\tilde{H}_\infty$ is a self-adjoint extension of H with the required properties.

First let us consider the case that $-\infty<a<b<\infty$. Without loss of generality we may assume that $(a,b)=(-\nu,\nu)$ for some positive constant ν. Let $f\in D(H_\infty)$. Choose $f_n\in D(H'_{\infty,0})$, $n\in\mathbb{N}$, such that

$$f_n\to f \quad\text{and}\quad H'_\infty f_n\to H'_\infty f=H_\infty f \quad\text{as}\quad n\to\infty.$$

For each $n\in\mathbb{N}$ we choose $l_n\in\mathbb{N}$ such that $f_n\in D(H'_{l_n})$ and denote by P_n the orthogonal projection in $\mathscr{H}$ onto the space $\bigoplus_{j=1}^{l_n}\mathscr{L}_{\lambda_j}$. Then we have

$$\|P_nf_n-P_nf\|\leqq\|f_n-f\|\to 0 \quad\text{as}\quad n\to\infty.$$

Since $f\perp\bigoplus_{n\in\mathbb{N}}\mathscr{L}_{\lambda_n}$ and therefore $P_nf=0$ for each $n\in\mathbb{N}$ this implies that

$$P_nf_n\to 0 \quad\text{as}\quad n\to\infty.$$

Thus we have

$$g_n:=f_n-P_nf_n\to f \quad\text{as}\quad n\to\infty.$$

For each $n\in\mathbb{N}$ we have $H'_\infty P_nf_n=P_nH'_\infty f_n$ since $\bigoplus_{j=1}^{l_n}\mathscr{L}_{\lambda_j}$ is a reducing subspace of H'_∞. Since $H_\infty f\perp\bigoplus_{n\in\mathbb{N}}\mathscr{L}_{\lambda_n}$ and $H'_\infty f_n\to H_\infty f$ as $n\to\infty$, this implies that

$$H'_\infty P_nf_n\to 0 \quad\text{as}\quad n\to\infty.$$

Thus

$$H'_\infty g_n\to H_\infty f \quad\text{as}\quad n\to\infty.$$

Since $g_n\in D(H_{l_n})$ for each $n\in\mathbb{N}$ it follows that

$$\|H_\infty f\|=\lim_{n\to\infty}\|H'_\infty g_n\|=\lim_{n\to\infty}\|H_{l_n}g_n\|\geqq\lim_{n\to\infty}\nu\|g_n\|=\nu\|f\|.$$

Hence H_∞ has a spectral gap $(-\nu,\nu)$ and therefore has a self-adjoint extension $\tilde{H}_\infty$ with $\sigma(\tilde{H}_\infty)\cap(-\nu,\nu)=\varnothing$. Thus we have proved the statement of the theorem in the case that $-\infty<a<b<\infty$.

In the semi-bounded case one only has to replace the estimate $\|H_\infty f\|\geqq\nu\|f\|$ by $\langle H_\infty f,f\rangle\geqq b\|f\|^2$. □

Let C be a conjugation on $\mathscr{H}$, H a C-real symmetric operator, and $\lambda\in\mathbb{R}$. Then $R(H-\lambda)$ is invariant under C, and therefore $\ker(H^*-\lambda)=R(H-\lambda)^\perp$ is invariant under C, too. Thus for every subspace $\mathscr{S}$ of $\mathscr{H}$ which is invariant under C it follows that $\ker(H^*-\lambda)\cap\mathscr{S}^\perp$ contains C-invariant subspaces of any dimension smaller than or equal to the dimension of $\ker(H^*-\lambda)\cap\mathscr{S}^\perp$. Using this fact we can easily prove the following theorem.

Theorem 2.3 *Let $\mathscr{E}$ be a countable set of real regular points of the closed symmetric operator H. Let $p\colon\mathscr{E}\to\mathbb{N}\cup\{\aleph_0\}$ be any mapping such that $\sum_{\lambda\in\mathscr{E}}p(\lambda)$ is smaller*

than or equal to the deficiency index of H. Suppose that H is C-real for some conjugation C. Then H has a C-real self-adjoint extension $\tilde{H}$ such that each $\lambda\in\mathscr{E}$ is an eigenvalue of $\tilde{H}$ with multiplicity larger than or equal to $p(\lambda)$.

Proof. We put $N:=\{n\in\mathbb{N}: n\leqq\sum_{\lambda\in\mathscr{E}}p(\lambda)\}$, choose a sequence $\{\lambda_n\}_{n\in N}$ in $\mathscr{E}$ such that every $\lambda\in\mathscr{E}$ occurs exactly $p(\lambda)$ times in this sequence and define inductively for $n\in N\cup\{0\}$ subspaces $\mathscr{L}_n$ of $\mathscr{H}$ and operators H_n in $\mathscr{H}$ as follows:

$n=0$: $H_0:=H$, $\mathscr{L}_0:=\{0\}$.

$n\Rightarrow n+1$: Let n, $n+1\in N\cup\{0\}$ and suppose that a C-real closed symmetric operator H_n and subspaces $\mathscr{L}_j$, $0\leqq j\leqq n$, with $\dim\operatorname{span}\bigcup_{j=0}^{n}\mathscr{L}_j=n$ and $D(H_n)=D(H)\dotplus\operatorname{span}\bigcup_{j=0}^{n}\mathscr{L}_j$ are defined. We have $\ker(H^*-\lambda_{n+1})\cap\left\{\operatorname{span}\bigcup_{j=0}^{n}\mathscr{L}_j\right\}^{\perp}\neq\{0\}$. Thus, by the considerations preceding the statement of the theorem, there exists a C-invariant one-dimensional subspace $\mathscr{L}_{n+1}$ of $\ker(H^*-\lambda_{n+1})\cap\left\{\operatorname{span}\bigcup_{j=0}^{n}\mathscr{L}_j\right\}^{\perp}$. We put

$$D(H_{n+1}):=D(H_n)\dotplus\mathscr{L}_{n+1},$$

$$H_{n+1}(f+g):=H_nf+\lambda_{n+1}g\qquad\forall f\in D(H_n)\ \forall g\in\mathscr{L}_{n+1}.$$

Obviously along with H_n the operator H_{n+1} is also C-real, closed and symmetric, and $H_j\subset H_n$ for all j, $n\in N$ with $j\leqq n$. We put

$$D(H_\infty):=\bigcup_{n\in N}D(H_n).$$

For each $n\in N$ and $h\in D(H_n)$ we put

$$H_\infty h=H_nh.$$

Then obviously H_∞ is a symmetric extension of H and each $\lambda\in\mathscr{E}$ is an eigenvalue of H_∞ with multiplicity larger than or equal to $p(\lambda)$. Moreover H_∞ is C-real since all operators H_n, $n\in N$, are C-real. Thus H_∞ has a C-real self-adjoint extension (cf. [5, Theorem 8.15]). □

For the case of infinite deficiency index we may extend the above result to operators which are not C-real.

Theorem 2.4 *Let $\mathscr{E}$ be a countable set of real regular points of the closed symmetric operator H with infinite deficiency index. Let p: $\mathscr{E}\to\mathbb{N}\cup\{\aleph_0\}$ be any mapping. Then H has a self-adjoint extension $\tilde{H}$ such that each $\lambda\in\mathscr{E}$ is an eigenvalue of $\tilde{H}$ with multiplicity at least equal to $p(\lambda)$.*

Proof. We choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathscr{E}$ such that each $\lambda\in\mathscr{E}$ occurs at least $p(\lambda)$ times in this sequence. We define, by induction, orthonormal sequences $\{g_n\}_{n\in\mathbb{N}}$, $\{f_n^+\}_{n\in\mathbb{N}}$ and $\{f_n^-\}_{n\in\mathbb{N}}$ as follows:

$n=1$: We choose normalized elements $g_1 \in \ker(H^* - \lambda_1)$ and $f_1^+ \in R(H+i)^\perp$ and $f_1^- \in R(H-i)^\perp$ with

$$\langle f_1^+, g_1 \rangle = 0 = \langle f_1^-, g_1 \rangle.$$

$n \Rightarrow n+1$: We choose a normalized element $g_{n+1} \in \ker(H^* - \lambda_{n+1})$ with

$$\langle g_{n+1}, g_j \rangle = 0 = \langle g_{n+1}, f_j^+ \rangle = \langle g_{n+1}, f_j^- \rangle \qquad \forall\, 1 \leqq j \leqq n.$$

In addition choose normalized elements $f_{n+1}^+ \in R(H+i)^\perp$ and $f_{n+1}^- \in R(H-i)^\perp$ with

$$\langle f_{n+1}^+, f_j^+ \rangle = 0 = \langle f_{n+1}^-, f_j^- \rangle \qquad \forall\, 1 \leqq j \leqq n,$$
$$\langle f_{n+1}^+, g_j \rangle = 0 = \langle f_{n+1}^-, g_j \rangle \qquad \forall\, 1 \leqq j \leqq n+1.$$

Then $\{g_n\}_{n\in\mathbb{N}}$, $\{f_n^+\}_{n\in\mathbb{N}}$ and $\{f_n^-\}_{n\in\mathbb{N}}$ are orthonormal sequences with the following properties:

(i) $g_n \in \ker(H^* - \lambda_n)\ \forall n \in \mathbb{N}$,
(ii) $f_n^+ \in R(H+i)^\perp$ and $f_n^- \in R(H-i)^\perp\ \forall n \in \mathbb{N}$,
(iii) $\langle g_j, f_n^+ \rangle = \langle g_j, f_n^- \rangle = 0\ \forall j, n \in \mathbb{N}$.

We put

$$D(H_\infty) := D(H) + \operatorname{span}\{g_n : n \in \mathbb{N}\},$$
$$H_\infty h := H^* h \qquad \forall h \in D(H_\infty).$$

Obviously H_∞ is a symmetric extension of H and each $\lambda \in \mathscr{E}$ is an eigenvalue of H_∞ with multiplicity at least equal to $p(\lambda)$. For every $n \in \mathbb{N}$ we have

$$\langle f_n^+, (H_\infty + i) f \rangle = \langle f_n^+, (H+i) f \rangle = 0 \qquad \forall f \in D(H),$$
$$\langle f_n^+, (H_\infty + i) g_j \rangle = \langle f_n^+, (\lambda_j + i) g_j \rangle = 0 \qquad \forall j \in \mathbb{N}.$$

It follows that $f_n^+ \in R(H_\infty + i)^\perp$ for all $n \in \mathbb{N}$. Thus we have

$$\dim R(H_\infty + i)^\perp = \dim R(H+i)^\perp.$$

Analogously it can be shown that

$$\dim R(H_\infty - i)^\perp = \dim R(H-i)^\perp.$$

Thus $\dim R(H_\infty + i)^\perp = \dim R(H_\infty - i)^\perp$ and consequently H_∞ has self-adjoint extensions. □

3 A supplement to von Neumann's theorem

It is a well known result due to J. von Neumann, that every symmetric operator H which commutes with a conjugation C has equal deficiency indices, and therefore allows self-adjoint extensions. The following theorem tells to which extend the converse holds:

Theorem 3.1 (i) *Let H be any closed symmetric operator (not necessarily with a spectral gap) with deficiency indices* (1, 1). *Then there exists a conjugation C commuting with H.*
(ii) *Let $N \in \mathbb{N} \cup \{\aleph_0\}$ with $N \geqq 2$. In every infinite-dimensional separable Hilbert space there exists a semibounded closed symmetric operator H with deficiency indices (N, N), such that H does not commute with any conjugation. (The proof can easily be generalized to non-separable Hilbert spaces and the case where N is any cardinal number less or equal to the cardinality of $\mathbb{R}$.)*

Proof. (i) Let $\hat{H}$ be any self-adjoint extension of H. Without loss of generality we may assume that $\hat{H}$ is the operator of multiplication with some real function F in a Hilbert space $L_2(M, \mu)$. We choose a normalized $g_+ \in R(H+i)^\perp$ and put

$$g_-(x) := (\hat{H}-i)(\hat{H}+i)^{-1} g_+(x) = \frac{F(x)-i}{F(x)+i} g_+(x) \quad \mu\text{-a.e.}$$

For each $f \in D(H)$ we have

$$\langle (H-i)f, g_- \rangle = \int (F(x)+i)\overline{f(x)} \frac{F(x)-i}{F(x)+i} g_+(x)\, \mu(dx) = \langle (H+i)f, g_+ \rangle = 0.$$

Thus $g_- \in R(H-i)^\perp$. We put

$$Cf(x) := \begin{cases} g_+(x)\overline{(g_-(x))}^{-1} \overline{f(x)}, & \text{if } g_-(x) \neq 0, \\ \overline{f(x)}, & \text{otherwise} \end{cases} \quad \mu\text{-a.e.}$$

for all $f \in L_2(M, \mu)$. Obviously C is a conjugation on $L_2(M, \mu)$ which commutes with $\hat{H}$. Since

$$Cg_- = g_+,$$

we have for $f \in D(H)$

$$\langle C(H+i)f, g_- \rangle = \langle Cg_-, (H+i)f \rangle = \langle g_+, (H+i)f \rangle = 0,$$

and therefore

$$C(H+i)f \in \{g_-\}^\perp = R(H-i).$$

Thus there exists a $g \in D(H)$ such that

$$(\hat{H}-i)g = (H-i)g = C(H+i)f = C(\hat{H}+i)f = (\hat{H}-i)Cf,$$

which implies $Cf = g \in D(H)$. Since $C\hat{H} = \hat{H}C$ it follows that C also commutes with H.
(ii)(a) We first consider the case $N = 2$: Without loss of generality for the underlying Hilbert space we choose

$$l_2(M) := \{f: M \to \mathbb{C}: \sum_{n \in M} |f(n)|^2 < \infty\} \quad \text{where} \quad M := \mathbb{N} \cup \{-1, 0\}$$

equipped with the scalar product $\langle f, g\rangle := \sum_{n\in M} \overline{f(n)}\, g(n)$. We define the operator H in $l_2(M)$ as follows:

$$D(H) := \left\{ f\in l_2(M): \sum_{n\in M} 2^{2n}\,|f(n)|^2<\infty, f(-1)=\sum_{n=1}^{\infty} a_n f(n), f(0)=\sum_{n=1}^{\infty} b_n f(n)\right\},$$

$$Hf(n) := \begin{cases} 2^n f(n), & \text{if } n\geqq 1,\\ 0, & \text{otherwise.}\end{cases}$$

The sequences $\{a_n\}$ and $\{b_n\}$ are given by

$$a_2=a_3=a_4=b_1=b_3=1, \quad a_1=b_2=0, \quad b_4=i,$$

$$a_n=\begin{cases} n, & n \text{ even},\\ 0, & n \text{ odd},\end{cases} \quad b_n=\begin{cases} 0, & n \text{ even},\\ n, & n \text{ odd},\end{cases} \quad \text{for } n\geqq 5.$$

We have

$$|f(-1)|^2=\left|\sum_{n=1}^{\infty} a_n 2^{-n} 2^n f(n)\right|^2 \leqq \sum_{n=1}^{\infty} a_n^2 2^{-2n} \sum_{n=1}^{\infty} 2^{2n}\,|f(n)|^2 = c\,\|Hf\|^2$$

for some finite constant c and for all $f\in D(H)$. Together with an analogous computation for $f(0)$ and the trivial fact that $\sum_{n=1}^{\infty} |f(n)|^2 \leqq \|Hf\|^2$ this yields that there exists a constant $b>0$ such that

$$\|Hf\| \geqq b\,\|f\| \qquad \forall f\in D(H). \tag{1}$$

Next we prove that $D(H)$ is dense in $l_2(M)$. Let $f\in l_2(M)$ with $f(n)=0$ for all $n>N$ for some $N\in M$. For each $k\in\mathbb{N}$ with $2k>\max\{4, N\}$ let

$$f_k(n) := \begin{cases} f(n), & \text{if } n\leqq N,\\ (2k)^{-1}\left(f(-1)-\sum_{n=1}^{N} a_n f(n)\right), & \text{if } n=2k,\\ (2k+1)^{-1}\left(f(0)-\sum_{n=1}^{N} b_n f(n)\right), & \text{if } n=2k+1,\\ 0, & \text{otherwise.}\end{cases}$$

Then obviously $f_k\in D(H)$ and $\|f_k-f\|\to 0$ as $k\to\infty$. Thus $D(H)$ is dense in $l_2(M)$.

By (1), H has a bounded inverse H^{-1} with

$$D(H^{-1})=R(H)=\{f\in l_2(M): f(-1)=f(0)=0\}=:\mathscr{R}.$$

Since $\mathscr{R}$ is closed it follows that H^{-1} and therefore also H is closed. Obviously $\langle Hf, g\rangle=\langle f, Hg\rangle$ for all $f, g\in D(H)$. Thus H is a closed symmetric operator with spectral gap $(-b, b)$ for some constant $b>0$; actually H is semibounded

by b, since it is a restriction of a positive multiplication operator. The above formula for $R(H)$ implies that the deficiency indices of H are equal to 2.

We shall prove by contradiction, that H does not commute with any conjugation. Assume that C is a conjugation on $l_2(M)$ and that $CH=HC$. Then obviously $C\mathscr{R}\subset\mathscr{R}$. For $f\in\mathscr{R}$ and $g\in\mathscr{N}:=\mathscr{R}^\perp$ we have

$$\langle Cg, Hf\rangle=\langle CHf, g\rangle=\langle HCf, g\rangle=0.$$

Thus $C\mathscr{N}\subset\mathscr{N}$ and therefore $C\mathscr{R}=\mathscr{R}$ and $C\mathscr{N}=\mathscr{N}$. This implies that C may be decomposed as

$$C=C_{\mathscr{R}}\oplus C_{\mathscr{N}},$$

where $C_{\mathscr{R}}$ and $C_{\mathscr{N}}$ denote the restrictions of C to $\mathscr{R}$ and $\mathscr{N}$, respectively. Obviously $C_{\mathscr{R}}$ and $C_{\mathscr{N}}$ are conjugations on $\mathscr{R}$ and $\mathscr{N}$, respectively. Consequently there exist constants c_{jk} $(1\leqq j, k\leqq 2)$ such that

$$\begin{pmatrix} C_{\mathscr{N}}f(-1)\\ C_{\mathscr{N}}f(0)\end{pmatrix}=\begin{pmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}\begin{pmatrix}\overline{f(-1)}\\ \overline{f(0)}\end{pmatrix}\quad \forall f\in\mathscr{N}. \tag{2}$$

Apparently we have

$$H^{-1}C=H^{-1}CHH^{-1}=H^{-1}HCH^{-1}=CH^{-1}.$$

With $A:=P_{\mathscr{R}}H^{-1}$ and $B:=P_{\mathscr{N}}H^{-1}$, where $P_{\mathscr{L}}$ denotes the orthogonal projection in $l_2(M)$ onto $\mathscr{L}$, this implies

$$C_{\mathscr{R}}A+C_{\mathscr{N}}B=C(A+B)=CH^{-1}=H^{-1}C=(A+B)C=AC_{\mathscr{R}}+BC_{\mathscr{R}},$$

and therefore

$$C_{\mathscr{R}}A=AC_{\mathscr{R}}\quad\text{and}\quad C_{\mathscr{N}}B=BC_{\mathscr{R}}. \tag{3}$$

Note that

$$Af(n)=2^{-n}f(n)\qquad \forall n\in M\ \forall f\in\mathscr{R}, \tag{4}$$

and

$$Bf(k)=\begin{cases}\sum\limits_{n=1}^{\infty}a_n2^{-n}f(n), & \text{if } k=-1,\\ \sum\limits_{n=1}^{\infty}b_n2^{-n}f(n), & \text{if } k=0,\\ 0, & \text{otherwise}\end{cases}\qquad \forall f\in\mathscr{N}. \tag{5}$$

Let C' be the conjugation on $\mathscr{R}$ given by

$$C'f(n)=\overline{f(n)}\qquad \forall n\in\mathbb{N}\ \forall f\in\mathscr{R}.$$

It immediately follows from (3) and (4) that $C_{\mathscr{R}}C'$ is a unitary operator on $\mathscr{R}$ which commutes with the multiplication operator A. Thus $C_{\mathscr{R}}C'$ is given by

$$C_{\mathscr{R}}C'\,g(n)=\theta(n)\,g(n) \qquad \forall\, n\in\mathbb{N}\ \forall\, g\in\mathscr{R},$$

where $\theta\colon \mathbb{N}\to S_1=\{c\in\mathbb{C}\colon |c|=1\}$. Therefore

$$C_{\mathscr{R}}f(n)=\theta(n)\overline{f(n)} \qquad \forall\, n\in\mathbb{N}\ \forall\, f\in\mathscr{R}. \tag{6}$$

Inserting $2^k e_k$ into (2), (3), (5) and (6) we get

$$c_{11}\overline{a_k}+c_{12}\overline{b_k}=a_k\,\theta(k) \quad \text{and} \quad c_{21}\overline{a_k}+c_{22}\overline{b_k}=b_k\,\theta(k) \qquad \text{for all } k\in\mathbb{N}.$$

Since $a_1=0=b_2$ and $b_1=a_2\neq 0$ it follows that $c_{12}=c_{21}$. Thus for $k=3$, 4 we have

$$\theta(k)=\frac{c_{11}\overline{a_k}}{a_k}=\frac{c_{22}\overline{b_k}}{b_k} \quad \text{and therefore} \quad \frac{c_{11}}{c_{22}}=\frac{a_k\overline{b_k}}{\overline{a_k}\,b_k}.$$

Thus, by our special choice of a_k and b_k $(k=3, 4)$, we have $\dfrac{c_{11}}{c_{22}}=1$ $(k=3)$ and $\dfrac{c_{11}}{c_{22}}=-1$ $(k=4)$ which is a contradiction, and therefore the assumption that H commutes with some conjugation is false.

(b) Let now $N\geqq 3$. For $j=3, 4, \ldots, N$ we define H_j in $\mathscr{H}_j=\ell_2(\mathbb{N}_0)$ by

$$D(H_j)=\{f\in\ell_2(\mathbb{N}_0)\colon \sum_{n\in\mathbb{N}_0}2^{2n}|f(n)|^2<\infty, f(0)=\sum_{n\in\mathbb{N}_0}f(n)\},$$

$$H_j f(n)=\begin{cases}\left(2^n+\dfrac{1}{j}\right)f(n) & \text{for } n\geqq 1,\\ 0 & \text{for } n=0.\end{cases}$$

Then H_j is a lower semi-bounded symmetric operator in $\ell_2(\mathbb{N}_0)$ with both deficiency indices equal to 1. Obviously

$$\mathscr{R}_j:=D(H_j^{-1})=\{f\in\ell_2(\mathbb{N}_0)\colon f(0)=0\} \qquad \text{for every } j\geqq 3.$$

We consider the operator

$$H_0:=H\oplus\left\{\bigoplus_{j=3}^{N}H_j\right\} \quad \text{in} \quad \mathscr{H}_0:=\ell_2(M)\oplus\left\{\bigoplus_{j=3}^{N}\ell_2(\mathbb{N}_0)\right\}$$

(with the operator H from part (a)) and assume that H_0 is C_0-real for some conjugation C_0 in $\mathscr{H}_0$. As in part (a) it follows that

$$\mathscr{R}_0:=D(H_0^{-1})=\mathscr{R}\oplus\left\{\bigoplus_{j=3}^{N}\mathscr{R}_j\right\}$$

reduces C_0, i.e. we have $C_0 = C_{\mathscr{R}_0} \oplus C_{\mathscr{N}_0}$ with

$$\mathscr{N}_0 := \mathscr{R}_0^{\perp} = \mathscr{N} \oplus \left\{ \bigoplus_{j=3}^{\infty} \mathscr{N}_j \right\}, \quad \text{where} \quad \mathscr{N}_j = \mathscr{R}_j^{\perp}.$$

Hence the operator

$$A_0 := A \oplus \left\{ \bigoplus_{j=3}^{N} A_j \right\} \quad \text{with} \quad A_j = P_{\mathscr{R}_j} H_j^{-1} \quad \text{for } j \geqq 3$$

commutes with $C_{\mathscr{R}_0}$. Since the self-adjoint operators A and A_j $(j \geqq 3)$ have mutually disjoint eigenvalues, it follows that $C_{\mathscr{R}_0}$ leaves their coordinates invariant. Looking at the equation $C_0 H_0 f = H_0 C_0 f$ for $f \in \mathscr{H}$ or $\mathscr{H}_j$ $(j \geqq 3)$ it follows easily, that C_0 is also reduced by $\mathscr{N}$ and $\mathscr{N}_j$ $(j \geqq 3)$. Therefore

$$C_0 = C \oplus \left\{ \bigoplus_{j=3}^{N} C_j \right\}$$

with conjugations C and C_j in $\mathscr{H}$ and $\mathscr{H}_j$, respectively, where C commutes with H. This contradicts part (a) of the proof. □

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Eigenvalue stability bounds via weighted Sobolev spaces

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1 Introduction

In 1983 and 1984 the author obtained some Schrödinger operator estimates of the form

$$\|X(-\Delta+V)^{-1}\| \leqq c$$

where V and X are non-negative potentials on R^N; see [5, 6]. Among other applications, these estimates were used to obtain norm bounds on the difference of two resolvent operators.

More recently Evans et al. [8] proved a similar bound for more general second order elliptic operators on a region in R^N. They also obtained an entirely new bound of the type

$$\int_{\mathrm{R}^N} \omega^2 |\nabla f|^2 d^N x \leqq c(\lambda) \int_{\mathrm{R}^N} |f|^2 d^N x$$

for a suitable ω and for an eigenfunction f of H with eigenvalue λ. The methods in these two papers are quite different.

In this paper we repeat the argument of [8] in a Riemannian manifold context. If f is an eigenfunction, then our estimates show better dependence upon the eigenvalue. However we are able to obtain bounds which are valid for any function in the domain of the relevant operator, and not just for eigenfunctions. This enables us to obtain local regularity properties of a function and of its first derivatives whenever it lies in the domain of a second order elliptic operator with measurable second order coefficients. Such results cannot be obtained by Sobolev embedding theorems, since the domain of such an operator is not equal to any of the Sobolev spaces.

We also give a new application of the bounds, to the problem of estimating the domain dependence of the spectrum of a variable coefficient second order elliptic operator on a region in R^N subject to Dirichlet boundary conditions. We obtain explicit upper bounds on the changes in the eigenvalues of such an operator when one makes a slight reduction in the size of the region. These estimates are particularly valuable for regions with fractal boundaries, when other methods are inapplicable.

We now specify the context in which our results are proved. Our assumptions are very similar to those of [5, 6, 8] but a precise comparison is not given; in applications any of the hypotheses are equally easy to verify. Let M be a possibly incomplete N-dimensional Riemannian manifold whose metric is locally bounded above and below, but which is not necessarily smooth. Let σ be a positive function on M such that $\sigma^{\pm 1}$ are both locally bounded. We define the weighted L^p spaces on M for $1 \leqq p < \infty$ by

$$\| f \|_p^p := \int_M |f|^p \sigma^2$$

where we have integrated with respect to the Riemannian volume element. We note that $C_c^\infty(M)$ is dense in each of these spaces. Let V be a locally L^1 potential on M such that

$$\int_M \{|\nabla f|^2 - (1+c_4) V_- |f|^2 + a c_3 |f|^2\} \sigma^2 \geqq 0 \tag{1}$$

for all $f \in C_c^\infty(M)$ and some constants $c_3 \geqq 0$ and $c_4 > 0$. If V is non-negative then we may take $c_3 = 0$ and $c_4 = \infty$ in what follows. These assumptions are sufficient by [4; 7, Theorem 1.8.2] to imply that the quadratic form defined on $C_c^\infty(M)$ by

$$Q(f) := \int_M \{|\nabla f|^2 + V|f|^2\} \sigma^2 \tag{2}$$

is semibounded and closable. By adding a suitable constant to V we assume in addition that Q is non-negative. We let H denote the non-negative self-adjoint operator associated with the closure of Q in the standard manner. We have the formal identity

$$Hf := -\sigma^{-2} \nabla \cdot (\sigma^2 \nabla f) + Vf$$

but do not assume that $C_c^\infty(M)$ is contained in the domain of H.

Our weighted estimates are expressed in terms of two functions on M, which are required to satisfy certain inequalities. We express these in general terms so as to capture the largest possible range of applications. We first suppose that $a \geqq 0$ and that X is a locally L^1 function on M which satisfies

$$0 \leqq X \leqq H + a \tag{3}$$

in the sense of quadratic forms, or equivalently

$$0 \leqq \int_M X |f|^2 \sigma^2 \leqq Q(f) + a \| f \|_2^2 := Q_a(f) \tag{4}$$

for all $f \in C_c^\infty(M)$. We suppose that ω is a positive, continuous and differentiable function on M which satisfies

$$0 \leqq W := \omega^{-2} |\nabla \omega|^2 \leqq c_1 (H + a) \tag{5}$$

in the sense of quadratic forms for some $0<c_1<1$. Note that this condition is not affected by multiplying ω by a constant. We assume that

$$\omega^2 \leqq c_2^2(H+a) \tag{6}$$

for some $c_2>0$. We also initially make the regularity assumptions

$$0 \leqq \omega^2 \leqq k_1 \tag{7}$$

and

$$0 \leqq |\nabla \omega|^2 \leqq k_2 X \tag{8}$$

for some constants k_1 and k_2. However we shall be interested in obtaining bounds which do not depend upon k_1 or k_2, and which persist under certain limiting procedures as they tend to infinity. Throughout this paper c_i will denote constants which depend only upon c_1, c_2, c_3 and c_4.

2 The basic estimates

The inequalities which we shall prove may be regarded as improvements upon Hardy's inequality, obtained under somewhat stronger assumptions on the functions involved. The proofs of the inequalities are not hard and their importance depends upon the fact that the basic assumptions are well adapted to a number of different applications.

Lemma 1 *The inequality*

$$\int_M |\nabla f|^2 \sigma^2 \leqq (1+c_3 c_4^{-1}+c_4^{-1}) Q_a(f)$$

holds for all $f \in \mathrm{Dom}(H^{1/2})$.

Proof. If $f \in C_c^\infty(M)$ then

$$\begin{aligned} c_4 \int |\nabla f|^2 \sigma^2 &\leqq \int \{(1+c_4)|\nabla f|^2 - (1+c_4) V_- |f|^2 + a c_3 |f|^2\} \sigma^2 \\ &\leqq (1+c_4) Q(f) + a c_3 \|f\|_2^2 \\ &\leqq (1+c_3+c_4) Q_a(f). \end{aligned}$$

This implies the stated inequality for all $f \in C_c^\infty(M)$, and the general result follows by approximation.

Lemma 2 *If* $f \in \mathrm{Dom}(H^{1/2})$ *then* $g=\omega f \in \mathrm{Dom}(H^{1/2})$.

Proof. If $f \in C_c^\infty(M)$ then

$$\begin{aligned} Q(g) &= \int \{|\nabla(\omega f)|^2 + V|\omega f|^2\}\sigma^2 \\ &= \int \{|\omega \nabla f + f\nabla\omega|^2 + V\omega^2|f|^2\}\sigma^2 \\ &\leqq \int \{2\omega^2|\nabla f|^2 + 2|f|^2|\nabla\omega|^2 + V\omega^2|f|^2\}\sigma^2 \\ &\leqq 2k_1 \int |\nabla f|^2\sigma^2 + 2k_2 \int X|f|^2\sigma^2 + k_1 \int V_+|f|^2\sigma^2 \\ &\quad + 2c_4^{-1}k_1 \int \{|\nabla f|^2 - (1+c_4)V_-|f|^2 + ac_3|f|^2\}\sigma^2 \\ &\leqq 2k_1(1+c_4^{-1}) \int \{|\nabla f|^2 + V|f|^2\}\sigma^2 \\ &\quad + 2k_2 Q_a(f) + 2c_4^{-1}k_1 ac_3 \int |f|^2\sigma^2 \\ &\leqq k_3 Q_a(f). \end{aligned}$$

The result again follows for $f \in \mathrm{Dom}(H^{1/2})$ by approximation.

Lemma 3 *If $f \in \mathrm{Dom}(H^{3/2})$ then $g = \omega f$ lies in $\mathrm{Dom}(H^{1/2})$ and*

$$Q_a(g) \leqq (1-c_1)^{-1} c_2^2 \|(H+a)^{3/2} f\|_2 \|(H+a)^{1/2} f\|_2.$$

Proof. If $f \in \mathrm{Dom}(H^{3/2})$ then $\omega^2 f \in \mathrm{Dom}(H^{1/2})$ by two applications of Lemma 2. Moreover

$$\begin{aligned} (9) \quad \langle (H+a)f, \omega^2 f\rangle &= \langle \omega(H+a)^{-1/2}.(H+a)^{3/2}f, \omega(H+a)^{-1/2}.(H+a)^{1/2}f\rangle \\ &\leqq c_2^2 \|(H+a)^{3/2} f\|_2 \|(H+a)^{1/2} f\|_2. \end{aligned}$$

Also

$$\begin{aligned} (10) \quad \langle (H+a)f, \omega^2 f\rangle &= \langle H^{1/2}f, H^{1/2}(\omega^2 f)\rangle + a\|g\|_2^2 \\ &= \int \{\nabla f \cdot \nabla(\omega^2 f) + Vf(\omega^2 f)\}\sigma^2 + a\|g\|_2^2 \\ &= \int \{\nabla(\omega^{-1}g)\cdot\nabla(\omega g) + V|g|^2\}\sigma^2 + a\|g\|_2^2 \\ &= \int \{|\nabla g|^2 - \omega^{-2}|\nabla\omega|^2|g|^2 + V|g|^2\}\sigma^2 + a\|g\|_2^2 \\ &= Q_a(g) - \langle Wg, g\rangle \\ &\geqq (1-c_1)Q_a(g) \end{aligned}$$

by the application of (5). The lemma follows by combining these bounds.

Lemma 4 *We have*

$$\int X\omega^2|f|^2\sigma^2 \leqq (1-c_1)^{-1}c_2^2 \|(H+a)^{3/2}f\|_2 \|(H+a)^{1/2}f\|_2$$

for all $f \in \mathrm{Dom}(H^{3/2})$. In particular if $Hf = \lambda f$ then

$$\int X\omega^2|f|^2\sigma^2 \leqq (1-c_1)^{-1}c_2^2(\lambda+a)^2 \|f\|_2^2.$$

Proof. We combine Lemma 3 with (3).

Lemma 5 *If $f \in \mathrm{Dom}(H^{3/2})$ then*

$$\begin{aligned} \int \omega^2|\nabla f|^2\sigma^2 &\leqq c_6 Q_a(\omega f) \\ &\leqq c_7 \|(H+a)^{3/2}f\|_2 \|(H+a)^{1/2}f\|_2. \end{aligned}$$

In particular if $Hf=\lambda f$ *then*

$$\int \omega^2 |\nabla f|^2 \sigma^2 \leqq c_5(\lambda+a)^2 \| f \|^2.$$

Proof. Putting $g=\omega f$ we have

$$\begin{aligned}\int \omega^2 |\nabla f|^2 \sigma^2 &= \int \omega^2 |\omega^{-1}\nabla g - \omega^{-2} g \nabla \omega|^2 \sigma^2 \\ &\leqq 2\int \{|\nabla g|^2 + \omega^{-2}|\nabla \omega|^2 |g|^2\} \sigma^2 \\ &= 2\int \{|\nabla g|^2 + W|g|^2\} \sigma^2 \\ &\leqq 2(1 + c_3 c_4^{-1} + c_4^{-1} + c_1) Q_a(g)\end{aligned}$$

by Lemma 1 and (5). The proof is completed by using Lemma 3.

The main weakness of Lemmas 4 and 5 is that they depend upon the regularity assumptions (7) and (8) via Lemma 2. We now replace these assumptions by the following.

Hypothesis. For any $u>3$ there exists a constant $k_u<\infty$ such that

$$|\nabla \omega(x)|^2 \leqq k_u X(x)$$

for all $x\in M$ satisfying $\omega(x)\leqq u$.

Given a positive function ω on M and a constant $u>3$ we define the function ω_u on M by

$$\omega_u(x) := f_u(\omega(x)) \tag{11}$$

where $f_u\colon (0,\infty)\to(0,\infty)$ is given by

$$f_u(s) := \begin{cases} s & \text{if } 0<s<u-2 \\ u-1-(s-u)^2/4 & \text{if } u-2\leqq s\leqq u \\ u-1 & \text{if } u<s<\infty. \end{cases}$$

Lemma 6 *If* ω *satisfies* (5), (6) *and the hypothesis then* ω_u *satisfies the conditions* (5), (6), (7) *and* (8), *with constants* c_1 *and* c_2 *which are the same as those for* ω.

Proof. The inequality $f_u(s)\leqq s$ implies that $\omega_u(x)^2 \leqq \omega(x)^2$ for all $x\in M$, while the inequality

$$sf'(s)/f(s)\leqq 1$$

valid for all $s>0$ implies that

$$\omega_u^{-2}|\nabla \omega_u|^2 \leqq \omega^{-2}|\nabla \omega|^2$$

for all $x\in M$. It follows from the hypothesis that the conditions (7) and (8) are valid.

Our first main result improves upon [6] and [8] and may be regarded as a strengthening of the Hardy inequality (4).

Theorem 7 *The conclusions of Lemmas* 3, 4 *and* 5 *are still valid for weights* ω *which satisfy* (5), (6) *and the above hypothesis.*

Proof. We use Lemmas 3, 4 and 5 to obtain the required bounds for the weights ω_u and then let $u \to \infty$.

Theorem 7 can be improved in two ways. The simplest involves strengthening one of the hypotheses.

Theorem 8 *Let ω satisfy* (5), *the hypothesis and the quadratic form inequality*

$$\omega^4 \leqq c_2^4(H+a). \tag{12}$$

Then the following bounds hold for all $f \in \mathrm{Dom}(H)$:

$$\int_M X\omega^2 |f|^2 \sigma^2 \leqq (1-c_1)^{-1} c_2^2 \|(H+a)f\|_2 \|(H+a)^{1/2} f\|_2$$

$$\int_M \omega^2 |\nabla f|^2 \sigma^2 \leqq c_8 \|(H+a)f\|_2 \|(H+a)^{1/2} f\|_2.$$

Proof. We need only replace (9) in the above calculations by

$$\begin{aligned} \langle (H+a)f, \omega^2 f\rangle &\leqq \|(H+a)f\|_2 \|\omega^2 f\|_2 \\ &\leqq \|(H+a)f\|_2 c_2^2 \|(H+a)^{1/2} f\|_2. \end{aligned}$$

The other method of improving Theorem 7 is by an iterative procedure. It does not necessitate the extra hypothesis (12) but leads to a weaker conclusion.

Theorem 9 *Let ω satisfy* (5), (6) *and the hypothesis. Then one has*

$$\int_M X\omega^2 |f|^2 \sigma^2 \leqq c_2^2 (1-c_1)^{-2} \|(H+a)f\|_2^2$$

for all $f \in \mathrm{Dom}(H)$.

Proof. We assume that $a > 0$; the case $a = 0$ is handled by taking limits. Lemma 3 and (6) state that

$$\omega^2 \leqq c_2^2 (H+a) \tag{13}$$

$$Q_a(\omega f) \leqq (1-c_1)^{-1} c_2^2 a^{-1} \|(H+a)^{3/2} f\|_2^2 \tag{14}$$

for all $f \in \mathrm{Dom}(H^{3/2})$. Suppose inductively that we have

$$Q_a(\omega f) \leqq c_2^2 a^{-\alpha(n)} (1-c_1)^{-\beta(n)} \|(H+a)^{\gamma(n)/2} f\|_2^2 \tag{15}$$

for all $f \in \mathrm{Dom}(H^{3/2})$, where $2 \leqq \gamma(n) \leqq 3$, $1 \leqq \beta(n) \leqq 2$, $0 \leqq \alpha(n) \leqq 2$. Equations (13) and (15) together imply the quadratic form inequality

$$\omega^4 \leqq c_2^4 a^{-\alpha(n)} (1-c_1)^{-\beta(n)} (H+a)^{\gamma(n)}.$$

We then deduce by Theorem 4.20 of [4] that

$$\omega^{8/\gamma(n)} \leqq c_2^{8/\gamma(n)} a^{-2\alpha(n)/\gamma(n)} (1-c_1)^{-2\beta(n)/\gamma(n)} (H+a)^2$$

or

$$\|\omega^{4/\gamma(n)} (H+a)^{-1}\| \leqq c_2^{4/\gamma(n)} a^{-\alpha(n)/\gamma(n)} (1-c_1)^{-\beta(n)/\gamma(n)}.$$

In a similar manner (13) yields

$$\|\omega^{2-4/\gamma(n)}(H+a)^{-(1-2/\gamma(n))}\| \leqq c_2^{2-4/\gamma(n)}.$$

Repeating the proof of Lemma 3 we have

$$\begin{aligned} Q_a(\omega f) &\leqq (1-c_1)^{-1}\langle (H+a)f, \omega^2 f\rangle \\ &= (1-c_1)^{-1}\langle \omega^{2-4/\gamma(n)}(H+a)f, \omega^{4/\gamma(n)}f\rangle \\ &\leqq c_2^2(1-c_1)^{-1-\beta(n)/\gamma(n)} a^{-\alpha(n)/\gamma(n)-1+2/\gamma(n)} \|(H+a)^{2-2/\gamma(n)}f\|_2^2. \end{aligned}$$

This implies a bound of the same type as (15) but with

$$\begin{aligned} \alpha(n+1) &:= 1+(\alpha(n)-2)/\gamma(n) \\ \beta(n+1) &:= 1+\beta(n)/\gamma(n) \\ \gamma(n+1) &:= 4-4/\gamma(n). \end{aligned}$$

The solution of these recurrence relations subject to the initial values $\alpha(0)=1$, $\beta(0)=1$ and $\gamma(0)=3$ is

$$\begin{aligned} \alpha(n) &= 2/(n+2) \\ \beta(n) &= 2(n+1)/(n+2) \\ \gamma(n) &= 2(n+3)/(n+2). \end{aligned}$$

It may be seen that

$$\lim_{n\to\infty} \alpha(n)=0, \qquad \lim_{n\to\infty} \beta(n)=2, \qquad \lim_{n\to\infty} \gamma(n)=2$$

and this leads to the inequality

$$Q_a(\omega f) \leqq c_2^2(1-c_1)^{-2}\|(H+a)f\|_2^2 \tag{16}$$

from which the theorem follows.

Theorem 10 *Let ω satisfy* (5), (6) *and the hypothesis. Then*

$$\int_M \omega^2 |\nabla f|^2 \sigma^2 \leqq c_9 \|(H+a)f\|_2^2$$

for all $f \in \mathrm{Dom}(H)$.

Proof. Once again it suffices to prove the inequality for all $f \in \mathrm{Dom}(H^{3/2})$. This is done by combining Lemma 5 with (16).

3 Applications

It may be seen that essentially every variable coefficient, self-adjoint, second order elliptic operator in divergence form can be treated within the framework of the last section. In the rest of this paper we restrict attention to applications of the above theory in which M is a bounded region in Euclidean space.

Example 11 We suppose that V is a non-negative locally L^1 potential on $M \subseteq R^N$, and that

$$H_0 := -\Delta, \qquad H := -\Delta + V$$

subject to Dirichlet boundary conditions. The following consequences of our earlier theorems should be compared with the results of [5, 6]. Although (17) and (18) below are not equivalent they have essentially the same force when applied to an eigenfunction of H.

Theorem 12 *If the potentials, V, Y, Z satisfy*

$$0 \leqq Y \leqq H_0$$
$$0 \leqq Z \leqq V + Y + a$$
$$Z^{-2} |\nabla Z|^2 \leqq 4c_1 (V + Y + a)$$

where $0 < c_1 < 1$, *then we have*

$$\int_M Z(V+Y+a)|f|^2 d^N x \leqq (1-c_1)^{-1} \|(H+a)^{3/2} f\|_2 \|(H+a)^{1/2} f\|_2 \tag{17}$$

for all $f \in \mathrm{Dom}(H^{3/2})$, *and*

$$\int_M Z(V+Y+a)|f|^2 d^N x \leqq (1-c_1)^{-2} \|(H+a) f\|_2^2 \tag{18}$$

for all $f \in \mathrm{Dom}(H)$.

Proof. The bound (17) follows from Lemma 4 upon putting $\omega = Z^{1/2}$, $X = V + Y + a$ and $c_2 = 1$. The bound (18) follows in a similar way from Theorem 9, and coincides exactly with Theorem 6 of [6]. Further bounds of this type in the case $Z = V$ and $Y = 0$ may be found in [5].

Example 13 Let M be a bounded region in R^N, and put

$$d(x) := \min\{|x-y| : y \notin M\}.$$

We suppose that M is regular in the sense that there exists a constant c with $0 < c \leqq \frac{1}{4}$ such that

$$c \int_M d^{-2} |f|^2 d^N x \leqq \int_M (|\bar{\nabla} f|^2 + a|f|^2) d^N x \tag{19}$$

for all $f \in C_c^\infty(M)$, where $\bar{\nabla}$ is the Euclidean gradient operator. There are many different criteria for the existence of such a Hardy inequality, and the size of the constant c depends upon the irregularity of the boundary [2; 6; 7, Theorem 1.5.7].

Now consider the Riemannian metric

$$ds^2 := \sum_{i,j=1}^{n} g_{i,j}(x) dx_i dx_j$$

on M, and assume that the inverse matrix $g^{i,j}(x)$ *satisfies*

$$1 \leqq g^{i,j}(x) \leqq \Lambda 1$$

in the sense of spectral theory, for some constant $\Lambda \geqq 1$ and all $x \in M$. Finally put

$$\sigma(x) := \det\{g_{i,j}(x)\}^{-1/4}.$$

Then the identities

$$\int_M |\nabla f|^2 \sigma^2 d\,\mathrm{vol} = \int_M \sum g^{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} d^N x$$

$$\int_M |f|^2 \sigma^2 d\,\mathrm{vol} = \int_M |f|^2 d^N x$$

show that operators on $L^2(M, d^N x)$ of the form

$$Hf := \sum \frac{\partial}{\partial x_i} \left\{ g^{i,j}(x) \frac{\partial f}{dx_j} \right\}$$

fall within the ambit of our general theory.

Theorem 14 *If $0 < \Lambda \alpha^2 < c$ where c satisfies* (19) *then there exist constants c_{10}, c_{11} such that*

$$\int_M d^{-2-2\alpha} |f|^2 d^N x \leqq c_{10} \|(H+a)f\|_2 \|(H+a)^{1/2} f\|_2 \tag{20}$$

$$\int_M d^{-2\alpha} |\bar{\nabla} f|^2 d^N x \leqq c_{11} \|(H+a)f\|_2 \|(H+a)^{1/2} f\|_2 \tag{21}$$

for all $f \in \mathrm{Dom}(H)$.

Proof. We apply Theorem 8 with $X(x) := c d(x)^{-2}$ and $\omega(x) := d(x)^{-\alpha}$. From the inequalities

$$|\nabla d|^2 \leqq \Lambda |\bar{\nabla} d|^2 \leqq \Lambda$$

one sees that condition (5) holds provided $\Lambda \alpha^2 \leqq c_1$ where $0 < c_1 < 1$. This implies $0 < \alpha < \frac{1}{2}$, from which it then follows that (12) holds. The validity of the hypothesis is immediate.

Bounds of the above type were obtained for eigenfunctions by [8], who commented that the value of α obtained was often optimal. If ∂M is smooth and $H = -\Delta$, so that $\Lambda = 1$, then one can show by the methods of this paper that

$$\int_M d^{-\beta} |f|^2 d^N x < \infty$$

for all $f \in \mathrm{Dom}(H)$ and $0 < \beta < 3$. Since it is known that in this case the ground state eigenfunction satisfies $f(x) \sim d(x)$ as $x \to \partial M$ the integral is not finite for $\beta = 3$.

Corollary 15 *Let M be a simply connected region in R^2 and let $H=-\Delta$. Then for any $0<\alpha<\frac{1}{4}$ there exist c_{10}, c_{11} such that* (20) *and* (21) *hold.*

Proof. By [2] or Theorem 1.5.10 of [7] we have $c=1/16$ and can also put $\Lambda=1$.

Example 16 Let $N\geqq 3$ and let M be any bounded region in R^N. Let

$$Hf=-\sum\frac{\partial}{dx_i}\left\{g^{i,j}\frac{\partial f}{dx_j}\right\}$$

subject to Dirichlet boundary conditions, where

$$1\leqq g^{i,j}(x)\leqq\Lambda 1$$

for all $x\in M$. If we fix $u\in M$ then $\{u\}$ has capacity zero so we get the same operator as before by replacing M by $M'=M\setminus\{u\}$. Putting $d(x):=|x-u|$ it is well known that

$$X:=\frac{(N-2)^2}{4d^2}\leqq H.$$

The following theorem restricts the possible local singularities of the gradient of a function in $\mathrm{Dom}(H)$, and appears to be the first result of its type.

Theorem 17 *If*

$$0\leqq\alpha<\frac{N-2}{2\Lambda^{1/2}},\qquad 0\leqq\alpha\leqq 1 \tag{22}$$

then there exist constants c_{13} *and* c_{14} *such that*

$$\int_M |u-x|^{-2\alpha}|\nabla f|^2 d^N x\leqq c_{13}\|(H+a)f\|_2^2$$

and

$$\int_M |u-x|^{-2-2\alpha}|f|^2 d^N x\leqq c_{14}\|(H+a)f\|_2^2$$

for all $f\in\mathrm{Dom}(H)$ and all $u\in M$.

Proof. We put $\omega(x):=d(x)^{-\alpha}$ and see that (5) holds for some c_1 satisfying $0<c_1<1$ provided

$$\Lambda\alpha^2<(N-2)^2/4$$

while (6) holds if $0\leqq\alpha\leqq 1$. We may now apply Theorem 9.

We deduce from the above a weak principle of equidistribution of energy.

Corollary 18 *Let α satisfy* (22), *let $Hf=\lambda f$ and let $B(u, \varepsilon)$ be the ball with centre u and radius ε. Then there exists a constant $c<\infty$ such that*

$$\int_{B(u,\varepsilon)} |\nabla f|^2 d^N x \leqq c\varepsilon^{2\alpha}\lambda^2 \|f\|_2^2.$$

Note 19 A version of the above two results exists subject to Neumann boundary conditions for regions with suitably regular boundaries.

The above theorems are most valuable when dealing with an operator H with measurable second order coefficients, so that the application of the standard Sobolev embedding theorems [1] is not possible. However we can guage the sharpness of the results by comparing them with known results in the case where $H=-\Delta$. In this case if $N=3$ then $f\in\mathrm{Dom}(H)$ implies that $f\in L^\infty$. Thus

$$\int_M |u-x|^{-\beta}|f(x)|^2 d^N x<\infty \tag{23}$$

for all $0<\beta<3$. Clearly no such bound can be valid for $\beta=3$. Thus Theorem 17 captures exactly what is possible for this example. Similarly if $N=4$ then $f\in\mathrm{Dom}(H)$ implies that $f\in L^p$ for all $2<p<\infty$ so (23) holds for all $\beta<4$. Once again this is exactly what we get from Theorem 17. If $N\geqq 5$ then $f\in\mathrm{Dom}(H)$ implies $f\in L^{2N/(2N-4)}$ by [1, p. 97] from which we easily see that (23) holds provided $\beta<4$. This result is of course known [10, p. 172; 11].

4 Approximation of eigenvalues

If M is a bounded region in R^N then the operator $H=-\Delta$ subject to Dirichlet boundary conditions has compact resolvent, without any restrictions on the regularity of the boundary. The finite element method enables one to compute the eigenvalues of H with error bounds when M is a polyhedral region, and can be extended to the case where the boundary is piecewise smooth [3, 9]. The reliable numerical evaluation of the eigenvalues is however not easy when the boundary is very irregular.

One strategy is to construct suitable polyhedral regions M^1 and M^2 such that $M^1\subseteq M\subseteq M^2$, so that one has $\lambda_n^1\geqq\lambda_n\geqq\lambda_n^2$ for all $n\geqq 1$, in an abvious notation. For regions with irregular or fractal boundaries it is often impractical to obtain good lower bounds on λ_n by this method. In the region of Fig. 1, given in polar coordinates by

$$0\leqq r(\theta)\leqq 2+\cos(50\theta)$$

the inner approximating disc yields good estimates of the first few eigenvalues while the outer approximating disc yields very poor estimates of the first few eigenvalues.

If M has an irregular or fractal boundary the following method of approximating the eigenvalues can also be used. Given $\varepsilon>0$ let $\tilde{M}$ be a polyhedral region such that $M^\varepsilon\subseteq\tilde{M}\subseteq M$, where

$$M^\varepsilon:=\{x\in M:\mathrm{dist}(x,\partial M)>\varepsilon\}.$$

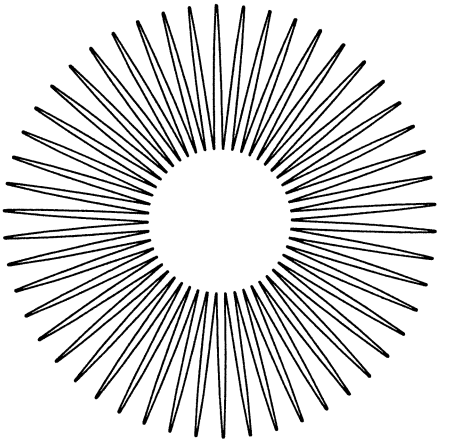


Fig. 1

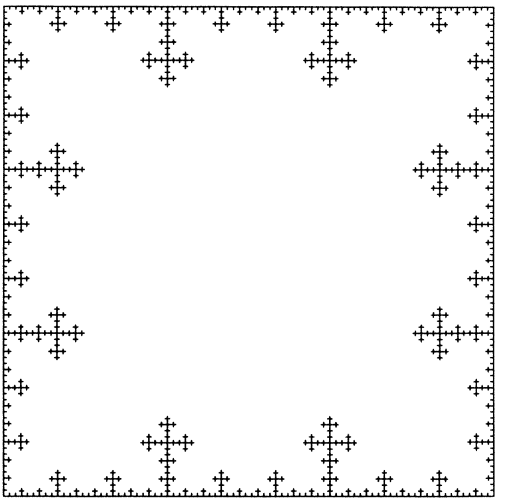


Fig. 2

Then the eigenvalues of these regions satisfy $\lambda_n^\varepsilon \geqq \tilde{\lambda}_n \geqq \lambda_n$. Taking the attitude that the eigenvalues $\tilde{\lambda}_n$ are computable by standard methods, we need only obtain a quantitative upper bound on the difference $\lambda_n^\varepsilon - \lambda_n$ as $\varepsilon \to 0$ in order to approximate λ_n effectively. For some regions, such as that whose boundary is the Koch snowflake, appropriate internal polygonal approximations are given by the very construction of the regions. This is in particular the case in Fig. 2, where the region is a union of similarly oriented squares of diameters 3^{-m}. Note that the interior and exterior Minkowski dimensions of the boundary of this region are not equal.

Although the obvious numerical applications are to operators with piecewise smooth coefficients, we assume below that we are in the more general context of Example 13. We also assume that the operator H on $L^2(M,d^Nx)$ satisfies (20) and (21) for positive constants c_{10}, c_{11} and α, where $0<\alpha<1$. Given M, $\tilde{M}$ and $\varepsilon>0$, we define τ on M by

$$\tau(x):=\begin{cases}0 & \text{if } x\notin\tilde{M}\\ \varepsilon^{-1}d(x,\mathrm{R}^N\backslash\tilde{M}) & \text{if } 0<d(x,\mathrm{R}^N\backslash\tilde{M})<\varepsilon\\ 1 & \text{otherwise}\end{cases}$$

so that $0\leqq\tau\leqq 1$, and $\tau(x)=1$ for all $x\in M^{2\varepsilon}$. Moreover

$$0\leqq|\nabla\tau|^2\leqq\Lambda|\bar{\nabla}\tau|^2\leqq\Lambda\varepsilon^{-2}$$

for all $x\in M$.

Lemma 20 *If $f\in\mathrm{Dom}(H)$ then*

$$|Q(\tau f)-Q(f)|\leqq c_{15}\varepsilon^{2\alpha}\|(H+a)f\|_2\|(H+a)^{1/2}f\|_2.$$

Proof. Putting $S:=M\backslash M^{2\varepsilon}$ we have

$$\begin{aligned}|Q(\tau f)-Q(f)|&=|\int_S\{|f\nabla\tau+\tau\nabla f|^2-|\nabla f|^2\}d^Nx|\\ &\leqq\int_S\{|\nabla f|^2+2|f|^2|\nabla\tau|^2\}d^Nx\\ &\leqq(2\varepsilon)^{2\alpha}\Lambda\int_S d^{-2\alpha}|\bar{\nabla}f|^2d^Nx\\ &\quad+2\Lambda\varepsilon^{-2}(2\varepsilon)^{2+2\alpha}\int_S d^{-2-2\alpha}|f|^2d^Nx\\ &\leqq(c_{11}(2\varepsilon)^{2\alpha}\Lambda+2\Lambda\varepsilon^{-2}(2\varepsilon)^{2+2\alpha}c_{10})\|(H+a)f\|_2\|(H+a)^{1/2}f\|_2.\end{aligned}$$

We now obtain an upper bound on $\tilde{\lambda}_n-\lambda_n$ by applying the minimax criterion to the linear subspaces

$$L_n:=\mathrm{lin}\{\phi_1,\ldots,\phi_n\},\qquad \tilde{L}_n:=\mathrm{lin}\{\tau\phi_1,\ldots,\tau\phi_n\}$$

where $H\phi_i=\lambda_i\phi_i$ and $\|\phi_i\|=1$ for $1\leqq i\leqq n$.

Lemma 21 *The map $f\to\tau f$ from L_n onto $\tilde{L}_n$ is one-one, contractive and nearly isometric for small $\varepsilon>0$. More precisely we have*

$$(1-c_{16}\varepsilon^{1+\alpha}(\lambda_n+a)^{3/4})\|f\|_2\leqq\|\tau f\|_2\leqq\|f\|_2$$

for all $f\in L_n$.

Proof. If $f\in L_n$ then

$$f=\sum_{i=1}^n\alpha_i\phi_i,\qquad Hf=\sum_{i=1}^n\alpha_i\lambda_i\phi_i.$$

Thus

$$(\lambda_1 + a)\| f \|_2 \leqq \|(H+a)f\|_2 \leqq (\lambda_n + a)\| f \|_2 .$$

Hence

$$\begin{aligned} |\|\tau f\|_2 - \| f \|_2|^2 &\leqq \|\tau f - f\|_2^2 \\ &\leqq \int_S |f|^2 d^N x \\ &\leqq (2\varepsilon)^{2+2\alpha} \int_S d^{-2-2\alpha} |f|^2 d^N x \\ &\leqq c_{10}(2\varepsilon)^{2+2\alpha} \|(H+a)f\|_2 \|(H+a)^{1/2} f\|_2 \\ &\leqq C_{16}^2 \varepsilon^{2+2\alpha} (\lambda_n + a)^{3/2} \| f \|_2^2 . \end{aligned}$$

The statement of the lemma follows.

Theorem 22 *For sufficiently small $\varepsilon > 0$, depending upon $n \geqq 1$, we have*

$$0 \leqq \tilde{\lambda}_n - \lambda_n = O(\varepsilon^{2\alpha}).$$

Proof. If we combine Lemma 20, Lemma 21 and the minimax bound

$$\begin{aligned} \tilde{\lambda}_n &\leqq \sup \{Q(g)/\|g\|_2^2 : g \in \tilde{L}_n\} \\ &= \sup \{Q(\tau f)/\|\tau f\|_2^2 : f \in L_n\} \end{aligned}$$

then we obtain

$$\tilde{\lambda}_n \leqq \frac{\lambda_n + c_{15}\varepsilon^{2\alpha}(\lambda_n + a)^{3/2}}{\{1 - c_{16}\varepsilon^{1+\alpha}(\lambda_n + a)\}^{3/4}}$$

for sufficiently small $\varepsilon > 0$, from which the theorem follows.

If $H = -\Delta$ and M is a plane domain with smooth boundary, then we can choose $\tilde{M}$ to be the union of a set of triangles of diameters h, where $\varepsilon = O(h^2)$. We may also choose α to be as close to $1/2$ as we like. Theorem 22 then yields

$$0 \leqq \tilde{\lambda}_n - \lambda_n = O(h^\gamma)$$

for any $0 < \gamma < 2$; the correct value in this instance is $\gamma = 2$.

The advantage of our method is that it applies in much more irregular situations. In two dimensions if M is a bounded, simply connected domain in $\mathbf{R}^2$ then by Corollary 15, without any knowledge of the degree of irregularity of the boundary, we can assert that Theorem 22 is valid for any α in the interval $0 < \alpha < 1/4$. For self-similar regions with fractal boundaries larger values of α may be appropriate, depending upon the particular boundary considered.

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On existence of harmonic maps from a surface to the space of it's normal bundle

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Let Σ be a closed surface, E be the total space of some two-dimensional vector bundle ν: $E \to \Sigma$ and f: $\Sigma \to E$ be the zero section of this bundle. We prove the following statement:

Theorem. *If the Euler characteristic, $\chi(\nu)$, of the two-dimensional vector bundle ψ: $E \to \Sigma$ does not equal zero and E carries a metric of nonpositive curvature, then there exists a harmonic map F: $\Sigma \to E$ homotopic to the zero section.*

The problem of existence of a harmonic map F: $M \to N$ which is homotopic to a given one, f: $M \to N$, was considered in a lot of articles, but mostly for the case in which N was a compact manifold of nonpositive curvature (see, for example, resent results in [LT]). Usually the heat flow method is used. We show that some simple geometric reasons make us sure that in our setting, in which N is not compact, the solution of the heat equation does not go to infinity, so we can deform by the heat flow the zero section to a harmonic map.

Proof of theorem. To find F we consider a heat equation:

$$\begin{cases} \partial f(x,t)/\partial t = \Delta f(x,t) \\ f(x,0) = f(x), x \in \Sigma. \end{cases}$$

According to [H] this equation has a solution defined on the interval $[0, \infty)$, and if for all t the image of the surface Σ under the map f_t: $\Sigma \to E$, where $f_t(x) = f(x, t)$, lies in some fixed compact region, then there exists a limit $F(x) = \lim f(x,t)$, $t \to \infty$ which is a harmonic map. The statement of the theorem will therefore follow from statements:

Lemma 1 *The diameter $d(t)$ of $\Sigma_t = f_t(\Sigma)$ in the metric induced on Σ_t by the inclusion $\Sigma_t \to E$ is a nonincreasing function.*

Lemma 2 *For every t*

$$\Sigma_t \cap \Sigma_0 \neq \emptyset.$$

In fact, this means that for all t Σ_t lies in some fixed compact – the closed $d(0)$-neighborhood of Σ_0.

Proof of Lemma 1 Let us use the main Lemma 2.1 of [H], which states that if $f(x,u)$: $\Sigma \times [0,a] \to E$ is a family of maps of the surface Σ into E, and $f(x,u,t)$ are solutions of the heat equation with initial conditions $f(x,u,0)=f(x,u)$, then

$$D(t)=a\sup_{x\in\Sigma}(\nabla_u f, \nabla_u f)$$

is a nonincreasing function in t, where by $(\cdot,\cdot)$ we denote the Riemannian metric on E and by ∇_u the corresponding covariant derivative. Let y_1 and y_2 be arbitrary two points on Σ_τ and $\gamma(s)$ is a minimal geodesic on Σ_τ connecting them: $y_1=\gamma(0)$, $y_2=\gamma(s_0)$. In some ε-neighborhood B_ε of $\gamma(s)$ on Σ_τ we can define a Fermi-coordinate system (z_1,z_2) and a smooth function ϕ on B_ε with the following properties: $\phi\equiv 1$ on $B_{\varepsilon/2}$, $\phi\equiv 0$ on $\Sigma_\tau\setminus B_\varepsilon$ and $\|\phi\|\leqq 1$. Let us also define the following vector field

$$X(z_1,z_2)=(1-K(z_2)^2)\,\phi(z_1,z_2)\frac{\partial}{\partial z_1}$$

on B_ε, and $X(z_1,z_2)\equiv 0$ on $\Sigma_\tau\setminus B_\varepsilon$, where

$$K\geqq\sup_{y\in\Sigma}|K(y)|$$

and $K(y)$ is the Gaussian curvature of the surface Σ_τ at the point y. The choice of the constant K make us sure that

$$\|X(z_1,z_2)\|\leqq 1.$$

Denote by Φ_u a family of maps generated by this vector field;

$$\frac{\partial}{\partial u}\Phi_u(x)=X(\Phi_u(x)).$$

Obviously $\Phi_u(\Sigma_\tau)\equiv\Sigma_\tau$, therefore for the solution $\tilde f(x,u,t)$ of

$$\begin{cases}\partial\tilde f(x,u,t)/\partial t=\Delta\tilde f(x,u,t), & t\geqq\tau,\\ \tilde f(x,u,0)=\Phi_u(f(x,\tau)), & x\in\Sigma,\end{cases}$$

we have $\tilde f(\cdot,u,t)(\Sigma)=\Sigma_t$ for every u and $t\geqq\tau$. So the curve $\gamma_t(u)=\tilde f(y_1,u,t)$, $0\leqq u\leqq s_0$, lies on Σ_t and connects points $y_1(t)=f(y_1,t)$ and $y_2(t)=f(y_2,t)$. But from the statement of the lemma 2.1 of [H] we see that

$$\|\dot\gamma_t(u)\|=\|\nabla_u\tilde f(y_1,u,t)\|\leqq\|X\|=1.$$

Therefore the length of the curve $\gamma_t(u)$ is not greater than s_0, i.e. the distance between $y_1(t)$ and $y_2(t)$ on Σ_τ is not greater than the distance between y_1 and y_2 on Σ. We have chosen points y_1 and y_2 on Σ arbitrarily, therefore $d(t)\leqq d(\tau)$ for all $t\geqq\tau$. Lemma 1 is proved.

Proof of Lemma 2 If the bundle v is oriented, then it's Euler characteristic $\chi(v)$ equals to the self-intersection number of it's zero section Σ_0, which we can calculate as the intersection number of Σ_0 and some other surface homotopic to Σ_0, say Σ_t. Therefore, in the oriented case, the theorem condition $\chi(v) \neq 0$ implies $\Sigma_0 \cap \Sigma_t \neq \emptyset$. If v is not orientable, we may consider it's two-fold oriented cover $\tilde{v} = (\tilde{E}, \tilde{\Sigma}, \tilde{\pi}\colon \tilde{E} \to \tilde{\Sigma})$. Then $\chi(\tilde{v}) = 2\chi(v) \neq 0$ and $\tilde{\Sigma}_0 \cap (\tilde{\Sigma}_t) \neq \emptyset$, where $(\tilde{\Sigma}_t)$ is the image of $\tilde{\Sigma}_0$ under the heat-flow. But E and $\tilde{E}$ are locally isometric, so $(\tilde{\Sigma}_t)$ are two folded covers of the corresponding surfaces Σ_t. Therefore $\tilde{\Sigma}_0 \cap (\tilde{\Sigma})_t \neq \emptyset$ for all t. This completes the proof of Lemma 2.

As was previously noted, the statement of the theorem follows from statements of Lemmas 1, 2.

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Product formulas for resultants and Chow forms

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1 Introduction

Sparse elimination theory is concerned with the algebraic and combinatorial study of multivariate polynomial systems with prescribed Newton polytopes. Its starting point lies in the large body of work by Gel'fand, Kapranov and Zelevinsky (see [8, 9]) on generalized hypergeometric functions, $\mathscr{A}$-discriminants and regular triangulations of Newton polytopes. In the subsequent work [11] the $\mathscr{A}$-resultant was introduced and shown to be equivalent to the Chow form of the associated toric variety. In the present paper we extend the results in [8] and [11] by establishing product formulas of Poisson type for general Chow forms, for $\mathscr{A}$-resultants, and for sparse mixed resultants.

We consider a system of $k+1$ generic Laurent polynomials

$$f_i := c_{i1}\,\mathbf{t}^{a_{i1}} + c_{i2}\,\mathbf{t}^{a_{i2}} + \ldots + c_{in_i}\,\mathbf{t}^{a_{in_i}} \in K[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}], \tag{1.1}$$

where $i = 0, \ldots, k$, $a_{ij} = (a_{ij1}, \ldots, a_{ijr}) \in \mathbf{Z}^r$, and $\mathbf{t}^{a_{ij}} := t_1^{a_{ij1}} t_2^{a_{ij1}} \ldots t_r^{a_{ijr}}$. The coefficients c_{ij} in (1.1) are algebraically independent indeterminates over $\mathbf{Q}$, and the field K is the algebraic closure of the rational function field $\mathbf{Q}(c_{01}, \ldots, c_{kn_k})$. The *support* of f_i is the set of exponent vectors $\mathscr{A}_i = \{a_{ij}\}_{j=1,\ldots,n_i} \subset \mathbf{Z}^r$, and its *Newton polytope* is the convex hull $P_i = \mathrm{conv}(\mathscr{A}_i)$ in $\mathbf{R}^r$. We assume that $\mathscr{A}_0, \mathscr{A}_1, \ldots, \mathscr{A}_k$ lie in a common k-dimensional affine sublattice of $\mathbf{Z}^r$. This sublattice is identified with $\mathbf{Z}^k$.

Each Laurent polynomial f_i can be viewed as a generic point $(c_{i1} : \ldots : c_{in_i})$ in a complex projective space $\mathbf{P}^{n_i - 1}$. Let $Z_0 = Z_0(\mathscr{A}_0, \ldots, \mathscr{A}_k)$ denote the set of all points (c_{ij}) in the product of projective spaces $\mathbf{P}^{n_0 - 1} \times \ldots \times \mathbf{P}^{n_k - 1}$ such that the system $f_0 = f_1 = \ldots = f_k = 0$ has a solution in the algebraic torus $(\mathbf{C}^*)^k$. Let $Z = Z(\mathscr{A}_0, \ldots, \mathscr{A}_k)$ denote the closure of Z_0 in $\mathbf{P}^{n_0 - 1} \times \ldots \times \mathbf{P}^{n_k - 1}$. In Sect. 2 we will show that Z is a proper irreducible subvariety, defined over $\mathbf{Q}$.

The *mixed resultant* $R_{\mathscr{A}_0, \ldots, \mathscr{A}_k}$ of the system (1.1) is an irreducible polynomial in $\mathbf{Z}[c_{01}, \ldots, c_{kn_k}]$. It is uniquely defined up to a sign factor ± 1 as follows. If $\mathrm{codim}(Z) = 1$ then $R_{\mathscr{A}_0, \ldots, \mathscr{A}_k}$ is the defining polynomial of the hypersurface

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Z. If $\operatorname{codim}(Z)>1$ then $R_{\mathscr{A}_0,\ldots,\mathscr{A}_k}$ is the constant 1. In [8, Proposition 1.3.1] the mixed resultant was called the $(\mathscr{A}_0, \ldots, \mathscr{A}_k)$-*resultant*. It is our objective to find an expression as a *Poisson product*

$$R_{\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_k}(f_0,f_1,\ldots,f_k)=R'\cdot\prod_{\gamma\in V(f_1,\ldots,f_k)} f_0(\gamma), \tag{1.2}$$

where the product is taken over the set of discrete solutions

$$V(f_1,\ldots,f_k):=\{\gamma\in(K^*)^k \mid f_1(\gamma)=\ldots=f_k(\gamma)=0\}. \tag{1.3}$$

The extraneous factor R' is a rational function in $\mathbf{Q}(c_{11},\ldots,c_{kn_k})$, which is to be determined. It is our goal to express R' as a product of lower dimensional resultants depending only on the distinct leading forms of $f_1,\ldots,f_k$. In order to state our formula, we need to introduce a number of definitions and conventions.

We fix an origin O in the lattice $\mathbf{Z}^k$, and we use the following notations throughout.

- We select a normalized volume form $\operatorname{vol}(\cdot)$ for each sublattice of interest, which assigns volume 1 to an elementary simplex of that lattice. The normalized volume $\operatorname{vol}(\cdot)$ for $\mathbf{Z}^k$ is related to the standard volume $\operatorname{Vol}(\cdot)$ on $\mathbf{R}^k$ by $\operatorname{vol}(\cdot)=k!\cdot\operatorname{Vol}(\cdot)$.
- Let S^{k-1} denote the unit $(k-1)$-sphere in $\mathbf{R}^k$. For $w\in S^{k-1}$ let P_i^w denote the face of P_i on which the linear functional $(w,\cdot)$ is maximized. We also refer to $\mathscr{A}_i^w=\mathscr{A}_i\cap P_i^w$.
- Given $w\in S^{k-1}$, then $f_i^w:=\sum_{a\in\mathscr{A}_i^w} c_a\,\mathbf{t}^a$ denotes the *leading form* of the Laurent polynomial f_i with respect to w. Note that f_i^w is a monomial for generic w.
- We abbreviate the pointwise sums

$$\mathscr{A}:=\mathscr{A}_0+\mathscr{A}_1+\ldots+\mathscr{A}_k,$$
$$\bar{\mathscr{A}}:=\quad\mathscr{A}_1+\ldots+\mathscr{A}_k.$$

The corresponding *Minkowski sums* of Newton polytopes are $\bar{P}:=P_1+\ldots+P_k=\operatorname{conv}(\bar{\mathscr{A}})$ and $P:=P_0+\bar{P}=\operatorname{conv}(\mathscr{A})$. Let $L(\mathscr{A})$, $L(\bar{\mathscr{A}})$, etc., denote the sublattice of Z^k affinely spanned by the points of $\mathscr{A}$, $\bar{\mathscr{A}}$, etc.

For each direction $w\in S^{k-1}$ we define a parallel vector $w[\mathscr{A}]=\lambda\cdot w$ as follows. If P^w is not a *facet* (=face of dimension 1) of P then $\lambda=0$ and $w[\mathscr{A}]$ is the zero vector. If P^w is a facet then we choose $w[\mathscr{A}]$ subject to the normalization condition:

$$\langle w[\mathscr{A}],x-a\rangle\cdot\operatorname{vol}_{\mathscr{A}^w}(P^w)=\operatorname{vol}(\operatorname{conv}(P^w\cup\{x\}))\quad\text{for all } a\in P^w, x\in\mathbf{Z}^k. \tag{1.4}$$

Here $\operatorname{vol}_{\mathscr{A}^w}(\cdot)$ denotes the $(k-1)$-dimensional volume form normalized with respect to the lattice $L(\mathscr{A}^w)$. Observe that $\langle w[\mathscr{A}],a\rangle$ does not depend on the choice of $a\in\mathscr{A}^w$, so that the abbrevation $\langle w[\mathscr{A}],\mathscr{A}^w\rangle$ is unambiguous. We similarly define $w[\bar{\mathscr{A}}]$, and we also use the notation $\langle w[\bar{\mathscr{A}}],\mathscr{A}_i^w\rangle$ for any other $\mathscr{A}_i$. Our main result is the following:

Theorem 1.1 *We have the following identity of rational functions in* $\mathbf{Q}(c_{01}, \ldots, c_{kn_k})$:

$$\prod_{\gamma \in V(f_1,\ldots,f_k)} f_0(\gamma) = R_{\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_k}(f_0, f_1, \ldots, f_k) \cdot \prod_{w \in S^{k-1}} R_{\mathscr{A}_1^w,\ldots,\mathscr{A}_k^w}(f_1^w, \ldots, f_k^w)^{\langle w[\tilde{\mathscr{A}}], \mathscr{A}_0 \rangle}. \tag{1.5}$$

The product over w is finite; it includes only directions w which are normal to facets of the Minkowski sum $\bar{P} = P_1 + \ldots + P_k$. *For all other directions the face resultants equal* 1.

In order to state Theorem 1.1 in a more precise fashion we need a notion of "evaluating f_0 at the roots of $f_1, \ldots, f_k$" which takes into account the structure of the inputs as polynomials in $r \geqq k$ variables. We translate the affine lattices $L(\mathscr{A}_i)$ into a common lattice of dimension k, and then find the coarsest common refinement up to translations. Call this lattice $L(\mathscr{A}_0, \ldots, \mathscr{A}_k)$.

We consider the semigroup algebra

$$S := K[\mathbf{t}^a \mid a \in L(\mathscr{A}_0, \ldots, \mathscr{A}_k)], \tag{1.6}$$

and we define $V(f_1, \ldots, f_k)$ to be the spectrum of the quotient algebra $R = S/\langle f_1, \ldots, f_k \rangle$. This algebra is a finite-dimensional K-vector space, and we consider the K-linear map

$$\mathrm{Times}_{f_0} : R \to R, \quad g \mapsto f_0 \cdot g. \tag{1.7}$$

We now apply the methods in [14] to see that the endomorphism (1.7) may be diagonalized over the algebraically closed field K. It has the distinct eigenvalues $f_0(\gamma)$, where $\gamma \in V(f_1, \ldots, f_k)$. We obtain

$$\det(\mathrm{Times}_{f_0}) = \prod_{\gamma \in V(f_1,\ldots,f_k)} f_0(\gamma). \tag{1.8}$$

This determinant is what we mean by the left hand side of (1.5).

Remark 1.2 The expression (1.8) can be evaluated using only computations over the rational numbers. We compute a Gröbner basis for the ideal $\langle f_1, \ldots, f_k \rangle$, and we construct a matrix for Times_{f_0} using Gröbner basis normal forms. Then we compute the determinant of that matrix. For an introduction to Gröbner bases we refer to the book of Cox et al. [6]. See also [1, 14, 16] and the references given there.

Example 1.3 We illustrate the construction in (1.6) for two sparse binary forms:

$$f_0(t_1, t_2) = c_{02} t_1^4 + c_{01} t_2^4 \quad \text{and} \quad f_1(t_1, t_2) = c_{11} t_1^2 + c_{12} t_2^2. \tag{1.9}$$

We embed the affine lattices $\mathbf{Z}[(2,0),(0,2)]$ and $\mathbf{Z}[(4,0),(0,4)]$ in the one-dimensional lattice $\mathbf{Z}^1$ by sending $(2,0) \mapsto 1$, $(0,2) \mapsto 0$. Under this embedding we get $f_0 \mapsto c_{01} x^2 + c_{02}$, $f_1 \mapsto c_{11} x + c_{12}$, and $S/\langle f_1 \rangle \simeq K[x]/\langle c_{11} x + c_{12} \rangle$, where x is a new variable. The spectrum of $S/\langle f_1 \rangle$ can be identified with the singleton $\{-c_{12}/c_{11}\}$. In this example the expression (1.5) or (1.8) equals

$$c_{01}(-c_{12}/c_{11})^2 + c_{02} = (c_{01} c_{12}^2 + c_{02} c_{11}^2) \cdot (1/c_{11}^2). \tag{1.10}$$

We see that the mixed resultant equals $R_{\mathscr{A}_0,\mathscr{A}_1}(f_0,f_1)=c_{01}c_{12}^2+c_{02}c_{11}^2$. This is the square root of the usual Sylvester resultant of the binary quartic f_0 and the binary quadric f_1.

2 Basic facts on mixed volumes and mixed resultants

Let $f_1,\ldots,f_k$ be Laurent polynomials as in the introduction. After an affine transformation we may assume that their supports $\mathscr{A}_1,\ldots,\mathscr{A}_k$ affinely span a k-dimensional lattice $\mathbf{Z}^k$. We recall the following fundamental result of Bernstein [2].

Theorem 2.1 (Bernstein) *The set of roots $V(f_1,\ldots,f_k)$ is finite, and its cardinality equals the mixed volume $\mathscr{M}(P_1,\ldots,P_k)$ of the given Newton polytopes $P_i=\mathrm{conv}(\mathscr{A}_i)$.*

See [7, Chap. 5] for a nice intersection-theoretic proof of Theorem 2.1. We now recall several equivalent definitions and basic properties of the *mixed volume*. For an introduction to mixed volumes and convexity in general we refer to Bonnesen and Fenchel [4].

The mixed volume $\mathscr{M}(\cdot,\cdot,\ldots,\cdot)$ is an operator on k polytopes in a k-dimensional space $\mathbf{R}^k$, which attains non-negative real values. It is symmetric, it is multilinear, and it is $\mathrm{SL}_k(\mathbf{R})$-invariant and translation invariant. These three requirements uniquely define the mixed volume, subject to the normalization condition

$$\mathscr{M}(P,P,\ldots,P)=\mathrm{Vol}(P)\cdot k!=\mathrm{vol}(P). \tag{2.1}$$

This implies that $\mathscr{M}(P_1,P_2,\ldots,P_k)$ is an integer whenever $P_1,\ldots,P_k$ are lattice polytopes. The mixed volume can be computed by any of the following four methods:

(a) For $\lambda_1,\ldots,\lambda_k>0$ consider the polytope $\lambda_1 P_1+\ldots+\lambda_k P_k$. Its volume is a homogeneous polynomial of degree k in $\lambda_1,\lambda_2,\ldots,\lambda_k$. The mixed volume $\mathscr{M}(P_1,P_2,\ldots,P_k)$ equals the coefficient of the monomial $\lambda_1\lambda_2\ldots\lambda_k$ in the expansion of the polynomial $\mathrm{Vol}(\lambda_1 P_1+\ldots+\lambda_k P_k)$.
(b) We have the inclusion-exclusion formula

$$\mathscr{M}(P_1,\ldots,P_k)=(-1)^k\cdot\sum_{I\subset\{1,\ldots,k\}}(-1)^{|I|}\cdot\mathrm{Vol}\Big(\sum_{i\in I}P_i\Big). \tag{2.2}$$

(c) Consider the linear map of polytopes

$$\phi: P_1\times\ldots\times P_k\to P_1+\ldots+P_k,(p_1,\ldots,p_k)\mapsto p_1+\ldots+p_k. \tag{2.3}$$

A *mixed subdivision* is a polyhedral subdivision Δ of $P_1+\ldots+P_k$ which is *induced* from the linear map ϕ as in [3]. Let Δ be any mixed subdivision. The mixed volume $\mathscr{M}(P_1,P_2,\ldots,P_k)$ is the sum of the volumes of all parallelotopes $E_1\times E_2\times\ldots\times E_k$ in Δ, where E_i is an edge of P_i. This representation appears to be most useful for practical computations. It also proves that $\mathscr{M}(P_1,P_2,\ldots,P_k)$ is non-negative.
(d) The *lattice distance* d_i^w from the face P_i^w to the origin O is defined to be the number of steps from O to the image of the face P_i^w in the one-dimensional

factor lattice $\mathbf{Z}^k/\langle w\rangle^\perp$. We have the following recursive formula for the mixed volume:

$$\mathcal{M}(P_1, P_2, \ldots, P_k) = \sum_{w\in S^{k-1}} d_1^w \cdot \mathcal{M}(P_2^w, \ldots, P_k^w). \tag{2.4}$$

In this formula it is understood that the faces P_i^w are projected perpendicularly into a common $(k-1)$-dimensional lattice, with its normalized mixed volume. The sum over w is finite; it includes only directions w which are normal to facets of $P_2+\ldots+P_k$. For all other directions the mixed volume on the right hand side is zero.

Lemma 2.2 *Using the notations introduced in Sect.* 1, *we have the following identities*:

$$\sum_{w\in S^{k-1}} \langle w[\mathcal{A}], \mathcal{A}^w\rangle \cdot \mathrm{vol}_{\mathcal{A}^w}(P^w) = -\mathrm{vol}(P) \tag{2.5}$$

$$\sum_{w\in S^{k-1}} \langle w[\bar{\mathcal{A}}], \mathcal{A}_0^w\rangle \cdot \mathcal{M}_{\bar{\mathcal{A}}^w}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) = -\mathcal{M}(P_0, \ldots, \hat{P}_i, \ldots, P_k). \tag{2.6}$$

Proof. Suppose that $x=0$ lies inside of P. Then summing (1.4) over $w\in S^{k-1}$ gives

$$\sum_w \langle w[\mathcal{A}], \mathcal{A}^w\rangle \cdot \mathrm{vol}_{\mathcal{A}^w}(P^w) = -\sum_w \mathrm{vol}(\mathrm{conv}(P^w \cup \{O\})) = -\mathrm{vol}(P).$$

Let $\mathrm{span}_{\mathbf{R}}(\bar{\mathcal{A}}^w)$ denote the real affine space spanned by the point configuration $\bar{\mathcal{A}}^w$. We abbreviate the restricted lattice $L(\bar{\mathcal{A}}, w) := \mathrm{span}_{\mathbf{R}}(\bar{\mathcal{A}}^w) \cap L(\bar{\mathcal{A}})$. To prove (2.6) we first observe that each $a\in\mathcal{A}_0^w$ satisfies the relation

$$\langle w[\bar{\mathcal{A}}], -a\rangle = \frac{\mathrm{vol}_{\bar{\mathcal{A}}}(\mathrm{conv}(\bar{P}^w \cup \{O\}))}{\mathrm{vol}_{\bar{\mathcal{A}}^w}(\bar{P}^w)} = [L(\bar{\mathcal{A}}, w): L(\bar{\mathcal{A}}^w)] \cdot d_0^w. \tag{2.7}$$

To see this, note that the index of the sublattice $L(\bar{\mathcal{A}}^w)$ in $L(\bar{\mathcal{A}}, w)$ measures precisely the ratio between the normalized volume forms in these two lattices. The mixed volume $\mathcal{M}(\cdot, \ldots, \cdot)$ computed with respect to the possibly finer lattice $L(\bar{\mathcal{A}}, w)$ is related to $\mathcal{M}_{\bar{\mathcal{A}}^w}$ by the same index. Using formulas (2.7) and (2.4), we find

$$\begin{aligned}
&\sum_w \langle w[\bar{\mathcal{A}}], \mathcal{A}_0^w\rangle \cdot \mathcal{M}_{\bar{\mathcal{A}}^w}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) \\
&\quad = -\sum_w d_0^w \cdot [L(\bar{\mathcal{A}}, w): L(\bar{\mathcal{A}}^w)] \cdot \mathcal{M}_{\bar{\mathcal{A}}^w}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) \\
&\quad = -\sum_w d_0^w \cdot \mathcal{M}(P_1^w, \ldots, \widehat{P_i^w}, \ldots, P_k^w) = -\mathcal{M}(P_0, P_1, \ldots, \widehat{P_i}, \ldots, P_k).
\end{aligned}$$

This completes the proof of Lemma 2.2. □

We will now show that the mixed resultant $R_{\mathcal{A}_0,\ldots,\mathcal{A}_k}$ is well-defined and irreducible. As in the introduction, let Z be the closure in $\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}$ of the set Z_0 of all systems (1.1) which have a root in the torus $(\mathbf{C}^*)^k$.

Proposition 2.3 *The variety Z is a proper subvariety of $\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}$. It is irreducible and defined over the rationals $\mathbf{Q}$.*

Proof. Consider the incidence variety

$$W=\{(\mathbf{t},(c_{ij}))\mid \sum_{j=1}^{n_i} c_{ij}\,\mathbf{t}^{a_{ij}}=0 \text{ for } i=0,1,\ldots,k\} \tag{2.8}$$
$$\subset(\mathbf{C}^*)^k\times(\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}).$$

It has two projections $\pi_1: W\to(\mathbf{C}^*)^k$ and $\pi_2: W\to\mathbf{P}^{n_0-1}\times\ldots\times\mathbf{P}^{n_k-1}$.

The first projection π_1 is surjective. The fiber $\pi_1^{-1}(\mathbf{t})$ over a torus element $\mathbf{t}\in(\mathbf{C}^*)^k$ consists of all systems (c_{ij}) such that $\mathbf{t}$ is a common root of (1.1). This fiber is the product of the perpendicular hyperplanes to $(\mathbf{t}^{a_{i1}},\ldots,\mathbf{t}^{a_{in_i}})$, for $i=0,1,\ldots\ldots,k$. Therefore the fibers of π_1 form a family of irreducible varieties of constant dimension. Clearly, the base space $(\mathbf{C}^*)^k$ is an irreducible quasiprojective variety. We can thus apply the proof of Theorem I.1.6.8 in [15] to conclude that W is an irreducible variety.

The image of the incidence variety W under the second projection π_2 is the quasi-projective variety Z_0. Since W is irreducible, $\pi_2(W)=Z_0$ is also irreducible, hence its closure Z is irreducible.

The fact that Z is a proper subvariety follows from Theorem 2.1 because the system $f_1(\gamma)=\ldots=f_k(\gamma)=0$ has only finitely many roots γ, and we can choose coefficients $c_{01},\ldots,c_{0n_0}$ such that $f_0(\gamma)\neq 0$ for all roots γ. Standard elimination theory implies that Z is defined over $\mathbf{Q}$. □

Bernstein's Theorem 2.1 implies that Z has codimension greater than 1 if and only if $\mathscr{M}(P_0,\ldots,\hat{P}_i,\ldots,P_k)=0$ for $i=0,\ldots,k$. It can be shown that this is equivalent to the proposition, $(\forall i)(\exists I\subseteq\{0,\ldots,\hat{i},\ldots,k\})$: $\dim(\sum_{j\in I}P_j)<|I|$. We therefore impose the following condition on our input polynomials:

$$\forall I\subset\{1,\ldots,n\},\quad |I|\leqq\dim(\sum_{j\in I}P_i), \tag{2.9}$$

where I is a proper subset of $\{1,\ldots,n\}$. With condition (2.9), the sparse mixed resultant $R_{\mathscr{A}_0,\ldots,\mathscr{A}_k}$ is a non-constant irreducible polynominal in $\mathbf{Z}[c_{01},\ldots,c_{kn}]$, separately homogeneous in the coefficients of each f_i, and we have the following corollary to Theorem 2.1.

Corollary 2.4 *The degree of the sparse mixed resultant in the coefficients $(c_{i1},\ldots,c_{in_i})$ of the i-th input polynomial f_i equals the mixed volume*

$$\deg_{f_i}(R_{\mathscr{A}_0,\ldots,\mathscr{A}_k})=\mathscr{M}(P_0,\ldots,\widehat{P_i},\ldots,P_k). \tag{2.10}$$

3 Product formula for the $\mathscr{A}$-resultant

We begin by restricting to the unmixed case $\mathscr{A}=\mathscr{A}_0=\mathscr{A}_1=\ldots=\mathscr{A}_k$. There is only one set of lattice points $\mathscr{A}=\{a_1,a_2,\ldots,a_n\}\subset\mathbf{Z}^k$, shared by the exponent vectors of all $k+1$ input polynomials. Here the resultant is called the *$\mathscr{A}$-resultant*

(or *unmixed resultant*) and denoted $R_{\mathscr{A}}$. Basic properties of $\mathscr{A}$-resultants were studied in [11] and [17].

We assume that $\mathscr{A}$ affinely spans the lattice $\mathbf{Z}^k$. Let $P=\text{conv}(\mathscr{A})$ be the Newton polytope, and abbreviate the normalized volume of P by $d:=\text{vol}(P)$ $=\text{Vol}(P)\cdot k!$. Our input polynomials in (1.1) have the following form:

$$
\begin{aligned}
f_0(\mathbf{t}) &= c_{01}\,\mathbf{t}^{a_1} + c_{02}\,\mathbf{t}^{a_2} + \ldots + c_{0n}\,\mathbf{t}^{a_n},\\
f_1(\mathbf{t}) &= c_{11}\,\mathbf{t}^{a_1} + c_{12}\,\mathbf{t}^{a_2} + \ldots + c_{1n}\,\mathbf{t}^{a_n},\\
&\ldots \quad \ldots\ldots \quad \ldots \quad \ldots \quad \ldots\\
f_k(\mathbf{t}) &= c_{k1}\,\mathbf{t}^{a_1} + c_{k2}\,\mathbf{t}^{a_2} + \ldots + c_{kn}\,\mathbf{t}^{a_n}.
\end{aligned} \tag{3.1}
$$

As before, the coefficients c_{ij} are taken to be indeterminates over $\mathbf{Q}$, and we let

$$
V = V(f_1, \ldots, f_k) = \{\gamma \in (K^*)^k \mid f_1(\gamma) = \ldots = f_k(\gamma) = 0\}. \tag{3.2}
$$

The set V consists of precisely d distinct points by a result of Kushnirenko [12].

We have the following product formula for the $\mathscr{A}$-resultant. Theorem 3.1 will be proved in Sect. 6. In Sect. 7 we will derive the more general Theorem 1.1 from Theorem 3.1.

Theorem 3.1 *We have the following identity of rational functions in* $\mathbf{C}(c_{01}, \ldots, c_{kn})$:

$$
\prod_{\gamma \in V} f_0(\gamma) = R_{\mathscr{A}}(f_0, f_1, \ldots, f_k) \cdot \prod_{w \in S^{k-1}} R_{\mathscr{A}^w}(f_1^w, \ldots, f_k^w)^{\langle w[\mathscr{A}], \mathscr{A}^w \rangle}. \tag{3.3}
$$

The right hand product is over all facets of the Newton polytope $P=\text{conv}(\mathscr{A})$. If w selects a facet P^w then the degree of $R_{\mathscr{A}^w}$ is equal to the volume of that facet, normalized with respect to $L(\mathscr{A}^w)$. The corresponding exponent $\langle w[\mathscr{A}], \mathscr{A}^w \rangle$ can be computed as the ratio of $\text{vol}(\text{conv}(\mathscr{A}^w \cup \{O\}))$ and this normalized facet volume.

The left hand side of (3.3) is a rational function of degree zero in the coefficients of $f_1, \ldots, f_n$. So is the right hand side, since the degree of $R_{\mathscr{A}}(f_0, f_1, \ldots, f_k)$ is d by Corollary 2.4, and the product over $w \in S^{k-1}$ has degree $-d$ by Eq. (2.5) in Lemma 2.2.

Example 3.2 (*A quadrangle in the plane*) Let $\mathscr{A}=\{(1,2), (3,2), (2,3), (1,3)\}$ and consider the system

$$
\begin{aligned}
f_0 &= c_{01}\,xy^2 + c_{02}\,x^3y^2 + c_{03}\,x^2y^3 + c_{04}\,xy^3,\\
f_1 &= c_{11}\,xy^2 + c_{12}\,x^3y^2 + c_{13}\,x^2y^3 + c_{14}\,xy^3,\\
f_2 &= c_{21}\,xy^2 + c_{22}\,x^3y^2 + c_{23}\,x^2y^3 + c_{24}\,xy^3.
\end{aligned} \tag{3.4}
$$

The Newton polytope $\text{conv}(\mathscr{A})$ is a quadrangle with normalized area 3. The $\mathscr{A}$-resultant equals

$$
R_{\mathscr{A}}(f_0, f_1, f_2) = [1\,2\,3]^2[1\,3\,4] + [1\,2\,4]^2[2\,3\,4], \tag{3.5}
$$

where $[ijk]$ abbreviates the determinant

$$\begin{vmatrix} c_{0i} & c_{0j} & c_{0k} \\ c_{1i} & c_{1j} & c_{1k} \\ c_{2i} & c_{2j} & c_{2k} \end{vmatrix}.$$

The two bracket monomials in (3.5) express the two triangulations of $\mathrm{conv}(\mathscr{A})$, as predicted by [11, Theorem 5.3]; see also [17, Theorem 3.1]. The extraneous factor on the right hand side of (3.3) is found to be

$$\prod_{w \in S^1} R_{\mathscr{A}^w}(f_1^w, f_2^w)^{\langle w[\mathscr{A}], \mathscr{A}^w \rangle} = [1\,2]^4 [2\,3]^{-5} [3\,4]^{-3} [1\,4], \tag{3.6}$$

where $[ij]$ abbreviates the determinant $\begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix}$. The factors in the product (3.6) express the boundary of the quadrangle $\mathrm{conv}(\mathscr{A})$, joined to the origin. (See Fig. 1). □

Remark 3.3 As an immediate application of Theorem 3.1 we obtain a new and elementary proof of the product formula for the *principal $\mathscr{A}$-determinant* given in the earlier work of Gel'fand, Kapranov and Zelevinsky [8, Theorem 2F.2]. To this end we apply the *toric specialization* of [11, Theorem 5.10] to both the right hand side and the left hand side of (3.3). The result of this specialization is the formula in [8, Theorem 2F.2].

4 Projections and intersections of projective varieties in terms of Chow forms

Our strategy in proving Theorem 3.1 is to utilize the known interpretation of the $\mathscr{A}$-resultant as the Chow form of the toric variety defined by $\mathscr{A}$ [11, §5]. In fact, we will give a completely general product formula (Theorem 5.1) for arbitrary projective varieties. Theorem 3.1 will then be derived as a special case in Sect. 6. In this section we show how to use toric deformations of Chow forms for computing coordinate projections and intersections of projective varieties.

Throughout Sects. 4 and 5 we let X be any irreducible, complex projective variety of dimension k and degree d in $\mathbf{P}^n$. We represent X by its *Chow form* R_X [11; 15, I.1.6.5]. The Chow form is an irreducible polynomial function of the coefficients c_{ij} of $k+1$ generic linear forms $\ell_i = c_{i0} x_0 + c_{i1} x_1 + \ldots + c_{in} x_n$; $i = 0, \ldots, k$. The Chow form vanishes under specialization of the c_{ij} if and only if the accordingly specialized linear forms ℓ_i have a common zero on X. In other words, R_X is the defining polynomial of the hypersurface of all $(n-k-1)$-planes in $\mathbf{P}^n$ which meet X. Note that R_X can also be expressed as a polynomial in *brackets*, i.e., in the $(k+1) \times (k+1)$-minors of the coefficient matrix (c_{ij}).

We fix the point $e_0 = (1:0:\ldots:0)$, the affine space $\mathbf{A}^n = \{x \in \mathbf{P}^n \mid x_0 \neq 0\}$, the hyperplane "at infinity" $\mathbf{P}^{n-1} = \{x \in \mathbf{P}^n \mid x_0 = 0\}$, and the projection $\pi: \mathbf{P}^n \backslash e_0 \to \mathbf{P}^{n-1}$ onto $\mathbf{P}^{n-1}$ with center at e_0. The projection π determines a mapping π_* of algebraic cycles. We are interested in computing the Chow forms of the image cycle $Y = \pi_*(X)$ and the intersection cycle $X \cap \mathbf{P}^{n-1}$ directly from the Chow form R_X. We assume that $X \cap \mathbf{P}^{n-1}$ has dimension $k-1$ and that Y

has dimension k. Then the degree of $X \cap \mathbf{P}^{n-1}$ equals d, while the degree $\tilde{d}$ of Y satisfies the inequality $\tilde{d} \leqq d$. Let $(X \cap \mathbf{P}^{n-1}) * e_0$ denote the cone over $X \cap \mathbf{P}^{n-1}$ with apex e_0. Thus both $(X \cap \mathbf{P}^{n-1}) * e_0$ and Y are algebraic cycles of the same dimension k in $\mathbf{P}^n$.

Let $w: \mathbf{C}^* \to (\mathbf{C}^*)^{n+1}$, $t \mapsto \operatorname{diag}(t^{w_0}, \ldots, t^{w_n})$ be the one-parameter subgroup of $(\mathbf{C}^*)^{n+1}$ defined by the weight vector $w = (1, 0, 0, \ldots, 0)$. The natural action of $w(t)$ on $\mathbf{P}^n$ extends to a covariant action on Chow forms via $c_{i0} \mapsto t \cdot c_{i0}$ (i.e., by scaling the first column). It extends to a contravariant action on the polynomial ring $\mathbf{C}[x_0, x_1, \ldots, x_n]$ via $x_0 \mapsto t^{-1} \cdot x_0$.

Let $I \subset \mathbf{C}[x_0, x_1, \ldots, x_n]$ denote the homogeneous prime ideal defining the variety X. For each parameter value $t \in \mathbf{C}^*$ we get an isomorphic ideal $w(t) \cdot I$ after the substitution $x_0 \mapsto t^{-1} \cdot x_0$. The *initial ideal* $\operatorname{init}_w(I)$ of I is the ideal generated by all leading forms f^w where f runs over I. Computing the initial ideal $\operatorname{init}_w(I)$ amounts to a *Gröbner basis* computation for I with respect to a term order which refines the partial ordering

$$\mathbf{x}^\alpha \leqq \mathbf{x}^\beta :\Leftrightarrow \langle w, \alpha \rangle \leqq \langle w, \beta \rangle. \tag{4.1}$$

By the results of Bayer et al. (see [1]), the initial ideal $\operatorname{init}_w(I)$ equals the limit $\lim_{t \to 0} w(t) \cdot I$ in the *Hilbert scheme*. Similary, we obtain the initial terms of the Chow form $\operatorname{init}_w(R_X)$ as the limit $\lim_{t \to 0} w(t) \cdot R_X$ in the *Chow variety*. Algebraically, the initial Chow form $\operatorname{init}_w(R_X)$ is the sum of all terms in R_X whose weight has maximum inner product with w. For details see [11, § 3; 17, Corollary 3.5].

Our main result in this section is the following geometric description of the two toric deformations of X defined by w and $-w$ respectively.

Proposition 4.1 *We have the following identities in the polynomial ring* $\mathbf{C}[c_{00}, \ldots, c_{kn}]$:

$$\operatorname{init}_{-w}(R_X) = R_{(X \cap \mathbf{P}^n) * e_0} \quad \text{and} \quad \operatorname{init}_w(R_X) = R_Y \cdot R_{Z'}, \tag{4.2}$$

where Z' is an algebraic cycle of the form $Z * e_0$ of degree $d - \tilde{d}$ and dimension k in $\mathbf{P}^n$.

Proof. We use the general fact that $\operatorname{init}_w(R_X) = R_{\operatorname{init}_w(I)}$, where $R_{\operatorname{init}_w(I)}$ denotes the Chow form of the algebraic cycle underlying the subscheme of $\mathbf{P}^n$ defined by the initial ideal $\operatorname{init}_w(I)$. This relation holds because the action of the one-parameter group $w(t)$ is equivariant with respect to the natural morphism from the Hilbert scheme to the Chow variety.

The initial ideal $\operatorname{init}_{-w}(I)$ is obtained from I by the substitution $x_0 \mapsto 0$, since in this instance the action is $x_0 \mapsto t \cdot x_0$, and we are giving x_0 arbitrarily small weight as $t \to 0$. The algebraic cycle underlying $\operatorname{init}_{-w}(I)$ equals $(X \cap \mathbf{P}^{n-1}) * e_0$. Note that we need to cone the intersection cycle with e_0 because the ideal $\operatorname{init}_{-w}(I)$ is generated by polynomials which depend only on $\bar{x} = (x_1, \ldots, x_n)$, yet we are looking at its cycle in $\mathbf{P}^n$. We get

$$\operatorname{init}_{-w}(R_X) = R_{\operatorname{init}_{-w}(I)} = R_{(X \cap \mathbf{P}^{n-1}) * e_0}. \tag{4.3}$$

The initial ideal $\text{init}_w(I)$ may be determined by computing a Gröbner basis G for I with respect to any elimination order which is lexicographic between $x_0 > \bar{x} = (x_1, \ldots, x_n)$. Such a Gröbner basis has the form

$$G = \{g_1(\bar{x}), \ldots, g_r(\bar{x}), h_1(x_0, \bar{x}), \ldots, h_s(x_0, \bar{x})\}. \tag{4.4}$$

Forgetting the ordering imposed on the $\bar{x}$ variables in the elimination order, we find that the desired initial ideal $\text{init}_w(I)$ is generated by

$$\text{init}_w(G) = \{g_1(\bar{x}), \ldots, g_r(\bar{x}), x_0^{m_1} q_1(\bar{x}), \ldots, x_0^{m_s} q_s(\bar{x})\}, \tag{4.5}$$

where $q_i(\bar{x})$ is the coefficient of the highest power of x_0 appearing in h_i. The variety defined by the polynomials in (4.5) decomposes as

$$\{x_0 = g_1(\bar{x}) = \ldots = g_r(\bar{x}) = 0\} \cup \{g_1(\bar{x}) = \ldots = g_r(\bar{x}) = q_1(\bar{x}) = \ldots = q_s(\bar{x}) = 0\}. \tag{4.6}$$

Using (4.6), we may express the cycle underlying $\text{init}_w(I)$ as the sum of two algebraic cycles. The first cycle in this decomposition is precisely the projection cycle $Y = \pi_*(X)$. Let Z' denote the second cycle. We see in (4.6) that the scheme underlying Z' is defined by a set of equations not involving the variable x_0 at all. Therefore Z' is the cone with apex e_0 over some cycle in $\mathbf{P}^{n-1}$; in symbols $Z' = Z * e_0$.

The Chow form of a sum of cycles is the product of the Chow forms of the summands:

$$\text{init}_w(R_X) = R_{\text{init}_w(I)} = R_{Y + Z * e_0} = R_Y \cdot R_{Z * e_0}. \tag{4.7}$$

This completes the proof of Proposition 4.1. □

Example 3.2 (*continued*) We illustrate Proposition 4.1 for the toric surface $X = X_{\mathcal{A} \cup \{(0,0)\}}$, where $\mathcal{A} = \{(1, 2), (3, 2), (2, 3), (1, 3)\}$. Here $n = 4$, $\bar{d} = 3$ and $d = 8$.

Let $I \subset \mathbf{C}[x_0, x_1, x_2, x_3, x_4]$ denote the ideal generated by the set

$$G = \{\underline{x_0 x_2 x_4^3} - x_1^4 x_3, \underline{x_0 x_3^3} - x_1^2 x_2 x_4, \underline{x_0 x_3 x_4} - x_1^3, \underline{x_1 x_3^2} - x_2 x_4^2\}. \tag{4.8}$$

Let X denote the projective variety in $\mathbf{P}^4$ defined by I. The set G is a Gröbner basis for I with respect to the purely lexicogrpahic term order induced from the variable ordering $x_0 > x_1 > x_2 > x_3 > x_4$. From this it can be deduced that X has dimension 2 and degree 8, using standard methods of Gröbner basis theory. To be more precise, using the notation of [18, §2], we find that the underlined initial monomials give rise to precisely eight *standard pairs*:

$$\begin{aligned}&(1, \{x_2, x_3, x_4\}), (1, \{x_1, x_2, x_4\}), (x_3, \{x_1, x_2, x_4\}), (1, \{x_0, x_1, x_4\}),\\ &(1, \{x_0, x_1, x_2\}), (x_3, \{x_0, x_1, x_2\}), (x_4, \{x_0, x_1, x_2\}), (x_4^2, \{x_0, x_1, x_2\}).\end{aligned} \tag{4.9}$$

These eight standard pairs will appear as the bracket factors in the second term of $\text{init}_{-w}(R_X) = R_{X,5}$ below.

The two initial ideals considered in the proof of Proposition 4.2 are

$$\begin{aligned}&\text{init}_w(I) = \langle x_0 x_2 x_4^3, x_0 x_3^3, x_0 x_3 x_4, x_1 x_3^2 - x_2 x_4^2 \rangle \quad \text{and}\\ &\text{init}_{-w}(I) = \langle x_1^4 x_3, x_1^2 x_2 x_4, x_1^3, x_1 x_3^2 - x_2 x_4^2 \rangle.\end{aligned} \tag{4.10}$$

We now determine the Chow form R_X. Consider three generic linear forms

$$\begin{aligned}\ell_0 &= c_{00}x_0 + c_{01}x_1 + c_{02}x_2 + c_{03}x_3 + c_{04}x_4 \\ \ell_1 &= c_{10}x_0 + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 \\ \ell_2 &= c_{20}x_0 + c_{21}x_1 + c_{22}x_2 + c_{23}x_3 + c_{24}x_4.\end{aligned} \tag{4.11}$$

By definition, the Chow form is the unique irreducible polynomial $R_X = R_X(c_{00}, \ldots, c_{24})$ in the coefficients of (4.11), which vanishes if and only if the system of linear equations $\ell_0(x) = \ell_1(x) = \ell_0(x) = 0$ has solution x in X.

For $0 \leqq i < j < k \leqq 4$ we abbreviate $[ijk] :=$

$$\det \begin{pmatrix} c_{0i} & c_{0j} & c_{0k} \\ c_{1i} & c_{1j} & c_{1k} \\ c_{2i} & c_{2j} & c_{2k} \end{pmatrix}.$$

The Chow form equals $R_X = R_{X,5} + R_{X,6} + R_{X,7} + R_{X,8}$, where

$$\begin{aligned}R_{X,5} &= (-[123]^2[134] - [124]^2[234])[012]^4[014], \\ R_{X,6} &= [012]^2(-[012]^2[013][034][234]^2 + 3[012]^3[023][034][134][234] \\ &\quad -[012][013][023][024][134][234] + 7[012][014][024]^2[234]^2 \\ &\quad -5[012][023]^2[024][134]^2 - 5[012][024]^3[134][234] \\ &\quad +4[013][023]^2[024][124][134] + 4[013][024]^3[124][234]) \\ R_{X,7} &= -7[012]^2[023]^2[024][034]^2[234] + [012][013][023]^4[034][134] \\ &\quad +3[012][013][023]^2[024]^2[034][234] \\ &\quad +5[012][023]^3[024]^2[034][134] + 7[012][023][024]^4[034][234] \\ &\quad -[013]^2[023]^4[024][134] - [013]^2[023]^2[024]^3[234] \\ &\quad -2[013][023]^3[024]^3[134] - 2[013][023][024]^5[234] \\ &\quad -[023]^2[024]^5[134] - [024]^7[234] \\ R_{X,8} &= [023]^5[034]^3.\end{aligned}$$

These bracket polynomials have been computed with the algorithm described in [17, Sect. 4.2]. The two initial Chow forms of the algebraic cycles defined by (4.10) are

$$\operatorname{init}_w(R_X) = R_{X,8} \quad \text{and} \quad \operatorname{init}_{-w}(R_X) = R_{X,5}. \tag{4.12}$$

The projection $Y = \pi_*(X)$ is a surface of degree three in $\mathbf{P}^3$. Its defining polynomial is $x_1 x_3^2 - x_2 x_4^2$. Therefore its Chow form equals

$$R_Y = [234][124]^2 - (-[134])(-[123])^2.$$

The intersection cycle $X \cap \mathbf{P}^3$ is supported on the union of the two lines $\{x_1 = x_2 = 0\}$ and $\{x_1 = x_4 = 0\}$ in $\mathbf{P}^3$. The cone over this reducible curve is a reducible surface in $\mathbf{P}^4$. It has the Chow form

$$R_{(X \cap \mathbf{P}^3) * e_0} = [023]^5[034]^3.$$

Finally, the extraneous cycle $Z' = Z * e_0$ has degree five and is given by the Chow form

$$R_{Z'} = [012]^4 [014].$$

Returning to our general discussion, we now state a lemma, to be used in the subsequent section, for computing the Chow form of the cone over a projective variety (or cycle). Let Z be any algebraic cycle of dimension $k-1$ in $\mathbf{P}^{n-1}$. Its Chow form R_Z is a polynomial in brackets $[i_1 \dots i_k]$, $1 \leqq i_1 < \dots < i_k \leqq n$. The cone $Z * e_0$ over Z is a cycle of dimension k in $\mathbf{P}^n$, and its Chow form R_{Z*e_0} is a polynomial in brackets $[j_0 j_1 \dots j_k]$ for $0 \leqq j_0 < j_1 < \dots < j_k \leqq n$.

Lemma 4.2 *The Chow form R_{Z*e_0} is gotten from the Chow form R_Z by the substitutions*

$$[i_1 \dots i_k] := [0 i_1 \dots i_k] \quad \text{for } 1 \leqq i_1 < \dots < i_k \leqq n. \tag{4.13}$$

Proof. Let L be a generic flat of codimension $k+1$ in $\mathbf{P}^n$, having Plücker coordinates $[j_0 j_1 \dots j_k]$, $0 \leqq j_0 < \dots < j_k \leqq n$. Consider its projection $\pi(L)$ into the coordinate hyperplane $\mathbb{P}^{n-1}$. The flat $\pi(L)$ has codimension k in $\mathbf{P}^{n-1}$. The formula (4.13) expresses the Plücker coordinates of $\pi(L)$ directly in terms of the Plücker coordinates of L.

It suffices to prove Lemma 4.2 for the case when Z is an irreducible variety. The cone $Z * e_0$ is the unique irreducible variety in $\mathbf{P}^n$ defined by the property that a point $x = (x_0 : x_1 : \dots : x_n) \in \mathbf{P}^n$ lies in $Z * e_0$ if and only if $\pi(x) = (x_1 : \dots : x_n) \in \mathbf{P}^n$ lies in Z. Therefore a flat L of codimension $k+1$ in $\mathbf{P}^n$ lies in the hypersurface defined by R_{Z*e_0} if and only if its projection $\pi(L)$ lies in the hypersurface defined by R_Z. □

5 Product formula for the Chow form

We retain the notations of Sect. 4. We now concentrate on the special case where the projection $\pi: X \to Y$ is a birational map. This means that the ideal I of X contains a homogeneous polynomial of the form $x_0 f(x_1, \dots, x_n) - g(x_1, \dots, x_n)$. The rational function $\dfrac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)}$, which inverts π, is homogeneous of degree 1.

For each generic form $\ell_i = c_{i0} x_0 + c_{i1} x_1 + \dots + c_{in} x_n$ we let $\tilde{\ell}_i = c_{i1} x_1 + \dots + c_{in} x_n$ denote its restriction to $\mathbf{P}^{n-1}$. The intersection cycle $X \cap \mathbf{P}^{n-1}$ is a cycle of dimension $k-1$ and degree d on $\mathbf{P}^{n-1}$. Its Chow form is a polynomial function on the space of k linear forms in $n-1$ variables, so $R_{X \cap \mathbf{P}^{n-1}}(\tilde{\ell}_1, \dots, \tilde{\ell}_k)$ is well-defined.

We consider the generic section $X' = \{\alpha \in X \mid \ell_1(\alpha) = \dots = \ell_k(\alpha) = 0\}$. This linear section consists of d distinct points, with coordinates in the field K. The Chow form of X can be factored as

$$R_X(\ell_0, \ell_1, \dots, \ell_k) = \prod_{l=1}^{d} (\alpha_{l0} c_{00} + \alpha_{l1} c_{01} + \dots + \alpha_{ln} c_{0n}). \tag{5.1}$$

(See [15, I.1.6.5 Eq. (2)], where we have absorbed the constant α into the α_{ij}.) The coefficients α_{lj} are the homogeneous coordinates of the points in the generic section:

$$X' = \{(\alpha_{l0} : \alpha_{l1} : \ldots : \alpha_{ln}) \mid l = 1, 2, \ldots, d\}. \tag{5.2}$$

The generic $(n-k-1)$-plane defined by $\ell_1 = \ldots = \ell_k = 0$ in $\mathbf{P}^{n-1} = \{x_0 = 0\}$ does not intersect the $(k-1)$-dimensional variety $X \cap \mathbf{P}^{n-1}$. Therefore the point configuration X' lies in the affine space $\mathbf{A}^n$ and we have $\alpha_{l0} \neq 0$ for $l = 1, 2, \ldots, d$. We will use local coordinates $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$ for all points in the affine space $\mathbf{A}^n$.

Let $Y = \pi_*(X)$ be the projection of Y into $\mathbf{P}^{n-1}$ as before. We consider its generic section $Y' = \{\beta \in Y \mid \tilde{\ell}_1(\beta) = \ldots = \tilde{\ell}_k(\beta) = 0\}$. This section is a zero-dimensional variety in $\mathbf{P}^{n-1}$, consisting of $\tilde{d}$ distinct points. By genericity, the form g does not vanish at any point in the linear section Y'. Therefore we can represent each element in Y' uniquely by a vector $\beta \in \mathbf{C}^n$ such that $x_0 = f(\beta)/g(\beta) = 1$. Our main result in this section is the following product formula. It relates the Chow form of the projection of X into $\mathbf{P}^{n-1}$ to the Chow form of the intersection of X with $\mathbf{P}^{n-1}$.

Theorem 5.1 *We have the following identity of rational functions in* $\mathbf{C}(c_{01}, c_{02}, \ldots, c_{kn})$:

$$\prod_{\beta \in Y', \frac{f(\beta)}{g(\beta)} = 1} \tilde{\ell}_0(\beta) = \frac{R_Y(\tilde{\ell}_0, \tilde{\ell}_1, \ldots, \tilde{\ell}_k) \cdot R_Z(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)}{R_{X \cap \mathbf{P}^{n-1}}(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)}. \tag{5.3}$$

Proof. Consider the following rational function on $\mathbf{P}^n \times \mathbf{P}^n \times \ldots \times \mathbf{P}^n (k+1 \text{ times})$:

$$\frac{\operatorname{init}_w(R_X)}{\operatorname{init}_{-w}(R_X)}(\ell_0, \ell_1, \ldots, \ell_k) = \frac{R_Y \cdot R_{Z * e_0}}{R_{(X \cap \mathbf{P}^{n-1}) * e_0}}(\ell_0, \ell_1, \ldots, \ell_k). \tag{5.4}$$

The identity in (5.4) is derived from Proposition 4.1. We claim that the right hand side of Eq. (5.3) is obtaind from the right hand side of Eq. (5.4) by taking the limit

$$\lim_{c_{00} \to \infty} (c_{00}^{\tilde{d}} \cdot (\text{r.h.s. (5.4)})). \tag{5.5}$$

This is a consequence of the following three observations.

(a) The factor $R_Y = R_Y(\ell_0, \ldots, \ell_k)$ does not contain any of the first column variables $c_{00}, c_{10}, \ldots, c_{k0}$ because Y lies in $\mathbf{P}^{n-1}$. Therefore we can write $R_Y = R_Y(\tilde{\ell}_0, \ldots, \tilde{\ell}_k)$.

(b) The factor $R_{Z * e_0}$ is a homogeneous polynomial of degree $d - \tilde{d}$ in brackets of the form $[0 i_1 \ldots i_k]$, by Lemma 4.2. Each occurring bracket is linear in the variable c_{00}, with leading coefficient equal to the complementary k-bracket $[i_1 \ldots i_k]$. Thus $R_{Z * e_0}$ is a polynomial of degree $d - \tilde{d}$ in c_{00}, and its leading coefficient equals $R_Z(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)$.

(c) The factor $R_{(X \cap \mathbf{P}^{n-1}) * e_0}$ is a polynomial of degree d in the variable c_{00}. By Lemma 4.2, its leading coefficient equals $R_{X \cap \mathbf{P}^{n-1}}(\tilde{\ell}_1, \ldots, \tilde{\ell}_k)$.

For each parameter value $t \in \mathbf{C}^*$ the Chow form $w(t) \cdot R_X$ represents the deformed irreducible variety $w(t) \cdot X$. The intersection of $w(t) \cdot X$ with $\ell_1 = \ldots = \ell_k = 0$ consists of d points $(\alpha_{l0}(t) : \alpha_{l1}(t) : \ldots : \alpha_{ln}(t))$, $1 \leqq l \leqq d$. These points may be viewed as a vector-valued algebraic function with d branches. As t tends to zero, $\tilde{d}$ of these points approach the points of a generic section of $Y \subset \mathbf{P}^{n-1}$, while the remaining $d - \tilde{d}$ branches approach the points of a generic section of $Z * e_0$. The factorization (5.1) for the Chow form implies that the left hand side of (5.4) equals

$$\lim_{t \to 0} \frac{t^{\tilde{d}} \prod_{i=1}^{d} \left(c_{00} + \frac{\alpha_{i1}}{\alpha_{i0}}(t)\, c_{01} + \ldots + \frac{\alpha_{in}}{\alpha_{i0}}(t)\, c_{0n} \right)}{\prod_{i=1}^{d} \left(c_{00} + \frac{\alpha_{i1}}{\alpha_{i0}}(t^{-1})\, c_{01} + \ldots + \frac{\alpha_{in}}{\alpha_{i0}}(t^{-1})\, c_{0n} \right)}. \tag{5.6}$$

Here we need the factor $t^{\tilde{d}}$ because $\tilde{d}$ of the first coordinates $\alpha_{l0}(t)$ tend to zero as $t \to 0$. We now consider (5.6) as an algebraic function in c_{00}, and we determine its leading term. This is the ratio of the leading term in the denominator with the leading term in the numerator. The former is simply c_{00}^{d}, while the latter is found to equal

$$\lim_{t \to 0} t^{\tilde{d}} \prod_{i=1}^{d} \left(c_{00} + \frac{\alpha_{i1}}{\alpha_{i0}}(t)\, c_{01} + \ldots + \frac{\alpha_{in}}{\alpha_{i0}}(t)\, c_{0n} \right) = c_{00}^{d-\tilde{d}} \prod_{\beta \in Y'} \tilde{\ell}_0(\beta) + o(c_{00}^{d-\tilde{d}}). \tag{5.7}$$

The product on the right hand side is over representatives $\beta \in Y'$ such that $f(\beta)/g(\beta) = 1$. Therefore $\lim_{c_{00} \to \infty} (c_{00}^{\tilde{d}} \cdot (5.6))$ is equal to both the left and the right hand side in (5.3). This completes the proof of Theorem 5.1. □

Example 3.2 (continued) The limit in (5.5) equals the product of (3.5) and (3.6):

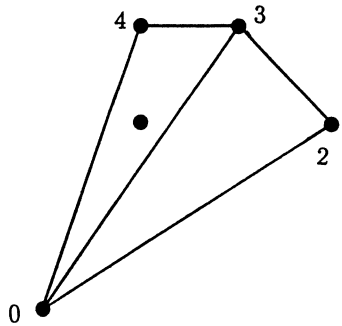
$$\lim_{c_{00} \to \infty} c_{00}^{3} \cdot \frac{([234][124]^2 + [134][123]^2)([012]^4[014])}{[023]^5[034]^3} = R_{\mathscr{A}} \cdot [12]^4 [23]^{-5} [34]^{-3} [14].$$

6 Toric varieties

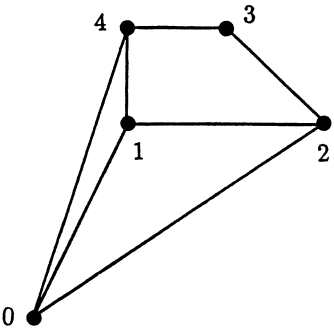
Fix a configuration of lattice points $\mathscr{A} = \{a_0, a_1, \ldots, a_n\}$ which affinely spans the lattice $\mathbf{Z}^k$. This set defines a k-dimensional toric variety $X_{\mathscr{A}}$ in $\mathbf{P}^n$, namely, $X_{\mathscr{A}}$ is the projective spectrum of the monoid algebra $\mathbf{C}[s\mathbf{t}^{a_0}, s\mathbf{t}^{a_1}, \ldots, s\mathbf{t}^{a_n}]$ which is graded by a new variable s. It has been observed in [11] that the Chow form of $X_{\mathscr{A}}$ equals the $\mathscr{A}$-resultant $R_{\mathscr{A}}$. We recall the following result from [11, §5]:

Theorem 6.1 *The following objects are in one-to-one-to-one-to-one correspondence:*

- *the toric degenerations of $X_{\mathscr{A}}$ in the Chow variety,*
- *the distinct initial forms of the $\mathscr{A}$-resultant $R_{\mathscr{A}}$ (= Chow form of $X_{\mathscr{A}}$),*
- *the regular polyhedral subdivisions of the polytope $P = \mathrm{conv}(\mathscr{A})$ with vertices in $\mathscr{A}$,*
- *the faces of the secondary polytope $\Sigma(\mathscr{A})$ (= Chow polytope of $X_{\mathscr{A}}$).*



$R_{(X\cap \mathbf{P}^{n)}*e_0} =$
$[023]^5[034]^3$



$R_Y \cdot R_{Z*e_0} = R_{Y+Z*e_0} =$
$([234][124]^2+[134][123]^2)\cdot[012]^4[014]$

Fig. 1. The two antipodal polyhedral subdivisions in Example 3.2

We apply the constructions from Sects. 5 and 6 to the toric variety $X = X_{\mathscr{A}}$ in $\mathbf{P}^n$. Its projection $Y = \pi_*(X)$ into $\mathbf{P}^{n-1}$ is again a toric variety. It corresponds to the point configuration $\tilde{\mathscr{A}} = \mathscr{A} \backslash \{a_0\}$ gotten by deleting one point; in symbols, $Y = X_{\tilde{\mathscr{A}}}$. We assume (without loss of generality) that the deleted point a_0 is the origin in the lattice $\mathbf{Z}^k$. Hence $\mathbf{t}^{a_0} = 1$. It is our goal to derive Theorem 3.1 for the configuration $\tilde{\mathscr{A}}$ by applying Theorem 5.1 to the toric varieties $X = X_{\mathscr{A}}$ and $Y = X_{\tilde{\mathscr{A}}}$.

Our assumption in Theorem 3.1 that the lattice $\mathbf{Z}^k$ is already affinely spanned by the subset $\tilde{\mathscr{A}}$ implies that the map $X \to Y$ is birational. Hence there exists a representation $a_0 = \sum_{i=1}^{n} \lambda_i a_i$ where $\lambda_1, \ldots, \lambda_n$ are integers satisfying $\sum_{i=1}^{n} \lambda_i = 1$. Then the birational inverse of the projection can be chosen to be

$$x_0 = \frac{g(x_1, \ldots, x_n)}{f(x_1, \ldots, x_n)} = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}. \tag{6.1}$$

For instance, in Example 3.2 we can take any of the following representations by (4.8):

$$x_0 = \frac{x_1^4 x_3}{x_2 x_4^3} = \frac{x_1^2 x_2 x_4}{x_3^3} = \frac{x_1^3}{x_3 x_4}. \tag{6.2}$$

Using Theorem 6.1, we now give a polyhedral description of the two toric degenerations $(X \cap \mathbf{P}^{n-1}) * e_0$ and $Y + Z * e_0$ of the toric variety X. The initial forms of the $\mathscr{A}$-resultant are the denominator and numerator in (5.4). They correspond to the two faces of the secondary polytope $\Sigma(\mathscr{A})$, which maximize the two antipodal linear functionals w and $-w$.

See Fig. 1 for an illustration of these two toric degenerations for the toric surface in $\mathbf{P}^4$ presented in Example 3.2.

To describe the two polyhedral subdivisions of $\mathscr{A}$ we employ the terminology introduced by Lee [13]. The cycle $(X \cap \mathbf{P}^{n-1}) * e_0$ corresponds to the regular subdivision of $\mathscr{A}$ gotten by *pulling* the point a_0. In this subdivision, which is denoted $p_{a_0}^-(\{\mathscr{A}\})$ in [13, §2], the point a_0 is joined with each facet of $\mathrm{conv}(\mathscr{A})$

not containing a_0. The cycle $Y+Z*e_0$ corresponds to the regular subdivision of $\mathscr{A}$ gotten by *placing* the point a_0. This subdivision is denoted $p^+_{a_0}(\{\mathscr{A}\})$ in [13, §2]. It includes the subpolytope $\text{conv}(\tilde{\mathscr{A}})$ as one of its cells, while the other cells are gotten by joining the point a_0 with each facet of $\text{conv}(\tilde{\mathscr{A}})$ which is visible from a_0. See [16, §5] for a discussion of the operations of pulling and placing in terms of Gröbner bases.

Proof of Theorem 3.1 The degree d equals the normalized volume of $\text{conv}(\mathscr{A})$, while $\tilde{d}$ is the normalized volume of the subpolytope $\text{conv}(\tilde{\mathscr{A}})$. Over the algebraically closed field K, the zero-dimensional variety

$$Y' = \{\beta = (\beta_1 : \ldots : \beta_n) \in Y \mid \tilde{f}_1(\beta) = \ldots = \tilde{f}_k(\beta) = 0\} \tag{6.3}$$

consists of $\tilde{d}$ distinct points. None of the coordinates of these points is zero. Therefore we can uniquely coordinatize each point of Y' by the requirement that $\beta^\lambda = \prod_{i=1}^{n} \beta_i^{\lambda_i} = 1$, with $\lambda_1, \ldots, \lambda_n$ as in (6.1). Using this normalization of coordinates, we find that the assignment

$$(\mathbf{C}^*)^k \to \mathbf{P}^{n-1},\ \gamma \mapsto (\gamma^{a_1}, \ldots, \gamma^{a_n}) = \beta \tag{6.4}$$

defines a bijection from the zero-dimensional variety $V(f_1, \ldots, f_k)$ in (3.2) to the zero-dimensional variety Y'. This follows from the fact that a_0 equals the origin and hence

$$\beta^\lambda = \prod_{i=1}^{n} (\gamma^{a_i})^{\lambda_i} = \gamma^{\sum_{i=1}^{n} \lambda_i a_i} = \gamma^{a_0} = 1. \tag{6.5}$$

Under this bijection between $V(f_1, \ldots, f_k)$ and Y', each scalar $\tilde{f}_0(\beta)$ appearing on the left hand side of (5.3) equals the scalar $f_0(\gamma)$ appearing on the left hand side of (3.3). Therefore the left hand side of (3.3) equals (5.3).

By our polyhedral interpretation arising from Theorem 6.1, the right hand side of (5.4) is an alternating product of $\mathscr{A}$-resultants corresponding to the maximal cells in two "antipodal" subdivisions of $\mathscr{A}$. If we apply the limit (5.5) to that alternating product we get precisely the right hand side of (3.3). Therefore the right hand side of (3.3) equals (5.3). This completes our derivation of Theorem 3.1 from Theorem 5.1. □

7 Factorization of mixed resultants

We now return to the original setting of $k+1$ forms f_i supported on $k+1$ distinct subsets $\mathscr{A}_i \subset \mathbf{Z}^k$, $i = 0, \ldots, k$. We will prove Theorem 1.1 in parallel with the following result.

Proposition 7.1 *The mixed resultant satisfies the following multiplicativity relation:*

$$\begin{aligned} &R_{\mathscr{A}_0, \ldots, \mathscr{A}_i + \mathscr{A}_{i'}, \ldots, \mathscr{A}_k}(f_0, \ldots, f_i' \cdot f_i'', \ldots, f_k) \\ &\quad = R_{\mathscr{A}_0, \ldots, \mathscr{A}_i, \ldots, \mathscr{A}_k}(f_0, \ldots, f_i', \ldots, f_k)^{[L(\mathscr{A}) : L(\mathscr{A}')]} \\ &\quad \times R_{\mathscr{A}_0, \ldots, \mathscr{A}_{i'}, \ldots, \mathscr{A}_k}(f_0, \ldots, f_i'', \ldots, f_k)^{[L(\mathscr{A}) : L(\mathscr{A}'')]} \end{aligned} \tag{7.1}$$

where $L(\mathscr{A})$, $L(\mathscr{A}')$ and $L(\mathscr{A}'')$ are lattices affinely generated by $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{A}''$.

Example 7.2 We illustrate Proposition 7.1 in a small one-dimensional example. Let $k=1$, $\mathscr{A}_0=\{0,6\}$, $\mathscr{A}_1'=\{0,2\}$, $\mathscr{A}_1''=\{0,3\}$, $\mathscr{A}_1=\{0,2,3,5\}$. Then $L(\mathscr{A}_1)=\mathbf{Z}^1$, $[L(\mathscr{A}):L(\mathscr{A}')]=3$ and $[L(\mathscr{A}):L(\mathscr{A}'')]=2$. We consider the generic polynomials

$$f_0 := c_{01}x^6 + c_{02},$$
$$f_1 := c_{11}x^5 + c_{12}x^3 + c_{13}x^2 + c_{14},$$
$$f_1' := a_0x^2 + a_1, \qquad f_1'' := b_0x^3 + b_1.$$

The mixed resultant of f_0 and f_1 coincides with the Sylvester resultant:

$$\begin{aligned}(7.2)\qquad R_{\mathscr{A}_0,\mathscr{A}_1}(f_0,f_1) = {} & c_{01}^5c_{14}^6 + 3c_{01}^4c_{02}c_{12}^2c_{14}^4 - 2c_{01}^4c_{02}c_{13}^3c_{14}^3 \\ & + 6c_{01}^3c_{02}^2c_{11}^2c_{13}c_{14}^3 - 18c_{01}^3c_{02}^2c_{11}c_{12}c_{13}^2c_{14}^2 \\ & + 3c_{01}^3c_{02}^2c_{12}^4c_{14}^2 + 6c_{01}^3c_{02}^2c_{12}^2c_{13}^3c_{14} \\ & + c_{01}^3c_{02}^2c_{13}^6 + 6c_{01}^2c_{02}^3c_{11}^3c_{12}c_{14}^2 \\ & - 18c_{01}^2c_{02}^3c_{11}^2c_{12}^2c_{13}c_{14} + 3c_{01}^2c_{02}^3c_{11}^2c_{13}^4 \\ & + 6c_{01}^2c_{02}^3c_{11}c_{12}^3c_{13}^3 + c_{01}^2c_{02}^3c_{12}^6 \\ & + 3c_{01}c_{02}^4c_{11}^4c_{13}^2 - 2c_{01}c_{02}^4c_{11}^3c_{12}^3 + c_{02}^5c_{11}^6.\end{aligned}$$

We now replace f_1 by the product $f_1'\cdot f_1''$, or equivalently, we replace $c_{11}\mapsto a_0b_0$, $c_{12}\mapsto a_1b_0$, $c_{13}\mapsto a_0b_1$, $c_{14}\mapsto a_1b_1$ in (7.2). As the result we obtain

$$\begin{aligned}(7.3)\qquad R_{\mathscr{A}_0,\mathscr{A}_1}(f_0,f_1'\cdot f_1'') &= (c_{01}a_1^3 - c_{02}a_0^3)^2\cdot(c_{01}b_1^2 + c_{02}b_0^2)^3 \\ &= R_{\mathscr{A}_0,\mathscr{A}_1'}(f_0,f_1')^2\cdot R_{\mathscr{A}_0,\mathscr{A}_1''}(f_0,f_1'')^3,\end{aligned}$$

as predicted by Proposition 7.1.

Proof of Theorem 1.1 *and Proposition* 7.1 We will prove both statements simultaneously by induction on k, the number of variables. The main idea is to reduce the mixed identiy (1.5) to the already established unmixed identity (3.3).

We form $k+1$ duplicates of each form using new indeterminate coefficients, and multiply these together as follows:

$$\begin{aligned}(7.4)\qquad u_0 &= f_{00}f_{01}f_{02}\cdots f_{0k}, & v_0 &= f_{01}f_{02}\cdots f_{0k}, \\ u_1 &= f_{10}f_{11}f_{12}\cdots f_{1k}, & v_1 &= f_{11}f_{12}\cdots f_{1k}, \\ \vdots &= \quad \vdots & \vdots &= \quad \vdots \\ u_k &= f_{k0}f_{k1}f_{k2}\cdots f_{kk}, & v_k &= f_{k1}f_{k2}\cdots f_{kk}.\end{aligned}$$

Each polynomial f_{ij}, $0\leqq i\leqq k$, appearing in the j-th column is supported on $\mathscr{A}_j$. Therefore each row product u_i is supported on $\mathscr{A}=\mathscr{A}_0+\mathscr{A}_1+\ldots+\mathscr{A}_k$. Similarly, each row product v_i is supported on $\bar{\mathscr{A}}=\mathscr{A}_1+\mathscr{A}_2+\ldots+\mathscr{A}_k$.

Step 1 Multiplicativity. We show how formula (1.5) in the statement of the main theorem implies (7.1). Since the resultant is symmetric up to sign in its inputs, we may assume without loss of generality that $i=0$ and let

$$\begin{aligned} \mathscr{A} &= (\mathscr{A}_0' + \mathscr{A}_0'') + \mathscr{A}_1 + \ldots + \mathscr{A}_k, \\ \mathscr{A}' &= \mathscr{A}_0' + \mathscr{A}_1 + \ldots + \mathscr{A}_k, \\ \mathscr{A}'' &= \mathscr{A}_0'' + \mathscr{A}_1 + \ldots + \mathscr{A}_k. \end{aligned} \tag{7.5}$$

Then

$$R(f_0' \cdot f_0'', f_1, \ldots, f_k) = \prod_{\gamma \in V} f_0'(\gamma) f_0''(\gamma) \prod_w R(f_1^w, \ldots, f_k^w)^{-\langle w[\bar{\mathscr{A}}], \mathscr{A}_0 \rangle}. \tag{7.6}$$

The number of roots $|V(f_1, \ldots, f_k)|$ depends on which lattice $L(\mathscr{A})$, $L(\mathscr{A}')$, or $L(\mathscr{A}'')$ is used to compute the mixed volumes, since we are normalizing with respect to different unit simplices. Let V, V', and V'' denote the three possible varieties arising from the three different lattices $L(\mathscr{A})$, $L(\mathscr{A}')$, and $L(\mathscr{A}'')$. (See the construction in (1.6)). We have the relations

$$|V| = [L(\mathscr{A}) : L(\mathscr{A}')] \cdot |V'| = [L(\mathscr{A}) : L(\mathscr{A}'')] \cdot |V''|. \tag{7.7}$$

Therefore

$$\prod_{\gamma \in V} f_0'(\gamma) f_0''(\gamma) = \prod_{\gamma \in V'} f_0'(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}')]} \cdot \prod_{\gamma \in V''} f_0''(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}'')]}. \tag{7.8}$$

We have $\mathscr{A}_0^w = (\mathscr{A}_0')^w + (\mathscr{A}_0'')^w$ and consequently

$$\langle w[\bar{\mathscr{A}}], \mathscr{A}_0^w \rangle = \langle w[\bar{\mathscr{A}}], (\mathscr{A}_0')^w \rangle + \langle w[\bar{\mathscr{A}}], (\mathscr{A}_0'')^w \rangle. \tag{7.9}$$

We rewrite the relation (7.6) as follows:

$$\begin{aligned} R(f_0' \cdot f_0'', f_1, \ldots, f_k) = & \prod_{\gamma \in V'} f_0'(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}')]} \cdot \prod_w R(f_1^w, \ldots, f_k^w)^{\langle w[\bar{\mathscr{A}}], (\mathscr{A}_0')^w \rangle} \\ & \times \prod_{\gamma \in V''} f_0''(\gamma)^{[L(\mathscr{A}) : L(\mathscr{A}'')]} \cdot \prod_w R(f_1^w, \ldots, f_k^w)^{\langle w[\bar{\mathscr{A}}], (\mathscr{A}_0'')^w \rangle}. \end{aligned} \tag{7.10}$$

In the formula (7.10) the facet resultants $R(f_1^w, \ldots, f_k^w)$ are regarded with respect to the lattice affinely generated by $\bar{\mathscr{A}}$, $\mathscr{A}_0$. In order for (1.5) to be applicable, the facet resultants must be regarded with respect to the possibly coarser lattices generated by $\bar{\mathscr{A}}$, $\mathscr{A}_0'$, and $\bar{\mathscr{A}}$, $\mathscr{A}_0''$ respectively. Therefore we must introduce extra exponents $[L(\mathscr{A}) : L(\mathscr{A}')]$ and $[L(\mathscr{A}) : L(\mathscr{A}'')]$. We obtain the following identity:

$$\begin{aligned} R(f_0' \cdot f_0'', f_1, \ldots, f_k) = & R(f_0', f_1, \ldots, f_k)^{[L(\mathscr{A}) : L(\mathscr{A}')]} \\ & \times R(f_0'', f_1, \ldots, f_k)^{[L(\mathscr{A}) : L(\mathscr{A}'')]}. \end{aligned} \tag{7.11}$$

Step 2 The induction. As before, let $\bar{\mathscr{A}} = \mathscr{A}_1 + \ldots + \mathscr{A}_k$ and $\mathscr{A} = \mathscr{A}_0 + \bar{\mathscr{A}}$. Each u_i is supported on $\mathscr{A}$, and each v_j is supported on $\bar{\mathscr{A}}$. Applying the "unmixed" Theorem 3.1 to the $\mathscr{A}$-resultant and the $\bar{\mathscr{A}}$-resultant, we get

$$(7.12)\quad R_{\mathscr{A}}(u_0, u_1, \ldots, u_k) = \prod_{\gamma \in V(u_1, \ldots, u_k)} u_0(\gamma) \cdot \prod_{w \in S^{k-1}} R_{\mathscr{A}^w}(u_1^w, \ldots, u_k^w)^{\langle w[\mathscr{A}], \mathscr{A}^w \rangle},$$

$$(7.13)\quad R_{\bar{\mathscr{A}}}(v_0, v_1, \ldots, v_k) = \prod_{\gamma \in V(v_1, \ldots, v_k)} v_0(\gamma) \cdot \prod_{w \in S^{k-1}} R_{\bar{\mathscr{A}}^w}(v_1^w, \ldots, v_k^w)^{\langle w[\bar{\mathscr{A}}], \bar{\mathscr{A}}^w \rangle}.$$

We now apply Proposition 7.1 inductively for lattices of dimension $\leqq k-1$. The right hand sides of (7.12) and (7.13) may be completely factored as products of lower dimensional mixed resultants with various exponents. We collect those factors which are functions only in the coefficients of u_0 and of the diagonal polynomials f_{ii}. This includes the factors

$$(7.14)\quad \prod_{\gamma \in V(f_{11}, \ldots, f_{kk})} u_0(\gamma), \quad \text{and} \quad \prod_{\gamma \in V(f_{11}, \ldots, f_{kk})} v_0(\gamma),$$

since only the "diagonal" roots $\gamma \in V(f_{11}, \ldots, f_{kk})$ depend *only* on the stated coefficients.

We now factor the left hand side of (7.12). One of the appearing factors (conceivably with an exponent $\nu > 1$) must be the mixed resultant $R(u_0, f_{11}, \ldots, f_{kk})$. This follows directly from our geometric definition in Section 1, because the solvability of the system $u_0 = f_{11} = \ldots = f_{kk} = 0$ implies the solvability of $u_0 = u_1 = \ldots = u_k = 0$.

A degree count using Lemma 2.2 shows that $\nu = 1$ and that $R(u_0, f_{11}, \ldots, f_{kk})$ is the only factor on the left hand side which depends only on the coefficients of u_0 and of the diagonal polynomials f_{ii}. The same arguments apply to (7.13). Collecting the factors of interest on both sides of (7.12) and (7.13), we obtain a new pair of equations:

$$(7.15)\quad R(u_0, f_{11}, \ldots, f_{kk}) = \prod_{\gamma} u_0(\gamma) \cdot \prod_{w} R(f_{11}^w, \ldots, f_{kk}^w)^{[L(\mathscr{A}^w) : L(\bar{\mathscr{A}}^w)] \cdot \langle w[\mathscr{A}], \mathscr{A}^w + \bar{\mathscr{A}}^w \rangle}$$

$$(7.16)\quad R(v_0, f_{11}, \ldots, f_{kk}) = \prod_{\gamma} v_0(\gamma) \cdot \prod_{w} R(f_{11}^w, \ldots, f_{kk}^w)^{\langle w[\bar{\mathscr{A}}], \bar{\mathscr{A}}^w \rangle},$$

where γ runs over $V(f_{11}, \ldots, f_{kk})$ and w runs over the $(k-1)$-sphere S^{k-1}. The index $[L(\mathscr{A}^w) : L(\bar{\mathscr{A}}^w)]$ appears in (7.15) because when extracting the diagonal factors $R(f_{11}^w, \ldots, f_{kk}^w)$ from $R(u_1^w, \ldots, u_k^w)$, we must take into account the refinement of the lattice caused by the additional dependence of the u_i on $\mathscr{A}_0$. Here we have inductively used the multiplicative formula (7.1) in lower dimensions. The same index for the second equation (7.16) is simply $1 = [L(\bar{\mathscr{A}}^w) : L(\bar{\mathscr{A}}^w)]$.

We divide Eq. (7.15) by Eq. (7.16). Using the facts that $[L(\mathscr{A}^w) : L(\bar{\mathscr{A}}^w)] \cdot w[\mathscr{A}] = w[\bar{\mathscr{A}}]$ and $u_0(\gamma)/v_0(\gamma) = f_{00}(\gamma)$, we obtain the identity

$$(7.17)\quad \frac{R(u_0, f_{11}, \ldots, f_{kk})}{R(v_0, f_{11}, \ldots, f_{kk})} = \prod_{\gamma \in V(f_{11}, \ldots, f_{kk})} f_{00}(\gamma) \cdot \prod_{w \in S^{k-1}} R_{\mathscr{A}^w, \ldots, \mathscr{A}^w}(f_{11}^w, \ldots, f_{kk}^w)^{\langle w[\bar{\mathscr{A}}], \mathscr{A}^w \rangle}.$$

When deriving (7.17) from (7.15) and (7.16), the exponents cancel as follows:

$$(7.18)\qquad [L(\mathscr{A}^w): L(\bar{\mathscr{A}}^w)]\cdot\langle w[\mathscr{A}], \mathscr{A}_0^w + \bar{\mathscr{A}}^w\rangle - \langle w[\bar{\mathscr{A}}], \bar{\mathscr{A}}^w\rangle$$
$$= \langle w[\bar{\mathscr{A}}], \mathscr{A}_0^w + \bar{\mathscr{A}}^w\rangle - \langle w[\bar{\mathscr{A}}], \bar{\mathscr{A}}^w\rangle = \langle w[\bar{\mathscr{A}}], \mathscr{A}_0^w\rangle.$$

All that remains is to observe that the quotient on the left hand side of (7.17) is a polynomial, since $v_0 = 0$ implies $u_0 = 0$. The denominator exactly divides, and the result is $R_{\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_k}(f_{00}, f_{11}, \ldots, f_{kk})$ by another degree count. We conclude that the product formula (1.5) holds for the "diagonal" polynomials f_{00}, $f_{11}, \ldots, f_{kk}$. These polynomials are entirely generic, so we are done.

We note that our induction begins in dimension $k = 0$ with constants, in which case both Theorem 1.1 and Proposition 7.1 are trivially satisfied. □

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The zeros of thetafunctions belonging to binary-hermitian forms over $\mathbb{Z}[i]$

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1 Statement of results

Let E, $G \in \mathbb{N}$, $\frac{F}{2} \in \mathbb{Z}[i]$ be numbers such that $N := EG - \left|\frac{F}{2}\right|^2$ is positive. Then $Q(x, y) = E|x|^2 + \frac{1}{2}\operatorname{tr}(F x \bar{y}) + G|y|^2 =: [E, F, G]$ is a positive binary hermitian form. ($\bar{x}$ = conjugate of the complex number x, $|x|^2 = x\bar{x}$, $\operatorname{tr}(x) = x + \bar{x}$.) Introducing $x = x_1 + ix_2$, $y = y_1 + iy_2$ and $F = F_1 + iF_2$, Q becomes the quaternary quadratic form

$$Q(x, y) = E(x_1^2 + x_2^2) + F_1(x_1 y_1 + x_2 y_2) + F_2(x_1 y_2 - x_2 y_1) + G(y_1^2 + y_2^2).$$

We set $M_Q = \begin{pmatrix} E & \frac{F}{2} \\ \frac{\bar{F}}{2} & G \end{pmatrix}$ and denote by $\kappa(Q)$ the greatest common divisor of E, G, F_1 and F_2. We always suppose that M_Q is primitive (not divisible by any natural number >1). This implies $\kappa(Q) \in \{1, 2\}$ and also that $\frac{F}{2}$ is not divisible by 2 in case of $\kappa(Q) = 2$. The group $Gl_2(\mathbb{Z}[i])$ is acting on the set of forms of fixed determinant N by

$$M_Q \mapsto A M_Q \bar{A}^t =: M_{\tilde{Q}}, \qquad A \in Gl_2(\mathbb{Z}[i]). \tag{1}$$

(A^t = transpose of A)

Forms Q and $\tilde{Q}$ as in (1) are called equivalent.

Let $\mu=(\mu_1,\mu_2)$, $\nu=(\nu_1,\nu_2)$ be pairs of numbers $\mu_i,\nu_i\in\{0,1\}$. For given ν and $x, y\in\mathbb{Z}[i]$ we set $\nu(x,y)=\nu_1(x_1+x_2)+\nu_2(y_1+y_2)$. The complex number $i+1$ will be denoted by ω. We are interested in the theta series

$$\vartheta(Q,\mu,\nu,z)=\sum_{x,y\in\mathbb{Z}[i]}(-1)^{\nu(x,y)}e^{\pi iQ\left(x+\mu_1\frac{\omega}{2},y+\mu_2\frac{\omega}{2}\right)z}.$$

These series converge absolutely for any $z\in H=\left\{z\in\mathbb{C},\ \frac{z-\bar{z}}{i}>0\right\}$ and thus define holomorphic functions on H. We always suppose that $\mu_1\nu_1+\mu_2\nu_2$ is even because otherwise $\vartheta(Q;\mu,\nu,z)$ is vanishing identically. There are ten even characteristics. If $\tilde{Q}$ is equivalent to Q then there is a characteristic $(\tilde{\mu},\tilde{\nu})$ such that $\vartheta(\tilde{Q},\tilde{\mu},\tilde{\nu},z)=\pm\vartheta(Q,\mu,\nu,z)$ holds. It is well-known that $\vartheta(Q,\mu,\nu,z)$ is a modular form of weight 2 for some level $2N$ – congruence subgroup $\Gamma(Q,\mu,\nu)$ of $Sl_2(\mathbb{Z})$. [5, 9.4.2, Theorem 5].

For given $\alpha,\gamma\in\mathbb{Z}$ not both odd we define

$$(\mu,\nu)[\alpha,\gamma]=\begin{cases}(1111) & \text{in case of } \alpha\equiv\gamma\equiv 0 \bmod 2\\ (1001) & \text{in case of } \alpha\equiv\gamma+1\equiv 0 \bmod 2\\ (0110) & \text{in case of } \alpha+1\equiv\gamma\equiv 0 \bmod 2.\end{cases}$$

Now we can state:

Proposition 1 *The theta series $\vartheta(Q,\mu,\nu,z)$ vanishes identically on H if and only if there are $\alpha,\gamma\in\mathbb{Z},\beta\in\mathbb{Z}[i]$ such that $|\beta|^2-4\alpha\gamma=2$, $\alpha E+\frac{1}{4}\operatorname{tr}(\bar{\beta}F)+\gamma G=0$ and $(\mu,\nu)=(\mu,\nu)[\alpha,\gamma]$ holds.*

We shall postpone the proof of Proposition 1 and first derive some consequences:

Corollary 1 *In case of $\kappa(Q)=2$ and $N\equiv 1 \bmod 2$ none of the $\vartheta(Q,\mu,\nu,z)$ is vanishing identically on H.*

Proof. There can be no (α,β,γ) as in Proposition 1 because $\beta\equiv\omega \bmod 2$ and $\left|\frac{F}{2}\right|^2\equiv 1 \bmod 2$ would imply $\frac{1}{4}\operatorname{tr}(\bar{\beta}F)\equiv 1 \bmod 2$ in contradiction to $E\equiv G\equiv 0 \bmod 2$.

Corollary 2 If G is the minimum of all numbers represented by Q and if neither $\frac{1}{2}(F_1+F_2)$ nor $\frac{1}{2}(F_1-F_2)$ is divisible by G then none of the $\vartheta(Q,\mu,\nu,z)$ is vanishing identically.

Proof. Suppose there is an identically vanishing ϑ. Then the conditions of Proposition 1 imply $2G=G|\beta|^2+\operatorname{tr}\left(2\alpha\bar{\beta}\frac{F}{2}\right)+E(2\alpha)^2$, $\beta\equiv\omega \bmod 2$. From this and the minimality of G follows $\alpha=0$, $\pm\beta\in\{\omega,\bar{\omega}\}$, and also $\frac{1}{2}(F_1\pm F_2)\equiv 0 \bmod G$.

It follows from Proposition 1 that for fixed Q there are at most three ϑ's vanishing identically. Our next proposition says that this actually occurs, but only for one class of Q's.

Proposition 2 *There are three characteristics (μ_i,ν_i), $i=1,2,3$ such that $\vartheta(Q,\mu_i,\nu_i,z)$ vanishes identically if and only if Q is equivalent to $[2,2\omega,2]$.*

This will be proven later.

Table 1

(μ, ν)	α	γ	A/N	C
(1111)	0	0	0	0
(0110)	1	0	0	0
(1001)	0	1	0	0
(0000)	1	1	1	1
(0011)	0	0	1	0
(0001)	0	1	1	0
(0010)	1	0	1	0
(1100)	0	0	0	1
(1000)	0	1	0	1
(0100)	1	0	0	1

As we shall see, every isolated zero of $\vartheta(Q, \mu, \nu, z)$ is imaginary quadratic. For given $Q=[E, F, G]$, μ, ν and imaginary quadratic $z \in H$ let $m(Q, \mu, \nu, z)$ be the number of quintuples (A, B, C, r, k) fulfilling the following conditions:

(2) $$z=\frac{B+\sqrt{\Delta}}{2A}$$

(3) $$A, C \in \mathbb{N}; \quad B, r \in \mathbb{Z}; \quad k \in \mathbb{Z}[i]$$

(4) $$A \equiv 0 \bmod N$$

(5) $$\Delta := B^2-4AC=\frac{1}{G^2}(r^2+|k|^2 N)-2N<0$$

(6) $$\alpha := \frac{r+BG}{2N} \in \mathbb{Z}; \quad \gamma := \frac{B-\alpha E-\frac{1}{4}\operatorname{tr}(\bar{\beta} F)}{G} \in \mathbb{Z}$$

(7) $$\beta := \frac{k-\alpha F}{G} \in \mathbb{Z}[i]$$

(8) $$\alpha, \gamma, \frac{A}{N}, \quad C \bmod 2 \text{ are as in Table 1 below}$$

The greatest common divisor of N, B and $\frac{A}{N} C$ (as in (3), (4)) is a $\Gamma_0(N)$-invariant of the binary-quadratic form $Ax^2+Bxy+Cy^2$. An easy calculation shows that for any quintuple such that (3)–(7) holds we have

(9) $$(N, B, \frac{A}{N} C)=(N, r, 2G^2-|k|^2).$$

Proposition 3 (a) *Suppose that $\vartheta(Q, \mu, \nu, z)$ is not vanishing identically on the upper half plane H. Then its vanishing order at the imaginary quadratic point $z \in H$ is $m(Q, \mu, \nu, z)$.*
(b) *We always have $m(Q, \mu, \nu, z) \leqq 2$.*
(c) *$m(Q, \mu_0, \nu_0, z)=2$ and $(\mu, \nu) \neq (\mu_0, \nu_0)$ imply $m(Q, \mu, \nu, z)=0$.*

A point $z \in H$ that is a solution of a quadratic equation $Az^2 - Bz + C = 0$ with $A, B, C \in \mathbb{Z}$ and $A \equiv 0 \bmod N$ is sometimes called a Heegner point. (See e.g. [3, p. 235–236]) Using this therminology we may conclude from Proposition 3.

Corollary 3 *The function $\vartheta(Q, \mu, \nu, z)$ either vanishes identically on the upper half plane or else all its zeros are Heegner points.*

Remark 1 As an application of Proposition 2 one may produce a complete list of $\Gamma(Q; \mu, \nu)$-nonequivalent zeros of $\vartheta(Q, \mu, \nu, z)$. Choosing a convenient fundamental domain for $\Gamma(Q, \mu, \nu)$.)

Remark 2 If Q is equivalent to a direct sum of two binary quadratic forms then Propositions 1–3 give us the results of [4] as a special case.

2 Proofs

To prove Propositions 1 to 3 we use a theorem of Freitag who determined the zeros of certain hermitian theta functions [2, Satz 1]. Let $\mathcal{H}_2$ be the hermitian halfspace

$$\mathcal{H}_2 = \{Z \in M_2(\mathbb{C}); \quad \frac{1}{i}(Z - \bar{Z}^t) > 0\}.$$

On $\mathcal{H}_2$ the group $\Omega_2 = \left\{M \in M_4(\mathbb{C}); \ \bar{M}^t I M = I; \ I = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}\right\}$ is acting by $Z \mapsto M(Z) = (AZ + B)(CZ + D)^{-1}$; $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Its discrete subgroup $\Gamma := \Omega_2 \cap \mathrm{M}_4(\mathbb{Z}[i])$ is acting discontinously. Reducing mod ω the coefficients of $M \in \Gamma$ we get a matrix $\bar{M} \in \mathrm{Sp}_2(\mathbb{F}_2)$. The kernel of the homomorphism $\Gamma \to \mathrm{Sp}_2(\mathbb{F}_2)$ defined by $M \mapsto \bar{M}$ will be denoted by $\Gamma(\omega)$. The functions

$$\theta(\mu, \nu, Z) = \sum_{x,\, y \in \mathbb{Z}[i]} (-1)^{\nu(x,y)} e^{\pi i Z\{(x,y) + \mu \frac{\omega}{2}\}}$$

($Z\{(a, b)\} = (a, b) Z (\bar{a}, \bar{b})^t$, μ, ν as above) are $\Gamma(\omega)$-modular forms of weight one and characters of order 4. The product $\Theta := \prod_{(\mu,\nu)\text{even}} \theta(\mu, \nu, Z)$ is a Γ-modular form of weight 10. The theorem of Freitag mentioned above states that the zeros of Θ are precisely those points on $\mathcal{H}_2$ that are Γ-equivalent to points $Z \in V$ where

$$V = \left\{Z = \begin{pmatrix} z_0 & z_1 \\ z_3 & z_2 \end{pmatrix} \in \mathcal{H}_2, \ \omega z_1 + \bar{\omega} z_3 = 0\right\}.$$

Let π denote the projection π: $\mathcal{H}_2 \to \mathcal{H}_2/\Gamma$. We may interprete $R_2 := \pi(V)$ as a 3-dimensional subspace of $X := \mathcal{H}_2/\Gamma$. For given Q we define maps $\phi_Q : \mathrm{H} \to \mathcal{H}_2$; $z \mapsto M_Q z$ and

$$\Phi_Q : Sl_2(\mathbb{R}) \mapsto \Omega_2; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} aE & bM_Q \\ cM_Q^{-1} & dE \end{pmatrix} \quad ((E = \text{unit-matrix}).$$

We have $\phi_Q(M(z))=\Phi_Q(M)(\phi_Q(z))$ for all $z\in H$ and all $M\in Sl_2(\mathbb{R})$ and furthermore $\Phi_Q(\Gamma_0(N))\subset\Gamma$. Therefore (ϕ_Q, Φ_Q) defines a holomorphic map $j_Q: H/\Gamma_0(N)\to X$. Because of $\vartheta(Q,\mu,\nu,z)=\theta(\mu,\nu,M_Q z)$ we will find the zeros of $\prod_{(\mu,\nu)\text{even}}\vartheta(Q,\mu,\nu,z)$ by determining the intersection of $j_Q(H/\Gamma_0(N))$ and R_2. To this end we would like to characterize the zeros of Θ in terms of equations. To given numbrs $\alpha,\gamma,\vartheta,\varepsilon\in\mathbb{Z}[i]$ such that $|\beta|^2-4\alpha\gamma-4\vartheta\varepsilon=2$ holds we associate the set

$$V(\alpha,\beta,\gamma,\delta,\varepsilon)$$
$$=\left\{\begin{pmatrix}z_0 & z_1\\ z_3 & z_2\end{pmatrix}\in\mathcal{H}_2;\quad \alpha z_0+\tfrac{1}{2}(\beta z_1+\bar{\beta}z_3)+\gamma z_2+\delta(z_1z_3-z_0z_2)+\varepsilon=0\right\}.$$

It is easy to see that for any $M\in\Gamma$ there exists $\tilde{\alpha},\ldots,\tilde{\varepsilon}$ such that $M(V(\alpha,\ldots,\varepsilon))=V(\tilde{\alpha},\ldots,\tilde{\varepsilon})$ holds, in other words Γ is acting on $\{V(\alpha,\beta,\gamma,\delta,\varepsilon),$ as above$\}$. We may prove in the same way as Franke does in [1, Theorem 3.1.9] that this action is transitive. Because of $V=V(0,\omega,0,0,0)$ the zero set of Θ is $\cup V(\alpha,\ldots,\varepsilon)$ the union running over all $\alpha,\ldots,\varepsilon$ as above. The group $\Gamma(\omega)$ is acting trivially on both the set of even characteristics and the subset of $\{(\alpha,\gamma,\delta,\varepsilon)\bmod 2\}\subset\mathbb{F}_2^4$ appearing in Table 1. Thus $\Gamma/\Gamma(\omega)\cong\mathrm{Sp}_2(\mathbb{F}_2)$ is acting on these two sets. Both actions are transitive and furthermore that subgroup of $\mathrm{Sp}_2(\mathbb{F}_2)$ fixing the characteristic (1111) is identical with the subgroup fixing the element $(0000)\in\{(\alpha,\gamma,\delta,\varepsilon)\bmod 2\}$. Freitag has also proven that $\Theta(11,11,Z)$ is the only $\Theta(\mu,\nu,Z)$ that vanishes on all of V, and that its vanishing order is one. From this and all we said before it follows that for given $\alpha,\ldots,\varepsilon$ there is precisely one $\Theta(\Theta(\mu,\nu,Z)$ that is vanishing on all of $V(\alpha,\ldots,\varepsilon)$, its vanishing order at $V(\alpha,\ldots,\varepsilon)$ is one and its characteristic is given by Table 1. Therefore the vanishing order of a $\vartheta(Q,\mu,\nu,z)$ that is not vanishing identically at $z\in H$ equals the number of quintuples $(\alpha,\beta,\gamma,\delta,\varepsilon)$, $\alpha,\gamma,\delta,\varepsilon\in\mathbb{Z}$, $\beta\in\mathbb{Z}[i]$, $|\beta|^2-4\alpha\gamma-4\delta\varepsilon=2$, $(\alpha,\gamma,\delta,\varepsilon)\bmod 2$ as in Table 1 such that $M_Q z$ lies in $V(\alpha,\ldots,\varepsilon)$. Now $M_Q z\in V(\alpha,\ldots,\varepsilon)$ implies $Az^2-Bz+C=0$ where $A=\delta N$, $B=\alpha E+\frac{1}{4}\,\mathrm{tr}(\bar{\beta}F)+\gamma G$, $C=-\varepsilon$, and $\{M_Q z, z\in H\}$ will be contained in $V(\alpha,\ldots,\varepsilon)$ iff $A=B=C=0$ or equivalently iff $\delta=\varepsilon=B=0$, $|\beta|^2-4\alpha\gamma=2$. From this follows Proposition 1, and also the proof of Proposition 3 (a) is now easy. The following lemma enters:

Lemma. *Let be given numbers* $\alpha,\gamma,\delta,\varepsilon,E\in\mathbb{Q}$, $G\in\mathbb{Q}\setminus\{0\}$, $\beta,F\in\mathbb{Q}(i)$. *Put* $r=\alpha\left(EG-2\left|\frac{F}{2}\right|^2\right)-\frac{1}{4}G\,tr(\bar{\beta}F)-\gamma G^2$, $k=\alpha F+\beta G$, $A=\delta N$, $C=-\varepsilon$, $B=\alpha E+\frac{1}{4}\,\mathrm{tr}\,(\beta F)+\gamma G$, $N=EG-\left|\frac{F}{2}\right|^2$. *Then we have*

$$\frac{1}{G^2}(r^2+N|k|^2)+(4AC-B^2)=N(|\beta|^2-4\alpha\gamma-4\delta\varepsilon).$$

This can be proven by direct computation.

Next we want to analyze the self-intersection $R_2\cdot R_2$. To this end we set $X_0=\mathcal{H}_2/\Gamma(\omega)$ and look at the covering map $\rho: X_0\to X$. $\rho^{-1}(R_2)$ has 10 components $R_2(\mu,\nu)$. Since ρ is a Galois covering it is enough to study the self-intersec-

tion in X_0 of one fixed $R_2(\mu_0, \nu_0)$ as well as the intersection of $R_2(\mu_0, \nu_0)$ and $\cup^* R_2 := \bigcup_{(\mu,\nu)\neq(\mu_0,\nu_0)} R_2(\mu, \nu)$. Let

$$\Gamma^* = \Gamma\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} \cup I\Gamma\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

be the paramodular group of level $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ extended by the involution $I = \begin{pmatrix}U & 0\\ 0 & U^{t^{-1}}\end{pmatrix}$, $U = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 2\\ 1 & 0\end{pmatrix}$. (See [2, Chap. 5] for a precise definition.) Set $T = \begin{pmatrix}1 & 0\\ 0 & \omega\end{pmatrix}$ and look at the maps $\Psi: \Gamma^* \to \Omega_2$; $M \to \begin{pmatrix}T & 0\\ 0 & \bar{T}^{-1}\end{pmatrix} M \begin{pmatrix}T^{-1} & 0\\ 0 & \bar{T}\end{pmatrix}$ and $\psi: S_2 \to \mathscr{H}_2$; $W \mapsto TW\bar{T}^t$. (S_2 = Siegel upper half space.)

As Freitag has shown $\Psi(\Gamma^*)$ is the stabilizer of V in Γ. The pair (ψ, Ψ) defines a holomorphic map $\hat{\psi}: S_2/\Gamma^* \to X$ whose image is R_2. There is a commutative diagram

$$\begin{array}{ccc} S_2/\Gamma^* & \xrightarrow{\hat{\psi}} & X \\ \uparrow{\scriptstyle \lambda} & & \uparrow{\scriptstyle h} \\ S_2/G & \xrightarrow{\eta} & X_0 \end{array}$$

($G = \Psi^{-1}(\Psi(\Gamma^*) \cap \Gamma(\omega))$.)

The self-intersection of $\rho^{-1}(R_2)$ in X_0 can be described by equations on V and via ψ also in terms of equations on S_2. By inspection of these equations one gets

$$\eta^{-1}(R_2(11, 11) \cdot R_2(11,11)) = \lambda^{-1}(\mathscr{F}_1)$$

and

$$\eta^{-1}(R_2(11, 11) \cap (\cup^* R_2)) = \lambda^{-1}(\mathscr{F}_4),$$

$\mathscr{F}_\Delta \subset S_2/\Gamma^*$ being the Humbert surface of discriminant Δ. (See [6, pp. 210–212]). It is easy to see that there is no self-intersection of $\mathscr{F}_1$ and also the intersection of $\mathscr{F}_1$ and $\mathscr{F}_4$ is empty.

That $\mathscr{F}_1$ has no self-intersection means that there are no three quintuples $(\alpha_i, \ldots, \varepsilon_i)$, $i = 1, 2, 3$ belonging to the same characteristic (in the sense of Table 1) such that $\bigcap_{i=1}^{3} V(\alpha_i, \ldots, \varepsilon_i)$ is not empty. In other words, we always have $m(Q, \mu, \nu, z) \leqq 2$. Similarily Proposition 3 (c) is implied by the fact that $\mathscr{F}_1$ and $\mathscr{F}_4$ do not intersect. Finally we come to the proof of Proposition 2. Let be given a Q of determinant N and three characteristics (μ_i, ν_i), $i = 1, 2, 3$ such that $\theta(Q, \mu_i, \nu_i, z)$ vanishes identically. Then we automatically have

$$\{(\mu_i, \nu_i); \quad i = 1, 2, 3\} = \{(1111), (0110), (1001)\}.$$

Furthermore $\phi_Q(H)$ must be a component of $\pi^{-1}(C)$ where

$$C := R_2(1111) \cap R_2(0110) \cap R_2(1001).$$

Comparing the stabilizer groups in Γ of $\phi_Q(H)$ and a component of $\pi^{-1}(C)$ we find that we must have $N=2$. There are only two classes of determinant 2, namely $Q_1=[2,0,1]$ and $Q_2=[2,2\omega,2]$ and the proof of Proposition 2 can now be accomplished by using Proposition 1.

3 Examples

Let p be an odd prime and $Q_p=[p+1, 2(1+i), 2]$. From the theta-transformation-formulas given in [2, p. 9] we see that all then $\theta(Q_p, \mu, \nu, z)$ are $\Gamma_0(2p)-$ modularforms of weight 2 and characters of order 4. (This is true for all $Q=[E,F,G]$ with $E\equiv G\equiv 0 \bmod 2$ and $\frac{F}{2}\equiv 1+i \bmod 2$.) From Proposition 1 we easily conclude that $\Theta(Q_p, \mu, \nu, z)$ vanishes identically on H iff $(\mu,\nu)\in\{(1001),(1111)\}$. Let Δ be the discriminant of a zero of one of the remaining eight Θ's. Then (in the context of Proposition 2) we must have $k=0$ and $\alpha\equiv 1 \bmod 2$. Looking at Table 1 we get that the four $\Theta(Q_p,\mu,\nu,z)$ with $(\mu,\nu)\in\{(0011),(0001),(1100),(1000)\}$ have no zero on the upper half plane. (Comparing vanishing orders at the 4 cusps it is easy to express them as rational functions in $\eta(mz)$, $m\in\{1,2,p,2p\}$, $\eta(z)$ being Dedekind's etafunction.)

The group $\Gamma_0(2p)$ can be extended by a group of 4 (Atkin-Lehner) involutions to give a group $\tilde{\Gamma}_0(2p)$ with only one cusp. Using arguments as in [4, 13] it can be proven that $f_p(z) := \Theta(Q_p, 0110, z)$, $g_p(z) := \Theta(Q_p, 0010, z)\cdot\Theta(Q_p, 0100, z)$ and $\Theta_p(z) := \Theta(Q_p, 0000, z)$ are $\tilde{\Gamma}_0(2p)-$ modularforms with non-trivial quadratic characters.

Now conditions (6) in Proposition 2 imply $B\equiv 0 \bmod 2$ and $t := \frac{r}{4}\in\mathbb{Z}$. Let be given numbers $A, C\in\mathbb{N}$; $B, t\in\mathbb{Z}$; $A\equiv 0 \bmod 2p$; $B\equiv 0 \bmod 2$ such that the following congruences mod 2 are fulfilled:

(μ,ν)	t	$A/2p$	C	$B/2$
(0110)	$\frac{p+1}{2}$	0	0	$\frac{p-1}{2}$
(0000)	$\frac{p-1}{2}$	1	1	$\frac{p+1}{2}$
(0010)	$\frac{p+1}{2}$	1	0	$\frac{p-1}{2}$
(0100)	$\frac{p+1}{2}$	0	1	$\frac{p-1}{2}$

Then for $z=\frac{B+\sqrt{\Delta}}{2A}$ we have $m(Q_p,\mu,\nu,z)=1$.

Let us look at the special case $p=3$. For $(\mu,\nu)=(0110)$ the table above tells us $t\equiv 0 \bmod 2$ and this implies together with the condition $p-t^2>0$ that we

have $t=0$. Thus $\Delta=-12$ is the only possible discriminant. But $B\equiv C\equiv\frac{A}{6}\equiv 0$ mod 2 implies $B^2-4AC\equiv 4$ *mod* 16, and this shows that f_3 has no zero on the upper half plane. (We have $f_3(z)\in\mathbb{C}\eta(z)\eta(2z)\eta(3z)\eta(6z)$). Similarily we find that every zero of $g_3(z)$ is equivalent mod $\tilde{\Gamma}_0(6)$ to $a=\frac{6+\sqrt{-12}}{12}$ and every zero of Θ_3 to $b=\frac{8+\sqrt{-8}}{12}$. Both a and b are fixed by some transformation in $\tilde{\Gamma}_0(6)\backslash\Gamma_0(6)$. Using this it is easy to prove that the field of modularfunctions for the group $\tilde{\Gamma}_0(6)$ is generated by $X=(g_3)^2(\Theta_3 f_3)^{-2}$. Let A (resp. B) be the image of a (resp. b) in the compactification of the Riemann surface $H/\tilde{\Gamma}_0(6)$. Then the divisor of X is $A-B$.

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On submanifolds of two manifolds

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Let $f\colon M^n \to \mathbf{R}^{n+1}$ be an n-dimensional Euclidean hypersurface and suppose that M^n admits an isometric immersion $g\colon M^n \to S_c^{n+p} \subset \mathbf{R}^{n+p+1}$ into Euclidean sphere. We showed recently in [D-T] that if $2 \leqq p \leqq n-3$, then g, considered as an Euclidean submanifold, can be always described as the result of the smooth attachment of compositions. More precisely, for each connected component $\mathcal{U}_\lambda$ of an open dense subset $\mathcal{U}$ of M^n, there exists an open subset $V_\lambda \subset \mathbf{R}^{n+1}$ and an isometric immersion $h_\lambda\colon V_\lambda \to \mathbf{R}^{n+p+1}$ such that

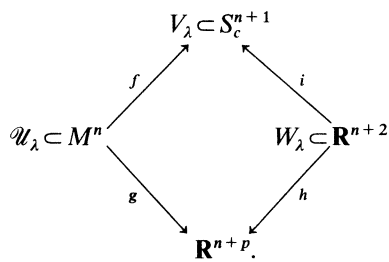
$$g|_{\mathcal{U}_\lambda} = h \circ f|_{\mathcal{U}_\lambda}.$$

Geometrically, this roughly means that M^n, f and g are produced by the intersection in $\mathbf{R}^{n+p+1}$ of $h(U_\lambda)$ with S_c^{n+p}.

This paper is mostly devoted to the "dual" case. Again, we assume $2 \leqq p \leqq n-3$, and consider a spherical hypersurface $f\colon M^n \to S_c^{n+1}$ and a second isometric immersion $g\colon M^n \to \mathbf{R}^{n+p}$. Suppose that at no point f has a principal curvature of multiplicity greater than or equal to $n-p$. If $p \leqq 6$, we show that g, restricted to each connected component $\mathcal{U}_\lambda$ of an open dense subset $\mathcal{U}$ of M^n, is a composition

$$g|_{\mathcal{U}_\lambda} = h \circ i \circ f|_{\mathcal{U}_\lambda}$$

where i denotes an umbilical inclusion of an open subset $V_\lambda \subset S_c^{n+1}$ into $\mathbf{R}^{n+2}$ and h is a local isometric immersion of $W_\lambda \subset \mathbf{R}^{n+2}$ into $\mathbf{R}^{n+p}$; namely,



If $p \geqq 7$, a second possibility arises. In this case, we may have k-ruled isometric immersions $F\colon N^{n+1} \to \mathbf{R}^{n+2}$ and $G\colon N^{n+1} \to \mathbf{R}^{n+p}$, with the same rulings of dimension $k \geqq n-p+4$, such that G cannot be obtained from F by a composition. Then M^n and f are locally produced by the intersection of $F(N^{n+1})$ with S_c^{n+1} contained in $\mathbf{R}^{n+2}$ as an umbilical hypersurface and g is just the composition with G of the inclusion of M^n into N^{n+1}.

Let Q_c^N denote an N-dimensional complete and simply connected Riemannian manifold of constant sectional curvature c. All of the above results remain valid in the case when f is an hypersurface in Q_c^{n+1} and g is an isometric immersion into $Q_{\tilde{c}}^{n+p}$ for $c \neq \tilde{c}$. Without the assumption on the multiplicities of the principal curvatures of the hypersurface f, our result does not hold. In fact, it can be shown (see [Ch]) that if M^n is a simply connected conformally flat submanifold of Q_c^{n+1}, then M^n can be isometrically immersed into $Q_{\tilde{c}}^{n+1}$ for any $\tilde{c} > c$.

We conclude this paper with some applications. First, we prove a rigidity result for minimal immersions. Then we extend the above mentioned result in [D-T] to the case when the hypersurface f is replaced by an isometric immersion of codimension 2.

1 The basic lemma

Given a symmetric bilinear form $\alpha\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^m$, we denote by $\nu_\alpha = \dim N(\alpha)$ the *nullity* of α where

$$N(\alpha) = \{X \in \mathbf{R}^n\colon \alpha(X, Y) = 0 \ \forall\, Y \in \mathbf{R}^n\},$$

and by ρ_α the *rank* of α defined by

$$\rho_\alpha = \min\{\operatorname{rank} A_\xi\colon \xi \in \mathbf{R}^m,\ \xi \neq 0\}$$

where A_ξ denotes the symmetric endomorphism of $\mathbf{R}^n$ defined by

$$\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle.$$

Lemma. *Let $\alpha\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^2$, $n \geqq 5$, be a symmetric bilinear form with $\nu_\alpha \leqq n-p-3$ and $\rho_\alpha \geqq p+1$ for some p satisfying $2 \leqq p \leqq n-3$. Let $\gamma\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^p$ be a symmetric bilinear form verifying*

$$\langle \alpha(X, Y), \alpha(Z, W)\rangle - \langle \alpha(X, W), \alpha(Z, Y)\rangle = \langle \gamma(X, Y), \gamma(Z, W)\rangle - \langle \gamma(X, W), \gamma(Z, Y)\rangle$$

for all $X, Y, Z, W \in \mathbf{R}^n$. Then there exists a 2-dimensional subspace $L \subset \mathbf{R}^p$ such that

$$\pi_L \circ \gamma = \alpha$$

where π_L denotes the orthogonal projection onto L.

Proof. Consider the symmetric bilinear form

$$\beta = \alpha \oplus \gamma\colon\ \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^2 \oplus \mathbf{R}^p \approx \mathbf{R}^{p+2}$$

where $\mathbf{R}^2 \oplus \mathbf{R}^p$ is endowed with the nondegenerate inner product $\langle\langle\ ,\ \rangle\rangle$ defined by

$$\langle\langle(\xi, \xi'), (\eta, \eta')\rangle\rangle = \langle \xi, \eta\rangle_{\mathbf{R}^2} - \langle \xi', \eta'\rangle_{\mathbf{R}^p}.$$

Then β is *flat* in the sense that it verifies for all $X, Y, Z, W \in \mathbf{R}^n$ that

$$\langle\langle\beta(X, Y), \beta(Z, W)\rangle\rangle - \langle\langle\beta(X, W), \beta(Z, Y)\rangle\rangle = 0.$$

Given $X \in \mathbf{R}^n$, consider the linear transformation $B(X)$: $\mathbf{R}^n \to \mathbf{R}^{p+2}$ defined by $B(X)(Z) = \beta(X, Z)$. We say that $X \in \mathbf{R}^n$ belongs to the set $\mathrm{RE}(\beta)$ of *regular elements* of β iff

$$\dim(\operatorname{im} B(X)) = \max\{\dim(\operatorname{im} B(Y)) : Y \in \mathbf{R}^n\}.$$

The set $\mathrm{RE}(\beta)$ is open and dense and for an element $X \in \mathrm{RE}(\beta)$ it holds that

$$\beta(Y, n) \subset \mathscr{U}(X) = \operatorname{im} B(X) \cap (\operatorname{im} B(X))^{\perp} \tag{1}$$

for all $n \in \ker B(X)$ and any $Y \in \mathbf{R}^n$. For the proof of the above statement and further facts and definitions we refer to [Mo], [dC-D] or [Da1].

First we prove the lemma under the assumption that the subspace $S(\beta) \subset \mathbf{R}^{p+2}$ defined by

$$S(\beta) = \operatorname{span}\{\beta(X, Y) : X, Y \in \mathbf{R}^n\}$$

is degenerate with respect to the metric induced by $\langle\langle\ ,\ \rangle\rangle$. In this case, by dimension reasons, the *null* subspace $W = S(\beta) \cap S(\beta)^{\perp}$ has dimension either 1 or 2.

If $\dim W = 2$, there exists a pseudo-orthonormal basis $\xi_1, \xi_2, \hat{\xi}_1, \hat{\xi}_2, \eta_1, \ldots, \eta_{p-2}$ of $\mathbf{R}^{p+2}$ such that $W = \operatorname{span}\{\xi_1, \xi_2\}$, and

$$\langle \xi_i, \xi_j\rangle = 0 = \langle \hat{\xi}_i, \hat{\xi}_j\rangle, \qquad \langle \xi_i, \hat{\xi}_i\rangle = 1,$$

$$\langle \xi_i, \eta_j\rangle = 0 = \langle \hat{\xi}_i, \eta_j\rangle, \qquad \langle \eta_i, \eta_j\rangle = \delta_{ij}.$$

We have

$$\beta = \sum_{i=1}^{2} \phi_i \xi_i \oplus \sum_{i=1}^{2} \hat{\phi}_i \hat{\xi}_i \oplus \sum_{j=1}^{p-2} \theta_j \eta_j.$$

From $\xi_i \perp S(\beta)$, we get that $\hat{\phi}_i = 0$. Thus

$$\beta = \sum_{i=1}^{2} \phi_j \xi_i \oplus \theta, \quad \text{where} \quad \theta = \sum_{j=1}^{p-2} \theta_j \eta_j.$$

Set $\xi_j = \delta_j + \mu_j$, $1 \leqq j \leqq 2$, where $\delta_1, \delta_2 \in \mathbf{R}^2$ and $\mu_1, \mu_2 \in \mathbf{R}^p$, and $L = \operatorname{span}\{\mu_1, \mu_2\}$. Notice that $\langle \delta_i, \delta_j\rangle = \langle \mu_i, \mu_j\rangle$, $1 \leqq i, j \leqq 2$. In this case the result follows from

$$0 = \langle\langle\beta(X, Y), \xi_j\rangle\rangle = \langle \alpha(X, Y), \delta_j\rangle - \langle \gamma(X, Y), \mu_j\rangle, \qquad 1 \leqq j \leqq 2.$$

Now suppose that $\dim W = 1$. Similarly, there exists a pseudo-orthonormal basis $\xi_1, \hat{\xi}_1, \eta_1, \ldots, \eta_p$ of $\mathbf{R}^{p+2}$ with $\eta_1 \in \mathbb{R}^2$ and $\eta_2, \ldots, \eta_p \in \mathbf{R}^p$ such that

$$\beta = \phi \xi_1 \oplus \theta, \quad \text{where} \quad \theta = \sum_{j=1}^{p} \theta_j \eta_j.$$

From $\dim W=1$, we have that $S(\theta)$ is a nondegenerate subspace of the Lorentzian space spanned by the η_j's. Also θ is flat. It follows from Corollary 2 of [Mo] that $\nu_\theta \geqq n-p$, which is in contradiction with our assumption that $\rho_\alpha \geqq p+1$.

To conclude the proof of the lemma it remains to show that $S(\beta)$ is degenerate. First observe that for any $X \in \mathrm{RE}(\beta)$, we have that $\operatorname{im} B(X)$ is a degenerate subspace. Otherwise, it follows easily from (1) that $\nu_\beta \geqq n-(p+2)$, which is in contradiction with $\nu_\alpha \leqq n-p-3$. On the other hand, if $\operatorname{im} B(X)$ is a null subspace for all $X \in \mathrm{RE}(\beta)$, we have from [dC-D, p. 969] that also $S(\beta)$ is null in which case we are done.

So assume that $S(\beta)$ is not degenerate and let $X \in \mathrm{RE}(\beta)$ be such that $\operatorname{im} B(X)$ is not null. We will only work out the case $\dim \mathscr{U}(X)=2$, being the case $\dim \mathscr{U}(X)=1$ similar and easier.

Let $\mathscr{U}(X)=\operatorname{span}\{\xi_1, \xi_2\}$. It is not difficult to see that it must exist Z_0, Z_1, $Z_2 \in \mathbf{R}^n$ so that

$$\langle \beta(Z_0, Z_1), \xi_1 \rangle \neq 0, \qquad \langle \beta(Z_0, Z_2), \xi_2 \rangle \neq 0.$$

For $n \in \ker B(X)$, it follows using (1) that

$$\beta(Y, n)=0 \quad \text{iff } \langle \beta(Y, n), \beta(Z_0, Z_1) \rangle = 0 = \langle \beta(Y, n), \beta(Z_0, Z_2) \rangle.$$

By flatness,

$$\beta(Y, n)=0 \quad \text{iff } \langle \beta(Z_0, n), \beta(Y, Z_1) \rangle = 0 = \langle \beta(Z_0, n), \beta(Y, Z_2) \rangle. \tag{2}$$

Consider the linear map $L\colon \ker B(X) \to \mathscr{U}(X)$ defined by $L(n)=\beta(Z_0, n)$. It follows from (2) that $\beta(Y, n)=0$ for $n \in \ker L$ and $Y \in \mathbf{R}^n$. Thus

$$\nu_\beta \geqq \dim \ker L \geqq \dim \ker B(X) - \dim \mathscr{U}(X) \geqq n-p-2,$$

a contradiction. This concludes the proof. □

2 The main results

Given an isometric immersion $g\colon M^n \to Q_c^{n+p}$, we define the *first normal space* $N_1^g(x)$ of g at $x \in M^n$ as the subspace of the normal space $T_x M^\perp$ given by

$$N_1^g(x)=\operatorname{span}\{\alpha_g(X, Y)\colon X, Y \in T_x M\}$$

where α_g stands for the vector valued second fundamental form of the immersion. Set $s_g(x)=\dim N_1^g(x)$.

Our statements in the introduction are a consequence of the following more general result.

Theorem 1 *Suppose that M^n, $n \geqq 5$, admits isometric immersions $f\colon M^n \to Q_c^{n+1}$ and $g\colon M^n \to Q_{\tilde{c}}^{n+p}$, $c>\tilde{c}$, with $s_g(x) \leqq n-3$ everywhere. Assume that f is an embedding which has no principal curvatures with multiplicity greater than or equal to $n-s_g(x)$ at any point, and denote by i_0 the umbilical inclusion of Q_c^{n+1} into $Q_{\tilde{c}}^{n+2}$. Then there exists an open dense subset $\mathscr{U} \subset M^n$ such that, on each connected component $\mathscr{U}_\lambda$ of $\mathscr{U}$, one of the following holds:*

1) *There exists a local isometric immersion $h=i_0 \circ H$ of Q_c^{n+1} into $Q_{\tilde{c}}^{n+p}$ such that $g=h \circ f$, where H is a local isometric immersion of $Q_{\tilde{c}}^{n+2}$ into $Q_{\tilde{c}}^{n+p}$.*

2) *There exist a Riemannian manifold* N^{n+1}, *an isometric immersion* $i: U_\lambda \to N^{n+1}$ *and* k*-ruled isometric immersions with the same rulings* $F: N^{n+1} \to Q_{\tilde{c}}^{n+2}$ *and* G: $N^{n+1} \to Q_{\tilde{c}}^{n+p}$, *with* $k \geqq n - s_g + 4$, *such that* $i_0 \circ f = F \circ i$ *and* $g = G \circ i$.
Moreover, case 2) *can only occur if* $s_g \geqq 7$ *on* $\mathcal{U}_\lambda$.

For the proof of the above we will make use of the following extension of Theorem 2 of [D-T] which is quite interesting by itself. For an isometric immersion f, let us denote by $\nu_f(x)$ and $\rho_f(x)$ the nullity and rank of the second fundamental form $\alpha_f(x)$.

Theorem 2 *Let* $f: M^n \to Q_c^{n+2}$ *and* $g: M^n \to Q_c^{n+p}$, $n \geqq 5$ *and* $p \geqq 2$, *be isometric immersions with* $s_g(x) \leqq n-3$ *for all* $x \in M^n$. *Assume further that* f *is an embedding, that* $\nu_f(x) \leqq n - s_g(x) - 3$ *and that* $\rho_f(x) \geqq s_g(x) + 1$ *everywhere. Then there exists an open dense subset* $\mathcal{U} \subset M^n$ *such that on each connected component* $\mathcal{U}_\lambda$ *of* $\mathcal{U}$ *one of the following holds:*
1) *There exists a local isometric immersion* h *of* Q_c^{n+2} *into* Q_c^{n+p} *such that* g *is a composition* $g = h \circ f$,
2) *The immersions* f *and* g *are both* k*-ruled with the same rulings where* $k \geqq n - s_g(x) + 3$,
3) *There exist a Riemannian manifold* N^{n+1}, *an isometric immersion* $i: \mathcal{U}_\lambda \to N^{n+1}$ *and* k*-ruled isometric immersions with the same rulings* $F: N^{n+1} \to Q_c^{n+2}$ *and* G: $N^{n+1} \to Q_c^{n+p}$, *where* $k \geqq n - s_g(x) + 4$, *so that* $f = F \circ i$ *and* $g = G \circ i$.
Moreover, cases 2) *and* 3) *can only occur if* $s_g(x) \geqq 7$ *on* $\mathcal{U}_\lambda$.

Proof. At $x \in M^n$, we have by the Gauss equations for both immersions that

$$\begin{aligned} &\langle \alpha_f(X, Y), \alpha_f(Z, W) \rangle - \langle \alpha_f(X, W), \alpha_f(Z, Y) \rangle \\ &\quad = \langle \alpha_g(X, Y), \alpha_g(Z, W) \rangle - \langle \alpha_g(X, W), \alpha_g(Z, Y) \rangle \end{aligned}$$

for all $X, Y, Z, W \in T_x M$. It follows from our assumptions and the lemma that $\alpha_g(x)$ decomposes orthogonally as

$$\alpha_g(x) = \alpha_f(x) \oplus \gamma(x)$$

where, again by the Gauss equations, $\gamma(x)$ has to be flat. In particular, we have from Corollary 1 of [Mo] that

$$\dim N(\gamma(x)) \geqq n - s_g(x) + 2. \tag{3}$$

A slight modification of the proof of Proposition 9 in [D-T] shows that the decomposition can be chosen to be smooth on any open subset $V \subset M$ where $s_g(x)$ is constant. More precisely, there exist a smooth vector subbundle $L \subset T_g M^\perp$ of rank 2 and a smooth bundle isometry τ: $T_f M^\perp \to L$ along V, such that $\alpha_g = \tau \circ \alpha_f \oplus \gamma$.

We claim that τ is parallel with respect to the connection on L induced by the normal connection $\nabla^{g\perp}$ of g. For each $X \in T_x M$, consider the skew-symmetric operator $K(X)$ on L given by

$$K(X)\eta = \tau \nabla_X^{f\perp} \tau^{-1} \eta - (\nabla_X^{g\perp} \eta)_L$$

where $(\ )_L$ denotes taking the L-component. We claim that the linear map K: $X \mapsto K(X)$ vanishes. Assume $K \neq 0$. Clearly, $\dim \ker K = n-1$. Comparing for $\eta \in L$ the Codazzi equations for both immersions yield

$$A_{K(X)\eta} Y - A_{(\nabla^{\perp}_X \eta)_{L^{\perp}}} Y = A_{K(Y)\eta} X - A_{(\nabla^{\perp}_X \eta)_{L^{\perp}}} X. \tag{4}$$

In particular,

$$\langle A_{K(X)\eta} Z, Y \rangle = 0$$

for all $X \in (\ker K)^{\perp}$, $Y \in \ker K$ and $Z \in N(\gamma)$. This together with (3) implies

$$\rho_f \leqq n - \dim N(\gamma) + 1 \leqq s_g - 1,$$

which is a contradiction and proves the claim.

Next, we show that for any $\eta \in L$ it holds that

$$(\nabla^{\perp}_Y \eta)_{L^{\perp} \cap N^g_1} = 0 \quad \text{for all } Y \in N(\gamma), \tag{5}$$

and

$$(\nabla^{\perp}_X \eta)_{N^g_1{}^{\perp}} = 0 \quad \text{for all } X \in TM. \tag{6}$$

Equation (5) follows from (4) and $K = 0$. To prove (6), suppose, on the contrary, that the linear functional ϕ given by

$$\phi(Y) = \langle \nabla^{\perp}_Y \delta, \eta_0 \rangle$$

is nonzero for vector fields $\delta \in N^g_1{}^{\perp}$ and $\eta_0 \in L$. Let Y_0 be a local unit vector field spanning $(\ker \phi)^{\perp}$. From the Codazzi equation for δ, we get

$$\langle A_{(\nabla^{\perp}_{Y_0} \delta)_L} W, Y \rangle = 0$$

for $Y \in \ker \phi$ and $W \in N(\gamma) \cap \{A_{\eta_0^{\perp}}, Y_0\}^{\perp}$, where $\eta_0^{\perp} \in L$ is orthogonal to η_0. This implies $\rho_f \leqq n - \dim N(\gamma) + 2 \leqq s_g$, a contradiction.

Given vector fields $Y, Z \in N(\gamma)$, $X \in TM$, $\xi \in L^{\perp}$, and using $(\nabla^{\perp}_Y \xi)_L = 0$ as follows from (5) and (6), the Codazzi equation for ξ yields

$$\langle \nabla_Y A_\xi X, Z \rangle + \langle \alpha_g(Y, Z), (\nabla^{\perp}_X \xi)_L \rangle = 0,$$

which is equivalent to

$$\langle (\tilde{\nabla}_X \xi)_{TM \oplus L}, \tilde{\nabla}_Y Z \rangle = 0, \tag{7}$$

where $\tilde{\nabla}$ stands for the connection in Q^{n+p}_c. At each point $x \in V$, define the subspace $W(x) \subset N(\gamma(x))^{\perp} \oplus L(x)$ by

$$W(x) = \operatorname{span}\{(\tilde{\nabla}_X \xi)_{TM \oplus L} : X \in TM, \xi \in L^{\perp}\}.$$

Clearly, $W(x)$ has codimension at most 2 in $N(\gamma(x))^{\perp} \oplus L(x)$. Consider an open subset $U \subset V$ where $N(\gamma(x))$ has constant dimension k and $W(x)$ has constant codimension ℓ in $N(\gamma(x))^{\perp} \oplus L(x)$. We have to treat separately three different possibilities.

Case $\ell=0$. From (7), we have

$$\langle \nabla_Y Z, X\rangle = 0 = \langle A_\eta Y, Z\rangle$$

for all $Y, Z \in N(\gamma)$, $X \in N(\gamma)^\perp$ and $\eta \in L$. Hence $N(\gamma)$ is an integrable distribution and its leaves are totally geodesic submanifolds of M^n, Q_c^{n+2} and Q_c^{n+p}. Thus f and g are both k-rules on U with $k \geqq n - s_g + 2$.

Case $\ell=1$. Assume $c=0$, being the proofs of the other cases similar. Let Γ be the line subbundle of $N(\gamma)^\perp \oplus L$ orthogonal to W along U and let T: $TM \oplus T_f M^\perp \to TM \oplus L$ be the vector bundle isometry defined by $T = I \oplus \tau$ where I denotes the identity map on TM. Set $\Omega = T^{-1}(\Gamma)$. Using $\Gamma \cap TM = \{0\}$, we have that the map $F: \Omega \to \mathbf{R}^{n+2}$ defined by

$$F(\xi(x)) = f(x) + \xi(x)$$

parametrizes a tubular neighborhood of $f(U)$ if restricted to a neighborhood N^{n+1} of the 0-section of Ω. We claim that the map $G: N^{n+1} \to \mathbf{R}^{n+p}$ given by

$$G(\xi(x)) = g(x) + T\xi(x)$$

is isometric with respect to the metric induced by F. Given a local section $\xi \in \Omega$, set $\xi = Z + \eta$, where $Z \in TM$ and $\eta \in T_f M^\perp$. Then

$$\begin{aligned} G_*(\xi(x))X &= g_*(x)X + \tilde{\nabla}_X T(Z+\eta) \\ &= g_*(x)(X + \nabla_X Z - A^g_{\tau(\eta)} X) + (\alpha_g(X, Z) + \nabla^{g\perp}_X \tau(\eta))_L \\ &= (g \circ f^{-1})_*(f_*(x)(X + \nabla_X Z - A^f_\eta X)) + \tau(\alpha_f(X, Z) + \nabla^{f\perp}_X \eta), \end{aligned}$$

and the claim follows easily.

From (6), we have that $(\nabla^{g\perp}_X \delta)_L = 0$ for all $\delta \in N_1^{g\perp}$ and any $X \in TM$. It follows that

$$N_1^{G\perp}(\xi(x)) = N_1^{g\perp}(x)$$

at any $x \in U$. Hence $s_G = s_g - 1$. Therefore,

$$\rho_F(\xi(x)) \geqq \rho_f \geqq s_g + 1 = s_G + 2.$$

By Theorem 2 in [D-T] either F and G are as in part 3) if restricted to an open subset of Ω or $G = h \circ F$ is a composition for a local isometric immersion $h: \mathbf{R}^{n+2} \to \mathbf{R}^{n+p}$.

Case $\ell=2$. In this case take Γ in Theorem 5 of [D-T] to be the rank 2 vector subbundle of $N(\gamma)^\perp \oplus L$ along U orthogonal to W. We obtain that g is a composition $g = h \circ f$, where h is a local isometric immersion of Q_c^{n+2} into Q^{n+p}.

It remains to show that if $k = n - s_g + 2$ on U then $\ell = 2$. By Proposition 7 of [D-T], α_g splits orthogonally and smoothly as a direct sum

$$\alpha_g = \alpha_f \oplus \gamma_1 \oplus \ldots \oplus \gamma_{s_g - 1},$$

where $\operatorname{rank} \gamma_j = 1$, $1 \leqq j \leqq s_g - 1$. Consider local frames $\xi_1, \ldots, \xi_{s_g-1}$ and $X_1, \ldots, X_{s_g-1}$, where $\operatorname{im} \gamma_j = \operatorname{span}\{\xi_j\}$ and each X_j is a unit eigenvector of A_{ξ_j} correspondent to the unique nonzero eigenvalue λ_j. By assumption,

$X_1, \ldots, X_{s_g-1}$ are linearly independent. From the Codazzi equations for f and g, and using $K=0$, we have

$$A_{(\nabla^\perp_X \eta)_{L^\perp}} Y = A_{(\nabla^\perp_Y \eta)_{L^\perp}} X$$

for any $\eta \in L$ and $X, Y \in TM$. Therefore,

$$\sum_{k=1}^{s_g-1} \lambda_k [\langle \nabla^\perp_{X_i} \eta, \xi_k \rangle \langle X_i, X_k \rangle - \langle \nabla^\perp_{X_j} \eta, \xi_k \rangle \langle X_j, X_k \rangle] X_k = 0,$$

which implies that

$$\langle \nabla^\perp_{X_i} \eta, \xi_k \rangle = \langle \nabla^\perp_{X_k} \eta, \xi_k \rangle \langle X_i, X_k \rangle.$$

Hence,

$$\langle \nabla^\perp_Z \xi_k, \eta \rangle = 0 \quad \text{for all } Z \perp X_k. \tag{8}$$

From (8), we get

$$W = \operatorname{span}\{(\tilde{\nabla}_{X_j} \xi_j)_{TM \oplus L} : 1 \leqq j \leqq s_g - 1\}$$

which implies that $\ell = 2$.

It is easy to see that there exists an open dense subset $\mathcal{U} \subset M$ such that on each connected component $\mathcal{U}_\lambda$ the functions $s_g(x)$, $k(x)$ and $\ell(x)$ are constant, and where $\ell = 1$, then either F and G are both k-ruled or they are non-ruled an any open subset of $\mathcal{U}_\lambda$. Finally, the last assertion of the theorem follows easily from the fact that a k-ruled hypersurface always satisfies $\rho_f \leqq 2(n-k)$. □

Remark. For $p=2$, the above rigidity result was obtained in [dC-D].

Proof of Theorem 1 The assumption on the principal curvatures of f implies that $\tilde{f} = i \circ f$ satisfies $\rho_{\tilde{f}}(x) \geqq s_g(x) + 1$ everywhere. Clearly, $\tilde{f}$ can not be ruled on any open subset of M^n and $\nu_{\tilde{f}} = 0$. The result follows from Theorem 2 and Theorem 1 of [D-T]. □

3 Applications

A consequence of our basic lemma is the following rigidity result for minimal immersions.

Theorem 3 *Let $f: M^n \to Q_c^{n+2}$ be an isometric immersion with $\nu_f(x) \leqq n-p-3$ and $\rho_f(x) \geqq p+1$ everywhere for $2 \leqq p \leqq 3$. Then any minimal isometric immersion $g: M^n \to Q_c^{n+p}$ is congruent to f in Q_c^{n+p}.*

Proof. By the lemma, at each point $x \in M^n$ there exists a 2-dimensional subspace $L \subset T_x M^\perp$ such that $\pi_L \circ \alpha_g = \alpha_f$. Thus $\beta = \pi_{L^\perp} \circ \alpha_g$ is flat by the Gauss equations for f and g and traceless from our assumption. We claim that $\beta = 0$. In fact, given an orthonormal basis $X_1, \ldots, X_n$ of $T_x M$, we have by flatness

$$0 = \left\langle \beta(X_i, X_i), \sum_{j=1}^{n} \beta(X_j, X_j) \right\rangle = \sum_{j=1}^{n} \|\beta(X_i, X_j)\|^2,$$

which proves our claim. From Theorem A of [D-R] we have that g reduces codimension to 2, and the remaining of the proof is now standard. □

The case when f is a minimal hypersurface was considered in [B-D-J]. The above result also improves Theorem 2 of [Da2] for the case of codimension 2. In fact, we do not have to assume that f is minimal and it is easy to see that the assumption there on the type number implies our assumption on ν_f. If f is real analytic, then it is sufficient to assume our hypothesis on ν and ρ at one point. Here we use that g is also real analytic because it is minimal and thus, since they are locally congruent, they must also be congruent globally.

As a consequence of Theorem 2, we have the following immediate extension to higher codimension of the result in [D-T] we discussed at the beginning of the introduction.

Theorem 4 *Suppose that* M^n, $n \geqq 5$, *admits isometric immersions* $f\colon M^n \to Q_c^{n+2}$ *and* $g\colon M^n \to Q_{\tilde{c}}^{n+p}$, $p \geqq 2$, *with* $c < \tilde{c}$ *and* $s_g(x) \leqq n-4$ *for all* $x \in M^n$. *Assume further that* $\rho_f(x) \geqq s_g(x)+2$ *everywhere. Set* $\tilde{g} = i_0 \circ g$, *where* $i_0\colon Q_{\tilde{c}}^{n+p} \to Q_c^{n+p+1}$ *is the umbilical inclusion. Then there exists an open dense subset* $\mathcal{U} \subset M^n$ *so that one of the following holds locally:*

1) *The immersion* $\tilde{g}$ *is a composition* $\tilde{g} = h \circ f$ *where* h *is a local isometric immersion of* Q_c^{n+2} *into* Q_c^{n+p+1}.

2) *There exist a Riemannian manifold* N^{n+1}, *an isometric immersion* $j\colon \mathcal{U} \subset M^n \to N^{n+1}$ *and* k*-ruled isometric immersions with the same rulings* $F\colon N^{n+1} \to Q_c^{n+2}$ *and* $G\colon N^{n+1} \to Q_c^{n+p+1}$, $k \geqq n - s_g(x) + 3$, *such that* $f|_{\mathcal{U}} = F \circ j$ *and* $\tilde{g}|_{\mathcal{U}} = G \circ j$.

Moreover, 2) can only occur if $s_g \geqq 7$ *on* $\mathcal{U}$.

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Norm convergence of random walks on compact hypergroups

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1 Introduction

Let G be a finite group with uniform (Haar) probability measure m. For a probability measure P on G, Aldous and Diaconis [1, 2] (see also [7, Chap. 4C]) introduced the n step "separation"

$$s(n)=|G|\max_{g\in G}\left\{\frac{1}{|G|}-P^n(g)\right\},$$

where $|G|$ is the order of G and P^n is the nth convolution power of P. They show that this is a natural tool for studying strong uniform times for random walks on G, and they show in Theorem 7 of [1] (stated as Theorem 5 in [7, Chap. 4C]) that if $\|P^n-m\|\to 0$ as $n\to\infty$, then $s(n)\to 0$ as $n\to\infty$ at a rate that depends on how fast $\|P^n-m\|$ converges to 0 but does not depend on the size of the group G. In particular, if $\|P^n-m\|$ is small then $s(2n)$ is small.

Since $s(n)=1-\min_{g\in G}\dfrac{P^n(g)}{m(g)}$, we can write

$$s(n)=1-\operatorname{essinf}(D(P^n)), \tag{1.1}$$

where $D(P^n)$ is the Radon-Nikodym derivative of P^n with respect to m. This definition is equally meaningful for a probability measure P on a compact group with normalized Haar measure m. Notice that for any measure μ on G, $D(\mu)$ denotes the Radon-Nikodym derivative with respect to m of the absolutely continuous component of μ. Also, for any measurable function f on G,

$$\operatorname{essinf}(f)=\sup\{\alpha\in(-\infty,\infty): m\{f>\alpha\}=1\},$$

where $\sup\emptyset=-\infty$.

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If we rewrite (1.1) as

$$\operatorname{essinf}(D(P^n)) = 1 - s(n),$$

then it is evident that $s(n) \to 0$ if and only if for each $\alpha \in (0, 1)$ there exists a positive integer q such that $m(\{D(P^q) > \alpha\}) = 1$ for $n \geqq q$. So the following theorem establishes the equivalence of $s(n) \to 0$ with the norm convergence of the convolution powers of a probability on a compact group.

Theorem 1.1 *Let P be a probability measure on a compact group G, with normalized Haar measure m, and suppose that $\|P^n - m\| \to 0$. Then for each $\alpha \in (0, 1)$, there exists a positive integer q such that $m(\{D(P^n) > \alpha\}) = 1$ for $n \geqq q$.*

This will follow immediately from the following lemma, which is adapted from Aldous and Diaconis [2].

Lemma 1.2 *If a measurable nonnegative function f on a compact group G, with normalized Haar measure m, satisfies $\int_G |f(x) - 1| \, dm(x) \leqq \varepsilon$, then*

$$\operatorname{essinf}(f * f) \geqq (1 - 2\sqrt{\varepsilon})(1 - \sqrt{\varepsilon})^2.$$

Proof. Let $A = \{x : f(x) \leqq 1 - \sqrt{\varepsilon}\}$. Then

$$\begin{aligned} \varepsilon &\geqq \int_G |f(x) - 1| \, dm(x) \\ &\geqq \int_A [1 - f(x)] \, dm(x) \\ &\geqq \sqrt{\varepsilon} m(A), \end{aligned}$$

and hence $m(A) \leqq \sqrt{\varepsilon}$. So for almost all $x \in G$,

$$\begin{aligned} (f * f)(x) &= \int_G f(xy) f(y^{-1}) dm(y) \\ &\geqq \int_{\{y : y \notin x^{-1}A \cup A^{-1}\}} f(xy) f(y^{-1}) dm(y) \\ &\geqq (1 - \sqrt{\varepsilon})^2 [1 - 2m(A)] \\ &\geqq (1 - \sqrt{\varepsilon})^2 (1 - 2\sqrt{\varepsilon}). \quad \square \end{aligned}$$

Proof of Theorem 1.1 Let $f = D(P^n)$, note that $\int_G |f(x) - 1| \, dm(x) \leqq \|P^n - m\|$, and apply Lemma 1.2 to conclude that $\operatorname{essinf}(D(P^{2^n})) \to 1$ as $\|P^n - m\| \to 0$. The lemma even gives the rate of convergence for $\operatorname{essinf}(D(P^{2^n})) \to 1$ in terms of the rate for $\|P^n - m\| \to 0$. $\square$

Theorem 1.1 shows, in particular, that if $\|P^n - m\| \to 0$ as $n \to \infty$, then some convolution power P^q of P satisfies

$$P^q \geqq \alpha m \quad \text{for some } \alpha \in (0, 1). \tag{1.2}$$

Thus we have the converse of a 1959 theorem of Kloss [15] which asserts that (1.2) implies $\|P^n - m\| \to 0$ as $n \to \infty$.

Theorem 1.1 implies that, for large n, the Radon-Nikodym derivative of P^n is never much below the Radon-Nikodym derivative 1, of the limit m. It is natural to wonder whether the following two-sided conclusion is possible:

(*) For each $\varepsilon\in(0,1)$, there exists a positive integer q so that $m(\{1-\varepsilon<D(P^n)<1+\varepsilon\})=1$ for $n\geqq q$.

Nothing like this is possible. There are absolutely continuous probabilities P on $[-\pi,\pi)$ so that none of the convolution powers are essentially bounded, i.e., $m(\{D(P^n)>\alpha\})>0$ for all $\alpha>0$. To create an example, for small positive δ we write $I(\delta)$ for the indicator function of $[-2\pi\delta,2\pi\delta]$ and show that the convolution powers $I(\delta)^n$ satisfy $I(\delta)^n\geqq\delta^{n-1}I(\delta)$ by induction. Then we select P so that $D(P)=\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\alpha_{ln}I(\delta_{ln})$ and $\|D(P)\|_1=2\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}(\alpha_{ln}\delta_{ln})=1$. We can also arrange that $\alpha_{ln}^n\delta_{ln}^{n-1}=l$ for all l and n, so that

$$\begin{aligned}m(\{D(P^n)\geqq l\})&\geqq m(\{\alpha_{ln}^n I(\delta_{ln})^n\geqq l\})\\&\geqq m(\{\alpha_{ln}^n\delta_{ln}^{n-1}I(\delta_{ln})\geqq l\})\\&=m(\{I(\delta_{ln})\geqq 1\})=2\delta_{ln}>0.\end{aligned}$$

Theorem 1.1 will be proved for compact hypergroups (as Theorem 3.1) in Sect. 3. A hypergroup K is a locally compact Hausdorff space in which the product of two elements is a probability measure p_x*p_y, where p_z for every $z\in K$ is the unit point mass at z. This convolution of point masses extends to a convolution $\mu*\nu$ of finite regular Borel measures. It is also assumed that K is provided with an involution $x\to x^-$, which is analogous to inversion in a group, and an identity element e that satisfies $p_e*\mu=\mu$ for all measures μ. The complete set of axioms is given in [12], our main reference for the basic theory of hypergroups, where hypergroups are called "convos". When for each pair of points x, y in K there exists a point $x\cdot y$ in K such that $p_x*p_y=p_{x\cdot y}$, then $\{K,\cdot\}$ is a locally compact group. Hypergroups arise in a natural way as a double coset space, as the space of orbits of a locally compact group, and as the dual of a compact group.

Much of harmonic analysis on groups carries over to hypergroups, but there are fundamental differences and care is necessary. Commutative hypergroups and compact hypergroups have an invariant measure m, which we will call Haar measure, but it is not known whether every hypergroup has a Haar measure. In general, the translations f_x of a function f only satisfy $\|f_x\|_p\leqq\|f\|_p$ for $1\leqq p\leqq\infty$ (3.3B in [12]) rather than equality. Similarly, compact sets A in K only satisfy $m(A)\leqq m(\{x\}*A)$ for $x\in K$ (3.3C in [12]) rather than equality.

The elegant simple proof of Aldous and Diaconis does not carry over to the case of compact hypergroups.

It is interesting to note that we were led to Theorem 3.1 in an attempt to extend to compact hypergroups a result in Mindlin [17] that is weaker than Theorem 1.1. We discovered Theorem 3.1 because some of Mindlin's arguments did not carry over to hypergroups. For more details, including the connection between Mindlin's results and earlier results in the literature, see Sect. 4.

2 Preparatory lemmas

Let K be a compact hypergroup with normalized Haar measure m and P a probability measure on K, and let P^n be the n-th convolution power of P.

For any measure μ on K, let μ_a and μ_s be the absolutely continuous and singular components, respectively, of μ with respect to m, and $D(\mu)$ the Radon-Nikodym derivative of μ_a with respect to m. We write $\mathrm{spt}(\mu)$ for the support of a measure μ, while $\mathrm{pos}(f)$ denotes the set $\{x: f(x)>0\}$ for a nonnegative function f. The other notations also follow [12].

Corollary 3.7 in [5] implies

Lemma 2.1 *If P^n converges weak-* to m, then for every nonempty open set $W \subseteq K$, there exists a positive integer n_0 so that*

$$\mathrm{spt}(P^n) \cap W \neq \emptyset$$

for $n \geqq n_0$.

Lemma 2.2 *Let $A, C \subseteq K$ and $x \in K$. If $C \cap (A * \{x\}) \neq \emptyset$, then $x \in A^- * C$.*

Proof. This is a consequence of 4.1 B in [12] with $B = \{x\}$. □

The following is the part of the variant of Lemma 4.2 D in [12], which we will need.

Lemma 2.3 *If μ is a non-negative Borel measure on K, and f a non-negative continuous function on K, then $f * \mu$ is a non-negative continuous function, and*

$$\mathrm{pos}(f) * \mathrm{spt}(\mu) \subseteq \mathrm{pos}(f * \mu).$$

Lemma 2.4 *For any probabilities P and Q on K,*

$$D(P) * D(Q) \leqq D(P) * Q \leqq D(P * Q), \tag{2.1}$$

and

$$D(P) * D(Q) \leqq P * D(Q) \leqq D(P * Q). \tag{2.2}$$

Proof. Since

$$D(P) * Q = D(P) * (Q_a + Q_s) = D(P) * Q_a + D(P) * Q_s,$$

and $D(P) * Q_s \geqq 0$ from 4.2 E in [12], we have

$$D(P) * Q \geqq D(P) * D(Q).$$

Also

$$P * Q = (P_a + P_s) * Q = P_a * Q + P_s * Q \geqq P_a * Q,$$

$P_a * Q$ is absolutely continuous with respect to m and $D(P_a * Q) = D(P) * Q$ by 3.3 E and 5.4 D in [12], so we have

$$D(P * Q) \geqq D(P) * Q.$$

Similarly, we can prove (2.2). □

Lemma 2.5 *If $m(D(P) \geqq \alpha_1) = 1$ for some $0 < \alpha_1 \leqq 1$ and if*

$$\alpha_2 = \alpha_1 (2 - \alpha_1), \tag{2.3}$$

then

$$\alpha_1 \leqq \alpha_2 \leqq 1 \quad \textit{and} \quad m(D(P*P) \geqq \alpha_2) = 1. \tag{2.4}$$

Proof. We have

$$P*P = P_a*P_a + P_a*P_s + P_s*P_a + P_s*P_s,$$

and hence from Lemma 2.4

$$D(P*P) \geqq f*f + f*P_s + P_s*f, \tag{2.5}$$

where $f \equiv D(P)$. Let $\beta = \int_K f\,dm = 1 - \|P_s\|$; then

$$(P_s*f)(x) = \int_K f(y^- * x)\,P_s(dy) \geqq \alpha_1(1-\beta).$$

A similar inequality holds for $f*P_s$ and so

$$f*P_s + P_s*f \geqq 2\alpha_1(1-\beta). \tag{2.6}$$

Now

$$\begin{aligned}(f*f)(x) &= \int_K f(x*y^-)f(y)m(dy) \\ &= \int_K (f(x*y^-)-\alpha_1)(f(y)-\alpha_1)m(dy) \\ &\quad + \alpha_1\{\int_K f(x*y^-)m(dy) + \int_K f(y)m(dy)\} - \alpha_1^2 \\ &\geqq \alpha_1\{\int_K f(x*y^-)m(dy) + \int_K f(y)m(dy)\} - \alpha_1^2,\end{aligned} \tag{2.7}$$

because $f \geqq \alpha_1$ a.e. So, combining (2.5)–(2.7),

$$\begin{aligned}D(P*P)(x) \geqq \alpha_1\{&\int_K f(x*y^-)m(dy) + \int_K f(y)m(dy)\} \\ &+ 2\alpha_1(1-\beta) - \alpha_1^2.\end{aligned}$$

From 7.2A in [12], K is unimodular, and hence, by 5.3B there,

$$m = m^-. \tag{2.8}$$

Thus, from (2.8), and 3.4 and 3.3F in [12],

$$\begin{aligned}\int_K f(x*y^-)m(dy) &= \int_K f_x(y^-)m(dy) \\ &= \int_K f_x\,dm \\ &= \int_K f\,dm \\ &= \beta.\end{aligned}$$

Therefore, (2.3) and (2.4) (the first inequality in (2.4) is trivial) are shown. □

Lemma 2.6 *For any probabilities P and Q on K, if $m(D(P)\geqq\alpha)=1$ for some $0<\alpha\leqq 1$, then $m(D(P*Q)\geqq\alpha)=1$.*

Proof. This is simply because we have, from (2.1),

$$D(P*Q)\geqq D(P)*Q\geqq\alpha*Q=\alpha \quad \text{a.e.} \quad \square$$

3 Main result

We will prove our main result in this section.

Theorem 3.1 *Let P be a probability measure on a compact hypergroup K with normalized Haar measure m, and suppose that $\|P^n-m\|\to 0$. Then for each $\alpha\in(0,1)$, there exists a positive integer q such that $m(\{D(P^n)>\alpha\})=1$ for $n\geqq q$.*

Proof. From the hypothesis it is clear that not all P^n can be singular with respect to m. Thus there exists an integer k such that $m(DP^k)>0)>0$. Choose $t>0$ so that $m(D(P^k)I_{\{D(P^k)<t\}}>0)>0$, and let

$$g=(D(P^k)I_{\{D(P^k)<t\}})*(D(P^k)I_{\{D(P^k)<t\}}). \tag{3.1}$$

Then g is non-negative, continuous (5.5D in [12]), and $g\not\equiv 0$ (5.5L in [12]). So $U=\{g>0\}$ is open and nonempty. Select $x_0\in U$; then $\{e\}*\{x_0\}\subseteq U$. Now apply 3.2D in [12] with $A=\{e\}$ and $B=\{x_0\}$ to obtain open sets W_4 and W_3 so that $e\in W_4$, $x_0\in W_3$ and $W_4*W_3\subseteq U$. In particular,

$$W_4*\{x_0\}\subseteq U. \tag{3.2}$$

Again apply 3.2D in [12] with $A=\{e\}$, $B=\{e\}$ and $U=W_4$ to obtain open sets W_1 and W_2 so that

$$e\in W_1\cap W_2$$

and $W_1*W_2\subseteq W_4$. Let $V=(W_1\cap W_2)\cap(W_1\cap W_2)^-$. Then

$$e\in V, \tag{3.3}$$

$V=V^-$ and $V*V\subseteq W_1*W_2\subseteq W_4$. Thus, from (3.2),

$$V*V*\{x_0\}\subseteq U. \tag{3.4}$$

From (3.3), $x\in V*\{x\}$ for each $x\in K$. By 4.1D in [12], each $V*\{x\}$ is open. So $\{V*\{x\}\colon x\in K\}$ is an open cover of K. Since K is compact, there exist a positive integer m and $x_1,\ldots,x_m\in K$ so that

$$\bigcup_{i=1}^{m} V*\{x_i\}=K. \tag{3.5}$$

Since $P^n \to m$ weak-*, we can apply Lemma 2.1 to the finite collection $\{\{x_0^-\} * V * \{x_i\} : i = 1, \ldots, m\}$ of open sets to obtain p so that

$$\text{spt}(P^p) \cap \{x_0^-\} * V * \{x_i\} \neq \emptyset \tag{3.6}$$

for $i = 1, \ldots, m$. By Lemma 2.2 and 3.4C in [12],

$$x_i \in (\{x_0^-\} * V)^- * \text{spt}(P^p) = V * \{x_0\} * \text{spt}(P^p). \tag{3.7}$$

Therefore, from (3.4), (3.5) and (3.7),

$$\begin{aligned} K &= \bigcup_{i=1}^{m} V * \{x_i\} \\ &\subseteq V * V * \{x_0\} * \text{spt}(P^p) \\ &\subseteq U * \text{spt}(P^p), \end{aligned}$$

which is the same as

$$K = \{g * P^p > 0\} \tag{3.8}$$

by Lemma 2.3. Let $l = 2k + p$. Lemma 2.4 and 5.5J in [12] imply that

$$D(P^l) \geqq D(P^k) * D(P^k) * P^p \geqq g * P^p \quad \text{a.e.} \tag{3.9}$$

Since $g * P^p$ is continuous by Lemma 2.3, (3.8) and (3.9) imply that there exists $\alpha_1 > 0$ such that $m(D(P^l) \geqq \alpha_1) = 1$. Lemma 2.5 gives us a sequence $\{\alpha_n\}$ in $(0, 1]$ so that

$$m(D(P^{2^{n-1}l}) \geqq \alpha_n) = 1$$

and

$$\alpha_{n+1} = \alpha_n(2 - \alpha_n). \tag{3.10}$$

The α_n's in (3.10) are nondecreasing with limit 1. Therefore, for every $\alpha \in [0, 1)$, we can find m so that $\alpha_n > \alpha$ for $n \geqq m$, and (2) is proved by applying Lemma 2.6. □

The hypothesis $\|P^n - m\| \to 0$ in Theorem 3.1 was only used in the proof to establish that $D(P^k)$ is nonzero for some positive integer k and that $P^n \to m$ weak-*. Since the conclusion of Theorem 3.1 clearly implies norm convergence, we have the following corollary.

Corollary 3.2 *For a probability measure P on a compact hypergroup K with normalized Haar measure m, norm convergence $\|P^n - m\| \to 0$ is equivalent to $D(P^k)$ is nonzero for some positive integer k, and P^n converges weak-* to m.*

4 Concluding remarks

For compact groups, Corollary 3.2 is well known. Since the literature about this and some related results is scattered and somewhat confusing, we will present

a brief account of what we know. We have been greatly assisted by Phillipe Bougerol, Vadim Kaimanovich, and Sadahiro Saeki. The next theorem will help put things into context.

Theorem 4.1 *For a probability measure P on a compact group G, the following are equivalent:*
(1) *$D(P^k)$ is nonzero for some positive integer k, and P^n converges weak-* to m.*
(2) *For each $\alpha\in(0,1)$, there exists a positive integer q so that $m(\{D(P^n)>\alpha\})=1$ for $n\geqq q$.*
(3) *$P^q\geqq\alpha m$ for some $\alpha\in(0,1)$ and positive integer q.*
(4) *$m(\{D(P^q)>0\})>1/2$ for some positive integer q.*
(5) *$\|P^n-m\|\to 0$ as $n\to\infty$ (in which case the convergence is exponential).*
(6) *$D(P^k)$ is nonzero for some positive integer k, and $\|f*P^n\|_1\to 0$ for all $f\in L^1(G)$ such that $\int_G f\,dm=0$.*

Condition (1) is obviously a necessary condition for $\|P^n-m\|\to 0$. Kawada and Itô [14], in 1940, and Stromberg [20], in 1960, showed that P^n converges weak-* to m if and only if P is aperiodic, i.e., $\mathrm{spt}(P)$ is not contained in any coset of any proper normal closed subgroup of G and the smallest closed subgroup of G containing $\mathrm{spt}(P)$ is G itself. Grenander (§ 3.2 in [11]) contains this result, as well as some related results and early history.

The implications (5)⇒(6)⇒(1) and (2)⇒(3)⇒(4)⇒(1) are obvious, the last one being a consequence of the Kawada-Itô theorem. Corollary 3.2 and Theorem 1.1 give the implications (1)⇒(5)⇒(2), completing the proof of Theorem 4.1.

As mentioned in § 1, Kloss [15] proved (3)⇒(5) in 1959. In 1972, Bhattacharya [3] showed (4)⇒(5). In 1978, Bougerol (Theorem 4 in [6]) showed (1)⇒(3) provided that G has finitely many connected components. In 1979, Major and Shlosman (Remark 3 in [16]) observed that if $D(P)\neq 0$ and P is aperiodic, then (5) holds; it is easy to get from this observation to the well-known implication (1)⇒(5); see also [13]. In 1990, Mindlin [17] observed that (1)⇒(4) follows from his work on random measures; this combines with Bhattacharya's work to give another proof that (1)⇒(5). Mindlin also corrects an error in [3].

The second property in (6),

$$\text{(MC)}\qquad \|f*P^n\|_1\to 0 \text{ for all } f\in L^1(G) \text{ such that } \int_G f\,dm=0,$$

is called mixing by convolutions by Rosenblatt [18]. This is a natural condition to study for noncompact groups, where convolution powers cannot be expected to converge to m in any sense. See [18] and its references. Property (MC) implies

$$\text{(EC)}\qquad \left\|\frac{1}{N}f*\sum_{n=1}^{N}P^n\right\|_1\to 0 \text{ as } N\to\infty \text{ for all } f\in L^1(G) \text{ such that } \int_G f\,dm=0,$$

which is called ergodic by convolutions or the Choquet-Deny property. In general, (EC) does not imply (MC), even for compact groups. However, Theorem 2 of Glasner [10] shows that (MC) and (EC) are equivalent for any aperiodic probability P on a locally compact group for which $D(P^k)$ is nonzero for some positive integer k. In view of the Kawada-Itô theorem, Glasner's theorem says

that for compact groups (MC) and (EC) are equivalent provided (1) of Theorem 4.1 holds. In fact, since (1)⇒(6), both properties hold in the presence of (1).

Theorem 4.1 also holds for compact hypergroups; see Theorem 3.1 and its corollary. The implication (4)⇒(1) follows from Bloom and Heyer's generalization of the Kawada-Itô theorem (Theorem 3.5 in [5]) and Theorem 3.3.3 in [4].

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Fixed sets and free subgroups of groups acting on metric spaces

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1 Introduction

Fixed point properties versus existence of free subgroups in groups of homeomorphisms of a topological space have been studied in various contexts. For example, this is present in some sense in the "Tits alternative" [T2] concerning subgroups of linear groups, see de la Harpe [dH]. Other examples are provided by infinite trees, see Tits [T1, T3], Halin [H2], Nebbia [N1, N2] and Pays and Valette [PV]: a group of automorphisms of a tree either contains a free group on two generators, or fixes a vertex, an edge, an end or a pair of ends. The situation is similar, when instead of a tree one considers an arbitrary infinite, connected graph, see Halin [H2], Woess [W1], Möller [Mö] and Jung [J2]. Similar results are also known for metric spaces which are hyperbolic in the sense of Gromov [Gr], see Ghys and de la Harpe [GdH], Coornaert et al. [CDP] or Short [Sh].

The many similar features of the end compactification of a graph and the hyperbolic compactification of a hyperbolic metric space were the main motivation for the work presented here. The purpose of this note is to describe an approach which unifies and also simplifies the above-mentioned results.

We consider the general case of a metric space X and a group G of automorphisms (i.e., self-isometries) of X. We assume that we have a compactification of X which is compatible with the action of G. The principal purpose of this paper is to provide a simple condition concerning compactification and group action, which – when satisfied – allows us to deduce the following results:

1. A classification of the elements of G into elliptic, parabolic or hyperbolic according to their fixed points on the boundary of X,
2. a description of the limit set of G on the boundary, and
3. an alternative between fixed sets under G and the presence of a free subgroup on two generators.

The condition, which we call "contractivity", is hidden between the lines in papers of Furstenberg [F1, F2, F3] in connection with the study of random walks on homogeneous spaces. (In [F3], the word "contractive" is also used,

but in a slightly different context.) Indeed, Furstenberg's method has been applied, with a direct use of contractivity in our sense, by Cartwright and Soardi [CS] and Woess [W2] to random walks on the automorphism groups of trees and vertex-transitive graphs, respectively. The usefulness of the approach proposed here should be seen in the simplicity, both in deducing the mentioned results and in verifying this property in typical cases.

Contractivity bears an evident similarity (but is not identical) with the theory of *convergence groups* acting on $\mathbb{R}^n$, introduced by Gehring and Martin [GM] and carried further by Martin and Skora [MS] and others. Indeed, when finishing the first version of this note, the author had been unaware of this related work, which was then kindly pointed out to him by G. Mess and by the referee.

In § 2, we present the main results (Definitions 1–2 and Theorems 1–3), which are proved in § 3. Various applications are then described in § 4:

We first indicate how Theorem 3 applies to random walks on G and the corresponding harmonic functions. Next, we consider infinite, locally finite, connected graphs: it is very easily seen that the end compactification of such a graph is contractive with respect to the automorphism group. Thus, the above mentioned results for locally finite graphs arise as corollaries of our general approach, proofs get more transparent (even slightly shorter) and make the inherent topological spirit of Halin's work [H2] explicit.

We then turn our attention to a case where, to the author's knowledge, the results are new at least in part: if X is a hyperbolic, geodesic metric space, then again it is easy to show that its hyperbolic compactification is contractive with respect to $\mathrm{Aut}(X)$. Thus, we obtain once more a comprehensible and short access to the (known) classification of the elements of $\mathrm{Aut}(X)$, but also the alternative "free subgroup-fixed set" for subgroups of $\mathrm{Aut}(X)$. The classical case of groups of Möbius transformations of the unit disc (see e.g. [dH]) is a typical example. Finally, we study the particular case of hyperbolic graphs whose boundary has more than two elements: if $G \leqq \mathrm{Aut}(X)$ acts quasi-transitively on X and is closed with respect to pointwise convergence, then G either contains a discrete free subgroup on two generators or is nonunimodular. Combining this with results of Soardi and Woess [SW] and Salvatori [Sa], we obtain that a quasitransitive, infinite hyperbolic graph either is a strip in the sense of Jung and Watkins [JW] or an expander, i.e., it satisfies a strong isoperimetric inequality. This was known only for (Cayley graphs of) hyperbolic groups.

2 Statement of results

Let (X, d) be a *proper* metric space, i.e., all closed balls in X are compact. By a *compactification* of X we mean a compact Hausdorff space $\hat{X} \supset X$ having countable neighbourhood bases, such that X is open and dense and the inclusion mapping $X \hookrightarrow \hat{X}$ is a homeomorphism. The *boundary* is $\partial X = \hat{X} \setminus X$. Let G be a group of autmorphisms (self-isometries) of (X, d). We say that $\hat{X}$ is a *G-compactification*, if every $g \in G$ extends to a homeomorphism of $\hat{X}$. (Most reasonable compactifications are $\mathrm{Aut}(X)$-compactifications.)

Definition 1 (1) A G-compactification $\hat{X}$ of (X, d) is called *projective*, if for sequences (x_n), (y_n) in X,

$$x_n \to \xi \in \partial X \quad \text{and} \quad d(y_n, x_n) \leqq M < \infty \quad \text{implies} \quad y_n \to \xi.$$

(2) We say that $\hat{X}$ is *contractive*, if in addition to (1) the following holds: whenever (g_n) is a sequence in G such that for some $x \in X$,

$$g_n x \to \xi \in \partial X \quad \text{and} \quad g_n^{-1} x \to \eta \in \partial X,$$

then $g_n z \to \xi$ uniformly outside every neighbourhood of η.

The last condition means that if U and V are neighbourhoods of ξ and η in $\hat{X}$, respectively, then there is $n_0 = n_0(U, V)$ such that $g_n z \in U$ for all $z \in \hat{X} \setminus V$, $n \geqq n_0$. Note that we allow $\xi = \eta$.

If $g \in G$ fixes a compact, nonempty subset of X, then we call g *elliptic*.

Theorem 1 *Let $\hat{X}$ be a contractive G-compactification of X. If $g \in G$ is non-elliptic, then there are two possibilities: either*

(1) *g fixes precisely two elements $\xi, \eta \in \partial X$,*

$$\lim_{n \to \infty} g^n x = \xi \quad \textit{and} \quad \lim_{n \to \infty} g^{-n} x = \eta$$

for every $x \in X$, or
(2) *g fixes a unique $\xi \in \partial X$,*

$$\lim_{n \to \infty} g^n x = \xi \quad \text{and} \quad \lim_{n \to \infty} g^{-n} x = \xi$$

for every $x \in X$.

In case (1), we say that g is *hyperbolic*, while in case (2), g is called *parabolic*. By Fix g we denote the set of fixed points of g on ∂X.

Definition 2 If $\hat{X}$ is a contractive G-compactification of X, then the *limit set* $L(G)$ of G is the set of accumulation points of an orbit Gx in ∂X. The *hyperbolic limit set* $H(G)$ is the set of points in ∂X which are fixed by some hyperbolic element of G.

Note that $L(G)$ does not depend on $x \in X$ by projectivity. Of course, $L(G)$ is compact and fixed by G, and $H(G) \subset L(G)$.

Theorem 2 *Let $\hat{X}$ be a contractive G-compactification of X.*

(i) *If $L(G)$ is nonempty, then it has one, two, or infinitely many elements.*
(ii) *If $L(G)$ has more than one element, then $H(G)$ is nonempty and dense in $L(G)$.*

If $G \leqq \mathrm{Aut}(X)$ as above, then we say that the action of G on X is *discontinuous*, if there is a nonempty open set $O \subset X$ such that $g(O) \cap O = \emptyset$ for every nonidentity element g of G.

Theorem 3 *If $\hat{X}$ is a contractive G-compactification of X then one of the following four cases holds:*

(a) *G contains two hyperbolic elements g_1, g_2 without common fixed points, g_1, g_2 freely generate a free subgroup of G whose action on X is discontinuous, and all nonidentity elements of $\langle g_1, g_2 \rangle$ are hyperbolic.*
(b) *G fixes a nonempty compact subset of X.*
(c) *G fixes a unique element of ∂X.*
(d) *G fixes a unique pair of elements of ∂X, and $L(G) = H(G)$ consists of this pair.*

In the last statement we mean that the pair of points is fixed as a set. We remark that in (c), it may well be that $L(G)$ is infinite. Also, it is not hard to construct examples where (c) holds with $|L(G)|=1$, while all elements of G are elliptic. If one of (b), (c), (d) holds, then we shall say that G has the *fixed set property*.

3 Proofs of the theorems

Proof of Theorem 1 Let $g\in G$ be non-elliptic. Choose $x\in X$. As X is proper, we must have $\lim_{n\to\infty} d(x, g^{\pm n}x)=\infty$. This is a consequence of the classification of monothetic groups (see Hewitt and Ross [HR, § 9]), applied to the closure of $\langle g\rangle$ in $\mathrm{Aut}(X)$ with respect to the topology of uniform convergence on compact sets: as $\langle g\rangle^-$ cannot be compact, it is infinite cyclic, whence $g^{\pm n}x$ tends to infinity, compare with [GdH, Lemme 8.18]. Hence there is a subsequence (n') of positive integers such that $g^{n'}x\to\xi\in\partial X$, and by compactness of $\hat{X}$ we may suppose that $g^{-n'}x\to\eta\in\partial X$.

We have by projectivity

$$g\xi=\lim_{n'} g^{n'}gx=\xi$$

and analogously $g\eta=\eta$.

By contractivity, $g^{n'}\zeta\to\xi$ for every $\zeta\in\hat{X}$ different from ξ, η. Hence g cannot fix other points than ξ and η. This and the argument used above for showing $g\xi=\xi$ yield that ξ and η are the only accumulation points of $\{g^nx\,|\,n\in\mathbb{Z}\}$ on the boundary.

First consider the case $\xi\neq\eta$. Let U, V be disjoint open neighbourhoods in $\hat{X}$ of ξ, η, respectively, not containing x. By contractivity, there is n_0 in (n') such that

$$g^{n_0}(U^-)\subset g^{n_0}(\hat{X}\setminus V)\subset U.$$

Inductively,

$$g^{kn_0}x\in U \qquad \text{for all } k>0.$$

Therefore, every accumulation point of the sequence $(g^{kn_0}x)$ must lie in $U\cap\partial X$, and hence must be ξ by the above. In other words, $g^{kn_0}x\to\xi$. By projectivity, also $g^{kn_0+r}x=g^{kn_0}(g^rx)$ tends to ξ as $k\to\infty$. Hence

$$\lim_{n\to\infty} g^nx=\xi.$$

Once more by projectivity, this is true for every $x\in X$. Analogously,

$$\lim_{n\to\infty} g^{-n}x=\eta.$$

This completes the proof in case (1).

Now suppose that $\xi=\eta$. By the above, this must be the only accumulation point of $\{g^nx\,|\,n\in\mathbb{Z}\}$, i.e.,

$$\lim_{n\to\infty} g^{\pm n}x=\xi. \quad \square$$

Proof of Theorem 2 We start with (ii). Let $\xi \in L(G)$. By assumption, there is $\xi' \in L(G)$, $\xi' \neq \xi$. Let $x \in X$. There are sequences (g_n) and (h_n) in G such that $g_n x \to \xi$ and $h_n x \to \xi'$. Without loss of generality, we may assume that $g_n^{-1} x \to \eta$, $h_n^{-1} x \to \eta' \in \partial X$.

Case 1 $\eta \neq \xi$. Let U, V be disjoint open neighbourhoods in $\hat{X}$ of ξ and η, respectively, which are at positive distance from x. By contractivity, there is an index n_0 such that

$$g_{n_0}(\hat{X} \setminus V) \subset U \quad \text{and} \quad g_{n_0}^{-1}(\hat{X} \setminus U) \subset V.$$

Consequently, $g_{n_0}^k x \in U$ and $g_{n_0}^{-k} x \in V$ for every $k > 0$. Therefore $d(x, g_{n_0}^{\pm k} x) \geqq d(x, (U \cup V) \cap X) > 0$ and so no subsequence of $(g_{n_0}^k x)_{k \in \mathbb{Z}}$ may converge in X. As in the proof of Theorem 1, we conclude that this sequence has accumulation points $\xi_1 \in U \cap \partial X$ and $\eta_1 \in V \cap \partial X$. By Theorem 1, g_{n_0} is hyperbolic, and $\xi_1 \in H(G) \cap U$.

Case 2 $\eta' \neq \xi'$. As in case 1, we can find n_0 such that h_{n_0} is hyperbolic with fixed points ξ_1 and η_1. One of them, say ξ_1, must be different from η. By contractivity, $g_n \xi_1 \to \xi$, and $g_n \xi_1$ is fixed by the hyperbolic element $g_n h_{n_0} g_n^{-1} \in G$: in other words, $U \cap H(G) \neq 0$.

Case 3 $\eta = \xi$ and $\eta' = \xi'$. We choose disjoint open $\hat{X}$-neighbourhoods U and U' of ξ and ξ', respectively, which are at positive distance from x. By contractivity, there is n_0 such that

$$g_{n_0}^{\pm 1}(\hat{X} \setminus U) \subset U \quad \text{and} \quad h_{n_0}^{\pm 1}(\hat{X} \setminus U') \subset U'.$$

Set $g = g_{n_0} h_{n_0}$. Then

$$g(\hat{X} \setminus U') \subset g_{n_0}(U') \subset U \quad \text{and} \quad g^{-1}(\hat{X} \setminus U) \subset h_{n_0}^{-1}(U) \subset U'.$$

As above, we conclude that g is hyperbolic with fixed points $\xi_1 \in U$ and $\eta_1 \in U'$. Once more, $U \cap H(G) \neq \emptyset$.

We now prove (i). Suppose that $|L(G)| \geqq 3$. Then, by (ii), there must be two hyperbolic elements $g, h \in G$ with $|\operatorname{Fix} g \cup \operatorname{Fix} h| \geqq 3$. Let $\xi \in \operatorname{Fix} g \setminus \operatorname{Fix} h$. Then, for every $n \geqq 0$, $h^n g h^{-n}$ is hyperbolic and fixes $h^n \xi$. As h is hyperbolic, all $h^n \xi$, $n \geqq 0$, must be distinct. This proves that $L(G)$ is infinite. □

The proof of Theorem 3 is now a combination of the following three propositions.

Proposition 1 *Let $G \leqq \operatorname{Aut}(X)$ and $h_1, h_2 \in G$. Suppose that for some contractive G-compactification of X, h_1 and h_2 are hyperbolic with no common fixed point. Then there is $k \geqq 1$ such that $g_1 = h_1^k$ and $g_2 = h_2^k$ freely generate a free subgroup of G, all nonidentity elements of $\langle g_1, g_2 \rangle$ are hyperbolic, and the action of $\langle g_1, g_2 \rangle$ on X is discontinuous.*

Proof. This a typical application of the "table tennis lemma", see e.g. [dH]. We need some details for showing hyperbolicity and discontinuity.

Let $\operatorname{Fix} h_i = \{\xi_i, \eta_i\}$, $i = 1, 2$. These sets are disjoint. We can find an open (nonempty) set $O \subset X$ and open $\hat{X}$-neighbourhoods U_i, V_i of ξ_i and η_i, respectively ($i = 1, 2$), such that all five sets are pairwise disjoint. We set $W_i = U_i \cup V_i$. By

hyperbolicity of h_i, there is an index n_0 (without loss of generality >0) such that

$$h_i^{n_0}(\hat{X}\setminus V_i)\subset U_i \quad \text{and} \quad h_i^{-n_0}(\hat{X}\setminus U_i)\subset V_i, \qquad i=1,2.$$

We set $g_i=h_i^{n_0}$. The g_i are hyperbolic with the same fixed points as h_i. We show that $\langle g_1, g_2\rangle=\langle g_1\rangle * \langle g_2\rangle$ (free product). Let $g\in G$ be represented as a nonempty word in integer powers of the g_i.

Case 1 $g=g_1^{k_1} g_2^{\ell_1} \dots g_1^{k_r} g_2^{\ell_r}$ with integers $k_j, \ell_j\neq 0$. Then

$$g(\hat{X}\setminus W_2)\subset g_1^{k_1} g_2^{\ell_1} \dots g_1^{k_r}(W_2)\subset \dots \subset g_1^{k_1}(W_2)\subset W_1.$$

Analogously, $g^{-1}(\hat{X}\setminus W_1)\subset W_2$. We conclude that g is hyperbolic and that $g(O)\cap O=\emptyset$. In particular, g is different from the identity.

The case $g=g_2^{\ell_1} g_1^{k_1} \dots g_2^{\ell_r} g_1^{k_r}$ (with $k_j, \ell_j\neq 0$) is completely analogous.

Case 2 $g=g_1^{k_0} g_2^{\ell_0} \dots g_1^{k_{r-1}} g_2^{\ell_{r-1}} g_1^{k_r}$ with integers $k_j, \ell_j\neq 0$ (or, analogously, $g=g_2^{\ell_0} g_1^{k_0} \dots g_2^{\ell_{r-1}} g_1^{k_{r-1}} g_2^{\ell_r}$).

As above, $g(\hat{X}\setminus W_1)\subset W_1$ (or $g(\hat{X}\setminus W_2)\subset W_2$, respectively). We see that $g(O)\cap O=\emptyset$ and that $g\neq \mathrm{id}$.

Furthermore, it is possible to write $g=hg'h^{-1}$, where $h\in G$ and g' is either of "even length" as in case 1 or a nonzero power of one of the g_i. Thus, g is hyperbolic. □

Proposition 2 *Under the assumptions of Theorem* 1, *let* $g_1, g_2\in G$. *Suppose that* g_1 *is non-elliptic and that* $g_2(\mathrm{Fix}\, g_1)\cap \mathrm{Fix}\, g_1=\emptyset$. *Then* $\langle g_1, g_2\rangle$ *contains two hyperbolic elements without common fixed points.*

Proof. (a) Suppose that g_1 is hyperbolic. Then $h_1=g_2 g_1 g_2^{-1}$ is hyperbolic by Theorem 1, and $\mathrm{Fix}\, h_1=g_2(\mathrm{Fix}\, g_1)$, disjoint from $\mathrm{Fix}\, g_1$.
(b) Suppose that g_1 and g_2 are parabolic with (unique) fixed points $\xi, \eta\in\partial X$, respectively ($\xi\neq\eta$). Let $x\in X$. Let U, V be disjoint open neighbourhoods in $\hat{X}$ of ξ, η, respectively, such that $d(x, U\cap X)$, $d(x, V\cap X)\geqq\varepsilon>0$. By Theorem 1 and contractivity,

$$g_1^{\pm k}(\hat{X}\setminus U)\subset U \quad \text{and} \quad g_2^{\pm k}(\hat{X}\setminus V)\subset V.$$

Consider $h_2=g_1^k g_2^k$. Then, exactly as in case 3 of the proof of Theorem 2, we conclude that h_2 is hyperbolic with fixed points $\xi_2\in U\cap\partial X$, $\eta_2\in V\cap\partial X$. As $U\cap V=\emptyset$, it cannot be that $\xi_2=\eta$ or $\eta_2=\xi$. Suppose that $\xi_2=\xi$. Then

$$\xi=h_2^{-1}\xi=g_2^{-\ell}\xi\in g_2^{-\ell}(\hat{X}\setminus V)\subset V,$$

a contradiction. Analogously $\eta_2\neq\eta$. Thus $g_1(\mathrm{Fix}\, h_2)$ is disjoint from $\mathrm{Fix}\, h_2$, and case (a) applies to h_2 and g_1.
(c) If g_1 is parabolic and g_2 is elliptic or hyperbolic then $h_3=g_2 g_1 g_2^{-1}$ is parabolic, and $\mathrm{Fix}\, h_3\neq\mathrm{Fix}\, g_1$. We may apply (b) to g_1 and h_3. □

Proposition 3 *Under the hypotheses of Theorem* 1, *assume that G contains hyperbolic elements and that every pair of hyperbolic elements of G has a common fixed point on* ∂X. *Then G fixes a unique point or a set of two points in* ∂X. *In the latter case, these two are the whole of* $L(G)$.

Proof. Choose a hyperbolic $g\in G$, with fixed points $\xi, \eta\in\partial X$.

(a) Assume that G does not fix $\{\xi, \eta\}$. If $h \in G$, then Proposition 2 implies that $h(\{\xi, \eta\}) \cap \{\xi, \eta\} \neq \emptyset$.

Suppose that there are $h_1, h_2 \in G$ such that

$$h_1(\{\xi, \eta\}) \cap \{\xi, \eta\} = \xi, \quad h_2(\{\xi, \eta\}) \cap \{\xi, \eta\} = \eta.$$

Then $g_i = h_i g h_i^{-1}$, $i = 1, 2$, are hyperbolic and must have a common fixed point. Hence, there is $\zeta \in \partial X$ such that

$$\operatorname{Fix} g_1 = \{\xi, \zeta\}, \quad \operatorname{Fix} g_2 = \{\eta, \zeta\}.$$

Let U, V be disjoint neighbourhoods in $\hat{X}$ of ξ and η, respectively. By hyperbolicity of g and contractivity of $\hat{X}$, there is an integer n such that $g^n \zeta \in U$ and $g^{-n} \zeta \in V$, so that $g^n \zeta \neq g^{-n} \zeta$. But then $g^n g_1 g^{-n}$ and $g^{-n} g_2 g^n$ are hyperbolic elements in G without common fixed points, a contradiction.

Hence, without loss of generality we may assume that $\xi \in h(\{\xi, \eta\})$ for all $h \in G$. We claim that G fixes ξ.

Let $h \in G$ with $h(\{\xi, \eta\}) \neq \{\xi, \eta\}$. Suppose $h\xi \notin \{\xi, \eta\}$, so that $h\eta = \xi$. Then $h^2 \eta = h\xi \neq \xi$ and hence $h^2 \xi = \xi$. This yields $h\xi = h^{-1} \xi = \eta$, a contradiction. Thus, we must have $h\xi = \xi$ and $h\eta \neq \eta$. Choose some $h_0 \in G$ with these properties (it exists by assumption).

Suppose that there is $h \in G$ with $h\xi \neq \xi$. By the above, it must be $h\xi = \eta$ and thus $h\eta = \xi$. But then $hh_0(\{\xi, \eta\}) = \{\eta, hh_0 \eta\}$ does not contain ξ, a contradiction: G fixes ξ.

As g fixes no points other than ξ and η, ξ is the unique fixed point of G.

(b) Suppose that G fixes $\{\xi, \eta\}$. If $L(G) \setminus \{\xi, \eta\} \neq \emptyset$, then by Theorem 2 there must be a hyperbolic $h \in G$ which fixes some $\zeta \neq \xi, \eta$. By Theorem 1, h cannot fix $\{\xi, \eta\}$, a contradiction. Hence $L(G) = \{\xi, \eta\}$. □

Proof of Theorem 3 (1) If G contains hyperbolic elements, then Propositions 1–3 show that one of (a), (c) or (d) holds.

(2) Otherwise, $|L(G)| \leqq 1$ by Theorem 2. If $L(G) = \emptyset$, then by properness of X all elements of G must be elliptic and fix the orbit Gx, $x \in X$. Once more using that X is proper, any orbit Gx is relatively compact in X, and (b) holds. On the other hand, if $|L(G)| = 1$ then (c) holds. Indeed, let $L(G) = \{\xi\}$. Then $g_n x \to \xi$ for some sequence (g_n) in G and some $x \in X$. If $h \in G$, then $hg_n x \to h\xi \in L(G)$. Hence $h\xi = \xi$. □

4 Applications

A Random walks and harmonic functions

Let G be a group of automorphisms of the proper metric space (X, d). We assume that G carries the structure of a locally compact, Hausdorff topological group such that the mapping $(g, x) \mapsto gx$ is jointly continuous on $G \times X$. In this case, disontinuous action of a subgroup becomes discreteness in the induced topology. (Typically, $\operatorname{Aut}(X)$ will carry the topology of uniform convergence on compact sets, and G will be a closed subgroup of $\operatorname{Aut}(X)$.)

In the sequel we always assume that $\hat{X}$ is a contractive G-compactification of X. If G does not have the fixed set property, then G contains a discrete free subgroup on two generators. In particular, G is nonamenable. (See e.g. Pier [Pi] for all questions related to amenability.)

Now let μ be a (Borel) probability measure on G, such that supp μ generates G as a closed group. We can construct a sequence of G-valued random variables Y_n, $n=1, 2, \ldots$, which are independent and identically distributed with law μ. We consider the *right random walk* on G with law μ:

$$S_0 = \mathrm{id}, \qquad S_n = Y_1 \ldots Y_n \quad (n \geqq 1).$$

For the following, recall that a probability measure ν on ∂X is *continuous*, if it carries no point masses, and that the convolution of μ with ν is given by

$$\int_{\partial X} f(\xi)\, \mu * \nu(d\xi) = \int_G \int_{\partial X} f(g\xi)\, \nu(d\xi)\, \mu(dg), \qquad f \in C(\partial X).$$

Corollary 1 *Under the above hypotheses, suppose that G does not have the fixed set property. Then for every $x \in X$,*

$$\lim_{n \to \infty} S_n x \in \partial X$$

exists almost surely in $\hat{X}$. The limit distribution on ∂X,

$$\nu(B) = \Pr[\lim S_n x \in B], \qquad B \text{ a Borel set in } \partial X,$$

is continuous and the unique measure satisfying $\mu * \nu = \nu$. In particular, $(\partial X, \nu)$ is a boundary of (G, μ) in the sense of Furstenberg.

This is derived from the methods of Furstenberg [F2, § 3] (see also [F1, § 1.3]) precisely as in the case of the end compactification of graphs, see [CS] and [W].

Now let us suppose in addition that X is a homogeneous space of G, i.e., G acts transitively on X and the mapping $g \mapsto gx$ is open for every $x \in X$. Denote by G_x the stabilizer in G of $x \in X$. Let μ be a probabilty measure on G as above. We assume that μ is *irreducible* (supp μ generates G as a closed semigroup) and that for some reference point $o \in X$, μ is *G_o-invariant*, i.e.,

$$\mu(h(A)) = \mu(A)$$

for every Borel set $A \subset G$ and every $h \in G_o$. If $g \in G$ then $(gS_n o)_{n \geqq 0}$ is a Markov chain on X starting at $x_0 = go$ with transition probabilities

$$p(x, U) = \mu(\{g \in G \mid hgo \in U\}),$$

where $h \in G$ with $ho = x$ (independent of the choice of such h).

If, as above, G does not have the fixed set property with respect to the contractive G-compactification $\hat{X}$, then Corollary 1 applies. Choosing $g \in G$ with $go = x$, we can define the probability measure

$$\nu_x(B) = \Pr[\lim gS_n o \in B], \qquad B \text{ a Borel set in } \partial X.$$

Once more, it is independent of the particular choice of g with $go=x$. Furthermore, $\nu_x = \int_X p(x, dy)\,\nu_y$. If h^* is a bounded Borel-measurable function on ∂X, then

$$(*) \qquad h(x) = \int_{\partial X} h^*\, d\nu_x$$

defines a function on X which, by the above, is *harmonic*:

$$h(x) = \int_X p(x, dy)\, h(y) \qquad \text{for every } x \in X.$$

We can now formulate the solution of the Dirichlet problem for harmonic functions of the random walk relative to $\hat{X}$.

Corollary 2 *Let μ and p be as above, and assume that G does not have the fixed set property with respect to $\hat{X}$. Then every continuous function h^* on ∂X has a unique continuous extension to $\hat{X}$ which is harmonic on X.*

The extension is of course given by (*). The proof is verbatim the same as in [W2, Theorem 5.2], replacing the end topology with the topology of $\hat{X}$.

B Ends of locally finite graphs

Recall that X is a *graph*, if it is equipped with a symmetric *neighbourhood relation* E (the *edge set*). If $x, y \in X$, then a *path* of *length* n from x to y is a sequence $x = x_0, x_1, \ldots, x_n = y$ of successive neighbours. If any pair of points is connected by a path, then we say that X is *connected*, and we can equip X with a discrete metric: $d(x, y)$ is the minimum length of a path from x to y. The graph is called *locally finite*, if every point has only finitely many neighbours.

Here, we assume that X is a locally finite, connected, infinite graph. Thus, (X, d) is proper and countable. The *end compactification* of X can be defined in various ways, see Freudenthal [Fr], Jung [J1] or Cartwright et al. [CSW].

If F is a finite set of edges, then by deleting F from E, X decomposes into a finite family $\mathscr{C}_F$ of connected subgraphs. If $F \subset F'$ (also finite), then every subgraph in $\mathscr{C}_{F'}$ is contained in a subgraph in $\mathscr{C}_F$. This defines a natural projection $\mathscr{C}_{F'} \to \mathscr{C}_F$. The projective limit of the system

$$\{\mathscr{C}_F \mid F \subset E, F \text{ finite}\}$$

is the end compactification of X.

Ends can be represented by equivalence classes of infinite paths: an *infinite path* is a one-sided sequence of successive neighbours in X without repetitions. Two infinite paths are *equivalent*, if there is a third one which meets each of the two infinitely often. (This is Halin's [H1]) definition of ends.)

$\mathrm{Aut}(X)$ is the group of all neighbourhood-preserving bijections $X \to X$, its topology is that of pointwise convergence, and every automorphism extends to a homeomorphism of the end compactification. Projectivity is obvious, and contractivity is an easy exercise, see [W2, Lemma 2.4].

Corollary 3 *For a locally finite, connected, infinite graph, Theorems* 1–3 *apply to every subgroup of* $\mathrm{Aut}(X)$.

This is of course well known, see the references given in the introduction. Indeed, Jung [J2] proves the analogue of Theorem 3 even for non locally finite graphs, a result which cannot be obtained by the methods presented here without serious modifications.

We briefly outline two examples where case (c) of Theorem 3 holds. Let X be a regular tree with degree at least 3. Select an end ξ of X and consider the group G of all automorphisms of X which fix ξ. Then $L(G)$ is the whole of ∂X, but G fixes just ξ. Next, choose an infinite path in X which represents ξ. Let H be the subgroup of G consisting of all automorphisms which leave all but finitely elements of this path fixed. Then $L(H)=\{\xi\}$, and all elements of H are elliptic.

Remark. Graphs with infinitely many ends (even vertex-transitive) may have a rich and complicated structure, well beyond the simple examples of trees or graphs which are very similar to trees. To get a flavour, see e.g. Dicks and Dunwoody [DD], Dunwoody [Du] or Thomassen and Woess [TW]. In particular, such graphs are in general far from being hyperbolic in the sense of the following subsections.

C Hyperbolic metric spaces

Given a metric space (X, d) and a reference point $o\in X$, we define

$$(x\,|\,y)=(x\,|\,y)_o=\tfrac{1}{2}(d(x,o)+d(y,o)-d(x,y)), \qquad x, y\in X.$$

Following Gromov [Gr], X is called *hyperbolic* if there is a $\delta\geqq 0$ (possibly large) such that for all $x, y, z\in X$

$$(x\,|\,z)\geqq \min\{(x\,|\,y),(y\,|\,z)\}-\delta.$$

We assume that X is proper and, in addition, *geodesic*: for every $x, y\in X$, there is an isometric embedding of the interval $[0, d(x, y)]$ into X which sends 0 to x and $d(x, y)$ to y.

We now describe the *hyperbolic compactification* $\hat{X}$ of X. For many details, see [Gr, § 1; CDP, § 2; GdH, § 7; Sh, § 4]. We use an adaptation of the approach of [GdH, § 7.3], as used for hyperbolic graphs by Kaimanovich and Woess [KW]. Compare also with [Sh, Definition 4.7].

Choose $a>0$ such that $a'=e^{2\delta a}-1<\sqrt{2}-1$ and define for $x, y\in X$

$$\rho_a(x,y)=\begin{cases}0, & \text{if } x=y;\\ e^{-a(x|y)}, & \text{otherwise.}\end{cases}$$

If we define for $x, y\in X$

$$\theta_a(x,y)=\inf\left\{\sum_{i=1}^{n}\rho_a(x_{i-1},x_i)\,|\,n\geqq 1, x=x_0, x_1, \ldots, x_n=y\in X\right\},$$

then θ_a is a metric on X, and

$$(1-2a')\,\rho_a(x,y) \leqq \theta_a(x,y) \leqq \rho_a(x,y) \quad \text{for all } x, y \in X.$$

This can be proved precisely as in [G-H, § 7.3]. Note that $(x|y) \leqq d(x,o)$, so that θ_a is discrete on X. The completion of (X, θ_a) gives rise to a set of new points, the *hyperbolic boundary* ∂X, which is independent of the particular choice of constant a and reference point o. The *hyperbolic topology* of $\hat{X} = X \cup \partial X$ is the original one (induced by d) on X, while a neighbourhood base at $\xi \in \partial X$ is given by the open balls $B(\xi, r)$, $r > 0$, with respect to θ_a.

Corollary 4 *The hyperbolic compactification of a hyperbolic, geodesic and proper metric space X is contractive, and Theorems* 1–3 *apply to every subgroup of* $\mathrm{Aut}(X)$.

Proof. (a) Let (x_n) be a sequence in X converging to some boundary point ξ, and let $d(x_n, y_n) \leqq M < \infty$. Then $d(x_n, o) \to \infty$ and $d(y_n, o) \to \infty$, and hence $(x_n | y_n) \to \infty$. Thus $\theta_a(x_n, y_n) \to 0$, and $\hat{X}$ is projective.

(b) Let (g_n) be a sequence in $\mathrm{Aut}(X)$ with $g_n o \to \xi$ and $g_n^{-1} o \to \eta$ $(\xi, \eta \in \partial X)$. As above, denote by $B(z, r)$ open θ_a-balls with radius r centered at $z \in \hat{X}$. Let U, V be open neighbourhoods in $\hat{X}$ of ξ and η, respectively. We can find $0 < \varepsilon < 1$ and $n_0 > 0$ such that

$$\{g_n o \mid n \geqq n_0\} \cup \{\xi\} \subset B(g_{n_0} o, \varepsilon/3) \subset B(g_{n_0} o, \varepsilon) \subset U$$

and

$$\{g_n^{-1} o \mid n \geqq n_0\} \cup \{\eta\} \subset B(g_{n_0}^{-1} o, \varepsilon/3) \subset B(g_{n_0}^{-1} o, \varepsilon) \subset V.$$

Let $z \in X$ with $\theta_a(g_{n_0}^{-1} o, z) \geqq 2\varepsilon/3$. Choose $n \geqq n_0$. Then $\theta_a(g_n^{-1} o, z) \geqq \varepsilon/3$ and hence

$$(g_n^{-1} o | z) \leqq M, \quad \text{where } M = -\frac{1}{a} \log \frac{\varepsilon}{3}.$$

But then

$$\begin{aligned}(g_n z | g_n o) &= \tfrac{1}{2} d(g_n o, o) + \tfrac{1}{2}(d(g_n z, o) - d(z, o)) \\ &= d(g_n o, o) - (g_n^{-1} o | z) \geqq d(g_n o, o) - M,\end{aligned}$$

which tends to infinity as $n \to \infty$. Hence there is $n_1 \geqq n_0$ such that for $n \geqq n_1$

$$\theta_a(g_n z, g_n o) \leqq e^{-a(d(g_n o, o) - M)} < \varepsilon/3$$

for every z as above. As every $\zeta \in \partial X \setminus V$ can be approximated by elements in X outside of $B(g_{n_0}^{-1} o, 2\varepsilon/3)$, contractivity follows. □

For hyperbolic spaces, Theorem 1 can be found in all of the above references, while Proposition 1 is in [CDP, § 11.3]. As for Theorem 2 and the second part of Theorem 3 (in the setting of hyperbolic spaces), this is known without doubt to the specialists, but the author could find in the literature only special cases such as that of finitely generated hyperbolic groups. In any case, the proofs in these references are burdened by "hyperbolic technicalities".

D Hyperbolic graphs

Suppose that X is a connected graph. It becomes a geodesic space, if the edges are considered as copies of the unit interval and the discrete metric is extended in the obvious way. Thus, assuming that (X, d) is hyperbolic and locally finite, Corollaries 1, 2 and 4 apply. (However, when speaking of $\hat{X}$, only vertices and boundary points are considered.) Note that the topology of $\hat{X}$ is completely described by the metric θ_a.

We recall that an (infinite) *geodesic* in a graph is a sequence $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n\in\mathbb{Z}}$ such that $d(x_n, x_m)=|n-m|$. In the first case, x_n converges to some $\xi\in\partial X$ as $n\to\infty$, and we call it a geodesic from x_0 to ξ. In the second case, there are distinct $\xi, \eta\in\partial X$ such that $x_n\to\xi$ and $x_{-n}\to\eta$ as $n\to\infty$, and we call it a geodesic from ξ to η. For every $z, w\in\hat{X}$, $z\neq w$, there is at least one geodesic from z to w. (In the finite case, this is just a shortest path between the two.) There is a constant $C=C_\delta\in\mathbb{N}$ such that the following holds:

If (x_n) and (y_n) are two geodesics from $z_1, z_2\in\hat{X}$ (respectively) to $\xi\in\partial X (z_i\neq\xi)$ then there is an index n_0 such that

$$d(x_n, \{y_m\})\leqq C \quad \text{for all } n\geqq n_0.$$

If $z_1=z_2$, then this holds for all x_n.

For details, see e.g. [GdH, § 7].

Here, we are interested in the case when X is *vertex-transitive*. Let G be a subgroup of $\mathrm{Aut}(X)$, closed in the topology of pointwise convergence, which acts transitively on X. The modular function of G is given by

$$\Delta(g)=|G_x\, g\, x|/|G_{gx}\, x|,$$

where $x\in X$ is arbitrary, see Schlichting [Sch], Trofimov [Tr] or Soardi and Woess [SW]. We then have the following.

Proposition 4 *If $|\partial X|>2$ and G has the fixed set property, then G is nonunimodular.*

Proof. X being transitive, there is a constant $D\in\mathbb{N}$ such that

$$|\{y\in X\,|\,d(x, y)\leqq n\}|\leqq D^n \quad \text{for all } n\geqq 0 \quad \text{and} \quad x\in X.$$

As G acts transitively, $\partial X=L(G)$ is infinite, and by Theorems 2 and 3, the fixed set property means that G fixes a unique $\xi\in\partial X$.

For every $N>0$, we can find distinct points $\eta_0, \ldots \eta_N\in\partial X\setminus\{\xi\}$. Let $(x_n^i)_{n\in\mathbb{Z}}$ be geodesics from η_i to ξ, $i=0, \ldots, N$. Relabeling the sequences if necessary, we may assume that

$$d(x_n^i, x_n^j)\leqq 3C \quad \text{for all } i, j=0, \ldots, N \quad \text{and} \quad n\geqq 0.$$

Also, we may assume that our reference point is $o=x_0^0$.

If K is sufficiently large, then there must be points $y_i\in\{x_n^i\,|\,n<0\}$ $(i=0, \ldots, N)$, such that

$$d(o, y_i)=K \quad \text{and} \quad y_i\neq y_j \quad \text{for all } i\neq j.$$

In particular, $y_0=x_{-K}^0$, and $y_i=x_{-\ell(i)}^i$ with

$$|K-\ell(i)|\leqq d(o, x_0^i)\leqq 3C.$$

For every $i=1, \ldots, N$, there is $g_i \in G$ with $g_i y_0 = y_i$. As g_i fixes ξ, $(g_i x^0_{n-K})_{n \geqq 0}$ is a geodesic from y_i to ξ. Also $(x^i_{n-\ell(i)})_{n \geqq 0}$ is a geodesic from y_i to ξ. Hence,

$$d(g_i o, o) \leqq d(g_i o, x^i_{K-\ell(i)}) + d(x^i_{K-\ell(i)}, x^i_0) + d(x^i_0, 0) \leqq 9C.$$

Now, $|\{x \in X \mid d(x, o) \leqq 9C\}| \leqq D^{9C}$. Hence, there must be an x in this set with

$$|I| \geqq N/D^{9C}, \quad \text{where } i = \{i \in \{1, \ldots, N\} \mid g_i o = x\}.$$

Choose and fix $i \in I$. Then $g_i^{-1} g_j \in G_0$ for every $j \in I$, and the points $g_i^{-1} g_j y_0$ are all distinct. We now choose $g \in G$ with $go = y_0$. Thus

$$|G_o go| \geqq N/D^{9C}.$$

On the other hand, let $h \in G_{y_0}$. Then $(hx^0_{n-K})_{n \geqq 0}$ is a geodesic from y_0 to ξ, and it must be $d(ho, o) \leqq 3C$. This yields that

$$|G_{go} o| \leqq D^{3C}.$$

We obtain

$$\Delta(g) \geqq N/D^{12C}.$$

As N may be chosen arbitrarily large, we find $g \in G$ with $\Delta(g) > 1$. In other words, G is nonunimodular. □

For a finite subgraph A of X, let Volume(A) be the number of points in A and Surface(A) the number of edges with precisely one endpoint in A. We say that X is an *infinite expander*, if there is a constant $\kappa > 0$ such that the isoperimetric inequality

$$\text{Surface}(A) \geqq \kappa \cdot \text{Volume}(A)$$

holds for every finite subgraph A of X. We refer to Dodziuk [Do] and Gerl [Ge] for significance and consequences of this property. If satisfied by the Cayley graph of a finitely generated group, this means that the group is nonamenable.

A *quasi-transitive* graph is one whose automorphism group acts with finitely many orbits. A *strip* is a quasi-transitive graph with two ends [JW]. Every strip is a hyperbolic non-expander (indeed, it has linear growth).

Corollary 5 *Let X be a quasi-transitive, locally finite, infinite hyperbolic graph. Then either $|\partial X| = 2$ and X is a strip or $|\partial X| = \infty$ and X is an expander.*

Proof. It is sufficient to prove this for vertex-transitive graphs, see [Sa].

Every vertex-transitive graph possesses a two-way infinite geodesic [Wa]. Hence $|\partial X| \geqq 2$. We apply Theorem 2. If $|\partial X| = \infty$ then by Corollary 1 and Proposition 5, Aut(X) is nonamenable or nonunimodular. By [SW], X is an expander.

Let $|\partial X| = 2$. For an arbitrary hyperbolic, locally finite graph it is easy to see that the identity on X extends to a continuous surjection of the hyperbolic compactification onto the end compactification. Furthermore, this mapping is one-to-one between connected components of ∂X and ends, see [GdH, § 7.5]. Hence, X has two ends. □

As a byproduct, the last corollary contains another proof of the fact that a finitely generated hyperbolic group either contains a free subgroup on two generators, or it has a cyclic subgroup of finite index.

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On the number of bound states of a bosonic N-particle Coulomb system

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1 Introduction

Let $H_{N,Z}$ be the Hamiltonian of N spinless particles of charge one in the field of a nucleus of charge Z, i.e.,

$$H_{N,Z} := \sum_{i=1}^{N} \left(-\Delta_i - \frac{Z}{|x_i|} \right) + \sum_{\substack{i,j=1 \\ i<j}}^{N} \frac{1}{|x_i - x_j|} \tag{1}$$

self-adjointly realized in $L^2(\mathbb{R}^{3N})$ with domain $H^2(\mathbb{R}^{3N})$. We wish to find an upper bound to the number of eigenvalues of $H_{N,Z}$ below its essential spectrum. Three cases can be distinguished according to whether no symmetry restriction is placed on the eigenfunctions (the *boltzon* case), the eigenfunctions are restricted to be symmetric (the *boson* case) or whether they are antisymmetric (the *fermion* case). The number of eigenvalues are denoted, respectively, by $\nu(N, Z)$, $\nu_b(N, Z)$, and $\nu_f(N, Z)$. (Note that the definition of $H_{N,Z}$ and its essential spectrum depends on the case.) The fermion case is, of course, the one of physical interest but our techniques are not strong enough to deal with it effectively. We can deal with the bosonic case, however, as Theorem 1 below shows. We regard this result as the first step in trying to understand the fermionic case. As for the boltzons, our techniques go only part way; some of the results in the rest of

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this paper apply to that case, but we cannot complete all the steps to achieve an analogue of Theorem 1.

Denote the bottom of the spectrum of $H_{N,Z}$ by $E(N, Z)$. By the HVZ theorem the bottom of the essential spectrum of $H_{N,Z}$ is given by $E(N-1, Z)$. It is known that $v(N, Z)$ is infinite for $N<Z+1$ (Zhislin [19]) and that $v(N, Z)$ is finite for $N \geqq Z+1$ (Uchiyama [16], Zhislin [20], Yafaev [18], and Vugal'ter and Zhislin [17]). Moreover $v(N, Z)$, $v_b(N, Z)$, and $v_f(N, Z)$ vanish for $2N+1 \geqq Z$ (see [11]). $v_b(N, Z)$ vanishes, if N exceeds $N_c(Z)$ which is about $1.21Z$ for large Z (see [3, 4, 14]). For fermions, $v_f(N, Z)$ vanishes if $N/Z>1$ and N is large enough (see [12, 6, 7]).

Our result is as follows:

Theorem 1 *Let* $n_c := \lim_{Z\to\infty} (N_c(Z)/Z)$. *Fix two positive constants* β *and* β', $\beta<\beta'<n_c-1$. *Then there exists a constant* K *such that for large* Z *and all* $N \in [(1+\beta)Z+1, (1+\beta')Z]$

$$v_b(N, Z) = O(Z^K)$$

in the bosonic case.

We note that we cannot expect phase space volume counting to be correct. In fact it turns out that the phase space volume is always infinite. This singularity stems on the one hand from those directions in the configuration space $\mathbb{R}^{3N}$ where the potential does not decay as a function in $\mathbb{R}^{3N}$, e.g., x_1 tending to infinity and $x_2, ., x_N$ staying bounded. On the other hand the phase space volume is infinite because of the Coulomb singularity which is aggravated by the high dimension (namely N) of the phase space. This would hold true even if the configuration space were restricted to be a compact set containing the origin. Therefore, we decompose our problem into an outer problem where no bound states are localized and an inner one where the counting will be done with the help of a suitable one-particle approximation.

For this purpose we define the following continuous, piecewise C^∞ function

$$g_d(t) := \begin{cases} 0 & t \leqq d \\ \sin\left(\frac{\pi}{2}\,\frac{t-d}{1-d}\right) & d<t<1 \\ 1 & 1 \leqq t \end{cases} \tag{2}$$

on the positive half line where d is a fixed positive number smaller than one. Let $f_d(t) := \sqrt{1-g_d(t)^2}$. Moreover, let R be some positive radius. Then define functions $A: \mathbb{R}^{3N} \to \mathbb{R}$ and $I: \mathbb{R}^{3N} \to \mathbb{R}$ by

$$A(x) := g_d(|x|_\infty/R) \quad \text{and} \quad I(x) := \sqrt{1-A(x)^2}$$

where $|x|_\infty = \max\{|x_1|, \ldots, |x_N|\}$, i.e., A localizes onto the outside and I onto the inside of a polyball of side length $2R$.

We shift the energy scale by $E(N-1, Z)$ so that we need to count only the negative eigenvalues of $H_{N,Z}-E(N-1, Z)$. With the help of the functions I and A we may decompose the operator as follows

$$
\begin{aligned}
(3) \qquad & H_{N,Z}-E(N-1, Z) \\
& = A(H_{N,Z}-E(N-1, Z))\, A \\
& \quad + I(H_{N,Z}-E(N-1, Z))\, I-(\nabla A)^2-(\nabla I)^2 \\
& \geqq A\left(H_{N,Z}-E(N-1, Z)-\frac{\pi^2\, \chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}\right) A \\
& \quad + I\left(H_{N,Z}-E(N-1, Z)-\frac{\pi^2\, \chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}\right) I.
\end{aligned}
$$

In Sect. 2 we show that there is no bound state supported outside a ball whose radius is of the order of $Z^{-1/2}$. In Sect. 3 we reduce that problem to a spectral problem for a one-particle Schrödinger operator with compact resolvent whose eigenvalues approximate our original one to sufficiently high accuracy. This reduces the problem to a combinatorial problem for the one-particle operator. As input we need some knowledge of the number of eigenvalues of this operator, which we obtain from the semiclassical bound on the number of eigenvalues [10]. The bosonic correlation estimate that allows us to compare the multiparticle Hamiltonian and the one-particle operator to the required accuracy is proved in Sect. 4. We remark that the correlation estimate derived in Theorem 2 holds independently of the special choice of the Hilbert space $\mathfrak{H}_N$ – in our case $L^2(\mathbb{R}^{3N})$ – in which $H_{N,Z}$ is defined as long as we restrict to the corresponding bosonic subspace. Moreover the correlation estimate holds also independently of the pair interaction of $H_{N,Z}$ provided the pair interaction obeys a certain positivity condition. The correlation estimate bounds the pair correlation of a state $\psi_N \in \mathfrak{H}_N$ in terms of $\gamma_\psi - N^{-1}\gamma_\psi^2$, where γ_ψ is the one-particle density matrix corresponding to ψ_N. This quantity vanishes for a product state $\psi_N = \varphi \otimes \ldots \otimes \varphi$, which would be the ground state for a one-particle model. Since $H_{N,Z}$ in (1) is well-approximated by a suitable one-particle Schrödinger operator, we can show $\gamma_\psi - N^{-1}\gamma_\psi^2$ to be small, if ψ_N is the ground state of $H_{N,Z}$. Finally note that all operator inequalities and expectations are meant in the sense of quadratic forms.

2 The outside problem

To show the positivity of

$$
A\left(H_{N,Z}-\frac{\pi^2\, \chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}-E(N-1, Z)\right) A
$$

we do a further partitioning. We shall cut this outside problem into $N+1$ regions as follows. There are N regions where the first, the second, ..., the N-th particle is far from the origin whereas all other particles are inside the ball of some radius δR with $\delta < 1$. The $(N+1)^{\text{th}}$ region is the remaining part.

For $k=1, \ldots, N$ denote the vector $(x_1, ., x_{k-1}, 0, x_{k+1}, ., x_N)$ by $\mathfrak{x}_k$ and define the localizing functions (with c and δ chosen to be less than one)

$$J_k(\mathfrak{x}) := f_c(|\mathfrak{x}_k|_\infty / (\delta\,|\mathfrak{x}|_\infty)) \tag{4}$$

and

$$J_{N+1}(\mathfrak{x}) := \sqrt{1 - J_1(\mathfrak{x})^2 - \ldots - J_N(\mathfrak{x})^2}. \tag{5}$$

We first note the following estimate

$$\begin{aligned} A\left(H_{N,Z} - E(N-1, Z) - \frac{\pi^2\,\chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}\right) A \\ \geqq \sum_{k=1}^{N+1} A J_k \left(H_{N,Z} - E(N-1, Z)\right. \\ \left. - \frac{\pi^2\,\chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2} - \frac{\pi^2(1+\delta^2)}{4\delta^2(1-c)^2\,|\mathfrak{x}|_\infty^2}\right) J_k A. \end{aligned} \tag{6}$$

Note that the localization error stays bounded in N, because the supports of the functions $J_1, ., J_N$ do not overlap each other in the support of A. Now, consider, e.g., the case where the first particle is outside. In the support of J_1 we have

$$|\mathfrak{x}_1|_\infty \leqq \delta\,|\mathfrak{x}|_\infty .$$

Since we assumed δ to be smaller than one, we can conclude that $|x_1| = |\mathfrak{x}|_\infty$. Thus

$$|x_2|, \ldots, |x_N| \leqq \delta\,|x_1| \tag{7}$$

and therefore $|x_k - x_1| \leqq (1+\delta)\,|x_1|$. This yields

$$\begin{aligned} A J_1 \left(H_{N,Z} - E(N-1, Z) - \frac{\pi^2\,\chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2} - \frac{\pi^2(1+\delta^2)}{4\delta^2(1-c)^2\,|\mathfrak{x}|_\infty^2}\right) J_1 A \\ \geqq A J_1 \left(-\Delta_1 - \frac{Z}{|x_1|} + \sum_{j=2}^{N} \frac{1}{|x_1 - x_j|} - \frac{\pi^2\,\chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2} - \frac{\pi^2(1+\delta^2)}{4\delta^2(1-c)^2\,|\mathfrak{x}|_\infty^2}\right. \\ \left. + 1 \otimes H_{N-1,Z} - E(N-1, Z)\right) J_1 A \\ \geqq A J_1 \left(-\Delta_1 + \frac{-Z + (N-1)/(1+\delta)}{|x_1|}\right. \\ \left. - \frac{\pi^2\,\chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2} - \frac{\pi^2(1+\delta^2)}{4\delta^2(1-c)^2\,|\mathfrak{x}|_\infty^2}\right) J_1 A \\ \geqq \frac{A^2 J_1^2}{|x_1|^2}\left(\left(-Z + \frac{N-1}{1+\delta}\right)|x_1| - \frac{\pi^2}{4}\left(\frac{1}{(1-d)^2} + \frac{1+\delta^2}{\delta^2(1-c)^2}\right)\right). \end{aligned} \tag{8}$$

Since $|x_1| \geqq dR$ in the support of J_1 the right hand side of (8) is positive, if we pick $R \geqq R_0$ where R_0 fulfills

$$dR_0 = \frac{\frac{\pi^2}{4}(1+\delta)\left(\frac{1}{(1-d)^2}+\frac{1+\delta^2}{\delta^2(1-c)^2}\right)}{N-Z-1-\delta Z},$$

and if $N > Z+1+\delta Z$ which is true, if we pick $\delta < \beta$. In particular we note, that R on the order of Z^{-1} suffices at this point. – Of course the same is true for $k=2, \ldots, N$.

Now we wish to treat the remaining outside region, i.e., the support of AJ_{N+1}. Let $U_{k,l}$ be the set where the k-th particle has the biggest distance from the origin and the l-th particle the second biggest, i.e.,

$$U_{k,l} = \{x \in \mathbb{R}^{3N} \,|\, |x|_\infty = |x_k|, |x_k|_\infty = |x_l|\}.$$

Note that the sets $U_{k,l}$, $k, l = 1, \ldots, N$, $k \neq l$ are – up to a set of measure zero – pairwise disjoint and cover the whole of $\mathbb{R}^{3N}$. Now, choose some $\rho > 0$ and let g_ρ be the normalized positive eigenfunction of the Laplacian with Dirichlet boundary condition on the polyball $\Gamma_\rho = \{x \in \mathbb{R}^{3N} \,|\, |x|_\infty < \rho\} = \{(x_1, ., x_N) \,|\, |x_i| < \rho$ for all $i\}$. Define

$$J_{i,j}(x) := \sqrt{\int_{U_{i,j}} g_\rho(x-\eta)^2 \, d^{3N}\eta}. \tag{9}$$

We have the following facts.

Lemma 1 *Let $x \in \operatorname{supp} J_{i,j}$. Then the following holds true:*
1. *For all $\eta \in U_{i,j}$ it holds that $|x-\eta|_\infty \leqq 2\rho$.*
2. $|x_i| \leqq |x|_\infty \leqq |x_i| + 4\rho$.
3. $|x_i|_\infty \leqq |x|_\infty \leqq |x_i|_\infty + 4\rho$.
4. $|x_i|_\infty \leqq |x_j| + 4\rho$.

The proof of these claims is immediate from the triangle inequality and the definition of the sets $U_{i,j}$.

Let us note, moreover, that by Schwarz's inequality and explicit computation for the three-dimensional Laplacian we have

$$\sum_{i,j=1, i\neq j}^{N} (\nabla J_{i,j}(x))^2 \leqq \frac{\pi^2}{4\rho^2} N.$$

Thus taking, e.g., $(i,j) = (1,2)$, we need to estimate

$$\begin{aligned} AJ_{N+1} J_{1,2} &(H_{N,Z} - E(N-1, Z) \\ &- \frac{\pi^2 \chi_{[dR,R]}(|x|_\infty)}{4(1-d)^2 R^2} - \frac{\pi^2}{2\delta^2(1-c)^2 |x|_\infty^2} - \frac{\pi^2 N}{4\rho^2}\Big) J_{1,2} J_{N+1} A \\ \geqq AJ_{N+1} J_{1,2} &\Big(-\Delta_1 - \Delta_2 - \frac{Z}{|x_1|} - \frac{Z}{|x_2|} + E(N-2, Z) - E(N-1, Z) \\ &- \frac{\pi^2 \chi_{[dR,R]}(|x|_\infty)}{4(1-d)^2 R^2} - \frac{\pi^2}{4\delta^2(1-c)^2 |x|_\infty^2} - \frac{\pi^2 N}{4\rho^2}\Big) J_{1,2} J_{N+1} A. \end{aligned} \tag{10}$$

Because of our assumption on N, namely $N \leqq (1+\beta)Z$, the difference $E(N-2, Z)-E(N-1, Z) \geqq \mu Z^2$ for some positive, Z independent constant μ (see [1]). Thus choosing $\rho = dR/16$ and $R = fZ^{-1/2}$ for sufficiently large constant f, we obtain positivity.

Summarizing the result of this section, we have

Lemma 2 *Let β and β' be two positive constants with $\beta < \beta' < n_c - 1$. For both the boson and boltzon cases, there exist constants c, $d \in (0, 1)$ and f, $Z_0 \in (0, \infty)$ such that for $Z > Z_0$ and for $(1+\beta)Z \leqq N \leqq (1+\beta')Z$ and $R = fZ^{-1/2}$*

$$A\left(H_{N,Z} - E(N-1, Z) - \frac{\pi^2 \chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}\right) A \geqq 0.$$

3 The inside problem

We begin by noting that by the minimax principle the number N_I of negative eigenvalues of

$$I\left(H_{N,Z} - E(N-1, Z) - \frac{\pi^2 \chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}\right) I$$

is bounded from above by the number of negative eigenvalues of

$$\left(H_{N,Z} - E(N-1, Z) - \frac{\pi^2 \chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}\right)\Big|_{\mathfrak{H}_R} \tag{11}$$

where $\mathfrak{H}_R$ is $H_0^2(\operatorname{supp} A)$ in the case of boltzons or the symmetric subspace of $H_0^2(\operatorname{supp} A)$ in the case of bosons. We may also introduce the projection P_I onto the negative eigenvalues of (11), i.e., N_I is also dominated by the number of negative eigenvalues of

$$H_I := P_I\left(H_{N,Z} - E(N-1, Z) - \frac{\pi^2 \chi_{[dR,R]}(|\mathfrak{x}|_\infty)}{4(1-d)^2 R^2}\right) P_I. \tag{12}$$

We dropped the restriction onto $\mathfrak{H}_R$ and view P_I as an operator on $\mathfrak{H}$ whose kernel includes all functions with support outside the ball of radius R. Within the ball of radius R we wish to replace the operator $H_{N,Z}$ by the effective Hamiltonian

$$H_{N,Z}^W := \sum_{i=1}^{N} h_i - D(\rho^W, \rho^W) \tag{13}$$

where $h_i := \underbrace{1 \otimes \ldots \otimes 1}_{i-1 \text{ factors}} \otimes h_\alpha^Z \otimes \underbrace{1 \otimes \ldots \otimes 1}_{N-i \text{ factors}}$ with

$$h_\alpha^Z := -\Delta - \frac{Z}{|x|} + \rho^W * |x|^{-1}. \tag{14}$$

The function ρ^W is the minimizer of the bosonic Weizsäcker functional

$$\mathscr{E}^W(\rho) = \int_{\mathbb{R}^3} (\nabla \sqrt{\rho}(x))^2 - \frac{Z}{|x|}\rho(x)\, d^3x + D(\rho, \rho)$$

under the constraint $\int_{\mathbb{R}^3} \rho \leqq \alpha Z := N-1$. As usual, D denotes the Coulomb scalar product, i.e.,

$$D(f, g) = \frac{1}{2} \int_{\mathbb{R}^6} \frac{\overline{f(x)}\, g(y)}{|x-y|}\, d^3x\, d^3y.$$

Let us recall that the minimal energy $E^W(N-1, Z)$ of $\mathscr{E}^W$ under the above constraint is bounded from below by $E(N-1, Z)$, that the smallest eigenvalues of h_α^Z can be written as $Z^2 e'(\alpha)$, and that $E^W(N-1, Z) + D(\rho^W, \rho^W) = Z^2 e'(\alpha)$ (see [4]). Therefore, by the correlation estimate (81), and the minimax principle we have that the number of negative eigenvalues of H_I is bounded from above by the number of negative eigenvalues of

$$P_I \left(H_{N,Z}^W - E^W(N-1, Z) - \frac{\pi^2}{4(1-d)^2 R^2} - \hat{C} \max\{Z^2, E(N-1, Z) - E(N, Z)\} \right) P_I,$$

and thus N_I – and therefore $\nu(N, Z)$ – is bounded from above by the number $\nu^W(N, Z)$ of negative eigenvalues of

$$H_{\text{eff}} := (H_{N,Z}^W - E^W(N-1, Z) - \text{const}\, Z^2)|_{H_0(\{x \in \mathbb{R}^3 \,|\, |x| < R\})}. \tag{15}$$

Note that up to the above step, namely the use of the correlation inequality, we were able to treat both the boltzonic and the bosonic case. It is at this point that we have to specialize to bosons only, since we have the correlation estimate for fermions [2] and bosons (see Sect. 4).

We are now in a position to use the fact, that $H_{N,Z}^W$ is a one-particle operator whose eigenvalues are given as sums of N eigenvalues of h_α^Z plus the constant $D(\rho^W, \rho^W) - \text{const}\, Z^2$. Abbreviate $h_\alpha^Z|_{H_0(\{x \in \mathbb{R}^3 \,|\, |x| < R\})}$ by $h_{\alpha,R}^Z$. Since the gap between the first and second eigenvalue of $h_{\alpha,R}^Z$ is bigger than a constant times Z^2 (one may use the scaling for h_α^Z, the fact that all eigenvalues increase because of the Dirichlet conditions and that the difference between the first eigenvalue of $h_{\alpha,R}^Z$ and the first eigenvalue of h_α^Z is $O(Z^1)$), it follows that of the above N eigenvalues of $h_{\alpha,R}^Z$ in the sum for the eigenvalues E of H_{eff}, at least $N-k$ of these have to be ground state eigenvalues of $h_{\alpha,R}^Z$, where k is a constant independent of N and Z.

Furthermore, we can estimate the number n of bound states of $h^Z_{\alpha,R}$ smaller than const Z^2 from above by

$$\text{const} \int_{|x|<R} (\text{const}\, Z^2 + Z/|x|)^{3/2}\, d^3x$$

by the Cwickel-Lieb-Rosenbljum bound [13, 10, 5]. Evaluating this bound yields $n = O(Z^{3/2})$. Thus $\nu(N, Z)$ can be estimated by const $Z^{\frac{3}{2}k}$ which proves our Theorem 1.

4 Correlations estimate for bosons

In this section we derive an estimate on the pair correlations in N-boson systems, similar to the correlation estimate for N-fermion systems in [1]. We then apply this estimate to the Coulomb interaction.

4.1 Boson density matrices

We recall definitions and basic properties of boson density matrices. For a separable Hilbert space $\mathfrak{H}$, let $\mathscr{F}(\mathfrak{H}) = \bigoplus_{N=0}^{\infty} \mathfrak{H}_N$ be the corresponding boson Fock space, i.e., $\mathfrak{H}_0 = \mathbb{C}$, and $\mathfrak{H}_N$ the symmetric subspace of the n-fold tensor product of $\mathfrak{H}$ (see [15]). The annihilation and creation operators are defined as usual and obey for all $f, g \in \mathfrak{H}$ the commutation relations

$$a(f)\, a^\dagger(g) - a^\dagger(g)\, a(f) =: [a(f), a^\dagger(g)] = \langle f | g \rangle, \tag{16}$$

$$[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0$$

(see, e.g., Thirring [15]). Note that we fix the following notation: $\langle f | g \rangle$ is the scalar product of two elements f and g in $\mathfrak{H}$. Moreover given $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ and complex numbers $c_{\mu,\nu}$ we define the operator $\sum_{\mu,\nu=1}^{\infty} c_{\mu,\nu} |f_\mu\rangle\langle g_\nu|$ to be given by

$$\langle \psi | \left(\sum_{\mu,\nu=1}^{\infty} c_{\mu,\nu} |f_\mu\rangle\langle g_\nu| \right) |\varphi\rangle := \sum_{\mu,\nu=1}^{\infty} c_{\mu,\nu} \langle \psi | f_\mu \rangle \langle g_\nu | \varphi \rangle,$$

if this sum converges absolutely for ψ and φ on a dense subspace of $\mathfrak{H}$. To express a certain symmetry in the case of a self-adjoint operator A we write also $\langle \psi | A | \varphi \rangle$ instead of $\langle \psi | A\, \varphi \rangle$.

Given an orthonormal basis $\{\varphi_i\}_{i \in \mathbb{N}}$ of $\mathfrak{H}$, we define $a_i := a(\varphi_i)$. Now, let ρ_N be an N-particle density matrix (N-pdm), i.e., $\rho_N = \sum_i \lambda_i |\Psi_{N,i}\rangle\langle\Psi_{N,i}|$ for some

orthonormal set $\{\Psi_{N,i}\}_{i\in\mathbb{N}}\subseteq\mathfrak{H}_N$ and nonnegative numbers $0\leqq\lambda_i\leqq 1$, $\sum_i \lambda_i = 1$. With $\langle A\rangle := \mathrm{tr}_N\{A\,\rho_N\}$ we define the 1-pdm

$$\gamma_\rho := \sum_{k;m}\langle a_k^\dagger a_m\rangle\,|\varphi_m\rangle\langle\varphi_k| \tag{17}$$

and the 2-pdm

$$\Gamma_\rho := \sum_{k,l;m,n}\langle a_k^\dagger a_l^\dagger a_m a_n\rangle\,|\varphi_m\otimes\varphi_n\rangle\langle\varphi_k\otimes\varphi_l| \tag{18}$$

corresponding to ρ_N, and we obtain

$$\langle H_{N,Z}\rangle = \mathrm{tr}_1\{h_0\gamma_\rho\} + \tfrac{1}{2}\,\mathrm{tr}_2\{V\Gamma_\rho\}, \tag{19}$$

where $h_0 := -\Delta - Z|x|^{-1}$ and $V := |x-y|^{-1}$. Obviously,

$$0\leqq\gamma_\rho\leqq\mathrm{tr}_1\{\gamma_\rho\} = N, \qquad 0\leqq\Gamma_\rho\leqq\mathrm{tr}_2\{\Gamma_\rho\} = N(N-1). \tag{20}$$

Moreover, if $\rho_N = |\Psi_N\rangle\langle\Psi_N|$ and $\Psi_N = \varphi\otimes\ldots\otimes\varphi = (N!)^{-1/2}\,a^\dagger(\varphi)\ldots a^\dagger(\varphi)\,|0\rangle$, one easily checks

$$\gamma_\rho = N\,|\varphi\rangle\langle\varphi|, \tag{21}$$

$$\begin{aligned}\Gamma_\rho &= N(N-1)\,|\varphi\otimes\varphi\rangle\langle\varphi\otimes\varphi| \\ &= \frac{N-1}{N}\,(\gamma_\rho\otimes\gamma_\rho).\end{aligned} \tag{22}$$

Since these $\Psi_N = \varphi\otimes\ldots\otimes\varphi$ correspond to the Slater determinants in the fermionic case, we accordingly define the *truncated* 1- and 2-pdm by

$$\gamma_\rho^T := \gamma_\rho - \frac{1}{N}\,\gamma_\rho^2, \tag{23}$$

$$\Gamma_\rho^T := \Gamma_\rho - \frac{N-1}{N}\,(\gamma_\rho\otimes\gamma_\rho). \tag{24}$$

4.2 *Correlation estimate for bosons*

With the above definitions we can formulate the main estimate.

Theorem 2 *Let ρ_N be an N-pdm and let $\gamma := \gamma_\rho$, $\Gamma := \Gamma_\rho$. Let $X = X^\dagger = X^2$ be an orthogonal projection on $\mathfrak{H}$. Then*

$$\mathrm{tr}_2\{(X\otimes X)\,\Gamma^T\} \geqq -\mathrm{tr}_1\{X\gamma\}\cdot\min\{1,\, 7.831\,[\mathrm{tr}_1\{X\gamma^T\}]^{1/2}\}. \tag{25}$$

Proof. Since our proof is very similar to the one of the correlation estimate for fermions [2, Theorem 4], we just give a brief outline with emphasis on the differences from the fermionic case.

We choose $\{\varphi_i\}_{i\in\mathbb{N}}$ to consist of the eigenfunctions of γ:

$$\langle a_k^\dagger a_m\rangle = \langle\varphi_m|\,\gamma\,|\varphi_k\rangle = \delta_{mk}\lambda_k, \tag{26}$$

$0\leqq\lambda_k\leqq N$, $\sum_{k=1}^{\infty}\lambda_k = N$, and denote $X_{mk} := \langle\varphi_m|\,X\,|\varphi_k\rangle$. We have

$$\mathrm{tr}_2\{(X\otimes X)\Gamma^T\} = \sum_{k,l,m,n} X_{mk}X_{nl}\left\{\langle a_n^\dagger a_m^\dagger a_k a_l\rangle - \frac{N-1}{N}\langle a_m^\dagger a_k\rangle\langle a_n^\dagger a_l\rangle\right\}. \tag{27}$$

Reproducing the proof of Lemma 2 in [2], we establish the left part of inequality (25):

Lemma 3

$$\mathrm{tr}_2\{(X\otimes X)\Gamma^T\} \geqq -\mathrm{tr}_1\{X\gamma\}. \tag{28}$$

Following [2], we fix a number $0<\tau<N$ and introduce the operators

$$c_p := \sum_{k<\tau} X_{kp}a_k, \qquad d_p := \sum_{k\geqq\tau} X_{kp}a_k, \tag{29}$$

where $k<\tau$ and $k\geqq\tau$ denote $\lambda_k<\tau$ and $\lambda_k\geqq\tau$, respectively. Then

$$\begin{aligned}
[d_p, d_q^\dagger] &= \sum_{k,l\geqq\tau} X_{kp}X_{ql}[a_k, a_l^\dagger] = \sum_{k\geqq\tau} X_{kp}X_{qk}, \\
[c_p, c_q^\dagger] &= \sum_{k,l<\tau} X_{kp}X_{ql}[a_k, a_l^\dagger] = \sum_{k<\tau} X_{pk}X_{kq}, \\
\langle d_p^\dagger d_q\rangle &= \sum_{k,l\geqq\tau} X_{pk}X_{lq}\langle a_k^\dagger a_l\rangle = \sum_{k\geqq\tau} X_{pk}X_{kq}\lambda_k, \\
\langle c_p^\dagger c_q\rangle &= \sum_{k,l<\tau} X_{pk}X_{lq}\langle a_k^\dagger a_l\rangle = \sum_{k<\tau} X_{pk}X_{kq}\lambda_k,
\end{aligned} \tag{30}$$

and all other commutators and expectation values vanish. Thus, we have the decomposition

$$\mathrm{tr}_2\{(X\otimes X)\Gamma\} = MP + R, \tag{31}$$

where

$$MP := \sum_{p,q}\langle d_p^\dagger(c_q^\dagger + d_q^\dagger)\,d_q(c_p + d_p)\rangle, \tag{32}$$

$$R := \sum_{p,q}(\langle c_p^\dagger c_q^\dagger c_q c_p\rangle + 4\,\mathrm{Re}\langle c_p^\dagger d_q^\dagger c_q c_p\rangle + 2\langle c_p^\dagger d_q^\dagger d_q c_p\rangle) \tag{33}$$

$$+\sum_{p,q}\langle c_p^\dagger d_q^\dagger c_q d_p\rangle \tag{34}$$

$$+\sum_{p,q} 2\,\mathrm{Re}\langle c_p^\dagger d_q d_q^\dagger d_p\rangle \tag{35}$$

$$+2\,\mathrm{Re}\langle(\sum_p c_p^\dagger d_p)^2\rangle. \tag{36}$$

Then, abbreviating $A := \sum_q [d_q^\dagger (c_q + d_q) - \langle d_q^\dagger (c_q + d_q)\rangle]$, we obtain

Lemma 4

$$MP - \frac{N-1}{N} \operatorname{tr}_2 \{(X \otimes X)(\gamma \otimes \gamma)\} - \langle A^\dagger A\rangle \geqq -\left(\frac{2N}{N-\tau} + \frac{1}{\tau}\right) \cdot \operatorname{tr}_1 \{X\gamma\} \cdot \operatorname{tr}_1 \{X\gamma^T\}. \tag{37}$$

Proof of Lemma 4 After commuting through $d_p^\dagger$, we get

$$\begin{aligned} MP &= \langle A^\dagger A\rangle + |\sum_p \langle d_p^\dagger (c_p + d_p)\rangle|^2 - \sum_{p,q} ([d_q, d_p^\dagger] \langle (c_q^\dagger + d_q^\dagger)(c_p + d_p)\rangle) \\ &= \langle A^\dagger A\rangle + (\sum_{k \geqq \tau} X_{kk} \lambda_k)^2 - \sum_{k \geqq \tau; l} |X_{kl}|^2 \lambda_l \\ &\geqq \langle A^\dagger A\rangle + (\sum_k X_{kk} \lambda_k)^2 \\ &\quad - 2(\sum_k X_{kk} \lambda_k)(\sum_{k<\tau} X_{kk} \lambda_k) - \sum_{k \geqq \tau; l} |X_{kl}|^2 \lambda_l . \end{aligned} \tag{38}$$

Hence, the left side of (37) is bounded below by

$$\begin{aligned} &-2(\sum_k X_{kk} \lambda_k)(\sum_{k<\tau} X_{kk} \lambda_k) - \sum_{k \geqq \tau; l} |X_{kl}|^2 \left(1 - \frac{\lambda_k}{N}\right) \lambda_l \\ &\geqq -\frac{2N}{N-\tau} (\sum_k X_{kk} \lambda_k) \left(\sum_{k<\tau} X_{kk} \left(\lambda_k - \frac{1}{N} \lambda_k^2\right)\right) \\ &\quad - \frac{1}{\tau} \sum_{k \geqq l; l} X_{kk} X_{ll} \lambda_l \left(\lambda_k - \frac{1}{N} \lambda_k^2\right), \end{aligned} \tag{39}$$

using $|X_{kl}|^2 \leqq X_{kk} X_{ll}$. Letting the sums run over their entire range, we arrive at the assertion. □

Next, we estimate R.

Lemma 5

$$\begin{aligned} R + \langle A^\dagger A\rangle &\geqq -\frac{6N^2}{\tau(N-\tau)} \operatorname{tr}_1 \{X\gamma\} \operatorname{tr}_1 \{X\gamma^T\} \\ &\quad - \frac{2N}{\tau^{1/2}(N-\tau)^{1/2}} \left(1 + \frac{N^2}{\tau(N-\tau)} \operatorname{tr}_1 \{X\gamma^T\}\right)^{1/2} \operatorname{tr}_1 \{X\gamma\} (\operatorname{tr}_1 \{X\gamma^T\})^{1/2}. \end{aligned} \tag{40}$$

Proof of Lemma 5 First, we trace everything back to estimating $\sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p\rangle$. Namely, note that

$$\sum_{p,q} (\langle c_p^\dagger d_q d_q^\dagger c_p\rangle - \langle c_p^\dagger d_q^\dagger d_q c_p\rangle) = \sum_{p,q} [d_q, d_q^\dagger] \langle c_p^\dagger c_p\rangle \geqq 0, \tag{41}$$

and, hence,

$$0 \leqq \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle \leqq \sum_{p,q} \langle c_p^\dagger d_q d_q^\dagger c_p \rangle \tag{42}$$
$$= \sum_{p,q} [d_q, d_q^\dagger] \langle c_p^\dagger c_p \rangle + \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle.$$

Furthermore,

$$|\sum_{p,q} \langle c_p^\dagger d_q^\dagger d_p c_p \rangle| \leqq \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle^{1/2} \langle c_q^\dagger d_p^\dagger d_p c_q \rangle^{1/2} \tag{43}$$
$$\leqq \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle,$$

and, thus,

$$|\sum_{p,q} \langle c_p^\dagger d_p d_q^\dagger c_p \rangle| \leqq \sum_{p,q} [d_p, d_q^\dagger] \langle c_p^\dagger c_q \rangle + \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle. \tag{44}$$

By means of (42), (43), and (44), we obtain

$$\text{rhs}(33) \geqq -2 \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle, \tag{45}$$

$$(34) \geqq -\sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle, \tag{46}$$

$$(35) = 2 \operatorname{Re} \langle (\sum_p c_p^\dagger d_p) A \rangle - 2 \sum_{p,q} \langle c_p^\dagger d_p d_q^\dagger c_q \rangle \tag{47}$$
$$\geqq -\langle A^\dagger A \rangle - 3 [\sum_{p,q} [d_p, d_q^\dagger] \langle c_p^\dagger c_q \rangle + \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle],$$

$$(36) \geqq -2 (\sum_{p,q} \langle c_p^\dagger d_p d_q^\dagger c_q \rangle)^{1/2} (\sum_{p,q} \langle d_q^\dagger c_q c_p^\dagger d_p \rangle)^{1/2} \tag{48}$$
$$\geqq -2 [\sum_{p,q} [d_p, d_q^\dagger] \langle c_p^\dagger c_q \rangle + \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle]^{1/2}$$
$$\cdot [\sum_{p,q} [c_q, c_p^\dagger] \langle d_q^\dagger d_p \rangle + \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle]^{1/2}.$$

Adding up (45)–(48) yields

$$R + \langle A^\dagger A \rangle \geqq -3 \sum_{p,q} [d_p, d_q^\dagger] \langle c_p^\dagger c_q \rangle - 6 \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle \tag{49}$$
$$-2 [\sum_{p,q} [d_p, d_q^\dagger] \langle c_p^\dagger c_q \rangle + \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle]^{1/2}$$
$$\cdot [\sum_{p,q} [c_p, c_p^\dagger] \langle d_q^\dagger d_p \rangle + \sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle]^{1/2}.$$

The commutator terms in (49) are easily estimated:

$$0 \leqq \sum_{p,q} [d_p, d_q^\dagger] \langle c_p^\dagger c_q \rangle \leqq \sum_{p,q} [d_p, d_p^\dagger] \langle c_q^\dagger c_q \rangle \tag{50}$$

$$= \Big(\sum_{k \geqq \tau} X_{kk}\Big)\Big(\sum_{k<\tau} \lambda_k X_{kk}\Big)$$

$$\leqq \frac{N}{\tau(N-\tau)} \Big(\sum_{k \geqq \tau} \lambda_k X_{kk}\Big)\left(\sum_{k<\tau} \left(\lambda_k - \frac{\lambda_k^2}{N}\right) X_{kk}\right)$$

$$\leqq \frac{N}{\tau(N-\tau)} \operatorname{tr}_1\{X\gamma\} \operatorname{tr}_1\{X\gamma^T\},$$

$$0 \leqq \sum_{p,q} [c_p, c_q^\dagger] \langle d_q^\dagger d_p \rangle = \sum_{k<\tau} \sum_{l \geqq \tau} \lambda_l |X_{kl}|^2 \leqq \operatorname{tr}_1\{X\gamma\}. \tag{51}$$

Now, suppose we also knew that

$$\sum_{p,q} \langle c_p^\dagger d_q^\dagger d_q c_p \rangle \leqq \frac{N(N-1)}{\tau(N-\tau)} \operatorname{tr}_1\{X\gamma\} \operatorname{tr}_1\{X\gamma^T\}. \tag{52}$$

Then, inserting (50), (51), and (52) into (49), we would arrive at (40).

Therefore, it remains to establish (52). Let $\phi_{N-1} \in \mathfrak{H}_{N-1}$ and set $B := \sum_{k;m} |\varphi_m\rangle\langle\phi_{N-1}| a_k^\dagger a_m |\phi_{N-1}\rangle\langle\varphi_k| \geqq 0$, and $P := \sum_{k \geqq \tau} |\varphi_k\rangle\langle\varphi_k|$. We observe $\operatorname{tr}_1\{B\} = (N-1)\langle\phi_{N-1}|\phi_{N-1}\rangle$. Hence, it follows that

$$\sum_q \langle\phi_{N-1}| d_q^\dagger d_q |\phi_{N-1}\rangle \tag{53}$$

$$= \sum_{k,m \geqq \tau} X_{mk} \langle\phi_{N-1}| a_k^\dagger a_m |\phi_{N-1}\rangle$$

$$= \operatorname{tr}_1\{PXPB\} \leqq (\operatorname{tr}_1\{(PXP)^2\})^{1/2} (\operatorname{tr}_1\{B^2\})^{1/2}$$

$$\leqq \operatorname{tr}_1\{PXP\} \operatorname{tr}_1\{B\} \leqq \frac{N-1}{\tau} \operatorname{tr}_1\{X\gamma\} \langle\phi_{N-1}|\phi_{N-1}\rangle.$$

Using this estimate for each p we obtain

$$\sum_p \langle c_p^\dagger (\sum_q d_q^\dagger d_q) c_p \rangle \leqq \frac{N-1}{\tau} \operatorname{tr}_1\{X\gamma\} \sum_p \langle c_p^\dagger c_p \rangle \tag{54}$$

$$\leqq \frac{N(N-1)}{\tau(N-\tau)} \operatorname{tr}_1\{X\gamma\} \operatorname{tr}_1\{X\gamma^T\}. \quad \square$$

Together, Lemma 3, 4 and 5 imply

$$\operatorname{tr}_2\{(X \otimes X) \Gamma^T\} \geqq -\operatorname{tr}_1\{X\gamma\} \tag{55}$$

$$\cdot \min\left\{1, \left[\left(\frac{2\tau g^2}{N} + \frac{(N-\tau) g^2}{N} + 6g^2\right) a + 2g(1+g^2 a^2)^{1/2}\right] a\right\}, \tag{56}$$

where $a := (\mathrm{tr}_1\{X\gamma^T\})^{1/2}$ and $g := \frac{N}{\tau^{1/2}(N-\tau)^{1/2}}$. We choose $\tau := \frac{N}{2} \geqq 1$, which yields

$$\mathrm{tr}_2\{(X\otimes X)\Gamma^T\} \geqq -\mathrm{tr}_1\{X\gamma\}\min\{1, [29a+4(1+4a^2)^{1/2}]a\} \geqq -\mathrm{tr}_1\{X\gamma\}\min\{1, \gamma_c a\}, \tag{57}$$

where $\gamma_c = 29\hat{a}+4(1+4\hat{a}^2)^{1/2}$; $\hat{a}$ is defined by $[29\hat{a}+4(1+4\hat{a}^2)^{1/2}]\hat{a}=1$. Numerically $\gamma_c = 7.831$. □

4.3 *Application to the Coulomb interaction*

Our next task is the application of the fairly abstract result of Sect. 4.2 to the Coulomb interaction in atoms. We proceed as in [2]. We specialize to $\mathfrak{H} = L^2(\mathbb{R}^3)$. Let $\mathfrak{Q}_N$ be the symmetric, i.e., bosonic, subspace of $H^1(\mathbb{R}^{3N})$ and denote the multiplication operator on $\mathfrak{H}$ corresponding to $\chi_{B(r,z)}$ by $X_{(r,z)}$ and $V := |x-y|^{-1}$, being the Coulomb interaction in $\mathfrak{H}\otimes\mathfrak{H}$. Here, $\chi_{B(r,z)}$ is the characteristic function of $\{x\in\mathbb{R}^3 \,|\, |x-z|\leqq r\}$. We use the decomposition

$$\frac{1}{|x-y|} = \frac{1}{\pi}\int d^3z \int_0^\infty \frac{dr}{r^5}\,\chi_{B(r,z)}(x)\,\chi_{B(r,z)}(y), \tag{58}$$

which was introduced by Fefferman and de la Llave [8]. Therefore,

$$\mathrm{tr}_2\{V\Gamma_\rho^T\} = \frac{1}{\pi}\int d^3z \int_0^\infty \frac{dr}{r^5}\,\mathrm{tr}_2\{(X_{(r,z)}\otimes X_{(r,z)})\,\Gamma_\rho^T\}. \tag{59}$$

We will use Theorem 2 to show

Lemma 6 *Let ρ_N be an N-*pdm *with finite kinetic energy. Then*

$$\mathrm{tr}_2\{V\Gamma_\rho^T\} \geqq -C(\mathrm{tr}_1\{-\Delta\gamma_\rho^T\})^{1/3}(\mathrm{tr}_1\{-\Delta\gamma_\rho\})^{1/6}N^{5/6}, \tag{60}$$

where $C := 3456\cdot 3^{-4/9}\cdot 2^{1/18}\cdot\pi\cdot(7.831)^{2/3}$.

Note that C is about 27307.

Proof. We apply Theorem 2 to (59) for any (r, z) and obtain

$$\mathrm{tr}_2\{V\Gamma_\rho^T\} \geqq -\frac{1}{\pi}\int d^3z \int_0^\infty \frac{dr}{r^5}\Big(\int_{|x-z|\leqq r}\rho(x)\,d^3x\Big) \cdot\min\Big\{1, \beta\Big(\int_{|x-z|\leqq r}\rho_T(x)\,d^3x\Big)^{1/2}\Big\}, \tag{61}$$

where $\rho(x) := \sum_i \lambda_i |\varphi_i(x)|^2$ is the one-particle density and $\rho_T(x) := \sum_i \left(\lambda_i - \frac{\lambda_i^2}{N}\right) |\varphi_i(x)|^2$ is the truncated one-particle density and $\beta = 7.831$. Denoting by M and M_T the maximal function of ρ and ρ_T, respectively, we arrive at

$$\text{(62)} \qquad \operatorname{tr}_2\{V\Gamma_\rho^T\} \geqq -3\left(\frac{4\pi}{3}\right)^{4/3} \beta^{2/3} \int M(z)\, M_T^{1/3}(z)\, d^3 z,$$

mimicking the steps towards (87) in [2]. Using the Hölder inequality, the maximal inequality, and again the Hölder inequality, we estimate the righthand as follows

$$\text{(63)} \qquad \begin{aligned} \int M(z)\, M_T^{1/3}(z)\, d^3 z &\leqq (\int M^{9/8}(z)\, d^3 z)^{8/9} (\int M_T^3(z)\, d^3 z)^{1/9} \\ &\leqq \left(\frac{96}{\pi}\right) 9^{8/9} \left(\frac{3}{2}\right)^{1/9} (\int \rho^{9/8}(x)\, d^3 x)^{8/9} (\int \rho_T^3(x)^{1/9}\, d^3 x)^{1/9} \\ &\leqq \frac{864}{6^{1/9}\pi} (\int \rho(x)\, d^3 x)^{5/6} (\int \rho^3(x)\, d^3 x)^{1/18} (\int \rho_T^3(x)^{1/9}\, d^3 x)^{1/9}. \end{aligned}$$

The assertion (60) follows now, provided we can show

$$\text{(64)} \qquad (\int \rho^3\, d^3 x)^{1/3} \leqq C_{\text{sob}} \operatorname{tr}_1\{-\Delta d\}$$

for any 1-pdm $0 \leqq d = \sum_k \lambda_k |\varphi_k\rangle\langle\varphi_k|$ with $\operatorname{tr}_1\{-\Delta d\} < \infty$, orthonormal $\varphi_k \in L^2(\mathbb{R}^3)$ and $C_{\text{sob}} = 2^{-1/3}\pi^{4/3}$. The function ρ_d is the one-particle density obtained from d. This inequality, however, follows by successive application of Sobolev's and Schwarz' inequality (Hoffmann-Ostenhof and Hoffmann-Ostenhof [9]). □

4.4 *Bounds on truncated quantities and the correlation error*

In this section we will derive bounds on $\operatorname{tr}\{-\Delta \gamma_\psi^T\}$ for Z^2-approximate ground states $\psi_N \in \mathfrak{Q}_N$. This, in turn, enables us to control the correlation error $\frac{1}{2}\operatorname{tr}_2\{V\Gamma_\psi^T\}$. We start with introducing some more notation. Throughout this section we assume $0 < \alpha := (N-1)/Z < \alpha_c = n_c - 1 \ldots$. Let $\varphi_\alpha \in H^2(\mathbb{R}^3)$, $\|\varphi_\alpha\| = 1$ be the positive solution of $h_\alpha \varphi_\alpha = e'(\alpha)\varphi_\alpha$, where $h_\alpha = -\Delta - |x|^{-1} + \alpha \int |\varphi_\alpha(y)|^2 |x-y|^{-1} |\phi_\alpha(y)|^2\, d^3 y$. We set $P_\alpha := |\varphi_\alpha\rangle\langle\varphi_\alpha|$. Since $e'(\alpha)$ is the lowest eigenvalue of h_α, there exists a gap $\Delta e'(\alpha) > 0$, such that $[h_\alpha - e'(\alpha) - \Delta e'(\alpha)] \cdot [1 - P_\alpha] \geqq 0$. By scaling $\varphi_\alpha^Z := Z^{3/2}\varphi_\alpha(Zx)$ and

$$\text{(65)} \qquad h_\alpha^Z := -\Delta - \frac{Z}{|x|} + (N-1) \int |x-y|^{-1} |\varphi_\alpha^Z(y)|^2\, d^3 y,$$

this translates into $h_\alpha^Z \varphi_\alpha^Z = Z^2 e'(\alpha) \varphi_\alpha^Z$ and

$$\text{(66)} \qquad [h_\alpha^Z - Z^2 e'(\alpha) - Z^2 \Delta e'(\alpha)] \cdot [1 - P_\alpha^Z] \geqq 0,$$

where $P_\alpha^Z := |\varphi_\alpha^Z\rangle\langle\varphi_\alpha^Z|$.

For $\psi_N \in \mathfrak{Q}_N$ we denote

$$\Delta E(\psi_N) := \langle\psi_N | H_{N,Z} - E_{N,Z} | \psi_N\rangle \geqq 0. \tag{67}$$

The basic estimate we will use is as follows

Lemma 7 *Let $\psi_N \in \mathfrak{Q}_N$ be normalized. Then*

$$\mathrm{tr}_1\{h_\alpha^Z \gamma_\psi\} \leqq N Z^2 e'(\alpha) - \tfrac{1}{2}\,\mathrm{tr}_2\{V\Gamma_\psi^T\} + \Delta E(\psi_N). \tag{68}$$

Proof. Define $\phi_N := \varphi_\alpha^Z \otimes \ldots \otimes \varphi_\alpha^Z \in \mathfrak{H}_N$. Then

$$\begin{aligned}\langle\phi_N | H_{N,Z} | \phi_N\rangle &= N \cdot \langle\varphi_\alpha^Z | -\Delta - \frac{Z}{|x|} + \frac{N-1}{2} \int |x-y|^{-1} |\varphi_\alpha^Z(y)|^2 d^3y \,|\varphi_\alpha^Z\rangle \\ &= N Z^2 e'(\alpha) - \frac{N-1}{2N}\,\mathrm{tr}_2\{V(\gamma_\phi \otimes \gamma_\phi)\}.\end{aligned} \tag{69}$$

By the variational principle,

$$\begin{aligned}\langle\phi_N | H_{N,Z} | \phi_N\rangle &\geqq E_{N,Z} \\ &= \mathrm{tr}_1\left\{\left(-\Delta - \frac{Z}{|x|}\right)\gamma_\psi\right\} + \frac{N-1}{2N}\,\mathrm{tr}_2\{V(\gamma_\psi \otimes \gamma_\psi)\} \\ &\quad + \frac{1}{2}\,\mathrm{tr}_2\{V\Gamma_\psi^T\} - \Delta E(\psi_N) \\ &\geqq \mathrm{tr}_1\{h_\alpha^Z \gamma_\psi\} - \frac{N-1}{2N}\,\mathrm{tr}_2\{V(\gamma_\phi \otimes \gamma_\phi)\} \\ &\quad + \frac{1}{2}\,\mathrm{tr}_2\{V\Gamma_\psi^T\} - \Delta E(\psi_N),\end{aligned} \tag{70}$$

and we obtain (68) by adding (69) and (70). □

Now, we show that Lemma 7 implies

Lemma 8 *Let $\psi_N \in \mathfrak{Q}_N$ be normalized. Then*

$$\mathrm{tr}_1\{\gamma_\psi^T\} \leqq \frac{1}{Z^2 \Delta e'(\alpha)}\left(-\mathrm{tr}_2\{V\Gamma_\psi^T\} + 2\Delta E(\psi_N)\right). \tag{71}$$

Proof. We first derive

$$N - \mathrm{tr}_1\{P_\alpha^Z \gamma_\psi\} \leqq \frac{1}{2Z^2 \Delta e'(\alpha)}\left(-\mathrm{tr}_2\{V\Gamma_\psi^T\} + 2\Delta E(\psi_N)\right) \tag{72}$$

from (68) and (66). Indeed,

$$N-\mathrm{tr}_1\{P_\alpha^Z\gamma_\psi\}=\mathrm{tr}_1\{(1-P_\alpha^Z)\gamma_\psi\} \tag{73}$$
$$\leqq\frac{1}{Z^2\,e'(\alpha)}\,\mathrm{tr}_1\{[h_\alpha^Z-Z^2\,e'(\alpha)][1-P_\alpha^Z]\gamma_\psi\}$$
$$=\frac{1}{Z^2\,e'(\alpha)}\,\mathrm{tr}_1\{[h_\alpha^Z-Z^2\,e'(\alpha)]\gamma_\psi\}$$
$$\leqq\frac{1}{2\,Z^2\,\Delta\,e'(\alpha)}\,(-\mathrm{tr}_2\{V\Gamma_\psi^T\}+2\Delta E(\psi_N)).$$

Now, we expand

$$\mathrm{tr}_1\{\gamma_\psi^T\}=\mathrm{tr}_1\{P_\alpha^Z(\gamma_\psi-N^{-1}\gamma_\psi^2)\}+\mathrm{tr}_1\{(1-P_\alpha^Z)(\gamma_\psi-N^{-1}\gamma_\psi^2)\} \tag{74}$$
$$\leqq\mathrm{tr}_1\{P_\alpha^Z(N-\gamma_\psi)\}+\mathrm{tr}_1\{(1-P_\alpha^Z)\gamma_\psi\}$$
$$=2(N-\mathrm{tr}_1\{P_\alpha^Z\gamma_\psi\}).\quad\square$$

We obtain a similar bound for the truncated kinetic energy

Lemma 9 *Let $\psi_N\in\mathfrak{Q}_N$ be normalized. Then*

$$\mathrm{tr}_1\{-\Delta\gamma_\psi^T\}\leqq Z^2\,\mathrm{tr}_1\{\gamma_\psi^T\}-\mathrm{tr}_2\{V\Gamma_\psi^T\}+2\Delta E(\psi_N). \tag{75}$$

Proof. Using $h_\alpha^Z-Z^2\,e'(\alpha)\geqq 0$, we obtain

$$\mathrm{tr}_1\{[h_\alpha^Z-Z^2\,e'(\alpha)]\gamma_\psi^T\}\leqq\mathrm{tr}_1\{[h_\alpha^Z-Z^2\,e'(\alpha)]\gamma_\psi\}\leqq-\tfrac{1}{2}\,\mathrm{tr}_2\{V\Gamma_\psi^T\}+\Delta E(\psi_N), \tag{76}$$

by means of Lemma 7. Therefore,

$$-\mathrm{tr}_2\{V\Gamma_\psi^T\}+2\Delta E(\psi_N)+Z^2\,e'(\alpha)\,\mathrm{tr}_1\{\gamma_\psi^T\} \tag{77}$$
$$\geqq\mathrm{tr}_1\{h_\alpha^Z\gamma_\psi^T\}$$
$$\geqq\frac{1}{2}\,\mathrm{tr}_1\{-\Delta\gamma_\psi^T\}+\frac{1}{2}\,\mathrm{tr}_1\left\{\left(-\Delta-\frac{2Z}{|x|}\right)\gamma_\psi^T\right\} \tag{78}$$
$$\geqq\frac{1}{2}\,\mathrm{tr}_1\{-\Delta\gamma_\psi^T\}+\frac{1}{2}\,Z^2\,\mathrm{tr}_1\{\gamma_\psi^T\}.\quad\square$$

We arrive at the final result

Theorem 3 *Let $\psi_N\in\mathfrak{Q}_N$ be a normalized state obeying $0\geqq\langle\psi_N|H_{N,Z}|\psi_N\rangle=E_{N,Z}+\Delta E(\psi_N)$. Then*

$$-\mathrm{tr}_2\{V\Gamma_\psi^T\}\leqq\hat{C}\max\{Z^{2/3},\Delta E(\psi_N)^{1/3}\}\cdot Z^{4/3} \tag{79}$$

$$\mathrm{tr}_1\{\gamma_\psi^T\}\leqq\frac{2\hat{C}}{Z^2\,\Delta\,e'(\alpha)}\max\{Z^2,\Delta E(\psi_N)\} \tag{80}$$

$$\langle\psi_N|H_{N,Z}|\psi_N\rangle\geqq\langle\psi_N|H_{N,Z}^W|\psi_N\rangle-\hat{C}\max\{Z^2,Z^{4/3}\Delta E(\psi_N)^{1/3}\}, \tag{81}$$

where $\hat{C}:=2C^{3/2}(1+e'(\alpha)^{-1})^{1/2}$ which is about $9025000(1+e'(\alpha)^{-1})^{1/2}$.

Proof. We insert Lemma 8 into Lemma 9, which gives

$$\text{(82)} \qquad \mathrm{tr}_1\{-\Delta\gamma_\psi^T\} \leqq \left(1+\frac{1}{\Delta e'(\alpha)}\right)(-\mathrm{tr}_2\{V\Gamma_\psi^T\}+2\Delta E(\psi_N)).$$

Furthermore, as in (77), $0 \geqq \langle\psi_N|H_{N,Z}|\psi_N\rangle$ implies $\mathrm{tr}_1\{-\Delta\gamma_\psi\} \leqq NZ^2$, and this, together with (82), leads in Lemma 6 to the estimate

$$\text{(83)} \quad -\frac{1}{2}\mathrm{tr}_2\{V\Gamma_\psi^T\} \leqq C\left(1+\frac{1}{\Delta e'(\alpha)}\right)^{1/3}(-\mathrm{tr}_2\{V\Gamma_\psi^T\}+2\Delta E(\psi_N))^{1/3}NZ^{1/3}.$$

Obviously, (79) follows from here for $N<\alpha_c Z$. Then, (80) follows directly from (79) and Lemma 8, and (81) is a consequence of (79), inserted into (70). □

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Note added in proof

Recently G.M. Graf and J.P. Solovej ("A Correlation Estimate with Applications to Quantum Systems with Coulomb Interactions", Preprint 1993, Rev. Math. Phys., to appear) proved a correlation inequality that extends our bosonic correlation inequality – among others – to the full Hilbert space regardless of symmetry (boltzon case). As explained above this implies that our Theorem 1 can be strengthed to include the boltzon case as well. The form of the estimate does not change. Based on the one-particle picture we expect it to be optimal for boltzons. Moreover, M.B. Ruskai – based on idea of B. Simon mentioned in I.M. Sigal, "How Many Electrons Can a Nucleus Bind?", Ann. Phys. **157**, 307–320 (1984) – showed recently in a note in preparation for publication that the localization error $\pi^2 N/(4\rho^2)$ in (10) may be substituted by $\text{const}(\log Z)^2/R^2$. This allows the choice of $R = \text{const}(\log Z)/Z$ and implies in the bosonic case the better bound $O((\log Z)^K)$ for some Z-independent constant K.

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Hyperbolic involutions

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Introduction

The Wedderburn-Artin theory shows that every central simple algebra A can be viewed as an algebra of endomorphisms of some vector space V over a division algebra. Moreover, involutions on A are then adjoint involutions with respect to some λ-hermitian form on V, and it is natural to define an involution to be hyperbolic if it corresponds to a hyperbolic λ-hermitian form.

The purpose of this paper is to show how hyperbolic involutions can be defined intrinsically in A. It turns out that an involution σ is hyperbolic if and only if A contains an idempotent e such that $\sigma(e)=1-e$: see Theorem 2.1. We also show that hyperbolic involutions are decomposable (Theorem 2.2) and we determine when an involution of the first kind becomes hyperbolic under a quadratic extension of scalars (Theorem 3.3). This result is then applied to give a decomposability criterion for involutions on central simple algebras of Schur index 2. In the final section, the following "odd cancellation" property is proved (Theorem 4.2): Suppose (A, σ) and (B, τ) are central simple algebras with involution of the first kind and the degree of B is odd. If $(A \otimes B, \sigma \otimes \tau)$ is hyperbolic, then (A, σ) is hyperbolic.

1 Involutions and hermitian forms

Throughout this paper the characteristic of the base field F is assumed to be different from 2 and all F-algebras are associative, finite-dimensional, with 1.

Let A be a central simple F-algebra and let V be an irreducible left A-module. By the Wedderburn-Artin theory, the F-algebra

$$D = \mathrm{End}_A(V)$$

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is a division algebra which is Brauer-equivalent to A. Moreover, V carries a natural right D-vector space structure, and

$$A = \mathrm{End}_D(V).$$

The *degree* $\deg A$ is the square root of the dimension of A (over F). The preceding equality thus shows:

$$\deg A = \deg D.dim_D V.$$

Our purpose in this first section is to show how geometric properties of the D-vector space V relate to ring-theoretic properties of the algebra A, particularly when A has an involution. Most of the results here are well-known, although precise references are difficult to find.

We first investigate the right ideals of A. If $W \subset V$ is a subspace, then the F-vector space $\mathrm{Hom}_D(V, W)$ of D-linear maps from V to W may be regarded as a subset of A by composing its elements on the left with the canonical inclusion map $W \hookrightarrow V$.

1.1. Proposition. *Every right ideal I of A is of the form*:

$$I = \mathrm{Hom}_D(V, W)$$

for some subspace $W \subset V$ and also of the form

$$I = eA$$

for some idempotent $e \in A$. In particular, for every $f \in A$,

$$fA = \mathrm{Hom}_D(V, \mathrm{im} f).$$

Proof. The equality $fA = \mathrm{Hom}_D(V, \mathrm{im} f)$ is clear and the fact that every right ideal of A is generated by an idempotent can be found in [4, § 5, Proposition 7]. The proposition follows, since from $I = eA$ we deduce: $I = \mathrm{Hom}_D(V, \mathrm{im}\, e)$. □

There is a corresponding result for left ideals, which is proven by analogous arguments. If $W \subset V$ is a subspace, we identify the F-vector space $\mathrm{Hom}_D(V/W, V)$ with a subset of A by composing its elements on the right with the canonical map $V \to V/W$.

1.2. Proposition. *Every left ideal I of A is of the form*

$$I = \mathrm{Hom}_D(V/W, V)$$

for some subspace $W \subset V$ and also of the form

$$I = Ae$$

for some idempotent $e \in A$. In particular, for every $f \in A$,

$$Af = \mathrm{Hom}_D(V/\ker f, V).$$

From the propositions above, it follows that for all $f \in A$,

$$\dim_F fA = \dim_D V . \dim_D \mathrm{im}\, f \dim_F D = \dim_F Af.$$

Therefore, $\dim_F fA$ is divisible by $\deg A = \dim_D V . \deg D$; we call the quotient the *rank* of f:

$$\mathrm{rk}\, f = \frac{\dim_F fA}{\deg A} = \frac{\dim_F Af}{\deg A} = \dim_D \mathrm{im}\, f . \deg D.$$

The advantage of this definition, which deviates from the usage in [1, p. 50], for example, is that the rank of an element in a central simple algebra is invariant under a scalar extension.

Similarly, we define the rank of a left or right ideal I by:

$$\mathrm{rk}\, I = \frac{\dim_F I}{\deg A}.$$

If A has an involution σ, then D has an involution $^-$: $d \mapsto \bar{d}$ which induces on F the same automorphism as σ; this holds because A and D are Brauer-equivalent: see [15, Corollary 8.8.3]. Recall that an involution σ on A is *of the first kind* if the induced automorphism on F is the identity; otherwise it is *of the second kind.* If σ is of the first kind, then the set $(A, \sigma)_+$ of symmetric (i.e. invariant) elements under σ is a vector space over F of dimension $\frac{n(n+1)}{2}$ or $\frac{n(n-1)}{2}$, where $n = \deg A$ (see [15, Definition 8.7.6]). We say that the *type* of σ is $+1$ in the first case, and -1 in the second case. Thus,

$$\dim_F (A, \sigma)_+ = \frac{n(n+\varepsilon)}{2} \quad \text{where } \varepsilon = \mathrm{type}\, \sigma.$$

(Involutions of type $+1$ (resp. -1) are also known as involutions of *orthogonal* (resp. *symplectic*) type).

We also recall that, given an involution $^-$ on D and $\lambda \in F$ such that $\lambda\bar{\lambda} = 1$, a λ-hermitian form on V is a bi-additive map h: $V \times V \to D$ satisfying:

$$h(x, yd) = h(x, y)\, d \quad \text{and} \quad h(y, x) = \lambda \overline{h(x, y)}$$

for every $x, y \in V$ and $d \in D$. It is *non-degenerate* if $h(x, y) = 0$ for all $y \in V$ implies $x = 0$. There is a well-known correspondence between non-degenerate λ-hermitian forms on V and involutions on A:

1.3. Theorem. *Let* $^-: d \mapsto \bar{d}$ *be a fixed involution on* D. *For every non-degenerate* λ*-hermitian form* h *on* V *there is an adjoint involution* I_h *on* A *defined through*:

$$h(fx, y) = h(x, I_h(f)\, y) \quad \text{for all } x, y \in V \text{ and } f \in A = \mathrm{End}_D(V).$$

The involution I_h *induces on* F *the same automorphism as* $^-$. *If* $^-$ *is of the first kind and of type* ε, *then the type of* I_h *is* $\lambda\varepsilon$. *Moreover, every involution* σ *on* A *which induces the same automorphism of* F *as* $^-$ *is of the form* $\sigma = I_h$ *for some* λ*-hermitian form* h *on* V, *which is uniquely determined up to a factor in* $F^\times$. *If* $^-$ *is of the second kind, then there exists* $\alpha \in F^\times$ *such that* αh *is an hermitian* (*i.e.* 1*-hermitian*) *form on* V.

Proof. See [15, p. 302] or [13] or [17, Chap. 6]. If $^-$ is of the second kind, use Hilbert's Theorem 90 to find α such that $\lambda = \bar{\alpha}\alpha^{-1}$. □

We now fix a non-degenerate λ-hermitian form h on V (with respect to the fixed involution $^-$ on D) and we denote by σ the adjoint involution: $\sigma = I_h$. For every right ideal I of A, we define the orthogonal $I^\perp$ (with respect to σ) as the right annihilator of $\sigma(I)$:

$$I^\perp = \{x \in A \mid \sigma(I)\, x = \{0\}\}.$$

This is clearly a right ideal of A.

1.4. Lemma. *For every subspace* $W \subset V$,

$$\mathrm{Hom}_D(V, W)^\perp = \mathrm{Hom}_D(V, W^\perp),$$

where $W^\perp$ *denotes the orthogonal of* W *with respect to* h.

Proof. Let $f \in \mathrm{Hom}_D(V, W)$ and $g \in \mathrm{Hom}_D(V, W^\perp)$. For all $x, y \in V$, we then have

$$h(x, \sigma(f)\, g\, y) = h(fx, fy) = 0,$$

hence $\sigma(f)\, g = 0$. This proves $\mathrm{Hom}_D(V, W^\perp) \subset \mathrm{Hom}_D(V, W)^\perp$. Conversely, if $f \in \mathrm{Hom}_D(V, W)$ is such that $f(V) = W$ and $g \in \mathrm{Hom}_D(V, W)^\perp$, then from $\sigma(f)\, g = 0$ it follows that $g(V) \subset W^\perp$, hence $\mathrm{Hom}_D(V, W)^\perp \subset \mathrm{Hom}_D(V, W^\perp)$. □

1.5. Corollary. *For every right ideal* $I \subset A$,

$$\mathrm{rk}\, I + \mathrm{rk}\, I^\perp = \deg A.$$

This follows from the lemma and from Proposition 1.1, since $\dim_D W + \dim_D W^\perp = \dim_D V$. □

It is now easy to translate into ring-theoretic terms the property for a subspace to be totally isotropic:

1.6. Proposition. *For a subspace* $W \subset V$ *with associated right ideal* $I = \mathrm{Hom}_D(V, W)$, *the following conditions are equivalent*:

(1) *W is totally isotropic, i.e. $W \subset W^\perp$.*
(2) *$I \subset I^\perp$, i.e. $\sigma(x)\, y = 0$ for all $x, y \in I$.*
(3) *$I = f A$ for some $f \in A$ such that $\sigma(f) = 0$.*

The characterization of non-singular subspaces is equally easy:

1.7. Proposition. *For a subspace $W \subset V$ with associated right ideal $I = \mathrm{Hom}_D(V, W)$, the following conditions are equivalent:*

(1) *W is non-singular, i.e. $W \cap W^\perp = \{0\}$.*
(2) *$I \cap I^\perp = \{0\}$, hence $A = I \oplus I^\perp$.*
(3) *$I = f A$ for some symmetric idempotent $f \in A$.*

Moreover, if these conditions hold, then the symmetric idempotent f such that $I = f A$ is unique and satisfies: $(1-f) A = I^\perp$.

Proof. The equivalence of (1) and (2) follows from Lemma 1.4. If condition (1) holds, then $V = W \oplus W^\perp$ and we define f to be the orthogonal projection on W. Suppose (3) holds. Then $I^\perp = \{x \in A \mid fx = 0\}$, hence $I \cap I^\perp = \{0\}$, proving (2), and moreover $(1-f) A \subset I^\perp$. Since $A = f A \oplus (1-f) A$, Corollary 1.5 implies that $(1-f) A$ and $I^\perp$ have the same rank and the equality follows.

Finally, suppose that f, f' are symmetric idempotents such that $I = f A = f' A$. Then

$$f - f' = (1-f') - (1-f) \in I \cap I^\perp,$$

hence $f = f'$. □

1.8. Corollary. *The following statements are equivalent:*

(1) *(V, h) is anisotropic, that is: for $x \in V$, $h(x, x) = 0$ implies $x = 0$.*
(2) *For $a \in A$, $\sigma(a)\, a = 0$ implies $a = 0$.*
(3) *Every right ideal of A is generated by a symmetric idempotent.*

Proof. This equivalence is a restatement of the fact that (V, h) is anisotropic if and only if V has no non-trivial totally isotropic subspace, if and only if every subspace of V is non-singular. □

The involution σ will be called *anisotropic* if these equivalent conditions hold.

2 Hyperbolic hermitian forms and involutions

Let D be a division algebra with involution $^-: d \mapsto \bar{d}$ over a field F. For any finite-dimensional right D-vector space S, let $S^* = \mathrm{Hom}_D(S, D)$ be the dual space, which is a *left* D-vector space, and let

$$\langle\, |\, \rangle : S^* \times S \to D$$

denote the canonical pairing:

$$\langle s^* | s \rangle = s^*(s) \quad \text{for } s^* \in S^* \text{ and } s \in S.$$

We define a *right* D-vector space structure on S^* by using the involution on D to "change hands"; we thus set:

$$s^* d = \bar{d} s^* \quad \text{for } s^* \in S^* \text{ and } d \in D.$$

Explicitly, $s^* d$ is defined by:

$$\langle s^* d | s \rangle = \bar{d} \langle s^* | s \rangle \quad \text{for } s \in S.$$

Given any $\lambda \in F$ such that $\lambda \bar{\lambda} = 1$ and any finite-dimensional right D-vector space S as above, we define a λ-hermitian form h_λ on the right D-vector space $H_\lambda(S) = S^* \oplus S$ by:

$$h_\lambda((s_1^*, s_1), (s_2^*, s_2)) = \langle s_1^* | s_2 \rangle + \lambda \overline{\langle s_2^* | s_1 \rangle}$$

for $s_1, s_2 \in S$ and $s_1^*, s_2^* \in S^*$. It is readily verified that the formula above defines a non-degenerate λ-hermitian form, and that $S^* \oplus 0$ and $0 \oplus S$ are totally isotropic subspaces. A λ-hermitian space is called *hyperbolic* if it is isometric to $H_\lambda(S)$ for some vector space S. Every vector space isomorphism $S \to T$ induces a natural isometry $H_\lambda(S) \to H_\lambda(T)$; therefore, two λ-hermitian hyperbolic spaces over D are isometric if and only if they have the same dimension.

Now let h be a non-degenerate λ-hermitian form on a finite-dimensional right D-vector space V. As in the preceding section, we let $A = \mathrm{End}_D(V)$ and we denote $\sigma = I_h$ the adjoint involution with respect to h.

2.1. Theorem. *The following statements are equivalent*:

(1) *(V, h) is hyperbolic.*
(2) *There exists a totally isotropic subspace $S \subset V$ with $\dim S = \frac{1}{2} \dim V$.*
(3) *There exists a right ideal $I \subset A$ such that $I^\perp = I$.*
(4) *There exists an idempotent $e \in A$ such that $\sigma(e) = 1 - e$.*

Proof. (1)⇒(2) is clear, and (2)⇒(3) follows from Corollary 1.5 and Proposition 1.6 (with $I = \mathrm{Hom}_D(V, S)$).
(3)⇒(4) From $I^\perp = I$ it follows that $\mathrm{rk}\, I = (\deg A)/2$, by Corollary 1.5. If $f \in A$ is an idempotent such that $I = fA$, then $\mathrm{rk}\, f = (\deg A)/2$ and $\sigma(f) f = 0$. Let $e = f - \frac{1}{2} f \sigma(f) \in I$. It is readily verified that e is an idempotent and that $\sigma(e) e = e \sigma(e) = 0$. Moreover, $f = ef \in eA$, hence $I = eA$ and $\mathrm{rk}\, e = (\deg A)/2$. Since e and $\sigma(e)$ are orthogonal idempotents, $e + \sigma(e)$ also is an idempotent, and $(e + \sigma(e)) A = eA \oplus \sigma(e) A$ is an ideal of rank $\deg A$. Therefore, $e + \sigma(e) = 1$, proving (4).
(4)⇒(1) Given e, we let $S = \mathrm{im}\, e \subset V$ and $T = \mathrm{im}\, \sigma(e) \subset V$, so that $eA = \mathrm{Hom}_D(V, S)$ and $\sigma(e) A = \mathrm{Hom}_D(V, T)$. Since $\sigma(e) e = e \sigma(e) = 0$, Proposition 1.6 shows that S and T are totally isotropic. Moreover,

$$\dim_D S = \frac{\mathrm{rk}\, e}{\deg D} = \frac{\mathrm{rk}\, \sigma(e)}{\deg D} = \dim_D T,$$

and since $\sigma(e)=1-e$,

$$V=S\oplus T.$$

Define $\theta: S\to T^*$ by the formula:

$$\langle\theta(s)|t\rangle=h(s,t) \quad \text{for } s\in S \text{ and } t\in T.$$

The map θ is clearly D-linear, and it is injective since h is non-degenerate. Therefore, θ is an isomorphism and the map

$$\theta\oplus I:\ V=S\oplus T\to T^*\oplus T=H_\lambda(T)$$

is an isometry carrying h to h_λ. □

We will say that the involution σ is *hyperbolic* if the equivalent conditions of the preceding proposition hold.

Note that hyperbolic λ-hermitian forms have been considered more generally in the literature, see e.g. [5, 7, 15, 16]. One can prove generally that if $S\subset V$ is a totally isotropic subspace, then $S\subset H\subset V$ for some subspace H such that $H\simeq H_\lambda(S)$. Consequently, for any given λ-hermitian space (V, h), there is a Witt decomposition:

$$(V,h)\simeq(H_\lambda(S),h_\lambda)\perp(V_0,h_0)$$

for some vector space S and some anisotropic λ-hermitian space (V_0, h_0).

Let us consider the smallest hyperbolic space. Let $V=F^2$ and define $b: V\times V\to F$ by:

$$b((x_1,x_2),(y_1,y_2))=\overline{x_1}\,y_2+\lambda\,\overline{x_2}\,y_1,$$

where $\lambda\in F$ is such that $\lambda\bar{\lambda}=1$. Then (V, b) is λ-hermitian, hyperbolic, and the adjoint involution I_b on $\mathrm{End}(V)=M_2(F)$, which we denote by Θ_λ, is:

$$\Theta_\lambda\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}\bar{d} & \bar{\lambda}\bar{b}\\ \lambda\bar{c} & \bar{a}\end{pmatrix}.$$

If $^-$ is the identity on F, then $\lambda=\pm1$; it is readily checked that Θ_λ is of type λ.

If (A, σ) and (B, τ) are central simple F-algebras with involutions, we use the notation $(A,\sigma)\simeq(B,\tau)$ to express the existence of an F-algebra isomorphism $f: A\to B$ such that $\tau=f\circ\sigma\circ f^{-1}$.

2.2. Theorem. *Let (A, σ) be a central simple algebra with involution such that $\sigma(x)=\bar{x}$ for all $x\in F$. If σ is of the first kind, let $\varepsilon=$ type σ; if σ is of the second kind, let $\varepsilon=1$. The involution σ is hyperbolic if and only if there exists a σ-invariant subalgebra $M\subset A$ such that $M\simeq M_2(F)$ and the restriction σ_M of σ to M satisfies:*

$$(M,\sigma_M)\simeq(M_2(F),\Theta_\varepsilon).$$

Moreover, if $\deg A$ *is divisible by* 4 *and* σ *is of the first kind and of type* ε, *these conditions are also equivalent to the existence of a* σ*-invariant subalgebra* $N \subset A$ *such that*

$$(N, \sigma_N) \simeq (M_2(F), \Theta_{-\varepsilon}).$$

Proof. Suppose first that A contains a subalgebra M (or N) as above. The image $e \in A$ of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(F)$ then satisfies: $\sigma(e) = 1 - e$, hence condition (4) of Theorem 2.1 holds, and σ is hyperbolic.

For the converse, we may suppose $A = \mathrm{End}_D(H_\lambda(S))$ for some right D-vector space S and some $\lambda = \pm 1$, and that σ is the adjoint involution with respect to the hyperbolic form h_λ. (More precisely, we have $\lambda = \varepsilon\delta$ where δ is the type of $^-$ if σ is of the first kind, and we may choose $\lambda = 1$ if σ is of the second kind). We choose a non-degenerate $(+1)$-hermitian form $\ell : S \times S \to D$, and we denote by $\theta : S \to S^*$ the corresponding isomorphism of right D-vector spaces, defined through:

$$\langle \theta(s) | t \rangle = \ell(s, t) \quad \text{for all } s, t \in S.$$

We then define $e_{11}, e_{12}, e_{21}, e_{22} \in A$ by:

$$e_{11}(s^*, s) = (s^*, 0) \qquad e_{12}(s^*, s) = (\theta(s), 0)$$
$$e_{21}(s^*, s) = (0, \theta^{-1}(s^*)) \qquad e_{22}(s^*, s) = (0, s).$$

Straightforward computations show that the F-vector space M spanned by these elements is a subalgebra of A isomorphic to $M_2(F)$ and that, for $a, b, c, d \in F$:

$$\sigma(a e_{11} + b e_{12} + c e_{21} + d e_{22}) = \bar{a} e_{22} + \lambda \bar{b} e_{12} + \lambda \bar{c} e_{21} + \bar{d} e_{11}.$$

Therefore, $(M, \sigma_M) \simeq (M_2(F), \Theta_\lambda)$, and the proof is complete if σ is of the second kind.

If $\deg A$ is divisible by 4, then at least one of $\deg D$, $\dim_D S$ is even, and we can find a non-degenerate (-1)-hermitian form $\ell' : S \times S \to D$. Using the corresponding isomorphism $\theta' : S \to S^*$ instead of θ, we get a subalgebra $M' \subset A$ such that $(M', \sigma_{M'}) \simeq (M_2(F), \Theta_{-\lambda})$. Thus, when σ is hyperbolic of the first kind and $\deg A$ is divisible by 4, we get σ-invariant subalgebras isomorphic to $(M_2(F), \Theta_1)$ and to $(M_2(F), \Theta_{-1})$, and the proof is complete in this case.

The only remaining case is when σ is hyperbolic of the first kind and $\deg A$ is not divisible by 4; we have found in this case a σ-invariant subalgebra M such that $(M, \sigma_M) \simeq (M_2(F), \Theta_\lambda)$ and it remains to see that $\lambda = \text{type } \sigma$. Let B be the centralizer of M in A; we then have:

$$(A, \sigma) = (M, \sigma_M) \otimes_F (B, \sigma_B)$$

and $\deg B$ is odd since $\deg A$ is not divisible by 4. Since the type of a tensor product of involutions of the first kind is the product of the types (see for instance [17, Chap. 6]), the involution σ_B is then necessarily of type $+1$, and type $\sigma = \text{type } \sigma_M = \lambda$. □

2.3. Corollary. *The (signed) discriminant of every hyperbolic involution of the first kind of type +1 is trivial.*

Proof. Recall that the discriminant of an involution of the first kind σ of type $+1$ on a central simple algebra A of even degree has been defined by Knus et al. [8] as the square class of the reduced norm of any invertible skew-symmetric element:

$$\operatorname{disc}(\sigma) = \operatorname{Nrd}_A(a)\, F^{\times 2} \quad \text{for any } a \in A^{\times} \text{ such that } \sigma(a) = -a.$$

If σ is hyperbolic then the preceding proposition shows that A contains a σ-invariant subalgebra isomorphic to $(M_2(F), \Theta_1)$. The image $a \in A$ of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then satisfies: $\sigma(a) = -a$, hence

$$\operatorname{disc}(\sigma) = \operatorname{Nrd}_A(a)\, F^{\times 2} = (-1)^{(\deg A)/2} \cdot F^{\times 2}. \quad \square$$

To conclude this section, we consider two special cases: the case where the base field F is real-closed and the case of involutions of type $+1$ on algebras of degree 4.

Let (A, σ) be a central simple algebra with involution of the first kind over a real-closed field R. The *signature* $\operatorname{sgn} \sigma$ of σ has been defined in [13] as the square root of the trace form T_σ on A defined by:

$$T_\sigma(x, y) = \operatorname{Trd}_A(\sigma(x)\, y).$$

2.4. Corollary. *With the hypotheses above, the involution σ is hyperbolic if and only if* $\operatorname{sgn} \sigma = 0$.

Proof. If A is split, i.e. $A = \operatorname{End}_R(V)$ for some R-vector space V, then σ is the adjoint involution with respect to some bilinear form b on V, and σ is hyperbolic if and only if b is hyperbolic. In particular, if b is alternating, then it is hyperbolic, hence σ is hyperbolic. On the other hand, the type of σ is -1, hence $\operatorname{sgn} \sigma = 0$ by [13, Theorem 1], so the corollary holds in this case. If b is symmetric, then $\operatorname{sgn} \sigma = |\operatorname{sgn} b|$, by [13, Theorem 1], and b is hyperbolic if and only if $\operatorname{sgn} b = 0$. The proof is thus complete if A is split.

If A is not split, then it is Brauer-equivalent to the unique non-split quaternion algebra Q over R. Let $(A, \sigma) \simeq (\operatorname{End}_Q(V), I_h)$ for some right Q-vector space V and some non-degenerate λ-hermitian form h on V with respect to quaternion conjugation $^-$ on Q. Since conjugation is an involution of type -1, we have: type $\sigma = -\lambda$. If σ is of type $+1$, we have $\operatorname{sgn} \sigma = 0$ by [13, Theorem 1] and h (hence also σ) is hyperbolic by [15, Theorem 10.3.7]. If σ is of type -1 (i.e. h is hermitian), we consider the quadratic form $q_h: V \to R$ defined by: $q_h(v) = h(v, v)$. The computations in [13, Corollary 2] show that $\operatorname{sgn} \sigma = \frac{1}{2} |\operatorname{sgn} q_h|$, hence $\operatorname{sgn} \sigma = 0$ if and only if q_h is hyperbolic. The latter condition holds if and only if h is hyperbolic, by [15, Theorem 10.1.7], hence the proof is complete. $\square$

We now turn to the case where A has degree 4. If σ is an involution of the first kind and type $+1$ on A, Knus et al. [9] have shown that σ is decomposable if and only if $\operatorname{disc}(\sigma) = 1$. Moreover, in this case, there is a unique decomposition:

$A = Q_1 \otimes_F Q_2$ such that σ induces on Q_1 and Q_2 the quaternion conjugation (see [21]), and the Clifford algebra $C(\sigma)$ as defined by Jacobson [6] (or by Tits [19]) is then:

$$C(\sigma) \simeq Q_1 \times Q_2$$

(see [18] or [10]).

2.5. Corollary. *An involution σ of the first kind and of type $+1$ on a central simple algebra A of degree 4 is hyperbolic if and only if* $\mathrm{disc}(\sigma) = 1$ and $C(\sigma) = Q_1 \times Q_2$ *where Q_1 or Q_2 is split.*

Proof. If σ is hyperbolic, then $\mathrm{disc}(\sigma) = 1$ by Corollary 2.3. Moreover, Theorem 2.2 shows that $A \simeq (M_2(F), \Theta_{-1}) \otimes (B, \sigma_B)$ for some quaternion algebra B, and then $C(\sigma) \simeq M_2(F) \times B$. Conversely, if $C(\sigma) \simeq M_2(F) \times B$, then $(A, \sigma) \simeq (M_2(F), \Theta_{-1}) \otimes (B, \bar{\ })$ since Θ_{-1} is the conjugation involution on the split quaternion algebra $M_2(F)$, hence σ is hyperbolic by Theorem 2.2. □

Remark. The "only if" part of the corollary above holds without restriction on the degree, as has been shown independently by Van Drooge (in the case of Schur index 2) [20, Theorem 2.3], Tits [19, Proposition 8] and Allen [2, Theorem 3]. (We are grateful to Max Knus for this observation).

3 Behavior under quadratic extensions

The determination of quadratic forms which become hyperbolic under a quadratic field extension is easy; we recall the result:

3.1. Proposition. *Suppose (V, q) is an anisotropic quadratic space over a field F and let $K = F(\sqrt{d})$ be a quadratic field extension of F. The following conditions are equivalent:*

(1) *$(V \otimes K, q_K)$ is hyperbolic.*
(2) *$q \simeq \langle 1, -d \rangle \otimes q'$ for some quadratic form q' over F.*
(3) *There exists $f \in \mathrm{End}_F(V)$ such that $f^2 = d$ and $I_q(f) = -f$, where I_q is the adjoint involution with respect to q.*

For an arbitrary quadratic form q, (2) *and* (3) *are equivalent and imply* (1).

Proof. (2)⇒(1) is clear and the elementary proof of (1)⇒(2) (for q anisotropic) can be found in the texts by Lam [11, p. 200] and Scharlau [15, p. 45]. The equivalence of (2) and (3) is in [17, Proposition 1.10]. We sketch a proof for the reader's convenience: If condition (2) holds, let $V = F^2 \otimes V'$ where F^2 is endowed with the quadratic form $\langle 1, -d \rangle$ and V' with the quadratic form q'. The map $f = \begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} \otimes I_{V'}$ then satisfies (3).

Conversely, if $f \in \mathrm{End}_F(V)$ satisfies the conditions in (3), then u is orthogonal to $f(u)$ for all $u \in V$. If U_1 is a maximal non-singular subspace of V such that U_1 and $f(U_1)$ are orthogonal, then $V = U_1 \perp f(U_1)$ and $q \simeq \langle 1, -d \rangle \otimes q'$ where q' is the restriction of q to U_1. □

We aim to prove an analogous result for arbitrary involutions of the first kind.

3.2. Lemma. *Suppose σ is an anisotropic involution of the first kind on a central simple F-algebra A and let $K=F(\sqrt{d})$ be a quadratic field extension of F. If the involution $\sigma_K=\sigma\otimes I_K$ on $A\otimes_F K$ is hyperbolic, then there exists $r\in A$ such that $r^2=d$ and $\sigma(r)=-r$.*

Proof. If σ_K is hyperbolic, then there exists $e\in A\otimes K$ with $e^2=e$ and $\sigma_K(e)=1-e$. Expressing $e=e_1\otimes 1+e_2\otimes\sqrt{d}$, we calculate that

$$\begin{aligned} e_1^2+de_2^2&=e_1 \qquad \sigma(e_1)=1-e_1\\ e_1e_2+e_2e_1&=e_2 \qquad \sigma(e_2)=-e_2. \end{aligned}$$

It now suffices to prove that e_2 is invertible, for then it is straightforward to check that $r=e_1e_2^{-1}$ fulfills the conditions. Since σ is anisotropic, Corollary 1.8 yields a symmetric idempotent f generating the right annihilator ideal

$$(e_2A)^{\perp}=\{x\in A\mid e_2x=0\}.$$

Geometrically, this means that f is the orthogonal projection to $\ker e_2$. Then $e_2f=0$ and multiplying the relations above by f we obtain:

$$e_1^2f=e_1f \quad\text{and}\quad e_2e_1f=0.$$

Therefore, $fe_1.\sigma(fe_1)=0$ and $\sigma(e_1f).e_1f=0$; since σ is anisotropic, these relations imply $fe_1=e_1f=0$. But then $0=\sigma(fe_1)=(1-e_1)f=f$. Therefore $(e_2A)^{\perp}=0$ and it follows that e_2 is invertible. □

In the proof of our main theorem, we use the Witt cancellation property for hermitian forms. This property has been proved in more general contexts using a variety of approaches: see for example [16] or [15, p. 268].

3.3. Theorem. *Let (A,σ) be a central simple F-algebra with involution of the first kind and let $K=F(\sqrt{d})$ be a quadratic field extension of F. If there exists $r\in A$ such that $r^2=d$ and $\sigma(r)=-r$, then $(A\otimes_F K,\sigma_K)$ is hyperbolic. The converse holds except in the case where A is split, σ is of type $+1$ and its Witt index is odd.*

Proof. Given r, let $t=d^{-1}r\otimes\sqrt{d}\in A\otimes_F K$. Note that $t^2=1$ and $\sigma_K(t)=-t$; therefore, the element $e=\frac{1}{2}(1+t)$ satisfies $e^2=e$ and $\sigma(e)=1-e$, so that $(A\otimes_F K,\sigma_K)$ is hyperbolic.

For the converse, let V be an irreducible left A-module and $D=\mathrm{End}_A(V)$, so that $A=\mathrm{End}_D(V)$. Choose an involution of the first kind $^-$ on D and let h be a λ-hermitian form on V with respect to which σ is the adjoint involution. So, $\lambda=\varepsilon.\mathrm{type}\,\sigma(=\pm 1)$ where ε is the type of $^-$. Consider a Witt decomposition:

$$(V,h)\simeq(H_\lambda(S),h_\lambda)\perp(V_0,h_0)$$

for some vector space S and some anisotropic λ-hermitian space (V_0,h_0). Let σ_λ (resp. σ_0) denote the adjoint involution with respect to h_λ (resp. h_0) on $\mathrm{End}_D(H_\lambda(S))$ (resp. on $\mathrm{End}_D(V_0)$). By the Witt cancellation property, (V_0,h_0)

becomes hyperbolic over K, hence Lemma 3.2 yields an element $r_0 \in \mathrm{End}_D(V_0)$ such that $r_0^2 = d$ and $\sigma_0(r_0) = -r_0$.

If $\deg \mathrm{End}_D(H_\lambda(S)) = 2 \deg D . \dim_D S$ is divisible by 4, or if A is split (i.e. $D = F$) and $\lambda = -1$, then Theorem 2.2 shows that $\mathrm{End}_D(H_\lambda(S))$ contains a σ_λ-invariant subalgebra M such that $(M, \sigma_\lambda) \simeq (M_2(F), \Theta_{-1})$. (The only excluded case is thus when A is split, σ is of type $+1$ and $\dim S$ is odd). Let $r_\lambda \in M \subset \mathrm{End}_D(H_\lambda(S))$ denote the image of $\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix}$; then $\sigma_\lambda(r_\lambda) = -r_\lambda$ and $r_\lambda^2 = d$. Finally, regarding $V = H_\lambda(S) \oplus V_0$, define $r \in A = \mathrm{End}_D(V)$ by:

$$r(x, y) = (r_\lambda(x), r_0(y)).$$

It is straightforward to check that $\sigma(r) = -r$ and $r^2 = d$. □

Observe that the property does not hold in the exceptional case we have singled out: Suppose $A = \mathrm{End}_F(V)$ and σ is the adjoint involution with respect to some symmetric bilinear form b of odd Witt index; if q denotes the quadratic form associated to b, we have:

$$q \simeq m\langle 1, -1\rangle \perp q_0$$

for some anisotropic quadratic form q_0 and some odd integer m. If q (hence also q_0) becomes hyperbolic over K, then Proposition 3.1 shows that

$$q_0 \simeq \langle 1, -d\rangle \otimes q_0'$$

for some quadratic form q_0' over F. On the other hand, if A contains an element r such that $\sigma(r) = -r$ and $r^2 = d$, Proposition 3.1 again yields:

$$q \simeq \langle 1, -d\rangle \otimes q'$$

for some quadratic form q' over F. Taking discriminants of both sides of the relation:

$$\langle 1, -d\rangle \otimes q' \simeq m\langle 1, -1\rangle \perp \langle 1, -d\rangle \otimes q_0',$$

we get: $d \in F^{\times 2}$, a contradiction.

As an application of this theorem, we determine when certain involutions admit invariant quaternion subalgebras:

3.4. Proposition. *Suppose $A \simeq M_n(D)$ where D is a quaternion division algebra over F. Let σ be an involution of the first kind on A and let $^- : d \mapsto \bar{d}$ denote the quaternion conjugation.*

(1) *If the type of σ is -1, then A contains a σ-invariant subalgebra E such that $(E, \sigma_E) \simeq (D, ^-)$.*
(2) *If the type of σ is $+1$ (and n is arbitrary), then A contains a σ-invariant subalgebra isomorphic to D if and only if σ becomes hyperbolic over some quadratic field extension K of F which splits D.*
(3) *If the type of σ is $+1$ and n is odd, then σ is not hyperbolic. Moreover, A contains a σ-invariant subalgebra isomorphic to D if and only if σ becomes hyperbolic over the quadratic field extension $F(\sqrt{-\mathrm{disc}\,\sigma})$.*

Proof. (1) Let $A=\mathrm{End}_D(V)$ and $\sigma=I_h$ where h is an hermitian form on V. The division algebra D acts naturally on V on the right, so that $a\cdot vd=av\cdot d$ for $a\in A$, $v\in V$ and $d\in D$. We now define a *left* action of D on V which commutes with the right action: Choose an orthogonal D-basis $(v_1, \ldots, v_s)$ of V; then every $v\in V$ has a unique expression: $v=\sum_i v_i d_i$ for some $d_i\in D$. For $x\in D$, define

$$x*v=\sum_i v_i\, x\, d_i.$$

Then $x*vd=(x*v)\cdot d$, so that the map $v\mapsto x*v$ is D-linear. We denote this map by $\mathscr{L}(x)$; then $\mathscr{L}: D\to \mathrm{End}_D(V)=A$ is a homomorphism of F-algebras. Furthermore,

$$h(x*v, w)=h(v, \bar{x}*w) \qquad \text{for all } v, w\in V \text{ and } x\in D$$

since $h(v, v)\in F$ for every $v\in V$. The image $\mathscr{L}(D)$ is the desired subalgebra E.
(2) Suppose σ is of type $+1$. If A contains a σ-invariant subalgebra isomorphic to D, then

$$(A, \sigma)\simeq(M_n(F), I_b)\otimes_F(D, \tau)$$

for some involutions I_b and τ. Let $z\in D^\times$ be such that $\tau(z)=-z$. The quaternion z is then pure, hence $z^2=d$ for some $d\in F^\times$. Theorem 2.2 shows that σ becomes hyperbolic over $K=F(\sqrt{d})$, which is a splitting field of D. Moreover, since $\sigma(z)=-z$, we have

$$\mathrm{disc}(\sigma)=\mathrm{Nrd}_A(z)\, F^{\times 2},$$

hence $\mathrm{disc}(\sigma)=-dF^{\times 2}$ if n is odd.

Conversely, suppose σ becomes hyperbolic over a quadratic extension $K=F(\sqrt{d})$ of F which splits D (hence also A). By Theorem 3.3, we may then find $r\in A$ with $r^2=d$ and $\sigma(r)=-r$. Let $B=C_A(r)$ be the centralizer of r in A; then B is a simple algebra with center $F(r)\simeq K$ and σ induces an involution σ_B of the second kind on B. Since K splits A and B is Brauer-equivalent to $A\otimes_F K$ (see [1, Theorem 4.13]), B must be split. Therefore,

$$(B, \sigma_B)\simeq(\mathrm{End}_K(W), I_h)$$

for some K-vector space W and some hermitian form h on W. Let $(w_1, \ldots, w_n)$ be an orthogonal basis for (W, h) and let V be the F-span of this basis in W. Let also $M=\mathrm{End}_F(V)\subset \mathrm{End}_K(W)=B$. Since $h(w, w)\in F$ for all $w\in W$, the form h induces a symmetric bilinear form on V; therefore, $\sigma_B(M)=M$ and $D\simeq C_A(M)\subset A$ provides the desired embedding. This proves (2).
(3) Suppose σ is of type $+1$ and n is odd. If σ is hyperbolic, then Theorem 2.2 shows that A contains a subalgebra M isomorphic to $M_2(F)$. The centralizer M' of M then has odd degree, hence it is necessarily split: $M'\simeq M_n(F)$. Therefore,

$$A\simeq M_2(F)\otimes M_n(F)\simeq M_{2n}(F).$$

This contradiction shows that σ is not hyperbolic. It also shows that if σ becomes hyperbolic over some extension K of F, then K splits A since we can repeat the same arguments after extending the scalars to K. Therefore, if $-\mathrm{disc}\,\sigma\notin F^{\times 2}$

and σ becomes hyperbolic over $F(\sqrt{-\operatorname{disc}\sigma})$, then (2) above shows that A contains a σ-invariant subalgebra isomorphic to D. The converse has been shown in the proof of (2). □

3.5. Corollary. *Suppose $A \simeq M_n(D)$ where D is a quaternion division algebra over F and let σ be an involution of type $+1$ on A. If n is odd and $\operatorname{disc}\sigma = (-1)F^{\times 2}$, then A does not contain any σ-invariant quaternion subalgebra. If moreover n is prime, then σ is indecomposable.*

Proof. The first part follows from the proposition. If n is prime, then any non-trivial stable decomposition of A involves a factor of degree n and a factor of degree 2. Such a decomposition is impossible since the factor of degree 2 must then be a σ-invariant subalgebra isomorphic to D. □

Example. Let D be a quaternion division algebra such that -1 is a reduced norm from D:

$$-1 = \operatorname{Nrd}_D(u) \quad \text{for some } u \in D.$$

Let D' denote the space of pure quaternions and let $c \in D'$, $c \neq 0$. Counting dimensions, one has

$$ucD' \cap D' \neq \{0\},$$

hence one can find $a, b \in D'$ such that $uca = b \neq 0$. We then have:

$$\operatorname{Nrd}_D(abc) \equiv \operatorname{Nrd}_D(u) \equiv -1 \bmod F^{\times 2}.$$

Consider then the -1-hermitian form on D^n defined by:

$$h((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = a\bar{x}_1 y_1 + b\bar{x}_2 y_2 + c\bar{x}_3 y_3 + c\bar{x}_4 y_4 + \ldots + c\bar{x}_n y_n,$$

where $\bar{}$ is the usual quaternion conjugation. Since $a, b, c \in D'$, the form h is -1-hermitian, hence its adjoint involution σ on D has type $+1$. The discriminant of σ is the square class of the reduced norm of the diagonal matrix

$$\begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & c & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & c \end{pmatrix}.$$

hence $\operatorname{disc}(\sigma) = (-1)F^{\times 2}$ if n is odd. Therefore the algebra $M_n(D)$ does not contain any σ-invariant quaternion subalgebra if n is odd, and σ is indecomposable if n is an odd prime number.

4 Behavior under odd-degree extensions

In this final section, we invetigate the behavior of involutions under odd-degree scalar extensions and also under tensor product with an involution on a central simple algebra of odd degree. In contrast with the preceding section, our

approach will be based exclusively on λ-hermitian forms, using Scharlau's transfer and the module structure of the Witt group of a division algebra on the Witt ring of the center, as in [14] and [3, Sect. 1].

We first note the following weak analogue of Springer's theorem for odd-degree field extensions:

4.1. Proposition. *Suppose (A, σ) is a central simple F-algebra with involution and K/F is an odd-degree field extension. If $(A \otimes_F K, \sigma_K)$ is hyperbolic, then (A, σ) is hyperbolic.*

Proof. Let V be an irreducible left A-module and $D = \mathrm{End}_A(V)$. Let also $^-$ be an involution on D and let h be a λ-hermitian form on V such that

$$(A, \sigma) \simeq (\mathrm{End}_D(V), I_h).$$

Proposition 1.2 of [3] shows that the natural map $W^\lambda(D, ^-) \to W^\lambda(D \otimes K, ^-)$ is injective. Therefore, if I_h is hyperbolic over K, then it is hyperbolic. □

The strong version of Springer's theorem remains open:

Question. If σ is an anisotropic involution on A and K/F is an odd degree field extension, then is σ_K necessarily anisotropic on $A \otimes_F K$?

The main result of this section is the following "odd-cancellation" property:

4.2. Theorem. *Suppose (A, σ) and (B, τ) are central simple F-algebras with involution of the first kind. If $\deg B$ is odd and $(A, \sigma) \otimes (B, \tau)$ is hyperbolic, then (A, σ) is hyperbolic.*

Proof. We mimic the proof of the corresponding result as given in [15, Corollary 2.6.5]. The major ingredient is the fact that the torsion in the Witt group $W^\lambda(D, ^-)$ is 2-primary [14, Theorem 5.1].

Since B has odd degree it is necessarily split and the involution τ is of type $+1$. Let $(B, \tau) \simeq (\mathrm{End}(W), I_q)$ for some quadratic space (W, q) of odd dimension. By the usual arguments it suffices to see that if (V, h) is a λ-hermitian space over an F-central division algebra D and if $(V \otimes W, h \otimes q)$ is hyperbolic then (V, h) is hyperbolic.

Suppose the claim fails for some λ-hermitian spaces over D and suppose $n \geqq 3$ is the minimal odd positive integer for which there exists a form q over F with $\dim q = n$ and there exists a λ-hermitian form h over D such that $h \not\sim 0$ but $h \otimes q \sim 0$, where $\sim$ denotes Witt-equivalence. We may assume q represents 1 and express $q \simeq \langle 1, a_2, \ldots, a_n \rangle$. Then

$$h \otimes \langle a_3, \ldots, a_n \rangle \sim h \otimes \langle -1, -a_2 \rangle$$

and we find that

$$(\langle 1, -a_2 \rangle \otimes h) \otimes \langle a_3, \ldots, a_n \rangle \sim 0.$$

By the minimality of n we conclude that $\langle 1, -a_2 \rangle \otimes h \sim 0$ so that $h \simeq \langle a_2 \rangle h$. This argument works for every slot a_i just as well and we find that

$$n.h = \langle 1, \ldots, 1 \rangle \otimes h \simeq q \otimes h \sim 0.$$

Since $h \neq 0$ and n is odd we have odd torsion in $W^{\lambda}(D, {}^{-})$ contrary to Scharlau's result. □

David Lewis has kindly suggested the following alternative argument: for every quadratic form q of odd dimension, [12, Theorem 1] yields a monic polynomial P with integral coefficients and odd constant term such that $P(q)=0$ in WF. Let $m \in \mathbb{Z}$ denote the constant term of P. If $h \otimes q$ is hyperbolic, then $h \otimes P(q)=m.h=0$ in $W^{\lambda}(D, {}^{-})$, hence $h=0$ since the torsion in the Witt group is 2-primary.

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Graphes planaires et présentations des groupes de tresses

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0 Introduction

Dans son travail séminal de 1925, Emil Artin [A] a défini le groupe des tresses B_n et a indiqué une présentation, devenue classique, dont les générateurs σ_k, $1 \leqq k \leqq n-1$ sont les tresses

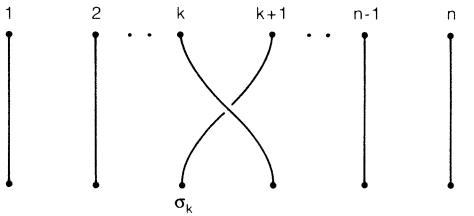


Fig. 1

et sont assujettis aux relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ si $|i-j| \geqq 2$, $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ si $1 \leqq i \leqq n-2$.

Fadell et Neuwirth [F-N] ont remarqué que le groupe des tresses peut être vu comme groupe fondamental d'un espace de configurations. Soit $\widetilde{\mathrm{Conf}}^n = (\mathbf{R}^2)^n - \{(x_i) | \exists i,j \text{ tel que } x_i = x_j\}$ l'ensemble des n-uples ordonnés de points distincts dans $\mathbf{R}^2$. Le groupe symétrique Σ_n agit sur $\widetilde{\mathrm{Conf}}^n$ par permutation des coordonnées et le quotient $\mathrm{Conf}^n = \Sigma_n \backslash \widetilde{\mathrm{Conf}}^n$ est l'espace des configurations de n points. Soit S un ensemble fixe de n points. Le groupe des tresses B_S est alors $\pi_1(\mathrm{Conf}^n; S)$.

Il est clair que le groupe B_S ne dépend pas du choix de l'ensemble S; il est isomorphe au groupe B_n. Lorsque l'ensemble S est infini, localement fini, on peut encore définir B_S comme limite inductive; nous laissons les détails au lecteur (voir aussi 1, démonstration).

A la mémoire de Wilhelm Magnus

L'observation principale de cet article est que lorsque les points de S sont les sommets d'un graphe planaire Γ, les arêtes produisent un système de générateurs de B_S dont les relations se lisent sur la géométrie du graphe.

La présentation d'Artin correspond alors au graphe:

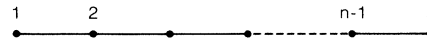


Fig. 2

Bien que celui-ci est le graphe le plus simple parmi ceux qui ont un ensemble de sommets donné, l'intérêt de ces nouvelles présentations est qu'elles apparaissent naturellement dans certains contextes géométriques.

Dans ce travail, on désignera par Γ un graphe connexe planaire localement fini sans boucle mais éventuellement avec arêtes multiples. Si $S(\Gamma)$ est l'ensemble des sommets de Γ, nous noterons B_Γ le groupe $B_{S(\Gamma)}$ introduit plus haut. En particulier, si deux graphes Γ_1 et Γ_2 ont même sommets $S(\Gamma_1)=S(\Gamma_2)$, alors $B_{\Gamma_1}=B_{\Gamma_2}$. A chaque arête σ de Γ on associe une tresse, $\bar{\sigma}$ dans B_Γ.

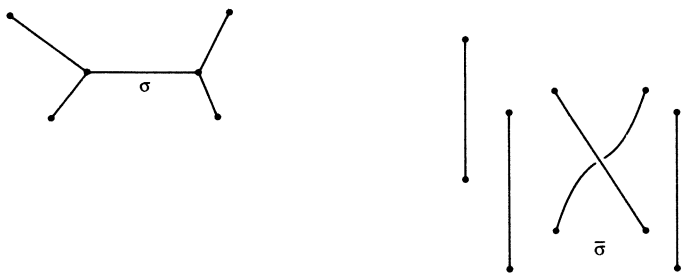


Fig. 3

Dans la suite, on utilisera la même notation σ pour une arête géométrique et pour l'élément correspondant du groupe libre engendré par les arêtes.

Notre résultat principal est le

Théorème. *Le groupe de tresses B_Γ a la présentation $\langle X_\Gamma, B_\Gamma\rangle$ ou $X_\Gamma=\{\sigma|\sigma$ arête de $\Gamma\}$ et R_Γ est formé des quatre types suivants de relations:*

(i) *disjonction: si σ_1 et σ_2 sont disjointes, alors*

$$\sigma_1\sigma_2=\sigma_2\sigma_1.$$

(ii) *adjacence: si σ_1 et σ_2 ont un sommet en commun alors:*

$$\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2.$$

(iii) *nodale: si σ_1, σ_2, σ_3 ont un seul sommet en commun et sont disposées dans le sens horaire alors*

$$\sigma_1\sigma_2\sigma_3\sigma_1=\sigma_2\sigma_3\sigma_1\sigma_2=\sigma_3\sigma_1\sigma_2\sigma_3.$$

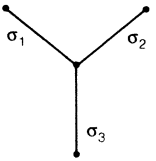


Fig. 4

(iv) *cyclique*: *si* $\sigma_1 \dots \sigma_n$ *est un pseudo-cycle irréductible* (*i.e.* $\sigma_1 \dots \sigma_n$ *décrivent dans le sens horaire un circuit fermé de* Γ *qui borde un disque ne contenant pas de point du graphe dans l'intérieur*) *et si* σ_1 *n'est pas l'aller ni* σ_n *le retour d'un retournement alors* $\sigma_1 \dots \sigma_{n-1} = \sigma_2 \dots \sigma_n$.

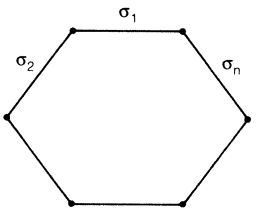


Fig. 5

$$\sigma_1 \sigma_2 \dots \sigma_{n-1} = \sigma_2 \sigma_3 \dots \sigma_n = \sigma_3 \dots \sigma_n \sigma_1 = \dots = \sigma_n \sigma_1 \dots \sigma_{n-2}$$

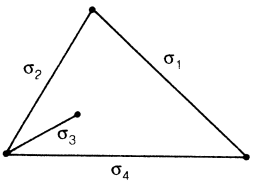


Fig. 6

$$\sigma_1 \sigma_2 \sigma_3^2 = \sigma_2 \sigma_3^2 \sigma_4 = \sigma_3^2 \sigma_4 \sigma_1$$
$$\sigma_3 \sigma_4 \sigma_1 \sigma_2 = \sigma_4 \sigma_1 \sigma_2 \sigma_3$$

(*sont exclues les relations*:

$$\sigma_3^2 \sigma_4 \sigma_1 = \sigma_3 \sigma_4 \sigma_1 \sigma_2, \ \sigma_4 \sigma_1 \sigma_2 \sigma_3 = \sigma_1 \sigma_2 \sigma_3^2).$$

Corollaire 0.1 *Si le graphe* Γ *est un arbre, le groupe* B_Γ *admet la présentation* $\langle X_\Gamma, R_\Gamma \rangle$ *où seules les relations* (i), (ii), (iii) *du théorème apparaissent.*

Une conséquence immédiate est le

Corollaire 0.2 *Les groupes de tresses* B_{n+1} *et* $B_\infty = \lim_{\rightarrow} B_n$ *admettent des automorphismes d'ordre n.*

Preuve. Il suffit de considérer les automorphismes induits par des rotations d'angle $\frac{2\pi}{n}$ sur les graphes:

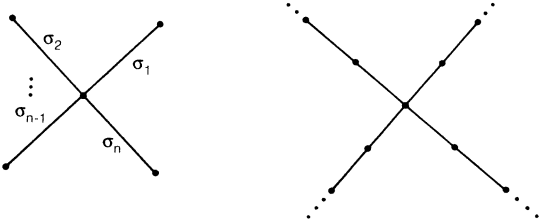


Fig. 7

Cet article est organisé ainsi: le théorème est démontré au par. 1, en admettant les lemmes 1, 2, 3 qui sont démontrés au par. 2. Le dernier paragraphe contient diverses remarques dont une présentation «universelle» (sans graphe) du groupe des tresses.

1 La preuve du théorème

Dans ce paragraphe nous démontrons le théorème, après avoir énoncé trois lemmes dont la preuve est donnée plus loin.

Lemme 1.1 (i) *Soit Γ_1 le graphe*

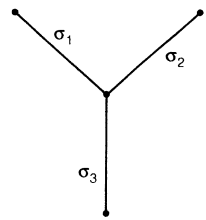


Fig. 8

Dans le groupe B_{Γ_1} on a $\bar{\sigma}_1\,\bar{\sigma}_2\,\bar{\sigma}_3\,\bar{\sigma}_1=\bar{\sigma}_2\,\bar{\sigma}_3\,\bar{\sigma}_1\,\bar{\sigma}_2=\bar{\sigma}_3\,\bar{\sigma}_1\,\bar{\sigma}_2\,\bar{\sigma}_3$.
(ii) *Soit Γ_2 le pseudo-cycle irréductible.*

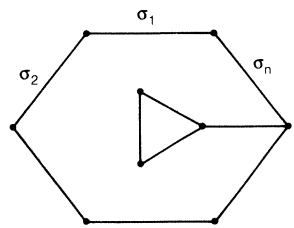


Fig. 9

Si σ_1 n'est pas l'aller et σ_n n'est pas le retour d'un retournement, alors $\bar{\sigma}_1 \ldots \bar{\sigma}_{n-1}=\bar{\sigma}_2 \ldots \bar{\sigma}_n$.

Lemme 1.2 *Soit σ_1 et σ_2 deux arêtes voisines d'un graphe qui ne font pas partie d'un cycle. Soit $\Gamma'=\Gamma\cup\{\tau\}$ le graphe obtenu en ajoutant une arête τ à Γ comme ci-dessous*
Si le théorème est vrai pour le groupe $B_{\Gamma'}$ alors il l'est aussi pour le groupe B_Γ.

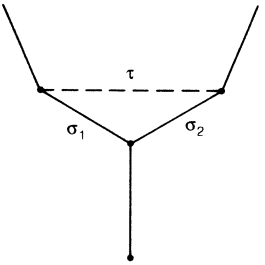


Fig. 10

Lemme 3 *Soit τ une arête d'un graphe Γ' qui borde un seul pseudo-cycle irréductible qui est un triangle $\tau\sigma_1\sigma_2$. Si le théorème est vrai pour le groupe $B_{\Gamma'}$, alors il l'est aussi pour B_Γ où $\Gamma = \Gamma' - \tau$* (voir la figure ci-dessus).

Introduisons d'abord quelques notions utiles dans la suite. Soit Γ un graphe.

Definition. Nous appellerons *nœud tout sommet commun* à au moins trois arêtes. La *valence réduite* de Γ, notée $v(\Gamma)$ est la somme des valences des nœuds. Le nombre total des pseudo-cycles irréductibles et appelée *complexité* de Γ et noté $b(\Gamma)$: c'est le premier nombre de Betti de Γ.

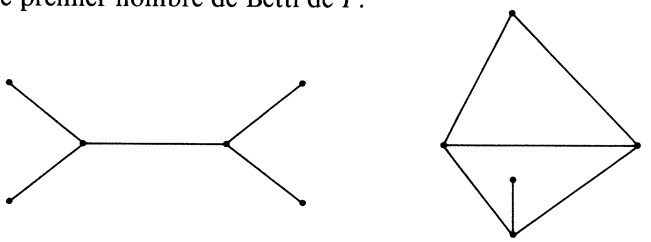


Fig. 11

Passons à la

Démonstration du théorème. (i) *Le cas d'un arbre fini.* Le théorème est vrai si $v(\Gamma)=0$: c'est le résultat d'Artin cité dans l'introduction. Supposons-le vrai pour les arbres de valence inférieure à $p>0$ et soit Γ de valence réduite égale à p.

Soit v_0 une extrémité de Γ. En parcourant l'arbre à partir de v_0 soit v_1 le sommet précédant le premier nœud et v_2 l'extrémité de l'arc décrit en tournant ensuite à droite à chaque nœud.

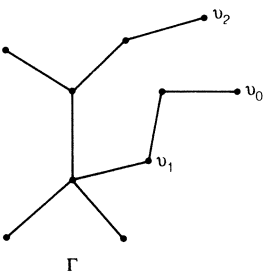


Fig. 12

En remplaçant l'arête entre v_1 et le premier nœud par une autre qui joint v_1 à v_2 on obtient un graphe Γ_1 et $v(\Gamma_1)<v(\Gamma)$. Le théorème est vrai pour Γ_1.

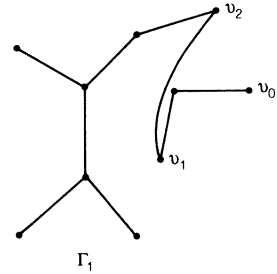


Fig. 13

D'après le lemme 2, le théorème restera vrai pour le graphe Γ_2 obtenu de Γ_1 en ajoutant une arête entre v_1 et l'autre sommet voisin de v_2:

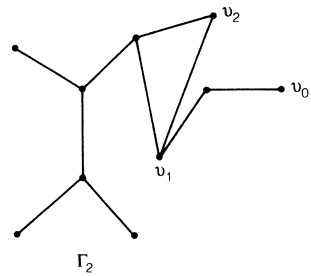


Fig. 14

Par le lemme 3, il reste vrai encore après avoir effacé dans Γ_2 l'arête $v_1 v_2$.

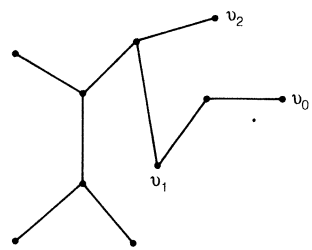


Fig. 15

L'application répétée des lemmes 2 et 3 conduit au résultat pour l'arbre initial Γ.

(ii) *Le cas d'un graphe fini* Γ. Si $b(\Gamma)=0$, Γ est un arbre. Supposons donc le résultat vrai pour les graphes de complexité inférieure à p et soit Γ tel que $b(\Gamma)=p$ et σ_1 une arête bordant un seul pseudo-cycle

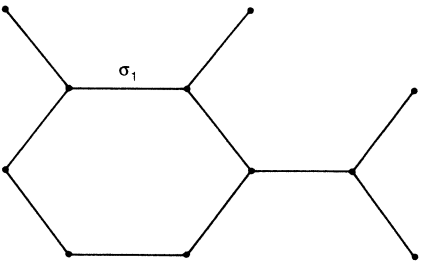


Fig. 16

Le théorème est donc vrai pour le graphe $\Gamma_1 = \Gamma - \{\sigma_1\}$ et $\langle X_{\Gamma_1}, R_{\Gamma_1} \rangle$ est une présentation de B_{Γ_1}. Par une transformation de Tietze [11] on voit que $\langle X_\Gamma, R_{\Gamma_1} \cup \{\sigma_1 \dots \sigma_{n-1} = \sigma_2 \dots \sigma_n\}\rangle$ est aussi une présentation de $B_{\Gamma_1} = B_\Gamma$. Le lemme 1 montre que la tresse $\bar{\sigma}_1$ vérifie les relations du théorème dans B_Γ. On en déduit que les relations dans R_Γ sont des conséquences de $R_{\Gamma_1} \cup \{\sigma_1 \dots \sigma_{n-1} = \sigma_2 \dots \sigma_n\}$.

(iii) *Le cas d'un graphe infini.* Précisons d'abord ce que signifie le groupe des tresses dans ce cas. L'ensemble $S = S(\Gamma)$ étant localement fini, pour tout disque D relativement compact, $S \cap D$ est un ensemble fini. Soit $B_{D, S\cap D}$ le groupe des tresses relatif à l'ouvert D. Alors B_S est défini comme égal à $\varinjlim_D B_{D, S\cap D}$.

Soit $\Gamma_1 \subset \Gamma_2 \subset \dots$ une suite croissante de sous-graphes connexes finis pleins telle que $\cup \Gamma_i = \Gamma$ et $D_1 \subset D_2 \subset \dots$ une suite de disques telle que $\cup D_i = \mathbf{R}^2$ et telle que Γ_i est un sous-graphe maximal de $\Gamma \cap D_i$. Il est facile de voir alors que $B_\Gamma = \varinjlim_D B_{\Gamma_i}$.

Le groupe libre de générateurs X_Γ se surjecte naturellement sur B_Γ. Toute relation fait intervenir un nombre fini de générateurs. Elle est donc valable dans un groupe B_{Γ_i} et est alors une conséquence des relations R_{Γ_i} d'après le cas (ii) et a fortiori de R_Γ. On a donc montré que $\langle X_\Gamma, R_\Gamma \rangle$ est une présentation de B_Γ. □

2 Démonstration des lemmes

Nous commençons par la

Démonstration du lemme 1.1 Considérons d'abord un triangle Γ:

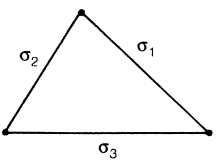


Fig. 17

Dans B_Γ on vérifie par un calcul géométrique direct la relation (du triangle): $\bar\sigma_2\,\bar\sigma_3=\bar\sigma_3\,\bar\sigma_1=\bar\sigma_1\,\bar\sigma_2$ (en composant les tresses comme des fonctions).

(i) Complétons le graphe Γ_1 par trois arêtes τ_1,τ_2,τ_3:

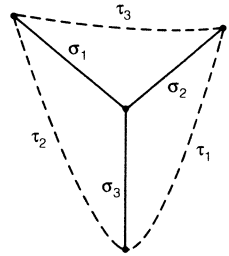


Fig. 18

La relation du triangle donne: $\bar\sigma_1\,\bar\sigma_2\,\bar\sigma_3\,\bar\sigma_1=\bar\sigma_2\,\bar\tau_3\,\bar\sigma_3\,\bar\sigma_1$, $\bar\sigma_2\,\bar\sigma_3\,\bar\sigma_1\,\bar\sigma_2=\bar\sigma_2\,\bar\sigma_3\,\bar\tau_3\,\bar\sigma_1$. Comme $\bar\tau_3\,\bar\sigma_3=\bar\sigma_3\,\bar\tau_3$ on en déduit que $\bar\sigma_1\,\bar\sigma_2\,\bar\sigma_3\,\bar\sigma_1=\bar\sigma_2\,\bar\sigma_3\,\bar\sigma_1\,\bar\sigma_2$ et de même $\bar\sigma_2\,\bar\sigma_3\,\bar\sigma_1\,\bar\sigma_2=\bar\sigma_3\,\bar\sigma_1\,\bar\sigma_2\,\bar\sigma_3$.

(ii) Nous faisons une récurrence sur la longueur du pseudocycle. Si $n=2$ le résultat est évident et si $n=3$ il s'agit d'un triangle. Si $n\geqq 4$, soit β une arête joignant les extrémités de σ_1 et σ_2 distinctes de leur sommet commun.

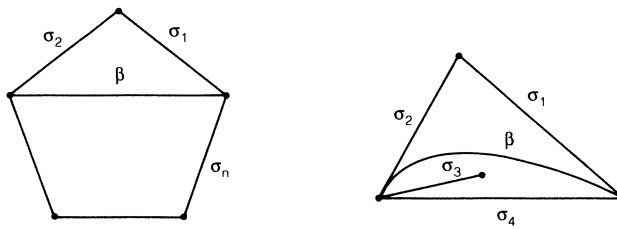


Fig. 19

Alors $\sigma_1\,\sigma_2=\sigma_2\,\beta=\beta\,\sigma_1$ et par l'hypothèse de récurrence $\beta\,\sigma_3\ldots\sigma_{n-1}=\sigma_3\ldots\sigma_n$, d'où $\sigma_2\ldots\sigma_n=\sigma_2\,\beta\,\sigma_3\ldots\sigma_{n-1}=\sigma_1\ldots\sigma_{n-1}$. □

Nous donnons maintenant la

Démonstration du lemme 1.2 Considérons la présentation $\langle X_{\Gamma'}, R_\Gamma\cup\{\tau=\sigma_1\,\sigma_2\,\sigma_1^{-1}\}\rangle$ de $B_\Gamma=B_{\Gamma'}$ obtenue à partir de $\langle X_\Gamma, R_\Gamma\rangle$ par une transformation de Tietze [11]. Les relations $R_{\Gamma'}$ étant vérifiées dans $B_{\Gamma'}=B_\Gamma$ d'après le lemme 1.1, on en déduit que ce sont des conséquences de $R_\Gamma\cup\{\tau=\sigma_1\,\sigma_2\,\sigma_1^{-1}\}$; aussi $\langle X_{\Gamma'}, R_{\Gamma'}\rangle$ est une présentation de $B_{\Gamma'}$. □

Finalement, on donne la

Démonstration du lemme 1.3 Dans le groupe libre de générateurs $X_{\Gamma'}$ on considère le sous-groupe normal engendré par $R_\Gamma\cup\{\tau=\sigma_1\,\sigma_2\,\sigma_1^{-1}\}$. Nous allons montrer que les relations dans $R_{\Gamma'}$ appartiennent à ce sous-groupe. Comme les pré-

sentations $\langle X_\Gamma, R_\Gamma\rangle$ et $\langle X_{\Gamma'}, R_\Gamma \cup \tau = \sigma_1\,\sigma_2\,\sigma_1^{-1}\rangle$ diffèrent par une transformation de Tietze, elles définissent le même groupe. D'où $\langle X_\Gamma, R_\Gamma\rangle$ et $\langle X_{\Gamma'}, R_{\Gamma'}\rangle$ définiront le groupe $B_\Gamma = B_{\Gamma'}$

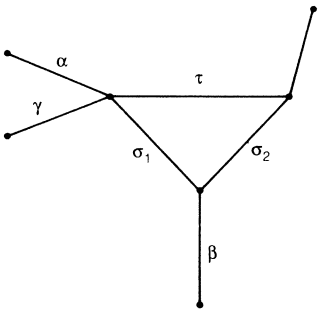


Fig. 20

Tout d'abord si $\tau\sigma_1 = \sigma_1\,\sigma_2$, alors $\sigma_2\,\tau\sigma_1 = \sigma_2\,\sigma_1\,\sigma_2 = \sigma_1\,\sigma_2\,\sigma_1$, d'où $\sigma_2\,\tau = \sigma_1\,\sigma_2$.

Soit β une arête disjointe de τ. Si β est disjointe de σ_1 et σ_2, alors elle commute avec $\tau = \sigma_1\,\sigma_2\,\sigma_1^{-1}$. Si β est contigue avec σ_1 et σ_2, la relation $\beta\tau = \tau\beta$ est équivalente à $\beta\sigma_1\,\sigma_2\,\sigma_1^{-1} = \sigma_2^{-1}\,\sigma_1\,\sigma_2\,\beta$, donc $\sigma_2\,\beta\sigma_1\,\sigma_2 = \sigma_1\,\sigma_2\,\beta\sigma_1$ qui est bien une relation de R_Γ.

Soit α une arête contigue à τ et, par exemple, à σ_1. Pour voir que $\alpha\tau\alpha = \tau\alpha\tau$, on écrit:

$$\alpha\tau\alpha = \alpha\sigma_2^{-1}\,\sigma_1\,\sigma_2\,\alpha = \sigma_2^{-1}\,\alpha\sigma_1\,\alpha\sigma_2 = \sigma_2^{-1}\,\sigma_1\,\alpha\sigma_1\,\sigma_2$$

et

$$\tau\alpha\tau = \sigma_2^{-1}\,\sigma_1\,\sigma_2\,\alpha\sigma_2^{-1}\,\sigma_1\,\sigma_2.$$

Il suffit donc de démontrer que $\alpha = \sigma_2\,\alpha\sigma_2^{-1}$; c'est une relation dans R_Γ.

Soit enfin γ une autre arête contigue à τ et σ_1. Pour avoir $\gamma\alpha\tau\gamma = \tau\gamma\alpha\tau$, écrivons que $\gamma\alpha\tau\gamma = \gamma\alpha\sigma_2^{-1}\,\sigma_1\,\sigma_2\,\gamma = \sigma_2^{-1}\,\sigma_1\,\gamma\alpha\sigma_1\,\sigma_2 = \sigma_2^{-1}\,\sigma_1\,\sigma_2\,\gamma\sigma_2^{-1}\,\sigma_1\,\sigma_2 = \tau\gamma\alpha\tau$. De la même façon on vérifie que $\tau\gamma\alpha\tau = \alpha\tau\gamma\alpha$. □

3 Remarques et applications

Les premières remarques de ce paragraphe concernent le système de relations donné par le théorème.

Remarque 3.1 Pour un graphe Γ, les relations R_Γ ne sont pas toutes nécessaires. La relation nodale

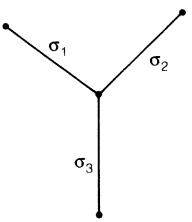


Fig. 21

$\sigma_1\sigma_2\sigma_3\sigma_1=\sigma_2\sigma_3\sigma_1\sigma_2$ et les relations d'adjacence $\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2$, $\sigma_2\sigma_3\sigma_2=\sigma_3\sigma_2\sigma_3$ entraînent que $\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1=\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2$ $=\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2=\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, d'où $\sigma_3\sigma_1\sigma_2\sigma_3=\sigma_2\sigma_3\sigma_1\sigma_2$.

De même, pour le cycle

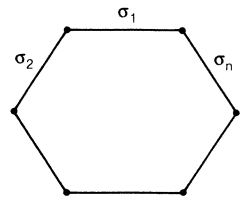


Fig. 22

si $\sigma_1\sigma_2\ldots\sigma_{n-1}=\sigma_2\ldots\sigma_n$, alors $\sigma_n\sigma_1\ldots\sigma_{n-1}=\sigma_n\sigma_2\ldots\sigma_n=\sigma_2\ldots\sigma_n\sigma_{n-1}\sigma_n$ $=\sigma_2\ldots\sigma_{n-1}\sigma_n\sigma_{n-1}$ d'où $\sigma_n\sigma_1\ldots\sigma_{n-1}=\sigma_2\ldots\sigma_{n-1}\sigma_n$; toutes les relations cycliques sont des conséquences d'une seule et des relations de disjonction et d'adjacence. □

Remarque 3.2 Une série de relations remarquables généralise la relation nodale. Pour le graphe

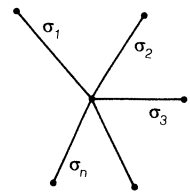


Fig. 23

on a: $\sigma_{i_1}\ldots\sigma_{i_k}\sigma_{i_1}=\sigma_{i_2}\ldots\sigma_{i_k}\sigma_{i_1}\sigma_{i_2}$ où $\sigma_{i_1}\ldots\sigma_{i_k}\sigma_{i_1}$ sont disposées dans le sens horaire. En effet, ceci est vrai si $k\leqq 3$. Supposons-le vrai lorsque $k-1$ arêtes sortent d'un nœud. Alors

$$\begin{aligned}\sigma_{i_1}\ldots\sigma_{i_k}\sigma_{i_1}&=\sigma_{i_1}\ldots\sigma_{i_{k-1}}\sigma_{i_1}\sigma_{i_1}^{-1}\sigma_{i_k}\sigma_{i_1}=\sigma_{i_2}\ldots\sigma_{i_{k-1}}\sigma_{i_1}\sigma_{i_2}\sigma_{i_1}^{-1}\sigma_{i_k}\sigma_{i_1}\\&=\sigma_{i_2}\ldots\sigma_{i_{k-1}}\sigma_{i_1}\sigma_{i_2}\sigma_{i_k}\sigma_{i_1}\sigma_{i_k}^{-1}=\sigma_{i_2}\ldots\sigma_{i_k}\sigma_{i_1}\sigma_{i_2}\sigma_{i_k}\sigma_{i_k}^{-1}\\&=\sigma_{i_2}\ldots\sigma_{i_k}\sigma_{i_1}\sigma_{i_2}\end{aligned}$$

en utilisant l'hypothèse de récurrence et les relations (i), (ii) et (iii) du théorème. □

Remarque 3.3 L'observation précédente suggère la

Question. Peut-on remplacer les relations nodales et d'adjacence par la relation $\sigma_1 \ldots \sigma_n \sigma_1 = \sigma_2 \ldots \sigma_n \sigma_1 = \ldots = \sigma_n \sigma_1 \ldots \sigma_n$ pour chaque sommet de valence n?

La réponse est négative comme le montre l'exemple suivant:

Exemple. Soit de nouveau le graphe

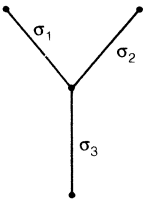


Fig. 24

et considérons le groupe $\mathscr{B}_4 = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1 \sigma_2 \sigma_3 \sigma_1 = \sigma_2 \sigma_3 \sigma_1 \sigma_2 = \sigma_3 \sigma_1 \sigma_2 \sigma_3 \rangle$. Il est bien connu que pour un groupe G ayant g générateurs et r relations, $H_2(G; \mathbf{Z})$ peut être engendré par $r - g + rgH_1(G)$ éléments [Bro, p. 46]. Pour $G = \mathscr{B}_4$, $H_1(\mathscr{B}_4) = \mathbf{Z}$, d'où $H_2(\mathscr{B}_4; \mathbf{Z}) = 0$. Comme pour le groupe de tresses B_4, $H_2(\mathscr{B}_4) = \mathbf{Z}_2$ [Bri, p. 36] on déduit que $\mathscr{B}_4 \neq B_4$.

Nous faisons maintenant deux remarques sur les groupes symétriques.

Remarque 3.4 Une version relative du théorème est valable pour les groupes de permutations. Sa démonstration est similaire.

Proposition 3.4 *Le groupe Σ_Γ des permutations à support fini des sommets du graphe planaire Γ est engendré par les générateurs X_Γ du théorème et les relations R'_Γ obtenues en ajoutant à R_Γ les relations $\sigma^2 = 1$ pour chaque arête σ.* □

Remarque 3.5 Dans le cas du graphe

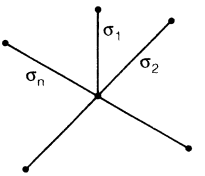


Fig. 25

dans le groupe Σ_Γ on a:

$$\sigma_{i_1} \sigma_{i_2} \sigma_{i_1} \sigma_{i_3} = \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_1} \sigma_{i_1} \sigma_{i_3} \sigma_{i_1} \sigma_{i_3} = \sigma_{i_3} \sigma_{i_1} \sigma_{i_2} \sigma_{i_1} \sigma_{i_3} \sigma_{i_3} = \sigma_{i_3} \sigma_{i_1} \sigma_{i_2} \sigma_{i_1}.$$

On en déduit que le groupe Σ_Γ a une présentation obtenue en remplaçant dans R'_Γ la relation $\sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_1} = \sigma_{i_3} \sigma_{i_1} \sigma_{i_2} \sigma_{i_3}$ par $\sigma_{i_1} \sigma_{i_2} \sigma_{i_1} \sigma_{i_3} = \sigma_{i_3} \sigma_{i_1} \sigma_{i_2} \sigma_{i_1}$.

En fait, Carmicael a montré (voir [MC]) que l'on peut se restreindre aux seules relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \sigma_i \sigma_{i+1} \sigma_i \sigma_j = \sigma_j \sigma_i \sigma_{i+1} \sigma_i,$$

$\sigma_i^2 = 1$, $i = 1, \ldots, n$, (ou $\sigma_{n+1} = \sigma_1$).

Remarque 3.6 Les différentes présentations du groupe B_n proposées plus haut dépendent du choix d'un graphe planaire dont les sommets sont les n points fixés. Nous allons donner aussi une présentation «universelle», ne dépendant pas d'un tel choix.

Soit $\mathscr{X}_n$ l'ensemble des classes d'isotopie à extrémités fixées de courbes qui joignent deux parmi n points donnés. Soit $\mathscr{R}_n$ l'ensemble des relations $\sigma\tau = \tau\sigma$ lorsque σ et τ sont disjointes et $\sigma\tau = \tau v = v\sigma$ lorsque σ, τ, v bordent un triangle sans point intérieur.

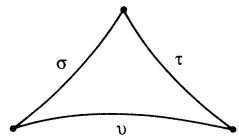


Fig. 26

Proposition 3.7 *Le groupe de tresses B_n admet la présentation $\langle \mathscr{X}_n, \mathscr{R}_n \rangle$.*

Démonstration. Soit $F_{\mathscr{X}_n}$ le groupe libre de générateurs $\mathscr{X}_n$. Les relations de $\mathscr{R}_n$ se trouvent dans le noyau de l'application de $F_{\mathscr{X}_n}$ sur B_n qui envoie une courbe σ sur la tresse $\bar{\sigma}$. Nous allons montrer que tout élément $\sigma_1 \ldots \sigma_n \in F_{\mathscr{X}_n}$ tel que $\bar{\sigma}_1 \ldots \bar{\sigma}_n = 0$ se trouve dans le sous-groupe normal engendré par $\mathscr{R}_n$. Nous aurons besoin du

Lemme 3.8 *L'élément $\sigma_1 \ldots \sigma_n$ est équivalent relativement à $\mathscr{R}_n$ à un élément $\tau_1 \ldots \tau_m$ tel que les intérieurs des supports des τ_i sont disjoints. Aussi $\tau_1, \ldots, \tau_m$ sont les arêtes d'un graphe.*

Terminons d'abord la démonstration. Le lemme permet de supposer que $\sigma_1 \ldots \sigma_n$ sont les arêtes d'un graphe Γ. Le mot $\bar{\sigma}_1 \ldots \bar{\sigma}_n$ est nul dans B_n donc dans B_Γ. Il se trouve alors dans le sous-groupe engendré par R_Γ. Il suffit de montrer que les relations de R_Γ sont des conséquences de celles de $\mathscr{R}_n$. Ceci est évident pour les relations de type (i).

Pour la relation de type (ii) $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ on considère le triangle

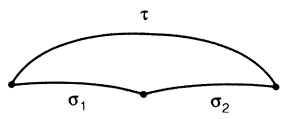


Fig. 27

Puisque $\tau\sigma_1 = \sigma_1\sigma_2 = \sigma_2\tau$ sont des relations de $\mathscr{R}_n$, elles entraînent que $\tau = \sigma_1 \sigma_2 \sigma_1^{-1} = \sigma_2^{-1} \sigma_1 \sigma_2$, d'où $\sigma_2 \sigma_1 \sigma_2 = \sigma_1 \sigma_2 \sigma_1$.

La relation nodale est une conséquence des relations de $\mathscr{R}_n$ comme on le voit en reprenant la démonstration du lemme 2.1 (i).

Pour la relation cyclique on remarque aussi que, d'après la démonstration du lemme 2.1 (ii), elle est une conséquence de l'application répétée de relations de $\mathscr{R}_n$. □

Nous complétons la preuve ci-dessus avec la

Démonstration du lemme 3.8 Faisons une récurrence sur le nombre de points k d'auto-intersection du chemin $\sigma_1 \dots \sigma_n$. Si $\kappa = 0$ il n'y a rien à démontrer. Supposons le résultat vrai lorsqu'il y a moins de κ points d'intersection. Si ce nombre est égal à κ, soit σ_i le premier chemin qui coupe un autre, soit σ_j, ou $j > i$.

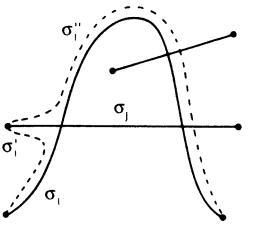


Fig. 28

On considère deux courbes σ_i', σ_i'' formant avec σ_i un triangle sans point intérieur, comme dans la figure ci-dessus. Si on remplace σ_i par $\sigma_i'^{-1}\, \sigma_i''\, \sigma_i'$ dans le mot $\sigma_1 \dots \sigma_n$ on obtient un mot équivalent par rapport au sous-groupe engendré par $\mathscr{R}_n$. Comme le nombre d'auto-intersection a diminué on peut conclure en utilisant l'hypothèse de récurrence. □

La présentation du théorème donne une démonstration rapide du

Corollaire 3.9 *Le groupe libre à deux générateurs agit naturellement comme automorphismes du groupe* B_∞.

Preuve. L'action par translation du groupe libre sur son graphe de Cayley préserve la présentation associé à ce graphe. Elle induit une action au niveau du groupe des tresses. □

Remarque 3.10 Le même argument montre que si Fg est une surface orientable, $\pi_1(Fg)$ agit comme automorphismes de B_∞. □

Remarque 3.11 Le résultat principal associe à tout graphe planaire une présentation du groupe de tresses *classique*.

Dans une direction différente, Johnson [J2] associe à n'importe quel graphe une présentation dont les relations sont proches de celles d'Artin. Il obtient ainsi une famille de groupes de tresses généralisés, distincte de celle de Brieskorn [Bri]. □

Pour terminer, nous mentionnons une situation plus subtile ou l'on construit un groupe d'automorphismes du groupe de tresses à partir d'une donnée arborescente.

Remarque 3.12 Dans [GS] nous avons mis en évidence que le groupe F des homéomorphismes linéaires par morceaux de l'intervalle [0, 1] de pentes $2^n (n \in \mathbf{Z})$

et dont les discontinuités de la dérivée sont dyadiques, agit «à l'infini» sur l'arbre binaire infini: chaque élément agit simplicialement en dehors d'un sous-arbre fini.

Cette action ne s'étend pas à l'arbre tout entier. Cependant si on considère le groupe de tresses sur les sommets, nous pouvons utiliser le théorème pour montrer que l'action de chaque élément s'étend (non-uniquement) à un automorphisme de B_∞ (voir aussi [GS]). On obtient ainsi une extension $1 \to B_\infty \to A \to F \to 0$ qui a des propriétés homologiques remarquables.

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Cohomogeneity and positive curvature in low dimensions

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The classification of compact postively curved Riemannian manifolds (CPCM's), is a basic and wide-open problem. To date, the problem has been completely solved only in dimensions 2 and 3 (Gauss-Bonnet and Hamilton [H], respectively). Recently, Hsiang and Kleiner [H-K] made some progress in dimension 4: they were able to show that in the presence of an effective, isometric S^1 action, the only simply-connected CPCM's are S^4 and CP^2. On the other hand, when we consider the list of known examples of simply-connected CPCM's, we note that except for one in dimension 6 and an infinite string in dimension 7 [E1,2] all such manifolds are homogeneous. The question we pose here is the following: if we decrease the allowed symmetry on a CPCM, will we obtain new examples of CPCM's, or will we show that the list we have now is exhaustive? Concretely, we define symmetry in terms of isometric group actions:

Definition. We say that a compact, connected Lie group G *acts by cohomogeneity* k on a Riemannian manifold M, if G acts effectively and isometrically with a principal orbit of codimension k.

We note that $S^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is of all cohomogeneities with respect to the canonical metric of constant curvature. We will prove the following theorem in this paper:

Theorem. *Suppose that* M^n, $2 \leqq n \leqq 6$, *is a simply-connected* CPCM *admitting a cohomogeneity one action with respect to this metric. Then* M^n *is diffeomorphic to either* S^n *or* $CP^{n/2}$.

Remark that two known examples of simply-connected CPCM's are missing from this list: $V^6 \cong \mathrm{SU}(3)/T^2$ [W], which is homogeneous, and $M^6 \cong S^1 \backslash \mathrm{SU}(3)/S^1$, which is inhomogeneous [E1]. It is known that the simply-connected CPCM's which are CROSSes (compact rank one symmetric spaces) are all cohomogeneity one spaces with respect to a metric of positive curvature. This then gives us the first example of a homogeneous CPCM which is not

cohomogeneity one with respect to a metric of positive curvature. This work forms a part of the author's PhD thesis at the University of Maryland at College Park, under the direction of Karsten Grove.

1 Preliminaries

For the sake of completeness, we will recall some facts about fixed point sets for orientation-preserving, isometric group actions on simply-connected CPCM's which will be used throughout. In the special case of an isometric circle action we know that the components of the fixed point set will be closed, hence compact, totally geodesic, orientable submanifolds of even codimension [Ko]. Thus the curvature of these submanifolds will also be strictly positive, and so if their dimension is greater than or equal to 2 they will also be CPCM's. If the dimension of the manifold is even, then the submanifolds will be simply-connected [Sy]. We also have the same results for orientation-preserving, isometric cyclic group actions on such manifolds [Ko]. Note further that in the case of an even dimensional simply connected CPCM, the fixed point set of a circle action will be non-empty, whereas in odd-dimensions this need not be true [Ko].

The following results from [Gr-S1,2] (cf. also [S]) will be used throughout:

Theorem 1.1 *Let S^1 act on M^n effectively and by isometries, where M^n is a simply-connected* CPCM. *Suppose further that* $\mathrm{Fix}(S^1; M^n)$ *is of codimension* 2. *Then M^n is diffeomorphic to S^n or $CP^{n/2}$.*

Theorem 1.2 *Let M^n be a simply-connected* CPCM. *If T^k acts on M^n effectively and by isometries, then* $k \leqq \left[\frac{n+1}{2}\right]$. *Furthermore, M^n is diffeomorphic to either S^n or $CP^{n/2}$ when* $k = \left[\frac{n+1}{2}\right]$.

Theorem 1.3 *Let* SU(2) *act almost effectively and by isometries on M^n, where M^n is a simply-connected* CPCM, *and further assume that* $\mathrm{Fix}(\mathrm{SU}(2); M^n)$ *is of codimension less than or equal to* 4, *then M^n is diffeomorphic to S^n, $CP^{n/2}$, or $HP^{n/4}$.*

Finally, we recall some facts about cohomogeneity one manifolds of positive curvature: by a theorem of Mostert [M] we know that the quotient space is homeomorphic to R, R^+, S^1 or I. In the case where M is compact, the circle or the closed interval are the only possible quotient spaces. With the further restriction that the manifold M be strictly positively curved, we are left with only the closed interval as quotient space, since otherwise $\pi_1(M)$ would be infinite, contradicting the Bonnet-Meyers theorem. In this case we have a decomposition of M into a union of two disc bundles over each singular orbit, where the singular orbits correspond to the inverse images of the endpoints of the closed interval I.

2 Six dimensional CPCM's of cohomogeneity 1

We begin with the proof of the main theorem for dimension six, since the techniques used for its proof will also be used in the proof for dimension five.

Theorem 2.1 *Let G, a compact, connected Lie group, act by cohomogeneity* 1 *on M^6, a simply-connected* CPCM. *Then M^6 is diffeomorphic to S^6 or CP^3.*

Proof. Note that if the rank(G) exceeds two, then by Theorem 1.2 we are done. The principal orbit here is of dimension 5, hence the dimension of G must be between 5 and 15 [M-Z]. Since all such groups have rank two or larger it suffices to consider only those of rank two. By allowing the action of G to be central-almost-effective, that is, where the kernel of the action is a finite central subgroup, the groups of interest then are $SU(2)\times SU(2)$, $SU(3)$, $Spin(5)$ and G_2, of dimensions 6, 8, 10 and 14 respectively.

We may rule out $Spin(5)$ and G_2 as such a group of isometries immediately, since in both cases there is no appropriate subgroup H which would give us a 5-dimensional principal orbit.

The strategy of proof for the remaining two groups is as follows: we plan to show that there will be either a codimension 4 submanifold fixed by an $SU(2)$ subgroup, or a codimension 2 submanifold fixed by a circle subgroup. We may then apply either Theorem 1.3 or Theorem 1.1, respectively, to obtain the result.

We consider first the group $SU(3)$. The only connected 3-dimensional subgroups of $SU(3)$ are $SU(2)$ and $SO(3)$. We may actually be more specific about the isotropy subgroups of the principal orbits: indeed, if $SU(3)$ acts by cohomogeneity one on M^6, where M^6 is as above, then the principal isotropy subgroup, H, cannot contain $SO(3)$. This follows directly from the condition that the principal orbit must fiber over its singular orbits with sphere fiber.

Lemma 2.1.1 *If* $SU(2)$ *is contained in the principal isotropy subgroup of a cohomogenity one action on M^6, where M^6 is as above, then its fixed point set will be non-empty and of codimension less than or equal to* 4.

Proof. First we note that if $SU(2)$ is contained in the isotropy subgroup of the principal orbit, it will act trivially on the normal 1-space to the points of isotropy in the principal orbit [Bre]. This implies that $SU(2)$ fixes at least an S^1 in M^6. In addition any circle subgroup of $SU(2)$ will then fix at least two dimensions in M^6, since the codimension of such a subgroup's fixed point set must be even. Remark further that $\mathrm{Fix}(SU(2); G/H) = \mathrm{Fix}(SU(2); M^6)\cap G/H$. And, we claim that $\mathrm{Fix}(SU(2); G/H)$ cannot be of dimension 0 in G/H, since the only two possibilities for a non-trivial representation of $SU(2)$ on R^5 are the following: either the action decomposes R^5 as $R^4\bigoplus R^1$, or as $R^3\bigoplus R^2$, and we note that the smallest irreducible representation of $SU(2)$ is of dimension 3. Thus, in either case we see that $SU(2)$ is fixing either one or two dimensions in G/H, and hence it fixes a total of two or three dimensions in M^6. □

Lemma 2.1.2 *If $G\cong SU(2)\times SU(2)$ acts by cohomogeneity one on M^6, where M^6 is as above, then either some circle subgroup of G will fix a codimension* 2 *submanifold in M^6, or some* $SU(2)$ *subgroup of G will fix a codimension* 4 *submanifold in M^6.*

Proof. Since the principal orbit is 5-dimensional, we have S^1 isotropy in the principal orbit. At any point of isotropy of this circle it will act trivially on the normal 1-space to the principal orbit. Since the codimension of the fixed point set of a circle action is even, this implies that the circle fixes at least a two-dimensional totally geodesic submanifold in M^6. If the fixed point set is of dimension 4, using Theorem 1.1, we are done, so we assume instead that

the fixed point set of any circle subgroup, denoted by C, of G is at most of dimension 2.

Since $\mathrm{Fix}(C; M^6)$ is 2-dimensional, it is a finite union of S^2's and possibly a finite number of isolated fixed points. Thus $\mathrm{Fix}(C; G/H)$ is 1-dimensional, hence a finite number of circles. We denote by S^1_a and S^1_b, respectively, any circle of the maximal torus containing C which lies exclusively in the first or the second factor of $\mathrm{SU}(2)_a \times \mathrm{SU}(2)_b$. S^1_a will act on any $S^2 \in \mathrm{Fix}(C; M^6)$, and either (a), it fixes the S^2, or (b), it fixes isolated points in the S^2. In case (a), $\mathrm{Fix}(S^1_a; G/H)$ is 1-dimensional and, since the actions commute, $\mathrm{SU}(2)_b$ acts on any circle in $\mathrm{Fix}(S^1_a; G/H)$ and can only do so trivially. However, since a T^2 cannot foliate an S^3 [Gro-Gr], $S^1_a \times \mathrm{SU}(2)_b$ cannot act almost freely on the unit normal S^3 to any point of the fixed circle, which implies that some circle in $S^1_a \times \mathrm{SU}(2)_b$ fixes 4 dimensions in the manifold. In case (b), S^1_a fixes only isolated points in $\mathrm{Fix}(C; M^6)$, and $\mathrm{SU}(2)_b$ acts on and fixes those points fixed by S^1_a. If $\mathrm{SU}(2)_b$ fixes something larger dimensional, then any of its circle subgroups will have at least a 2-dimensional fixed point set, and so we are back to case (a). So we assume these are isolated fixed points for both $\mathrm{SU}(2)$ actions. We note that the action of each $\mathrm{SU}(2)$ on the normal R^6 to such points is then such that any of its circle subgroups is fixing 2 dimensions. But then we are back to case (a), again. □

With these two lemmas, we may now apply Theorem 1.1 or Theorem 1.3 to imply that M^6 is diffeomorphic to either S^6 or CP^3, thus concluding the proof of Theorem 2.1.

In addition, we note that both of these groups do admit actions on both S^6 and CP^3 by cohomogeneity one.

3 Five dimensional CPCM's of cohomogeneity 1

We now proceed with the argument for dimension 5. As mentioned previously, we will use the techniques of the last section, as well as a generalization of techniques used in [H-K] to the 4-dimensional quotient space of the manifold in question.

Theorem 3.1 *Suppose that M^5 is a simply-connected* CPCM *admitting a cohomogeneity* 1 *action with respect to this metric. Then M^5 is diffeomorphic to S^5.*

Proof. In this case we have a principal orbit of dimension 4. The dimension of the groups to be considered is then between 4 and 10 [M-Z]. Note that by Theorem 1.2 it suffices to consider groups of rank 2 only. Again, by allowing the action of G to be central-almost-effective, the only groups of interest are: $\mathrm{SU}(2) \times S^1$, $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{SU}(3)$, and $\mathrm{Spin}(5)$, which are of dimensions 4, 6, 8, and 10 respectively. Note that in all dimensions except 4 we have at least a T^2 contained in the isotropy subgroup. Since the dimension of the isotropy subgroup is always even and at least 2 in all these cases, the normal 1-space to the principal orbit at a point of isotropy is fixed by this T^2 (if not some larger subgroup). Again, the action of T^2 on the unit normal S^3 to any point of its 1-dimensional fixed point set cannot be almost free, and thus some circle subgroup must be fixing three dimensions, and by Theorem 1.1 we obtain the result.

It only remains to consider the group $\mathrm{SU}(2) \times S^1$. We claim that if this group acts by cohomogeneity one on a 5-dimensional simply-connected CPCM,

then the dimensions of the singular orbits are either 1 or 3. This follows directly from the fact that the principal orbit fibers over the singular orbits with sphere fiber. In fact, by considering the Mayer-Vietoris sequence of the space written as a union of two disc bundles over the singular orbits, we may rule out all combinations for singular orbits except the following: case (a), where both orbits are 3-dimensional, that is, both are S^3/F, where F is a finite group, possibly trivial (although not necessarily the same F), or one is an S^3/F and the other is $S^2 \times S^1$ and case (b), where one orbit is 3-dimensional and one is 1-dimensional, that is where one singular orbit is an S^1 and the other is an S^3/F, where F is as above.

We will show how to rule out case (a). In case (b), it is easy to show that there is an SU(2) with a codimension 4 fixed point set, and hence using Theorem 1.3, M^5 is diffeomorphic to S^5.

Case (a) Both singular orbits are 3-dimensional. Note that for all the possible combinations the manifold in question will be a homology $S^2 \times S^3$. We proceed by enumerating the possible ways in which $S^1_b \subseteq \mathrm{SU}(2)_a \times S^1_b$ may act on M^5 as follows: (i): where the fixed point set of S^1_b is non-empty, (ii): where S^1_b acts freely, and (iii): where S^1_b acts almost freely.

Lemma 3.1.1 *Let M^5 be as above, and suppose that $\mathrm{SU}(2) \times S^1$ acts by cohomogeneity 1 on M^5, and both singular orbits of the action are of dimension 3. Then the S^1_b action must be almost free, and, in fact S^1_b will act freely on all of M^5, except for on one singular orbit, which its isotropy subgroup will fix.*

Proof. It is an easy calculation to show that only case (iii) is consistent with both singular orbits being of dimension 3. Denote the isotropy subgroup of the S^1_b action by Z_k. Since the codimension of $\mathrm{Fix}(Z_k; M^5)$ is even we have two cases to consider: (1) $\mathrm{Fix}(Z_k; M^5)$ is 1-dimensional or (2) $\mathrm{Fix}(Z_k; M^5)$ is 3-dimensional. In case (1), by the commutativity of the actions, $\mathrm{SU}(2)_a$ will act on $\mathrm{Fix}(Z_k; M^5)$ and it can only do so trivially. But then $\mathrm{SU}(2)_a$ fixes at least a 1-dimensional submanifold in M^5, and so we may apply either Theorem 1.1 or 1.3 to conclude that M^5 is diffeomorphic to S^5.

In case (2): we note first that there are no points of Z_k isotropy in the principal orbits, since if there were, we would then just divide out this isotropy to obtain an effective S^1_b action, and thus it follows that $\mathrm{Fix}(Z_k; M^5)$ lies entirely within one or both of the singular orbits. We also note that $\mathrm{SU}(2) \times S^1$ acts on $\mathrm{Fix}(Z_k; M^5)$, and that it must do so transitively, and thus $\mathrm{Fix}(Z_k; M^5)$ is an orbit in M^5. However, by Frankel's theorem [F], there can only be one component of dimension 3 of $\mathrm{Fix}(Z_k; M^5)$ in M^5, and further, there can be no other 3-dimensional component of different isotropy in the other orbit, since if there were, it would have to intersect $\mathrm{Fix}(Z_k; M^5)$ nontrivially. Thus there can be 3-dimensional isotropy in only one singular orbit of dimension 3. Further, there cannot be finite isotropy with respect to the S^1_b action in the other singular orbit, since it would have to be 1-dimensional, and we ruled out this possibility in case (1). So, given another singular orbit of dimension three, the action of S^1_b on this singular orbit is either free, or it has non-empty fixed point set, and the latter case is not consistent with having two singular orbits of dimension three. □

We now want to understand the quotient space M^5/S^1_b. Denote by $Y^3_1 = \mathrm{Fix}(Z_k; M^5)$ the singular orbit on which S^1_b acts with finite isotropy, where

Z_k is the largest isotropy subgroup of S_b^1 fixing Y_1^3, and by Y_2^3 the singular orbit on which S_b^1 acts freely. Note that Y_1^3/S_b^1 and Y_2^3/S_b^1 are topological manifolds. Further, it is clear that Y_1^3/S_b^1 and Y_2^3/S_b^1 are both S^2's (where Y_1^3/S_b^1 is positively curved and Y_2^3/S_b^1 is positively curved if Y_2^3 was).

Let $\pi: M^5 \to M^5/S_b^1$. It follows that $M^5/S_b^1 = X^4$ is a compact Aleksandrov space of strictly positive curvature [B-G-P], where we define an Aleksandrov space with curvatures bounded below to be a complete, finite Hausdorff dimensional, inner metric space with lower curvature bound in the (local) distance comparision sense (the distance from any vertex to a point on the opposite side of a geodesic triangle in the space is greater than or equal to the corresponding distance in the triangle with the same sidelengths in constant curvature) [B-G-P, Pl]. Note further that we have a central-almost-effective and isometric action of $SU(2)_a$ on X^4.

Claim 3.1.2 *With X^4 and S_a^1 as above, S_a^1 fixes 2 isolated points in $\pi(Y_1^3)$ and 2 isolated points in $\pi(Y_1^3)$ for a total of 4 isolated points in X^4.*

Proof. Recall that $\pi(Y_2^3)$ is a positively curved S^2 and thus an isometric circle action will have non-empty fixed point set, consisting either of all of $\pi(Y_1^3)$, or 2 isolated points. In order to show that $\mathrm{Fix}(S_a^1; \pi(Y_2^3))$ is non-empty, we consider the fixed point set of the isotropy subgroup of Y_2^3 in M^5. In the case where Y_2^3 is $S^2 \times S^1$, it is clear that its isotropy subgroup, a circle subgroup, C, of $SU(2)_a$, fixes two disjoint circles. Since S_a^1 and S_b^1 are in the normalizer of C, they will both act on any fixed circles contained in $\mathrm{Fix}(C; Y_2^3)$. Thus S_a^1 will be fixing the 2 isolated points which correspond to the images of these two disjoint circles in X^4. A similar argument can be made in the case where Y_2^3 is a Berger sphere.

We now claim that S_a^1 must be fixing a total of 4 isolated points in $\pi(Y_1^3)$ and $\pi(Y_2^3)$. Suppose instead that these points are not isolated. In this case, we see that S_a^1 will then fix all of either $\pi(Y_1^3)$ or $\pi(Y_2^3)$, and so $\pi^{-1}(S_a^1) \cong T^2$ has circle orbits in one of the singular orbits in M^5, which implies that some circle in this T^2 must have been fixing an entire singular orbit to begin with, contrary to our assumptions. Thus all those points fixed by S_a^1 must be isolated fixed points. □

We now want to show that any isometric circle action in X^4 cannot have more than 3 isolated points fixed, and thus that the situation of case (a) could not have occurred.

We generalize an argument of Hsiang and Kleiner to our space X^4. Recall that Toponogov's theorem holds in Aleksandrov spaces with curvatures bounded below [B-G-P]. Denote by $\{p_i, p_j, p_k, p_l\} \in X^4$ the set of 4 isolated fixed points of the S_a^1 action. Let $l_{ij} = \mathrm{dist}(p_i, p_j)$ be the length of a minimizing geodesic segment from p_i to p_j. Define $C_{ij} = \{\gamma: [0, l_{ij}] \to X^4 | \gamma$ is a minimizing geodesic segment from p_i to $p_j\}$, $1 \leqq i, j \leqq 4$. For each triple of distinct integers $1 \leqq i, j, k \leqq 4$ set $\alpha_{ijk} = \min \{ \angle(\gamma_j'(0), \gamma_k'(0)) | \gamma_j \in C_{ij}, \gamma_k \in C_{ik}\}$. Since X^4 is compact, a minimum exists.

Lemma 3.1.3 *With X^4 and p_i, p_j, p_k, p_l as above, we have for each quadruple of distinct integers $1 \leqq i, j, k, l \leqq 4$,*

$$\sum_{\substack{1 \leqq j < k \leqq 4 \\ j, k \neq i}} \alpha_{ijk} \leqq \pi.$$

Proof. We first consider the inverse image of such a point, p in X^4, in M^5, with image p' in X^4/S^1_a. By assumption, $\pi^{-1}(p)$ is a circle, and it is acted upon by $\pi^{-1}(S^1_a) \cong T^2$. Thus some circle in T^2, denoted by C', must be fixing it. Consider the unit normal bundle over $\pi^{-1}(p)$, which is isomorphic to $S^3(1) \times \pi^{-1}(p)$. Now the space of directions, $\Sigma^2_{p'}$ at p', is isometric to $(S^3(1) \times \pi^{-1}(p))/T^2 \cong (S^3(1) \times \{p\}/C') \cong S^2$ and thus we may use a theorem of Hsiang and Kleiner to obtain the result. □

On the other hand, since Toponogov's theorem holds in Aleksandrov spaces with curvatures bounded below, this tells us that for any triple of integers we have $\alpha_{ijk} + \alpha_{kij} + \alpha_{jki} > \pi$. Then, we see that

$$\sum_{1 \leqq i \leqq 4} \sum_{\substack{1 \leqq j < k \leqq 4 \\ j,k \neq i}} \alpha_{ijk} > 4\pi,$$

in contradiction with 3.1.3. Hence there could not have been 4 isolated fixed points in X^4, thus concluding the proof of Theorem 3.1.

Note that all of the groups mentioned above, with the exception of SU(3) do act on S^5 by cohomogeneity one. Combining the results in Sects. 2 and 3 with [H-K, H], and Gauss-Bonnet yields the classification theorem in the introduction.

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Smooth contractible hypersurfaces in $\mathbf{C}^n$ and exotic algebraic structures on $\mathbf{C}^3$

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1 Introduction

In this paper we study complex affine algebraic varieties that are diffeomorphic to $\mathbf{R}^{2n}$ but not isomorphic to $\mathbf{C}^n$. According to [Z1–Z3] we shall refer to these varieties as to exotic algebraic structures on $\mathbf{R}^{2n}$ or on $\mathbf{C}^n$. It was Ramanujam who first tried to find these structures. His famous theorem [R] implies that the only complex algebraic structure on $\mathbf{R}^4$ is the standard structure of $\mathbf{C}^2$. In the case of higher dimensions Zaidenberg [Z1–Z3] constructed examples of exotic algebraic structures on $\mathbf{C}^n$ ($n \geqq 3$). Moreover, he proved that these exotic structures on $\mathbf{C}^n$ are not biholomorphically equivalent to $\mathbf{C}^n$. His proof is based on some analytic version of the cancellation theorem. For $n=3$ all varieties in his examples have Kodaira logarithmic dimension $-\infty$. Petrie has informed us that he and tom Dieck also knew about the existence of exotic algebraic structures. In their paper [PtD] there is a list of all smooth contractible hypersurfaces in $\mathbf{C}^3$ of Kodaira logarithmic dimension 1. The authors note in this paper that an analogue of their construction holds for higher dimensions, but all contractible hypersurfaces in $\mathbf{C}^n$ with $n \geqq 4$ are diffeomorphic to $\mathbf{R}^{2(n-1)}$ due to the h-cobordism theorem. Thus they are exotic structures when their Kodaira logarithmic dimension is 1. It is worth mentioning that a combination of Zaidenberg's approach and the examples from [PtD] give curious polynomials on $\mathbf{C}^4$. Consider a polynomial $p(x, y, z) = z^2 x^3 + 3 z x^3 + 3 x - z y^2 - 2 y$ in three variables. The zero fiber of this polynomial has a singular point and all other fibers are contractible hypersurfaces [PtD]. If we consider p in $\mathbf{C}^4$ with the coordinate system (x, y, z, t), then, by the h-cobordism theorem, all nonzero fibers are diffeomorphic to $\mathbf{R}^6$. As Zaidenberg mentions in [Z2] these fibers must be exotic algebraic structures on $\mathbf{C}^3$ due to the strong cancellation theorem [IF], i.e., the polynomial p is not equivalent to a linear polynomial, but all its nonzero fibers are diffeomorphic to the Euclidean space.

Our main aim is to construct new examples of exotic algebraic structures on $\mathbf{C}^3$. Our examples have Kodaira logarithmic dimension 2 and they admit a natural $\mathbf{C}^*$-action with one fixed point compared with the above examples.

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All of them are hypersurfaces in $\mathbf{C}^4$ (for instance, the hypersurface $x+x^4y^{46}+y^{92}z^3+t^5=0$ is such an exotic algebraic structure on $\mathbf{C}^3$).

The study of exotic algebraic structures is closely connected with the study of contractible hypersurfaces in complex Euclidean spaces. Some interesting examples of such hypersurfaces in $\mathbf{C}^{n+1}$ was known to Libgober in 1977, who found them in connection with another problem. He proved [L] that the hypersurface Z in $\mathbf{CP}^{n+1}$ given by $w_{n+1}^d+w_0w_1^{d-1}+w_1w_2^{d-1}+\ldots+w_{n-1}w_n^{d-1}=0$ is a homology projective space (where $(w_0, \ldots, w_{n+1})$ is a homogeneous coordinate system). It follows from this paper that the affine hypersurface $Z-\{w_n=0\}$ is contractible when n is odd and $\geqq 4$. Thus, it is diffeomorphic to $\mathbf{R}^{2n}$, by the h-cobordism theorem.

A. Dimca increased this list of contractible hypersurfaces [D]. Let n, a and d are natural numbers such that $a<d-1$, $(a, d)=1$, $(a, d-1)=1$, and even $n\geqq 4$. Then A. Dimca has proved that the following hypersurfaces

$$p(x_1, \ldots, x_n)=x_1^a x_2^{d-a}+x_2x_3^{d-1}+\ldots+x_{n-2}x_{n-1}^{d-1}+x_{n-1}+x_n^d=0$$

in $\mathbf{C}^n$ are diffeomorphic to $\mathbf{C}^{n-1}$ [D]. When $a=1$ and $n=4$, A. Dimca proved that these hypersurfaces were isomorphic to $\mathbf{C}^3$. Moreover he proved that p was a coordinate function in this case. We would like to mention that for $d=3$ it also follows from the fact that $x_1x_2^2+x_2x_3^2+x_3$ is a coordinate function of Nagata's automorphism [Na]. If $a>1$, then it is easy to check that p is not equivalent to a linear polynomial up to polynomial automorphisms of $\mathbf{C}^n$.

We shall show how to generalize the examples of Libgober and Dimca (this is the second aim of our paper). The following observation is useful here. Let $x'=(x_1, \ldots, x_{n-1})$ and $q(x')=x_1^a x_2^{d-a}+x_2x_3^{d-1}+\ldots+x_{n-2}x_{n-1}^{d-1}+x_{n-1}$. Put $X=\{(x_1, x_2, \ldots, x_n)\in\mathbf{C}^n \mid q(x')+x_n^d=0\}$. Hence one can see that there is a $\mathbf{Z}_d$-action on X and the set $V=\{q(x')=0\}\cap X$ is the set of fixed points for this action. In virtue of the Smith theory, since X is diffeomorphic to $\mathbf{C}^{n-1}$, the set V must be $\mathbf{Z}_k$-acyclic for every prime divisor k of d (e.g., see [FV]). In particular all senior homology groups must be finite and the order of every element of these groups is relatively prime with d. We shall show that this fact is very essential in such examples. The main result of Sect. 2 is the following theorem.

Theorem A *Let G be a regular $\mathbf{C}^*$-action on $\mathbf{C}^{n-1}$ $(n\geqq 2)$ and a polynomial $q(x')$ is a quasi-invariant of a positive weight l under this action. Suppose that the fiber $F_0=\{q(x')=0\}$ is smooth and non-multiple, and $\pi_1(\mathbf{C}^{n-1}-F_0)=\mathbf{Z}$. Suppose that for a certain natural k the manifold F_0 is $\mathbf{Z}_s$-acyclic for every prime divisor s of k, and the pair (k, l) is coprime. Then the hypersurface $X=\{q(x')+x_n^k=0\}$ in $\mathbf{C}^n$ is diffeomorphic to $\mathbf{R}^{2n-2}$.*

In Sect. 3 we construct nontrivial contractible hypersurfaces in $\mathbf{C}^4$. In the last section we present examples of exotic structures which belong to the list of hypersurfaces in $\mathbf{C}^4$ from Sect. 3.

2 Proof of Theorem A

In this section we shall fix the following notations: $x=(x_1, \ldots, x_n)$ is a coordinate system in $\mathbf{C}^n$ and $x'=(x_1, \ldots, x_{n-1})$. Put $\mathbf{C}^*=\mathbf{C}-0$. Recall that a homomorphism G of the group $\mathbf{C}^*$ in the group of polynomial automorphisms of $\mathbf{C}^n$ is called

a $\mathbf{C}^*$-action on $\mathbf{C}^n$. A polynomial $p(x)$ is a quasi-invariant under the $\mathbf{C}^*$-action if $p(G_\lambda(x))=\lambda^m p(x)$ (where G_λ is the polynomial automorphism that corresponds to $\lambda\in\mathbf{C}^*$). The number m is called the weight of p.

Lemma 1 *Let p be a polynomial on $\mathbf{C}^n$, and $X=\{p(x)=0\}$ be a smooth irreducible hypersurface in $\mathbf{C}^n$. Suppose $n>3$, $\pi_1(X)=0$, and $H_\ell(X,\mathbf{Z})=0$ for each $\ell>1$. Then X is diffeomorphic to $\mathbf{C}^{n-1}$.*

Proof. Let B be an open ball in $\mathbf{C}^n$ with center at the origin and with a sufficiently large radius. Then $X\cap B$ is diffeomorphic to X and the embedding $X\cap\partial B\to X$ induces an isomorphism $\pi_1(x\cap\partial B)\to\pi_1(X)$ [H, Theorem 2]. Thus $\pi_1(X\cap\partial B)=0$. By Hurewicz's theorem $\pi_k(X)=\pi_k(X\cap B)=0$ for every $k\geqq 1$. Hence $X\cap B$ is diffeomorphic to a ball of real dimension $2n-2$ according to [S, Theorem 5.1]. Therefore, X is diffeomorphic to $\mathbf{C}^{n-1}$. □

Lemma 2 *Under the assumption of Theorem* A $\pi_1(X)=0$.

Proof. Put $F_c=\{q(x')=c\}$. Since q is quasi-invariant under G with nonzero weight, for every c, $c'\in\mathbf{C}^*$ there exists λ such that $F_{c'}=G_\lambda(F_c)$ according to Nullstellensatz. Hence the mapping q: $\mathbf{C}^{n-1}-F_0\to\mathbf{C}^*$ is a holomorphic fibration. Let F be the generic fiber of this fibration. Since the multiplicity of q on F_0 is 1, one can choose a loop $\gamma\subset\mathbf{C}^{n-1}-F_0$ in a small neighborhood of a point $x\in F_0$ so that the loop $q(\gamma)$ generates the group $\pi_1(\mathbf{C}^*)=\mathbf{Z}$. Hence in the exact sequence

$$0\to\pi_1(F)\to\pi_1(\mathbf{C}^{n-1}-F_0)\xrightarrow{j}\pi_1(\mathbf{C}^*)\to$$

the mapping j is an epimorphism. Since $\pi_1(\mathbf{C}^{n-1}-F_0)=\mathbf{Z}$, it is an isomorphism and, therefore, $\pi_1(F)=0$. Put $p(x)=x_n^k+q(x')$, then $X=\{p(x)=0\}$. Consider $\Gamma_c=X\cap\{x_n=c\}$ and the mapping x_n: $X-\Gamma_0\to\mathbf{C}^*$. Clearly, it is a holomorphic fibration and its generic fiber is isomorphic to F. Let $y\in\Gamma_0$. Same arguments as above show that there is a loop η in a small neighborhood of y so that its image $x_n(\eta)$ under the mapping x_n can be viewed as a generator in the group $\pi_1(\mathbf{C}^*)$. Since $\pi_1(F)=0$, the mapping i in the exact sequence

$$0\to\pi_1(F)\to\pi_1(X-\Gamma_0)\xrightarrow{i}\pi_1(\mathbf{C}^*)\to 0$$

is an isomorphisms. Hence the loop η may be considered as a generator of $\pi_1(X-\Gamma_0)$. But η is contractible in X. Therefore $\pi_1(X)=0$. □

Consider pairs $(\mathbf{C}^{n-1},F_0)$ and (X,Γ_0), and the quotient spaces $W=\mathbf{C}^{n-1}/F_0$ and $Y=X/\Gamma_0$. Then $H_*(\mathbf{C}^{n-1},F_0)=\tilde{H}_*(W)$, $H_*(X,\Gamma_0)=\tilde{H}_*(Y)$, and the exact homology sequence of pair implies

Lemma 3 *For every $k\geqq 0$ the homology groups $\tilde{H}_{k+1}(W)$ and $\tilde{H}_k(F_0)$ coincide.*

Lemma 4 *Let k and l be the same as in Theorem* A. *Suppose that $k=1+ls$ for a certain integer $s>0$. Then the topological sets W and Y are homeomorphic.*

Proof. Let y be a point in Y that corresponds to Γ_0 and w be a point in W that corresponds to F_0. Then one can identify $Y-y$ with $X-\Gamma_0$ and $W-w$ with $\mathbf{C}^{n-1}-F_0$. Consider the mapping f: $\mathbf{C}^{n-1}-F_0\to X\to\Gamma_0$ given by the formula $x'\to(G_{\lambda(x')}(x'),x_n(x'))$, where $\lambda(x')=(q(x')/|q(x')|)^s$ and $x_n(x')=(-q(G_{\lambda(x')}(x'))^{1/k}$. The restriction of f to a fiber $\{q(x')=c\}\subset\mathbf{C}^{n-1}-F_0$ is an isomorphism between this fiber and the fiber $\{x_n=c/|c|^{(k-1)/k}\}\subset X-\Gamma_0$. Thus

f is a homeomorphism. Let $P=\{v_i\}$ be a sequence of points in $\mathbf{C}^{n-1}-F_0$ that approaches a point $v\in F_0$. Since $|\lambda(v_i)|=1$ for every i, the closure of $f(P)$ is a compact, and the limiting set for $f(P)$ belongs to Γ_0. Hence the homeomorphism f between $Y-y$ and $W-w$ can be extended to a homeomorphism between W and Y. □

The projection $\rho: x\to x'$ generates the mapping $\tau: Y\to W$.

Lemma 5 *Under the assumptions of Lemma* 4 *the mapping* $\tau: Y\to W$ *generates an isomorphism* $\tau_*: H_*(Y)\to H_*(W)$.

Proof. Since the mapping τ is k-sheeted, for every $\sigma\in H_*(W)$ the element $k\sigma\in\tau_*(H_*(Y))$. Recall that under the assumptions of Theorem A the set Y is $\mathbf{Z}_s$-acyclic for every prime divisor s of k, i.e. the order of σ and k are relatively prime. Hence there exists $\sigma'\in H_*(W)$ so that $\sigma=k\sigma'$ and, therefore, τ_* is an epimorphism. By Lemma 4 the groups $H_*(Y)$ and $H_*(W)$ are isomorphic. Thus τ_* is an isomorphism. □

Lemma 6 *Under the assumptions of Lemma* 4 $\tilde{H}_\ell(X)=0$ *for every* $\ell\geqq 1$.

Proof. The mapping τ generates the following commutative diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & \tilde{H}_\ell(\Gamma_0) & \longrightarrow & \tilde{H}_\ell(X) & \longrightarrow & \tilde{H}_\ell(Y) & \xrightarrow{j_\ell} & \tilde{H}_{\ell-1}(\Gamma_0) & \longrightarrow & \tilde{H}_{\ell-1}(X) & \longrightarrow \\
 & \downarrow{\scriptstyle \varphi_\ell} & & \downarrow{\scriptstyle \psi_\ell} & & \downarrow{\scriptstyle \tau_\ell} & & \downarrow{\scriptstyle \varphi_{\ell-1}} & & \downarrow{\scriptstyle \psi_{\ell-1}} & \\
\longrightarrow & \tilde{H}_\ell(F_0) & \longrightarrow & \tilde{H}_\ell(\mathbf{C}^{n-1}) & \longrightarrow & \tilde{H}_\ell(W) & \xrightarrow{i_\ell} & \tilde{H}_{\ell-1}(F_0) & \longrightarrow & \tilde{H}_{\ell-1}(\mathbf{C}^{n-1}) & \longrightarrow
\end{array}$$

By construction, $\varphi_{\ell-1}$ is an isomorphism and τ_ℓ is an isomorphism by Lemma 5. Since $\tilde{H}_*(\mathbf{C}^{n-1})=0$, the mapping i_ℓ is an isomorphism. Hence j_ℓ is an isomorphism and $\tilde{H}_\ell(X)=0$. □

Now we can prove Theorem A. Since l and k are relatively prime, there exist natural s and t so that $1+ls=kt$. These numbers s and t are not unique. We can consider $s'=s+mk$ and $t'=t+ml$ instead of s and t for every integer $m>0$. Since t and l are relatively prime, one can choose t' so that it is relatively prime with each element of a finite set of natural numbers D. The algebraic variety F_0 is $\mathbf{Z}_k$-acyclic, and thus all senior homology groups of F_0 are finite. Let D consist of the orders of all these groups and the number k. Put $k'=kt'$. Since t' is relatively prime with each element of D, the set F_0 is $\mathbf{Z}_{k'}$-acyclic. Consider the hypersurface $X'=\{q(x')+x_n^{k'}=0\}$. A combination of Lemmas 1, 2, and 6 shows that X' is diffeomorphic to $\mathbf{R}^{2n-2}$. Consider the natural t'-sheeted mapping from X' to X. Since X' is contractible, the order of every group $H_i(X)$ for $i>0$ must be a divisor of t'. Now choose another $t''=t+m'l$ which is relatively prime with every element of D and t'. Same argument shows that the order of every group $H_i(X)$ $(i>0)$ must be a divisor of t''. Hence all these groups are trivial and Lemmas 1 and 2 imply the desired conclusion.

3 Polynomials satisfying the assumptions of Theorem A

The next three lemmas show how to construct a nontrivial polynomial satisfying the assumption of Theorem A.

Lemma 7 [De, F] *Let q, F_0 be the same as in Theorem* A. *Suppose that $\mathbf{C}^{n-1} \hookrightarrow \mathbf{CP}^{n-1}$ is a natural embedding, $E = \mathbf{CP}^{n-1} - \mathbf{C}^{n-1}$, and $\bar{F}_0$ is the closure of F_0 in $\mathbf{CP}^{n-1}$. If F_0 is irreducible and $\bar{F}_0$ meets E normally outside an algebraic subvariety of codimension* 3, *then $\pi_1(\mathbf{C}^{n-1} - F_0) = \mathbf{Z}$.*

In other words Lemma 7 says that for every polynomial p in general position $\pi_1(\mathbf{C}^{n-1} - \{p(x)=0\}) = \mathbf{Z}$. In some case this fundamental group may be $\mathbf{Z}$ although p is not in general position. Let g be a polynomial and $\Gamma_c = \{x \in \mathbf{C}^n \mid g(x) = c\}$, where $c \in \mathbf{C}$. Recall that a fiber Γ_c is generic iff there is a neighborhood U of c in $\mathbf{C}$ such that the mapping g: $g^{-1}(U) \to U$ is a fibration. There is a finite set $S \subset \mathbf{C}$, for which the fiber Γ_c is generic when $c \notin S$. The polynomial g is called primitive if its generic fibers are connected.

Lemma 8 *Let Γ_c be a generic fiber of a primitive polynomial g. Then the fundamental group of $\mathbf{C}^n - \Gamma_c$ is $\mathbf{Z}$.*

Proof. Consider two connected simply connected domains V, $U \subset \mathbf{C}$ so that $c \in U$, $S \subset V$, the sets $W = U \cap V$ and $T = U \cup V$ are connected simply connected, and $S \cap W = \emptyset$. Since V is homotopy equivalent to $\mathbf{C}$ and every fiber Γ_s is generic as $s \notin V$, the set $V' = g^{-1}(V)$ homotopy equivalent to $\mathbf{C}^n$. Hence $\pi_1(V') = 0$. Put $U' = g^{-1}(U - c)$. Since $g^{-1}(U)$ is diffeomorphic to the direct product of U and Γ_c, the group $\pi_1(U')$ is isomorphic to $\pi_1(\Gamma_c) \oplus \pi_1(U - c) = \pi_1(\Gamma_c) \oplus \mathbf{Z}$. Note that $V' \cap U' = g^{-1}(W)$. Taking into consideration the description of W, one can see that $\pi_1(V' \cap U') = \pi_1(\Gamma_c)$. Put $T^* = T - c$. Since T^* is homotopy equivalent to $\mathbf{C} - c$ and every fiber Γ_s is generic as $s \notin T$, the set $T' = g^{-1}(T^*)$ is homotopy equivalent to $\mathbf{C}^n - \Gamma_c$. Thus $\pi_1(T') = \pi_1(\mathbf{C}^n - \Gamma_c)$. The set $T' = V' \cup U'$. By the van Kampen-Seifert theorem $\pi_1(T') = \mathbf{Z}$ [FV, p. 109]. □

Put $X_a = \{p(x) = a\}$, where $p = q(x') + x_n^k$ and q satisfies the assumptions of Theorem A. Recall that the fiber F_0 is the only degenerate fiber of the family $\{F_c = \{q(x') = c\}\}$. Denote the generic fiber of this family by F.

Definition. We shall say that the contractible hypersurface X_0 is *nontrivial* if for some $a \neq 0$ the hypersurface X_a are not contractible.

Lemma 9 *If F is not contractible and $k \neq 1$, then for every $a \neq 0$ the hypersurface X_a is not contractible.*

Proof. (We are indebted the referee for suggesting us this natural and simple proof). By the global Sebastiani-Thom theorem for polynomial maps [Ne], X_a is homotopy equivalent to the joint space of F and $\mathbf{Z}_k$. Hence X_a is not contractible whenever F is not contractible and $k > 1$. □

Theorem 10 *Let a polynomial $q(x, y, z) = x + \alpha y^m x^k + \beta y^l z^s$ ($\alpha, \beta \in \mathbf{C}^*$) and let $p(x, y, z, t) = q(x, y, z) + t^n$, where $m \geq 1$, $k \geq 2$, $l \geq 0$, s, $n \geq 1$ are integers. Suppose that the pairs $(l(k-1)+m, s)$ and (ms, n) are relatively prime. The hypersurface $X = \{p(x, y, z, t) = 0\} \subset \mathbf{C}^4$ is diffeomorphic to $\mathbf{R}^6$. For k, $s \geq 2$ this hypersurface is nontrivial.*

Proof. Without loss of generality put $q(x, y, z) = x - y^m x^k + y^l z^s$. It is easy to check that $F_0 = \{q(x, y, z) = 0\}$ is smooth, but it does not meet the hyperplane at infinity normally. Nevertheless we shall show that the fundamental group of the manifold $V = \mathbf{C}^3 - F_0$ is $\mathbf{Z}$. Fix $a \neq 0$ and put $M_a = F_0 \cap \{y = a\}$. The equation of F_0 implies that $z = (a^{m-l} x(x^{k-1} - a^{-m}))^{1/s}$ on M_a. Let $\omega(a)$ be the set of $(k-1)$-roots of a^{-m}. The mapping x: $M_a \to \mathbf{C}$ has k branch points $x = 0$ and

$x \in \omega(a)$ of order s. Put $g_a(x, z) = a^l z^s + x - a^m x^k$. Consider a fiber $S_c = \{g_a(x, z) = c\}$ of the polynomial g_a. The branch points of the mapping $x\colon S_c \to \mathbf{C}$ coincide with the roots of the polynomial $h_c(x) = x - a^m x^k - c$. The polynomials h_c have multiple roots only for a finite number of c's. If there is no multiple root the mapping $x\colon S_c \to \mathbf{C}$ has k different branch points. Consider a diffeomorphism f_c of $\mathbf{C}$ that depends continuously on the parameter c from a neighborhood of zero and for which $f_c(\{0, \omega(a)\})$ coincides with the set of roots of h_c. It generates the diffeomorphism $\phi_c\colon M_a \to S_c$ so that $x \circ \phi_c = f_c \circ x$, and this diffeomorphism also depends on c continuously. In particular, $M_a = S_0$ is a generic fiber of g_a. Put $L_a = \mathbf{C}^2 - M_a$ and $V^* = V - \{y = 0\}$. Note that the polynomial g_a is primitive. Hence the fundamental group of L_a is $\mathbf{Z}$ by Lemma 8. It is easy to check that the mapping $y\colon V^* \to \mathbf{C}^*$ is a fibration and L_a is its generic fiber. From the exact homotopy sequence

$$0 \to \pi_1(L_a) \to \pi_1(V^*) \to \pi_1(\mathbf{C}^*) \to 0$$

we see that $\pi_1(V^*)$ is generated by two elements γ and η so that $y_*(\eta)$ is a generator of $\pi_1(\mathbf{C}^*)$. Since the loop η is contractible in V, we have $\pi_1(V) = \mathbf{Z}$.

It is easy to check that the polynomial q is quasi-invariant under the effective $\mathbf{C}^*$-action $G\colon (x, y, z) \to (\lambda^{ms} x, \lambda^{-(k-1)s} y, \lambda^{m+l(k-1)} z$ and its weight is ms. Thus it is enough to prove that F_0 is $\mathbf{Z}_n$-acyclic. For the rest of the proof we shall follow [Ru]. Consider the smooth surface $F' = \{x - x^k y^m + y^l z = 0\} \subset \mathbf{C}^3$. Since every fiber of the function y on F' is isomorphic to $\mathbf{C}$, the surface F' is isomorphic to $\mathbf{C}^2$. Let $\tau = \omega(s)$ acts on F_0 by $(x, y, z) \to (x, y, \lambda^{m+l(k-1)} z)$ with $\lambda \in \omega(s)$. Then $F_0/\tau = F'$ and τ can be viewed as a subgroup of $\mathbf{C}^*$ acting on F_0 via G. The mapping $F_0 \to F'$ generates a mapping $H_i(F_0, \mathbf{Z})^\tau \to H_i(F', \mathbf{Z})$ such that its kernel and cokernel are $\mathbf{Z}_s$-torsion groups [B]. Since G (and hence τ) acts on $H_i(F_0, \mathbf{Z})$ trivially, we have $H_i(F_0, \mathbf{Z})^\tau = H_i(F_0, \mathbf{Z})$, i.e. $H_i(F_0, \mathbf{Z})$ is a $\mathbf{Z}_s$-torsion group for $i > 0$. This means that F_0 is $\mathbf{Z}_n$-acyclic. Application of the Theorem A shows that X is diffeomorphic to $\mathbf{R}^6$. Using Lemma 9 and Suzuki's formula for Euler characteristics [Su] it is easy to check that for k, $s \geqq 2$ nonzero fibers of p are not contractible, i.e., the hypersurface X is nontrivial in this case. □

Remark. Initially the list of these examples was shorter. In particular, the author missed the simplest example $\{x + x^2 y + z^3 + t^2 = 0\}$. This hypersurface appeared in [Ru], where one can find a remarkable construction of contractible hypersurfaces in $\mathbf{C}^4$ (though this construction satisfies Theorem A it is more explicit).

4 Exotic structures

We believe that all non-trivial contractible hypersurfaces described in Theorem 10 are exotic algebraic structures. In this section we shall show that this is true at least for some of them.

Theorem B *Let* $p(x, y, z, t) = x + x^k y^m + y^l z^{k-1} + t^n$, *where* $k \geqq 4$, $(k-1, m) = 1$, $(m(k-1), n) = 1$, $n > k$, *and* $m = m' r$, $l = l' r$ *with* $r > s := (k-1)^2 n$. *Then the hypersurface* $X = \{p(x, y, z, t) = 0\} \subset \mathbf{C}^4$ *is an exotic algebraic structure on* $\mathbf{R}^6$.

We need some facts about Kodaira logarithmic dimension. First recall some notations and terminology from [I] and [Sa]. Let $\bar{V}$ be a completion of a

smooth algebraic variety V for which the divisor $D=\bar{V}-V$ is of simple normal crossing type. Then the Kodaira logarithmic dimension of V is $\bar{k}V=k(K+D, \bar{V})$, where K is a canonical divisor on $\bar{V}$ and $k(B, \bar{V})=\overline{\lim_{m\to\infty}} \log(\dim H^0(mB, \bar{V}))/\log m$ for every divisor B in $\bar{V}$. Sakai introduced another kind of Kodaira dimension of V that is $k_c V=\lim_{m\to\infty} \log(\dim H^0(mK+(m-1)D, \bar{V}))/\log m$. When $k_c V\geqq 0$, then $k_c V=\bar{k}V$. Recall that V is a variety of hyperbolic type if $\bar{k}V=\dim V$, and in this case $k_c V=\bar{k}V$ [Sa]. We shall also need Formula (R) from [I]. Let $\bar{W}$ be a completion of a smooth algebraic variety W for which the divisor $C=\bar{W}-W$ is of simple normal crossing type, and let K' be a canonical divisor on $\bar{W}$. Suppose that $\dim W=\dim V$ and that there is a dominant morphism $f: W\to V$ so that it can be extended to a morphism $\bar{f}: \bar{W}\to\bar{V}$ with $\bar{f}^{-1}(D)\subset C$. Then Formula (R) says that

$$K'+C\sim\bar{f}^*(K+D)+R_f$$

where R_f is an effective divisor which is called the logarithmic ramification divisor of f.

Recall that a divisor $E\subset\bar{W}$ is exceptional for $\bar{f}$ if $\operatorname{codim}\bar{f}(E)\geqq 2$. Note that R_f in Formula (R) is exceptional for $\bar{f}$ when f is étale. We shall need the following theorem of Iitaka [I, Lemma 1]

$$k(\bar{f}^*B+E, \bar{W})=k(B, \bar{V})$$

for every divisor B on $\bar{V}$ and effective divisor E that is exceptional for $\bar{f}$.

Let F be a smooth irreducible divisor on V. Suppose that $f^*F=\sum s_i F_i$, where F_i are irreducible. We say that the multiplicity of f on $f^{-1}(F)$ is at least $s=\min_i(s_i)$. If $s_i=s$ for each i then we say that f has pure multiplicity s on $f^{-1}(F)$.

Lemma 11 *Let V be a smooth algebraic variety and let $F_1, \ldots, F_k$ be smooth irreducible divisors such that $F=F_1+\ldots+F_k$ is of simple normal crossing type. Suppose that $f: W\to V$ is a dominant morphism of equidimensional smooth algebraic varieties such that it is étale above $V-F$ and its multiplicity on $f^{-1}(F_i)$ is at least s_i for each i. Then*

$$\bar{k}(W)\geqq\bar{k}\left(\left(K+D+\sum_i\left(1-\frac{1}{s_i}\right)\bar{F}_i\right), \bar{V}\right)$$

where K is a canonical divisor on a completion $\bar{V}=V\cup D$ of V such that the divisor $B=D+\bar{F}$ has normal simple crossing, and $\bar{F}_i$ is the closure of F_i in $\bar{V}$. Moreover, if f has pure multiplicity s_i for each i (in this case one can easily see that f is finite), then this inequality becomes an equality.

Proof. One may suppose that $F'=f^{-1}(F)$ is of simple normal crossing type (since blow-ups of $\bar{W}$ obviously do not reduce the multiplicities of f on $f^{-1}(F)$). We can find a completion $\bar{W}=W\cup D'$ of W such that the divisor $B'=\bar{F}'\cup D'$

has normal simple crossing (where $\bar{F}'$ is the closure of F' in $\bar{W}$) and there is a commutative diagram

$$\begin{array}{ccc} \bar{W} & \supset & W \\ \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ \bar{V} & \supset & V. \end{array}$$

By Formula (R)

$$K'+B' \sim \bar{f}^*(K+B)+R_f$$

where R_f is exceptional for $\bar{f}$ and effective, and K' is a canonical divisor for $\bar{W}$. Suppose $\bar{f}^*(\bar{F}_i)=\sum n_{ij}F'_{ij}$ with F'_{ij} irreducible. Put $a_{ij}=1$, $s_{ij}=n_{ij}$ if $F'_{ij}\cap W\neq\emptyset$ and $a_{ij}=0$, $s_{ij}=\infty$ otherwise. Note that $\bar{F}'\sim\sum_j a_{ij}F'_{ij}$ and $s_i=\min_j(s_{ij})$.

Hence

$$K'+D' \sim \bar{f}^*(K+B)+R_f-\bar{F}' \sim f^*(K+B)+R_f+S-f^*\left(\sum_i \frac{1}{s_i}\bar{F}_i\right),$$

where $S=\sum\left(\frac{n_{ij}}{s_i}-a_{ij}\right)F'_{ij}$, i.e. S is effective.

Thus $K'+D' \sim f^*\left(K+D+\sum_i\left(1-\frac{1}{s_i}\right)\bar{F}_i\right)+S+R_f$. Hence, by Iitaka's theorem,

$$\begin{aligned} \bar{k}(W)&=\bar{k}(K'+D',\bar{W}) \\ &\geqq \bar{k}\left(\bar{f}^*\left(K+D+\sum_i\left(1-\frac{1}{s_i}\right)\bar{F}_i\right)+R_f,\bar{W}\right) \\ &=\bar{k}\left(K+D+\sum_i\left(1-\frac{1}{s_i}\right)\bar{F}_i,\bar{V}\right). \end{aligned}$$

If $s_{ij}=s_i$ for all i,j, then $S\sim 0$ and we have the desired equality. □

The next two corollaries and Theorem 14 are not related to the main purpose of this paper, but they are interesting by themselves (cf. [I2]).

Corollary 12 *Let V, F be the same as in Lemma* 11. *Suppose that Kodaira dimension $k_c(V-F)\geqq 0$. There exists an integer s such that if W and f are the same as in Lemma* 11 *and the multiplicity of f on each component of $f^{-1}(F)$ is greater than s, then $\bar{k}W\geqq\bar{k}(V-F)$.*

Proof. Since the complete linear system $|s(K+B)-B,\bar{V}|$ is non-empty for some s, the completer linear system $\left|(K+B)-\frac{1}{s}\bar{F},\bar{V}\right|$ is also non-empty (where B, $\bar{F}$, $\bar{V}$ are the same as in the formulation and the proof of Lemma 11). □

Corollary 13 *Let V, F be the same as in Lemma* 11. *Suppose that the variety $V-F$ is of hyperbolic type. There exists an integer s such that if W and f are the same as in Lemma* 11 *and the multiplicity of f on each component of $f^{-1}(F)$ is greater than s, then W is of hyperbolic type.*

Proof. Since $V-F$ has hyperbolic type, $\bar{k}(V-F)=k_c(V-F)$. □

Theorem 14 *Let V, F, f, W, $\bar{V}$, D, K be the same as in Lemma 11. Suppose that the divisor $-(K+D)$ is ample, f is a finite morphism, and $\bar{k}W\geqq 0$. Then $V-F$ is of hyperbolic type.*

Proof. Let $\bar{f}$, $\bar{F}_i$, $\bar{W}$, D', K', B, B', $\bar{F}'$ be the same as in the proof of Lemma 11. Since f is finite, $\bar{f}^*(\bar{F}_i)=\sum_j s_{ij}F'_{ij}+E'_i$ where E'_i is effective and exceptional for $\bar{f}$, $E'_i\cap W=\varnothing$, F'_{ij} are irreducible and $F'_{ij}\cap W\neq\emptyset$. Put $t_i=\max_j(s_{ij})$. Then

$$\begin{aligned}K'+D'&\sim\bar{f}^*(K+B)+R_f-\bar{F}'\\&\sim\bar{f}^*(K+B)+R_f+\sum_{i,j}\left(\frac{s_{ij}}{t_i}-1\right)F'_{ij}+\sum_i\frac{1}{t_i}E'_i\\&\quad-\bar{f}^*\left(\sum_i\frac{1}{t_i}F_i\right)\sim f^*\left(K+\sum_i\left(1-\frac{1}{t_i}\right)F_i\right)+E-S,\end{aligned}$$

where R_f is the ramification divisor for the mapping $f|_{W-F'}$, S is effective, and E is effective and exceptional for $\bar{f}$. Hence, by Iitaka's theorem,

$$\begin{aligned}&\bar{k}\left(K+\sum_i\left(1-\frac{1}{t_i}\right)F_i, V\right)\\&\quad=\bar{k}\left(f^*\left(K+\sum_i\left(1-\frac{1}{t_i}\right)F_i\right)+E, W\right)\geqq\bar{k}W\geqq 0.\end{aligned}$$

Put $t=\max_i t_i$. Note that the complete linear system $\left(\left(1-\frac{1}{t}\right)(F+K), V\right)$ contains a divisor $-\frac{1}{t}K+G$, where G is effective. Since $-K$ is ample and $t<\infty$, we have the desired conclusion. □

Lemma 15 *Let V, F, W, F_i, f be the same as in Lemma 11. Suppose that f has pure multiplicity s_i on $f^{-1}(F_i)$ for each i. Let $g\colon U\to V$ be another dominant morphism of smooth equidimensional algebraic varieties étale above $V-F$ and such that its multiplicity on $g^{-1}(F_i)$ is at least $r_i\geqq s_i$ for each i. Then $\bar{k}W\leqq\bar{k}U$.*

Proof. By Lemma 11,

$$\bar{k}U\geqq\bar{k}\left(K+D+\sum\left(1-\frac{1}{r_i}\right)F_i, \bar{V}\right)\geqq k\left(K+D+\sum\left(1-\frac{1}{s_i}\right)F_i, \bar{V}\right)=\bar{k}W$$

(where $\bar{V}$, K, and D are the same as in Lemma 11). □

Lemma 16 *Let $F'\subset\mathbf{C}^2\subset\mathbf{CP}^2$ be a plane affine curve of degree n for which its closure F in $\mathbf{CP}^2$ is smooth.*

(i) *If $n\geqq 3$, then the variety $\mathbf{C}^2-F$ is of hyperbolic type.*
(ii) *If $n\geqq 4$ and $\bar{V}$ is a smooth completion of $\mathbf{C}^2$ for which the divisor $D=(\bar{V}-\mathbf{C}^2)\cup F^*$ has normal simple type (where F^* is a closure of F in $\bar{V}$), then the image of $\bar{V}$ under the mapping associated with the complete linear system $|K+D, \bar{V}|$*

has dimension 2. *Moreover, the complete linear system* $|n(K+D)-F^*, \bar{V}|$ *is not empty.*

Proof. Let $E=\mathbf{CP}^2-\mathbf{C}^2$. For the sake of simplicity we shall consider the case when $E\cap F$ is one point a. If E meets F non-normally, then in order to get $\bar{V}$ we have to blow up $\mathbf{CP}^2$ at a and infinitely near points. Then the inverse image of F is $F^*+\sum_{m=1}^{l} mE_m$, and the inverse image E^* of E is $E^*+\sum_{m=1}^{l} mE_m$, where F^* and E^* are the proper transforms of F and E respectively and the curves $\{E_m\}$ belong to the inverse image of a. Let H be a line in $\mathbf{CP}^2$ that does not contain a and let H^* be its proper transform in $\bar{V}$. Then $H^*\sim E^*+\sum_{m=1}^{l} mE_m$, and $K=-3H^*+\sum_{m=1}^{l} mE_m$ is a canonical divisor. Note that $D=E^*+F^*+\sum E_m$ and $nH^*\sim F^*+\sum_{m=1}^{l} mE_m$. Hence $D+K=(n-3)H^*+\sum_{m=1}^{l} E_m+E^*$. Therefore, for $n=3$ the divisor $H^*+\sum_{m=1}^{l}(l-m)E_m+(l-1)E^*$ belongs to the complete linear system $|l(D+K), \bar{V}|$. One can see that for $n\geqq 4$ there also exist effective divisors J and J' for which the divisor H^*+J belongs to $|(D+K), \bar{V}|$ and the divisor F^*+J' belongs to $|n(D+K), \bar{V}|$. This implies (i) and (ii). □

Now we can prove Theorem B. Consider the curve F' in $\mathbf{C}^2$ given by $x+x^k+z^{k-1}=0$. Suppose that $\bar{W}$ is a smooth completion of $\mathbf{C}^2$ for which the divisor $D=(\bar{W}-\mathbf{C}^2)\cup\bar{F}'$ has simple normal crossing type. Then $\mathbf{C}^2-F'$ is of hyperbolic type, the image of $\mathbf{C}^2$ under the rational mapping associated with $|K+D, \bar{W}|$ is 2, and the complete linear system $|k(K+D)-\bar{F}', \bar{W}|$ is not empty due to Lemma 16. By Lemma 11, the surface $S=\{x+x^k+z^{k-1}+t^n=0\}$ is of hyperbolic type when $n>k$. Consider the threefold $V=\{x+x^k y^{m'}+y^{l'} z^{k-1}+t^n=0\}$. Then y is a quasi-invariant of weight $-s$ under the natural $\mathbf{C}^*$-action on V. Consider the threefold $V_s=\{x+y^{m's}x^k+y^{l's}z^{k-1}+t^n=0\}$. The mapping $V_s\to V$ given by $(x, y, z, t)\to(x, y^s, z, t)$ is a s-sheeted mapping with the ramification over the smooth surface $\{y=0\}\cap V$. There exists a natural $\mathbf{C}^*$-action G on V_s and the weight of y under G is -1. Hence $V_s/G=V_s\cap\{y=1\}=S$. Application of Kawamata's addition theorem [Ka] to the quotient morphism $V_s\to S$ shows that $\bar{k}V_s\geqq\bar{k}S=2$. The equation of V_s does not satisfy the assumptions of Theorem 10, since s and n are not relatively prime, but one can consider $V_r=\{x+x^k y^{m'r}+y^{l'r}z^{k-1}+t^n=0\}$ with $r>s$ for which $(r, s)=1$ and then these assumptions hold for V_r. Thus V_r is diffeomorphic to $\mathbf{R}^6$. By Lemma 15, $\bar{k}V_r\geqq 2$ (in fact it is exactly 2 due to Lemma 2 of [I]), i.e., V_r is an exotic structure.

Note that these examples of exotic structures are different from Zaidenberg's examples in dimension 3, since for the last ones their Kodaira logarithmic dimension is always $-\infty$.

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Conformally non-spherical 2-complexes

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1 Introduction

Suppose a 2-complex is constructed by gluing together polygons along their edges. Then the links of vertices become graphs in a natural way where the edges correspond to the corners of the 2-cells. If the corners of each polygon are assigned "angles" (i.e., real numbers that are not necessarily positive), then every path in the link of a vertex obtains a weight. A 2-complex of this type is said to be (conformally) non-spherical if each 2-cell can be assigned angles whose sum is not more than the angle sum of a geodesic polygon in the euclidean plane with the same number of sides, and with these angles, no reduced closed path in the link of a vertex has weight less than 2π; see Sect. 3 for a more careful definition.

Piecewise Euclidean (PE) 2-complexes introduced by Gersten and Stallings [6] are a special type of non-spherical 2-complexes where each 2-cell can be given the geometry of a geodesic polygon in the euclidean plane such that whenever a pair of 2-cells has an edge in common, then the two lengths of the edge determined by each 2-cell agree, and the above link condition holds for these angle assignments. In this case one obtains a metric by taking shortest paths and this metric turns out to be non-positively curved, provided there are only finitely many types of 2-cells. It then follows from the Bruhat-Tits Fixed Point Theorem that any finite group acting on a simply connected PE 2-complex of this type has a fixed point, see [6].

In general for a non-spherical 2-complex, we do not assume that the 2-cells can be given a euclidean geometry producing the angles such that the geometries agree on common edges, in fact angles are allowed to be non-positive. Hence there is no metric and one must rely on combinatorial notions of curvature.

This work is concerned with generalizing the above result of Gersten and Stallings to non-spherical 2-complexes in general. As a step toward this we prove that for a group acting on a simply connected, non-spherical 2-complex, each element of finite order has a fixed point. We do not require any assumption on the number of types of 2-cells.

We begin with some preliminaries in Sect. 2 about conformal structures on 2-complexes. Then in Sect. 3 an analogue of Lyndon's Curvature Formula is

proved for non-spherical diagrams. The main result mentioned above is then proved in Sect. 4. And in Sect. 5, (conformally) hyperbolic 2-complexes are discussed and it is shown that the finite ones have fundamental groups that are word hyperbolic in the sense of Gromov.

Definitions. As in [3], a continuous map $f: X \to Y$ between CW-complexes shall be called *combinatorial* if it maps each open cell of X homeomorphically onto an open cell of Y. And a 2-complex X is called *combinatorial* if each 2-cell σ of X is attached to the 1-skeleton along a non-trivial edge path (i.e., the attaching map of σ is a combinatorial map $S^1 \to X$ for some subdivision of S^1).

Let X be a combinatorial 2-complex. The *degree* of the 2-cell σ of X, denoted $d(\sigma)$, is defined to be the number of edges in the attaching loop of σ. Note that $d(\sigma)$ counts the edges in $\partial\sigma$ with multiplicity. The *degree* (or *valence*) of a vertex v, denoted $d(v)$, is defined to be the number (possibly infinite) of directed edges e with initial vertex $\iota(e)=v$. Note that an undirected edge that is a loop at a vertex v, contributes 2 to $d(v)$.

We also make use of the notion of a corner of X; see [3, Sect. 2]. Basically, a *corner* is a small neighborhood in a 2-cell about a vertex. Then we view the link of a vertex v in X as a graph $L_v = L(v, X)$ whose vertices are identified with the directed edges e of X such that $\iota(e)=v$ and whose edges are identified with the corners of X at v.

2 Conformal structures on 2-complexes

Let X be a combinatorial 2-complex. A *conformal structure* (or *weight*) on X is a real function that assigns an angle (i.e., real number) to each unoriented corner of X. Equivalently a conformal structure assigns weights to the edges of the link graphs L_v, for each vertex v of X.

Given a conformal structure m on X and an edge path γ in some link L_v, we let $m(\gamma)$ denote the sum of the weights on the edges of γ. And for a 2-cell σ of X, we write $m(\sigma)$ for the sum of the angles of σ.

Diagrams

Let X be a combinatorial 2-complex. A *diagram* in X shall mean a combinatorial map $f: D \to X$ where D is a finite 2-complex embedded in an orientable surface. In the case D is the entire 2-sphere, we call it a *spherical diagram.* If D is properly contained in S^2, we call it a *planar diagram.* A *van Kampen diagram* is a connected, simply connected, planar diagram. Note that a connected, simply connected diagram is either a spherical diagram or a van Kampen diagram.

A diagram $f: D \to X$ is said to be *reduced* if for each vertex v of D, the induced map on the links $L(v, D) \to L(f(v), X)$ is a local embedding, i.e., reduced paths in $L(v, D)$ map to reduced paths in $L(f(v), X)$. This is easily seen to be equivalent to the notion of reduced given in [5] (provided no two distinct 2-cells

are attached along the same path and no 2-cell is attached along a path which is a proper power).

By the *boundary* of D, denoted ∂D, we mean the topological boundary (i.e., the frontier) of D in the containing surface.

Curvature and area

Let m be a fixed conformal structure on X. Given a diagram $f\colon D \to X$, put the pullback conformal structure $\mu = f^*(m)$ on D. Define the *curvature* of ∂D as

$$K(\partial D) = \sum_{\partial D} [\pi - \mu(v)]$$

summing over the vertices in ∂D where $\mu(v)$ denotes the sum of the weights on the corners at the vertex v. If σ is a 2-cell of D, define the *curvature* of σ by $K(\partial\sigma) = \sum[\pi - \mu(\alpha)]$, summing over the corners of σ. Thus $K(\partial\sigma) = d(\sigma)\pi - \mu(\sigma)$. Then analogous to the Gauss-Bonnet formula for polygons in the hyperbolic plane, define the *area* of σ by $A(\sigma) = K(\partial\sigma) - 2\pi$. Finally the area of D is obtained by summing over the 2-cells of D:

$$A(D) = \sum_{D} A(\sigma).$$

3 Non-spherical 2-complexes

For convenience, a combinatorial 2-complex X shall be called (*conformally*) *non-spherical* if it admits a conformal structure m satisfying the two Axioms:

(A1) $m(\gamma) \geqq 2\pi$ for each non-trivial reduced loop γ in L_v and vertex v of X.
(A2) $m(\sigma) \leqq (d(\sigma) - 2)\pi$ for each 2-cell σ of X.

By scaling all the angles one sees that the number π in conditions (A1) and (A2) is irrelevant and can be taken to be any positive constant. Indeed π is taken to be 1 in [3]. We include π in the conditions to make them look more intuitive. In particular, if π is regarded as having its usual value, then Axiom (A2) says that the sum of the angles of σ is no larger than it would be if σ were a geodesic polygon in the euclidean plane.

Let X be a non-spherical 2-complex and fix a conformal structure m satisfying conditions (A1) and (A2). Then given a diagram $f\colon D \to X$ in X, let $\mu = f^*(m)$ be the pullback conformal structure on D. Observe that μ satisfies condition (A2). Thus the area $A(\sigma) \geqq 0$ for each 2-cell of D, and hence, $A(D) \geqq 0$. Furthermore, if f is a reduced diagram, then a reduced path in the link of a vertex of D is mapped to a reduced path in the link of some vertex of X. Hence in this case μ also satisfies (A1) so it is a non-spherical conformal structure on D.

Notice that the links of the interior vertices of D are circles as these are manifold points. And the links of the boundary vertices contain no reduced closed curves. It follows that μ is non-spherical if and only if it satisfies:

(i) $\mu(v) \geqq 2\pi$ for each interior vertex v of D, and
(ii) $\mu(\sigma) \leqq (d(\sigma)-2)\pi$ for each 2-cell σ of D.

We now prove a combinatorial version of the Gauss-Bonnet formula.

Curvature Lemma. *Suppose μ is a non-spherical conformal structure on D and each component of D contains at least one edge. Then $K(\partial D) \geqq 2\pi\chi(D)$ where $\chi(D)$ is the Euler characteristic of D.*

Area Lemma. *With the same hypothesis, $A(D) \leqq K(\partial D) - 2\pi\chi(D)$.*

Proof. As noted above, $A(D) \geqq 0$. Hence the Curvature Lemma follows directly from the Area Lemma. Moreover, since the area, curvature and Euler characteristic are all additive on components, it suffices to only consider the case when D is connected.

So assume D is connected and let V, E, and F denote the number of vertices, edges, and 2-cells of D respectively, and let V_∞ and E_∞ be the number of boundary vertices and boundary edges respectively. By definition,

$$(\dagger) \qquad A(D) = \sum [K(\partial\sigma) - 2\pi] = \sum [d(\sigma)\pi - \mu(\sigma)] - 2\pi F$$

where the sums are over all 2-cells of D. But

$$\begin{aligned} \sum_D \mu(\sigma) = \sum_D \mu(v) &= \sum_{\text{int}(D)} \mu(v) + \sum_{\partial D} \mu(v) \\ &\geqq 2\pi(V - V_\infty) + \sum_{\partial D} \mu(v) \\ &= 2\pi V - \pi V_\infty - K(\partial D). \end{aligned}$$

And

$$\sum d(\sigma)\pi = \pi(2E - E_\infty - E_0)$$

where E_0 is the number of edges of D with "both sides" on the boundary. Substituting into (†), we get

$$\begin{aligned} A(D) &\leqq K(\partial D) - 2\pi(V - E + F) + \pi(V_\infty - E_\infty) - \pi E_0 \\ &= K(\partial D) - 2\pi\chi(D) + \pi\chi(\partial D) - \pi E_0. \end{aligned}$$

Now if no component of ∂D is a tree, then $\chi(\partial D) \leqq 0$ and the area formula follows. In case some boundary component of D is a tree, it must be that D itself is a tree (as D is connected and contained in a surface). Thus D has no 2-cells, so $A(D) = 0$ and $K(\partial D) - 2\pi\chi(D) = \pi V - 2\pi \geqq 0$ by the hypothesis that D contains an edge. □

Gersten's Lemma [3, Proposition 4.4] *No combinatorial 2-sphere admits a conformal structure satisfying conditions* (A1) *and* (A2).

Proof. If D is a non-spherical, combinatorial 2-sphere, then $\partial D = \emptyset$, so $K(\partial D) = 0$. But $\chi(D) = 2$ and the Curvature Lemma is violated. □

Remark. It follows from Gersten's Lemma that non-spherical 2-complexes are aspherical, that is, their universal covers are contractible; see [3].

Example. Small cancellation 2-complexes. Let X be a combinatorial 2-complex. A non-trivial edge path b in the 1-skeleton of X is called a *piece* if there exists 2-cells σ_1 and σ_2 in X such that b occurs as a subpath of a boundary cycle of σ_1 and σ_2. We allow σ_1 to equal σ_2, but in this case we require b to have two distinct occurrences in the boundary.

Consider the following conditions for positive integers p and q.

Condition C(p). No 2-cell of X has a boundary cycle which is a product of fewer than p pieces.

Condition T(q). No simple closed edge path in the link of a vertex of X has length ℓ such that $2<\ell<q$.

We call X a *small cancellation 2-complex* if it satisfies C(p) and T(q) for some positive integers satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. The only possibilities for (p, q) are (6,6), (4,4) or (3,6). Note that T(3) is no condition at all.

To make the connection with small cancellation groups, we simply mention that if a group has a presentation satisfying the small cancellation conditions C(p) and T(q) as stated in [5, Chap. V] where (p, q) is one of the above three pairs, then the Cayley 2-complex associated to this presentation satisfies the above conditions C(p) and T(q). However if a small cancellation presentation contains relators that are proper powers, then the 2-complex associated to the presentation is not itself a small cancellation 2-complex in the above sense. (Recall that the Cayley 2-complex is a sort of substitute for the universal cover of a presentation; the group acts on the Cayley 2-complex and the only non-trivial isotropy groups that occur are finite cyclic groups at the centers of the 2-cells corresponding to relators that are proper powers, see [5, Chap. III].)

Small cancellation 2-complexes are not necessarily non-spherical but they behave the same. To be precise, we show that a small cancellation 2-complex X admits a conformal structure m such that for each reduced diagram $f: D \to X$, the pull-back conformal structure $\mu = f^*(m)$ on D satisfies (i) and (ii) after interior vertices of degree 2 are removed. Hence all the results of this section and the following sections hold for X. In particular, small cancellation 2-complexes are aspherical.

The desired conformal structure m on X is defined as follows. Let α be a corner, say of the 2-cell σ. If both the edges in the boundary of σ that form the corner α are pieces, then define $m(\alpha)=\frac{2\pi}{q}$. Otherwise put $m(\alpha)=0$.

Now let $f: D \to X$ be a reduced diagram in X and give D the pull-back conformal structure μ. Remove interior vertices of degree 2 in D by uniting the two adjacent edges into one. (Notice that when an interior vertex of degree 2 is removed, two corners are also removed.) Now if v is an interior vertex of D, then each corner α at v is of the type where $\mu(\alpha)=\frac{2\pi}{q}$ and it follows by condition T(q) that $d(v) \geqq q$. Hence $\mu(\gamma) \geqq 2\pi$ and (i) holds.

To verify (ii), let σ be a 2-cell of D. If some edge in the boundary of σ is not mapped to a piece of X, then at least two corners are assigned the

angle 0. Hence $\mu(\sigma) \leqq (d(\sigma)-2)\frac{2\pi}{q} \leqq (d(\sigma)-2)\pi$, as required. So suppose that all the edges in the boundary of σ are mapped to pieces of X. Then $d(\sigma) \geqq p$ by condition C(p) and

$$\begin{aligned}\mu(\sigma) &= d(\sigma)\cdot\frac{2\pi}{q}\\ &= \left[d(\sigma)-2\frac{d(\sigma)}{p}\right]\pi \quad \left[\text{as } \frac{2}{q}=1-\frac{2}{p}\right]\\ &\leqq (d(\sigma)-2)\pi \quad [\text{as } d(\sigma)\geqq p].\end{aligned}$$

Hence (ii) holds in any case.

4 Non-spherical G-2-complexes

Let G be a group. In this section, a G-2-complex shall always mean a combinatorial 2-complex X and a combinatorial action of G on X without inversions. Acting without inversions means that if a cell is fixed, it is fixed pointwise. This is an insignificant restriction since given any combinatorial action, the action on the barycentric subdivision is without inversions.

Theorem. *Let X be a non-spherical G-2-complex. If X is connected and simply connected, then each element of finite order in G fixes a vertex in X.*

Proof. We start by making two simple observations. First notice that since G acts on X without inversions, if suffices to show only that a torsion element has a fixed point. It then follows that a vertex is also fixed, namely any vertex in the boundary of the cell containing the fixed point. Second we observe that it can be assumed that X is a simplicial complex as any combinatorial 2-complex becomes a simplicial complex after barycentrically subdividing twice and the barycentric subdivision of a non-spherical 2-complex is non-spherical by [3, Proposition 6.16].

So suppose X is a simplicial complex and let g be an element of finite order in G, say of order n. Let v be a vertex in X. If g fixes v, we are done. So suppose $g\cdot v \neq v$ and choose an edge path γ in X from v to $g\cdot v$. Then $\gamma(g\gamma)(g^2\gamma)\ldots(g^{n-1}\gamma)$ is a cyclic path in X which is g-invariant. Cyclically reduce this path to get a reduced, g-invariant cyclic path ω in X.

Since X is simply connected, ω bounds a van Kampen diagram $f\colon D \to X$, i.e., D is a 1-connected simplicial 2-complex embedded in the plane such that for some boundary cycle δ of D, $f(\delta)=\omega$. For each integer k, let $g^k(\delta)$ be the boundary cycle of D such that $f(g^k\delta)=g^k(\omega)$.

Next we form a finite, simply connected 2-complex Y as follows. Take n copies $D_0, D_1, \ldots, D_{n-1}$ of D and define $f_k\colon D_k \to X$ to be the composition $g^k \circ f$. Make the cyclic group $H=\langle g\rangle$ act on the disjoint union of the disks D_k by cyclically permuting them. Let Y be the 2-complex obtained by identifying the boundaries of the D_k as follows. Glue ∂D_k to ∂D_0 by attaching the boundary cycle δ of D_k to the boundary cycle $g^k(\delta)$ of D_0. It is easy to see that the 2-complex Y is simply connected since whenever two points on the boundary

of some D_i are identified, the loop formed is homotopically trivial in the D_k that forced the identification.

Notice that the H-action of the disjoint union projects to an action on Y. Furthermore the maps f_k paste together to give a map $\bar{f}$: $Y \to X$ which is H-equivariant.

Define an *H-fold* of Y as follows. Suppose that σ_1 and σ_2 are 2-simplexes in Y that have an edge e in common such that $\bar{f}(\sigma_1)$ and $\bar{f}(\sigma_2)$ are the same 2-simplex of X. Then for each $h \in H$, identify the pair of simplexes $h(\sigma_1)$ and $h(\sigma_2)$ in Y (by folding them together along the edge $h(e)$). Notice that $\bar{f}$ projects to a map on the quotient complex, and since an entire orbit of 2-simplexes was identified with another such orbit, the H-action also projects. Moreover the quotient complex is obviously still simply connected. We say the quotient 2-complex was obtained from Y by an H-fold.

Now perform H-folds on Y until $\bar{f}$: $Y \to X$ is a reduced map, i.e., no further H-folding can be done. Rename the resulting reduced complex Y. It follows that this Y is non-spherical since a reduced path in the link $L(x, Y)$ of a vertex x in Y must be mapped to a reduced path in $L(f(x), X)$ as otherwise there would exist an H-fold. Hence Y is a simply connected, non-spherical 2-complex, so it is contractible; see the remark in Sect. 3 following Gersten's Lemma. Thus g has a fixed point y in Y by the Lefschetz Fixed Point Theorem. Since $\bar{f}$: $Y \to X$ is H-equivariant, it follows that $\bar{f}(y)$ is a fixed point of g in X. □

Corollary. *Let X be a small cancellation G-2-complex. If X is connected and simply connected, then each torsion element of G fixes a vertex in X.*

Proof. Following the same proof as given for the theorem, we produce a finite, simply connected, small cancellation 2-complex Y. Hence Y is contractible (see the example in Sect. 3) and the Lefschetz Fixed Point Theorem again gives a fixed point. □

Applications. One can now prove torsion theorems for small cancellation groups and the fundamental groups of labelled 2-complexes of groups.

For example if G is a small cancellation group over a free group, free product with amalgamation, HNN-extension, or in general over any graph of groups (Γ, F), then it can be shown that G acts on a simply connected, small cancellation 2-complex such that each vertex stabilizer is conjugate to a vertex group F_v, v a vertex of Γ, see [2]. To simplify the discussion, assume no relator is a proper power. Then the action is without inversions and it follows that any element of finite order in G is conjugate to an element of a vertex group of F.

Similarly, it is shown in [1] that the fundamental group $\pi_1(X, G, \varphi)$ of a labelled 2-complex of groups satisfying the hypothesis of [1, Theorem 5.3] acts without inversions on a simply connected, non-spherical 2-complex such that each vertex stabilizer is conjugate to a vertex group G_v, v a vertex of X. Again it follows from the Theorem that the torsion elements in the fundamental group are precisely the conjugates of the torsion elements of the vertex groups.

5 Hyperbolic 2-complexes

Hyperbolic 2-complexes were introduced by Gersten in [3] and the finite ones were shown to have solvable word and conjugacy problems. In this section

we prove that finite hyperbolic 2-complexes satisfy a linear isoperimetric inequality and hence have word hyperbolic fundamental groups in the sense of Gromov.

Recall that a 2-complex is (conformally) hyperbolic if it admits a nonnegative conformal structure satisfying the axioms:

(A1) $m(\gamma) \geqq 2\pi$ for each simple closed curve γ in L_v and vertex v.
(A2)′ $m(\sigma) < (d(\sigma) - 2)\pi$ for each 2-cell σ.

Just as for non-spherical 2-complexes, we see that a hyperbolic conformal structure on X pulls back to one on any reduced diagram $f\colon D \to X$ in X. Fixing a hyperbolic conformal structure m on X, we define the curvature and area of D using the pullback structure $\mu = f^*(m)$. Notice that condition (A2)′ implies the area of each 2-cell of D is positive and hence $A(D) > 0$.

Since hyperbolic 2-complexes are non-spherical, the curvature and area formulas hold for D. In fact, in the hyperbolic case, the same proof shows the inequalities are strict. Thus if the boundary of D is empty (i.e., D is a closed surface), then the strict curvature formula implies $\chi(D) < 0$. In particular, no combinatorial 2-sphere or 2-torus is hyperbolic.

We next show that the area formula yields a quick proof that finite hyperbolic 2-complexes have word hyperbolic fundamental groups.

Linear isoperimetric inequality. *Suppose X is a finite hyperbolic 2-complex. Then there exists a constant C (depending only on X) such that for each reduced van Kampen diagram $f\colon D \to X$ in X,*

$$(\textit{number of faces of } D) \leqq C \cdot \ell(\partial D)$$

where $\ell(\partial D)$ is the number of edges in the boundary of D counted with multiplicity (i.e., the length of a boundary cycle).

Proof. Fix a conformal structure m on X satisfying (A1) and (A2)′. It makes sense to define the area of the 2-cells of X just as we did for diagrams in Sect. 2. Since X has only a finite number of 2-cells, there is one with minimum area. Let b be this minimum area. Then given a van Kampen diagram $f\colon D \to X$, b is a lower bound for the area of the faces of D as the area of each face of D is equal to the area of its image in X. Thus $b \cdot$(number of faces of $D) \leqq \sum A(\sigma) = A(D) \leqq K(\partial D) - 2\pi\chi(D)$ by the area formula. But directly from the definition of curvature we see that $K(\partial D) \leqq \pi V_\infty \leqq \pi \ell(\partial D)$ where V_∞ is the number of boundary vertices. (Here the nonnegativity of the conformal structure was used.) Combining with the fact that $\chi(D) = 1$ for a van Kampen diagram, we get the desired inequality with $C = \pi/b$. □

Now a standard argument (see [5, p. 262]) yields a solution to the word problem for $\pi_1(X)$. And since annular diagrams have Euler characteristic zero, the proof shows that the linear isoperimetric inequality also holds for annular diagrams. It follows that $\pi_1(X)$ has a solvable conjugacy problem. But more can be said.

A finitely generated group G is called *word hyperbolic* if the metric space obtained by endowing G with the word metric determined by some (and hence any) finite generating set is a hyperbolic space in the following sense of Gromov. There exists a constant $\delta \geqq 0$ such that for each $x, y, z \in G$,

$$(x \cdot y) \geqq \min\{(x \cdot z), (z \cdot y)\} - \delta$$

where $(x \cdot y) = \frac{1}{2}(d(1, x) + d(1, y) - d(x, y))$. It is known that a group G is word hyperbolic if and only if G has a finite presentation satisfying a linear isoperimetric inequality [4, Sect. 2.3]. This gives us the following observation.

Corollary. *If X is a finite hyperbolic 2-complex then $\pi_1(X)$ is word hyperbolic.*

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Geleitwort

Zum 75. Geburtstag der Mathematischen Zeitschrift

Im Jahre 1918 gründeten Konrad Knopp, Leon Lichtenstein, Erhard Schmidt und Issai Schur mit Lichtenstein als geschäftsführenden Herausgeber die *Mathematische Zeitschrift*, die erste mathematische Zeitschrift des Springer-Verlages. Unter den deutschen mathematischen Zeitschriften ist sie die drittälteste, an Alter nur übertroffen von dem „Journal für Reine und Angewandte Mathematik" und den „Mathematischen Annalen", die erst seit 1920 im Springer Verlag erscheint. In den 75 Jahren ihres Bestehens entwickelte sich die *Mathematische Zeitschrift* zu einem Publikationsorgan mit einer der höchsten internationalen Auflagenzahlen innerhalb der Mathematik. Viele herausragende Mathematiker haben in ihr publiziert. Stellvertretend sei Hermann Weyl genannt, der zwischen 1918 und 1952 14 Publikationen in der *Mathematischen Zeitschrift* erscheinen ließ, darunter in den Bänden 23 und 24 seine bahnbrechenden Arbeiten zur „Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformationen".

Die gegenwärtigen Herausgeber der *Mathematischen Zeitschrift* werden die nunmehr 75jährige Tradition dieser Zeitschrift weiterführen: die *Mathematische Zeitschrift* dient der Pflege der Reinen und Angewandten Mathematik. Bei der Reinen Mathematik ist besonders an aktuelle Gebiete gedacht, die einen Aufschwung erleben oder aus anderen Gründen das allgemeine Interesse auf sich ziehen. Raum für neuere Entwicklungen zu geben, bedeutet dabei nicht, klassische Fragestellungen zu verdrängen. All dieses gilt im wesentlichen auch für die Angewandte Mathematik. Darüber hinaus versuchen die Herausgeber, der Tradition von Leon Lichtenstein, dem Gründer der *Mathematischen Zeitschrift*, zu folgen und bemühen sich um die Betonung der Anwendung von Methoden aus der Reinen Mathematik auf Probleme, die ihre Motivation aus der mathematischen Physik oder aus anderen Anwendungsbereichen beziehen. Die Auswahl der zu publizierenden Manuskripte soll dem Leser die Möglichkeit geben, durch hervorragende, richtungsweisende Arbeiten die modernen Entwicklungen in den von der *Mathematischen Zeitschrift* betreuten Teilgebieten der Mathematik zu verfolgen.

Die Geschichte der *Mathematischen Zeitschrift* weist immer wieder besondere Herausforderungen für Herausgeber und Verlag auf. Bereits 5 Jahre nach der Gründung der Zeitschrift stellte sich im Jahre 1923 die erste große Bewährungsprobe. In dieser Zeit der Weltwirtschaftskrise mußten mehrere Publikationshäu-

ser aus wirtschaftlichen Gründen den Druck wissenschaftlicher Literatur einstellen oder gar wissenschaftliche Organe verkaufen. Im Springer Verlag wurde trotz der negativen Vorzeichen der damaligen Zeit weiter investiert. Herausgeber und Verlag ließen sich nicht durch sinkende Abonnentenzahlen aus dem Ausland und steigende Herstellungskosten abschrecken. Wenn auch im Umfang reduziert, so erschienen doch die *Mathematische Zeitschrift* und die „Mathematischen Annalen" regelmäßig. Der hier abgedruckte Brief der Herausgeber dieser beiden Zeitschriften aus dem Jahre 1923 ist vor dem Hintergrund der wirtschaftlichen Krise geschrieben und zeigt deutlich den Willen aller Beteiligten, den großen Schwierigkeiten erfolgreich zu begegnen.

In der Zeit des Nationalsozialismus erschien die *Mathematische Zeitschrift* jährlich mit Ausnahme von 1941. Die Herausgeber waren mit großem Erfolg bemüht, die Internationalität der Zeitschrift möglichst lange zu bewahren. Die jüdische Herkunft des ersten geschäftsführenden Herausgebers Leon Lichtenstein, gestorben 1933, und die Aufnahme von Arbeiten jüdischer Autoren führten zwar zu Problemen, die jedoch nicht die Einstellung der Zeitschrift nach sich zogen, da Mathematik nicht zu den ideologisch belegten Wissenschaften zählte und generell der Vertrieb wissenschaftlicher Zeitschriften zudem dringend benötigte Devisen einbrachte. Nach dem 50. Band im Jahre 1944 erschien die *Mathematische Zeitschrift* erst wieder im Jahre 1949 mit Band 51. Dieser Publikationsausfall war wesentlich in der Streichung des Papierkontingents wegen Rohstoffmangels begründet. Ab 1949 erscheint die *Mathematische Zeitschrift* wieder regelmäßig, gegenwärtig mit 3 Bänden und insgesamt ca. 2000 Seiten pro Jahr.

Der 75. Geburtstag der *Mathematischen Zeitschrift* fällt in eine Zeit eines gesellschaftlich-technologischen Wandels mit weitreichenden, noch nicht voll abzuschätzenden Konsequenzen für die wissenschaftlichen Zeitschriften und Verlage. Enormer technischer Fortschritt bei der Entwicklung der elektronischen Kommunikation, Stagnation oder gar Rückgang öffentlicher Mittel für wissenschaftliche Bibliotheken und rasches Anwachsen der Zahl von Wissenschaftlern haben in den letzten Jahren zur Ausbildung neuer Publikationsformen und -märkte geführt. Die klassische Rolle der wissenschaftlichen Zeitschrift scheint in Frage gestellt zu sein, wissenschaftliche Zeitschriften und Verlage sehen sich einer neuartigen Herausforderung gegenüber.

Die Herausgeber der *Mathematischen Zeitschrift* sind davon überzeugt, daß im gesamten Wissenschaftsprozeß den wissenschaftlichen Zeitschriften auch weiterhin eine zentrale Bedeutung zukommt, die möglicherweise noch wachsen wird. Angesichts einer ständig steigenden Zahl von Publikationen bedarf es verstärkt eines Forums für Forschungsergebnisse in ausgereifter Form, für die Orientierung über den gegenwärtigen Stand der Wissenschaft und zukünftige Entwicklungslinien und für die Herausbildung von Standards. Die Herausgeber der *Mathematischen Zeitschrift* sind bemüht, den damit verbundenen Ansprüchen gerecht zu werden. Zusätzlich wenden sie in enger Zusammenarbeit mit dem Verlag der drängenden Kostenfrage und den neuen technologischen Möglichkeiten ihre volle Aufmerksamkeit zu.

Herausgeber und Verlag wünschen der *Mathematischen Zeitschrift* eine weitere lebendige Entwicklung für viele kommende Jahre.

Die Herausgeber und der Verlag

Editorial

The 75th Anniversary of the Mathematische Zeitschrift

In 1918 Konrad Knopp, Leon Lichtenstein, Erhardt Schmidt and Issai Schur founded the "Mathematische Zeitschrift", Springer-Verlag's first mathematical journal. Leon Lichtenstein became the first managing editor. Among the German mathematical journals the "Mathematische Zeitschrift" is the third oldest, surpassed in age only by the "Journal für Reine und Angewandte Mathematik" and the "Mathematische Annalen"; the latter has been published by Springer Verlag since 1920. In the 75 years since its foundation the "Mathematische Zeitschrift" has developed into a journal with one of the largest circulations among mathematical journals in the world. Many outstanding mathematicians have published in its pages. Hermann Weyl can serve as their representative: between 1918 and 1952, 14 of his publications appeared in this journal, among them his epoch-making papers "Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformation", published in volumes 23 and 24.

The "Mathematische Zeitschrift" purports to cover pure and applied mathematics. In pure mathematics the focus is particularly on fields that are either experiencing a surge of research activity or otherwise attracting attention. Providing space for more recently emerging topics does not at all mean that the classical ones must be neglected. All the above is true in essence for applied mathematics as well. In addition, the editors are trying to follow the traditions established by Leon Lichtenstein, one of the founders of the "Mathematische Zeitschrift", by putting emphasis on the application of methods from pure mathematics to problems arising in mathematical physics or other areas. The refereeing and selection process is such that the reader can obtain, from the selection of manuscripts published, an indication of what are the modern directions of research being pursued in the fields covered by "Mathematische Zeitschrift".

In the history of the "Mathematische Zeitschrift" the editors and the publisher have often had to cope with special challenges. As little as five years after its foundation, in 1923 the journal was put to the first big test: the world recession of that year forced several publishers to abandon the printing of scientific books and journals or even to sell them off. The Springer publishing house, however, continued to invest despite the negative prospects. The editors and publisher resisted the discouragement occasioned by the decreasing number of subscribers from abroad and the rapidly increasing production expenses. The "Mathematische

Annalen" and the "Mathematische Zeitschrift" continued to appear regularly, albeit reduced in size. The letter reproduced here, signed by the editors of these two journals, was written in 1923 in the face of this far-reaching crisis. It displays clearly the strong will of all parties concerned to overcome the great difficulties.

In the period of National Socialism the "Mathematische Zeitschrift" appeared annually with the exception of 1941. The editors had tried with great success to preserve the international character of the journal as long as possible. The Jewish origins of the first managing editor Leon Lichtenstein, who died in 1933 and the publication of the work of Jewish authors gave rise to some problems. However, this did not result in the suspension of the journal, since mathematics was not considered an ideologically critical science and, quite generally, the sale of scientific literature yielded urgently needed foreign currencies. After volume 50 in 1944, the "Mathematische Zeitschrift" did not appear again until 1949 when volume 51 came out. This was due mainly to the cancellation of the paper quota as a result of lack of raw materials. Starting in 1949 the "Mathematische Zeitschrift" was published regularly again, at present in 3 volumes per year, comprising approximately 2000 pages altogether.

The present 75th anniversary takes place in a period of a sociological and technological change entailing far-reaching consequences for scientific journals and publishing houses; consequences which can hardly be anticipated fully at present. In recent years, enormous technological progress in the development of electronic communication, stagnation and even a decrease in public spending for scientific libraries and the fast growth of the number of scientists have led to the establishment of new markets and channels for scientific publication. The classical role of scientific journal and publishing houses seems to be becoming questionable. Editors and publishers are facing a new challenge.

The editors of the "Mathematische Zeitschrift" are convinced that scientific journals will continue to maintain their central importance in the whole process of science. Their importance could possibly even grow. In view of an ever increasing number of publications, there is a strong need now for a forum for the presentation of scientific results in mature form, a forum that may be used to find orientation as to the current status of a science or about future perspectives and in order to set standards. The editors of the "Mathematische Zeitschrift" endeavour to meet the corresponding demands. In addition, in close cooperation with the publishing house, they are giving their full attention to the urgent matter of costs and the opportunities afforded by recent technological developments.

It is our express hope that the "Mathematische Zeitschrift" will continue on a course of energetic evolution for many years to come.

The editors and the publisher



Sehr verehrter Herr Dr. Springer!

Die unterzeichneten Redaktionsmitglieder der „Mathematischen Annalen" und „Mathematischen Zeitschrift" möchten den Tag, an welchem der fünfundzwanzigste in Ihrem Verlage erscheinende Band mathematischer Zeitschriften herausgegeben wird, nicht vorübergehen lassen, ohne Ihnen gegenüber zum Ausdruck zu bringen, wie sehr sie von der Wichtigkeit Ihres Eintretens für unsere Wissenschaft durchdrungen sind und wie hoch sie dieses Eintreten einzuschätzen wissen. Wenn nicht Ihre opferbereite Unternehmungslust und Ihre umsichtige Energie sich der Sache der mathematischen Wissenschaft angenommen hätte, so würde heute die Mathematik in Deutschland nicht mehr lebensfähig sein. Dass sie es noch ist, und dass die deutschen mathematischen Journale nach wie vor zu den angesehensten der Welt gehören, haben die deutschen Mathematiker und damit die Wissenschaft überhaupt wesentlich Ihnen zu verdanken.

Nehmen Sie an diesem Tage mit unseren Glückwünschen den Ausdruck der Hoffnung entgegen, dass der beschrittene Weg unter gegenseitigem Vertrauen und Verständnis unbeirrt weiter gegangen wird.

Berlin und
Göttingen, den 17. 7. 23.

Klein D. Hilbert C. Carathéodory R. Courant

W. Blaschke M. Born Bieberbach Knopp A. Einstein.

Edm. Schmidt I. Schur O. Toeplitz G. Herglotz L. Lichtenstein

C. Neumann. F. Schur. A. Kneser E. Hecke. Wirtinger

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New surfaces of constant mean curvature

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0 Introduction

Immersed surfaces of constant mean curvature 1 in Euclidean three-space (henceforth: H-surfaces) have been found by many authors. We would like to mention the following. In 1841 Delaunay determined all surfaces of revolution [D], which are simply periodic except for the sphere. In 1970 Lawson described two doubly periodic surfaces [L]. In 1984 Wente discovered immersed H-tori, and others contributed to a further study of H-tori, namely Pinkall and Sterling achieved a classification [PS]. Karcher obtained some triply periodic H-surfaces in 1989 [Ka]. In 1990 Kapouleas proved existence of a wealth of H-surfaces, compact and non-compact [Kp1, Kp2]. Karcher's and Kapouleas' surfaces come in continuous families of the same topological type.

In this work we extend Karcher's method – which is in turn based on Lawson's original ideas – to a broader class of surfaces, namely to surfaces with ends. We get the full one-parameter family of certain symmetric surfaces. The limiting cases for these families are as follows: surfaces whose Delaunay-shaped building blocks have small necks and the centres are spherical, as described by Kapouleas on the one hand, and surfaces with again small necks but a new type of centre (which asymptotically is n-noid-shaped) on the other hand. In between we find surfaces with maximal neck-size. For example we can embed Lawson's surfaces into a continuous one-parameter family of surfaces with the same symmetry and we are able to prove that they are exactly those in the family having maximal neck-size (Theorem 3.3). We obtain a similar result for the symmetric surfaces with n onduloid ends, see Theorem 6.1.

The work is organized as follows. The conjugate surface construction for H-surfaces is given in Sect. 1. Basically this is a geometric transformation which reduces the free bundary value problem for desired fundamental patches of H-surfaces to a Plateau problem for a geodesic polygon in the three-sphere S^3. If this polygon is embedded in the boundary of an H-convex set, Morrey's solution to the Plateau problem in S^3 followed by conjugation yields the desired Euclidean H-surface patch. Since the Morrey solution minimizes we get H-surfaches provided their fundamental patch is small enough, i.e. we have to suppose sufficiently high symmetry.

In Sect. 2 we study particular surfaces in S^3. First we describe the surfaces associated to the Delaunay surfaces, the spherical helicoids. Their particular significance is that any embedded end of an H-surface is asymptotic to a Delaunay surface [KKS]. A foliation of S^3 without two great circles leads to a similar result for spherical ends in Subsect. 4.4. Then we consider tori and Clifford tori. The solid Clifford torus serves as an H-convex set in this work and replaces the intersection of hemispheres used by Lawson and Karcher. In order to embed arbitrarily long boundary contours we use the universal covering of solid Clifford tori instead of subsets of S^3.

The periodic H-surfaces we obtain are described in Sect. 3. For each surface we specify the associated boundary contour and the H-convex set containing it.

In the remaining sections we prove existence of surfaces with ends. Their spherical boundary polygons are infinitely long. In Sect. 4 we approximate them by a sequence of bounded polygons and prove convergence of their Plateau solutions, using curvature estimates by R. Schoen and establishing local area bounds. In Sect. 5 we apply this scheme to obtain H-surfaces with cylinder ends, for which the H-convex barriers are easy to describe. In terms of the parameter these surfaces are most distant to those of Kapouleas. In Sect. 6 we give examples of H-surfaces with ends of general Delaunay type.

1 Constant mean curvature surfaces in $\mathbb{R}^3$ and associated minimal surfaces in S^3

1.1 Associated surfaces

Locally a *surface* in an oriented three-dimensional Riemannian manifold N with metric g is an immersion $f: \Omega^2 \to N^3$ of class C^2, where Ω is a domain in $\mathbb{R}^2$. f induces a metric $\langle v, w\rangle := g(df(v), df(w))$ on Ω. By the orientation of Ω and the induced metric the rotation by 90 degrees, $R^{90}: T\Omega \to T\Omega$, is given by $R^{90}e_1 = e_2$ and $R^{90}e_2 = -e_1$ for a positively oriented orthonormal base e_1, e_2 of the tangent space. We use the same notation R^{90} for the induced 90° rotation in $df(T\Omega) \subset TN$, $R^{90}df(v) = df(R^{90}v)$.

The second fundamental tensor $S: T\Omega \to T\Omega$ with respect to a continuous choice of normal $\nu: \Omega \to TN$ is defined by

$$\langle Sv, w\rangle = g(-\nabla_{df(v)}\nu, df(w)),$$

where ∇ denotes the covariant derivative in N. The *mean curvature* of the immersion is

$$H = \tfrac{1}{2}\operatorname{trace} S = \tfrac{1}{2}\langle Se_1, e_1\rangle + \tfrac{1}{2}\langle Se_2, e_2\rangle,$$

for an orthonormal base e_1, e_2 of the surface. f is an immersion of constant mean curvature or *H-surface* if $H \equiv 1$ on Ω, and a *minimal surface* if $H \equiv 0$. Under these conventions the unit sphere $S^2 \subset \mathbb{R}^3$ together with its inner normal becomes an H-surface.

Basic for this work is Lawson's

Theorem 1.1 [L, p. 364] *Let $f: \Omega^2 \to N^3(1)$ be a minimal immersion of a simply connected domain into a manifold of constant sectional curvature 1 with metric g and second fundamental tensor S. Then there exists an associated H-surface $\tilde{f}: \Omega^2 \to \mathbb{R}^3$ with metric $\langle\langle .,. \rangle\rangle = \langle .,. \rangle$ and second fundamental tensor*

$$\tilde{S} = R^{90} S + \mathrm{id}. \tag{1.1}$$

On the other hand for each H-surface in $\mathbb{R}^3$ there is an associated minimal surface in the 3-sphere $\mathbb{S}^3$ with $S = R^{-90}(\tilde{S} - \mathrm{id})$.

Proof [Ka]. We verify the equations of Gauß and Codazzi for the data of $\tilde{f}$. The existence of the surfaces is then a consequence of the fundamental theorem for hypersurfaces [doC, p. 236].

By the symmetry of S we have $\mathrm{trace}(R^{90} S) = 0$, and therefore

$$\begin{aligned} \det \tilde{S} &= \det(R^{90} S + \mathrm{id}) \\ &= \det(R^{90} S) + \mathrm{trace}(R^{90} S) + 1 \\ &= \det S + 1. \end{aligned} \tag{1.2}$$

On the other hand f satisfies the Gauß equation $K = \det S + R = \det S + 1$, where K is Gauß curvature of the surface, and R sectional curvature of the space. It follows $K = \det \tilde{S}$. But the two surface metrics coincide and they determine the Gauß curvature, hence we have $K = \tilde{K}$. This establishes the Gauß equation $\tilde{K} = \det \tilde{S} + 0$ for $\tilde{f}$. For Codazzi's equations we verify $\nabla_X \tilde{S} Y = \nabla_Y \tilde{S} X$ (wlog. $[X, Y] = 0$). Since $\nabla_X SY = \nabla_Y SX$ and R^{90} commutes with ∇, we have $\nabla_X R^{90} SY = \nabla_Y R^{90} SX$. Adding $\nabla_X Y = \nabla_Y X$, the claim follows.

Finally $\mathrm{trace}\, \tilde{S} = \mathrm{trace}(R^{90} S + \mathrm{id}) = 2$ by the symmetry of S and thus $\tilde{f}$ is an H-surface. By the same calculations the converse of the theorem holds using the fundamental theorem for hypersurfaces in $\mathbb{S}^3$. □

Remarks. 1. The same method yields H-surfaces in $N^3(c-1)$ from minimal surfaces in $N^3(c)$, e.g. hyperbolic H-surfaces from Euclidean minimal surfaces.

2. Every rotation of the second fundamental tensor, $\tilde{S}^{\phi} = R^{\phi} S + \mathrm{id}$, describes an H-surface since (1.2) holds. In analogy to minimal surfaces this family of H-surfaces is called the *associated family*. However $\phi = \pm 90°$ leads to two different H-surfaces, which are obtained as well by the two choices of ν in (1.1) and likewise by $R^{90} S \pm \mathrm{id}$.

3. $\mathbb{S}^2 \subset \mathbb{R}^3$ with $\tilde{S} = \mathrm{id}$ is the associated surface of the great sphere $\mathbb{S}^2 \subset \mathbb{S}^3$ with $S = 0$. The cylinder in $\mathbb{R}^3$ with $\tilde{S} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is associated to the Clifford torus with $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The Delaunay surfaces are associated to the spherical helicoids, see Subsect. 2.1.

1.2 *Associated boundary curves*

We denote by $\tilde{M}^2 \subset \mathbb{R}^3$ an H-surface and by $M^2 \subset \mathbb{S}^3$ its associated minimal surface. Let c be a curve in Ω.

Lemma 1.2 *$\tilde\gamma(s)=\tilde f(c(s))$ is a curvature line and geodesic of the surface $\tilde M\subset\mathbb{R}^3$ iff the associated curve $\gamma(s)=f(c(s))$ in M describes a geodesic of S^3, that is a great circle arc.*

Proof. We choose a normal ν resp. $\tilde\nu$ in M resp. $\tilde M$. Let us use $\cdot$ for the scalar product in $\mathbb{R}^3$ and $\mathsf{S}^3\subset\mathbb{R}^4$ and denote the covariant derivative $\nabla_{\dot\gamma}$ along a geodesic by d/ds in $\mathbb{R}^3$ and by D/ds in S^3. Since $d\tilde\nu/ds$ is parallel to $\dot{\tilde\gamma}$ we have

$$
\begin{aligned}
(1.3)\qquad 0&=\frac{d}{ds}\tilde\nu\cdot R^{90}\dot{\tilde\gamma}\\
&=-\langle\tilde S\dot c, R^{90}\dot c\rangle=-\langle(R^{90}S+\mathrm{id})\dot c, R^{90}\dot c\rangle=-\langle S\dot c,\dot c\rangle\\
&=\frac{D}{ds}\nu\cdot\dot\gamma=-\nu\cdot\frac{D}{ds}\dot\gamma.
\end{aligned}
$$

As $g=\tilde g$, γ is a geodesic of the surface iff $\tilde\gamma$ is. Hence $D\dot\gamma/ds$ is parallel to ν, and by (1.3) $D\dot\gamma/ds=0$, that is γ is a geodesic of S^3, and vice versa. □

$\tau:=g(\nabla_{\dot\gamma}\nu, R^{90}\dot\gamma)$ is called *torsion* and $\kappa:=g(\nabla_{\dot\gamma}\dot\gamma,\nu)$ *curvature* of the curve γ; Eq. (1.3) then reads

$$\tilde\tau=-\kappa.$$

The torsion of a geodesic in N^3 is the rotation speed of the normal and thus depends on the surface. We compute this speed in terms of the associated curve:

$$
\begin{aligned}
(1.4)\qquad \tilde\kappa&=\tilde\nu\cdot\frac{d}{ds}\dot{\tilde\gamma}\\
&=\langle\tilde S\dot c,\dot c\rangle=\langle(R^{90}S+\mathrm{id})\dot c,\dot c\rangle=-\langle S\dot c, R^{90}\dot c\rangle+1\\
&=\frac{D}{ds}\nu\cdot R^{90}\dot\gamma+1=\tau+1.
\end{aligned}
$$

Since the sign of curvature and torsion depends on the normal let us fix orientations.

Definition 1.3 (i) (*Orientation of* S^3) r, v, $w\in T_p\mathsf{S}^3$ are positively oriented, if r, v, w, $p\in\mathbb{R}^4$ are positively oriented.

(ii) (*Choice of normal*) The normal ν of a surface $f:\Omega\to N^3$ orients $df(v)$, $df(w)$, ν positively if v, $w\in T\Omega$ are positively ortiented. In particular $df(v)$, $R^{90}df(v)$, ν are positively oriented.

(iii) Let γ be a geodesic in N^3 and v a vector field along γ. Then v *right (left) rotates* with respect to the axis $\dot\gamma$ if $\dot\gamma$, v, $\nabla_{\dot\gamma}v$ are positively (negatively) oriented.

1.3 Geometric data of H-surfaces

We construct H-surfaces invariant under a group of planar symmetries. A *fundamental patch* generates the comlete surface by the group of reflections. We assume the patch is simply connected. Its boundary consists of planar curves $\tilde\gamma_i$ which must be symmetry lines ($\tilde S\dot{\tilde\gamma}_i\parallel\dot{\tilde\gamma}_i$). See images in Subsect. 3.1 for an example.

A fundamental patch, bounded by n curvature lines $\tilde{\gamma}_i$ (parameterized with unity speed), defines the following *geometric data:*

(i) The *length* $\tilde{l}_i$ of the curvature line $\tilde{\gamma}_i$, also denoted by $|\tilde{\gamma}_i|$.

(ii) The *vertex angle* $\tilde{e}_i = \pi/(m_i+1)$, $m_i \in \mathbb{N}$, of two edges $\tilde{\gamma}_i$, $\tilde{\gamma}_{i+1}$ satisfies

$$\cos \tilde{e}_i = \cos \angle(\tilde{\gamma}_i, \tilde{\gamma}_{i+1}) = -\dot{\tilde{\gamma}}_i(\tilde{l}_i) \cdot \dot{\tilde{\gamma}}_{i+1}(0).$$

(iii) The *tilting angle* $\tilde{t}_i \in \mathbb{R}$ of the normal $\tilde{\nu}$,

$$\tilde{t}_i = \int_0^{\tilde{l}_i} \frac{d}{ds} \tilde{\nu} \cdot \dot{\tilde{\gamma}}_i \, d\sigma.$$

This angle measures the total turn of the normal, $\cos \tilde{t}_i = \tilde{\nu}(\tilde{\gamma}_i(0)) \cdot \tilde{\nu}(\tilde{\gamma}_i(l_i))$.

Vertex and tilting angle modulo π are determined by the symmetry type, that is by the boundary polygon. However the lengths are unknown. If in addition the position of the normal at the vertices is prescribed then the tilting angle can only be chosen modulo 2π.

By Lemma 1.2 the associated contour is a geodesic polygon in S^3 with the same lengths $l_i = \tilde{l}_i$ and the same vertex angles

$$e_i = \tilde{e}_i \tag{1.5}$$

because the metrics coincide. An integration of (1.4) along a geodesic arc γ_i yields:

$$\begin{aligned} \tilde{t}_i &= \int_0^{l_i} \frac{d}{ds} \tilde{\nu} \cdot \dot{\tilde{\gamma}}_i \, d\sigma \\ &= -\int_{\tilde{\gamma}_i \subset \mathbb{R}^3} \tilde{\kappa} \, d\sigma \\ &= -\int_{\gamma_i \subset S^3} (r+1) \, d\sigma \\ &= -\int_0^{l_i} \frac{D}{ds} \nu \cdot R^{90} \dot{\gamma}_i \, d\sigma - l_i \\ &= r_i - l_i. \end{aligned} \tag{1.6}$$

Here we used the notation

$$r := -\int_0^{l} \frac{D}{ds} \nu(\gamma) \cdot R^{90} \dot{\gamma} \, d\sigma$$

for the *rotation angle* of the tangent plane along a geodesic arc $\gamma \subset S^3$. Indeed $r \in \mathbb{R}$ measures the total rotation of the normal with respect to parallel transport. r is positive for right rotations ($\dot{\gamma}, \nu, D\nu/ds = -R^{90}\dot{\gamma}$ positively oriented). Given $\tilde{l}_i, \tilde{e}_i, \tilde{t}_i$ the associated contour in S^3 can be found arcwise: Prescribing a length $l_i = \tilde{l}_i$ to γ_i, the position of the normal $\nu(\gamma_i(l_i))$ at the endpoint is determined

by (1.6). Then (1.5) gives the correct vector $\dot{\gamma}_{i+1}(0)$ in the tangent plane $\nu^{\perp}(\gamma_i(l_i))$. In Subsect. 1.5 we prove there exist lengths, such that the boundary contour closes.

1.4 Hopf vector fields

Karcher [Ka] introduced the idea to measure the angles of rotation in S^3 with respect to rotating vector fields. In terms of such fields the right hand side of (1.6) does not contain the unknown length any more. In S^3 such vector fields exist:

Definition 1.4 A vector field $\tau: S^3 \to TS^3$ is called *Hopf vector field* if $\tau(x) = T \cdot x$ with a skew symmetric orthogonal 4×4-matrix $T = -{}^tT = -T^{-1}$.

According to Definition 1.3(iii) T right rotates along the great circle $\cos s\, x + \sin s\, y$, if y, Tx, Ty, x is positively oriented for $x \perp y$. By a homotopy argument this orientation agrees for all x, y it is defined for. Hence *right* and *left rotating Hopf vector fields* are well defined.

Lemma 1.5 (i) *Hopf vector fields are Killing fields whose integral curves are great circles.*

(ii) *The angle of a Hopf field with a great circle is constant.*

(iii) *If a Hopf field is not tangent to a great circle it rotates with constant speed and once around the great circle.*

(iv) *Furthermore two right rotating Hopf fields make a constant angle with each other and are thus determined by their values at one point* $x \in S^3$.

Proof. (i) Since $Tx \perp x$ the integral curves of T are the circles $s \mapsto (\cos s)\, x + \sin s)\, Tx$.

(ii) Let $\gamma(s) = -\ddot{\gamma}(s)$ be the great circle. Then we have

$$\frac{d}{ds}(T\gamma \cdot \dot{\gamma}) = T\dot{\gamma} \cdot \dot{\gamma} + T\gamma \cdot \ddot{\gamma} = 0.$$

(iii) By (ii) $T\gamma$ rotates around $\dot{\gamma}$ provided $\dot{\gamma} \neq \pm T\gamma$. The velocity vector $\frac{d}{ds} T\gamma = T\dot{\gamma}$ projected to the tangent space of S^3, $\frac{D}{ds} T\gamma = T\dot{\gamma} - (T\dot{\gamma} \cdot \gamma)\, \gamma$, is of constant length

$$\left| \frac{D}{ds} T\gamma \right|^2 = |T\dot{\gamma}|^2 - 2(T\dot{\gamma} \cdot \gamma)^2 + (T\dot{\gamma} \cdot \gamma)^2 |\gamma|^2. \tag{1.7}$$

On the other hand $T\gamma$ has a component orthogonal to $\dot{\gamma}$ with length

$$|T\gamma - (T\gamma \cdot \dot{\gamma})\, \dot{\gamma}|^2 = 1 - 2(T\gamma \cdot \dot{\gamma})^2 + (T\gamma \cdot \dot{\gamma})^2 |\dot{\gamma}|^2. \tag{1.8}$$

By the skew symmetry of T (1.8) and (1.7) coincide. Therefore the rotation speed of the orthogonal component of $T\gamma$ is ± 1.

(iv) is a consequence of (ii) and (iii). □

We can find an orthogonal basis of three right rotating fields

$$(1.9)\quad A=\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\quad B=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\quad C=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},$$

where Ax, Bx, Cx, x are positively orientated. For $\alpha^2+\beta^2+\gamma^2=1$ a linear combination $\alpha A+\beta B+\gamma C$ again is a right rotating Hopf field. In the following all Hopf vector fields are right rotating unless stated otherwise.

Using property (iv) of the preceding lemma we define angles between Hopf fields pointwise:

$$\angle(A, B):=\alpha,$$

if $A\cdot B=\cos\alpha$ for $0\leqq\alpha\leqq 180°$. Let the oriented angle of two fields B, C with respect to a third field $A\perp(B, C)$ be $\angle_A(B, C):=\pm\angle(B, C)$, depending on Ax, Bx, Cx, x being positively resp. negatively oriented in $\mathbb{R}^4$. Finally let the oriented angle of two fields B, C with respect to a linear independent field A be defined by the projection of B, C to $A^\perp$:

$$\angle_A(B, C):=\angle_A(B-(A\cdot B)\,A, C-(A\cdot C)\,A).$$

Thus $\angle_A(B, C)$ is the dihedral angle of the planes $\operatorname{span}\{A, B\}$ and $\operatorname{span}\{A, C\}$. If a right rotation by $+90°$ leads from B to C with respect to the axis A in the sense of Definition 1.3(iii) then $\angle_A(B, C)=+90°$, provided A, B, C are positively oriented. All computations with respect to oriented angles are modulo 2π so that we have identities like $\angle_A(B, C)=\angle_A(B, D)+\angle_A(D, C)$.

1.5 Existence of closed associated boundary polygons

Theorem 1.6 *Let $n\geqq 4$ angles $0<e_i<180°$ and n angles $t_i\in\mathbb{R}$ be given and suppose $e_{n-2}=e_{n-1}=90°$ and $t_{n-1}\neq 0 \bmod \pi$. Then there is a closed polygon Γ consisting of n geodesic arcs γ_i in $\mathbb{S}^3$, with n normals in the vertices, having*

(i) *vertex angle $\angle(\gamma_i, \gamma_{i+1})=e_i$ (indices* mod n*) and*

(ii) *rotation angle of the tangent plane along an arc γ_i satisfying $r_i-|\gamma_i|=t_i \bmod 2\pi$.*

(iii) *We can prescribe any lengths $0<l_1, \ldots, l_{n-3}<\pi$ except for isolated values to have Γ embedded.*

Proof. We claim that if we take γ_i to be an integral curve of A_i with $\gamma_i(0)=\gamma_{i-1}(l_{i-1})$, where A_i satisfies

$$(1.10)\qquad \angle(-A_i, A_{i+1})=e_i \quad\text{and}\quad \angle_{A_i}(-A_{i-1}, A_{i+1})=t_i \bmod 2\pi,$$

then (i) and (ii) hold, no matter which lengths $0<l_i=|\gamma_i|<\pi$ we choose.

Clearly the first equation of (1.10) implies (i). To compute r_i we split the rotation into two parts: A rotation of A_{i-1} to A_{i+1} at $\gamma_i(0)$ and the rotation

of $A_{i+1}(\gamma_i(s))$ along $\gamma_i(s)$. For the latter we let $A^\perp$ be the Hopf field orthogonal to $A_i = \dot\gamma_i$ and A_{i+1}. We obtain

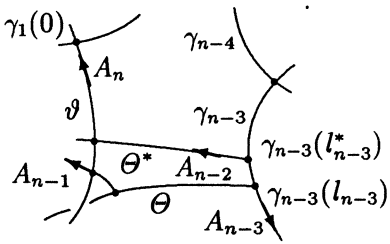
$$r_i = \angle_{A_i}(-A_{i-1}, A_{i+1}) - \int_0^{l_i} \frac{D}{ds} A^\perp \cdot R^{90}\dot\gamma_i \, d\sigma \bmod 2\pi.$$

Since both r_i and $\angle_\cdot(.,.)$ are positive for right rotations with respect to the axis A_i, the sign of the first term is correct. But $\frac{D}{ds}(A^\perp\gamma_i) = A^\perp\dot\gamma_i$ and since $A^\perp\dot\gamma_i$ is orthogonal to $\dot\gamma_i, A^\perp\gamma_i, \gamma_i$ we must have $A^\perp\dot\gamma_i = \pm R^{90}\dot\gamma_i$. In fact $A^\perp\dot\gamma_i = -R^{90}\dot\gamma_i$ since $A^\perp$ is right rotating, that is $\dot\gamma_i, A^\perp\gamma_i, A^\perp\dot\gamma_i, \gamma_i$ is positively orientated as is $\dot\gamma_i, R^{90}\dot\gamma_i, \nu = A^\perp\gamma_i, \gamma_i$ by Definition 1.3(ii). This yields (ii) for the claim:

$$r_i = \angle_{A_i}(-A_{i-1}, A_{i+1}) + \int_0^{l_i} A^\perp\dot\gamma_i \cdot A^\perp\dot\gamma_i \, d\sigma = t_i + l_i \bmod 2\pi.$$

We now prove we can satisfy (1.10) and find lengths such that the polygon closes. To start we choose A_n and A_1 such that $\angle(-A_n, A_1) = e_n$. Then we determine fields A_{i+1} for $2 \leqq i+1 \leqq n-1$ from (1.10). We fix a point $\gamma_1(0)$ and choose lengths $0 < l_1,\ l_2 < \pi$. Furthermore we choose lengths $l_3, \ldots, l_{n-5}$ and $l^*_{n-4},\ l^*_{n-3}$ small enough so that no self-intersections occur. Let ϑ be the great circle passing through $\gamma_1(0)$ with tangent vector A_n and Θ^* be the great circle through $\gamma_{n-3}(l^*_{n-3})$ tangent to A_{n-2}. We look for an integral of A_{n-1} meeting both ϑ and Θ^* in an angle of $e_{n-2} = e_{n-1} = 90°$ and defining thereby lengths l_{n-2}, l_{n-1}, l_n.

First we rule out that ϑ and Θ^* intersect eachover. By assumption on t_{n-1} they do not coincide. By Lemma 2.6 either we can find a length l_{n-3} (close to l^*_{n-3}), such that Θ, defined as the great circle along A_{n-2} passing through $\gamma_{n-3}(l_{n-3})$, does not intersect ϑ. Or γ_{n-3}, Θ^*, ϑ are contained in a torus ∂T_r. However this can occur only for $n \geqq 5$, since γ_{n-3} is then Clifford-parallel to ϑ. Thus there is an edge γ_{n-4}. If γ_{n-4} is contained in ∂T_r then γ_{n-4} left rotates with respect to ∂T_r and changing l^*_{n-4} to l_{n-4} gives $\gamma_{n-3} \not\subset \partial T_r$. The same conclusion holds if $\gamma_{n-4} \not\subset \partial T_r$. Hence we can assume $\vartheta \cap \Theta = \varnothing$ for lengths l_{n-3} and l_{n-4}. The same argument shows we obtain an embedded polygon for $l_1, \ldots, l_{n-3}$ any numbers less than π except for isolated values.



Two disjoint great circles ϑ, Θ have two perpendicular great circles ϕ and Φ, which meet the former in a distance of $\pi/2$. Namely let $\vartheta(s)$ and $\Theta(t)$ be two points in minimal distance and let ϕ be the great circle containing these

two points. ϕ meets ϑ orthogonally in $\vartheta(s)$ and therefore $\phi \subset \{\vartheta(s+\pi/2)\}^{\perp}$. In particular $\vartheta(s+\pi/2) \perp \Theta(t)$ and likewise $\Theta(t+\pi/2) \perp \vartheta(s)$. That is the great circle $\Phi := \{\vartheta(s), \Theta(t)\}^{\perp}$ is also perpendicular.

ϕ and Φ are orthogonal to the linear independent fields A_{n-2} and A_n. By assumption on e_{n-2}, e_{n-1} they are necessarily integrals of A_{n-1} – this is the main idea of the proof. Four great circle arcs, two of ϕ and two of Φ, join the great circles ϑ and Θ in correct orientation $+A_{n-1}$. For the respective lengths l_{n-2}, l_{n-1}, l_n the polygon closes. Possibly readjusting the lengths $l_1, \ldots, l_{n-3}$ again, this polygon is seen to be embedded. □

1.6 Adding handles

We can easily enlarge the spherical polygon to obtain H-surfaces with an additional handle.

Theorem 1.7 *Let Γ be a closed polygon consisting of $n \geqq 3$ geodesics $\gamma_1, \ldots, \gamma_n$ with tangent Hopf vector fields $A_1, \ldots, A_n$. Suppose the three edges γ_n, γ_1, γ_2 are not contained in a torus (see 2.3), that is*

$$|\gamma_1| \neq \tfrac{1}{2} \angle_{A_1}(-A_n, \pm A_2) \quad \text{or} \quad \angle(-A_n, A_1) \neq \angle(-A_1, \pm A_2). \tag{1.11}$$

Then there is an enlarged closed polygon $\Gamma^* = \{\gamma_0^*, \gamma_1^*, \gamma_2^*, \gamma_3, \ldots, \gamma_{n-1}, \gamma_n^*\}$ along the Hopf fields $A_0, A_1, \ldots, A_n$, where $A_0 = \{A_1, A_n\}^{\perp}$, with the following properties:

(i) *The vertex angles of the new edge are $\angle(\gamma_n^*, \gamma_0^*) = \angle(\gamma_0^*, \gamma_1^*) = 90°$, the other vertex angles remain the same.*

(ii) *The rotation angle along the new edge γ_0^* is the vertex angle of γ_n, γ_1: $\angle_{A_0}(-A_n, A_1) = \pm \angle(\gamma_n, \gamma_1)$. The rotation angles of the two adjacent edges change by 90°:*

$$\angle_{A_n}(-A_{n-1}, A_0) = \angle_{A_n}(-A_{n-1}, A_1) \mp 90°$$

and

$$\angle_{A_1}(-A_0, A_2) = \angle_{A_1}(-A_n, A_2) \mp 90°.$$

All other rotation angles remain unchanged.

(iii) *Γ^* can be chosen such that the new edge is arbitrarily short and the lengths of the adjacent edges change by a small amount.*

Proof. We choose a length $l_2^* < l_2$. Then the integral of A_1 passing through $\gamma_2(l_2^*)$ does not intersect the great circle which contains γ_n by assumption (1.11). We take γ_0^* to be the common perpendicular of the great circles γ_1^* and γ_n. Now we proceed as in the proof of Theorem 1.6. The other sign in (ii) relates to the choice $l_2^* > l_2$. □

Depending on the sign chosen in (ii) the associated H-surface has a handle to the inside or outside which closes after $2m-1$ reflections if $\angle_{A_0}(A_n, A_1) = \pm \pi/m$. Theorem 1.7 allows to construct polygons with many right angles by the successive insertion of handles.

1.7 Solution of Plateau's problem in H-convex sets

We want to solve Plateau's problem for the boundary polygon found in Subsect. 1.5. By the results of [H2] the Plateau problem is solvable in C^2-*H-convex* Riemannian manifolds, that is in manifolds whose boundary ∂N is of class C^2 and has non-negative mean curvature with respect to the inner normal. In order to control the boundary behaviour of the surface we need to enclose the boundary polygon in the H-convex boundary and therefore we want to allow for edges of N.

Let M be a compact minimal surface in $\mathbb{R}^3$ with boundary Γ. By the maximum principle a complete embedded minimal surface S, moved from ∞ towards M, cannot touch the interior of M before S intersects the boundary Γ. It is therefore convenient to extend the notion of C^2-H-convexity to the following:

Definition 1.8 A closed manifold N with boundary is *H-convex*, if there are closed submanifolds N_i, $0 \leqq i \leqq I$, and embedded C^2-surfaces $S_i \subset N_{i-1}$ with boundary satisfying:

(i) N_0 is C^2-H-convex and $N_1 = N$.

(ii) N_i is (the closure of) a connected component of $N_{i-1} - S_i$ for $i \geqq 1$.

(iii) $\partial S_i \subset \partial N_{i-1}$ and S_i has non-negative mean curvature w.r.t. the inner normal of N_i.

(iv) For each S_i there exists a continuous family (w.r.t. distance of surfaces) of H-convex embedded C^2-surfaces $S_i^\sigma \subset N_{i-1} - \operatorname{int} N_i$, with boundary $\partial S_i^\sigma \subset N_{i-1}$, such that $S_i^0 = S_i$ and either

(a) $0 \leqq \sigma \leqq 1$ and $S_i^1 \subset \partial N_{i-1}$

(b) or (applicable to non-compact N_{i-1} only) $0 \leqq \sigma < \infty$ and $\operatorname{dist}(S_i^\sigma, S_i) \geqq \sigma$.

We call the sets S_i *barriers*. Our definition of H-convexity is slightly tighter than the usual one [MY]. However the solvability of Plateau's problem in N is immediate, since by the maximum principle the solution is contained in each N_i and therefore in N:

Theorem 1.9 *Let N^3 be an H-convex manifold with boundary, and Γ a Jordan curve in N which bounds a topological disk of finite area. Then there exists a map of the disk $f: D \to N$, such that* (i) *in the interior of D, f is a smooth minimal immersion,* (ii) *$f(\partial D) = \Gamma$ is continuous and monotone, and* (iii) *f has minimal area among all maps of the same topological type.*

f is an immersion by the results of [O] and [G].

Theorem 1.10 (Boundary regularity [H1]) *If $f: D \to N$ is a minimal surface of class $C^0(\bar{D}, N) \cap C^2(D, N)$ which maps a boundary* arc $\delta \subset \partial D$ *to an analytic Jordan* arc $\gamma \subset N$, *then f is analytic on $D \cup \delta$.*

Lemma 1.11 (Reflection principle [L, Proposition 3.1]) *Let f be a spherical minimal surface of class $C^2(D^+ \cup I, N^3(c))$, where $D^+ = \{(x, y) \in D \mid y > 0\}$ and $I = (-1, 1) \times \{0\} \subset \partial D^+$. If $f(I)$ is either contained in a geodesic γ, or perpendicular to a totally geodesic plane σ, then f can be extended to D by reflection to a minimal surface f^* of class $C^2(D, N)$.*

This reflection is either 180°-rotation around γ or plane reflection in σ. For H-surfaces a similar theorem holds [DHKW, 3.4 Theorem 2].

If $f: D^+ \to N$ is a minimum of area with geodesic boundary $f(I)$ the extension by reflection f^* is free of *branch points* on I, i.e. of points with $|\nabla f^*|=0$; this is a result of [GL]. If two adjacent geodesic arcs of the boundary curve make an angle of $e_i=\pi/(m_i+1)$, $m_i \in \mathbb{N}$, we can extend the surface to a complete neighbourhood of the vertex by Lemma 1.11. This neighbourhood can be reparameterized regularly according to Theorem 1.10, however (true) branch points may occur.

If $f: D \to S^3$ parameterizes a spherical minimal surface without branch points, then the associated H-surface $\tilde{f}: D \to \mathbb{R}^3$ is regular, since the second fundamental tensor $\tilde{S}$ is bounded. Higher regularity follows from the H-surface equation.

In the next theorem we embed the boundary curve of the H-surface into the boundary of an H-convex set in order to exclude vertex branch points and to prescribe the rotation angle t_i as a real number instead of mod 2π.

Theorem 1.12 *Let Γ be a closed polygon, embedded in an H-convex manifold $N^3(1)$. Let the arcs be integrals of the Hopf fields $A_1, \ldots, A_n$, and e_i, t_i be angles such that* (1.10) *is satisfied. Suppose Γ bounds a disk of finite area in N and satisfies*:

(i) *Every vertex of the polygon is contained in two different H-convex barriers, i.e. in an edge of ∂N.*

(ii) *There is a vector field v_i along γ_i contained in the boundary tangent space $T_{\gamma_i} N - \operatorname{span}\{\dot{\gamma}_i\}$, coinciding with $-\dot{\gamma}_{i-1}(l_{i-1})$ and $\dot{\gamma}_{i+1}(0)$ at the vertices, with rotation angle $r_i(v_i)=t_i+l_i$.*

Then there exists a fundamental H-surface patch $\tilde{M} \subset \mathbb{R}^3$ with the geometric data e_i and t_i, whose reflection in $\mathbb{R}^3$ extends to a complete immersed H-surface.

Proof. We take the Plateau solution M given by Theorem 1.9. With the help of the maximum principle we see from assumption (i) that the surface extended by reflection is free of branch points. □

This theorem reduces the existence proof for H-surfaces to the construction of suitable barriers for the associated boundary curve. By the positive curvature of S^3 this becomes more difficult the bigger the boundary polygon is.

2 Some surfaces in S^3

2.1 *Spherical helicoids and Delaunay surfaces*

The helicoids are the associated surfaces of the Delaunay surfaces. We choose a great circle $(\cos u, \sin u, 0, 0)$ and an orthogonal great circle $(0, 0, \cos au, \sin au)$ parameterized with speed $a \in \mathbb{R}$. Joining corresponding points with great circles we obtain a ruled surface $f^a = f: \mathbb{R}^2 \to S^3 \subset \mathbb{R}^4$,

$$f(u,v)=\begin{pmatrix} \cos v \cos u \\ \cos v \sin u \\ \sin v \cos au \\ \sin v \sin au \end{pmatrix}. \tag{2.1}$$

f^0 parameterizes a great sphere, f^1 is a Clifford torus (see 2.3). The tangent space is spanned by

$$f_u(u,v)=\begin{pmatrix}-\cos v\sin u\\ \cos v\cos u\\ -a\sin v\sin au\\ a\sin v\cos au\end{pmatrix},\qquad f_v(u,v)=\begin{pmatrix}-\sin v\cos u\\ -\sin v\sin u\\ \cos v\cos au\\ \cos v\sin au\end{pmatrix}.$$

f^a is an immersion called (*spherical*) *helicoid* provided $a\neq 0$, with metric

$$\begin{pmatrix} f_u^2 & f_u\cdot f_v\\ f_u\cdot f_v & f_v^2\end{pmatrix}=\begin{pmatrix}\cos^2 v+a^2\sin^2 v & 0\\ 0 & 1\end{pmatrix}.$$

In the following we suppose $a\neq 0$ and write $\gamma(v):=|f_u|>0$. The normal

$$n(u,v)=\frac{1}{\gamma(v)}\begin{pmatrix}-a\sin v\sin u\\ a\sin v\cos u\\ \cos v\sin au\\ -\cos v\cos au\end{pmatrix}$$

orients $\frac{1}{\gamma}f_u, f_v, n, f\in\mathbb{R}^4$ positively. The second fundamental tensor is

$$Sf_u=-n_u=-\frac{a}{\gamma}f_v\quad\text{and}\quad Sf_v=-n_v=-\frac{a}{\gamma^3}f_u.$$

f_u/γ and f_v are orthonormal and thus $2H=\operatorname{trace}S=S\frac{1}{\gamma}f_u\cdot\frac{1}{\gamma}f_u+Sf_v\cdot f_v=0$, i.e. the helicoids are minimal surfaces.

By Lemma 1.2 all v-lines are curvature lines on the associated H-surfaces. Thus there is a continuous group of reflections and the H-surfaces are either rotationally symmetric (if $a\neq -1$ we have $\tilde{S}f_u=(1+a/\gamma^2)f_u\not\equiv 0$) or translation invariant and cylinders ($\tilde{S}f_u\equiv 0$ in case $a=-1$). The Delaunay surfaces are characterized as the only non-compact rotationally symmetric H-surfaces [D]. We want to show that all Delaunay surfaces can be derived this way.

Suppose $a>-1$. The boundary of the helicoid $f^a:[0,\pi/(2+2a)]\times[0,\pi/2]\to S^3$, $\pm a>0$, consists of integrals of the Hopf fields $\mp A$, C, A, $-B$ where A, B, C are as in (1.9). Indeed we have

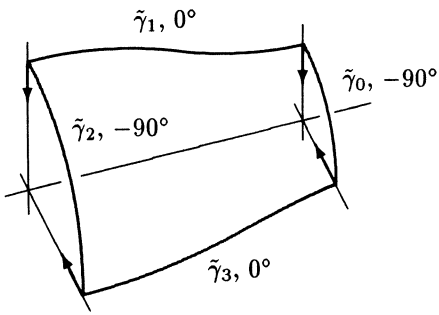
$$-af=\frac{d}{du}f\left(-u,\frac{\pi}{2}\right),\qquad Cf=\frac{d}{dv}f(0,-v),\qquad Af=\frac{d}{du}f(u,0),$$

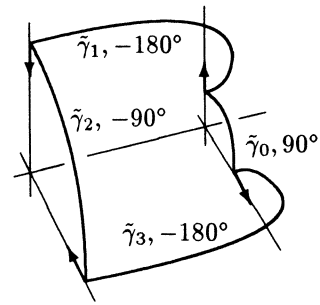
and using $\cos\pi/(2+2a)=\sin a\pi/(2+2a)$

$$-Bf=\frac{d}{dv}f\left(\frac{\pi}{2+2a},v\right).\tag{2.2}$$

We label the arcs in this order by γ_0, γ_1, γ_2, γ_3. We have four rectangular vertex angles and four tilting angles $-90°$, $0°$, $-90°$, $0°$ in case $a>0$ resp. $90°$, $-180°$, $-90°$, $-180°$ if $-1<a<0$, e.g. the first tilting angle is $\angle_{\mp A}(B,C)$

$= \mp 90°$ since A, B, C is positively oriented. These are exactly the geometric data of half a period of a 90° *segment* of the Delaunay surfaces; either the embedded *onduloid*, or the immersed *nodoid* (for $a >$ resp. < 0):





The u-lines are circles in $\mathbb{R}^3$ of the length $4\gamma(v)\pi/(2+2a)$, which is extremal for $v = 0 \bmod \pi$, where $\gamma = 1$, and $v = \pi/2 \bmod \pi$, where $\gamma = |a|$. Thus the ratio of the radii is

$$\frac{r_{\min}}{r_{\max}} = |a| \quad \text{for } -1 < a \leqq 1$$

resp. $1/|a|$ otherwise. It therefore suffices to consider $0 < a \leqq 1$ for the onduloids and $-1 < a < 0$ for the nodoids to obtain all Delaunay surfaces. Note $a = \pm 1$ is associated to the cylinder of radius $1/2$ and $a = 0$ to $\mathbf{S}^2$.

Theorem 2.1 *Each Delaunay surface is the associated H-surface of a helicoid $f^a(u, v)$ with $-1 < a \leqq 1$ and $a \neq 0$ in $\mathbf{S}^3$. A segment of angle ϕ of the Delaunay surface, containing k periods is associated to a piece of the helicoid with $0 \leqq u < \phi/(1+a)$ and $0 \leqq v \leqq k\pi$.*

2.2 *Foliation of* $\mathbf{S}^3$ *with helicoids*

Let $a \neq 0$ and set $\alpha := \dfrac{\pi}{2} \dfrac{1}{1+|a|}$,

$$K := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \operatorname{sign} a & 0 \\ 0 & -\operatorname{sign} a & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

For $a > 0$ the field K is the right rotating Hopf field B from (1.9) whereas K is left rotating in case $a < 0$.

Lemma 2.2 *$Kf(u, v) \cdot n(u, v) > 0$ for $|u| < \alpha$ and $= 0$ for $u = \pm\alpha$.*

Proof.

$$Kf\cdot n=\begin{pmatrix}\sin v\sin au\\ \operatorname{sign} a\sin v\cos au\\ -\operatorname{sign} a\cos v\sin u\\ -\cos v\cos u\end{pmatrix}\cdot\frac{1}{\gamma(v)}\begin{pmatrix}-a\sin v\sin u\\ a\sin v\cos u\\ \cos v\sin au\\ -\cos v\cos au\end{pmatrix}$$

$$=\frac{1}{\gamma(v)}(a\sin^2 v(-\sin u\sin au+\operatorname{sign} a\cos u\cos au)$$

$$+\cos^2 v(-\operatorname{sign} a\sin u\sin au+\cos u\cos au))$$

$$=\frac{1}{\gamma(v)}(|a|\sin^2 v(-\sin u\sin|a|u+\cos u\cos au)$$

$$+\cos^2 v(-\sin u\sin|a|u+\cos u\cos au))$$

$$=\frac{\cos^2 v+|a|\sin^2 v}{\sqrt{\cos^2 v+a^2\sin^2 v}}\cos(u+|a|u)$$

$$>0,\quad \text{if } (1+|a|)|u|<\pi/2.\ \square$$

Thus the flow of the helicoids $f^a((-\alpha,\alpha)\times[0,2\pi))$, by the field K gives rise to a foliation of $\mathbb{S}^3$ without the two great circles $\eta^a_\pm(t):=f^a(\pm\alpha,t)$. Namely $\Psi^a=\Psi\colon\mathbb{R}^3\to\mathbb{S}^3$, $\Psi(\phi,u,v):=\cos\phi f^a(u,v)+\sin\phi Kf^a(u,v)$ satisfies:

Lemma 2.3 (i) $\Psi(\phi,\pm\alpha,v)=\Psi(0,\pm\alpha,v\mp\phi)=\eta_\pm(v\mp\phi)$

(ii) $\Psi(\phi+\pi,u,v)=\Psi(\phi,u,v+\pi)$

(iii) $\Psi\colon D\to\mathbb{S}^3$ *is injective on* $D:=[0,\pi)\times(-\alpha,\alpha)\times[0,2\pi)$ *and*

(iv) *the image of* D *is* $\mathbb{S}^3-\eta_\pm=\mathbb{S}^3-(\eta_+\cup\eta_-)$.

Proof. (i) For $a>0$ we have $\dot\eta_\pm=\mp B\eta_\pm$ by (2.2) and the result follows. For $a<0$ the calculation is similar.

(ii) Use $f(u,v+\pi)=-f(u,v)$.

(iii) $(u,v)\mapsto\Psi(\phi,u,v)$ with $|u|<\alpha$ is a minimal surface with boundary $\eta_\pm$ for each ϕ. We set

$$\phi_0:=\inf\{\phi>0\,|\,\Psi(\phi.U,V)=f(u,v)\ \text{and}\ (u,v)\neq(U,V);|u|,|U|<\alpha;0\leqq v,V<2\pi\}$$

and claim $\phi_0=\pi$.

$\mu_0\leqq\pi$ by (ii). We show $\phi_0>0$. $f\colon[-\alpha,\alpha]\times[0,2\pi)\to\mathbb{S}^3$ is injective. Namely if $f(u,v)=f(U,V)$ the quotient of the first two components is $\tan u=\tan U$ and since $\alpha<\pi/2$ it follows $u=U$ and $v=V$. Hence $\cos tf+\sin tn$ is injective for $0\leqq t<\varepsilon$ and $(u,v)\in[-\alpha,\alpha]\times[0,2\pi)$. Finally, by the preceding lemma Ψ is an injective reparameterization of the above on $[0,\varepsilon(\delta))\times[-\alpha+\delta,\alpha-\delta]\times[0,2\pi)$ for any $\delta>0$. Since f is locally minimizing $\varepsilon(\delta)$ is bounded away from 0. Either $\Psi(\phi_0,U,V)$ intersects $f(u,v)$ in the interior $(|u|,|U|<\alpha)$. Because the touching is locally on one side the surfces coincide by the maximum principle. Or the two surfaces are tangential in a point of the common boundary without interior

intersection. By the boundary maximum principle they coincide. In order to prove $\phi_0 \geqq \pi$ it is therefore sufficient to conclude from

$$\Psi(\phi, 0, 0) = \begin{pmatrix} \cos\phi \\ 0 \\ 0 \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \cos v \cos u \\ \cos v \sin u \\ \sin v \cos au \\ \sin v \sin au \end{pmatrix} = f(u, v)$$

with $0 \leqq \phi < \pi$ and $-\alpha < u < \alpha$, $0 \leqq v < 2\pi$ that ϕ, u, v are equal to 0. However, all other cases lead to a contradiction:

$$v \neq 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi \Rightarrow \cos v, \sin v \neq 0 \Rightarrow \sin u, \cos au = 0,$$

$$v \in \left\{\frac{\pi}{2}, \frac{3}{2}\pi\right\} \Rightarrow \phi = \pi/2 \Rightarrow \sin au = 1,$$

$$v = \pi \Rightarrow u = 0 \Rightarrow \cos\phi = -1.$$

(iv) The image of D under Ψ is open and closed in $\mathbf{S}^3 - \eta_{\pm}$. □

By the maximum principle a uniqueness theorem follows:

Theorem 2.4 *Let $M \subset \mathbf{S}^3 - \eta_{\pm}$ be a compact minimal surface with boundary satisfying $\partial M \subset \Psi(0,.,.)$ and $\Psi(\phi_0,.,.) \cap M = \varnothing$ for a ϕ_0. Then M is contained in the sheet $\Psi(0,.,.)$.*

Corollary 2.5 *The helicoids $W_\xi^a = \{f^a(u, v) | -\xi \leqq u \leqq \xi, 0 \leqq v < 2\pi\}$ are stable for $\xi < \alpha$.*

By Theorem 2.1 W_α^a belongs to a 180° segment of the onduloid if $a > 0$.

2.3 Tori and Clifford tori

Let us define $F: \mathbb{R}^3 \to \mathbf{S}^3$ by

$$F(r, x, y) := \begin{pmatrix} \cos r \cos(x+y) \\ \cos r \sin(x+y) \\ \sin r \cos(y-x) \\ \sin r \sin(y-x) \end{pmatrix}. \tag{2.3}$$

The image of $0 \leqq r \leqq \pi/2$, $0 \leqq x < 2\pi$, $0 \leqq y < \pi$ parameterizes the whole sphere since

$$F(r, x+2\pi, y) = F(r, x, y) \quad \text{and} \quad F(r, x, y+\pi) = F(r, x+\pi, y). \tag{2.4}$$

The set $T_r := \{F(\rho, x, y) | 0 \leqq \rho \leqq r\}$ is called a *solid torus* resp.

$$T := T_{45} = \{x \in \mathbf{S}^3 | x_3^2 + x_4^2 \leqq 1/2\} \tag{2.5}$$

the *solid Clifford torus*. These are tubular neighbourhoods of the *soul*, i.e. the great circle $F(0, x, 0) = F(0, 0, x)$. We call the boundary ∂T_r for $0 < r < 90°$ a *torus*, resp. $\partial T = \partial T_{45}$ the *Clifford torus*.

$x \mapsto F(r, x, y)$ and $y \mapsto F(r, x, y)$ are great circles. They have a constant distance r to the soul and are called *r-Clifford parallels* to the soul. Their tangent vectors are

$$\frac{\partial}{\partial x} F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} F \quad \text{and} \quad \frac{\partial}{\partial y} F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} F.$$

Thus

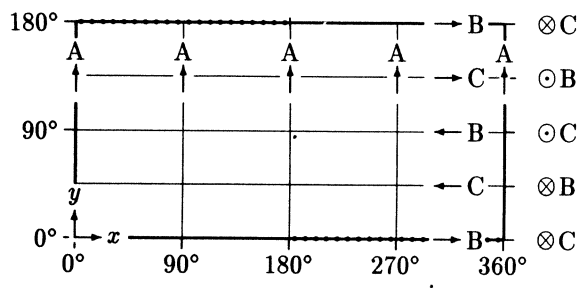
$$\frac{\partial}{\partial x} F(r, x, y) \cdot \frac{\partial}{\partial y} F(r, x, y) = \cos^2 r - \sin^2 r = \cos 2r \tag{2.6}$$

and two intersecting great circles on ∂T_r make an angle of $2r$. In particular the Clifford torus has orthogonal asymptotic lines and therefore is a minimal surface.

By Definition 1.4 $\partial F/\partial y$ is a right rotating Hopf vector field, whereas $\partial F/\partial x$ is a left rotating Hopf field. Indeed using (1.9) we have $\partial F/\partial y = A \cdot F$. Along an A-line the field $\partial F/\partial x$ once left rotates w.r.t. parallel fields resp. twice left rotates w.r.t. right rotating fields. We describe this left rotating field on the Clifford torus with the right rotating fields from (1.9):

$$\frac{\partial}{\partial x} F(45°, x, y) = (\cos 2y\, B - \sin 2y\, C)\, F(45°, x, y). \tag{2.7}$$

Drawing the Hopf fields into the domain we get (identification of the boundary as indicated by the dots, see (2.4)).



The inner normal of the solid Clifford torus T,

$$n(x, y) = -\frac{\partial}{\partial r} F(r, x, y)\Big|_{r=45°} = (\cos 2y\, C + \sin 2y\, B)\, F(45°, x, y) \tag{2.8}$$

is indicated on the right: Fields with $\otimes$ point in the normal direction, those with $\odot$ in the opposite direction.

The following fact for tori was used in 1.5 and 1.6:

Lemma 2.6 *Let γ and ϑ be two great circles to Hopf fields A and C. Let Θ_s be the great circle along $B(\neq \pm A$ and $\neq \pm C)$ with $\Theta_s(0) = \gamma(s)$. Suppose that*

for $s=0$ we have $\Theta_0(l)=(0)$, that is the arc Θ_0 *of length $0<l<2\pi$ ends in ϑ. Then*

(i) *either $\Theta_s(l)$ is a point of ϑ for all $0\leqq s<2\pi$ and γ, Θ_s, ϑ are contained in the same torus ∂T_r. This case is equivalent to either*

$$l=\tfrac{1}{2}\angle_B(A,-C) \quad \text{and} \quad \angle(A,B)=\angle(B,C),$$

or

$$l=\tfrac{1}{2}\angle_B(A,C) \quad \text{and} \quad \angle(A,B)=180°-\angle(B,C).$$

(ii) *Or $\vartheta\cap\Theta_s$ for small $s\neq 0$.*

Proof. There is a torus ∂T_r which contains γ and Θ_s for all s. Namely we take A to be one of the left rotating fields, B right rotating, and $\cos 2r=A\cdot B$, see (2.6). Now suppose for (i) that ϑ is also contained in ∂T_r. Since $B\neq\pm C$ then $\pm C$ belongs to the left rotating fields. Let us first assume that $+C$ arises from the right rotation of A along the arc Θ_0. Because the rotation speed is 2 w.r.t. right rotating fields we have $2l=\angle_B(A,-C)$ and $\angle(A,B)=\angle(B,C)$. Otherwise the same equations hold with $-C$.

The intersection of ϑ with ∂T_r is either transverse or tangent. In the latter case and if (i) does not hold the intersection is in a non-asymptotic direction of ∂T_r. So either a first or a second order comparison proves that ϑ intersects the torus (and therefore Θ_s) in a neighbourhood of Θ_0 in the points $\Theta_0(l)$ and $\Theta_0(l+\pi)$ only. □

2.4 *Intersection of solid Clifford torus and great sphere*

Lemma 2.7 *Let T be a solid Clifford torus and c a geodesic with $c(0)\in\partial T$, whose tangent vector $\dot{c}(0)$ is contained in the plane spanned by the inner normal $n(c(0))$ of the torus and one asymptotic direction of the torus in the point $c(0)$. If $\dot{c}(0)\cdot n>0$ then $c(t)\in T$ for $0\leqq t\leqq 90°$.*

Proof. By a rotation of S^3 we assume that $p:=c(0)=F(45°,0,0)$ with normal $n(p)=n(0,0)$ as in (2.3) and (2.8) and we take the great circle $c^*(t)=F(45°,0,t)\subset\partial T$ to be the asymptotic direction. The great circle which makes an angle $\beta(0<\beta<180°)$ with the asymptotic direction is

$$\begin{aligned} c(t)&=\cos t\,p+\sin t(\cos\beta\, c^*(90°)+\sin\beta\, n(0,0))\\ &=\cos t\begin{pmatrix}\frac{1}{2}\sqrt{2}\\0\\\frac{1}{2}\sqrt{2}\\0\end{pmatrix}+\sin t\,\tfrac{1}{2}\sqrt{2}\begin{pmatrix}\sin\beta\\\cos\beta\\-\sin\beta\\\cos\beta\end{pmatrix}. \end{aligned}$$

By (2.5) the claim follows from the inequality

$$\begin{aligned} \tfrac{1}{2}\geqq c_3^2(t)+c_4^2(t)&=\tfrac{1}{2}(\cos t-\sin t\,\sin\beta)^2+\tfrac{1}{2}\sin^2 t\cos^2\beta \\ &=\tfrac{1}{2}-\cos t\,\sin t\,\sin\beta. \quad \square \end{aligned} \tag{2.9}$$

Corollary 2.8 *Let $G \subset S^3$ be a great sphere containing an asymptotic line c^* of a Clifford torus ∂T. Then $G \cap \partial T = \{c^*\} \cup \{c^{**}\}$, where $c^{**} \subset \partial T$ is another asymptotic line, which intersects c^* in two antipodal points.*

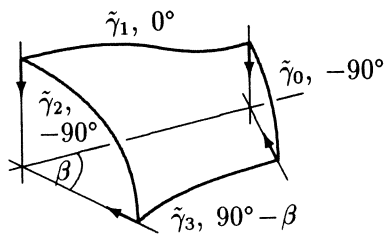
Proof. In (2.9) equality holds for $\beta = 0$ and $t = 90°$, that is the great circles $c^*(t)$ and $c^{**}(\beta) = \cos\beta c^*(90°) + \sin\beta n(p)$ are contained in ∂T. □

3 Periodic H-surfaces

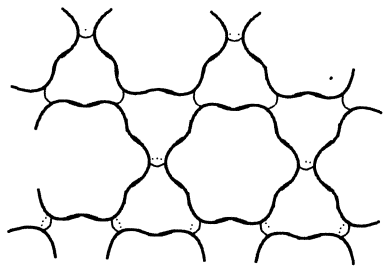
In this section we prove existence of various compact H-surface patches; by reflection we obtain doubly and triply periodic surfaces.

3.1 *The simplest doubly periodic surfaces*

We consider an H-surface patch $\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ with vertex angles 90°, $0 < \beta < 90°$, 90°, 90° and tilting angles $-90°$, 0°, $-90°$, $90° - \beta$. Note that with these tilting angles we restrict attention to the onduloid case.



The surfaces extended by reflection are doubly periodic with triangular, quadratic, or hexagonal lattice, if $\beta = 30°$, 45°, or 60°. E.g. in the hexagonal case the surface will look like:



We can choose the Hopf fields of $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ to be $-A$, C, $\sin\beta A - \cos\beta C$, $-B$ (with A, B, C orthogonal). It is easy either to check (1.10) or to determine the fields from the degenerate situation $|\gamma_0| = 0$, which is a spherical triangle, using Theorem 1.7.

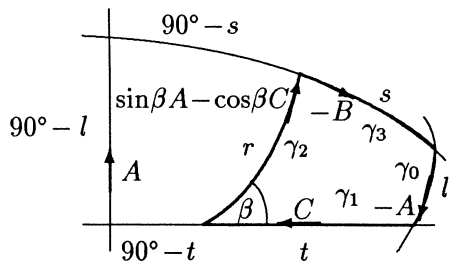
Lemma 3.1 (i) *Let* $0<l, t, r, s<90°$ *denote the lengths of a quadrilateral along the Hopf fields* $-A$, C, $\sin\beta A-\cos\beta C$, $-B$ *and* $0<\beta<90°$. *Then*

$$0<l\leqq\frac{\beta}{2}.$$

(ii) *There exists a continuous one-parameter family* Γ_t, $0<t<90°$, *of quadrilaterals with the Hopf fields considered. The family consists of* small quadrilaterals *whose lengths* l, t, r, s *increase monotonously from all lengths* 0 *to* $\beta/2$, 45°, 45°, $\beta/2$ *and continues with* large quadrilaterals *where* l *decreases monotonously to* 0 *again, but* t, r, s *increase monotonously to* $t=90°$, $r=90°$, $s=\beta$. *The two degenerate cases are a point for* Γ_0 *and a spherical triangle for* Γ_{90}.

Proof. (i) We need the cosine law of spherical geometry $\cos c=\cos a\cos b+\cos\gamma\sin a\sin b$, which holds in a triangle with an angle $0<\gamma<180°$ enclosed by the sides a and b.

At the ends on γ_2 we extend the edges γ_1 and γ_3 to 90°. The resulting endpoints can be joined by an A-perpendicular. We recognize the Hopf fields and the lengths of the extended arcs γ_0, γ_1, γ_2 from the onduloid case of the spherical helicoids, i.e. $a>0$ (see Subsect. 2.1) and therefore the length of the perpendicular arc is $90°-l$. In stereographic projection:



In the given quadrilateral we consider the diagonal passing through the angle β. The cosine law yields for the two rectangular triangles

$$\cos s\cos r=\cos l\cos t. \tag{3.1}$$

Similarly in the other quadrilateral:

$$\sin s\cos r=\sin l\sin t. \tag{3.2}$$

Dividing (3.2) by (3.1) we get

$$\tan s=\tan l\tan t. \tag{3.3}$$

For the other two diagonals we compute:

$$\cos s\cos l=\cos r\cos t+\cos\beta\sin r\sin t \tag{3.4}$$

$$\sin s\sin l=\cos r\sin t-\cos\beta\sin r\cos t. \tag{3.5}$$

Since $\cos l \cdot (3.1) = \cos r \cdot (3.4)$ we can eliminate s

$$\cos^2 l \cos t = \cos^2 r \cos t + \cos\beta \sin r \cos r \sin t. \tag{3.6}$$

In the same way from $\sin l \cdot (3.2) = \cos r \cdot (3.5)$,

$$\sin^2 l \sin t = \cos^2 r \sin t - \cos\beta \sin r \cos r \cos t. \tag{3.7}$$

Adding $\sin t \cdot (3.6) + \cos t \cdot (3.7)$ we get

$$\tfrac{1}{2} \sin 2t = \sin 2t \cos^2 r - \cos\beta \sin r \cos r \cos 2t$$

or $\sin 2t \cos 2r = \cos\beta \sin 2r \cos 2t$. So $t = 45°$ is equivalent to $r = 45°$ and otherwise

$$\tan 2t = \cos\beta \tan 2r. \tag{3.8}$$

Subtracting $\sin t \cdot (3.6) - \cos t \cdot (3.7)$ we obtain

$$\cos 2l \sin t \cos t = \cos\beta \sin r \cos r \Rightarrow \cos 2l = \cos\beta \sin 2r \frac{1}{\sin 2t}$$

and finally by (3.8)

$$\cos 2l = \cos\beta \sin 2r \sqrt{1 + \frac{1}{\cos^2\beta \tan^2 2r}} = \sqrt{\cos^2\beta \sin^2 2r + \cos^2 2r}, \tag{3.9}$$

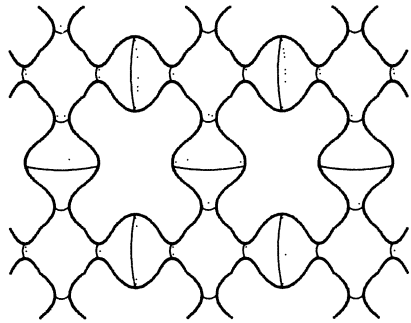
that is $\cos 2l \geqq \cos\beta$.

(ii) We can reverse (i) instead of applying Theorem 1.6: For each $0 < r < 90°$ we get a length $0 < l < \beta/2$ by (3.9), $0 < t < 90°$ by (3.8) and $0 < s < 90°$ by (3.3), and obtain a quadrilateral for which (3.1) (3.2) (3.4) (3.5) hold. Therefore it has the desired Hopf fields. □

A geodesic quadrilateral whose edges are shorter than 180° is contained in the boundary of an H-convex set: Any three vertices of Γ_t are contained in a great sphere and the fourth vertex is in one of the two hemi-S^3. Thereby we get four hemi-S^3 whose intersection is H-convex. From Theorem 1.12 we conclude the existence of the periodic surfaces.

We can lengthen γ_1 and γ_3 by integer multiples of 90° to obtain *long quadrilaterals* with the same Hopf fields: We fix γ_2 and let $\Gamma^{k,\beta} := \{\gamma_0^k, \gamma_1^k, \gamma_2, \gamma_3^k\}$ be the quadrilateral with edge lengths $k\pi/2 < |\gamma_1^k|$, $|\gamma_3^k| \leqq (k+1)\pi/2$, $k \in \mathbb{N}_0$, and with angle $0 < \angle(\gamma_1^k, \gamma_2) = \beta \leqq 90°$. Since short and long perpendiculars alternate, we then have $0 < |\gamma_0^k| \leqq \beta/2$ for $k/2 \in \mathbb{N}_0$, and $90° - \beta/2 \leqq |\gamma_0^k| < 90°$ for $(k+1)/2 \in \mathbb{N}$.

However, only for $k=1$ the lengths do not exceed 180° and the same existence proof works. We obtain surfaces with an additional bubble. $\tilde{M}^{1,45^\circ}$ might look like this:



3.2 H-convex sets for long helicoids

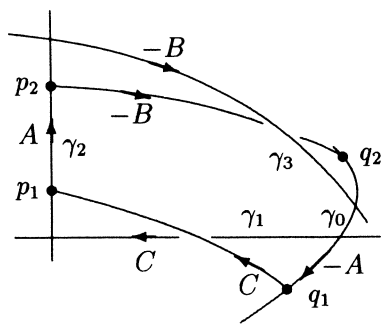
In the onduloid case, 90°-segments $W^a_{\xi/2}$ (see Theorem 2.4) can be embedded in solid Clifford tori.

Lemma 3.2 *Let* $a \neq 0$ *and* $\xi \leqq \alpha = \frac{\pi}{2}\frac{1}{1+|a|}$.

(i) *The helicoid* $W^a_{\xi/2} = \{f^a(u,v) | 0 \leqq u \leqq \xi, 0 \leqq v < 2\pi\}$ *is contained in a solid Clifford torus* T_1.

(ii) *There exists an H-convex set* T *with* $W^a_{\xi/2} \subset T$ *and* $\partial W^a_{\xi/2} \subset \partial T$.

Proof. (i) It suffices to take $\xi = \alpha$. We consider the helicoid $f^a([0,\alpha] \times [0,\pi/2])$ with its boundary Hopf fields given in 2.1. Let p_1 and q_1 be two points on the A-lines, being 45° away of the $-B$-line, towards γ_0 resp. γ_2. The great circle passing through p_1 and q_1 is again a C-line, since it belongs to the boundary of a quadrilateral with $l = 45°$ (i.e. $a = \pm 1$). *Let* T_1 be the Clifford torus with this C-soul.



Being a 45°-Clifford parallel the $-B$-line is contained in ∂T_1. The three other boundary curves of the quadrilateral are also contained in T_1: the C-line is

$|l-45°|$-Clifford-parallel to the soul; the two A-lines are perpendicular to the soul and do not contain points in a distance greater than 45° since $\xi \leqq \alpha < \pi/2$. In fact T_1 encloses the complete half period of the helicoids: every great circle arc $v \mapsto f(u, v)$ is at most in a distance of the soul given by the length of the longer common perpendicular. The perpendiculars are the A-lines and their length is at most 45°.

(ii) We define a second torus T_2 w.r.t. two points p_2 and q_2 on the A-lines in 45°-distance of the C-line. Its soul is a $-B$-line and the C-line is contained in ∂T_2. $T = T_1 \cap T_2$ is H-convex: The solid Clifford torus T_1 is C^2-H-convex and the family of surfaces needed for property (iv. a) of Definition 1.8 consists of Clifford tori ∂T_σ, whose soul interpolates between the one of T_1 and the one of T_2. □

Topologically T is again a solid torus. Let T^c be the universal covering of this torus. Topologically T^c is a solid cylinder and again H-convex. We shall use T^c to embed the long quadrilaterals. Let us describe the remaining barriers in the easiest case of $\Gamma^{k,90}$, which leads to an existence proof for long 90° segments of the onduloids ($a > 0$) with the help of Theorem 1.12.

To embed γ_0^k and γ_2 in an H-convex boundary we use great spheres. We call a rectangular geodesic 2-gon bounded by two 180° arcs a *quarter great sphere*. By Corollary 2.8 two asymptotic lines of a Clifford torus bound a quarter great sphere.

Let G_1 be a quarter great sphere with boundary in T_1^c defined as follows. Let one boundary curve be a 180°-arc of the $-B$-line and let the A-line γ_2 be contained in the interior of G_1. The boundary curve $\Gamma^{k,90}$ is seen to be completely contained in one of the two components of $T^c - G_1$ by the following argument: Each Clifford parallel to the soul only intersects a quarter sphere once, since in general a great circle only intersects a great sphere in two antipodal points. However int G_1 does not contain any of its antipodal points. The great circle γ_3 bounding the quarter sphere and the Clifford parallel to the soul γ_1 have the same sense of rotation and therefore the long edges of the quadrilateral lie on the same side of the quarter great sphere. Condition (iv. b) of Definition 1.8 can be satisfied by moving G_1 from ∞ to its position.

In the same way we obtain a quarter sphere $G_2^k \subset T_1^c$ at the other end of the quadrilateral. Then the compact connected component $N^{k,90}$ of $T^c - G_1 - G_2^k$ is an H-convex set, satisfying the requirements of Theorem 1.12.

3.3 Doubly periodic surfaces with many bubbles on the edges

We modify the H-convex set $N^{k,90}$ presented in the last section for the boundary curve $\Gamma^{k,\beta}$. By Lemma 2.7 the $(\sin\beta A - \cos\beta C)$-line is contained in T_1. Indeed it lies in the plane of the torus normal $-A$ and the asymptotic line C (since T_1 has a C-soul, C is asymptotic direction on the whole of ∂T_1); finally its length is $r \leqq 90°$. Similarly this line is contained in T_2, where A is normal and C asymptotic line.

We can take over G_2^k from 3.2. We intersect T_1^c on the side of γ_2 with a quarter great sphere G_1^β, bounded by a 180° arc of the $-B$ asymptotic line and with the $(\sin\beta A - \cos\beta C)$-line in its interior. Again we can move G_1^β from

∞ to satisfy Definition 1.8(iv. b). The argument given in 3.2 implies that the boundary curve is completely contained in a connected component $N^{k,\beta}$ of $T^c - G_1^\beta - G_2^k$. Theorem 1.12 provides a solution $M^{k,\beta}$ of Plateau's problem for the long quadrilaterals. The solution $M^{k,\beta}$ has k *bubbles* on an edge, that is the curvature lines along the edges have length in between $k\pi$ and $(k+1)\pi$.

Theorem 3.3 *Given an angle $\beta \in \{\pi/3, \pi/4, \pi/6\}$ there is a family $\tilde{M}_t^{k,\beta}$ of H-surfaces, with any number of bubbles for the edges $k \in \mathbb{N}$. The family is continuous with respect to the parameter $0 < t < \pi/2$. The surfaces are contained in a slab of $\mathbb{R}^3$ and their fundamental domain is contained in a triangle with angles β, $90°$, $90° - \beta$ times the ray $[0, \infty)$.*

Proof. It remains to prove continuity of the family. From Lemma 3.1 we know that $t \to \Gamma_t$ is continuous. Suppose the spherical Plateau solutions M_s do not converge to M_t for $s \to t$ w.r.t. distance of surfaces. By the technique of Theorem 4.4 we get another minimal surface M bounded by the same curve $\Gamma = \Gamma_t^{k,\beta}$ as M_t. Therefore continuity for M is reduced to uniqueness. But a continuous convergence is also in C^2 (or C^∞) in the interior and hence the associated surfaces $\tilde{M}_s$ converge w.r.t. distance (or in C^∞) in the interior; applying this argument to the surfaces extended by reflection yields the convergence in the desired sense up to the boundary.

To prove uniqueness we specify a family of isometries Ψ_σ of S^3 with $\Psi_0 = \mathrm{id}$ and $-\sigma_0 < \sigma < \sigma_0$, and find an H-convex set $N \supset \Gamma$ such that for every $0 < |\sigma| < \sigma_0$ there are neighbourhoods of Γ and $\Psi_\sigma(\Gamma)$ disjoint to $\Psi_\sigma(N) \cap N$. Then the two Plateau solutions M and $\Psi_\sigma(M)$ do not intersect each other in a neighbourhood of their boundaries. We claim they cannot intersect in the interior. Both surfaces are minimizing, and a replacement of any piece of one surface cut out by a piece of the other surface does not reduce area. But the intersection is transveral at all but finitely many points in the intersection set, so that smoothing out the edge made by a piece of one and a piece of the other surface does reduce area. Thus M is enclosed in between $\Psi_\sigma(M)$ and $\Psi_{-\sigma}(M)$ for any $\sigma \neq 0$.

For Ψ_σ we take the flow to the Killing field $B - C$. The Clifford torus ∂T_1^c contains the arc γ_3, being a $-B$ integral and it contains C-lines as asymptotic lines, because the T_1-soul is a C-line. Thus the field $B - C$ is tangent along γ_3. No other bundary arc is contained in ∂T_1^c and hence sufficiently small rotations of T_1^c with axis γ_3 also contain the quadrilateral Γ. We take N_1 to be the intersection of two such tori rotated in a different sense. Then N_1 does not contain (a neighbourhood of) the $\pm(B-C)$-direction along the arc γ_3. In the same way we define N_2 for the arc γ_1 along which the direction $B - C$ is again tangent to T_2^c.

We now describe two pairs of great spheres for the remaining arcs. This is more technical because we have to convert Hopf fields into parallel fields. We give the barriers for $k = 0$ so that we can use hemi-S^3; for the general case they have to be replaced by the appropriate component in the covering space. We start with the arc γ_0. The great sphere G_2 from 3.2 has a normal $\nu(\gamma_0(0)) = -B$. On a $90°$ arc γ along A the field B rotates to C (see Definition 1.3). That is a parallel translation of $-B$ from $\gamma(0)$ to $\gamma(90°)$ yields C. Viewing the $-A$-line γ_0 in decreasing parameterization we conclude that the normal of G_2 along γ_0 is $\nu(\gamma_0(s)) = -\cos s B + \sin s C$. But γ_0 is shorter than $90°$, and so $\nu(\gamma_0(l_0)) \cdot \dot{\gamma}_1$ is positive for the C-line γ_1. γ_1 and γ_2 are no longer than $90°$ and

hence contained in the hemi-$S^3 = \{x \in S^3 \mid x \cdot v \geqq 0\}$. Thus Γ is contained in that hemi-S^3.

We consider a great sphere G_3 containing γ_0 and γ_1. We claim that Γ is contained in the hemi-S^3 defined by $v(\gamma_0(l_0)) = B$. Indeed, $v(\gamma_0(s)) = -\sin(l_0 - s)\, C + \cos(l_0 - s)\, B$ has positive scalar product with B in 0. Again $v(\gamma_0(0)) \cdot (-\dot{\gamma}_3(l_3)) > 0$ as the latter direction is B and $l_0 < 90°$.

Now we have to show that the intersection N_3 of the two hemispheres does not contain the vector $\pm(B - C)$ along the arc γ_0. That is the product of $\pm(B - C)$ with v has to be negative for at least one normal for each sign of $B - C$. For the normal of G_2 we compute $v(\gamma_0(s)) \cdot (B - C) = -\cos s - \sin s$ and $v(\gamma_0(s)) \cdot (B - C) = \cos(l_0 - s) + \sin(l_0 - s)$ for the normal of G_3. Clearly the first product is negative and the second product is positive for $0 \leqq s \leqq l_0 < 90°$, so the second one is negative taken with $-(B - C)$.

We need similar calculations along γ_2. A normal of G_1^β at $\gamma_2(l_2)$ is $v(\gamma_2(l_2)) = -\cos\beta A - \sin\beta C$. As $\angle_{\sin\beta A - \cos\beta C}(-B, -\cos\beta A - \sin\beta C) = +90°$ the parallel translation of v to the point $\gamma_2(l_2 - 90°)$ is $+B$. Now $v(\gamma_2(s)) = \sin(l_2 - s)\, B + \cos(l_2 - s)(-\cos\beta A - \sin\beta C)$ and the product with $-C$ is positive so that Γ lies in the hemi-S^3 defined by v.

For a great sphere G_4^β defined by the normal $-B$ in the point $\gamma_2(0)$ the normal is $v(\gamma_2(s)) = -\cos s B + \sin s(\cos\beta A + \sin\beta C)$. Again $v(\gamma_2(l_2)) \cdot (-B) > 0$ and Γ is on the side of v. Finally we compute the products $\pm(B - C) \cdot v$ along γ_2. For the normal of G_1^β we have $(B - C) \cdot v(\gamma_2(s)) = \sin(l_2 - s) + \cos(l_2 - s)\sin\beta$ and $(B - C) \cdot v(\gamma_2(s)) = -\cos s - \sin s \sin\beta$ for G_4^β. The second product is negative for the given sign of $(B - C)$ as the first one is for the other sign. Hence the intersection of these two hemi-S^3, N_4, with N_1, N_2, and N_3 is an H-convex set with the property claimed.

When we compare the surface M with itself moved along the flow $\Psi_\sigma(M)$ this proof recovers M as a graph w.r.t. the $B - C$ direction. □

Remark. The H-surfaces have a different asymptotic behaviour for large and small quadrilaterals.

(i) The large quadrilaterals belong to H-surfaces with *spherical centres*. In fact, taking the limit $t \to 90°$ the large quadrilaterals $\Gamma_t^{0,\beta}$ converge to a triangle contained in a great sphere $S^2 \subset S^3$ and the associated H-surfaces converge to punctured spheres which touch each other with the given symmetry. The same applies to the centre spheres of the surfaces $\tilde{M}_t^{k,\beta}$. For a small neck perimeter $4|\gamma_0|$ the existence of the surfaces with a spherical centre was proved by Kapouleas [Kp1].

However, for the small quadrilaterals the cases $k = 0$ and $k \in \mathbb{N}$ are different.

(ii) Computing the lengths of the small quadrilaterals $\Gamma_t^{0,\beta}$ in terms of t, we see from (3.8) and (3.9) that r and l shrink of order $O(t)$, whereas s shrinks of order $O(t^2)$ by (3.3). Asymptotically the patches are therefore triangles. Hence when we blow up the associated surfaces in a way that t and the lattice is of constant size ($H \to 0$), the patches tend to flat triangles and the complete H-surfaces $\dfrac{1}{O(t)} \tilde{M}_t^{0,\beta}$ tend to punctured double planes. That is $\tilde{M}_t^{0,\beta}$ itself lies in a slab of height $O(t^2)$ and converges the more to a plane of multiplicity 2, but the singular points become dense.

(iii) The associated H-surfaces $\tilde{M}_t^{k,\beta}$ for $k > 0$ have k additional bubbles on the edges of the lattice. The blow up of the surfaces belonging to small quadrilaterals

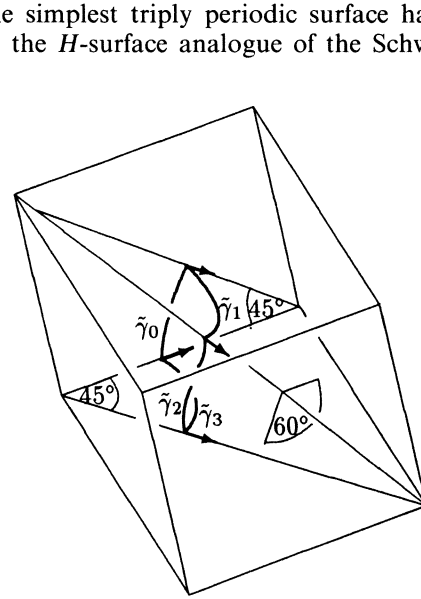
can be shown to converge to the Jorge-Meeks n-noids. Indeed, a blow up by $1/O(t)$ in the space $S^3_{1/O(t)}$ has two lines γ_1 and γ_3 of unbounded length $1/O(t)$ and one line γ_2 of bounded length $O(1)$. The limit boundary contour coincides with the associated boundary contour of the Euclidean n-noids. By an appropriate maximum principle at infinity (for minimal surfaces with boundary going to infinity) it can be proved that minimal surfaces spanned by the limit contour are unique. A proof will appear elsewhere. We say these surfaces have *n-noidal centres.*

The hexajonal surface $\tilde{M}^{0,\pi/3}_{45}$ and the quadratic surface $\tilde{M}^{0,\pi/4}_{45}$ with maximal neck size in their family were found by Lawson [L, Theorem 9]. They join the surfaces to large and small quadrilaterals in the family.

Centre-spheres and centre n-noids within the same surface. A rotation by 180° with axis γ^k_0 of the quadrilaterals $\Gamma^{k,\beta}_t$ is contained in the same topological torus T^c from 3.2 which contains $\Gamma^{k,\beta}_t$ itself. The Plateau solutions are associated to doubled fundamental H-patches and yield the same surfaces $\tilde{M}^{k,\beta}_t$ as before. By Lemma 3.1 the quadrilaterals $\Gamma^{k,\beta}_t$ and $\Gamma^{k,\beta}_{90-t}$ have the same length $|\gamma_0|$. We can glue them together along γ_0 and obtain H-surfaces each k-bubble edge of which joins a spherical with an n-noidal centre in case $\beta=45°$ and $\beta=60°$. Of course the same barriers work. This yields another one parameter family, being continuous w.r.t. to t with the same proof as in Theorem 3.3. One of the symmetries of $\tilde{M}^{k,\beta}_t$ is removed and the family under consideration coincides with the former family in the surface with the parameter $t=45°$.

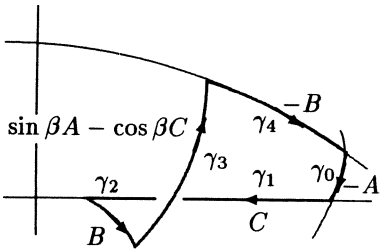
3.4 *Triply periodic surfaces*

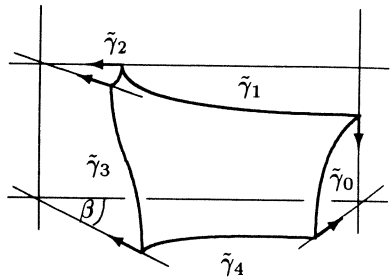
(a) Cubic lattice. The simplest triply periodic surface has the symmetry of a cubical lattice and is the H-surface analogue of the Schwarz P-surface. Again



there may be k bubbles on the edges. The boundary curve $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ has Hopf fields $-A$, $\frac{1}{2}\sqrt{2}(C+B)$, $\frac{1}{2}\sqrt{2}(A-C)$, $-B$. By the Handle-Theorem 1.7 the boundary curve closes for small $|\gamma_0|$. We are in the case of $\xi=\alpha/2$ of Lemma 3.2 and the respective tori as well as great spheres similar to those defined in the last section serve as barriers.

(b) Layers of doubly periodic surfaces joined by handles. We replace the vertex at the angle β in the quadrilateral from 3.3, $0<\beta\leqq 90°$, by a fifth edge along the field B to obtain an additional outward handle of the H-surface patch ($\angle_B(-C, \sin\beta A-\cos\beta C)=-\beta$):

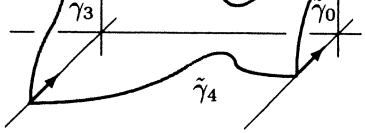


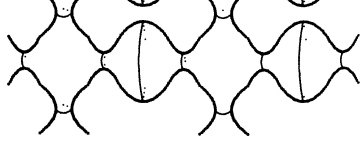


The Handle Theorem 1.7 ensures existence of the boundary curve with a small handle length $|\gamma_2|$ for any length of γ_0 less than $\beta/2$; only in case $l=\beta/2$ assumption (1.11) is not satisfied since $\gamma_1, \gamma_3, \gamma_4$ are then contained in a Clifford torus. In fact the handle length can be any value in $(0, \beta/2-l]$.

We can use the barriers from 3.2 and 3.3 to embed the boundary curve in an H-convex boundary. For $|\gamma_2|$ small the pentagon is certainly contained in T_1^c. The Clifford torus ∂T_2^c contains γ_2 and thus the pentagon is contained in T_2^c. To embed γ_3 we choose a great sphere containing this curve and the handle γ_2. Since the latter is an asymptotic line for T_2^c this great sphere cuts out a quarter sphere of T_2^c and γ_1 and γ_4 are on the same side of this great sphere by the argument given in 3.2. For γ_0 we use the quarter great sphere G_2^k defined in 3.2. Every vertex of the polygon is contained in two of these four barriers.

The complete H-surface consists of layers of doubly periodic surfaces, similar to those described before, whose vertices are vertically joined by handles. In case $\beta=45°$ the Schwarz P-surface is generalized in a less symmetric fashion than in (a). For $\beta=30°$ and $\beta=60°$ we obtain an H-surface analogue to A. Schoen's H'-T-surface. For $\beta=90°$ we get a merely doubly periodic H-surface: The lattice consists of rectangles the long edge of which takes k additional bubbles. E.g. for $k=1$ the fundamental domain and a portion of surface can be sketched as follows:





(c) Every second vertex joined by a handle. In the same way as in 3.3 we can consider boundary curves of double length. A handle on one side only yields a surface every second vertex of which is joined to the next layer. This makes sense for $\beta = 45°$ and $\beta = 60°$.

4 Convergence of ends

In Subsect. 3.3 we found Plateau solutions $M^{k,\beta}$ to quadrilaterals Γ^k of length approximately $k\pi$. For instance in case $\beta = 60°$ these are associated to H-surfaces with a triangular lattice and have k bubbles on the edges of the lattice (see second figure in Subsect. 3.1). To obtain a "Delaunay-surface with 3 ends" we let k tend to ∞. Convergence is proved for the spherical patches. In general these are not graphs and we use a local description. Interior convergence of the sequence is implied by curvature estimates and locally uniform area bounds. No regularity problems arise at the piecewise geodesic boundary. However to know the limiting position of the normal at the boundary of the end we have to prove uniqueness of the end in Subsect. 4.4.

4.1 Curvature estimate and graph lemma

A minimal surface is *stable*, if the second variation of the area is non-negative. R. Schoen proved that stable surfaces, in particular solutions to Plateau's problem, have uniformly bounded curvature outside a neighbourhood of the boundary curve:

Theorem 4.1 [S, Theorem 3] *Let M be a stable minimal surface immersed in a 3-manifold N and $x \in M$ be a point, such that the geodesic ball $B_r^2(x)$ is contained in M $(0 < r \leqq 1)$. Then the norm of the second fundamental form is bounded,*

$$|A(x)|^2 := \langle Se_1, e_1\rangle^2 + 2\langle Se_1, e_2\rangle^2 + \langle Se_2, e_2\rangle^2 \leqq C(N, r), \tag{4.1}$$

where C depends on bounds for the sectional curvature of N and its derivative.

Locally hypersurfaces can be described as graphs. In manifolds we can take normal coordinates in a geodesic ball $B_r^3(p) \cap M$ over the tangent plane $T_p M$. The radius $r > 0$ is uniform in p, if the curvature of the hypersurface is bounded:

Lemma 4.2 [Wh, Lemma 1; KKS; S] *Suppose N^3 is a manifold with injectivity radius bounded away from 0, whose metric has bounded C^2-norm in normal coordinates. Let M be a minimal surface in N whose second fundamental form is bounded on a domain $\Omega \subset M$,*

$$|A(x)| \leqq A_0 \quad \text{for all } x \in \Omega.$$

Then there exists a radius $\rho(A_0, N)$ such that for a point p with $B_{2\rho}^2(p) \subset \Omega$ (i) the connected component of $M \cap B_\rho^3(p)$ which contains p is a disk and (ii) is graph w.r.t. the tangent plane $T_p M$. (iii) Furthermore choosing $\rho(A_0, N, \alpha)$ the coordinates u of the graph have bounded Hölder norm on $B_\rho^2(0) \subset T_p M$:

$$\|u\|_{C^{2,\alpha}} \leqq C(A_0, N, \alpha).$$

Note this lemma also allows to describe a surface as graph over slightly tilted planes close to the tangent plane.

4.2 Convergence of compact minimal surfaces

Definition 4.3 A sequence M_k of C^1-surfaces satisfies *uniform local area bounds* if there is a radius r such that for each ball with $B^3_\rho(x) \cap \partial M_k = \varnothing$ and all $\rho < r$ holds

$$|M_k \cap B_\rho(x)| < C.$$

In Subsect. 4.3 we give uniform local area bounds for helicoid ends. Assuming area bounds we solve improper Plateau problems, that is problems with non-closed boundary curve, using an exhaustion:

Theorem 4.4 *Let N^3 be a non-compact closed manifold with boundary (whose curvature tensor is C^1-bounded) and Γ a connected non-closed curve in N, being piecewise C^1. Assume there exist compact sets N_n with $N_{n-1} \subset N_n$ and $\bigcup_{n \in \mathbb{N}} N_n = N$, and closed Jordan curves $\Gamma_n \subset N_n$ with $\Gamma_k \cap N_n = \Gamma \cap N_n$ for all $k > n$. Finally let the Plateau problems for Γ_n be solvable with a sequence of minimal disks $M_n \subset N_n$ satisfying uniform local area bounds. Then there exists a minimal surface $M \subset N$ of the type of the disk which is regular in the interior and whose boundary is Γ.*

Proof. Let $N_n^r := N_n - U_r(\bigcup_{k>n} \Gamma_k)$ where U_r denotes an open r-neighbourhood of a set. Because Γ is piecewise C^1 we have $\bigcup_{r>0} N_n^r = N_n - \Gamma$ and N_n^r is non-empty for small r. The Plateau solutions M_k accumulate in int N_n^r, for sufficiently small r; indeed, if there were accumulation points in ∂N_n only (for all but finitely many n), the maximum principle would yield a minimal surface in ∂N bounded by Γ and the proof finished. Thus we find a point $p \in \text{int } N_n^r$, such that a subsequence of $k \mapsto M_k$ converges in distance to p. Taking a further subsequence we assume the normals to converge as well.

For each $x \in N_n^{2r} \cap M_k$, $k > n$, we have $\text{dist}(x, \partial M_k) \geqq 2r$, and thus Theorem 4.1 provides a curvature bound on N_n^r, $|A(x)|^2 \leqq C(N, r)$ for all $x \in N_n^r \cap M_k$. By the graph Lemma 4.2 we find a radius $\rho (\leqq r/2$ wlog.), such that $M_k \cap B^3_{2\rho}(p)$ is graph over the tangent plane $T_p M_k$ provided $p \in M_k \cap N_n^{2r}$. Therefore a connected component of a subsequence of M_k converges in $C^{2,\alpha}(B_\rho(p))$ as graph w.r.t. the limit tangent plane $T_p N$.

We cover N_n^{2r} with balls $B^3_\rho(y_i) \subset N_n^r$, $i = 1, \ldots, I$. For an i_1 we have proved convergence of a connected component of M_k in $B_\rho(y_{i_1}) \subset B_{2\rho}(p)$. Since a subsequence converges in the whole ball $B_{2\rho}(p)$ there is a point $q \in B_{2\rho}(p) - B_\rho(y_{i_1})$ of convergence. Let q be in $B_\rho(y_{i_2})$, $i_1 \neq i_2$. By another application of the graph lemma we obtain convergence of a connected component of a further subsequence in $B_{2\rho}(q)$ and thus in $B_\rho(y_{i_2})$. We continue to consider further balls $B_\rho(y_{i_l})$ such that either $i_l \neq i_1, \ldots, i_{l-1}$ or for $i_l = i_j (l > j)$ another connected component of the surface is considered. By the uniform area bound the procedure ends after finitely many steps. Therefore we obtain $C^{2,\alpha}$-convergence of a subsequence of M_k, again denoted by $M_k^{(n)}$ to a surface $M_\infty^{(n)}$, such that $\partial M_\infty^{(n)} \cap N_n^r = \varnothing$.

Now we consider a sequence $N_n^{r_n}$ with $r_n \searrow 0$. With finitely many balls of smaller radius $\rho_n/2 < r_n$, given by curvature estimate and graph lemma, we extend the covering of $N_{n-1}^{r_{n-1}}$ to $N_n^{r_n}$. From the sequence $M_k^{(n-1)}$ we choose a subsequence $M_k^{(n)}$ converging on $N_n^{r_n}$. The diagonal sequence $M_n^{(n)}$ converges on each compact set $K \subset N - \Gamma = \bigcup_{n \in \mathbb{N}} N_n^{r_n}$ to $M = \bigcup_n M_\infty^{(n)}$ in $C^{2,\alpha}$. Since M has no boundary points in the interior of N and each subsequence accumulates in Γ we have $\partial M = \Gamma$.

Since the convergence is uniform in distance on each compact set we can compare M with a sufficiently close minimizing surface M_n. Hence M is a stable minimal surface and each curve in M is contractible. For connected Γ there is only one connected component M. □

If Γ consists of geodesic arcs making an angle of $\pi/(m+1)$, $m \in \mathbb{N}$, then M is smooth at the boundary (see Subsect. 1.7). If N is H-convex and all vertices are contained in two barriers as in Theorem 1.12 M is free of boundary branch points.

4.3 Uniform local area bounds for finite helicoid ends

First we describe coordinates on the covering of a solid Clifford torus induced by the foliation of Subsect. 2.2.

Let a helicoid $W_\xi^a = f^a([-\xi, \xi], [0, 2\pi))$ for $\xi < \alpha$ and $-1 < a \leqq 1$, $a \neq 0$, be given together with its induced foliation $\Psi: \mathbb{R} \times (-\alpha, \alpha) \times \mathbb{R} \to S^3$. We consider closed sets $N \subset S^3$ satisfying three properties: (i) N contains the helicoids W_ξ^a, and $\zeta_\pm(v) := f^a(\xi, v)$ is contained in ∂N. (ii) The class of the curves $\zeta_\pm$ is the only non-trivial homotopy class in N and (iii) N contains no curve linked with $\eta_\pm = f^a(\pm a, .)$. Let X denote the connected component of $\Psi^{-1}(N - \eta_\pm)$ containing $(0, 0, 0)$. Because of (iii) and Lemma 2.3(ii) if $(\phi, u, v) \in X$ then $(\phi \pm \pi, u, v) \notin X$, so that $X \subset (-\pi, \pi) \times [-\xi, \xi] \times \mathbb{R}$. By (ii) and Lemma 2.3(iii), (iv) the parameterization Ψ lifts to a bijection $\Psi^c: X \to N^c - \eta_\pm^c$. By (i) X contains the set $0 \times [-\xi, \xi] \times \mathbb{R}$. We omit the suberscript c for N and $\eta_\pm$.

Let $N_x^y := \{\Psi^c(\phi, u, v) \in N - \eta_\pm \mid x \leqq v < y\}$ be the portion of N between x and y, $D_x := \{\Psi^c(\phi, u, x) \in N\}$ be the lateral boundary of N and $D_x^y := D_x \cup D_y$. We prove that k-times the area of the helicoid $w_\xi^a := |f^a([-\xi, \xi] \times [0, \pi))|$ is a lower bound for the area of any piece of length $2\pi k$ of the surfaces $M = M_l$ defined in Sect. 3:

Lemma 4.5 *Let $M \subset N_0^{\pi k}$ be an immersed C^1-surface with* finite helicoid end, *i.e.*

(i) *$\partial M \subset \zeta_\pm^c \cup D_0^{\pi k}$ with $\zeta_\pm$ as above.*
(ii) *Every closed curve in M is contractible in $M \cup D_0^{\pi k}$.*
(iii) *The two connected components of $(\partial N_x^y - D_x^y) - \zeta_\pm^c$ are contained in different components N_+ and N_- of $N - M$.*

Then we have the following area bound, where $c := 2|D_0^{\pi k}|$:

$$|M| \geqq k w_\xi^a - c. \tag{4.2}$$

Proof. We find an embedded comparison surface of the type of the disk which has the boundary of the helicoid. Let $M_0 := M \cap \partial N_+$. Rounding off corners and edges of M_0 yields an embedded surface M_1 with $|M_1| \leqq |M_0|$. We discard

for all connected components of M_1 except for the one containing $\zeta^c_\pm$. $M_1 \cap D_0$ is a disjoint union of closed Jordan curves as well as another curve with end points $\zeta^c_\pm(0)$, being homotopic to a boundary curve of a helicoid $\gamma_0 \subset D_0 \cap W^a_\xi$. The area of this homotopy is at most $|D_0|$. The closed curves in $M_1 \cap D_0$ bound disjoint disks with total area less than $|D_0|$. Proceeding at the other end $D_{\pi k}$ in the same way, we obtain an embedded surface of the type of the disk by (ii) and with boundary of the helicoids. From Corollary 2.5, applied to k periods of the helicoid, we infer (4.2). □

Lemma 4.6 *Let M_k be a sequence of minima of area fulfilling the hypothesis of Lemma 4.5. Then M_k satisfies uniform local area bounds.*

Proof (idea of E. Kuwert). Γ_k bounds a surface of area $k w^a_\xi + c_1$, where c_1 is the area of a disk spanned by the difference of the curve Γ_k to the helicoid boundary. This together with (4.2), applied to outer portions of M_k, yields an estimate for the area of a middle part $|M_k \cap N^{\pi(l+m)}_{\pi l}|$, where $l+m+n=k$:

$$\begin{aligned} |M_k \cap N^{\pi(l+m)}_{\pi l}| &= |M_k \cap N^{\pi(l+m+n)}_0| - |M_k \cap N^{\pi l}_0| - |M_k \cap N^{\pi(l+m+n)}_{\pi(l+m)}| \\ &\leqq (l+m+n)\, w^a_\xi + c_1 - l w^a_\xi + c - n w^a_\xi + c \\ &= m w^a_\xi + c_1 + 2c. \quad \square \end{aligned}$$

4.4 Uniqueness of ends and convergence of boundary normals

We call a non-compact minimal surface M contained in a half cylinder $N \cap \{v \geqq 0\}$ a *helicoid end* if ∂M is a Jordan curve consisting of two geodesic rays $\zeta^c_\pm(s)$, $s \in \mathbb{R}^+_0$, joined by a compact curve ζ_0. Here again $\zeta^c_\pm(s)$ denotes a covering of $f^a(\pm\xi, s)$ with nonzero $-1 < a \leqq 1$ and $\xi < \alpha$. We can pull back the end in its H-convex barriers to obtain a limit surface which is periodic. By Corollary 2.4 the limiting helicoid type is uniquely determined by $\zeta^c_\pm$. We say an end satisfies uniform local area bounds if this holds for the pull back sequence $M_k := \Xi_{-k} M$, where the pull back $\Xi: N \to N$ is defined by $\Xi_{-k}(p) = \Psi^c((\Psi^c)^{-1}(p) - (0, 0, k\pi))$.

Theorem 4.7 *Let M be a minimizing (w.r.t. compactly supported variations) helicoid end, satisfying uniform local area bounds and contained in an H-convex set N. Then M_n converges in distance to a helicoid W^c uniquely determined by $\zeta^c_\pm$.*

Proof. Similarly to Theorem 4.4 we get convergence of a subsequence of M_k to a minimal surface M_0, using area and curvature bounds. The convergence is in $C^{2,\alpha}$ in the interior.

Let us first note that we can assume the coordinates Ψ^c to be defined in a neighbourhood of any point $p \in \operatorname{int} M_0$. Otherwise p is contained in the boundary ∂N and by the maximum principle either $\operatorname{int} M_0 \subset \operatorname{int} N$ or M_0 coincides with a subset of ∂N bounded by $\zeta^c_\pm$, which must be W^c by minimality of M.

Suppose for a moment in addition to our assumptions that

$$\Xi_k M \text{ is a barrier for } M, \tag{4.3}$$

then $\Xi_{-k} M$ and $\Xi_{-k-1} M$ intersect in the boundary rays $\zeta^c_\pm$ only. Thus M_k converges monotonely to M_0 and therefore the whole sequence converges and M_0 is invariant under covering transformations, $\Xi M_0 = \lim \Xi_{-k} M = M_0$, that

is M_0 projects to S^3. From Theorem 2.4 we obtain the uniqueness $\Pi M_0 = W^a_\xi$. (4.3) holds for all surfaces with cylinder ends we describe.

In general consider

$$\lim_{k\to\infty} \sup\{\phi \mid \Psi^c(\phi, u, v) \in M \cap N^\infty_{\pi k}\} =: \mu.$$

Since $\Psi^c(0, \pm\xi, v) \in \zeta^c_\pm \subset \partial M$ and Ψ^c is bounded by π we have $0 \leqq \mu \leqq \pi$. Suppose $\mu > 0$. Then there is a sequence of points $p_k := \Psi^c(\phi_k, u_k, v_k) \in M \subset N^\infty_{\pi k}$ with $\phi_k \to \mu$. Replacing $M_k := M \cap B^3_\varepsilon(p_k)$ by $\{\Psi^c(\phi - \varepsilon_k, u, v) \mid \Psi^c(\phi, u, v) \in M_k\}$, where $\varepsilon_k \searrow 0$, if necessary, we assume $\phi_k \leqq \mu$ for all $\Psi^c(\phi, u, v) \in M_k$. But for any subsequence M_{k_i} such that $(u_k, v_{k_i} \bmod 2\pi)$ converges to (u_0, v_0), ΠM_{k_i} is a minimal surface which converges in a uniform neighbourhood of $\Psi(\mu, u_0, v_0)$ to a M_0 by curvature estimate and graph lemma. In particular by the maximality of μ the normals converge. Since M_0 lies locally on one side of the minimal surface $\Psi(\mu, u, v)$ and touches in the interior point $\Psi(\mu, u_0, v_0)$ the surfaces locally agree by the maximum principle. Hence any subsequence M_{k_i} converges to $\Psi(\mu, u_0, v_0)$ in a uniform neighbourhood (depending on the distance to the boundary). Repeating these arguments for any accumulation point $\Psi(\mu, u, v)$ we infer that $\Xi_n M \to \Psi^c(\mu, u, v)$. By the boundary values we obtain $\mu = 0$. Applying the same arguments to $\lim_{k\to\infty} \inf\{\dots\}$ proves the result. □

Now we have proved that the spherical ends converge in distance a helicoid. Next we show C^1-convergence at the boundary: when moving towards infinity the normals at the boundary converge to those of the helicoids.

Since we know convergence in distance, we can use a local barrier argument. Take a point $p = f^a(u, v) \in \partial W^a$ in the boundary of a helicoid. Extend the helicoid to a full neighbourhood W^a_* of p and let $D := B^3_\varepsilon \cap W^a_*$ be a minimal disk. Now we rotate D first around the normal $n(p)$ by an angle $\pm\psi$. Since the principal curvatures of the helicoid do not vanish the rotated disks do not intersect the boundary of the helicoid ∂W^a except for the point p itself (for small ψ and ε). The tangent planes in p agree. Thus a further rotation of the disks around the boundary circle ∂W^a by the angle ψ yields two disks $D^\pm_\psi$ that do not intersect the helicoid, provided the rotations have the right sense.

Let M_k be a sequence of minimal surfaces converging in distance to the helicoid and with boundary of the helicoid. If M_k is close enough to the helicoid in a neighbourhood of a point p then ∂M_k is on one side of $D^\pm_\psi$. By the maximum principle the tangent halfplane at $T_p M_k$ is contained in the wedge between the tangent halfplanes of D^+_ψ and D^-_ψ. As ψ tends to 0, $T_p M_k$ converges to the tangent plane of $D^\pm_0 = W^a$ in p. A similar construction works for points with vanishing principal curvatures. Slightly generalizing we get

Lemma 4.8 *Let M_k be a sequence of minimal surfaces with the same C^2-boundary portion Γ which converges in distance to a minimal surface M of class C^2 including Γ. Then at each interior point of Γ the normals of the surfaces M_k converge to the normal of M.*

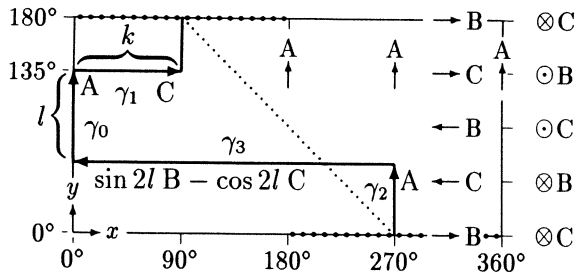
Thus we can read off the geometric data from the ends of the minimizing helicoid defined by $\zeta_\pm$ in case $\xi < \alpha$.

5 *H*-surfaces with cylinder ends and their periodic cousins

5.1 *Surfaces of Pinkall and Sterling*

The following surfaces are parameterized by a torus and described in [PS] where an explicit representation by solutions of the sinh-Gordon-equation is given. See also [We].

Simply periodic. The simply periodic cousins of these surfaces have immersed ends not converging to Delaunay surfaces. Take a solid Clifford torus T with Hopf fields as in Subsect. 2.3. Let $\Gamma^{k,l} \subset \partial T$ for $k>0$, $0<l<180°$ be a curve in ∂T, being the image of the following curve in parameter space (in the sketch we take $k=90°$):



$\Gamma^{k,l}$ is embedded in the Clifford cylinder T^c and has geometric data:

$$
\begin{aligned}
&|\gamma_0^{k,l}| = l, && \angle_A(\cos 2l\, C - \sin 2l\, B, C) = -2l,\\
&|\gamma_1^{k,l}| = k\pi, && \angle_C(-A, A) = 180°,\\
&|\gamma_2^{k,l}| = \pi - l, && \angle_A(-C, \sin 2l\, B - \cos 2l\, C) = 2l - 360°,\\
&|\gamma_3^{k,l}| = \pi + k, && \angle_{\sin 2l\, B - \cos 2l\, C}(-A, A) = -180°.
\end{aligned}
$$

The given rotational Hopf angles are measured inside T in the sense of Theorem 1.12(ii). Namely along γ_1 the inner normal of the torus is $-B$ by (2.8). Therefore $-A$ rotates through $-B$ to A. Now $\angle_C(-A, -B) = \angle_C(-B, A) = 90°$ and we get $t_1 = 90° + 90° = 180°$ as claimed. Similarly we get t_2 and t_3.

We have to verify that $\Gamma^{k,l}$ bounds a disk in T^c. We show that $\Gamma^{k,l}$ is contractible. $\Gamma^{k,l}$ is homotopic to the dotted curve $y + x = 270°$. The image of the map $[0, 45°] \to T$, $\rho \mapsto F(\rho, x, x-\pi)$ is the dotted curve for $\rho = 45°$ and shrinks to a point for $\rho = 0$. This curve is also contractible in the covering cylinder T^c.

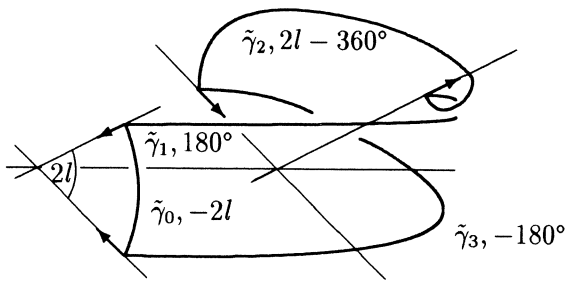
We need barriers in addition to ∂T^c. We define two great spheres: Extending γ_0 to 180° yields a new endpoint of γ_0 on γ_3. The arc $\gamma_3^k(t)$, $k \leqq t \leqq k+\pi$, closes γ_0 to a 2-gon. This 2-gon is contractible and thus defines a quarter sphere Q_1. In the same way the extended arc γ_2 and an arc of γ_3 bounds a quarter sphere Q_2. Then $\Gamma^{k,l}$ is contained in the compact component of $N^k := T^c - Q_1 - Q_2$ and all vertices of $\Gamma^{k,l}$ are contained in two different barriers.

By Theorem 1.12 we get existence of an H-surface patch with 4 edges and tilting angles $-2l$, 180°, $2l - 360°$, $-180°$. The surfaces have different symmetry properties for l less or greater than 90°. The case $l = 90°$ is exceptional: The

boundary curve $\Gamma^{k,90}$ spans helicoids of negative a, whose associated surfaces are 180°-segments of the nodoids (see 2.1). Indeed, 90° arcs through $\gamma_3(t)$ in direction of $-\cos\frac{t}{1+k/\pi}A+\sin\frac{t}{1+k/\pi}B$ give the ruling which is contained in T by Lemma 2.7.

We claim that for $l\neq 90°$ the curves $\tilde{\gamma}_0^{k,l}$ and $\tilde{\gamma}_2^{k,l}$ are contained in different planes. Let $M^{k,l}$ be the Plateau solution of $\Gamma^{k,l}$. Any translation of $M^{k,l}$ in x-direction is a barrier for $M^{k,l}$ itself. The translation leaves the Hopf directions of γ_1 and γ_3 invariant and thus the rotational Hopf angle is monotone along γ_1 and γ_3. It follows that $\tilde{\gamma}_1$ and $\tilde{\gamma}_3$ are convex planar curves. On the other hand the nodoid $W=M^{k,90}$ with the same edge $\gamma_1^{k,90}$ is a barrier for $M^{k,l}$. But for the nodoid $\tilde{\gamma}_0^{k,90}$ and $\tilde{\gamma}_2^{k,90}$ are contained in the same plane. Thus the claim follows form the Comparison Lemma 5.1.

Reflection at the symmetry planes yields a simply periodic immersed H-surface $\tilde{M}^{k,l}$. If $2l\cdot m=\pi$ the surface closes through $2m-1$ reflections. The fundamental H-patch looks as follows:



The simply periodic H-surface $\tilde{M}^{k,l}$ is a cylinder with lobes of m spheres attached in a constant distance. $k>0$ is a parameter of this distance. If more generally $2l\cdot m/\mu=\pi$ (m, $\mu\in\mathbb{N}$ without common divisor), the cylindrical tubes are covered μ-times and lobes of $m\cdot\mu$ spheres are attached. For $90°<l<180°$ the roles of γ_0 and γ_2 swap and we obtain the same surfaces as for $90°-l$.

With ends. By Lemma 4.6 the sequence $M^{k,l}$ of minimal surfaces satisfies uniform area bounds and thus we obtain from Theorem 4.4 a minimal surface $M^{\infty,l}\subset T^c$, bounded by the two rays $\gamma_1^{\infty,l}$, $\gamma_3^{\infty,l}$ and by the arc γ_2^l. According to Subsect. 4.3 the solid Clifford torus T is foliated by Clifford tori, such that $\eta_\pm$ are the great circles $F(45°, x, 135°-l/2\pm 45°)$. In case $l\neq 90°$ we can apply Theorem 4.7 (note (4.3) holds) and obtain that the end is asymptotically the strip $S^l=\{F(45°,x,y)|135°-l<y<135°\}\subset\partial T$. For $0<l<90°$ the arc γ_0 is contained in this strip and the geometric data of the surface with end $M^{\infty,l}$ are those of the approximating surfaces $M^{k,l}$ by Lemma 4.8. Thus we obtain an H-surface $\tilde{M}^{\infty,l}$ with two cylinder ends in case $2l\cdot m=\pi$. To this cylinder a lobe of m spheres is attached in the middle. As in the periodic case the cylinder ends are covered μ-times, if $2l\cdot m/\mu=\pi$. For $90°<l<180°$ the attached spheres on $\tilde{M}^{k,l}$ disappear to infinity as $k\to\infty$ and $\tilde{M}^{\infty,l}$ is the ordinary cylinder.

The surface $\tilde{M}^{\infty,45}$ is pictured in the paper of Pinkall and Sterling [PS].

5.2 Comparison of planar curves

Lemma 5.1 *Let $\gamma, \Gamma\colon [0,l] \to \mathbb{R}^2$ be curves with $\gamma(0)=\Gamma(0)=0$. Suppose for*

$$\dot\gamma(s)=\begin{pmatrix}\cos t(s)\\ \sin t(s)\end{pmatrix} \quad \text{and} \quad \dot\Gamma(s)=\begin{pmatrix}\cos T(s)\\ \sin T(s)\end{pmatrix}$$

and $s\in(0,l)$ holds

$$t(0)=T(0)=0\leqq t(s)<T(s)\leqq 180°=t(l)=T(l). \tag{5.1}$$

Then at the end points the first components of the curves satisfy

$$\Gamma_1(l)<\gamma_1(l).$$

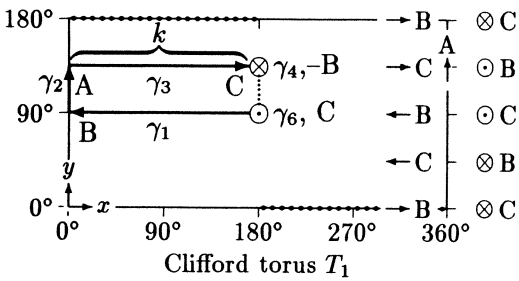
Proof.

$$\Gamma_1(l)=\int_0^l \dot\Gamma_1(s)\,ds=\int_0^l \cos T(s)\,ds<\int_0^l \cos t(s)\,ds=\int_0^l \dot\gamma_1(s)\,ds=\gamma_1(l). \quad \square$$

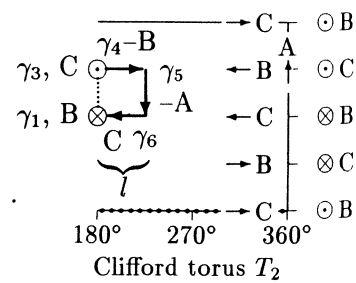
Let $\gamma=\Gamma$ be a geodesic arc in S^3 bounding two minimal surfaces M_1, M_2 which have the same tangent planes in $\gamma(0)$ and $\gamma(l)$ but do not intersect in the interior. By the boundary maximum principle the rotation angles r and R of M_1 and M_2 along γ satisfy $r(s)<R(s)$ for $0<s<l$ and we get by (1.6) $t(s)<T(s)$. The other inequalities in (5.1) certainly hold for convex curves. We conclude that if the end points of $\tilde\gamma$ have the same height then the end points of $\tilde\Gamma$ don't.

5.3 Doubly periodic pile of cylinders

In the next two sections we construct H-surfaces with umbillic points, which are parameterized by surfaces of higher genus. Take two Clifford tori ∂T_1 and ∂T_2, such that ∂T_2 is the 90°-rotation of ∂T_1 about the great circle $x=0°, 180°$.



Clifford torus T_1



Clifford torus T_2

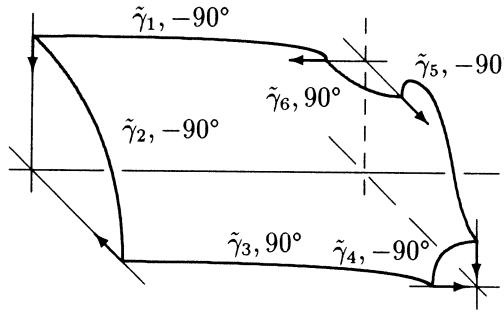
The curve $\Gamma^{k,l}$ has the geometric data ($k>0, 0<l\leqq 45°$),

$$\begin{aligned}
&|\gamma_1^{k,l}|=k, && \angle_B(-C,A)=-90°, && |\gamma_4^{k,l}|=l, && \angle_{-B}(-C,-A)=-90°,\\
&|\gamma_2^{k,l}|=45°, && \angle_A(-B,C)=-90°, && |\gamma_5^{k,l}|=45°, && \angle_{-A}(B,C)=-90°,\\
&|\gamma_3^{k,l}|=k, && \angle_C(-A,-B)=90°, && |\gamma_6^{k,l}|=l, && \angle_C(A,B)=90°.
\end{aligned}$$

$\Gamma^{k,l}$ is contained in the solid torus T_1. Indeed γ_5 is a $45°-l$ Clifford parallel to the A-soul of T_1, and γ_4 as well as γ_6 are no longer than 90°. In the covering torus T_1^c the curve $\Gamma^{k,l}$ is contractible.

Let T_3 be a solid Clifford torus with soul γ_3. γ_1 and γ_6 are contained in ∂T_3, and γ_2 and γ_4 are in T_3 since they are orthogonal to the soul and of length less than 45°. γ_5 is orthogonal to γ_6 and contained in T_3 by Lemma 2.7. In the same way the torus T_4 with soul γ_1 contains Γ. Let N be the H-convex component of $T_1^c \cap T_3^c \cap T_4^c$ which contains $\Gamma^{k,l}$.

By Theorem 1.12 we obtain an H-surface-patch with one interior handle $\tilde{\gamma}_6$ of width l and one exterior handle of width $\tilde{\gamma}_4$, looking as follows ($\tilde{\gamma}_5$ contains an umbilic point):



By reflection we get a doubly periodic surface $\tilde{M}^{k,l}$ with cylindrical tubes of length approximately k. By Theorem 4.4 we obtain the limiting surface for $k \to \infty$. This surface is doubly periodic and has cylinder ends according to Theorem 4.7 (note that (4.3) holds).

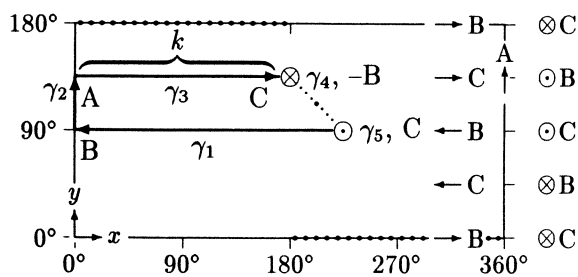
Dividing the complete surface by all translations yields a surface of genus 2. If we choose $|\gamma_2|$ and $|\gamma_5|$ to be 30°, we get a doubly periodic surface, each cylinder of which is joined with handles to six nearby cylinders (genus 3).

5.4 *Cylinder ends in a halfspace*

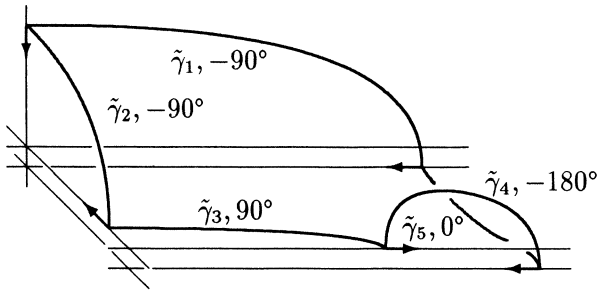
Let the curve Γ^k, $k>0$, with the data

$$
\begin{aligned}
&|\gamma_1^k| = k+45°, && \angle_B(-C, A) = -90°,\\
&|\gamma_2^k| = 45°, && \angle_A(-B, C) = -90°,\\
&|\gamma_3^k| = k, && \angle_C(-A, -B) = 90°,\\
&|\gamma_4^k| = 45°, && \angle_{-B}(-C, C) = -180° \quad \text{(over } -A\text{)},\\
&|\gamma_5^k| = 45°, && \angle_C(B, B) = 0°
\end{aligned}
$$

be given on a Clifford torus T as follows:



The arcs γ_4 and γ_5 have length 45° and meet each other exactly in a point of the soul of T. A disk G of radius 45° of a great sphere containing these curves (and orthogonal to the soul) furnishes a barrier. Using the cylinder barriers from the last section we obtain an H-convex set N^k, being the component of $T^c \cap T_3^c \cap T_4^c - G$ containing Γ, and the requirements of Theorem 1.12 are satisfied. The generating patch of the H-surface $\tilde{M}^k$ looks like:



The end points of $\tilde{\gamma}_4$ must sit in different planes. The periodic H-surface $\tilde{M}^k$ is contained in a slab between two parallel planes. The surface $\tilde{M}$ obtained by Theorem 4.4 as the limiting surface has cylinder ends in a halfspace by Theorem 4.7.

5.5 Non-existence results

Instead of listing more difficult examples we give non-existence proofs in certain symmetric classes. Suppose we had a cylinder as in 5.3 but with two handles only. A fundamental patch (a 90°-segment of one cylinder end) would be associated to a surface in S^3 with an approximating boundary curve $\Gamma = \gamma_1, \ldots, \gamma_5$ along, say, the Hopf fields B, A, C, $-A \pm C$. γ_5 is the handle and $|\gamma_2| = 45°$ is the curve approaching the cylinder end. Hence γ_1, γ_2, γ_3, γ_4 are contained in a cylinder ∂T and γ_5 is orthogonal. In conclusion the length of γ_5 must be a multiple of 90° in order to close the boundary curve. A shorter handle is not possible. (However with the lengths k, l, $k+90°$, $l+90°$, 90° the boundary curve closes such that for $l = 45°$ we have $\dot{\gamma}_5 = +C$.)

In the same way a cylinder as in 5.3 with two handles to the outside does not exist. Indeed we had to find lengths in 5.3 such that γ_6 is tangent to $-C$: as above we require $|\gamma_2| = 45°$ in order to obtain a 90°-segment of the cylinder end. But for $0 < |\gamma_4|, |\gamma_6| < 90°$ the end point of $\gamma_4(l_4)$ is contained in T_1 (independently of the lengths of γ_1 and γ_3) whereas $\gamma_6(0)$ is in $S^3 - T_1$. The A-great circles are either contained in T_1 or in $S^3 - T_1$ (see Subsect. 2.3) and therefore $\gamma_4(l_4)$ and $\gamma_6(0)$ can not be joined by an A-line γ_5.

6 *H*-surfaces with Delaunay-ends

In this section we describe H-surfaces with general ends of onduloid type ($0 < a < 1$). The existence of these surfaces follows immediately from the existence of the periodic surfaces given in Sect. 3 by the limit methods of Sect. 4.

6.1 n-ends-surface

In Subsect. 3.3 we described barriers for doubly periodic surfaces with k bubbles on the edges. Using Lemma 4.6 and the H-convex sets from Subsect. 3.3 we see Theorem 4.4 applies and hence the surfaces $M^{k,\beta}$, bounded by $\Gamma^{k,\beta}$, converge to a surface M^{β} as $k\to\infty$. By Theorem 4.7 the ends converge to a helicoid f^a defined by γ_1 and γ_3. Since γ_0^k is in the helicoid f^a for all $k\in\mathbb{N}$ the geometric data of the end coincide with those of the finite quadrilaterals, see Lemma 4.8. Thus the H-surfaces have $n\geqq 3$ ends in a plane making the same angle with each other, provided $\beta=\pi/n$. The centres are spherical or n-noidal, depending on whether we start with a small or large quadrilateral. From Lemma 3.1 and Theorem 3.3 we conclude uniqueness and existence:

Theorem 6.1 (i) *Every n-ends-surface which lies between two parallel planes of* $\mathbb{R}^3$, *has maximal symmetry, and whose ends converge to an embedded Delaunay surface has a quotient of radii with*

$$0<a\leqq\frac{1}{n-1}, \tag{6.1}$$

if the boundary arc of finite length in the fundamental domain is no longer than 90°.

(ii) *There is a continuous one-parameter family of surfaces* $M_t^{\pi/n}$, $0<t<90°$, *with the properties mentioned above. In the family exactly the two different surfaces* $M_t^{\pi/n}$ *and* $M_{90-t}^{\pi/n}$ *for* $0<t<45°$ *have asymptotically the same Delaunay ends. Each value of a in (6.1) is taken twice except for the maximal value* $1/(n-1)$ *which is only taken by* $M_{45}^{\pi/n}$.

Remark. The surface $\tilde{M}_t^{\pi/n}$ for $0<t<45°$ has an n-noidal centre, in the terminology of the Remark in Subsect. 3.3. The k-th bubble is closer to the centre when compared with the surface with spherical centre $\tilde{M}_{90-t}^{\pi/n}$, because t is the length of the curvature line. Certainly these surfaces have the asymptotic behaviour for $t\to 0°$ and 90° claimed in Remark (i) and (iii) of Subsect. 3.3. It eases the imagination of the family to consider the two degenerate situations of $t\to 0$ and $t\to 90°$, when the ends tend to a chain of spheres.

Take a ray of spheres and mark the end point on the axis. To glue the rays with the n marked points onto a centre sphere $\mathbb{S}^2$ in a regular fashion yields the limit for $t\to 90°$. To glue these points onto the origin of $\mathbb{R}^3$ gives the limit $t\to 0$; in this case the n spheres closest to the origin intersect each other, so that by continuity the surfaces $\tilde{M}_t^{\pi/n}$ cannot be embedded for small t.

Proof. (i) We consider the spherical fundamental path $M\subset\mathbb{S}^3$ associated to a 90°-segment of an end. This is bounded by two great circle rays γ_1 and γ_3, and a great circle arc γ_2. By the symmetry of the surface we can take γ_1, γ_2, γ_3 to be integrals of the Hopf fields C, $\sin\beta A-\cos\beta C$, $-B$ as in Subsect. 3.1. By the assumption on the length $|\gamma_2|$ we are in the situation of Lemma 3.1 and the perpendiculars γ_1 and γ_3 have length $0<l\leqq\beta/2$. Thus the quotient of radii is

$$0<\frac{l}{90°-l}\leqq\frac{\beta/2}{90°-\beta/2}=\frac{1}{180°/\beta-1}=\frac{1}{n-1},$$

for $\beta=\pi/n$.

(ii) We get continuity with the double-barrier technique from the proof of Theorem 3.3. However, in this non-compact case we need a maximum principle at infinity to rule out a non-compact intersection of a minimal surface M with $\Psi_\sigma(M)$. But this is given through Theorem 4.7: Both surfaces converge to the same helicoid, so there is no intersection of M with $\Psi_\sigma(M)$ in a neighbourhood of infinity. □

6.2 Ends having the symmetry of the regular polyhedra

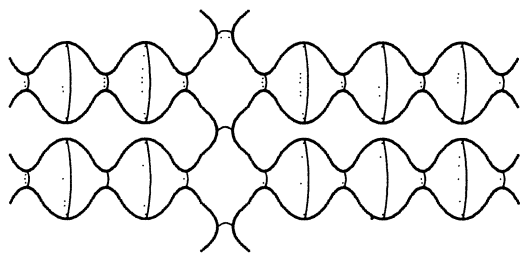
In 3.4(a) we found a surface patch with cubical symmetry. This can be generalized to the symmetry of any regular polyhedron. Since the segment needed is always less than 90° (in fact 36°, 45°, or 60°), it is easy to find the barriers.

6.3 n-ends-surfaces joined with handles

Taking the limit $k \to \infty$ of the triply periodic surface from Subsect. 3.4(b) we get a simply periodic surface with ends: It consists of (vertical) translations of n-ends-surfaces ($n \geqq 3$), where two adjacent translates are joined by a handle.

6.4 A fence of Delaunay surfaces joined by handles

Taking the doubly periodic surface of Subsect. 3.4(b) with $\beta = 90°$ the limit yields Delaunay surfaces, one of each bubbles is joined with a handle to the next surface. This can be considered the surface for $n = 2$ of the preceeding section. By the remark at the end of 3.4(b) each a-parameter for the end occurs except for $a = 1$, the cylinder end. We can similarly construct doubly periodic surfaces with Delaunay ends.



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Holomorphic mappings between spaces of different dimensions. I

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0 Introduction

In this paper we generalize some results for holomorphic mappings between equidimensional spaces to a nonequidimensional case which will contain generalizations of Schwarz lemma, a Kodaira's theorem and a Griffiths' theorem and discuss some of its applications.

1 A generalization of Schwarz lemma

In this section we use all notations and terminology of Sect. 14 of Stoll [9] and present some results on Jacobian sections. Let M and N be complex manifolds of dimensions m and n, respectively, and let θ and ω be the associated 2-forms of hermitian metrics ds_M^2 and ds_N^2 on M and N, respectively. We will study a holomorphic mapping

$$f: M \to N. \tag{1.1}$$

Lemma 1.1 *Take* $M = \mathbb{C}^m$ *and define*

$$Z_i := f_*(\partial/\partial z_i) \qquad i = 1, 2, \ldots, m. \tag{1.2}$$

Assume $q = n - m > 0$. *Then a holomorphic field* φ *on* f *over* $\mathbb{C}^m$ *of degree* q *is effective for* f *if and only if*

$$Z := Z_{1f} \wedge \ldots \wedge Z_{mf} \wedge \varphi \not\equiv 0. \tag{1.3}$$

Further, we have $\operatorname{div}(F_\varphi) = \operatorname{div}(Z)$.

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Proof. Write

$$\varphi = \sum_{\mu \in \mathfrak{T}(q,n)} \varphi_\mu \, \omega^*_{\mu f}, \qquad Z_i = \sum_{k=1}^{n} f_i^k \, \omega_k^* \tag{1.4}$$

where $\omega_k^* = \partial/\partial w_k$ and $f_i^k = \partial w_k / \partial z_i$ relative to a local coordinates w_k on $U \subset N$. Then

$$Z_{1f} \wedge \ldots \wedge Z_{mf} = \sum_{\nu \in \mathfrak{T}(m,n)} A_\nu \, \omega^*_{\nu f} \tag{1.5}$$

$$Z = \sum_{\mu \in \mathfrak{T}(q,n)} \operatorname{sign}(\mu^\perp, \mu) \, \varphi_\mu \, A_{\mu^\perp} \, \omega_f^* \tag{1.6}$$

where $A_\mu = \det(f_i^{\nu(j)})$, $\omega^* = \omega_1^* \wedge \ldots \wedge \omega_n^*$.

On the other hand, setting $dw = dw_1 \wedge \ldots \wedge dw_n$, we know [9, p. 115]

$$F_\varphi[dw] = \sum_{\mu \in \mathfrak{T}(q,n)} \operatorname{sign}(\mu^\perp, \mu) \, \varphi_\mu \, A_{\mu^\perp} \, dz. \tag{1.7}$$

Combining (1.6), we obtain $\operatorname{div}(F_\varphi) \cap f^{-1}(U) = \operatorname{div}(Z) \cap f^{-1}(U)$. Q.E.D.

Lemma 1.2 *Assume that $q = n - m > 0$, and that f has strict rank m. Let S be the set of all $x \in M$ such that the rank of the Jacobian of f at x is smaller than m.*

(i) *If T_N is generated by its holomorphic sections, then there exist holomorphic vector fields $X_1, \ldots, X_q$ on N such that a holomorphic field $\varphi = (X_1 \wedge \ldots \wedge X_q)_f$ on f over M of degree q is effective for f.*

(ii) *If there exist holomorphic vector fields $X_1, \ldots, X_n$ on N such that $X := X_1 \wedge \ldots \wedge X_n \not\equiv 0$, and if $f(M) \not\subseteq F(S) \cup \operatorname{div}(X)$, then there exists $j \in \mathfrak{T}(q, n)$ such that a holomorphic field $\varphi = X_{jf}$ on f over M of degree q is effective for f.*

The proof of Lemma 1.2 can be completed as that of Theorem 14.2 in Stoll [9]. In terms of a coframe $(\omega_1, \ldots, \omega_n)$ on N and a coframe $(\theta_1, \ldots, \theta_m)$ on M, we have

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum \omega_\alpha \wedge \bar{\omega}_\alpha \qquad 1 \leqq \alpha \leqq n, \qquad \theta = \frac{\sqrt{-1}}{2\pi} \sum \theta_k \wedge \bar{\theta}_k \qquad 1 \leqq k \leqq m. \tag{1.8}$$

Let us set

$$f^* \omega_\alpha = \sum_k a_{\alpha k} \, \theta_k, \qquad A = \sum_{\alpha, k} |a_{\alpha k}|^2, \tag{1.9}$$

and define a function u_0 by

$$f^*(\omega^b) \wedge \theta^{m-b} = u_0 \, \theta^m \qquad b = \min(m, n). \tag{1.10}$$

Lemma 1.3 *Assume $q = n - m > 0$. For the Jacobian section F_φ induced by a holomorphic field φ on f over M of degree q, define a function g by*

$$F_\varphi[\omega^n] = g \, f^*(\omega^m). \tag{1.11}$$

Then $g \leqq |\varphi|^2$ and $u_0 \leqq \binom{n}{m}(A/m)^m$.

Proof. Relative to the dual frame $\omega_1^*, \ldots, \omega_n^*$ of $\omega_1, \ldots, \omega_n$, then φ can be expressed in form of (1.4). For all $\mu \in \mathfrak{T}(m, n)$, set

$$F^*(\omega_\mu) = A_\mu \theta_1 \wedge \ldots \wedge \theta_m$$

where $A_\mu = \det(a_{\mu(i)j})$. Then

$$u_0 = \sum_{\mu \in \mathfrak{T}(m,n)} |A_\mu|^2. \tag{1.12}$$

By the Hadamard's determinant inequality,

$$|A_\mu|^{2/m} \leqq \frac{1}{m} \sum_{i,j} |a_{\mu(i)j}|^2 \leqq A/m$$

combining (1.12), we obtain one of results. By using (14.2) and (14.4) of Stoll [9], we have

$$u := g u_0 = |\sum_{\mu \in \mathfrak{T}(q,n)} \operatorname{sign}(\mu^\perp, \mu)\, \varphi_\mu\, A_{\mu^\perp}|^2. \tag{1.13}$$

By the Schwarz's inequality, we obtain

$$u \leqq u_0 |\varphi|^2, \tag{1.14}$$

which implies $g \leqq |\varphi|^2$. Q.E.D.

In a similar fashion we can prove

Lemma 1.4 *Assume $q = m - n \geqq 0$. For the Jacobian section F_φ induced by a holomorphic form φ of degree q on M, define a function g by*

$$F_\varphi[\omega^n] = g f^*(\omega^n) \wedge \theta^{m-n}. \tag{1.15}$$

Then $g \leqq \binom{m}{q} |\varphi|^2$ and $u_0 \leqq \binom{m}{q}^{-1} \binom{m}{n} (A/n)^n$.

According to $q = m - n \geqq 0$ or $q = n - m > 0$, we let φ denote a holomorphic form of degree q on M or a holomorphic field on f over M of degree q, respectively. Let F_φ be the Jacobian section induced by φ and define a function u by

$$F_\varphi[\omega^n] = u \theta^m. \tag{1.16}$$

Then $u = g u_0$. By Lemma 1.3 and 1.4, we have

$$b u^{1/b} \leqq \left(\binom{a}{b} |\varphi|^2\right)^{1/b} A \qquad a = \max(m, n). \tag{1.17}$$

By the Stoll's formula [9, Theorem 14.4], one has

$$dd^c \log u = f^*(\mathrm{Ric}\,\omega^n) - \mathrm{Ric}\,\theta^m \tag{1.18}$$

where $u>0$, which implies that the Laplacian of $\log u$ is

$$\Delta \log u = \tfrac{1}{2}(R - \sum_{\alpha,\beta,k} S_{\alpha\beta}\, a_{\alpha k}\, \bar{a}_{\beta k}) \tag{1.19}$$

where R is the scalar curvature of M, and $S_{\alpha\beta}$ is the Ricci tensor of N.

Theorem 1.5 *Let M be complete Kähler manifold with scalar curvature bounded from below by K_1. Let N be hermitian manifold with Ricci curvature bounded from above by a negative constant K_2. Suppose the Ricci curvature of M is bounded from below. Then existence of an effective φ for f implies $K_1 \leqq 0$ and*

$$0 < \sup u \leqq (K_1/(bK_2))^b \binom{a}{b} \sup|\varphi|^2. \tag{1.20}$$

Proof. Let $p=1/(2b)$ and let $h=(u+c)^{-1}$ for any positive number c. Then direct computation shows

$$h^p \Delta h^p = -p h^{2p+1} \Delta u + 4\left(1+\frac{1}{p}\right)|\mathrm{grad}\, h^p|. \tag{1.21}$$

Let $\varepsilon>0$ be any number. Then, by Theorem 1 of Yau [10], there is a point $x \in M$ such that at x

$$|\mathrm{grad}\, h^p| < \varepsilon,\ \Delta h^p > -\varepsilon \quad \text{and} \quad h^p(x) < \inf h^p + \varepsilon. \tag{1.22}$$

Since $\Delta u \geqq u \Delta \log u$ for $u>0$, by our assumptions, then (1.19), (1.21) and (1.22) imply, at x

$$\frac{p}{2} u h^{2p+1}(K_1 - AK_2) < \varepsilon(\inf h^p + \varepsilon) + 4\left(1+\frac{1}{p}\right)\varepsilon. \tag{1.23}$$

When $\varepsilon \to 0$, h^p goes to its infimum and u goes to its supremum. Therefore, if there exists an effective φ for f which implies that u is not identically zero, by (1.17) and (1.23), then K_1 is non-positive and (1.20) follows. Q.E.D.

If $m=n$ and $\varphi=1$, Chern [3] obtained the formula (1.19), and S. Yau [10] proved that $\sup u \leqq K_1/K_2$.

Take $M=\mathbb{C}^m$ and let ds_M^2 be the Euclidian metric. Then $R=K_1=0$. If rank $f=b$ and if either $m \geqq n$ or the conditions in Lemma 1.2 hold, then there exist an effective φ for f such that $\sup|\varphi|^2 < \infty$, the latter is assumed that N is compact.

Corollary 1.6 *Let $M=\mathbb{C}^m$ and let N be a hermitian manifold with Ricci curvature bounded from above by a negative constant. Then we have (i) rank $f<n$; (ii) rank $f<m$ if N is compact and if T_N is generated by its holomorphic sections, in particular, N is hyperbolic; and (iii) T_N is not ample.*

Proof. The (i) and (ii) are direct consequence of Theorem 1.5. We prove (iii) only. Assume, to the contrary, that T_N is ample. Then the anticanonical line

bundle K_N^* is ample. By a theorem of Mori [8], N contains a rational curve V. Let $f: \mathbb{P}^1 \to V$ be the resolution. Then T_{Nf} is a sum of ample line bundles (see [8]). By Lemma 1.1 and Theorem 14.2 of Stoll [9], there exists a holomorphic field φ on f over $\mathbb{P}^1$ of degree $n-1$ such that $\varphi|_{\mathbb{C}}$ is effective for $f|_{\mathbb{C}}: \mathbb{C}=\mathbb{P}^1-\{\infty\} \to N$. Since $\mathbb{P}^1$ is compact, under the induced metric from T_N we have $|\varphi| \leqq \text{const}$. Hence, using the Euclidian metric in $\mathbb{C}$, Theorem 1.5 implies $u \equiv 0$. This is a contradiction. Q.E.D.

Lang [7] conjectured that if N is hyperbolic then the canonical class is ample. Corollary 1.6 gives a part of supporting evidence to the conjecture.

2 A generalization of Kodaira's theorem

We continue to study a holomorphic mapping $f: M \to N$. From now on, we assume that M is parabolic, and N is compact. Let $\tau: M \to [0, \infty)$ be a parabolic exhaustion function and set

$$v = dd^c \tau, \qquad \sigma = d^c \log \tau \wedge (dd^c \log \tau)^{m-1}.$$

For a subvariety A of pure dimension $k (\leqq m)$ in M, a (p, p)-form χ on M with $0 \leqq p \leqq k$, and a non-negative function h on M, define

$$A(r; \chi) := r^{2p-2k} \int_{A[r]} \chi \wedge v^{k-p}, \qquad A(r, s; \chi) := \int_s^r A(t; \chi) \frac{dt}{t}$$

$$m(r; h) := \int_{M\langle r \rangle} (\log h)\, \sigma, \qquad m(r, s; h) := m(r; h) - m(s; h)$$

where $A[r] = \{x \in A \mid \tau(x) \leqq r^2\}$ and $A\langle r \rangle = \{x \in A \mid \tau(x) = r^2\}$. We will assume the existence of a non-negative holomorphic line bundle $L \geqq 0$ on N together with a positive integer k satisfying the following condition:

$$P_k(L) := \dim H^0(N, K_N^k \otimes L^*) > 0$$

Now choose a non-trivial section $\alpha \in H^0(N, K_N^k \otimes L^*)$, and a metric ρ in $K_N^k \otimes L^*$ such that

$$|\alpha(x)| \leqq 1 \quad \text{for all } x \in N. \tag{2.1}$$

Also take a metric κ in L such that the Chern form $c(L, \kappa) \geqq 0$. Abbreviate

$$A_f(r, L) := M(r; f^* c(L, \kappa)), \quad T_f(r, s, L) := M(r, s; f^* c(L, \kappa)).$$

Then there exists a volume form Ω on N such that

$$c(K_N^k \otimes L^*, \rho) = k \operatorname{Ric} \Omega - c(L, \kappa).$$

If $f(M) \not\subseteq D_\alpha := \operatorname{div}(\alpha)$, then first main theorem for divisor D_α (see [9]) implies

$$kM(r, s; f^* \operatorname{Ric} \Omega) - T_f(r, s, L) = D_{\alpha f}(r, s; 1) - m(r, s; |\alpha \circ f|) \tag{2.2}$$

where $D_{\alpha f}=f^{-1}(D_\alpha)$. For a Jacobian section F of f, define a function h by

$$F[\Omega]=h v^m. \tag{2.3}$$

If F is effective, then one has the Stoll's Formula [9, Theorem 15.5]

$$M(r,s;f^*\operatorname{Ric}\Omega)+D_F(r,s;1)=\operatorname{Ric}_\tau(r,s)+\tfrac{1}{2}m(r,s;h) \tag{2.4}$$

where $D_F=\operatorname{div}(F)$.

For the rest of this section, take $M=\mathbb{C}^m$ having the parabolic exhaustion function τ defined by $\tau(z)=|z|^2$. Then one has $\operatorname{Ric}_\tau(r,s)=0$. Set

$$\deg_p(r,F):=\mathbb{C}^m(r;h^{1/p})/(\int_N \Omega)^{1/p} \quad \text{for } p>0.$$

Theorem 2.1 Let $f:\mathbb{C}^m\to N$ be a holomorphic mapping into a compact complex manifold N. If there exists an effective Jacobian section F of f such that for a positive number p,

$$\lim_{r\to\infty}\inf \deg_p(r,F)=0, \tag{2.5}$$

either $f(\mathbb{C}^m)$ is contained in a divisor of N or for all $L\geqq 0$,

$$P_k(L)=0 \quad \text{for } k=1,2,\ldots \tag{2.6}$$

Proof. Assume, to the contrary, that $f(\mathbb{C}^m)$ is not contained in any divisor of N, and that there exists a non-negative line bundle $L\geqq 0$ on N together with a positive integer k such that $P_k(L)>0$. Then for any effective Jacobian section F and a non-trivial $\alpha\in H^0(N, K_N^k\otimes L^*)$, (2.2) and (2.4) hold. Thus we obtain

$$m(r;h)\geqq m(s;h)+\frac{2}{k}m(s;|\alpha\circ f|)=c_1.$$

Since $\log x$ is a concave function of x, $x>0$, then for any positive number p we have

$$c_1\leqq m(r;h)\leqq p\,\log(m(r;\exp(h^{1/p}))$$

where $m(r;e)=\mathbb{C}^m(r;1)=1$ for $r>0$, which gives .

$$\begin{aligned} r^{2m}\,\mathbb{C}^m(r;h^{1/p}) &= m\int_{\mathbb{C}^m[r]} h^{1/p}\,\tau^{m-1}\,d\tau\wedge\sigma \\ &= 2m\int_0^r t^{2m-1}m(t;\exp(h^{1/p}))\,dt\geqq c_2+2m\exp(c_1/p)\int_s^r t^{2m-1}\,dt \\ &= c_3+\exp(c_1/p)\,r^{2m} \end{aligned}$$

which contradicts (2.5). Q.E.D.

If $m\geqq n=\operatorname{rank} f$, then the image $f(\mathbb{C}^m)$ contains an open subset of N. By Stoll [9, Theorem 14.1], there exists a effective holomorphic form φ for f of degree $m-n$. Hence if (2.5) holds for $F=F_\varphi$, then (2.6) holds for all $L\geqq 0$. In

particular, if $m=n$, $p=1$, $\varphi=1$ and if L is trivial, this is the Kodaira's theorem [6].

Corollary 2.2 *Let* $f: \mathbb{C}^m \to N$ be a holomorphic mapping into the compact complex manifold N of dimension n such that for a positive (1,1)-form ω on N

$$\liminf_{r\to\infty} r^{-2}\mathbb{C}^m(r; f^*\omega)=0. \tag{2.7}$$

(i) If rank $f=n$, then (2.6) holds for all $L\geqq 0$.
(ii) If $m<n$ and if there exists an effective holomorphic field φ on f over $\mathbb{C}^m$ of degree $n-m$ such that $|\varphi|\leqq$ const, then either $f(\mathbb{C}^m)$ is contained in a divisor of N or (2.6) holds.

Proof. Define a function σ_q by

$$f^*(\omega^q)\wedge v^{m-q}=\sigma_q v^m. \tag{2.8}$$

Then a pointwise relation among the σ_q's is provided by Newton's inequality (see [1, p. 239])

$$\sigma_q^{1/q}\leqq c_{pq}\sigma_p^{1/p} \qquad (q\geqq p). \tag{2.9}$$

For case (i) or (ii), there always exists a holomorphic form or field φ which induces the effective Jacobian section F_φ such that $|\varphi|\leqq c(=\text{const})$. By Lemma 1.3 and 1.4, we have

$$h v^m=F_\varphi[\omega^n]\leqq c f^*(\omega^b)\wedge v^{m-b}=c\sigma_b v^m \qquad b=\min(m,n)$$

which implies

$$h^{1/b} v^m\leqq c_1\sigma_b^{1/b} v^m\leqq c_2\sigma_1 v^m=c_2 f^*(\omega)\wedge v^{m-1}$$

and

$$\mathbb{C}^m(r; h^{1/b})\leqq c_2 r^{-2}\mathbb{C}^m(r; f^*\omega).$$

Hence it follows from Theorem 2.1. Q.E.D.

If $\omega>0$ is the Chern form of a positive line bundle L on N, we know that f is rational if and only if

$$A_f(\infty, L)=\lim_{r\to\infty} A_f(r, L)<\infty \tag{2.10}$$

(see [4, 9]). Hence Lemma 1.2 and Corollary 2.2 yield

Corollary 2.3 *Let* $f: \mathbb{C}^m \to N$ *be a holomorphic mapping of rank* $\min(m, n)$ *into a smooth projective variety of dimension n. If f is rational, then*

(i) *if* $m\geqq n$, *(2.6) holds for all* $L\geqq 0$.
(ii) *if* $m<n$ *and if there exist holomorphic vector fields* $X_1, \ldots, X_n$ on N such that $X_1\wedge\ldots\wedge X_n\not\equiv 0$, then either $f(\mathbb{C}^m)$ *is contained in a proper subvariety of* N *or* (2.6) *holds for* $L\geqq 0$.

Now let $f:\mathbb{C}^m \to N$ be a holomorphic map of rank $\min(m, n)$ into a complex torus N. Then there exist n holomorphic vector fields $X_1, \ldots, X_n$ which are linearly independent at every point of N. Hence there exists an effective φ for f such that $|\varphi| \leqq c$. Since K_N is trivial, by using (2.4) for $F = F_\varphi$, in a similar fashion as the proof of Theorem 2.1 we have

$$\liminf_{r\to\infty} \deg_p(r, F_\varphi) > 0. \tag{2.11}$$

Hence by the proof of Corollary 2.2, we obtain

Proposition 2.4 *If N is an n-dimensional complex torus and if $f: \mathbb{C}^m \to N$ is a holomorphic map of rank* $\min(m, n)$, *then for a positive* (1,1)*-form ω on N,*

$$\liminf_{r\to\infty} r^{-2}\, \mathbb{C}^m(r; f^*\omega) > 0. \tag{2.12}$$

Further, if N is an abelian variety, then f is not rational.

3 A generalization of a Griffiths' theorem

We continue with the situation $f: M \to N$ of § 2. Let $\omega > 0$ be the Chern form of a positive line bundle L on N. For a Jacobian section F of f, define a function g by

$$F[\omega^n] = g f^*(\omega^b) \wedge v^{m-b} \qquad b = \min(m, n). \tag{3.1}$$

Theorem 3.1 *If F is effective, and if $f(M) \nsubseteq D_\alpha$ for a non-trivial $\alpha \in H^0(N, K_N^k \otimes L^*)$, then for $\delta > 0$, we have*

$$\|_\varepsilon \frac{1}{k}(1-\delta)\, T_f(r, s, L) < \mathrm{Ric}_\tau(r, s) + \tfrac{1}{2} m(r; g) + c\varepsilon \log r \tag{3.2}$$

where $c > 0$ is a constant, while the notation $\|_\varepsilon$ means that the inequality holds except on an open set I with $\int_I r^\varepsilon dr < \infty$ for $\varepsilon > 0$.

Proof. By (2.1), (2.2) and (2.4), we have

$$\frac{1}{k} T_f(r, s, L) \leqq \mathrm{Ric}_\tau(r, s) + \tfrac{1}{2} m(r; h) + O(1) \tag{3.3}$$

where h is defined by $F[\omega^n] = h v^m$. Set $M^+ = \{x \in M \mid v(x) > 0\}$ and define a function σ_q on M^+ by (2.8). Then (2.9) also holds on M^+. We obtain

$$(h/g)^{1/b} v^m = \sigma_b^{1/b}\, v^m \leqq c_{1,b}\, \sigma_1\, v^m = c_{1,b} f^*(\omega) \wedge v^{m-1} \tag{3.4}$$

on M^+, which implies

$$M(r, s; (h/g)^{1/b} v) \leqq c_{1,b}\, T_f(r, s, L). \tag{3.5}$$

By using Lemma 7 of Hu [5], we have

$$\|_{\varepsilon}\, m(r; h) = m(r; g) + b\, m(r; (h/g)^{1/b}) \leqq m(r; g) + c((1+2\varepsilon)\log T_f(r, s, L) + \varepsilon \log r + O(1)). \tag{3.6}$$

Since $T_f(r, s, L) \to +\infty$ as $r \to \infty$ (see [9]), then

$$c((1+2\varepsilon)\log T_f(r, s, L) + O(1)) < \frac{\delta}{k} T_f(r, s, L)$$

if r large enough. Hence (3.2) follows from (3.3)–(3.6). Q.E.D.

If K_N is positive, by using (2.4), we can prove directly

Theorem 3.2 *If $K_N > 0$ and if F is effective, then for $\delta > 0$*

$$\|_{\varepsilon}(1-\delta)\, T_f(r, s, K_N) < \mathrm{Ric}_{\tau}(r, s) + \tfrac{1}{2} m(r; g) + c\varepsilon \log r. \tag{3.7}$$

If M is a smooth affine algebraic variety, Stoll [9, p. 165] proved

$$R(\tau) := \lim_{r \to \infty} \mathrm{Ric}_{\tau}(r, s)/\log r < \infty$$

and further, if rank $f = n$, then there exists a holomorphic form φ of degree m–n on M which is effective for f such that for a parabolic exhaustion τ of M, $g \leqq \mathrm{const}$ (see [9, Proposition 18.6]). Hence Theorem 3.1 gives

$$A_f(\infty, L) = \lim_{r \to \infty} T_f(r, s, L)/\log r \leqq k R(\tau). \tag{3.8}$$

Also by using Lemma 1.2, we obtain

Corollary 3.3 *Let M be a smooth affine algebraic variety and N smooth projective algebraic variety. Let $f: M \to N$ be a holomorphic map of rank* $\min(m, n)$. *Then*

(i) *if $m \geqq n$, either f is rational or* (2.6) *holds for all $L > 0$*
(ii) *if $m < n$ and if there exist holomorphic vector fields $X_1, \ldots, X_n$ on N such that $X_1 \wedge \ldots \wedge X_n \not\equiv 0$, either f is rational, or $f(M)$ is contained in a proper subvariety of N, or* (2.6) *holds for all $L > 0$.*

For case (i), see [4, 6, 9].

4 A defect relation

We continue with the situation $f: M \to N$ of § 2. Let L be a positive holomorphic line bundle on N and let $D_1, \ldots, D_k \in |L|$ be divisors such that $D = D_1 + \ldots + D_k$ has simple normal crossing. Take $s_i \in H^0(N, L)$ such that $D_i = \mathrm{div}(s_i)$. Given a positive number λ such that the Chern classes of L^k and K_N in $H^2_{\mathrm{DR}}(N, R)$ satisfy

$$\lambda c(L^k) + c(K_N) > 0, \tag{4.1}$$

Carlson-Griffiths-King [2, 4] and Stoll [9] showed that there is a volume form Ω on N and there exists a metric κ in L such that

$$\Psi = Q_\lambda \Omega, \qquad Q_\lambda = \left\{ \prod_{i=1}^{k} |s_i|^{2\lambda} (\log |s_i|^2)^2 \right\}^{-1} \tag{4.2}$$

satisfies

$$\operatorname{Ric} \Psi > 0, \qquad (\operatorname{Ric} \Psi)^n \geqq Q_1 \Omega, \qquad \int_N (\operatorname{Ric} \Psi)^n < \infty. \tag{4.3}$$

Thus a positive function G on N is defined by

$$(\operatorname{Ric} \Psi)^n = G\Omega \tag{4.4}$$

with $G \geqq Q_1$. For a Jacobian section F of f, define a function by

$$F[\omega^n] = g f^*(\omega^b) \wedge v^{m-b} \qquad (\omega = \operatorname{Ric} \Psi). \tag{4.5}$$

Theorem 4.1 *If F is effective and if $f(M) \nsubseteq D_i$ for all i, we have*

$$\begin{aligned} \|_\varepsilon D_F(r, s; 1) + \sum_{i=1}^{k} (T_f(r, s, L) - D_{if}(r, s; 1)) \leqq & - T_f(r, s, K_N) \\ & + \operatorname{Ric}_\tau(r, s) + \tfrac{1}{2} m(r; g) + c(\log T_f(r, s, L) + \varepsilon \log r). \end{aligned} \tag{4.6}$$

Proof. Write $G_f = G \circ f$, etc. By (2.4) and FMT for divisors,

$$\begin{aligned} \sum_{i=1}^{k} (T_f(r, s, L) - D_{if}(r, s, 1)) &= - \sum_{i=1}^{k} m(r; |s_{if}|) + O(1) \\ &= \tfrac{1}{2} m(r; Q_{1f}) + S(r, f) \leqq \tfrac{1}{2} m(r; G_f) + S(r, f) \\ &= - T_f(r, s, K_N) - D_F(r, s; 1) + \operatorname{Ric}_\tau(r, s) + \tfrac{1}{2} m(r; hG_f) + S(r, f) \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} S(r, f) &= \sum_{i=1}^{k} m(r; \log |s_{if}|^2) + O(1) \\ &\leqq c \log T_f(r, s, L) + O(1) \end{aligned} \tag{4.8}$$

see [4, Lemma 7.30; 9, Proposition 18.12]. Note that

$$hG_f v^m = G_f F[\Omega] = F[G\Omega] = F[\omega^n].$$

Similar to the proof of Theorem 3.1, we can obtain

$$\|_\varepsilon m(r; hG_f) \leqq m(r; g) + c((1 + 2\varepsilon) \log M(r, s; f^* \omega) + \varepsilon \log r). \tag{4.9}$$

By using the following inequality [9, Proposition 18.12],

$$M(r, s; f^* \omega) \leqq c T_f(r, s, L) + c' \tag{4.10}$$

then (4.6) follows from (4.7)–(4.10). Q.E.D.

Similar to Corollary 3.3, Theorem 4.1 implies

Corollary 4.2 [2, 4] *If M is affine and if rank $f=n$, then*

$$\text{(D.R.)} \qquad \sum_{i=1}^{k} \delta_f(D_i) \leqq c(K_N^*)/c(L) + R_f$$

where

$$R_f := \lim_{r \to \infty} \sup \mathrm{Ric}_\tau(r, s)/T_f(r, s, L).$$

The singular positive form $\omega = \mathrm{Ric}\,\Psi$ induces a metric ρ on $N-D$ which has "singularity" in D. By using the "singular" metric on N and using Lemma 1.2, Lemma 7 in Hu [5], we have

Corollary 4.3 *If $f(M) \nsubseteq D_i$, $i=1, \ldots, k$, and if there exists a holomorphic field φ on f over M of degree $n-m(>0)$ which is effective for f such that for some positive numbers c, α, β,*

$$(4.11) \qquad \|_\varepsilon M(r, s; |\varphi|^{2/\alpha} v) \leqq c(T_f(r, s, L)^\beta + r^\varepsilon),$$

then (D.R.) *holds.*

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Canonical equiaffine hypersurfaces in $\mathbb{R}^{n+1}$

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1 Introduction

An important problem in affine differential geometry is to classify all the affine hyperspheres with constant sectional curvature (abbreviated CSC) Blaschke metric. This problem has been extensively studied in the recent years.

The classification has been made in dimension 2 due to works of Radon [14], Li and Penn [8], Magid and Ryan [9] and Simon [17].

The first result in high dimension for local affine hyperspheres with positive definite Blaschke metric was given by Li [7], who proved that an affine hypersphere with vanishing scalar curvature is either the elliptic paraboloid or affinely equivalent to the hypersphere defined by

$$x_1 x_2 x_3 \dots x_{n+1} = 1. \tag{1.1}$$

Then Yu [23] could show that an CSC affine hypersphere in $\mathbb{R}^4$ is either a quadric or the hypersphere (1.1) with $n=3$. Finally Vrancken et al. [21] could generalize Yu's result to all dimensions by using a technique to simplify the Fubini-Pick cubic form. Thus the classification of CSC affine hyperspheres with positive definite Blaschke metric was completed.

The classification problem becomes much more difficult if the Blaschke metric is indefinite and $n \geqq 3$. The only known result was given by Magid and Ryan [10]. They showed that an CSC affine hypersphere in $\mathbb{R}^4$ with indefinite metric and non-zero Pick invariant is affinely equivalent to one of the hyperspheres

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1, \tag{1.2}$$

$$(x_1^2 + x_2^2)(x_3^2 - x_4^2) = 1. \tag{1.3}$$

The case $n \geqq 4$ remains open. Magid and Ryan gave the following

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Conjecture. *An* CSC *affine hypersphere in* $\mathbb{R}^{n+1}$ *with non-zero Pick invariant is affinely equivalent to one of the hyperspheres*

$$(1.4)\qquad (x_1^2+x_2^2)\dots(x_{2s-1}^2+x_{2s}^2)x_{2s+1}\dots x_{n+1}=1,\quad 0\leqq s\leqq\left[\frac{n+1}{2}\right].$$

We note that the hyperspheres in (1.4) have flat Blaschke metric and parallel Fubini-Pick form. It motivates us to study a class of hyperspheres called canonical hyperspheres. A hypersurface $f\colon M\to\mathbb{R}^{n+1}$ is said to be canonical if its Blaschke metric G is flat and its Fubini-Pick form C is parallel with respect to G. By the Bokan-Nomizu-Simon theorem [2] we know that any canonical hypersurface is an affine hypersphere. Thus the conjecture of Magid and Ryan is equivalent to the following two assertions:

Assertion (*i*) Any CSC affine hypersphere with non-zero Pick invariant is proper and canonical.

Assertion (*ii*) Any proper canonical affine hypersphere is affinely equivalent to one of the hyperspheres given by (1.4).

Our purpose in this paper is to study the canonical hyperspheres. We reduce the classification problem of proper canonical hyperspheres to an algebraic classification problem of n mutually commutative self-adjoint linear operators in $\mathbb{R}^{n+1}$ (with an indefinite inner product) which satisfy some algebraic conditions. Then we can show the main

Theorem. *Let* $f\colon \mathbb{R}^n\to\mathbb{R}^{n+1}$ *be a proper canonical affine hypersphere with Blaschke metric* G. *If the dimension of the maximal negative definite subspace of* G *is* 1, *then* f *is affinely equivalent to the hypersphere* $(x_1^2+x_2^2)x_3\dots x_{n+1}=1$.

The key point for the proof of this theorem is the following algebraic lemma, which is also important in linear algebra.

Lemma. *Let* $\langle\ ,\ \rangle$ *be the indefinite inner product in* $\mathbb{R}^{n+1}$ *defined by*

$$(1.5)\qquad \langle x,x\rangle=-x_1^2+x_2^2+\dots+x_{n+1}^2,\qquad x={}^t(x_1,x_2,\dots,x_{n+1})\in\mathbb{R}^{n+1}.$$

Let $A_1, A_2, \dots, A_m$ *be finite mutually commutative self-adjoint linear operators in* $\mathbb{R}^{n+1}$. *Then either there is an orthonormal basis* $\{e_1,e_2,\dots,e_{n+1}\}$ *for* $\langle\ ,\ \rangle$ *such that we have the matrix representations*

Type I. $$A_i=\begin{pmatrix} a_{i1} & & & 0\\ & a_{i2} & & \\ & & \ddots & \\ 0 & & & a_{in+1}\end{pmatrix}=(a_{i1})\oplus(a_{i2})\oplus\dots\oplus(a_{in+1}),\quad i=1,2,\dots,m,\ a_{i\alpha}\in\mathbb{R};$$

or

Type II. $$A_i=\begin{pmatrix} a_{i1} & a_{i2}\\ -a_{i2} & a_{i1}\end{pmatrix}\oplus(a_{i3})\oplus\dots\oplus(a_{in+1}),\quad i=1,2,\dots,m,\ a_{i\alpha}\in\mathbb{R}$$
and $a_{j2}\neq 0$ *for some* j;

or

there exist a basis $\{x, y, e_3, \ldots, e_{n+1}\}$ *with* $\langle x, x\rangle = \langle y, y\rangle = \langle x, e_\alpha\rangle = \langle y, e_\alpha\rangle = 0$, $\langle x, y\rangle = 1$ *and* $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$, $3 \leqq \alpha, \beta \leqq n+1$, *and a positive integer* v, $2 \leqq v \leqq n+1$, *such that we have the matrix representations*

$$\text{Type III.} \quad A_i = \begin{pmatrix} a_{i1} & a_{i2} & a_{i3} & a_{i4} & \ldots & a_{iv} \\ 0 & a_{i1} & 0 & 0 & \ldots & 0 \\ 0 & a_{i3} & a_{i1} & & & \\ 0 & a_{i4} & & a_{i1} & & 0 \\ \vdots & \vdots & & & \ddots & \\ 0 & a_{iv} & & 0 & & a_{i1} \end{pmatrix} \oplus (a_{iv+1}) \oplus \ldots \oplus (a_{in+1}),$$

$i = 1, 2, \ldots, m$, $a_{i\alpha} \in \mathbb{R}$ *and* $a_{iv} \neq 0$ *for some* i. *Moreover, if* $v = 2$, *then there is* j *with* $a_{j2} = \pm 1$; *if* $v \geqq 3$, *there is* j *such that* $a_{j3} = 1$ *and* $a_{j2} = a_{j4} = \ldots = a_{jv} = 0$.

This paper is organized as follows. In § 2 we study the canonical proper hyperspheres in $\mathbb{R}^{n+1}$. In § 3 we prove the algebraic lemma. In § 4 we give the proof of the main theorem.

2 Proper canonical hyperspheres in $\mathbb{R}^{n+1}$

Let $f\colon M \to \mathbb{R}^{n+1}$ be a proper canonical affine hypersphere. By the definition we know that its Blaschke metric G is flat and its Fubini-Pick form C is parallel with respect to G. We can choose a local coordinate system $(u^1, u^2, \ldots, u^n)$ for M such that

$$G = \sum_{i,j=1}^{n} G_{ij} du^i du^j = \sum_{i=1}^{n} \varepsilon_i (du^i)^2, \qquad \varepsilon_i \in \{+1, -1\}. \tag{2.1}$$

Thus we have a parallel basis $\{e_1, e_2, \ldots, e_n\}$ for TM, $e_i = \dfrac{\partial f}{\partial u^i}$, satisfying

$$G_{ij} = G(e_i, e_j) = \varepsilon_i \delta_{ij}. \tag{2.2}$$

Let $\{C_{ijk}\}$ be the components of the symmetric cubic form C with respect to the basis $\{e_i\}$. Since $\{e_i\}$ and C are parallel with respect to G, we know that

$$C_{ijk} = C_{ikj} = C_{jik} = \text{const.}, \qquad \forall i, j, k. \tag{2.3}$$

We denote by e_{n+1} the equiaffine normal for M in $\mathbb{R}^{n+1}$. In general there are two possibilities to choose the equiaffine normal. But in order to fix the Blaschke metric G we need to fix e_{n+1} such that it is in the same direction of $f - f_0$, where f_0 is the center of f. Then the structure equations read:

$$de_i = \sum_{j,k=1}^{n} C_{ij}^k du^i e_k + \varepsilon_i du^i e_{n+1}, \qquad i = 1, 2, \ldots, n, \tag{2.4}$$

$$de_{n+1} = -H \sum_{i=1}^{n} du^i e_i, \tag{2.5}$$

where $C_{ij}^k=\varepsilon_k C_{ijk}$ and H is the mean curvature of M in $\mathbb{R}^{n+1}$. By the choice of e_{n+1} we know that $H<0$, so by an affine transformation we may assume that $H=-1$. The apolarity condition gives

$$\sum_{i=1}^{n} C_{ij}^i=0, \quad j=1,2,\ldots,n. \tag{2.6}$$

We denote by $(e_1, e_2, \ldots, e_{n+1})$ the matrix for the basis $\{e_\alpha\}$, then (2.4) and (2.5) can be written by

$$d(e_1, e_2, \ldots, e_{n+1})=(e_1, e_2, \ldots, e_{n+1})\left(\sum_{i=1}^{n} A_i du^i\right), \tag{2.7}$$

where $\{A_i\}$ are the constant matrices given by

$$A_i=\begin{pmatrix} C_{i1}^1 & \ldots & C_{ii}^1 & \ldots & C_{in}^1 & 0 \\ \cdot & \ldots & \cdot & \ldots & \cdot & \ldots \\ C_{i1}^i & \ldots & C_{ii}^i & \ldots & C_{in}^i & 1 \\ \cdot & \ldots & \cdot & \ldots & \cdot & \ldots \\ C_{i1}^n & \ldots & C_{ii}^n & \ldots & C_{in}^n & 0 \\ 0 & \ldots & \varepsilon_i & \ldots & 0 & 0 \end{pmatrix}, \quad i=1,2,\ldots,n, \tag{2.8}$$

where both the $(n+1)$th column and row have only one zon-zero element. By differentiating (2.7) we get

$$[A_i, A_j]=A_iA_j-A_jA_i=0, \quad \forall i,j, \tag{2.9}$$

(cf. Li [7]). From (2.6) we have

$$\operatorname{tr}(A_i)=0, \quad i=1,2,\ldots,n. \tag{2.10}$$

In order to rewrite (2.3) as a restricting condition on $\{A_i\}$ we introduce in $\mathbb{R}^{n+1}$ the indefinite inner product $\langle\ ,\ \rangle$,

$$\begin{aligned} \langle x, x\rangle &= \varepsilon_1 x_1^2+\varepsilon_2 x_2^2+\ldots+\varepsilon_n x_n^2+x_{n+1}^2, \\ x &= {}^t(x_1, x_2, \ldots, x_{n+1})\in\mathbb{R}^{n+1}, \end{aligned} \tag{2.11}$$

where $\{\varepsilon_i\}$ are given by (2.1). Then we can easily verify that

$$\langle A_i x, y\rangle=\langle x, A_i y\rangle, \quad \forall x, y\in\mathbb{R}^{n+1}, \ \cdot i=1,2,\ldots,n, \tag{2.12}$$

i.e., $\{A_i\}$ are self-adjoint with respect to $\langle\ ,\ \rangle$. Moreover, if we denote $\xi={}^t(0,\ldots,0,1)\in\mathbb{R}^{n+1}$, then $\{A_i\}$ has the property that $\{A_1\xi, A_2\xi, \ldots, A_n\xi, \xi\}$ is an orthonormal basis for $\langle\ ,\ \rangle$ with determinant 1.

Conversely, we can show that

Theorem 2.1 *Let* $\langle x, x\rangle=\sum_{i=1}^{n}\varepsilon_i x_i^2+x_{n+1}^2$ *be an inner product in* $\mathbb{R}^{n+1}$ *with* $\varepsilon_i\in\{+1,-1\}$. *If there exist n matrices* $\{A_i\}$ *satisfying* (i) $\langle A_i x, y\rangle=\langle x, A_i y\rangle$, $\forall x, y, i$; (ii) $[A_i, A_j]=0$, $\forall i,j$; (iii) $\operatorname{tr}(A_i)=0$, $\forall i$; (iv) $\exists\xi\in\mathbb{R}^{n+1}$ *such that*

$\{A_1\xi, A_2\xi, \ldots, A_n\xi, \xi\}$ is an orthonormal basis for $\langle\ ,\ \rangle$ with determinant 1, then there exists a canonial affine hypersphere $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ with the Blaschke metric $G=\sum_{i=1}^{n} \varepsilon_i (du^i)^2$, where $(u^1, u^2, \ldots, u^n)$ are the coordinates for $\mathbb{R}^n$.

Proof. Let $S=(A_1\xi, \ldots, A_n\xi, \xi)$. We define $B_i = S^{-1}A_iS$, $i=1,2,\ldots,n$. By (iv) we know that S is an orthogonal matrix with respect to $\langle\ ,\ \rangle$, so $\{B_i\}$ also have the properties (i), (ii) and (iii). Moreover, if we denote $v={}^t(0,\ldots,0,1)\in\mathbb{R}^{n+1}$, then

$$B_i v = S^{-1}A_i(Sv) = S^{-1}(A_i\xi) = {}^t(0,\ldots,0,1,0,\ldots,0), \quad i=1,2,\ldots,n. \tag{2.13}$$

From (2.13) and the fact that B_i is self-adjoint with respect to $\langle\ ,\ \rangle$ we know that

$$B_i = \begin{pmatrix} C^1_{i1} & \ldots & C^1_{ii} & \ldots & C^1_{in} & 0 \\ \cdot & \ldots & \cdot & \ldots & \cdot & \ldots \\ C^i_{i1} & \ldots & C^i_{ii} & \ldots & C^i_{in} & 1 \\ \cdot & \ldots & \cdot & \ldots & \cdot & \ldots \\ C^n_{i1} & \ldots & C^n_{ii} & \ldots & C^n_{in} & 0 \\ 0 & \ldots & \varepsilon_i & \ldots & 0 & 0 \end{pmatrix}, \quad i=1,2,\ldots,n, \tag{2.14}$$

for some constants $\{C^k_{ij}\}$ with $\varepsilon_j C^j_{ik} = \varepsilon_k C^k_{ij}$. From $[B_i, B_j]=0$ we get in particular $\varepsilon_i C^i_{jk} = \varepsilon_j C^j_{ik}$. Thus we know that $C_{ijk} = \varepsilon_k C^k_{ij}$ are totally symmetric. Now we consider the linear system

$$d(e_1, e_2, \ldots, e_{n+1}) = (e_1, e_2, \ldots, e_{n+1})\left(\sum_{i=1}^{n} B_i du^i\right) \tag{2.15}$$

with the initial value condition

$$\det(e_1(o), e_2(o), \ldots, e_{n+1}(o)) = 1. \tag{2.16}$$

This system is completely integrable because of $[B_i, B_j]=0$. Given any initial values satisfy (2.16) we get an unique solution $(e_1, e_2, \ldots, e_{n+1})$ of (2.15), which is determined by $\{B_i\}$ up to linear transformations in $\mathrm{SL}(n+1)$. Since $C^k_{ij} = C^k_{ji}$, we have $\dfrac{\partial e_i}{\partial u^j} = \dfrac{\partial e_j}{\partial u^i}$, so we can get an unique solution $f: \mathbb{R}^n \to \mathbb{R}^{n+1}$ (up to constant vectors in $\mathbb{R}^{n+1}$) from the equation

$$df = \sum_{i=1}^{n} e_i du^i. \tag{2.17}$$

It is clear that f is determined by $\{B_i\}$ up to equiaffine transformations in $\mathbb{R}^{n+1}$. One can easily see that f is a proper canonial affine hypersphere with Blaschke metric $G=\sum_{i=1}^{n} \varepsilon_i (du^i)^2$. Q.E.D.

Example 2.2 Let J_s be the $(n+1)\times(n+1)$ matrix

$$J_s=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\oplus(1)\ldots\oplus(1) \tag{2.18}$$

with s copies of $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $s=0,1,\ldots,\left[\frac{n+1}{2}\right]$. We define $\langle\ ,\ \rangle$ in $\mathbb{R}^{n+1}$ by

$$\langle x,x\rangle={}^t x J_s x=(-x_1^2+x_2^2)+\ldots+(-x_{2s-1}^2+x_{2s}^2)+x_{2s+1}^2+\ldots+x_{n+1}^2 \tag{2.19}$$

for $x={}^t(x_1,x_2,\ldots,x_{n+1})\in\mathbb{R}^{n+1}$. Let

$$\xi=\frac{1}{\sqrt{n+1}}{}^t((0,\sqrt{2}),\ldots,(0,\sqrt{2}),1,\ldots,1)\in\mathbb{R}^{n+1}$$

with s copies of $(0,\sqrt{2})$. Then we have $\langle\xi,\xi\rangle=1$. So we can extend ξ to an orthonormal basis $(\eta_1,\ldots,\eta_n,\xi)$ for $\langle\ ,\ \rangle$ with determinant 1. We write

$$\eta_i=\frac{1}{\sqrt{n+1}}{}^t(\sqrt{2}b_{i1},\sqrt{2}a_{i1}),\ldots,(\sqrt{2}b_{is},\sqrt{2}a_{is}),a_{i\,2s+1},\ldots,a_{in+1}), \tag{2.20}$$
$$i=1,2,\ldots,n,$$

and define

$$A_i=\begin{pmatrix}a_{i1} & b_{i1}\\ -b_{i1} & a_{i1}\end{pmatrix}\oplus\ldots\oplus\begin{pmatrix}a_{is} & b_{is}\\ -b_{is} & a_{is}\end{pmatrix}\oplus(a_{i\,2s+1})\oplus\ldots\oplus(A_{in+1}). \tag{2.21}$$

then $(A_1\xi,A_2\xi,\ldots,A_n\xi,\xi)=(\eta_1,\eta_2,\ldots,\eta_n,\xi)$ is an orthonormal basis for $\langle\ ,\ \rangle$. Moreover, one can easily verify that $\{A_i\}$ are mutually commutative and self-adjoint with respect to $\langle\ ,\ \rangle$, i.e., ${}^tA_iJ_s=J_sA_i$, and

$$\operatorname{tr}(A_i)=(2a_{i1}+\ldots+2a_{is}+a_{i\,2s+1}+\ldots+a_{in+1})=(n+1)\langle\eta_i,\xi\rangle=0.$$

Thus by Theorem 2.1 we know that $(A_1,A_2,\ldots,A_n)$ define a proper canonical affine hypersphere $x\colon\ \mathbb{R}^n\to\mathbb{R}^{n+1}$ with $G=\sum_{i=1}^{s}[-(du^{2i-1})^2+(du^{2i})^2]+\sum_{i=2s+1}^{n}(du^i)^2$.

Now we show that x is affinely equivalent to the hypersphere given by (1.4). Let

$$S=(A_1\xi,\ldots,A_n\xi,\xi),\qquad B_i=S^{-1}A_iS \tag{2.22}$$

as in the proof of Theorem 2.1. We get a solution

$$(e_1,e_2,\ldots,e_{n+1})=\exp\left(\sum_{i=1}^{n}A_iu^i\right)S \tag{2.23}$$

of (2.15) with the initial value condition $\det(e_1(o), e_2(o), \ldots, e_{n+1}(o)) = \det(S) = 1$. Since x is an affine hypersphere with $H = -1$, we may assume that $x = e_{n+1}$. By (2.22) and (2.23) we have

$$x = {}^t(x_1, x_2, \ldots, x_{n+1}) = e_{n+1} = \exp\left(\sum_{i=1}^{n} A_i u^i\right)\xi. \tag{2.24}$$

Thus from (2.21) we get

$$\begin{pmatrix} x_{2j-1} \\ x_{2j} \end{pmatrix} = \frac{1}{\sqrt{n+1}} \exp\left(\sum_{i=1}^{n} \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix} u^i\right) \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}, \quad j = 1, 2, \ldots, s;$$

$$x_j = \frac{1}{\sqrt{n+1}} \exp\left(\sum_{i=1}^{n} a_{ij} u^i\right), \quad j \geqq 2s+1.$$

So we have

$$(x_{2j-1}^2 + x_{2j}^2) = (x_{2j-1}, x_{2j}) \begin{pmatrix} x_{2j-1} \\ x_{2j} \end{pmatrix} = \frac{2}{n+1} \exp\left(2 \sum_{i=1}^{n} a_{ij} u^i\right), \quad j = 1, 2, \ldots, s,$$

and

$$\begin{aligned}(x_1^2 + x_2^2) \ldots (x_{2s-1}^2 + x_{2s}^2) x_{2s+1} \ldots x_{n+1} &= 2^s (n+1)^{-\frac{n+1}{2}} \exp\left(\sum_{i=1}^{n} \operatorname{tr}(A_i) u^i\right) \\ &= 2^s (n+1)^{-\frac{n+1}{2}}.\end{aligned}$$

Note that $s = 0$ gives the hypersphere (1.1).

Remark 2.3 From the fundamental theorem of equiaffine geometry we know that to determine all the canonical proper hyperspheres is equivalent to determine all the constant solutions $\{C_{ijk}\}$ and $\{\varepsilon_i\} \in \{+1, -1\}$ for the quadratic equation system

$$\begin{aligned} &\sum_{r=1}^{n} \varepsilon_r (C_{ikr} C_{jmr} - C_{ijr} C_{kmr}) = -\varepsilon_i \varepsilon_k (\delta_{ij} \delta_{km} - \delta_{im} \delta_{kj}) \\ &\sum_{r=1}^{n} \varepsilon_r C_{rrj} = 0 \\ &C_{ijk} = C_{jik} = C_{ikj}. \end{aligned} \tag{2.25}$$

For any $s = 1, 2, \ldots, \left[\frac{n+1}{2}\right]$ we get solutions $\{C_{ijk}\}$ and $\{\varepsilon_i\}$ for (2.25) by (2.19), (2.21), (2.22) and (2.14). Thus the assertion (ii) in § 1 claims that all solutions for (2.25) can be defined by this way.

3 The proof of the algebraic lemma

In order to prove the main theorem we establish first the algebraic lemma stated in § 1.

Let $\{A_i\}=\{A_1, A_2, \ldots, A_m\}$ be finite mutually commutative and self-adjoint linear operators in $\mathbb{R}^{n+1}$ with respect to the Lorentz inner product $\langle\ ,\ \rangle$,

$$\langle x, x\rangle = -x_1^2+x_2^2+\ldots+x_{n+1}^2, \qquad x={}^t(x_1, x_2, \ldots, x_{n+1})\in\mathbb{R}^{n+1}. \tag{3.1}$$

Let $\mathbb{C}^{n+1}=\mathbb{R}^{n+1}\otimes\mathbb{C}$ be the complexification of $\mathbb{R}^{n+1}$. We can extend $\langle\ ,\ \rangle$ to the Hermitian Lorentz inner product in $\mathbb{C}^{n+1}$ defined by

$$\langle z, w\rangle = -z_1\overline{w_1}+z_2\overline{w_2}+\ldots+z_{n+1}\overline{w_{n+1}} \tag{3.2}$$

for $z={}^t(z_1, z_2, \ldots, z_{n+1})$ and $w={}^t(w_1, w_2, \ldots, w_{n+1})$ in $\mathbb{C}^{n+1}$. We extend $\{A_i\}$ to the real linear operators in $\mathbb{C}^{n+1}$ by

$$A_i(x+\sqrt{-1}\,y)=A_ix+\sqrt{-1}\,A_iy, \qquad \forall x, y\in\mathbb{R}^{n+1}. \tag{3.3}$$

It is easy to see that for any z, w in $\mathbb{C}^{n+1}$ we have

$$\langle A_iz, w\rangle=\langle z, A_iw\rangle, \qquad i=1, 2, \ldots, m, \tag{3.4}$$

i.e., $\{A_i\}$ are real self-adjoint operators in $\mathbb{C}^{n+1}$.

Proposition 3.1 *Let $W\subset\mathbb{C}^{n+1}$ be a complex invariant subspace of $\{A_i\}$ with* $\dim W\geqq 1$. *Then for any j, given any eigenvalue λ_j of A_j there exists a common eigenvector $\eta\in W$ of $\{A_i\}$ such that $A_j\eta=\lambda_j\eta$.*

Proof. We may assume that $j=1$. *Let* $x_1\in W$ be an λ_1-eigenvector of A_1, i.e., $A_1x_1=\lambda_1x_1$. We define

$$W_1=\mathrm{span}_{\mathbb{C}}\{x_1, A_2x_1, \ldots, A_2^kx_1, \ldots\}.$$

Then W_1 is a subspace of W with $\dim W_1\geqq 1$. Since $A_2W_1\subset W_1$, A_2 has an eigenvector $x_2\in W_1$ and $x_2=p_2(A_2)x_1$ for some polynomial p_2. Similarly A_3 has an eigenvector $x_3=p_3(A_3)x_2=p_3(A_3)p_2(A_2)x_1$ in the invariant subspace $W_2=\mathrm{span}_{\mathbb{C}}\{x_2, A_3x_2, \ldots, A_3^kx_2, \ldots\}$. Since $\{A_i\}$ are finite, by repeating this process we will get an eigenvector x_m of A_m, $x_m=p_m(A_m)x_{m-1}=\cdots=p_m(A_m)p_{m-1}(A_{m-1})\ldots p_2(A_2)x_1$. Since $\{A_i\}$ are commutative and x_i is an eigenvector of A_i, $i=1, 2, \ldots, m$, we know that x_m is a common eigenvector of $\{A_i\}$ such that $A_1x_m=\lambda_1x_m$. Q.E.D.

Proposition 3.2 *Let $\xi, \eta\in\mathbb{C}^{n+1}$ be λ-, μ-eigenvalue of A_j respectively. If $\lambda\neq\bar{\mu}$, then $\langle\xi, \eta\rangle=0$.*

Proposition 3.3 *Let $V\subset\mathbb{R}^{n+1}$ be an invariant subspace of $\{A_i\}$. Let $\eta\in V\otimes\mathbb{C}\subset\mathbb{C}^{n+1}$ be a common eigenvector of $\{A_i\}$. If $\langle\eta, \eta\rangle\neq 0$, then there is a real common eigenvector $x\in V$ ($x=\mathrm{Re}\,\eta$ or $\mathrm{Im}\,\eta$) such that $A_ix=\lambda_ix$, $\lambda_i\in\mathbb{R}$. $i=1, 2, \ldots, m$, and $\langle x, x\rangle\neq 0$. Moreover, we have the orthonogal decomposition $V=\mathbb{R}x\oplus x^{\perp}$, where $x^{\perp}=\{y\in V\mid\langle y, x\rangle=0\}$ which is also an invariant subspace of $\{A_i\}$.*

Proposition 3.2 and 3.3 follow immediately from the fact that $\{A_i\}$ are self-adjoint and that $\langle\ ,\ \rangle$ is a Hermitian inner product.

Proposition 3.4 *Suppose that* $\langle z, z\rangle = \langle z, w\rangle = \langle w, w\rangle = 0$ *for* z, w *in* $\mathbb{C}^{n+1}$, *then* z *and* w *are linearly dependent.*

Proof. Let $z = {}^t(z_1, z_2, \ldots, z_{n+1})$ and $w = {}^t(w_1, w_2, \ldots, w_{n+1})$. Then we can find $(k_1, k_2) \neq 0$ in $\mathbb{C}^2$ such that $k_1 z_1 + k_2 w_1 = 0$. Since $\langle k_1 z + k_2 w, k_1 z + k_2 w\rangle = 0$, we conclude from (3.2) that $k_1 z + k_2 w = 0$. Q.E.D.

It is clear that we have an irreducible orthonogal decomposition

$$\mathbb{R}^{n+1} = V_1 \oplus V_2 \oplus \ldots \oplus V_\gamma, \tag{3.5}$$

where V_k, $k = 1, 2, \ldots, \gamma$, are irreducible invariant subspaces of $\{A_i\}$ with $\dim V_k \geqq 1$.

Proposition 3.5 *There exists at most one* k *such that* $\dim V_k \geqq 1$.

Proof. Since $\langle\ ,\ \rangle$ is nondegenerate in each V_k and the maximal negative definite subspace of $\langle\ ,\ \rangle$ has dimension 1, we know that all but one V_k are positive definite subspaces. Since any positive definite invariant subspace of $\{A_i\}$ with dimension greater than 1 is reducible, so there is at most one k with $\dim V_k > 1$. Q.E.D.

Thus we can find an orthonormal subbasis $\{e_{\nu+1}, e_{\nu+2}, \ldots, e_{n+1}\}$ for $\mathbb{R}^{n+1}$ consisting of common eigenvectors of $\{A_i\}$ such that we have the orthonogal decomposition

$$\mathbb{R}^{n+1} = V_0 \oplus \mathbb{R}e_{\nu+1} \oplus \ldots \oplus \mathbb{R}e_{n+1}, \tag{3.6}$$

where V_0 is an irreducible non-positive definite invariant subspace of $\{A_i\}$ of dimension ν.

If $\dim V_0 = 1$, then $\{A_i\}$ is of Type I as in the lemma in § 1, i.e., $\{A_i\}$ can be simultaneously diagonalized. In the rest of this section we will always assume that $\dim V_0 \geqq 2$. It follows from Proposition 3.3 that

Proposition 3.6 *Let* $\eta \in V_0 \otimes \mathbb{C}$ *be any common eigenvector of* $\{A_i\}$, *then* $\langle \eta, \eta\rangle = 0$.

Proposition 3.7 *Each* $A_j : V_0 \to V_0$ *has either* (i): *only two non-real dual eigenvalues* λ_j *and* $\bar{\lambda}_j$; *or* (ii): *only one real eigenvalue* λ_j.

Proof. Let λ_j and μ_j $(\lambda_j \neq \mu_j)$ be two eigenvalues of $A_j : V_0 \to V_0$. By Proposition 3.1 we have two common eigenvectors ξ and η of $\{A_i\}$ in $V_0 \otimes \mathbb{C}$ such that $A_j \xi = \lambda_j \xi$, $A_j \eta = \mu_j \eta$. Then Proposition 3.6 implies that $\langle \xi, \xi\rangle = \langle \eta, \eta\rangle = 0$. Thus by Proposition 3.4 we have $\langle \xi, \eta\rangle \neq 0$. So by Proposition 3.2 we have $\lambda_j = \bar{\mu}_j$. Therefore, $A_j : V_0 \to V_0$ has either (i): only two non-real dual eigenvalues λ_j and $\bar{\lambda}_j$; or (ii): only one eigenvalue λ_j. In case (ii) λ_j must be real since A_j is real. Q.E.D.

Definition 3.8 $\{A_i\}$ is said to be (i): *of Type* II if there is j such that $A_j : V_0 \to V_0$ has two non-real dual eigenvalues λ_j and $\bar{\lambda}_j$; (ii): *of Type* III if each $A_i : V_0 \to V_0$ has only one real eigenvalue λ_i.

Proposition 3.9 *If $\{A_i\}$ is of Type* II, *then* $\dim V_0=2$. *Moreover, there exists a basis $\{e_1, e_2\}$ for V_0 with $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = -1$ and $\langle e_1, e_2\rangle = 0$ such that*

$$A_i(e_1, e_2)=(e_1, e_2)\begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix}, \quad i=1, 2, \ldots, m; \tag{3.7}$$

where $a_i, b_i \in \mathbb{R}$ and at least one b_i is non-zero.

Proof. Let $\eta \in V_0 \otimes \mathbb{C}$ be a common eigenvector of $\{A_i\}$. We may assume that $A_i\eta=\lambda_i\eta$, $i=1, 2, \ldots, m$. We write $\eta=x+\sqrt{-1}\,y$ with $x, y \in V_0$. Since some of λ_i are non-real, we know that x and y are linearly independent. By Proposition 3.6 we have $\langle \eta, \eta\rangle = \langle x, x\rangle + \langle y, y\rangle = 0$. Then it follows from Proposition 3.4 that $\langle \eta, \bar{\eta}\rangle = \langle x, x\rangle - \langle y, y\rangle + 2\sqrt{-1}\langle x, y\rangle \neq 0$. By letting $\xi=\rho\eta$ with $\rho^4=\dfrac{\langle \bar{\eta}, \eta\rangle}{\langle \eta, \bar{\eta}\rangle}$ we have $\langle \operatorname{Re}\xi, \operatorname{Im}\xi\rangle = 0$. Since $\langle \xi, \xi\rangle = 0$, $\langle \operatorname{Re}\xi, \operatorname{Re}\xi\rangle = -\langle \operatorname{Im}\xi, \operatorname{Im}\xi\rangle$ is non-zero. Thus we may assume $\xi = e_1 + \sqrt{-1}\,e_2$ and $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = -1$, $\langle e_1, e_2\rangle = 0$. By letting $\lambda_i = a_i + \sqrt{-1}\,b_i$ we get (3.7). Since $V=\operatorname{span}_{\mathbb{R}}\{e_1, e_2\}$ is an invariant subspace of $\{A_i\}$ and $V \cap V^{\perp} = \{0\}$, by the irreducibility of V_0 we know that $V_0 = V$. Q.E.D.

In the rest of this section we assume that $\{A_i\}$ is of Type III. Let V_0 be as in (3.6) and λ_i as in Definition 3.8 we have

Proposition 3.10 *Let γ be the smallest positive integer such that $(A_i-\lambda_i I)^{\gamma}=0$ in V_0 for all $i=1, 2, \ldots, m$, then $\gamma \leqq 3$.*

Proof. Let $x \in V_0 \otimes \mathbb{C}$ be a common eigenvector of $\{A_i\}$, then $A_i x = \lambda_i x$, $i=1, 2, \ldots, m$. Since $\{A_i\}$ and $\{\lambda_i\}$ are real, we may assume that $x \in V_0$. By the definition of γ we can find j and $u \in V_0$ such that $(A_j-\lambda_j I)^{\gamma-1}u \neq 0$. We define $y=(A_j-\lambda_j I)^{\gamma-2}u$. If $\gamma \geqq 4$, then $2\gamma-4 \geqq \gamma$, we have $\langle y, y\rangle = \langle u, (A_j-\lambda_j I)^{2\gamma-4}u\rangle = 0$ and $\langle y, x\rangle = 0$. But Proposition 3.6 implies $\langle x, x\rangle = 0$, we conclude from Proposition 3.4 that $y=kx$ for some $k \in \mathbb{R}$. Thus $(A_j-\lambda_1 I)y=(A_j-\lambda_j I)^{\gamma-1}u=0$, we get a contradiction. Q.E.D.

Proposition 3.11 *Suppose that $\gamma=2$, then* $\dim V_0=2$. *Let j satisfy $(A_j-\lambda_j I)\neq 0$ in V_0, then there is a basis $\{x, y\}$ for V_0 with $\langle x, y\rangle = 1$ and $\langle x, x\rangle = \langle y, y\rangle = 0$ such that*

$$A_i(x, y)=(x, y)\begin{pmatrix} \lambda_i & a_i \\ 0 & \lambda_i \end{pmatrix}, \quad i=1, 2, \ldots, m; \quad a_i \in \mathbb{R}; \quad a_j = \pm 1. \tag{3.8}$$

Proof. Let $x \in V_0$ be a common eigenvector of $\{A_i\}$. Then $A_i x = \lambda_i x$, $i=1, 2, \ldots, m$, and $\langle x, x\rangle = 0$. Let $u \in V_0$ with $\langle u, x\rangle \neq 0$, then u and x are linearly independent. Since $\langle (A_i-\lambda_i I)u, x\rangle = 0$ and $\langle (A_i-\lambda_i I)u, (A_i-\lambda_i I)u\rangle = 0$, we have by Proposition 3.4 that $(A_i-\lambda_i I)u = b_i x$, $i=1, 2, \ldots, m$; $b_i \in \mathbb{R}$. Thus $V=\operatorname{span}_{\mathbb{R}}\{u, x\}$ is an invariant subspace of $\{A_i\}$ with $V \cap V^{\perp} = \{0\}$. By the irreducibility of V_0 we have $V_0 = V$, so $\dim V_0 = 2$. Since $(A_j-\lambda_j I)\neq 0$ in V_0, we have $(A_j-\lambda_j I)u = b_j x \neq 0$. By modifying u we may assume $\langle (A_j-\lambda_j I)u, u\rangle = \varepsilon \in \{+1, -1\}$. Then by modifying x we have $\langle u, x\rangle = 1$. Now let $y = u - \dfrac{\langle u, u\rangle}{2}x$, we get $\langle x, x\rangle = \langle y, y\rangle = 0$

and $\langle x, y\rangle = 1$. Moreover, $(A_i - \lambda_i I)y = a_i x$ for some $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, and $a_j = \varepsilon$. Q.E.D.

Proposition 3.12 *Suppose that $\gamma = 3$ and j satisfies $(A_j - \lambda_j I)^2 \neq 0$ in V_0. Then there is a basis $\{x, y, e_3, \ldots, e_n\}$ for V_0 with $\langle x, x\rangle = \langle y, y\rangle = \langle x, e_\alpha\rangle = \langle y, e_\alpha\rangle = 0$, $\langle x, y\rangle = 1$ and $\langle e_\alpha, e_\beta\rangle = \delta_{\alpha\beta}$, $3 \leqq \alpha, \beta \leqq r$, such that we have the matrix representations*

$$A_i = \begin{pmatrix} \lambda_i & a_{i2} & a_{i3} & a_{i4} & \ldots & a_{iv} \\ 0 & \lambda_i & 0 & 0 & \ldots & 0 \\ 0 & a_{i3} & \lambda_i & & & \\ 0 & a_{i4} & & \lambda_i & & \\ \vdots & \vdots & & & & \\ 0 & a_{iv} & & 0 & & \lambda_i \end{pmatrix}, \quad i = 1, 2, \ldots, m, \tag{3.9}$$

and $a_{ik} \in \mathbb{R}$ with $a_{j3} = 1$, $a_{j2} = a_{j4} = \ldots = a_{jr} = 0$.

Proof. Let $x \in V_0$ be a common eigenvector for $\{A_i\}$, then $A_i x = \lambda_i x$ for all i and $\langle x, x\rangle = 0$. Let $u \in V_0$ with $(A_j - \lambda_j I)^2 u \neq 0$. Since $\langle (A_j - \lambda_j I)u, x\rangle = 0$, we know by Proposition 3.4 that $v = (A_j - \lambda_j I)u$ satisfies $\langle v, v\rangle \neq 0$. We claim that $\langle v, v\rangle > 0$. Otherwise we have $\langle v, v\rangle < 0$, then we have the orthonogal decomposition $V_0 = \mathbb{R}v \oplus v^\perp$. Since $\langle\ ,\ \rangle$ is negative definite in $\mathbb{R}v$, it must be positive definite in $v^\perp$. But $x \in v^\perp$ and $\langle x, x\rangle = 0$, we get a contradiction. Thus by modifying u we may assume that $\langle v, v\rangle = 1$. By Proposition 3.4 and the fact that $\gamma = 3$ we know that $(A_j - \lambda_j I)^2 u = kx \neq 0$ for some $k \in \mathbb{R}$, so by modifying x we may assume that $k = 1$, i.e., $\langle u, x\rangle = \langle v, v\rangle = 1$. Now let $y = u - \frac{1}{2}\langle u, v\rangle v - [\frac{1}{2}\langle u, u\rangle - \frac{3}{8}\langle u, v\rangle^2]x$ and $e_3 = v - \frac{1}{2}\langle u, v\rangle x$, we have

$$\langle x, x\rangle = \langle y, y\rangle = \langle x, e_3\rangle = \langle y, e_3\rangle = 0, \quad \langle x, y\rangle = \langle e_3, e_3\rangle = 1. \tag{3.10}$$

We define $W = \mathrm{span}_{\mathbb{R}}\{x, y, e_3\}$, then $W \cap W^\perp = \{0\}$ and $V_0 = W \oplus W^\perp$. Since $\langle\ ,\ \rangle$ is indefinite in W, it must be positive definite in $W^\perp$. So $A_j\colon W^\perp \to W^\perp$ is diagonalizable. We can choose an orthonormal basis $\{e_4, e_5, \ldots, e_v\}$ for $W^\perp$ such that $A_j e_\alpha = \lambda_j e_\alpha$, $\alpha = 4, 5, \ldots, v$. Thus $\{x, y, e_3, e_4, \ldots, e_v\}$ is a basis for V_0. By Proposition 3.4 we know that $(A_i - \lambda_i I)^2 e_\alpha = k_{i\alpha} x$ for some $k_{i\alpha} \in \mathbb{R}$, $i = 1, 2, \ldots, m$; so $\langle (A_i - \lambda_i I)e_\alpha, (A_i - \lambda_i I)e_\alpha\rangle = \langle e_\alpha, k_{i\alpha} x\rangle = 0$. Again by Proposition 3.4 we get

$$(A_i - \lambda_i I)e_\alpha = a_{i\alpha} x, \quad a_{i\alpha} \in \mathbb{R}, \quad i = 1, 2, \ldots, m; \quad \alpha = 3, 4, \ldots, v. \tag{3.11}$$

Using the formula $A_i y = \langle A_i y, y\rangle x + \langle A_i y, x\rangle y + \sum_{\alpha=3}^{v} \langle A_i y, e_\alpha\rangle e_\alpha$ and (3.11) we get

$$A_i y = a_{i2} x + \lambda_i y + \sum_{\alpha=3}^{v} a_{i\alpha} e_\alpha, \quad i = 1, 2, \ldots, m, \quad a_{i2} = \langle A_i y, y\rangle. \tag{3.12}$$

Thus (3.11) and (3.12) imply (3.9). Furthermore, we have $(A_j - \lambda_j I)y = v - \frac{1}{2}\langle u, v\rangle x = e_3$, so we get $a_{j3} = 1$ and $a_{j2} = a_{j4} = \ldots = a_{jv} = 0$. Q.E.D.

Thus the algebraic lemma follows from the orthonogal decomposition (3.6), Proposition 3.9, 3.11 and 3.12.

4 The proof of the main theorem

Let $f: M \to \mathbb{R}^{n+1}$ be a proper canonical affine hypersphere and G its Blaschke metric. We assume that the maximal negative definite subspaces of G have dimension 1. Then as in § 2 we know that there exist n mutually commutative linear operators $\{A_i\}$ in $\mathbb{R}^{n+1}$ which are self-adjoint with respect to the Lorentz inner product $\langle\ ,\ \rangle$,

$$\langle x, x\rangle = -x_1^2 + x_2^2 + \ldots + x_{n+1}^2, \qquad x = {}^t(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}. \tag{4.1}$$

Moreover, $\{A_i\}$ satisfy also the properties (i): $\operatorname{tr}(A_i)=0$, $i=1,2,\ldots,n$; and (ii): $\exists\, \xi \in \mathbb{R}^{n+1}$ such that $\{A_1\xi, A_2\xi, \ldots, A_n\xi, \xi\}$ is an orthonormal basis for $\mathbb{R}^{n+1}$ with respect to $\langle\ ,\ \rangle$.

We know from the algebraic lemma in § 1 that $\{A_i\}$ have three possible types, namely the Type I, II and III.

If $\{A_i\}$ is of Type I, then there is an orthonormal basis $\{e_1, e_2, \ldots, e_{n+1}\}$ for $\mathbb{R}^{n+1}$ such that we have the matrix representations

$$A_i = (a_{i1}) \oplus (a_{i2}) \oplus \ldots \oplus (a_{in+1}), \qquad i=1,2,\ldots,n. \tag{4.2}$$

By the property (ii) we know that $A_1, A_2, \ldots, A_n$ are linearly independent, so the matrix

$$\mathbb{T} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n+1} \\ a_{21} & a_{22} & \ldots & a_{2n+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn+1} \end{pmatrix} \tag{4.3}$$

has rank n, i.e., the kernal of $\mathbb{T}: \mathbb{R}^{n+1} \to \mathbb{R}^n$ has dimension 1. Now let $\xi = \sum_{\alpha=1}^{n+1} \xi_\alpha e_\alpha$. By the property (ii) we have

$$\langle A_i\xi, \xi\rangle = a_{i1}(-\xi_1^2) + \sum_{\alpha=2}^{n+1} a_{i\alpha}(\xi_\alpha)^2 = 0, \qquad i=1,2,\ldots,n.$$

Thus ${}^t(-\xi_1^2, \xi_2^2, \ldots, \xi_{n+1}^2) \in \ker \mathbb{T}$. But the property (i) implies that ${}^t(1,1,\ldots,1) \in \ker \mathbb{T}$, so there is $k \in \mathbb{R}$ such that $(-\xi_1^2, \xi_2^2, \ldots, \xi_{n+1}^2) = k(1,1,\ldots,1)$. It is impossible for $\langle \xi, \xi\rangle = 1$. Thus $\{A_i\}$ cannot be of Type I.

Let $\{A_i\}$ be of Type II. Then there is an orthonormal basis $\{e_1, e_2, \ldots, e_{n+1}\}$ for $\mathbb{R}^{n+1}$ such that we have the matrix representations

$$A_i = \begin{pmatrix} a_{i1} & a_{i2} \\ -a_{i2} & a_{i1} \end{pmatrix} \oplus (a_{i3}) \oplus \ldots \oplus (a_{in+1}), \qquad i=1,2,\ldots,n. \tag{4.4}$$

Let $\xi = \sum_{\alpha=1}^{n+1} \xi_\alpha e_\alpha$. By the property (ii) we have

$$\langle A_i\xi, \xi\rangle = a_{i1}(\xi_2^2 - \xi_1^2) + a_{i2}(-2\xi_1\xi_2) + \sum_{\alpha=3}^{n+1} a_{i\alpha}(\xi_\alpha)^2 = 0, \qquad i=1,2,\ldots,n.$$

We define $\mathbb{T}$ by (4.3), then ${}^t(\xi_2^2-\xi_1^2, -2\xi_1\xi_2, \xi_3^2, \ldots, \xi_{n+1}^2)\in\ker\mathbb{T}$. But the property (i) implies that ${}^t(2,0,1,\ldots,1)\in\ker\mathbb{T}$, so there is $k\in\mathbb{R}$ such that $(\xi_2^2-\xi_1^2, -2\xi_1\xi_2, \xi_3^2, \ldots, \xi_{n+1}^2)=k(2,0,1,\ldots,1)$. Thus $\xi_1=0$. From $\langle\xi,\xi\rangle=1$ we get $k=(n+1)^{-1}$. By changing e_α to $-e_\alpha$ if necessary we may assume that $\xi=\frac{1}{\sqrt{n+1}}(\sqrt{2}e_2+e_3+\ldots+e_{n+1})$. As we know in Example 2.2 that $\{A_i\}$ determine the proper canonical hypersphere

$$(x_1^2+x_2^2)x_3\ldots x_{n+1}=2(n+1)^{-\frac{n+1}{2}}. \tag{4.5}$$

Finally we assume that $\{A_i\}$ is of Type III. Then by the algebraic lemma we have a basis $\{x, y, e_3, \ldots, e_{n+1}\}$ such that we have the matrix representations as in the lemma in § 1. We write

$$A_i=a_{i1}T_1+a_{i2}T_2+\ldots+a_{in+1}T_{n+1}, \qquad i=1,2,\ldots,n, \tag{4.6}$$

and $\xi=\xi_1 x+\xi_2 y+\sum_{\alpha=3}^{n+1}\xi_\alpha e_\alpha$. By the property (ii) we have

$$\langle A_i\xi,\xi\rangle=\sum_{\alpha=1}^{n+1}a_{i\alpha}\langle T_\alpha\xi,\xi\rangle=0, \qquad i=1,2,\ldots,n.$$

Thus ${}^t(\langle T_1\xi,\xi\rangle,\ldots,\langle T_{n+1}\xi,\xi\rangle)\in\ker\mathbb{T}$, where $\mathbb{T}$ is defined by (4.3). But the property (i) implies that ${}^t(v,0,\ldots,0,1,\ldots,1)\in\ker\mathbb{T}$, and we know that $\dim(\ker\mathbb{T})=1$, so there is $k\in\mathbb{R}$ such that $(\langle T_1\xi,\xi\rangle,\ldots,\langle T_{n+1}\xi,\xi\rangle)=k(v,0,\ldots,0,1,\ldots,1)$, where we have $v-1$ copies of 0. Since $v\geqq 2$, we get $0=\langle T_2\xi,\xi\rangle=\langle\xi_2 x,\xi\rangle=\xi_2^2=0$, i.e., $\xi_2=0$. So we have $\langle\xi,x\rangle=\xi_2=0$ and $\langle A_i\xi,x\rangle=\langle\xi,a_{i1}x\rangle=0$, $i=1,2,\ldots,n$. It is impossible because of the property (ii).

Thus we complete the proof of the main theorem.

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Applications of the Ein-Lazarsfeld criterion for spannedness of adjoint bundles

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0 Introduction

In this article we apply the striking new criteria of Ein and Lazarsfeld [1] for the spannedness of the adjoint bundle, $K_X \otimes L$, of a nef and big line bundle, L, on a complex, projective 3-fold, X.

The Ein-Lazarsfeld results are very strong and allow them to settle some outstanding questions on spannedness. Qualitatively the Ein-Lazarsfeld criteria say that for L^3 sufficiently large, the failure of spannedness of $K_X \otimes L$ results in the existence of effective surfaces $S \subset X$ and/or effective curves $C \subset X$, with $L^2 \cdot S$ and $L \cdot C$ respectively sufficiently small. The main difference between their result and the classical Reider Theorem [12] is the lack of control of the normal bundles of the bad surfaces, S, and curves, C.

In this article we give a few geometric conditions on X and L that allow us to rule out bad curves and surfaces, and thus show spannedness.

Theorem 11 shows that given an ample and spanned vector bundle, $\mathscr{E}$, on a smooth 3-fold, X, it follows that $K_X \otimes \det(\mathscr{E})$ is spanned if $\operatorname{rank}(\mathscr{E}) \geqq 3$ and $L^3 \gg 0$, e.g. $\geqq 850$, and for $\operatorname{rank}(\mathscr{E}) \geqq 13$ in all cases. The theorem rests on a number of inequalities given in § 1 for the Chern classes of ample and spanned vector bundles. The spannedness of $\mathscr{E}$ is necessary in these results. For all $r \geqq 1$ and $n \geqq 1$, we give examples in § 2 of ample vector bundles, $\mathscr{V}$, of rank r on a smooth n-fold, X, such that $K_X \otimes \det(\mathscr{V})$ is not spanned.

Theorem 15 of § 4 is:

Theorem 15 *Let X be a smooth projective n-fold, $n \geqq 3$, with no rational curves. Let L be a line bundle that is very ample, and assume that $L^n \geqq 850$. Then $K_X \otimes L^{n-2}$ is spanned by global sections.*

By rational curve we mean an irreducible reduced curve whose desingularization is of genus 0. Note that if X is hyperbolic in the sense of Kobayashi, or if X has a nef cotangent bundle, then the condition on no rational curves is

satisfied. (A vector bundle, $\mathscr{E}$, is said to be *numerically effective*, or *nef* for short, if the tautological bundle on $\mathbb{P}(\mathscr{E})$ is nef.) The result rests on Lemma 14 of § 3 which shows that surfaces of degree $\leqq 4$ contain rational curves.

1 Bounds for Chern numbers of ample and spanned vector bundles

By $s_j(\mathscr{E})$ for a vector bundle $\mathscr{E}$ on a projective variety, X, we mean the j-th Segre class of $\mathscr{E}$, i.e. $\pi_* \xi^{r+j-1}$ where $\pi: \mathbb{P}(\mathscr{E}) \to X$ is the bundle projection from the projectivization of $\mathscr{E}$ to X, ξ is the first Chern class of the tautological bundle on $\mathbb{P}(\mathscr{E})$, and r is the rank of $\mathscr{E}$.

Combining the main result of [7] with Corollary 1.5 of [13] (note the proof in [13] is given for singular varieties, though it is stated for smooth varieties) gives the following lemma.

Lemma 1 *Let $\mathscr{E}$ be an ample and spanned vector bundle with $r := \operatorname{rank}(\mathscr{E}) \geqq 2$, on an irreducible and reduced, projective surface, S. Then $c_1^2(\mathscr{E}) \geqq 2 + r$.*

Proof. Let ξ be the first Chern class of the tautological bundle of $\mathbb{P}(\mathscr{E})$. From [13, (1.5)] we get $\xi^{r+1} = c_1^2(\mathscr{E}) - c_2(\mathscr{E}) \geqq r$. If $c_2(\mathscr{E}) = 1$ we know from [7, (2.1)] that $(S_v, v^*\mathscr{E}) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)^{\oplus 2})$, where $v: S_v \to S$ is the normalization of S. Since the inequality is obvious in this case, we can assume that $c_2(\mathscr{E}) \geqq 2$ without loss of generality. This gives the lemma. Q.E.D.

Remark 2 We also have immediately from Corollary 1.5 of [13] that given an ample and spanned vector bundle, $\mathscr{E}$, with $r := \operatorname{rank}(\mathscr{E}) \geqq 2$, on an irreducible and reduced, curve, C, then $c_1(\mathscr{E}) \geqq r$.

Using the representation of Chern classes by effective subvarieties of a variety, we get the following useful estimate.

Corollary 3 *Let $\mathscr{E}$ be an ample and spanned, rank r vector bundle on an irreducible projective variety, X, of dimension n. Any Chern number with $c_1(\mathscr{E})$ as a factor is bounded below by r, and any Chern number with $c_1^2(\mathscr{E})$ as a factor is bounded below by $r+2$. In particular if $r \geqq n$, then $(c_1 \cdot c_{n-1})(\mathscr{E}) \geqq r$.*

The following result generalizes [13, Theorem 0.1] when the base of the vector bundle is smooth of dimension $\geqq 2$.

Theorem 4 *Let $\mathscr{E}$ be an ample and spanned, rank r vector bundle, $r \geqq 2$, on a smooth projective n-fold, X, $n \geqq 1$. Then $s_n(\mathscr{E}) \geqq n + r - 1$. If $\dim H^1(X, \mathbb{C}) \neq 0$ then $s_n(\mathscr{E}) \geqq n + r$.*

Proof. Since $\mathscr{E}$ is ample and spanned, there exist $n+r$ sections of $\mathscr{E}$ spanning $\mathscr{E}$ at all points. Let $\phi: \mathbb{P}(\mathscr{E}) \to \mathbb{P}^{n+r-1}$ be the associated finite to one map. By Lazarsfeld's Barth-Lefschetz Theorem [8] for branched coverings

$$\dim H^2(\mathbb{P}(\mathscr{E}), \mathbb{C}) = \dim H^2(\mathbb{P}^{n+r-1}, \mathbb{C}) = 1 \quad \text{if } 2 \leqq (n+r-1)+1-d$$

where d is the degree of the branched covering, ϕ. Since $\mathbb{P}(\mathscr{E}) \to X$ is a projective space bundle with positive dimensional fiber, we have that $\dim H^2(\mathbb{P}(\mathscr{E}), \mathbb{C}) \geqq 2$ by the Leray-Hirsch Theorem. Therefore we conclude that $d \geqq n + r - 1$. Note that $s_n(\mathscr{E}) = d$.

The proof when $\dim H^1(X, \mathbb{C}) \neq 0$ follows by the same argument. Q.E.D.

Theorem 5 *Let $\mathscr{E}$ be a nef vector bundle on an irreducible n dimensional projective variety, X. Let H be the class of a nef line bundle on X, and let W be an irreducible w-dimensional subvariety of X. If $a+b=w$, then:*

$$[s_a(\mathscr{E})\cdot W\cdot H^b]^2 \geqq [s_{a-1}(\mathscr{E})\cdot W\cdot H^{b+1}][s_{a+1}(\mathscr{E})\cdot W\cdot H^{b-1}].$$

Proof. Let ξ denote the first Chern class of the tautological line bundle on $\mathbb{P}(\mathscr{E})$. Since we can restrict to W, it suffices to prove the theorem with $W=X$. Using the definition of Segre classes, it therefore suffices to prove, if $r=\operatorname{rank}(\mathscr{E})$ and H' denotes the pullback of H under the bundle projection,

$$[\xi^{r+a-1}\cdot H'^b]^2 \geqq [\xi^{r+a-2}\cdot H'^{b+1}][\xi^{r+a}\cdot H'^{b-1}].$$

This is the usual Hodge Index Theorem, and follows by a reduction to the Hodge Index Theorem on surfaces. See [2, Proposition 1.2]. Q.E.D.

Corollary 6 *Let $\mathscr{E}$ be a spanned vector bundle on X. The above result holds with the i-th Chern class, $c_i(\mathscr{E})$, of $\mathscr{E}$ in place of $s_i(\mathscr{E})$.*

Proof. Let $\mathscr{F}$ denote the dual of the vector bundle kernel of the evaluation map $X\times\Gamma(\mathscr{E})\to\mathscr{E}$. Note that $\mathscr{F}$ is nef and $s_i(\mathscr{F})=c_i(\mathscr{E})$. Q.E.D.

Remark 7 Probably the corollary is true for any nef bundle. Indeed it is easy to check for $n\leqq 3$ that replacing the Segre classes by their expressions in terms of Chern classes, the inequalities for the Segree classes are equivalent to the inequalities for the Chern classes.

Remark 8 As an amusing example of the type of inequalities the corollary gives, let $\mathscr{E}$ be an ample and spanned vector bundle on a variety X. Take $H=c_1(\mathscr{E})$. Let W be a subvariety of dimension w representing a product of Chern classes, $c_I:=\prod_{i\in I}c_i(\mathscr{E})$. If $a+b=w$, then

$$[c_1(\mathscr{E})^b\cdot c_a(\mathscr{E})\cdot c_I(\mathscr{E})]^2 \geqq [c_1(\mathscr{E})^{b-1}\cdot c_{a+1}(\mathscr{E})\cdot c_I(\mathscr{E})]\cdot[c_1(\mathscr{E})^{b+1}\cdot c_{a-1}(\mathscr{E})\cdot c_I(\mathscr{E})].$$

For an ample and spanned rank $r\geqq 3$ vector bundle, $\mathscr{E}$, on a 3-fold, X, we get:

$$c_1^3(\mathscr{E}) \geqq (c_1\cdot c_2)(\mathscr{E})+\sqrt{s_3(\mathscr{E})\,c_1^3(\mathscr{E})}. \tag{1}$$

$$\begin{aligned} c_1^3(\mathscr{E}) &= s_3(\mathscr{E})+(c_1\cdot c_2)(\mathscr{E})+((c_1\cdot c_2)(\mathscr{E})-c_3(\mathscr{E})) \\ &\geqq (r+2)+r+1=2r+3. \end{aligned} \tag{2}$$

Note that the inequality (1) is just Theorem 5 for 3-folds. Take $a=2$, $W=X$, $b=1$, and $H=c_1(\mathscr{E})$. The second is the expression for the 3rd Segre class combined with the inequality from Theorem 4, with the estimate $(c_1\cdot c_2)(\mathscr{E})\geqq r$ from Corollary 3, and the positivity of $(c_1\cdot c_2)(\mathscr{E})-c_3(\mathscr{E})$, see e.g. [4, Example 12.1.7]. Q.E.D.

Theorem 9 *For an ample and spanned vector bundle, $\mathscr{E}$, of rank, $r\geqq 3$, on a smooth 3-fold, X, we have:*

$$c_1^3(\mathscr{E}) \geqq \frac{3r+2+\sqrt{5r^2+12r+4}}{2}.$$

Proof. Let $t=\sqrt{c_1^3(\mathscr{E})}$. Then from Theorem 4, Corollary 3, and the inequality (1), we get

$$t^2-t\sqrt{2+r}-r\geqq 0.$$

This gives $t\geqq\dfrac{\sqrt{2+r}+\sqrt{5r+2}}{2}$, and the desired inequality. Q.E.D.

Remark 10 With further argument we can show $(c_1\cdot c_2)(\mathscr{E})\geqq r+1$, and improve the lower bound of Theorem 9 to $c_1^3(\mathscr{E})\geqq\dfrac{3r+4+\sqrt{5r^2+16r+12}}{2}$.

2 Spannedness of $K_X\otimes\det\mathscr{E}$ for a rank $\geqq 3$ ample and spanned vector bundle, $\mathscr{E}$

Given an ample and spanned vector bundle, $\mathscr{E}$, of rank $r\geqq n$, on a smooth n-fold, X, we expect that $K_X\otimes\det(\mathscr{E})$ is spanned by global sections with the exception of $r=n$, $X\cong\mathbb{P}^n$, and $\mathscr{E}\cong\mathscr{O}_{\mathbb{P}^n}(1)^{\oplus n}$. For $n=1$ this is obvious using Theorem 4 or Remark 2. For $n=2$ this is a result of Lanteri and Maeda [6, Theorem A] which follows from Reider's Theorem [12]. (For the nefness of bundles such as $K_X+\det(\mathscr{E})$ for $\mathscr{E}$ ample, see [14, 15, 3].) Using [1] and the results of § 1 we have:

Theorem 11 *Let $\mathscr{E}$ be an ample and spanned vector bundle on a smooth, projective 3-fold, X. If $\operatorname{rank}(\mathscr{E})\geqq 3$ then $K_X\otimes\det(\mathscr{E})$ is spanned at all points by global sections if one of the following conditions is satisfied:*

(1) $\operatorname{rank}(\mathscr{E})=3$ *and* $c_1^3(\mathscr{E})\geqq 850$;

(2) $\operatorname{rank}(\mathscr{E})\geqq 4$ *and* $c_1^3(\mathscr{E})\geqq 162$; *or*

(3) $\operatorname{rank}(\mathscr{E})\geqq 13$.

Proof. Let $L=\det(\mathscr{E})$. Let S be an irreducible surface on X and let C be an irreducible curve on X. We have $L\cdot C\geqq r$ by Remark 2, and $L^2\cdot S\geqq r+2$ by Lemma 1. A numerical check using [1, Theorem 1*] gives the $r=3$ and $r\geqq 4$ results. The $r\geqq 13$ result follows from a numerical check by again using [1, Theorem 1*] and Theorem 9. Q.E.D.

Spannedness is needed in the above result. To show this we need a lemma.

Lemma 12 *Given any smooth projective curve, C, of genus $g\geqq 2$, any $r>0$, and any point $x\in C$, there exists an ample, stable, rank r vector bundle, E, with $\det(E)\cong\mathscr{O}_C(x)$.*

Proof. Since the lemma is trivial if $r=1$, we can assume without loss of generality that $r\geqq 2$.

Take a stable rank $r-1$ vector bundle, F, on C with $\det(F)\cong\mathscr{O}_C$ [10, 11]. Choose a point $x\in C$. Since F is stable of degree 0, $H^0(F(-x))=(0)$. Hence

$$h^1(F(-x))=-\chi(F(-x))=-(r(1-g)-r)=rg.$$

Take a nontrivial extension, $0\to F\to E\to\mathscr{O}_C(x)\to 0$. Note that $\deg(E)=1$ and $\operatorname{rank}(E)=r$.

Claim. *E is stable.*

Proof of claim. Take $\mathscr{S}$ a locally free, proper subsheaf of E (we use the same symbol for a bundle and its sheaf of germs of sections). We must show that

$$\frac{\deg(\mathscr{S})}{\operatorname{rank}(\mathscr{S})} < \frac{\deg(E)}{\operatorname{rank}(E)} = \frac{1}{r},$$

i.e. $\deg(\mathscr{S}) \leqq 0$.

If the composition $\mathscr{S} \to E \to \mathscr{O}_C(x)$ is zero, we get an injection $\mathscr{S} \to F$ and have

$$\frac{\deg(\mathscr{S})}{\operatorname{rank}(\mathscr{S})} < \frac{\deg(F)}{\operatorname{rank}(F)} = 0.$$

This implies that $\deg(\mathscr{S}) < 0$.

If the composition $\mathscr{S} \to E \to \mathscr{O}_C(x)$ is nonzero, look at the sequence,

$$0 \to K \to \mathscr{S} \to \operatorname{Im}(\mathscr{S} \to \mathscr{O}_C(x)) \to 0.$$

Since $0 \to K \to F$, we have $\deg(K) < 0$ by stability of F. Hence $\deg(\mathscr{S}) \leqq 1 + \deg(K) \leqq 0$. This shows that E is stable and proves the claim.

To finish the proof note that any positive degree, stable bundle on a positive genus curve is ample by a result of Hartshorne [5].

Example 13 Let C be a smooth curve of genus $\geqq 2$, and let E be an ample vector bundle of rank r on C with $\det(E) = \mathscr{O}_C(x)$ where $x \in C$. Let $X := C \times M$ with product projections, $p: X \to C$, $q: X \to M$, where M is any $(n-1)$-dimensional projective manifold. Given any ample line bundle, L on M, the bundle $\mathscr{V} := p^*E \otimes q^*L$ is ample of rank r and not spanned. Note that $K_X \otimes \det(\mathscr{V})$ is not spanned since for any $y \in M$, $(K_X \otimes \det(\mathscr{V}))_{q^{-1}(y)} \cong K_C \otimes \mathscr{O}_C(x)$ is not spanned.

Let $\mathscr{L} := \det(\mathscr{V})$ and let $n = 3$. Note that given any two positive real numbers, σ_2, σ_3, we can choose M and L so that $\mathscr{L}^3 > \sigma_3^3$ and given any effective surface, $S \subset X$, we have $\mathscr{L}^2 \cdot S > \sigma_2^2$. Thus the conditions of the Ein-Lazarsfeld Theorems [1] fail only for curves, and in fact curves not in the base locus of $K_X \otimes \mathscr{L}$.

3 Rational curves on singular surfaces of degree $\leqq 4$

The following is probably known, but we know no reference.

Lemma 14 *Let L be a very ample line bundle on a reduced, projective surface, S. If $d := L \cdot L \leqq 4$, then there is a rational curve on S.*

Proof. If S is of degree $d = 1$ or $d = 2$ the lemma is immediate. Therefore we can assume that $d \geqq 3$.

If $\deg S = 3$, then an hyperplane section of S has arithmetic genus, g, equal to 0 or 1 by Castelnuovo's bound for the arithmetic genus of an irreducible and reduced curve. It can be assumed that $g = 1$ since otherwise we are done. Using Castelnuovo's bound again gives that $h^0(L) \leqq 4$. Since $d \geqq 2$ implies that $h^0(L) \geqq 4$ we conclude that $|L|$ embeds S in $\mathbb{P}^3$. If S is smooth there are lines

on S. If $\deg S=3$ and S is singular we can find a sequence of lines in $\mathbb{P}^3$, one on each of a sequence of smooth cubic surfaces approaching S. Passing to a subsequence if necessary, the sequence of lines converges to a line on the singular cubic surface.

Thus we can assume that S is a quartic surface.

By Castelnuovo's bound for the arithmetic genus of an irreducible and reduced curve, we see that the arithmetic genus, g, of a hyperplane section is at most 3, with equality happening only if $|L|$ embeds S into $\mathbb{P}^3$. To see that equality must happen assume otherwise. Let $\pi: S' \to S$ be a desingularization of S. If S has a hyperplane section of genus $g \leqq 2$, then the genus of a general $C \in |\pi^* L|$ is $\leqq 2$. We conclude from the adjunction formula and $d=4$, that $K_{S'} \cdot \pi^* L \leqq -2$. This implies that S' is of negative Kodaira dimension and thus is covered by rational curves. In particular S is covered by rational curves.

Thus we can assume that S is a quartic surface in $\mathbb{P}^3$. If S is smooth, then it contains a rational curve of degree 4 by "Mumford's Theorem on curves on algebraic K 3 surfaces", [9, Appendix]. By taking limits we conclude that every surface of degree 4 in $\mathbb{P}^3$ contains a rational curve. Q.E.D.

4 Spannedness of the adjoint bundle on n-folds with no rational curves

In this section we will study projective n-folds, X, with no rational curves. For example, this happens if X is hyperbolic, or if the cotangent bundle of X is nef. Using Mori theory this condition implies that the canonical bundle K_X of X is nef.

Theorem 15 *Let X be a smooth projective n-fold, $n \geqq 3$, with no rational curves. Let L be a line bundle that is very ample, and assume that $L^n \geqq 850$. Then $K_X \otimes L^{n-2}$ is spanned by global sections.*

Proof. By the Kodaira Vanishing Theorem, we are immediately reduced to the case when $\dim X = 3$.

Using [1, Theorem 1*] we are reduced to showing that:

(1) $L \cdot C \geqq 3$ for all irreducible curves, C, on X;

(2) $L^2 \cdot S \geqq 5$ for all irreducible surfaces, S, on X.

The first condition is immediate since $L \cdot C \leqq 2$ implies that C is a rational curve. The second condition is immediate from Lemma 14 since $L^2 \cdot S \leqq 4$ implies that S contains a rational curve. Q.E.D.

Remark 16 Note that the argument in § 3 combined with the fact that every K 3-surface is covered by elliptic curves shows that all surfaces of degree $\leqq 4$ are covered by rational and elliptic curves. Thus using the fact that the divisors and curves in [1, Theorem 1*] meet the base locus of the adjoint bundle, we conclude that given a very ample line bundle, L, on a smooth projective n-fold, X, with $n \geqq 3$, and $L^n \geqq 850$, it follows that $\Gamma(K_X \otimes L^{n-2})$ spans $(K_X \otimes L^{n-2})_{X-\mathscr{E}}$ where $\mathscr{E}$ is the union of all elliptic and rational curves on X.

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Critical Riemannian 4-manifolds

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0 Introduction

Let (M, g) be a compact oriented n-dimensional smooth Riemannian manifold. Consider the following quadratic Riemannian functional

$$SR(g) = \int_M |R_{ijkl}(g)|^2 \, d\mu$$

which is homogeneous of degree $\frac{n}{2} - 2$, where $R_{ijkl}(g)$ is the curvature tensors of (M, g) and $d\mu$ is the volume element measured by g. A critical point of $SR(g)$ is called a critical metric on M, that is, the Ricci tensor satisfies the critical equations grad $SR_g = 0$. In particular, for a compact 4-manifold M, every Einstein metric is a critical metric for SR on M.

Recently, Gao [G1] proved that the space of Einstein metrics on a compact 4-manifold M has a certain compactness property. In this paper, we propose an extension of the compactness property for Einstein metrics to critical metrics on a compact smooth Riemannian 4-manifold M. First we consider the subspace $I(M)$ of all critical metrics on M with the injectivity radius bounded from below by a constant $i_0 > 0$ and diameter bounded from above by d. Then we are able to prove that $I(M)$ is compact as a subset of moduli space of critical metrics in the C^∞-topology. More precisely, we have the following theorem:

Theorem A (Theorem 5.1) *Let $I(M)$ be the subspace of all critical metrics on a compact 4-manifold M such that*

(i) $Rc(g) \geqq -Hg$, $\int_M |Rc|^2 \, d\mu \leqq H$

(ii) $i_M \geqq i_0 > 0$

(iii) $\operatorname{diam}(M) \leqq d$

for some positive constants H, i_0 and d.

Then $I(M)$ is compact in the C^∞-topology.

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Remark. The author heartily appreciates the referee to point out the relevant paper of [AC] on C^α-compactness under the injectivity radius and Ricci bound.

We briefly describe the methods used in the proofs. In Sect. 2 and Sect. 3, we use the critical equations satisfied by the Ricci tensors of critical manifolds and elliptic version of Moser iteration to obtain the local estimates, which show that the Ricci tensors are bounded on the ball with small L^2-norm of curvature. Moreover, we can also show that the curvature tensors are bounded. Then by the Gauss-Bonnet formula for the 4-dimensional manifolds show that the curvature tensors can be only concentrated on a finite number of points, away from these points, the curvature tensors are uniformly bounded.

In Sect. 4, we use the similar argument of harmonic coordinates to prove that these critical Riemannian manifolds converge to a critical manifold with a finite number of isolated singular points [G1, GW].

In Sect. 5, we study the neighborhood of a singularity. That is, we study the tangent cone metric at each singular point by blowing up at this point, then show that this tangent cone metric is the flat Euclidean metric, combining the local form of Gromov compactness theorem, it is proved that the curvature tensors of critical manifolds are uniformly bounded. Then apply the Gromov compactness theorem, we prove the main compactness theorem in the subspace $I(M)$ of critical metrics for SR on a compact smooth 4-manifold M. [A, BKN, G1].

1 Notations and conventions

We will follow the old-fashioned index notations for tensors as in [H]. We denote vectors as v^i, covectors as v_j, and mixed tensors as T^{ij}_{kl} etc. The summation conventions will always hold. The Riemannian metric is g_{ij}, its inverse is g^{ij}, and the induced measure is $d\mu_g = \sqrt{g}\,dx$, where $\sqrt{g} = \sqrt{\det g_{ij}}$. The Ricci curvature tensor is Rc

$$R_{ij} = g^{kl} R_{ikjl} = R^l_{ilj}. \tag{1.1}$$

The scalar curvature is

$$R = g^{ij} R_{ij}. \tag{1.2}$$

For any tensor T such as T^k_{ij}, we define its length by

$$|T^k_{ij}|^2 = g_{kl}\, g^{im}\, g^{jn}\, T^k_{ij}\, T^l_{mn} \tag{1.3}$$

and we define the Laplacian ΔT by

$$\Delta T^k_{ij} = g^{pq} \nabla_p \nabla_q T^k_{ij}. \tag{1.4}$$

Let (M, g) be a compact connected smooth Riemannian manifold.

$$C_p = \{t_v \cdot v \in T_p M \mid \|v\| = 1, p \in M\}$$

and

$$C(p) = \exp(C_p)$$

where $t_v = \sup\{t>0 \mid d_g(p, \exp tv) = t\}$. The injectivity radius at p is defined as

$$\text{inj}_p = i_p := d_g(p, C(p)). \tag{1.5}$$

The injectivity radius of M is defined as

$$\text{inj}(M) = i_g(M) := \inf\{i_p \mid p \in M\}. \tag{1.6}$$

2 The equations of critical metrics

Consider a critical metric g_{ij} of a Riemannian quadratic functional SR on a compact 4-manifold M, that is,

$$\text{grad } SR_g = 0, \tag{2.1}$$

where

$$SR(g) = \int_M |Rm|^2 \, d\mu_g. \tag{2.2}$$

Then we have the corresponding critical equations [Bes, Ber].

Lemma 2.1 *For a critical metric g_{ij}, the Ricci curvature tensor R_{ij} satisfies the following equations*:

$$4\Delta R_{ij} + 4R_{ikjl}R^{kl} + 2R_{ikhl}R_j^{khl} - 4R_{il}R_j^l - \tfrac{1}{2}|Rm|^2 g_{ij} = 0.$$

Remarks. (i) From the proof, we have $\text{tr}(\text{grad } SR) = -2\Delta R$. Then every critical metric for SR has constant scalar curvature R.
(ii) For any curvature tensor in dimension 4, we have

$$R_{ikhl}R_j^{khl} - \frac{1}{4}|Rm|^2 g_{ij} = \left(\frac{R}{3}\right) Z_{ij} + 2W_{ikjl}Z^{kl}, \tag{2.3}$$

where $Z_{ij} = R_{ij} - \dfrac{R}{4} g_{ij}$, and W_{ijkl} is the Weyl tensor, i.e.

$$W_{jkl}^i = R_{jkl}^i - \frac{1}{2}(R_{jk}\delta_l^i - R_{jl}\delta_k^i + g_{jk}R_l^i - g_{jl}R_k^i) + \frac{R}{6}(g_{jk}\delta_l^i - g_{jl}\delta_k^i). \tag{2.4}$$

(iii) If A and B are two tensors, we write $A*B$ for any linear combination of tensors formed by contraction on $A_{i\ldots j}B_{k\ldots l}$ using the g^{ik}.

Combine Lemma 2.1 and the above remarks, we get the following theorem:

Theorem 2.2 *For any critical metric of SR on a compact smooth oriented 4-manifold, we have the following critical equation*

$$\Delta Rc = Rm * Rc.$$

Corollary 2.3 *There is a constant $c_1>0$ such that*

$$|\langle \Delta Rc, Rc\rangle| \leqq c_1 |Rm||Rc|^2.$$

By using Ricci identity, one obtains the following theorem:

Theorem 2.4

$$\Delta(\nabla^l Rc) = \sum_{k=0}^{l} \nabla^k Rm * \nabla^{l-k} Rc.$$

3 Local curvature estimates

In this section, following [G1] and [U], we will give the local estimate of the curvature and its covariant derivative of critical metrics of SR on the ball with small L^2-norm of curvature.

Consider a critical metric g_{ij} on a compact 4-manifold M. Let $f=|Rc|$, by Corollary 2.3, we then have

$$\Delta f \geqq -c_1 |Rm||Rc|.$$

Thus, we get the following lemma:

Lemma 3.1 *For any critical metric g_{ij} on M. Let*

$$f=|Rc|, \qquad b=c_1|Rm|.$$

Then

$$-\Delta f \leqq bf.$$

From [C], we have the following Sobolev inequality

Lemma 3.2 *If M is a complete n-dimensional Riemannian manifold and $i(M)>i_0>0$. Let $B_r(x)$ be a geodesic ball of radius r with $r \leqq \frac{i_0}{2}$. Then there is a constant $c_2(n)>0$ such that, for any $f \in C_0^\infty(B_r(x))$*

$$\left(\int_{B_r(x)} f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \leqq c_2(n) \int_{B_r(x)} |\nabla f|^2 d\mu.$$

We will derive the local Ricci curvature estimate. Let $B(x_0, \rho)$ be a geodesic ball in M with center $x_0 \in M$ and $0<\rho<\frac{i_0}{2}$.

Theorem 3.3 *For any critical metric g in M and any $1>\eta>0$, there exists a $\kappa_1>0$ such that, if*

$$\int_{B(x_0,\rho)} |Rm|^2 d\mu \leqq \kappa_1$$

Then

$$\sup_{B(x_0,(1-\eta)\rho)} |Rc| \leqq c_3(\rho, \eta, \kappa_1).$$

Proof. Define $\phi_\rho : [0, \rho] \mapsto [0, 1]$ be a C^∞-function by

$$\phi_\rho(t) = \phi\left(\frac{t}{\rho}\right), \quad \phi_\rho' \leqq 0, \quad |\phi_\rho'| \leqq \frac{c_4(\varepsilon)}{\rho}.$$

where $\phi : (-\infty, \infty) \mapsto [0, 1]$ is a C^∞-function with $\phi(t) = 0$, for $t \geqq 1$, $\phi(t) = 1$ for $t \leqq 1 - \varepsilon$.

Let $f = |Rc|$, $B_\rho = V(x_0, \rho)$, set $u(x) = \phi_\rho(r(x))$ and $r(x) = d(x, x_0)$, we may assume $f > 0$, take $p \geqq 1$, and $u^2 f^{2p-1}$ as a test function, then integrating by parts, one obtains

$$\begin{aligned}
&\int_{B_\rho} \langle \Delta f, u^2 f^{2p-1} \rangle \, d\mu \\
&\quad = -\int_{B_\rho} \langle \nabla f, \nabla (u^2 f^{2p-1}) \rangle \, d\mu \\
&\quad = -\int_{B_\rho} \langle \nabla f, f^{2p-1} \nabla u^2 + (2p-1) u^2 f^{2p-2} \nabla f \rangle \, d\mu \\
&\quad = -\frac{2p-1}{p^2} \int_{B_\rho} |\nabla (u f^p)|^2 \, d\mu + \frac{p-1}{2p^2} \int_{B_\rho} \langle \nabla u^2, \nabla f^{2p} \rangle \, d\mu \\
&\qquad + \frac{2p-1}{p^2} \int_{B_\rho} |\nabla u|^2 f^{2p} \, d\mu,
\end{aligned}$$

using lemma and Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{2p-1}{p^2} \int_{B_\rho} |\nabla (u f^p)|^2 \, d\mu \\
&\quad \leqq \int_{B_\rho} b u^2 f^{2p} \, d\mu + \frac{p-1}{2p^2} \int_{B_\rho} \langle \nabla u^2, \nabla f^{2p} \rangle \, d\mu \\
&\qquad + \frac{2p-1}{p^2} \int_{B_\rho} |\nabla_u|^2 f 2p \, d\mu \\
&\quad \leqq \Big(\int_{B_\rho} b^2 \, d\mu \Big)^{\frac{1}{2}} \Big(\int_{B_\rho} (u f^p)^4 \, d\mu \Big)^{\frac{1}{2}} \\
&\qquad + \frac{p-1}{2p^2} \int_{B_\rho} \langle \nabla u^2, \nabla f^{2p} \rangle \, d\mu \\
&\qquad + \frac{2p-1}{p^2} \int_{B_\rho} |\nabla u|^2 f^{2p} \, d\mu \\
&\quad \leqq -\frac{p-1}{2p^2} \int_{B_\rho} u^2 \Delta f^{2p} \, d\mu + \frac{2p-1}{p^2} \int_{B_\rho} |\nabla u|^2 f^{2p} \, d\mu \\
&\qquad + \Big(\int_{B_\rho} b^2 \, d\mu \Big)^{\frac{1}{2}} \Big(\int_{B_\rho} (u f^p)^4 \, d\mu \Big)^{\frac{1}{2}} \\
&\quad \leqq -\frac{p-1}{2p^2} \int_{B_\rho} u^2 (2p(2p-1) f^{2p-2} |\nabla f|^2 + 2p f^{2p-1} \Delta f) \, d\mu \\
&\qquad + \Big(\int_{B_\rho} b^2 \, d\mu \Big)^{\frac{1}{2}} \Big(\int_{B_\rho} (u f^p)^4 \, d\mu \Big)^{\frac{1}{2}}
\end{aligned}$$

$$+\frac{2p-1}{p^2}\int_{B_\rho}|\nabla u|^2 f^{2p}\,d\mu$$

$$\leqq\frac{p-1}{p}\int_{B_\rho}u^2 bf^{2p}\,d\mu+\frac{2p-1}{p^2}\int_{B_\rho}|\nabla u|^2 f^{2p}\,d\mu$$
$$+\left(\int_{B_\rho}b^2\,d\mu\right)^{\frac{1}{2}}\left(\int_{B_\rho}(uf^p)^4\,d\mu\right)^{\frac{1}{2}}$$

$$\leqq\frac{2p-1}{p}\left(\int_{B_\rho}b^2\,d\mu\right)^{\frac{1}{2}}\left(\int_{B_\rho}(uf^p)^4\,d\mu\right)^{\frac{1}{2}}$$
$$+\frac{2p-1}{p^2}\int_{B_\rho}|\nabla u|^2 f^{2p}\,d\mu.$$

Thies implies

$$\int_{B_\rho}|\nabla(uf^p)|^2\,d\mu\leqq p\left(\int_{B_\rho}(uf^p)^4\,d\mu\right)^{\frac{1}{2}}\left(\int_{B_\rho}b^2\,d\mu\right)^{\frac{1}{2}}+\int_{B_\rho}|\nabla u|^2 f^{2p}\,d\mu. \tag{3.1}$$

By Lemma 3.2, we have the following Sobolev inequality for $n=4$

$$\left(\int_{B_\rho}(uf^p)^4\,d\mu\right)^{\frac{1}{2}}\leqq c_2\int_{B_\rho}|\nabla(uf^p)|^2\,d\mu. \tag{3.2}$$

From (3.1) and (3.2), one obtains

$$\left(\int_{B_\rho}(uf^p)^4\,d\mu\right)^{\frac{1}{2}}\leqq pc_2\left(\int_{B_\rho}(uf^p)^4\,d\mu\right)^{\frac{1}{2}}\left(\int_{B_\rho}b^2\,d\mu\right)^{\frac{1}{2}}+c_2\int_{B_\rho}|\nabla u|^2 f^{2p}\,d\mu$$
$$\leqq pc_2\left(\int_{B_\rho}(uf^p)^4\,d\mu\right)^{\frac{1}{2}}\left(\int_{B_\rho}b^2\,d\mu\right)^{\frac{1}{2}}+\frac{c_4}{\rho^2}\int_{B_\rho}f^{2p}\,d\mu.$$

If $\left(\int_{B_\rho}b^2\,d\mu\right)^{\frac{1}{2}}$ is small, then we have

$$\left(\int_{B_{(1-\varepsilon)^2\rho}}(f^p)^4\,d\mu\right)^{\frac{1}{2}}\leqq c_5\int_{B_\rho}f^{2p}\,d\mu,$$

set $(1-\varepsilon)^2=1-\eta$, that is,

$$\left(\int_{B_{(1-\eta)\rho}}f^{4p}\,d\mu\right)^{\frac{1}{2}}\leqq c_5\left(\int_{B_\rho}f^{2p}\,d\mu\right)^{\frac{1}{2}}. \tag{3.3}$$

For $p=1$, we have

$$\left(\int_{B_{(1-\eta)\rho}}f^4\,d\mu\right)^{\frac{1}{2}}\leqq c_6\left(\int_{B_\rho}f^2\,d\mu\right)^{\frac{1}{2}}.$$

Now using Moser iteration, one obtains

$$\| f \|_\infty \leqq c_6^{c(k)} \| f \|_2 .$$

That is,

$$\sup_{B(x_0,(1-\eta)\rho)} |Rc| \leqq c_3(\rho, \eta, \kappa_1). \tag{3.4}$$

This completes the proof of Theorem 3.3.

In order to estimate the higher order covariant derivative of Ricci curvature tensor, we need the existence of harmonic coordinate on the ball with small L^2-norm of curvature.

Lemma 3.4 (Lemma 1.1 [G2]) *Given a small $0<\eta<1$, there exists a small constant $\kappa=\kappa(H, i_0, n, \eta)>0$ such that, for any Riemannian manifold (M, g) of dimension n, $x_0 \in M$, and $0<\rho<\frac{i_0}{4}$ with $|Rc| \leqq H$, $i(M) \geqq i_0$.*

$$\int_{B(x_0,\rho(1+2\eta))} |Rm|^{\frac{n}{2}} d\mu \leqq \kappa .$$

Then, for any $0<\delta<\frac{1}{3}$, and $x \in B(x_0, \rho(1+2\eta))$, so that $d(x_0, x)<1-\delta-2\eta$, there exists a harmonic coordinate F with domain $D(F) \supset B(x, \delta\rho)$ and $\operatorname{Im} F \supset B(\delta(1-\eta)\rho)=\{x \in \mathbf{R}^n \mid |x| \leqq \delta(1-\eta)\rho\}$, such that

(i) $$F^{-1}(B(\delta(1-\eta)\rho)) \supset B(x, \delta(1-\eta)\rho).$$

Let $h_{ij}=\langle \nabla h_i, \nabla h_j \rangle$ be the metric tensor in such harmonic coordinate, $F=(h_1, \ldots, h_n)$, $n \geqq 4$.

(ii) $$|h_{ij}-\delta_{ij}|_{C^0} \leqq \frac{\eta^2}{100n}.$$

(iii) $$|dh_{ij}|_{C^\alpha} \leqq c_7(H, i_0, n, \eta, \rho), \qquad 0<\alpha<1$$

(iv) $$\|h_{ij}\|_{L^{2,p}} \leqq c_8(H, i_0, n, \eta, \rho), \qquad p>n.$$

Theorem 3.5 *For any critical metric g on M, let κ_2 be small. Then, for any $0<\eta<1$, and*

$$\int_{B_\rho} |Rm|^2 d\mu \leqq \kappa_2 ,$$

we have

$$\sup_{B(x_0,(1-\eta)^2\rho)} |\nabla Rc| \leqq c_9(\rho, \eta, \kappa_2, i_0).$$

Before we prove the theorem, we need the following lemma:

Lemma 3.6

$$\int_{B_{(1-\eta)^2\rho}} |\nabla Rc|^2 d\mu \leqq c_{10}(\rho, \eta, \kappa, i_0).$$

Proof.

$$\begin{aligned}
&\int_{B_{(1-\eta)^2\rho}} |\nabla Rc|^2\, d\mu \\
&\leqq \int_{B_{(1-\eta)\rho}} u^2 |\nabla Rc|^2\, d\mu \\
&= -\int_{B_{(1-\eta)\rho}} u^2 \langle \Delta Rc, Rc\rangle\, d\mu - 2\int_{B_{(1-\eta)\rho}} u\nabla u \langle \nabla Rc, Rc\rangle\, d\mu \\
&\leqq c_1 \int_{B_{(1-\eta)\rho}} u^2 (|Rm||Rc|^2)\, d\mu + \frac{2c_4}{\rho}\int_{B_{(1-\eta)\rho}} u|\nabla Rc||Rc|\, d\mu \\
&\leqq \frac{2c_4}{\rho}\Big[\int_{B_{(1-\eta)\rho}} \frac{1}{\varepsilon}|Rc|^2\, d\mu + \int_{B_{(1-\eta)\rho}} \varepsilon u^2 |Rc|^2\, d\mu\Big] + c_1\kappa_2 c_3^2 \\
&\leqq \frac{1}{2}\int_{B_{(1-\eta)\rho}} u^2 |\nabla Rc|^2\, d\mu + \frac{8c_4^2}{\rho^2}\int_{B_{(1-\eta)\rho}} |Rc|^2\, d\mu + c_1 c_3^2 \kappa_2,
\end{aligned}$$

for $\varepsilon = \dfrac{\rho}{4c_4}$, that is,

$$\int_{B_{(1-\eta)^2\rho}} |\nabla Rc|^2\, d\mu \leqq c_{10}. \tag{3.5}$$

Proof of Theorem 3.5. Again, integrating by parts, one obtains

$$\begin{aligned}
&\int_{B_\rho} \langle \Delta\nabla Rc, u^2\nabla Rc|\nabla Rc|^{2p-2}\rangle\, d\mu \\
&= -\int_{B_\rho} u^2 |\nabla Rc|^{2p-2}|\nabla^2 Rc|^2\, d\mu - \int_{B_\rho} u^2 \langle \nabla^2 Rc, \nabla Rc\nabla|\nabla Rc|^{2p-2}\rangle\, d\mu \\
&\quad -2\int_{B_\rho} u\nabla u\langle \nabla^2 Rc, \nabla Rc|\nabla Rc|^{2p-2}\rangle\, d\mu \\
&= -\int_{B_\rho} u^2|\nabla^2 Rc|^2|\nabla Rc|^{2p-2}\, d\mu - \frac{1}{2}\int_{B_\rho} u^2\nabla|\nabla Rc|^2\nabla|\nabla Rc|^{2p-2}\, d\mu \\
&\quad -2\int_{B_\rho} u\nabla u\langle \nabla^2 Rc, \nabla Rc|\nabla Rc|^{2p-2}\rangle\, d\mu \\
&\leqq -2\int_{B_\rho} u\nabla u\langle \nabla^2 Rc, \nabla Rc|\nabla Rc|^{2p-2}\rangle\, d\mu \\
&\quad -\int_{B_\rho} u^2|\nabla^2 Rc|^2|\nabla Rc|^{2p-2}\, d\mu \\
&\leqq -\int_{B_\rho} u^2|\nabla^2 Rc|^2|\nabla Rc|^{2p-2}\, d\mu + 2\int_{B_\rho} u|\nabla u||\nabla^2 Rc||\nabla Rc|^{2p-1}\, d\mu,
\end{aligned}$$

this implies that

$$\begin{aligned}
&\int_{B_\rho} u^2|\nabla^2 Rc|^2|\nabla Rc|^{2p-2}\, d\mu \\
&\leqq \int_{B_\rho} |\langle \Delta\nabla Rc, u^2\nabla Rc|\nabla Rc|^{2p-2}\rangle|\, d\mu \\
&\quad +2\int_{B_\rho} u|\nabla u||\nabla^2 Rc||\nabla Rc|^{2p-1}\, d\mu.
\end{aligned} \tag{3.6}$$

Since $2ab \leqq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$, for $a, b \geqq 0$, $\varepsilon > 0$. We have

$$2 \int_{B_\rho} u|\nabla u||\nabla^2 Rc||\nabla Rc|^{2p-1}\, d\mu$$
$$\leqq \varepsilon \int_{B_\rho} u^2 |\nabla^2 Rc|^2 |\nabla Rc|^{2p-2}\, d\mu + \frac{1}{\varepsilon} \int_{B_\rho} |\nabla Rc|^{2p}\, d\mu,$$

choose ε small enough, (3.6) then implies that

$$\int_{B\rho} u^2 |\nabla^2 Rc|^2 |\nabla Rc|^{2p-2}\, d\mu \tag{3.7}$$
$$\leqq \int_{B_\rho} |\langle \Delta \nabla Rc, u^2 \nabla Rc |\nabla Rc|^{2p-2}\rangle|\, d\mu + \frac{1}{\varepsilon(1-\varepsilon)} \int_{B_\rho} |\nabla Rc|^{2p}\, d\mu.$$

By Theorem 2.4, we obtain

$$\int_{B_{(1-\eta)\rho}} |\langle \Delta \nabla Rc, u^2 \nabla Rc |\nabla Rc|^{2p-2}\rangle|\, d\mu$$
$$\leqq \int_{B_{(1-\eta)\rho}} u^2 |\langle \nabla Rm * Rc, \nabla Rc\rangle||\nabla Rc|^{2p-2}\, d\mu + \int_{B_{(1-\eta)\rho}} u^2 |\langle Rm * \nabla Rc, \nabla Rc\rangle||\nabla Rc|^{2p-2}\, d\mu$$
$$\leqq c_{11} \int_{B_{(1-\eta)\rho}} u|\nabla u||Rm||Rc||\nabla Rc|^{2p-1}\, d\mu + c_{11} \int_{B_{(1-\eta)\rho}} u^2 |Rm||Rc||\nabla^2 Rc||\nabla Rc|^{2p-2}\, d\mu + c_{11} \int_{B_{(1-\eta)\rho}} u^2 |Rm||\nabla Rc|^{2p}\, d\mu$$
$$\leqq c_{11} c_3 \int_{B_\rho} u^2 |Rm||\nabla^2 Rc||\nabla Rc|^{2p-2}\, d\mu + \frac{c_{11} c_4}{\rho} \int_{B_\rho} u|Rm||\nabla Rc|^{2p-1}\, d\mu + c_{11} \int_{B_\rho} u^2 |Rm||\nabla Rc|^{2p}\, d\mu.$$

From (3.7), one obtains

$$\int_{B_{(1-\eta)\rho}} u^2 |\nabla^2 Rc|^2 |\nabla Rc|^{2p-2}\, d\mu$$
$$\leqq \frac{1}{\varepsilon(1-\varepsilon)} \int_{B_{(1-\eta)\rho}} |\nabla Rc|^{2p}\, d\mu + c_{11} \int_{B_{(1-\eta)\rho}} u^2 |Rm||\nabla Rc|^{2p}\, d\mu + \frac{c_{11} c_4}{\rho} \int_{B_{(1-\eta)\rho}} u|Rm||\nabla Rc|^{2p-1}\, d\mu + c_{11} c_3 \int_{B_{(1-\eta)\rho}} u^2 |Rm||\nabla^2 Rc||\nabla Rc|^{2p-2}\, d\mu.$$

Since

$$\int_{B_{(1-\eta)\rho}} u^2|Rm||\nabla^2 Rc||\nabla Rc|^{2p-2}\,d\mu$$

$$\leqq\frac{\varepsilon}{2}\int_{B_{(1-\eta)\rho}} u^2|\nabla^2 Rc|^2|\nabla Rc|^{2p-2}\,d\mu+\frac{1}{2\varepsilon}\int_{B_{(1-\eta)\rho}} u^2|Rm|^2|\nabla Rc|^{2p-2}\,d\mu,$$

we have

$$\left(1-\frac{\varepsilon c_{11}c_3}{2}\right)\int_{B_{(1-\eta)\rho}} u^2|\nabla^2 Rc|^2|\nabla Rc|^{2p-2}\,d\mu$$

$$\leqq\frac{1}{\varepsilon(1-\varepsilon)}\int_{B_{(1-\eta)\rho}}|\nabla Rc|^{2p}\,d\mu+c_{11}\int_{B_{(1-\eta)\rho}} u^2|Rm||\nabla Rc|^{2p}\,d\mu$$

$$\frac{c_{11}c_4}{\rho}\int_{B_{(1-\eta)\rho}} u|Rm||\nabla Rc|^{2p-1}\,d\mu$$

$$+\frac{c_{11}c_3}{2\varepsilon}\int_{B_{(1-\eta)\rho}} u^2|Rm|^2|\nabla Rc|^{2p-2}\,d\mu$$

It follows that

$$\left(\int_{B_{(1-\eta)^2\rho}}(u|\nabla Rc|^p)^4\,d\mu\right)^{\frac{1}{2}}$$

$$\leqq c_2\left(\int_{B_{(1-\eta)^2\rho}} u^2|\nabla|\nabla Rc|^p|^2\,d\mu+\int_{B_{(1-\eta)^2\rho}}|\nabla u|^2|\nabla Rc|^{2p}\,d\mu\right)$$

$$\leqq c_2\int_{B_{(1-\eta)^2\rho}} u^2|\nabla^2 Rc|^2|\nabla Rc|^{2p-2}\,d\mu+\frac{c_4^2c_2}{\rho^2}\int_{B_{(1-\eta)^2\rho}}|\nabla Rc|^{2p}\,d\mu$$

$$\leqq c_{12}\int_{B_{(1-\eta)^2\rho}}|\nabla Rc|^{2p}\,d\mu+c_{13}\int_{B_{(1-\eta)^2\rho}} u^2|Rm|^2|\nabla Rc|^{2p}\,d\mu$$

$$\leqq\left(c_{12}+\frac{1}{\varepsilon}\right)\int_{B_{(1-\eta)^2\rho}}|\nabla Rc|^{2p}\,d\mu$$

$$+c_{13}\varepsilon\left(\int_{B_{(1-\eta)^2\rho}}|Rm|^8\,d\mu\right)^{\frac{1}{2}}\left(\int_{B_{(1-\eta)^2\rho}} u^4|\nabla Rc|^{4p}\,d\mu\right)^{\frac{1}{2}}.$$

From Theorem 3.3 and Lemma 3.4, if ε is small enough, we then have

$$\left(\int_{B_{(1-\eta)^2\rho}}(u|\nabla Rc|^p)^4\,d\mu\right)^{\frac{1}{2}}\leqq c_{14}\int_{B_{(1-\eta)^2\rho}}|\nabla Rc|^{2p}\,d\mu,$$

that is,

$$\left(\int_{B_{(1-\eta)^4\rho}}|\nabla Rc|^{4p}\,d\mu\right)^{\frac{1}{2}}\leqq c_{14}\int_{B_{(1-\eta)^2\rho}}|\nabla Rc|^{2p}\,d\mu. \tag{3.8}$$

Using Moser iteration again, this implies Theorem 3.5.

Theorem 3.3 and Lemma 3.4 imply the existence of a harmonic coordinate $h=(h_1,\ldots,h_n)$. In such coordinate system, we have

Lemma 3.7

$$\int_{B_\rho} |\nabla Rm|^p d\mu \leqq c, \qquad p>n.$$

Proof. In the harmonic coordinate,

$$\Delta h^{ij} = 2h^{il} h^{jk} R_{kl} + 2\Gamma^i_{pq} \Gamma^j_{kl} h^{pk} h^{ql}, \tag{3.9}$$

that is,

$$\Delta h = 2h*Rc + 2\nabla h*\nabla h.$$

Using the same method of Lemma 2.4 we obtain

$$\begin{aligned}\Delta\nabla h = {} & \nabla h*Rc + h*\nabla Rc + \nabla h*\nabla^2 h \\ & + \nabla h*Rc + h*\nabla Rm,\end{aligned}$$

by Theorem 3.3, Theorem 3.5 and Ricci identity, it follow that

$$\begin{aligned}|\Delta\nabla h| &\leqq c + c|Rm| + c|\nabla^2 h| \\ &\leqq c + c|\nabla^2 h|.\end{aligned}$$

Since $h \in L^{2,p}$, combines Lemma 3.4, it is easily to prove that

$$\|\nabla^3 h\|_{L^p} \leqq c, \qquad p>n.$$

Hence

$$\int_{B_\rho} |\nabla Rm|^p d\mu \leqq c, \qquad p>n.$$

Continue this way, we obtain

Corollary 3.8 *If* $|\nabla^l Rc| \leqq c$ *on* $B(x_0, \rho)$, $l=0, 1, 2, \ldots$ *Then*

$$\int_{B(x_0,\rho)} |\nabla^l Rm|^p d\mu \leqq c, \qquad p>n.$$

Theorem 3.9 *For any critical metric* g *on* M, *let* κ_3 *be small. Then, for any* $0<\eta<1$ *and*

$$\int_{B_\rho} |Rm|^2 d\mu \leqq \kappa_3,$$

we have

$$\sup_{B(x_0,(1-\eta)^{2l+2}\rho)} |\nabla^l Rc| \leqq c_{15}(\rho, \eta, i_0), \qquad l=0, 1, \ldots$$

Proof. We have proved the case $l=0, 1$. By induction, suppose it is true for $l-1$, then

$$(3.10)\quad \Big(\int_{B_{(1-\eta)^{2l}\rho}} (u|\nabla^l Rc|^p)^4\, d\mu\Big)^{\frac{1}{2}}$$
$$\leqq c\Big(\int_{B_{(1-\eta^{2l}\rho}} u^2|\nabla|\nabla^l Rc|^p|^2\, d\mu + \int_{B_{(1-\eta)^{2l}\rho}} |\nabla u|^2|\nabla^l Rc|^{2p}\, d\mu\Big)$$
$$\leqq c\int_{B_{(1-\eta)^{2l}\rho}} u^2|\nabla^{l+1} Rc|^2|\nabla^l Rc|^{2p-2}\, d\mu + c\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^{2p}\, d\mu$$
$$\leqq c\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^{2p}\, d\mu + c\int_{B_{(1-\eta)^{2l}\rho}} u^2|\nabla^{l+1} Rc|^2|\nabla^l Rc|^{2p-2}\, d\mu,$$

on the other hand,

$$\int_{B_{(1-\eta)^{2l}\rho}} u^2|\nabla^{l+1} Rc|^2|\nabla^l Rc|^{2p-2}\, d\mu$$
$$\leqq c\int_{B_{(1-\eta)^{2l}\rho}} u^2|\langle \Delta\nabla^l Rc, \nabla^l Rc|\nabla^l Rc|^{2p-2}\rangle|\, d\mu$$
$$+ c\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^{2p}\, d\mu$$
$$\leqq c\int_{B_{(1-\eta)^{2l}\rho}} u^2|\langle \nabla^k Rm * \nabla^{l-k} Rc, \nabla^l Rc\rangle||\nabla^l Rc|^{2p-2}\, d\mu$$
$$+ c\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^{2p}\, d\mu$$
$$\leqq c\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^{2p}\, d\mu$$
$$+ c\varepsilon\Big(\sum_{k=0}^{l-1}\Big(\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^k Rm|^8\, d\mu\Big)^{\frac{1}{2}}\Big)$$
$$\times\Big(\int_{B_{(1-\eta)^{2l}\rho}} u^4|\nabla^l Rc|^{4p}\, d\mu\Big)^{\frac{1}{2}}$$
$$+ c\varepsilon\int_{B_{(1-\eta)^{2l}\rho}} u^2|\nabla^{l+1} Rc|^2|\nabla^l Rc|^{2p-2}\, d\mu,$$

that is, for small ε,

$$(3.11)\quad \int_{B_{(1-\eta)^{2l}\rho}} u^2|\nabla^{l+1} Rc|^2|\nabla^l Rc|^{2p-2}\, d\mu$$
$$\leqq c\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^{2p}\, d\mu$$
$$+ c\varepsilon\Big(\sum_{k=0}^{l-1}\Big(\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^k Rm|^8\, d\mu\Big)^{\frac{1}{2}}\Big)\Big(\int_{B_{(1-\eta)^{2l}\rho}} u^4|\nabla^l Rc|^{4p}\, d\mu\Big)^{\frac{1}{2}}.$$

Thus, from (3.10), (3.11) and Corollary 3.8, we have, for small ε

$$(3.12)\quad \Big(\int_{B_{(1-\eta)^{2l}\rho}} (u|\nabla^l Rc|^p)^4\, d\mu\Big)^{\frac{1}{2}} \leqq c\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^{2p}\, d\mu.$$

On the other hand, as the same method of the proof of Lemma 3.6, one obtains

$$\int_{B_{(1-\eta)^{2l}\rho}} |\nabla^l Rc|^2 \, d\mu \leqq c. \tag{3.13}$$

Using Moser iteration, (3.12) and (3.13) imply Theorem 3.9.

Now we are ready to derive the local estimate of curvature tensors of critical metrics.

Lemma 3.10 (Lemma 7.2 [H]) For any metric g_{ij}, the curvature tensor satisfies the following identity

$$\Delta Rm = Rm * Rm + \nabla^2 Rc.$$

Corollary 3.11

$$\Delta \nabla^l Rm = \sum_{k=0}^{l} \nabla^k Rm * \nabla^{l-k} Rm + \nabla^{l+2} Rc.$$

Theorem 3.12 For any critical metric g on M, let κ_4 be small. Then, for any $0<\eta<1$, and

$$\int_{B_\rho} |Rm|^2 \, d\mu \leqq \kappa_4,$$

we have

$$\sup_{B(x_0,(1-\eta)^{2l+2}\rho)} |\nabla^l Rm| \leqq c_{16}(\rho, \eta, \kappa_4, i_0), \qquad l=0, 1, 2, \ldots$$

Proof. From Lemma 3.10, we have

$$\begin{aligned} -\Delta |Rm| &\leqq |Rm|^2 + |\nabla^2 Rc| \\ &\leqq |Rm||Rm| + c_{15}. \end{aligned}$$

Let $f=|Rm|$, $b=|Rm|$, that is,

$$-\Delta f \leqq bf + c_{15} \tag{3.14}$$

then, using the same method of the proof of Theorem 3.3, this implies the theorem for $l=0$. For the higher order, it follow from Lemma 3.11 and Theorem 3.9. This completes the proof of Theorem 3.12.

Remark. Let g_{ij} be a C^∞-metric on a fixed compact 4-manifold M. We have the following Gauss-Bonnet formula for the Euler characteristic $\chi(M)$

$$8\pi^2 \chi(M) = \int_M |Rm|^2 \, d\mu - 4 \int_M |Rc|^2 \, d\mu + \int_M R^2 \, d\mu.$$

Thus, there exists a constant $c_{17}=c_{17}(\chi(M), H)>0$ such that

$$\int_M |Rm|^2 \, d\mu \leqq c_{17}. \tag{3.15}$$

4 Weak limit of critical metrics

Let M be a fixed compact 4-manifold. Consider the following space of Riemannian metrics

$$I(M)=\{g^k \mid g^k \text{ is a critical } C^\infty\text{-metric of } SR \text{ on } M, \text{ such that } \operatorname{inj}(M,g^k)\geqq i_0>0, \operatorname{diam}(M,g^k)\leqq d, Rc(g)\geqq -Hg, \int_M |Rc|^2\, d\mu^k \leqq H\}.$$

In this section, we employ the covering argument of [SU] and combines with the argument of [P] or [GW] to prove a certain weak limit of critical manifolds. See also [G1].

A A weak convergence of critical metrics

From Theorem 3.12, we obtain

Theorem 4.1 *Let $\{g^k\}$, $k=1,2,\ldots$, be a sequence in $I(M)$, $M^k=(M,g^k)$. Then there exists a subsequence, say $\{g^k\}$, a sequence $\{r_l\}$, $r_l\to 0$ as $l\to\infty$, such that*

(i) *For each l, we have open subsets $F_k(r_l)\subset M^k$, and an open subset $D(r_l)$ in M.*
(ii) *For each l, we have diffeomorphism*

$$f^k(r_l): D(r_l)\mapsto F_k(r_l), \qquad k\geqq l$$

such that $f^k(r_l)^* g^k$ converges to a critical C^∞-metric $g(r_l)$ on $D(r_l)$ in C^2-norm.
(iii)

$$F_k(r_l)\subset F_k(r_{l+1}).$$

(iv)

$$\bigcup_{l=1}^{\infty} F_k(r_l)=M^k-\{m_1,\ldots,m_h\}, h(r_l)\leqq c_{20}(i_0,\chi(M)).$$

Proof. It follows more or less the same from Gao's work [G1].

B Construct a critical manifold M' with isolated singularities

Definition 4.2 Let $\{g^k\}$ be a sequence of metrices on M, we say $\{g^k\}$ converges weakly in C^2-norm to a Riemannian manifold (M',g') with C^∞-metric g', if for each k, there are finite number of points $\{m_1^k,\ldots,m_h^k\}$ in M, h is independent of k, such that for every open subset $D\subset M'$, there exist C^∞-embeddings

$$f^k: D\mapsto M_k-\{m_1^k,\ldots,m_h^k\}$$

such that $f^k: D\mapsto f^k(D)$ is a diffeomorphism and

$$(f^k)^* g^k \xrightarrow{C^2} g'$$

on D. g' is called the weak C^2-limit of $\{g^k\}$ on M.

With this definition, we can rewrite theorem 4.1 as

Theorem 4.3 *Let $\{g^k\}$ be a sequence in $I(M)$. Then there exists a cristical manifold (M', g') such that $\{g^k\}$ be a subsequence which converges weakly in C^2-norm to a C^∞-critical metric g' on a manifold M'.*

C *The topology of M'*

Theorem 4.4. *M' is diffeomorphic to $M-\{v_1, \ldots, v_{s+1}\}$, for a finite number of points $\{v_1, \ldots, v_{s+1}\}$ in M, and each v_j is a isolated singularity of the metric g'.*

Proof. Recall

$$F_k(r_l) \cup \bigcup_{j=Q'(r_l)+1}^{Q(r_l)} B^k(x_{j,l}, r_l) = M_k$$

and

$$\bigcup_{j=Q'(r_l)+1}^{Q(r_l)} B^k(x_{j,l}, r_l) \supset \bigcup_{j=Q'(r_l+1)+1}^{Q(r_{l+1})} B^k(x_{j,l+1}, r_{l+1}), \tag{4.1}$$

where $B^k(x_{j,l}, r_l)$'s are geodesic balls in (M, g^k). We will replace the balls $\{B^k(x_{j,l}, r_l)\}$ by a family of balls which are pairwise disjoint.

Lemma 4.5 *There is a family of balls*

$$\{B^k(x_{i,l}, s_i^l r_l), i=1, \ldots, K_l, K_l \leqq N\}$$

which are pairwise disjoint such that

$$\bigcup_{j=1}^{K_{l+1}} B^k(x_{j,l+1}, s_j^{l+1} r_{l+1}) \subset \bigcup_{i=1}^{K_l} B^k(x_{i,l}, s_i^l r_l)$$

where $s_i^l \leqq 3^{K_l} \leqq 3^N$.

Now, for each fixed l, set

$$B_l^k = \bigcup_{j=1}^{K_l} B^k(x_{j,l}, s_j^l r_l). \tag{4.2}$$

Since $s_j^l r_l \leqq 3^N r_l \leqq \frac{i_0}{800}$, B_l^k are disjoint union of geodesic balls, each of these geodesic balls is diffeomorphic to the unit ball in $\mathbf{R}^4$, and has C^∞-boundary which is diffeomorphic to the unit sphere S^3 in $\mathbf{R}^4$. Set

$$A_l^k = (f^k(r_l))^{-1}(F_k(r_l) - B_l^k).$$

Thus A_l^k is a submanifold of M' with C^∞-boundary which is the disjoint union of finite C^∞-sphere in M', and $A_l^k \subset A_{l+1}^k$. Next, consider

$$\tilde{B}_l^k = \bigcup_{j=1}^{K_l} B^k\left(x_{j,l}, s_j^l\left(\frac{479}{480}\right) r_l\right). \tag{4.3}$$

Thus

$$F_k(r_l) \cup \tilde{B}_l^k = M, \qquad \tilde{B}_{l+1}^k \subset \tilde{B}_l^k$$

and the balls in $\tilde{B}_l^k$ are pairwise apart by a distance at least $\frac{1}{240} r_l$. Set

$$Y_l = (f^{k_l}(r_l))^{-1}(F_{k_l}(r_l) - \tilde{B}_l^{k_l}) \subset D(r_l)$$

for $k_l \geqq N(r_l)$ such that for $k \geqq k_l$

$$(1-\varepsilon_l)\, g(r_l) \leqq f^k(r_l)^*\, g^k \leqq (1+\varepsilon_l)\, g(r_l), \tag{4.4}$$

here ε_l will be determined later on. From the construction, we have

Lemma 4.6

$$Y_l \subset D(r_l) \subset Y_{l+1}, \qquad \cup_{l=1}^\infty Y_l = \cup_{l=1}^\infty D(r_l).$$

Let $\partial Y_l = \coprod_{j=1}^{\bar{p}} S_j^l$ is the disjoint union of 3-spheres in M', $K_l = \bar{p}$. From (4.3) and (4.4), we have

$$d^{k_l}(r_l)(f^{k_l}(r_l)(S_i^l), \quad f^{k_l}(r_l)(S_j^l)) \geqq \frac{r_l}{240}, \; i \neq j$$

and

$$d(r_l)(S_i^l, S_j^l) \geqq \frac{r_l}{240} \frac{1}{1+\varepsilon_l}, \; i \neq j.$$

Set

$$\Phi_l = f^{k_{l+1}}(r_{l+1}) : D(r_{l+1}) \mapsto F_{k_{l+1}}(r_{l+1}).$$

Since

$$\operatorname{diam}(F^{k_l}(S_j^l)) \leqq 3^N r_l.$$

Hence

$$\operatorname{diam}(S_j^l) \leqq \frac{1}{1-\varepsilon_l} 3^N r_l.$$

Take $\varepsilon_l \leqq \frac{1}{3}$, we have

$$\operatorname{diam}(S_j^l) \leqq \tfrac{2}{3} 3^N r_l. \tag{4.5}$$

On the other hand,

$$\operatorname{diam}(\Phi_l(S_j^l)) \leqq \frac{1+\varepsilon_{l+1}}{1-\varepsilon_l} 3^N r_l \leqq 2 \cdot 3^N r_l. \tag{4.6}$$

Let $B_{l,j}=B^{k_{l+1}}(x_{l,j}, r_{l,j})$ be the smallest geodesic ball in $(M, g^{k_{l+1}})$ such that $\overline{B_{l,j}} \supset \Phi_l(S_j^l)$, then

$$r_{l,j} \leqq \operatorname{diam}(\Phi_l(S_j^l)) \leqq 2 \cdot 3^N r_l.$$

This implies that

$$2r_{l,j} \leqq 4 \cdot 3^N r_l \leqq \frac{i_0}{100}$$

and

$$\Phi_l(S_j^l) \subset \overline{B_{l,j}} \subset B^{k_{l+1}}\left(x_{l,j}, \frac{i_0}{100}\right).$$

That is, $\Phi_l(S_j^l)$ bounds a ball $B_{l,j} \subset B^{k_{l+1}}\left(x_{l,j}, \frac{i_0}{100}\right)$ in M and $B_{l,j}$ is homeomorphic to the unit ball in $\mathbf{R}^4$ and $\{B_{l,j}; j=1, \ldots, \bar{p}\}$ are pairwise disjoint. Then we have

(4.7)
$$\Phi_l(Y_l) \cup \bigcup_{j=1}^{\bar{p}} B_{l,j} = M$$

$$\Phi_l(Y_l) \cap \bigcup_{j=1}^{\bar{p}} B_{l,j} = \emptyset$$

$$\Phi_l(Y_l) \subset \operatorname{Int} \Phi_l(Y_{l+1})$$

$$\Phi_l(Y_{l+1} - \operatorname{Int} Y_l) \subset \bigcup_{j=1}^{\bar{p}} B_{l,j}$$

Note

$$\partial(Y_{l+1} - \operatorname{Int} Y_l) = \prod_{j=1}^{\bar{p}} S_j^l \cup \prod_{j=1}^{\bar{p}} S_j^{l+1}.$$

Thus $Y_{l+1} - \operatorname{Int} Y_l$ is homeomorphic to

$$\bigcup_{j=1}^{\bar{p}} \overline{B_{l,1}} - \bigcup_{j=1}^{\bar{p}} B_{l+1,j}.$$

From Lemma 4.6, (4.5), (4.7), letting $r_l \to 0$, these clearly imply Theorem 4.4.

5 Compactness of critical metrics

Let M_1 be the manifold which is homeomorphic to M, and obtained from M' by adding $\{v_1, \ldots, v_{s+1}\}$ to M' with the topology induced by the distance fuction of g' on M'. $M_1 - \{v_1, \ldots, v_{s+1}\}$ is isometric to (M', g'). Fix $v=v_i$, $1 \leqq i \leqq s+1$, let $r(x)=d(x, v)$ for $x \in M_1$, take N to be a neighbourhood of v in M_1, such that $N \cap \{v_1, \ldots, v_{s+1}\} = \{v\}$. We may assume that $N \supset B(v, 2\rho_0) = \{x \in M_1 \mid r(x) < 2\rho_0\}$ for some $\rho_0 > 0$.

In this section, we will prove the following main theorem.

Theorem 5.1 *$I(M)$ is compact in the C^∞- topology.*

First, we will give the basic estimate of the curvature $|Rm|$ on N.

Lemma 5.2

$$|Rm| \leqq \frac{\varepsilon(r)}{r^2}, \qquad \varepsilon(r) \to 0 \quad as \quad r \to 0.$$

Proof. For each small $\rho > 0$, define the metric $\tilde{g} = \frac{1}{\rho^2} g'$, then we have distance function $\tau = -\log r$ for metric g'. Since $SR(g)$ is conformally invariant, we have $|\tilde{R}m| = \rho^2 |Rm|$. Now fix $x_0 \in N$, such that $r(x_0) = \rho < \rho_0$. From Theorem 4.3, g' is the weak limit, one obtains

$$\frac{1}{8\pi^2} \int_N \cdot |Rm|^2 \, d\mu \leqq \frac{1}{8\pi^2} \liminf_{k \to \infty} \int_M |Rm|^2 \, d\mu^k$$

$$\leqq c_{17}.$$

This implies

$$\int_N |\tilde{R}m|^2 \, d\tilde{\mu} \leqq 8\pi^2 c_{17}. \tag{5.1}$$

First observe that the local Sobolev constant on $B(x_0, \rho)$ is still uniformly bounded, in fact, suppose $f \in C_0^\infty(B(x_0, \rho))$, we have

$$\left(\int_{B(x_0,\rho)} f^4 \, d\mu \right)^{\frac{1}{2}} \leqq c_2 \left(\int_{B(x_0,\rho)} |\nabla f|^2 \, d\mu + \int_{B(x_0,\rho)} f^2 \, d\mu \right).$$

$\tilde{g} = \frac{1}{\rho^2} g'$, one obtains

$$\left(\int_{B(x_0,\rho)} f^4 \, d\tilde{\mu} \right)^{\frac{1}{2}}$$

$$\leqq c_2 \left(\int_{B(x_0,\rho)} |\tilde{\nabla} f|^2 \, d\tilde{\mu} + \int_{B(x_0,\rho)} f^2 \, d\tilde{\mu} + \int_{B(x_0,\rho)} r^2 f^2 \, d\tilde{\mu} \right).$$

Take $\rho \leqq 1$ small, we have

$$\left(\int_{B(x_0,\rho)} f^4 \, d\tilde{\mu} \right)^{\frac{1}{2}} \leqq \bar{c}_2 \left(\int_{B(x_0,\rho)} |\tilde{\nabla} f|^2 \, d\tilde{\mu} + \int_{B(x_0,\rho)} f^2 \, d\tilde{\mu} \right). \tag{5.2}$$

Next, consider

$$\int_{\tilde{B}(x_0,\frac{1}{2})} |\tilde{R}m|^2 \, d\tilde{\mu} = \int_{B(x_0,\frac{1}{2}\rho)} |Rm|^2 \, d\mu. \tag{5.3}$$

For $\rho > 0$ small, such that $\int_{B(x_0,\frac{1}{2}\rho)} |Rm|^2 \, d\mu \leqq \kappa$, using (5.1), (5.2), (5.3), and combines the argument as Sect. 3, it follows that

$$|\tilde{R}m| \leqq \bar{c}_{16} \quad \text{on} \quad \tilde{B}(x_0, \tfrac{1}{2}).$$

On the other hand,

$$\tilde{V}(r)=\int_{\tilde{B}_r} d\tilde{\mu}=\rho^4 \int_{B^r_\rho} d\mu=\rho^4\, V\left(\frac{r}{\rho}\right),$$

that is,

$$\frac{\tilde{V}(r)}{r^4}=\frac{\rho^4\, V\left(\frac{r}{\rho}\right)}{r^4}=\frac{V\left(\frac{r}{\rho}\right)}{\left(\frac{r}{\rho}\right)^4}. \tag{5.4}$$

Since $\mathrm{inj}(g^k)\geqq i_0>0$, for small $\rho>0$, we have

$$\mathrm{Vol}(B^k(x_0,\tfrac{1}{2}\rho))\geqq c_0\,\rho^4 \tag{5.5}$$

for c_0 independent of ρ. By taking limit, it follows that $\mathrm{Vol}(B(x_0,\frac{1}{2}\rho))\geqq c_0\,\rho^4$. From (5.5), this implies

$$\mathrm{Vol}(\tilde{B}(x_0,\tfrac{1}{2}))\geqq c_0. \tag{5.6}$$

From (5.3), (5.4) and (5.6), one obtains $|\tilde{R}m(x)|\to 0$ as $r(x)\to 0$. This implies Lemma 5.2.

Let

$$A(\rho,\bar{\rho})=B(v,2\rho)-B(v,\bar{\rho})=\{v\in M\,|\,\bar{\rho}<r(x)<2\rho\} \tag{5.7}$$

here $A(\rho,\bar{\rho})$ is an open Riemnannian submanifold of M'. Consider $\frac{1}{\rho}A(\rho,\bar{\rho})$ to be the Riemannian manifold $A(\rho,\bar{\rho})$ with metric $\frac{1}{\rho^2}g'$.

Lemma 5.3 (Theorem 5.8 [G1])

$$\left(\frac{1}{\rho}A(\rho,0),x_\rho\right)\xrightarrow{C^2}(D^4-\{0\},e)$$

where $D^4=\{w\in\mathbf{R}^4\,||w|<2\}$ *is the Euclidean ball of radius* 2.

Proof. For any fixed large number K, by Lemma 5.2, one obtains

$$|\tilde{R}m|=\rho^2|Rm|\leqq\frac{\varepsilon(2\rho)}{\left(\frac{\rho}{K}\right)^2}\rho^2\leqq K^2\,\varepsilon(2\rho) \tag{5.8}$$

on $\frac{1}{\rho}A\left(\rho,\frac{\rho}{K}\right)$. Let $x_\rho\in\frac{1}{\rho}A(\rho,0)$, such that $r(x_\rho)=\rho$. Then

$$\left\{\left(\frac{1}{\rho}A(\rho,0),x_\rho\right)\right\}$$

is a family of critical manifolds, and for fixed K, $|\tilde{R}m| \to 0$ uniformly as $\rho \to 0$ on $\frac{1}{\rho} A\left(\rho, \frac{\rho}{K}\right)$. First, we show that

$$\left(\frac{1}{\rho} \overline{A\left(\rho, \frac{\rho}{K}\right)}, x_\rho\right)$$

converges to a flat manifold

$$\left(\overline{D\left(\frac{1}{K}\right)}, e_K\right)$$

For this we need to control the injectivity radius of $\frac{1}{\rho} A\left(\rho, \frac{\rho}{K}\right)$ from below.

Lemma 5.4 (Theorem 4.7 [CGT]) *Let M be a complete manifold with bounded section curvature $H \leqq K_M \leqq K$. For fix r, r_0, s with $r_0 + 2s < \frac{\pi}{\sqrt{K}}$, $r_0 \leqq \frac{\pi}{4\sqrt{K}}$. Then*

$$\operatorname{inj}(x) \geqq \frac{r_0}{2} \frac{1}{1 + \left(\frac{V^H_{r_0+s}}{V_r(p)}\right)\left(\frac{V^H_{\rho+r}}{V^H_s}\right)},$$

where $\rho = d(p, x)$, V^H_r is the volume of $B(p, r) \subset M^H$, M^H is the simply connected space form of constant curvature H.

By Lemma 5.4 and (5.4), one obtains

$$\operatorname{inj}(y_\rho) \geqq c_{31}\, r(y_\rho), \qquad y_\rho \in \frac{1}{\rho} A\left(\rho, \frac{\rho}{K}\right) \tag{5.9}$$

for a constant c_{31} independent of $r(y_\rho)$. From (5.9) and (5.8), as in the proof of Theorem 4.1, it follows that there exists a subsequence of $\left(\frac{1}{\rho} \overline{A\left(\rho, \frac{\rho}{K}\right)}, x_\rho\right)$ converges to a flat manifold $\left(\overline{D\left(\frac{1}{K}\right)}, e_K\right)$. Now we can show that for each subsequence $\left\{\left(\frac{1}{\rho_k} A(\rho_k, 0), x_{\rho_k}\right)\right\}$ of $\left\{\left(\frac{1}{\rho} A(\rho, 0), x_\rho\right)\right\}$, there is a subsequence of $\left\{\left(\frac{1}{\rho_k} A(\rho_k, 0), x_{\rho_k}\right)\right\}$ which converges to $(D^4 - \{0\}, e)$. In order to complete the proof of Lemma 5.3, it suffices to show the following lemma.

Lemma 5.5 *$A(\rho, \bar{\rho})$ is simply connected for small $0 < \bar{\rho} < \rho$.*

Proof. Let α be a closed curve in $A(\rho, \bar{\rho})$, since α is compact, there exists $\eta > 0$, such that $\bar{\rho} + \eta < r|_\alpha < \rho - \eta$, as in the proof of Theorem 4.1, choose r_l small

such that $A(\rho, \bar{\rho}) \subset D(r_l)$ and $\operatorname{diam}(\partial D(r_l)) < \delta$ for small δ. Since $f^k(r_l)^* g^k \to g'$ in C^2-topology, take a point $m_l \in \partial D(r_l) \cap N$, then

$$|f^k(r_l)^* d^k(r_l)(y, m_l) - d(r_l)(y, m_l)| < \delta$$

on $A(\rho, \bar{\rho})$ for large k. For small δ, we have

$$B_{g'}(m_l, \rho + \delta) \supset f^k(r_l)^{-1}(B^k(f^k(r_l)(m_l), \rho)) \supset B_{g'}(m_l, \rho - \delta)$$

on the other hand, by $\operatorname{diam}(\partial D(r_l)) < \delta$ and the definition of $r(x)$, one obtains

$$|r(y) - d(r_l)(m_l, y)| < \delta.$$

This implies

$$f^k(r_l)^{-1}(B^k(m_l, \rho - \eta + 2\delta) - \overline{B^k(m_l, \bar{\rho} + \eta - 2\delta)})$$

is simply connected and

$$\alpha \subset f^k(r_l)^{-1}(B^k(m_l, \rho - \eta + 2\delta) - B^k(m_l, \bar{\rho} + \eta - 2\delta)).$$

Thus, α can be contracted to a point in $A(\rho, \bar{\rho})$. This implies Lemma 5.5.

We will need the local version of Gromov compactness theorem.

Definition 5.6 A sequence of Riemannian manifolds $\{M_i\}$ converges uniformly on compact subsets in the $C^{k,\alpha}$-topology to a Riemannian manifold M if, for any compact domain $D \subset M$ and i sufficiently large, there exist compact domains $D_i \subset M_i$ and $C^{k,\alpha}$- diffeomorphisms

$$f^i : D \mapsto D_i$$

such that

$$(f^i)^*(g^i) \xrightarrow{C^{k,\alpha}} g$$

on D, g is the metric on $D \subset M$.

Lemma 5.7 [A, Kas, Gr] *Let $\{M_i\}$ be a sequence of complete smooth Riemannian manifolds and $\{\Omega_i\}$ a sequence of domains in M_i with smooth boundary $\partial\Omega_i$. Assume, for all i*

(i) $$\|\nabla^l Rm\| \leqq c(l) \quad \text{on} \quad \Omega_i, l = 0, 1, \ldots, k$$

(ii) $$\operatorname{inj}(x) \geqq i_0 > 0, \qquad x \in \Omega_i$$

(iii) $$0 < v_0 < \operatorname{Vol}(\Omega_i) < v_1.$$

Then, given $\varepsilon > 0$, the Riemannian manifolds

$$\Omega_i(\varepsilon) = \{x \in \Omega_i \mid d(x, \partial\Omega_i) > \varepsilon\}$$

have a subsequence which converges uniformly on compact subsets in the $C^{k+1,\alpha}$-topology to a $C^{k+1,\alpha}$-Riemannian manifold $\Omega_\infty(\varepsilon)$. In particular, for i sufficiently large, there exist diffeomorphisms

$$f^i : \Omega_\infty(\varepsilon) \mapsto \Omega_i(\varepsilon)$$

such that

$$(f^i)^* g^i \xrightarrow{C^{k+1,\alpha}} g$$

on $\Omega_\infty(\varepsilon)$.

Now we are ready to prove the main lemma of this section.

Lemma 5.8 *The norm $|Rm|$ of curvature tensor Rm of metric g' is bounded.*

Proof. We will prove this lemma by contradiction method [A]. It suffices to show that

$$\sup_M |Rm(g')| \leqq c. \tag{5.10}$$

Suppose (5.10) is not true, then there is a sequence of critical Riemannian manifolds $\{(M_i, g^i)\}$ such that

$$\sup_{Mi} |Rm| = R_i \to \infty \quad \text{as.} \tag{5.11}$$

It will be shown that this leads to a contradiction. Suppose the supremum (5.11) is achieved at $x_i \in M_i$, $x_i \to v$, consider the sequence of rescaled metrics $\{(M_i, h^i)\}$ with

$$h^i = R_i g^i. \tag{5.12}$$

Then the sequence $\{(M_i, h^i)\}$ satisfies

$$\begin{aligned} &\text{(i)} \quad |Rm(h^i| \leqq 1 \\ &\text{(ii)} \quad |Rm(h^i)(x_i)| = 1 \\ &\text{(iii)} \quad \int_{M_i} |Rm|^2 \, d\mu < \infty \\ &\text{(iv)} \quad i_{M_i}(h^i) \geqq i_0 R_i^{\frac{1}{2}} \to \infty \end{aligned} \tag{5.13}$$

Combines (5.13), (5.2), (5.3) and (5.4), one obtains that there are metrics h^i on M_i such that

$$\begin{aligned} &\text{(i)} \quad |Rm(h^i(x_i)| = 1 \\ &\text{(ii)} \quad \int_{M_i} |Rm|^2 \, d\mu < \infty \\ &\text{(iii)} \quad |\nabla^k Rm(h^i)| < c(k), k = 0, 1, \ldots \\ &\text{(iv)} \quad i_{h^i} \to \infty \end{aligned} \tag{5.14}$$

We apply Lemma 5.7 the local version of Gromov compactness theorem to the manifolds (M_i, h^i), then we have the sequence $(B(x_i, R), h^i)$ of Riemannian manifolds has a subsequence which converges uniformly in the $C^{k,\alpha}$-topology for any k, to a C^∞-Riemannian manifold $(B(x_\infty, R), h^\infty)$. By choosing a subsequence $\{R_i\} \to \infty$ and taking a diagonal subsequence, if follows that a subse-

quence of $\{(M_i, h^i)\}$ converges smoothly on compact subsets, to a complete C^∞-Riemannian manifold N with smooth metric h_N satisfying

(5.15) (i) $\mathrm{Vol}(B_r) \leqq C r^4$, for some constant C

(ii) $\int_N |Rm|^2 \, d\mu < \infty$

(iii) $|Rm(z)| = 1$, for some $z \in N$

(iv) $i_N \to \infty$.

Remark. For (i), it follows from the condition $Rc(h^i) \geqq -Hh^i$. For (iv), we only need $i_M \geqq i_0$ and the convergence is in the C^1-topology.

From Lemma 5.3 and (5.15) (i) (ii) (iv), we will show that N is isometric to the standard Euclidean space $\mathbf{R}^4$ by using a theorem of [BKN]. Thus, N must be flat, contradicting (3.15) (iii). In fact, the Riemannian manifold N has a special geometric structure, called asymptotically flat ([LP]).

Definition 5.9 A Riemannian manifold N with C^∞-metric h is called *asymptotically flat of order* $\tau > 0$, if there exist a decomposition $N = N_0 \cup N_\infty$ and a diffeomorphism $N_\infty \mapsto \mathbf{R}^n - B_R$ for some $R > 0$ with N_0 compact, satisfying

$$h_{ij} = \delta_{ij} + O(\rho^{-\tau})$$
$$\partial_k h_{ij} = O(\rho^{-\tau-1})$$
$$\partial_k \partial_l h_{ij} = O(\rho^{-\tau-2})$$

as $|z| = \rho \to \infty$ in the coordinates $\{z^i\}$ induced on N_∞. The coordinates $\{z^i\}$ are called *asymptotic coordinates.* Note that the asymptotically flat structure is determined by the metric alone.

Definition 5.10 Given an asymptotically flat Riemannian manifold (N, h) with asymptotic coordinate $\{z^i\}$, define the mass as follows:

$$m(h) = \lim_{R \to \infty} \int_{S_R} \mu \rfloor dz$$

where μ is the mass-density vector field defined on N_∞:

$$\mu = \left(\frac{\partial h_{ij}}{\partial z^i} - \frac{\partial h_{ii}}{\partial z^j} \right) \frac{\partial}{\partial z^j}.$$

Note that if $\tau > \dfrac{n-2}{2}$, then $m(h)$ exists.

Now we continue to prove Lemma 5.8. Consider the annulus

$$A\left(nr, \frac{1}{n} r\right) = \left\{ x \in N \mid \frac{1}{n} r < r(x) < nr \right\}.$$

Since

$$\int_N |Rm|^2 \, d\mu < \infty,$$

we have

$$\int_{A(nr,\frac{1}{2}r)} |Rm|^2 \, d\mu \to 0$$

as $r \to \infty$. Then, from 5.15 (iv) and the proof of Lemma 5.2, it follows that

$$(A^r, g^r) = \left(A\left(nr, \frac{1}{n}r\right), \frac{g}{r^2}\right)$$

have sectional curvature satisfying

$$|Rm^r| \to 0 \tag{5.16}$$

on A^r, for fixed n. By taking a sequence $n_i \to \infty$, we have

$$(A^r, g^r) \xrightarrow{C^2} B_\infty(0, \infty) \tag{5.17}$$

where B_∞ is a flat Riemannian manifold. Now, from 5.15 (iv), we have bound Sobolev constant and the volume growth [C], Vol $B(z, t) \geqq Vt^n$, for some $V > 0$, $z \in N$. Moreover, from (5.16), we have $|Rm| r^2 \to 0$ as $r \to \infty$. Then, following the proof of Proposition 6.1 of [BKN], we get B_∞ is the flat cone over S^3/Γ for some finite subgroup $\Gamma \subset O(4)$. Thus the metric h on N approaches the flat metric on $(R_0, \infty) \times S^3/\Gamma$, for some $R_0 > 0$. But $i_N \to \infty$, it follows from Corollary 6.7 of [BKN] that (N, h_N) is isometric to the standard Euclidean space. That is, N must be flat, but this leads to a contradiction with 5.15 (iii). Then we complete the proof of lemma 5.8.

Proof of Theorem 5.1 From Lemma 5.8, it follows that we don't have so called bad balls at all. Then, we have

$$\sup_M |\nabla^p Rm| \leqq c_{33}(p, i_0, \chi(M)), \qquad p = 0, 1, 2, \ldots$$

Using Gromov compactness theorem [GW] and the standard elliptic theory, we obtain that $\{g^k\}$ converges to a critical metric g in C^∞-norm.

It follows that $I(M)$ is compact in the C^∞-topology.

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On the character variety of group representations in SL(2, $\mathbb{C}$) and PSL(2, $\mathbb{C}$)

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1 Introduction

Let G be a finitely generated group. For a representation $\rho: G \to \mathrm{SL}(2, \mathbb{C})$ define $\mathrm{tr}\,\rho: G \to \mathbb{C}$ where $\mathrm{tr}\,\rho(g)$ is the trace of $\rho(g)$. Equivalent representations define the same *character* $\mathrm{tr}\,\rho$. The set $R(G)$ of representations forms an affine algebraic set in $\mathbb{C}^{4n}$: for a given ρ, define the point $(\rho(g_1), \ldots, \rho(g_n)) \in \mathbb{C}^{4n}$, where g_1, g_2, ..., g_n is a set of generators of G. For each $g \in G$ define a function $\tau_g: R(G) \to \mathbb{C}$ by $\tau_g(\rho) = \mathrm{tr}\,\rho(g)$. It turns out that the ring T generated by the functions τ_g, $g \in G$, is finitely generated. Let $\gamma_1, \ldots, \gamma_m$ be a set of elements of G such that $\tau_{\gamma_1}, \ldots, \tau_{\gamma_m}$ generate T. Define a map $t: R(G) \to \mathbb{C}^m$ by $t(\rho) = (\tau_{\gamma_1}(\rho), \ldots, \tau_{\gamma_m}(\rho))$. Let $X(G) = t(R(G))$. The following theorem is proved in [CS]:

Theorem A *$X(G)$ is a closed algebraic set.* □

The authors of [CS] have the following to say about the proof of Theorem A (see [CS, p. 112]). "The proof that the characters form a closed set is surprisingly difficult, and requires the Burnside Lemma from representation theory". In this paper we want to give an elementary (but long) proof of Theorem A. As a matter of fact, our "elementary proof" completes the work by Magnus in [M]. For the benefit of the reader we avoid borrowing from [M] and give complete proofs. In [M] the reader can find relevant references to this work. Magnus remarks that Vogt [V], in 1889, found all the results contained in §2 of [M]. We consider this as a too generous remark. The method of [V] does not seem to us enough to solve the problem in general, while the method developed by Magnus [M] is the one that we have sharpened in this paper to complete the proof of Theorem A. This paper is in part a completion of [M, §2].

Our proof has the advantage, with respect to the Culler-Shalen proof, that it gives an explicit set of polynomials defining the closed algebraic set $X(G)$. We give an example of our construction in §6, in which we obtain $X(G)$ for the group of the figure eight knot, and discuss the representations of G into SL(2, $\mathbb{C}$) (see also [HLM1]). This method has been employed also in [HLM2]

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to obtain the character variety for the group of the Whitehead link. People working in hyperbolic geometry are interested in representations of G into $\mathrm{PSL}(2,\mathbb{C})$, because this last group can be identified with the group of orientation preserving isometries of $\mathbb{H}^3$, the hyperbolic space. Therefore we add a section (§2) explaining under what conditions representations into $\mathrm{PSL}(2,\mathbb{C})$ lift to $\mathrm{SL}(2,\mathbb{C})$.

Section §3 contains definitions and statement of main theorems. Theorem A for the case of a free group G is proved in §§4, 5. Theorem A for the general case follows from this, and an argument given in §3. We are indebted to Raquel Díaz for her help with Case 2.2, and also for pointing us the argument of [W] which we have developed into the proof of our Theorem 3.2.

2 Lifting representations $G \to \mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$

We will study the problem of deciding when a homomorphism $\gamma\colon G \to \mathrm{PSL}(2,\mathbb{C})$ lifts to $\mathrm{SL}(2,\mathbb{C})$. The lifting problem when G is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and γ is the inclusion has been solved by Culler who proved that γ lifts to $\mathrm{SL}(2,\mathbb{C})$ if and only if G has no elements of order 2 [Cu]. (We thank M. Kapovich for this information.)

Here the conditions that we will impose on G will be of an algebraic nature. Typically we will postulate the existence of a homomorphism ψ from a free product of free abelian groups S_i into G such that ψ induces an epimorphism on second homology with Z_2 coefficients. Then a homomorphism $\gamma\colon G \to \mathrm{PSL}(2,\mathbb{C})$ lifts to $\mathrm{SL}(2,\mathbb{C})$ if and only if, for each i, $\gamma\psi(S_i)$ is not isomorphic to Z_2^2 (see Corollary 2.4). Examples of such G are link groups where the S_i can be taken to be peripheral subgroups and $\psi | S_i$ inclusions.

Consider a central extension $A \rightarrowtail B \twoheadrightarrow C$. It corresponds to an element $\chi \in H^2(C; A)$. Let $\gamma\colon \hat{C} \to C$ be a group homomorphism. We say that γ *lifts* to B if there is a homomorphism $\tilde{\gamma}\colon \hat{C} \to B$ such that $\varphi\tilde{\gamma} = \gamma$.

2.1. Lemma. *γ lifts to B if and only if $\gamma^*(\chi) = 0$ in $H^2(C; A)$.*

Proof. Let $\hat{B} = \{(b, x) \in B \times \hat{C} \mid \varphi(b) = \gamma(x)\}$. Define the group homomorphisms $\hat{\varphi}\colon \hat{B} \to \hat{C}$ and $\beta\colon \hat{B} \to B$ by $\hat{\varphi}(b, \chi) = \chi$ and $\beta(b, \chi) = b$. Notice that $\mathrm{Ker}\,\hat{\varphi}$ can be identified with A and that we have a commutative diagram

$$\begin{array}{ccccc} A & \rightarrowtail & \hat{B} & \twoheadrightarrow & \hat{C} \\ \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A & \rightarrowtail & B & \twoheadrightarrow & C \end{array}$$

whose rows are central extensions. The upper row corresponds to $\gamma^*(\chi) \in H^2(\hat{C}; A)$. If $\gamma^*(\chi) = 0$ then there is a splitting $\hat{\sigma}\colon \hat{C} \to \hat{B}$ and, if we define $\tilde{\gamma} = \beta\hat{\sigma}$, we have $\varphi\tilde{\gamma} = \gamma$. Conversely, if there is a lifting $\tilde{\gamma}\colon \hat{C} \to B$ of γ then we can define a splitting $\hat{\sigma}\colon \hat{C} \to \hat{B}$ by $\hat{\sigma}(\chi) = (\tilde{\gamma}(\chi), \chi)$ and so $\gamma^*(\chi) = 0$. □

In particular, if $H^2(\hat{C}; A) = 0$ then any homomorphism $\gamma\colon \hat{C} \to C$ lifts to B. This is the case if $\hat{C}$ is a knot group.

We will say that a group homomorphism $\psi\colon D \to \hat{C}$ is $H^2(_; A)$-*injective* (resp. $H(_; A)$-*surjective*) if $\psi^*\colon H^2(\hat{C}; A) \to H^2(D; A)$ is a monomorphism (resp.

$\psi_*: H_2(D; A) \to H_2(\hat{C}; A)$ is an epimorphism). By the universal coefficient theorem and left-exactness of Hom, $\psi: D \to \hat{C}$ is $H^2(_; A)$-injective if and only if $\mathrm{Hom}(H_2(\hat{C})/\psi_* H_2(D), A) = 0$ and $\mathrm{Ext}(H_1(\hat{C}), A) \to \mathrm{Ext}(H_1(D), A)$ is injective. If A is a field $H^2(_; A)$-injectivity is equivalent to $H_2(_; A)$-surjectivity.

The reason for introducing these notions is that, if $\psi: D \to \hat{C}$ is $H^2(_; A)$-injective, a class in $H^2(\hat{C}; A)$ is trivial if and only if its image under ψ^* is trivial. Thus, in the situation of Lemma 2.1, if $\psi: D \to \hat{C}$ is $H^2(_; A)$-injective, then γ lifts to B if and only if $\gamma\psi$ lifts to B.

It is well known that any class in the second homology of a space with Z_2 (resp. Z) coefficients can be represented by a singular closed (resp. closed orientable) surface. It follows that any class in $H_2(\hat{C}; Z_2)$ (resp. $H_2(\hat{C})$) is of the form $\psi_*(\mu)$ where ψ is a homomorphism of a surface group S into $\hat{C}$ and μ is a generator of $H_2(S; Z_2)$ (resp. $H_2(S)$). By a surface group we mean the fundamental group of a closed surface. As a consequence, there is a free product of surface groups $D = \underset{i \in I}{*} S_i$ and a homomorphism $\psi: D \to \hat{C}$ such that $\mathrm{Hom}(H_2(\hat{C})/\psi_* H_2(D), A) = 0$ where, if $\hat{C}$ is finitely presented, the index set I can be taken finite. If, in addition, $\mathrm{Ext}(H_1(\hat{C}), A) \to \mathrm{Ext}(H_1(D), A)$ is injective, then ψ is $H^2(_; A)$-injective.

These remarks reduce the general lifting problem to the problem of lifting homomorphisms defined on surface groups. The case where the surface group is Z^2 can be handled using the following lemma.

2.2. Lemma. *Let $A \rightarrowtail B \xrightarrow{\varphi} \!\!\!\!\rightarrow C$ be a central extension and let $\hat{C}$ be a free abelian group. A momomorphism $\gamma: \hat{C} \to C$ lifts to B if and only if $\varphi^{-1}(\gamma(\hat{C}))$ is abelian.*

Proof. If $\tilde{\gamma}: \hat{C} \to B$ is a lifting of γ then $\varphi^{-1}(\gamma(\hat{C}))$ is generated by the abelian subgroup $\tilde{\gamma}(\hat{C})$ and the central subgroup A. Hence $\varphi^{-1}(\gamma(\hat{C}))$ is abelian.

Conversely, if $\varphi^{-1}(\gamma(\hat{C}))$ is abelian then γ clearly can be lifted, $\hat{C}$ being free abelian. □

Remarks. If $A \rightarrowtail K \twoheadrightarrow G$ is the central extension corresponding to $\kappa \in H^2(G; A)$ and if G is abelian, then K is abelian if and only if κ lies in the image of the canonical monomorphism $\mathrm{Ext}(G, A) \rightarrowtail H^2(G; A)$ or equivalently, by the universal coefficient theorem, in the kernel of $H^2(G; A) \twoheadrightarrow \mathrm{Hom}(H_2(G), A)$. Thus, in the situation of Lemma 2.2, γ can be lifted to B if and only if $i^*(\chi)$ is in the kernel of $H^2(\gamma(\hat{C}); A) \twoheadrightarrow \mathrm{Hom}(H_2(\gamma(\hat{C})), A)$ where $i: \gamma(\hat{C}) \to C$ is the inclusion and $\chi \in H^2(C; A)$ corresponds to $A \rightarrowtail B \twoheadrightarrow C$. For example, if $\gamma(\hat{C})$ is cyclic then γ can be lifted to B.

We now restrict to the case of the central extension $Z_2 \rightarrowtail \mathrm{SL}(2, \mathbb{C}) \twoheadrightarrow \mathrm{PSL}(2, \mathbb{C})$. One may remark that Lemma 2.3, Corollary 2.4 and Corollary 2.5 hold for the central extensions $Z_2 \rightarrowtail \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ and $\mathbb{C}^* \rightarrowtail \mathrm{GL}(2, \mathbb{C}) \twoheadrightarrow \mathrm{PSL}(2, \mathbb{C})$.

2.3. Lemma. *Consider the central extension $Z_2 \rightarrowtail \mathrm{SL}(2, \mathbb{C}) \xrightarrow{\varphi} \!\!\!\!\rightarrow \mathrm{PSL}(2, \mathbb{C})$. Let $\hat{C}$ be free abelian. Then a homomorphism $\gamma: \hat{C} \to \mathrm{PSL}(2, \mathbb{C})$ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if $\gamma(\hat{C})$ is not isomorphic to Z_2^2.*

Proof. By Lemma 2.2, it suffices to prove that $\varphi^{-1}(\gamma(\hat{C}))$ is abelian if and only if $\gamma(\hat{C}) \ncong Z_2^2$.

If $\gamma(\hat{C})$ is cyclic we pointed out above that $\varphi^{-1}(\gamma(\hat{C}))$ is abelian.

If $\gamma(\hat{C})$ contains a parabolic element then $\gamma(\hat{C})$ is conjugate to a subgroup of $\varphi(T)$ where T is the group of upper triangular matrices $\begin{pmatrix} \varepsilon & b \\ 0 & \varepsilon^{-1} \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{C})$. T is abelian and $\varphi^{-1}(\varphi(T)) = T$. Hence $\varphi^{-1}(\gamma(\hat{C}))$ is abelian.

If $\gamma(\hat{C})$ contains a non parabolic element of order greater than 2 then $\gamma(\hat{C})$ is conjugate to a subgroup of $\varphi(D)$ where D is the group of diagonal matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{C})$. Again D is abelian, $\varphi^{-1}(\varphi(D)) = D$, and, therefore, $\varphi^{-1}(\gamma(\hat{C}))$ is abelian.

If $\gamma(\hat{C})$ has exponent 2 then $\gamma(\hat{C}) \approx Z_2^2$ and $\gamma(\hat{C})$ is conjugate to $\varphi(Q_8)$ where Q_8 is the quaternion group generated by $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now $\varphi^{-1}(\gamma(\hat{C})) \approx Q_8$ which is not abelian.

Hence, γ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if $\gamma(\hat{C}) \not\approx Z_2^2$.

For the case where $\hat{C}$ is a Klein bottle group we will use an ad hoc argument (which can also be used for the case $\hat{C} = Z^2$). The group $\hat{C}$ has a presentation $\|x, y: y^{-1} x y = x^{-1}\|$. Suppose $\gamma(x) = \varphi(X)$ and $\gamma(y) = \varphi(Y)$; we have $Y^{-1} X Y = \pm X^{-1}$. The homomorphism γ does not lift if and only if $Y^{-1} X Y = -X^{-1}$. (Notice that replacing X by $-X$ of Y by $-Y$ also yields $Y^{-1} X Y = -X^{-1}$.) This occurs if and only if X, Y and XY all have trace 0 which is equivalent to the condition that $\varphi(x)$ and $\varphi(y)$ generate a Klein 4-group. Hence γ lifts if and only if $\gamma(\hat{C}) \not\approx Z_2^2$. □

The case where $\hat{C} = Z_2$ is easy: a homomorphism $\gamma: Z_2 \to \mathrm{PSL}(2, \mathbb{C})$ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if it is trivial. This is a consequence of the fact that the only element of order 2 in $\mathrm{SL}(2, \mathbb{C})$ is $-I$. If $\hat{C}$ is the Klein bottle group $|x, y: y^{-1} x y = x^{-1}|$ one can show that a homomorphism $\gamma: \hat{C} \to \mathrm{PSL}(2, \mathbb{C})$ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if the order of $\gamma(x)$ is different from 2 or $\gamma(\hat{C}) \approx Z_2^2$.

It seems more difficult to obtain similar lifting results in the case when $\hat{C}$ is a hyperbolic surface group. We can, however, make the following observation. Suppose $\hat{C} = |x_1, \ldots, x_m: r_1(x_1, \ldots, x_m), \ldots, r_n(x_1, \ldots, x_m)|$ and that, for every i and j, the sum of the exponents of i in $r_j(x_1, \ldots, x_m)$ is even. (Surface groups have such a presentations with $n = 1$.) Let $\gamma: \hat{C} \to \mathrm{PSL}(2, \mathbb{C})$ be a homomorphism and let $X_i \in \mathrm{SL}(2, \mathbb{C})$ be such that $\gamma(x_i) = \varphi(X_i)$ $(i = 1, \ldots, m)$. We have then $r_j(X_1, \ldots, X_m) = \pm I$ for every j. The homomorphism γ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if $r_j(X_1, \ldots, X_m) = I$ for every j.

2.4. Corollary. *Suppose* $\psi: \underset{i \in I}{*} S_i \to \hat{C}$ *is* $H_2(_; Z_2)$*-surjective and every* S_i *is free abelian. Then a homomorphism* $\gamma: \hat{C} \to \mathrm{PSL}(2, \mathbb{C})$ *lifts to* $\mathrm{SL}(2, \mathbb{C})$ *if and only if* $\gamma\psi(S_i) \not\approx Z_2^2$ *for all* $i \in I$.

Proof. First recall that $H_2(_; Z_2)$-surjectivity is equivalent to $H^2(_; Z_2)$-injectivity. Hence γ lifts if and only if $\gamma\psi$ lifts. This, in turn, happens if and only if $\gamma\psi(S_i) \not\approx Z_2^2$ for $i \in I$ by Lemma 2.3. □

2.5. Corollary. *Let* W *be a connected* CW*-complex and let* F^2 *be a 2-manifold every component of which is a torus. Let* $f: F^2 \to W$ *be a map such that* $f_*: H_2(F^2; Z_2) \to H_2(W; Z_2)$ *is surjective. Then a homomorphism* $\gamma: \pi_1 W \to \mathrm{PSL}(2, \mathbb{C})$ *lifts to* $\mathrm{SL}(2, \mathbb{C})$ *if and only if, for every component* F_i *of* F*, the image of* $\gamma(f|F_i): \pi_1(F_i) \to \mathrm{PSL}(2, \mathbb{C})$ *is not isomorphic to* Z_2^2.

Proof. Take $S_i=\pi_1(F_i)$ and let ψ: $\underset{i\in I}{*} S_i \to \pi_1(W)$ be such that $\psi|S$ is induced by $f|F_i$. Since f_*: $H_2(F^2; Z_2)\to H_2(W; Z_2)$ and the natural homomorphism $H_2(W; Z_2)\to H_2(\pi_1 W; Z_2)$ are surjective ψ is $H_2(_; Z_2)$-surjective. The result then follows from Corollary 2.4. □

We will apply Corollary 2.5 to the situation where W^3 is the exterior of an n-component link in a Z_2-homology 3-sphere, F^2 is the union of $n-1$ components of ∂W^3 and f is the inclusion.

One can generalize Corollary 2.5 allowing projective planes and Klein bottles as components of F^2. Necessary and sufficient conditions for the lifting of γ would then be:

(i) $\gamma(f|F_i)(\pi_1 F_i)\ncong Z_2^2$ if F_i is a torus component of F.
(ii) $\gamma(f|F_i)(\pi_1 F_i)=1$ if F_i is a projective plane component of F and
(iii) $\gamma(f|F_i)(\pi_1 F_i)\approx Z_2^2$ or $\gamma(f|S_i^1)(\pi_1 S_i^1)\ncong Z_2$ if F_i is a Klein bottle component of F and S_i^1 is a 1-sphere in F_i with $F_i-S_i^1$ orientable.

This generalization can be used to study the lifting problem for $\pi_1 W^3 \to \mathrm{PSL}(2, \mathbb{C})$ when W^3 is a Seifert fibered space with orbit space B. One would take as F the inverse image of a suitable disjoint union of circles in B and as f: $F\to W^3$ the inclusion.

3 The variety of characters

Let G be a finitely generated group and let π: $F_n\to G$ be a surjection where F_n is the free group on the generators $y_1, y_2, \ldots, y_n$. Let $a_i=\pi(y_i)$ $(i=1, \ldots, n)$. For a representation ρ: $G\to \mathrm{SL}(2, \mathbb{C})$ we define $\mathrm{tr}\,\rho$: $G\to\mathbb{C}$ by $\mathrm{tr}\,\rho(g)=$ trace$(\rho(g))$. Denote $\mathrm{tr}\,\rho(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$ by $t_{i_1 i_2 \ldots i_m}(=t_{i_1 i_2 \ldots i_m}(\rho))$ where $1\leqq i_1 < i_2 < \ldots < i_m \leqq n$. We may view $t_{i_1 i_2 \ldots i_m}(\rho)$ as a function associating to every representation ρ an element of $\mathbb{C}$. If we fix $g\in G$ then $\mathrm{tr}\,\rho(g)$, as a function of ρ, is a polynomial with rational coefficients in the $p=\dfrac{n(n^2+5)}{6}$ functions $t_{i_1 i_2 \ldots i_m}(\rho)$, where $m\leqq 3$, ([V, see Corollary 4.14]). If we order lexicographically the p numbers $t_{i_1 i_2 \ldots i_m}(\rho)$, $(m\leqq 3)$, we obtain a point $\bar{\rho}\in\mathbb{C}^p$. Thus, $\mathrm{tr}\,\rho$ is determined by $\bar{\rho}$. Define $X(G)$ to be the locus of all points $\bar{\rho}\in\mathbb{C}^p$. Strictly speaking $X(G)$ depends on the choice of π. However, if $X'(G)$ is defined using a different epimorphism π': $F_m\to G$, there is a natural bijection f: $X(G)\to X'(G)$ such that each component of f and of f^{-1} is a polynomial with integral coefficients (see [CS, p. 113]).

3.1. Theorem (Compare [V, M]). *$X(F_n)$ is a closed algebraic set. In fact, $X(F_n)$ is the locus of the zeroes of the polynomials p such that $p(\bar{\rho})=0$, for all ρ: $F_n\to \mathrm{SL}(2, \mathbb{C})$.*

Proof. See in §4 and §5 a proof of a stronger statement.

By the remark preceding Theorem 3.1, $X(F_n)$ is a closed algebraic set for any choice of generators of F_n and in fact $X(F_n)$ is well defined up to canonical isomorphism.

Let G and π: $F_n\to G$ be as in the beginning of this Sect. 3. Let $W=\{w_j\}_{j\in J}$ be a subset of F that normally generates $\ker\pi$.

Now for $w_i \in W$ and $y_j \in \{y_0 = 1, y_1, y_2, \ldots, y_n\}$ $\operatorname{tr} \rho(w_i y_j) - \operatorname{tr} \rho(y_j)$ is a function of ρ which can be written as a polynomial $p_{i,j}$ with rational coefficients in the variables $t_{i_1 i_2 \ldots i_m}(\rho)$, $m \leqq 3$.

3.2. Theorem. $X(G) = \{x \in X(F_n) \mid p_{ij}(x) = 0$, *for all* $i \in J$, $j = 0, 1, \ldots, n\}$. *Therefore,* $X(G)$ *is a closed algebraic set.*

Proof (Compare [W]). Let $x \in \{x \in X(F_n) \mid p_{ij}(x) = 0, \forall i \in J, \forall j = 0, 1, \ldots, n\}$. Then $x = \bar{\rho}$, where $\rho\colon F_n \to \mathrm{SL}(2, \mathbb{C})$ is a representation. Let $\rho(y_i)$ be A_i and $\rho(y_0) = \rho(1) = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We have

$$\operatorname{tr}(w_i(A_1, \ldots, A_n) A_j) - \operatorname{tr}(A_j) = 0 \quad \forall i, j. \tag{1}$$

Then, either $w_i(A_1, \ldots, A_n) = I$ for every i, or there exists i (say $i = 1$) such that $w_1(A_1, \ldots, A_n)$ is parabolic ($\neq I$). Conjugating w_1 with a suitable matrix V we have

$$V w_1(A_1, \ldots, A_n) V^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now, modify the choice of $A_1, \ldots, A_n$, conjugating them by V, to obtain the new set

$$\{V A_1 V^{-1}, V A_2 V^{-1}, \ldots, V A_n V^{-1}\} := \{\hat{A}_1, \ldots, \hat{A}_n\}$$

which still satisfies (1) and also

$$w_1(\hat{A}_1, \ldots, \hat{A}_n) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We then have, for $j = 1, \ldots, n$, that

$$\begin{aligned} &\operatorname{tr}(w_1(\hat{A}_1, \ldots, \hat{A}_n) \hat{A}_j) - \operatorname{tr}(\hat{A}_j) = 0 \\ &\Leftrightarrow \operatorname{tr}\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}\right) - \operatorname{tr}\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix} = 0 \\ &\Leftrightarrow \alpha_j + \delta_j + \gamma_j = \alpha_j + \delta_j \Leftrightarrow \gamma_j = 0, \qquad j = 1, \ldots, n. \end{aligned}$$

Therefore, all the elements of $\{\hat{A}_1, \ldots, \hat{A}_n\}$ are upper triangular matrices. Let us modify the set $\{\hat{A}_1, \ldots, \hat{A}_n\}$ by putting zero in the entry "β". We thus obtain a new set $\{\bar{A}_1, \ldots, \bar{A}_n\}$ which defines a new representation $\rho_1\colon F_n \to \mathrm{SL}(2, \mathbb{C})$ such that $\bar{\rho}_1 = \bar{\rho} = x$ and such that still accomplishes (1). This is a direct consequence of the fact that

$$\operatorname{tr}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & d \end{pmatrix} = a a' + d d' = \operatorname{tr}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix}.$$

But since

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix}=\begin{pmatrix} a\,a' & 0 \\ 0 & d\,d' \end{pmatrix}$$

the equation

$$\operatorname{tr}(w_1(\bar{A}_1, \ldots, \bar{A}_n))=\operatorname{tr}(I)=2$$

implies

$$w_i(\bar{A}_1, \ldots, \bar{A}_n)=I, \qquad \forall i\in J.$$

Therefore, ρ_i defines a representation $\rho_1: G\to \mathrm{SL}(2, \mathbb{C})$ such that $\bar{\rho}_1=x$. □

As with $X(F)$, $X(G)$ is a closed algebraic set, well defined up to canonical isomorphism, which we call the *character variety* of G.

4 Preliminaries for the proof of Theorem 3.1

Given a representation $\rho: F_n\to \mathrm{SL}(2, \mathbb{C})$ we will denote $\rho(y_a)$ by A_a. Recall that $\operatorname{tr}\rho(y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is denoted by $t_{i_1 i_2 \ldots i_m}(\rho)$, and also by $t_{i_1 i_2 \ldots i_m}$ if there is no risk of confusion.

We have the following identities that can be obtained by direct computation

$$\operatorname{tr}(A_a^{-1})=\operatorname{tr}(A_a)$$

$$\operatorname{tr}(AB)=\operatorname{tr}(A)\operatorname{tr}(B)-\operatorname{tr}(AB^{-1}).$$

4.1. Lemma [V]. *For every* $\rho: F_n\to \mathrm{SL}(2, \mathbb{C})$, *the following identity holds:*

$$t_1 t_2 t_3-t_1 t_{23}-t_2 t_{13}-t_3 t_{12}+t_{123}+t_{321}=0$$

where t_i, t_{ij}, t_{ijk} *mean* $t_i(\rho)$, $t_{ij}(\rho)$, $t_{ijk}(\rho)$.

4.1.1. Lemma [V]. *For every* $\rho: F_n\to \mathrm{SL}(2, \mathbb{C})$ *the following relation holds:*

$$\begin{aligned} 2t_{1234}=&t_{13}t_2t_4-t_1t_2t_{34}-t_2t_3t_{41}-t_{13}t_{24}\\ &+t_{12}t_{34}+t_{23}t_{41}-t_{132}t_4+t_1t_{234}+t_2t_{341}+t_3t_{412} \end{aligned}$$

where $t_{i_1 i_2 \ldots i_m}$ *means* $t_{i_1 i_2 \ldots i_m}(\rho)$.

Proof. From Lemma 4.4 we have

$$A=t_{13}t_2t_4-t_{13}t_{24}-t_2t_{134}-t_4t_{132}+t_{1324}+t_{4213}=0$$

$$B=t_1t_2t_{34}-t_1t_{234}-t_2t_{134}-t_{34}t_{12}+t_{1234}+t_{2134}=0$$

$$C=t_{41}t_2t_3-t_{41}t_{23}-t_2t_{413}-t_3t_{412}+t_{4123}+t_{3241}=0$$

$$\begin{aligned} 0=B+C-A=&-t_{13}t_2t_4+t_1t_2t_{34}+t_2t_3t_{41}+t_{13}t_{24}\\ &-t_{12}t_{34}-t_{23}t_{41}+t_{132}t_4-t_1t_{234}-t_2t_{341}-t_3t_{412}+2t_{1234}. \end{aligned}$$ □

4.1.2. Corollary. *The trace of a word* $w(A_1, \ldots, A_n)$ *in the matrices* $A_1, \ldots, A_n$ *of* $\mathrm{SL}(2, \mathbb{C})$ *is a polynomial with rational coefficients in the traces*

$$t_a, \qquad a=1, \ldots, n$$

$$t_{ab}, \qquad 1 \leqq a < b \leqq n$$

$$t_{abc}, \qquad 1 \leqq a < b < c \leqq n.$$

Proof. Use Lemmas 4.1, 4.1.1 and induction. □

Let $\mathbb{C}^p$ have the coordinates $\{t_1, \ldots, t_n; t_{ab}, 1 \leqq a < b \leqq n; t_{abc}, 1 \leqq a < b < c \leqq n\}$ so that $p = \dfrac{n(n^2+5)}{6}$. Every representation $\rho: F_n \to \mathrm{SL}(2, \mathbb{C})$ give rise to a point of $\mathbb{C}^p$, namely: $\bar{\rho} = (t_1(\rho), \ldots, t_n(\rho); t_{ab}(\rho), 1 \leqq a < b \leqq n; t_{abc}(\rho), 1 \leqq a < b < c \leqq n)$.

Given a point $P \in \mathbb{C}^p$ we will introduce some functions M_{ab}, H_{ab} and T_{abc} as follows (compare [V, M]):

$$M_{ab}(P) = M_{ba}(P) := 2t_{ab} - t_a t_b, \quad \text{if } a < b$$

$$M_{aa}(P) = t_a^2 - 4$$

$$H_{ab}(P) := \begin{vmatrix} M_{aa}(P) & M_{ab}(P) \\ M_{ba}(P) & M_{bb}(P) \end{vmatrix} = -4(t_a^2 + t_b^2 + t_{ab}^2 - t_a t_b t_{ab} - 4),$$

$$T_{abc}(P) := \begin{vmatrix} M_{aa}(P) & M_{ab}(P) & M_{ac}(P) \\ M_{ba}(P) & M_{bb}(P) & M_{bc}(P) \\ M_{ca}(P) & M_{cb}(P) & M_{cc}(P) \end{vmatrix}.$$

The adjoint of the matrix whose determinant is $T_{abc}(P)$ will be represented by the symmetric matrix:

$$\begin{bmatrix} H_{bc}(P) & A_{ab,c}(P) & A_{ac,b}(P) \\ A_{ba,c}(P) & H_{ac}(P) & A_{bc,a}(P) \\ A_{ca,b}(P) & H_{cb,a}(P) & H_{ab}(P) \end{bmatrix}.$$

Frequently we will abbreviate $M_{ab}(P)$, $H_{ab}(P)$, $T_{abc}(P)$, $A_{ab,c}(P)$ by M_{ab}, H_{ab}, T_{abc}, or $A_{ab,c}$, and the meaning will be clear by the context.

4.2. Proposition. $M_{11}(\bar{\rho}) = 0 \Leftrightarrow A_1$ *is the identity matrix or it is parabolic.*

4.3. Proposition (compare [M]). *We have:*

(i) $H_{12}(\bar{\rho}) = -4(\mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) - 2)$

(ii) $H_{12}(\bar{\rho}) = 0 \Leftrightarrow \mathrm{tr}(A_1 A_2 A_1^{-1} A_2^{-1}) = 2 \Leftrightarrow \{A_1, A_2\}$ *is conjugate in* $\mathrm{SL}(2, \mathbb{C})$ *to a pair of upper triangular matrices.*

(iii) $H_{12}(\bar{\rho}) \neq 0$, $M_{11}(\bar{\rho}) = M_{22}(\bar{\rho}) = 0 \Leftrightarrow \{A_1, A_2\}$ *is conjugate in* $\mathrm{SL}(2, \mathbb{C})$ *to a pair of the form:*

$$\left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \neq 0; \pm \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, b \neq 0 \right\}.$$

(iv) $H_{12}(\bar{\rho})\neq 0$, $M_{11}(\bar{\rho})\neq 0 \Leftrightarrow \{A_1, A_2\}$ *is conjugate in* $\mathrm{SL}(2,\mathbb{C})$ *to a pair of the form:*

$$\left\{\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, \lambda^2\neq 1; \begin{pmatrix}a & 1\\ c & d\end{pmatrix}, c=ab-1\neq 0\right\}.$$

4.4. Proposition (compare [M]). *We have*
(i) $T_{123}(\bar{\rho})=-4(t_{123}(\rho)-t_{132}(\rho))^2$
(ii) $T_{123}(\bar{\rho})$ *is the discriminant of the polynomial equation* $z^2-Pz+Q=0$, *where*

$$P=t_1t_{23}+t_2t_{13}+t_3t_{12}-t_1t_2t_3$$

$$Q=t_1^2+t_2^2+t_3^2+t_{12}^2+t_{13}^2+t_{23}^2+t_{12}t_{13}t_{23}-t_1t_2t_{12}-t_1t_3t_{13}-t_2t_3t_{23}-4.$$

Here $t_{i_1i_2\ldots i_m}$ *means* $t_{i_1i_2\ldots i_m}(\rho)$. *The polynomial equation has roots* $t_{123}(\rho)$ *and* $t_{132}(\rho)$.
(iii) $H_{12}(\bar{\rho})=H_{23}(\bar{\rho})=H_{13}(\bar{\rho})=T_{123}(\bar{\rho})=0 \Leftrightarrow \{A_1, A_2, A_3\}$ *is conjugate in* $\mathrm{SL}(2,\mathbb{C})$ *to a triple of upper triangular matrices.*

The proof follows in part from Lemma 4.1.

4.5. Proposition. *Consider* $T_{1ab}(P)=0$ *as a second degree equation in the variable* $M_{ab}(P)$. *Then the coefficient of* M_{ab}^2 *is* M_{11} *and the discriminant is* $H_{1a}H_{1b}$. *Moreover, the following identity holds:*

$$M_{11}\,T_{abc}=-\begin{vmatrix} H_{1a} & A_{ab,1}\\ A_{ab,1} & H_{1b}\end{vmatrix}$$

where $A_{ab,1}=M_{1a}M_{1b}-M_{ab}M_{11}$.

Proof. $T_{1ab}=0$ is

$$M_{11}M_{ab}^2-2(M_{1a}M_{1b})M_{ab}-M_{11}M_{aa}M_{bb}+M_{1b}^2M_{aa}+M_{1a}^2M_{bb}=0$$

whose discriminant is

$$M_{1a}^2M_{bb}^2-M_{11}(M_{1b}^2M_{aa}+M_{1a}^2M_{bb}-M_{11}M_{aa}M_{bb})=H_{1a}H_{1b}. \quad \square$$

4.5.1. Proposition. *We have:*
(i) *If* $H_{12}(P)=H_{23}(P)=H_{13}(P)=0$; *then* $T_{123}(P)\neq 0$ *if and only if* $A_{12,3}(P)A_{23,1}(P)A_{13,2}(P)\neq 0$.
(ii) *If* $H_{12}(P)=0$; *then* $T_{123}(P)=0$ *if and only if* $A_{13,2}(P)=A_{23,1}(P)=0$.

Proof. If $T_{123}(P)\neq 0$, the determinant of the adjoint matrix is also $\neq 0$. But this determinant is

$$2A_{12,3}(P)\,A_{23,1}(P)\,A_{13,2}(P)$$

because $H_{12}(P)=H_{23}(P)=H_{13}(P)=0$. This shows (i).

To see (ii) use the identity of Proposition 4.5. If $T_{123}(P)=0$ we have

$$0=M_{11}(P)\,T_{123}(P)=A_{23,1}^2(P)$$

$$0=M_{22}(P)\,T_{213}(P)=A_{13,2}^2(P).$$

Now, if $A_{13,2}(P)=A_{23,1}(P)=0$, $T_{123}(P)=M_{13}(P)\,A_{13,2}(P)+M_{23}(P)\,A_{23,1}(P)+M_{33}(P)\,H_{12}(P)=0$. □

4.5.2. Corollary. $T_{123}(P)\neq 0$, $H_{12}(P)=H_{23}(P)=H_{13}(P)=0 \Rightarrow M_{ij}\neq 0$ *for every* $i<j$.

Proof. We have by Proposition 4.5.1(i):

$$A_{12,3}(P)=M_{13}M_{23}-M_{22}M_{33}\neq 0$$

$$A_{13,2}(P)=M_{12}M_{23}-M_{13}M_{22}\neq 0$$

$$A_{23,1}(P)=M_{12}M_{13}-M_{23}M_{11}\neq 0.$$

Assume $M_{11}=0$. Then $H_{12}=H_{13}=0$ imply $0=M_{11}M_{22}-M_{12}^2=M_{11}M_{33}-M_{13}^2 \Rightarrow M_{12}=M_{13}=0 \Rightarrow A_{23,1}(P)=0$ which is not the case. Thus $M_{11}\neq 0$ and analogously $M_{22}\neq 0$, $M_{33}\neq 0$. From $M_{12}^2=M_{11}M_{22}$ follows now $M_{12}\neq 0$, etc. □

The following lemma sharpens a result by Magnus [M]:

4.6. Lemma. *Let* $A_1, \ldots, A_4, B_1, \ldots, B_4 \in \mathrm{SL}(2,\mathbb{C})$. *Then*

$$|\mathrm{tr}(A_i B_j)-\mathrm{tr}(A_i B_j^{-1})|=0.$$

Proof. Put $C_i=\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$,

$$M(C_1, C_2, C_3, C_4)=M(C_i):=\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix},$$

$$J=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J^*=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and one can check the following matrix equality:

$$(\mathrm{tr}(A_i B_j))=M(A_i)\,J\,\dot{M}(B_j)^t$$

where t means transpose. Now

$$M(B_j^{-1})=M(B_j)\,J^*.$$

Therefore

$$(\mathrm{tr}(A_i B_j^{-1}))=M(A_i)\,J\,J^*\,M(B_j)^t.$$

Then

$$(\mathrm{tr}(A_i B_j)-\mathrm{tr}(A_i B_j^{-1}))=M(A_i)\,J(I-J^*)\,M(B_j)^t.$$

This proves Lemma 4.6 because $|I-J^*|=0$. □

4.7. *Notation.* Given $P \in \mathbb{C}^p$ put

$$Q_{abcd}(P) := \begin{vmatrix} M_{aa} & M_{ab} & M_{ac} & M_{ad} \\ M_{ba} & M_{bb} & M_{bc} & M_{bd} \\ M_{ca} & M_{cb} & M_{cc} & M_{cd} \\ M_{da} & M_{db} & M_{dc} & M_{dd} \end{vmatrix}.$$

4.8. Proposition. *For every* $\rho\colon F_n \to \mathrm{SL}(2, \mathbb{C})$, $Q_{abcd}(\bar{\rho}) = 0$. *Considering* $Q_{12ab}(P) = 0$ *as a quadratic equation in the variable* M_{ab}, *the coefficient of* M_{ab}^2 *is* H_{12} *and the discriminant is* $T_{12a} T_{12b}$. *Therefore, the following identity holds:*

$$T_{12a} T_{12b} = (A_{1a,2} M_{1b} + A_{2a,1} M_{2b} + H_{12} M_{ab})^2.$$

Proof. $Q_{1234}(\bar{\rho}) = |\operatorname{tr}(A_i A_j) - \operatorname{tr}(A_i A_j^{-1})|$ because $\operatorname{tr}(A_i A_j) = \operatorname{tr}(A_i)\operatorname{tr}(A_j) - \operatorname{tr}(A_i A_j^{-1})$; hence $\operatorname{tr}(A_i A_j) - \operatorname{tr}(A_i A_j^{-1}) = 2 t_{ij}(\rho) - t_i(\rho)\, t_j(\rho) = M_{ij}(\bar{\rho})$.

Now, $Q_{1234}(\bar{\rho}) = 0$ is a consequence of Lemma 4.6.

For the second part of the proof, put

$$\begin{aligned} Q_{12ab} = {} & H_{12} M_{ab}^2 + 2(M_{12} M_{2a} M_{1b} - M_{22} M_{1a} M_{1b} \\ & - M_{11} M_{2a} M_{2b} + M_{12} M_{1a} M_{2b})\, M_{ab} - (M_{bb} T_{12a} - M_{22} M_{aa} M_{1b}^2 \\ & - 2 M_{1a} M_{2b} M_{1b} M_{2b} + 2 M_{12} M_{aa} M_{1b} M_{2b} \\ & - M_{11} M_{aa} M_{2b}^2 + M_{2a}^2 M_{1b} + M_{1a}^2 M_{2b}^2) = 0 \end{aligned}$$

whose discriminant is $T_{12a} T_{12b}$. □

4.8.1. Proposition. $T_{123}(P) \neq 0$, $H_{12}(P) = H_{23}(P) = H_{13}(P) = 0$, $T_{124}(P) = 0 \Rightarrow H_{14}(P) = H_{24}(P) = 0$.

Proof. Using the identity of Proposition 4.8 we have

$$0 = T_{142} T_{14a} = (A_{12,4} M_{1a} + A_{42,1} M_{4a} + H_{14} + M_{2a})^2, \qquad a = 2, 3.$$

From Proposition 4.5.1(ii), $H_{12}(P) = 0$, $T_{124}(P) = 0 \Rightarrow A_{14,2}(P) = A_{24,1}(P) = 0$. Therefore,

$$A_{12,4} M_{1a} = -H_{14} M_{2a}.$$

Assume $H_{14} \neq 0$ and remember that $M_{1a} \neq 0$, $M_{2a} \neq 0$ (Corollary 4.5.2). We obtain, for $a = 2, 3$:

$$\frac{M_{12}}{M_{13}} = \frac{A_{12,4}}{A_{12,4}} \frac{M_{12}}{M_{13}} = \frac{-M_{14} M_{22}}{-H_{14} M_{23}} = \frac{M_{22}}{M_{23}}.$$

On the other hand $0 = H_{23} = M_{22} M_{33} - M_{23}^2$ imply

$$\frac{M_{12}}{M_{13}} = \frac{M_{22}}{M_{23}} = \frac{M_{23}}{M_{33}} \Rightarrow T_{123} = 0$$

which is not the case. Analogously $H_{24} = 0$. □

4.8.2. Proposition. $T_{123}(P) \neq 0$, $T_{124}(P) \neq 0$, $T_{134}(P) \neq 0$, $H_{ij}(P) = 0$ *for every* i, j *in* $\{1, 2, 3, 4\} \Rightarrow Q_{1234}(P) \neq 0$.

Proof. The inverse of the matrix which defines $T_{123}(P)$ is of the form (Proposition 4.5.1(i)):

$$\begin{pmatrix} 0 & \hat{a} & \hat{b} \\ \hat{a} & 0 & \hat{c} \\ \hat{b} & \hat{c} & 0 \end{pmatrix} \quad \hat{a}\neq 0, \quad \hat{b}\neq 0, \quad \hat{c}\neq 0.$$

Hence T_{123} is of the form:

$$T_{123}=\begin{pmatrix} -c^2 & bc & ac \\ bc & -b^2 & ab \\ ac & ab & -a^2 \end{pmatrix} a\neq 0, \quad b\neq 0, \quad c\neq 0.$$

Also

$$T_{124}=\begin{pmatrix} -c^2 & bc & dc \\ bc & -b^2 & db \\ dc & db & -d^2 \end{pmatrix}, \qquad T_{134}=\begin{pmatrix} -c^2 & ac & dc \\ ac & -a^2 & ad \\ dc & ad & -d^2 \end{pmatrix}$$

where $d\neq 0$. These are the necessary and sufficient conditions for the hypothesis of the Proposition to be true. Now

$$Q_{1234}(P)=\begin{vmatrix} -c^2 & bc & ac & dc \\ bc & -b^2 & ab & db \\ ac & ba & -a^2 & da \\ dc & bd & ad & -d^2 \end{vmatrix} = -16\,a^2 b^2 c^2 d^2 \neq 0. \quad \square$$

4.9. Proposition [M]. *For every representation* $\rho\colon F_n \to \mathrm{SL}(2,\mathbb{C})$ *we have*

$$\begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{1a} \\ M_{21} & M_{22} & M_{23} & M_{2a} \\ M_{31} & M_{32} & M_{33} & M_{3a} \\ M_{b1} & M_{b2} & M_{b3} & M_{ba} \end{vmatrix} = 0,$$

where M_{ij} *means* $M_{ij}(\bar{\rho})$.

This expression is a linear polynomial in the variable $M_{ab}(P)$. *The coefficient of* $M_{ab}(P)$ *is* $T_{123}(P)$.

Proof. Apply Lemma 4.6. $\square$

4.10. Lemma. $|\mathrm{tr}(A_i B_j)| = -|M(A_i)|\,|M(B_j)|$.

Proof. $|\mathrm{tr}(A_i B_j)| = |M(A_i)\,J\,M(B_j)| = |J|\,|M(A_i)|\,|M(B_j)|$, but $|J|=-1$. $\square$

4.11. Proposition (compare [M])

$$|M(A_1, A_2, A_3, I)| = t_{123}(\rho) - t_{132}(\rho).$$

Proof. Direct computation. $\square$

4.12. Corollary. *The following holds, for every* $\rho\colon F_n \to \mathrm{SL}(2,\mathbb{C})$

$$(t_{123}-t_{132})(2\,t_{abc}-t_a t_b t_c - t_a t_{bc} - t_b t_{ac} - t_c t_{ab}) = \begin{vmatrix} t_1 & t_{1a} & t_{1b} & t_{1c} \\ t_2 & t_{2a} & t_{2b} & t_{2c} \\ t_3 & t_{3a} & t_{3b} & t_{3c} \\ 2 & t_a & t_b & t_c \end{vmatrix},$$

where $t_{ii} = t_i^2 - 2$, *and* $t_{i_1 i_2 \ldots i_m}$ *means* $t_{i_1 i_2 \ldots i_m}(\rho)$.

Proof. Put $(A_1, A_2, A_3, A_4)=(A_1, A_2, A_3, I)$, $(B_1, B_2, B_3, B_4)=(A_a, A_b, A_c, I)$ in Lemma 4.10, and use Proposition 4.11 to obtain:

$$|\mathrm{tr}(A_i B_j)| = -(t_{123}-t_{132})(t_{abc}-t_{acb}).$$

Use the identity of Lemma 4.1 and change columns in $(\mathrm{tr}(A_i B_j))$ to obtain the formula. □

Remark. Corollary 4.12 gives an expression for $T_{123}(\bar{\rho})$, when one puts a, b, $c=1, 2, 3$.

$$T_{123} = -\begin{vmatrix} t_1^2-2 & t_{12} & t_{13} & t_1 \\ t_{12} & t_2^2-2 & t_{23} & t_2 \\ t_{13} & t_{23} & t_3^2-2 & t_3 \\ t_1 & t_2 & t_3 & 2 \end{vmatrix}.$$

5 Proof of Theorem 3.1

Let $\mathcal{J}$ denote the ideal of polynomials f in the $p=\dfrac{n(n^2+5)}{6}$ variables $\{t_1, \ldots, t_n;$ $t_{ab},\ 1\leqq a<b<c\leqq n;\ t_{abc},\ 1\leqq a<b<c\leqq n\}$ and complex coefficients, such that $f(\bar{\rho})=0$, for every representation $\rho: F_n \to \mathrm{SL}(2, \mathbb{C})$.

It was essentially shown by Vogt and by Magnus (compare §4) that $\mathcal{J}$ contains the following finite set of polynomials:

P(1) $t_{abc}^2 - P t_{abc} + Q$, $1\leqq a<b<c\leqq n$, given in Proposition 4.4(ii).

P(2) Q_{12a}, $3\leqq a<b\leqq n$. This is a quadratic polynomial in the variable M_{ab}. The coefficient of M_{ab}^2 is H_{12}; the discriminant is $T_{12a}T_{12b}$; the coefficients are functions of $t_1, t_2, t_a, t_b; t_{12}, t_{1a}, t_{1b}; t_{2a}, t_{2b}$ (Proposition 4.8).

P(3) $$\begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{1a} \\ M_{21} & M_{22} & M_{23} & M_{2a} \\ M_{31} & M_{32} & M_{33} & M_{3a} \\ M_{b1} & M_{b2} & M_{b3} & M_{ba} \end{vmatrix}, \qquad 3<a<b\leqq n$$

which is a linear polynomial in M_{ab} with coefficient T_{123}. The coefficients are function of $t_1, t_2, t_3; t_{12}, t_{13}, t_{1a}, t_{1b}; t_{23}, t_{2a}, t_{2b}; t_{3a}, t_{3b}$ (Proposition 4.9).

P(4) The linear polynomial in t_{abc}, $a<b<c$, with coefficient $+\sqrt{T_{123}}$ $(:=t_{123}-t_{132}=2t_{123}-t_1t_{23}-t_2t_{13}-t_3t_{12}+t_1t_2t_3)$ given by Corollary 4.12. The rest of the coefficients are function of t_a, $1\leqq a\leqq n$; t_{ab}, $1\leqq a<b\leqq n$.

Let $L\subset\mathbb{C}^p$ be the locus of the zeroes of the polynomials P(1) to P(4).

Vogt and Magnus proved [V, M] that

$$X(F_n)\subset L.$$

They also sketched a proof that some points (Case 1 below) of L belong to $X(F_n)$. We want to prove now that, in fact,

$$L\subset X(F_n)$$

i.e. given $P\in L$ there exists $\rho: F_n \to \mathrm{SL}(2, \mathbb{C})$ such that $P=\bar{\rho}$.

Now since by definition

$$L \supset \{(z_1, \ldots, z_p) \in \mathbb{C}^p \mid f(z_1, \ldots, z_p) = 0 \ \forall f \in \mathscr{J}\} \supset X(F_n)$$

these three sets will be equal, thus sharpening the statement of Theorem 3.1.

The proof proceeds case by case. For the benefit of the reader we give below a table of cases. In this table, $H_{12} \neq 0$, for instance, means that $H_{12} \neq 0$, where P is a given point of L; etc.

Case 1 $H_{12} \neq 0$, $T_{123} \neq 0$ (see [V] and [M]).

Case 1.1 $M_{11} \neq 0$.

Case 1.2 $M_{11} = M_{22} = 0$.

Case 2 $T_{i_1 i_2 i_3} \neq 0$ implies $H_{i_1 i_2} = H_{i_2 i_3} = H_{i_3 i_1} = 0$.

Case 2.1 $T_{i_1 i_2 i_3} = 0$, for every i_1, i_2, i_3.

Case 2.1.1 $H_{j_1 j_2} = 0$, for every j_1, j_2.

Case 2.1.1.1 $M_{11} \neq 0$.

Case 2.1.1.2 $M_{11} = M_{22} = \ldots = M_{nn} = 0$.

Case 2.1.2 $H_{12} \neq 0$.

Case 2.1.2.1 $M_{11} \neq 0$.

Case 2.1.2.2 $H_{ij} \neq 0$ implies $M_{ii} = M_{jj} = 0$. Moreover $H_{12} \neq 0$.

Case 2.2 $T_{i_1 i_2 i_3} \neq 0$ implies $H_{i_1 i_2} = H_{i_2 i_3} = H_{i_3 i_1} = 0$. Moreover $T_{123} \neq 0$.

Let P be a point $(z_1, \ldots, z_p) \in \mathbb{C}^p$ belonging to L, i.e. a point whose coordinates satisfy the polynomials P(1), P(2), P(3), P(4). We want to show that $P = \bar{\rho}$, where $\rho\colon F_n \to \mathrm{SL}(2, \mathbb{C})$ is a representation; ρ will be defined by a set of n matrices $A_1, \ldots, A_n$.

Case 1 $H_{12}(P) \neq 0$, $T_{123}(P) \neq 0$.

We distinguish two subcases:

Case 1.1 $M_{11}(P) \neq 0$.

Up to conjugation we can choose

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda^2 \neq 1, \quad A_2 = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \quad c = ad - 1 \neq 0$$

(Proposition 4.3(iv)). We have $t_1 = \lambda + \lambda^{-1}$. Therefore $\lambda^2 - t_1 \lambda + 1 = 0$.

Hence

$$\lambda = \frac{t_1 \pm \sqrt{t_1^2 - 4}}{2}, \quad t_1^2 - 4 \neq 0$$

has two values. Without lost of generality we choose for λ anyone of the two, because conjugating $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we obtain $\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$. Therefore A_1 is perfectly determined by t_1.

Now $a+d=t_2$. Therefore

$$d=t_2-a, \quad c=at_2-a^2.$$

To determine a, we use $t_{12}=a(\lambda-\lambda^{-1})+\lambda^{-1}t_2$. Thus

$$a=\frac{t_{12}-\lambda^{-1}t_2}{\lambda-\lambda^{-1}}, \quad (\lambda\neq\lambda^{-1}).$$

Therefore t_1, t_2, t_{12} determine A_1 and A_2. Note that

$$d=\frac{t_{12}-\lambda t_2}{\lambda^{-1}-\lambda}, \quad c\neq 0.$$

Now, putting $A_3=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $\alpha\delta-\beta\gamma=1$, we have

$$\delta=t_3-\alpha.$$

Also $t_{13}=\alpha(\lambda-\lambda^{-1})+\lambda^{-1}t_3$, i.e.

$$a=\frac{t_{13}-\lambda^{-1}t_3}{\lambda-\lambda^{-1}}$$

$$t_{23}=(a-d)\alpha+c\beta+\gamma+dt_3$$

hence

$$c\beta+\gamma=t_{23}-dt_3-(a-d)\alpha.$$

On the other hand, $\alpha\delta-\beta\gamma=1$ implies

$$(c\beta)\gamma=c(\alpha\delta-1).$$

Therefore $c\beta$ and γ are the solutions θ_1, θ_2 of the quadratic equation

$$X^2-(t_{23}-dt_3-(a-d)\alpha)X+c(\alpha\delta-1)=0 \tag{1}$$

whose coefficients are functions of $t_1, t_2, t_3, t_{12}, t_{13}$ and t_{23}. Therefore these six coordinates of P determine two triples for A_1, A_2, A_3 that we display in two rows:

$$A_1=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, \quad \lambda^2\neq 1; \quad A_2=\begin{pmatrix}a & 1\\ c & d\end{pmatrix}, \quad c=ad-1\neq 0; \quad A_3=\begin{pmatrix}\alpha & \theta_1/c\\ \theta_2 & \delta\end{pmatrix},$$

$$A_1'=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}, \quad \lambda^2\neq 1; \quad A_2=\begin{pmatrix}a & 1\\ c & d\end{pmatrix}, \quad c=ad-1\neq 0; \quad A_3=\begin{pmatrix}\alpha & \theta_2/c\\ \theta_1 & \delta\end{pmatrix}.$$

These two triples yield the two solutions, $\mathrm{tr}(A_1A_2A_3)$ and $\mathrm{tr}(A_3A_2A_1)=\mathrm{tr}(A_1^{-1}A_2^{-1}A_3^{-1})$, of the polynomial P(1) (see 4.4(ii)):

$$t_{123}^2-Pt_{123}+Q=0.$$

In fact conjugating the second triple by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we get:

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} d & -c \\ -1 & a \end{pmatrix}, \quad \begin{pmatrix} \delta & -\theta_1 \\ -\theta_2/c & \alpha \end{pmatrix}.$$

And conjugating again by $\begin{pmatrix} c^{-1/2} & 0 \\ 0 & c^{1/2} \end{pmatrix}$ we get

$$A_1^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_2^{-1} = \begin{pmatrix} d & -1 \\ -c & a \end{pmatrix}, \quad A_3^{-1} = \begin{pmatrix} \delta & -\theta_1/c \\ -\theta_2 & \alpha \end{pmatrix}.$$

Therefore $t_{123}(P)$, which is a solution of P(1), determines one of the two triples. But we have obtained more. In fact from the two triples above we obtain:

$$\operatorname{tr}(A_1 A_2 A_3) = \lambda a \alpha + \lambda \theta_2 + \lambda^{-1} \theta_1 + \lambda^{-1} d \delta$$
$$\operatorname{tr}(A_1' A_2' A_3') = \lambda a \alpha + \lambda \theta_1 + \lambda^{-1} \theta_2 + \lambda^{-1} d \delta.$$

Therefore (4.4(ii)):

$$T_{123} = 0 \Leftrightarrow \operatorname{tr}(A_1 A_2 A_3) = \operatorname{tr}(A_1' A_2' A_3') \Leftrightarrow (\lambda - \lambda^{-1})(\theta_2 - \theta_1) = 0$$
$$\Leftrightarrow \theta_2 = \theta_1.$$

Thus, *the discriminant of* (1) is T_{123}.
Combining all that we have obtained at this point, we have:

$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \quad A_2 = \left(\begin{array}{c|c} \lambda_2 & 1 \\ \hline \lambda_2 t_2 - \lambda_2^2 & t_2 - \lambda_2 \end{array}\right)$$

$$\lambda_1 = \frac{t_1 + \sqrt{t_1^2 - 4}}{2}, \quad \lambda_2 = \frac{t_{12} - \lambda_1^{-1} t_2}{\lambda_1 - \lambda_1^{-1}}, \quad c_1 := \lambda_2 t_2 - \lambda_2^2 \neq 0$$

$$A_3 = \begin{pmatrix} \lambda_3 & \beta_1 \\ \gamma_1 & t_3 - \lambda_3 \end{pmatrix}, \quad \lambda_3 = \frac{t_{13} - \lambda_1^{-1} t_3}{\lambda_1 - \lambda_1^{-1}},$$

and

(2)
$$\gamma_1(c_1 \beta_1) = c_1(\lambda_3 t_3 - \lambda_3^2 - 1)$$
$$\gamma_1 + c_1 \beta_1 = t_{23} - (t_2 - \lambda_2) t_3 - \lambda_3(2\lambda_2 - t_2).$$

(2) gives a quadratic equation with discriminant $T_{123} \neq 0$. Its solutions, $\theta_1 \neq \theta_2$, give two possibilities for A_3:

$$\begin{array}{ll} \beta_1 = \theta_1/c_1, & \theta_2/c_1 \\ \gamma_1 = \theta_2 & \theta_1 \end{array},$$

which realize the two possible values of the coordinate $t_{123}(P)$ (solutions of the polynomial $z^2 - Pz + Q = 0$).

Put now

$$A_4=\begin{pmatrix}\lambda_4 & \beta_2\\ \gamma_2 & t_4-\lambda_4\end{pmatrix}$$

and we have analogously, using t_{24}:

$$\lambda_4=\frac{t_{14}-\lambda_1^{-1}t_4}{\lambda_1-\lambda_1^{-1}}$$

$$\gamma_2(c_1\beta_2)=c_1(\lambda_4 t_4-\lambda_4^2-1)$$

$$\gamma_2+c_1\beta_2=t_{24}-(t_2-\lambda_2)t_4-\lambda_4(2\lambda_2-t_2)$$

which yields a quadratic equation, whose solutions θ_1, θ_2 give two possibilities for A_4:

(3)
$$\begin{matrix}\beta_2=\theta_1/c_1; & \theta_2/c_1\\ \gamma_2=\theta_2 & \theta_1.\end{matrix}$$

These two possibilities yield two values for t_{34}, say t'_{34}, t''_{34}. They are:

$$t'_{34}=2\lambda_3\lambda_4+t_3t_4-\lambda_4t_3-\lambda_3t_4+\beta_1\theta_2+\gamma_1\frac{\theta_1}{c_1}$$

$$t''_{34}=2\lambda_3\lambda_4+t_3t_4-\lambda_4t_3-\lambda_3t_4+\beta_1\theta_1+\gamma_1\frac{\theta_2}{c_1}.$$

Therefore

$$t'_{34}=t''_{34}\Leftrightarrow(c_1\beta_1-\gamma_1)(\theta_2-\theta_1)=0\Leftrightarrow\theta_2=\theta_1$$

because $c_1\beta_1\neq\gamma_1$.

We already know that the discriminant of the polynomial given by (3) (compare with (2) is T_{124}. Now, since $T_{123}\neq0$, we have

(4)
$$t'_{34}=t''_{34}\Leftrightarrow T_{123}T_{124}=0.$$

Now, t'_{34}, t''_{34}, as well as the coordinate $t_{34}(P)$, satisfy the same quadratic equation P(2): $Q_{1234}=0$ (Proposition 4.8, where $H_{12}\neq0$) whose discriminant is $T_{123}T_{134}$. The relation (4) shows that t_{34} must coincide with one of t'_{34}, t''_{34} (when they are different). In this way we obtain A_4.

The determination of $A_5, \ldots, A_n$ follows the same pattern as for A_4. In each case we will have two possibilities for the matrix A_b, $b\geqq4$ (which reduce to only one iff $T_{12b}=0$) which realize the two possible values of the coordinate t_{3b} of P, and which are the solutions of a quadratic polynomial $Q_{123b}=0$.

The representation ρ, i.e. the n-tuple $A_1, \ldots, A_n$, is then well determined by the coordinates $(t_1, \ldots, t_n; t_{12}, \ldots, t_{1n}; t_{23}, \ldots, t_{2n}; t_{34}, \ldots, t_{3n}, t_{123})$ of P. Then necessarily, $t_{ab}(\rho)=t_{ab}(P)$, $3<a<b\leqq n$ and $t_{abc}(\rho)=t_{abc}(P)$, $1\leqq a<b<c\leqq n$ because, by hypothesis, P (as well as $\bar{\rho}$) satisfies the polynomials P(1), P(2), P(3), P(4); and in particular, its coordinates $t_{ab}(P)$ ($t_{abc}(P)$, resp.) satisfy the linear polynomials P(3) (resp. P(4)) whose non independent coefficient is $\neq0$ (because $T_{123}\neq0$). This shows that there exists $\rho\colon F_n\to\mathrm{SL}(2,\mathbb{C})$ such that $P=\bar{\rho}$.

Case 1.2 $M_{11}(P)=0$, $M_{22}(P)=0$.

Up to conjugation we can choose

$$A_1=\varepsilon_1\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad \varepsilon_1=\pm 1;\qquad A_2=\varepsilon_2\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix},\quad \varepsilon_2=\pm 1,\quad b\neq 0$$

(Proposition 4.3(iii)).

Here $t_1=\pm 2$, $t_2=\pm 2$ determine ε_1, ε_2. Since $t_{12}=(2+b)\varepsilon_1\varepsilon_2$, $b=\varepsilon_1\varepsilon_2 t_{12}-2\neq 0$. Put

$$A_3=\begin{pmatrix}\lambda_3 & \beta_1\\ \gamma_1 & t_3-\lambda_3\end{pmatrix}$$

$t_{13}=\varepsilon_1(\gamma_1+t_3)$, hence $\gamma_1=\varepsilon_1 t_{13}-t_3$

$$t_{23}=\varepsilon_2(t_3+(\varepsilon_1\varepsilon_2 t_{12}-2)\beta_1),\qquad \beta_1=\frac{\varepsilon_2 t_{23}-t_3}{\varepsilon_1\varepsilon_2 t_{12}-2}.$$

The determinant of A_3 is 1:

$$\lambda_3^2-t_3\lambda_3+\beta_1\gamma_1+1=0. \tag{5}$$

Since

$$\gamma_3+(t_3-\lambda_3)=t_3$$
$$\lambda_3(t_3-\lambda_3)=\beta_1\gamma_1+1$$

if we choose a solution θ for (5), the values for λ_3, $t_3-\lambda_3$ are

$$\begin{matrix}\lambda_3=\theta\\ t_3-\lambda_3=t_3-\theta\end{matrix}\quad\text{or}\quad\begin{matrix}t_3-\theta\\ \theta.\end{matrix}$$

These values yield triples with equal traces $t_1, t_2, t_3, t_{12}, t_{13}, t_{23}$, and traces t'_{123}, t''_{123}, which (as well as the coordinate $t_{123}(P)$) are solutions of (6) $z^2-Pz+Q=0$. If the solution t'_{123} of (6) corresponds to $\left\{A_1, A_2, A_3=\begin{pmatrix}\theta & \beta_1\\ \gamma_1 t_3 & -\theta\end{pmatrix}\right\}$, the triple

$$A_1^{-1}=\varepsilon_1\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix},\quad A_2^{-1}=\varepsilon_2\begin{pmatrix}1 & 0\\ -b & 1\end{pmatrix},\quad A_3^{-1}=\begin{pmatrix}t_3-\theta & -\beta_1\\ -\gamma_1 & \theta\end{pmatrix}$$

or its conjugate by $\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$ in $\mathrm{SL}(2,\mathbb{C})$, i.e.:

$$A_1=\varepsilon_1\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad A_2=\varepsilon_2\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix},\quad A_3''=\begin{pmatrix}t_3-\theta & -\beta_1\\ \gamma_1 & \theta\end{pmatrix}$$

gives (by Proposition 4.4(ii)) the other solution of (6). But $\mathrm{tr}(A_1A_2A_3'')=t''_{123}$. Hence t'_{123}, t''_{123}, are the *two different* solutions of (6) (remember that $T_{123}(P)\neq 0$). This shows that the discriminant of (5) is T_{123}, and also that we can choose A_1, A_2, A_3 realizing the coordinates $(t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123})$ of P.

Put now $A_4=\begin{pmatrix}\lambda_4 & \beta_2\\ \gamma_2 & t_4-\lambda_4\end{pmatrix}$

$$\gamma_2=\varepsilon_1 t_{14}-t_4, \qquad \beta_2=\frac{\varepsilon_2 t_{24}-t_4}{\varepsilon_1\varepsilon_2 t_{12}-2}.$$

Determinant of A_4 equal 1, yields two solutions for λ_4 (which coincide if and only if $T_{124}(P)=0$, as we have shown just before) which give two values t'_{34}, t''_{34} such that

$$t'_{34}=t_{34} \Leftrightarrow T_{124}(P)=0.$$

Again, by Proposition 4.8, t'_{34}, t''_{34} are the two possible values that can assume the coordinate $t_{34}(P)$. The rest of the argument is as in Case 1.1.

Case 2 For each inequality $T_{abc}(P)\neq 0$, *it holds* $H_{ab}(P)=H_{bc}(P)=H_{ca}(P)=0$.

We distinguish two subcases:

Case 2.1 $T_{abc}(P)=0$ *for every triple a, b, c.*

Case 2.1.1 $T_{abc}(P)=0$ *for every triple a, b, c and* $H_{ij}(P)=0$ *for every pair i, j.*

By Proposition 4.4(iii) follows that the n-tuple $A_1, \ldots, A_n$ can be choosen of upper triangular matrices

$$A_1=\begin{pmatrix}\lambda_i & a_i\\ 0 & \lambda_i^{-1}\end{pmatrix}, \qquad i=1, \ldots, n.$$

Notice that $\operatorname{tr}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}a' & b'\\ 0 & d'\end{pmatrix}=\operatorname{tr}\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}a' & 0\\ 0 & d'\end{pmatrix}$.

Therefore, without loss of generality we can assume that $a_i=0$, $i=1, \ldots, n$. Of course, if we find a solution ρ such that $\bar{\rho}=P$, there are infinitely many more, nonconjugate solutions which can be obtained by changing the parameters $a_1, a_2, \ldots, a_n$. There are two subcases:

Case 2.1.1.1 $M_{11}(P)\neq 0$.

We fix $A_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix}$ using $t_1(P)$. Then each $A_a=\begin{pmatrix}\lambda_a & 0\\ 0 & \lambda_a^{-1}\end{pmatrix}$ admits two possibilities, A_a and $A'_a=\begin{pmatrix}\lambda_a^{-1} & 0\\ 0 & \lambda_a\end{pmatrix}$, which yield two values, t'_{1a} and t''_{1a}, for $\operatorname{tr}(A_1 A_a)$:

$$t'_{1a}=\lambda_1\lambda_a+\lambda_1^{-1}\lambda_a^{-1}$$
$$t''_{1a}=\lambda_1\lambda_a^{-1}+\lambda_1^{-1}\lambda_a.$$

Now $t'_{1a}=t''_{1a} \Leftrightarrow (\lambda_1-\lambda_1^{-1})(\lambda_a-\lambda_a^{-1})=0 \Leftrightarrow \lambda_a=\lambda_a^{-1} \Leftrightarrow M_{aa}=0$. Since $H_{1a}(P)=0$, the coordinate $t_{1a}(P)$ is a solution of the quadratic equation $M_{1a}^2=M_{11}M_{aa}$, which has exactly one solution iff $M_{aa}=0$. Therefore t'_{1a}, t''_{1a} realize the two possible values that the coordinate $t_{1a}(P)$ can assume.

Thus, the coordinates $(t_1, \ldots, t_n, t_{12}, \ldots, t_{1n})$ of P determine the representation ρ, i.e. the n-tuple $A_1, \ldots, A_n$.

Now $T_{1ab}(P)=T_{1ab}(\bar{\rho})=0$. Here T_{1ab} is a quadratic polynomial in the variable M_{ab}, $1<a<b\leqq n$, for which the discriminant is $H_{1a}H_{1b}=0$, and the coefficient

of M_{ab}^2 is $M_{11}\neq 0$. The other coefficients of this polynomial are functions of the coordinates $(t_1, t_a, t_b, t_{1a}, t_{1b})$. Therefore $M_{ab}(P)=M_{ab}(\bar{\rho})$. Hence $t_{ab}(P)=t_{ab}(\rho)$, $1<a<b\leqq n$.

By hypothesis $(P\in L)$, and by 4.4(ii), $t_{abc}(P)$, and $t_{abc}(\rho)$, satisfy $t_{abc}^2-P\,t_{abc}+Q=0$, whose discriminant is $T_{abc}=0$; the coefficients P and Q are functions of $(t_1, \ldots, t_n; t_{ab}, 1\leqq a<b\leqq n)$. Therefore $t_{abc}(P)=t_{abc}(\rho)$, for $1\leqq a<b\leqq n$.

Case 2.1.1.2 $M_{11}(P)=M_{22}(P)=\ldots=M_{nn}(P)=0$.

Then $t_{ii}=\pm 2$, $i=1, \ldots, n$. Hence $(t_1, \ldots, t_n)$ determine the representation ρ, i.e. the matrices $A_1, \ldots, A_n$ (but not uniquely, because there exist the parameters $a_1, \ldots, a_n$).

Since, by hypothesis $H_{ab}(P)=0$, and by construction $H_{ab}(\bar{\rho})=0$, $1\leqq a<b\leqq n$, we have $M_{ab}^2=M_{aa}M_{bb}=0$; hence, $M_{ab}=0$. Thus $t_{ab}(P)=t_{ab}(\rho)$. Also $t_{abc}(P)=t_{abc}(\rho)$ as in the previous case.

Case 2.1.2 $T_{abc}(P)=0$ *for every triple* a, b, c *but* $H_{12}(P)\neq 0$. We consider two subcases.

Case 2.1.2.1 $M_{11}(P)\neq 0$.

Since $H_{12}(P)\neq 0$, $T_{123}(P)=0$, we can assume that A_1, A_2 have the form (Proposition 4.3(iv)):

$$A_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix},\quad \lambda_1^2\neq 1;\qquad A_2=\begin{pmatrix}\lambda_2 & 1\\ c & t_2-\lambda_2\end{pmatrix},\quad c\neq 0.$$

Put $A_j=\begin{pmatrix}\alpha_i & \beta_j\\ \gamma_j & \delta_j\end{pmatrix}$. Since $T_{12j}=0$, we use Proposition 4.11 to have

$$T_{12j}=0\Leftrightarrow\begin{vmatrix}\lambda_1 & 0 & 0 & \lambda_1^{-1}\\ \lambda_2 & 1 & c & t_2-\lambda_2\\ \alpha_j & \beta_j & \gamma_j & \delta_j\\ 1 & 0 & 0 & 1\end{vmatrix}=0\Leftrightarrow(\gamma_j-c\,\beta_j)(\lambda_1-\lambda_1^{-1})=0\Leftrightarrow\gamma_j=c\,\beta_j.$$

Therefore we put $A_1=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_1^{-1}\end{pmatrix}$, $\lambda_1^2\neq 1$, λ_1 a fixed solution of $\lambda_1^2-t_1\lambda_1+1=0$

$$A_2=\begin{pmatrix}\lambda_2 & 1\\ c & t_2-\lambda_2\end{pmatrix},\qquad \lambda_2=\frac{t_{12}-\lambda_1^{-1}t_2}{\lambda_1-\lambda_1^{-1}},\qquad c=-\lambda_2^2+\lambda_2 t_2-1\neq 0$$

$$A_3=\begin{pmatrix}\lambda_3 & \beta_1\\ c\,\beta_1 & t_3-\lambda_3\end{pmatrix},\qquad \lambda_3=\frac{t_{13}-\lambda_1^{-1}t_3}{\lambda_1-\lambda_1^{-1}},\qquad \beta_1^2=\frac{\lambda_3(t_3-\lambda_3)-1}{c}.$$

Thus, there are two possible values β_1', β_1'' for β_1, which give two matrices A_3', A_3'' and two values t_{23}', t_{23}'' for trace $(A_2\,A_3)$:

$$t_{23}'=\lambda_2\lambda_3+2c\,\beta_1'+(t_2-\lambda_2)(t_3-\lambda_3)$$

$$t_{23}''=\lambda_2\lambda_3+2c\,\beta_1''+(t_2-\lambda_2)(t_3-\lambda_3).$$

Then

$$t_{23}'=t_{23}''\Leftrightarrow\beta_1'=\beta_1''\Leftrightarrow\lambda_3(t_3-\lambda_3)=1\Leftrightarrow t_{13}t_3t_1-t_3^2-t_{13}^2-t_1^2+4=0\Leftrightarrow H_{13}=0.$$

Now, the matrices (A_1, A_2, A_3'), (A_1, A_2, A_3'') and also the coordinates of P satisfy $T_{123}=0$. By Proposition 4.5 this is a quadratic polynomial in M_{23} such that the coefficient of M_{23}^2 is $M_{11}\neq 0$, and the discriminant is $H_{12}H_{13}$. Since $H_{12}\neq 0$, the solutions coincide iff $H_{13}=0$. Thus, the solutions are t_{23}', t_{23}''. In this way, the two values of β_1 give two triples $\{A_1, A_2, A_3\}$ which realize the possible values of the coordinate $t_{23}(P)$.

The argument is the same for the rest of the matrices; and, at the end, the coordinates $(t_1, \ldots, t_n, t_{12}, \ldots, t_{1n}, t_{23}, \ldots, t_{2n})$ are enough to determine the n-tuple $(A_1, \ldots, A_n)$, i.e. the representation ρ.

It remains to show that the other coordinates of P coincide with the corresponding coordinates of $\bar{\rho}$.

Certainly t_{abc}, $1\leqq a<b<c\leqq n$, is uniquely determined as double solution of

$$z^2-Pz+Q=0$$

(remember $T_{abc}=0$). Let us consider t_{ab}, $3\leqq a<b\leqq n$. M_{ab} satisfies the quadratic polynomial P(2) in which the term M_{ab}^2 has coefficient $H_{12}\neq 0$, and the discriminant is $T_{12a}T_{12b}=0$; the coefficients of the polynomial are function of the traces already known, namely $t_1, \ldots, t_{2n}$. Thus t_{ab} is uniquely determined.

Case 2.1.2.2 For each inequality of the form $H_{ij}(P)\neq 0$ *we have* $M_{ii}(P)=M_{jj}(P)=0$. *Moreover* $H_{12}(P)\neq 0$.

Since $H_{12}(P)\neq 0$, $T_{123}(P)=0$, we can assume that A_1, A_2 have the form (Proposition 4.3(iii)):

$$A_1=\varepsilon_1\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad 2\varepsilon_1=t_1; \quad A_2=\varepsilon_2\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}, \quad b\neq 0, \quad 2\varepsilon_2=t_2.$$

Put $A_j=\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix}$. Since $T_{12j}=0$, we use Proposition 4.11 to obtain

$$T_{12j}=0 \Leftrightarrow \varepsilon_1\varepsilon_2\begin{vmatrix}1 & 1 & 0 & 1\\ 1 & 0 & b & 1\\ \alpha_j & \beta_j & \gamma_j & \delta_j\\ 1 & 0 & 0 & 1\end{vmatrix}=0 \Leftrightarrow (\delta_j-\alpha_j)\,b=0 \Leftrightarrow \delta_j=\alpha_j.$$

Therefore we put

$$A_1=\varepsilon_1\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}; \quad A_2=\varepsilon_2\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}; \quad A_j=\begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \alpha_j\end{pmatrix}$$

where $b=\varepsilon_1\varepsilon_2 t_{12}-2\neq 0$.

If $M_{jj}\neq 0$ for some index j, then the hypothesis of this case implies $H_{1j}=H_{2j}=0$. Since $M_{11}=M_{22}=0$, we then have $M_{1j}=M_{2j}=0$. But then, the proof of Proposition 4.5 puts the equation $T_{12j}=0$ in the form:

$$M_{11}M_{2j}^2-2(M_{12}M_{1j})M_{2j}-M_{11}M_{22}M_{jj}+M_{1j}^2M_{22}+M_{12}^2M_{jj}=0$$
$$\Leftrightarrow M_{12}^2M_{jj}=0;$$

which is impossible because we are assuming $M_{jj} \neq 0$, and $H_{12} \neq 0$ implies $M_{12} \neq 0$ (since $M_{11} = M_{22} = 0$). Therefore $M_{11} = M_{22} = \ldots = M_{nn} = 0$ in the present case.

Since $M_{jj} = 0$ we have $2\alpha_j = \pm 2$ or $\alpha_j = \pm 1$. This implies that β_j or γ_j is zero. Therefore A_j has one of two forms:

$$A_j = \varepsilon_j \begin{pmatrix} 1 & 0 \\ \gamma_j & 1 \end{pmatrix} \quad \text{or} \quad \varepsilon_j \begin{pmatrix} 1 & \beta_j \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon_j = \pm 1.$$

Now, Proposition 4.5, just mentioned above, yields the equation:

$$-2 M_{12} M_{1j} M_{2j} = 0$$

which implies $M_{1j} M_{2j} = 0$ (since $M_{12} \neq 0$). If, for instance, P has $M_{1j} = 0$, then $H_{1j} = M_{11} M_{jj} - M_{1j}^2 = 0$, which implies that

$$A_j = \varepsilon_j \begin{pmatrix} 1 & \beta_j \\ 0 & 1 \end{pmatrix}.$$

Then, $t_{2j} = \varepsilon_2 \varepsilon_j (1 + b\beta_j + 1)$ gives

$$\beta_j = \frac{\varepsilon_2 \varepsilon_j t_{2j} - 2}{b}$$

while $M_{1j} = 0$ implies that $t_{1j} = \frac{t_1 t_j}{2}$, and t_{1j} is uniquely determined. If P has $M_{2j} = 0$, then t_{2j} is uniquely determined, while t_{2j} determines γ_j in A_j. To see that the other coordinates of P coincide with the corresponding coordinates of $\bar{\rho}$ use the same method as in the previous Case 2.1.2.1.

Case 2.2 For each inequality $T_{i_1 i_2 i_3}(P) \neq 0$ *holds* $H_{i_1 i_2}(P) = H_{i_2 i_3}(P) = H_{i_3 i_1}(P) = 0$. *Moreover* $T_{123}(P) \neq 0$.

Since $M_{11}(P) \neq 0$ (Corollary 4.5.2), we can assume that A_1, has the form: $A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$, $\lambda_1^2 \neq 1$, λ_1 is a solution of $X^2 - t_1 X + 1 = 0$.

Since $H_{12}(P) = 0$, A_2 has a zero in its second diagonal.

There are four cases (up to conjugation):

$$(7) \qquad A_2' = \begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_2^{-1} \end{pmatrix} \quad A_2'' = \begin{pmatrix} \lambda_2^{-1} & 1 \\ 0 & \lambda_2 \end{pmatrix}$$

$$A_2''' = \begin{pmatrix} \lambda_2 & 0 \\ 1 & \lambda_2^{-1} \end{pmatrix} \quad A_2^{\mathrm{iv}} = \begin{pmatrix} \lambda_2^{-1} & 0 \\ 1 & \lambda_2 \end{pmatrix}$$

where $\lambda_2^2 \neq 1$ because $M_{22}(P) \neq 0$. There are four traces $\mathrm{tr}(A_1 A_2)$ namely t_{12}', t_{12}'', t_{12}''', t_{12}^{iv}. Clearly $t_{12}' = t_{12}'''$, $t_{12}'' = t_{12}^{\mathrm{iv}}$. Clearly $t_{12}' = t_{12}''$ if and only if $(\lambda_1 - \lambda_1^{-1})(\lambda_2 - \lambda_2^{-1}) = 0$ which is not the case. Therefore t_{12}', t_{12}'' realize the two possible values that $t_{12}(P)$ can assume (namely: the solutions of the quadratic equation $H_{12}(P) = 0 \Leftrightarrow M_{11}(P) M_{22}(P) = M_{12}^2(P)$). Thus $t_{12}(P)$ serves to detect

the column of (7) in which A_2 falls. Without lost of generality, assume $A_2 = A_2'$ or A_2'''.

Since $H_{13}(P)=0$, A_3 has a zero in its second diagonal. Again, there are four cases, and $t_{13}(P)$ serves to detect the order of the diagonal elements of A_3. Without lost of generality, assume A_3 is of one of two forms:

$$A_3' = \begin{pmatrix} \lambda_3 & \gamma_3 \\ 0 & \lambda_3^{-1} \end{pmatrix} \quad \text{or} \quad A_3''' = \begin{pmatrix} \lambda_3 & 0 \\ \gamma_3 & \lambda_3^{-1} \end{pmatrix}, \qquad \lambda_3^2 \neq 1,$$

since $M_{33}(P) \neq 0$.

But, being $T_{123}(P) \neq 0$, there are two possibles triples (use Proposition 4.11) that we put in two rows:

$$A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \quad \lambda_1^2 \neq 1; \quad A_2 = \begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \quad \lambda_2^2 \neq 1; \quad A_3 = \begin{pmatrix} \lambda_3 & 0 \\ \gamma_3 & \lambda_3^{-1} \end{pmatrix}, \quad \lambda_3^2 \neq 1.$$

$$A_1' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \quad \lambda_1^2 \neq 1; \quad A_2' = \begin{pmatrix} \lambda_2 & 0 \\ 1 & \lambda_2^{-1} \end{pmatrix}, \quad \lambda_2^2 \neq 1; \quad A_3' = \begin{pmatrix} \lambda_3 & \gamma_3 \\ 0 & \lambda_3^{-1} \end{pmatrix}, \quad \lambda_3^2 \neq 1.$$

The value of γ_3 is determined by $t_{23}(P)$. In fact

$$t_{23}(P) = \lambda_2 \lambda_3 + \lambda_2^{-1} \lambda_3^{-1} + \gamma_3 .$$

The triples (A_1, A_2, A_3), (A_1', A_2', A_3') give the two solutions $\operatorname{tr}(A_1 A_2 A_3)$ and $\operatorname{tr}(A_3 A_2 A_1) = \operatorname{tr}(A_1^{-1} A_2^{-1} A_3^{-1})$ of

$$t_{123}^2 - P t_{123} + Q = 0.$$

In fact, conjugating the second triple by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we get

$$\left\{ \begin{pmatrix} \lambda_1^{-1} & 0 \\ 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} \lambda_2^{-1} & -1 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda_3^{-1} & 0 \\ -\gamma_3 & \lambda_3 \end{pmatrix} \right\} = \{A_1^{-1}, A_2^{-1}, A_3^{-1}\}.$$

Therefore $t_{123}(P)$ determines the triple. Let us assume that it is the first. We want to determine A_i, $i > 3$. First, let us prove that $H_{1a}(P) = H_{2a}(P) = H_{3a}(P) = 0$, for every a. In fact if $T_{12a}(P) \neq 0$, the hypothesis of this case implies $H_{1a}(P) = H_{2a}(P) = 0$. If $T_{12a}(P) = 0$, use 4.8.1 to conclude that $H_{1a}(P) = H_{2a}(P) = 0$. Analogously, $T_{13a}(P) \neq 0$ implies $H_{3a}(P) = 0$; but if $T_{13a}(P) = 0$, again 4.8.1 implies $H_{3a}(P) = 0$. Therefore $H_{1i}(P) = 0$, $i > 3$. Thus A_i has a zero in the second diagonal. The values of $t_i(P)$ and $t_{1i}(P)$ determine that A_i has one of three forms:

$$A_i' = \begin{pmatrix} \lambda_i & \gamma_i \\ 0 & \lambda_i^{-1} \end{pmatrix}, \quad \lambda_i^2 \neq 1; \quad A_i'' = \begin{pmatrix} \lambda_i & 0 \\ \gamma_i & \lambda_i^{-1} \end{pmatrix}, \quad \lambda_i^2 \neq 1; \quad A_i''' = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}$$

(the value of $t_{1i}(P)$ serves to determine the order of the elements on the principal diagonal).

Now, we have $T_{12i}(P)\, T_{13i}(P) = 0$. In fact, if $T_{12i}(P) \neq 0$, $T_{13i}(P) \neq 0$, the hypotheses of this case imply $H_{12}(P) = H_{2i}(P) = H_{1i}(P) = H_{13}(P) = H_{3i}(P) = 0$. Then, 4.8.2 implies $Q_{123i}(P) \neq 0$ for $i > 3$, which is not true for $P \in L$.

If $T_{12i}(P) = 0$, $T_{13i}(P) \neq 0$, then $A_i = A_i'$; if $T_{12i}(P) \neq 0$, $T_{13i}(P) = 0$, then $A_i = A_i''$; and if $T_{12i}(P) = T_{13i}(P) = 0$, then $A_i = A_i'''$. If $A_i = A_i'$, $t_{3i}(P)$ determines $\gamma_i (\lambda_3 \lambda_i$

$+\lambda_3^{-1}\lambda_i^{-1}+\gamma_3\gamma_i=t_{3i}(P)$), while $t_{2i}(P)$ is uniquely determined by $T_{12i}(P)=0$ (4.5, with $M_{11}(P)\neq 0$, $H_{12}(P)H_{1i}(P)=0$). If $A_i=A_i'$, $t_{2i}(P)$ determines γ_i, while $t_{3i}(P)$ is uniquely determined by $T_{13i}(P)=0$. In the case $A_i=A_i'''$, both $t_{2i}(P)$, $t_{3i}(P)$ are uniquely determined by $T_{12i}(P)=0$ and $T_{13i}(P)=0$. In this way the n-tuple $A_1, \ldots, A_n$, i.e. the representation ρ, is determined. We have to see that the other coordinates of P coincide with the corresponding ones of $\bar{\rho}$. The value of t_{ab}, $3<a<b\leqq n$ is uniquely determined by P(3) ($T_{123}\neq 0$). The value of t_{abc}, $1\leqq a<b<c\leqq n$ (except t_{123}) is uniquely determined by P(4) ($T_{123}\neq 0$).

This ends the proof of the theorem.

6 Example (compare with [HLM3])

A finite system of equations for $X(F_n)$ has been obtained in §4 and §5. Using the procedure in the proof of Theorem 3.2 it is possible to exhibit a finite set of equations for $X(G)$ where $G=\pi_1(S^3\backslash K)$, and K is a 2-bridge knot or link (see [W] and [Bu], [HLM1, HLM2, HLM3]). For instance, if K is the figure-eight knot, the variety

$$L:=X(G)$$

is (see [W], compare [HLM1] and [HLM3]):

$$\{(y,z)\in\mathbb{C}^2 \mid (y-z-2)(z^2-(1+y)z+2y-1)=0\}$$

where $y=t_1^2$, $z=t_{12}$. Note that $\pi_1(S^3\backslash K)$ has a presentation with 2 generators (meridians), and a relation of even length. Therefore if $\rho\colon \pi_1(S^3\backslash K)\to \mathrm{SL}(2,\mathbb{C})$ is a representation, so is $\rho'(m_i)=-\rho(m_i)$ $i=1,2$. Since $t_1(\rho)=-t_1(\rho')$ while $t_{12}(\rho)=t_{12}(\rho')$, we see that the equation of L is invariant under the involution $(t_1,t_{12})\to(-t_1,t_{12})$. Thus t_1 appears with even powers in the equation. This is the reason for selecting $y=t_1^2$ in the above equation.

Note that the equation has real coefficients (in fact, rational). This agrees with the fact that if ρ is a representation so is $\hat{\rho}$ defined by $\hat{\rho}(m_i)=\overline{\rho(m_i)}$ which conjugates the entries of $\rho(m_i)$. Thus L is invariant under $(t_1,t_2)\to(\overline{t_1},\overline{t_{12}})$ and so, the coefficients of the above equation are real.

Every knot has the *abelian representations:*

$$\pi_1(S^3\backslash K)\to H_1(S^3\backslash K)\cong Z$$

obtained by $\rho(m_i)=A$, a matrix of $\mathrm{SL}(2,\mathbb{C})$, for every meridian m_i. In our figure-eight knot we have

$$z=t_{12}(\rho)=\mathrm{trace}\,\rho(m_1)\rho(m_2)=\mathrm{trace}\,A^2=(\mathrm{trace}\,A)^2-2=t_1(\rho)^2-2=y-2.$$

This shows that the line $y-z-2=0$ of L corresponds to the abelian representations.

The *reducible representations* (see [CS]) correspond to

$$(y,z)\in L\cap\{H_{12}=0\}:=R.$$

Now $H_{12}(y,z)=\begin{vmatrix} M_{11} & M_{12}\\ M_{12} & M_{22}\end{vmatrix}=\begin{vmatrix} y-4 & 2z-y\\ 2z-y & y-4\end{vmatrix}=-4(z^2-yz+2y-4).$

The it is easy to check that R is $y-z-2=0$. Certainly the abelian representations are reducible, but there might be more; namely the so called *reducible metabelian representations.* In fact, if $\rho\colon \pi_1(S^3\backslash K)=G\to\mathrm{SL}(2,\mathbb{C})$ is reducible the commutator G' is represented by parabolic elements. The cases in which

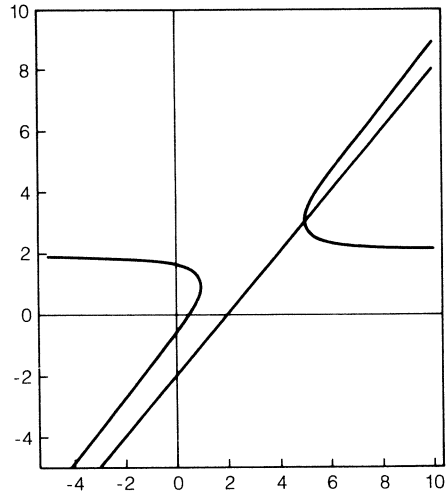


Fig. 1

$\rho(G')=0$ are the abelian representations. If $\rho(G')\neq 0$, the commutator of $\rho(G)$ is abelian, we say that ρ is *reducible metaabelian*. To obtain these representations, put

$$G=|a, c, d: a^{-1}ca=d;\ a^{-1}da=dc^{-1}d^2|$$

where $c, d\in G'$, a is a meridian of K. Then

$$\rho(c)=\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix},\quad \rho(d)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad \rho(a)=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}.$$

Then

$$\left.\begin{aligned} &a^{-1}ca=d \Rightarrow \lambda B^2=1\\ &a^{-1}da=dc^{-1}d^2 \Rightarrow B^2=3-\lambda\end{aligned}\right\}\Rightarrow \lambda^2-3\lambda+1=0$$

then $(\lambda, \lambda^{-1})=\left(\frac{3\pm\sqrt{5}}{2}, \frac{3\mp\sqrt{5}}{2}\right)$.

The image of the meridians $m_1=a$ and $m_2=c^{-1}a$ is

$$\rho(m_1)=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix},\quad \rho(m_2)=\begin{pmatrix}A & -\lambda B\\ 0 & B\end{pmatrix}$$

where

$$A=\pm\sqrt{\frac{3\mp\sqrt{5}}{2}},\quad B=\pm\sqrt{\frac{3\pm\sqrt{5}}{2}}.$$

Then $y=t_1^2=(A+B)^2=\frac{3\mp\sqrt{5}}{2}+\frac{3\pm\sqrt{5}}{2}+2=5$

$$z=t_{12}=A^2+B^2=3.$$

Therefore there are essentially two reducible metabelian representations of $\pi_1(S^3\backslash K)$ in $\mathrm{SL}(2, \mathbb{C})$ which correspond to the point $(5, 3)\in L$. This point is the intersection of the line $(y-z-2)=0$ with the hyperbola:

$$z^2-(1+y)z+2y-1=0.$$

The *irreducible representations correspond* to the points of the hyperbola, except (5, 3). In fact it is impossible that the same point of L corresponds, simultaneously to a reducible and an irreducible representation [CS]. As a curiosity note that there exist an irreducible metabelian representation (see [HLM1]).

For the irreducible representations we consider two cases.

Case 1 $M_{11}(P)=0$. Thus $\rho(m_1)=\rho(m_2)$ are both parabolic. This happens for $y=4$. Then $z^2-5z+7=0 \Rightarrow z=\dfrac{5\pm i\sqrt{3}}{2}$. The representation is $\rho(m_1)=\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\rho(m_2)=\pm\begin{pmatrix}1 & 0\\ \omega & 1\end{pmatrix}$, where $\dfrac{5\pm i\sqrt{3}}{2}=2+\omega \Rightarrow \omega=\dfrac{1\pm i\sqrt{3}}{2}$.

These are four representations in $SL(2,\mathbb{C})$, which yield two in $PSL(2,\mathbb{C})$. These two are equal up to complex-conjugation (orientation reversing isometry in hyperbolic 3-space), in concordance with Mostow's Theorem (note that these parabolic representations are the only faithful and discrete representation of G in $SL(2,\mathbb{C})$).

Case 2 $M_{11}(P)\neq 0$. Then $\rho(m_1)=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$, $(\lambda+\lambda^{-1})^2=y \Leftrightarrow \lambda=\dfrac{\sqrt{y}+\sqrt{y-4}}{2}$;

$\rho(m_2)=\begin{pmatrix}\mu & 1\\ c & \sqrt{y}-\mu\end{pmatrix}$ where $\mu=\dfrac{z-\lambda^{-1}\sqrt{y}}{\lambda-\lambda^{-1}}$, $c=\mu(\sqrt{y}-\mu)-1$.

Note that the representation degenerates iff $c=0 \Leftrightarrow \mu=\lambda$, i.e. for the reducible metabelian representation found before.

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Minimally knotted embeddings of planar graphs

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A graph Γ is *planar* if it can be embedded into a 2-sphere. An embedding φ of Γ into 3-sphere S^3 is *planar* or *unknotted* if there is a 2-sphere in S^3 containing $\varphi(\Gamma)$. Otherwise it is knotted. An embedding φ of Γ into S^3 is called *minimally knotted* if $\varphi(\Gamma)$ is knotted, but all proper subgraphs of $\varphi(\Gamma)$ are unknotted. A free edge of Γ is an edge e such that e has distinct endpoints, and one of its endpoints is incident to no other edges of Γ. Clearly, if Γ has a free edge, or if it has an isolated vertex, then it has no minimally knotted embeddings. It was conjectured [2, 3] that all other planar graphs admit minimally knotted embeddings. A proof of this conjecture was given by Kawauchi in [1], using "hyperbolic imitations".

Theorem 1 *If Γ is a planar graph with no free edges or isolated vertices, then it admits a minimally knotted embedding into S^3.*

Notice that the graph in the theorem is not necessarily connected. From the proof it is clear that there are actually infinitely many minimally knotted embeddings. Some special cases have also been proved by Simon and Wolcott [3]. The present note is to give an alternative proof of this theorem, which seems simpler and more natural. The idea is to consider some surfaces in the graph exterior, and show that they are incompressible. It then follows that the graph is unknotted only if each manifold piece obtained by cutting the graph exterior along these surfaces has free fundemental groups. This will reduce the knottedness of the embedding to a "local" problem.

We need some basic facts about decomposable graphs. A graph G in S^3 is called decomposable if there is a 2-sphere S in S^3 such that $S \cap G$ is a vertex v_0 of G, and both sides of S contain edges of G incident to v_0. We call v_0 a decomposing vertex, and S a decomposing sphere of G. Choose a regular neighborhood $N(v)$ for each vertex v of G, and choose a regular neighborhood $N(e)$ for each edge, so that the disks $N(e) \cap \partial N(v)$ are mutually disjoint. Then $N(G) = (\cup N(e)) \cup (\cup N(v))$ is a regular neighborhood of G in S^3. Denote by $E(G) = S^3 - \operatorname{Int} N(G)$ the exterior of G, and by $\delta(v_0)$ the punctured sphere $\partial N(v_0) - \operatorname{Int} N(G)$ on the boundary of $E(G)$. A decomposing sphere of Γ at v_0 gives

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rise to a compressing disk of $\delta(v_0)$ in $E(G)$. Conversely, any compressing disk of $\delta(v_0)$ can be extended to a decomposing sphere of Γ at v_0. Therefore we have the following

Lemma 1 *v_0 is a decomposing vertex if and only if $\delta(v_0)$ is compressible in $E(G)$.*

Now suppose e_1, e_2 are two edges of G. They are called parallel if there is a map φ: $D^2 \to S^3$ such that $\varphi(\partial D^2) = e_1 \cup e_2$, and $\varphi|_{\text{Int } D^2}$ is an embedding of Int D^2 into $S^3 - G$. A subgraph H of G is a reduction of G if $G - H$ consists of some edges, each of which is parallel to some edge in H. We say that H is obtained from G by removing some parallel edges. An edge of G is a cycle if its two endpoints are on the same vertex.

Lemma 2 *Let H be reduction of a graph G, and suppose every component of G has at least 2 vertices. If G is decomposable, then H is also decomposable.*

Proof. By induction we need only to prove the case that H is obtained from G by removing a single edge e' which is parallel to some edge e''. Let S be a decomposing sphere of G, and let B_1, B_2 be the 3-balls with $B_1 \cap B_2 = S$. Suppose $v_0 = S \cap G$. By definition, there are edges e_i in B_i incident to v_0. Since any component has at least two vertices, we can choose e_2, say, to be a noncycle edge. If $e' \neq e_i$, $(i = 1, 2)$, then it is clear that after removing e' the sphere S is still a decomposition sphere, and we are done. So suppose $e' = e_1$. (The other case is similar). Since e'' is parallel to e', by an isotopy of e'' we may assume that e'' also lies in B_1. Now e_1 and e_2 are not parallel because e_2 must have an endpoint in the interior of B_2, while e_1 has both endpoints in B_1. Thus $e_2 \neq e''$. The two edges e_2 and e'' of H lie on different sides of S, therefore S is a decomposing sphere of H. □

Lemma 3 *Let e be a cycle edge of G bounding an embedded disk D in S^3 so that* Int D *intersects G transversely at a single point v on some edge e'. Let H be the graph obtained from $G \cup D$ by shrinking D into a single vertex. If G is decomposable, then so is H.*

Proof. Let S be a decomposing sphere of G, intersecting G at a vertex v_0, isotoped so that $S \cap D$ is a union of some circles, and different circles intersect only possibly at $v' = \partial e$. We choose S so that the number of circles in $S \cap D$ is minimal.

Suppose $S \cap \text{Int } D \neq \emptyset$. Let C be a circle of $S \cap D$ which is innermost on the sphere S, i.e. it bounds a disk D' in S with interior disjoint from D. Denote by D'' the disk bounded by C in D.

If D'' intersects e', then after shrinking D to a point $D' \cup D''$ becomes a sphere $\bar{S}$ such that the two edges of H coming from e' are on different sides of $\bar{S}$, so $\bar{S}$ is a decomposing sphere of H in S^3, and the result follows. If D'' does not intersect e', then $D' \cup D'' = S'$ is a sphere in S^3 which is either disjoint from G or intersects G at v_0. We get contradictions as follows:

If on both sides of S' there are edges incident to v_0, then S' is a decomposing sphere of G which can be isotoped off the interior of D, contradicting the choice of S. So we assume S' bounds a 3-ball B such that no edges of G in B is incident to v_0. Using B one can find an isotopy of S in S^3 which fixes v_0 and pushes the rest of $S \cap B$ (including D') out of B through the disk D'', resulting in a sphere $\bar{S}$ whose intersection with D has fewer circles than S does. This isotopy may intersect the components of G lying in B. But since none of these components has edges incident to v_0, any two edges of G lying on different

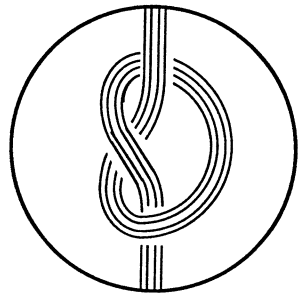


Fig. 1

sides of S are still on different sides of $\bar{S}$, so $\bar{S}$ is a decomposing sphere of G. It contradicts the minimality of the number of circles in $S \cap D$.

We now suppose $S \cap \text{Int } D = \emptyset$. If S is also disjoint from $v' = \partial e$, then $S \cap D = \emptyset$, and it is obvious that S remains a decomposing sphere of H. If S intersects v', then $v' = v_0 = S \cap G$. Let B_1, B_2 be the 3-balls such that $B_1 \cap B_2 = S$, and suppose D lies in B_1. By assumption there are edges of G in B_2 which are incident to v_0. These edges are also edges of H. After shrinking D into a point, the edge e' becomes two edges, both lie in B_1 and are incident to v_0. Since on both sides of S there are edges of H incident to v_0, S is a decomposing sphere of H. □

An n-strand tangle is a pair (B, A), where $B = D^2 \times I$ is a 3-ball, and A is the union of n properly embedded, mutually disjoint arcs with $\partial A \subset \partial D^2 \times \{1/2\}$. Usually we present a diagram of a tangle by drawing the graph $A \cup (\partial D^2 \times \{1/2\})$ so that $\partial D^2 \times \{1/2\}$ is the outside circle of the diagram. Two tangles (B, A) and (B, A') are equivalent if A is isotopic to A' rel ∂A. A tangle (B, A) is trivial if it has a diagram with no crossings, i.e. A can be isotoped (rel ∂A) to lie in $D^2 \times \{1/2\}$.

Let T_n be a tangle (B, A), where A consists of 2^n parallel arcs, each of which is a "trefoil" arc in B, i.e. an arc whose exterior in B is homeomorphic to the trefoil knot exterior in S^3. See Fig. 1 for a diagram of T_2. Let $M_n = B - \text{Int } N(A)$, and let F_n be the punctured sphere $\partial B \cap M_n$.

Lemma 4 *F_n is incompressible in M_n, and $\pi_1 M_n$ is not a free group.*

Proof. Let X be the exterior of a trefoil knot in S^3, let $Y = P_n \times I$, where P_n is a disk with 2^n holes, and I is the unit interval. Denote by C the annulus (*outer circle of* P_n) $\times I$. Then M_n is homeomorphic to $X \cup_C Y$, where C is identified with a regular neighborhood of a meridian on ∂X. By an innermost circle outermost arc argument it can be shown that any compressing disk of F_n in M_n can be isotoped off C, so it lies in either X or Y. But clearly $F_n \cap X$ and $F_n \cap Y$ are both incompressible. Therefore F_n is incompressible in M_n. Also, since C is incompressible in M_n, the trefoil knot group $\pi_1 X$ is a subgroup of $\pi_1 M_n$. Since $\pi_1 X$ is not a free group, $\pi_1 M_n$ is not either. □

Construct a tangle $Q_n = (B_n, A_n) = (B_n, a_1 \cup \ldots \cup a_n)$ as in Fig. 2, where V_k is the tangle T_k above when $k < n$, and $V_n = T_{n-1}$. Notice that if we replace a_1 by a trivial arc a_1' near ∂B_n with $\partial a_1' = a_1$, we get a trivial tangle $U_n = (B_n, A_n') = (B_n, a_1' \cup a_2 \cup \ldots \cup a_n)$.

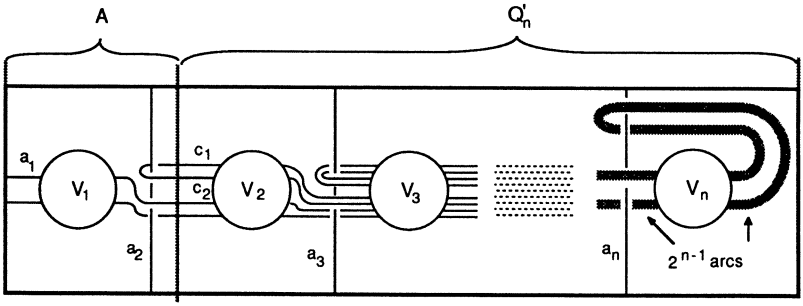


Fig. 2

Lemma 5 *The tangle* $(B_n, A_n - a_i)$ *is equivalent to the trivial tangle* $(B_n, A'_n - a_i)$ *for all* $i \geqq 2$.

Proof. This is obvious when $n=2$. In general we need to show that when removing a_i from B_n, the arc a_1 can be isotoped into a'_1 without intersecting the other a_j's$(j \neq 1, i)$. We notice the following relation between Q_n and Q_{n-1}: Let $b_1, a_3, \ldots a_n$ be the arcs of Q_{n-1}, where b_1 is the one with endpoints on the left hand side of B_{n-1}. Replacing b_1 by two parallel arcs c_1, c_2, we get the tangle $Q'_n=(B'_n, C_n)=(B'_n, c_1 \cup c_2 \cup a_3 \cup \ldots a_n)$ in Fig. 2. So Q_n is obtained by gluing the tangle A in Fig. 2 to the left hand side of Q'_n.

By induction, after removing a_i from $Q'_n (i \geqq 3)$, the arcs c_1, c_2 are isotopic to some trivial arcs c'_1, c'_2 near the left hand side boundary of B'_n. By another induction, it can be shown that c'_1, c'_2 may be chosen to have no intersection in the projection diagram, i.e. $(B'_n, C_n - a_i)$ is a trivial tangle. So a_1 can be further isotoped off the tangle V_1, and becomes a trivial arc in Q_n. Also, since c_1, c_2 are parallel, there is a disk D in Q'_n with $\partial D = c_1 \cup c_2 \cup \gamma_1 \cup \gamma_2$, where γ_1, γ_2 are two small arcs on $\partial Q'_n$ connecting the ends of c_1 to that of c_2, and $\operatorname{Int} D \cap a_j = \emptyset$, $j \neq 1$. Hence when deleting a_2 from Q_n, the arc a_1 can also be isotoped through D so that it is disjoint from Q'_n, and a further isotopy will modify it into a trivial arc near the left hand side of B_n. □

Now let Γ be a planar graph in S^3 with no free edges. We want to find another embedding of Γ which is minimally knotted. If some component of Γ has only one vertex, we can add a vertex to the middle of an edge in that component, dividing it into two edges. This will not affect any planarity, and a minimally knotted embedding of this new graph corresponds to a minimally knotted embedding of Γ. Therefore, without loss of generality, we may further make the following

Assumption. *Every component of* Γ *has at least two vertices and two edges.*

Label the edges as $e_1, \ldots, e_n$, so that e_1 and e_n have a common endpoint. Let p_i be the middle point of e_i. Choose an arc C embedded in S^3, so that C passes through $p_1, \ldots, p_n$ successively with endpoints on p_1 and p_n, and has no other intersections with Γ. We consider $\Gamma \cup C$ as a graph in the natural way.

Lemma 6 *If* Γ *has no free edges, then* $\Gamma \cup C$ *is indecomposable.*

Proof. Since $\partial C = \{p_1, p_n\}$ are midpoints of e_1, e_n. Since e_1 and e_n have a common vertex, we can connect p_1 and p_n by an arc in $e_1 \cup e_n$ to get an embedded

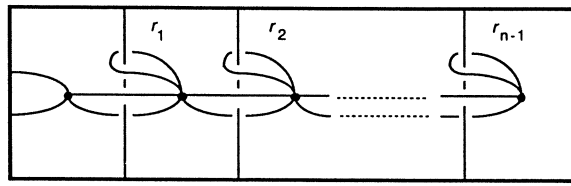


Fig. 3

circle J in $\Gamma \cup C$ containing C. Let S be a decomposing sphere of $\Gamma \cup C$ containing a vertex v_0 of $\Gamma \cup C$, let B_1, B_2 be 3-balls such that $B_1 \cap B_2 = S$. Since J is a circle, it is contained in B_1, say. Therefore $(\Gamma \cup C) \cap B_2 = \Gamma \cap B_2$ is a subset of Γ since the vertices of $\Gamma \cup C$ lie in Γ.

If v_0 is a vertex of Γ, then v_0 is not a point of C, so C is disjoint from any edge of Γ in B_2, contradicting the construction of C. If v_0 is not a vertex of Γ, then it is the middle point of some edge e of Γ because these are the only vertices of $\Gamma \cup C$ which are not vertices of Γ. Since C is disjoint from $\operatorname{Int} B_2$, there are no other edges in B_2. So if e is not completely contained in B_2, then it is a free edge because the end of e in $\operatorname{Int} B_2$ can not be incident to any other edges. This is impossible by the hypothesis of the lemma. If e lies completely in B_2, then e is a component of Γ because v_0 is not a vertex of Γ. This contradicts the above assumption that every component of Γ has at least two edges. □

Proof of Theorem 1 Choose an arc C as above. Let $B = N(C)$ be a regular neighborhood of C, and let b_i be the arc $e_i \cap B$. Then $(B, b_1 \cup \ldots b_n)$ is a trivial tangle. Recall the trivial tangle $U_n = (B_n, A_n')$ defined before Lemma 5. Choose a homeomorphism $\varphi\colon B \to B_n$ so that $\varphi(b_i) = a_i$ for $i \geqq 2$, and $\varphi(b_1) = a_1'$. Replacing b_1 by $\varphi^{-1}(a_1)$, we get another embedding G of Γ in S^3. We want to show that G is a minimally knotted embedding of Γ in S^3.

Identity $(B, G \cap B)$ with Q_n by the map φ. Denote by $e_1' = (e_1 - b_1) \cup a_1$ the edge in G corresponding to e_1 in Γ. Clearly, $G - e_1' = \Gamma - e_1$ is unknotted. Also, by Lemma 5, when deleting any edge $e_i (i > 1)$ from G, the edge e_1' can be isotoped to e_1, so $G - \operatorname{Int} e_i$ is isotopic to $\Gamma - \operatorname{Int} e_i$ in S^3 for all i. Hence any proper subgraph of G is unknotted in S^3. It remains to show that G is knotted.

Consider $E(G) = S^3 - \operatorname{Int} N(G)$, the exterior of G. It suffices to show that $E(G)$ is not a handlebody. Recall the tangles V_i in B_n, (see Fig. 2). We use the same notation V_i for the underlying 3-ball of V_i. Let M_i be the manifold $V_i \cap E(G)$, and let $X = E(G) - \cup \operatorname{Int} M_i$. We have $E(G) = X \cup (\cup M_i)$. The surface $F_i = X \cap M_i$ is a punctured sphere, which by Lemma 4 is incompressible in M_i. We want to show that it is also incompressible in X.

Assume F_i were compressible in X. Shrinking each 3-ball V_i into a single vertex v_i, we get a new graph G_1. It is clear that $X = E(G_1)$ and the surface F_i is exactly the punctured sphere $\delta(v_i) = \partial N(v_i) \cap E(G_1)$. By Lemma 1, F_i is compressible in X if and only if v_i is a decomposing vertex of G_1, so G_1 would be decomposable..

Denote by G_2 the graph obtained from G_1 by successively removing parallel edges of G_1 in the 3-ball B_n until there are no parallel edges in B_n. The part of G_2 in B_n is shown in Fig. 3. There are $n-1$ cycles $r_1, \ldots, r_{n-1}$ in G_2, each

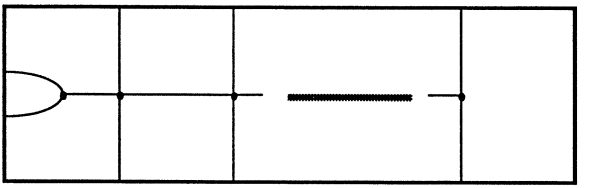


Fig. 4

of which bounds a disk D_i in B intersecting G_2 at a single point. Let G_3 be obtained from G_2 by shrinking every D_i into a single vertex. By further deleting parallel edges of G_3 in B, we get a graph G_4. The part of G_4 in B is shown in Fig. 4.

It is now clear that $G_4 = \Gamma \cup C$. By Lemma 2 and Lemma 3, if G_1 is decomposable then so is G_4. On the other hand, Lemma 6 says that $\Gamma \cup C$ is indecomposable. This is a contradiction, therefore F_i is incompressible in X.

The manifold $E(G)$ is the union of X and the M_i's along the surfaces F_i. Since every F_i is incompressible in both M_i and X, $\pi_i M_i$ is a subgroup of $\pi_1 E(G)$. By Lemma 4, $\pi_1 M_i$ are not free groups, so $\pi_1 E(G)$ is not a free group. But this certainly implies that $E(G)$ is not a handlebody, and so G is a knotted graph.

Therefore, G is a minimally knotted embedding of Γ. □

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A local Martinelli-Bochner formula on hypersurfaces

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1 Introduction

Let M be an oriented real hypersurface of class C^2 in $\mathbb{C}^n$, i.e. $M=\{z\in\theta\colon \varrho(z)=0\}$, where θ is an open subset of $\mathbb{C}^n$ and ϱ is a real C^2 function on θ with $d\varrho(z)\neq 0$ for all $z\in\theta$. For $z\in M$ and $\xi\in\mathbb{C}^n$, we denote by $\delta(\xi, z)$ the Euclidean distance between ξ and the complex tangent plane of M at z. The aim of this paper is to prove the following theorem:

Theorem 1.1 *Suppose, for some $z_0\in M$, the restriction of the Levi form of ϱ at z_0 to the complex tangent plane of M at z_0 has at least one positive and at least one negative eigenvalue. Then there exist an open neighbourhood $M_0\subseteq M$ of z_0 and a continuous differential form $K(z,\xi)$ defined and continuous for all $(z,\xi)\in\bar{M}_0\times\bar{M}_0$ with $z\neq\xi$ such that:*

(i) *$K(z,\xi)$ is of degree zero in z and of bidegree $(n, n-2)$ in ξ.*
(ii) *$d_\xi K(z,\xi)=0$ for all $(z,\xi)\in M_0\times M_0$ with $z\neq\xi$.*
(iii) *There is a constant $C>0$ such that*

$$\|K(z,\xi)\|\leqq C\frac{1+|\ln|\xi-z\|}{(\delta(\xi,z)+|\xi-z|^2)|\xi-z|^{2n-3}} \tag{1}$$

for all $\xi, z\in M_0$ with $\xi\neq z$.
(iv) *For each $0<\alpha<1$, the coefficients of $K(z,\xi)$ are of class $C^{\alpha,1/2}_{z,\xi}$ for all $(z,\xi)\in M_0\times M_0$ with $z\neq\xi$ (for the definition of $C^{\alpha,1/2}_{z,\xi}$ cf. the end of Sect. 2).*
(v) *Let $\Omega\subset\subset M_0$ be a domain with piecewise C^1 boundary. If f is a C^1 function on $\bar{\Omega}$, then*

$$f(z)=\int\limits_{\xi\in b\Omega} f(\xi)\,K(z,\xi)-\int\limits_{\xi\in\Omega} df(\xi)\wedge K(z,\xi) \tag{2}$$

for all $z\in\Omega$.

Remark 1.2 From estimate (1) it follows that $\|K(z,\xi)\|$ is integrable with respect to ξ and z. More precisely, it is easy to see that the following estimates hold:

Denote by $d\lambda$ the Euclidean volume form on M. Then there is a constant $C>0$ such that

$$\int\limits_{\substack{\xi\in M_0\\ |\xi-z|<\varepsilon}} \|K(z,\xi)\|\, d\lambda(\xi)\leqq C\varepsilon(1+|\ln\varepsilon|^2) \tag{3}$$

for all $z\in\bar{M}_0$ and $\varepsilon>0$, and

$$\int\limits_{\substack{z\in M_0\\ |z-\xi|<\varepsilon}} \|K(z,\xi)\|\, d\lambda(z)\leqq C\varepsilon(1+|\ln\varepsilon|^2) \tag{4}$$

for all $\xi\in\bar{M}_0$ and $\varepsilon>0$.

To obtain the kernel $K(z,\xi)$ in Theorem 1.1 we proceed as follows: Consider the Martinelli-Bochner kernel

$$B(z,\zeta):=\frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^{n}(-1)^{j+1}\frac{\bar{\zeta}_j-\bar{z}_j}{|\zeta-z|^{2n}}d\bar{\zeta}_1\wedge\ldots_j\ldots\wedge d\bar{\zeta}_n\wedge d\zeta_1\wedge\ldots\wedge d\zeta_n \tag{5}$$

and a sufficiently small open ball $U\subseteq\mathbb{C}^n$ centered at z_0. Set $U_+:=\{\zeta\in U:\varrho(\zeta)<0\}$ and $U_-:=\{\zeta\in U:\varrho(\zeta)>0\}$. Then, in view of the hypothesis on the Levi form of ϱ, it follows from the Andreotti-Grauert theory that, for fixed $z\in M$, one can solve the equations

$$\bar{\partial}K_+(z,\cdot)=-B(z,\cdot)\quad\text{on } U_+$$

and

$$\bar{\partial}K_-(z,\cdot)=-B(z,\cdot)\quad\text{on } U_-.$$

We prove that this can be done with appropriate uniform estimates so that $K_+(z,\xi)$ and $K_-(z,\xi)$ extend to $(U\cap M)\backslash\{z\}$ and $K(z,\xi):=K_+(z,\xi)-K_-(z,\xi)$ has the required properties. For that we use a version of the classical integral operators constructed by Grauert and Lieb [G-L], Henkin [H 1] and W. Fischer and Lieb [FW-L].

Formula (2) is an analogon of the Martinelli-Bochner formula in $\mathbb{C}^n$. At the end of this paper (Sect. 6) we want to show that this analogy extends also to some of the applications of the Martinelli-Bochner formula: using the kernel $K(z,\xi)$, we prove strengthened versions of some of the results on the tangential Cauchy-Riemann equation obtained by Henkin in [H 2] and [H 3] (see the regularity Theorems 6.6 and 6.8, the solvability Theorem 6.10 for (0, 1)-currents with small support, and the Hartogs-Bochner extension Theorem 6.11).

2 Preliminaries

Let $K\subset\subset\mathbb{C}^n$ be a compact set. Then $C^0(K)$ is the Banach space of all continuous complex functions on K. For $0<\alpha<1$, $C^\alpha(K)$ is the Banach space of all complex functions which are Hölder continuous with exponent α on K. The norm in $C^\alpha(K)$, $0\leqq\alpha<1$ will be denoted by $\|\cdot\|_{\alpha,K}$. That means $\|\cdot\|_{0,K}$ is the max-norm and for $0<\alpha<1$, $\|\cdot\|_{\alpha,K}$ is the Hölder norm with exponent α.

Let $D \subset\subset \mathbb{C}^n$ be a domain and $0 \leqq \alpha < 1$. Then $C^\alpha_*(\bar{D})$ is the Banach space of differential forms whose coefficients belong to $C^\alpha(\bar{D})$. The norm in $C^\alpha_*(\bar{D})$ will be denoted by $\| \cdot \|_{\alpha, \bar{D}}$. By $C^\alpha_{(s,r)}(\bar{D})$ we denote the subspace of forms in $C^\alpha_*(\bar{D})$ which are of bidegree (s, r). By $L^1_*(D)$ we denote the Banach space of all differential forms whose coefficients are integrable on D. The norm in $L^1_*(D)$ will be denoted by $\| \cdot \|_{L^1(D)}$ and $L^1_{(s,r)}(D)$ is the subspace of all forms in $L^1_*(D)$ which are o bidegree (s, r).

Proposition 2.1 *If $A, B \subset\subset \mathbb{C}^n$ are two compact sets, $f(z, \xi)$ is a complex function defined for $(z, \xi) \in A \times B$ and $0 \leqq \alpha, \beta < 1$ then it is easy to see that the following two conditions are equivalent*:

(i) *$f(z, \cdot) \in C^\beta(B)$ for all $z \in A$ and the assignment $A \ni z \to f(z, \cdot)$ is Hölder continuous with exponent α as a map with values in $C^\beta(B)$.*
(ii) *$f(\cdot, \xi) \in C^\alpha(A)$ for all $\xi \in B$ and the assignment $B \ni \xi \to f(\cdot, \xi)$ is Hölder continuous with exponent β as a map with values in $C^\alpha(A)$.*

Let Z be an arbitrary subset of $\mathbb{C}^n \times \mathbb{C}^n$, $f(z, \xi)$ a complex function defined for $(z, \xi) \in Z$ and let $0 \leqq \alpha, \beta < 1$. Then we say that $f(z, \xi)$ is of class $C^{\alpha,\beta}_{z,\xi}$ on Z if for each pair of compact sets $A, B \subseteq \mathbb{C}^n$ with $A \times B \subseteq Z$ the both equivalent conditions (i) and (ii) in Proposition 2.1 are fulfilled.

3 Local q-convex C^2 domains

If φ is a real C^2 function in some neighbourhood of a point $z \in \mathbb{C}^n$ then we denote by $L_\varphi(z)$ the Levi form and by $H_\varphi(z)$ the Hessian form of φ at z. That means

$$L_\varphi(z)\, t := \sum_{j,k=1}^{n} \frac{\partial^2 \varphi(z)}{\partial z_j \, \partial \bar{z}_k} t_j \bar{t}_k \qquad \text{for } t \in \mathbb{C}^n$$

and

$$H_\varphi(z)\, t := \frac{1}{2} \sum_{\nu,\mu=1}^{2n} \frac{\partial^2 \varphi(z)}{\partial x_\nu \, \partial x_\mu} x_\nu(t)\, x_\mu(t) \qquad \text{for } t \in \mathbb{C}^n$$

where $x_1, \ldots, x_{2n}$ are the real coordinates on $\mathbb{C}^n$ with $z_j = x_j(z) + i x_{j+n}(z)$ if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Definition. Let $0 \leqq q \leqq n-1$ be an integer.

(i) If $G \subset\subset \mathbb{C}^n$ is a C^2 domain then we say that G is strictly convex with respect to the real coordinates of $z_1, \ldots, z_{q+1}$ if there exists a real C^2 function ϱ is a neighbourhood $U_{\bar{G}}$ of $\bar{G}$ such that $G = \{z \in U_{\bar{G}} : \varrho(z) < 0\}$ and $d\varrho(z) \neq 0$ for $z \in bG$ and ϱ is strictly convex with respect to the real coordinates of $z_1, \ldots, z_{q+1}$, i.e.

$$H_\varrho(\zeta)\, t > 0 \tag{6}$$

for all $\zeta \in U_{\bar{G}}$ and $t \in \mathbb{C}^n$ with $t_{q+2} = \ldots = t_n = 0$.

(ii) A local q-convex C^2 domain is a C^2 domain $D \subset\subset \mathbb{C}^n$ for which there exists a biholomorphic map h from a neighbourhood of $\bar{D}$ onto an open set in $\mathbb{C}^n$ such that $h(D)$ is strictly convex with respect to the real coordinates of $z_1, \ldots, z_{q+1}$.

Lemma 3.1 *Let $0 \leqq q \leqq n-1$ be an integer. Further let $\theta \subseteq \mathbb{C}^n$ be an open set, ϱ a real C^2 function on θ with $d\varrho(z) \neq 0$ for $z \in \theta$ and let $M = \{z \in \theta: \varrho(z) = 0\}$. Set $\theta_+ = \{z \in \theta: \varrho(z) < 0\}$ and suppose that for some $z_0 \in M$ the restriction of $L_\varrho(z_0)$ to the complex tangent plane of M at z_0 has at least q positiv eigenvalues. Then there exist a local q-convex C^2 domain D and a neighbourhood U of z_0 such that*

$$U \cap \theta_+ \subseteq D \subseteq \theta_+ . \tag{7}$$

Proof. Choose a real C^2 function φ on θ with $d\varphi(z) \neq 0$ for $z \in \theta$ and $\theta_+ = \{z \in \theta: \varphi(z) < 0\}$ such that $L_\varphi(z_0)$ has at least $q+1$ positive eigenvalues (see Proposition 5.8 in [H-Le2]). Then the restriction of φ to a certain $(q+1)$-dimensional complex submanifold through z_0 is strictly plurisubharmonic and non-critical. Therefore in view of the Narasimhan lemma (see Theorem 1.4.14 in [H-Le1]) we may assume that φ is strictly convex with respect to the real coordinates of $z_1, \ldots, z_{q+1}$. Fix $r > 0$ so small that for the ball $B_r(z_0) := \{z \in \mathbb{C}^n: |z - z_0| < r\}$ we have $\bar{B}_r(z_0) \subseteq \theta$, $d\varphi(z) \neq 0$ for all $z \in \bar{B}_r(z_0)$ and the intersection of $bB_r(z_0)$ and the surface $\{\varphi = 0\}$ is transversal.

Now let $\beta > 0$, $\tau(z) := \max_\beta(\varphi(z), |z - z_0|^2 - r^2)$ and $D := \{z \in \theta: \tau(z) < 0\}$ where $\max_\beta(\cdot, \cdot)$ is the smoothing of the function $\max(\cdot, \cdot)$ from Definition 4.12 in [H-Le2]. By Lemma 4.13 in [H-Le2] $\max_\beta(\cdot, \cdot)$ is convex and has non negative first order derivatives at least one of which is positive. Therefore τ is strictly convex with respect to the real coordinates of $z_1, \ldots, z_{q+1}$ for any $\beta > 0$. Moreover by Lemma 4.13 in [H-Le2]

$$\max(t_1, t_2) \leqq \max_\beta(t_1, t_2) \leqq \max(t_1, t_2) + \beta$$

and

$$\max(t_1, t_2) = \max_\beta(t_1, t_2) \quad \text{for } |t_1 - t_2| \geqq \beta.$$

Therefore it is clear that for each neighbourhood $U \subset\subset B_r(z_0)$ *of* z_0 (7) will be satisfied if β is sufficiently small.

It remains to prove that $d\tau(z) \neq 0$ for all $z \in bD$ if β is sufficiently small. For that first we observe that $d\tau$ is a non-trivial linear combination of $d\varphi(z)$ and $d|z - z_0|^2$ (see the proof of Lemma 4.13 in [H-Le2]). Since the intersection of $bB_r(z_0)$ and $\{\varphi = 0\}$ is transversal this implies that for some neighbourhood V of this intersection $d\tau(z) \neq 0$ for all $z \in V$. Finally we observe that since $\max_\beta(t_1, t_2) = \max(t_1, t_2)$ if $|t_1 - t_2| \geqq \beta$ we can choose β so small that for all z in some neighbourhood of $bD \setminus V$ either $\tau(z) = \varphi(z)$ or $\tau(z) = |z - z_0|^2 - r^2$. □

Lemma 3.2 *Let $G \subset\subset \mathbb{C}^n$ be a C^2 domain which is strictly convex with respect to the real coordinates of $z_1, \ldots, z_{q+1}$. Let $0 \leqq q \leqq n-1$ and let $\varrho: U_{\bar{G}} \to \mathbb{R}$ be as in part* (i) *of the definition. Further let $\delta > 0$ be so small that the neighbourhood*

$$V_{\bar{G}} := \{z \in U_{\bar{G}}: \varrho(z) < \delta\}$$

of $\bar{G}$ is relatively compact in $U_{\bar{G}}$. Then there exist constants α, $A>0$ such that

$$2\mathrm{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)+A\sum_{j=q+2}^{n}|\zeta_j-z_j|^2\geqq\varrho(\zeta)-\varrho(z)+\alpha|\zeta-z|^2 \tag{8}$$

for all $z,\zeta\in V_{\bar{G}}$.

Proof. Set $t'=(t_1,\ldots,t_{q+1},0,\ldots,0)$ and $t''=(0,\ldots,0,t_{q+2},\ldots,t_n)$ if $t\in\mathbb{C}^n$. Then by (6) there is a constant $\beta>0$ such that

$$H_\varrho(\zeta)\,t'\geqq 3\beta|t'|^2 \tag{9}$$

for all $\zeta\in\overline{V}_{\bar{G}}$ and $t\in\mathbb{C}^n$. Using the inequality $2ab=2(\varepsilon a)(b/\varepsilon)\leqq\varepsilon^2a^2+b^2/\varepsilon^2$ we can choose a constant $C>0$ such that

$$|H_\varrho(\zeta)\,t-H_\varrho(\zeta)\,t'|\leqq\beta|t'|^2+(C-2\beta)|t''|^2 \tag{10}$$

for $\zeta\in\overline{V}_{\bar{G}}$ and $t\in\mathbb{C}^n$. Since by Taylor's theorem

$$2\mathrm{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)=\varrho(\zeta)-\varrho(z)+H_\varrho(\zeta)(\zeta-z)+o(|\zeta-z|^2)$$

it follows from (9) and (10) that for some $\varepsilon>0$ we have the estimate

$$2\mathrm{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)+C|\zeta''-z''|^2\geqq\varrho(\zeta)-\varrho(z)+\beta|\zeta-z|^2 \tag{11}$$

if $z,\zeta\in\overline{V}_{\bar{G}}$ with $|\zeta-z|\leqq\varepsilon$.

Now let $z,\zeta\in\overline{V}_{\bar{G}}$ with $|\zeta-z|\geqq\varepsilon$ and $\zeta''=z''$. Set

$$z^\varepsilon=\left(1-\frac{\varepsilon}{|\zeta-z|}\right)\zeta+\frac{\varepsilon}{|\zeta-z|}z.$$

Since ϱ is strictly convex with respect to the real coordinates of $z_1,\ldots,z_q$ we get $z^\varepsilon\in V_{\bar{G}}$ and

$$\varrho(z^\varepsilon)\leqq\left(1-\frac{\varepsilon}{|\zeta-z|}\right)\varrho(\zeta)+\frac{\varepsilon}{|\zeta-z|}\varrho(z)$$

and since $|\zeta-z^\varepsilon|=\varepsilon$ it follows from (11) that

$$\begin{aligned}2\mathrm{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)&\geqq\frac{|\zeta-z|}{\varepsilon}(\varrho(\zeta)-\varrho(z^\varepsilon)+\beta\varepsilon^2)\\&\geqq\varrho(\zeta)-\varrho(z)+\beta\varepsilon|\zeta-z|.\end{aligned}$$

Hence we can find $\varepsilon'>0$ so small that

$$2\mathrm{Re}\sum_{j=1}^{n}\frac{\partial\varrho(\zeta)}{\partial\zeta_j}(\zeta_j-z_j)\geqq\varrho(\zeta)-\varrho(z)+\frac{\beta\varepsilon}{2}|\zeta-z|$$

for all $z, \zeta \in \overline{V}_{\bar{G}}$ with $|\zeta - z| \geqq \varepsilon$ and $|\zeta'' - z''| \leqq \varepsilon'$. Clearly this implies that for sufficiently large constants $B > 0$ we have

$$2\,\mathrm{Re} \sum_{j=1}^{n} \frac{\partial \varrho(\zeta)}{\partial \zeta_j}(\zeta_j - z_j) + B|\zeta'' - z''|^2 \geqq \varrho(\zeta) - \varrho(z) + \frac{\beta\varepsilon}{2}|\zeta - z| \tag{12}$$

for all $z, \zeta \in \overline{V}_{\bar{G}}$ with $|\zeta - z| \geqq \varepsilon$. (8) now follows from (12) and (11) if we set $A = \max(C, B)$ and $\alpha = \min\limits_{\substack{z,\zeta \in \overline{V}_{\bar{G}} \\ |\zeta - z| \geqq \varepsilon}} \frac{\beta\varepsilon}{2|\zeta - z|}$. □

4 Certain new estimates for $\bar{\partial}$

In this section q is an integer with $0 \leqq q \leqq n-1$ and $D \subset\subset \mathbb{C}^n$ is a local q-convex C^2 domain. Then we have by definition a C^2 domain $G \subset\subset \mathbb{C}^n$ which is strictly convex with respect to the real coordinates of $z_1, \ldots, z_{q+1}$ and a biholomorphic map h from a neighbourhood $U_{\bar{D}}$ of $\bar{D}$ onto a neighbourhood $U_{\bar{G}}$ of $\bar{G}$ such that $h(D) = G$. After shrinking these neighbourhoods we may also assume that there is a C^2 function $\varrho: U_{\bar{G}} \to \mathbb{R}$ as in the first part of the Definition in Sect. 3. Further let $V_{\bar{G}}, A, \alpha$ be as in Lemma 3.2. Before we come to the announced estimates we construct an integral operator which gives a homotopy formula for (n, r)-forms with $n - q \leqq r \leqq n$.

For all $(\xi, \zeta) \in \mathbb{C}^n \times U_{\bar{G}}$ we set

$$w_j(\xi, \zeta) := \begin{cases} 2\dfrac{\partial \varrho(\zeta)}{\partial \zeta_j} & \text{for } 1 \leqq j \leqq q+1 \\ 2\dfrac{\partial \varrho(\zeta)}{\partial \zeta_j} + A(\overline{\zeta}_j - \overline{\xi}_j) & \text{for } q+2 \leqq j \leqq n, \end{cases}$$

$$w(\xi, \zeta) := (w_1(\xi, \zeta), \ldots, w_n(\xi, \zeta)),$$

$$\Phi(\xi, \zeta) := \langle w(\xi, \zeta), \zeta - \xi \rangle - 2\varrho(\zeta).$$

Then by (8)

$$\mathrm{Re}\, \Phi(\xi, \zeta) \geqq -\varrho(\zeta) - \varrho(\xi) + \alpha|\zeta - \xi|^2 \tag{13}$$

for all $\xi, \zeta \in \overline{V}_{\bar{G}}$. In particular $\Phi(\xi, \zeta) \neq 0$ if $\xi, \zeta \in G$ and for all $(\xi, \zeta, \lambda) \in \bar{G} \times \bar{G} \times [0, 1]$ with $\xi \neq \zeta$ we can define

$$\eta(\xi, \zeta, \lambda) := (1 - \lambda)\frac{w(\xi, \zeta)}{\Phi(\xi, \zeta)} + \lambda \frac{\overline{\zeta} - \overline{\xi}}{|\zeta - \xi|^2}$$

and

$$\hat{H}^G(\xi, \zeta, \lambda) := \frac{n!}{(2\pi i)^n} d\eta_1(\xi, \zeta, \lambda) \wedge \ldots \wedge d\eta_n(\xi, \zeta, \lambda) \wedge d\xi_1 \wedge \ldots \wedge d\xi_n$$

where $\eta_1, \ldots, \eta_n$ are the components of η and d is the exterior differential operator with respect to ξ, ζ, λ. For $\zeta \neq \xi$, $\hat{H}^G(\xi, \zeta, \lambda)$ is of class C^∞ in ξ, λ and

all derivatives with respect to ξ, λ are continuous in ξ, ζ, λ. Moreover if we consider only the part of $\hat{H}^G(\xi, \zeta, \lambda)$ which is of degree 1 in λ then we see that the singularity at $\xi = \zeta$ of this form is of order $\leqq 2n-1$.

Hence for each $g \in L^1_*(G) \cap C^0_*(G)$ the integrals

$$H^G g(\xi) := \int\limits_{(\zeta, \lambda) \in G \times [0,1]} g(\zeta) \wedge \hat{H}^G(\xi, \zeta, \lambda) \quad \text{for } \xi \in G$$

converge (for the definition of such integrals see for instance Sect. 0.2 in [H-Le2]) and in this way we obtain a form $H^G g \in C^0_*(G)$. Denote by $\hat{H}(\xi, \zeta, \lambda)$ the pull back of the form $\hat{H}^G(\xi, \zeta, \lambda)$ to $U_{\bar{D}} \times U_{\bar{D}} \times [0,1]$ with respect to the biholomorphic map h. That means

$$\hat{H}(\xi, \zeta, \lambda) = (h^*_\xi \times h^*_\zeta)\, \hat{H}^G(\xi, \zeta, \lambda). \tag{14}$$

Further let

$$H = h^* \circ H^G \circ (h^{-1})^*$$

be the pull back of the operator H^G to the domain D with respect to h. Then H is a linear operator from $L^1_*(D) \cap C^0_*(D)$ to $C^0_*(D)$ and for each $f \in L^1_*(D) \cap C^0_*(D)$ we have

$$Hf(\xi) = \int\limits_{(\zeta, \lambda) \in D \times [0,1]} f(\zeta) \wedge \hat{H}(\xi, \zeta, \lambda) \quad \text{for } \xi \in D.$$

Note that for $r = 1, \ldots, n$

$$H(L^1_{(n,r)}(D) \cap C^0_{(n,r)}(D)) \subseteq C^0_{(n,r-1)}(D). \tag{15}$$

Theorem 4.1 *If $n-q \leqq r \leqq n$ and if $f \in L^1_{(n,r)}(D) \cap C^0_{(n,r)}(D)$ such that df also belongs to $L^1_*(D) \cap C^0_*(D)$ then*

$$f = \begin{cases} dHf & \text{for } r = n \\ dHf + Hdf & \text{for } n-q \leqq r \leqq n-1. \end{cases} \tag{16}$$

Theorem 4.2 *There is a constant $C > 0$ such that for each bounded $f \in C^0_*(D)$, Hf is Hölder continuous on $\bar{D}$ with exponent $1/2$ on $\bar{D}$ and*

$$\|Hf\|_{1/2, \bar{D}} \leqq C \sup_{\zeta \in D} \|f(\zeta)\|.$$

Essentially these theorems are contained already in the works of Grauert and Lieb [G-L], Henkin [H1] and W. Fischer and Lieb [FW-L] where certain versions of the operator H with boundary integrals are used. To obtain proofs precisely for the statements formulated here one can use many different sources in the literature. We restrict ourselves to the following remarks: The idea to use operators without boundary integrals is due to Henkin, Lieb and Range (see [L-R] or (H-Le1]); Theorem 4.1 can be proved by the same arguments as Theorem 4.11 in [La-Le]; Theorem 4.2 can be proved by the same arguments as Theorem 3.1 in [FB].

Theorem 4.2 admits generalisations to forms satisfying different uniform growth conditions [L-R, FB]. For example in [FB] the case is studied where for a smooth submanifold N of bD

$$\|f(\zeta)\| \leqq [\operatorname{dist}(\zeta, N)]^{-\beta} \quad \text{for } \zeta \in D$$

where $0 \leqq \beta < 2n - \dim_{\mathbb{R}} N$. In the present paper we need the following improvement of this result for the case when N consists only of one point and $\beta = 2n-1$: Set

$$\tau(\xi, z) := \sum_{j=1}^{n} \frac{\partial \varrho \circ h(z)}{\partial z_j}(\xi_j - z_j) \tag{17}$$

for $z \in U_{\bar{D}}$ and $\xi \in \mathbb{C}^n$. Note that for $z \in bD$, $\tau(\xi, z)$ is proportional to the Euclidean distance $\delta(\xi, z)$ between ξ and the complex tangent plane of bD at z.

Theorem 4.3 *There is a constant $C > 0$ such that the following holds: If $z \in U_{\bar{D}} \setminus D$ (in particular $z \in bD$ is admitted) and $f \in C^0_*(D)$ satisfies the estimate*

$$\|f(\zeta)\| \leqq \frac{1}{|\zeta - z|^{2n-1}} \tag{18}$$

for all $\zeta \in D$ then Hf belongs to $C^{1/2}_*(\bar{D} \setminus \{z\})$ and moreover

$$\|Hf(\xi)\| \leqq C \frac{1 + |\ln|\xi - z\|}{(\tau(\xi, z) + |\xi - z|^2)|\xi - z|^{2n-3}} \tag{19}$$

for all $\xi \in \bar{D} \setminus \{z\}$.

Proof. We may assume that $D = G$ and h is the identical map. Let $z \in U_{\bar{D}} \setminus D$ and $f \in C^0_*(D)$ with (18) be given. That Hf belongs to $C^{1/2}_*(\bar{D} \setminus \{z\})$ then follows from Theorem 4.2 and the fact that for $\zeta \neq \xi$ the derivatives of $\hat{H}(\xi, \zeta, \lambda)$ with respect to ξ are continuous in ξ, ζ, λ.

Now we are going to prove estimate (19). During this proof by C, C_1, C_2 we denote positive constants which are independent of f and z. The constant C used in different places may have different values there. Observe that as usual (see for instance Sect. 3.2.7 in [H-Le 1]) we obtain that

$$\|Hf(\xi)\| \leqq C(I_0(\xi) + I_1(\xi) + I_2(\xi)) \quad \text{for } \xi \in \bar{D} \tag{20}$$

where

$$I_k(\xi) := \int_{\zeta \in D} \frac{d\sigma}{|\Phi(\xi, \zeta)|^k |\zeta - \xi|^{2n-1-k} \zeta - z|^{2n-1}}$$

and $d\sigma$ is the Lebesgue measure. We omit the elementary arguments which show that

$$|I_0(\xi)| \leqq \frac{C}{|\xi - z|^{2n-2}} \quad \text{for } \xi \in \bar{D}. \tag{21}$$

To estimate $I_1(\xi)$ and $I_2(\xi)$ we first give some auxiliary estimates. From the definition of Φ it is clear that

$$|\Phi(\xi, z)| \geqq 2\tau(\xi, z) - A|\xi - z|^2 - 2\varrho(z) \quad \text{for } \xi \in \bar{D}$$

and

$$|\Phi(\xi, z) - \Phi(\xi, \zeta)| \leqq C|\zeta - z| \quad \text{for } \xi, \zeta \in \bar{D}.$$

Hence

$$|\Phi(\xi, \zeta)| \geqq 2\tau(\xi, z) - C_1(|\zeta - z| + |\xi - z|^2) \tag{22}$$

for all $\xi, \zeta \in \bar{D}$. Further we introduce the abbreviation $t(\xi, \zeta) := \operatorname{Im} \Phi(\xi, \zeta)$ and recall the fact that $d_\zeta t(\xi, \zeta)_{\zeta = \xi} \wedge d\varrho(\zeta) \neq 0$ if $\zeta \in bD$. Choose a neighbourhood U_{bD} of bD and the number $\varepsilon > 0$ so small that

$$d_\xi t(\xi, \zeta) \wedge d\varrho(\zeta) \neq 0 \tag{23}$$

for all $\xi \in U_{bD}$ and $\zeta \in \mathbb{C}^n$ with $|\zeta - \xi| \leqq \varepsilon$. Note also that by (13)

$$|\Phi(\xi, \zeta)| \geqq |t(\xi, \zeta)| + |\varrho(\zeta)| + |\varrho(\xi)| + \alpha|\zeta - \xi|^2 \tag{24}$$

for all $\zeta, \xi \in \bar{D}$. It follows from (24) and (21) that

$$I_k(\xi) \leqq C I_0(\xi) \leqq \frac{C}{|\xi - z|^{2n-2}} \quad \text{for } \xi \in D \setminus U_{bD} \tag{25}$$

and

$$\int\limits_{\substack{\zeta \in D \\ |\zeta - \xi| > \varepsilon}} \frac{d\sigma_{2n}}{|\Phi(\xi, \zeta)|^k |\zeta - \xi|^{2n-1-k} |\zeta - z|^{2n-1}} \leqq C \tag{26}$$

for all $\xi \in \bar{D} \setminus \{z\}$ and $k = 1, 2$. Set

$$I_{k,\varepsilon}(\xi) := \int\limits_{\substack{\zeta \in D \\ |\zeta - \xi| < \varepsilon}} \frac{d\sigma_{2n}}{|\Phi(\xi, \zeta)|^k |\zeta - \xi|^{2n-1-k} |\zeta - z|^{2n-1}}$$

for $\xi \in (\bar{D} \cap U_{bD}) \setminus \{z\}$. Since

$$\tau(\xi, z) \leqq C_2 |\xi - z| \tag{27}$$

for all $\xi \in \bar{D}$ now by (20), (21), (25) and (26) it remains to prove that

$$I_{k,\varepsilon}(\xi) \leqq C \frac{1 + |\ln|\xi - z||}{|\xi - z|^{2n-1}} \tag{28}$$

and

$$I_{k,\varepsilon}(\xi) \leqq C \frac{1 + |\ln|\xi - z||}{\tau(\xi, z)|\xi - z|^{2n-3}} \tag{29}$$

for all $\xi\in(\bar D\cap U_{bD})\backslash\{z\}$ and $k=1, 2$. In doing so we use the following notation: If $\xi\in(\bar D\cap U_{bD})\backslash\{z\}$, $W(\xi)\subseteq D$ and $k\in\{1, 2\}$ then

$$I_{k,\varepsilon}(W(\xi)):=\int\limits_{\substack{\zeta\in W(\xi)\\ |\zeta-\xi|<\varepsilon}}\frac{d\sigma_{2n}}{|\Phi(\xi,\zeta)|^k|\zeta-\xi|^{2n-1-k}|\zeta-z|^{2n-1}}.$$

Proof of (28) For $\xi\in(\bar D\cup U_{bD})\backslash\{z\}$ we set

$$W'(\xi)=\{\zeta\in D: |\zeta-z|<|\xi-z|/2\}$$

and

$$W''(\xi)=\{\zeta\in D: |\zeta-z|>|\xi-z|/2\}.$$

Then

$$I_{k,\varepsilon}(\xi)=I_{k,\varepsilon}(W'(\xi))+I_{k,\varepsilon}(W''(\xi)) \tag{30}$$

for all $\xi\in(\bar D\cap U_{bD})\backslash\{z\}$ and $k\in\{1, 2\}$. Since $|\zeta-\xi|>|\xi-z|/2$ if $\zeta\in W'(\xi)$ and by (24) we have

$$I_{k,\varepsilon}(W'(\xi))\leqq\frac{C}{|\xi-z|^{2n-1-k}}\int\limits_{\substack{\zeta\in W'(\xi)\\ |\zeta-\xi|<\varepsilon}}\frac{d\sigma_{2n}}{(|t(\xi,\zeta)|+|\varrho(\zeta)|+|\xi-z|^2)^k|\zeta-z|^{2n-1}}$$

for all $\xi\in(\bar D\cap U_{bD})\backslash\{z\}$ and $k\in\{1, 2\}$. By (23) ϱ and $t(\xi,\cdot)$ may be considered as local coordinates. Hence

$$\begin{aligned} I_{k,\varepsilon}(W'(\xi))&\leqq\frac{C}{|\xi-z|^{2n-1-k}}\int\limits_{x\in\mathbb{R}^{2n}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_1|+|x_2|+|\xi-z|^2)^k|x|^{2n-1}}\\ &\leqq\frac{C}{|\xi-z|^{2n-1+k}}\int\limits_{\substack{x\in\mathbb{R}^{2n}\\ |x|<|\xi-z|^2}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{|x|^{2n-1}}\\ &\quad+\frac{C(1+|\ln|\xi-z||)}{|\xi-z|^{2n-1-k}}\int\limits_{\substack{x\in\mathbb{R}^{2n-k}\\ |x|>|\xi-z|^2}}\frac{dx_1\wedge\ldots\wedge dx_{2n-k}}{|x|^{2n-1}}\\ &\leqq C\frac{1+|\ln|\xi-z||}{|\xi-z|^{2n-1}}. \end{aligned} \tag{31}$$

for all $\xi\in(\bar D\cap U_{bD})\backslash\{z\}$ and $k\in\{1, 2\}$. By similar arguments we obtain that

$$\begin{aligned} I_{k,\varepsilon}(W''(\xi))&\leqq\frac{C}{|\xi-z|^{2n-1}}\int\limits_{x\in\mathbb{R}^{2n}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_1|+|x_2|+|x|^2)^k|x|^{2n-1-k}}\\ &\leqq\frac{C}{|\xi-z|^{2n-1}}\int\limits_{x\in\mathbb{R}^{2n-k}}\frac{(1+|\ln|x||)\,dx_1\wedge\ldots\wedge dx_{2n-k}}{|x|^{2n-1-k}}\\ &\leqq\frac{C}{|\xi-z|^{2n-1}} \end{aligned} \tag{32}$$

for all $\xi\in(\bar{D}\cap U_{bD})\backslash\{z\}$ and $k\in(1, 2\}$. Estimate (28) now follows from (30)–(32).

Proof of (29) Let $C_3=2(C_1+C_2)$ where C_1 and C_2 are the same constants as in (22) and (27), and set

$$W^0(\xi)=\{\zeta\in D: |\zeta-z|<\tau(\xi, z)/C_3\},$$
$$W^1(\xi)=\{\zeta\in D: |\zeta-z|>\tau(\xi, z)/C_3\},$$
$$W^{10}(\xi)=\{\zeta\in W^1(\xi): |\zeta-z|<|\xi-z|/2\},$$
$$W^{11}(\xi)=\{\zeta\in W^1(\xi): |\zeta-z|>|\xi-z|/2\},$$
$$W^{110}(\xi)=\{\zeta\in W^{11}(\xi): |\zeta-\xi|<|\xi-z|\},$$
$$W^{111}(\xi)=\{\zeta\in W^{11}(\xi): |\zeta-\xi|>|\xi-z|\}.$$

Then

$$I_{k,\varepsilon}(\xi)=I_{k,\varepsilon}(W^0(\xi))+I_{k,\varepsilon}(W^{10}(\xi))+I_{k,\varepsilon}(W^{110}(\xi))+I_{k,\varepsilon}(W^{111}(\xi)) \tag{33}$$

for all $\xi\in(\bar{D}\cap U_{bD})\backslash\{z\}$ and $k\in\{1, 2\}$. Since $|\zeta-\xi|\geqq|\xi-z|/2$ if $\zeta\in W^{10}(\xi)$ and by (24) and (23) we obtain that

$$I_{k,\varepsilon}(W^{10}(\xi))\leqq\frac{C}{|\xi-z|^{2n-1-k}}\int\limits_{\substack{x\in\mathbb{R}^{2n}\\ \tau(\xi,z)/C_3<|x|<|\xi-z|/2}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_1|+|x_2|+|\xi-z|^2)^k|x|^{2n-1}}$$
$$\leqq\frac{C(1+|\ln|\xi-z||)}{|\xi-z|^{2n-1-k}}\int\limits_{\substack{x\in\mathbb{R}^{2n-k}\\ \tau(\xi,z/C_3<|x|<|\xi-z|/2}}\frac{dx_1\wedge\ldots\wedge dx_{2n-k}}{|x|^{2n-1}}$$

for all $\xi\in(\bar{D}\cap U_{bD})\backslash\{z\}$ and $k\in\{1, 2\}$. Hence

$$I_{1,\varepsilon}(W^{10}(\xi))\leqq\frac{C(1+|\ln|\xi-z||)}{\tau(\xi, z)|\xi-z|^{2n-2}}\int\limits_{\substack{x\in\mathbb{R}^{2n-1}\\ |x|<|\xi-z|/2}}\frac{dx_1\wedge\ldots\wedge dx_{2n-1}}{|x|^{2n-2}} \tag{34}$$
$$\leqq\frac{C(1+|\ln|\xi-z||)}{\tau(\xi, z)|\xi\text{-}z|^{2n-3}}$$

and

$$I_{2,\varepsilon}(W^{10}(\xi))\leqq\frac{C(1+|\ln|\xi-z||)}{|\xi-z|^{2n-3}}\int\limits_{\substack{x\in\mathbb{R}^{2n-2}\\ \tau(\xi,z)/C_3<|x|}}\frac{dx_1\wedge\ldots\wedge dx_{2n-2}}{|x|^{2n-1}} \tag{35}$$
$$\leqq\frac{C(1+|\ln|\xi-z||)}{\tau(\xi, z)|\xi-z|^{2n-3}}$$

for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$. Further it follows from (24), (23) and (27) that

$$(36)\qquad I_{k,\varepsilon}(W^{110}(\xi))\leqq\frac{C}{|\xi-z|^{2n-1}}\int\limits_{\substack{x\in\mathbb{R}^{2n}\\|x|<|\xi-z|}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_1|+|x_2|+|x|^2)^k|x|^{2n-1-k}}$$

$$\leqq\frac{C}{|\xi-z|^{2n-1}}\int\limits_{\substack{x\in\mathbb{R}^{2n-k}\\|x|<|\xi-z|}}\frac{(1+|\ln|x||)\,dx_1\wedge\ldots\wedge dx_{2n-k}}{|x|^{2n-1-k}}$$

$$\leqq\frac{C(1+|\ln|\xi-z||)}{|\xi-z|^{2n-2}}\leqq\frac{C(1+|\ln|\xi-z||)}{\tau(\xi,z)|\xi-z|^{2n-3}}$$

for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$ and $k\in\{1,2\}$. Since $|\zeta-z|\geqq|\xi-z|/2$ and $|\zeta-\xi|\geqq|\xi-z|$ imply $|\zeta-\xi|\geqq(1/2)|\zeta-z|$ we get

$$(37)\qquad I_{k,\varepsilon}(W^{111}(\xi))\leqq C\int\limits_{\zeta\in W^{111}(\xi)}\frac{d\sigma_{2n}}{|\Phi(\xi,\zeta)|^k|\zeta-\xi|^{2n-3}|\zeta-z|^{2n+1-k}}$$

$$\leqq\frac{C}{|\xi-z|^{2n-3}}\int\limits_{\substack{x\in\mathbb{R}^{2n-k}\\\tau(\xi,z)/C_3<|x|}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{(|x_1|+|x_2|+|x|^2)^k|x|^{2n+1-k}}$$

$$\leqq\frac{C(1+|\ln|\xi-z||)}{|\xi-z|^{2n-3}}\int\limits_{\substack{x\in\mathbb{R}^{2n}\\\tau(\xi,z)/C_3<|x|}}\frac{dx_1\wedge\ldots\wedge dx_{2n-k}}{|x|^{2n+1-k}}$$

$$\leqq\frac{C(1+|\ln|\xi-z||)}{\tau(\xi,z)|\xi-z|^{2n-3}}$$

for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$ and $k\in\{1,2\}$. Finally we consider the integrals $I_{k,\varepsilon}(W^0(\xi))$. It follows from estimate (28) which is already proved that

$$(38)\qquad I_{k,\varepsilon}(W^0(\xi))\leqq\frac{C(1+|\ln|\xi-z||)}{\tau(\xi,z)|\xi-z|^{2n-3}}$$

for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$ with $\tau(\xi,z)\leqq C_3|\xi-z|^2$ and $k\in\{1,2\}$. Therefore it remains to estimate $I_{k,\varepsilon}(W^0(\xi))$ for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$ with

$$(39)\qquad \tau(\xi,z)\geqq C_3|\xi-z|^2.$$

It follows from (27) that $|\zeta-z|\leqq|\xi-z|/2$ and therefore $|\zeta-\xi|\geqq|\xi-z|/2$ for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$ and $\zeta\in W^0(<\xi)$. Moreover it follows from (22) that $|\Phi(\xi,\zeta)\geqq\tau(\xi,z)$ for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$ satisfying (39) and $\zeta\in W^0(\xi)$. Hence

$$(40)\qquad I_{k,\varepsilon}(W^0(\xi))\leqq\frac{C}{(\tau(\xi,z))^k|\xi-z|^{2n-1-k}}\int\limits_{\substack{x\in\mathbb{R}^{2n}\\|x|<\tau(\xi,z)/C_3}}\frac{dx_1\wedge\ldots\wedge dx_{2n}}{|x|^{2n-1}}$$

$$\leqq\frac{C}{(\tau(\xi,z))^{k-1}|\xi-z|^{2n-1-k}}$$

$$\leqq\frac{C}{\tau(\xi,z)|\xi-z|^{2n-3}}$$

for all $\xi\in(\bar{D}\cap U_{bD})\setminus\{z\}$ satisfying (39) and $\zeta\in W^0(\xi)$ (for $k=1$ we used (27)).

Estimate (29) now follows from (33)–(38) and (40). □

5 Construction of the kernel

We start this section with a corollary to Sect. 4.

Corollary 5.1 *Let $D \subset\subset \mathbb{C}^n$ be a local 1-convex C^2 domain and let H be the operator constructed in Sect. 4 for D. Set*

$$K_D(z, \xi) := [H(B(z, \cdot))](\xi)$$

for all $z \in \mathbb{C}^n \setminus D$ and $\xi \in D$ where $B(z, \xi)$ is the Martinelli-Bochner kernel (5). By Theorem 4.3, $K_D(z, \xi)$ is defined and continuous even for all $z \in \mathbb{C}^n \setminus D$ and $\xi \in \bar{D}$ with $z \neq \xi$. Moreover this form has the following properties:

(i) *$K_D(z, \xi)$ is of bidegree $(n, n-2)$ in ξ and of degree zero in z.*
(ii) *$d_\xi K_D(z, \xi) = B(z, \xi)$ for all $z \in \mathbb{C}^n \setminus D$ and $\xi \in \bar{D}$ with $z \neq \xi$.*
(iii) *There is a constant $C > 0$ such that*

$$\|K_D(z, \xi\| \leqq C \frac{1 + |\ln |\xi - z||}{(\tau(z, \xi) + |\xi - z|^2)|\xi - z|^{2n-3}} \tag{41}$$

for all $z \in U_{\bar{D}} \setminus D$ and $\xi \in \bar{D}$ with $z \neq \xi$ where $\tau(z, \xi)$ is defined by (17).
(iv) *For each $z \in \mathbb{C}^n \setminus \bar{D}$, the form $K_D(z, \cdot)$ belongs to $C^{1/2}_{(n, n-2)}(\bar{D})$ and the assignement $z \to K_D(z), \cdot)$ is of class C^∞ as a map from $\mathbb{C}^n \setminus \bar{D}$ with values in the Banach space $C^{1/2}_{(n, n-2)}(\bar{D})$.*
(v) *For any $0 < \alpha < 1$, $K_D(z, \xi)$ is of class $C^{\alpha, 1/2}_{z, \xi}$ for all $z \in \mathbb{C}^n \setminus D$ and $\xi \in \bar{D}$ with $z \neq \xi$.*

Proof. (i) follows from (15), (ii) follows from Theorem 4.1 and (iii) follows from Theorem 4.3. Since the Martinelli-Bochner kernel is of class C^∞ outside the diagonal and, by Theorem 4.2, H acts continuously from $C^0_{(n, n-1)}(\bar{D})$ to $C^{1/2}_{(n, n-2)}(\bar{D})$, (iv) is also clear.

It remains to prove (v). Fix $0 < \alpha < 1$, $z_0 \in \mathbb{C}^n \setminus D$ and $\xi_0 \in \bar{D}$ with $z_0 \neq \xi_0$. Set $\gamma = |z_0 - \xi_0|/5$ and

$$B(z_0) = \{z \in \mathbb{C}^n \setminus D : |z - z_0| < \gamma\},$$
$$B(\xi_0) = \{\xi \in \bar{D} : |\xi - \xi_0| < \gamma\}.$$

It is sufficient to prove that $K_D(z, \xi)$ is of class $C^{\alpha, 1/2}_{z, \xi}$ for $(z, \xi) \in B(z_0) \times B(\xi_0)$. For that we choose a real C^∞ function χ on $\mathbb{C}^n$ with $\chi(\zeta) = 1$ if $|\zeta - \zeta_0| < 2\gamma$ and $\chi(\zeta) = 0$ if $|\zeta - \xi_0| > 3\gamma$. Set

$$K^\chi_D(z, \xi) = [H(\chi B(z, \cdot))](\xi),$$
$$K^{1-\chi}_D(z, \xi) = [H((1 - \chi) B(z, \cdot))](\xi)$$

for $z \in \mathbb{C}^n \setminus D$ and $\xi \in \bar{D}$ with $z \neq \xi$. Since $\chi(\zeta) B(z, \zeta)$ is of class C^∞ for $(z, \zeta) \in B(z_0) \times \mathbb{C}^n$ and H acts continuously and linearly from $C^0_{(n, n-1)}(\bar{D})$ to $C^{1/2}_{(n, n-2)}(\bar{D})$, we see that the map $z \to K^\chi_D(z, \cdot)$ is C^∞ from $B(z_0)$ to $C^{1/2}_{(n, n-2)}(\bar{D})$. Hence in particular, $K^\chi_D(z, \xi)$ is of class $C^{\alpha, 1/2}_{z, \xi}$ for $(z, \xi) \in B(z_0) \times B(\xi_0)$. It remains to prove that

$K_D^{1-\chi}(z,\xi)$ is of class $C^{\alpha,1/2}_{z,\xi}$ for $(z,\xi)\in B(z_0)\times B(\xi_0)$. For that we consider the form

$$f(\xi,\zeta):=\int\limits_{\lambda\in[0,1]}(1-\chi(\zeta))\,\hat{H}(\xi,\zeta,\lambda)$$

(see (14) for the definition of $\hat{H}(\xi,\zeta,\lambda)$). Since $1-\chi(\zeta)=0$ if $\zeta\in B(\xi_0)$ the map $\xi\to f(\xi,\cdot)$ is C^∞ from $B(\xi_0)$ to $C^0_*(\bar{D})$. Since

$$K_D^{1-\chi}(z,\xi)=\pm\int\limits_{\zeta\in D}f(\xi,\zeta)\wedge B(z,\zeta)$$

for $z\in\mathbb{C}^n\setminus D$ and $\xi\in B(\xi_0)$ and since the Martinell-Bochner integral induces a continuous linear operator from $C^0_*(\bar{D})$ to $C^\alpha_*(\bar{B}(z_0))$ this implies that the map $\xi\to K_D^{1-\chi}(\cdot,\xi)$ is C^∞ from $B(\xi_0)$ to $C^\alpha_*(\bar{B}(z_0))$. This completes the proof. □

Proof of Theorem 1.1 Choose an open ball $B\subset\subset\mathbb{C}^n$ centered at z_0 so small that $B\setminus M$ consists of precisely two connected components and $B\cap M$ is relatively compact in M. The two connected components of $B\setminus M$ we denote by B_+ and B_- so that on $B\cap M$ the orientations of M and ∂M_+ coincide. In view of Lemma 3.1 we can find local 1-convex C^2 domains D_+ and D_- and open balls $B_0\subset\subset B_1\subset\subset B$ centered at z_0 such that $B_1\cap B_\pm\subseteq D_\pm\subseteq B_\pm$. Set $M_0:=M\cap B_0$ and denote by H_+ and H_- the operators defined in Sect. 4 for D_+ and D_- respectively. Set

$$K_\pm(z,\xi):=-[H_\pm\,B(z,\cdot)](\xi)$$

for all $z\in\mathbb{C}^n\setminus D_\pm$ and $\xi\in D_\pm$ with $z\neq\xi$. By Corollary 5.1(v) $K_\pm(z,\xi)$ is defined and continuous for all $z\in\mathbb{C}^n\setminus D_\pm$ and $\xi\in\bar{D}_\pm$ with $z\neq\xi$. Therefore by setting

$$K(z,\xi):=K_+(z,\xi)|_{M_0\times M_0}-K_-(z,\xi)|_{M_0\times M_0}$$

we obtain a differential form defined and continuous for all $(z,\xi)\in M_0\times M_0$ with $z\neq\xi$. It follows immediately from the statements (i), (ii), (iii) and (v) in Corollary 5.1 that $K(z,\xi)$ has the properties (i)–(iv) formulated in Theorem 1.1.

Now we prove part (v). Let $\Omega\subset\subset M_0$ be a domain with piecewise C^1 boundary.

First we consider a C^1 function f on Ω with compact support. Then there is a C^1 function $\tilde{f}$ on $\mathbb{C}^n$ with $\tilde{f}(\xi)=f(\xi)$ if $\xi\in\Omega$ and

$$\operatorname{supp}\tilde{f}\subset\subset D_+\cup D_-\cup\Omega=:D$$

and since, by Corollary 5.1(ii), $d_\xi K_\pm(z,\xi)=B(z,\xi)$, it follows from Stokes theorem and the Martinelli-Bochner formula that

$$\begin{aligned}-\int\limits_{\xi\in\Omega}df(\xi)\wedge K(z,\xi)&=\int\limits_{\xi\in D_+}d\tilde{f}(\xi)\wedge d_\xi K_+(z,\xi)+\int\limits_{\xi\in D_-}d\tilde{f}(\xi)\wedge d_\xi K_-(z,\xi)\\&=-\int\limits_{\xi\in D}df(\xi)\wedge B(z,\xi)=\tilde{f}(z)=f(z)\end{aligned}$$

for all $z\in\Omega$. That is (2) is proved in the case when f has compact support.

Now let f be an arbitrary C^1 function on $\bar{\Omega}$. Fix $z\in\Omega$ and choose a C^1 function χ_z on M_0 with $\operatorname{supp}\chi_z\subset\subset\Omega$ and $\chi_z\equiv1$ in some neighbourhood of

z. Then $(1-\chi_z)fK(z,\cdot)$ is a continuous form on $\bar{\Omega}$ which is identically zero in a neighbourhood of z and since $d_\xi K(z,\xi)=0$ for $\xi \neq z$ we have the relation

$$\begin{aligned} d[(1-\chi_z)fK(z,\cdot)] &= d[(1-\chi_z)f]\wedge K(z,\cdot) \\ &= \bar{\partial}_M f\wedge K(z,\cdot)-\bar{\partial}_M(\chi_z f)\wedge K(z,\cdot) \end{aligned}$$

on $\bar{\Omega}$. Therefore $d[1-\chi_z)fK(z,\cdot)]$ is also continuous on $\bar{\Omega}$ and Stokes theorem implies that

$$\int_{b\Omega} f\wedge K(z,\cdot)=\int_{\Omega}\bar{\partial}_M f\wedge K(z,\cdot)-\int_{\Omega}\bar{\partial}_M(\chi_z f)\wedge K(z,\cdot).$$

Since formula (2) is already proved for $\chi_z f$ and therefore

$$-\int_{\Omega}\bar{\partial}_M(\chi_z f)\wedge K(z,\cdot)=\chi_z(z)f(z)=f(z)$$

this completes the proof of (2). □

6 Further properties of the kernel $K(z,\xi)$ and applications

In this section we assume that ϱ, M, z_0, M_0 and $K(z,\xi)$ are as in Theorem 1.1 and B_0, B, B_+, B_-, $K_+(z,\xi)$ and $K_-(z,\xi)$ are as in Sect. 5. Moreover we shall assume that the ball B_0 is chosen sufficiently small so that the following two propositions hold:

Proposition 6.1 *Any continuous* CR*-function defined on an open set* $\Omega\subseteq M_0$ *extends to a holomorphic function in some* $\mathbb{C}^n$*-neighbourhood of* Ω.

Proposition 6.2 *If* $B(z)\subseteq B_0$ *is an open ball centered at some point* $z\in M_0$, *then any continuous and closed* $(n, n-2)$*-form on* $\overline{B_+\cap B(z)}$ *respectively* $\overline{B_-\cap B(z)}$ *can be approximated uniformly on* $\overline{B_+\cap B(z)}$ *respectively* $\overline{B_-\cap B(z)}$ *by* $\bar{\partial}$*-exact* $C^\infty_{(n,n-2)}$*-forms on* $\mathbb{C}^n$.

That this is possible follows from the hypothesis on the Levi form of ϱ: Proposition 6.1 is a consequence of the Levi extension theorem (see e.g., Theorem 1.3.8 in [H-Le2]), since in the sense of distributions, any continuous CR-function on a hypersurface is the jump of two holomorphic functions (the latter assertion can be proved by means of the Martinelli-Bochner-Koppelmann formula). Since $\overline{B_\pm\cap B(z)}$ is starshaped if B_0 is sufficiently small, Proposition 6.2 follows from the Andreotti-Grauert-Hörmander approximation theorem (see, e.g., Theorem 8.1 in [H-Le2]).

Further for each open $\Omega\subseteq M_0$ we use the following notations:

Spaces of forms. $C^k_{(n,r)}(\Omega)(0\leqq r\leqq n-1, k=0, 1, 2)$ is the space of $C^k_{(n,r)}$-forms on Ω endowed with the topology of uniform convergence together with all derivatives of order $\leqq k$ on the compact subsets of Ω. By $D^k_{(n,r)}(\Omega)$ we denote the space of all $f\in C^k_{(n,r)}(\Omega)$ with compact support endowed with the test-function-topology of order k: a sequence f_ν converges in $D^k_{(n,r)}(\Omega)$ if it converges in $C^k_{(n,r)}(\Omega)$ and moreover there is a compact set $\omega\subset\subset\Omega$ with $\operatorname{supp} f_\nu\subseteq\omega$ for all ν. By $L^\infty_{(n,r)}(\Omega)(0\leqq r\leqq n-1)$ we denote the Banach space of (n, r)-forms with bounded measurable coefficients on Ω endowed with the sup-norm.

Spaces of currents. $C^k_{(n,r)}(\Omega)'$ and $D^k_{(n,r)}(\Omega)'$ are the spaces of continuous linear forms on $C^k_{(n,r)}(\Omega)$ and $D^k_{(n,r)}(\Omega)$ respectively, i.e. the elements in $D^k_{(n,r)}(\Omega)'$ are the $(0, n-r-1)$-currents of order k on Ω, and the elements in $C^k_{(n,r)}(\Omega)'$ are the $(0, n-r-1)$-currents of order k with compact support on Ω.

If f is a differential form with locally integrable coefficients and of degree s on Ω then we denote by $\langle f \rangle$ the current in $D^0_{(n,n-s-1)}(\Omega)'$ defined by

$$\langle f \rangle(\varphi) := \int_\Omega f \wedge \varphi \quad \text{for } \varphi \in D^0_{(n,n-s-1)}(\Omega).$$

The operator $\bar\partial_\Omega$. For $0 \leqq r \leqq n-1$ and $k=0, 1$ we denote by $\bar\partial_\Omega$ the operator

$$\bar\partial_\Omega \colon D^k_{(n,r+1)}(\Omega)' \to D^{k+1}_{(n,r)}(\Omega)'$$

defined by $(\bar\partial_\Omega T)\,\varphi := (-1)^{n-r-1} T(d\varphi)$ for $T \in D^k_{(n,r+1)}(\Omega)'$ and $\varphi \in D^{k+1}_{(n,r)}(\Omega)$.

Definition. Let $\Omega \subseteq M_0$ be an open set. Set

$$K_\Omega f(\xi) := \int_{z\in\Omega} f(z) \wedge K(z, \xi)$$

for $f \in L^\infty_{(n,n-1)}(\Omega)$ and $\xi \in \Omega$. It follows from estimate (4) and since $H(z, \xi)$ is continuous for $z \neq \xi$ that in this way a continuous linear operator

$$K_\Omega \colon L^\infty_{(n,n-1)}(\Omega) \to C^0_{(n,n-2)}(\Omega)$$

is defined. Denote by K^*_Ω the operator from $C^0_{(n,n-2)}(\Omega)'$ to $D^0_{(n,n-1)}(\Omega)'$ defined by

$$K^*_\Omega T(\varphi) = T(K_\Omega \varphi)$$

for $T \in C^0_{(n,n-2)}(\Omega)'$ and $\varphi \in D^0_{(n,n-1)}(\Omega)$. Denote by $L^1(\Omega)$ the Banach space of integrable functions on Ω and set $\langle L^1(\Omega) \rangle := \{\langle f \rangle : f \in L^1(\Omega)\}$. Then it follows from estimate (4) and the fact that $K(z, \xi)$ is continuous for $z \neq \xi$ that the values of K^*_Ω belong to $\langle L^1(\Omega) \rangle$ and the map

$$K^*_\Omega \colon C^0_{(n,n-2)}(\Omega)' \to \langle L^1(\Omega) \rangle$$

is continuous if we identify $\langle L^1(\Omega) \rangle$ with $L^1(\Omega)$.

Theorem 6.3 *Let $\Omega \subseteq M_0$ be an open set and $f \in L^\infty_{(n,n-1)}(\Omega)$. Then*

$$dK_\Omega f = f.$$

Proof. If φ is a C^1 function with compact support on Ω then, by formula (2), it is

$$\int_\Omega \varphi f = \int_{z\in\Omega} \int_{\xi\in\Omega} d\varphi(\xi) \wedge K(z, \xi) \wedge f(z) = -\int_\Omega d\varphi \wedge K_\Omega f. \quad \square$$

Lemma 6.4 (i) *Let $\varphi \in D^1_{(n,n-2)}(M_0)$. Then the form $\varphi - K_{M_0} d\varphi$ can be approximated in $C^0_{(n,n-2)}(M_0)$ by $\bar\partial$-exact $C^\infty_{(n,n-2)}$-forms on $\mathbb{C}^n$.*
(ii) *Let $z \in M_0$ and $B' \subset\subset B_0$ an open ball such that $z \notin \bar B'$. Then the form $K(z, \cdot)$ can be approximated uniformly on $M_0 \cap \bar B'$ by $\bar\partial$-exact $C^\infty_{(n,n-2)}$-forms on $\mathbb{C}^n$.*

Both assertions of this lemma are special cases of an approximation theorem of Henkin for arbitrary continuous $\bar{\partial}$-closed $(n, n-2)$-forms (see the arguments proving relation (6) in [H2]). Since the proof of this general theorem is not so easy let us give direct proofs:

Proof of Lemma 6.4(i) Set

$$K^{\pm}_{M_0} d\varphi(\xi) := \int\limits_{z \in M_0} d\varphi(z) \wedge K_{\pm}(z, \xi) \quad \text{for } \xi \in B_0 \cap B_{\pm}.$$

Then it follows from estimate (41) and since $K_{\pm}(z, \xi)$ is continuous for $z \neq \xi$ that the forms $K^{\pm}_{M_0} d\varphi$ admit continuous extensions onto $(B_0 \cap B_{\pm}) \cup M_0$. Further we set

$$\varphi_{\pm}(\xi) := \int\limits_{z \in M_0} \varphi(z) \wedge B_1(z, \xi) \quad \text{for } \xi \in B_0 \cap B_{\pm},$$

where $B_1(z, \xi)$ is the part of the Martinelli-Bochner-Koppelman kernel which is of bidegree $(0, 1)$ in z. Since φ is Hölder continuous (it is even C^1) it is well known that also the forms $\varphi_{\pm}$ admit continuous extensions onto $(B_0 \cap B_{\pm}) \cup M_0$. Moreover it follows from the Martinelli-Bochner-Koppelman formula that $\varphi = \varphi_+|_{M_0} - \varphi_-|_{M_0}$ and therefore

$$\varphi - K_{M_0} d\varphi = (\varphi_+ - K^+_{M_0} d\varphi)|_{M_0} - (\varphi_- - K^-_{M_0} d\varphi)|_{M_0}.$$

Using the relations $d_\xi B_1(z, \xi) = -\bar{\partial}_z B(z, \xi)$ and $d_\xi K^{\pm}_{M_0}(z, \xi) = -B(z, \xi)$ we see that the forms $\varphi_{\pm} - K^{\pm}_{M_0} d\varphi$ are $\bar{\partial}$-closed on $B_0 \cap B_{\pm}$. The required assertion on approximation now follows from Proposition 6.2. □

Proof of Lemma 6.4(ii) Since B' is pseudoconvex and $z \notin B'$ we can solve the equation $dG = B(z, \cdot)$ with some continuous $(n, n-2)$-form G on B'. Since $dK_{\pm}(z, \cdot) = -B(z, \cdot)$ the forms $K_{\pm}(z, \cdot) + G$ are closed on $B' \cap B_{\pm}$ and the assertion follows from Proposition 6.2 and the representation

$$K(z, \cdot)|_{M_0 \cap B'} = (K_+(z, \cdot) + G)|_{M_0 \cap B'} - (K_-(z, \cdot) + G)|_{M_0 \cap B'}. \quad \square$$

Definition. Let $\Omega \subseteq M_0$ be an open set and let f be a continuous 1-form with compact support on Ω. Then we define

$$K'_\Omega f(z) := \int\limits_{\xi \in \Omega} f(\xi) \wedge K(z, \xi) \quad \text{for } z \in \Omega.$$

It follows from estimate (3) that $K'_\Omega f$ is a continuous function on Ω.

Remark 6.5 Let $\Omega \subseteq M_0$ be an open set and let f be a continuous 1-form with compact support on Ω. Then it follows from Fubini's theorem that $K^*_\Omega \langle f \rangle = \langle K'_\Omega f \rangle$.

Theorem 6.6 *Let $\Omega \subseteq M_0$ be an open set and let $T \in C^0_{(n, n-1)}(\Omega)'$. If $\bar{\partial}_\Omega T \in C^0_{(n, n-2)}(\Omega)'$, that means if $\bar{\partial}_\Omega T$ is also of order 0, then*

$$T = -K^*_\Omega \bar{\partial}_\Omega T.$$

In particular then T is defined by an L^1 function on Ω.

Proof. If $\varphi \in D^1_{(n,n-1)}(\Omega)$ then by Theorem 6.3

$$T(\varphi) = T(dK_\Omega \varphi) = -\bar\partial_\Omega T(K_\Omega \varphi) = -K^*_\Omega \bar\partial_\Omega T(\varphi).$$

Since $D^1_{(n,n-1)}(\Omega)$ is dense in $D^0_{(n,n-1)}(\Omega)$ this implies the assertion. □

Remark 6.7 Let $\Omega \subseteq M_0$ be an open set and let $T \in C^0_{(n,n-2)}(\Omega)'$. Then it is easy to see that

$$f(z) := T(K(z,\cdot)), \qquad z \in \bar\Omega \setminus \operatorname{supp} T,$$

is a continuous function and, on $\bar\Omega \setminus \operatorname{supp} T$, $K^*_\Omega T$ is defined by f. Hence for each $T \in C^0_{(n,n-2)}(\Omega')$, $K^*_\Omega T$ is defined by an L^1 function on Ω which is continuous on $\bar\Omega \setminus \operatorname{supp} T$.

Theorem 6.8 *Let $\Omega \subseteq M_0$ be an open set and let $T \in D^0_{(n,n-1)}(\Omega)'$. If $\bar\partial_\Omega T$ is defined by a continuous 1-form on Ω then T is defined by a continuous function on Ω.*

Proof. Let $\omega \subset\subset \Omega$ be an open and relatively compact subset of Ω. It is sufficient to find a continuous function g on ω with

$$T(\varphi) = \int_\Omega g\varphi \qquad \text{for all } \varphi \in D^0_{(n,n-1)}(\omega). \tag{42}$$

Choose a C^1 function χ with compact support on Ω such that $\chi = 1$ in a neighbourhood of $\bar\omega$. Then by Theorem 6.6 we have

$$T(\varphi) = \chi T(\varphi) = -K^*_\Omega(\bar\partial_\Omega(\chi T))(\varphi) = -K^*_\Omega(\chi \bar\partial_\Omega T)(\varphi) - K^*_\Omega(d\chi \wedge T)(\varphi)$$

for all $\varphi \in D^0_{(n,n-1)}(\omega)$. In view of Remarks 6.5 and 6.7 this implies (42) if we set

$$g(z) = -(K'_\Omega(\chi f))(z) - T(d\chi \wedge K(z,\cdot)) \qquad \text{for } z \in \omega,$$

where f is the continuous 1-form defining $\bar\partial_\Omega T$. □

Corollary 6.9 *Let $\Omega \subseteq M_0$ be open and $T \in D^0_{(n,n-1)}(\Omega)'$ such that $\bar\partial_\Omega T = 0$. Then T is holomorphic in a $\mathbb{C}^n$-neighbourhood of Ω, that means there exists a holomorphic function h in some $\mathbb{C}^n$-neighbourhood of Ω such that*

$$T(\varphi) = \int_\Omega h\varphi \qquad \textit{for all } \varphi \in D^0_{(n,n-1)}(\Omega).$$

Proof. This follows from Theorem 6.8 and Proposition 6.1. □

Corollary 6.9 was obtained by Henkin (see Theorem 3 in [H 3]). Note that Theorem 6.8 does not follow from Corollary 6.9 (as the corresponding statement for $\bar\partial$) because under the given hypothesis on the Levi form of ϱ the tangential Cauchy-Riemann equation for (0, 1)-currents on M_0 cannot be solved locally (see [A-F-N]).

Theorem 6.10 *Let $T \in C^0_{(n,n-2)}(M_0)'$ such that $\bar\partial_{M_0} T = 0$. Denote by ω_T the connected component of $M_0 \setminus \operatorname{supp} T$ whose boundary contains the boundary of M_0. Then*

$$T = -\bar\partial_{M_0} K^*_{M_0} T \tag{43}$$

and

$$\operatorname{supp} K^*_{M_0} T \subseteq M_0 \setminus \omega_T. \tag{44}$$

That under the hypothesis of Theorem 6.10 there exists a L^1 function u on Ω with $\bar\partial_{M_0}\langle u\rangle = T$ and $\operatorname{supp} u \subseteq M_0 \setminus \omega_T$ was proved by Henkin (see Theorem 1′ in [H2]). The new information contained in Theorem 6.10 consists in the representation

$$\langle u\rangle = -K^*_{M_0} T. \tag{45}$$

Although the validity of this representation follows immediately from Theorem 6.6 let us give also a proof of Theorem 6.10 which is independent of Henkins result:

Proof of Theorem 6.10 Since $\bar\partial_{M_0} T = 0$ it follows from Lemma 6.4(i) that for each $\varphi \in D^1_{(n,n-2)}(M_0)$, $T(\varphi - K_{M_0} d\varphi) = 0$ and therefore

$$-\bar\partial_{M_0} K^*_{M_0} T(\varphi) = T(K_{M_0} d\varphi) = T(\varphi).$$

Since $D^1_{(n,n-2)}(M_0)$ is dense in $D^0_{(n,n-2)}(M_0)$ this proves (43).

From (43) and Corollary 6.9 it follows that on $M_0 \setminus \operatorname{supp} T$, $K^*_{M_0} T$ is defined by some holomorphic function h. Choose a open ball $B' \subset\subset B_0$ centered at z_0 such that $\operatorname{supp} T \subseteq B'$. Then, by Lemma 6.4(ii), for each $\varphi \in D^0_{(n,n-1)}(M_0 \setminus \bar B')$, the form $K_{M_0}\varphi$ can be approximated uniformly on $M_0 \cap \bar B'$ by $\bar\partial$-exact $C^\infty_{(n,n-2)}$-forms on $\mathbb{C}^n$. Since $\bar\partial_{M_0} T = 0$ and $\operatorname{supp} T \subseteq B'$ this implies that

$$\int_{M_0} h\varphi = K^*_\Omega T(\varphi) = T(K_{M_0}\varphi) = 0$$

for all such φ. Hence $h = 0$ on $M_0 \setminus \bar B'$ and, by uniqueness of holomorphic functions, $h = 0$ on ω_T, that means (44) is also proved. □

It was observed by Henkin (see Theorem 1 in [H2]) that in the case of sufficiently smooth functions Theorem 6.10 leads to an Hartogs-Bochner extension theorem on M_0 using the same arguments as in Ehrenpreis' proof of the classical Hartogs extension theorem (see the proof of Theorem 2.3.2 in [Hö]). We want to show that using representation (45) and estimate (1) one can prove this theorem also in the case of Hölder continuous functions. Let $\Omega \subset\subset M_0$ be a domain with C^2-bounday. A continuous function f on $b\Omega$ will be called a CR-function if

$$\int_{b\Omega} f\, d\varphi = 0 \tag{46}$$

for all $C^\infty_{(n,n-3)}$-forms φ on $\mathbb{C}^n$.

Theorem 6.11 *Suppose $M_0 \setminus \bar\Omega$ is connected and let f be a Hölder continuous* CR*-function on $b\Omega$. Then there exists a (unique) continuous function F on $\bar\Omega$*

which extends holomorphically to some $\mathbb{C}^n$-neighbourhood of Ω such that $F(z)=f(z)$ for all $z\in b\Omega$. For $z\in\Omega$ this function is given by

$$F(z)=\int\limits_{\xi\in b\Omega} f(\xi)\,K(z,\xi). \tag{47}$$

Proof. (All positive constants will be denoted by the same letter C.) First we note that

$$\int\limits_{\xi\in b\Omega} K(z,\xi)=\begin{cases}1 & \text{for } z\in\Omega\\ 0 & \text{for } z\in M_0\setminus\bar\Omega.\end{cases} \tag{48}$$

If $z\in\Omega$ this follows from (2) and for $z\in M_0\setminus\bar\Omega$ this follows from Stokes' theorem and the fact that $d_\xi K(z,\xi)=0$. Denote by $T\in C^0_{(n,n-2)}(M_0)'$ the current defined by

$$T(\varphi)=\int\limits_{\beta\Omega} f\varphi \qquad \text{for } \varphi\in C^0_{(n,n-2)}(M_0).$$

Then by (46), $\bar\partial_{M_0} T=0$ and it follows from Theorem 6.10 that $\operatorname{supp} K^*_{M_0} T\subseteq\bar\Omega$ ($M_0\setminus\bar\Omega$ is connected) and $T=-\bar\partial_{M_0} K^*_{M_0} T$. Since by Remark 6.7 on $M_0\setminus b\Omega$, $K^*_{M_0} T$ is defined by the function

$$z\to\int\limits_{\xi\in b\Omega} f(\xi)\,K(z,\xi)$$

this implies that

$$\int\limits_{\xi\in b\Omega} f(\xi)\,K(z,\xi)=0 \qquad \text{for } z\in M_0\setminus\bar\Omega \tag{49}$$

and, by Corollary 6.9, the function F defined by (47) extends holomorphically to some $\mathbb{C}^n$-neighbourhood of Ω.

It remains to prove that

$$\lim_{\Omega\ni z\to\xi_0} F(z)=f(\xi_0) \qquad \text{for all } \xi_0\in b\Omega.$$

For $z\in M_0$ denote by ξ_z a point in $b\Omega$ with $|z-\xi_z|=\operatorname{dist}(z,b\Omega)$ (ξ_z is uniquely determined if z is close to $b\Omega$). Then $f(\xi_z)\to f(\xi_0)$ if $z\to\xi_0$. Therefore it is sufficient to prove that

$$\lim_{\Omega\ni z\to\xi_0}(F(z)-f(\xi_z))=0 \qquad \text{for all } \xi_0\in b\,\Omega. \tag{50}$$

To prove (50) we fix some $\xi_0\in b\Omega$. Denote by $B_r(\xi_0)$, $r>0$ the open ball of radius r centered at ξ_0. Set

$$I_r(z):=\int\limits_{\xi\in b\Omega\cap B_r(\xi_0)} |f(\xi)-f(\xi_z)|\,\|K(z,\xi)|_{b\Omega}\|\,d\lambda(\xi)$$

for $r>0$ and $z\in M_0\setminus(b\Omega\cap B_r(\xi_0))$, where $d\lambda(\xi)$ is the Euclidean volume form of $b\Omega$. Since $|\xi-\xi_z|\leqq 2|\xi-z|$ and f is Hölder continuous there exists $0<\alpha_0<1$ with

$$|f(\xi)-f(\xi_z)|\leqq C|\xi-z|^{\alpha_0} \tag{51}$$

for all $\xi\in b\Omega$ and $z\in M_0$. Fix $0<\alpha<\alpha_0$ and prove that then

$$I_r(z)\leqq Cr^{\alpha} \tag{52}$$

for all $r>0$ and $z\in M_0\setminus(b\Omega\cap B_r(\xi_0))$.

Proof of estimate (52) Since $K(z,\xi)$ is of maximal holomorphic degree in ξ one has

$$\|K(z,\xi)|_{b\Omega}\|\leqq C\,\|K(z,\xi)\|\,\|\partial\varrho(\xi)|_{b\Omega}\| \tag{53}$$

for all $\xi\in b\Omega$ and $z\in M_0$ with $z\neq\xi$. Set

$$u(z,\xi):=\operatorname{Im}\sum_{j=1}^{n}\frac{\partial\varrho(\xi)}{\partial\xi_i}(\xi_i-z_i).$$

Then

$$|u(z,\xi)|\leqq C\,\delta(z,\xi) \tag{54}$$

and

$$\|\partial\varrho(\xi)|_{b\Omega}\|\leqq C(\|d_\xi u(z,\xi)|_{b\Omega}\|+|\xi-z|) \tag{55}$$

for all $\xi\in b\Omega$ and $z\in M_0$. Set $\varepsilon=(\alpha_0-\alpha)/2$. Then it follows from (51)–(55) and (1) that

$$\begin{aligned} I_r(z)\leqq C&\int_{\xi\in b\Omega\cap B_r(\xi_0)}\frac{\|d_\xi u(z,\xi)|_{b\Omega}\|\,d\lambda(\xi)}{(|u(z,\xi)|+|\xi-z|^2)|\xi-z|^{2n-3-\alpha-\varepsilon}}\\ &+C\int_{\xi\in b\Omega\cap B_r(\xi_0)}\frac{d\lambda(\xi)}{|\xi-z|^{2n-2-\alpha}}\end{aligned} \tag{56}$$

for all $r>0$ and $z\in M_0\setminus(b\Omega\cap B_r(\xi_0))$. It is clear that the second integral in (56) is bounded by Cr^{α}. To estimate the first integral we use the trick of Range and Siu (see the proof of Proposition (3.7) in [R-S]), which allows us to consider $u(z,\cdot)$ as a local coordinate. So we obtain that this integral is bounded by

$$C\int_{\substack{x\in\mathbb{R}^{2n-2}\\ |x|<r}}\frac{dx_1\wedge\ldots\wedge dx_{2n-2}}{(|x_1|+|x|^2)|x|^{2n-3-\alpha-\varepsilon}}.$$

Integrating first with respect to x_1 we see that the last integral is also bounded by Cr^{α}. Hence estimate (52) is proved.

End of proof of (50) For $r>0$ we set

$$H_r(z)=\int\limits_{\xi\in b\Omega\cap B_r(\xi_0)}(f(\xi)-f(\xi_z))\,K(z,\xi)$$

if $z\in M_0\setminus(b\Omega\cap B_r(\xi_0))$ and

$$G_r(z)=\int\limits_{\xi\in b\Omega\cap B_r(\xi_0)}(f(\xi)-f(\xi_z))\,K(z,\xi)$$

if $z\in M_0\setminus(b\Omega\setminus B_r(\xi_0))$. Then by (52) it is

$$|H_r(z)|\leqq Cr^\alpha \tag{57}$$

for all $r>0$ and $z\in M_0\setminus(b\Omega\cap B_r(\xi_0))$. Since by (48) and (49), $H_r(z)+G_r(z)=0$ if $z\in M_0\setminus\bar\Omega$ and G_r is continuous on $M_0\cap B_r(\xi_0)$ this implies that

$$|\lim_{\Omega\ni z\to\xi_0}G_r(z)|=|G_r(\xi_0)|\leqq Cr^\alpha \tag{58}$$

for all $r>0$. Moreover it follows from (48) that

$$H_r(z)+G_r(z)=F(z)-f(\xi_z)$$

if $z\in\Omega$. In view of (57) and (58) this implies that for all $r>0$ we have

$$\limsup_{\Omega\ni z\to\xi_0}|F(z)-f(\xi_z)|\leqq Cr^\alpha.\quad\square$$

Remarks to Theorem 6.11 (i) It follows from this theorem (by standard arguments) that

$$|F(z)|\leqq\max_{\xi\in b\Omega}|f(\xi)|\qquad\text{for all } z\in\Omega.$$

Hence the assertion of the theorem holds for each continuous CR-function f on $b\Omega$ which can be approximated uniformly by Hölder continuous CR-functions. It is not clear if this is possible for all continuous CR-functions on $b\Omega$.

(ii) We do not assume that the boundary $b\Omega$ is a CR-manifold. Note however that, by the hypothesis on the Levi form of ϱ, the set of points in $b\Omega$ with complex tangent space is nowhere dense in $b\Omega$. .

(iii) The hypothesis that $b\Omega$ is of class C^2 is necessary for the Range-Siu trick in the proof.

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On a general class of elliptic equations in divergence form

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1 Introduction

Much is known for special types of quasilinear non-uniformly elliptic equations (cf. for example [F, GT, S1, S2]). The premiere example has for a long time been the minimal surface equation and many results proved originally for minimal surfaces have later been proven to hold in some generality. On the other hand a general theory is still very much lacking. In particular, regularity theorems which do not exploit some special structure of the coefficients do not appear to be known.

My purpose here is of extending a technique that goes back to De Giorgi in order to prove a partial regularity theorem for a very general class of equations in divergence form. It should be noted that in this context one cannot in general expect solutions to be regular everywhere (cf. § 2). Similar results have been obtained in the study of variational problems. See for example [AG], where a partial regularity theorem is proved for the local minimums of convex functionals with linear growth, whose Euler-Lagrange equations are of the type considered by us.

The plan of the paper is as follows. In § 2 we describe the class of equations which are the object of our study and establish the notation. In § 3 we derive a priori estimates for the supremum of a solution $u(x)$ and for the L^1 norm of its gradient. In § 4 we prove an excess decay lemma. As consequences of this lemma we derive in § 5 a partial regularity theorem and a Liouville type result.

2 Preliminaries

We are going to study a class of quasilinear non-uniformly elliptic partial differential equations in divergence form whose coefficients are continuously differenti-

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able with respect to the gradient of $u(x)$. To be precise, let Ω be a domain in $\mathbf{R}^n$, $x=(x_1, \ldots, x_n)$ and $u(x)$ a function defined on Ω. Let us consider the equation

$$\sum_{i=1}^{n} D_i A^i(p)=0, \tag{1}$$

where $p=(p_1, \ldots, p_n)$ is the gradient vector $(D_1 u(x), \ldots, D_n u(x))$. We only assume that the coefficients $A^i(p)$ belong to $C^1(\mathbf{R}^n)$ and are bounded in such a norm:

$$|A^i(p)| \leqq C, \quad |\partial_{pj} A^i(p)| \leqq C \quad i,j=1, \ldots, n \tag{2}$$

for all $p \in \mathbf{R}^n$, where C is a fixed constant. Note that if the functions $A^i(p)$ satisfy (2), Eq. (1) cannot be uniformly elliptic; therefore we only require pointwise ellipticity

$$\sum_{i,j=1}^{n} \partial_{pj} A^i(p)\, \xi_i\, \xi_j > 0 \quad \forall \xi \in \mathbf{R}^n \sim 0 \quad \text{and} \quad \forall p \in \mathbf{R}^n.$$

However, it can then be assumed without loss of generality that $A^i(0)=0$ and that the following inequalities hold

$$\sum_{i=1}^{n} A^i(p)\, p_i \geqq \begin{cases} c|p| & \text{for } |p| \leqq 1 \\ c|p|^2 & \text{for } |p|<1 \end{cases} \tag{3}$$

(see [F] or [S2]). One should keep in mind that if we use the notation

$$a_{ij}(p)=\int_0^1 \partial_{pj} A^i(tp)\, dt,$$

we can write Eq. (1) as

$$\sum_{i,j=1}^{n} D_i(a_{ij}(p)\, p_j)=0.$$

Because of the continuity assumptions we also have

$$\sup_{i,j=1,\ldots,n} |a_{ij}(p)-a_{ij}(0)| \leqq \omega(|p|) \quad \text{with} \quad \limsup_{|p| \to 0} \omega(|p|)=0. \tag{4}$$

We often make use of the following rescaling:

$$u_R(x)=\frac{1}{R} u(x_0+Rx). \tag{5}$$

Clearly if $u(x)$ is a solution of Eq. (1) on the ball $B_R(x_0)$, then $u_R(x)$ is a solution over $B_1(0)$. Therefore we will prove many estimates on the unit ball and use (5) to obtain the properly scaled inequalities.

Note that the minimal surface equation is a particular case of Eq. (1) where we take

$$A^i(p)=\frac{p_i}{\sqrt{1+|p|^2}}.$$

On the other hand, (1) includes many examples of equations not of minimal surface type. The conditions (2) defining our class of equations are natural and easy to state. However the conclusions of this paper continue to hold if we allow more general coefficients $A^i(x, z, p)$ and relax the differentiability requirement, so long as the structural conditions (3) and (4) are satisfied uniformly for $x\in\Omega$, $z\in\mathbf{R}$. An important example to which our arguments apply verbatim is

$$A^i(x, p)=\frac{\alpha(x)\, p_i}{\sqrt{1+\alpha(x)|p|^2}}, \tag{6}$$

where $\alpha(x)=\alpha(x_1)$ satisfies $0<c_1\leqq\alpha(x)\leqq c_2$. It is easily checked that

$$u(x)=u(x_1)=\int_0^{x_1}\frac{k}{\sqrt{\alpha(t)\,(\alpha(t)-k^2)}}\,dt$$

is a solution of (1), (6) for any value of the constant k. Therefore if we choose

$$\alpha(x)=k^2+\min(|x_1|^{2\beta}, \lambda^2),$$

where k, $\lambda\neq 0$ and $0<\beta<1$, we have $u(x)\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^n)$ with $u(x)\approx x_1^{1-\beta}$ in a neighbourhood of the plane $x_1=0$ and $u(x)=O(|x|)$ as $|x|$ goes to infinity.

The previous example shows that in the generality allowed by our tools we can have $W^{1,1}$ solutions which are not C^1. However we are not aware of an analogous example for Eq. (1) as stated (that is when the coefficients A^i depend solely on p).

Our work was partially motivated by [S2] where a Liouville type theorem is proved for C^2 solutions of (1) in the two-dimensional case. In that paper, Simon asks whether in higher dimensions every entire solution of Eq. (1) with linear growth is affine. In view of (6) the answer is negative if we allow the coefficients to depend on x, however we will show that every entire solutions with sublinear growth is constant.

We should also mention that in [T] Trudinger studied a class of equations substantially overlapping with ours and proved Hanack inequalities for $W^{2,n}$ solutions (with constants depending on the particular solution).

3 The sup bound

Let $u(x)$ be in the Sobolev space $W^{1,1}(\Omega)$. We will say that $u(x)$ is a weak solution of Eq. (1) if it satisfies one of the following two equivalent weak formulations:

$$\int_\Omega \sum_{i=1}^n A^i(Du)\, D_i\zeta(x)\,dx=0 \tag{7}$$

or

$$\int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(du)\, D_j u(x)\, D_i \zeta(x)\, dx = 0 \tag{8}$$

for every test function $\zeta(x) \in C_0^1(\Omega)$. In this section we will prove the following estimate for the supremum of $u(x)$:

Proposition 1 *Let $u(x) \in W^{1,1}(B_R(x_0))$ be a (weak) solution of Eq. (1). Then*

$$\sup_{x \in B_{R/2}(x_0)} |u(x)| \leqq C_1 R \left(\frac{1}{R^{n+1}} \int_{B_R(x_0)} |u(x)|\, dx \right)$$

and

$$\int_{B_{R/2}(x_0)} |Du(x)|\, dx \leqq \frac{C_2}{R} \int_{B_R(x_0)} |u(x)|\, dx$$

for fixed constants C_1, C_2 depending only on the bounds (2), (3).

Remark. It is not difficult to derive the estimate

$$\sup_{x \in B_{R/2}(x_0)} |u(x)| \leqq CR \left(\frac{1}{R^{n+1}} \int_{B_R(x_0)} |u(x)|\, dx + K \right) \tag{9}$$

from the standard Moser iteration argument (see for istance [GT]). However it is crucial for our applications to eliminate the constant K appearing in (9). In this form the sup bound seems to be new.

Proof. The proof is still based on the Moser iteration argument. Because of the non-uniform ellipticity, we will have to distinguish the two regions $|p| < 1$ and $|p| \geqq 1$. We can assume without loss of generality that $x_0 = 0$ and $R = 1$.

Let $u_+(x) = \max(u(x), 0)$. We first use the test function $\zeta(x) = u_+^{\alpha}(x)\ \varphi^2(x)$, where $\varphi(x) \in C_0^1(B_1)$ and α is a positive number to be determined later. From Eq. (7) we obtain

$$\alpha \int_{B_1} \sum_{i=1}^{n} A^i(Du)\, u_+^{\alpha-1}(x)\, D_i u_+(x)\, \varphi^2(x)\, dx$$

$$+ 2 \int_{B_1} \sum_{i=1}^{n} A^i(Du)\, u_+^{\alpha}(x)\, \varphi(x)\, D_i \varphi(x)\, dx = 0,$$

which by (2) and (3) gives

$$\int_{B_1 \cap \{|Du| \geqq 1\}} |D u\, |u_+^{\alpha-1}(x)\, \varphi^2(x)\, dx \leqq c \int_{B_1} u_+^{\alpha}(x) |D\varphi|\, \varphi(x)\, dx. \tag{10}$$

An analogous estimate on the part of the unit ball where $|Du|$ is small is obtained by using the test function $\zeta(x)=u_+^{\alpha-1}(x)\,\varphi^2(x)$ in (8):

$$(\alpha-1)\int_{B_1}\sum_{i,j=1}^{n}a_{ij}(Du)\,D_j u D_i u u_+^{\alpha-2}(x)\,\varphi^2(x)\,dx$$

$$+2\int_{B_1}\sum_{i,j=1}^{n}a_{ij}(Du)\,D_j u u_+^{\alpha-1}(x)\,D_i\varphi\,\varphi(x)\,dx=0.$$

This yields (after using the Cauchy-Schwartz inequality)

$$\int_{B_1}\sum_{i,j=1}^{n}a_{ij}(Du)\,D_i u D_j u u_+^{\alpha-2}(x)\,\varphi^2(x)\,dx \tag{11}$$

$$\leqq c\int_{B_1}\sum_{i,j=1}^{n}a_{ij}(Du)\,D_i\varphi\,D_j\varphi\,u_+^{\alpha}(x)\,dx$$

and hence

$$\int_{B_1\cap\{|Du|<1\}}|Du|^2u_+^{\alpha-2}(x)\,\varphi^2(x)\,dx\leqq c\int_{B_1}u_+^{\alpha}(x)|D\varphi|^2\,dx. \tag{12}$$

We can now use (12) together with the Cauchy-Schwartz inequality to get

$$\int_{B_1\cap\{|Du|<1\}}|Du|\,u_+^{\alpha-1}(x)\,\varphi^2(x)\,dx \tag{13}$$

$$\leqq\int_{B_1\cap\{|Du|<1\}}|Du|^2u_+^{\alpha-2}(x)\,\varphi^2(x)\,dx+\int_{B_1\cap\{|Du|<1\}}u_+^{\alpha}(x)\,\varphi^2(x)\,dx$$

$$\leqq c\int_{B_1}u_+^{\alpha}(x)\,(\varphi^2(x)+|D\varphi|^2)\,dx.$$

Combining (10) and (13) we derive our main estimate

$$\int_{B_1}|Du|\,u_+^{\alpha-1}(x)\,\varphi^2(x)\,dx\leqq c\int_{B_1}u_+^{\alpha}(\varphi^2(x)+|D\varphi|^2+\varphi(x)|D\varphi|)\,dx.$$

If φ is a standard cut-off function in the unit ball, that is $\varphi(x)$ is identically one in the ball $B_{1/2}$, zero outside B_1, and $\sup|D\varphi|\leqq 2$, the last estimate can be written as

$$\int_{B_{1/2}}|Du|\,u_+^{\alpha-1}(x)\,dx\leqq c\int_{B_1}u_+^{\alpha}(x)\,dx$$

for every real number $\alpha>1$. Of course one can repeat the same computation with $u_-(x)=\max(0,\,-u(x))$ in place of $u_+(x)$. Combining the results we have

$$\int_{B_{1/2}}|Du|\,|u(x)|^{\alpha-1}\,dx\leqq c\int_{B_1}|u(x)|^{\alpha}\,dx. \tag{14}$$

This inequality, together with the Sobolev inequality, is exactly what is needed to implement the Moser iteration scheme. The remainder of the argument that

gives the supremum estimate is standard; for details one can, for example, check [G].

In order to obtain the gradient estimate we use (10) with $\alpha=1$ to get

$$\int_{B_1\cap\{|Du|>1\}} |Du|\,\varphi(x)\,dx \leqq c\int_{B_1} |u(x)|\,dx \leqq c\Big(\int_{B_1} |u(x)|^2\,dx\Big)^{1/2}.$$

Also from the Hölder inequality and (12) with $\alpha=2$, we have

$$\begin{aligned}\int_{B_1\cap\{|Du|<1\}} |Du|\,\varphi(x)\,dx &\leqq c\Big(\int_{B_1\cap\{|Du|<1\}} |Du|^2\,\varphi^2(x)\,dx\Big)^{1/2}\\ &\leqq c\Big(\int_{B_1} |u(x)|^2\,dx\Big)^{1/2}.\end{aligned}$$

Combining the two inequalities we arrive at

$$\int_{B_1} |Du|\,\varphi(x)\,dx \leqq C\Big(\int_{B_1} |u(x)|^2\,dx\Big)^{1/2}$$

However it is a standard consequence of the sup bound (see [G] for details) that one can bound the L^2 norm of $u(x)$ by its L^1 norm on a bigger domain. The gradient bound follows immediately. □

4 An excess decay lemma

The main estimate we derive in this paper is an excess decay type of lemma. This approach to regularity goes back to De Giorgi and it has been used successfully in main situations.

The excess is a quantitative expression of how close the graph of $u(x)$ is to a plane over a given domain. The idea consists in noticing that if the excess is small enough, the function $u(x)$ can be well approximated by the solution of a suitable constant coefficent equation. One can then use elliptic estimates for such a solution to show that the excess decays at a fixed rate as the domain shrinks.

For our problem we define the excess E by

$$E(x,\rho,\ell)=\frac{1}{\rho^{n+1}}\int_{B_\rho(x)} |u(x)-\ell(x)|\,dx,$$

where $\ell(x)$ is a linear function

$$\ell(x)=\sum_{i=1}^{n} a_i x_i + b.$$

We need the following technical lemma, giving precise conditions under which we can approximate a function $f(x)$ in the L^1 norm by the solution of a certain constant coefficent equation. (A similar lemma can be found in [S3].)

Lemma 1 *Given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $f(x) \in W^{1,1}(B)$ satisfies*

$$\int_B |Df(x)|\, dx \leqq 1$$

and

$$\left| \int_B \sum_{i,j=1}^{n} a_{ij}(0)\, D_i f\, D_j \zeta\, dx \right| \leqq \delta \sup_{x \in B} |D\zeta(x)|$$

for any $\zeta(x) \in C_0^\infty(B)$, then there exists a smooth function $w(x)$ on B which satisfies $\sum_{i,j=1}^{n} a_{ij}(0)\, D_{ij}\, w(x) = 0$ and such that

$$\int_B |f(x) - w(x)|\, dx < \varepsilon, \qquad \int_B |Dw(x)|\, dx \leqq 1.$$

Proof. Suppose the lemma does not hold. Then there exists $\varepsilon > 0$ and a sequence $\{f_k(x)\} \in W^{1,1}(B)$ such that

$$\left| \int_B \sum_{i,j=1}^{n} a_{ij}(0)\, D_i f_k\, D_j \zeta\, dx \right| \leqq k^{-1} \sup_{x \in B} |D\zeta(x)|$$

for any $\zeta(x) \in C_0^\infty(B)$ and

$$\int_B |Df_k(x)|\, dx \leqq 1,$$

but so that

$$\int_B |f_k(x) - w(x)|\, dx \geqq \varepsilon \tag{15}$$

whenever $w(x)$ is a solution of $\sum_{i,j=1}^{n} a_{ij}(0)\, D_{ij}\, w(x) = 0$ with

$$\int_B |Dw(x)|\, dx \leqq 1.$$

If we let $\lambda_k = \omega_n^{-1} \int_B f_k(x)\, dx$, the Poincaré inequality implies

$$\int_B |f_k(x) - \lambda_k|\, dx \leqq c \int_B |Df_k(x)|\, dx \leqq c,$$

and hence, by Rellich's theorem there exists a subsequence $\{f_{k'}(x) - \lambda_{k'}\}$ that converges in $L^1(B)$ to a function $f(x)$ of bounded variation with

$$\int_B |Df| \leqq 1.$$

Since we have

$$\int_B \sum_{i,j=1}^{n} a_{ij}(0) f(x) D_{ij} \zeta(x)\, dx = 0$$

for every cut-off function $\zeta(x)$, standard elliptic regularity theory tells us that $f(x)$ is actually a smooth solution of the equation $\sum_{i,j=1}^{n} a_{ij}(0) D_{ij} f(x)$. The L^1 convergence means that

$$\int_B |f_{k'}(x) - \lambda_{k'} - f(x)|\, dx \to 0 \quad \text{as } k \uparrow \infty.$$

But the functions $\{f(x)+\lambda_{k'}\}$ are also solutions of $\sum_{i,j=1}^{n} a_{ij}(0) D_{ij} w(x)$, which contradicts (15). □

Remark. Recall that if $f(x)$ is a solution of the constant coefficent equation $\sum_{i,j=1}^{n} a_{ij}(0) D_{ij} f(x)$, elliptic regularity theory [GT] gives the following estimates:

$$\sup_{B_{1/2}} |D^{(q)} f| \leqq c \| Df \|_{L^1(B)}$$

for every integer $q \geqq 2$, with $c = c(q, n)$. In particular, this implies by a Taylor expansion that

$$\sup_{B_\eta} |f - g| \leqq c\eta^2 \| Df \|_{L^1(B)} \tag{16}$$

if g is the linear function $g(x) = f(0) + \sum_{i=1}^{n} x^i D_i f(0)$.

We can now prove the excess decay lemma:

Lemma 2 *There exist fixed* $\varepsilon > 0$, $\eta \in (0, 1/4)$ *and* $\alpha \in (0, 1)$ *such that if* $u(x)$ *is a* $W^{1,1}$ *solution of* (1) *with* $E(x_0, \rho, \ell_0) < \varepsilon$, *then*

$$E(x_0, \eta\rho, \ell_1) \leqq \eta^\alpha E(x_0, \rho, \ell_0)$$

for some linear function $\ell_1(x)$, *where* ε, η *and* α *depend on* $|D\ell_0|$ *and on the constants in* (2), (3), (4).

Proof. Note that the function $u(x) - \ell(x)$ also satisfies an equation of type (1) with $\tilde{A}^i(p) = A^i(p + D\ell) - A^i(D\ell)$. We can therefore assume without loss of generality that $x_0 = 0$ and $\ell_0 = 0$, keeping in mind that the constants (2), (3), (4) for the new equation may now depend on $|D\ell_0|$. In order to apply Lemma 1 we

need an estimate for $\left| \int_{B_{\rho/4}} \sum_{i,j=1}^{n} a_{ij}(0)\, D_i u D_j \zeta \, dx \right|$ where $\zeta(x) \in C_0^\infty(B_{\rho/4})$. Using (4) we can write

$$(17) \quad \int_{B_{\rho/4}} \sum_{i,j=1}^{n} a_{ij}(0)\, D_i u D_j \zeta \, dx = \int_{B_{\rho/4}} \sum_{i,j=1}^{n} (a_{ij}(0) - a_{ij}(Du))\, D_i u D_j \zeta \, dx$$

$$\leqq \sup_{x \in B_{\rho/4}} |D\zeta| \int_{B_{\rho/4}} \sum_{i,j=1}^{n} |a_{ij}(0) - a_{ij}(Du)|\, |Du| \, dx$$

$$\leqq \sup_{x \in B_{\rho/4}} |D\zeta| \left(c \int_{B_{\rho/4} \cap \{|Du| \geqq L\}} |Du| \, dx + \omega(L) \int_{B_{\rho/4} \cap \{|Du| < L\}} |Du| \, dx \right).$$

We can estimate the first term on the right hand side by using Eq. (11) with $\alpha = 2$ and the sup bound

$$\int_{B_{\rho/4} \cap \{|Du| \geqq L\}} |Du| \, dx \leqq (c(L)/\rho^2) \int_{B_{\rho/2}} |u(x)|^2 \, dx \leqq c(L)\, E^2(0, \rho, 0)\, \rho^n.$$

The gradient estimate of Proposition 1 takes care of the second term in Eq. (17):

$$\int_{B_{\rho/4} \cap \{|Du| < L\}} |Du| \, dx \leqq (c/\rho) \int_{B_\rho} |u(x)| \, dx = cE(0, \rho, 0)\, \rho^n.$$

Hence we get

$$\left| \int_{B_{\rho/4}} \sum_{i,j=1}^{n} a_{ij}(0)\, D_i u D_j \zeta \, dx \right| \leqq c(c(L)\, E + \omega(L))\, E(0, \rho, 0)\, \rho^n \sup_{x \in B_{\rho/4}} |D\zeta|.$$

With an appropriate choice of L and E, the quantity $c(L)\, E + \omega(L)$ can be made as small as we like. In particular, we can make it smaller than the δ of Lemma 1. Hence, letting $f(x) = u(x)/E$ and rescaling properly, we can conclude that there exists a function $w(x)$, solution of the equation $\sum_{i,j=1}^{n} a_{ij}(0)\, D_{ij} w(x) = 0$, such that

$$\int_{B_{\rho/4}} |u(x) - w(x)| \, dx \leqq \delta E(0, \rho, 0)\, \rho^{n+1}$$

and

$$(18) \quad \int_{B_{\rho/4}} |Dw(x)| \, dx \leqq cE(0, \rho, 0)\, \rho^n,$$

provided $E(0, \rho, 0)$ is small enough. We now choose $\ell(x) = w(0) + \sum_{i=1}^{n} x_i D_i w(0)$ and use the suitable version of (16):

$$\sup_{x \in B_{\eta\sigma}} |w(x) - \ell(x)| \leqq (c\eta^2/\sigma^{n-1}) \int_{B_\sigma} |Dw| \, dx.$$

Finally, we are ready to estimate $E(0, \eta\rho/4, \ell)$ (in the following computation we set $\sigma = \rho/4$).

$$\begin{aligned}
&\frac{1}{(\eta\sigma)^{n+1}} \int_{B_{\eta\sigma}} |u(x) - \ell(x)|\, dx \\
&\leqq \frac{1}{(\eta\sigma)^{n+1}} \int_{B_{\eta\sigma}} |u(x) - w(x)|\, dx + \frac{1}{(\eta\sigma)^{n+1}} \int_{B_{\eta\sigma}} |w(x) - \ell(x)|\, dx \\
&\leqq \frac{c\delta}{\eta^{n+1}} E(0, \rho, 0) + \frac{c\eta^2}{\sigma^{n-1}} \frac{(\eta\sigma)^n}{(\eta\sigma)^{n+1}} \int_{B_\sigma} |Dw|\, dx \\
&\leqq \frac{c\delta}{\eta^{n+1}} E(0, \rho, 0) + c\eta E(0, \rho, 0).
\end{aligned}$$

Choosing constants $\tilde{\eta} \in (0, 1/4)$, $\alpha \in (0, 1)$, $\delta > 0$ such that

$$4c\tilde{\eta} \leqq \frac{\tilde{\eta}^\alpha}{2} \quad \text{and} \quad \frac{c\delta}{(4\tilde{\eta})^{n+1}} \leqq \frac{\tilde{\eta}^\alpha}{2},$$

we find

$$E(0, \tilde{\eta}\rho, \ell) \leqq \tilde{\eta}^\alpha E(0, \rho, 0)$$

for $\eta = 4\tilde{\eta}$, giving the conclusion of the lemma. □

5 A partial regularity theorem

Once we have established the excess decay lemma, it is fairly standard to derive a partial regularity theorem.

Theorem 1 *Let $u(x) \in W^{1,1}(\Omega)$ be a solution of* (1). *There exists $\varepsilon > 0$ such that if $E(x_0, \rho_0, \ell_0) < \varepsilon$ for some $x_0 \in \Omega$, $\rho_0 > 0$ and ℓ_0 linear, then*

$$|u(x) - \ell_1(x)|_{C^{1,\alpha}(B_{\rho_0/4}(x_0))} \leqq c E(x_0, \rho_0, \ell_0)$$

for some $\ell_1(x)$. Here ε is a fixed constant depending on $|D\ell_0|$ and on the constants in (2), (3), (4).

Proof. Again without loss of generality take $x_0 = 0$ and $\ell_0 = 0$. If $y \in B_{\rho/4}$, we have

$$\frac{1}{(\rho/2)^{n+1}} \int_{B_{\rho/2}(y)} |u(x)|\, dx \leqq \frac{1}{(\rho/2)^{n+1}} \int_{B_\rho(0)} |u(x)|\, dx \leqq 2^{n+1} E(0, \rho, 0).$$

Therefore, for $E(0, \rho, 0)$ small enough, there exists some linear function $\ell_1^y(x)$ such that

$$E(y, \eta\rho/2, \ell_1^y) < \eta^\alpha E(y, \rho/2, 0),$$

which we can iterate to find

$$E(y, \eta^k \rho/2, \ell_k^y) < \eta^{k\alpha} E(y, \rho/2, 0) \leqq c \eta^{k\alpha} E(0, \rho, 0). \tag{19}$$

It follows from the previous discussion that one can take

$$\ell_{k+1}^y(x) - \ell_k^y(x) = w_{k+1}(y) + \sum_{i=1}^{n} x_i D_i w_{k+1}(y)$$

where the functions $w_k(x)$ are solutions of constant coefficent equations. Standard estimates (see (18), (19)) imply that

$$|w_k(y)| \leqq c \eta^{k\alpha} E(0, \rho, 0)\, \eta^k \rho$$

and

$$|D w_k(y)| \leqq c \eta^{k\alpha} E(0, \rho, 0)$$

for every point $y \in B_{\rho/4}$. Hence

$$\sup_{x \in B_r(y)} |\ell_k^y(x) - \ell_{k-1}^y(x)| \leqq c \eta^{k\alpha} E(0, \rho, 0)(\eta^k \rho + r).$$

Therefore the sequence ℓ_k^y of approximating planes centered at y converges to some plane $\ell^y(x)$ and we have

$$\sup_{x \in B_r(y)} |\ell_k^y(x) - \ell^y(x)| \leqq c \eta^{k\alpha} E(0, \rho, 0)(\eta^k \rho + r).$$

Given any $r > 0$, pick k such that $\eta^{k+1} \rho < r \leqq \eta^k \rho$ and evaluate:

$$\begin{aligned}
&\frac{1}{r^{n+1}} \int_{B_r(y)} |u(x) - \ell^y(x)|\, d\,x \\
&\quad \leqq \frac{1}{r^{n+1}} \int_{B_r(y)} |u(x) - \ell_k^y(x)|\, d\,x + \frac{1}{r^{n+1}} \int_{B_r(y)} |\ell_k^y(x) - \ell^y(x)|\, d\,x \\
&\quad \leqq \frac{1}{(\eta^{k+1}\rho)^{n+1}} \int_{B_{\eta^k \rho}(y)} |u(x) - \ell_k^y(x)|\, d\,x + \frac{1}{(\eta^{k+1}\rho)^{n+1}} \int_{B_{\eta^k \rho}(y)} |\ell^y(x) - \ell_k^y(x)|\, d\,x \\
&\quad \leqq c \eta^{k\alpha} E(0, \rho, 0) \leqq c\,(r/\rho)^\alpha E(0, \rho, 0).
\end{aligned}$$

We have proved that for all $y \in B_{\rho/4}$ there exists a linear space $\ell^y(x)$ such that

$$E(y, r, \ell^y) \leqq c(r/\rho)^\alpha E(0, \rho, 0).$$

This immediately gives a Hölder estimate for Du. Let $\beta_y = D\ell^y(x)$. Since $u(x) - \ell^y(x)$ also satisfies an equation of type (1), it follows from our gradient estimate that

$$\int_{B_\sigma(y)} |Du - \beta_y|\, d\,x \leqq (c/\sigma) \int_{B_{2\sigma}(y)} |u(x) - \ell^y(x)|\, d\,x \leqq c\, \sigma^{n+\alpha} E(0, \rho, 0)$$

for every $\sigma<\rho/4$ and for every $y\in B_{\rho/4}$. Since β_y is a constant vector we can apply Campanato's lemma (see [G]) to obtain

$$|Du(x)-Du(y)|\leqq cE(0,\rho,0)\frac{|x-y|^\alpha}{\rho^\alpha} \tag{20}$$

for every $x, y\in B_{\rho/4}$. □

As an immediate consequence of the partial regularity theorem and the Lebesgue theorem we obtain:

Corollary 1 *Let $u(x)$ be a $W^{1,1}$ solution of Eq. (1) in Ω. Then $u(x)\in C^{1,\alpha}(\Omega_0)$, where Ω_0 is an open set such that $H^n(\Omega\sim\Omega_0)=0$.*

Proof. Let $u_{x_0,\rho}$ be the average of $u(x)$

$$u_{x_0,\rho}=\frac{1}{|B_\rho|}\int_{B_\rho(x_0)} u(x)\,dx.$$

For any constant vector $\lambda=(\lambda_1,\ldots,\lambda_n)$ the Poincaré inequality implies

$$\frac{1}{\rho^{n+1}}\int_{B_\rho(x_0)}|u(x)-u_{x_0,\rho}-\lambda\cdot(x-x_0)|\,dx\leqq\frac{c}{\rho^n}\int_{B_\rho(x_0)}|Du-\lambda|\,dx.$$

Therefore x_0 is in the regular set if, for some finite λ, the integral on the right hand side is small enough. In other words, if

$$L\equiv\{x_0\in\Omega:\liminf_{\rho\to 0}\frac{1}{|B_\rho|}\int_{B_\rho(x_0)}|Du(x)-(Du)_{x_0,\rho}|\,dx=0\}$$

and

$$F\equiv\{x\in\Omega:\limsup_{\rho\to 0}(Du)_{x,\rho}\neq\infty\},$$

we have

$$L\cap F\subset\mathrm{Reg}(u)$$

and the Lebesgue theorem implies that the regular set of $u(x)$ has full measure. □

In view of our discussion of example (6) we can have $|Du(x)|\uparrow\infty$ on a set of positive H^{n-1} measure. Therefore a natural question is whether Corollary 1 could be extended to say $H^{n-1+\varepsilon}(\Omega\sim\Omega_0)=0$.

Corollary 2 *Let $u(x)$ be a $W^{1,1}$ entire solution of Eq. (1) with sublinear growth, then $u(x)$ is constant.*

Proof. We say that $u(x)$ is sublinear if $|u(x)|\leqq c|x|^\alpha$, with $\alpha<1$, for $|x|\geqq R_0$. Then

$$(1/\rho^{n+1})\int_{B_\rho}|u(x)|\,dx\leqq c(\rho^{n+\alpha}/\rho^{n+1})\leqq c\rho^{\alpha-1}.$$

Hence $E(0, \rho, 0) < \varepsilon$ for all ρ big enough and we can apply the excess decay lemma. In particular we obtain from (20) that the gradient of $u(x)$ is constant, immediately giving our claim. □

It is clearly enough to assume that $u(x) = o(|x|)$ or even that for some linear function $\ell(x)$ we have $|u(x) - \ell(x)| = o(|x|)$ in the previous argument to deduce that $u(x)$ is affine.

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