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SOME RESULTS CONCERNING A GENERAL SET THEORETICAL APPROACH TO LOGIC

ANTON JENSEN

1. Introduction.

The meaning of compound propositions in a logical system is usually defined recursively by defining the truth or falsehood of propositions as dependent on the truth or falsehood of already meaningful propositions. The basis for the definition is a set of atomic propositions which are considered meaningful in advance. A relation of consequence is defined by stating that a proposition is a consequence of a set of propositions if it is true whenever all propositions of the set are true.

The aim of this paper is to study the general aspects of a similar approach to logic. The idea is also recursively to attach meanings to compound propositions and to define relations of consequence in accordance with these meanings.

The observation which leads to the theory is that the meaning of e.g. the conditional $b \Rightarrow c$ is given if $b \Rightarrow c$ is defined to be a proposition which together with b yields c. Of course many propositions may have this property, but no other one can be defined by this property alone.

So, in our concept of logic, a meaning is attached to the elements of a system of propositions by defining how each proposition together with already meaningful propositions implies other already meaningful propositions.

It is possible to give a definition of relations of consequence (derivations) which in a natural way expresses the above formulated idea. The main topics of the paper are the minimal derivations, which correspond to the intuitionistic logic, the weak classical derivations, which are the smallest derivations with the property that the double negation of a proposition always implies the proposition itself, and finally the relations of consequence derived by the above mentioned "truth-table" method. The technique of characterizing relations of consequence as minimal relations (derivations) having certain properties is due to L. Henkin [2].

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A general set-theoretical presentation is used in order to simplify notation and proofs. All propositions are considered elements of a set-theory, and the compound propositions are defined in the following way: If Δ_i, $i \in I$, are sets of propositions and y_i, $i \in I$, are propositions, then $\{\langle \Delta_i, y_i \rangle \mid i \in I\}$ is the proposition the meaning of which is derived from the fact that the proposition, for all $k \in I$, together with Δ_k implies y_k.

The main result is that the minimal, the weak classical, and the strong classical derivations can be given general deductive characterizations, which resemble the formal deductive systems of Gentzen [1]. These deductive characterizations seem well suited for a kind of model theoretical treatment of formal logic, which is roughly sketched in the last section.

I wish to express my gratitude to professor L. Henkin for his valuable help in preparing this paper.

2. Set-theoretical preliminaries.

All objects of the theory which will be presented in the following sections are elements of a set-theory of the von Neumann–Bernays–Gödel-type with the axiom of regularity and possibly with individuals.

Below are listed some conventions which are not generally used or understandable:

V is the universal class.
$\emptyset$ is the empty set.
$S(A)$ is the class of all subsets of A.
$\langle a,b \rangle = \{\{a,b\},\{b\}\}$.
$\langle a,b,c \rangle = \langle \langle a,b \rangle, c \rangle$.
$\times\{a_\nu \mid \nu \in N\}$ = the class of functions from N to V such that $f(\nu) \in a_\nu$.

A very important convention is the following: An element $a \in V$ is called *basic with respect to a predicate* Φ if $\Phi(a)$ and $\Phi(x)$ does not hold for any x which is an element of the transitive closure $a \cup Ua \cup UUa \cup \ldots$ of a. It is a consequence of the normally used axioms of set theory that, if Φ is satisfied by some $b \in V$, then there exists an a which is basic with respect to Φ.

3. Fundamental definitions.

DEFINITION 3.1. An element x of the universe V is called *compound* if $x \in S(S(V) \times V)$, that is, if x is a set of the form $\{\langle \Delta_i, y_i \rangle \mid i \in I\}$, where $\Delta_k \subseteq V$ for all $k \in I$. For any class K, let $C(K)$ be the class of all compound elements of K.

DEFINITION 3.2. An element x of the universe V is called *atomic* if $x \notin C(V)$. For any class K, let $A(K)$ be the class of all atomic elements of K.

DEFINITION 3.3. A *prelogic* is a set L with the *property* that

$$\{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L)$$

implies $\Delta_k \subseteq L$, $y_k \in L$ for all $k \in I$.

NOTATION. The letter L will always denote a prelogic. For the relations $\vdash$, $\nvdash$, etc., we shall use the normal notation, writing $\Gamma \vdash x$ instead of $\langle \Gamma, x\rangle \in \vdash$, $\Gamma, \Delta \vdash x$ instead of $\langle \Gamma \cup \Delta, x\rangle \in \vdash$, $\Gamma, y \vdash x$ instead of $\langle \Gamma \cup \{y\}, x\rangle \in \vdash$, etc.

DEFINITION 3.4. The relation $\vdash$ is a *derivation on* L, if $\vdash \subseteq S(L) \times L$ and the following four conditions are satisfied:

(0) If $\Gamma \vdash z$ for all $z \in \Delta$ and $\Delta \vdash x$, then $\Gamma \vdash x$.
(1) If $x \in \Gamma \subseteq L$, then $\Gamma \vdash x$.
(2) If $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L)$, then $x, \Delta_k \vdash y_k$ for all $k \in I$.
(3) If $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L)$ and $\Gamma, \Delta_k \vdash y_k$ for all $k \in I$, then $\Gamma \vdash x$.

Conditions (0) and (1) express fundamental properties of any relation of consequence. Condition (2) expresses that $\{\langle \Delta_i, y_i\rangle \mid i \in I\}$ is to be considered as a proposition, the assumption of which enables one, for each $k \in I$, to draw the conclusion y_k provided all propositions of Δ_k are assumed. Condition (3) expresses that the meaning of $\{\langle \Delta_i, y_i\rangle \mid i \in I\}$ is to be derived solely from this property; that is, if the set of propositions Γ permits one, for each $k \in I$, to draw the conclusion y_k provided all propositions of Δ_k are assumed, then $\{\langle \Delta_i, y_i\rangle \mid i \in I\}$ is itself a consequence of Γ.

The following three definitions are listed at this point in order to facilitate the interpretation of the theory. Notice that the defined sets are not necessarily elements of the prelogic L.

DEFINITION 3.5. If $x \in V$, then $\neg_L x = \{\langle \{x\}, y\rangle \mid y \in A(L)\}$.

DEFINITION 3.6. If $x, y \in V$, then

$$\begin{aligned} x \Rightarrow y &= \{\langle \{x\}, y\rangle\}, \\ x \wedge y &= \{\langle \emptyset, x\rangle, \langle \emptyset, y\rangle\}, \\ x \vee_L y &= \{\langle \{x \Rightarrow z, y \Rightarrow z\}, z\rangle \mid z \in A(L)\}. \end{aligned}$$

DEFINITION 3.7. If $\Gamma \in S(V)$, then

$$\forall \Gamma = \{\langle \emptyset, x\rangle \mid x \in \Gamma\},$$
$$\exists_L \Gamma = \{\langle \{x \Rightarrow y \mid x \in \Gamma\}, y\rangle \mid y \in A(L)\}.$$

Below are listed without proof five simple consequences of the definitions. In the later sections these propositions will generally be used without any references.

PROPOSITION 3.1. *If* $\vdash$ *is a derivation on* L *and* $\Gamma' \subseteq \Gamma \subseteq L$, *then* $\Gamma' \vdash x$ *implies* $\Gamma \vdash x$.

PROPOSITION 3.2. *If* $\vdash$ *is a derivation on* L, $\langle \Delta, y\rangle \in \bigcup C(\Gamma)$, $\Gamma \vdash z$ *for all* $z \in \Delta$, *and* $\Gamma, y \vdash x$, *then* $\Gamma \vdash x$.

PROPOSITION 3.3. *If* $\vdash$ *is a derivation on* L *and*

$$\Gamma \vdash \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L),$$

then $\Gamma, \Delta_k \vdash y_k$ *for all* $k \in I$.

PROPOSITION 3.4. *The intersection of a non-empty set of derivations on* L *is a derivation on* L.

PROPOSITION 3.5. $S(L) \times L$ *is a derivation on* L.

4. The minimal derivations.

DEFINITION 4.1. $\Vdash_L$, *the minimal derivation on* L, is the intersection of all derivations on L.

DEFINITION 4.2. A *minimal deduction* is a triple $\langle \Gamma_0, x_0, \pi_0\rangle$ which is an element of a set P of triples $\langle \Gamma, x, \pi\rangle$ each of which satisfies the condition $\pi \subseteq P$ and one of the conditions:

(1) $\pi = \emptyset$ and $x \in \Gamma$.

(2) $\pi = \{\langle \Gamma, z, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma \cup \{y\}, x, \pi'\rangle\}$, where $\langle \Delta, y\rangle \in \bigcup C(\Gamma)$.

(3) $\pi = \{\langle \Gamma \cup \Delta_i, y_i, \pi_i\rangle \mid i \in I\}$, where $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$.

DEFINITION 4.3. $\Gamma \Vdash x$ means that there exists a minimal deduction $\langle \Gamma, x, \pi\rangle$.

THEOREM 4.1. *The relation* $\Vdash$ *satisfies the conditions*:

(0) *If* $\Gamma \Vdash z$ *for all* $z \in \Delta$ and $\Delta \Vdash x$, *then* $\Gamma \Vdash x$.

(1) *If* $x \in \Gamma \in V$, *then* $\Gamma \Vdash x$.

(2) *If* $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$, *then* $x, \Delta_k \Vdash y_k$ *for all* $k \in I$.

(3) *If* $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$ *and* $\Gamma, \Delta_k \Vdash y_k$ *for all* $k \in I$, *then* $\Gamma \Vdash x$.

PROOF. (1) If $x \in \Gamma \in V$, then $\langle \Gamma, x, \emptyset \rangle$ is a minimal deduction.

(2) If $x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$ and $k \in I$, then

$$\left\langle \{x\} \cup \Delta_k, y_k, \left\{\langle \{x\} \cup \Delta_k, z, \emptyset \rangle \mid z \in \Delta_k\right\} \cup \left\{\langle \{x\} \cup \Delta_k \cup \{y_k\}, y_k, \emptyset \rangle\right\} \right\rangle$$

is a minimal deduction.

(3) If $\pi = \{\langle \Gamma \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$ is a set of minimal deductions and $x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$, then $\langle \Gamma, x, \pi \rangle$ is a minimal deduction.

(0) is proved by lemmas 4.2, 4.3 and 4.4.

LEMMA 4.2. *$\Gamma' \subseteq \Gamma \in V$ and $\Gamma' \vdash x$ imply $\Gamma \vdash x$.*

PROOF. Assume $\delta_0 = \langle \Gamma_0', x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma', x, \pi \rangle$ is a minimal deduction and there exists a $\Gamma \in V$ such that $\Gamma' \subseteq \Gamma$ and $\Gamma \vdash x$ is false. Assume $\Gamma_0 \in V$, $\Gamma_0' \subseteq \Gamma_0$ and $\Gamma_0 \vdash x_0$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0'$. In this case $\langle \Gamma_0, x_0, \emptyset \rangle$ is a minimal deduction.

(2) $\pi_0 = \{\langle \Gamma_0', z, \pi_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0' \cup \{y\}, x_0, \pi' \rangle\right\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0')$. There exists a set $\bar{\pi}_0 = \{\langle \Gamma_0, z, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0 \cup \{y\}, x_0, \bar{\pi}' \rangle\right\}$ of minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

(3) $\pi_0 = \{\langle \Gamma_0' \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. There exists a set $\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \bar{\pi}_i \rangle \mid i \in I\}$ of minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

LEMMA 4.3. *$\Gamma \vdash \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$ implies $\Gamma, \Delta_k \vdash y_k$ for all $k \in I$.*

PROOF. Assume $\delta_0 = \langle \Gamma_0, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, x, \pi \rangle$ is a minimal deduction, $x \in C(V)$, and $\Gamma, \Delta \vdash y$ is false for some $\langle \Delta, y \rangle \in x$. Assume $\langle \Delta_0, y_0 \rangle \in x_0$ and $\Gamma_0, \Delta_0 \vdash y_0$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. In this case

$$\langle \Gamma_0 \cup \Delta_0, y_0, \{\langle \Gamma_0 \cup \Delta_0, z, \emptyset \rangle \mid z \in \Delta_0\} \cup \left\{\Gamma_0 \cup \Delta_0 \cup \{y_0\}, y_0, \emptyset \rangle\right\} \rangle$$

is a minimal deduction.

(2) $\pi_0 = \{\langle \Gamma_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0 \cup \{y\}, x_0, \pi' \rangle\right\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$. Using lemma 4.2, we get the existence of a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_0, z, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \left\{\langle \Gamma_0 \cup \Delta_0 \cup \{y\}, y_0, \bar{\pi}' \rangle\right\}$$

of minimal deductions, and $\langle \Gamma_0 \cup \Delta_0, y_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\}$. There is a $k \in I$ such that $\langle \Delta_0, y_0 \rangle = \langle \Delta_k, y_k \rangle$, and $\langle \Gamma_0 \cup \Delta_0, y_0, \pi_k \rangle$ is a minimal deduction.

LEMMA 4.4. *If $\Gamma \vdash z$ for all $z \in \Delta$ and $\Gamma, \Delta \vdash x$, then $\Gamma \vdash x$.*

PROOF. Assume Δ_0 basic with respect to Φ: $\Phi(\Delta)$ iff there exist Γ and x such that $\Gamma \vdash x$ is false although $\Gamma \vdash z$ for all $z \in \Delta$ and $\Gamma, \Delta \vdash x$.

Assume $\delta_0 = \langle \Gamma_0 \cup \Delta_0, x_0, \pi_0 \rangle$ basic with respect to Ψ: $\Psi(\delta)$ iff $\delta = \langle \Gamma \cup \Delta_0, x, \pi \rangle$ is a minimal deduction such that $\Gamma \vdash z$ for all $z \in \Delta_0$ and $\Gamma \vdash x$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0 \cup \Delta_0$. If $x_0 \in \Gamma_0$, then $\langle \Gamma_0, x_0, \emptyset \rangle$ is a minimal deduction. If $x_0 \in \Delta_0$, then $\Gamma_0 \vdash x_0$ is assumed.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \Delta_0 \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in C(\Gamma_0) \cup C(\Delta_0)$. Using that δ_0 is basic with respect to Ψ and lemma 4.2, we get the existence of a set

$$\bar{\pi}_0 = \{\langle \Gamma_0, z, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, x_0, \bar{\pi}' \rangle\}$$

of minimal deductions. If $\langle \Delta, y \rangle \in C(\Gamma_0)$, then $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction. If $\langle \Delta, y \rangle \in C(\Delta_0)$, then lemma 4.3 implies $\Gamma_0, \Delta \vdash y$. Since $\Gamma_0 \vdash z$ for all $z \in \Delta$ and Δ_0 is basic with respect to Φ, we get $\Gamma_0 \vdash y$. Finally, since $\Gamma_0, y \vdash x_0$, we get $\Gamma_0 \vdash x_0$, using on $\{y\}$ that Δ_0 is basic with respect to Φ.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. Using that δ_0 is basic with respect to Ψ and lemma 4.2, we get the existence of a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \bar{\pi}_i \rangle \mid i \in I\}$$

of minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a minimal deduction.

THEOREM 4.5. *$\Gamma \vdash_L x$ iff $\Gamma \subseteq L$, $x \in L$ and $\Gamma \vdash x$.*

PROOF. Let $\Gamma \vdash' x$ mean that $\Gamma \subseteq L$, $x \in L$ and $\Gamma \vdash x$. It is an immediate consequence of theorem 4.1 that $\vdash'$ is a derivation on L. In order to prove that $\vdash' \subseteq \vdash_L$, assume $\delta_0 = \langle \Gamma_0, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, x, \pi \rangle$ is a minimal deduction, $\Gamma \subseteq L$, $x \in L$ and $\Gamma \vdash_L x$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. Then $\Gamma_0 \vdash_L x_0$.

(2) $\pi_0 = \{\langle \Gamma_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$. We have $\Gamma_0 \vdash_L z$ for all $z \in \Delta$ and $\Gamma_0, y \vdash_L x_0$, and proposition 3.2 gives $\Gamma_0 \vdash_L x_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. We have $\Gamma_0, \Delta_k \vdash_L y_k$ for all $k \in I$, but then $\Gamma_0 \vdash_L x_0$.

The conditions in definition 4.2 of minimal deductions can be strengthened in various ways without decreasing the class of pairs $\langle \Gamma, x \rangle$ for which deductions $\langle \Gamma, x, \pi \rangle$ can be found. The following definition can be further strengthened, but is sufficient for our purpose.

DEFINITION 4.4. A *regular minimal deduction* is a triple $\langle \Gamma_0, x_0, \pi_0 \rangle$, which is an element of a set P of triples $\langle \Gamma, x, \pi \rangle$, each of which satisfies the condition $\pi \subseteq P$ and one of the conditions:

(1) $\pi = \emptyset$ and $x \in A(\Gamma)$.

(2) $\pi = \{\langle \Gamma, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma \cup \{y\}, x, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma)$, and $y = x$ if $y \in A(V)$.

(3) $\pi = \{\langle \Gamma \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$.

PROPOSITION 4.6. *$\Gamma \vdash x$ implies the existence of a regular minimal deduction $\langle \Gamma, x, \pi \rangle$.*

PROOF. Assume x_0 basic with respect to Φ: $\Phi(x')$ iff there exists a minimal deduction $\delta' = \langle \Gamma', x', \pi' \rangle$ which is basic with respect to Ψ: $\Psi(\delta)$ iff $\delta = \langle \Gamma, x, \pi \rangle$ is a minimal deduction, and no regular minimal deduction $\langle \Gamma, x, \pi'' \rangle$ exists. Assume $\langle \Gamma_0, x_0, \pi_0 \rangle$ is basic with respect to Ψ.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. If $x_0 \in A(V)$, then $\langle \Gamma_0, x_0, \emptyset \rangle$ is a regular minimal deduction. If $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\}$, then the assumption that x_0 is basic with respect to Φ implies the existence of sets

$$\pi_i = \{\langle \Gamma_0 \cup \Delta_i, z, \pi_{i,z} \rangle \mid z \in \Delta_i\} \cup \{\langle \Gamma \cup \Delta_i \cup \{y_i\}, y_i, \pi_i' \rangle\}, \quad i \in I,$$

of regular minimal deductions, and

$$\langle \Gamma_0, x_0, \{\langle \Gamma_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\} \rangle$$

is a regular minimal deduction.

(2) $\pi_0 = \{\langle \Gamma_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$. There exists a set

$$\bar{\pi}_0 = \{\langle \Gamma_0, z, \bar{\pi}_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, x_0, \bar{\pi}' \rangle\}$$

of regular minimal deductions. If $y \in C(V)$ or $y = x$, then $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a regular minimal deduction. If $y \in A(V)$ and $y \neq x$, then lemma 4.7 below can be used.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. There exists a set $\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, y_i, \bar{\pi}_i \rangle \mid i \in I\}$ of regular minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0 \rangle$ is a regular minimal deduction.

LEMMA 4.7. *If $\pi = \{\langle \Gamma, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma \cup \{y\}, x, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma)$ and $y \in A(V)$, is a set of regular minimal deductions, then there exists a regular minimal deduction $\langle \Gamma, x, \pi'' \rangle$.*

PROOF. Assume $\langle \Gamma_0 \cup \{y_0\}, x_0, \pi_0' \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \{y\}, x. \pi' \rangle$ is a regular minimal deduction, $\langle \Delta, y \rangle \in \bigcup C(\Gamma)$, $y \in A(V)$, and there exist regular minimal deductions $\langle \Gamma, z, \pi_z \rangle$ for all $z \in \Delta$, but no regular minimal deduction $\langle \Gamma, x, \pi \rangle$.

Assume that $\{\langle \Gamma_0, z, \pi_z \rangle \mid z \in \Delta\}$ is a set of regular minimal deductions, and that no minimal deduction $\langle \Gamma_0, x_0, \pi_0 \rangle$ exists.

(1) $\pi_0' = \emptyset$. If $x_0 \in \Gamma_0$, then $\langle \Gamma_0, x_0, \emptyset \rangle$ is a regular minimal deduction. If $x_0 = y_0$, then

$$\langle \Gamma_0, x_0, \{\langle \Gamma_0, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y_0\}, x_0, \pi_0' \rangle\} \rangle$$

is a regular minimal deduction.

(2) $\pi_0' = \{\langle \Gamma_0 \cup \{y_0\}, u, \pi_u' \rangle \mid u \in \Delta'\} \cup \{\langle \Gamma_0 \cup \{y_0\} \cup \{v\}, x_0, \pi'' \rangle\}$, where $\langle \Delta', v \rangle \in \bigcup C(\Gamma_0)$. Using the obvious generalization of lemma 4.2 to regular minimal deductions, we get the existence of a set

$$\bar{\pi}_0' = \{\langle \Gamma_0, u, \bar{\pi}_u' \rangle \mid u \in \Delta'\} \cup \{\langle \Gamma_0 \cup \{v\}, x_0, \bar{\pi}'' \rangle\}$$

of regular minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0' \rangle$ is a regular minimal deduction.

(3) $\pi_0' = \{\langle \Gamma_0 \cup \Delta_i \cup \{y_0\}, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(V)$. Using again the generalization of lemma 4.2 to regular minimal deductions, we get a set

$$\bar{\pi}_0' = \{\langle \Gamma_0 \cup \Delta_i, y_i, \bar{\pi}_i \rangle \mid i \in I\}$$

of regular minimal deductions, and $\langle \Gamma_0, x_0, \bar{\pi}_0' \rangle$ is a regular minimal deduction.

5. q-minimal derivations.

The following proposition is easily proved by theorem 4.1:

PROPOSITION 5.1. *If $q \in V$, then the relation $\Vdash_L^q$ defined by*

$$\Gamma \Vdash_L^q x \quad \textit{iff} \quad \Gamma \in L,\ x \in L \text{ and } \Gamma, q \Vdash x,$$

is a derivation on L.

DEFINITION 5.1. $\Vdash_L^q$ is called the q-minimal derivation on L.

THEOREM 5.2. *If $q = \{\langle \Gamma_\nu, x_\nu \rangle \mid \nu \in N\} \subseteq S(L) \times L$, then $\Vdash_L^q$ is the intersection of all derivations $\vdash$ on L having the property that $\Gamma_\nu \vdash x_\nu$ for all $\nu \in N$, and $\Vdash_L^q$ has this property itself.*

PROOF. Theorem 4.1 implies that $\Gamma_\nu, q \Vdash x_\nu$ for all $\nu \in N$; therefore, all we have to do is to prove, that $\Vdash_L^q \subseteq \vdash$ for all derivations $\vdash$ on L satisfying $\Gamma_\nu \vdash x_\nu$ for all $\nu \in N$. Let $\vdash$ be such a derivation.

Assume $\delta_0 = \langle \Gamma_0 \cup \{q\}, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \{q\}, x, \pi \rangle$ is a minimal deduction, $\Gamma \subseteq L$, $x \in L$, and $\Gamma \vdash' x$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. But then $\Gamma_0 \vdash' x_0$.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \{q\}, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\} \cup \{q\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0) \cup q$. We have $\Gamma_0 \vdash' z$ for all $z \in \Delta$, and $\Gamma_0, y \vdash' x_0$. If $\langle \Delta, y \rangle \in$

$\bigcup C(\Gamma_0)$, then proposition 3.2 implies $\Gamma_0 \vdash' x_0$. If $\langle \Delta, y\rangle \in q$, then $\Delta \vdash' y$, and consequently $\Gamma_0 \vdash' x_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \{q\} \cup \Delta_i, y_i, \pi_i\rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(V)$. Here $\Gamma_0, \Delta_k \vdash' y_k$ for all $k \in I$, and consequently $\Gamma_0 \vdash' x_0$.

6. Properties of the negation $\neg_L$.

PROPOSITION 6.1. *$x, \neg_L x \Vdash y$ for all $x \in V$ and $y \in L$.*

PROOF. Assume y_0 basic with respect to Φ: $\Phi(y)$ iff $y \in L$, and $x, \neg_L x \Vdash y_0$ is false.

If $y_0 \in A(L)$, then $\langle \{x\}, y_0\rangle \in \neg_L x$, and $x, \neg_L x \Vdash y_0$. If

$$y_0 = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L)\,,$$

then $x, \neg_L x \Vdash y_k$ for all $k \in I$.

Lemma 4.2 gives $x, \neg_L x, \Delta_k \Vdash y_k$ for all $k \in I$, and consequently $x, \neg_L x \Vdash y_0$.

PROPOSITION 6.2. *$\Gamma, x \Vdash y$ implies $\Gamma, \neg_L y \Vdash \neg_L x$.*

PROOF. $\Gamma, \neg_L y, x \Vdash y$ and $\Gamma, \neg_L y, x \Vdash \neg_L y$; consequently $\Gamma, \neg_L y, x \Vdash z$ for all $z \in A(L)$, and this implies $\Gamma, \neg_L y \Vdash \neg_L x$.

PROPOSITION 6.3. *$x \Vdash \neg_L \neg_L x$ for all $x \in V$.*

PROOF. $x, \neg_L x \Vdash z$ for all $z \in A(L)$, and consequently $x \Vdash \neg_L \neg_L x$.

PROPOSITION 6.4. *$\neg_L \neg_L \neg_L x \Vdash \neg_L x$ for all $x \in V$.*

PROOF. Proposition 6.2 applied to $x \Vdash \neg_L \neg_L x$ gives

$$\neg_L \neg_L \neg_L x \Vdash \neg_L x\,.$$

PROPOSITION 6.5. *If $A(L)$ contains more that one element, then $\neg_L \neg_L x \Vdash x$ is false for all $x \in A(L)$.*

PROOF. Assume $\langle \{\neg_L \neg_L x\}, x, \pi\rangle$ is a regular minimal deduction, and $x \in A(L)$. Then

$$\pi = \{\langle \{\neg_L \neg_L x\}, \neg_L x, \pi'\rangle, \langle \{\neg_L \neg_L x, x\}, x, \pi''\rangle\}\,.$$

This implies $\neg_L \neg_L x \Vdash \neg_L x$, and we have $x, \neg_L \neg_L x \Vdash y$ for all $y \in A(L)$. Since $x \Vdash \neg_L \neg_L x$, we have $x \Vdash y$ for some $y \in A(L) \setminus \{x\}$, which contradicts the obvious fact that no minimal deduction $\langle \{x\}, y, \pi\rangle$ exists, where $x, y \in A(V)$ and $x \neq y$.

7. The weak classical derivations.

DEFINITION 7.1. With $w_L = \{\langle\{\neg_L \neg_L x\}, x\rangle \mid x \in A(L)\}$, the relation $\vdash_L^{w_L}$ is called the *weak classical derivation on* L.

LEMMA 7.1. $\neg_L \neg_L x, w_L \vdash x$ *for all* $x \in L$.

PROOF. Obvious if $x \in A(L)$. Assume $x_0 = \{\langle \Delta_i, x_i\rangle \mid i \in I\}$ basic with respect to Φ: $\Phi(x)$ iff $x \in L$ and $\neg_L \neg_L x, w_L \vdash x$ is false. We have that $\Delta_k, x_0 \vdash y_k$ for all $k \in I$. Using proposition 6.2 twice, we obtain $\Delta_k, \neg_L \neg_L x_0 \vdash \neg_L \neg_L y_k$ for all $k \in I$, and since $\neg_L \neg_L y_k$, $w_L \vdash y_k$ for all $k \in I$, we have $\neg_L \neg_L x_0, \Delta_k, w_L \vdash y_k$ for all $k \in I$, and consequently $\neg_L \neg_L x_0, w_L \vdash x_0$.

THEOREM 7.2. *If* $\neg_L x \in L$ *for all* $x \in L$, *then* $\vdash_L^{w_L}$ *is the intersection of all derivations* $\vdash$ *on* L *having the property that* $\neg_L \neg_L x \vdash x$ *for all* $x \in L$, *and* $\vdash_L^{w_L}$ *has this property itself.*

PROOF. From theorem 5.2 it follows that $\vdash_L^{w_L}$ is the smallest derivation $\vdash$ with the property that $\neg_L \neg_L x \vdash x$ for all $x \in A(L)$; and from lemma 7.1 it follows that $\neg_L \neg_L x \vdash_L^{w_L} x$ for all $x \in L$.

DEFINITION 7.2. A weak classical deduction is a triple $\langle \Gamma_0, \theta_0, \pi_0\rangle$, which is an element of a set P of triples $\langle \Gamma, \theta, \pi\rangle$ each of which satisfies the condition $\pi \subseteq P$ and one of the conditions:

(1) $\pi = \emptyset$ and $\Gamma \cap \theta \neq \emptyset$.
(2) $\pi = \{\langle \Gamma, \theta \cup \{z\}, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma \cup \{y\}, \theta, \pi'\rangle\}$, where $\langle \Delta, y\rangle \in \bigcup C(\Gamma)$.
(3) $\pi = \{\langle \Gamma \cup \Delta_i, \theta \cup \{y_i\}, \pi_i\rangle \mid i \in I\}$, where $x = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(\theta)$.

DEFINITION 7.3. $\Gamma \vdash^w \theta$ means that there exists a weak classical deduction $\langle \Gamma, \theta, \pi\rangle$.

NOTATION. We shall write $\Gamma \vdash^w x$ instead of $\Gamma \vdash^w \{x\}$, $\Gamma \vdash^w x, \theta$ instead of $\Gamma \vdash^w \{x\} \cup \theta$, etc.

THEOREM 7.3. $\Gamma \vdash_L^{w_L} x$ iff $\Gamma \subseteq L$, $x \in L$ and $\Gamma \vdash^w x$.

PROOF. The theorem is proved by a series of lemmas of which lemma 7.7 and lemma 7.8 directly imply the theorem.

LEMMA 7.4. $\Gamma_0 \vdash^w \theta_0$ *implies* $\Gamma_0, \Gamma_1 \vdash^w \theta_0, \theta_1$ *for all* $\Gamma_1, \theta_1 \subseteq V$.

PROOF. Analogous to the proof of lemma 4.2.

LEMMA 7.5. $\Gamma, \neg_L x \vdash^w \theta, x$ *implies* $\Gamma \vdash^w \theta, x$.

PROOF. Assume $\delta_0 = \langle \Gamma_0 \cup \{\neg_L x_0\}, \theta_0 \cup \{x_0\}, \pi_0\rangle$ basic with respect to Φ:

$\Phi(\delta)$ iff $\delta=\langle \Gamma\cup\{\neg_L x\},\theta\cup\{x\},\pi\rangle$ is a weak classical deduction, and $\Gamma \nvdash^w \theta,x$ is false.

(1) $\pi_0=\emptyset$ and $(\Gamma_0\cup\{\neg_L x_0\})\cap(\theta_0\cup\{x_0\})\neq\emptyset$. If $\neg_L x_0\in\theta_0$, then $\langle \Gamma_0,\theta_0\cup\{x_0\},\{\langle \Gamma_0\cup\{x_0\},\theta_0\cup\{x_0\}\cup\{y\},\emptyset\rangle \mid y\in A(L)\}\rangle$ is a weak classical deduction. If $\neg_L x\notin\theta_0$, then $\langle \Gamma_0,\theta_0\cup\{x_0\},\emptyset\rangle$ is a weak classical deduction.

(2) $\pi_0=\{\langle \Gamma_0\cup\{\neg_L x_0\},\theta_0\cup\{x_0\}\cup\{z\},\pi_z\rangle \mid z\in\Delta\}\cup\{\langle \Gamma_0\cup\{\neg_L x_0\}\cup\{y\},\theta_0\cup\{x_0\},\pi'\rangle\}$, where $\langle\Delta,y\rangle\in UC(\Gamma_0\cup\{\neg_L x_0\})$. We have a set

$$\bar{\pi}_0=\{\langle \Gamma_0,\theta_0\cup\{x_0\}\cup\{z\},\bar{\pi}_z\rangle \mid z\in\Delta\}\cup\{\langle \Gamma_0\cup\{y\},\theta_0\cup\{x_0\},\bar{\pi}'\rangle\}$$

of weak classical deductions. If $\langle\Delta,y\rangle\in\bigcup C(\Gamma_0)$, then $\langle \Gamma_0,\theta_0\cup\{x_0\},\bar{\pi}_0\rangle$ is a weak classical deduction. If $\langle\Delta,y\rangle\in\neg_L x_0$, then $\langle \Gamma_0,\theta_0\cup\{x_0\},\bar{\pi}_{x_0}\rangle$ is a weak classical deduction.

(3) $\pi_0=\{\langle \Gamma_0\cup\{\neg_L x_0\}\cup\Delta_i,\theta_0\cup\{x_0\}\cup\{y_i\},\pi_i\rangle \mid i\in I\}$, $\{\langle\Delta_i,y_i\rangle \mid i\in I\}\in C(\theta_0\cup\{x_0\})$. There exists a set

$$\bar{\pi}_0=\{\langle \Gamma_0\cup\Delta_i,\theta_0\cup\{x_0\}\cup\{y_i\},\bar{\pi}_i\rangle \mid i\in I\}$$

of weak classical deductions, and $\langle \Gamma_0,\theta_0\cup\{x_0\},\bar{\pi}_0\rangle$ is a weak classical deduction.

LEMMA 7.6. *$\Gamma \nvdash^w \theta, \neg_L\neg_L x$ implies $\Gamma \nvdash^w \theta,x$ for all $x\in A(L)$.*

PROOF. Assume $\delta_0=\langle \Gamma_0,\theta_0\cup\{\neg_L\neg_L x_0\},\pi_0\rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta=\langle \Gamma,\theta\cup\{\neg_L\neg_L x\},\pi\rangle$ is a weak classical deduction, $x\in A(L)$, and $\Gamma \nvdash^w \theta,x$ is false.

(1) $\pi_0=\emptyset$ and $\Gamma_0\cap(\theta_0\cup\{\neg_L\neg_L x_0\})\neq\emptyset$. If $\Gamma_0\cap\theta_0\neq\emptyset$, then $\langle \Gamma_0,\theta_0\cup\{x_0\},\emptyset\rangle$ is a weak classical deduction. If $\neg_L\neg_L x_0\in\Gamma_0$, then

$$\langle \Gamma_0\cup\{\neg_L x_0\},\theta_0\cup\{x_0\},\{\langle \Gamma_0\cup\{\neg_L x_0\},\theta_0\cup\{x_0\}\cup\{\neg_L x_0\},\emptyset\rangle,\ \langle \Gamma_0\cup\{\neg_L x_0\}\cup\{x_0\},\theta_0\cup\{x_0\},\emptyset\rangle\}\rangle$$

is a weak classical deduction. Therefore, $\Gamma_0,\neg_L x_0 \nvdash^w \theta_0,x_0$, and lemma 7.5 gives $\Gamma_0 \nvdash^w \theta_0,x_0$.

(2) $\pi_0=\{\langle \Gamma_0,\theta_0\cup\{\neg_L\neg_L x_0\}\cup\{z\},\pi_z\rangle \mid z\in\Delta\}\cup\{\langle \Gamma_0\cup\{y\},\theta_0\cup\{\neg_L\neg_L x_0\},\pi'\rangle\}$, where $\langle\Delta,y\rangle\in\bigcup C(\Gamma_0)$. There exists a set

$$\bar{\pi}_0=\{\langle \Gamma_0,\theta_0\cup\{x_0\}\cup\{z\},\bar{\pi}_z\rangle \mid z\in\Delta\}\cup\{\langle \Gamma_0\cup\{y\},\theta_0\cup\{x_0\},\bar{\pi}'\rangle\}$$

of weak classical deductions, and $\langle \Gamma_0,\theta_0\cup\{x_0\},\bar{\pi}_0\rangle$ is a weak classical deduction.

(3) $\pi_0=\{\langle \Gamma_0\cup\Delta_i,\theta_0\cup\{\neg_L\neg_L x_0\}\cup\{y_i\},\pi_i\rangle \mid i\in I\}$, where

$$\{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(\theta_0 \cup \{\neg_L \neg_L x_0\}) .$$

There exists a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \bar{\pi}_i\rangle \mid i \in I\}$$

of weak classical deductions. If $\{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(\theta_0)$, then

$$\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0\rangle$$

is a weak classical deduction. If $\{\langle \Delta_i, y_i\rangle \mid i \in I\} = \neg_L \neg_L x_0$, then $x_0 = y_k$ for some $k \in I$, and $\langle \Gamma_0 \cup \{\neg_L x_0\}, \theta_0 \cup \{x_0\}, \bar{\pi}_k\rangle$ is a weak classical deduction. This proves that $\Gamma_0, \neg_L x_0 \vdash^w \theta_0, x_0$, and lemma 7.5 gives $\Gamma_0 \vdash^w \theta_0, x_0$.

LEMMA 7.7. *$\Gamma, \{\neg_L y \mid y \in \theta\}, w_L \vdash x$ implies $\Gamma \vdash^w \theta, x$.*

PROOF. Let $\bar{\theta} = \{\neg_L y \mid y \in \theta\}$ and assume $\delta_0 = \langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\}, x_0, \pi_0\rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \bar{\theta} \cup \{w_L\}, x, \pi\rangle$ is a *regular* minimal deduction and $\Gamma \vdash^w \theta, x$ is false.

(1) $\pi_0 = \emptyset$ and $x_0 \in \Gamma_0$. Then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \emptyset\rangle$ is a weak classical deduction.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\}, z, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\} \cup \{y\}, x_0, \pi'\rangle\}$, where $\langle \Delta, y\rangle \in \bigcup C(\Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\})$. Using lemma 7.4 we get a set

$$\bar{\pi}_0 = \{\langle \Gamma_0, \theta_0 \cup \{x_0\} \cup \{z\}, \bar{\pi}_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, \theta_0 \cup \{x_0\}, \bar{\pi}'\rangle\}$$

of weak classical deductions. If $\langle \Delta, y\rangle \in \bigcup C(\Gamma_0)$, then $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0\rangle$ is a weak classical deduction. If $\langle \Delta, y\rangle \in \bigcup \bar{\theta}_0$, then $\Delta = \{z_0\}$, where $z_0 \in \theta_0$, and $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0\rangle$ is a weak classical deduction. If $\langle \Delta, y\rangle \in w_L$, then $\langle \Delta, y\rangle = \langle \{\neg_L \neg_L x_0\}, x_0\rangle$ and $x_0 \in A(L)$. Consequently $\Gamma_0 \vdash^w \theta_0, x_0$, $\neg_L \neg_L x_0$, and lemma 7.6 gives $\Gamma_0 \vdash^w \theta_0, x_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \bar{\theta}_0 \cup \{w_L\} \cup \Delta_i, y_i, \pi_i\rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i\rangle \mid i \in \} \in C(V)$. Using lemma 7.4. we get a set

$$\bar{\pi}_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \bar{\pi}_i\rangle\}$$

of weak classical deductions, and $\langle \Gamma_0, \theta_0 \cup \{x_0\}, \bar{\pi}_0\rangle$ is a weak classical deduction.

LEMMA 7.8. *$x \in L$ and $\Gamma \vdash^w \theta, x$ imply $\Gamma, \{\neg_L y \mid y \in \theta\}, w_L \vdash x$.*

PROOF. Let $\bar{\theta} = \{\neg_L y \mid y \in \theta\}$, and assume $\delta_0 = \langle \Gamma_0, \theta_0 \cup \{x_0\}, \pi_0\rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, \theta \cup \{x\}, \pi\rangle$ is a weak classical deduction, $x \in L$ and $\Gamma, \bar{\theta}, w_L \vdash x$ is false.

(1) $\pi_0 = \emptyset$ and $\Gamma_0 \cap [\theta_0 \cup \{x_0\}] \neq \emptyset$. If $x_0 \in \Gamma_0$, then $\Gamma_0, \bar{\theta}_0, w_L \vdash x_0$. If $\Gamma_0 \cap \theta_0 \neq \emptyset$, then proposition 6.1 implies $\Gamma_0, \bar{\theta}_0, w_L \vdash x_0$.

(2) $\pi_0 = \{\langle \Gamma_0, \theta_0 \cup \{x_0\} \cup \{z\}, \pi_z\rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{y\}, \theta_0 \cup \{x_0\}, \pi'\rangle\}$, where $\langle \Delta, y\rangle \in \bigcup C(\Gamma_0)$. We have $\Gamma_0, \bar{\theta}_0, \neg_L x_0, w \vdash z$ for all $z \in \Delta$ and consequently $\Gamma_0, \bar{\theta}_0, \neg_L x_0, w_L \vdash y$. Furthermore, $\Gamma_0, \bar{\theta}_0, y, w_L \vdash x_0$, and proposition 6.2 gives $\Gamma_0, \bar{\theta}_0, \neg_L x_0, w_L \vdash \neg_L y$. Now proposition 6.1 gives $\Gamma_0, \bar{\theta}_0, w_L, \neg_L x \vdash z$ for all $z \in A(L)$, and this implies $\Gamma_0, \bar{\theta}_0, w_L \vdash \neg_L \neg_L x_0$, and finally lemma 7.1 gives $\Gamma_0, \bar{\theta}_0, w_L \vdash x_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i, \theta_0 \cup \{x_0\} \cup \{y_i\}, \pi_i\rangle \mid i \in I\}$, where

$$\{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(\theta_0 \cup \{x_0\}) .$$

We have $\Gamma_0, \bar{\theta}_0, \neg_L x_0, w_L, \Delta_k \vdash y_k$ for all $k \in I$, and consequently $\Gamma_0, \bar{\theta}_0, \neg_L x_0, w_L \vdash \{\langle \Delta_i, y_i\rangle \mid i \in I\}$. Since $\neg_L \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in \bar{\theta}_0 \cup \{\neg_L x_0\}$, we have $\Gamma_0, \bar{\theta}_0, \neg_L x_0, w_L \vdash z$ for all $z \in A(L)$, and consequently $\Gamma_0, \bar{\theta}_0, w_L \vdash \neg_L \neg_L x_0$. Now lemma 7.1 gives $\Gamma_0, \bar{\theta}_0, w_L \vdash x_0$.

8. The strong classical derivations.

DEFINITION 8.1. Using the notations

$$\tilde{\Delta}_L = \Delta \cup \{\neg_L y \mid y \in A(L) \setminus \Delta\} \quad \text{for all } \Delta \subseteq A(L) ,$$
$$s_L = \{\langle \{\langle \tilde{\Delta}_L, y\rangle \mid \Delta \subseteq A(L)\}, y\rangle \mid y \in A(L)\} ,$$

$\vdash_L^{s_L}$ is called the *strong classical derivation on* L.

Intuitively s_L states that one of the sets $\tilde{\Delta}_L$, $\Delta \subseteq A(L)$, contains nothing but true sentences. (Compare with the definitions of $x \vee_L y$ and $\exists_L \Gamma$ in section 3.)

PROPOSITION 8.1. *If* $\Delta \subseteq A(L)$, *then* $\tilde{\Delta}_L \vdash s_L$.

PROPOSITION 8.2. *If* $\Delta \subseteq A(L)$ *and* $x \in L$, *then* $\tilde{\Delta}_L \vdash x$ *or* $\tilde{\Delta}_L \vdash \neg_L x$.

PROOF. Obvious if $x \in A(L)$. Assume $x_0 = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L)$ basic with respect to Φ: $\Phi(x)$ iff $\tilde{\Delta}_L \vdash x$ and $\tilde{\Delta} \vdash \neg_L x$ are both false. Since $\tilde{\Delta}_L \vdash x_0$ is false, we have $\Delta_k, \tilde{\Delta}_L \vdash y_k$ is false for some $k \in I$. Consequently $\tilde{\Delta}_L \vdash y_k$ is false, and $\tilde{\Delta}_L \vdash \neg_L y_k$ is true. Since $\tilde{\Delta}_L \vdash \neg_L z$ for some $z \in \Delta_k$ would imply $\Delta_k, \tilde{\Delta}_L \vdash y_k$, we have $\tilde{\Delta}_L \vdash z$ for all $z \in \Delta_k$, and from this $\tilde{\Delta}_L, x_0 \vdash y_k$ follows. But $\tilde{\Delta}_L, x_0 \vdash y_k$ and $\tilde{\Delta}_L, x_0 \vdash \neg_L y_k$ imply $\tilde{\Delta}_L, x_0 \vdash z$ for all $z \in A(L)$, and consequently $\tilde{\Delta}_L \vdash \neg_L x_0$.

DEFINITION 8.2. A *truth-assignment on* L is a set $M \subseteq L$ with the property that

$$\{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(M)$$

iff no $k \in I$ exists such that $\Delta_k \subseteq M$ and $y_k \notin M$.

PROPOSITION 8.3. *If* $\Lambda \subseteqq A(L)$, *then* $M = \{x \mid x \in L \text{ and } \tilde{\Lambda}_L \vdash x\}$ *is the only truth-assignment on* L *with the property that* $A(M) = \Lambda$.

PROOF. $A(M) \supseteqq \Lambda$ is obvious. To prove $A(M) \subseteqq \Lambda$, assume $\delta_0 = \langle \tilde{\Lambda}_L, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \tilde{\Lambda}_L, x, \pi \rangle$ is a *regular* minimal deduction and $\neg_L x \in \tilde{\Lambda}_L$. Then

$$\pi_0 = \left\{ \langle \tilde{\Lambda}_L, y_0, \pi_0' \rangle, \langle \tilde{\Lambda}_L \cup \{x_0\}, x_0, \pi_0'' \rangle \right\},$$

where $\neg_L y_0 \in \tilde{\Lambda}_L$, and this contradicts that δ_0 is basic with respect to Φ.

To prove that M is a truth-assignment on L, assume

$$x = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L) .$$

If $\tilde{\Lambda}_L \vdash x$ and $\tilde{\Lambda}_L \vdash z$ for all $z \in \Delta_k$, where $k \in I$, then $\tilde{\Lambda}_L \vdash y_k$; and if no $k \in I$ exists such that $\tilde{\Lambda}_L \vdash z$ for all $z \in \Delta_k$ and $\tilde{\Lambda}_L \vdash y_k$ is false, then $\Delta_k, \tilde{\Lambda}_L \vdash y_k$ for all $k \in I$ because of proposition 8.2, and consequently $\tilde{\Lambda}_L \vdash x$.

Finally, assume that M' is a truth-assignment on L, and that $A(M') = A(M) = \Lambda$. Assume that $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$ is basic with respect to Ψ: $\Psi(x)$ iff either $x \in M$ and $x \notin M'$ or $x \notin M$ and $x \in M'$. Then $x_0 \in C(M)$ iff no $k \in I$ exists such that $\Delta_k \subseteqq M$ and $y_k \notin M$ iff no $k \in I$ exists such that $\Delta_k \subseteqq M'$ and $y_k \notin M'$ iff $x_0 \in C(M')$.

DEFINITION 8.3. A *strong classical deduction* is a triple $\langle \Gamma_0, \theta_0, \pi_0 \rangle$ which is an element of a set P of triples $\langle \Gamma, \theta, \pi \rangle$ each of which satisfies the condition $\pi \subseteqq P$ and one of the conditions:

(1) $\pi = \emptyset$ and $\Gamma \cap \theta \neq \emptyset$.

(2) $\pi = \left\{ \langle \Gamma \cup \Gamma_f, \theta \cup \theta_f, \pi_f \rangle \mid f \in \times \{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N\} \right\}$, where

$\Gamma_f = \{y_\nu \mid \nu \in N \text{ and } f(\nu) = \Delta_\nu\}$, $\theta_f = \{f(\nu) \mid \nu \in N \text{ and } f(\nu) \in \Delta_\nu\}$ and $\langle \Delta_\nu, y_\nu \rangle \in \cup C(\Gamma)$ for all $\nu \in N$.

(3) $\pi = \{\langle \Gamma \cup \Gamma_g, \theta \cup \theta_g, \pi_g \rangle \mid g \in \times \{x_\nu \mid \nu \in N\}$, where $\Gamma_g = \cup \{\Delta_{g,\nu} \mid \nu \in N\}$, $\theta_g = \{y_{g,\nu} \mid \nu \in N\}$, $g(\nu) = \langle \Delta_{g,\nu}, y_{g,\nu} \rangle$ and $x_\nu \in C(\theta)$ for all $\nu \in N$.

Notice that the weak classical deductions can be considered as strong classical deductions, where the index sets N are always singletons. Intuitively each step in a strong classical deduction is the simultaneous execution of a number of weak classical steps; one step for each element of the index set N. It is not hard to prove that any strong classical deduction with a finite index set N can be replaced by a weak classical deduction. As we shall see in the next section, this does not hold for strong classical deductions in general.

THEOREM 8.4. *The following four conditions are equivalent*:

(a) $\Gamma \vdash_L^{sL} x$.

(b) $\Gamma \subseteq L$, $x \in L$, *and* $\tilde{\Lambda}_L, \Gamma \vdash x$ *for all* $\Lambda \subseteq A(L)$.
(c) $\Gamma \subseteq L$, $x \in L$, *and* $\Gamma \subseteq M$ *implies* $x \in M$ *for all truth-assignments* M *on* L.
(d) $\Gamma \subseteq L$, $x \in L$, *and there exists a strong classical deduction* $\langle \Gamma, \{x\}, \pi \rangle$.

PROOF. The theorem is an immediate consequence of the following five lemmas, which prove (a) → (c), (c) → (b), (b) → (a), (d) → (c) and (c) → (d).

LEMMA 8.5. *If* $\Gamma \vdash_L^{s_L} x$, *then no truth-assignment* M *on* L *exists such that* $\Gamma \subseteq M$ *and* $x \notin M$.

PROOF. Assume $\delta_0 = \langle \Gamma_0 \cup \{s_L\}, x_0, \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma \cup \{s_L\}, x, \pi \rangle$ is a regular minimal deduction, and there exists a truth-assignment M on L such that $\Gamma \subseteq M$ and $x \notin M$. Assume $M_0 = \{x \mid x \in L \text{ and } \tilde{\Lambda}_L \vdash x\}$, where $\Lambda \subseteq A(L)$, $\Gamma_0 \subseteq M_0$ and $x_0 \notin M_0$.

(1) $\pi_0 = \emptyset$ and $x_0 \in A(\Gamma_0)$. Since $x_0 \in \Gamma_0$, $\Gamma_0 \subseteq M_0$ and $x_0 \notin M_0$ cannot both be true.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \{s_L\}, z, \pi_z \rangle \mid z \in \Delta\} \cup \{\langle \Gamma_0 \cup \{s_L\} \cup \{y\}, x_0, \pi' \rangle\}$, where $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0 \cup \{s_L\})$, and $x_0 = y$ if $y \in A(L)$. We have $z \in M_0$ for all $z \in \Delta$ because δ_0 is basic with respect to Φ. If $\langle \Delta, y \rangle \in \bigcup C(\Gamma_0)$, then $y \in M_0$; but $\Gamma_0 \cup \{y\} \subseteq M_0$ implies $x_0 \in M_0$ since δ_0 is basic with respect to Φ. If $\langle \Delta, y \rangle \in s_L$, then $y = x_0$ and $\langle \tilde{\Lambda}_L \cup \Gamma_0 \cup \{s_L\}, x_0, \pi'' \rangle \in \pi_z$ for some $z \in \Delta$, and since $\tilde{\Lambda}_L \vdash u$ for all $u \in \Gamma_0 \cup \{s_L\}$, we have $\tilde{\Lambda}_L \vdash x_0$, which implies $x_0 \in M_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Delta_i \cup \{s_L\}, y_i, \pi_i \rangle \mid i \in I\}$, where $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$. Since $x_0 \notin M_0$, a $k \in I$ exists such that $\Delta_k \subseteq M_0$ and $y_k \notin M_0$. But since $\Gamma_0 \cup \Delta_k \subseteq M_0$ and δ_0 is basic with respect to Φ, we have $y_k \notin M_0$, which is a contradiction.

LEMMA 8.6. *If* $\Lambda \subseteq A(L)$, $\Gamma \subseteq L$, $x \in L$, *and* $\tilde{\Lambda}_L, \Gamma \vdash x$ *does not hold, then* $\Gamma \subseteq M$ *and* $x \notin M$, *where* $M = \{y \mid y \in L \text{ and } \tilde{\Lambda}_L \vdash y\}$.

PROOF. We have $\tilde{\Lambda}_L \vdash z$ for all $z \in \Gamma$, since otherwise $\tilde{\Lambda}_L \vdash \neg_L z$ and $\tilde{\Lambda}_L, \Gamma \vdash x$; consequently $\Gamma \subseteq M$. Since $\tilde{\Lambda}_L, \Gamma \vdash x$ is false, $\tilde{\Lambda}_L \vdash x$ is false, and $x \notin M$.

LEMMA 8.7. *If* $\Gamma \subseteq L$, $x \in L$, *and* $\tilde{\Lambda}_L, \Gamma \vdash x$ *for all* $\Lambda \subseteq A(L)$, *then* $\Gamma, s_L \vdash x$.

PROOF. Assume x_0 basic with respect to Φ: $\Phi(x)$ iff $x \in L$, there exists a $\Gamma \subseteq L$ such that $\tilde{\Lambda}_L, \Gamma \vdash x$ for all $\Lambda \subseteq A(L)$, but $\Gamma, s_L \vdash x$ does not hold. Assume $\Gamma_0 \subseteq L$ and $\tilde{\Lambda}_L, \Gamma_0 \vdash x_0$ for all $\Lambda \subseteq A(L)$, but $\Gamma_0, s_L \vdash x_0$ is false.

If $x_0 \in A(L)$, then $\Gamma_0 \vdash \{\langle \tilde{\Delta}_L, x_0\rangle \mid \Delta \subseteq A(L)\}$ and thus $\Gamma_0, s_L \vdash x_0$. If $x_0 = \{\langle \Delta_i, y_i\rangle \mid i \in I\} \in C(L)$, then $\tilde{\Delta}_L, \Delta_k, \Gamma_0 \vdash y_k$ for all $\Delta \subseteq A(L)$ and all $k \in I$; consequently $\Delta_k, \Gamma_0, s_L \vdash y_k$ for all $k \in I$, and $\Gamma_0, s_L \vdash x_0$.

LEMMA 8.8. *If $\Gamma \subseteq L$, $\theta \subseteq L$, and there exists a strong classical deduction $\langle \Gamma, \theta, \pi\rangle$, then no truth-assignment M on L exists, for which $\Gamma \subseteq M$ and $\theta \subseteq L \setminus M$.*

PROOF. Assume $\delta_0 = \langle \Gamma_0, \theta_0, \pi_0\rangle$ basic with respect to Φ: $\Phi(\delta)$ iff $\delta = \langle \Gamma, \theta, \pi\rangle$ is a strong classical deduction, $\Gamma \subseteq L$, $\theta \subseteq L$, and there exists a truth-assignment M on L such that $\Gamma \subseteq M$ and $\theta \subseteq L \setminus M$. Assume M_0 is a truth-assignment on L, $\Gamma_0 \subseteq M_0$ and $\Theta_0 \subseteq L \setminus M_0$.

(1) $\pi_0 = \emptyset$ and $\Gamma_0 \cap \theta_0 \neq \emptyset$. Impossible since $\Gamma_0 \subseteq M_0$ and $\theta_0 \cap M_0 = \emptyset$.

(2) $\pi_0 = \{\langle \Gamma_0 \cup \Gamma_f, \theta_0 \cup \theta_f, \pi_f\rangle \mid f \in \times\{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N\}\}$, where

$$\begin{aligned} \Gamma_f &= \{y_\nu \mid \nu \in N \text{ and } f(\nu) = \Delta_\nu\}, \\ \theta_f &= \{f(\nu) \mid \nu \in N \text{ and } f(\nu) \in \Delta_\nu\}, \\ \langle \Delta_\nu, y_\nu\rangle &\in \bigcup C(\Gamma_0) \quad \text{for all } \nu \in N. \end{aligned}$$

At least one $f_0 \in \times\{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N\}$ has the property that $y_\nu \in M_0$ if $f_0(\nu) = \Delta_\nu$, and $f_0(\nu) \in L \setminus M$ if $f_0(\nu) \in \Delta_\nu$. But then $\Gamma_0 \cup \Gamma_{f_0} \subseteq M_0$ and $\theta_0 \cup \theta_{f_0} \subseteq L \setminus M_0$.

(3) $\pi_0 = \{\langle \Gamma_0 \cup \Gamma_g, \theta \cup \theta_g, \pi_g\rangle \mid g \in \times\{x_\nu \mid \nu \in N\}\}$, where $\Gamma_g = \bigcup\{\Delta_{g,\nu} \mid \nu \in N\}$, $\theta_g = \{y_{g,\nu} \mid \nu \in N\}$, $g(\nu) = \langle \Delta_{g,\nu}, y_{g,\nu}\rangle$, and $x_\nu \in C(\theta_0)$ for all $\nu \in N$. At least one $g_0 \in \times\{x_\nu \mid \nu \in N\}$ has the property that $\Delta_{g_0,\nu} \subseteq M_0$, $y_{g_0,\nu} \notin M_0$ for all $\nu \in N$. But then $\Gamma_0 \cup \Gamma_{g_0} \subseteq M_0$, $\theta_0 \cup \theta_{g_0} \subseteq L \setminus M_0$.

LEMMA 8.9. *If $\Gamma \subseteq L$, $\theta \subseteq L$, and no strong classical deduction $\langle \Gamma, \theta, \pi\rangle$ exists, then there is a truth-assignment M on L such that $\Gamma \subseteq M$ and $\theta \subseteq L \setminus M$.*

PROOF. If no classical deduction $\langle \Gamma, \theta, \pi\rangle$ exists, then there exists a sequence $\{\langle \Gamma_n, \theta_n\rangle \mid n = 0, 1, 2, \ldots\}$, where $\Gamma_0 = \Gamma$, $\theta_0 = \theta$, $\Gamma_{n+1} = \Gamma_n \cup \Gamma_{h_n}$, $\theta_{n+1} = \theta_n \cup \theta_{h_n}$ for $n = 0, 1, 2, \ldots$, and such that for all $k = 0, 1, 2, \ldots$:

(1) $h_{2k} \in \times\{\Delta_\nu \cup \{\Delta_\nu\} \mid \nu \in N_{2k}\}$,

$$\begin{aligned} \Gamma_{h_{2k}} &= \{y_\nu \mid \nu \in N_{2k} \text{ and } h_{2k}(\nu) = \Delta_\nu\}, \\ \theta_{h_{2k}} &= \{h_{2k}(\nu) \mid \nu \in N_{2k} \text{ and } h_{2k}(\nu) \in \Delta_\nu\}, \\ N_{2k} &= \bigcup C(\Gamma_{2k}), \\ \langle \Delta_\nu, y_\nu\rangle &= \nu \quad \text{for all } \nu \in N_{2k}, \end{aligned}$$

and no strong classical deduction $\langle \Gamma_{2k} \cup \Gamma_{h_{2k}}, \theta_{2k} \cup \theta_{h_{2k}}, \pi\rangle$ exists.

(2) $h_{2k+1} \in \times\{x_\nu \mid \nu \in N_{2k+1}\}$,
$\Gamma_{h_{2k+1}} = \cup\{\Delta_\nu \mid \nu \in N_{2k+1}\}$,
$\theta_{h_{2k+1}} = \{y_\nu \mid \nu \in N_{2k+1}\}$,
$N_{2k+1} = C(\theta_{2k+1})$,
$\langle \Delta_\nu, y_\nu \rangle = h_{2k+1}(\nu)$ for all $\nu \in N_{2k+1}$,

and no strong classical deduction $\langle \Gamma_{2k+1} \cup \Gamma_{h_{2k+1}}, \theta_{2k+1} \cup \theta_{h_{2k+1}}, \pi \rangle$ exists.

Let $\Gamma_\omega = \cup\{\Gamma_n \mid n = 0,1,2,\ldots\}$ and $\theta_\omega = \cup\{\theta_n \mid n = 1,2,\ldots\}$. Now we claim that $\Gamma_\omega \subseteq M$ and $\theta_\omega \subseteq L \setminus M$, where $M = \{x \mid x \in L \text{ and } \tilde{\Delta}_L \vdash x\}$ and $\Delta = A(\Gamma_\omega)$.

Assume that x_0 is basic with respect to Φ: $\Phi(x)$ iff either $x \in \Gamma_\omega$ and $x \notin M$, or $x \in \theta_\omega$ and $x \in M$.

$x_0 \in A(L)$ is impossible: If $x_0 \in A(\Gamma_\omega)$, then $x_0 \in M$ by the definition of M, and if $x_0 \in A(\theta_\omega) \cap M$, then there exists an n such that $x_0 \in A(\Gamma_n \cap \theta_n)$, and consequently $\langle \Gamma_n, \theta_n, \emptyset \rangle$ is a strong classical deduction.

Assume that $x_0 = \{\langle \Delta_i, y_i \rangle \mid i \in I\} \in C(L)$. Chose k such that $x_0 \in \Gamma_{2k} \cup \theta_{2k}$. If $x_0 \in \Gamma_{2k}$, then $\Delta_j \subseteq M$ implies $h_{2k}(\langle \Delta_j, y_j \rangle) = \Delta_j$ and consequently $y_j \in \Gamma_{h_{2k}}$ and $y_j \in M$, for all $j \in I$, and we have $x_0 \in M$. If $x_0 \in \theta_{2k}$, then $\Delta \subseteq M$ and $y \notin M$, where $\langle \Delta, y \rangle = h_{2k+1}(x_0)$, and consequently $x_0 \notin M$.

9. Relations between the weak and the strong classical derivations.

THEOREM 9.1. $s_L \vdash w_L$ and $\vdash_L{}^{s_L} \supseteq \vdash_L{}^{w_L}$.

PROOF. The theorem follows from the fact that the weak classical deductions are special cases of strong classical deductions.

PROPOSITION 9.2. *$w_L \vdash s_L$ iff $A(L)$ is finite.*

PROOF. That $w_L \vdash s_L$ if $A(L)$ is finite, is a consequence of the next theorem 9.3.

Assume that $A(L)$ is infinite and $w_L \vdash s_L$. Define

$$\bar{x} = \{\langle \tilde{\Delta}_L, x \rangle \mid \Delta \subseteq A(L)\} \quad \text{for all } x \in A(L).$$

Then we have $\bar{x}, w_L \vdash x$ for all $x \in A(L)$, and consequently there exists a weak classical deduction $\langle \{\bar{x}_0\}, \{x_0\}, \bar{\pi} \rangle$, where $x_0 \in A(L)$. Assume $\delta_0 = \langle \{\bar{x}_0\} \cup \Gamma_0, \theta_0' \cup \theta_0'', \pi_0 \rangle$ basic with respect to Φ: $\Phi(\delta)$ iff

$$\delta = \langle \{\bar{x}_0\} \cup \Gamma, \theta' \cup \theta'', \pi \rangle$$

is a weak classical deduction, Γ, θ', θ'' are finite sets,

$$\Gamma \subseteq A(L), \quad \theta' \subseteq A(L), \quad \theta'' = \{\neg_L y \mid y \in \Gamma'\}$$

for some $\Gamma' \subseteq \Gamma$, and $\Gamma \cap \theta' = \emptyset$. Notice that we have assumed

$$\Phi(\langle\{\bar{x}_0\},\{x_0\},\bar{\pi}\rangle)\ .$$

(1) $\pi_0=\emptyset$ and $[\{\bar{x}_0\}\cup\Gamma_0]\cap[\theta_0'\cup\theta_0'']\neq\emptyset$. This contradicts that $\Phi(\delta_0)$.

(2) $\pi_0=\{\langle\{\bar{x}_0\}\cup\Gamma_0,\theta_0'\cup\theta_0''\cup\{z\},\pi_z\rangle \mid z\in\Delta\}\cup$
$\cup\{\langle\{\bar{x}_0\}\cup\Gamma_0\cup\{x_0\},\theta_0'\cup\theta_0'',\pi'\rangle\}$,

where $\Delta=\tilde{\Delta}_L$ for some $\Delta\subseteq A(L)$. Since Γ_0, θ_0' and θ_0'' are finite sets and $A(L)$ is infinite, there exists a $z_0\in\Delta$ such that

$$\Phi(\langle\{\bar{x}_0\}\cup\Gamma_0,\theta_0'\cup\theta_0''\cup\{z_0\},\pi_{z_0}\rangle)\ ,$$

which contradicts that δ_0 is basic with respect to Φ.

(3) $\pi_0=\{\langle\{\bar{x}_0\}\cup\Gamma_0\cup\{y_0\},\theta_0'\cup\theta_0''\cup\{z\},\pi_z\rangle \mid z\in A(L)\}$, where $\neg_L y_0\in\theta_0''$. Since θ_0' and θ_0'' are finite and $A(L)$ is infinite, there exists a $z_0\in A(L)$ such that $\Phi(\langle\{\bar{x}_0\}\cup\Gamma_0\cup\{y_0\},\theta_0'\cup\theta_0''\cup\{z_0\},\pi_{z_0}\rangle)$, which contradicts that δ_0 is basic with respect to Φ.

THEOREM 9.3. *If* $\bigcup C(L)$ *is finite or denumerable, then* $\vdash_L^{wL}=\vdash_L^{sL}$.

PROOF. Let $\langle\Delta_n,y_n\rangle$, $n=0,1,2,\ldots$, be an enumeration of the elements of $\bigcup C(L)$, which repeats each element infinitely many times.

Since $\vdash_L^{wL}\subseteq\vdash_L^{sL}$ is generally true, we only have to prove that $\vdash_L^{wL}\supseteq\vdash_L^{sL}$. Assume that $\Gamma\vdash_L^{wL}x$ does not holds. Then there exists a sequence $\langle\Gamma_n,\theta_n,u_n\rangle$, $n=0,1,2,\ldots$, such that weak classical deductions $\langle\Gamma_n,\theta_n,\pi\rangle$ do not exist for any $n=0,1,2,\ldots$, $\langle\Gamma_0,\theta_0,n_0\rangle=\langle\Gamma,\{x\},x\rangle$, and for all $n=0,1,2,\ldots$ the following conditions hold:

$\langle\Gamma_{2n+1},\theta_{2n+1},u_{2n+1}\rangle = \langle\Gamma_{2n},\theta_{2n},u_{2n}\rangle$, if $u_n\in A(L)$;
$= \langle\Gamma_{2n}\cup\Delta,\theta_{2n}\cup\{y\},y\rangle$, where $\langle\Delta,y\rangle\in u_n$, if $u_n\in C(L)$.

$\langle\Gamma_{2n+2},\theta_{2n+2},u_{2n+2}\rangle = \langle\Gamma_{2n+1},\theta_{2n+1},u_{2n+1}\rangle$, if $\langle\Delta_n,y_n\rangle\notin\bigcup C(\Gamma_{2n+1})$;
$= \langle\Gamma_{2n+1}\cup\{y_n\},\theta_{2n+1},u_{2n+1}\rangle$, if $\langle\Delta_n,y_n\rangle\in\bigcup C(\Gamma_{2n+1})$ and no weak classical deduction $\langle\Gamma_{2n+1}\cup\{y_n\},\theta_{2n+1},u_{2n+1}\rangle$ exists;
$= \langle\Gamma_{2n+1},\theta_{2n+1}\cup\{z\},z\rangle$, where $z\in\Delta_n$, if $\langle\Delta_n,y_n\rangle\in\bigcup C(\Gamma_{2n+1})$ and there exists a weak classical deduction $\langle\Gamma_{2n+1}\cup\{y_n\},\theta_{2n+1},u_{2n+1}\rangle$.

Let $\Gamma_\omega=\bigcup\{\Gamma_n \mid n=0,1,2,\ldots\}$ and $\theta_\omega=\bigcup\{\theta_n \mid n=0,1,2,\ldots\}$. We claim that $\Gamma_\omega\subseteq M$ and $\theta_\omega\subseteq L\setminus M$, where $M=\{y \mid y\in L \text{ and } \tilde{\Delta}_L\vdash y\}$ and $\Delta=A(\Gamma_\omega)$.

Assume x_0 basic with respect to Φ: $\Phi(x)$ iff $x\in\Gamma_\omega$ and $x\notin M$, or $x\in\theta_\omega$ and $x\in M$.

$x_0\in A(L)$ is impossible: If $x_0\in A(\Gamma_\omega)$, then $x_0\in M$ by the definition of M, if $x_0\in A(\theta_\omega)\cap M$, then there exists an n such that $x_0\in A(\Gamma_n\cap\theta_n)$, and consequently $\langle\Gamma_n,\theta_n,\emptyset\rangle$ is a weak classical deduction.

If $x_0 \in C(\theta_\omega)$, then $x_0 = u_n$ for some n, and $\Delta \subseteqq \Gamma_{2n+1}$, $y \in \theta_{2n+1}$ for some $\langle \Delta, y\rangle \in x_0$. But then $\Delta \subseteqq M$, $y \in L \setminus M$, and consequently $x_0 \notin M$.

If $x_0 \in C(\Gamma_\omega)$ and $\langle \Delta, y\rangle \in x_0$, then there exists an n such that $\langle \Delta, y\rangle = \langle \Delta_n, y_n\rangle$, and $x_0 \in \Gamma_{2n+1}$. We have either $y \in \Gamma_{2n+2}$ or $z \in \theta_{2n+2}$ for some $z \in \Delta$, and since $\langle \Delta, y\rangle$ was an arbitrary element of x_0 this proves that $x_0 \in M$.

THEOREM 9.4. *If $A(L)$ is finite, then $\Vdash_L{}^{w_L} = \Vdash_L{}^{s_L}$.*

PROOF. Follows from the fact that $w_L \Vdash s_L$ (proposition 9.2).

10. Application to first order logic.

The purpose of this final section is briefly to indicate how the theory of the previous sections may be applied to formal logic, and to show how some of the theorems are related to well known results. In order to simplify the exposition, only first order logic with a single binary relation R and variables $x_1, x_2, \ldots$, but with no constants or functions, is considered. N denotes the set of positive natural numbers.

DEFINITION 10.1. $\mathscr{P}$ is the smallest set of formulas which satisfies all of the conditions:

(1) $R(x_i, x_j)$ belongs to $\mathscr{P}$ for all $i, j \in \mathsf{N}$.

(2) If A belongs to $\mathscr{P}$ then $\neg A$ belongs to $\mathscr{P}$.

(3) If A and B belong to $\mathscr{P}$ then $(A \wedge B)$, $(A \vee B)$ and $(A \Rightarrow B)$ belong to $\mathscr{P}$.

(4) If A belongs to $\mathscr{P}$ then $\forall x_i A$ and $\exists x_i A$ belong to $\mathscr{P}$ for all $i \in \mathsf{N}$.

PROPOSITION 10.1. *Let M be a set such that $M \times M$ is a prelogic containing only atomic elements, and let $\mathscr{F}$ be the set of functions $\mathsf{N} \to M$. Then there exist exactly one function μ and one prelogic L such that (definitions 3.5, 3.6 and 3.7 are used):*

(1) $A(L) = M \times M$.

(2) *μ maps $\mathscr{F} \times \mathscr{P}$ onto L.*

(3) $\mu\langle f, R(x_i, x_j)\rangle = \langle f(i), f(j)\rangle$ *for all* $f \in \mathscr{F}$, $i, j \in \mathsf{N}$.

(4) $\mu\langle f, \neg A\rangle = \neg_L \mu\langle f, A\rangle$ *for all* $f \in \mathscr{F}$, $A \in \mathscr{P}$.

(5) $\mu\langle f, (A \wedge B)\rangle = \mu\langle f, A\rangle \wedge \mu\langle f, B\rangle$ *for all* $f \in \mathscr{F}$, $A, B \in \mathscr{P}$.

(6) $\mu\langle f, (A \vee B)\rangle = \mu\langle f, A\rangle \vee_L \mu\langle f, B\rangle$ *for all* $f \in \mathscr{F}$, $A, B \in \mathscr{P}$.

(7) $\mu\langle f, (A \Rightarrow B)\rangle = \mu\langle f, A\rangle \Rightarrow \mu\langle f, B\rangle$ *for all* $f \in \mathscr{F}$, $A, B \in \mathscr{P}$.

(8) $\mu\langle f, \forall x_i A\rangle = \forall \{\mu\langle g, A\rangle \mid g \in \mathscr{F}$ and $g(j) = f(j)$ for all $j \in \mathsf{N} \setminus \{i\}\}$ *for all* $f \in \mathscr{F}$, $i \in \mathsf{N}$, $A \in \mathscr{P}$.

(9) $\mu\langle f, \exists x_i A\rangle = \exists_L \{\mu\langle g, A\rangle \mid g \in \mathscr{F}$ and $g(j) = f(j)$ for all $j \in \mathsf{N} \setminus \{i\}\}$ *for all* $f \in \mathscr{F}$, $i \in \mathsf{N}$, $A \in \mathscr{P}$.

DEFINITION 10.2. An *interpretation* of $\mathscr{P}$ is a quadruple $\langle M,f,\varphi,L'\rangle$, where M satisfies the conditions of proposition 10.1, and (using the terminology introduced in proposition 10.1) $f\in\mathscr{F}$; φ is the function $\mathscr{P}\to L$ such that $\varphi[A]=\mu\langle f,A\rangle$ for all $A\in P$, and L' is the smallest prelogic containing the image of $\mathscr{P}$ by φ.

DEFINITION 10.3. Let $\Gamma\subseteqq\mathscr{P}$ and $A\in\mathscr{P}$. The formula A is a *minimal consequence* of Γ (written $\Gamma\vdash A$), if $\varphi[\Gamma]\vdash\varphi[A]$ for all interpretations $\langle M,f,\varphi,L\rangle$. The formula A is a *weak classical consequence* of Γ (written $\Gamma\vdash^{w}A$) if $\varphi[\Gamma]\vdash_{L}^{wL}\varphi[A]$ for all interpretations $\langle M,f,\varphi,L\rangle$. The formula A is a *strong classical consequence* of Γ (written $\Gamma\vdash^{s}A$) if $\Gamma\vdash_{L}^{sL}A$ for all interpretations $\langle M,f,\varphi,L\rangle$.

THEOREM 10.2. *Let $\Gamma\subseteqq\mathscr{P}$, $A\in\mathscr{P}$. If there exitists an intuitionistic Gentzen type proof of the sequent $\Gamma\to A$ (see* [1], *or* [3, *chapter XV*]), *then $\Gamma\vdash A$. If there exists a classical Gentzen type proof of the sequent $\Gamma\to A$, then $\Gamma\vdash^{w}A$.*

THEOREM 10.3. *Let $\langle M,f,\varphi,L\rangle$ be any interpretation of $\mathscr{P}$ such that f is univalent and M contains a denumerable set of elements which are not images of* N *by f, let $\Gamma\subseteqq\mathscr{P}$ and $A\in\mathscr{P}$. If $\varphi[\Gamma]\vdash\varphi[A]$, then there exists an intuitionistic Gentzen type proof without cut of the sequent $\Gamma\to A$. If $\varphi[\Gamma]\vdash_{L}^{wL}\varphi[A]$, then there exists a classical Gentzen type proof without cut of the sequent $\Gamma\to A$.*

Theorem 10.3 may be proved using strong variants of theorem 4.1 and theorem 7.3. (Proposition 4.6 is an example of a strong variant of theorem 4.1.)

THEOREM 10.4. *The weak and the strong classical relations of consequence are identical on $\mathscr{P}$.*

Theorem 10.4 is proved by applying theorem 9.3 to an interpretation $\langle M,f,\varphi,L\rangle$ which satisfies the conditions of theorem 10.3 with the extra condition that M is denumerable. Theorem 10.4 is a variant of the Gödel completeness theorem, and it is natural to consider theorem 9.3 as a generalisation.

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ON THE BASES OF FREE ALGEBRAS

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We consider a fixed class $\mathscr{C}$ of universal algebras of the same species. All the concepts we introduce will be relative to the class $\mathscr{C}$.

Let F be an algebra of the same species as the algebras in $\mathscr{C}$, and S a non-empty set of generators of F. We say that S is a *$\mathscr{C}$-free set of generators of F*, if any mapping of S into an algebra $A \in \mathscr{C}$ may be extended to a homomorphism of F into A. Assume that this is the case; then any mapping of S into $A \in \mathscr{C}$ is, in fact, uniquely extended to a homomorphism of F into A. The notations being as above, we say that F is a *$\mathscr{C}$-free algebra* and that S is a *$\mathscr{C}$-free base of F*, if S is a $\mathscr{C}$-free set of generators of F and $F \in \mathscr{C}$. It is easily seen that two $\mathscr{C}$-free algebras with equipotent $\mathscr{C}$-bases are isomorphic.

Let A be an algebra of the same species as the algebras in $\mathscr{C}$, and S a subset of A. We say that S is *$\mathscr{C}$-independent* if either $S=\emptyset$ or the subalgebra of A generated by S is $\mathscr{C}$-freely generated by S. S is said to be *$\mathscr{C}$-dependent* if it is not $\mathscr{C}$-independent. $\mathscr{C}$-independence in this sense obviously includes the notions of independence studied by E. Marczewski [4] and J. Schmidt [5].

Observe that any subset of an algebra A containing a set of generators of A is again a set of generators of A, and that any subset of a $\mathscr{C}$-independent set is $\mathscr{C}$-independent. Note that a $\mathscr{C}$-base is both a minimal non-empty set of generators and a maximal non-empty $\mathscr{C}$-independent set. There is thus a duality between "generator sets" and "$\mathscr{C}$-independent sets".

We consider the following properties of the class $\mathscr{C}$, letting F denote an *arbitrary* $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base containing n elements:

(Ia) *Any set of generators of F contains at least n elements.*

(Ib) *Any $\mathscr{C}$-independent subset of F contains at most n elements.*

(IIa) *Any set of generators of F containing precisely n elements is a $\mathscr{C}$-base of F.*

(IIb) *Any $\mathscr{C}$-independent subset of F containing precisely n elements is a $\mathscr{C}$-base of F.*

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REMARK. If F has a $\mathscr{C}$-base consisting of exactly one element and F is generated by the empty set, then, strictly speaking, property (Ia) fails. But we shall admit this possible exception even when we state explicitly that $\mathscr{C}$ satisfies property (Ia).

Evidently, (IIa) $\Rightarrow$ (Ia) and (IIb) $\Rightarrow$ (Ib); and, moreover, either (Ia) or (Ib) implies that any two $\mathscr{C}$-bases of a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base have the same number of elements. Sufficient conditions for $\mathscr{C}$ to satisfy properties (Ia) and (IIb) were given by B. Jónsson and A. Tarski [2], and H. Jerome Keisler [3]. For the invariance property of the number of elements in a $\mathscr{C}$-base of a $\mathscr{C}$-free algebra, see [1] [6].

In the present note, we characterize properties (IIa) and (IIb) and derive sufficient conditions for properties (Ia) and (IIa), or properties (Ib) and (IIb).

LEMMA a. *Let F be a $\mathscr{C}$-free algebra with S as a $\mathscr{C}$-base, and A an algebra in $\mathscr{C}$ with T as a set of generators. Assume that there is a surjection $\varphi : S \to T$. Then φ is uniquely extended to an epimorphism $f : F \to A$.*

PROOF. Since $A \in \mathscr{C}$, φ is uniquely extended to a homomorphism $f : F \to A$. $f(F)$ is the subalgebra of A generated by $f(S) = \varphi(S)$. Since φ is surjective, $\varphi(S) = T$. But T generates A. Hence $f(F) = A$ and f is surjective.

LEMMA b. *Let F be a $\mathscr{C}$-free algebra with S as a $\mathscr{C}$-base, and A an algebra in $\mathscr{C}$ with T as a $\mathscr{C}$-independent subset. Assume that there is an injection $\varphi : S \to T$. Then φ is uniquely extended to a monomorphism $f : F \to A$.*

PROOF. Since $\varphi : S \to T$ is an injection, there is a surjection $\psi : T \to S$ such that $\psi \circ \varphi$ is the identity mapping of S, by the axiom of choice. Let B be the subalgebra of A generated by T. Then B is $\mathscr{C}$-freely generated by T. Since S is a $\mathscr{C}$-base of F and $A \in \mathscr{C}$, φ is uniquely extended to a homomorphism $f : F \to A$. Since $f(S) = \varphi(S) \subseteq T$ and B is generated by T, we have $f(F) \subseteq B$. Thus, we may regard f as a homomorphism of F into B. On the other hand, since T is a $\mathscr{C}$-independent subset of A and $F \in \mathscr{C}$, ψ is uniquely extended to a homomorphism $g : B \to F$. If $s \in S$, then

$$(g \circ f)(s) = g\big(\varphi(s)\big) = (\psi \circ \varphi)(s) = s .$$

Hence both $g \circ f$ and the identity mapping of F are homomorphisms extending $\psi \circ \varphi$. S being a $\mathscr{C}$-base of F, $g \circ f$ is the identity mapping of F. Therefore f is injective.

THEOREM a. *$\mathscr{C}$ satisfies property* (IIa) *if an only if any surjective endomorphism of a $\mathscr{C}$-free algebra with finite $\mathscr{C}$-base is an automorphism.*

PROOF. Let F be a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base containing n elements.

Assume that any surjective endomorphism of F is an automorphism. Let T be a set of generators of F containing precisely n elements. Then there is a surjection $\varphi : S \to T$, which is uniquely extended to a surjective endomorphism f of F, by Lemma a. By assumption, f is an automorphism of F. Therefore $T = \varphi(S) = f(S)$ is a $\mathscr{C}$-base of F.

Conversely, assume that $\mathscr{C}$ satisfies property (IIa). Then $\mathscr{C}$ also satisfies property (Ia). Let f be a surjective endomorphism of F. Then $f(S)$ contains at most n elements. On the other hand, $f(S)$ is a set of generators of $f(F) = F$ and, as such, must contain at least n elements, by property (Ia). Hence $f(S)$ contains precisely n elements and is consequently a $\mathscr{C}$-base of F, by property (IIa). But f maps S bijectively onto $f(S)$. In particular, this means that f is also injective, by Lemma b. Therefore f is an automorphism.

THEOREM b. *$\mathscr{C}$ satisfies property* (IIb) *if an only if any injective endomorphism of a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base is an automorphism.*

PROOF. Let F be a $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base containing n elements.

Assume that any injective endomorphism of F is an automorphism. Let T be a $\mathscr{C}$-independent subset of F containing precisely n elements. Then there exists a bijection $\varphi : S \to T$, which is uniquely extended to an injective endomorphism of F, by Lemma b. By assumption, f is an automorphism of F. Therefore $T = \varphi(S) = f(S)$ is a $\mathscr{C}$-base of F.

Conversely, assume that $\mathscr{C}$ satisfies property (IIb). Let f be an injective endomorphism of F. Then $f(S)$ contains precisely n elements. Let A be the subalgebra of F generated by $f(S)$. Then f may be regarded as an isomorphism of F onto A, so that f^{-1} exists and is an isomorphism of A onto F. Let φ be any mapping of $f(S)$ into an algebra $C \in \mathscr{C}$. Denote by $f \mid S$ the restriction of f to S; then $f \mid S$ maps S into $f(S)$ so that $\varphi \circ (f \mid S)$ is defined and maps S into C. S being a $\mathscr{C}$-base of F, $\varphi \circ (f \mid S)$ is uniquely extended to a homomorphism $g : F \to C$. It follows that $g \circ f^{-1}$ is a homomorphism of A into C. Moreover, if $t \in f(S)$, then $t = f(s)$ for some $s \in S$, so that

$$\begin{aligned}(g \circ f^{-1})(t) &= (g \circ f^{-1})\big(f(s)\big) \\ &= \big((g \circ f^{-1}) \circ f\big)(s) \\ &= \big(g \circ (f^{-1} \circ f)\big)(s) = g(s) = \big(\varphi \circ (f \mid S)\big)(s) \\ &\qquad = \varphi\big((f \mid S)(s)\big) = \varphi\big(f(s)\big) = \varphi(t);\end{aligned}$$

hence $g \circ f^{-1}$ extends φ. Thus, $f(S)$ generates A $\mathscr{C}$-freely and is consequently a $\mathscr{C}$-independent subset of F. Hence $f(S)$ is a $\mathscr{C}$-base of F, by property (IIb). But f maps S bijectively onto $f(S)$. In particular, this means that f is also surjective, by Lemma a. Therefore f is an automorphism.

COROLLARY. a. *Assume that the lattice of congruences of any $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base satisfies the ascending chain condition. Then $\mathscr{C}$ satisfies property* (IIa).

COROLLARY b. *Assume that the lattice of subalgebras of any $\mathscr{C}$-free algebra with a finite $\mathscr{C}$-base satisfies the descending chain condition. Then $\mathscr{C}$ satisfies property* (IIb).

There exist classes of algebras not satisfying any of the properties we considered (cf. [2] [6]). There exists a class $\mathscr{C}$ of algebras satisfying (Ia) but not (IIa) (cf. [2]). The class of groups satisfies (Ia) and (IIa), but not (Ib) and (IIb). The class of abelian groups satisfies (Ia), (IIa) and (Ib), but not (IIb).

The applications of our results to the class of distributive lattices, the class of Boolean algebras, and the class of left modules over a ring satisfying the ascending or descending chain condition for left ideals are immediate.

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A QUASI-SPECTRAL OPERATOR

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In [1] we defined a quasi-spectral operator and stated sufficient conditions to insure that an operator be quasi-spectral. The operator $Tf(x) = xf(x)$ on $C[0,1]$ was used as a guiding example, but it is clear that for any continuous monotone function g on $[0,1]$, such that $g(0)=0$, $g(1)=1$, $Tf(x)=g(x)f(x)$ satisfies the sufficient conditions of [1] if we pick $\langle f, \Phi(\lambda)\rangle = f(g^{-1}(\lambda))$. This choice is necessary in order to make $\overline{(\lambda I - T)X}$ the nullspace of $\Phi(\lambda)$. Therefore $Tf(x)=g(x)f(x)$ on $C[0,1]$ is a quasi-spectral operator.

The restrictions of monotonicity and that $g(0)=0$, $g(1)=1$ seem to be rather strong, so it is natural to attempt to remove them. The restriction that $g(0)=0$, $g(1)=1$ can easily be removed by a change of scale. To study monotonicity, let us consider an example. Let $g(x)$ be a continuous function on $[0,1]$ with a single relative maximum. In order to fix our ideas, let us say that for some z, $0<z<1$, $g(z)=1$, $g(0)=0$, $g(1)=0$, g is monotone increasing from 0 to z, and monotone decreasing from z to 1. We now define projections $E(a,b)$ for intervals (a,b) or $(a,b]$ or $[a,b)$ or $[a,b]$ on the range of g, which is the spectrum of T. As before, we use $E(0,b)$ whether 0 is in the spectrum of T or not. Let

$$\delta(a,b) = g^{-1}(a,b) = \{x \mid a < g(x) < b\}.$$

Then

$$\delta(a,b) = (a',b') \cup (a'',b'').$$

Define, for $a \neq 0$

$$\begin{aligned} E(a,b)f(x) &= 0 && x \leqq a' \\ &= f(x)-f(a') && a' \leqq x \leqq b' \\ &= f(b')-f(a') && b' \leqq x \leqq a'' \\ &= f(x)-f(a'')+f(b')-f(a') && a'' \leqq x \leqq b'' \\ &= f(b'')-f(a'')+f(b')-f(a') && b'' \leqq x. \end{aligned}$$

For $a=0$,

$$\begin{aligned} E(0,b)f(x) &= f(x) && x \leqq b' \\ &= f(b') && b' \leqq x \leqq a'' \\ &= f(x)-f(a'')+f(b') && a'' \leqq x. \end{aligned}$$

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Since g is monotone on the intervals $[0,z]$ and $[z,1]$, it determines two invertable functions $g_1(x)$ on $[0,z]$ and $g_2(x)$ on $[z,1]$. Given an x in the range of g, let $x'=g_1^{-1}(x)$ and $x''=g_2^{-1}(x)$.

THEOREM 1. *Let* $0=x_0<x_1<\ldots<x_n=1$ *be a partition of* $[0,1]$. *Let* $\delta_i=(x_{i-1},x_i)$. *Then* $\sum_{i=1}^{n}E(\delta_i)=I$.

PROOF. Let f and x be arbitrary, and pick k so that $x_{k-1}\leqq g(x)<x_k$. Suppose first that $x<z$. Then

$$\begin{aligned}\sum_{i=1}^{n}E(\delta_i)f(x) &= f(x_1')+\sum_{i=2}^{k-1}[f(x_i')-f(x_{i-1}')]+f(x)-f(x_{k-1}')+0\\ &= f(x)\,.\end{aligned}$$

Next suppose $x>z$. Then

$$\begin{aligned}\sum_{i=1}^{n}E(\delta_i)f(x) &= f(x_1')+\sum_{i=2}^{k-1}[f(x_i')-f(x_{i-1}')]+f(x)-f(x_k'')+\\ &\quad +f(x_k')-f(x_{k-1}')+\sum_{i=k+1}^{n}[f(x_{i-1}'')-f(x_i'')+f(x_i')-f(x_{i-1}')]\\ &= f(x)+f(x_n')-f(x_n'')\\ &= f(x)+f(z)-f(z)\\ &= f(x)\,.\end{aligned}$$

Since $\sum_{i=1}^{n}E(\delta_i)f(x)=f(x)$ for all x in $[0,1]$ and for all f in $C[0,1]$, $\sum_{i=1}^{n}E(\delta_i)=I$.

THEOREM 2. *Let* δ_i *be as in Theorem* 1. *Then*

$$\sum_{i=1}^{n}\lambda_iE(\delta_i)=T-N$$

where $\lambda_i\in\delta_i$, *and where*

$$\begin{aligned}Nf(x) &= \int_0^x f(t)\,dg(t) && x\leqq z\,,\\ &= \int_0^z f(t)\,dg(t)-\int_z^x f(t)\,dg(t) && z\leqq x\,.\end{aligned}$$

PROOF. Let f and x be arbitrary, and pick k so that $x_{k-1}\leqq g(x)<x_k$. Suppose first that $x<z$. Then

$$\sum_{i=1}^{n} \lambda_i E(\delta_i) f(x)$$
$$= \lambda_1 f(x_1') + \sum_{i=2}^{k-1} \lambda_i [f(x_i') - f(x_{i-1}')] + \lambda_k [f(x) - f(x_{k-1}')]$$
$$= g(\lambda_1') f(x_1') + \sum_{i=2}^{k-1} g(\lambda_i') [f(x_i') - f(x_{i+1}')] + g(\lambda_k') [f(x) - f(x_{k-1}')] .$$

Rearrangement of this sum by partial summation gives

$$\sum_{i=1}^{k-1} f(x_i') [g(\lambda_i') - g(\lambda_{i-1}')] + g(\lambda_k') f(x) .$$

Passage to the limit gives

$$\int_{\sigma(T)} \lambda E(d\lambda) f(x) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \lambda_i E(\delta_i) f(x)$$
$$= -\int_0^x f(t)\, dg(t) + g(x) f(x) = Tf(x) - Nf(x) .$$

Next suppose $x > z$. Then

$$\sum_{i=1}^{n} \lambda_i E(\delta_i) f(x) = \lambda_1 f(x_1') + \sum_{i=2}^{k-1} \lambda_i [f(x_i') - f(x_{i-1}')] +$$
$$+ \lambda_k [f(x) - f(x_k'') + f(x_k') - f(x_{k-1}')] +$$
$$+ \sum_{i=k+1}^{n} \lambda_i [f(x_{i-1}'') - f(x_i'') + f(x_i') - f(x_{i-1}')] .$$

Rearrangement of this sum by partial summation gives

$$\sum_{i=1}^{n-1} f(x_i') [\lambda_i - \lambda_{i+1}] + \lambda_n f(x_n') + \sum_{i=k}^{n-1} f(x_i'') [\lambda_{i+1} - \lambda_i] - \lambda_n f(x_n'') + \lambda_k f(x)$$
$$= \sum_{i=1}^{n-1} f(x_i') [g(\lambda_i') - g(\lambda_{i+1}')] + \sum_{i=k}^{n-1} f(x_i'') [g(\lambda_{i+1}'') - g(\lambda_i'')] + g(\lambda_k'') f(x) .$$

Passage to the limit gives

$$\int_{\sigma(T)} \lambda E(d\lambda) f(x) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \lambda_i E(\delta_i) f(x)$$
$$= g(x) f(x) - \int_0^z f(t)\, dg(t) + \int_z^x f(t)\, dg(t) .$$

Thus in either case

$$\int_{\sigma(T)} \lambda E(d\lambda) f(x) = Tf(x) - Nf(x) ,$$

and since f and x are arbitrary,

$$\int_{\sigma(T)} \lambda E(d\lambda) = T - N .$$

Thus T is a quasi-spectral operator, where $Tf(x) = g(x)f(x)$ for $g \in C[0,1]$ such that $g(0) = 0$, $g(1) = 0$, and such that g has a single relative maximum at z, $g(z) = 1$. Again by a change of scale we can remove the restriction on the values of $g(0)$, $g(z)$, and $g(1)$. Also, it is clear that only notational difficulties would be introduced by allowing g to have a finite number of extreme points. Thus we have the following theorem.

THEOREM 3. *Let $Tf(x) = g(x)f(x)$ where g is continuous and has a finite number of extreme points. Then T is a quasi-spectral operator on $C[0,1]$.*

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ÜBER DIE α-KAPAZITÄT EINER VERALLGEMEINERTEN CANTORSCHEN PUNKTMENGE

ÅGOT E. RIIBER

1.

M. Ohtsuka hat in [2] die α-Kapazität und die logarithmische Kapazität einer symmetrischen verallgemeinerten Cantorschen Punktmenge F im Euklidischen n-Raum untersucht. Nachdem ich diese Abhandlung gelesen hatte, bekam ich den Einfall, die α-Kapazität der in meiner Arbeit [3] definierten verallgemeinerten Cantorschen Punktmenge zu studieren. ([3] behandelt die logarithmische Kapazität.) Bei vorliegender Untersuchung benutzte ich auch die Abhandlung [1] von G. af Hällström, und ich bin ihm auch sonst für wertvolle Hilfe grossen Dank schuldig.

Die Punktmenge in [3] erhält man folgendermassen: Wir gehen von einer in der z-Ebene liegenden, nicht degenerierten, beschränkten, abgeschlossenen, zusammenhängenden Punktmenge M_0 mit dem Durchmesser d_0 aus. In M_0 wählen wir k_1 neue paarweise punktfremde, nicht degenerierte, abgeschlossene, zusammenhängende Punktmengen $A_{1,\nu}$, $\nu = 1, 2, \ldots, k_1$. Ihre Vereinigungsmenge M_1 ist demnach eine Untermenge von M_0. Der Prozess wird fortgesetzt, so dass M_n aus $k_1 k_2 \ldots k_n$ paarweise punktfremden, nicht degenerierten, abgeschlossenen, zusammenhängenden Punktmengen $A_{n,\nu}$ besteht. Diese entstehen dadurch, dass man k_n Punktmengen $A_{n,\nu}$, $k_n \geqq 2$, in jedes $A_{n-1,\nu}$ legt. Wir nennen zwei $A_{n,\nu}$ zueinander *assoziiert*, wenn sie zu demselben $A_{n-1,\nu}$ $(A_0 = M_0)$ gehören, und $A^*_{n,\nu}$ soll die Vereinigungsmenge aller $A_{n+1,\lambda} \subset A_{n,\nu}$ sein. Den Durchmesser von $A_{n,\nu}$ nennen wir $D_{n,\nu}$, und das Maximum d_n der Grössen $D_{n,\nu}$ soll die Bedingung $\lim_{n\to\infty} d_n = 0$ erfüllen. Setzen wir (der zu (1) entsprechende Zusammenhang in [3] lautet etwas anders)

$$1/p_n = k_n d_n / d_{n-1}, \qquad n = 1, 2, \ldots, \tag{1}$$

$$P_n = p_1 p_2 \ldots p_n \tag{2}$$

und

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(3) $$N_n = k_1 k_2 \ldots k_n ,$$

so wird

(4) $$d_n = d_0 (P_n N_n)^{-1}, \qquad n = 1,2,\ldots .$$

Unsere verallgemeinerte Cantorsche Punktmenge ist nun $M = \bigcap_{n=1}^{\infty} M_n$.

2.

Zuerst werden wir einen Satz über die verschwindende α-Kapazität beweisen.

SATZ 1. *Falls*

(5) $$\sum_{\nu=1}^{\infty} \frac{(p_1 p_2 \ldots p_\nu)^\alpha}{(k_1 k_2 \ldots k_\nu)^{1-\alpha}} = \infty, \qquad 0 < \alpha < 2 ,$$

ist, so ist M von der α-Kapazität Null.

BEWEIS. Das α-Potential und die entsprechende Kapazität sind Spezialfälle der Grössen bei Ugaheri [4], wo man folgende Definition findet: Wir haben eine beschränkte Borelmenge E in einem Euklidischen Raum R. Sei $\mu(e)$ eine vollständig additive und nicht-negative Funktion der Borelmengen e auf E; dann nennen wir $\mu(e)$ eine positive Massenbelegung auf E mit Totalmasse $\mu(E)$. Ist $\nu(e)$ eine Massenbelegung auf einer Borelschen Untermenge E' von E, so setzen wir $\mu(e) = \nu(E'e)$ für eine Borelmenge e auf E und erhalten dann eine positive Massenbelegung auf E.

Nach Ugaheri [4, S. 155, Satz 1] gibt es immer eine positive Massenbelegung mit positiver Totalmasse auf einer beschränkten Borelmenge von positiver Kapazität, so dass das von dieser Belegung erzeugte Potential beschränkt ist. Wir werden zeigen, dass man keine Massenverteilung auf M mit beschränktem α-Potential finden kann, wenn (5) gilt.

Zu diesem Zwecke verteilen wir ganz willkürlich auf M eine positive Einheitsmasse und studieren das α-Potential $g(z)$ dieser Masse, $0 < \alpha < 2$. In jedem $A_{n,\nu}$ nehmen wir einen Punkt $z_{n,\nu}$ ausserhalb $A^*_{n,\nu}$ und suchen eine untere Schranke des Mittelwertes

$$\bar{g}_n = \frac{1}{N_n} \sum_{\nu=1}^{N_n} g(z_{n,\nu}) = \frac{1}{N_n} \sum_{h=1}^{N_n} \left[\sum_{\nu=1}^{N_n} \int_{A^*_{n,h}} \frac{1}{|z_{n,\nu} - x|^\alpha} \, d\mu(x) \right].$$

Wir betrachten nun die nicht-negative Masse $\mu_{n,h} = \mu(A^*_{n,h})$. Der Abstand eines jeden $A^*_{n,h}$-Punktes von $z_{n,h}$ ist höchstens gleich d_n. Der Abstand von den anderen $k_n - 1$ Punkten $z_{n,\nu}$, die in demselben $A_{n-1,\xi}$ wie $A^*_{n,h}$ liegen, ist höchstens gleich d_{n-1}. Der Abstand von den $(k_{n-1} - 1)k_n$ Punkten $z_{n,\nu}$, die in demselben $A_{n-2,\eta}$ wie $A^*_{n,h}$ liegen, und die noch

nicht mitgenommen sind, ist höchstens gleich d_{n-2} u. s. w. Daher ist für $n \geqq 3$

$$\bar{g}_n > \frac{1}{N_n} \sum_{h=1}^{N_n} \mu_{n,h} \left[\frac{1}{d_n^\alpha} + \frac{k_n - 1}{d_{n-1}^\alpha} + \frac{(k_{n-1}-1)k_n}{d_{n-2}^\alpha} + \ldots + \frac{(k_2-1)k_3 \ldots k_n}{d_1^\alpha} \right].$$

Wir benutzen hier $k_\nu - 1 \geqq \frac{1}{2} k_\nu$ und erhalten mit Hilfe von (2), (3) und (4)

$$\bar{g}_n > \sum_{h=1}^{N_n} \mu_{n,h} \left[\frac{1}{2 d_0^\alpha} \sum_{\nu=1}^{n} \frac{(p_1 p_2 \ldots p_\nu)^\alpha}{(k_1 k_2 \ldots k_\nu)^{1-\alpha}} \right].$$

Also wird

$$\bar{g}_n > \frac{1}{2 d_0^\alpha} \sum_{\nu=1}^{n} \frac{(p_1 p_2 \ldots p_\nu)^\alpha}{(k_1 k_2 \ldots k_\nu)^{1-\alpha}},$$

weil

$$\sum_{h=1}^{N_n} \mu_{n,h} = 1$$

ist. Setzen wir nun (5) voraus, so gibt unsere willkürliche Belegung kein beschränktes α-Potential, und die Kapazität muss nach Ugaheri Null sein.

3.

Wir wenden Satz 1 auf die Punktmengen $F = F_1$ und $F = F_1 \times F_2$ mit $F_1 = F_2$ (siehe [2]) an.

Die Punktmenge F_1 liegt auf der reellen Achse und ist ein Spezialfall von M. Die Menge $A_{n,\nu}$, $n = 1, 2, \ldots$, ist dann ein Intervall dieser Achse, und wir haben $D_{n,\nu} = d_n$. Die Menge $A^*_{n,\nu}$ entsteht aus $A_{n,\nu}$ dadurch, dass man $k_{n+1} - 1$ offene Strecken von derselben Länge entfernt. Wir bezeichnen für F_1 die Grösse k_ν mit k_ν' und p_ν mit p_ν'. Satz 1 gibt unmittelbar in dem linearen Fall die hinreichende Bedingung für verschwindende α-Kapazität des Ohtsukaschen Satzes [2, S. 152]. Ohtsuka hat hier $\alpha < 1$ vorausgesetzt.

Die Punktmenge F_2 ist in derselben Weise wie F_1 konstruiert und liegt auf der imaginären Achse. Um Satz 1 auf die Punktmenge $F = F_1 \times F_2$ (mit $F_1 = F_2$) anwenden zu können, setzen wir $k_n = (k_n')^2$ und $p_n = p_n'/k_n'$. Dann erhält man folgendes: *Falls*

$$\sum_{\nu=1}^{\infty} \frac{(p_1' p_2' \ldots p_\nu')^\alpha}{(k_1' k_2' \ldots k_\nu')^{2-\alpha}} = \infty, \quad 0 < \alpha < 2,$$

ist, so ist $F = F_1 \times F_2$ (mit $F_1 = F_2$) von der α-Kapazität Null.

Dieses Ergebnis ist auch in dem oben erwähnten Satz von Ohtsuka enthalten.

4.

Um eine hinreichende Bedingung für positive α-Kapazität zu finden, betrachten wir den kleinsten Abstand δ_n zwischen zwei assoziierten $A_{n,\lambda}$ und setzen $r_n = \delta_n / d_{n-1}$, $n = 1, 2, \ldots$.

SATZ 2. *Falls*

$$(6) \qquad \sum_{\nu=1}^{\infty} \frac{(p_1 p_2 \ldots p_\nu)^\alpha}{(r_{\nu+1})^\alpha (k_1 k_2 \ldots k_\nu)^{1-\alpha}} < \infty, \quad 0 < \alpha < 2,$$

ist, so hat M positive α-Kapazität.

BEWEIS. Nach Ugaheri [4, S. 162, Satz 3] gibt es immer auf einer kompakten Punktmenge E von der Kapazität Null eine positive Massenverteilung mit positiver Totalmasse, welche ein unendliches Potential in jedem E-Punkt erzeugt. Weiter gilt für das Potential $u(z)$ dieser Belegung $\lim_{z \to z_0} u(z) = \infty$, wenn z_0 ein beliebiger E-Punkt ist (siehe [4, S. 152, Korollar]). Wir werden mit Hilfe von (6) zeigen, dass man mit einer willkürlichen Einheitsverteilung auf M eine Punktfolge $\{\xi_m\}$ konstruieren kann, die sich einem M-Punkte nähert, und für welche das α-Potential beschränkt ist.

Sei $\mu_{n,\nu} \geqq 0$ die Masse auf $A^*_{n,\nu}$ unserer willkürlichen Einheitsverteilung. Man zeigt leicht, dass es in jedem $A_{n,\nu}$ einen Punkt $z_{n,\nu}$ gibt, dessen Abstand von $A^*_{n,\nu}$ mindestens gleich $\delta_{n+1}/3$ ist.

Wir suchen zuerst eine obere Schranke des Mittelwertes $\bar{g}_n$ aller $g(z_{n,\nu})$, $\nu = 1, 2, \ldots, N_n$, wo $g(z)$ das α-Potential der Verteilung ist.

Auch in diesem Beweise studieren wir zuerst das α-Potential, das in den verschiedenen Punkten $z_{n,\nu}$ von der Masse $\mu_{n,h}$ erzeugt wird. Der Abstand zwischen $A^*_{n,h}$ und $z_{n,h}$ ist mindestens gleich $\delta_{n+1}/3$. Die $k_n - 1$ Punkte $z_{n,\nu}$ der zu $A_{n,h}$ assoziierten $A_{n,\nu}$ haben von $A^*_{n,h}$ einen Abstand, der mindestens gleich δ_n ist. Die k_n genannten $A_{n,\nu}$ gehören zu demselben $A_{n-1,\lambda}$, welches $k_{n-1} - 1$ assoziierte $A_{n-1,\lambda}$ hat. In diesen Punktmengen liegen $(k_{n-1} - 1)k_n$ Punkte $z_{n,\nu}$. Der Abstand eines jeden Punktes von $A^*_{n,h}$ ist mindestens gleich δ_{n-1} u. s. w. Wir erhalten also für $n \geqq 2$

$$\bar{g}_n \leqq \frac{1}{N_n} \sum_{h=1}^{N_n} \mu_{n,h} \left[\frac{3^\alpha}{r_{n+1}^\alpha d_n^\alpha} + \frac{k_n - 1}{r_n^\alpha d_{n-1}^\alpha} + \frac{(k_{n-1} - 1)k_n}{r_{n-1}^\alpha d_{n-2}^\alpha} + \ldots + \frac{(k_1 - 1)k_2 \ldots k_n}{r_1^\alpha d_0^\alpha} \right].$$

Wegen (2), (3), (4) und der Ungleichung $k_\nu - 1 < k_\nu$ wird

$$\bar{g}_n < \sum_{h=1}^{N_n} \mu_{n,h} \left[\frac{1}{(d_0 r_1)^\alpha} + \left(\frac{3}{d_0} \right)^\alpha \sum_{\nu=1}^{n} \frac{(p_1 p_2 \ldots p_\nu)^\alpha}{r_{\nu+1}^\alpha (k_1 k_2 \ldots k_\nu)^{1-\alpha}} \right].$$

Weil auch hier

$$\sum_{h=1}^{N_n} \mu_{n,h} = 1$$

ist, so wird also

$$\bar{g}_n < \frac{1}{(d_0 r_1)^\alpha} + \left(\frac{3}{d_0}\right)^\alpha \sum_{\nu=1}^{n} \frac{(p_1 p_2 \dots p_\nu)^\alpha}{r_{\nu+1}^\alpha (k_1 k_2 \dots k_\nu)^{1-\alpha}}.$$

Falls (6) gilt, haben wir daher $\bar{g}_n < C$, wo C eine von n unabhängige Konstante ist. Für mindestens einen Punkt $z_{n,\nu}$, etwa z_n, ist $g(z_n) < C$. Die Folge $\{z_n\}$ mit beschränktem α-Potential nähert sich M und muss auf dieser Menge einen Häufungspunkt haben. Daher existiert eine Folge $\{\xi_m\}$ mit den früher erwähnten Eigenschaften, und die Kapazität muss positiv sein.

5.

Satz 1 und Satz 2 gibt

SATZ 3. *Falls für alle* ν

(7) $$r_\nu \geqq r > 0$$

ist, so hat M *dann und nur dann die* α*-Kapazität Null, wenn*

$$\sum_{\nu=1}^{\infty} \frac{(p_1 p_2 \dots p_\nu)^\alpha}{(k_1 k_2 \dots k_\nu)^{1-\alpha}} = \infty, \quad 0 < \alpha < 2,$$

gilt.

6.

Setzen wir für $F = F_1$ und $F = F_1 \times F_2$ (mit $F_1 = F_2$)

(8) $$k_\nu' \leqq k, \quad p_\nu' \geqq p > 1, \quad \nu = 1, 2, \dots,$$

voraus, so gilt (7). Daher sind in diesem Fall die früher erwähnten hinreichenden Bedingungen für Nullkapazität auch notwendig. Der Ohtsukasche Satz ist hier schärfer, denn er braucht nicht (8) anzunehmen.

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TECHNISCHE HOCHSCHULE, TRONDHEIM, NORWEGEN

MATH. SCAND. 16 (1965), 38–40

ON SETS OF VECTORS

PETER SCHERK

Given n non-void sets

$$A_1,\ldots,A_n \tag{1}$$

of vectors in a finite or infinite dimensional vector space V over an arbitrary field. The letters a_i, b_i, c_i denote elements of A_i, and $\dim A_i$ denotes the dimension of the subspace spanned by A_i, $i=1,\ldots,n$. Let

$$0 \leqq m \leqq n\,.$$

We consider the following two statements:

$\mathrm{P}_{n,m}$: *n vectors $a_1,\ldots,a_n$ always span a subspace of dimension $\leqq m$.*

$\mathrm{Q}_{n,m}$: *There exists an integer h, $0 \leqq h \leqq m$, and an h-space containing $h+(n-m)$ of the sets* (1).

Obviously, $\mathrm{Q}_{n,m}$ implies $\mathrm{P}_{n,m}$.

The purpose of this note is to show that conversely $\mathrm{P}_{n,m}$ *implies* $\mathrm{Q}_{n,m}$.

This assertion is trivial for $n=1$. We assume it has been proved up to $n-1$. From now on let $n>1$ be fixed. The case $m=0$ being trivial, we may assume $m>0$.

If some $n-1$ of the sets (1) satisfy $\mathrm{P}_{n-1,m-1}$, then they satisfy $\mathrm{Q}_{n-1,m-1}$ by induction assumption and the sets (1) themselves will satisfy $\mathrm{Q}_{n,m}$. Thus we may assume that no $n-1$ of the sets (1) satisfy $\mathrm{P}_{n-1,m-1}$, and hence, in particular, that $m<n$.

By the last assumption, there are $n-1$ vectors $b_1,\ldots,b_{n-1}$ spanning a subspace V_m of dimension $\geqq m$. By $\mathrm{P}_{n,m}$, we have $\dim V_m = m$ and every vector a_n lies in V_m. This yields $\dim A_m \leqq m$; more generally,

$$\dim A_i \leqq m\,, \qquad i=1,\ldots,n\,.$$

In particular, we may assume V to be finite dimensional.

Suppose $A_n=\{0\}$. Then $A_1,\ldots,A_{n-1}$ satisfy $\mathrm{P}_{n-1,m}$. If $m=n-1$, then $\mathrm{Q}_{n,m}$ is trivial with $h=0$. If $m<n-1$, then our induction assumption implies $\mathrm{Q}_{n-1,m}$ for the sets $A_1,\ldots,A_{n-1}$ and the sets (1) satisfy $\mathrm{Q}_{n,m}$. Thus we may assume

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$$A_i \neq \{0\}, \qquad i = 1, \ldots, n . \tag{2}$$

LEMMA. *Suppose there is an integer k, $1 \leqq k \leqq m$, and there is a k-space containing k of the sets* (1). *Then $Q_{n,m}$ holds.*

PROOF. Without loss of generality, we may assume that the sets

$$A_1, \ldots, A_k \tag{3}$$

lie in a k-space V_k. We may also assume that k is minimal. Thus either $k = 1$ or every h-space contains fewer than h of the sets (3) if $0 < h < k$. Thus $Q_{k,k-1}$ and hence $P_{k,k-1}$ are false for the sets (3) if $k > 1$. There are, therefore, k linearly independent vectors

$$c_1, \ldots, c_k .$$

They form a base of V_k. (If $k = 1$, this remark follows directly from (2).)

We now project V parallel to V_k onto a subspace complementary to V_k. Dashes denote projections. For every choice of $a_{k+1}, \ldots, a_n$, the vectors

$$c_1, \ldots, c_k, a_{k+1}, \ldots, a_n$$

span a space of dimension $\leqq m$. Hence the projections

$$a'_{k+1}, \ldots, a_n'$$

always span a subspace of dimension $\leqq m-k$ and the projections

$$A'_{k+1}, \ldots, A_n' \tag{4}$$

satisfy $P_{n-k,m-k}$ and hence $Q_{n-k,m-k}$. Thus there is an integer g, $0 \leqq g \leqq m-k$, and there are

$$g + (n-k) - (m-k) = g + (n-m)$$

distinct sets

$$A'_{i_1}, \ldots, A'_{i_{g+n-m}}$$

among the sets (4) which lie in a g-space. The $k+g+n-m$ sets

$$A_1, \ldots, A_k, A_{i_1}, \ldots, A_{i_{g+n-m}}$$

then lie in the $(g+k)$-space through V_k and that g-space. Thus our lemma is proved with $h = g+k$.

We now complete the induction proof of $Q_{n,m}$. The integer n was fixed. Put

$$f = \sum_1^n \dim A_i .$$

For $f < n$, our assertion is trivial; cf. (2). Suppose it has been proved up to $f-1$. On account of the Lemma, we may assume that

$$\dim A_n > 1 .$$

Let the set B_n consist of one single element $b_n \in A_n$, $b_n \neq 0$. Thus

$$\dim B_n = 1 < \dim A_n .$$

The sets

(5) $$A_1, \ldots, A_{n-1}, B_n$$

satisfy $P_{n,m}$. Also

$$\sum_1^{n-1} \dim A_i + \dim B_n < f .$$

Hence by our induction assumption for f and by (2), there is an integer k, $1 \leqq k \leqq m$, and a k-space containing $k+(n-m)$ of the sets (5). Thus it contains $k+(n-m-1) \geqq k$ of the sets (1). Our lemma now yields $Q_{n,m}$.

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ON CERTAIN INTERPOLATION SPACES RELATED TO GENERALIZED SEMI-GROUPS

JÖRGEN LÖFSTRÖM

Introduction.

Let $D(A)$ be the domain of the infinitesimal generator A of a strongly continuous semi-group $G(\sigma)$ of bounded linear operators in a Banach-space E (see [2], [3]). In [4] Lions has described interpolation spaces between E and $D(A)$. Later these results were extended to interpolation spaces between E and $D(A^m)$ $(m \geqq 1)$, (see [6], [7]). The description was made by means of the first order difference $G(\sigma)-1$ and the m-th order difference

$$(G(\sigma)-1)^m ,$$

respectively.

The purpose of this paper is to describe interpolation spaces between $D(A^{m-1})$ and $D(A^m)$ $(m \geqq 1)$, where now A is the infinitesimal generator of a generalized semi-group of a certain class (see [8]; see also [5], [9]). Since differences are not bounded operators they cannot be used any longer. We use instead the remainder of order m in Taylor's formula, i.e.

$$\frac{1}{(m-1)!}\int_0^s (s-\sigma)^{m-1} G(\sigma) A^m \, d\sigma .$$

The precise result will be found in section 3.

The first two sections are devoted to a brief presentation of definitions and results in the theory of interpolation spaces and generalized semi-groups, based on [6] and [7] and [8] (cf. also [5], [9]), respectively. In section 4 the result obtained in section 3 is used to extend theorems of Butzer and Tillmann [1] to generalized semi-groups. Finally, section 5 contains a further generalization of the results of the preceding sections.

The subject of this paper was suggested to me by professor Jaak Peetre. I thank him for valuable advice and great interest in my work.

1. Preliminaries; interpolation spaces.

Let X_0 and X_1 be Banach spaces, continuously embedded in a Banach space E. The corresponding norms are denoted by

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$$a \to \|a\|_{X_0}$$

and

$$a \to \|a\|_{X_1},$$

respectively. In X_0+X_1 we consider the family of norms

$$K(s,a) = \inf_{a=a_0+a_1} (\|a_0\|_{X_0} + s\|a_1\|_{X_1}), \qquad 0<s<\infty.$$

Further, let Φ be a function norm, i.e. a positive functional defined on the set of positive functions on $(0,\infty)$ which are measurable with respect to the measure dt/t such that:

(1.1) $$\Phi(\varphi) = 0 \quad \text{if and only if} \quad \varphi = 0 \text{ a.e.}$$

(1.2) $$\Phi(\varphi) < \infty \quad \text{implies} \quad \varphi < \infty \text{ a.e.}$$

(1.3) $$\Phi(\alpha\varphi) = \alpha\Phi(\varphi) \quad \text{if} \quad \alpha \geqq 0.$$

(1.4) $$\varphi \leqq \sum_{n=1}^{\infty} \varphi_n \text{ a.e. implies } \Phi(\varphi) \leqq \sum_{n=1}^{\infty} \Phi(\varphi_n).$$

Then

$$\mathsf{K}(\Phi) = \mathsf{K}(\Phi; X_0, X_1) = \{a \mid a \in X_0+X_1,\ \Phi\big(K(s,a)\big) < \infty\}$$

is a Banach space with the norm

$$a \to \Phi\big(K(s,a)\big).$$

It can be shown that $\mathsf{K}(\Phi)$ is an interpolation space (see [6], [7]).

If

(1.5) $$\Phi\big(\min(1,s)\big) < \infty,$$

then

$$X_0 \cap X_1 \subset \mathsf{K}(\Phi) \subset X_0+X_1,$$

(see [7]). On the other hand, if (1.5) does not hold, then $\mathsf{K}(\Phi)=\{0\}$. In the sequel we will therefore assume (1.5).

In the special case that we will consider, we have

(1.6) $$\|a\|_{X_0} \leqq \|a\|_{X_1},$$

so that $X_0 \supset X_1$.

From (1.6) it follows that

(1.7) $$K(s,a) = \|a\|_{X_0} \quad \text{if } s \geqq 1.$$

Indeed, obviously $K(s,a) \leqq \|a\|_{X_0}$ for all $s>0$, and

$$\|a\|_{X_0} \leqq \|a_0\|_{X_0} + \|a_1\|_{X_1} \leqq \|a_0\|_{X_0} + s\|a_1\|_{X_1},$$

if $a=a_0+a_1$, $s \geqq 1$. From the first inequality we also get

$$s\,\|a\|_{X_0} \leqq K(s,a) \quad \text{if } s<1\,,$$

and thus

$$K(s,a) \geqq \min(1,s)\,\|a\|_{X_0}\,. \tag{1.8}$$

REMARK 1.1. This formula also shows that $\mathsf{K}(\Phi)=\{0\}$ if (1.5) does not hold.

2. Preliminaries; generalized semi-groups.

Let $\mathscr{D}_+$ be the space of infinitely differentiable complex-valued functions with compact supports contained in $\mathsf{R}_+=\{t \mid t>0\}$. Moreover, let $\mathscr{L}(E)$ be the algebra of bounded linear operators in E. A generalized semi-group on E is a linear mapping G from $\mathscr{D}_+$ into $\mathscr{L}(E)$ such that:

$$G(\varphi*\psi) = G(\varphi)G(\psi), \qquad \varphi,\psi \in \mathscr{D}_+\,,$$

$$G(\varphi) \to G(\varphi_0) \text{ uniformly} \quad \text{when} \quad \varphi \to \varphi_0 \text{ in } \mathscr{D}_+$$

(see [5], [8], [9]).

For example, let $G(t)$ be a strongly continuous semi-group, that is, a mapping from R_+ into $\mathscr{L}(E)$ such that:

$$G(s+t) = G(s)G(t), \qquad G(s) \to G(s_0) \text{ strongly} \quad \text{when} \quad s \to s_0\,.$$

Then, if we define G on $\mathscr{D}_+$ by the formula

$$G(\varphi)a = \int_0^\infty \varphi(t)G(t)a\,dt\,,$$

G is a generalized semi-group.

Now it is possible to extend the domain of G to the space $\overline{\mathscr{E}}_+{}'$ of distributions with compact supports in $\overline{\mathsf{R}}_+=\{t \mid t\geqq 0\}$, so that:

$$G(T*S)a = G(T)G(S)a, \qquad S,T \in \overline{\mathscr{E}}_+{}'\,,$$

$$G(T)a \to G(T_0)a, \qquad T \to T_0 \text{ in } \overline{\mathscr{E}}_+{}'\,,$$

where

$$a \in R = \{a \mid a=\textstyle\sum G(\varphi_n)a_n,\ \varphi_n \in \mathscr{D}_+,\ a_n \in E\}\,,$$

which is assumed dense in E (see [8], [9]). Thus $G(T)$ is, in general, an unbounded operator with domain R.

Put

$$G(\delta_\sigma)a = G(\sigma)a, \qquad a \in R,\ \sigma>0\,,$$

where δ_σ is the Dirac measure in the point σ. If $T \in \overline{\mathscr{E}}_+{}'$ then, as can be shown,

$$G(T)a = \langle T(\sigma); G(\sigma)a\rangle = \int_0^\infty T(\sigma)G(\sigma)a\, d\sigma\,, \tag{2.1}$$

where $a \in R$. Write $\overline{G(T)}$ for the closure of $G(T)$. Then, by definition,

$$A = -\overline{G(\delta_0')}$$

is the infinitesimal generator and

$$A^m = (-1)^m\overline{G(\delta_0^{(m)})}, \qquad m \geqq 1\,. \tag{2.2}$$

Now we say that the semi-group G is of class $\sigma(k)$, if for every $\varphi \in \mathscr{D}_+$ we have that

$$\|G(\varphi)\| \leqq C\int_0^\infty t^k|\varphi^{(k)}(t)|\, dt\,. \tag{2.3}$$

Then the same inequality holds for $\varphi \in \overline{\mathscr{D}}_+{}^k$, where $\overline{\mathscr{D}}_+{}^k$ is the space of k-times continuously differentiable functions with compact supports in R_+, and even for φ in the space M_k of measures T with compact supports in R_+ such that $|T|\{0\}=0$ and such that the restriction of $T^{(k)}$ to R_+ is a measure with

$$\int_{+0}^\infty t^k|T^{(k)}(t)|\, dt < \infty$$

(see [8]). After a change of variable (2.3) becomes

$$\|G(\varphi_s)\| \leqq C\int_{+0}^\infty t^k|\varphi^{(k)}(t)|\, dt, \qquad \varphi \in M_k\,, \tag{2.4}$$

where

$$\varphi_s(t) = s^{-1}\varphi(ts^{-1})\,.$$

From (2.2) it follows that

$$\frac{d^n}{ds^n}G(s)a = G(s)\,A^n a \tag{2.5}$$

(cf. section 5). Hence

$$G(s)a - a = \int_0^s G(\sigma)\,Aa\, d\sigma\,.$$

Iteration of this leads to Taylor's formula

$$G(s)a = \sum_{n=0}^{m-1}\frac{s^n}{n!}A^n a + R_m(s)a\,, \tag{2.6}$$

where

$$R_m(s)a = \int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!} G(\sigma)\, A^m a\, d\sigma, \qquad a \in D(A^m)\,. \tag{2.7}$$

3. Interpolation between $D(A^{m-1})$ and $D(A^m)$.

In this section we put

$$X_0 = D(A^{m-1}), \qquad X_1 = D(A^m)\,.$$

The norms in $D(A^{m-1})$ and $D(A^m)$ will be denoted

$$\|a\|_{m-1} \quad \text{and} \quad \|a\|_m\,,$$

respectively, and are defined by

$$\|a\|_{m-1} = \|a\| + \|A^{m-1}a\|, \qquad \|a\|_0 = \|a\|\,,$$
$$\|a\|_m = \|a\| + \|A^m a\|\,,$$

where $\|\cdot\|$ is the norm in E. Then we may suppose

$$\|a\|_{m-1} \leqq \|a\|_m\,. \tag{3.1}$$

(We have $\|a\|_{m-1} \leqq C\|a\|_m$. By replacing $\|a\|_m$ with the equivalent norm $C\|a\|_m$ we get (3.1)). We also put (cf. (2.7))

$$\varrho_m(s,a) = s^{1-m} \sup_{0<\sigma\leqq s} \|R_m(\sigma)a\|\,.$$

THEOREM 3.1. *If G is of class $\sigma(k)$ and if $m > k$, then for $s > 0$,*

$$C^{-1}K(s,a) \leqq \varrho_m(s,a) + \min(1,s)\,\|a\|_{m-1} \leqq CK(s,a)\,,$$

where C is independent of a and s.

PROOF. (i) The second inequality:

Let $a = a_0 + a_1$, $a_0 \in D(A^{m-1})$, $a_1 \in D(A^m)$, be a given decomposition of a. Then

$$R_m(\sigma)a_1 = \frac{\sigma^m}{(m-1)!}\frac{1}{\sigma}\int_0^\sigma (1-\tau\sigma^{-1})^{m-1}\, G(\tau)\, A^m a_1\, d\tau\,.$$

But

$$\frac{1}{\sigma}\int_0^\sigma (1-\tau\sigma^{-1})^{m-1}\, G(\tau)\, b\, d\tau = G(\varphi_\sigma)b\,,$$

if

$$\varphi(t) = \begin{cases} (1-t)^{m-1}, & 0 \leqq t < 1\,, \\ 0, & t \geqq 1\,. \end{cases}$$

(cf. (2.1)). According to (2.4) we obtain

$$\|R_m(\sigma)a_1\| \leqq \frac{\sigma^m}{(m-1)!}\|G(\varphi_\sigma)A^m a_1\| \leqq C\sigma^m\|A^m a_1\|\,,$$

and thus

$$\varrho_m(s,a_1) \leqq Cs\|A^m a_1\| \leqq Cs\|a_1\|_m\,.$$

Now we have

$$R_m(\sigma)a_0 = R_{m-1}(\sigma)\,a_0 - \frac{\sigma^{m-1}}{(m-1)!}A^{m-1}a_0\,, \tag{3.2}$$

(we put $R_0(\sigma)b = G(\sigma)b$). Since

$$R_{m-1}(\sigma)a_0 = \frac{\sigma^{m-1}}{(m-2)!}\frac{1}{\sigma}\int_0^\sigma (1-\tau\sigma^{-1})^{m-2}\,G(\tau)\,A^{m-1}a_0\,d\tau\,,$$

we again obtain from (2.4) that

$$\|R_{m-1}(\sigma)a_0\| \leqq C\sigma^{m-1}\|A^{m-1}a_0\|\,.$$

Hence

$$\|R_m(\sigma)a_0\| \leqq C\sigma^{m-1}\|A^{m-1}a_0\|\,,$$

and thus

$$\varrho_m(s,a_0) \leqq C\|A^{m-1}a_0\| \leqq C\|a_0\|_{m-1}\,.$$

Finally we obtain

$$\varrho_m(s,a) \leqq \varrho_m(s,a_0)+\varrho_m(s,a_1) \leqq C(\|a_0\|_{m-1}+s\|a_1\|_m)\,,$$

from which follows

$$\varrho_m(s,a) \leqq CK(s,a)\,.$$

From (1.8) we get

$$\min(1,s)\,\|a\|_{m-1} \leqq K(s,a)\,,$$

and hence the second inequality of the theorem is proved.

(ii) The first inequality: Put

$$a_1 = ms^{-m}\int_0^s (s-\sigma)^{m-1}\,G(\sigma)a\,d\sigma = ms^{-1}\int_0^s (1-\sigma s^{-1})^{m-1}\,G(\sigma)a\,d\sigma$$

and

$$a_0 = a - a_1\,.$$

Then, according to (2.4),

$$\|a_1\| \leqq C\|a\| \leqq C\|a\|_{m-1}\,. \tag{3.3}$$

Moreover we have

$$a_0 = -ms^{-m}\int_0^s (s-\sigma)^{m-1}(G(\sigma)-1)a\,d\sigma$$

$$= -ms^{-m}\int_0^s (s-\sigma)^{m-1}\left(\int_0^\sigma G(\tau)\,Aa\,d\tau\right)d\sigma$$

$$= -ms^{-m}\int_0^s \left(\int_\tau^s (s-\sigma)^{m-1}\,d\sigma\right)G(\tau)\,Aa\,d\tau$$

$$= -s^{-m}\int_0^s (s-\tau)^m\,G(\tau)\,Aa\,d\tau = -\int_0^s (1-\tau s^{-1})^m\,G(\tau)\,Aa\,d\tau$$

if $m>1$. Thus

(3.4) $$\|a_0\| \leqq Cs\,\|Aa\| \leqq Cs\,\|a\|_{m-1}, \qquad m>1\,.$$

If $m=1$ we easily get

(3.4') $$\|a_0\| = s^{-1}\left\|\int_0^s (G(\sigma)-1)a\,d\sigma\right\| \leqq \varrho_1(s,a)\,.$$

Now

(3.5) $$A^m a_1 = m!\,s^{-m}\int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!}\,G(\sigma)A^m a\,d\sigma = m!\,s^{-m}R_m(s)a\,,$$

and thus

(3.6) $$s\|A^m a_1\| \leqq C\varrho_m(s,a)\,.$$

Moreover we have

$$A^{m-1}a_1 = m!\,s^{-m}\int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!}\,G(\sigma)A^{m-1}a\,d\sigma$$

$$= m!\,s^{-m}\int_0^s\int_0^\sigma \frac{(\sigma-\tau)^{m-2}}{(m-2)!}\,G(\tau)A^{m-1}a\,d\tau d\sigma = m!\,s^{-m}\int_0^s R_{m-1}(\sigma)a\,d\sigma\,.$$

It follows that

$$A^{m-1}a_0 = A^{m-1}a - A^{m-1}a_1 = m!\,s^{-m}\int_0^s \left[\frac{\sigma^{m-1}}{(m-1)!}\,A^{m-1} - R_{m-1}(\sigma)\right]a\,d\sigma$$

$$= -m!\,s^{-m}\int_0^s R_m(\sigma)a\,d\sigma\,,$$

which gives

(3.7) $$\|A^{m-1}a_0\| \leqq C\varrho_m(s,a)\,.$$

From (3.3) and (3.6) we obtain

$$s\|a\|_m \leqq C\big(\varrho_m(s,a)+s\|a\|_{m-1}\big)\,, \tag{3.8}$$

and from (3.4) (or (3.4')) and (3.7)

$$\|a_0\|_{m-1} \leqq C\big(\varrho_m(s,a)+s\|a\|_{m-1}\big)\,. \tag{3.9}$$

Finally

$$K(s,a) \leqq \|a_0\|_{m-1}+s\|a_1\|_m \leqq C\big(\varrho_m(s,a)+s\|a\|_{m-1}\big)\,,$$

which is the first inequality for $0<s<1$. But if $s\geqq 1$ it follows from (1.7) that

$$K(s,a) = \|a\|_{m-1} \leqq \varrho_m(s,a)+\|a\|_{m-1}\,.$$

Hence the first inequality is proved.

Now we put

$$W_\Phi{}^m = \{a \mid a\in D(A^{m-1}),\ \Phi\big(\varrho_m(s,a)\big)<\infty\}\,,$$

where Φ is a function norm such that (1.5) holds. Then $W_\Phi{}^m$ is a Banach space with the norm

$$a \to \Phi\big(\varrho_m(s,a)\big)+\|a\|_{m-1}\,.$$

From theorem 3.1 we immediately get

COROLLARY 3.1. *If G is of class $\sigma(k)$ and if $m>k$ then*

$$W_\Phi{}^m = \mathsf{K}\big(\Phi; D(A^{m-1}), D(A^m)\big)\,,$$

except for an equivalence of norms.

REMARK 3.1. For $k=0$, $m=1$ we get that the space

$$W_\Phi{}^1 = \{a \mid a\in E,\ \Phi\big(\sup_{0<\sigma\leqq s}\|G(\sigma)a-a\|\big)<\infty\}$$

is identical with the interpolation space $\mathsf{K}\big(\Phi; E, D(A)\big)$, except for an equivalence of norms. This is the description made in [7, pp. 9–10] if

$$\Phi(\varphi) = \Phi_{\theta,p}(\varphi) = \left(\int_0^\infty \left(\frac{\varphi(t)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p}, \qquad 0<\theta<1,\ 1\leqq p\leqq\infty\,,$$

(cf. also [4] and [6, pp. 55–57]).

4. Extension of theorems of Butzer and Tillmann.

If G is of class $\sigma(k)$ and if $m>k$, then

$$m s^{-m}\int_0^s (s-\sigma)^{m-1}\,G(\sigma)b\,d\sigma \to b, \qquad s\to +0\,,$$

(see [8]). This implies that if $a \in D(A^m)$, then

$$m!\, s^{-m} R_m(s)a = m!\, s^{-m} \int_0^s (s-\sigma)^{m-1} G(\sigma)\, A^m a\, d\sigma \to A^m a\,.$$

We have the following converse:

THEOREM 4.1. *If G is of class $\sigma(k)$, if $m > k$ and if there is an element $b \in E$ such that*

$$\lim_{s \to +0} \| m!\, s^{-m} R_m(s)a - b \| = 0, \qquad a \in D(A^{m-1})\,, \tag{4.1}$$

then

$$a \in D(A^m) \qquad \text{and} \qquad b = A^m a\,.$$

REMARK 4.1. This and the following theorem were proved by Butzer and Tillmann [1] for ordinary semi-groups. After a slight modification their proofs can be carried over to generalized semi-groups. Here we give alternative proofs, along the lines of theorem 3.1.

PROOF OF THEOREM 4.1. In the proof of theorem 3.1 we constructed a decomposition $a = a_0 + a_1$, such that

$$A^m a_1 = m!\, s^{-m} R_m(s)a \tag{4.2}$$

(formula (3.5)), and such that ((3.8) and (3.9))

$$\|a_0\|_{m-1} + s\|a_1\|_m \leqq C(\varrho_m(s,a) + s\,\|a\|_{m-1})\,.$$

But according to the assumptions we have

$$\varrho_m(s,a) = s \sup_{0<\sigma\leqq s} \|s^{-m} R_m(\sigma)a\| = O(s), \qquad s \to +0\,.$$

Hence

$$\|a_0\|_{m-1} + s\|a_1\|_m = O(s), \qquad s \to +0\,,$$

so that

$$a_0 \to 0 \quad \text{in } D(A^{m-1})\,,$$

which implies

$$a_1 \to a \quad \text{in } D(A^{m-1})\,.$$

But, by (4.1) and (4.2),

$$A^m a_1 \to b\,.$$

Since A^m is a closed operator the conclusion follows.

THEOREM 4.2. *If E is reflexive, if G is of class $\sigma(k)$ and if $m > k$, then*

$$\sup_{0<s\leqq 1} \| m!\, s^{-m} R_m(s)a \| < \infty, \qquad a \in D(A^{m-1})\,,$$

implies

$$a \in D(A^m)\,.$$

PROOF. Since $\varrho_m(s,a)=O(s)$ we get from theorem 3.1 that there is a decomposition $a=a_{0s}+a_{1s}$ such that

$$\|a_{0s}\|_{m-1}+s\|a_{1s}\|_m \leqq CK(s,a) = O(s), \qquad s \to 0 .$$

Hence $a_{0s} \to 0$ in $D(A^{m-1})$ and

$$\sup_{0<s<1} \|a_{1s}\|_m < \infty .$$

The space $D(A^m)$ with the graph-norm $\|a\|+\|A^m a\|$ can be identified with the graph of A^m. It is then a closed sub-space of $E \times E$, which is reflexive. This implies that $D(A^m)$ is reflexive. Then the set $\{a_{1s}, 0<s<1\}$, being bounded, is weakly sequentially compact. Therefore the sequence $a_{1,\,1/n}$ has a weakly convergent subsequence b_ν. Then, if $b_\nu \to b$ weakly in $D(A^m)$, we easily obtain $a=b$ and $a \in D(A^m)$. (Concerning these general results on reflexive spaces see e.g. [2, pp. 65–69].)

5. Generalization of the preceding results.

Let $T \in M_k$ and G be of class $\sigma(k)$. We recall that

$$T_s(x) = s^{-1}T(xs^{-1}) .$$

Now we have

$$\frac{d}{ds}T_s(x) = -s^{-2}T(xs^{-1})-s^{-1}\frac{d}{dx}T(xs^{-1}) = -\frac{d}{dx}(xT)_s = -\delta_0'*(xT)_s ,$$

and in general

$$\frac{d^n}{ds^n}T_s = (-1)^n\delta_0^{(n)}*(x^nT)_s .$$

From (2.2) it follows that

$$\frac{d^n}{ds^n}G(T_s)a = G\left(\frac{d^n}{ds^n}T_s\right)a = G\big((x^nT)_s\big)A^n a ,$$

and from this we obtain Taylor's formula:

$$G(T_s)a = \sum_{n=0}^{m-1}\frac{C_n(T)s^n}{n!}A^n a + R_m(T_s), \tag{5.1}$$

where

$$R_m(T_s) = \int_0^s \frac{(s-\sigma)^{m-1}}{(m-1)!}G\big((x^mT)_\sigma\big)A^m a\, d\sigma$$

and

$$C_n(T) = \int_0^\infty x^n T(x)\, dx$$

(cf. (2.6)). We put

$$\varrho_m(T; s, a) = s^{1-m} \sup_{0<\sigma\leqq s} \|R_m(T_\sigma)a\| .$$

The proof of theorem 3.1 depends essentially on Taylor's formula (5.1) (for $T=\delta_1$) and on (2.4). We can thus prove the following generalization.

THEOREM 5.1. *If G is of class $\sigma(k)$, if $T \in M_k$ and if $m > k$, $C_m(T) \neq 0$, then*

$$C^{-1}k(s,a) \leqq \varrho_m(T; s,a) + \varrho_m(xT; s,a) + \min(1; s)\, \|a\|_{m-1} \leqq Ck(s,a) .$$

REMARK 5.1. The special case $m=1$ can be found in [8].

PROOF. (i) The second inequality:

Let $a = a_0 + a_1$, $a_0 \in D(A^{m-1})$, $a_1 \in D(A^m)$ be a decomposition of a. Since

$$R_m(T_\sigma)a_1 = \frac{\sigma^m}{(m-1)!} \frac{1}{\sigma} \int_0^\sigma (1-\tau\sigma^{-1})^{m-1} G\big((x^m T)_\tau\big) A^m a_1\, d\tau ,$$

we have, according to (2.5),

$$\|R_m(T_\sigma)a_1\| \leqq C\sigma^{m-1} \|A^m a_1\| .$$

Moreover

$$R_m(T_\sigma)a_0 = R_{m-1}(T_\sigma)a_0 - \frac{C_{m-1}(T)\sigma^{m-1}}{(m-1)!} A^{m-1}a_0 ,$$

which by (2.4) gives

$$\|R_m(T_\sigma a_0\| \leqq C\sigma^{m-1} \|A^{m-1}a_0\| .$$

Hence we easily get

$$\varrho_m(T; s,a) \leqq CK(s,a) .$$

By substituting xT for T we get

$$\varrho_m(xT; s,a) \leqq CK(s,a) .$$

Now the second inequality follows by applying (1.8).

(ii) The first inequality: Put

$$a_1 = m s^{-m} \int_0^s (s-\sigma)^{m-1}\, G\big((x^m T)_\sigma\big)\, a\, d\sigma, \qquad a_0 = a - a_1 ,$$

and suppose that $C_m(T) = 1$. From (2.5) we obtain

(5.2) $$\|a_1\| \leqq C\|a\|_{m-1} .$$

A simple calculation, analogous to the corresponding one in the proof of theorem 3.1 gives

$$\begin{aligned} -a_0 &= ms^{-m}\int_0^s (s-\sigma)^{m-1}\big(G\big((x^mT)_\sigma\big)-1\big)a\,d\sigma \\ &= ms^{-m}\int_0^s (s-\sigma)^{m-1}\left(\int_0^\sigma G\big((x^{m+1}T)_\tau\big)Aa\,d\tau\right)d\sigma \\ &= \int_0^s (1-\tau s^{-1})^m G\big((x^{m+1}T)_\tau\big)Aa\,d\tau, \qquad m>1 , \end{aligned}$$

and thus

(5.3) $$\|a_0\| \leqq Cs\|a\|_{m-1}, \qquad m>1 .$$

If $m=1$ we easily get

(5.3′) $$\|a_0\| \leqq C\varrho_1(xT,s,a) .$$

Since

$$A^m a_1 = m!\, s^{-m} R_m(T_s)a ,$$

we obtain

(5.4) $$\|A^m a_1\| \leqq C\varrho_m(T;s,a) .$$

We also have

$$\begin{aligned} A^m a_0 &= m!\, s^{-m}\int_0^s \left[\frac{\sigma^{m-1}}{(m-1)!}A^{m-1} - R_{m-1}\big((xT)_\sigma\big)\right]a\,d\sigma \\ &= -m!\, s^{-m}\int_0^s R_m\big((xT)_\sigma\big)a\,d\sigma . \end{aligned}$$

Hence

(5.5) $$\|A^m a_0\| \leqq C\varrho_m(xT;s,a) .$$

As in the proof of theorem 3.1 it follows that

$$K(s,a) \leqq C\big(\varrho_m(T;s,a) + \varrho_m(xT;s,a) + s\|a\|_{m-1}\big) .$$

From (1.7) we obtain

$$K(s,a) = \|a\|_{m-1} \quad \text{if } s \geqq 1 ,$$

and hence the first inequality follows.

Now, let $W^m_{\Phi,T}$ be the Banach space of all $a \in E$ such that the norm

$$\|a\|_{W^m_{\Phi,T}} = \Phi\big(\varrho_m(T;s,a)\big) + \Phi\big(\varrho_m(xT;s,a)\big) + \|a\|_{m-1} ,$$

is finite. If (1.5) holds, theorem 5.1 implies that

$$C^{-1}\Phi\big(K(s,a)\big) \leqq \|a\|_{W^m_{\Phi,T}} \leqq C\Phi\big(K(s,a)\big)\,.$$

We have proved

COROLLARY 5.1. *Except for an equivalence of norms we have*

$$W^m_{\Phi,T} = \mathsf{K}\big(\Phi;\, D(A^{m-1}),\, D(A^m)\big) = W_\Phi{}^m\,.$$

Obviously we also have

COROLLARY 5.2. $a \in \mathsf{K}\big(\Phi;\, D(A^{m-1}),\, D(A^m)\big)$ *if and only if*

$$a \in D(A^{m-1}) \quad \textit{and} \quad \Phi\big(\varrho_m(T;s,a)\big) < \infty\,,$$

for all $T \in M_k$ (cf. [8, p. 96–98]).

From theorem 5.1 also follows

THEOREM 5.2. *Under the assumptions of theorem* 5.1 *and if there is a* $b \in E$ *such that*

$$\lim_{s\to+0} \|m!\, s^{-m} R_m(T_s)a - b\| = 0, \qquad a \in D(A^{m-1})\,, \tag{5.6}$$

then $a \in D(A^m)$, $b = A^m a$. *Conversely, if* $a \in D(A^m)$, *then* (5.6) *holds with* $b = A^m a$.

THEOREM 5.3. *If* E *is reflexive and if*

$$\sup_{0<s\leqq 1} \|m!\, s^{-m} R_m(T_s)a\| < \infty, \qquad a \in D(A^{m-1})\,,$$

then the assumptions of theorem 5.1 *imply that* $a \in D(A^m)$.

The proofs of these theorems are analogous to those of theorems 4.1 and 4.2 and will be omitted.

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ON WEAK SEPARATION OF CONVEX SETS AND ON α-UBIQUITOUS SETS

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1. Introduction.

We say that a linear functional f on a real vector space X *weakly separates* two subsets A and B of X iff for some real number k, the functional f is $\geqq k$ on one of the subsets and $\leqq k$ on the other but is not identically $= k$ on $A \cup B$. We say that a subset A of X is *radial* at a point x of X or that x is an *inner point* of A iff for each $y \in X$ there exists a positive number t so that $x + [0,t)y \subset A$.

The most fundamental result concerning weak separation of convex sets in real vector spaces is the following

THEOREM A. *If A and B are disjoint convex subsets of a real vector space X and if one of these subsets has an inner point, then there exists a linear functional weakly separating A and B.*

This theorem is a direct consequence of the Hahn–Banach Theorem which is in turn implied by this separation theorem. Any attempt to weaken the hypotheses of this separation theorem may therefore be regarded as an attempt to generalize the Hahn–Banach Theorem.

A number of years ago A. P. Morse proposed the following

PROBLEM. *If A and B are disjoint convex subsets of a real vector space X, does there exist a linear functional weakly separating A and B provided that some member of A is an inner point of $X \setminus B$?*

The author has shown in a part of his dissertation [9] as yet unpublished that the separation cannot in general be effected under these hypotheses in spaces of infinite dimension. More precisely it was shown that

THEOREM B. *If X is a real vector space of linear dimension $\aleph_0$, then there exists a convex subset C of X such that the origin θ, is an inner point of*

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$X \setminus C$ and such that any linear functional which is non-negative on C is identically zero on X.

The *linear dimension* referred to in this theorem is the cardinal number of a Hamel basis for X.

In light of subsequent remarks in this introduction it will be seen that this theorem is considerably generalized in Theorem 3.2 of this paper.

The separation problems under discussion can be rephrased in terms of the *lin* operation which is defined for any subset A of a real vector space X by

$$\operatorname{lin} A = A \cup \{x : \exists y \neq x,\ x + (0,1]y \subset A\}.$$

(Here $(0,1]$ denotes a half-open interval on the real line.) That is to say $x \in \operatorname{lin} A$ iff either $x \in A$ or x is the endpoint of a segment contained in A. One further defines

$$\operatorname{lin}^0 A = A$$

and for each ordinal $\alpha > 0$

$$\operatorname{lin}^\alpha A = \begin{cases} \operatorname{lin} \operatorname{lin}^{\alpha-1} A & \text{if } \alpha \text{ is a successor ordinal,} \\ \bigcup_{\beta<\alpha} \operatorname{lin}^\beta A & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Clearly $\operatorname{lin}^\beta A \subset \operatorname{lin}^\alpha A$ if $\beta < \alpha$. One finally defines the *order* of a subset A of a real vector space X by

$$\operatorname{order} A = \text{the smallest ordinal } \alpha \text{ for which } \operatorname{lin}^{\alpha+1} A = \operatorname{lin}^\alpha A.$$

The following results have been obtained by O. M. Nikodym [7] [8] and subsequently demonstrated somewhat more simply by V. L. Klee [4]. In each of the statements enumerated below, X is a real vector space and C a convex subset of X.

1) Order $C \leqq \Omega$ where, Ω denotes the first non-countable ordinal.
2) If $\dim X = \aleph_0$, then order $C < \Omega$.
3) If $\dim X = \aleph_0$, then for each ordinal $\alpha < \Omega$ there exists a convex set C with order $C = \alpha$.
4) If $\dim X < \aleph_0$, then $\operatorname{lin} C = \bar{C}$ where $\bar{C}$ is the closure of C in the Euclidean topology for X, so that order $C = 0$ or 1 according as whether C is closed or not.
5) If C is convex, then for each ordinal α, $\operatorname{lin}^\alpha C$ is convex.

It is emphasized that 4) does not hold in general in the absence of the hypothesis that C is convex.

In order to see how the lin operation relates to the separation problem, refer to Theorem B and note that, in view of the convexity of C, the con-

dition that θ is an inner point of $X \setminus C$ is equivalent to the condition that $\theta \notin \operatorname{lin} C$. Further note that for any set C, convex or not, if a linear functional f is $\geqq k$ on C then f is $\geqq k$ on $\operatorname{lin}^\alpha C$ for each ordinal α. Thus it is seen that Theorem B will follow if we can demonstrate:

THEOREM C. *If X is a real vector space of dimension $\aleph_0$, then there exists a convex subset C of X such that $\theta \notin \operatorname{lin} C$ while $\operatorname{lin}^2 C = X$.*

It was by proving essentially this theorem that the author obtained Theorem B in [9].

In this paper a more general theorem than Theorem C will be obtained. Klee [2] defines a set, A, to be *ubiquitous* iff $\operatorname{lin} A = X$ and shows that a space of dimension $\aleph_0$, can be represented as the union of two (or in fact infinitely many) disjoint ubiquitous convex sets. [It is worthy of note that this result affords one way of demonstrating the impossibility in spaces of infinite dimension of weakly separating two convex sets under the sole hypothesis that they be disjoint. That weak separation can always be effected under this hypothesis in spaces of finite dimension has long been known.] In [4] Klee proposes the problem: for what (if any) ordinals α other than 0 and 1 do there exist convex sets C of order α such that $\operatorname{lin}^\alpha C = X$.

The answer to this question is provided in the principal result of this paper, Theorem 3.2. If X is a real vector space with $\dim X = \aleph_0$ and α is an ordinal with $\alpha < \Omega$, then there exists a convex set C of order α such that $\operatorname{lin}^\alpha C = X$.

It is clear that this theorem solves the cited problem of Klee and provides a negative answer to the separation problem of Morse implying as it does Theorems C and B.

We will define a subset A of a vector space X to be *α-ubiquitous* iff order $A = \alpha$ and $\operatorname{lin}^\alpha A = X$.

Since the submission of this paper, some of the results have been obtained by V. L. Klee using different methods.

2. Preliminary definitions and lemmas.

The proofs of the main theorems of this paper depend on finding general conditions under which for a family Γ of convex sets we may conclude that

$$\operatorname{lin}^\alpha \langle \bigcup \{C : C \in \Gamma\} \rangle = \langle \bigcup \{\operatorname{lin}^\alpha C : C \in \Gamma\} \rangle , \tag{2.1}$$

where $\langle S \rangle$ denotes the *convex hull* of S, that is $\bigcap \{C : S \subset C \wedge C \text{ convex}\}$. This relation does not hold in general even in finite dimensional spaces when $\alpha = 1$. We give two examples in E_2.

EXAMPLE 2.1. For $n=1,2,3,\ldots$, let C_n be the segment of the line through the origin with argument n lying in the closed unit disc. Here $\operatorname{lin}\langle\bigcup\{C_n:\ n=1,2,\ldots\}\rangle$ is closed while $\langle\bigcup\{\operatorname{lin}C_n:\ n=1,2,\ldots\}\rangle$ is not.

EXAMPLE 2.2. Let $C_1=\{\theta\}$ and $C_2=\{(x,y):\ x\geqq 0, y=1\}$. Here

$$\operatorname{lin}\langle\bigcup\{C_n:\ n=1,2\}\rangle = \{(x,y):\ x\geqq 0, 0\leqq y\leqq 1\}$$

while

$$\langle\bigcup\{\operatorname{lin}C_n:\ n=1,2\}\rangle = \{\theta\}\cup\{(x,y):\ x\geqq 0, 0<y\leqq 1\}.$$

In the first example the conclusion (2.1) fails because the set Γ is infinite; in the second example because the members of Γ are not linearly bounded, where a convex set is said to be *linearly bounded* iff it contains no ray. If neither of these exceptions occurs then the conclusion holds with $\alpha=1$ in spaces of finite dimension. That is

REMARK 2.1. If $C_1,C_2,\ldots,C_n$ are linearly bounded convex subsets of a finite dimensional vector space, then

$$\operatorname{lin}\langle\bigcup\{C_i:\ i=1,2,\ldots,n\}\rangle = \langle\bigcup\{\operatorname{lin}C_i:\ i=1,2,\ldots,n\}\rangle.$$

This is easily proved using compactness, see for example [4]. This remark inspires the following Definition 2.1. Before stating this fundamental definition we introduce some notation which will be used throughout the remainder of this section and the next.

NOTATION: We let X denote a real vector space with $\dim X=\aleph_0$. We let I denote the set of non-negative integers, I' the set of positive integers, R the set of real numbers, P the interval $[1,\infty)$, Q the interval $(0,1]$. For use in discussing convex combinations we let K be the set of functions t satisfying:

(1) range $t\subset[0,1]$,

(2) $\{i:\ t(i)>0\}$ is finite,

(3) $\sum\{t(i):\ i\in\text{domain } t\}=1$.

We further agree that if b is a real number, A a set of real numbers, y a vector and C a set of vectors, then

$$AC=\{ax:\ a\in A\wedge x\in C\},\qquad bC=\{bx:x\in C\},\qquad Ay=\{ay:a\in A\}.$$

For each set S of subsets of X we define $\sigma S=\bigcup\{H:H\in S\}$. We use $S\subset\subset T$ to mean $\forall A\in S\,\exists B\in T\,(A\subset B)$. The *natural cone* of a basis $\mathscr{B}$ for X is defined to be the set of vectors which when expressed as linear combinations of members of $\mathscr{B}$ involve only non-negative coefficients.

DEFINITION 2.1. We say that a convex set C is *enveloped* (respectively, *delineated*) *by* Γ iff

(1) Γ is a family of linearly bounded convex sets (respectively, segments) with $C=\langle\sigma\Gamma\rangle$;
(2) for each finite dimensional subspace, Y, of X there is a finite dimensional subspace, Z, of X and a finite subset, Δ, of Γ such that

$$Y\cap C\subset\langle\cup\{Z\cap G: G\in\Delta\}\rangle.$$

We say that C is *enveloped* (respectively, *delineated*) iff for some Γ, C is enveloped (respectively, delineated) by Γ. We say Γ is an *enveloping* (respectively, *delineating*) *family* iff $\langle\sigma\Gamma\rangle$ is enveloped (respectively, delineated) by Γ. (It should be emphasized that it is *not* required of an enveloping family Γ that each finite dimensional subspace of X should intersect only finitely many members of Γ.)

It is clear that if C is delineated by Γ then C is enveloped by Γ.

Partial motivation for this definition is furnished by the following lemma.

LEMMA 2.1. *If Γ is an enveloping family, then*

$$\operatorname{lin}\langle\cup\{G: G\in\Gamma\}\rangle=\langle\cup\{\operatorname{lin}G: G\in\Gamma\}\rangle.$$

It is noted that the family in Example 2.1 above fails to satisfy condition (2) of the above definition while that in Example 2.2 fails to satisfy condition (1).

PROOF. Let $C=\langle\cup\{G: G\in\Gamma\}\rangle$ and let $x\in\operatorname{lin}C$ with $x+Qy\subset C$. Let Y be the (at most two-dimensional) subspace of X spanned by x and y. Let Z be a finite dimensional subspace of X and Δ a finite subset of Γ so that

$$Y\cap C\subset\langle\cup\{Z\cap G: G\in\Delta\}\rangle.$$

The family $\{Z\cap G: G\in\Delta\}$ is a finite family of linearly bounded convex sets contained in the finite dimensional space Z so that

$$\operatorname{lin}\langle\cup\{Z\cap G: G\in\Delta\}\rangle=\langle\cup\{\operatorname{lin}(Z\cap G): G\in\Delta\}\rangle$$

by Remark 2.1. Now

$$x+Qy\subset Y\cap C\subset\langle\cup\{Z\cap G: G\in\Delta\}\rangle$$

so that

$$\begin{aligned}x\in\operatorname{lin}\langle\cup\{Z\cap G: G\in\Delta\}\rangle&=\langle\cup\{\operatorname{lin}(Z\cap G): G\in\Delta\}\rangle\\&\subset\langle\cup\{\operatorname{lin}G: G\in\Gamma\}\rangle.\end{aligned}$$

This shows that

$$\operatorname{lin}\langle \bigcup \{G : G \in \Gamma\}\rangle \subset \langle \bigcup \{\operatorname{lin} G : G \in \Gamma\}\rangle .$$

The inclusion the other way follows immediately from the observation that for each $G \in \Gamma$,

$$\operatorname{lin} G \subset \operatorname{lin}\langle \bigcup \{G : G \in \Gamma\}\rangle$$

and from the fact that $\operatorname{lin} A$ is convex whenever A is convex. The proof is therefore complete.

DEFINITION 2.2: C is *α-enveloped* (respectively, *α-delineated*) *by* F iff:

(1) F is a function on the set of ordinals β with $\beta \leqq \alpha$, whose values are families of subsets of X;
(2) for each ordinal $\beta \leqq \alpha$, $\operatorname{lin}^{\beta} C$ is enveloped (respectively, delineated) by $F(\beta)$;
(3) for $\gamma < \beta \leqq \alpha$, $F(\gamma) \subset \subset F(\beta)$.

We say that C is *α-enveloped* (respectively, *α-delineated*) iff for some function F, C is α-enveloped (respectively, α-delineated) by F. We say that F is an *α-enveloping* (respectively, *α-delineating*) function iff $\langle \sigma F(0)\rangle$ is α-enveloped (respectively, α-delineated) by F. If conditions (1), (2) and (3) hold with $\beta \leqq \alpha$ replaced by $\beta < \alpha$ in each of these conditions then we say that C is *α-subenveloped* (respectively, *α-subdelineated*) by F, etc.

The following remarks are self-evident consequences of the above definition.

REMARKS 2.1. 1. If C is α-enveloped (respectively, α-delineated) by F and $\beta < \alpha$ then C is β-enveloped (respectively, β-delineated) by $F \mid \{\gamma : \gamma \leqq \beta\}$.

2. If C is α-enveloped (respectively, α-delineated) by F and order $C \leqq \alpha$ then for each ordinal $\beta > \alpha$, C is β-enveloped (respectively, β-delineated) by G where $G(\gamma) = F(\gamma)$ for $\gamma \leqq \alpha$ and $G(\gamma) = F(\alpha)$ for $\alpha < \gamma \leqq \beta$.

3. If α is a successor ordinal then C is α-subenveloped (respectively, α-subdelineated) by F iff C is $(\alpha - 1)$-enveloped (respectively $(\alpha - 1)$-delineated) by F.

4. If C is α-subenveloped (respectively, α-subdelineated) by F then for each ordinal $\beta < \alpha$, C is β-enveloped (respectively, β-delineated) by $F \mid \{\gamma : \gamma \leqq \beta\}$.

5. If C is α-enveloped (respectively, α-delineated) by F then C is α-subenveloped (respectively, α-subdelineated) by $F \mid \{\beta : \beta < \alpha\}$.

6. If C is α-subenveloped (respectively, α-subdelineated) by F and $\operatorname{lin}^{\alpha} C$ is enveloped (respectively, delineated) by Γ and for each ordinal

$\beta<\alpha$, $F(\beta)\subset\subset\Gamma$, then C is α-enveloped (respectively, α-delineated) by G where $G(\beta)=F(\beta)$ for $\beta<\alpha$ and $G(\alpha)=\Gamma$.

7. If C is α-delineated by F then C is α-enveloped by F.

We next present some lemmas which give conditions under which

$$P\lin^{\alpha}C = \lin^{\alpha}(PC)\,.$$

The following example suffices to show that this is not in general true even when $\alpha=1$ and X is two-dimensional.

EXAMPLE 2.3. Let $C=\{(x,y):(x-1)^2+y^2\leqq 1\}$. Then

$$\lin(PC) = \{(x,y): x\geqq 0\}$$

while

$$P\lin C = \{\theta\}\cup\{(x,y): x>0\}\,.$$

However,

LEMMA 2.2. *If C is a convex set in two dimensional Euclidean space, Y, and either $\theta\notin\lin C$ or $\lin C$ is a polygonal region, then*

$$P\lin C = \lin(PC)\,.$$

Before proving this lemma we state and prove

LEMMA 2.3. *For all $C\subset X$ and for all $\alpha<\Omega$*

$$P\lin^{\alpha}C \subset P\lin^{\alpha}(PC) = \lin^{\alpha}(PC)\,.$$

PROOF. The inclusions $P\lin^{\alpha}C\subset P\lin^{\alpha}(PC)$ and $\lin^{\alpha}(PC)\subset P\lin^{\alpha}(PC)$ are obvious. It only remains to show that $P\lin^{\alpha}(PC)\subset\lin^{\alpha}(PC)$ which we do first for $\alpha=1$. Let $x\in P\lin(PC)$. Then for some $k\in Q$, $kx\in\lin(PC)$ so that for some y, $kx+Qy\subset PC$. Thus since $1/k\in P$ we have

$$x+Q\frac{1}{k}y = \frac{1}{k}(kx+Qy)\subset PC\,,$$

whence $x\in\lin(PC)$ which yields the desired inclusion for $\alpha=1$. Suppose now that the lemma has been demonstrated for all ordinals less than α where $\alpha>1$. Then if α is a successor ordinal,

$$\begin{aligned} P\lin^{\alpha}(PC) = P\lin\lin^{\alpha-1}(PC) &\subset \lin P\lin^{\alpha-1}(PC)\\ &\subset \lin\lin^{\alpha-1}(PC) = \lin^{\alpha}(PC)\,. \end{aligned}$$

If α is a limit ordinal then,

$$\begin{aligned} P\lin^{\alpha}(PC) = P\bigcup_{\beta<\alpha}\lin^{\beta}(PC) &= \bigcup_{\beta<\alpha}P\lin^{\beta}(PC)\\ &\subset \bigcup_{\beta<\alpha}\lin^{\beta}(PC) = \lin^{\alpha}(PC)\,. \end{aligned}$$

This completes the proof.

PROOF OF LEMMA 2.2. In light of Lemma 2.3 it need only be shown that $\lin(PC) \subset P \lin C$.

Assume first that $\theta \notin \lin C$. Let x and y be such that $x \in \lin(PC)$ and $x + Qy \subset PC$. Suppose that $x \notin P \lin C$. Then $[0,1]x \subset Y \setminus \lin C$. Since $[0,1]x$ is compact and $\lin C$ is closed in the Euclidean topology for Y, there is an open set U with $[0,1]x \subset U \subset Y \setminus \lin C$ and a $z \in x + Qy$ such that $[0,1]z \subset U$. Hence $Qz \cap \lin C = \emptyset$ or equivalently $z \notin P \lin C$ whence a fortiori $z \notin PC$. This contradiction assures us that $x \in P \lin C$ which completes the proof in the case that $\theta \notin \lin C$.

Next assume that $\lin C$ is a polygonal region. In view of the part of the proof already given we need only consider the case that $\theta \in \lin C$. If θ is an interior point of $\lin C$ then $P \lin C = Y$ which gives the desired result. Finally consider the case that θ is a boundary point of $\lin C$. Let S be the smallest sector (convex cone) with vertex at the origin containing $\lin C$. Since $\lin C$ is a polygonal region there are segments contained in $\lin C$ on each of the bounding rays of S. Thus $P \lin C = S$. Since $PC \subset S$ we have $\lin(PC) \subset \lin S = S$. This completes the proof.

Lemmas 2.2 and 2.3 facilitate the proofs of the important lemmas 2.4 and 2.6.

LEMMA 2.4. *If C is convex, $\alpha < \Omega$ and $\theta \notin \lin^\alpha C$, then $P \lin^\alpha C = \lin^\alpha(PC)$.*

PROOF (by transfinite induction). From Lemma 2.3 we know that $P \lin^\alpha C \subset \lin^\alpha(PC)$. Let x and y be such that $x \in \lin(PC)$ and $x + Qy \subset PC$. Let Y be the (at most two-dimensional) subspace of X spanned by x and y and let $C' = Y \cap C$. Now C' is a convex subset of Y with $\theta \notin \lin C' \subset \lin C$. Since $x + Qy \subset Y \cap (PC) = PC'$ we see that $x \in \lin(PC')$. Invoking Lemma 2.2 we obtain $x \in P \lin C' \subset P \lin C$. This proves the lemma in the case that $\alpha = 1$. Assuming the lemma to have been proved for all ordinals less than α, we obtain the result for α by means of the following familiar computations. If α is a successor ordinal, then

$$\begin{aligned} (2.2) \qquad \lin^\alpha(PC) = \lin \lin^{\alpha-1}(PC) &= \lin P \lin^{\alpha-1} C \\ &= P \lin \lin^{\alpha-1} C = P \lin^\alpha C , \end{aligned}$$

while if α is a limit ordinal

$$\begin{aligned} (2.3) \qquad \lin^\alpha(PC) = \bigcup_{\beta<\alpha} \lin^\beta(PC) &= \bigcup_{\beta<\alpha} P \lin^\beta C \\ &= P \bigcup_{\beta<\alpha} \lin^\beta C = P \lin^\alpha C . \end{aligned}$$

This completes the proof.

LEMMA 2.5. *If C is delineated, then $\lin(PC) = P \lin C$.*

PROOF. Again, by Lemma 2.3, we need only show that $\lin(PC) \subset P \lin C$. Let Γ be a family which delineates C. Let x and y be such that $x \in \lin(PC)$ and $x + Qy \subset PC$. Let Y be the space (at most two-dimensional) spanned by x and y and let Z be such a finite dimensional subspace of X and Δ such a finite subset of Γ that

$$Y \cap C \subset \bigcup \langle \{Z \cap G : G \in \Delta\} \rangle . \tag{2.4}$$

Letting $C' = Y \cap C$, (2.4) yields

$$C' = Y \cap C \subset Y \cap \langle \bigcup \{Z \cap G : G \in \Delta\} \rangle \subset Y \cap C = C' . \tag{2.5}$$

Since the family $\{Z \cap G : G \in \Delta\}$ is a finite collection of points and segments, the set $\lin\langle \bigcup \{Z \cap G : G \in \Delta\} \rangle$ is a polytope so that $C' = Y \cap C$ is by (2.5) a polygonal region in Y. Now $x + Qy \subset Y \cap (PC) = PC'$. Thus $x \in \lin(PC')$ whence, by Lemma 2.2, $x \in P \lin C'$. This completes the proof.

LEMMA 2.6. *If C is α-subdelineated, then* $\lin^\alpha(PC) = P \lin^\alpha C$.

PROOF (by transfinite induction). The case $\alpha = 1$ is covered by the preceding lemma. Assume that the lemma holds for all ordinals less than α where $\alpha > 1$. A repetition of the arguments (2.2) and (2.3) completes the proof.

The following lemma which will be employed in the final stages of the proof of the main theorem of Section 3 is so obvious as to require no proof.

LEMMA 2.7. *If C is contained in the natural cone of a basis $\mathscr{B}$ for X, then for each ordinal α,* $\lin^\alpha C$ *is contained in this cone.*

A few words of orientation may be helpful to the reader at this point. In the proof of the main Theorem 3.2 in the case that α is a limit ordinal there will be constructed a convex set C of order α such that θ is an inner point of $\lin^\alpha C$. The steps in the argument require that C be linearly bounded. We next pass to the set PC which is not linearly bounded. It is clear from Lemma 2.3 that $X = P \lin^\alpha C \subset \lin^\alpha(PC)$. It will next be necessary to show that order (PC) is not less than α. This will be achieved by use of Lemma 2.6 applied to C. The hypotheses of the said lemma require that C have a certain "polygonal" character namely that C be α-subdelineated. The notions of delineated, α-delineated, and α-subdelineated sets are required solely for the proof of Theorem 3.2 in the case that α is a limit ordinal as are Lemmas 2.5, 2.6, 2.7 and the subsequent Lemma 3.5 and the parts of Lemmas 3.2 and 3.4 which deal

with delineated sets. The proof of Theorem 3.2 in the case that α is a limit ordinal also involves Theorem 3.1 in the case that α is a successor ordinal which in turn employs all the lemmas not mentioned above. It is seen then that the proof of Theorem 3.2 in the case that α is a limit ordinal requires a good deal more machinery than in the case that α is a successor ordinal.

3. Main results.

NOTATION. We denote by $Y \oplus Z$ the direct sum of the subspaces Y and Z and by $\sum^d \mathscr{X}$ the direct sum of the family, $\mathscr{X}$, of subspaces of X. If $M \subset \sum^d \mathscr{X}$ then for each $Z \in \mathscr{X}$ we define

$$\operatorname{proj} M \text{ on } Z(\mathscr{X}) = \{x : x \in Z \wedge \exists y \in \sum^d (\mathscr{X} \setminus \{Z\})\,(x+y \in M)\}\,.$$

We write simply $\operatorname{proj} M$ on Z when the family $\mathscr{X}$ is clearly understood. We denote by $[M]$ the space spanned by M.

LEMMA 3.1. *Suppose that* $X = \sum^d \{X_i : i \in I\}$. *Suppose that for each* $i \in I'$, C_i *is a linearly bounded convex subset of* X_i *and that* C_0 *is a linearly bounded, finite dimensional convex set. Let* $u_0 = \theta$ *and for* $k \in I'$ *let*

$$u_k \in \sum^d \{X_i : i = 0, 1, \ldots, k-1\}\,.$$

Further suppose that $\{u_k : \theta \in C_k\} \subset C_0$. *Then the set*

$$C = \langle \bigcup \{u_i + C_i : i \in I\} \rangle$$

is enveloped by the family $\{u_i + C_i : i \in I\}$ *so that*

$$\operatorname{lin} C = \langle \bigcup \{u_i + \operatorname{lin} C_i : i \in I\} \rangle\,.$$

PROOF. The first of the conditions of Definition 2.1 is clearly satisfied. In order to establish condition (2) of this definition let Y be a finite dimensional subspace of X. Let m be the smallest integer such that

$$C_0 \cup Y \subset \sum^d \{X_i : i = 0, 1, \ldots, m\}\,.$$

Let $W = [Y, C_0, u_1, u_2, \ldots, u_m]$. Note that W is finite dimensional and that

$$W \subset \sum^d \{X_i : i = 0, 1, \ldots, m\}\,.$$

Now let

$$Z = \sum^d \{\operatorname{proj} W \text{ on } X_i : i = 0, 1, \ldots, m\}\,.$$

Again Z is finite dimensional and

$$W \subset Z \subset \sum^d \{X_i : i = 0, 1, \ldots, m\}\,.$$

Next let $x \in Y \cap C$. Then there exists $t \in K$ with domain $t = I$ so that

$$x = \sum \{t_i(u_i + x_i) \ : i \in I\}$$

where $x_i \in C_i$ when $t_i \neq 0$ and $x_i = \theta$ when $t_i = 0$. Let

$$I_0 = \{i : \ x_i = \theta \wedge t_i \neq 0\}, \qquad I_1 = \{i : \ x_i \neq \theta \wedge t_i \neq 0\}.$$

Now I_0 and I_1 are disjoint finite sets of non-negative integers and recall that for each $i \in I_0$,

$$u_i \in C_0 \subset W \subset \sum^d \{X_i : \ i = 0, 1, \ldots, m\}.$$

Furthermore

$$x = \sum \{t_i u_i : i \in I_0\} + \sum \{t_i(u_i + x_i) \ : i \in I_1\}. \tag{3.1}$$

Observe that $u_i \in C_0$ for $i \in I_0$ and use the convexity of C_0 to write

$$\sum (t_i u_i : i \in I_0\} = t_0' x_0'$$

where

$$x_0' \in C_0, \qquad t_0' = \sum \{t_i : i \in I_0\}.$$

Now we have

$$x = t_0' x_0' + \sum \{t_i(u_i + x_i) \ : i \in I_1\}. \tag{3.2}$$

Let $k = \max I_1$, and using (3.2) write

$$t_k x_k = x - t_0' x_0' - t_k u_k - \sum \{t_i(u_i + x_i) \ : \ i \in I_1 \wedge i < k\}. \tag{3.3}$$

It is readily seen that each term on the righthand side of (3.3) belongs either to $\sum^d \{X_i : i \leqq m\}$ or to $\sum^d \{X_i : i < k\}$. Since $t_k \neq 0$ and $x_k \in X_k \setminus \{\theta\}$ we see that $k \leqq m$. It therefore follows that for $i \in I_1$ we have $u_i \in W$ and since $x \in Y \subset W$ and $x_0' \in C_0 \subset W$ we have from (3.2)

$$\sum \{t_i x_i : i \in I_1\} = x - t_0' x_0' - \sum \{t_i u_i : i \in I_1\} \ \in \ W. \tag{3.4}$$

The definition of Z together with (3.4) now assures us that $x_i \in Z$ for $i \in I_1$. And since as we have already noted, $u_i \in W \subset Z$ for $i \in I_1$ we find that

$$u_i + x_i \in Z \cap (u_i + C_i) \qquad \text{for} \quad i \in I_1$$

And now, since $t_0' + \sum \{t_i : i \in I_1\} = 1$, $x_0' \in C_0$, and $\max I_1 \leqq m$ we have

$$\begin{aligned} x &= t_0' x_0' + \sum \{t_i(u_i + x_i) \ : i \in I_1\} \\ &\in \langle \bigcup \{Z \cap (u_i + C_i) \ : \ i \in I_1 \vee i = 0\} \rangle \\ &\subset \langle \bigcup \{Z \cap (u_i + C_i) \ : \ i = 0, 1, 2, \ldots, m\} \rangle. \end{aligned}$$

Thus we see that

$$Y \cap C \subset \langle \bigcup \{Z \cap (u_i + C_i) \ : \ i = 0, 1, 2, \ldots, m\} \rangle.$$

Thus C is enveloped by the family $\{u_i+C_i : i \in I\}$. That $\operatorname{lin} C = \langle \bigcup \{u_i + \operatorname{lin} C_i : i \in I\}\rangle$ follows from Lemma 2.1. This completes the proof.

LEMMA 3.2. *If all the hypotheses of Lemma* 3.1 *hold and in addition, for each* $i \in I$, C_i *is enveloped (respectively, delineated) by a family* Γ_i, *then* C *is enveloped (respectively, delineated) by the family* $\{u_i+G : G \in \Gamma_i \wedge i \in I\}$.

PROOF. It is clear that we only need verify condition (2) of Definition 2.1. To this end let Y be a finite dimensional subspace of X and employ Lemma 3.1 to find a finite dimensional subspace Z' of X and an integer m so that

$$Y \cap C \subset \langle \bigcup \{Z' \cap (u_i+C_i) \ : \ i=0,1,2,\ldots,m\}\rangle . \tag{3.5}$$

For each $i=0,1,2,\ldots,m$ we have

$$Z' \cap (u_i+C_i) = u_i+(Z'-u_i) \cap C_i \subset u_i+[Z',u_i] \cap C_i . \tag{3.6}$$

And, since C_i is enveloped (respectively, delineated) by Γ_i, there exists a finite dimensional subspace, Z_i, of X_i and a finite subset, Δ_i, of Γ_i so that

$$[Z',u_i] \cap C_i \subset \langle \bigcup \{Z_i \cap G : G \in \Delta_i\}\rangle . \tag{3.7}$$

Now (3.6) and (3.7) yield for $i=0,1,2,\ldots,m$

$$\begin{aligned} Z' \cap (u_i+C_i) &\subset u_i+[Z',u_i] \cap C_i \\ &\subset u_i+\langle \bigcup \{Z_i \cap G : G \in \Delta_i\}\rangle \\ &\subset \langle \bigcup \{[Z_i,u_i] \cap (u_i+G) \ : G \in \Delta_i\}\rangle . \end{aligned} \tag{3.8}$$

Let $Z=[Z_0,Z_1,\ldots,Z_m,u_1,u_2,\ldots,u_m]$ and note that Z is finite dimensional. Now combining (3.5) and (3.8) we have

$$\begin{aligned} Y \cap C &\subset \langle \bigcup \{Z' \cap (u_i+C_i) \ : \ i=0,1,2,\ldots,m\}\rangle \\ &\subset \langle \bigcup \{[Z_i,u_i] \cap (u_i+G) \ : \ G \in \Delta_i \wedge i=0,1,2,\ldots,m\}\rangle \\ &\subset \langle \bigcup \{Z \cap (u_i+G) \ : \ G \in \Delta_i \wedge i=0,1,2,\ldots,m\}\rangle . \end{aligned}$$

The finiteness of each of the Δ_i, $i=0,1,2,\ldots,m$, yields the desired result.

LEMMA 3.3. *Let* X_i, C_i, u_i *for* $i \in I$ *and* C *satisfy the hypotheses of Lemma* 3.1. *Let* α *be an ordinal* >0 *and define by induction for each ordinal* $\beta \leqq \alpha$

$$C_0^{\beta} = \begin{cases} C_0 & \text{if } \beta=0 , \\ \langle \operatorname{lin} C_0^{\beta-1} \cup \{u_i : \ \theta \in \operatorname{lin}^{\beta} C_i\}\rangle & \text{if } \beta \text{ is a successor ordinal} , \\ \bigcup_{\gamma<\beta} C_0^{\gamma} & \text{if } \beta \text{ is a limit ordinal} . \end{cases}$$

Suppose that for each $\beta < \alpha$ *the sets* C_0^{β} *and* $\operatorname{lin}^{\beta} C_i$, $i \in I$, *are linearly bounded and that* C_0^{β} *is finite dimensional. Then* C *is* α*-subenveloped by* F *where*

$$F(\beta) = \{C_0^{\beta}\} \cup \{u_i + \lin^{\beta} C_i : i \in I'\} \quad \textit{for} \quad \beta < \alpha .$$

Furthermore

$$\lin^{\alpha} C = \langle C_0^{\alpha} \cup \bigcup \{u_i + \lin^{\alpha} C_i : i \in I'\} \rangle . \tag{3.9}$$

PROOF. It is readily checked that Lemma 3.1 is just the case $\alpha = 1$ of this lemma. It is also easily checked that for $\gamma < \beta < \alpha$, $F(\gamma) \subset\subset F(\beta)$. Assume that the lemma has been demonstrated for all ordinals $\delta < \alpha$ where α is an ordinal greater than 1. It therefore follows that for $\delta < \alpha$, C is δ-subenveloped by $F \mid \{\beta : \beta < \delta\}$ and that

$$\lin^{\delta} C = \langle C_0^{\delta} \cup \bigcup \{u_i + \lin^{\delta} C_i : i \in I'\} \rangle .$$

Moreover, since $\delta < \alpha$, the hypotheses assure us that C_0^{δ} is finite dimensional and linearly bounded and that each of the sets $\lin^{\delta} C_i$, $i \in I$, is linearly bounded. Reference to the definition of C_0^{δ} assures us that the hypotheses of Lemma 3.1 are satisfied with C_0 replaced by C_0^{δ} and C_i replaced by $\lin^{\delta} C_i$. Therefore $\lin^{\delta} C$ is enveloped by

$$\{C_0^{\delta}\} \cup \bigcup \{u_i + \lin^{\delta} C_i : i \in I'\} .$$

Therefore (see for example Remark 2.1.6) C is δ-enveloped by $F \mid \{\beta : \beta \leqq \delta\}$. This shows that C is α-subenveloped by F. In order to establish (3.9) first consider the case that α is a successor ordinal. Now

$$\lin^{\alpha-1} C = \langle C_0^{\alpha-1} \cup \bigcup \{u_i + \lin^{\alpha-1} C_i : i \in I'\} \rangle .$$

Since the hypotheses of Lemma 3.1 are satisfied with C_0 replaced by $C_0^{\alpha-1}$, C_i replaced by $\lin^{\alpha-1} C_i$ for $i \in I'$ and C replaced by $\lin^{\alpha-1} C$ we have by that lemma

$$\begin{aligned} \lin^{\alpha} C = \lin \lin^{\alpha-1} C &= \langle \lin C_0^{\alpha-1} \cup \bigcup \{u_i + \lin^{\alpha} C_i : i \in I'\} \rangle \\ &= \langle C_0^{\alpha} \cup \bigcup \{u_i + \lin^{\alpha} C_i : i \in I'\} \rangle , \end{aligned}$$

where the last equality is obtained by reference to the definition of C_0^{α}. Thus (3.9) holds for α a successor ordinal.

In the case that α is a limit ordinal,

$$\begin{aligned} \lin^{\alpha} C = \bigcup_{\beta<\alpha} \lin^{\beta} C &= \bigcup_{\beta<\alpha} \langle C_0^{\beta} \cup \bigcup \{u_i + \lin^{\beta} C_i : i \in I'\} \rangle \\ &= \langle \bigcup_{\beta<\alpha} C_0^{\beta} \cup \bigcup \{u_i + \lin^{\alpha} C_i : i \in I'\} \rangle . \\ &= \langle C_0^{\alpha} \cup \bigcup \{u_i + \lin^{\alpha} C_i : i \in I'\} \rangle . \end{aligned}$$

This establishes (3.9) when α is a limit ordinal and completes the proof of the lemma.

LEMMA 3.4. *Suppose that the hypotheses of Lemma* 3.3 *hold and that in addition for each* $i \in I'$ *there is a function* H_i *so that* C_i *is* α*-subenveloped (respectively,* α*-subdelineated) by* H_i. *Further suppose that there is a function* H_0 *on the set of ordinals less than* α *so that for each* $\beta < \alpha$, $H_0(\beta)$ *is a finite family of linearly bounded convex sets (respectively, segments) with* $C_0^\beta = \langle \sigma H_0(\beta) \rangle$ *and that for* $\gamma < \beta < \alpha$, $H_0(\gamma) \subset\subset H_0(\beta)$. *Then* C *is* α*-subenveloped (respectively,* α*-subdelineated) by* H *where*

$$H(\beta) = \{u_i + G : G \in H_i(\beta) \wedge i \in I\}.$$

PROOF. Conditions (1) and (3) of Definition 2.2 are clearly satisfied. Moreover for $\beta < \alpha$

$$\begin{aligned} \lin^\beta C &= \langle C_0^\beta \cup \bigcup \{u_i + \lin^\beta C_i : i \in I'\} \rangle \\ &= \langle u_0 + \langle \sigma H_0(\beta) \rangle \cup \bigcup \{u_i + \langle \sigma H_i(\beta) \rangle : i \in I'\} \rangle \\ &= \langle \bigcup \{u_i + \sigma H_i(\beta) : i \in I\} \rangle = \langle \sigma H(\beta) \rangle . \end{aligned}$$

Note that for each $\beta < \alpha$ the hypotheses of Lemma 3.2 are satisfied with C replaced by $\lin^\beta C$, C_0 replaced by C_0^β and with C_i replaced by $\lin^\beta C_i$ for $i \in I'$, Γ_i replaced by $H_i(\beta)$ for $i \in I$ and with Γ replaced by $H(\beta)$. Applying Lemma 3.2 we find therefore that $\lin^\beta C$ is enveloped (respectively, delineated) by

$$\{u_i + G : G \in H_i(\beta), i \in I\} .$$

This completes the proof.

LEMMA 3.5. *If* $X = \sum^d \{X_i : i \in I'\}$ *and* C_i *is a convex subset of* X_i *for* $i \in I'$ *with* $\theta \notin \lin^\Omega C_i$ *and* $\lin^\Omega C_i$ *linearly bounded and* $C = \langle \bigcup \{C_i : i \in I'\} \rangle$, *then*

$$\operatorname{order} C = \sup \{\operatorname{order} C_i : i \in I'\} .$$

PROOF. Let $\beta = \operatorname{order} C$ and $\alpha = \sup \{\operatorname{order} C_i : i \in I'\}$. Apply Lemma 3.3 with $C_0 = \emptyset$ and $u_0 = u_1 = u_2 = \ldots = \theta$ to find for each ordinal $\gamma < \Omega$ that

$$\lin^\gamma C = \langle \bigcup \{\lin^\gamma C_i : i \in I'\} \rangle .$$

Thus by the definition of α,

$$\begin{aligned} \lin^{\alpha+1} C &= \langle \bigcup \{\lin^{\alpha+1} C_i : i \in I'\} \rangle \\ &= \langle \bigcup \{\lin^\alpha C_i : i \in I'\} \rangle = \lin^\alpha C . \end{aligned}$$

It therefore follows that $\beta \leqq \alpha$. It will suffice now to show that for $\gamma < \Omega$

$$(\lin^\gamma C) \cap X_n = \lin^\gamma C_n \qquad \text{for each } n \in I' .$$

Accordingly let $x \in (\lin^\gamma C) \cap X_n$. Then

$$x = \sum\{t_i x_i : i \in I'\},$$

where $x_i \in \lin^\gamma C_i \subset X_i$, $t \in K$ and $\operatorname{domain} t = I'$. Since $\theta \notin \lin^\gamma C_i$, we can only have $x \in X_n$ if $t_i = 0$ for $i \neq n$, whence $t_n = 1$ and $x = x_n \in \lin^\gamma C_n$. Therefore for $n \in I'$ and $\gamma < \Omega$,

$$\lin^\gamma C_n \subset (\lin^\gamma C) \cap X_n \subset (\lin^\beta C) \cap X_n \subset \lin^\beta C_n$$

so that for each $n \in I'$, order $C_n \leqq \beta$, whence $\alpha \leqq \beta$. As we have already shown that $\beta \leqq \alpha$ the proof is complete.

THEOREM 3.1. *If* $\dim X = \aleph_0$, *then for each* $\alpha < \Omega$ *there is an* α*-delineated convex set* C *of order* α *lying in the natural cone of some basis* $\mathscr{B}$ *for* X. *Moreover in the case that* α *is a successor ordinal we may further require either that* C *satisfy*

$$\theta \notin \lin^\alpha C \tag{3.10}$$

or

$$\lin^\alpha C = \langle\{\theta\} \cup \lin^{\alpha-1} C\rangle . \tag{3.11}$$

PROOF (by transfinite induction). For $\alpha = 1$ let $u \in X$ with $u \neq \theta$. Then $(1,2]u$ and $(0,1]u$ are 1-delineated sets of order 1 satisfying respectively (3.10) and (3.11). Furthermore if u is chosen as a basis element, these sets are in the natural cone regardless of how the other basis elements are chosen. Suppose next that the theorem has been proved for all ordinals $< \alpha$ where $\alpha > 1$. Let $X = \sum^d\{X_i : i \in I\}$ where $X_0 = Ru$, $u \neq \theta$ and for $i \in I'$, $\dim X_i = \aleph_0$. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be a sequence of successor ordinals satisfying:

$\alpha_i = \alpha - 1$ if α and $\alpha - 1$ are successor ordinals;

otherwise

$\alpha_1, \alpha_2, \alpha_3, \ldots$ is a strictly increasing sequence with limit α or $\alpha - 1$ whichever is a limit ordinal.

For $i \in I'$ let C_i be an α_i-delineated convex subset of X_i which lies in the natural cone with respect to some basis $\mathscr{B}_i$ of X_i and such that order $C_i = \alpha_i$. Let $\mathscr{B}$ be the basis for X defined by $\mathscr{B} = \{u\} \cup \bigcup\{\mathscr{B}_i : i \in I'\}$. We now divide the remainder of the proof into two cases.

CASE I. α *is a successor ordinal.* Require that

$$\lin^{\alpha_i} C_i = \langle\{\theta\} \cup \lin^{\alpha_i - 1} C_i\rangle .$$

Let $a_1, a_2, a_3, \ldots$ be a strictly decreasing sequence of positive numbers and let $a_0 = \lim a_n$. Let

$$C = \langle \bigcup\{a_i u + C_i : i \in I'\}\rangle .$$

It is clear that C lies in the natural cone of $\mathscr{B}$. We note that by Remark 2.1.2 each of the C_i, $i \in I'$, is $(\alpha+1)$-delineated. Check that the hypotheses of Lemma 3.4 are satisfied with α replaced by $\alpha+1$, with u_i replaced by $a_i u$, and with

$$C_0^{\beta} = \begin{cases} \langle \{a_i u : \alpha_i \leqq \beta\} \rangle & \text{for } \beta < \alpha; \\ [a_0, a_1]u & \text{for } \beta = \alpha, \alpha+1 . \end{cases} \tag{3.12}$$

It follows from Lemma 3.4 that C is $\alpha+1$-subdelineated and by Remark 2.1.4 that C is α-delineated. The hypotheses of Lemma 3.3 are satisfied with u_i replaced by $a_i u$ and with C_0^{β} defined for $\beta \leqq \alpha$ by (3.12) so that the conclusion (3.9) of that lemma yields

$$\begin{aligned} \operatorname{lin}^{\alpha} C &= \langle C_0^{\alpha} \cup \bigcup \{a_i u + \operatorname{lin}^{\alpha} C_i \ : i \in I'\} \rangle \\ &= \langle [a_0, a_1]u \cup \bigcup \{a_i u + \operatorname{lin}^{\alpha_i} C_i \ : i \in I'\} \rangle . \end{aligned}$$

The hypotheses of Lemma 3.1 are seen to hold with C_0 replaced by $[a_0, a_1]u$ with u_i replaced by $a_i u$ and with C_i replaced by $\operatorname{lin}^{\alpha_i} C_i$, $i \in I'$. Therefore

$$\begin{aligned} \operatorname{lin}^{\alpha+1} C = \operatorname{lin} \operatorname{lin}^{\alpha} C &= \langle \operatorname{lin}([a_0, a_1]u) \cup \bigcup \{a_i u + \operatorname{lin} \operatorname{lin}^{\alpha_i} C_i \ : i \in I'\} \rangle \\ &= \langle [a_0, a_1]u \cup \bigcup \{a_i u + \operatorname{lin}^{\alpha_i} C_i \ : i \in I'\} \rangle = \operatorname{lin}^{\alpha} C . \end{aligned}$$

whence order $C \leqq \alpha$. Application of Lemma 3.3 with α replaced by β where $\beta < \alpha$ yields

$$\operatorname{lin}^{\beta} C = \langle C_0^{\beta} \cup \bigcup \{a_i u + \operatorname{lin}^{\beta} C_i\} \rangle \qquad \text{for } \beta < \alpha .$$

It is now easily verified that $a_0 u \notin \operatorname{lin}^{\beta} C$ for $\beta < \alpha$ and since $a_0 u \in \operatorname{lin}^{\alpha} C$ we find that order $C \geqq \alpha$. Thus order $C = \alpha$. The condition (3.10) or (3.11) is satisfied according to whether $a_0 > 0$ or $a_0 = 0$. This establishes the result in the case that α is a successor ordinal.

Case II. *α is a limit ordinal.* Require that $\theta \notin \operatorname{lin}^{\alpha_i} C_i$ for $i \in I'$. Let

$$C = \langle \bigcup \{C_i : i \in I'\} \rangle .$$

Lemma 3.5 assures us that order $C = \alpha$ and it is obvious that C is contained in the natural cone of $\mathscr{B}$.

Since each of the sets C_i is an α_i-delineated set of order α_i, it follows from Remark 2.1.2 that for each $i \in I'$, C_i is $(\alpha+1)$-delineated. Applying Lemma 3.4 we find that C is $\alpha+1$-subdelineated and hence by Remark 2.1.4, C is α-delineated. This completes the proof.

It might shed some light on the procedure employed in this paper to observe that in the proof of case I of the above theorem we used the fact that

$$\lin^\beta C = \langle C_0^\beta \cup \bigcup \{a_i u + \lin^\beta C_i \ : i \in I'\}\rangle$$

whence it is easily checked that

$$\lin^\beta C = \langle \bigcup \{a_i u + \lin^\beta C_i \ : i \in I'\}\rangle .$$

It might be thought therefore that the sets C_0^β could be dispensed with; this is not the case however since $\lin^\beta C$ is *not* for all $\beta \leqq \alpha$ enveloped by the family $\{a_i u + \lin^\beta C_i \ : i \in I'\}$ as required for the induction. A similar remark applies to Theorem 3.2.

THEOREM 3.2. *If* $\dim X = \aleph_0$, *then for each ordinal* α *with* $\alpha < \Omega$ *there is a convex* C' *of order* α *such that*

$$\lin^\alpha C' = X .$$

PROOF. CASE I. α *is a successor ordinal.* Let

$$X = X_0 \oplus \sum{}^d \{X_n{}^i : \ i = 1,2;\ n \in I'\} .$$

For each $i = 1,2$ and each $n \in I'$, let $C_n{}^i$ be a linearly bounded convex set of order α with

$$C_n{}^i \subset X_n{}^i \quad \text{and} \quad \lin^\alpha C_n{}^i = \langle \lin^{\alpha-1} C_n{}^i \cup \{\theta\}\rangle . \tag{3.13}$$

Let $e_1, e_2, e_3, \ldots$ be a basis for X with

$$e_n \in X_0 \cup \bigcup \{X_k{}^i : \ i = 1,2 \wedge k = 1,2,\ldots,n-1\} .$$

Let

$$C = \langle \bigcup \{(-1)^i e_n + C_n{}^i \ : \ i = 1,2 \wedge n \in I'\}\rangle .$$

By Lemma 3.3, for $\beta < \alpha$ we have

$$\lin^\beta C = \langle \bigcup \{(-1)^i e_n + \lin^\beta C_n{}^i \ : \ i = 1,2 \wedge n \in I'\}\rangle$$

and by the same lemma

$$\begin{aligned} \lin^\alpha C = \langle & \bigcup \{(-1)^i e_n \ : \ i = 1,2 \wedge n \in I'\} \cup \\ & \cup \bigcup \{(-1)^i e_n + \lin^\alpha C_n{}^i \ : \ i = 1,2 \wedge n \in I'\}\rangle . \end{aligned} \tag{3.14}$$

Thus $\lin^\alpha C$ contains each basis element e_n and its negative, hence also θ. It may be seen however that $\theta \notin \lin^\beta C$ for $\beta < \alpha$ as follows. Suppose on the contrary that for some $\beta < \alpha$ we have $\theta \in \lin^\beta C$. Then θ can be expressed in the form

$$\theta = \sum \{t_{ni} x_n{}^i : \ i = 1,2 \wedge n \in I'\} ,$$

where $x_n{}^i \in \big((-1)^i e_n + C_n{}^i\big)$ and $t \in K$ with domain $t = I' \times \{1,2\}$. Let m

be the largest integer n for which $t_{n1}+t_{n2}>0$ and let j be the largest integer i for which $t_{mi}>0$ and let $j'=3-j$. Set

$$x_m{}^j = (-1)^j e_m + y_m{}^j \quad \text{with} \quad y_m{}^j \in \lin^\beta C_m{}^j \subset X_m{}^j$$

and note that $y_m{}^j$ cannot be θ by (3.13). Now

$$\begin{aligned} t_{mj} y_m{}^j &= -t_{mj}(-1)^j e_m - t_{mj'} x_m{}^{j'} - \sum \{t_{ni} x_n{}^i : i=1,2 \wedge n=1,2,\ldots,m-1\} \\ &\in X_m{}^{j'} \oplus X_0 \oplus \textstyle\sum^d \{X_n{}^i : i=1,2 \wedge n=1,2,\ldots,m-1\} \end{aligned}$$

which is impossible since $t_{mj} \neq 0$ and $\theta \neq y_m{}^j \in C_m{}^j \subset X_m{}^j$.

Since $\theta \notin \lin^\beta C$ for $\beta<\alpha$ we see by Lemma 2.4 that

$$\theta \notin P \lin^\beta C = \lin^\beta(PC) \quad \text{for} \quad \beta<\alpha .$$

However, applying (3.14) and Lemma 2.3 we find

$$X = \langle \textstyle\bigcup \{Re_n : n \in I'\} \rangle \subset P \lin^\alpha C \subset \lin^\alpha(PC) \subset X .$$

Therefore PC has order α and $\lin^\alpha(PC)=X$. This completes the proof in the case that α is a successor ordinal.

Case II. *α is a limit ordinal.* Again let

$$X = X_0 \oplus \textstyle\sum^d \{X_n{}^i : i=1,2 \wedge n \in I'\}$$

where for $n \in I'$ and $i=1,2$, $\dim X_n{}^i = \aleph_0$. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be a strictly increasing sequence of successor ordinals greater than zero with $\lim \alpha_n = \alpha$. For each $i=1,2$ and each $n \in I'$ let $C_n{}^i$ be an α_n-delineated set of order α_n contained in the natural cone of a basis $\mathscr{B}_n{}^i$ for $X_n{}^i$ with

$$\lin^{\alpha_n} C_n{}^i = \langle \{\theta\} \cup \lin^{\alpha_{n-1}} C_n{}^i \rangle .$$

Let $\mathscr{B}_0$ be a basis for X_0 and let $e_1, e_2, e_3, \ldots$ be a basis for X with

$$e_n \in \mathscr{B}_0 \cup \textstyle\bigcup \{\mathscr{B}_k{}^i : i=1,2 \wedge k=1,2,\ldots,n-1\} .$$

Let

$$C = \langle \textstyle\bigcup \{(-1)^i e_n + C_n{}^i : i=1,2 \wedge n \in I'\} \rangle .$$

The hypotheses of Lemma 3.3 are satisfied where

$$C_0{}^\beta = \langle \textstyle\bigcup \{[-1,1]e_n : \alpha_n \leqq \beta\} \rangle .$$

It is clear that for $\beta<\alpha$, $C_0{}^\beta$ is finite dimensional and linearly bounded. Letting G_0 be the function on the ordinals $<\alpha$ defined by

$$G_0(\beta) = \{[-1,1]e_n : \alpha_n \leqq \beta\}$$

we see that for $\beta<\alpha$, $G_0(\beta)$ is a finite family of segments, that $C_0{}^\beta = \langle \sigma G_0(\beta) \rangle$ and that for $\gamma<\beta<\alpha$, $G_0(\gamma) \subset\subset G_0(\beta)$. Since, furthermore, each $C_n{}^i$ is α-delineated, Lemma 3.4 assures us that C is α-subdelineated.

Now Lemma 3.3 yields for $\beta < \alpha$

$$\text{(3.15)} \qquad \lin^{\beta} C = \langle \langle \bigcup \{[-1,1]e_n \,:\, \alpha_n \leqq \beta\} \rangle \cup \cup \bigcup \{(-1)^i e_n + \lin^{\beta} C_n{}^i \,:\, i = 1,2 \wedge n \in I'\} \rangle .$$

Furthermore

$$\lin^{\alpha} C = \bigcup_{\beta<\alpha} \lin^{\beta} C \supset \langle \bigcup \{[-1,1]e_n \,:\, n \in I'\} \rangle .$$

Now by Lemma 2.3

$$\lin^{\alpha}(PC) \supset P \lin^{\alpha} C \supset \langle \bigcup \{Re_n : n \in I'\} \rangle = X .$$

It now only remains to be shown that

$$\lin^{\beta}(PC) \neq X \quad \text{for} \quad \beta < \alpha .$$

To this end we will first show that for $r > 0$, $-re_{n+1} \notin \lin^{\alpha_n} C$. Suppose on the contrary that for some $r > 0$, $-re_{n+1} \in \lin^{\alpha_n} C$. Then by (3.15) we find

$$\text{(3.16)} \qquad -re_{n+1} = t_0 y_0 + \sum \{t_{hi}((-1)^i e_h + y_h{}^i) \,:\, i = 1,2 \wedge h \in I'\}$$

where $y_0 \in \langle \bigcup \{[-1,1]e_h \,:\, h = 1,2,\ldots,n\} \rangle$ and $y_h{}^i \in \lin^{\alpha_n} C_h{}^i \subset X_h{}^i$ and $t \in K$ with domain $t = \{0\} \cup (I' \times \{1,2\})$.

Recalling that $C_h{}^i$ (and hence by Lemma 2.7 also $\lin^{\alpha_n} C_h{}^i$) is contained in the natural cone of $\mathscr{B}$, we see that when the $y_h{}^i$ are expressed as linear combinations of the basis elements e_m, the coefficients are therefore $\geqq 0$. Consequently (3.16) can hold only if $t_{n+1,1} > 0$. Thus, letting k be the largest of the integers h for which $t_{h1} + t_{h2} > 0$ we see that $k > n$. And since $\theta \notin \lin^{\alpha_n} C_k{}^i$ we have $y_k{}^i \neq \theta$. But $y_k{}^i$ when expressed in terms of the basis elements e_m involves only e_m for which $m > k \geqq n+1$. (This is because $y_k{}^i \in X_k{}^i$ and

$$e_m \in X_0 \oplus \textstyle\sum^d \{X_h{}^i \,:\, i = 1,2 \wedge h = 1,2,\ldots,m-1\}.)$$

These terms cannot drop out of the sum on the right hand side of (3.16) since all the terms with negative coefficients involve only e_h with $h \leqq k$. This contradiction shows that

$$-re_{n+1} \notin \lin^{\alpha_n} C .$$

Using the fact that C is β-delineated for each $\beta < \alpha$ we have by Lemma 2.6 that for $\beta < \alpha$

$$\lin^{\beta}(PC) = P \lin^{\beta} C .$$

Thus $-Pe_{n+1} \cap \lin^{\alpha_n} C = \emptyset$ for $n \in I'$. As we have already seen that

$$\lin^{\alpha}(PC) = X$$

we see that PC is a convex set of order α for which $\lin^{\alpha}(PC) = X$. This completes the proof.

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SOME OPEN MAPPING THEOREMS IN LF-SPACES AND THEIR APPLICATION TO EXISTENCE THEOREMS FOR CONVOLUTION EQUATIONS

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It is a well known fact that the operation of differentiation can be made continuous if we are prepared to accept the price: the resulting topology has some decidedly unpleasant properties. If E is an inductive limit of a sequence E_n, then some but unfortunately not all topological properties are inherited by E if they are satisfied on each E_n; thus, e.g., a linear mapping of E into a convex space is continuous if (and only if) its restriction to each E_n is continuous or, a hyperplane $H \subset E$ is closed in E if (and only if) its intersection with each E_n is closed in E_n. However if we replace hyperplanes by arbitrary subspaces, the last equivalence is no longer true. Similar complications occur in the study of openness of mappings.

It is the purpose of the present paper to present a few remarks concerning the openness of mappings of one LF-space into another. It turns out that a continuous linear mapping T of an LF-space E into an LF-space F need not be open even if it behaves as one in each F_j. Some additional assumptions are necessary to make T open and the study of this problem is the object of this communication. The main idea of the present remark consists in using quotient mappings (see e.g. proposition (4.5)). Although the results are far from satisfactory we are able to improve an earlier result of B. Malgrange [3] and strenghten slightly a theorem on the range of convolution transforms in distribution spaces obtained by L. Hörmander [2]. Hörmander's work is concerned with existence theorems for equations $Tx = y$ where T is a convolution transform and $x \in \mathscr{D}'(\Omega_2)$, $y \in \mathscr{D}'(\Omega_1)$. Earlier results in this direction, especially those of L. Ehrenpreis, are mentioned in the bibliography of [2].

1. Notation and auxiliary results.

The term "convex space" is used for "locally convex Hausdorff topological linear space". Similarly a convex topology is the topology of a

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convex space. If E is a convex space we shall denote by $\boldsymbol{U}(E)$ the system of all closed absolutely convex neighbourhoods of zero in E. If E and F are two linear spaces, a linear mapping T from E into F has a domain of definition $D(T)$ which is a linear subspace of E; its range will be denoted by $R(T)$. We shall use the preposition "of" to distinguish mappings which are everywhere defined. Thus a mapping T of E into F necessarily has $D(T)=E$ while, for a mapping T from E into F, $D(T)$ may be different from E.

Let E and F be two convex spaces and A a linear mapping from E into F defined on a dense subspace $D(A)$ of E. We define $G(A)$, the graph of A, as the set of all $[x,Ax]\in E\times F$ for $x\in D(A)$. We shall denote by $D(A')$ the set of those $y'\in F'$ for which the scalar product $\langle Ax,y'\rangle$ is continuous on $D(A)$. If $y'\in D(A')$ there exists an $x'\in E'$ such that $\langle Ax,y'\rangle=\langle x,x'\rangle$ for all $x\in D(A)$. Since $D(A)$ is dense in E, there is exactly one such x'; we may thus write $x'=A'y'$. The mapping A', the adjoint of A, is clearly linear. If E' and F' are equipped with the topologies $\sigma(E',E)$ and $\sigma(F',F)$, then $D(A')$ will be dense in F' iff A is closable, i.e. iff the closure of $G(A)$ in $E\times F$ is again the graph of a mapping B from E into F. In this case B is an extension of A and clearly $B'=A'$, hence $D(B')=D(A')$.

The rest of this section is devoted to some simple results concerning weakly continuous and weakly open mappings which will be needed later.

DEFINITION (1,1). *A linear mapping T from a convex space E into a convex space F is said to be weakly continuous if it is continuous as a mapping of $\big(D(T),\sigma(D(T),E')\big)$ into $\big(F,\sigma(F,F')\big)$.*

PROPOSITION (1,2). *Let A be a linear mapping from E into F with $D(A)=E$. Then the following conditions are equivalent:*

1° *if Q is a closed subspace of F, then $A^{-1}(Q)$ is closed in E;*

2° *if Q is a closed hyperplane in F, then $A^{-1}(Q)$ is closed in E;*

3° $D(A')=F'$;

4° *A is continuous as a mapping of $\big(E,\sigma(E,E')\big)$ into $\big(F,\sigma(F,F')\big)$.*

PROOF. The implication 1° $\Rightarrow$ 2° being obvious, assume 2° and take an element $y'\in F'$. The linear form $f(x)=\langle Ax,y'\rangle$ defined on E will be continuous iff $f^{-1}(0)$ is closed in E. Since clearly $f^{-1}(0)=A^{-1}(Q)$ when Q is the zero hyperplane of y', $f^{-1}(0)$ will be closed by 2° whence $y'\in D(A')$. It follows that $D(A')=F'$. Suppose now that 3° is satisfied and take $y_1',\ldots,y_n'\in F'$. Since $D(A')=F'$, we may form $x_i'=A'y_i'$. If $|\langle x,x_i'\rangle|\leqq 1$ for $1\leqq i\leqq n$, we have $|\langle Ax,y_i'\rangle|\leqq 1$ for $1\leqq i\leqq n$ which proves 4°. The implication 4° $\Rightarrow$ 1° is obvious.

DEFINITION (1,3). *A linear mapping T from a convex space E into a convex space F is said to be weakly open if it is open as a mapping of $\big(D(T),\sigma(D(T),E')\big)$ into $\big(F,\sigma(F,F')\big)$.*

If T is closed we have the following equivalence.

PROPOSITION (1,4). *Let T be a linear mapping from E into F with $D(T)=E$ the graph of which is closed in $E\times F$. Then the following conditions are equivalent:*

1° *$R(T')$ is closed in E';*
2° *$R(T')=T^{-1}(0)^0$;*
3° *$T^{-1}(0)^0\subset R(T')$;*
4° *if P is a closed subspace of E with $P\supset T^{-1}(0)$ then TP is closed in TE;*
5° *if P is a closed hyperplane in E with $P\supset T^{-1}(0)$ then TP is closed in TE;*
6° *T is open as a mapping of $\big(E,\sigma(E,E')\big)$ into $\big(F,\sigma(F,F')\big)$.*

PROOF. The inclusion $R(T')\subset T^{-1}(0)^0$ is obvious. Further, if $x\in R(T')^0$, we have $\langle Tx,D(T')\rangle=0$ whence $Tx=0$ the subspace $D(T')$ being dense in F'. Hence $R(T)^0\subset T^{-1}(0)$. It follows that for any mapping T we have

$$R(T')\subset T^{-1}(0)^0\subset R(T')^{00}.$$

The implication 1° $\Rightarrow$ 2° follows immediately from these inclusions. The implication 2° $\Rightarrow$ 3° is immediate. Now assume 3° and let P be a closed subspace of E containing $T^{-1}(0)$. Let $y\in TE$, y non$\in TP$. It follows that $y=Tx$ for some x for which x non$\in P$. Since P is closed there exists an $x'\in E'$ such that $x'\in P^0$ and $\langle x,x'\rangle\neq 0$. Since $x'\in P^0\subset T^{-1}(0)^0$ we have $x'=T'y'$ for some y', whence

$$\langle TP,y'\rangle=\langle P,x'\rangle=0 \quad\text{and}\quad \langle y,y'\rangle=\langle Tx,y'\rangle=\langle x,x'\rangle\neq 0.$$

The implication 4° $\Rightarrow$ 5° is immediate.

Let us show now that 5° implies 1°. Since $R(T')\subset T^{-1}(0)^0$, it suffices to prove the inclusion $T^{-1}(0)^0\subset R(T')$ only. To do this, take an $x'\in T^{-1}(0)^0$; if $x'=0$, we have $x'=T'y'$ for $y'=0$. We may thus assume $x'\neq 0$ so that there exists an $x_0\in E$ which $\langle x_0,x'\rangle=1$. Denote by P the set of all $x\in E$ for which $\langle x,x'\rangle=0$; we have $T^{-1}(0)\subset P$ since $x'\in T^{-1}(0)^0$ and it follows from 5° that TP is closed in TE. We show next that Tx_0 non$\in TP$. Indeed, if $Tx_0=Tp$ for some $p\in P$, we have $x_0-p\in T^{-1}(0)$ whence $\langle x_0-p,x'\rangle=0$ so that $1=\langle x_0,x'\rangle=\langle p,x'\rangle=0$, a contradiction. Since TP is closed in TE and Tx_0 is outside TP, there exists a $y'\in F$ with $\langle TP,y'\rangle=0$ and $\langle Tx_0,y'\rangle=1$. Let us show that $x'=T'y'$. If $x\in E$, we have $x=\langle x,x'\rangle x_0+p$ with $p\in P$ whence

$$\langle Tx,y'\rangle = \langle\langle x,x'\rangle Tx_0+Tp,y'\rangle$$
$$= \langle x,x'\rangle\langle Tx_0,y'\rangle+\langle Tp,y'\rangle = \langle x,x'\rangle$$

and the proof is complete.

The equivalence of the first five conditions is thus established. To complete the proof, we intend to prove $1^\circ \Rightarrow 6^\circ \Rightarrow 3^\circ$.

To prove 6°, take $x_1',\ldots,x_n' \in E'$ and denote by U the set of all $x \in E$ for which $|\langle x,x_i\rangle| \leqq 1$, $1 \leqq i \leqq n$. We intend to prove by induction that TU is a neighbourhood of zero in $(TE,\sigma(F,F'))$. If $n=1$, we shall distinguish two cases.

I. $x' \in R(T')$. We have $x'=T'y'$ for some $y' \in D(T')$. Suppose that $y \in TE$ and $|\langle y,y'\rangle| \leqq 1$; $y=Tx$ for some x whence

$$|\langle x,x'\rangle| = |\langle x,T'y'\rangle = |\langle Tx,y'\rangle| \leqq 1$$

so that $x \in U$ and $y \in TU$.

II. x' non $\in R(T')$. The last set being closed in E', there exists an $x_0 \in E$ with $\langle x_0,R(T')\rangle=0$ and $\langle x_0,x'\rangle=1$. Since $D(T')$ is dense in E', $\langle x_0,R(T')\rangle=0$ implies $Tx_0=0$. Let us show now that $TU=TE$. If $x \in E$ take $z=x-\langle x,x'\rangle x_0$. We have $Tz=Tx$ and $z \in U$.

Now let $n>1$ and suppose the statement proved for $n-1$ functionals. Let us denote by M the smallest linear subspace of E' containing $R(T')$ and $x_1',\ldots,x_{n-1}'$. Since $R(T')$ is closed, M is closed as well. We shall distinguish two cases:

I. $x_n' \in M$. We have $x_n'=\beta_1x_1'+\ldots+\beta_{n-1}x_{n-1}'+T'y'$ for some β_i and a suitable $y' \in D(T')$. Let V be the set of those $x \in E$ for which $|\langle x,x_i'\rangle| \leqq 1$, $1 \leqq i \leqq n-1$. By induction hypothesis TV is a weak neighbourhood of zero in TE.

Take a number $\beta \geqq 1$, $\beta \geqq |\beta_i|$, $1 \leqq i \leqq n-1$. We intend to show that $y \in (n\beta)^{-1}TV$ and $|\langle y,y'\rangle| \leqq (n\beta)^{-1}$ imply $y \in TU$. If $y \in (n\beta)^{-1}TV$ we have, for a suitable x, $y=(n\beta)^{-1}Tx$ and $|\langle x,x_i'\rangle| \leqq 1$, $1 \leqq i \leqq n-1$. At the same time

$$(n\beta)^{-1}|\langle x,T'y'\rangle| = |\langle y,y'\rangle| \leqq (n\beta)^{-1}.$$

We have

$$\langle (n\beta)^{-1}x,x_n'\rangle = (n\beta)^{-1}\sum_1^{n-1}\beta_i\langle x,x_i'\rangle+(n\beta)^{-1}\langle x,T'y'\rangle,$$

whence

$$|\langle (n\beta)^{-1}x,x_n'\rangle| \leqq (n\beta)^{-1}\sum_1^{n-1}|\beta_i|+(n\beta)^{-1} \leqq 1$$

so that $(n\beta)^{-1}x \in U$.

II. x_n' non $\in M$. Since M is closed, there exists an $x_0 \in E$ such that $\langle x_0,M\rangle=0$ and $\langle x_0,x_n'\rangle=1$. Again we have $Tx_0=0$. We intend to show

that $TU = TV$. Indeed if $x \in V$, take $z = x - \langle x, x_n' \rangle x_0$. We have $Tx = Tz$ and $\langle z, x_n' \rangle = 0$. If $i < n$, $\langle x_0, x_i' \rangle = 0$ so that $\langle z, x_i' \rangle = \langle x, x_i' \rangle$. It follows that $z \in U$ and the proof is complete.

The cycle of implications will be complete if we prove the following proposition.

PROPOSITION (1,5). *Let T be a linear mapping from E into F which is open as a mapping of $\big(D(T), \sigma(D(T), E')\big)$ into $\big(F, \sigma(F, F')\big)$. If $x' \in T^{-1}(0)^0$ then there exists a $y' \in F'$ so that $\langle x, x' \rangle = \langle Tx, y' \rangle$ for $x \in D(T)$.*

PROOF. Let U be the set of those x for which $|\langle x, x' \rangle| \leqq 1$. Since $T\big(U \cap D(T)\big)$ is a neighbourhood of zero in $R(T)$, there exist $y_1', \ldots, y_n' \in F'$ so that $x \in D(T)$ and $|\langle Tx, y_i' \rangle| \leqq 1$ imply $Tx \in TU$. Since $x' \in T^{-1}(0)^0$ it is easy to see that $Tx \in TU$ implies $x \in U$. Especially, $x \in D(T)$ and $\langle Tx, y_i' \rangle = 0$ imply $\langle x, x' \rangle = 0$. It follows that there exist scalars $\lambda_1, \ldots, \lambda_n$ such that

$$\langle x, x' \rangle = \sum \lambda_i \langle Tx, y_i' \rangle = \langle Tx, \sum \lambda_i y_i' \rangle$$

for $x \in D(T)$ which proves the theorem.

PROPOSITION (1,6). *Let E and F be two topological spaces, φ a mapping of E into F. Then the following conditions are equivalent:*

1° *the mapping φ is continuous and open;*

2° *the mapping φ satisfies the following two conditions:*

2.1° *if G is open in E then $\varphi^{-1}\varphi G$ is open in E;*

2.2° *a set $F \subset \varphi E$ is closed in φE iff $\varphi^{-1}F$ is closed in E.*

PROOF. 1° $\Rightarrow$ 2.1°. Let G be open in E. Since φ is open the set φG is open in φE whence $\varphi G = H \cap \varphi E$ for a suitable H open in F. Since φ is continuous $\varphi^{-1}H$ is open in E. Clearly $\varphi^{-1}H = \varphi^{-1}\varphi G$.

1° $\Rightarrow$ 2.2°. Let B be closed in φE so that $B = H \cap \varphi E$ for a suitable H closed in F. Since φ is continuous, $\varphi^{-1}H$ is closed in E. Clearly $\varphi^{-1}H = \varphi^{-1}B$. If B is a subset of φE such that $\varphi^{-1}B$ is closed in E, the set $\varphi(E - \varphi^{-1}B)$ will be open in φE since φ is open. Clearly $\varphi E - B = \varphi(E - \varphi^{-1}B)$.

Now assume that 2.1° and 2.2° are satisfied. Let $B \subset F$ be closed in F. Since $\varphi^{-1}B = \varphi^{-1}(B \cap \varphi E)$ and $B \cap \varphi E$ is closed in φE, the set $\varphi^{-1}B$ will be closed in E by 2.2°. It follows that φ is continuous. If G is open in E, we have $\varphi^{-1}\varphi G$ open in E by 2.1° so that $\varphi^{-1}(\varphi E - \varphi G) = E - \varphi^{-1}\varphi G$ is closed in E. It follows from 2.2° that $\varphi E - \varphi G$ is closed in φE hence φG is open in φE. The mapping φ is thus seen to be open.

PROPOSITION (1,7). *Let T be a linear mapping from E into F with $D(T) = E$. Then the following conditions are equivalent:*

1° *if Q is a subspace of TE, then Q is closed in TE iff $T^{-1}(Q)$ is closed in E;*

2° *if Q is a hyperplane in TE, then Q is closed in TE iff $T^{-1}(Q)$ is closed in E;*

3° *T is a homomorphism of $(E,\sigma(E,E'))$ into $(F,\sigma(F,F'))$.*

PROOF. The implication 1° $\Rightarrow$ 2° being immediate, assume 2° and let us prove 3°. If P is a closed hyperplane in F then $T^{-1}(P)=T^{-1}(P\cap TE)$ and the last set is closed by 2° since $P\cap TE$ is closed in TE. It follows that T is weakly continuous. If $T^{-1}(0)\subset H\subset E$ and H is a closed hyperplane in E then by 2°, TH will be closed in TE iff $T^{-1}TH$ is closed in E. We have, however, $T^{-1}TH=H$ since $T^{-1}(0)\subset H$. It follows that T is weakly open.

Now assume 3° and let us prove 1°. If $Q\subset TE$ is closed in TE, the space $T^{-1}(Q)$ will be closed in E since T is weakly continuous. Suppose that $Q\subset TE$ and $T^{-1}(Q)$ is closed in E. Since $T^{-1}(0)\subset T^{-1}(Q)$ and T is weakly open, $Q=TT^{-1}(Q)$ will be closed in TE. The proof is complete.

2. Closed mappings.

This section describes some methods of generating closed mappings. The results, although of an auxiliary character, seem to be interesting enough on their own to be stated separately.

PROPOSITION (2,1). *Let E, E_1, E_2, be convex spaces, h_1 a linear mapping from E into E_1, h_2 a linear mapping from E into E_2. Suppose that*

1° *h_1 is weakly open and $R(h_1)=E_1$;*

2° *h_2 is closed in $E\times E_2$;*

3° *$h_1^{-1}(0)\subset h_2^{-1}(0)$.*

Let H be the subset of $E_1\times E_2$ consisting of $[x_1,x_2]$ such that $x_1=h_1x$ and $x_2=h_2x$ for some $x\in D(h_1)\cap D(h_2)$.

Then H is closed in $E_1\times E_2$.

PROOF. For the proof we shall adopt the following convention: if T is a linear mapping from E into F, not necessarily densely defined, T' is not uniquely defined. We shall nevertheless use the symbol $D(T')$ for the set of those $y'\in F'$ for which $\langle Tx,y'\rangle$ is continuous on $D(T)$.

First we shall prove the following statement: if $x_2'\in D(h_2')$ and $z'\in E'$ is such that $\langle h_2x,x_2'\rangle=\langle x,z'\rangle$ for $x\in D(h_2)$ then there exists an $x_1'\in D(h_1')$ with $\langle h_1x,x_1'\rangle=\langle x,z'\rangle$ for $x\in D(h_1)$; the functional $[-x_1',x_2']$ annihilates H.

Indeed, since $h_1^{-1}(0)\subset h_2^{-1}(0)$, we have $\langle h_1^{-1}(0),z'\rangle=0$ so that, h_1 being

weakly open, there exists by Theorem (1,5) an $x_1' \in D(h_1')$ with $\langle h_1x, x_1'\rangle = \langle x, z'\rangle$ for $x \in D(h_1)$; if $x \in D(h_1) \cap D(h_2)$, we have

$$\langle [h_1x, h_2x][-x_1', x_2']\rangle = -\langle h_1x, x_1'\rangle + \langle h_2x, x_2'\rangle = 0 .$$

Suppose now that $[x_1, x_2]$ non $\in H$. Since $R(h_1) = E_1$ we have $x_1 = h_1x_0$ for some $x_0 \in D(h_1)$. Then $[x_0, x_2]$ non $\in G(h_2)$ hence there exists $[x', x_2']$ such that $\langle [x, h_2x], [x', x_2']\rangle = 0$ for $x \in D(h_2)$ and $\langle [x_0, x_2], [x', x_2']\rangle \neq 0$. It follows that $\langle x, x'\rangle + \langle h_2x, x_2'\rangle = 0$ for $x \in D(h_2)$ so that $x_2' \in D(h_2')$. According to what has been proved above there exists an $x_1' \in D(h_1')$ with $\langle h_1x, x_1'\rangle = -\langle x, x'\rangle$ for $x \in D(h_1)$. Now $[-x_1', x_2']$ annihilates H and

$$\begin{aligned} \langle [x_1, x_2][-x_1', x_2']\rangle &= -\langle x_1, x_1'\rangle + \langle x_2, x_2'\rangle \\ &= -\langle h_1x_0, x_1'\rangle + \langle x_2, x_2'\rangle \\ &= \langle x_0, x'\rangle + \langle x_2, x_2'\rangle = \langle [x_0, x_2], [x', x_2']\rangle \neq 0 . \end{aligned}$$

The proof is complete.

The meaning of the preceding theorem is obvious: the set H is the graph of a mapping from E_1 into E_2. Indeed, if $[0, x_2] \in H$, we have $0 = h_1x$ and $x_2 = h_2x$ for some $x \in D(h_1) \cap D(h_2)$. It follows that $x \in h_1^{-1}(0) \subset h_2^{-1}(0)$, whence $x_2 = h_2x = 0$.

The following result is, in a certain sense, dual to the preceding one.

PROPOSITION (2,2). *Let E_1, E_2, E be convex spaces, f_1 a linear mapping from E_1 into E, f_2 a linear mapping from E_2 into E. Suppose that*

1° *f_1 is weakly continuous and $D(f_1) = E_1$,*

2° *f_2 is closed in $E_2 \times E$.*

Let F be the subset of $E_1 \times E_2$ consisting of $[x_1x_2]$ such that $f_1x_1 = f_2x_2$.

Then F is closed in $E_1 \times E_2$.

PROOF. Define a mapping f from $E_1 \times E_2$ into E in the following manner: $D(f)$ will be the set of all $[x_1, x_2]$ where

$$x_2 \in D(f_2) \quad \text{and} \quad f(x_1, x_2) = f_1x_1 - f_2x_2 .$$

We intend to show that f is closed in $E_1 \times E_2 \times E$; this also proves our theorem since clearly $F = f^{-1}(0)$.

Take a point $[x_1^0\, x_2^0\, x^0]$ which does not belong to $G(f)$. It follows that $[x_2^0, f_1(x_1^0) - x^0]$ non $\in G(f_2)$ so that there exists a point $[x_2', x'] \in E_2' \times E'$ with the following properties:

$$\langle [x_2, f_2(x_2)], [x_2', x']\rangle = 0 \quad \text{for all } x_2 \in D(f_2)$$

and

$$\langle [x_2^0, f_1(x_1^0) - x^0], [x_2', x']\rangle \neq 0 .$$

It follows that

$$\langle x_2, x_2' \rangle + \langle f_2(x_2), x' \rangle = 0 \quad \text{for all } x_2 \in D(f_2) .$$

Since f_1 is weakly continuous, $D(f_1') = E'$ and hence $f_1'(x')$ is defined. If $p = [f_1'(x'), x_2', -x']$, we have

$$\begin{aligned} \langle G(f), p \rangle &= \langle x_1, f_1'(x') \rangle + \langle x_2, x_2' \rangle - \langle f_1(x_1) - f_2(x_2), x' \rangle \\ &= \langle x_1 f_1'(x') \rangle - \langle f_1(x_1), x' \rangle + \langle x_2, x_2' \rangle + \langle f_2(x_2), x' \rangle = 0 . \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle [x_1^0, x_2^0, x^0], p \rangle &= \langle x_1^0, f_1'(x') \rangle + \langle x_2^0, x_2' \rangle - \langle x^0, x' \rangle \\ &= \langle x_2^0, x_2' \rangle + \langle f_1(x_1^0) - x^0, x' \rangle \\ &= \langle [x_2^0, f_1(x_1^0) - x^0], [x_2', x'] \rangle \neq 0 . \end{aligned}$$

This completes the proof.

The set F is the graph of a mapping from E_1 into E_2 if f_2 is injective. Indeed, if $[0, x_2] \in F$ we have $f_2 x_2 = 0$ whence $x_2 = 0$. The domain of this mapping will be the whole of E_1 if $R(f_1) \subset R(f_2)$.

PROPOSITION (2,3). *Let h be a linear mapping of (E, u) onto (B, v). Suppose that (E, u) is continuously imbedded in a space (E^*, u^*). Suppose that*

1° *h is weakly open;*

2° *$h^{-1}(0)$ is closed in (E, u^*).*

Then the closure of $G(h)$ in $(E^*, u^*) \times (B, v)$ is again the graph of a mapping from (E^*, u^*) onto (B, v).

PROOF. Suppose that $[0, y_0]$ belongs to the closure of $G(h)$ in $(E^*, u^*) \times (B, v)$. Since $R(h) = B$, $y_0 = h(x_0)$ for some $x_0 \in E$. Suppose that $y_0 \neq 0$. It follows that x_0 non $\in h^{-1}(0)$. The last set being closed in (E, u^*) there exists a $z' \in (E, u^*)$ such that $\langle h^{-1}(0), z' \rangle = 0$ and $\langle x_0, z' \rangle \neq 0$. The mapping h being weakly open there exists by (1,5) an element $b' \in (B, v)'$ such that $\langle hx, b' \rangle = \langle x, z' \rangle$ for all $x \in E$. If $x \in E$, we have

$$\langle [x, h(x)], [z', -b'] \rangle = \langle x, z' \rangle - \langle h(x), b' \rangle = 0$$

and

$$\langle [0, y_0], [z', -b'] \rangle = -\langle y_0, b' \rangle = -\langle h(x_0), b' \rangle = -\langle x_0, z' \rangle \neq 0 .$$

This contradiction proves the theorem.

3. Sequentially open mappings.

Let E and F be two LF-spaces and T a continuous linear mapping of E into F. Let $U \in \boldsymbol{U}(E)$ and consider the set TU. If $R(T) = F$ it is

not difficult to show [1] that TU will be a neighbourhood of zero in F. If $R(T)$ is different from F, the situation is much more complicated:

TU need not be a relative neighbourhood of zero in $R(T)$ even if $TU \cap F_j$ is a relative neighbourhood of zero in $R(T) \cap F_j$ for each j. In the present section we intend to study mappings which are open only in this weakened sequential sense.

PROPOSITION (3,1). *Let E and F be two LF-spaces and T a continuous linear mapping of E into F. Then for each i there exists a j such that $TE_i \subset F_j$.*

PROOF. Since $E_i = \bigcup_j (E_i \cap T^{-1}(F_j))$ it follows from Baire's theorem that $E_i \subset T^{-1}(F_j)$ for some j whence $TE_i \subset F_j$.

THEOREM (3,2). *Let E and F be two LF-spaces and T a continuous linear mapping of E into F. Then the following conditions are equivalent:*

1° *for each j the intersection $TE \cap F_j$ is closed;*

2° *for each j the intersection $TE \cap F_j$ is closed and there exists a defining sequence H_j of E such that $TH_j = TE \cap F_j$;*

3° *for each neighbourhood of zero U in E and each j the set TU is a neighbourhood of zero in $TE \cap F_j$ and there exists a defining sequence H_r of E such that for each r the restriction of T to H_r is open and $H_r + T^{-1}(0)$ closed in E.*

PROOF. 1° $\Rightarrow$ 2°. Let j be fixed. The set $TE \cap F_j$ is complete since it is closed in F_j. Since it may be represented as the union of $TE_i \cap F_j$ the set $TE_p \cap F_j$ will be of the second category in $TE \cap F_j$ for some p. Consider the mapping T of $E_p \cap T^{-1}(F_j)$ into $TE \cap F_j$. Its range $TE_p \cap F_j$ is of the second category in $TE \cap F_j$, so that we have $TE_p \cap F_j = TE \cap F_j$ by the open mapping theorem.

We see thus that for each j there exists a p such that $T^{-1}(F_j) \subset E_p + T^{-1}(0)$. Let us denote by $p(j)$ the minimal p of this property. We shall distinguish two cases:

I. The sequence $p(j)$ is bounded: $p(j) \leqq q$ for all j. It follows that $E = E_q + T^{-1}(0)$ whence $TE = TE_q$ so that $TE = TE_q \subset F_k$ for some k. Since TE is closed in F_k, the mapping T considered as a mapping of E_q onto $TE_q = TE$ is open. Hence T is open as a mapping of E onto TE. If $j \geqq \max(q,k)$ we have

$$TE_j = TE = TE \cap F_j$$

so that E_j for $j \geqq \max(q,k)$ is a defining sequence of the required property.

II. The sequence $p(j)$ is unbounded. Clearly $p(j) \leqq p(j+1)$, hence $p(j)$ tends to infinity. If j is given, put

$$H_j = E_{p(j)} \cap T^{-1}(F_j) .$$

Let us show first that H_j is a defining sequence.

(1) If j is given, we have $H_j \subset E_{p(j)}$ by definition.

(2) Let i be given. By (3,1) there exists a j such that $E_i \subset T^{-1}(F_j)$. Since $p(r)$ tends to infinity, there exists a $k > j$ such that $p(k) > i$. It follows that

$$E_i \subset T^{-1}(F_j) \subset T^{-1}(F_k) \quad \text{and} \quad E_i \subset E_p(k) .$$

Combining these inclusions we obtain

$$E_i \subset E_{p(k)} \cap T^{-1}(F_k) = H_k .$$

The sequence H_j is thus seen to be a defining sequence for E.

To show that $TH_j = TE \cap F_j$ let us observe first that $T^{-1}(F_j) \subset E_{p(j)} + T^{-1}(0)$. Each $x \in T^{-1}(F_j)$ may thus be written in the form $x = y + z$ with $y \in E_{p(j)}$ and $z \in T^{-1}(0)$. Since $y = x - z \in T^{-1}(F_j)$ we have $y \in H_j$ so that $T^{-1}(F_j) \subset H_j + T^{-1}(0)$. It follows that

$$H_j \subset T^{-1}(F_j) \subset H_j + T^{-1}(0) ,$$

whence $TH_j = TE \cap F_j$.

$2° \Rightarrow 3°$. Let H_j be a defining sequence of E such that $TH_j = TE \cap F_j$; we have $H_j + T^{-1}(0) = T^{-1}(F_j)$ so that $H_j + T^{-1}(0)$ is closed in E for each j.

Further, consider the mapping T as a mapping of H_j onto $TE \cap F_j$. The last set being closed in F_j and hence complete, it follows that T is an open mapping of H_j into F_j. If U is an arbitrary neighbourhood of zero in E, we have $TU \cap F_j \supset T(U \cap H_j) \cap F_j$ and the last set is a neighbourhood of zero in $TE \cap F_j$.

$3° \Rightarrow 1°$. Let H_j be a defining sequence for E such that $H_j + T^{-1}(0)$ is closed in E and the restriction of T to H_j is open for each j. Let j be given and let us show first that

$$TE \cap F_j = TH_p \cap F_j \quad \text{for some } p .$$

Suppose that for each p there exists an x_p such that $Tx_p \in F_j$ and Tx_p non $\in TH_p$. Taking, if necessary, suitable multiples of the x_j, we may assume the sequence Tx_p to be bounded. Since Tx_p non $\in TH_p$, we have x_p non $\in H_p + T^{-1}(0)$ so that there exists an $x_p' \in E'$ such that $\langle H_p + T^{-1}(0), x_p' \rangle = 0$ and $\langle x_p, x_p' \rangle = 1$. Let U_p be the set of all $x \in E$ for which $|\langle x, x_p' \rangle| \leqq 1/p$. Clearly $U = U_1 \cap U_2 \cap \ldots$ is a neighbourhood of zero in E. Let $m > 0$ be given and suppose that $x_p \in mU + T^{-1}(0)$ for all p. If $p > m$, we have $x_p \in mU_p + T^{-1}(0)$ whence $1 = \langle x_p, x_p' \rangle \leqq m/p < 1$ which is a contradiction. It follows that, given m, the inclusion $Tx_p \in mTU$ can never be satisfied for all p. Since TU is a relative neighbour-

hood of zero in F_j and Tx_p is bounded, we arrive at a contradiction. We have thus shown that

$$TE \cap F_j = TH_p \cap F_j \quad \text{for some } p .$$

Since $TH_p \subset F_k$ for some k by (3,1), the set TH_p is closed in F_k, the mapping T being open when considered as a mapping of H_p. It follows that $TE \cap F_j = TH_p \cap F_j$ is closed in E and the proof is complete.

If $R(T) = F$, the situation is very simple. Indeed, condition 1° of theorem (3,2) is automatically satisfied. It follows that, for each $U \in \boldsymbol{U}(E)$ and each j, the set $TU \cap F_j$ is a neighbourhood of zero in F_j so that TU is a neighbourhood of zero in F. A continuous linear mapping of an LF-space E onto an LF-space F is thus seen to be open. This result has been obtained first by J. Dieudonné and L. Schwartz [1].

THEOREM (3,3). *Let E and F be two LF-spaces and T a continuous linear mapping of E onto F. Then T is open.*

PROOF. If j is fixed we have $F_j = \bigcup T(E_p \cap T^{-1}(F_j))$ so that the set $T(E_p \cap T^{-1}(F_j))$ will be of the second category in F_j for some p. It follows from the open mapping theorem that $F_j = T(E_p \cap T^{-1}(F_j))$ for this p and that the restriction of the mapping T to $E_p \cap T^{-1}(F_j)$ is open. If U is a neighbourhood of zero in E, the set $U \cap E_p \cap T^{-1}(F_j)$ is a neighbourhood of zero in $E_p \cap T^{-1}(F_j)$, hence

$$TU \cap F_j \supset T\big(U \cap E_p \cap T^{-1}(F_j)\big)$$

is a neighbourhood of zero in F_j. Since j was arbitrary this shows that T is open.

4. Open mappings.

A sequentially open mapping need not be open. Indeed, suppose T is a continuous and sequentially open mapping of an LF-space E into an LF-space F with range R. If U is a neighbourhood of zero in E we know that, for each j, there exists a neighbourhood of zero V_j in F such that $V_j \cap R \cap F_j \subset TU$. It does not follow that the V_j can be put together somehow to form a neighbourhood V of zero in F so as to have $V \cap R \subset TU$. It is necessary to indroduce additional assumptions to be able to do this; in the present section we intend to discuss some sufficient conditions for T to be open.

First let us mention the following rather strong condition due to B. Malgrange ([3, Prop. 6, p. 315]).

THEOREM (4,1). *Let F be an LF-space and M an absolutely convex subset of F. Let $Z \subset M$ be absolutely convex and such that for each j the set $Z \cap F_j$ is an open neighbourhood of zero in $M \cap F_j$. Suppose that $M \cap F_j$ is compact for each j.*

Then Z is a neighbourhood of zero in M.

PROOF. Let $V_j \in \boldsymbol{U}(F)$ be such that

$$V_j \cap M \cap F_j \subset Z .$$

We intend to show that there exists a $V_{j+1} \in \boldsymbol{U}(F)$ such that

1° $V_{j+1} \cap M \cap F_{j+1} \subset Z$;

2° $V_j \cap F_j \subset V_{j+1} \subset V_j$.

Denote by $\boldsymbol{V}$ the system of all $V \in \boldsymbol{U}(F)$ which satisfy $V_j \cap F_j \subset V \subset V_j$ and suppose that no $V \in \boldsymbol{V}$ satisfies $V \cap M \cap F_{j+1} \subset Z$. It follows that the set $V \cap M \cap F_{j+1} \cap (F - Z)$ is nonvoid for each $V \in \boldsymbol{V}$. Since $\boldsymbol{V}$ has the finite intersection property and the sets under consideration are compact, there exists a point x_0 which satisfies

$$x_0 \in V \cap M \cap F_{j+1} \cap (F - Z)$$

for each $V \in \boldsymbol{V}$. Since $V_j \in \boldsymbol{V}$, we have $x_0 \in V_j$.

Take now a point y non $\in F_j$. There exists a point $x' \in F'$ such that $\langle F_j, x' \rangle = 0$ and $\langle y, x' \rangle = 1$. Let S be the set of those $x \in F$ for which $|\langle x, x' \rangle| \leqq 1$. Clearly $V_j \cap S \in \boldsymbol{V}$ and $V_j \cap S$ does not contain y. It follows that $x_0 \in F_j$. We have thus $x_0 \in V_j \cap M \cap F_j \subset Z$ which is a contradiction.

Now take a $V_1 \in \boldsymbol{U}(F)$ such that $V_1 \cap M \cap F_1 \subset Z$ and construct by induction a sequence $V_j \in \boldsymbol{U}(F)$ with the properties 1° and 2°. It is easy to see that $V = \cap V_j$ belongs to $\boldsymbol{U}(F)$ and satisfies $V \cap M \subset Z$. This completes the proof.

The preceding compactness condition may be somewhat loosened. For that purpose a slightly more refined construction is required the idea of which is due to B. Malgrange (l.c. p. 316).

LEMMA (4,2). *Let F be an LF-space and $W \subset R$ two absolutely convex subsets of F. Suppose that, for each j, there exists a $V_j \in \boldsymbol{U}(F)$ such that*

$$V_j \cap R \cap F_j \subset W .$$

Suppose that the following condition is satisfied: Given j, ε, V_j such that $0 < \varepsilon < 1$, $V_j \in \boldsymbol{U}(F)$ and $V_j \cap R \cap F_j \subset W$ there exists a $V_{j+1} \in \boldsymbol{U}(F)$ such that

1° $V_{j+1} \cap R \cap F_{j+1} \subset W$;

2° $(1 - \varepsilon)(V_j \cap F_j) \subset V_{j+1} \subset V_j$.

Then there exists a $V \in \boldsymbol{U}(F)$ so that $V \cap R \subset W$.

PROOF. Take a sequence $0<\varepsilon_j<1$ such that $\eta=\prod(1-\varepsilon_j)$ is positive. Choose a $V_1\in \boldsymbol{U}(F)$ so that $V_1\cap R\cap F_1\subset W$ and construct by induction a sequence $V_j\in \boldsymbol{U}(F)$ so that

$$V_j\cap R\cap F_j\subset W \quad \text{and} \quad (1-\varepsilon_j)(V_j\cap F_j)\subset V_{j+1}\subset V_j$$

Put $V=\bigcap V_j$ so that $R\cap V\subset W$. To see that V is a neighbourhood of zero in F it suffices to show that $\eta(V_j\cap F_j)\subset V$ or, in other words, that $\eta(V_j\cap F_j)\subset V_k$ for each k. With view so the inclusions $V_1\supset V_2\supset\ldots$ it is sufficient to prove this for $k>j$ only. Put $H_j=V_j\cap F_j$; we have

$$(1-\varepsilon_j)H_j\subset V_{j+1}\cap F_{j+1}=H_{j+1}$$

so that $\eta_jH_j\subset\eta_{j+1}H_{j+1}$ where $\eta_j=(1-\varepsilon_j)(1-\varepsilon_{j+1})\ldots$. It follows that, for each $p>0$,

$$\eta H_j\subset\eta_jH_j\subset\eta_{j+p}H_{j+p}\subset V_{j+p}\,,$$

and the proof is complete.

PROPOSITION (4,3). *Let F be an LF-space and R a subspace of F. Let $H\in \boldsymbol{U}(F)$ and put $M=R\cap H$. Let $Z\subset M$ be an absolutely convex set such that for each j the set $Z\cap F_j$ is an open neighbourhood of zero in $M\cap F_j$. Suppose that, for each j, the following condition is satisfied: if $X\subset M\cap F_{j+1}$ and X/F_j is bounded, then $X/R\cap F_j$ is relatively compact in $M/R\cap F_j$. Then Z is a neighbourhood of zero in M.*

PROOF. Let $V_j\in \boldsymbol{U}(F)$ be such that $V_j\subset H$ and

$$V_j\cap M\cap F_j\subset Z$$

and let $0<\varepsilon<1$ be given. We intend to show that then exists a $V_{j+1}\in \boldsymbol{U}(F)$ such that

1° $V_{j+1}\cap M\cap F_{j+1}\subset Z$;

2° $(1-\varepsilon)(V_j\cap F_j)\subset V_{j+1}\subset V_j$.

Denote by $\boldsymbol{V}$ the system of all $V\in \boldsymbol{U}(F)$ which satisfy

$$(1-\varepsilon)(V_j\cap F_j)\subset V\subset V_j$$

and suppose that no $V\in \boldsymbol{V}$ satisfies $V\cap M\cap F_{j+1}\subset Z$. Clearly there exists a convex topology w on F with the following properties:

(1) w is coarser than the topology v of F;

(2) w coincides with v on F_{j+1};

(3) w has a countable complete system of neighbourhoods of zero.

Let us denote by Q the canonical quotient mapping of F modulo F_j and put $\tilde{w}=Qw$. The topology $\tilde{w}$ has a countable complete system of neighbourhoods of zero $\tilde{W}_n\in \boldsymbol{U}(F/F_j,\tilde{w})$. For each n put

$$V^{(n)} = (1-\varepsilon)V_j \cap Q^{-1}\tilde{W}_n .$$

Let us show first that $V^{(n)} \in \boldsymbol{V}$ for each n. Since $\tilde{W}_n \in \boldsymbol{U}(F/F_j,\tilde{w})$ we have $Q^{-1}\tilde{W}_n \in \boldsymbol{U}(F)$ whence $V^{(n)} \in \boldsymbol{U}(F)$. The inclusion

$$V^{(n)} = (1-\varepsilon)V_j \cap Q^{-1}\tilde{W}_n \subset (1-\varepsilon)V_j \subset V_j$$

being obvious, take an $x \in (1-\varepsilon)(V_j \cap F_j)$. Since

$$x \in F_j = Q^{-1}(0) \subset Q^{-1}\tilde{W}_n ,$$

we have $x \in V^{(n)}$ which completes the proof of $V^{(n)} \in \boldsymbol{V}$.

Let $y_n \in V^{(n)} \cap M \cap F_{j+1} \cap (F-Z)$. Since $Qy_n \in \tilde{W}_n$, $y_n \in F_{j+1}$ and w coincides with v on F_{j+1}, the set of the y_n is bounded in F_{j+1} modulo F_j. It follows from our assumption that there is a subsequence z_n of y_n, a point $y_0 \in M \cap F_{j+1}$ and a sequence $r_n \in R \cap F_j$ such that $z_n - r_n \to y_0$. Since

$$Qy_n \to 0 \quad \text{and} \quad Qz_n = Q(z_n - r_n) \to Qy_0 ,$$

we have $Qy_0 = 0$ so that $y_0 \in F_j$. Hence $y_0 \in R \cap F_j$ and $q_n = r_n + y_0 \in R \cap F_j$. Now $q_n = z_n + (q_n - z_n)$ and $(q_n - z_n) \to 0$ so that

$$q_n \in (q_n - z_n) + (1-\varepsilon)V_j \subset (1-\tfrac{1}{2}\varepsilon)V_j \subset (1-\tfrac{1}{2}\varepsilon)H \quad \text{for large } n .$$

It follows that

$$q_n \in (1-\tfrac{1}{2}\varepsilon)(V_j \cap H \cap R \cap F_j) \subset (1-\tfrac{1}{2}\varepsilon)Z$$

for large n.

Now both z_n and q_n belong to $R \cap F_{j+1}$ and z_n do not belong to Z. The last set being a relative neighbourhood of zero in $R \cap F_{j+1}$, it follows that, for large n, $q_n \notin (1-\tfrac{1}{2}\varepsilon)Z$ which is a contradiction.

THEOREM (4,4). *Let (E,u) and (F,v) be two LF-spaces. A continuous linear mapping T of E into F will be open if the following condition is satisfied*:

There exist defining sequences E_j and F_j such that $T^{-1}F_j \subset E_j + T^{-1}(0)$ and an $H \in \boldsymbol{U}(F)$ with the following properties: if $x_n \in E_{j+1}$, $Tx_n \in H$ and $Tx_n \to 0 \bmod F_j$, then $x_n \to 0 \bmod E_j + T^{-1}(0)$.

PROOF. Denote by R_j the set TE_j and by Q the canonical mapping of F onto F/F_j. Let w be a convex topology on F which is coarser than v, coincides with v on R_{j+1} and has a countable complete system of neighbourhoods of zero. Put $\tilde{w} = Qw$ and take a countable complete system of neighbourhoods of zero $\tilde{W}_n \in \boldsymbol{U}(F/F_j,\tilde{w})$. Let $U \in \boldsymbol{U}(E)$ and suppose we are given a $V_j \in \boldsymbol{U}(F)$ such that $V_j \subset H$ and $R_j \cap V_j \subset TU$. Let $0 < \varepsilon < 1$ be given. Put

$$V^{(n)} = (1-\varepsilon)V_j \cap Q^{-1}\tilde{W}_n$$

and let us show that at least one $V^{(n)}$ satisfies the following two conditions:

1° $(F_j \cap V^{(n)}) = (1-\varepsilon)(F_j \cap V_j)$;

2° $R_{j+1} \cap V^{(n)} \subset TU$.

First of all, $V^{(n)} \in \boldsymbol{U}(F)$ since Q is continuous and w is coarser than v. Further it is easy to see that 1° is satisfied for any n. Indeed, $V^{(n)} \subset (1-\varepsilon)V_j$ whence

$$F_j \cap V^{(n)} \subset (1-\varepsilon)(F_j \cap V_j)\,.$$

On the other hand, if $y \in (1-\varepsilon)(F_j \cap V_j)$ we have $y \in (1-\varepsilon)V_j$ and $Qy = 0$ since $y \in F_j$. Hence $y \in Q^{-1}\tilde{W}_n$ so that $y \in V^{(n)}$ and $y \in F_j \cap V^{(n)}$.

Suppose now that the inclusion 2° is not satisfied for any n. It follows that there exists a sequence $x_n \in E_{j+1}$ such that

$$Tx_n \in (1-\varepsilon)V_j \subset H, \qquad QTx_n \in \tilde{W}_n, \qquad Tx_n \notin TU\,.$$

Since $Tx_n \in R_{j+1}$ and w coincides with v on R_{j+1}, we have $Tx_n \to 0 \bmod F_j$. It follows from our assumption that we have $x_n - z_n \to 0$ for a suitable sequence $z_n = p_n + q_n$ with $p_n \in E_j$ and $q_n \in T^{-1}(0)$. Since

$$Tp_n = T(x_n - q_n) - T(x_n - p_n - q_n) = Tx_n - T(x_n - z_n)\,,$$

$Tp_n \in (1-\frac{1}{2}\varepsilon)V_j$ for large n. Since $Tp_n \in R_j$, it follows that, for large n,

$$Tp_n \in (1-\tfrac{1}{2}\varepsilon)(V_j \cap R_j) \subset (1-\tfrac{1}{2}\varepsilon)TU$$

so that there exist $r_n \in T^{-1}(0)$ such that $p_n + r_n \in (1-\frac{1}{2}\varepsilon)U$. Hence

$$x_n - q_n + r_n = p_n + r_n + (x_n - z_n) \in U$$

for large n, so that $Tx_n = T(x_n - q_n + r_n) \in TU$ for large n, which is a contradiction.

The conclusion follows from lemma (4,2).

THEOREM (4,5). *Let (E,u) and (F,v) be two LF-spaces and T a continuous linear mapping of E into F. The mapping T will be open if the following two conditions are satisfied*:

1° *there exist defining sequences E_j and F_j such that $T^{-1}(F_j) \subset E_j + T^{-1}(0)$ for each j*;

2° *there exists an $H \in \boldsymbol{U}(F)$ such that, for each j, the mapping $Q_j \circ T_{j+1}$ is open in H (here Q_j is the canonical quotient mapping of $F \bmod F_j$ and T_{j+1} is the restriction of T to E_{j+1}).*

PROOF. An immediate consequence of Proposition (4,4).

5. Convolution operators.

In this section we intend to give an application of Theorem (4,4) to the case of a convolution operator in a space of type $\mathscr{D}(\Omega)$. Let Ω_1 and Ω_2 be two open subsets of the euclidean space E_n and let $S \in \mathscr{E}'(E_n)$ be such that $\operatorname{supp}(S*\varphi) \subset \Omega_2$ for each $\varphi \in \mathscr{D}(\Omega_1)$. The following facts are well known: if $\varphi \in \mathscr{D}(\Omega_1)$ then $S*\varphi \in \mathscr{D}(\Omega_2)$ and the mapping T defined by $T\varphi = S*\varphi$ is an injective linear and continuous mapping of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$. The mapping T will be called a convolution operator of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$.

The operator T may be extended in an obvious manner to a mapping A of $\mathscr{E}'(\Omega_1)$ into $\mathscr{E}'(\Omega_2)$. Let us recall now a result of Hörmander [2]: if T^{-1} is sequentially continuous then $T'\mathscr{E}(\Omega_2) = \mathscr{E}(\Omega_1)$. Especially it follows from this result that, for a sequentially open convolution operator T, the mapping A is an injective continuous and open extension of T' to $\mathscr{E}'(\Omega_1)$. We shall apply this result in the proof of Theorem (5,1).

THEOREM (5,1). *Let Ω_1 and Ω_2 be two open subsets of the euclidean space E_n and let T be a convolution operator of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$. Suppose that the following two conditions are satisfied*:

1° *T is sequentially open*;

2° *for each compact $K_2 \subset \Omega_2$ there exists a compact $K_1 \subset \Omega_1$ with the following property: if $\varphi \in \mathscr{E}'(\Omega_1)$, $A\varphi$ is continuous on Ω_2 and infinitely differentiable outside K_2, then φ is infinitely differentiable outside K_1.*

Then T is open.

PROOF. We intend to show that the assumptions of Theorem (4,4) are satisfied.

1. Let K_2^j be an increasing sequence of compact sets with union Ω_2. It follows from Theorem (3,2) and condition 2° of the present theorem that there exists an increasing sequence of compact sets $K_1^j \subset \Omega_1$ with union Ω_1 such that (a) if $\varphi \in \mathscr{D}(\Omega_1)$ and $\operatorname{supp} T\varphi \subset K_2^j$ then $\operatorname{supp}\varphi \subset K_1^j$ (b) if $\varphi \in \mathscr{E}'(\Omega_1)$, $A\varphi$ is continuous on Ω_2 and infinitely differentiable outside K_2^j then φ is infinitely differentiable outside K_1^j. Now take an arbitrary j and keep it fixed. We shall denote by h_{j+1} the natural imbedding of $C(K_2^{j+1})$ into $\mathscr{E}'(\Omega_2)$. Let q_2^j be the canonical mapping of $\mathscr{D}(K_2^{j+1})$ modulo $\mathscr{D}(K_2^j)$ and let Q_2^j be its natural extension to $\mathscr{E}'(\Omega_2)$ in the sense of Proposition (2,3). The mappings q_1^j and Q_1^j are defined in a similar manner.

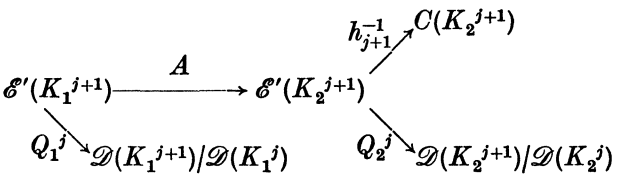
2. Consider now the space $\mathscr{E}'(K_2^{j+1})$; clearly h_{j+1}^{-1} is an open mapping from $\mathscr{E}'(K_2^{j+1})$ onto $C(K_2^{j+1})$. By its definition, Q_2^j is a mapping from $\mathscr{E}'(K_2^{j+1})$ onto $\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^j)$ the graph of which is closed in

$$\mathscr{E}'(K_2^{j+1}) \times \big(\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^{j})\big)$$

It follows from theorem (2,1) that the set

$$P \subset C(K_2^{j+1}) \times \big(\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^{j})\big)$$

of those pairs $[x,y]$ for which there exists a $\xi \in D(h_{j+1}^{-1}) \cap D(Q_2^j)$ with $h_{j+1}^{-1}\xi = x$ and $Q_2^j\xi = y$ is closed in $C(K_2^{j+1}) \times \big(\mathscr{D}(K_2^{j+1})/\mathscr{D}(K_2^{j})\big)$.



3. Clearly the mapping p from $\mathscr{E}'(K_2^{j+1})$ into P defined for $\xi \in D(p) = D(h_{j+1}^{-1}) \cap D(Q_2^j)$ as

$$p(\xi) = [h_{j+1}^{-1}\xi, Q_2^j\xi]$$

is open and $R(p) = P$. Further, we have assumed T to be sequentially open so that A is an open mapping of $\mathscr{E}'(\Omega_1)$ into $\mathscr{E}'(\Omega_2)$. Using these facts, it is easy to see that the mapping $p \circ A$ defined on $A^{-1}\big(D(p)\big)$ with range P is open as well. Since $D(p) \subset \mathscr{E}'(K_2^{j+1})$, we have $A^{-1}\big(D(p)\big) \subset \mathscr{E}'(K_1^{j+1})$. We have thus an open mapping $p \circ A$ from $\mathscr{E}'(K_1^{j+1})$ into P with domain $A^{-1}\big(D(p)\big)$ and range P. Further, the graph of the mapping Q_1^{j+1} is closed in $\mathscr{E}'(K_1^{j+1}) \times \big(\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^j)\big)$. It follows from theorem (2,1) that the set M of those $[u,v] \in P \times \big(\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^j)\big)$ for which there exists a $z \in A^{-1}(D(p)) \cap D(Q_1^j)$ with $p\big(A(z)\big) = u$ and $Q_1^j z = v$, is closed in $P \times \big(\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^j)\big)$.

4. Since $p \circ A$ is one-to-one, the set M is the graph of a mapping g from P into $\mathscr{D}(K_1^{j+1})/\mathscr{D}(K_1^j)$. Now we use assumption 2°. It follows that $A^{-1}\big(D(p)\big) \subset D(Q_1^j)$ so that g is defined on the whole of P. By the closed graph theorem the mapping g is continuous.

5. We intend to show now that the assumption of theorem (4,4) is satisfied. Denote by H the set of those $y \in \mathscr{D}(\Omega_2)$ for which

$$\max |y(t)| \leqq 1, \qquad t \in \Omega_2 .$$

It follows that $H \in \mathfrak{U}\big(\mathscr{D}(\Omega_2)\big)$. Suppose now that we have a sequence $x_n \in \mathscr{D}(K_1^{j+1})$ such that $Tx_n \in H$ and $Q_2^j Tx_n \to 0$. It follows that pTx_n is bounded so that $Q_1^j x_n = gpTx_n$ is bounded as well.

Since pTx_n is bounded, the sequence Tx_n is bounded in $\mathscr{E}'(K_2^{j+1})$ so that, A being open, x_n is bounded in $\mathscr{E}'(K_1^{j+1})$. Since $Q_1^j x_n$ is bounded, there exists a sequence $y_n \in \mathscr{D}(K_1^{j+1})$ such that $Q_1^j y_n = Q_1^j x_n$ and y_n is

bounded in $\mathscr{D}(K_1^{j+1})$, a Montel space. Let $y_0 \in \mathscr{D}(K_1^{j+1})$ be an arbitrary limit point of the sequence y_n and let W be an infinite set of natural numbers such that $\lim y_n = y_0$ for $n \in W$. Since x_n is bounded in $\mathscr{E}'(K_1^{j+1})$ there exists an infinite $S \subset W$ and an $x_0 \in \mathscr{E}'(K_1^{j+1})$ such that $\lim x_n = x_0$ for $n \in S$ in $\mathscr{E}'(K_1^{j+1})$. It follows that, for $n \in S$, we have $\lim Tx_n = Ax_0$. Since $Q_2^j Tx_n \to 0$, we have $Ax_0 \in \mathscr{E}'(K_2^j)$ whence $x_0 \in \mathscr{E}'(K_1^j)$. Since $Q_1^j(y_n - x_n) = 0$ and $y_n - x_n$ converges for $n \in S$ to $y_0 - x_0$ in the topology of $\mathscr{E}'(K_1^{j+1})$, we have $Q_1^j(y_0 - x_0) = 0$. Since $\mathscr{E}'(K_1^j) \subset D(Q_1^j)$ it follows that

$$Q_1^j y_0 = Q_1^j x_0 + Q_1^j(y_0 - x_0) = 0 .$$

Hence, for $n \in W$,

$$\lim Q_1^j x_n = \lim Q_1^j y_n = 0 .$$

Since y_0 was arbitrary, we have $\lim Q_1^j x_n = 0$.

Theorem (5,1) represents a slight improvement of a result of Hörmander [2]. Hörmander's theorem requires that, for each $K_2 \subset \Omega_2$ there exists a $K_1 \subset \Omega_1$ with the following property: if $\varphi \in \mathscr{E}'(\Omega_1)$ and $A\varphi$ is infinitely differentiable outside K_2 then φ is infinitely differentiable outside K_1. In (5,1) we require this for continuous $A\varphi$ only. This (very mild) generalization can be pushed a little further.

THEOREM (5,2). *Let Ω_1 and Ω_2 be two open subsets of the euclidean space E_n and let T be a convolution operator of $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$. Suppose that T is sequentially open. Suppose we have two sequences of compact sets $K_2^j \subset \Omega_2$ and $K_1^j \subset \Omega_1$ and a nondecreasing sequence of nonnegative integers $n(i)$ with the following properties:*

1° *$K_2^0 = 0$, the sequence K_2^j is increasing with union Ω_2, the sets K_2^j are closures of their interiors; similarly for K_1^j;*

2° *if $\varphi \in \mathscr{E}'(\Omega_1)$, $A\varphi$ has continuous derivatives up to order $n(i)$ outside K_2^i with continuous extensions to the set $\Omega_2 - \operatorname{int} K_2^i$ and if $A\varphi$ is infinitely differentiable outside K_2^j then φ is infinitely differentiable outside K_1^j.*

Then T is open.

PROOF. We shall introduce first, for each j, a space C_{j+1} defined as follows. The elements of C_{j+1} will be those $x \in C(K_2^{j+1})$ which possess the following properties: for $i = 0, 1, \ldots, j$ the function x has continuous derivatives up to the order $n(i)$ on $\operatorname{int} K_2^{j+1} - K_2^i$ and these have continuous extensions to $K_2^{j+1} - \operatorname{int} K_2^i$. The topology of C_{j+1} will be defined by means of the following $j+1$ pseudonorms:

$$p_i(x) = \max_{|D| \leq n(i)} \{\max |x^D(t)|,\ t \in K_2^{j+1} - \operatorname{int} K_2^i\}$$

for $i = 0, 1, \ldots, j$. Clearly C_{j+1} is complete.

Further, let us denote by h_{j+1}^* the natural imbedding of C_{j+1} into $\mathscr{E}'(\Omega_2)$ so that h_{j+1}^* is continuous. We shall denote by H^* the set of those $y \in \mathscr{D}(\Omega_2)$ which satisfy the following inequalities

$$\max_{|D| \leqq n(i)} |x^D(t)| \leqq 1, \qquad t \in \Omega_2 - \operatorname{int} K_2{}^i$$

for $i = 0, 1, 2, \ldots$. It follows that $H \in \boldsymbol{U}(\mathscr{D}(\Omega_2))$.

The proof of the present theorem is identical with the proof of Theorem (5,1). It suffices to replace $C(K_2{}^{j+1})$ by C_{j+1}, h_{j+1}, by h_{j+1}^* and H by H^*.

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ON THE DIRECTIONS OF ALMOST PERIODICITY OF ENTIRE FUNCTIONS

PER ASISOFF

The purpose of this note is to prove two theorems concerning entire functions almost periodic in several directions. The theorems state, roughly speaking, that there cannot exist too many such directions. The existence of entire functions almost periodic in several directions is proved in the paper by H. Tornehave [4].

Let $f(z)$ be an entire function and let S denote the set of numbers φ in the interval $0 \leqq \varphi < \pi$, for which $f(z)$ is almost periodic in every strip $-\infty < a < \mathrm{Re}\,(ze^{-i\varphi}) < b < \infty$. With these notations we have the following theorems:

THEOREM 1. *If the entire function $f(z)$ is not a constant, the complementary set of S with respect to the interval $0 \leqq \varphi < \pi$ is everywhere dense.*

THEOREM 2. *If the entire function $f(z)$ is of finite order and not a constant, the set S is finite.*

Let α and β be real numbers satisfying $0 < \beta - \alpha < 2\pi$. Let P denote the point set

$$\{z = re^{i\varphi} \mid r \geqq 0, \alpha \leqq \varphi \leqq \beta\}.$$

With these notations we have the following theorem:

THEOREM 3. *Let $f(z)$ be analytic in a domain containing P. Let $f(z)$ be bounded in P and let for some γ in the interval $\alpha < \gamma < \beta$ the function $f_\gamma(r) = f(re^{i\gamma})$ be the restriction to the half line $r \geqq 0$ of an almost periodic function $g(r)$, $-\infty < r < \infty$. Then $f(z)$ is a constant.*

We shall first prove that the theorems 1 and 2 follow from theorem 3. Afterwards, we shall prove theorem 3.

THEOREM 3 IMPLIES THEOREM 1. If theorem 1 is not true, we can choose α_1 and β_1 such that $0 < \beta_1 - \alpha_1 < 2\pi$, and such that for every φ satisfying $\alpha_1 \leqq \varphi \leqq \beta_1$ the function $f_\varphi(r) = f(re^{i\varphi})$ is almost periodic and,

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hence, bounded for $-\infty < r < \infty$. From a classical theorem of W. F. Osgood [3] applied to the family of functions $f_\varphi(r)$ for $r \geqq 0$, $\alpha_1 \leqq \varphi \leqq \beta_1$ it follows that there exist numbers α_2, β_2 satisfying $\alpha_1 \leqq \alpha_2 < \beta_2 \leqq \beta_1$ such that the family is uniformly bounded for $\alpha_2 \leqq \varphi \leqq \beta_2$. Then theorem 3 implies that $f(z)$ is a constant in contradiction to the assumptions in theorem 1.

THEOREM 3 IMPLIES THEOREM 2. If theorem 2 is not true, we can for every $\delta > 0$ choose real numbers α, β, γ such that $\alpha < \gamma < \beta$, $\beta - \alpha < \delta$, and such that each of the functions $f(re^{i\alpha})$, $f(re^{i\beta})$, $f(re^{i\gamma})$ is almost periodic. If $f(z)$ is not a constant, theorem 3 implies that $f(z)$ is not bounded for $r \geqq 0$, $\alpha \leqq \varphi \leqq \beta$. According to Phragmén–Lindelöf's theorem the order of $f(z)$ is then at least $\pi(\beta - \alpha)^{-1}$. Since $\beta - \alpha$ can be chosen arbitrarily small, this contradicts the assumption that the order of $f(z)$ is finite.

PROOF OF THEOREM 3. We may assume that $\gamma = 0$, $-\frac{1}{2}\pi < \alpha < 0$ and $0 < \beta < \frac{1}{2}\pi$. We introduce the Laplace transforms

$$F_\varphi(s) = \int_0^\infty f(re^{i\varphi})\, e^{-sre^{i\varphi}}\, dre^{i\varphi}$$

for $\varphi = 0$, α, β. Since $f(z)$ is bounded in P, these Laplace transforms are analytic when $\mathrm{Re}(se^{i\varphi}) > 0$. The almost periodic function $g(r)$ has a Fourier series

$$g(r) \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n r}, \quad \lambda_1 = 0\,, \tag{1}$$

where

$$a_n = \lim_{T\to\infty} \frac{1}{T} \int_0^T g(r)\, e^{-i\lambda_n r}\, dr\,.$$

By means of a theorem of abelian type [2, p. 193], we conclude that $(s - i\lambda_n)F_0(s) \to a_n$ for $s = \sigma + i\lambda_n$ and $\sigma \to 0$ through positive values. Hence $i\lambda_n$ is a singular point for $F_0(s)$ if $a_n \neq 0$. (As a matter of fact the closure of the set $\{i\lambda_n \mid a_n \neq 0\}$ is the set of singular points for $F_0(s)$ on the imaginary axis, see [1]).

Let $\varkappa$ denote the boundary of the triangle

$$\{z \mid 0 \leqq \mathrm{Arg}\, z \leqq \beta,\ 0 \leqq \mathrm{Re}\, z \leqq T\}$$

orientated in the positive direction. From Cauchy's theorem it follows that

$$\int_\varkappa f(z)\, e^{-sz}\, dz = 0\,.$$

If s is a positive number, the integral along the vertical side tends to 0 when $T \to \infty$ since $f(z)$ is bounded. Hence we have $F_\beta(s) = F_0(s)$ for positive values of s. According to the theorem on uniqueness of analytic continuation this implies that the functions $F_\beta(s)$ and $F_0(s)$ are identical. Similarly, we prove that $F_\alpha(s)$ and $F_0(s)$ are identical. However, $F_\alpha(s)$ is analytic for $\mathrm{Re}(se^{i\alpha}) > 0$ and $F_\beta(s)$ is analytic for $\mathrm{Re}(se^{i\beta}) > 0$. We have thus proved that no points of the imaginary axis except 0 are singular points for $F_0(s)$. According to the preceding remark this implies that the Fourier series (1) consists only of the constant term. Hence, $g(r)$ is constant, and this implies that $f(z)$ is constant. This completes the proof of theorem 3.

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ON A CARTAN FORMULA FOR SECONDARY COHOMOLOGY OPERATIONS

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1. Introduction.

The purpose of the present paper is to prove a Cartan formula for secondary cohomology operations. By a Cartan formula we mean an expansion $\Phi(xy)=\sum\Phi'(x)\,\Phi''(y)$, valid for a certain set of pairs of cohomology classes $\{(x,y)\}$. The existence of such a formula was proved by Adams [1]. Later Adem [2] gave a formula for certain operations Φ.

The present paper is a continuation of the paper [3] in which a definition of secondary operations based on a study of cochain operations was given.

In Section 2 we continue the study of cochain operations which we began in [3]. Let us recall that a cochain operation θ in one variable is a sequence $\{\theta_n\}$ of natural transformations $\theta_n\colon C^n\to C^{n+i}$, where $i=\deg\theta$. Everywhere we shall assume Z_2 coefficients. In [3] we proved the exactness of the sequence

$$\mathcal{O}\xrightarrow{\Delta} Z(\mathcal{O})\xrightarrow{\varepsilon} A\longrightarrow 0\,,$$

where $\mathcal{O}$ is the set of cochain operations, Δ is the boundary operator $\Delta\theta=\delta\theta+\theta\delta$, $Z(\mathcal{O})=\operatorname{Ker}(\Delta)$ and A is the Steenrod algebra (mod 2).

Here we consider cochain operations G in two variables satisfying

$$\deg G(x,y) = \deg(x)+\deg(y)+\deg G \tag{1}$$

and

$$G(x,0) = G(0,y) = 0\,. \tag{2}$$

The Z_2-module of such operations is denoted by Q. In Q there is a boundary operator ∇ defined by

$$(\nabla G)(x,y) = \delta G(x,y)+G(\delta x,y)+G(x,\delta y)\,.$$

We prove that the following sequence is exact (Theorem 2.2):

$$Q\xrightarrow{\nabla} Z(Q)\longrightarrow A\otimes A\longrightarrow 0\,, \tag{1.1}$$

where $Z(Q)=\operatorname{Ker}(\nabla)$.

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Sections 3–5 are devoted to the proof of the Cartan formula mentioned above. The main theorem is Theorem 4.1 (in Section 4). The main tool in the proof is the sequence (1.1).

Another application of (1.1) is to cohomology operations associated with Cartan formulas. Operations of this type have been studied by Schweitzer [4]. This application is the topic of a forthcoming paper.

2. Theorem on cochain operations.

In [3] we have considered cochain operations of the following kind. A cochain operation of degree i is a sequence $\theta=\{\theta_n\}$ of natural transformations of the cochain functor into itself

$$\theta_n\colon\ C^n \to C^{n+i}, \qquad -\infty < n < \infty\ ,$$

preserving the zero element but not in general the additive structure. The integer i is called the degree of θ. If we define $\Delta\theta=\delta\theta+(-1)^i\theta\delta$, then Δ is a boundary operator in the graded Z_2-module $\mathcal{O}$ of all cochain operations. Let $Z(\mathcal{O})$ denote the Δ-cycles, then there is a map $\varepsilon\colon Z(\mathcal{O}) \to A$, where A denotes the mod 2 Steenrod algebra. Composition defines a multiplication in $\mathcal{O}$. This multiplication is left distributive but not right distributive. We shall, however, allow ourselves to say that $\mathcal{O}$ is an algebra. The mapping ε is then an algebra mapping. One can also consider cochain operations in k variables. The set of all these, $\mathcal{O}^{(k)}$, has in a similar way a boundary operator

$$(\Delta\theta)(x_1,\ldots,x_k) = \delta\theta(x_1,\ldots,x_k) + \theta(\delta x_1,\ldots,\delta x_k)\ .$$

Later in this section we shall consider cochain operations of a different kind. Therefore, in what follows we shall denote the above operations as operations of the first kind. In [3] we proved

THEOREM 2.1. *The sequence*

$$\mathcal{O}^{(k)} \xrightarrow{\Delta} Z(\mathcal{O}^{(k)}) \xrightarrow{\varepsilon} A\oplus A\oplus\ldots\oplus A \to 0, \qquad k \text{ copies of } A\ ,$$

is exact.

Now let us consider operations of the second kind. These are operations $H=\{H_{m,n}\}$ of two variables with the properties

$$(2.1)\qquad H_{m,n}\colon\ C^m\oplus C^n \to C^{m+n+i}, \qquad H(x,0) = H(0,y) = 0\ ,$$

where i is an integer independent of m and n. This integer is called the degree of H, $\deg(H)=i$. As earlier we do not assume additivity.

Let Q denote the set of all cochain operations of the second kind. Then Q is a graded module over Z_2. If H and K are of degree i, then

$$(H_{m,n}+K_{m,n})(x,y) = H_{m,n}(x,y)+K_{m,n}(x,y) .$$

A boundary operator ∇ in Q is defined by

$$(\nabla H)(x,y) = \delta H(x,y)+H(\delta x,y)+H(x,\delta y) .$$

It is obvious that $\nabla\nabla = 0$.

Similar to the case of operations of the first kind there is a mapping from the ∇-cycles $Z(Q)$ into $A\otimes A$,

$$\varepsilon\colon\ Z(Q) \to A\otimes A . \tag{2.2}$$

The mapping ε can be defined on a larger class of operations than $Z(Q)$. This goes as follows.

An operation $F(x,y)$ is in what follows called special if $F(x,y)=0$ on each pair (x,y) of cocycles. If $\delta x=0$ or $\delta y=0$ implies that $F(x,y)=0$, then we call F very special. The mapping ε can be defined on an operation G if ∇G is very special. This is done by considering the Eilenberg–MacLane complexes $K(Z_2,n)$ and $L(Z_2,n)$. For short we shall denote these complexes by K_n and L_n. We recall that in K_n (L_n) there is a basic cocycle z_n (cochain c_n) with the property that to an arbitrary cocycle (cochain) a in a css-complex X there is one and only one mapping $f\colon X \to K_n$ ($f\colon X \to L_n$) such that

$$f^{\#}(z_n) = a \qquad (f^{\#}(c_n) = a) ,$$

$f^{\#}$ denoting the induced cochain transformation. The inclusion of K_n in L_n is denoted by $i\colon K_n \to L_n$ and the projection $L_n \to K_{n+1}$ is denoted by p. They have the properties

$$i^{\#}(c_n) = z_n \qquad \text{and} \qquad p^{\#}(z_{n+1}) = \delta c_n .$$

Since ∇G is very special, we have for each pair (m,n) of integers that $G(z_m,z_n)$ is a cocycle. This means that for each pair (m,n) we have an element

$$\{G(z_m,z_n)\} \in H^*(K_m\times K_n) .$$

We shall examine the connection between these cohomology classes for various pairs (m,n). First, let us consider the pairs (m,n) and $(m+1,n)$. We have

$$\delta G(c_m,z_n) = G(\delta c_m,z_n) = (p\times 1)^{\#}\big(G(z_{m+1},z_n)\big) .$$

This implies that

$$\{G(z_m,z_n)\} = (\sigma\otimes 1)\{G(z_{m+1},z_n)\} ,$$

where σ is the cohomology suspension $H^i(X) \to H^{i-1}(\Omega X)$ and

$$\{G(z_{m+1}, z_n)\} \in H^*(K_{m+1} \times K_n) \cong H^*(K_{m+1}) \otimes H^*(K_m) .$$

In a similar fashion one sees that

$$\{G(z_m, z_n)\} = (1 \otimes \sigma)\{G(z_m, z_{n+1})\} .$$

It is now obvious that there are elements $\alpha_i, \beta_i \in A$ such that

$$\{G(z_m, z_n)\} = \sum \alpha_i(\{z_m\})\, \beta_i(\{z_n\}) . \tag{2.3}$$

The mapping ε of (2.2) is defined by

$$\varepsilon(G) = \sum \alpha_i \otimes \beta_i . \tag{2.4}$$

By the Künneth formula and the well-known structure of the cohomology of K_n (see Serre [5]) the element $\sum \alpha_i \otimes \beta_i \in A \otimes A$ is uniquely determined.

The object of the present section is to establish a theorem similar to Theorem 2.1.

THEOREM 2.2. *The sequence*

$$Q \xrightarrow{\nabla} Z(Q) \xrightarrow{\varepsilon} A \otimes A \to 0$$

is exact. Hence the cohomology of Q with respect to ∇ is $A \otimes A$.

PROOF. First, let us show that $\varepsilon \circ \nabla = 0$. Let $F = \nabla G$. Then by the definition of ε we must consider $\{F(z_m, z_n)\}$. Since $F = \nabla G$, we have

$$\{F(z_m, z_n)\} = \{\delta G(z_m, z_n) + G(\delta z_m, z_n) + G(z_m, \delta z_n)\} = \{\delta G(z_m, z_n)\} = 0 .$$

Next we show that ε is onto. Let $\alpha, \beta \in A$ and let a, b be cochain operations of the first kind representing α and β respectively. It is enough to show that $\alpha \otimes \beta$ is in the image of ε. Define a cochain operation H by

$$H(x, y) = a(x)\, b(y) ;$$

then obviously $\nabla H = 0$ and $\varepsilon(H) = \alpha \otimes \beta$.

Finally, we must show that if $F \in Z(Q)$ and $\varepsilon(F) = 0$, then there is a cochain operation G with $\nabla G = F$. The construction of G is somewhat more involved than the first part of the proof.

First, let us note that a cochain operation H defines a system of cochains, one cochain $H(c_m, c_n)$ in $L_m \times L_n$ for each pair (m, n). The dimension of $H(c_m, c_n)$ is $m + n + i$, where i is the degree of H, and the restrictions of the cochain to each of the factors L_m and L_n are zero. On the other hand, if we have given a system of cochains satisfying the above conditions, then by naturality there is a unique cochain operation de-

fined on all pairs of cochains such that the associated system of cochains is equal to the given system of cochains. These facts are quite obvious. In what follows we shall go back and forth between the two concepts, cochain operation and system of cochains, quite freely without mentioning so each time.

Before we give the proof, it will be useful to have a look at special examples of cochain operations. First, let us define a cochain operation $G_1 = G_1{}^{m,n}$ by

$$G_1(c_s, c_t) = \begin{cases} 0 & \text{for } t \neq n, \\ 0 & \text{for } t = n \text{ and } s \leqq m, \\ \alpha(c_s)\beta_1(c_n)\dots\beta_\nu(c_n) & \text{for } t = n \text{ and } s > n, \end{cases}$$

where $\alpha, \beta_1, \dots, \beta_\nu \in Z(\mathcal{O})$, $\nu \geqq 1$. Then we get

$$\nabla G_1(c_s, z_t) = \begin{cases} 0 & \text{for } (s,t) \neq (m,n), \\ \alpha(\delta c_s)\beta_1(z_n)\dots\beta_\nu(z_n) & \text{for } (s,t) = (m,n). \end{cases} \tag{2.6}$$

We see that ∇G_1 is special, that is, $\nabla G_1(z_s, z_t) = 0$ for all (s,t). We therefore have that $\nabla G_1(c_s, z_t)$ is a cocycle in $(L_s, K_s) \times (K_t, *)$ and hence determines a cohomology class in that complex. From (2.6) one easily reads off the cohomology class in question.

Next, let $G_2 = G_2{}^{m,n}$ be defined by

$$G_2(c_s, c_t) = \begin{cases} 0 & \text{for } (s,t) \neq (m,n), \\ \alpha_1(c_m)\dots\alpha_\mu(c_m)\beta_1(c_n)\dots\beta_\nu(c_n) & \text{for } (s,t) = (m,n), \end{cases}$$

where $\alpha_1, \dots, \alpha_\mu, \beta_1, \dots, \beta_\nu \in Z(\mathcal{O})$, $\mu > 1$ and $\nu \geqq 1$.

As before, ∇G_2 is special and $\nabla G_2(c_s, z_t)$ determines the following cohomology classes in $(L_s, K_s) \times (K_t, *)$:

$$\{\nabla G_2(c_s, z_t)\} = \begin{cases} 0 & \text{for } (s,t) \neq (m,n), \\ \{\delta[\alpha_1(c_m)\dots\alpha_\mu(c_m)]\}\,\beta_1(z_n)\dots\beta_\nu(z_n) & \text{for } (s,t) = (m,n). \end{cases}$$

Using the facts that ∇ is additive and that $H^{n+i}(K_n)$ is a finite dimensional vector space and that it is generated by products $\hat{\alpha}_1(\{z_n\})\,\hat{\alpha}_2(\{z_n\})\dots\hat{\alpha}_\nu(\{z_n\})$, we get the following lemma.

LEMMA 2.3. *For any system $\{Z_{s,t}\}$ of cohomology classes in $H^{s+t+i}((L_s, K_s) \times (K_t, *))$ there is an operation G of degree $i-1$ with ∇G special such that*

$$\{\nabla G(c_s, z_t)\} = Z_{s,t}$$

for each (s,t).

To prove the lemma we of course have to add together a number of operations G_1 and G_2. Note that although we must add an infinite number

of operations, only a finite number of them are non zero in a particular dimension (m,n).

Next, we give examples of operations G with $\nabla G(c_s,z_t)=0$. Let an operation $G_3=G_3{}^n$ be defined by

$$G_3(c_s,c_t) = \begin{cases} \alpha(c_s)\,\beta(c_t) & \text{for } t>n\,, \\ 0 & \text{for } t\leqq n\,, \end{cases} \tag{2.7}$$

where $\alpha,\beta\in Z(\mathcal{O})$. Then $\nabla G_3(c_s,z_t)=0$ for all (s,t). From (2.7) we get

$$(\nabla G_3)(z_s,c_t) = \begin{cases} 0 & \text{for } t\neq n\,, \\ \alpha(z_s)\,\beta(\delta c_n) & \text{for } t=n\,. \end{cases} \tag{2.8}$$

Since $\nabla G_3(z_s,z_t)=0$ we see that $(\nabla G_3)(z_s,c_t)$ is a cocycle in $(K_s,*)\times(L_t,K_t)$. The cohomology class determined by $\nabla G_3(z_s,c_t)$ in $(K_s,*)\times(L_t,K_t)$ can be read off from (2.8).

Let G_4 be defined by

$$G_4(c_s,c_t) = \begin{cases} \alpha(c_s)\,\beta_1(c_n)\ldots\beta_\nu(c_n) & \text{for } t=n\,, \\ 0 & \text{for } t\neq n\,, \end{cases}$$

where $\alpha,\beta_1,\ldots,\beta_\nu\in Z(\mathcal{O})$, $\nu>1$. Then $\nabla G_4(c_s,z_t)=0$ for all (s,t), and the cohomology classes in $(K_s,*)\times(L_t,K_t)$ determined by $\nabla G_4(z_s,c_t)$ are

$$\{\nabla G_4(z_s,c_t)\} = \begin{cases} \{\alpha(z_s)\,\delta[\beta_1(c_n)\ldots\beta_\nu(c_n)]\} & \text{for } t=n\,, \\ 0 & \text{for } t\neq n\,. \end{cases}$$

As above, we collect this information in a lemma. First, let us introduce the mappings

$$\mu\colon\quad A\otimes H^*(L_t,K_t)\to H^*(K_s)\otimes H^*(L_t,K_t)\cong H^*\big((K_s,*)\times(L_t,K_t)\big)$$

which lets the first factor act on the basic class in $(K_s,*)$.

LEMMA 2.4. *Let Z_n be a sequence of elements in $A\otimes H^*(L_n,K_n)$, $\deg Z_n=n+i$, with usual grading of $A\otimes H^*(L_n,K_n)$. Then there is an operation G of degree $i-1$ such that, for each pair (s,t), $\nabla G(c_s,z_t)=0$ and*

$$\mu(Z_t) = \{\nabla G(z_s,c_t)\} \in H^*\big((K_s,*)\times(L_t,K_t)\big)\,.$$

Finally, we shall give examples of operations G with ∇G very special, i.e. with $\nabla G(z_s,c_t)=0$, $\nabla G(c_s,z_t)=0$ for all (s,t):

$$G_5(c_s,c_t) = \begin{cases} 0 & \text{for } t\neq n \text{ and for } t=n,\ s\leqq m\,, \\ \alpha(c_s)\,\delta[\beta_1(c_n)\ldots\beta_\nu(c_n)] & \text{for } t=n \text{ and } s>m\,, \end{cases}$$

$$G_6(s_s,c_t) = \begin{cases} 0 & \text{for } s\neq m \text{ and for } s=m,\ t\leqq n\,, \\ \delta[\alpha_1(c_m)\ldots\alpha_\lambda(c_m)]\,\beta(c_t) & \text{for } s=m \text{ and } t>n\,, \end{cases}$$

$$G_7(c_s,c_t) = \begin{cases} 0 & \text{for } (s,t) \neq (m,n)\,, \\ \alpha_1(c_m)\dots\alpha_\lambda(c_m)\,\delta[\beta_1(c_n)\dots\beta_\nu(c_n)] & \text{for } (s,t)=(m,n)\,, \end{cases}$$

where $\lambda > 1$ and $\nu > 1$ in the case of G_7. One easily sees that ∇G_j, $j = 5,6,7$, is very special. It follows that $\nabla G_j(c_s,c_t)$ is a cocycle in $(L_s,K_s)\times(L_t,K_t)$. The cohomology classes determined by $\nabla G_j(c_j,c_t)$ in the three cases are $\{\nabla G_j(c_s,c_t)\} = 0$ for $(s,t) \neq (m,n)$ and

$$\begin{aligned} \{\nabla G_5(c_m,c_n)\} &= \{\alpha(\delta c_m)\,\delta[\beta_1(c_n)\dots\beta_\nu(c_n)]\}\,, \\ \{\nabla G_6(c_m,c_n)\} &= \{\delta[\alpha_1(c_m)\dots\alpha_\lambda(c_m)]\,\beta(\delta c_n)\}\,, \\ \{\nabla G_7(c_m,c_n)\} &= \{\delta[\alpha_1(c_m)\dots\alpha_\lambda(c_m)]\,\delta[\beta_1(c_n)\dots\beta_\lambda(c_n)]\}\,. \end{aligned}$$

As before, one derives a lemma.

LEMMA 2.5. *Let $\{Z_{m,n}\}$ be a system of cohomology classes,*

$$Z_{m,n} \in H^{m+n+i}\big((L_m,K_m)\times(L_n,K_n)\big)\,.$$

Then there is an operation G of degree $i-1$ such that

$$\{\nabla G(c_m,c_n)\} = Z_{m,n} \in H^*\big((L_m,K_m)\times(L_n,K_n)\big)$$

for all (m,n).

Now, let us continue the proof of the theorem. We are given F with $\nabla F = 0$ and $\varepsilon(F) = 0$, and we must determine a G with $\nabla G = F$.

Since $\varepsilon(F) = 0$, it follows that for each (m,n)

$$\{F(z_m,z_n)\} = 0 \in H^*(K_m\times K_n, K_m \vee K_n)\,.$$

It follows that there are cochains $\alpha_{m,n}$ in $(K_m\times K_n, K_m\vee K_n)$ with

$$\delta\alpha_{m,n} = F(z_m,z_n) \quad \forall (m,n)\,.$$

Let $\beta_{m,n}$ be a cochain in $(L_m\times L_n, L_m\vee L_n)$ which restricts to $\alpha_{m,n}$ in $(K_m\times K_n, K_m\vee K_n)$. Let H be the cochain operation associated with this system of cochains. Then the cochain operation $F-\nabla H$ is special, that is, $F(z_m,z_n)-\nabla H(z_m,z_n) = 0$ for each pair (m,n). This means that to prove the theorem it is enough to show that special operations in the kernel of ε are in the image of ∇. Therefore, in what follows we shall assume that $\nabla F = 0$ and that F is special (which implies that $\varepsilon(F) = 0$).

For each pair (m,n) the cochain $F(c_m,z_n)$ in $L_m\times K_n$ restricts to zero in $K_m\times K_n$ (since $F(z_m,z_n) = 0$). Furthermore, we have that $F(c_m,z_n)$ is a cocycle in $(L_m,K_m)\times(K_n,*)$. This means that F determines a double-sequence of cohomology classes

$$\{F(c_m,z_n)\} \in H^*\big((L_m,K_m)\times(K_n,*)\big)\,.$$

By Lemma 2.3 there is a cochain operation H with ∇H special and with

$$\{\nabla H(c_m,z_n)\} = \{F(c_m,z_n)\} \quad \forall(m,n) .$$

From this we see that besides $\nabla F = 0$ and F special, we can assume that

$$\{F(c_m,z_n)\} = 0 \in H^*\big((L_m,K_m)\times(K_n,*)\big) \quad \forall(m,n) .$$

Let, for all pairs (m,n), $\alpha_{m,n}$ be a cochain in $(L_m,K_m)\times(K_n,*)$ with

$$\delta\alpha_{m,n} = F(c_m,z_n) .$$

Let $H(c_m,c_n)$ be a cochain in $L_m\times L_n$ which restricts to $\alpha_{m,n}$ in $L_m\times K_n$ and to zero in L_n. The cochain operation H defined by this has

$$(\nabla H)(c_m,z_n) = \delta\alpha_{m,n}+(p\times 1)^{\#}(0) = F(c_m,z_n) .$$

This means that in what follows we can assume that $\nabla F = 0$ and $F(c_m,z_n)=0$ for all pairs (m,n). We see that $F(z_m,c_n)$ is a cocycle in $(K_m,*)\times(L_n,K_n)$. Since

$$\delta F(c_m,c_n) = F(\delta c_m,c_n) = (p\times 1)^{\#}F(z_{m+1},c_n) ,$$

it follows that

$$(\sigma\otimes 1)\{F(z_{m+1},c_n)\} = \{F(z_m,c_n)\} \in H^*\big((K_m,*)\times(L_n,K_n)\big) .$$

It follows that there is a $Z_n \in A\otimes H^*\big((L_n,K_n)\big)$ such that

$$\mu(Z_n) = \{F(z_m,c_n)\} \in H^*\big((K_m,*)\times(L_n,K_n)\big) \quad \forall m$$

(see Lemma 2.4). By Lemma 2.4 there is a G such that

$$\{(F-\nabla G)(z_m,c_n)\} = 0 \in H^*\big((K_m,*)\times(L_n,K_n)\big) \quad \forall(m,n) .$$

Therefore, in what follows we can assume that F itself has the properties

$$\nabla F = 0, \qquad F(c_m,z_n) = 0, \qquad F(z_m,c_n) \sim 0 ,$$

in $(K_m,*)\times(L_n,K_n)$. Let $\alpha_{m,n}$ be a cochain in $(K_m,*)\times(L_n,K_n)$ with $\delta\alpha_{m,n}=F(z_m,c_n)$. We extend $\alpha_{m,n}$ to a cochain $H(c_m,c_n)$ in $(L_m,*)\times(L_n,K_n)$. Then

$$\nabla H(c_m,z_n) = 0 \qquad \text{and} \qquad \nabla H(z_m,c_n) = \delta\alpha_{m,n} = F(z_m,c_n) .$$

It follows that $F-\nabla H$ is very special. Therefore, in what follows we shall assume F itself to be very special. In that case $F(c_m,c_n)$ is a cocycle in $(L_m,K_m)\times(L_n,K_n)$. By Lemma 2.5 there is a cochain operation G such that $F-\nabla G$ is very special and $(F-\nabla G)(c_m,c_n)\sim 0$ in $(L_m,K_m)\times(L_n,K_n)$ for each pair (m,n).

Now we see that to prove Theorem 2.2 we only have to show that, if

F is very special, $\nabla F = 0$ and $F(c_m, c_n) \sim 0$ in $(L_m, K_m) \times (L_n, K_n)$ for all (m,n), then F is in the image of ∇. To see this, let $a_{m,n}$ be a cochain in $(L_m, K_m) \times (L_n, K_n)$ with $\delta a_{m,n} = F(c_m, c_n)$. Define $G(c_m, c_n) = a_{m,n}$. Then $\nabla G = F$. This proves the theorem.

3. Some lemmas.

As earlier, let A denote the mod 2 Steenrod algebra. It is well known that A is a Hopf algebra with diagonal ψ: $A \to A \otimes A$. If $\hat{a} \in A$ and $\psi(\hat{a}) = \sum \hat{a}' \otimes \hat{a}''$, then for each pair u, v of cohomology classes

$$\hat{a}(uv) = \sum \hat{a}'(u)\, \hat{a}''(v) . \tag{3.1}$$

The Cartan formula for primary cohomology operations can then be expressed by

$$\psi(Sq^k) = \sum Sq^i \otimes Sq^j, \qquad i + j = k .$$

In what follows we shall be working both with cohomology operations and with cochain operations. We shall use letters a, b, c etc. to denote cochain operations in one variable and commuting with δ. If a is a cochain operation, then $\hat{a}$ is the corresponding cohomology operation.

First, let us consider the equation (3.1) on the level of cochain operations.

Let a, a' and a'' be cochain operations representing $\hat{a}$, $\hat{a}'$ and $\hat{a}''$ respectively. Consider the cochain operation B of the second kind defined by

$$B(x,y) = a(xy) + \sum a'(x)\, a''(y) + d(a; \delta x y, x \delta y) + \\ + n(x)\, d(a; x \delta y, x \delta y) , \tag{3.2}$$

where $n(x)$ is one of the functions $n(x) = \deg(x)$ or $n(x) = \deg(x) + 1$, and d is a cochain operation (of the first kind) in two variables defined in [3]. The essential property of d is that it gives the deviation from additivity of the cochain operation a:

$$a(u+v) + a(u) + a(v) = \delta d(a; u, v) + d(a; \delta u, \delta v) .$$

For t variables $x_1, \ldots, x_t$ we have the property

$$d(a; x_1, \ldots, x_t) + d(a; \delta x_1, \ldots, \delta x_t) = a(x_1 + \ldots + x_t) + \sum_{j=1} a(x_j) . \tag{3.3}$$

The existence of d can be derived from the exact sequence in Theorem 2.1. The right hand side of (3.3) defines an element of $Z(\mathcal{O}^{(t)})$ in the kernel of ε. Hence there is $d'(a) \in \mathcal{O}^{(t)}$ with

$$(\Delta d'(a))(x_1, \ldots, x_t) = a(\sum x_j) + \sum_j a(x_j) .$$

Putting

$$d(a,x_1,\ldots,x_t) = d'(a)(x_1,\ldots,x_t) + \sum_j d'(a)(0,\ldots,0,x_j,0,\ldots,0)\,,$$

d satisfies (3.3) and has the property

$$d(a;0,\ldots,0,x_j,\ldots,0) = 0\,.$$

This property we shall make use of later in this paper.

It is easy to see that B is a cochain operation of the second kind and that $\nabla B = 0$. We can therefore ask about the image of B under the mapping $\varepsilon\colon Z(Q) \to A \otimes A$, discussed in Section 2. We can determine $\varepsilon(B)$ by evaluating B on pairs (x,y) of cocycles. Since by (3.1)

$$B(x,y) = a(xy) + \sum a'(x)\,a''(y) \sim 0\,,$$

we see that $\varepsilon(B) = 0$. By Theorem 2.2 there is a cochain operation $T \in Q$ with

$$\nabla T = B\,.$$

We formulate this in the following lemma.

LEMMA 3.1. *Let $\hat{a} \in A$ and let $\psi(\hat{a}) = \sum \hat{a}' \otimes \hat{a}''$. Let a, a' and a'' be cochain operations representing $\hat{a}$, $\hat{a}'$ and $\hat{a}''$ respectively. Then there is a cochain operation T of the second kind with*

$$(\nabla T)(x,y) = a(xy) + \sum a'(x)a''(y) + d(a;\delta x y, x\delta y) + \\ + n(x)\,d(a;x\delta y, x\delta y)$$

for each pair (x,y) of cochains.

For the next lemma we need the following setup which shall also be used in Section 4 of this paper. This algebraic machinery was used by Adams [1] in his description of secondary cohomology operations.

Let C_0 be a free (graded) A-module on one generator c_0, and let C_1 be a free A-module on a finite set of generators $\{c_1{}^i \mid i \in I\}$. Let a mapping

$$d\colon\ C_1 \to C_0 \tag{3.4}$$

of graded A-modules be given. We recall that a secondary cohomology operation is determined by an element of the kernel of d.

Let us assume that besides the mapping d of (3.4) we have in the same fashion two other mappings

$$d'\colon\ C_1' \to C_0', \qquad d''\colon\ C_1'' \to C_0'' \tag{3.5}$$

such that $\deg c_0 = \deg c_0' + \deg c_0''$. The tensor product $C_0' \otimes C_0''$ (over Z_2) is an $A \otimes A$-module. Using the diagonal mapping in A, we obtain an

induced A-module structure in $C_0'\otimes C_0''$. There is a unique mapping $\psi_0\colon C_0 \to C_0'\otimes C_0''$ with the property $\psi(c_0)=c_0'\otimes c_0''$. Assume that there is a mapping

$$\psi_1\colon\ C_1 \to C_0'\otimes C_1'' + C_1'\otimes C_0''$$

such that the following diagram is commutative:

$$\begin{array}{ccc} C_1 & \xrightarrow{\ d\ } & C_0 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} \\ C_0'\otimes C_1''\oplus C_1'\otimes C_0'' & \xrightarrow{\ d^{\otimes}\ } & C_0'\otimes C_0'' \end{array} \tag{3.6}$$

where $d\otimes = 1\otimes d'' + d'\otimes 1$. Let us consider an element $z\in \mathrm{Ker}(d)$ such that

$$\psi_1(z) = \Sigma\hat{\alpha}_m' c_0'\otimes z_m'' + \Sigma z_n'\otimes\hat{\alpha}_n'' c_o'' , \tag{3.7}$$

where $z_n'\in \mathrm{Ker}(d')$, $z_m''\in \mathrm{Ker}(d'')$ and $\alpha_m', \alpha_n''\in A$. Let

$$z = \Sigma\hat{a}_i {c_1}^i, \qquad \hat{a}_i\in A ,$$

and let a_i be a cochain operation representing $\hat{a}_i$. Also, let

$$d({c_1}^i) = \hat{b}_i c_0, \qquad \hat{b}_i = \{b_i\}\in A ,$$

where $b_i\in Z(\mathcal{O})$. Then (by Theorem 2.1) there is a cochain operation θ with

$$\Delta\theta = \Sigma a_i b_i = r .$$

In a similar fashion we put

$$d'(c_1'^j) = \hat{b}_j' c_o', \qquad d''(c_1''^k) = \hat{b}_k'' c_0'' ,$$

where $\{c_1'^j \mid j\in J\}$ and $\{c_1''^k \mid k\in K\}$ are generators for C_1' and C_1'' respectively, and $\hat{b}_j' = \{b_j'\}\in A$, $\hat{b}_k'' = \{b_k''\}\in A$. Also,

$$z_n' = \Sigma\hat{a}_{nj}' c_1'^j, \qquad z_m'' = \Sigma\hat{a}_{mk}'' c_1''^k .$$

Corresponding to these cycles we have

$$\Delta\theta_n' = \Sigma a_{nj}' b_j' = r_n', \qquad \Delta\theta_m'' = \Sigma a_{mk}'' b_k'' = r_m'' .$$

Finally, we need notation for $\psi_1\colon C_1 \to C_0'\otimes C_1''\oplus C_1'\otimes C_0''$. Let us put

$$\psi_1({c_1}^i) = \Sigma\hat{e}_{ik}' c_0'\otimes\hat{f}_{ik}'' c_1''^k + \Sigma\hat{f}_{ij}' c_1'^j\otimes\hat{e}_{ij}'' c_0'' .$$

Let us consider the cochain operation H defined by

$$\begin{aligned} H(x,y) = {} & \theta(\delta x y + x\delta y) + \theta(\delta x y) + \theta(x\delta y) + \\ & + d(r;\delta x y, x\delta y) + n(x)\, d(r;x\delta y, x\delta y) . \end{aligned} \tag{3.8}$$

Then $\nabla H = 0$ and $\varepsilon(H) = 0$ as is easily seen. Hence there is a cochain operation $D = D(\theta)$ with

$$\nabla D = H \,. \tag{3.9}$$

Let us consider a cochain operation E defined by

$$E(x,y) = \theta(xy) + \Sigma\theta'_n(x)\,\alpha''_n(y) + \Sigma\alpha'_m(x)\,\theta''_m(y) + T(r)(x,y) + D(\theta)(x,y)\,, \tag{3.10}$$

where $T(r)$ is a cochain operation given by Lemma 3.1 with a replaced by r, and

$$\psi(\hat{r}) = \Sigma\hat{r}'_n \otimes \hat{\alpha}''_n + \Sigma\hat{\alpha}'_m \otimes \hat{r}''_m$$

($\hat{r}$, $\hat{r}'_n$ and $\hat{r}''_m$ are equal to zero, but the cochain operations r etc. are usually not zero). Since

$$\nabla T(r)(x,y) = r(xy) + \Sigma r'_n(x)\,\alpha''_n(y) + \Sigma\alpha'_m(x)\,r''_m(y) + d(r;\,\delta x y, x\delta y) + n(x)d(r;\,x\delta y, x\delta y)\,, \tag{3.11}$$

it follows that $\nabla E = 0$. Hence there are cochain operations β'_j and β''_j in $Z(\mathcal{O})$ such that

$$\varepsilon(E) = \Sigma\hat{\beta}'_j \otimes \hat{\beta}''_j \,.$$

Applying Theorem 2.3 we get

LEMMA 3.2. *Let notation be as above. There are cochain operations β'_j and β''_j in $Z(\mathcal{O})$ such that for each pair (x,y) of cocycles*

$$\theta(xy) + \Sigma\theta'_n(x)\alpha''_n(y) + \Sigma\alpha'_m(x)\theta''_m(y) + D(\theta)(x,y) + T(r)(x,y) \sim \Sigma\beta'_j(x)\,\beta''_j(y)\,.$$

This lemma is designed for proving the Cartan formula for secondary cohomology operations. This will be done in Section 4.

4. The Cartan formula.

In this section we shall prove a Cartan formula for secondary operations. This formula gives an expansion of $\Phi(xy)$, where Φ is a secondary operation and x and y are cohomology classes, in a sum of terms $\Phi'(x)\,\Phi''(y)$, where Φ' and Φ'' are cohomology operations. This formula was first treated by Adams [1]. He proved the existence of such an expansion but did not give the actual form of the terms Φ' and Φ''. Later Adem [2] proved that for certain Φ we have $\Phi(xy) = \Phi(x)\,y + x\,\Phi(y)$.

THEOREM 4.1. *Let d, d' and d'' be A-module mappings as described in* (3.4) *and* (3.5) *and, as described by* (3.6), *let a commutative diagram*

$$\begin{array}{ccc} C_1 & \xrightarrow{d} & C_0 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} \\ C'_0\otimes C''_1 \oplus C'_1\otimes C''_0 & \xrightarrow{d^{\otimes}} & C'_0\otimes C''_0 \end{array}$$

be given. Let $z \in \operatorname{Ker}(d)$ *such that*

$$\psi_1(z) = \Sigma \hat{\alpha}'_m c'_0 \oplus z''_m + \Sigma z'_n \oplus \hat{\alpha}''_n c''_o$$

with $z'_n \in \operatorname{Ker} d'$ *and* $z''_m \in \operatorname{Ker} d''$. *Then there are primary operations* $\hat{\beta}'_k$, $\hat{\beta}''_k$ *such that modulo the total indeterminacy*

$$\Phi(\hat{x}\,\hat{y}) = \Sigma \Phi'_n(\hat{x})\hat{\alpha}''_n(\hat{y}) + \Sigma \hat{\alpha}'_m(\hat{x})\Phi''_m(\hat{y}) + \Sigma \hat{\beta}'_k(\hat{x})\hat{\beta}''_k(\hat{y}),$$

where Φ, Φ'_n *and* Φ''_m *are secondary operations associated with* z, z'_n *and* z''_m *respectively, and where* $\hat{x}$ *and* $\hat{y}$ *are arbitrary cohomology classes satisfying that* d'*-operations are defined on* $\hat{x}$ *and* d''*-operations are defined on* $\hat{y}$.

REMARK. Before the proof, let us remark, that in [3] we defined secondary operations associated with relations $\sum \alpha_\nu a_\nu + b = 0$ in A containing an unfactorized term b. These operations are only defined in dimensions less than the excess of b. Hence they are not stable in the usual sense. They are, however, very close to being stable. In the statement of Theorem 4.1 we have only considered stable operations. Minor additions to the proof of Theorem 4.1, however, give a Cartan formula for the almost stable operations mentioned. I believe that in certain applications it will be useful (if not necessary) to take the almost stable operations into consideration.

We hope that the unknown stable primary operations $\hat{\beta}'_k$ and $\hat{\beta}''_k$ can be determined by methods similar to the ones used in this paper.

A simple example where the theorem applies is as follows. Let

$$p_i = Sq^{(0,0,\ldots,1)}, \qquad i-1 \text{ zeroes},$$

be an element in the Milnor basis for the Steenrod algebra. Then

$$p_i p_j + p_j p_i = 0,$$
$$\Phi(p_i) = 1\otimes p_i + p_i \otimes 1.$$

Let $i \neq j$. Let C_1 be generated by ${c_1}^i$ and ${c_1}^j$, and C_0 by c_0. The mapping

$$d\colon\ C_1 \to C_0$$

is given by $d({c_1}^i) = p_i c_0$ and $d({c_1}^j) = p_j c_0$. Then

$$\psi_1\colon\ C_1 \to C_0\otimes C_1 \oplus C_1 \otimes C_0$$

is given by

$$\psi_1(c_1{}^k) = c_0 \otimes c_1{}^k + c_1{}^k \otimes c_0, \qquad k = i \text{ or } j .$$

The element $z = p_i c_1{}^j + p_j c_1{}^i$ determines the operation Φ. Since

$$\psi_1(z) = z \otimes c_0 + c_0 \otimes z$$

the theorem applies.

PROOF OF THEOREM 4.1. We shall use the algebraic setup from Section 3. If $\hat{x} = \{x\}$ and $\hat{y} = \{y\}$, then $b'_j(x) \sim 0$ and $b''_k(y) \sim 0$. Let us choose bounding cochains

$$\delta w'_j = b'_j(x), \qquad \delta w''_k = b''_k(y) .$$

Since $\psi_1 \colon C_1 \to C'_0 \otimes C''_1 \oplus C'_1 \otimes C''_0$ is given by

$$\psi_1(c_1{}^i) = \Sigma \hat{e}'_{ik} c'_0 \otimes \hat{f}''_{ik} c''_1{}^k + \Sigma \hat{f}'_{ij} c'_1{}^j \otimes \hat{e}''_{ij} c''_0 ,$$

we get

$$\hat{b}_i(\hat{x}\hat{y}) = \Sigma \hat{e}'_{ik}(\hat{x}) \hat{f}''_{ik} \hat{b}''_k(\hat{y}) + \Sigma \hat{f}'_{ij} \hat{b}'_j(\hat{x}) \hat{e}''_{ij}(\hat{y}) = 0 .$$

On the cochain level we get

$$\delta w_i = b_i(xy)$$

with w_i given by

$$w_i = \Sigma e'_{ik}(x) f''_{ik}(w''_k) + \Sigma f'_{ij}(w'_j) e''_{ij}(y) + T_i(x,y) , \tag{4.1}$$

where T_i is a cochain operation given by Lemma 3.1,

$$\begin{aligned} \nabla T_i(u,v) = b_i(uv) + \Sigma e'_{ik}(u) f''_{ik} b''_k(v) + \Sigma f'_{ij} b'_j(u) e''_{ij}(v) + \\ + d(b_i; \delta u\, v, u\, \delta v) + n(u) d(b_i; u\, \delta v, u\, \delta v) . \end{aligned} \tag{4.2}$$

Since $\Delta\theta = \Sigma a_i b_i$, a cocycle representing $\Phi(\hat{x}\hat{y})$ is given by

$$\theta(xy) + \Sigma a_i(w_i) .$$

To prove the theorem we must examine the expression

$$\begin{aligned} Z = \theta(xy) + \Sigma a_i(w_i) + \Sigma_n [\theta'_n(x) + \Sigma a'_{nj}(w'_j)] \alpha''_n(y) + \\ + \Sigma_m \alpha'_m(x) [\theta''_m(y) + \Sigma a''_{mk}(w''_k)] . \end{aligned}$$

By Lemma 3.2

$$\begin{aligned} Z \sim \Sigma a_i(w_i) + \Sigma a'_{nj}(w'_j) \alpha''_n(y) + \Sigma \alpha'_m(x) a''_{mk}(w''_k) + \\ + D(\theta)(x,y) + T(r)(x,y) + \Sigma \beta'(x) \beta''(y) . \end{aligned} \tag{4.3}$$

By the expression (4.1) for w_i we get

$$(4.4)\qquad a_i(w_i) \sim \sum a_i^{(1)} e'_{ik}(x)\, a_i^{(2)} f''_{ik}(w''_k) + \\ + \sum a_i^{(1)} f'_{ij}(w'_j)\, a_i^{(2)} e''_{ij}(y) + a_i T_i(x,y) + D_i(x,y) + \\ + \sum S_i\big(e'_{ik}(x), f''_{ik} b''_k(y)\big) + \sum S_i\big(f'_{ij} b'_j(x), e''_{ij}(y)\big)\,,$$

where $\Psi(\hat{a}_i) = \sum \hat{a}_i^{(1)} \otimes \hat{a}_i^{(2)}$ and S_i the corresponding cochain operation given by Lemma 3.1,

$$(4.5)\qquad \nabla S_i(u,v) = a_i(uv) - \sum a_i^{(1)}(u)\, a_i^{(2)}(v) + d(a_i;\, \delta u\, v, u\, \delta v) + \\ + n(u) d(a_i;\, u\,\delta v, u\,\delta v)\,.$$

Also,

$$D_i(x,y) = d\big(a_i;\, \ldots, e'_{ik}(x) f''_{ik} b''_k(y), \ldots, f'_{ij} b'_j(x)\, e''_{ij}(y), \ldots \\ \ldots, b_i(xy) + \sum e'_{ik}(x) f''_{ik} b''_k(y) + \sum f'_{ij} b'_j(x)\, e''_{ij}(y)\big)\,,$$

where $d(a_i; \ldots)$ is deviation from additivity of a_i as defined in Section 3.

Next, we shall compare some of the terms in (4.3) and (4.4). It is clear that

$$\sum \hat{a}_i^{(1)} \hat{e}'_{ik} c'_0 \otimes \hat{a}_i^{(2)} \hat{f}''_{ik} c''^k_1 = \sum \hat{\alpha}'_m c'_0 \otimes \hat{a}''_{mk} c''^k_1\,.$$

If we keep k fixed, this equation is still true. Hence, for each k there is a cochain operation U_k such that

$$(4.6)\qquad (\nabla U_k)(u,v) = \sum a_i^{(1)} e'_{ik}(u)\, a_i^{(2)} f''_{ik}(v) - \sum \alpha'_m(u)\, a''_{mk}(v), \qquad k \text{ fixed}\,,$$

where (u,v) is an arbitrary pair of cochains. Similarly,

$$(4.7)\qquad (\nabla V_j)(u,v) = \sum a_i^{(1)} f'_{ij}(u)\, a_i^{(2)} e''_{ij}(v) - \sum a'_{nj}(u)\, \alpha''_n(v), \qquad j \text{ fixed}\,.$$

This and (4.4) simplify (4.3). We get

$$(4.8)\qquad Z \sim K(x,y)\,,$$

where

$$(4.9)\qquad K(x,y) = \sum U_k\big(x, b''_k(y)\big) + \sum V_j\big(b'_j(x), y\big) + \sum a_i T_i(x,y) + \\ + D(\theta)(x,y) + T(r)(x,y) + \sum S_i\big(e'_{ik}(x), f''_{ik} b''_k(y)\big) + \\ + \sum S_i\big(f'_{ij} b'_j(x), e''_{ij}(y)\big) + \sum D_i(x,y) + \sum \beta'(x)\beta''(y) + \\ + \sum d\big(a_i;\, \delta T_i(x,y), T_i(\delta x, y), T_i(x, \delta y)\big) + \\ + d\big(a_i;\, \nabla T_i(x,y), d(b_i;\, \delta x\, y, x\,\delta y), n(x) d(b_i;\, x\,\delta y, x\,\delta y)\big).$$

The terms $D(\theta)$ and $T(r)$ are defined in (3.9) and (3.11), respectively. The last two terms we have added for convenience. They are, of course, zero when x and y are cocycles.

It will be convenient to consider x and y as variable cochains. Then the expression (4.9) defines a cochain operation K in Q. We shall examine ∇K. A straightforward compution shows that ∇K is a sum of two terms: a very special operation (i.e. an operation which is zero on (x,y)

if $\delta x = 0$ or $\delta y = 0$) and the following special operation (i.e. zero on (x,y) if $\delta x = 0$ and $\delta y = 0$):

$$n(x)\,[\sum a_i \varkappa(b_i)(x\,\delta y) + \sum \varkappa(a_i)\big(e'_{ik}(x)\,f''_{ik}b''_k(\delta y)\big) + \sum \varkappa(a_i)\,(f'_{ij}\,b'_j(x)\,e''_i(\delta y))]\,,$$

where we have denoted the deviation from additivity $d(a;z,z)$ by $\varkappa(a)(z)$ as it was done in [3], and $n(x) = \deg(x)$. It is now very easy to complete the proof of the theorem. Before doing so we shall, however, give a few relevant remarks about $\varkappa$. These things are proved in the following section. The reason for treating $\varkappa$ in a special section is the feeling that $\varkappa$ is of general interest not only in usual cohomology theory but also in connection with general cohomology theories based on (general) cochain functors.

The defining property of the deviation from additivity at once gives that

$$\delta\varkappa(a)(z) = \varkappa(a)(\delta z)\,,$$

or $\varkappa(a) \in Z(\mathcal{O})$. We therefore get an element $\varepsilon\big(\varkappa(a)\big) \in A$. It turns out that $\varepsilon\big(\varkappa(a)\big)$ depends only on $\varepsilon(a)$ and not on the choices of representatives a for $\varepsilon(a)$ and derivation $d(a;-,-)$. Hence we have a mapping $\hat{\varkappa}\colon A \to A$,

$$\hat{\varkappa}(\hat{a}) = \varepsilon\big(\varkappa(a)\big) \in A\,,$$

where $\hat{a} = \varepsilon(a)$. Furthermore, it follows that $\hat{\varkappa}$ is a derivation in A of degree -1

$$\hat{\varkappa}(\hat{a}\hat{b}) = \hat{\varkappa}(\hat{a})\,\hat{b} + \hat{a}\,\hat{\varkappa}(\hat{b})\,,$$

and that

$$\hat{\varkappa}(Sq^i) = Sq^{i-1}, \qquad i = 0,1,\ldots\,.$$

This last property is not needed in the proof of the theorem, but is added here just to show that $\hat{\varkappa}$ is not zero.

To complete the proof of Theorem 4.1 we must determine K on a pair of cocycles. Since ∇K is special, that is, $(\nabla K)(x,y) = 0$ if $\delta x = 0$ and $\delta y = 0$, it follows that $K(x,y)$ is a cocycle if $\delta x = 0$ and $\delta y = 0$. The $K(z_s,z_t)$ therefore determines an element in $H^*\big(K(Z_2,s) \times K(Z_2,t)\big)$ for each pair (s,t). As in Section 2 when we defined the mapping ε on very special operations, we shall here compare the classes $\{K(z_s,z_t)\}$ for different pairs (s,t). This, however, is quite easy. First, let us compare $\{K(z_s,z_t)\}$ and $\{K(z_{s+1},z_t)\}$. Since $(\nabla K)(c_s,z_t) = 0$ in $L_s \times K_t$, we have

$$\delta K(c_s,z_t) = (p \times 1)^{\#} K(z_{s+1},z_t)\,,$$

where p is the projection $L(Z_2,s) \to K(Z_2,s+1)$. It follows that

$$\{K(z_s,z_t)\} = (\sigma \otimes 1)\{K(z_{s+1},z_t)\}\,,$$

where σ is the cohomology suspension $H^n(X) \to H^{n-1}(\Omega X)$. To compare $\{K(z_s, z_{t+1})\}$ and $\{K(z_s, z_{t+1})\}$ we note that

$$\begin{aligned}(\nabla K)(z_s, c_t) &= s[\sum a_i \varkappa(b_i)(z_s \delta c_t) + \\ &\quad + \sum \varkappa(a_i)\big(e'_{ik}(z_s)\, f''_{ik} b''_k(\delta c_t)\big) + \sum \varkappa(a_i)\big(f'_{ij} b'_j(z_s)\, e''_i(\delta c_t)\big)]\end{aligned}$$

or

$$\begin{aligned}\delta[K(z_s, c_t) &+ s[\sum a_i \varkappa(b_i)(z_s c_t) + \sum \varkappa(a_i)\big(e'_{ik}(z_s)\, f''_{ik} b''_k(c_t)\big) + \\ &\quad + \sum \varkappa(a_i)\big(f'_{ij}\, b'_j(z_s)\, (e''_i(c_t))\big)] = (1 \times p)^{\#} K(z_s, z_{t+1}) .\end{aligned}$$

It follows that

$$\begin{aligned}(1 \otimes \sigma)\{K(z_s, z_{t+1})\} &= \{K(z_s, z_t)\} + s[\sum \hat{a}_i \hat{\varkappa}(b_i)\{z_s z_t\} + \\ &\quad + \sum \hat{\varkappa}(\hat{a}_i)(\hat{e}'_{ik})(\{z_s\})\, \hat{f}''_{ik} \hat{b}''_k(\{z_t\}) + \sum \hat{\varkappa}(\hat{a}_i)(\hat{f}'_{ij}\, \hat{b}'_j(\{z_s\}) \hat{e}''_i(\{z_t\})] \\ &= \{K(z_s, z_t)\} + s[\sum \hat{a}_i \hat{\varkappa}(\hat{b}_i)(\{z_s z_t\}) + \sum \hat{\varkappa}(\hat{a}_i)(\hat{b}_i\{z_s z_t\})] \\ &= \{K(z_s, z_t)\} + s\, \hat{\varkappa}(\sum \hat{a}_i \hat{b}_i)(\{z_s z_t\}) \\ &= \{K(z_s, z_t)\} .\end{aligned}$$

This follows from the properties of $\varkappa$ and the fact that $\sum \hat{a}_i \hat{b}_i = 0$. Now, of course, it follows easily that there are cochain operations β'_k and β''_k in $Z(\mathcal{O})$ such that for each pair (x, y) of cocycles $K(x, y) \sim \sum \beta'_k(x)\, \beta''_k(y)$. This completes the proof of the theorem.

5. A derivation in the Steenrod algebra.

As in Section 4, let $a \in Z(\mathcal{O})$ and $d(a; -, -)$ be an operation giving the deviation from additivity of a, satisfying $d(a; x, 0) = 0$ and $d(a; 0, y) = 0$. Then, as before, a cochain operation $\varkappa(a, d) \in Z(\mathcal{O})$ is defined by $\varkappa(a, d)(x) = d(a; x, x)$. We shall show that $\varepsilon\big(\varkappa(a, d)\big) \in A$ depends only on $\varepsilon(a)$. Let $d'(a; x, y)$ be another deviation from additivity associated with a. Then

$$D(x, y) = d(a; x, y) - d'(a; x, y)$$

belongs to $Z(\mathcal{O}^{(2)})$. Since $D(x, 0) = D(0, y) = 0$, it follows that $\varepsilon(D) = 0 \in A \oplus A$. Therefore (Theorem 2.1) there is a $B \in \theta^{(2)}$ with $\Delta B = D$. Since

$$\delta B(x, x) + B(\delta x, \delta x) = \varkappa(a, d)(x) - \varkappa(a, d')(x) ,$$

it follows that $\varepsilon\big(\varkappa(a, d)\big) = \varepsilon\big(\varkappa(a, d')\big)$. Because of that we shall use the notation $\varkappa(a) = \varkappa(a, d)$. Now, let $a, b \in Z(\mathcal{O})$ with $\varepsilon(a) = \varepsilon(b)$ and let $d(a; -, -)$ be associated with a. Since $\varepsilon(a - b) = 0$, there is an operation $\theta \in \mathcal{O}$ with $\Delta\theta = a - b$. Let us define

$$d(b; x, y) = d(a; x, y) + \theta(x + y) - \theta(x) - \theta(y) .$$

Then it is easy to see that $d(b;-,-)$ gives the deviation from additivity for b. Using this, we get

$$\varkappa(b)(x) = \varkappa(a)(x)$$

and therefore so much more $\varepsilon(\varkappa(b)) = \varepsilon(\varkappa(a))$. We define $\hat{\varkappa}(\hat{a}) = \varepsilon(\varkappa(a))$, where $\varepsilon(a) = \hat{a} \in A$. It is obvious that $\hat{\varkappa}$ preserves addition in A and that it decreases degrees by one. To see that $\hat{\varkappa}$ is a derivation, we proceed as follows: Let $a, b \in Z(\mathcal{O})$ and let $d(a;-,-)$ and $d(b;-,-)$ be associated with these; then we define a deviation from additivity for ba by

$$\begin{aligned} d(ba;x,y) = {} & bd(a;x,y) + d\big(b;a(x),a(y)\big) + \\ & + d\big(b;\delta d(a;x,y),d(a;\delta x,\delta y)\big) + \\ & + d\big(b;a(x+y),a(x)+a(y)\big) + \\ & + d\big(b;a(x),a(x)\big) + d\big(b;a(y),a(y)\big) . \end{aligned}$$

One easily checks that $d(ba;-,-)$ has the desired properties. Putting $x = y$ and $\delta x = 0$, we get

$$\varkappa(ba)(x) = b\big(\varkappa(a)(x)\big) + \varkappa(b)\big(a(x)\big)$$

which shows that $\hat{\varkappa}$ is a derivation.

To show that $\hat{\varkappa}(Sq^i) = Sq^{i-1}$, we simply refer to the explicit formula for $d(sq^i,-,-)$ given on p. 60 in [3]:

$$d(sq^i;x,y) = \varphi(e_{n-i+1} \otimes xy + e_{n-i+2} \otimes \delta x\, y), \qquad \deg x = n .$$

If $\delta x = 0$, this shows that

$$\hat{\varkappa}(Sq^i)(\{x\}) = \{\varphi(e_{n-i+1} \otimes x^2)\} = Sq^{i-1}(\{x\}) .$$

It is curious that one can obtain information about the multiplicative structure in the Steenrod algebra in this way. It is, for instance, possible to derive all the Adem relations from the relations $Sq^{2k-1}Sq^k = 0$ by iterated application of $\hat{\varkappa}$; for example

$$Sq^3 Sq^2 = 0 \Rightarrow \hat{\varkappa}(Sq^3 Sq^2) = Sq^2 Sq^2 + Sq^3 Sq^1 = 0 .$$

Using the structure of the dual Hopf algebra A^* of A we can obtain $\hat{\varkappa}$ in a different way. Let $\xi_1 \in A^*$ be the non-zero element of degree one. Multiplication with ξ_1 defines a mapping of A^* into itself of degree one. Since ξ_1 is primitive, it follows that the dual of this mapping is a derivation. This mapping is of degree -1 and it sends Sq^i into Sq^{i-1}. It is therefore equal to $\hat{\varkappa}$.

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ON ENTIRE FUNCTIONS ALMOST PERIODIC IN SEVERAL DIRECTIONS

HANS TORNEHAVE

1.

An example by R. Petersen [2] proved the existence of entire functions almost periodic in two directions. H. Tornehave [3] proved by further examples that the almost periodicity in the two directions may be of very different type. It was stated without proof in the same paper that an entire function might be almost periodic in three directions. In all these cases almost periodicity in a certain direction means almost periodicity in the whole plane in this direction.

In the present paper we shall present a method for construction of entire functions almost periodic in an arbitrary finite number of directions and even in a denumerable everywhere dense set of direction. In a recent paper P. Asisoff [1] proved that, on the other hand, the set of directions of non-almost-periodicity must be everywhere dense and, for a function of finite order, there exists only a finite set of directions of almost periodicity.

2.

Let ω_1 and ω_2 be complex numbers such that $\omega_1^{-1}\omega_2$ is not real. Let $(p_1,q_1),(p_2,q_2),\ldots$ be a sequence of mutually different pairs of relatively prime integers, the numbers p_ν being non negative. We shall assume that $(p_1,q_1)=(1,0)$ and $(p_2,q_2)=(0,1)$. Under these assumptions we shall prove the following statement:

There exists an entire function almost periodic in the complex plane in each of the directions determined by $p_\nu\omega_1+q_\nu\omega_2$, $\nu=1,2,\ldots$.

The existence will thus be proved for every bundle of directions with the property that every subset consisting of 4 directions has a rational cross-ratio. It is an open question yet, whether there exists an entire function almost periodic in 4 directions with irrational cross ratio.

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It will be convenient first to prove the theorem for a finite sequence $(p_1,q_1),\ldots,(p_n,q_n)$ and afterwards describe the changes in the procedure necessary in the general case.

3.

The set Ω of all numbers $p\omega_1+q\omega_2$ is a subgroup of the additive group of complex numbers. Let m be a natural number. The set $m\Omega$ is then a subgroup of Ω. We shall call two complex numbers equivalent with respect to $m\Omega$ if they belong to the same coset of $m\Omega$ considered as a subgroup of the additive group of complex numbers. If A is a set of complex numbers, the set $A+m\Omega$ is the set of numbers equivalent to elements of A. In particular, if A is a straight line, the set $A+m\Omega$ will be a system of straight lines parallel to A. If A has one of the directions $p_\nu\omega_1+q_\nu\omega_2$ the set of lines $A+m\Omega$ will be discrete.

The parallelogram

$$P_m = \{m(t_1\omega_1+t_2\omega_2) \mid 0\leqq t_1<1,\ 0\leqq t_2<1\}$$

is divided by the systems of parallels equivalent with respect to $m\Omega$ to the straight lines through 0 in the directions $p_\nu\omega_1+q_\nu\omega_2$, $\nu=1,\ldots,n$, into a finite number of convex polygons. Each parallelogram equivalent to P_m with respect to $m\Omega$ is subdivided by the same system of lines, the subdivision being congruent to the subdivision of P_m. If we replace m by another natural number m' we obtain a subdivision of $P_{m'}$ homothetic to the subdivision of P_m. These phenomena are illustrated in Fig. 1, where we have chosen $\omega_2=i\omega_1$ and the directions $(1,0)$, $(0,1)$, $(3,2)$, $(1,3)$ and $(1,-2)$. The small squares correspond to $m=1$ and are subdivided by all the lines. The thin lines do not belong to the subdivision of the large square corresponding to $m=2$.

4.

As in the paper [2] our entire function will be constructed as the limit of a sequence of elliptic functions. We shall now introduce the particular class of elliptic functions which will be used for our present purpose.

Let $a_{m1},\ldots,a_{mk}$ be the geometric centers of gravity of the polygons of the subdivision of P_m. Let $\boldsymbol{v}=(v_{\varkappa\lambda})$ be a k by l complex matrix. We introduce

$$Q_\varkappa(\boldsymbol{v};z) = v_{\varkappa 1}z^{-3}+\ldots+v_{\varkappa l}z^{-l-2}\,.$$

In [3, § 2] it was proved that we have for $\varrho>0$ an estimate

$$|Q_\varkappa(\boldsymbol{v};z)| \leqq K|z|^{-3}$$

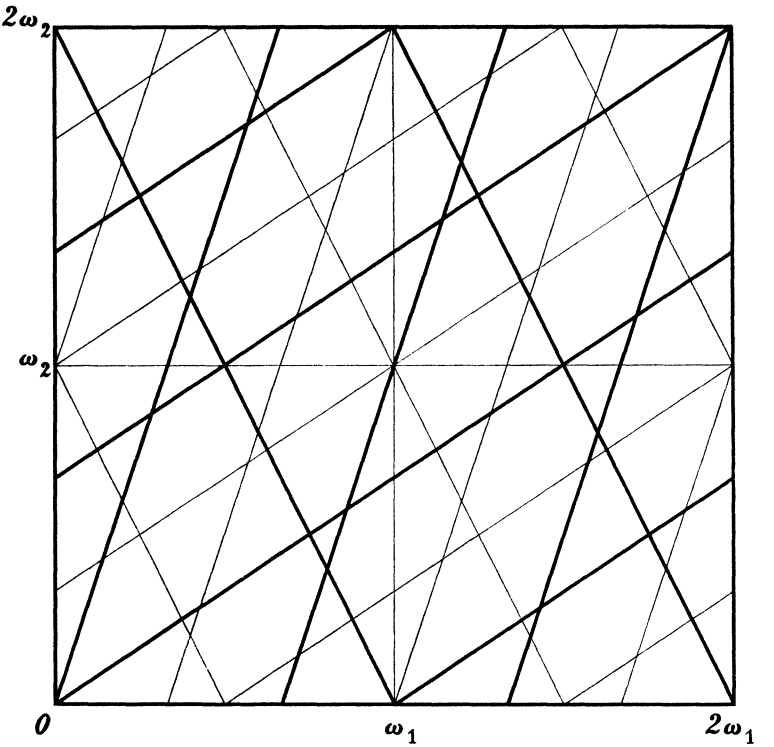


Fig. 1

valid for $|z| \geqq \varrho$. It follows that the function

$$F_m(v; z) = \sum_{\varkappa=1}^{k} \sum_{\omega \in m\Omega} Q_\varkappa(v; z - a_{m\varkappa} - \omega) \tag{1}$$

is an elliptic function with $m\Omega$ as its group of periods.

Let δ denote the smallest distance between the set of points $a_{1\varkappa}$, $\varkappa = 1, \ldots, k$, and the union S of the straight lines through 0 with directions $p_\nu\omega_1 + q_\nu\omega_2$, $\nu = 1, \ldots, n$. The number δ will be positive. Let S_m denote the set of all points with distance $< m\delta$ from the set S. The set S_m is the union of n strips, and $F_m(v; z)$ is holomorphic and periodic in each of these strips.

We shall rewrite (1) extending the summation with respect to $\varkappa$ over the $4k$ points

$$a_{m\varkappa}, \quad a_{m\varkappa} + m\omega_1, \quad a_{m\varkappa} + m\omega_2, \quad a_{m\varkappa} + m\omega_1 + m\omega_2, \qquad \varkappa = 1, \ldots, k\,.$$

This requires an obvious change of notation, which we indicate by replacing $a'_{m\varkappa}$ by $a'_{m\varkappa}$ and v by v':

$$F_m(\boldsymbol{v}; z) = \sum_{\varkappa=1}^{4k} \sum_{\omega \in 2m\Omega} Q_\varkappa(\boldsymbol{v}'; z - a'_{m\varkappa} - \omega) .$$

For an arbitrary set $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_{4k})$ of complex numbers we shall use the notation

$$F_m(\boldsymbol{v}; \boldsymbol{\tau}; z) = \sum_{\varkappa=1}^{4k} \sum_{\omega \in 2m\Omega} Q_\varkappa(\boldsymbol{v}'; z - a'_{m\varkappa} - \tau_\varkappa - \omega) .$$

If every $a'_{m\varkappa} + \tau_\varkappa + \omega$ is outside S_m and $|\tau_\varkappa| \leqq \frac{1}{2}\delta$, $\varkappa = 1, \ldots, 4k$, and η_1 is a positive number, we can choose $\boldsymbol{w}' = (w'_{\varkappa\lambda})_{4k,l}$ such that

$$\left| \sum_{\varkappa=1}^{4k} \left(Q_\varkappa(\boldsymbol{w}'; z - a'_{m\varkappa} - \tau_\varkappa - \omega) - Q_\varkappa(\boldsymbol{v}'; z - a'_{m\varkappa} - \omega) \right) \right| \leqq \eta_1 |z - a'_{m\varkappa} - \omega|^{-3}$$

when $|z - a'_{m\varkappa} - \omega| \geqq \delta$, $\varkappa = 1, \ldots, 4k$, $\omega \in 2m\Omega$. This follows immediately from [3, § 3, lemma 6], since $\boldsymbol{v}'$ is fixed in the present case. Hence, there exists a constant K depending only on m, Ω and k such that

$$|F_m(\boldsymbol{w}'; \boldsymbol{\tau}; z) - F_m(\boldsymbol{v}; z)| \leqq K\eta_1 \tag{2}$$

for every $z \in S_{m-1}$. Since we have used no properties of the points $a'_{m\varkappa}$ except that they and every $a'_{m\varkappa} + \omega$; $\omega \in 2m\Omega$, are outside S_m, we can repeat the procedure N times, and obtain the inequality (2) with $NK\eta_1$ instead of $K\eta_1$; when $|\tau_\varkappa| \leqq \frac{1}{2}N\delta$ and the segment from $a'_{m\varkappa} + \omega$ to $a'_{m\varkappa} + \omega + \tau_\varkappa$ is outside S_m for $\varkappa = 1, \ldots, 4k$; $\omega \in 2m\Omega$.

Each of the numbers $a_{2m,\varkappa} = 2a_{m,\varkappa}$ is the geometric center of gravity of a polygon of the subdivision of P_{2m}, and this polygon contains finitely many of the points $a_{m,\varkappa'}$. For each of these we can choose $\tau_{\varkappa'} = a_{2m,\varkappa} - a_{m,\varkappa'}$. We can then choose the number N such that (2) is satisfied with $NK\eta_1$ instead of $K\eta_1$. Since N and K are independent of η_1, we have proved that for $\eta > 0$ we can choose $\boldsymbol{w} = (w_{\varkappa,\lambda})_{k,l''}$ such that

$$|F_{2m}(\boldsymbol{w}; z) - F_m(\boldsymbol{v}; z)| \leqq \eta \qquad \text{for} \qquad |z| \in S_{m-1} .$$

We shall now start with $m = 2$. It does not matter how we choose $\boldsymbol{v} = \boldsymbol{v}^{(1)}$ except that it should have at least one element $\neq 0$. We can then construct the sequence of functions $F_{2^m}(\boldsymbol{v}^{(m)}; z)$ such that

$$|F_{2^{m+1}}(\boldsymbol{v}^{(m+1)}; z) - F_{2^m}(\boldsymbol{v}^{(m)}; z)| \leqq 2^{-m}\varepsilon \qquad \text{for} \qquad z \in S_{2^m-1} .$$

Here ε is an arbitrary positive number. The sequence of functions is then uniformly convergent in every finite strip in each of the prescribed directions. This implies that the limit function is entire and that it is limit periodic in each strip. If ε is small enough, the limit function will obviously not be constant. This completes the proof.

The case where there is a sequence of directions is treated similarly, except that the periods are not simply multiplied by 2 in each step but

by an integer ν chosen separately for each step. Further, the straight lines in the directions $p_\nu\omega_1+q_\nu\omega_2$, $\nu=3,4,\ldots$, are introduced gradually, one new direction in each step. This has the effect that the number k increases with each step, and in each step the distance δ is replaced by a smaller distance δ_1. However, if we multiply the periods by a number greater than $2\delta_1^{-1}\delta$, the width of each of the periodicity strips will be doubled in each step.

The displacement is carried out according to the same rules as in the finite case. It may, however, happen that some of the old poles are situated on sides of polygons in the new subdivision. We may then choose arbitrarily whether we want to consider them as belonging to one or the other of the adjoining polygons. Except for the changes mentioned here, the proof is carried out exactly as in the finite case.

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EINE FÜR BELIEBIGE CALL-PROZESSE GELTENDE VERALLGEMEINERUNG DER PALMSCHEN FORMELN

WERNER FIEGER

Ist $x(t)$ ein stationärer, ordinärer Call-Prozeß mit endlichem Parameter λ, so gelten die Palmschen Formeln

$$p\big(x(b)-x(a)=0\big) = 1 - \lambda \int_0^{b-a} \varphi_0(\tau)\, d\tau\,,$$

$$p\big(x(b)-x(a)=k\big) = \lambda \int_0^{b-a} \varphi_{k-1}(\tau)\, d\tau - \lambda \int_0^{b-a} \varphi_k(\tau)\, d\tau, \qquad k=1,2,\ldots\,,$$

für $0 \leqq a < b < +\infty$, wobei

$$\varphi_i(t) := \lim_{\tau \searrow 0} p\big(x(t+\tau)-x(\tau)=i \mid x(\tau)-x(0) \geqq 1\big)$$

für $i=0,1,2,\ldots$ ist; einen Beweis für diese Formeln findet man in [7, Kap. 3]. Mit Methoden, die einfache Verallgemeinerungen der von Khintchine in [7] angewandten Hilfsmittel sind, ist in [2] eine Verallgemeinerung dieser Formeln für stationäre, nicht-ordinäre Call-Prozesse hergeleitet. In [2] ist außerdem der nicht-stationäre, ordinäre Fall mittels eines einfachen integrationstheoretischen Satzes (Hilfssatz 3), der ein Spezialfall einer tiefergehenden Aussage über Burkill-Integrale ist, behandelt.

In § 3 der vorliegenden Arbeit werden die Palmschen Formeln mit Hilfe von Sätzen aus der Theorie der Burkill-Integrale für beliebige Call-Prozesse, die nur einer einfachen Endlichkeitsbedingung genügen müssen, verallgemeinert.

Als Anwendung dieser verallgemeinerten Palmschen Formeln erhalten wir eine Charakterisierung der Call-Prozesse ohne Nachwirkungen und der Poisson-Prozesse mit nicht-stationärer Parameterfunktion (§ 4). Ferner zeigen wir, daß ein Satz von Korolyuk auch für nicht-stationäre Call-Prozesse gilt (§ 5). In § 1 und § 2 sind die integrationstheoretischen Sätze zusammengestellt, die in den Beweisen benötigt werden.

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1. Das Burkill-Unterteilungsintegral über einer beliebigen Grundmenge.

In § 1 und § 2 werden nur die Sätze über das Burkill-Unterteilungsintegral angegeben, die wir später benötigen. Auf Zusammenhänge mit anderen Sätzen über das Burkill-Unterteilungsintegral und mit entsprechenden Sätzen über das Burkill-Normintegral gehen wir nicht ein. Man vergleiche hierzu [6, Abschnitt X] und die dort angegebene Literatur.

Die folgenden Begriffe sind teilweise aus [6] übernommen; auf wesentliche Abweichungen ist im Text hingewiesen.

Es sei Ω eine beliebig vorgegebene Grundmenge. Ein System $\mathfrak{L}$ von Teilmengen der Menge Ω heißt Intervallsystem, falls gilt:

a) es gibt Mengen $I_1,\ldots,I_n \in \mathfrak{L}$ mit $I_\nu \cap I_\mu = 0$ für $\nu \neq \mu$ und mit $\Omega = I_1 \cup \ldots \cup I_n$;
b) aus $I,\ I' \in \mathfrak{L}$, $I \cap I' \neq \emptyset$ folgt $I \cap I' \in \mathfrak{L}$;
c) $\mathfrak{L} \neq \emptyset$.

Ein endliches Teilsystem $Z(J) = \{I_1,\ldots,I_r\}$ von paarweise disjunkten Mengen aus $\mathfrak{L}$ mit $J = I_1 \cup \ldots \cup I_r$ nennen wir eine Intervallunterteilung von J. Eine Intervallunterteilung $Z'(J) = \{I_1',\ldots,I_s'\}$ heißt feiner als $Z(J) = \{I_1,\ldots I_r\}$ (Bezeichnung: $Z'(J) \geq Z(J)$), wenn es zu jedem $I_\sigma' \in Z'(J)$ ein $I_\varrho \in Z(J)$ mit $I_\sigma' \subset I_\varrho$ gibt.

Es bezeichne ${}^K\mathfrak{L}$ den kleinsten Mengenkörper, der $\mathfrak{L}$ umfaßt, ${}^B\mathfrak{L}$ den kleinsten σ-Körper, der $\mathfrak{L}$ enthält.

Über $\mathfrak{L}$ sei die reelle Funktion $F(I)$ gegeben. Zur Abkürzung führen wir die Bezeichnung

$$F(Z(J)) := \sum_{\varrho=1}^{r} F(I_\varrho)$$

ein.

Definition. Die »Intervallfunktion« F heißt $\mathfrak{L}$-Burkill-unterteilungsintegrabel (kurz: $\mathfrak{L}$-Burkill-integrabel, oder: B-integrabel) über $J \in {}^K\mathfrak{L}$ mit dem Integralwert $\int_J F(I)$, wenn es zu jedem $\varepsilon > 0$ eine Intervallunterteilung $Z_0(J)$ mit

$$\left| F(Z(J)) - \int_J F(I) \right| < \varepsilon \quad \text{für alle} \quad Z(J) \geq Z_0(J)$$

gibt.

Der Integralwert einer über J B-integrablen Funktion F ist eindeutig bestimmt. Ist F über J B-integrabel, so auch über jedem $J' \in {}^K\mathfrak{L}$ mit $J' \subset J$. Ist F B-integrabel über Ω, so ist $f(J) := \int_J F(J)$ über ${}^K\mathfrak{L}$ additiv.

Wir betrachten nun über Ω einen σ-Körper $\mathfrak{B}$ und ein endliches Maß $m \mid \mathfrak{B}$. Ein System $\mathfrak{K}$ von Teilmengen von Ω nennen wir ein starkes Vitalisches System, wenn folgende Bedingungen erfüllt sind:

a) $\mathfrak{K} \subset \mathfrak{B}$;

b) $\mathfrak{K} \neq \emptyset$;

c) zu jedem $S \subset \Omega$ und zu jedem $\varepsilon > 0$ gibt es endlich oder abzählbar unendlich viele, paarweise disjunkte Mengen N; $K_1, K_2, \ldots \in \mathfrak{K}$ mit

$$S \subset \bigcup_{\nu \geqq 1} K_\nu \cup N \quad \text{und} \quad \underline{m}\left(\bigcup_{\nu \geqq 1} K_\nu - S\right) < \varepsilon,$$

wobei $m(N) = 0$ ist.

Mit $\underline{m}(R)$, $\overline{m}(R)$ bezeichnen wir das innere resp. äußere m-Maß von R. In Abweichung von der Definition in [6] ist hier die Forderung »$m(K) > 0$ für alle $K \in \mathfrak{K}$« durch »$\emptyset \notin \mathfrak{K}$« ersetzt.

Entsprechend übernehmen wir aus [6] die Definition der Ableitungsbasis $\mathfrak{A}$ (Forderung $m(V) > 0$ ersetzt durch $\emptyset \notin \mathfrak{g} \in \mathfrak{G}$), die Definition der starken Vitalischen Ableitungsbasis und die dazu gehörenden Begriffe (vgl. hierzu [6, S. 210–214]).

Wir sagen, daß die Ableitungsbasis $\mathfrak{A}$ die Eigenschaft T^{**} hat, wenn es zu jedem $K \in \bigcup_{\mathfrak{g} \in \mathfrak{A}} \mathfrak{g}$ eine m-Nullmenge N gibt derart, daß zu jedem $\omega \in K - N$ und zu jeder dem Punkt ω zugehörigen Ableitungsfamilie $\mathfrak{g} \in \mathfrak{A}$ ein Ende $\mathfrak{g}_e$ von $\mathfrak{g}$ mit $K' \subset K$ für jedes $K' \in \mathfrak{g}_e$ gibt.

Es sei nun F erklärt über $\bigcup_{\mathfrak{g} \in \mathfrak{A}} \mathfrak{g}$. Als obere $\mathfrak{A}$-Ableitung von F bezüglich m im Punkt ω definiert man

$$\overline{D}F(\omega) := \sup\left\{\overline{\lim_{K \in \mathfrak{g}}} \frac{F(K)}{m(K)} : \mathfrak{g} \text{ zu } \omega \text{ gehörige Ableitungsfamilie}\right\},$$

wobei

$$\frac{F(K)}{m(K)} = \begin{cases} +\infty & \text{für} \quad F(K) \geqq 0, \; m(K) = 0 \\ -\infty & \text{für} \quad F(K) < 0, \; m(K) = 0 \end{cases}$$

gesetzt ist, und wobei $\overline{\lim}_{K \in \mathfrak{g}}$ als oberer Limes bezüglich der gerichteten Familie $\mathfrak{g}$ zu nehmen ist. Ebenso erklärt man die untere Ableitung

$$\underline{D}F(\omega) := \inf\left\{\underline{\lim_{K \in \mathfrak{g}}} \frac{F(K)}{m(K)} : \mathfrak{g} \text{ zu } \omega \text{ gehörige Ableitungsfamilie}\right\}.$$

Falls $\underline{D}F(\omega) = \overline{D}F(\omega)$ ist, heißt F in ω bezüglich m differenzierbar mit der Ableitung $DF(\omega) := \underline{D}F(\omega)$.

Ist $\mathfrak{L} \subset \mathfrak{B}$, und gibt es zu jedem $\varepsilon > 0$ ein $\delta > 0$, so daß es zu beliebigem $J = I_1 \cup \ldots \cup I_n$ mit $I_1, \ldots, I_n \in \mathfrak{L}$ und mit $m(J) < \delta$ stets ein $Z_0(J)$ mit $|F(Z(J))| < \varepsilon$ für alle $Z(J) \geq Z_0(J)$ gibt ,so heißt F totalstetig bezüglich m.

Wir nehmen nun an, daß $\mathfrak{B}$ ein σ-Körper über Ω ist, $m \mid \mathfrak{B}$ ein endliches Maß und $\mathfrak{A}$ eine starke Vitalische Ableitungsbasis mit der Eigenschaft T^{**}; $\mathfrak{L} := \bigcup_{g \in \mathfrak{A}} g$ sei ein Intervallsystem. Unter diesen Voraussetzungen gilt:

SATZ 1. *Ist die m-totalstetige Intervallfunktion F_1 $\mathfrak{L}$-Burkill-integrabel über Ω, so existiert $DF_1(\omega)$ m-fast überall. Ist F_2 ebenfalls m-totalstetig und $\mathfrak{L}$-Burkill-integrabel über Ω, und gilt*

$$\int_J F_1(I) = \int_J F_2(I) \qquad \text{für alle} \quad J \in {}^K\mathfrak{L},$$

so ist

$$DF_1(\omega) = DF_2(\omega)$$

m-fast überall.

BEMERKUNG. Man vergleiche die letzte Anmerkung in [10]. Dort ist jedoch die — für Teile der Theorie wesentliche — Voraussetzung gemacht, daß es beliebig maßfeine Unterteilungen gibt. — Den Beweis für Satz 1 kann man auch ähnlich wie in [11] führen, wenn man die dort auftretende Forderung B^* durch unsere Eigenschaft T^{**} ersetzt.

BEWEIS VON SATZ 1. 1) Es seien F_1, F_2 m-totalstetig und $\mathfrak{L}$-Burkill-integrabel über Ω mit $f_\nu(J) := {}_J\int F_\nu(I)$. Es genügt zu zeigen, daß aus der Gleichheit

$$f_1(J) = f_2(J) \qquad \text{für alle} \quad J \in {}^K\mathfrak{L}$$

die Relation

$$\overline{m}\big(\{\omega\colon \overline{D}F_1(\omega) > \underline{D}F_2(\omega)\}\big) = 0$$

oder, was damit gleichwertig ist,

$$\overline{m}\big(\{\omega\colon \overline{D}F_1(\omega) > \alpha > \beta > \underline{D}F_2(\omega)\}\big) = 0$$

für alle reellen α, β mit $\alpha > \beta$ folgt.

2) Es seien α, β reell vorgegeben mit $\alpha > \beta$. Wir setzen

$$R := \{\omega\colon \overline{D}F_1(\omega) > \alpha > \beta > \underline{D}F_2(\omega)\}$$

und nehmen an, daβ $\overline{m}(R) > 0$ gilt. Zu

$$\varepsilon := \min\left\{1, \frac{\alpha - \beta}{2(|\beta| + 4)}\right\} \overline{m}(R)$$

wählen wir $\delta > 0$ so, daß für beliebige paarweise fremde $K_1, \ldots, K_m \in \mathfrak{L}$ aus $m(K_1 \cup \ldots \cup K_m) < \delta$ immer

$$|F_\nu(Z(K_1 \cup \ldots \cup K_m))| < \varepsilon \qquad \text{für} \quad \nu = 1, 2$$

folgt, sofern nur $Z(K_1 \cup \ldots \cup K_m)$ genügend fein ist. Dazu gibt es abzählbar viele, paarweise fremde $K_1', K_2', \ldots \in \mathfrak{L}$ und eine m-Nullmenge N mit

$$R \subset \bigcup_{\varrho=1}^{\infty} K_\varrho' \cup N, \qquad \underline{m}\Big(\bigcup_{\varrho=1}^{\infty} K_\varrho' - R\Big) < \tfrac{1}{2}\eta$$

für $\eta := \min\{\varepsilon, \delta\}$, da $\mathfrak{L}$ ein starkes Vitalisches System ist. Folglich existiert eine natürliche Zahl n so, daß für $J := \bigcup_{\varrho=1}^{n} K_\varrho'$ gilt:

(*) $\quad \overline{m}(R) - \tfrac{1}{2}\eta < \overline{m}(R \cap J) \leqq m(J) < \overline{m}(R) + \tfrac{1}{2}\eta < \overline{m}(R \cap J) + \eta$.

3) Da F_1, F_2 über J B-integrabel sind, gibt es eine Intervallunterteilung $Z_0(J) = \{I_1, \ldots, I_r\}$ mit

(**) $\quad |F_\nu(Z(J)) - f_\nu(J)| < \varepsilon \qquad$ für $Z(J) \geq Z_0(J)$ und $\nu = 1, 2$.

4) Wegen der Eigenschaft T^{**} gibt es eine feste m-Nullmenge N_1 und zu jedem $\omega \in R \cap J - N_1 =: R_1$ zwei Ableitungsfamilien $\mathfrak{g}_1(\omega)$, $\mathfrak{g}_2(\omega)$ mit

$$F_1(K') \geqq \alpha m(K'), \qquad K' \subset I_\varrho \text{ für ein } \varrho \in \{1, \ldots, r\} \text{ für alle } K' \in \mathfrak{g}_1(\omega)$$

und mit

$$F_1(K'') \leqq \beta m(K''), \qquad K'' \subset I_{\varrho'} \text{ für ein } \varrho' \in \{1, \ldots, r\} \text{ für alle } K'' \in \mathfrak{g}_2(\omega) .$$

Da $\mathfrak{A}$ eine starke Vitalische Ableitungsbasis ist, existieren abzählbar viele, disjunkte

$$I_1', I_2', \ldots \in \bigcup_{\omega \in R_1} \mathfrak{g}_1(\omega)$$

und abzählbar viele, disjunkte

$$I_1'', I_2'', \ldots \in \bigcup_{\omega \in R_1} \mathfrak{g}_2(\omega)$$

mit

$$R \cap J \subset \bigcup_{\sigma=1}^{\infty} I_\sigma' \cup N_3 \subset J ,$$
$$R \cap J \subset \bigcup_{\sigma=1}^{\infty} I_\sigma'' \cup N_4 \subset J . \qquad \big(m(N_3) = m(N_4) = 0\big)$$

5) Zu jedem s gibt es paarweise disjunkte $\tilde{I}_1^s, \ldots, \tilde{I}_{t(s)}^s \in \mathfrak{L}$ und paarweise disjunkte $\tilde{\tilde{I}}_1^s, \ldots, \tilde{\tilde{I}}_{u(s)}^s \in \mathfrak{L}$ mit

$$I_1' \cup \ldots \cup I_s' \cup \tilde{I}_1^s \cup \ldots \cup \tilde{I}_{t(s)}^s = J ,$$
$$(I_1' \cup \ldots \cup I_s') \cap (\tilde{I}_1^s \cup \ldots \cup \tilde{I}_{t(s)}^s) = \emptyset ,$$
$$\{I_1', \ldots, I_s', \tilde{I}_1^s, \ldots, \tilde{I}_{t(s)}^s\} > Z_0(J)$$

und entsprechend für $\tilde{\tilde{I}}^s_\nu$. Dabei gilt weiter

$$\bigcap_{s=1}^{\infty} (\tilde{I}_1{}^s \cup \ldots \cup \tilde{I}^s_{t(s)}) \subset J - R \cap J$$

und ebenso

$$\bigcap_{s=1}^{\infty} (\tilde{\tilde{I}}_1{}^s \cup \ldots \cup \tilde{\tilde{I}}^s_{u(s)}) \subset J - R \cap J ,$$

also wegen (*)

$$\lim_{s\to\infty} m(\tilde{I}_1{}^s \cup \ldots \cup \tilde{I}^s_{t(s)}) \leqq \underline{m}(J - R \cap J) < \delta ,$$
$$\lim_{s\to\infty} m(\tilde{\tilde{I}}_1{}^s \cup \ldots \cup \tilde{\tilde{I}}^s_{u(s)}) \leqq \underline{m}(J - R \cap J) < \delta .$$

Ohne Einschränkung der Allgemeinheit können wir daher annehmen, daß die $\tilde{I}^s_\varrho$, $\tilde{\tilde{I}}^s_\varrho$ so gewählt sind, daß

$$\overline{\lim}_{s\to\infty} |F_1(\tilde{I}_1{}^s) + \ldots + F_1(\tilde{I}^s_{t(s)})| < \varepsilon$$

und

$$\overline{\lim}_{s\to\infty} |F_2(\tilde{\tilde{I}}_1{}^s) + \ldots + F_2(\tilde{\tilde{I}}^s_{u(s)})| < \varepsilon .$$

gilt.

6) Aus (**) folgern wir

$$\begin{aligned} f_1(J) + \varepsilon &> F_1(I_1') + \ldots + F_1(I_s') + F_1(\tilde{I}_1{}^s) + \ldots + F_1(\tilde{I}^s_{t(s)}) \\ &\geqq \alpha\, m(I_1' \cup \ldots \cup I_s') + F_1(\tilde{I}_1{}^s) + \ldots + F_1(\tilde{I}^s_{t(s)}) , \\ f_2(J) - \varepsilon &< F_2(I_1'') + \ldots + F_2(I_s'') + F_2(\tilde{\tilde{I}}_1{}^s) + \ldots + F_2(\tilde{\tilde{I}}^s_{u(s)}) \\ &\leqq \beta\, m(I_1'' \cup \ldots \cup I_s'') + F_2(\tilde{\tilde{I}}_1{}^s) + \ldots + F_2(\tilde{\tilde{I}}^s_{u(s)}) \end{aligned}$$

und damit für $s \to \infty$ unter Beachtung von (*)

$$\begin{aligned} f_1(J) + \varepsilon &> \alpha\, m(\textstyle\bigcup_{s=1}^{\infty} I_s') - \varepsilon \\ &\geqq \alpha\, \overline{m}(R \cap J) - \varepsilon \\ &> (\alpha - \beta)(\overline{m}(R) - \tfrac{1}{2}\eta) + \beta\, \overline{m}(J) - |\beta|\eta - \varepsilon \\ &\geqq \tfrac{1}{2}(\alpha - \beta)\overline{m}(R) - (|\beta| + 1)\varepsilon + \beta\, m(\textstyle\bigcup_{s=1}^{\infty} I_s'') \\ &> f_2(J) - (|\beta| + 3)\varepsilon + \tfrac{1}{2}(\alpha - \beta)\overline{m}(R) , \end{aligned}$$

also wegen $f_1(J) = f_2(J)$

$$(|\beta| + 4)\varepsilon > \tfrac{1}{2}(\alpha - \beta)\overline{m}(R) ,$$

was ein Widerspruch zur Wahl von ε ist.

2. Das Burkill-Integral über endlichen Teilintervallen des R^1.

Als Anwendung betrachten wir nun eine über einem endlichen Intervall gegebene Intervallfunktion.

Es sei $-\infty < a < b < +\infty$. Für $a \leqq t < t' \leqq b$ sei die reelle Funktion $\psi(t,t')$ erklärt. Setzen wir

$$\psi(I) := \psi(t,t') \qquad \text{für} \qquad I = (t,t'] \text{ und } \Omega' := (a,b],$$

so ist ψ eine für alle links offenen, rechts abgeschlossenen Teilintervalle I von Ω' erklärte Funktion. ψ ist genau dann über Ω' B-integrabel mit dem Integralwert $\int_a^b \psi(I)$, wenn es zu jedem $\varepsilon > 0$ Zwischenpunkte $t_1, \ldots, t_n$ $(n = n(\varepsilon))$ mit

$$a =: t_0 < t_1 < \ldots < t_n < t_{n+1} := b$$

gibt derart, daß für jede Unterteilung

$$a =: t_0' < t_1' < \ldots < t_{m+1}' := b$$

mit

$$\{t_1, \ldots, t_n\} \subset \{t_1', \ldots, t_m'\}$$

die Ungleichung

$$\left| \sum_{\mu=1}^{m+1} \psi(t'_{\mu-1}, t_\mu') - \int_a^b \psi(I) \right| < \varepsilon$$

gilt. Wir sprechen im folgenden auch kurz davon, daß ψ über $[a,b]$ Burkill-unterteilungsintegrabel (oder: B-integrabel) ist.

Die durch die endlich vielen Punkte $t_1, \ldots, t_n$ mit $a < t_1 < \ldots < t_n < b$ gegebene Intervallunterteilung bezeichnen wir wieder mit $Z = \{t_1, \ldots, t_n\}$, ferner setzen wir $Z_1 \geq Z$, falls

$$Z_1 = \{t_1', \ldots, t_m'\} \supset \{t_1, \ldots, t_n\},$$

und

$$\psi(Z) := \sum_{\nu=1}^{n+1} \psi(t_{\nu-1}, t_\nu).$$

Es sei $s(t)$ eine über $[a,b]$ monoton nicht fallende Funktion; die Punkte von [a, b] klassifizieren wir nach dem Stetigkeitsverhalten von $s(t)$:

$$\begin{aligned} S^{L,R} &:= \{t \in (a,b):\ s(t-0) < s(t) < s(t+0)\} \\ S^{L} &:= \{t \in (a,b]:\ s(t-0) < s(t) = s(t+0)\}, \\ S^{R} &:= \{t \in [a,b):\ s(t-0) = s(t) < s(t+0)\} \end{aligned}$$

mit $s(a-0) := s(a)$, $s(b+0) := s(b)$ und

$$T := [a,b] - (S^{L,R} \cup S^{L} \cup S^{R}).$$

Zur Vereinfachung führen wir noch die Mengen

$$C(t) := \begin{cases} \{1\} & \text{für} \quad t \in T \\ \{2\} & \text{für} \quad t \in S^L \\ \{3\} & \text{für} \quad t \in S^R \\ \{2,3\} & \text{für} \quad t \in S^{L,R} \end{cases}$$

für $t \in [a,b]$ ein. Nun setzen wir

$$\Omega := \{(t,\lambda)\colon\ a \leqq t \leqq b,\ \lambda \in C(t)\}\,,$$

betrachten das »Intervall«

$$\langle \alpha,\beta\rangle := \{(t,\lambda)\colon\ \alpha < t < \beta,\ \lambda \in C(t)\} \cup \{(\alpha,\lambda)\colon\ \lambda \in C(\alpha) \cap \{3\}\} \cup \\ \cup \{(\beta,\lambda)\colon\ \lambda \in C(\beta) \cap \{1,2\}\}\,,$$

und setzen

$$m(\langle \alpha,\beta\rangle) := s(\beta) - s(\alpha)\,,$$
$$\mathfrak{L} := \{\langle \alpha,\beta\rangle\colon\ a \leqq \alpha < \beta \leqq b\}\,,$$
$$\mathfrak{B} := {}^{B}\mathfrak{L}\,.$$

Es ist m σ-additiv und endlich über $\mathfrak{L}$; also läßt sich m zu einem ebenfalls mit m bezeichneten Maß über $\mathfrak{B}$ erweitern.

Durch diese Konstruktion, die nur eine Trennung der beidseitigen Unstetigkeitspunkte von $s(t)$ bewirkt, haben wir der monoton nicht fallenden Funktion $s(t)$ ein Maß über Ω zugeordnet. Die Intervallfunktion $\psi(t,t')$ überpflanzen wir durch

$$\psi(\langle \alpha,\beta\rangle) := \psi(\alpha,\beta)\,.$$

Ein durch die Teilmengenrelation gerichtetes System von Intervallen $\langle \alpha_\gamma,\beta_\gamma\rangle$ (Γ beliebige Indexmenge, $\gamma \in \Gamma$) nennen wir eine zugelassene Ableitungsfamilie zu $\omega = (t,\lambda)$, wenn $\lim(\beta_\gamma - \alpha_\gamma) = 0$ und wenn für $\lambda = 1$ (d.h. $t \in T$) entweder $\alpha_\gamma = t$ für alle $\gamma \in \Gamma$ oder $\beta_\gamma = t$ für alle $\gamma \in \Gamma$ bzw. für $\lambda = 2$, $\beta_\gamma = t$ für alle $\gamma \in \Gamma$ bzw. für $\lambda = 3$, $\alpha_\gamma = t$ für alle $\gamma \in \Gamma$ gilt. Die Gesamtheit $\mathfrak{A}$ aller zu irgendeinem Element $\omega \in \Omega$ zugelassenen Ableitungsfamilien bildet eine starke Vitalische Ableitungsbasis. (Den Beweis für diese Aussage führt man etwa mit der in [12] angegebenen Überpflanzung der Menge Ω auf das abgeschlossene Intervall $[s(a),s(b)]$.) Unmittelbar einzusehen ist, daß die Ableitungsbasis $\mathfrak{A}$ die Eigenschaft T^{**} hat.

Mit diesen Begriffen ist der nächste Satz ohne Schwierigkeiten aus Satz 1 zu folgern.

SATZ 2. *Es sei $s(t)$ eine über $[a,b]$ monoton nicht fallende Funktion; σ_0 bezeichne das vom Stetigkeitsanteil von $s(t)$ über dem σ-Körper aller Borel-*

schen Teilmengen von $[a,b]$ *definierte Maß. Die für* $a \leqq t < t' \leqq b$ *erklärte reelle Funktion* $\psi(t,t')$ *genüge der Ungleichung*

$$|\psi(t,t')| \leqq s(t') - s(t) .$$

Ist $\psi(t,t')$ *über* $[a,b]$ *B-integrabel, so existieren*

$$\lim_{\tau \nearrow t} \frac{\psi(\tau,t)}{s(t)-s(\tau)}$$

für alle $t \in S^L \cup S^{L,R}$ *und für* σ_0*-fast alle* $t \in T$ *und*

$$\lim_{\tau \searrow t} \frac{\psi(t,\tau)}{s(\tau)-s(t)}$$

für alle $t \in S^R \cup S^{L,R}$ *und für* σ_0*-fast alle* $t \in T$*; für* σ_0*-fast alle* $t \in T$ *sind diese beiden Grenzwerte gleich.*

BEWEIS. Die von $\psi(t,t')$ und $s(t)$ induzierten Funktionen $\psi \mid \mathfrak{L}$, $m \mid \mathfrak{B}$ erfüllen bezüglich der starken Vitalischen Ableitungsbasis $\mathfrak{A}$ die entsprechenden Voraussetzungen von Satz 1. Daher existiert $D\psi(\omega)$ m-fast überall. Wegen der speziellen Form von $\mathfrak{A}$ und wegen

$$\frac{\psi(\langle\alpha,\beta\rangle)}{m(\langle\alpha,\beta\rangle)} = \frac{\psi(\alpha,\beta)}{s(\beta)-s(\alpha)}$$

ergibt sich daraus die Behauptung.

Mit $\mathfrak{B}(a,b)$ bezeichnen wir den σ-Körper der Borelschen Teilmengen von $[a,b]$, mit $\mathfrak{B}(0,+\infty)$ den σ-Körper der Borelschen Teilmengen von $[0,+\infty)$. — Aus [1] übernehmen wir

DEFINITION. Die Funktion $g(t)$ heißt über $[a,b]$ $s(t)$-meßbar, wenn $g(t)$ über $[a,b]-S^{L,R}$ eindeutig und $\mathfrak{B}(a,b)$-meßbar ist, und wenn $g(t)$ für jedes $\tau \in S^{L,R}$ zwei nicht notwendig verschiedene Funktionswerte $g^L(\tau)$, $g^R(\tau)$ besitzt. Weiter heißt $g(t)$ $s(t)$-integrabel, wenn die über Ω definierte Funktion

$$g(\omega) := \begin{cases} g(t) & \text{für} \quad \omega=(t,\lambda) \text{ mit } t \in [a,b]-S^{L,R} \\ g^L(t) & \text{für} \quad \omega=(t,\lambda) \text{ mit } t \in S^{L,R},\ \lambda=2 \\ g^R(t) & \text{für} \quad \omega=(t,\lambda) \text{ mit } t \in S^{L,R},\ \lambda=3 \end{cases}$$

m-integrabel ist; der Wert des $s(t)$-Integrals ist

$$\int_a^b g(t)\, s(dt) := \int_\Omega g(\omega)\, m(d\omega) .$$

Einige Eigenschaften des $s(t)$-Integrals, die in folgendem teilweise benutzt werden, sind in [1, § 4] zusammengestellt.

$g(t)$ ist genau dann $s(t)$-meßbar, wenn $g(\omega)$ $\mathfrak{B}$-meßbar ist. Erfüllen $s(t)$, $\psi(t,t')$ die Voraussetzungen von Satz 2, so gibt es eine $s(t)$-integrable Funktion $\varphi(t)$ mit

$$\int_a^t \psi(I) = \int_a^t \varphi(\tau)\, s(d\tau);$$

$\varphi(t)$ ist bis auf eine σ_0-Nullmenge von Stetigkeitspunkten von $s(t)$ gleich den in Satz 2 auftretenden Limites, genauer:

$$\varphi(t) = \lim_{\tau \nearrow t} \frac{\psi(\tau,t)}{s(t)-s(\tau)} = \lim_{\tau \searrow t} \frac{\psi(t,\tau)}{s(\tau)-s(t)}$$

für σ_0-fast alle $t \in T$,

$$\left.\begin{matrix}\varphi(t)\\ \varphi^L(t)\end{matrix}\right\} = \lim_{\tau \nearrow t} \frac{\psi(\tau,t)}{s(t)-s(\tau)} \qquad \begin{cases}\text{für} & t \in S^L\\ \text{für} & t \in S^{L,R}\end{cases}$$

und

$$\left.\begin{matrix}\varphi(t)\\ \varphi^R(t)\end{matrix}\right\} = \lim_{\tau \searrow t} \frac{\psi(t,\tau)}{s(\tau)-s(t)} \qquad \begin{cases}\text{für} & t \in S^R\\ \text{für} & t \in S^{L,R}\end{cases}$$

(siehe [1, Satz (4.6)]).

Eine unmittelbare Folgerung aus Satz 2 ist

SATZ 3. *Sind $\psi_1(t,t')$, $\psi(t,t')$ für $a \leqq t < t' \leqq b$ erklärt mit $0 \leqq \psi_1(t,t') \leqq \psi(t,t')$, sind beide Intervallfunktionen über $[a,b]$ B-integrabel, und definieren wir $s(t) := \int_a^t \psi(I)$ und damit σ_0, S^L etc. wie früher, so existieren*

$$\lim_{\tau \nearrow t} \frac{\psi_1(\tau,t)}{\psi(\tau,t)}$$

für alle $t \in S^L \cup S^{L,R}$ und für σ_0-fast alle $t \in T$ und

$$\lim_{\tau \searrow t} \frac{\psi_1(t,\tau)}{\psi(t,\tau)}$$

für alle $t \in S^R \cup S^{L,R}$ und für σ_0-fast alle $t \in T$.

Beide Grenzwerte sind für σ_0-fast alle $t \in T$ gleich.

SATZ 4. *Sind $\psi(t,t')$, $\psi_1(t,t'),\ldots$ über $[a,b]$ nicht-negativ und B-integrabel, ist*

$$\psi(t,t') = \sum_{\mu=1}^{\infty} \psi_\mu(t,t') \qquad \text{für} \quad a \leqq t < t' \leqq b\,,$$

und gilt

$$\left| \sum_{\mu=m_\varepsilon}^{\infty} \psi_\mu(Z) \right| < \varepsilon \quad \text{für} \quad Z \geq Z_\varepsilon$$

für jedes $\varepsilon > 0$, *so ist*

$$\int_a^b \psi(I) = \sum_{\mu=1}^{\infty} \int_a^b \psi_\mu(I) .$$

Weiter erhalten wir mit $s(t) := \int_a^t \psi(I)$

$$\sum_{\mu=1}^{\infty} \lim_{\tau \nearrow t} \frac{\psi_\mu(\tau,t)}{\psi(\tau,t)} = 1$$

für $t \in S^L \cup S^{L,R}$ *und für* σ_0*-fast alle* $t \in T$, *und*

$$\sum_{\mu=1}^{\infty} \lim_{\tau \searrow t} \frac{\psi_\mu(t,\tau)}{\psi(t,\tau)} = 1$$

für $t \in S^R \cup S^{L,R}$ *und für* σ_0*-fast alle* $t \in T$.

BEWEIS. Wir setzen

$$\varphi_m(t,t') := \sum_{\mu=1}^{m} \psi_\mu(t,t') .$$

Zu $\varepsilon > 0$ gibt es ein Z_0 mit

$$\left| \psi(Z) - \int_a^b \psi(I) \right| < \varepsilon \quad \text{für} \quad Z \geq Z_0$$

und nach Voraussetzung ein m_ε und ein Z_ε mit

$$\psi(Z) - \varepsilon < \varphi_{m_\varepsilon}(Z) \leqq \psi(Z) \quad \text{für} \quad Z \geq Z_\varepsilon .$$

Für $m \geq m_\varepsilon$, $Z \geq Z_\varepsilon$, $Z \geq Z_0$ gilt dann

$$\int_a^b \psi(I) - 2 \cdot \varepsilon < \psi(Z) - \varepsilon < \varphi_{m_\varepsilon}(Z) < \varphi_m(Z) \leqq \psi(Z) < \int_a^b \psi(I) + \varepsilon ;$$

wir erhalten daher

$$\int_a^b \psi(I) = \lim_{m \to \infty} \int_a^b \varphi_m(I) ,$$

was wegen

$$\int_a^b \varphi_m(I) = \sum_{\mu=1}^{m} \int_a^b \psi_\mu(I)$$

die erste Behauptung ergibt. Sind $g_\mu(t)$ die durch

$$\int_a^t g_\mu(\tau)\, s(d\tau) = \int_a^t \psi_\mu(I)$$

bis auf σ_0-Nullmengen $N_\mu \subset T$ eindeutig festgelegten, $s(t)$-integrablen Funktionen, so gilt die Gleichung

$$\int_a^b 1\, s(d\tau) = \sum_{\mu=1}^{\infty} \int_a^b g_\mu(\tau)\, s(d\tau) = \int_a^b \Big(\sum_{\mu=1}^{\infty} g_\mu(\tau) \Big)\, s(d\tau)\,,$$

wegen $0 \leqq \sum_{\mu=1}^{\infty} g_\mu(t) \leqq 1$ und des für $s(t)$-Integrale gültigen Satzes von Lebesgue. Daraus folgt, daß bis auf eine σ_0-Nullmenge von Punkten aus T stets $\sum_{\mu=1}^{\infty} g_\mu(t) = 1$ gilt. Nach der vor Satz 3 gemachten Bemerkung erhalten wir schließlich die zweite Behauptung.

Als letzten Satz dieses Paragraphen formulieren wir folgende einfache Existenzaussage:

Satz 5. *Ist $\psi(t,t')$ über $[a,b]$ B-integrabel, und gilt für die Funktion $\psi_1(t,t')$ die Ungleichung*

$$\left| \psi_1(t_0,t_n) - \sum_{\nu=1}^{n} \psi_1(t_{\nu-1},t_\nu) \right| \leqq \left| \psi(t_0,t_n) - \sum_{\nu=1}^{n} \psi(t_{\nu-1},t_\nu) \right|$$

für beliebige $t_0,t_1,\ldots,t_{n-1}$, t_n mit $a \leqq t_0 < t_1 < \ldots < t_{n-1} < t_n \leqq b$, so ist ψ_1 über $[a,b]$ ebenfalls B-integrabel.

3. Die verallgemeinerten Palmschen Formeln.

Es sei $x(t)$ ein in $0 \leqq t < +\infty$ definierter Call-Prozeß, d.h. ein stochastischer Prozeß mit

$$\sum_{i=0}^{\infty} p\big(x(t_2) - x(t_1) = i\big) = 1$$

für beliebige t_1, t_2 mit $0 \leqq t_1 < t_2 < +\infty$. ($x(t_2) - x(t_1)$ ist mit Wahrscheinlichkeit Eins eine nicht negative, *endliche*, ganze Zahl. Über den Wert $i = \infty$ ist nicht zu summieren.) Weiter setzen wir für diesen Paragraphen voraus, daß

$$s(t) := \sup \left\{ \sum_{\nu=1}^{n} p\big(x(t_\nu) - x(t_{\nu-1}) \geqq 1\big) \colon\ 0 = t_0 < t_1 < \ldots < t_n = t \right\} < +\infty$$

für jedes $t > 0$ ist. (Wegen der Bedeutung dieser Voraussetzung siehe [3] und [4]. In [7, § 7] ist ein ordinärer, stationärer Call-Prozeß konstruiert, der diese Voraussetzung nicht erfüllt.) Führen wir die Abkürzung

$$\Psi_i(t',t'') := p\big(x(t'') - x(t') \geqq i\big) \quad \text{für} \quad 0 \leqq t' < t''\,,$$

$i=1,2,\ldots$, ein, so besagt diese letzte Bedingung nur, daß die Intervallfunktion $\Psi_1(t,t')$ über jedem endlichen Intervall $[a,b]\subset[0,+\infty)$ B-integrabel ist, da

$$\Psi_1(t,t') \leqq \sum_{\nu=1}^{n} \Psi_1(t_{\nu-1},t_\nu)$$

für $t=t_0<t_1<\ldots<t_n=t'$ gilt. ($\Psi_1(t,t')$ ist unteradditiv.) Wegen

$$0 \leqq \Psi_1(t',t'') \leqq E\big(x(t'')-x(t')\big)$$

folgt aus $E\big(x(t)\big)<+\infty$ für jedes $t\geqq 0$ die B-Integrabilität von $\Psi_1(t,t')$ über jedem endlichen Teilintervall der positiven Halbachse. Ist $x(t)$ ein stationärer Call-Prozeß, d.h.

$$\Psi_i(t,t') = \Psi_i(0,t'-t)\,,$$

so liegt B-Integrabilität von $\Psi_1(t,t')$ genau dann vor, wenn der Parameter $\lambda:=\lim_{t\searrow 0}\Psi_1(0,t)/t$ des Call-Prozesses endlich ist; $s(t)$ hat in diesem Fall die Form $s(t)=\lambda t$, wie sich sofort aus Satz 1 ergibt. Für einen Call-Prozeß mit B-integrablen $\Psi_1(t,t')$ braucht natürlich der Erwartungswert $E\big(x(t)\big)$ nicht zu existieren. Wir definieren die Intervallfunktionen

$$\Psi_{i,k}(t',t'';b) := p\big(x(t'')-x(t')\geqq i,\, x(b)-x(t'')\geqq k\big)$$

($0\leqq t'<t''\leqq b$); für jedes $b\geqq t''$ ist $\Psi_{i,0}(t',t'';b)=\Psi_i(t',t'')$. Zunächst beweisen wir

LEMMA. *Für $i=1,2,\ldots$; $k=0,1,2,\ldots$; $0\leqq t_0<t_1<\ldots<t_n\leqq b$ gilt*

$$\left|\Psi_{i,k}(t_0,t_n;b) - \sum_{\nu=1}^{n}\Psi_{i,k}(t_{\nu-1},t_\nu;b)\right| \leqq \sum_{\nu=1}^{n}\Psi_1(t_{\nu-1},t_\nu) - \Psi_1(t_0,t_n)\,.$$

BEWEIS. a) Es sei $n=2$. Mit den Abkürzungen

$$z := x(b)-x(t_2),\qquad z_1 := x(t_1)-x(t_0),\qquad z_2 := x(t_2)-x(t_1)$$

gilt

$$\begin{aligned}\Psi_1(t_0,t_1)+\Psi_1(t_1,t_2) &= p(z_1\geqq 1)+p(z_2\geqq 1)\\ &= p(z_1\geqq 1,z_2=0) + p(z_1\geqq 1,z_2\geqq 1) + p(z_2\geqq 1)\\ &= p(z_1+z_2\geqq 1) + p(z_1\geqq 1,z_2\geqq 1)\\ &= \Psi_1(t_0,t_2) + p(z_1\geqq 1,z_2\geqq 1)\,.\end{aligned}$$

Weiter ist

$$\begin{aligned}\Psi_{i,k}(t_0,t_2;b) &= p(z_1+z_2\geqq i, z\geqq k)\\ &\leqq p(z_1\geqq i,z_2=0,z\geqq k)+p(z_1=0,z_2\geqq i,z\geqq k)\\ &\quad + p(z_1\geqq 1,z_2\geqq 1,z_1+z_2\geqq i,z\geqq k)\\ &\leqq \Psi_{i,k}(t_0,t_1;b) + \Psi_{i,k}(t_1,t_2;b)+p(z_1\geqq 1,z_2\geqq 1)\\ &= \sum_{\nu=1}^{2}\Psi_{i,k}(t_{\nu-1},t_\nu;b) + \sum_{\nu=1}^{2}\Psi_1(t_{\nu-1},t_\nu) - \Psi_1(t_0,t_2)\end{aligned}$$

und

$$\sum_{\nu=1}^{2} \Psi_{i,k}(t_{\nu-1},t_\nu;\, b)$$
$$= p(z_1 \geqq i, z_2+z \geqq k) + p(z_2 \geqq i, z \geqq k)$$
$$= p(z_1 \geqq i, z_2=0, z \geqq k) + p(z_1 \geqq i, z_2 \geqq 1, z_2+z \geqq k) + p(z_2 \geqq i, z \geqq k)$$
$$\leqq p(z_1+z_2 \geqq i, z \geqq k) + p(z_1 \geqq 1, z_2 \geqq 1)$$
$$= \Psi_{i,k}(t_0,t_2;\, b) + \sum_{\nu=1}^{2} \Psi_1(t_{\nu-1},t_\nu) - \Psi_1(t_0,t_2)\,.$$

Für $n=2$ ist also die Aussage des Lemmas bewiesen.

b) Wenden wir nun vollständige Induktion an, so erhalten wir wegen

$$\left| \Psi_{i,k}(t_0,t_n;\, b) - \sum_{\nu=1}^{n} \Psi_{i,k}(t_{\nu-1},t_\nu;\, b) \right|$$
$$\leqq |\Psi_{i,k}(t_0,t_n;\, b) - \Psi_{i,k}(t_0,t_1;\, b) - \Psi_{i,k}(t_1,t_n;\, b)| +$$
$$+ \left| \Psi_{i,k}(t_1,t_n;\, b) - \sum_{\nu=2}^{n} \Psi_{i,k}(t_{\nu-1},t_\nu;\, b) \right|$$
$$\leqq \Psi_1(t_0,t_1) + \Psi_1(t_1,t_n) - \Psi_1(t_0,t_n) + \sum_{\nu=2}^{n} \Psi_1(t_{\nu-1},t_\nu) - \Psi_1(t_1,t_n)$$

sofort die Behauptung.

Aus diesem Lemma und aus Satz 5 folgt, daß die Intervallfunktionen $\Psi_{i,k}(t,t';\, b)$, $\Psi_i(t,t')$ und damit auch

$$\psi_{i,k}(t',t'';\, b) := p\big(x(t'')-x(t')=i, x(b)-x(t'')=k\big)$$
$$= \Psi_{i,k}(t',t'';\, b) - \Psi_{i,k+1}(t',t'';\, b) - \Psi_{i+1,k}(t',t'';\, b) + \Psi_{i+1,k+1}(t',t'';\, b)$$

und

$$\psi_i(t',t'') := p\big(x(t'')-x(t')=i\big) = \Psi_i(t',t'') - \Psi_{i+1}(t',t'')$$

über jedem endlichen Teilintervall von $[0,b]$ bzw. von $[0,+\infty)$ B-integrabel sind.

Bezeichnen wir mit σ_i das Maß über $\mathfrak{B}(0,+\infty)$, das vom Stetigkeitsanteil von

$$s_i(t) := \int_0^t \psi_i(I), \qquad i=1,2,\ldots,$$

definiert wird und mit σ_0 das vom Stetigkeitsanteil von $s(t)$ definierte Maß, so existieren wegen

$$0 \leqq \psi_{i,k}(t',t'';\, b) \leqq \psi_i(t',t'') \leqq \Psi_i(t',t'') \leqq \Psi_1(t',t'')$$

und wegen

$$0 \leqq \Psi_{i,k}(t',t'';\, b) \leqq \Psi_1(t',t'')$$

nach Satz 3

$$\lim_{\tau \nearrow t} p\big(x(t)-x(\tau)=i \mid x(t)-x(\tau)\geqq 1\big) = \lim_{\tau \nearrow t} \frac{\psi_i(\tau,t)}{\Psi_1(\tau,t)}$$

und

$$\lim_{\tau \nearrow t} p\big(x(b)-x(t)=k \mid x(t)-x(\tau)\geqq 1\big) = \lim_{\tau \nearrow t} \frac{\Psi_{1,k}(\tau,t;b)-\Psi_{1,k+1}(\tau,t;b)}{\Psi_1(\tau,t)}$$

für σ_0-fast alle $t \in T$ und alle $t \in S^L \cup S^{L,R}$ und analog

und

$$\lim_{\tau \searrow t} p\big(x(\tau)-x(t)=i \mid x(\tau)-x(t)\geqq 1\big)$$

$$\lim_{\tau \searrow t} p\big(x(b)-x(\tau)=k \mid x(\tau)-x(t)\geqq 1\big)$$

für σ_0-fast alle $t \in T$ und alle $t \in S^R \cup S^{L,R}$. Auf T sind die Grenzwerte $\tau \nearrow t$ σ_0-fast überall gleich den Grenzwerten bei $\tau \searrow t$. Ebenso existieren die Grenzwerte von

$$p\big(x(b)-x(t)=k \mid x(t)-x(\tau)=i\big) = \frac{\psi_{i,k}(\tau,t;b)}{\psi_i(\tau,t)}$$

für $\tau \nearrow t$ für σ_i-fast alle t und für alle t mit $s_i(t-0)<s_i(t)$ und die entsprechenden Grenzwerte für $\tau \searrow t$ bis auf ebensolche Ausnahmemengen.

Eine Funktion $\varphi(t;b)$, die $s(t)$-integrabel in t (bei festem b) in jedem endlichen Teilintervall von $[0,+\infty)$ bzw. $[0,b]$ ist, nennen wir kurz $s(t)$-integrabel. Es gilt

SATZ 5. *Es existieren $s(t)$-integrable Funktionen*

$$p_i(t) \mid 0 \leqq t < +\infty\,,$$
$$\varphi_{\cdot,k}(t;b) \mid 0 \leqq t \leqq b < +\infty\,,$$
$$\varphi_{i,k}(t;b) \mid 0 \leqq t \leqq b < +\infty$$

mit folgenden Eigenschaften:

a) *Es gibt eine σ_0-Nullmenge $N \subset T$ derart, daß für $i=1,2,\ldots$ gilt:*

$$p_i(t) = \lim_{\tau \nearrow t} p\big(x(t)-x(\tau)=i \mid x(t)-x(\tau)\geqq 1\big)$$
$$= \lim_{\tau \searrow t} p\big(x(\tau)-x(t)=i \mid x(\tau)-x(t)\geqq 1\big)$$

für alle $t \in T-N$,

$$\left.\begin{matrix} p_i(t) \\ p_i^L(t) \end{matrix}\right\} = \lim_{\tau \nearrow t} p\big(x(t)-x(\tau)=i \mid x(t)-x(\tau)\geqq 1\big) \quad \begin{matrix} \text{für } t \in S^L \\ \text{für } t \in S^{L,R} \end{matrix}$$

und

$$\left.\begin{matrix} p_i(t) \\ p_i^R(t) \end{matrix}\right\} = \lim_{\tau \searrow t} p\big(x(\tau)-x(t)=i \mid x(\tau)-x(t)\geqq 1\big) \quad \begin{matrix} \text{für } t \in S^R \\ \text{für } t \in S^{L,R}\,. \end{matrix}$$

b) *Zu jedem b mit $0<b<+\infty$ gibt es eine σ_0-Nullmenge $N(b)\subset T\cap(0,b)$, so daß für $k=0,1,2,\ldots$*

$$\begin{aligned}\varphi_{\cdot,k}(t;b) &= \lim_{\tau\nearrow t} p\big(x(b)-x(t)=k \mid x(t)-x(\tau)\geqq 1\big)\\ &= \lim_{\tau\searrow t} p(x(b)-x(\tau)=k \mid x(\tau)-x(t)\geqq 1)\end{aligned}$$

für alle $t\in T\cap(0,b)-N(b)$ gilt, und die entsprechenden Gleichungen für $t\in S^L$, $t\in S^R$ und $t\in S^{L,R}$ bestehen, und für $i=1,2,\ldots$; $k=0,1,2,\ldots$ die Gleichungen

$$\begin{aligned}\varphi_{i,k}(t;b) &= \lim_{\tau\nearrow t} p\big(x(b)-x(t)=k \mid x(t)-x(\tau)=i\big)\\ &= \lim_{\tau\searrow t} p\big(x(b)-x(\tau)=k \mid x(\tau)-x(t)=i\big)\end{aligned}$$

für alle $t\in T\cap(0,b)-N(b)$ mit $p_i(t)\neq 0$,

$$\varphi_{i,k}(t;b) = \lim_{\tau\nearrow t} p\big(x(b)-x(t)=k \mid x(t)-x(\tau)=i\big)$$

für alle $t\in S^L$ mit $p_i(t)\neq 0$,

$$\varphi^L_{i,k}(t;b) = \lim_{\tau\nearrow t} p\big(x(b)-x(t)=k \mid x(t)-x(\tau)=i\big)$$

für alle $t\in S^{L,R}$ mit $p_i{}^L(t)\neq 0$,

$$\varphi^R_{i,k}(t;b) = \lim_{\tau\searrow t} p\big(x(b)-x(\tau)=k \mid x(\tau)-x(t)=i\big)$$

für alle $t\in S^{L,R}$ mit $p_i{}^R(t)\neq 0$, und die entsprechende Gleichung für alle $t\in S^R$ mit $p_i(t)\neq 0$ erfüllt sind.

Dieser Satz ist wegen $s_i(t)=\int_0^t p_i(\tau)\,s(d\tau)$ nur eine Zusammenfassung des Vorhergehenden.

SATZ 6. *Für die in Satz 5 eingeführten $s(t)$-integrablen Funktionen gelten bis auf eine σ_0-Nullmenge bzw. σ_i-Nullmenge die folgenden Beziehungen:*

a) $0\leqq p_i(t)\leqq 1$, $\quad\sum_{i=1}^{\infty} p_i(t)=1$;

b) $0\leqq\varphi_{\cdot,k}(t;b)\leqq 1$, $\quad\sum_{k=0}^{\infty}\varphi_{\cdot,k}(t;b)=1$,

$0\leqq\varphi_{i,k}(t;b)\leqq 1$, $\quad\sum_{k=0}^{\infty}\varphi_{i,k}(t;b)=1$;

c) $\sum_{i=1}^{\infty} p_i(t)\,\varphi_{i,k}(t;b)=\varphi_{\cdot,k}(t;b)$.

BEWEIS. Nach dem am Anfang dieses Paragraphen angegebenen Lemma ist

$$\Psi_{i,k}(Z;b) \leqq \Psi_{i,k}(Z_1;b)+\Psi_1(Z)-\Psi_1(Z_1) \qquad \text{für} \quad Z \geq Z_1 .$$

Wählen wir zu vorgegebenem $\varepsilon>0$ die Intervallunterteilung Z_1 von $[0,b]$ so, daß $|\Psi_1(Z)-\Psi_1(Z_1)|<\varepsilon$ für alle $Z \geq Z_1$ gilt, und zu Z_1 die natürliche Zahl m_ε so, daß

$$\Psi_{m_\varepsilon,0}(Z_1;b) < \varepsilon, \qquad \Psi_{1,m_\varepsilon}(Z_1;b) < \varepsilon$$

ist, so erhalten wir für jede Intervallunterteilung Z von $[0,b]$ mit $Z \geqq Z_1$ und für $m \geqq m_\varepsilon$

$$|\Psi_{m,k}(Z;b)| < 2\varepsilon \qquad \text{und} \qquad |\Psi_{i,m}(Z;b)| < 2\varepsilon ,$$

$k=0,1,\ldots;\ i=1,2,\ldots$. Wegen

$$\Psi_m(t,t') = \sum_{i=m}^{\infty} p\big(x(t')-x(t)=i\big) ,$$

$$\Psi_{1,m}(t,t';b) = \sum_{k=m}^{\infty} p\big(x(b)-x(t')=k,\ x(t')-x(t)\geqq 1\big) ,$$

$$\Psi_{i,m}(t,t';b)-\Psi_{i+1,m}(t,t';b) = \sum_{k=m}^{\infty} p\big(x(b)-x(t')=k,\ x(t')-x(t)=i\big)$$

und

$$\Psi_{m,k}(t,t';b)-\Psi_{m,k+1}(t,t';b) = \sum_{i=m}^{\infty} p\big(x(b)-x(t')=k,\ x(t')-x(t)=i\big)$$

ergibt Satz 4 die nichttrivialen Behauptungen von Satz 6. Für die Aussage c) ist jedoch eine einfache Modifikation von Satz 4 notwendig.

Im nächsten Satz können wir nun die verallgemeinerten Palmschen Formeln zusammenstellen.

SATZ 7. *Für jeden Call-Prozeß mit B-integrablem* $\Psi_1(t',t'')$ *gilt für* $0 \leqq a < b < +\infty$

$$p\big(x(b)-x(a)=0\big) = 1 - \int_a^b \varphi_{\cdot,0}(t;b)\, s(dt) ,$$

$$p\big(x(b)-x(a)=k\big) = \sum_{i=1}^{k} \int_a^b p_i(t)\, \varphi_{i,k-i}(t;b)\, s(dt) - \int_a^b \varphi_{\cdot,k}(t;b)\, s(dt) ,$$

$k=1,2,\ldots$.

BEWEIS. $p\big(x(b)-x(t)=i\big)$ ist als Funktion von t totalstetig bezüglich $s(t)$, $i=0,1,\ldots$. Für $0 \leqq t' < t'' \leqq b$ gilt

$$p\big(x(b)-x(t')=0\big) = p\big(x(b)-x(t'')=0,\, x(t'')-x(t')=0\big)$$
$$= p\big(x(b)-x(t'')=0\big) - p\big(x(b)-x(t'')=0,\, x(t'')-x(t')\geqq 1\big)$$

und

$$p\big(x(b)-x(t')=k\big) = \sum_{j=0}^{k} p\big(x(b)-x(t'')=k-j,\, x(t'')-x(t')=j\big)$$
$$= \sum_{j=1}^{k} p\big(x(b)-x(t'')=k-j,\, x(t'')-x(t')=j\big) +$$
$$+ p\big(x(b)-x(t'')=k\big) - p\big(x(b)-x(t'')=k,\, x(t'')-x(t')\geqq 1\big).$$

Gehen wir zur Grundmenge Ω über, und bilden wir die $\mathfrak{A}$-Ableitungen $D\chi_0$, $D\chi_k$ der Intervallfunktionen

$$\chi_0(\langle t',t''\rangle) := \psi_0(t'',b)-\psi_0(t',b), \qquad \chi_k(\langle t',t''\rangle) := \psi_k(t'',b)-\psi_k(t',b)$$

bezüglich m, so erhalten wir (mit den Bezeichnungen von Seite 129)

$$D\chi_0(\omega) = \varphi_{\cdot,0}(\omega; b),$$
$$D\chi_k(\omega) = \varphi_{\cdot,k}(\omega; b) - \sum_{j=1}^{k} p_j(\omega)\cdot\varphi_{j,k-j}(\omega; b)$$

für m-fast alle $\omega\in\Omega$. Wegen der Totalstetigkeit von $p\big(x(b)-x(t)=i\big)$ folgt daraus

$$p\big(x(b)-x(a)=i\big)-p\big(x(b)-x(b)=i\big) = -\int_{\Omega} D\chi_i(\omega)\, m(d\omega),$$

also nach Übergang zu den $s(t)$-Integralen:

$$p\big(x(b)-x(a)=0\big) = 1 - \int_a^b \varphi_{\cdot,0}(t; b)\, s(dt),$$

$$p\big(x(b)-x(a)=k\big) = \sum_{j=1}^{k} \int_a^b p_i(t)\,\varphi_{j,k-j}(t; b)\, s(dt) - \int_a^b \varphi_{\cdot,k}(t; b)\, s(dt)$$

für $k=1,2,\ldots$, was zu beweisen war.

4. Zwei Sätze von Khintchine.

In diesem Paragraphen sei $x(t)$ ein Call-Prozeß ohne Nachwirkungen (d.h. $x(t_2)-x(t_1)$ und $x(t_4)-x(t_3)$ sind unabhängig für $0\leqq t_1<t_2\leqq t_3<t_4<+\infty$). Zunächst beweisen wir (in [5] ist unter anderen Voraussetzungen eine ähnliche Aussage bewiesen)

SATZ 8. *Für einen Call-Prozeß $x(t)$ ohne Nachwirkungen gilt*

$$\sup\left\{\sum_{\nu=1}^{n} p\big(x(t_\nu)-x(t_{\nu-1})\geqq 1\big):\ a=t_0<t_1<\ldots<t_n=b\right\}<+\infty$$

für beliebige a, b mit $0\leqq a<b<+\infty$.

BEWEIS. Aus der Annahme

$$\sup\left\{\sum_{\nu=1}^{n} p\big(x(t_\nu)-x(t_{\nu-1})\geqq 1\big):\ a=t_0<t_1<\ldots<t_n=b\right\}=+\infty$$

folgt die Existenz von $\tau_1,\ldots,\tau_{n_0-1}$ mit $a=:\tau_0<\tau_1<\ldots<\tau_{n_0}:=b$ und mit

$$\sum_{\nu=1}^{n_0} p\big(x(\tau_\nu)-x(\tau_{\nu-1})\geqq 1\big)\geqq 1\,.$$

Ferner gibt es ein $\nu_0\in\{1,\ldots,n_0\}$ mit

$$\sup\left\{\sum_{\mu=1}^{m} p(x(t_\mu)-x(t_{\mu-1})\geqq 1):\ \tau_{\nu_0-1}=t_0<t_1<\ldots<t_m=\tau_{\nu_0}\right\}=+\infty\,.$$

Es sei

$$\begin{aligned}
z_1 &:= x(\tau_1)-x(a)\,,\\
z_2 &:= x(\tau_2)-x(\tau_1)\,,\\
&\vdots\\
z_{\nu_0-1} &:= x(\tau_{\nu_0-1})-x(\tau_{\nu_0-2})\,,\\
z_{\nu_0} &:= x(\tau_{\nu_0+1})-x(\tau_{\nu_0})\,,\\
&\vdots\\
z_{n_0-1} &:= x(b)-x(\tau_{n_0-1})\,.
\end{aligned}$$

Ebenso gibt es wieder $\tau_{11},\ldots,\tau_{1m_0-1}$ mit $\tau_{\nu_0-1}=\tau_{10}<\tau_{11}<\ldots<\tau_{1m_0}=\tau_{\nu_0}$ und mit

$$\sum_{\mu=1}^{m_0} p\big(x(\tau_{1\mu})-x(\tau_{1\mu-1})\geqq 1\big)\geqq 2;$$

ferner existiert ein μ_0 derart, daß Ψ_1 über $[\tau_{1\mu_0-1},\tau_{1\mu_0}]$ nicht B-integrabel ist. Setzen wir nun wieder

$$\begin{aligned}
z_{n_0} &:= x(\tau_{11})-x(\tau_{\nu_0-1})\\
&\vdots\\
z_{n_0+\mu_0-2} &:= x(\tau_{1\mu_0-1})-x(\tau_{1\mu_0-2})\\
z_{n_0+\mu_0-1} &:= x(\tau_{1\mu_0+1})-x(\tau_{1\mu_0})\\
&\vdots\\
z_{n_0+m_0-2} &:= x(\tau_{\nu_0})-x(\tau_{1m_0-1})
\end{aligned}$$

und wenden auf das Intervall $[\tau_{1\mu_0-1}, \tau_{1\mu_0}]$ wieder das gleiche Verfahren an usw., so erhalten wir eine Folge $z_1, z_2, \ldots$ von unabhängigen Zufallsvariablen mit

$$\sum_{\varrho=1}^{\infty} p(\{z_\varrho \geqq 1\}) = +\infty .$$

Nach dem Borel-Cantellischen Lemma treten daher mit Wahrscheinlichkeit Eins unendlich viele der Ereignisse $\{z_\varrho \geqq 1\}$ ein; folglich ist $p\big(x(b) - x(a) = +\infty\big) = 1$, was aber ein Widerspruch zur Definition des Call-Prozesses ist.

Wegen

$$\begin{aligned} p\big(x(b) - x(t'') = k \mid x(t'') - x(t') \geqq 1\big) &= p\big(x(b) - x(t'') = k \mid x(t'') - x(t') = i\big) \\ &= p\big(x(b) - x(t'') = k\big) \end{aligned}$$

erhalten wir

$$\varphi_{\cdot,k}(t; b) = \varphi_{i,k}(t; b) = p\big(x(b) - x(t) = k\big)$$

für $t \in S^L \cap (0, b]$ und für σ_0-fast alle $t \in [0, b] - (S^L \cup S^R \cup S^{L,R})$,

$$\varphi_{\cdot,k}(t; b) = \varphi_{i,k}(t; b) = \lim_{\tau \searrow t} p\big(x(b) - x(\tau) = k\big)$$

für $t \in S^R \cap [0, b)$ und

$$\varphi^L_{\cdot,k}(t; b) = \varphi^L_{i,k}(t; b) = p\big(x(b) - x(t) = k\big)$$

für $t \in S^{L,R} \cap (0, b]$,

$$\varphi^R_{\cdot,k}(t; b) = \varphi^R_{i,k}(t; b) = \lim_{\tau \searrow t} p\big(x(b) - x(\tau) = k\big)$$

für $t \in S^{L,R} \cap (0, b)$.

Definieren wir für $|z| \leqq 1$, z reell, die erzeugenden Funktionen

$$\Pi(z; t, b) := \sum_{k=0}^{\infty} p\big(x(b) - x(t) = k\big)\, z^k$$

und

$$\Phi(z; t, b) := \sum_{k=0}^{\infty} \varphi_{\cdot,k}(t; b)\, z^k ,$$

für $0 \leqq t \leqq b$, so ist $\Phi(z; t, b)$ bei festen z und bei festem b über jedem Teilintervall von $[0, b]$ $s(t)$-integrabel, ferner

$$\begin{aligned} \Phi(z; t, b) &= \Pi(z; t, b) && \text{für} && \text{alle } t \in [0, b] - (S^R \cup S^{L,R}) , \\ \Phi(z; t, b) &= \Pi(z; t+0, b) && \text{für} && t \in [0, b) \cap S^R , \\ \Phi^L(z; t, b) &= \Pi(z; t, b) && \text{für} && t \in (0, b] \cap S^{L,R} , \\ \Phi^R(z; t, b) &= \Pi(z; t+0, b) && \text{für} && t \in (0, b) \cap S^{L,R} . \end{aligned}$$

Aus den verallgemeinerten Palmschen Formeln ergibt sich

$$\Pi(z;t,b) = 1 - \int_t^b \varphi_{\cdot,0}(\tau;b)\, s(d\tau) + \\ + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}\int_t^b p_i(\tau)\,\varphi_{\cdot,k-i}(\tau;b)\,z^k\, s(d\tau) - \int_t^b \varphi_{\cdot,k}(\tau;b) z^k\, s(d\tau)\right)$$

und nach Vertauschung der Summationsreihenfolge

$$(*) \qquad \Pi(z;t,b) = 1 + \int_t^b \Phi(z;\tau,b)\, P(z;\tau)\, s(d\tau)$$

mit

$$P(z;\tau) := \sum_{k=1}^{\infty} p_k(\tau) z^k - 1 .$$

Definieren wir für $0<z<\frac{1}{2}$ und für $0\leqq t\leqq b$ die Funktion

$$\Theta(z;t,b) := \ln \Pi(z;t,b) ,$$

so gilt wegen

$$\Pi(z;t_1,b) = \Pi(z;t_1,t_2)\,\Pi(z;t_2,b) \qquad \text{für} \quad 0\leqq t_1\leqq t_2\leqq b$$

zunächst

$$\Theta(z;t_2,b)-\Theta(z;t_1,b) = -\ln \Pi(z;t_1,t_2) .$$

Weiter ist für $|z|\leqq\frac{1}{2}$ stets

$$|1-\Pi(z;t_1,t_2)| \leqq p\big(x(t_2)-x(t_1)\geqq 1\big)\sum_{k=0}^{\infty}|z|^k \leqq 2\big(s(t_2)-s(t_1)\big)$$

und damit für t_1,t_2 mit $s(t_2)-s(t_1)\leqq\frac{1}{4}$ auch

$$|\Theta(z;t_1,b)-\Theta(z;t_2,b)| = \left|\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\big(1-\Pi(z;t_1,t_2)\big)^n}{n}\right| \\ \leqq \sum_{n=1}^{\infty}\big[2\big(s(t_2)-s(t_1)\big)\big]^n \\ \leqq \big[s(t_2)-s(t_1)\big]\frac{2}{1-\frac{1}{2}} = 4\big[s(t_2)-s(t_1)\big];$$

das bedeutet, daß die Intervallfunktion $\Theta(z;t_2,b)-\Theta(z;t_1,b)$ bezüglich $s(t_2)-s(t_1)$ totalstetig ist. Folglich existiert eine $s(t)$-integrable Funktion $\vartheta(z;t,b)$ mit

$$\Theta(z;t,b) = \int_t^b \vartheta(z;\tau,b)\, s(d\tau) \qquad \text{für} \quad 0\leqq t\leqq b$$

(vgl. hierzu die Ausführungen vor Satz 3 und [1, § 4]). Aus (*) können wir $\vartheta(z;t,b)$ berechnen.

Ist $t \in (0,b] \cap T$, so ist auch $\Pi(z;t-0,b) = \Pi(z;t,b)$ und damit bis auf eine σ_0-Nullmenge von t-Werten

$$\begin{aligned}\vartheta(z;t,b) &= -\lim_{\tau \nearrow t} \frac{\Theta(z;t,b)-\Theta(z;\tau,b)}{s(t)-s(\tau)} \\ &= -\lim_{\tau \nearrow t} \frac{\ln \Pi(z;t,b)-\ln \Pi(z;\tau,b)}{\Pi(z;t,b)-\Pi(z;\tau,b)} \lim_{\tau \nearrow t} \frac{\Pi(z;t,b)-\Pi(z;\tau,b)}{s(t)-s(\tau)} \\ &= \frac{-1}{\Pi(z;t,b)} \lim_{\tau \nearrow t} \frac{\Pi(z;t,b)-\Pi(z;\tau,b)}{s(t)-s(\tau)} = P(z;t)\,,\end{aligned}$$

da der letzte Limes für σ_0-fast alle Stetigkeitspunkte von $s(t)$ gleich

$$-\Phi(z;t,b)\cdot P(z;t) = -\Pi(z;t,b)\,P(z;t)$$

ist. Ist $t \in (0,b] \cap (S^L \cup S^{L,R})$, so folgt aus (*)

$$\Pi(z;t-0,b) = \begin{cases} \Pi(z;t,b)\,[1+P(z;t)\,(s(t)-s(t-0))], & t \in S^L\,, \\ \Pi(z;t,b)\,[1+P^L(z;t)\,(s(t)-s(t-0))], & t \in S^{L,R}\,, \end{cases}$$

und damit

$$\begin{aligned}&\lim_{\tau \nearrow t} \frac{\Theta(z;t,b)-\Theta(z;\tau,b)}{s(t)-s(\tau)} \\ &\quad = \lim_{\tau \nearrow t} \frac{1}{s(t)-s(\tau)} \ln \frac{\Pi(z;t,b)}{\Pi(z;\tau,b)} \\ &\quad = \frac{-1}{s(t)-s(t-0)} \begin{cases} \ln[1+P(z;t)\cdot(s(t)-s(t-0))], & t \in S^L\,, \\ \ln[1+P^L(z;t)\cdot(s(t)-s(t-0))], & t \in S^{L,R}\,, \end{cases}\end{aligned}$$

also

$$\vartheta(z;t,b) = \frac{1}{s(t)-s(t-0)} \ln[1+P(z;t)\cdot(s(t)-s(t-0))], \quad t \in S^L\,,$$

und

$$\vartheta^L(z;t,b) = \frac{1}{s(t)-s(t-0)} \ln[1+P^L(z;t)\cdot(s(t)-s(t-0))], \quad t \in S^{L,R}\,.$$

Ebenso erhält man

$$\vartheta(z;t,b) = \frac{1}{s(t+0)-s(t)} \ln[1+P(z;t)\,(s(t+0)-s(t))], \quad t \in S^R\,,$$

und

$$\vartheta^R(z;t,b) = \frac{1}{s(t+0)-s(t)} \cdot \ln[1+P^R(z;t)\,(s(t+0)-s(t))], \quad t \in S^{L,R}\,.$$

Bildet man das $s(t)$-Integral der Funktion $\vartheta(z;t,b)$, so ergibt sich die Behauptung des nächsten Satzes; die im Beweis notwendige Einschränkung $0<z<\frac{1}{2}$ kann man nunmehr wieder durch $|z|\leqq 1$ ersetzen, da die auftretenden Potenzreihen in diesem Bereich konvergieren.

SATZ 9. *Für jeden Call-Prozeß ohne Nachwirkungen hat die erzeugende Funktion $\Pi(z;a,b)$ die Form*

$$\Pi(z;a,b) = \exp\left\{\int_{(a,b]\cap T} P(z;t)\,\sigma_0(dt)\right\} U_1(z;a,b)\,U_2(z;a,b)$$

mit

$$P(z;t) = \sum_{k=1}^{\infty} p_k(t)\,z^k - 1\,,$$

$$U_1(z;a,b) := \prod_{\tau\in(a,b]\cap S^L}\big(1+P(z;\tau)\,[s(\tau)-s(\tau-0)]\big)\cdot \cdot \prod_{\tau\in(a,b]\cap S^{L,R}}\big(1+P^L(z;\tau)\,[s(\tau)-s(\tau-0)]\big)$$

und

$$U_2(z;a,b) := \prod_{\tau\in[a,b)\cap S^R}\big(1+P(z;\tau)\,[s(\tau+0)-s(\tau)]\big)\cdot \cdot \prod_{\tau\in[a,b)\cap S^{L,R}}\big(1+P^R(z;\tau)\,[s(\tau+0)-s(\tau)]\big)\,.$$

Khintchine hat in [8] diesen Satz unter der zusätzlichen Voraussetzung »$E\big(x(t)\big)<+\infty$ für jedes $t\geqq 0$« mit anderen Hilfsmitteln bewiesen. Die Umkehrung kann man aus [8] übernehmen. Für die in [8] beim regulären Fall auftretenden Funktionen $\chi_k(\beta)$ erhält man

$$\chi_k(\beta) = \begin{cases} \int_0^{\beta} p_k(t)\,s(dt) = \int_0^{\beta} \psi_k(I), & k\geqq 1 \\ -s(\beta) = -\int_0^{\beta}\psi_1(I), & k=0; \end{cases}$$

die Darstellung der $\chi_k(\beta)$ als Burkill-Integrale ist bereits in [14] enthalten.

Als einfachstes Beispiel erhalten wir für $p_1(t)=1$, $p_k(t)=0$ für $k\geqq 2$, $s(t)=\lambda t$ den Poisson-Prozeß mit der erzeugenden Funktion

$$\Pi(z;a,b) = e^{(z-1)\lambda(b-a)}\,.$$

Einfach ist nun der Beweis für folgenden Satz, der eine Präzisierung des Ergebnisses von [9] ist.

SATZ 10. *Für einen Call-Prozeß $x(t)$ gilt*

$$p\big(x(b)-x(a)=k\big) = e^{-(\Lambda(b)-\Lambda(a))}\frac{[\Lambda(b)-\Lambda(a)]^k}{k!}$$

für $k=0,1,\ldots$ mit monoton nicht fallendem $\Lambda(t)$ genau dann, wenn folgende Bedingungen erfüllt sind:

a) *$x(t)$ ist ohne Nachwirkungen;*

b) *der Stetigkeitsanteil von $\Lambda(t)$ ist bis auf eine additive Konstante gleich dem Stetigkeitsanteil von $s(t)$; das Stetigkeitsverhalten von $\Lambda(t)$ ist in jedem Punkt gleich dem Stetigkeitsverhalten von $s(t)$;*

c) $\Lambda(t)-\Lambda(t-0)=-\ln\{1-\big(s(t)-s(t-0)\big)\}$ *für* $t\in S^L\cup S^{L,R}$,
$\Lambda(t+0)-\Lambda(t)=-\ln\{1-\big(s(t+0)-s(t)\big)\}$ *für* $t\in S^R\cup S^{L,R}$;

d) $p_1(t)=1$, $p_k(t)=0$ *für* $k\geqq 2$ *σ_0-fast überall auf T;*

e) $$\left.\begin{matrix}p_k(t)\\ p_k{}^L(t)\end{matrix}\right\} = \frac{1}{s(t)-s(t-0)}e^{-(\Lambda(t)-\Lambda(t-0))}\frac{[\Lambda(t)-\Lambda(t-0)]^k}{k!}$$

für $t\in S^L$ resp. $t\in S^{L,R}$;

$$\left.\begin{matrix}p_k(t)\\ p_k{}^R(t)\end{matrix}\right\} = \frac{1}{s(t+0)-s(t)}e^{-(\Lambda(t+0)-\Lambda(t))}\frac{[\Lambda(t+0)-\Lambda(t)]^k}{k!}$$

für $t\in S^R$ resp. $t\in S^{L,R}$.

BEWEIS. 1) $p\big(x(b)-x(a)=k\big)$ hat genau dann die angegebene Gestalt, wenn $x(b)-x(a)$ die erzeugende Funktion $\Pi(z;a,b)=\exp\{\big(\Lambda(b)-\Lambda(a)\big)(z-1)\}$ hat.

2) Erfüllt der Call-Prozeß $x(t)$ die Forderungen a)–e), so hat nach Satz 9 $x(b)-x(a)$ die erzeugende Funktion $\exp\{\big(\Lambda(b)-\Lambda(a)\big)(z-1)\}$.

3) Hat $x(t')-x(t)$ für $0\leqq t<t'<+\infty$ die erzeugende Funktion $\exp\{\big(\Lambda(t')-\Lambda(t)\big)(z-1)\}$, so ist $x(t)$ ohne Nachwirkungen. Die erzeugende Funktion $\Pi(z;a,b)$ hat daher die in Satz 9 angegebene Gestalt.

Zerlegt man beide Darstellungen der erzeugenden Funktion $\Pi(z;a,b)$ in den Anteil von $S^L\cap(a,b]$:

$$\prod_{\tau\in S^L\cap(a,b]}\Pi(z;\tau-0,\tau)\,,$$

in den entsprechenden Anteil von $S^R\cap[a,b)$, in den Anteil von $S^{L,R}\cap[a,b]$:

$$\Big(\prod_{\tau\in S^{L,R}\cap(a,b]}\Pi(z;\tau-0,\tau)\Big)\Big(\prod_{\tau\in S^{L,R}\cap[a,b)}\Pi(z;\tau,\tau+0)\Big)$$

und in den Anteil von $T\cap[a,b]$, so erhält man durch Vergleichen

$$1 + P(z;\tau)\,[s(\tau)-s(\tau-0)] = \exp\{\big(\Lambda(\tau)-\Lambda(\tau-0)\big)(z-1)\}$$

für $\tau \in S^L$ und entsprechend für $\tau \in S^R$ und $\tau \in S^{L,R}$ und für den Anteil von $T \cap [a,b]$ schließlich

$$\exp\left\{\int_{(a,b]\cap T} P(z;\tau)\,\sigma_0(d\tau)\right\} = \exp\{\big(\Lambda_0(b)-\Lambda_0(a)\big)(z-1)\}\,,$$

wobei zur Abkürzung

$$\Lambda_0(t) := \Lambda(t) - \sum_{\tau\in S^L(t)} \big(\Lambda(\tau)-\Lambda(\tau-0)\big) - \sum_{\tau\in S^R(t)} \big(\Lambda(\tau+0)-\Lambda(\tau)\big)$$

mit

$$S^L(t) := (S^L \cup S^{L,R}) \cap (0,t] \quad \text{und} \quad S^R(t) := (S^R \cup S^{L,R}) \cap [0,t)$$

gesetzt ist. Daraus ergeben sich unmittelbar die Behauptungen b)–e).

5. Ein Satz von Korolyuk.

Nach einem Satz von Korolyuk (siehe [7]) gilt für einen stationären, ordinären Call-Prozeß $x(t)$ stets

$$E\big(x(b)-x(a)\big) = \lambda(b-a), \qquad \text{wobei} \quad \lambda = \lim_{\tau\searrow 0} \Psi_1(\tau)/\tau$$

der Parameter des Call-Prozesses $x(t)$ ist. Mit den hier entwickelten Methoden können wir allgemein $E\big(x(b)-x(a)\big)$ berechnen:

SATZ 11. *Ist $x(t)$ ein Call-Prozeß mit B-integrablem Ψ_1, so existiert der Erwartungswert von $x(b)-x(a)$ genau dann, wenn $\sum_{k=1}^{\infty} k\,p_k(t)$ über $[a,b]$ $s(t)$-integrabel ist. Wenn $E\big(x(b)-x(a)\big)$ existiert, gilt*

$$E\big(x(b)-x(a)\big) = \int_a^b \sum_{k=1}^{\infty} k\,p_k(t)\,s(dt)\,.$$

BEWEIS. 1) Sind x_1, x_2 zwei Zufallsvariablen, die mit Wahrscheinlichkeit Eins nicht negative ganze Zahlen annehmen, so ist für $m = 1,2,\ldots$

$$\begin{aligned}\sum_{k=1}^{m} k\,p(x_1+x_2=k) &= \sum_{\substack{i_1,i_2\geqq 0\\ i_1+i_2\leqq m}} (i_1+i_2)\,p(x_1=i_1,x_2=i_2)\\ &= \sum_{i_1=1}^{m} i_1 \sum_{i_2=0}^{m-i_1} p(x_1=i_1,x_2=i_2) + \sum_{i_2=1}^{m} i_2 \sum_{i_1=0}^{m-i_2} p(x_1=i_1,x_2=i_2)\\ &\leqq \sum_{i_1=1}^{m} i_1 p(x_1=i_1) + \sum_{i_2=1}^{m} i_2 p(x_2=i_2)\,.\end{aligned}$$

2) Es existiere $E(x(b)-x(a))$. Nach 1) gilt für $m=1,2,\ldots$ und für jede Intervallunterteilung Z von $[a,b]$

$$\sum_{k=1}^{m} k\,p(x(b)-x(a)=k) \leqq \sum_{k=1}^{m} k\,\psi_k(Z) \leqq E(x(b)-x(a)),$$

also auch

$$\sum_{k=1}^{m} k\,p(x(b)-x(a)=k) \leqq \sum_{k=1}^{m} \int_a^b k\,\psi_k(I) = \int_a^b \sum_{k=1}^{m} k\,p_k(t)\,s(dt)$$

$$\leqq E(x(b)-x(a)) < +\infty$$

und damit für $m \to \infty$ nach dem Satz von Lebesgue

$$E(x(b)-x(a)) = \sum_{k=1}^{\infty} k\,p(x(b)-x(a)=k)$$

$$\leqq \int_a^b \sum_{k=1}^{\infty} k\,p_k(t)\,s(dt) \leqq E(x(b)-x(a)).$$

3) Ist umgekehrt $\sum_{k=1}^{\infty} k\,p_k(t)$ $s(t)$-integrabel über $[a,b]$, so gilt für $m = 1,2,\ldots$

$$+\infty > \int_a^b \sum_{k=1}^{\infty} k\,p_k(t)\,s(dt) \geqq \int_a^b \sum_{k=1}^{m} k\,p_k(t)\,s(dt) \geqq \sum_{k=1}^{m} k\,p(x(b)-x(a)=k).$$

Daraus folgt die Existenz von $E(x(b)-x(a))$. Beweisteil 2) ergibt dann wieder die zweite Behauptung des Satzes.

Für stationäre Call-Prozesse ohne Nachwirkungen ist dieser Satz in [7] bewiesen. Eine unmittelbare Folgerung ist

SATZ 12. *Für einen Call-Prozeß $x(t)$ mit B-integrablem Ψ_1 gilt $E(x(b)-x(a))=s(b)-s(a)$ genau dann, wenn $p_k(t)=0$ für $k=2,\ldots$ und $p_1(t)=1$ für alle $t\in[a,b]$ mit Ausnahme einer σ_0-Nullmenge von Stetigkeitspunkten von $s(t)$ ist.*

Nennt man abweichend von der in [2], [9] und [14] gebrauchten Terminologie einen Call-Prozeß $x(t)$ ordinär, wenn $p_k(t)=0$ für $k=2,\ldots$ mit Ausnahme einer σ_0-Nullmenge von Stetigkeitspunkten von $s(t)$ gilt, so lautet Satz 12 wie folgt:

Die Ordinarität von $x(t)$ ist notwendig und hinreichend für die Gültigkeit der Gleichung $E(x(b)-x(a))=s(b)-s(a)$ $(0\leqq a<b<+\infty)$.

Diese Formulierung enthält als Spezialfälle den Satz von Korolyuk, die von Khintchine [7] und Zitek [13] bewiesenen Umkehrungen und, sofern man von der Abweichung in der Definition der Ordinarität absieht, auch die von Zitek [14] für nicht-stationäre Call-Prozesse ohne Nachwirkungen angegebene Verallgemeinerung.

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MAXIMAL-FASTPERIODIZITÄT VON GRUPPEN I

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Einleitung.

Will man die Theorie der fastautomorphe Funktionen auf höhere Modulgruppen übertragen, so ist es interessant zu wissen, ob es überhaupt fastperiodische Funktionen auf diesen Gruppen gibt. Für die gewöhnliche (=elliptische) Modulgruppe ist diese Frage erledigt durch die Existenz von treuen unitären 2-dim. Darstellungen, das heißt sie ist maximal-fastperiodisch (max.-f.p.) (daher auch ihre Untergruppen $\Gamma(n)$; speziell also alle freien Gruppen F_m). Die Maximal-fastperiodizität (Max.-f.p.) der Hilbert'schen und der Siegel'schen Modulgruppe ergibt sich im Folgenden in Satz 1.

Sei nämlich Γ_n der Ring der ganzen Zahlen eines algebraischen Zahlkörpers K_n, so wird bewiesen, daß die Gruppe $\mathfrak{M}_k(\Gamma_n)$, der $k \times k$-Matrizen mit Determinante 1 und Koeffizienten aus Γ_n, max.-f.p. ist. Dabei ist es möglich, ein System von Normalteilern $G_\mathfrak{p}$ mit endlichem Index und Durchschnitt $\bigcap G_\mathfrak{p} = 1$ zu finden. Dann wird die Max.-f.p. einiger Gruppen aufgezeigt, für welche ein Normalteilersystem mit den eben angegebenen Eigenschaften nicht unmittelbar ersichtlich ist. In einigen Fällen können wir auf Grund von Satz 1 auf die Max.-f.p. der Faktorgruppen bei Kongruenzmoduln schließen, und damit auf die der Gruppe selbst. In der Arbeit werden Beispiele von max.-f.p. Gruppen mit unendlich vielen Erzeugenden und Relationen und sogar solche Gruppen mit endlich vielen Erzeugenden und unendlich vielen Relationen gegeben. Sämtliche bisher bekannten minimal-f.p. Gruppen sind einfache Gruppen. Wir zeigen jedoch, daß die Gruppe $\mathfrak{M}_2(R[x])$, welche unendlich viele Normalteiler besitzt, min.-f.p. ist. Weiter geben wir ein Beispiel einer diskreten max.-f.p. Gruppe, welche keine treue endliche unitäre Darstellung besitzt. Lemma 3 und 4 haben als Aussagen über endliche unitäre Matrizen selbständiges Interesse. Besondere Beachtung verdient Satz 7.

1. Unimodulare Gruppen.

Wir wiederholen kurz die Definition von Max.-f.p. Sei G_0 der Normalteiler der Gruppe G, bestehend aus allen Elementen $a \in G$ mit $D(a) = 1$

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für jede endliche unitäre Darstellung D von G. Dann heißt G max.-f.p., falls $G_0 = 1$.

SATZ 1. *Die Gruppe* $\mathfrak{M}_k(\Gamma_n)$, *der* $k \times k$*-Matrizen mit Elementen aus dem Ring* Γ_n *und der Determinante* 1, *ist* max.-f.p.

BEMERKUNG. Da jede Untergruppe einer max.-f.p. Gruppe wieder max.-f.p. ist, so ist die in der Arbeit gegebene Formulierung (der Sätze 1, 2, 3, 4, 9, 10) als Kurzform der folgenden aufzufassen:

»Jede Matrizen-Gruppe, bestehend aus Matrizen mit Determinante 1 und Koeffizienten aus dem Ring ..., ist max.-f.p.«

Ist G_1 der Durchschnitt aller Normalteiler N von G mit endlichem Index, so gilt folgendes Lemma (vgl. Neumann–Wigner [4]):

LEMMA 0. *Es gilt stets* $G_0 \subseteqq G_1$.

BEWEIS DES SATZES 1. Es sei $\mathfrak{p} \in K_n$ ein beliebiges Primideal. Wir bilden die Modulgruppe G_A von $\mathfrak{M}_k(\Gamma_n) \equiv G_A \pmod{\mathfrak{p}}$. Da G_A eine (endliche) Gruppe ist, bilden die Matrizen $M \in \mathfrak{M}_k(\Gamma_n)$ mit $M \equiv E \pmod{\mathfrak{p}}$, wobei E die Einheitsmatrix, einen Normalteiler $G_\mathfrak{p}$ von endlichem Index A mit $\mathfrak{M}_k(\Gamma_n)/G_\mathfrak{p} \cong G_A$. Dabei ist $A = (N\mathfrak{p})^{k^2-1} \cdot (1 - N_\mathfrak{o}^{-2}) \ldots (1 - N_\mathfrak{o}^{-k})$. (Der genaue Wert ist natürlich hier unwesentlich.) Wir wählen eine beliebige Matrix $M \in G_1$. Für diese muß dann gelten:

$$M \equiv E \pmod{\mathfrak{p}}$$

für jedes Primideal aus K_n. Nach dem Fundamentalsatz der Idealtheorie folgt aber

$$M = E .$$

Damit ist $G_1 = 1$, also erst recht $G_0 = 1$.

Spezialfälle: (1) $K_n = R$, Körper der rationalen Zahlen. $\mathfrak{M}_k(\Gamma)$ ist die unimodulare Gruppe mit ganz rat. Koeffizienten; sie enthält für $k = 2m$ als Untergruppe die Siegel'sche Modulgruppe.

(2) K_n = total reeller algebraischer Zahlkörper, und $k = 2$. Die Gruppe $\mathfrak{M}_2(\Gamma_n)$ ist isomorph zur Hilbert'schen Modulgruppe.

Aus einem Satz von Balcerzyk–Mycielski [1] folgt die *Maximalfastperiodizität des freien Produktes*

$$G = \prod_{t \in T}^{*} G_t$$

einer beliebigen Menge T (mit Mächtigkeit $\leqq 2^{\aleph_0}$) *von Gruppen* G_t *der Gestalt*:

Zyklische Gruppen Z_n, Diedergruppen D_{2n}, Tetraedergruppe A_4, Oktaedergruppe S_4, Ikosaedergruppe A_5.

Dabei darf $n=\infty$ sein und jede dieser Gruppen mehrmals als Faktor auftreten.

Im Spezialfall $G=Z_2*Z_3$ bzw. $F_n=Z_\infty*\ldots*Z_\infty$ erhalten wir die in der Einleitung zitierten Gruppen aufs Neue.

Wir beweisen folgendes Lemma, auf welches sich die folgenden Beweise stützen werden.

LEMMA 1. *Ist N ein Normalteiler von G mit* max.-f.p. *Faktorgruppe, dann ist $G_0 \subseteq N$. Weiter ist G_0 selbst der Durchschnitt über alle solche Normalteiler.*

BEWEIS. Da G/N max.-f.p. ist, so gibt es zu je zwei Elementen a_j, a_k aus zwei verschiedenen Restklassen von $G=\bigcup_i Na_i$ eine endliche unitäre Darstellung D, so daß sie getrennt werden, das heißt $D(a_j) \neq D(a_k)$. Alle von 1 nicht trennbaren Elemente liegen daher in einer Restklasse, somit in N, also $G_0 \subseteq N$. Nach Definition von G_0 ist G/G_0 max.-f.p. Faktorgruppe, und G_0 ist daher selbst ein Normalteiler N_1. Somit $\bigcap N \subseteq G_0$; wegen $G_0 \subseteq N$ folgt dann aber $\bigcap N = G_0$.

BEMERKUNG. Zur Untersuchung einer Gruppe G auf Max.-f.p. stehen uns also jedenfalls Normalteilersysteme a) mit endlichen, b) mit treu unitär darstellbaren, c) mit max.-f.p. Faktorgruppen zur Verfügung.

Sei Γ der Ring der ganz rationalen Zahlen und $\Gamma[x]$ der Polynomring über Γ. Dann gilt folgender Satz:

SATZ 2. *Die Gruppe $\mathfrak{M}_k(\Gamma[x])$, der $k\times k$-Matrizen mit Determinante* 1 *und Koeffizienten aus $\Gamma[x]$, ist* max.-f.p.

Wir geben für diesen Satz zwei Beweise.

ERSTER BEWEIS. Wir bilden mittels $p(x) \in \Gamma[x]$, welches irreduzibel sei und ersten Koeffizienten $a_0=1$ habe, die Modulgruppe $\mathfrak{M}_{p(x)}$:

$$\mathfrak{M}_k(\Gamma[x]) \equiv \mathfrak{M}_{p(x)} \mod p(x) .$$

Für diese (unendliche) Gruppe gilt nun folgendes Lemma:

LEMMA 2. *Die Gruppe $\mathfrak{M}_{p(x)}$ ist isomorph einer Untergruppe von $\mathfrak{M}_k(\Gamma_n)$, wobei K_n der durch $p(x)=0$ definierte algebraische Zahlkörper ist.*

BEWEIS DES LEMMAS. Sei α definiert durch $p(\alpha)=0$, so ist α nach den Voraussetzungen über $p(x)$ eine ganz algebraische Zahl. Jedes Element $m(x)$ von $M(x) \in \mathfrak{M}_k(\Gamma[x])$ ist aber ein ganz rationales Polynom in x, das heißt aus $\Gamma[x]$, somit jedes Element $\tilde{m}$ von $\tilde{M} \in \mathfrak{M}_{p(x)}$ ganz rationales

Polynom in α. Daher ist $\tilde{m}$ ganz algebraisch $\in R(\alpha)=K_n$ und da $\mathfrak{M}_{p(x)}$ eine Gruppe, ist $\mathfrak{M}_{p(x)}$ isomorph einer Untergruppe von $\mathfrak{M}_k(\Gamma_n)$.

Um den Beweis von Satz 2 zu beenden, wählen wir jetzt irgendwelche unendlich viele $p(x)$ mit oben angegebenen Eigenschaften. Zu jedem $p(x)$ gehört dann ein Normalteiler $G_{p(x)}$, so daß

$$\mathfrak{M}_k(\Gamma[x])/G_{p(x)} \cong \mathfrak{M}_{p(x)}$$

wird. Aus Lemma 2 und dem Satz 1 folgt, daß $\mathfrak{M}_{p(x)}$ max.-f.p. ist. Wir wollen jetzt noch zeigen, daß stets $\bigcap G_{p(x)}=1$ für unendlich viele $p(x)$. Sei $M(x)\in\bigcap G_{p(x)}$, das heißt $M(x)\equiv E \bmod p(x)$ für unendlich viele $p(x)$. Jedes Element $a(x)$ der Matrix $M(x)-E$ hat aber eine eindeutige Zerlegung in Primpolynome, insbesondere jedes $a(x)\neq 0$ nur endlich viele Primpolynome als Teiler (endlicher Grad), so daß aus $a(x)\equiv 0 \bmod p(x)$ für unendliche viele $p(x)$ notwendig folgt

$$a(x) = 0, \qquad \text{also} \qquad M(x)-E = 0\,.$$

Nach Lemma 1 ist aber $G_0\subseteq G_{p(x)}$, also erst recht $G_0=1$.

Man könnte sich bei diesem Beweis auf die Polynome $p(x)=x-\alpha$, $\alpha\in\Gamma$, beschränken.

Zweiter Beweis. Wir bilden zu jeder Primzahl $p\in\Gamma$ die Modulargruppe $\mathfrak{M}_p(x)$

$$\mathfrak{M}_k(\Gamma[x]) \equiv \mathfrak{M}_p(x) \mod p\,.$$

$\mathfrak{M}_p(x)$ ist wieder eine unendliche Gruppe; wir behaupten jedoch

Lemma 2a. *Die Modulargruppe $\mathfrak{M}_p(x)$ ist* max.-f.p.

Beweis des Lemmas. Sei $\pi(x)\in\Gamma[x] \bmod p$ irreduzibel über GF[p] und vom Grade n. Wir bilden $\mathfrak{M}_p(x)\equiv\mathfrak{M}_p(\alpha) \bmod \pi(x)$. Jede Matrix $M\in\mathfrak{M}_p(\alpha)$, wo $\pi(\alpha)=0$, besteht aus Elementen

$$m = m_0+m_1\alpha+\ldots+m_{n-1}\alpha^{n-1}$$

mit $m_i\in$ GF[p], das heißt $m\in$ GF[p^n]. Somit ist $\mathfrak{M}_p(\alpha)$ endliche Gruppe. Den zu $\pi(x)$ gehörenden Normalteiler bezeichnen wir mit $G_{\pi(x)}$ und wählen $M(x)\in\bigcap G_{\pi(x)}$, das heißt $M(x)\equiv E \bmod \pi(x)$. Wie beim ersten Beweis folgt $M(x)=E$, also $\mathfrak{M}_p(x)$ max.-f.p.

Nun können wir den zweiten Beweis leicht zu Ende führen. Ist nämlich G_p der Normalteiler zur Primzahl p, so daß

$$\mathfrak{M}_k(\Gamma[x])/G_p \cong \mathfrak{M}_p(x)$$

wird und $M(x)\in\bigcap G_p$, so folgt aus $M(x)\equiv E \bmod p$, daß $M(x)=E$ das heißt $G_0=1$. Wegen der Max.-f.p. der Gruppen $\mathfrak{M}_p(x)$ folgt daraus die Behauptung des Satzes.

BEMERKUNG. Für jedes $k \geqq 2$ hat $\mathfrak{M}_k(\Gamma[x])$ unendlich viele Erzeugende und Relationen. $\mathfrak{M}_2(\Gamma[x])$ enthält die sogenannten Hecke-gruppen

$$\mathfrak{H}(x) = \left\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$$

als Untergruppen.

Eine leichte Verallgemeinerung von Satz 2 gibt uns der Satz 3, bei dessen Beweis wir uns nun kürzer fassen können.

SATZ 3. *Die Gruppe* $\mathfrak{M}_k(\Gamma_n[x])$, *der* $k\times k$*-Matrizen mit Determinante* 1 *und Koeffizienten aus* $\Gamma_n[x]$, *ist* max.-f.p.

BEWEIS (analog wie bei Satz 2). Man wähle unendlich viele irreduzible $p(x) \in \Gamma_n[x]$ mit $a_0 = 1$. Dann ist α, mit $p(\alpha) = 0$, ganz algebraische Zahl und daher die Modulargruppe $\mathfrak{M}_{p(x)}$, welche durch

$$\mathfrak{M}_k(\Gamma_n[x]) \equiv \mathfrak{M}_{p(x)} \mod p(x)$$

definiert ist, isomorph einer Untergruppe von $\mathfrak{M}_k(\hat{\Gamma})$; dabei ist $\hat{\Gamma}$ der Ring der ganzen Zahlen des Körpers $\hat{K} = K_n(\alpha)$. Speziell wird für die Polynome $p(x) = x - \alpha$, $\alpha \in \Gamma_n$, die Modulargruppe $\mathfrak{M}_{p(x)}$ isomorph einer Untergruppe von $\mathfrak{M}_k(\Gamma_n)$. Da im Polynomring $\Gamma_n[x]$ eindeutige Zerlegung in Primpolynome herrscht, läßt sich der Beweis leicht beenden.

Wir benützen nun Satz 3, um mittels Induktion dieselbe Aussage für die Gruppe $\mathfrak{M}_k(\Gamma_n[x_1 \ldots x_m])$ zu beweisen.

SATZ 4. *Die Gruppe* $\mathfrak{M}_k(\Gamma_n[x_1 \ldots x_m])$, *der* $k\times k$*-Matrizen mit Determinante* 1 *und Koeffizienten aus dem Polynomring* $\Gamma_n[x_1 \ldots x_m]$ *ist* max.-f.p.

BEWEIS. Mittels vollständiger Induktion nach m. Für $m = 1$ ist Satz 4 mit Satz 3 identisch, also bewiesen. Nun sei er bereits für $m - 1$ richtig! Wir wählen unendlich viele (irreduzible) Polynome

$$p(x_1, \ldots, x_m) = x_m - p_1(x_1, \ldots, x_{m-1}),$$

mit $p_1(x_1 \ldots x_{m-1}) \in \Gamma_n[x_1 \ldots x_{m-1}]$. Als Modulargruppe erhalten wir

$$\mathfrak{M}_k(\Gamma_n[x_1 \ldots x_m]) \equiv \mathfrak{M}_{p(x_1 \ldots x_m)} \mod p(x_1, \ldots, x_m).$$

$\mathfrak{M}_{p(x_1 \ldots x_m)}$ ist also wegen unserer Wahl von $p(x_1, \ldots, x_m)$ isomorph einer Untergruppe von $\mathfrak{M}_k(\Gamma_n[x_1, \ldots, x_{m-1}])$ das heißt aber max.-f.p. nach Induktionsvoraussetzung. Für die zugehörigen Normalteiler $G_{p(x_1, \ldots, x_m)}$ gilt aber

$$\bigcap G_{p(x_1,\ldots,x_m)} = 1 ,$$

da aus $M(x_1 \ldots x_m) \equiv E \bmod p(x_1,\ldots,x_m)$ für unendlich viele $p(x_1,\ldots,x_m)$, folgt

$$M(x_1,\ldots,x_m) = E$$

(ZPE-Satz im Polynomring $\Gamma_n[x_1 \ldots x_m]$). Nach Lemma 1 folgt $G_0 = 1$.

2. Beispiel einer nichteinfachen minimal-f.p. Gruppe.

Wir wollen selbstredend von den direkten Produkten einfacher Gruppen absehen und geben ein Beispiel einer min.-f.p. Gruppe mit unendlich vielen Normalteilern.

R sei der Körper der rationalen Zahlen und $R[x]$ der Polynomring über R. Dann gilt folgender Satz.

SATZ 5. *Die Gruppe* $\mathfrak{M}_2(R[x])$, *der* 2×2-*Matrizen mit Determinante* 1 *und Elementen aus* $R[x]$, *ist* min.-f.p. *Die Gruppe hat unendlich viele Normalteiler.*

BEWEIS. Die Gruppe wird erzeugt von den Elementen

$$\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & b(x) \\ 0 & 1 \end{pmatrix}\right\}$$

wobei die $b(x) \in R[x]$ ein Erzeugendensystem der additiven Gruppe der Polynome aus $R[x]$ bilden sollen. Die Elemente

$$V_n = \begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}, \qquad n = \text{natürliche Zahl} ,$$

liegen in $\mathfrak{M}_2(R[x])$. Bezeichnen wir mit S_b das Element

$$S_b = \begin{pmatrix} 1 & b(x) \\ 0 & 1 \end{pmatrix}, \qquad \text{so gilt} \qquad V_n S_b V_n^{-1} = S_b^{n^2} .$$

Daraus folgt nun (siehe Neumann-Wigner [4, Lemma 1]), daß für jede endliche unitäre Darstellung $D(S_b) = E$, also $S_b \in G_0$. Mit

$$T_b = \begin{pmatrix} 1 & 0 \\ b(x) & 1 \end{pmatrix} \qquad \text{gilt} \qquad V_n^{-1} T_b V_n = T_b^{n^2} ,$$

und daher $D(T_b) = E$. Insbesondere ist

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G_0 .$$

Also sind sämtliche Erzeugende von $\mathfrak{M}_2(R[x])$ in G_0; das bedeutet aber

gerade $G_0 = \mathfrak{M}_2(R[x])$. Mit den unendliche vielen irreduziblen Polynomen $p(x) \in \Gamma[x]$ können wir unendlich viele (nicht isomorphe) Modulargruppen bilden:

$$\mathfrak{M}_2(R[x]) \equiv \mathfrak{M}_{p(x)} \mod p(x) .$$

Wir erhalten also unendlich viele verschiedene Normalteiler. Alle Faktorgruppen sind natürlich min.-f.p.

Der Satz läßt sich auf Körper erweitern, die den rationalen Zahlkörper enthalten, und von der Dimension 2 auf beliebige Dimension $n \geqq 2$.

Um den nächsten Satz beweisen zu können, benötigen wir ein Lemma, aus welchem sich sofort das oben zitierte Neumann–Wigner'sche Lemma ergibt.

LEMMA 3. *Jede endliche unitäre Matrix U, welche mit einer Potenz $\geqq 2$ von sich selbst konjugiert ist, hat endliche Ordnung, mit anderen Worten, gibt es eine invertierbare Matrix V, so daß $VUV^{-1} = U^t$, mit $|t| \geqq 2$, t ganzrational, dann folgt $U^m = 1$ für eine natürliche Zahl m.*

BEMERKUNG. Allgemein lautet dieses Lemma offenbar so: Gibt es zu einer endlichen Matrix U eine invertierbare Matrix V, so daß $VUV^{-1} = U^t$, mit $|t| \geqq 2$, dann sind die Eigenwerte von U lauter Einheitswurzeln. Beweis wie unten.

BEWEIS VON LEMMA 3. Wir setzen $V_k = V^k$, $k = 1, 2, 3, \ldots$. Aus $VUV^{-1} = U^t$ folgt sofort $V_k U V_k{}^{-1} = U^{t^k}$ für jedes natürliche k. Die Dimension der Matrizen sei n und $\alpha_1, \ldots, \alpha_n$ seien die Eigenwerte der Matrix U. Nun hat $V_k U V_k{}^{-1}$ dieselben Eigenwerte wie U, während U^{t^k} die Eigenwerte $\alpha_1{}^{t^k}, \ldots, \alpha_n{}^{t^k}$ hat; diese müssen also eine Permutation der α_i sein. Da wir aber für k unendlich viele Möglichkeiten haben, so gibt es für jedes $i = 1, 2, \ldots, n$ zwei verschiedene k mit $\alpha_i{}^{t^{k'}} = \alpha_i{}^{t^{k''}}$, das heißt $\alpha_i{}^{t^{k'} - t^{k''}} = 1$. Jedes α_i ist also Einheitswurzel, somit $U^m = 1$ für geeignetes m.

Der Fall $t = 0$ ist trivial. Im Falle $|t| = 1$ ist das Lemma offensichtlich falsch; für $t = 1$ wähle man Diagonalmatrizen, für $t = -1$

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

mit λ transzendent und $|\lambda| = 1$.

KOROLLAR. *Jede Gruppe, welche ein Element von unendlich hoher Ordnung hat, das mit einer Potenz $\geqq 2$ von sich in ein und derselben Klasse konjugierter Elemente liegt, besitzt keine treue endliche unitäre Darstellung.*

Das Korollar gilt insbesondere für die Gruppen $G=\{a,b\}$ mit der einzigen Relation $bab^{-1}=a^n$, $n\geqq 2$. Also gilt (für die letzte Behauptung siehe Satz 1).

SATZ 6. *Es gibt Gruppen mit zwei Erzeugenden und einer einzigen Relation, welche keine treue endliche unitäre Darstellung besitzen. Im Gegensatz dazu ist die freie Gruppe mit zwei Erzeugenden treu unitär darstellbar durch* 2×2*-Matrizen.*

LEMMA 4. *Seien* U, U_2 *zwei unitäre* $n\times n$*-Matrizen, welche mit* $K=U_1U_2U_1^{-1}U_2^{-m}$, $m\geqq 1$, *vertauschbar sind. Dann hat* K *endliche Ordnung, und für* $m\geqq 2$ *auch* U_2.

BEWEIS. Wir bezeichnen den n-dimensionalen Vektorraum, in dem die Matrizen wirken, mit $\mathfrak{W}$; mit α einen Eigenwert von K und mit $\mathfrak{W}_\alpha$ den Teilraum von $\mathfrak{W}$, definiert durch

$$\mathfrak{W}_\alpha = \{x \mid Kx=\alpha x\}.$$

Wegen $KU_1=U_1K$, $KU_2=U_2K$ gilt

$$U_1(\mathfrak{W}_\alpha)\subseteq\mathfrak{W}_\alpha \qquad \text{und} \qquad U_2(\mathfrak{W}_\alpha)\subseteq\mathfrak{W}_\alpha.$$

In $\mathfrak{W}_\alpha$ ist aber $K=\alpha\cdot 1$, das heißt $U_1U_2U_1^{-1}=\alpha U_2^m$.

Zuerst sei $m=1$. Dann haben U_2 und αU_2 als Transformationen von $\mathfrak{W}_\alpha$ dieselben Eigenwerte: $\lambda,\alpha\lambda,\alpha^2\lambda,\ldots$, $\lambda\neq 0$. Wegen $\dim(\mathfrak{W}_\alpha)\leqq n$ muß $\lambda\alpha^{k'}=\lambda\alpha^{k''}$ sein, für zwei verschiedene k. Somit $\alpha^{k'-k''}=1$; jeder Eigenwert von K ist Einheitswurzel und $K^s=1$, mit geeignetem s.

Im Falle $m\geqq 2$ folgt, daß U_2 und αU_2^m dieselben Eigenwerte haben. Somit sind $\lambda,\alpha\lambda^m,\alpha^2\lambda^{m^2},\alpha^3\lambda^{m^3},\ldots,\lambda\neq 0$, die Eigenwerte von U_2. Wir setzen

$$U_2=\sigma W, \qquad \sigma=\alpha^{-1/(m-1)}.$$

Dann ist W unitär, weil $|\alpha|=1$, und $U_1WU_1^{-1}=W^m$. Nach Lemma 3 ist jeder Eigenwert von W Einheitswurzel, also $\lambda=\sigma\mu$, wobei μ eine Einheitswurzel ist. Wegen $\dim(\mathfrak{W}_\alpha)\leqq n$ gibt es zwei ganze Zahlen $k_1<k_2$ für welche $\alpha^{k_1}\lambda^{m^{k_1}}=\alpha^{k_2}\lambda^{m^{k_2}}$. Also ist

$$\alpha^{(m^{k_1}-m^{k_2})/(m-1)-(k_1-k_2)}=\mu^{m^{k_1}-m^{k_2}}.$$

Da

$$\frac{m^{k_1}-m^{k_2}}{m-1}=m^{k_1-1}+\ldots+m^{k_2}>k_1-k_2, \qquad m\geqq 2,$$

ist α Einheitswurzel.

Im Falle $m=0$ ist das Lemma offenbar falsch (vgl. Lemma 3), und für $K=1$ ist es in Lemma 3 enthalten.

Mit Lemma 4 ist bewiesen, daß die Gruppen $G=\{a,b\}$ mit den Relationen

$$ac = ca, \qquad bc = cb, \qquad c = aba^{-1}b^{-n},$$

$n \geqq 1$, nicht treu unitär darstellbar sind durch endliche Matrizen.

Wir beweisen nun den folgenden, weitergehenden Satz.

SATZ 7. *Es gibt Gruppen mit endlich vielen Erzeugenden und Relationen, welche* min.-f.p. *sind, das heißt deren sämtliche beschränkte, endliche Darstellungen trivial sind.*

BEWEIS. Wir zeigen, daß die (unendliche) Gruppe $G=\{a,b,c,d\}$ mit den Relationen

$$a^{-1}ba = b^2, \qquad b^{-1}cb = c^2, \qquad c^{-1}dc = d^2, \qquad d^{-1}ad = a^2$$

min.-f.p. ist. Nach Lemma 3 hat nämlich bei jeder endlichen unitären Darstellung von G das Element $\bar{a}=D(a)$ endliche Ordnung α; analog haben die Elemente $\bar{b}$, $\bar{c}$, $\bar{d}$ endliche Ordnungen β, γ, δ. Daraus folgt nun (siehe [2]), daß $\alpha=\beta=\gamma=\delta=1$. Damit ist die dargestellte Gruppe stets 1, also G min.-f.p.

Es ist interessant zu bemerken, daß die ähnlich definierte (unendliche) Gruppe $G=\{a,b,c\}$ mit

$$a^{-1}ba = b^{-1}, \qquad b^{-1}cb = c^{-1}, \qquad c^{-1}ac = a^{-1}$$

treu unitär darstellbar ist. (In diesem Falle $t=-1$ ist Lemma 3 ja auch falsch). Zum Beweis der Behauptung wähle man drei Darstellungen

$$D_1(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D_1(b) = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \qquad D_1(c) = E$$

mit $|\beta|=1$ transzendent; analog sind D_2, D_3 mit $D_2(b)=D_3(a)=E$ und $|\alpha|=|\gamma|=1$ ebenfalls transzendent. Die unitäre Darstellung

$$D(x) = D_1(x) \oplus D_2(x) \oplus D_3(x)$$

ist treu, wie man an der Normalform der Elemente $x \in G$ sieht.

Eine teilweise Verschärfung von Satz 7, welche jedoch nicht in den Rahmen dieser Arbeit gehört, erhalten wir so (weiter-gehende Resultate in meiner Arbeit: »Über die Nichtlinearität einer gewissen Gruppe«, welche demnächst erscheint, und [4]):

SATZ 8. *Es gibt Gruppen mit endlich vielen Erzeugenden, welche keine (nichttriviale) endliche Darstellung besitzen.*

Beim Beweis stützen wir uns auf folgendes

LEMMA 5. *Es gibt eine einfache, unendliche, periodische Gruppe F vom Exponent 73 mit endlich vielen Erzeugenden.*

BEWEIS. Siehe [3, Seite 71ff].

Andererseits folgern wir aus den Burnside-Kriterium über periodische Matrizengruppen, daß bei jeder endlichen Darstellung von F die (treu) dargestellte Gruppe endlich ist. Wegen der Einfachheit von F also stets gleich 1. Es ist jedoch nicht bekannt, ob die Gruppe F auch endlich viele Relationen besitzt.

Wir gehen nun über zur Untersuchung der Gruppe $\mathfrak{M}_2(\Gamma_{(p)})$. Mit $\Gamma_{(p)}$ bezeichnen wir den Ring der p-rationalen Zahlen, das heißt die Menge der rationalen Zahlen

$$a = \sum_{\nu=-n}^{n} a_\nu p^\nu, \qquad 0 \leqq a_\nu \leqq p-1 ,$$

wobei p eine Primzahl sein soll. Dann ist $\mathfrak{M}_2(\Gamma_{(p)})$ die Gruppe der 2×2-Matrizen mit Determinante 1 und Elementen aus $\Gamma_{(p)}$.

SATZ 9. *Die Gruppe $\mathfrak{M}_2(\Gamma_{(p)})$ ist* max.-f.p. *Die Gruppe besitzt jedoch keine treue, endliche, unitäre Darstellung.*

BEWEIS. Der zweite Teil der Aussage folgt sofort aus dem Korollar. Die beiden Matrizen

$$V = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

liegen nämlich in $\mathfrak{M}_2(\Gamma_{(p)})$, S hat unendliche Ordnung, und es gilt

$$VSV^{-1} = S^{p^2}, \qquad \text{also} \quad t = p^2 > 2 .$$

Wir bilden nun mit den Zahlen $N_m = p^{2m} - 1$, $m = 1, 2, 3, \ldots$, die Modulargruppen

$$\mathfrak{M}_2(\Gamma_p) \equiv G_m \mod N_m .$$

Diese sind endliche Gruppen, da wegen $p^{-1} \equiv p^{2m-1} \bmod N_m$ jede Zahl $a \in \Gamma_p$ einer ganz rationalen Zahl $(\bmod N_m)$ kongruent ist. Für eine Matrix M aus dem Durchschnitt der zugehörigen Normalteiler aber muß $M \equiv E \bmod N_m$, $m = 1, 2, 3, \ldots$, gelten. Wegen $N_m \to \infty$ für $m \to \infty$ ist also $M = E$.

Zum Schluß deuten wir noch den Beweis eines Satzes an, der eine Art Verallgemeinerung des vorherigen darstellt.

SATZ 10. *Die Gruppe $\mathfrak{M}_k(\Gamma_n[x, x^{-1}])$, der $k \times k$-Matrizen mit Determinante 1 und Elementen aus dem Polynomring $\Gamma_n[x, x^{-1}]$ ist* max.-f.p.

BEWEIS. Man wähle unendlich viele verschiedene irreduzible Polynome $p(x) \in \Gamma[x]$, so daß α mit $p(\alpha)=0$ eine »Einheit« ist, und also α und α^{-1} ganz-algebraische Zahlen sind. Die mittels $p(x)$ gewonnene Modulargruppe,

$$\mathfrak{M}_k(\Gamma[x,x^{-1}]) \equiv \mathfrak{M}_{p(x)} \mod p(x),$$

ist daher isomorph einer Untergruppe von $\mathfrak{M}_k(\hat{\Gamma}_n)$, wobei $\hat{\Gamma}$ der Ring der ganzen Zahlen des Körpers $\hat{K}_n = K_n(\alpha)$ ist. Der Durchschnitt der zugehörigen Normalteiler ist notwendig 1, da wir unendlich viele verschiedene $p(x)$ zur Verfügung haben.

FOLGERUNG. Es gibt Gruppen mit endlich vielen Erzeugenden und unendlich vielen Relationen, welche max.-f.p. sind.

Zum Beweis dieser Behauptung bemerken wir, daß die Gruppe

$$N = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Untergruppe von $\mathfrak{M}_2(\Gamma[x,x^{-1}])$ ist. Sie besitzt aber, wie in [5] bewiesen, unendlich viele definierende Relationen.

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MAXIMAL-FASTPERIODIZITÄT VON GRUPPEN II

W. FLUCH

Einleitung.

Im ersten Teil der Arbeit [3] war u.a. gezeigt worden, daß die spezielle lineare Gruppe über dem Polynomring $\Gamma_n[x_1,\ldots,x_m]$ max.-f.p. ist. Dies soll nun verschärft werden zu dem Satz, daß sie sogar über dem Ring $\Gamma_n^{(\mathfrak{p})}[x_1,\ldots,x_m]$ max.-f.p. ist (Definition von $\Gamma_n^{(\mathfrak{p})}$ in § 1).

Weiter wurde in Teil I eine endlich-erzeugte Gruppe (mit endlich vielen Relationen) angegeben, welche min.-f.p. ist. Wir beweisen als Hauptergebnis den Satz, daß *jede endlich-erzeugbare Matrizengruppe* max.-f.p. *ist.* Als Korollar folgert man den für die Theorie der fastautomorphen Funktionen interessanten Satz, daß jede endlich-erzeugbare Funktionsgruppe max.-f.p. ist.

Offen bleiben die Fragen, ob die spezielle lineare Gruppe über dem Ring *aller* ganzalgebraischen Zahlen noch max.-f.p. ist bzw. wann die allgemeine lineare Gruppe max.-f.p. ist.

1. Spezielle lineare Gruppen.

Sei wieder Γ_n der Ring der ganzen Zahlen eines algebraischen Zahlkörpers K_n vom Grade n und $\mathfrak{p}$ ein Primideal aus Γ_n. Mit $\Gamma_n^{(\mathfrak{p})}$ bezeichnen wir den Ring derjenigen Zahlen aus K_n, welche zu $\mathfrak{p}$ teilerfremde Nenner besitzen. Mittels der Ideale $\mathfrak{p}^r$ bildet man die Homomorphismen

$$\varphi_r\colon\ \mathrm{SL}(k,\Gamma_n^{(\mathfrak{p})}) \to \mathrm{SL}(k,R_r)\,,$$

wobei mit R_r der endliche Restklassen-Ring $\Gamma_n^{(\mathfrak{p})} \bmod \mathfrak{p}^r$, $r=1,2,3,\ldots$, bezeichnet sei. Für die zugehörigen Normalteiler N_r gilt: eine Matrix M ist dann und nur dann in N_r, falls $M \equiv E \bmod \mathfrak{p}^r$. Somit ist der Durchschnitt $\bigcap_{r=1}^{\infty} N_r = 1$. Wir erhalten den

SATZ 1a. *Die Gruppe* $\mathrm{SL}(k,\Gamma_n^{(\mathfrak{p})})$ *ist* max.-f.p.

BEMERKUNG. Analog wie beim Beweis von Satz 5 in Teil I kann man einsehen, daß $\mathrm{SL}(2,K_n)$ min.-f.p. ist. Man darf also im obigen Satz das Primideal $\mathfrak{p}$ nicht ohne weiteres weglassen.

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SATZ 1b. *Die Gruppe* $\mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})}[x])$ *ist* max.-f.p.

Zum Beweis wähle man sich unendlich viele verschiedene Zahlen $\alpha \in \Gamma_n$, so daß also $\Gamma_n^{(\mathfrak{p})}[\alpha] \subset \Gamma_n^{(\mathfrak{p})}$ für jedes α gilt. Die Polynome $p(x) = x - \alpha$ definieren uns Homomorphismen

$$\mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})}[x]) \to \mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})}[\alpha]) ,$$

wobei $\mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})}[\alpha]) \subset \mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})})$, also für jedes α max.-f.p. ist. Daß der Durchschnitt der Kern dieser Homomorphismen aber 1 sein muß, folgt aus dem ZPE-Satz im Ringe $\Gamma_n^{(\mathfrak{p})}[x]$.

SATZ 1. *Die Gruppe* $\mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})}[x_1, \ldots, x_m])$ *ist* max.-f.p.

Den Beweis dieses Satzes führt man mittels vollständiger Induktion nach m. Dazu wähle man unendlich viele Polynome

$$p(x_1, \ldots, x_m) = x_m - p_1(x_1, \ldots, x_{m-1})$$

mit

$$p_1(x_1, \ldots, x_{m-1}) \in \Gamma_n^{(\mathfrak{p})}[x_1, \ldots, x_{m-1}] .$$

Der Ring

$$\Gamma_n^{(\mathfrak{p})}[x_1, \ldots, x_m] \mod p(x_1, \ldots, x_m)$$

ist dann isomorph einem Teilring von $\Gamma_n^{(\mathfrak{p})}[x_1, \ldots, x_{m-1}]$ und

$$\mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})}[x_1, \ldots, x_{m-1}])$$

nach Induktionsvoraussetzung max.-f.p. Daß der Durchschnitt der zu den Homomorphismus gehörenden Normalteiler 1 ist, folgt wie oben.

KOROLLAR. *Jede Funktionsgruppe mit endlich vielen Erzeugenden ist* max.-f.p.

Der Begriff der Funktionsgruppe ist in [2] erläutert. Nach Definition ist eine solche Gruppe F Untergruppe von $\mathrm{SL}(2, C)$. Wegen Satz 1 genügt es zu zeigen, daß F bereits Untergruppe einer geeigneten

$$\mathrm{SL}(k, \Gamma_n^{(\mathfrak{p})}[x_1 \ldots x_m])$$

ist. Seien $a_1, \ldots, a_\mu$ die erzeugenden Matrizen von F, welche m transzendente Zahlen $x_1, \ldots, x_m$ und s algebraische Zahlen $\alpha_1, \ldots, \alpha_s \neq 0$ als Elemente enthalten mögen $(m + s \leqq 4\mu)$. Dann ist also

$$F \subset \mathrm{SL}(2, \Gamma[\alpha_1 \ldots \alpha_s, x_1, \ldots, x_m]) .$$

Da nur endlich viele algebraische Zahlen $\alpha_1, \ldots, \alpha_s$ in den a_σ vorhanden sind, so gibt es einen algebraischen Zahlkörper K_n mit genügend hohem Grad n, so daß alle in diesem einen Körper liegen. Weiter gibt es in dem

zu K_n gehörenden Ring Γ_n sicherlich ein Primideal $\mathfrak{p}$, das teilerfremd zu den Nennern der α_i ist, da durch die Nenner insgesamt nur eine endliche Menge von Primidealen ausgeschlossen ist, das heißt

$$\Gamma[\alpha_1 \dots \alpha_s x_1 \dots x_n] \subset \Gamma_n^{(\mathfrak{p})}[x_1, \dots, x_m] .$$

Damit ist das Korollar bereits bewiesen.

2. Endlich-erzeugbare Matrizengruppen.

Satz 7 in Teil I besagt, daß es endlich-erzeugbare Gruppen (sogar mit endlich vielen Relationen) gibt, welche min-f.p. sind. Im Gegensatz dazu soll nun als Hauptergebnis der Arbeit der Satz bewiesen werden, daß jede endlich-erzeugbare Matrizengruppe max.-f.p. ist (Matrizengruppe heißt jede endl.-dim. treu darstellbare Gruppe). Dieser Satz ist deshalb von selbständigem Interesse, weil er die Untersuchung einer endlich-erzeugbaren Gruppe nach Max.-f.p. zurückführt auf die Frage nach einer treuen (endl.-dim.) Darstellung dieser Gruppe. Letzteres läßt sich in einigen Fällen entscheiden. Wir bringen dazu Beispiele im nächsten Paragraphen.

SATZ 2. *Jede endlich-erzeugbare Matrizengruppe ist* max.-f.p.

BEWEIS. Sei k der Grad der Darstellung und $a_1, \dots, a_\mu$ die erzeugenden Matrizen von G mit $\mathrm{Det}(a_i) = d_i \neq 0$. In den a_i mögen m transzendente Zahlen $x_1, \dots, x_m$ und s algebraische Zahlen $\alpha_1, \dots, \alpha_s \neq 0$ als Elemente auftreten. Dann gilt $m + s \leqq k^2 n$. Die Zahlen $d_i = d_i(\alpha, x)$ sind Polynome aus $\Gamma[\alpha, x]$ mit

$$\alpha = (\alpha_1, \dots, \alpha_s) \qquad \text{und} \qquad x = (x_1, \dots, x_m) .$$

Man wähle sich nun unendlich viele verschiedene m-tupeln $\beta = (\beta_1 \dots, \beta_m)$ von algebraischen Zahlen β_i, so daß $d_i(\alpha, \beta) \neq 0$ wird. Jede Substitution $x \to \beta$ liefert einen Homomorphismus

$$\eta\colon G \to \bar{G} \qquad \text{mit} \qquad \bar{G} \subset GL(k, \Gamma[\alpha, \beta]) .$$

HILFSSATZ. *Für jede Wahl der β_j mit $d_i(\alpha, \beta) \neq 0$ ist $\bar{G}$* max.-f.p.

Die alg. Zahlen α_i, β_j, $d_k(\alpha, \beta)$ liegen nun in einem Zahlkörper genügend hohen Grades n, den wir mit K_n bezeichnen. Die Zähler und Nenner der eben genannten alg. Zahlen bilden eine endliche Menge A von Zahlen aus Γ_n. Es gibt daher unendlich viele Primideale $\mathfrak{p} \subset \Gamma_n$, welche zu allen Zahlen aus A teilerfremd sind. Bezeichnen wir $\Gamma[\alpha, \beta] \bmod \mathfrak{p}$ mit R für ein solches $\mathfrak{p}$. R ist ein endlicher Restklassenring und $\mathfrak{p}$ induziert den Homomorphismus

$$\varphi\colon\ GL(k,\Gamma[\alpha,\beta]) \to GL(k,R)\ ,$$

bei welchem $\bar{G}$ auf eine endliche Gruppe abgebildet wird. Der Durchschnitt der Kerne der Homomorphismus φ bzw. η ist aber 1, da stets unendlich viele solche Homomorphismen (φ bzw. η) zur Verfügung stehen und in den entsprechenden Ringen der ZPI- bzw. ZPE-Satz gilt. Damit ist Satz 2 bewiesen.

3. Beispiele.

Wir wollen als erste Anwendung zeigen, daß die Fundamentalgruppe einer geschlossenen orientierbaren bzw. nichtorientierbaren Fläche vom Geschlecht p max.-f.p. ist. Dieser Satz wurde für die orientierbaren Flächen zuerst von K. Fung bewiesen (siehe [5]).

Nun kann man zeigen (siehe [1]), daß diese Fundamentalgruppen Untergruppen der Spiegelungsgruppe $S=\{R_1,R_2,R_3\}$ mit den Relationen

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^m = (R_2R_3)^n = (R_3R_1)^2 = E$$

sind für geeignete m und n. Damit hat man aber eine treue Darstellung durch 4×4-Matrizen, also wegen Satz 2 den

SATZ 3. *Die Fundamentalgruppe einer geschlossenen orientierbaren bzw. nicht-orientierbaren Fläche vom Geschlecht p ist* max.-f.p. *Sie besitzt eine treue Darstellung durch* 4×4-*Matrizen.*

Eine treue Darstellung durch 3×3-Matrizen wurde von J. Mennicke [6] angegeben.

Um eine weitere Anwendung geben zu können, brauchen wir den Hilfssatz, daß eine Gruppe G mit treu darstellbarem Normalteiler von endlichem Index selbst treu darstellbar ist, welcher in meiner Arbeit [4] bewiesen wurde. Als Resultat erhält man den

SATZ 4. *Jede endlich-erzeugbare Gruppe G mit treu darstellbarem Normalteiler von endlichem Index ist* max.-f.p.

Weitere, jedoch speziellere Beispiele lassen sich leicht finden.

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UNIVERSITÄT GÖTTINGEN, DEUTSCHLAND

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GRUPPEN OHNE ENDLICH-DIMENSIONALE DARSTELLUNGEN

W. FLUCH

Eine Gruppe $G \neq 1$, deren sämtliche endlich-dimensionalen Darstellungen (über dem Körper C der komplexen Zahlen) trivial sind, wollen wir nichtlinear nennen. In [2] wurde die Existenz endlich-erzeugbarer, nichtlinearer Gruppen nachgewiesen und in [3] konnte dies verschärft werden zu: es gibt sogar nichtlineare Gruppen mit endlich vielen Erzeugenden und Relationen. In dieser Arbeit werden nun sämtliche endlich-erzeugbaren, nichtlinearen Gruppen charakterisiert durch den

SATZ 1. *Eine endlich-erzeugbare Gruppe G ist genau dann nichtlinear, falls ihre sämtlichen Faktorgruppen $F \neq 1$ unendlich sind.*

Überabzählbar viele Beispiele dazu sind die endlich erzeugten, unendlichen, einfachen Gruppen von R. Camm (siehe [1]).

Falls G eine endliche Faktorgruppe $F \neq 1$ hat, so liefert ja die reguläre Darstellung von F eine nichttriviale Darstellung von G. Im anderen Fall aber folgt die Behauptung aus dem unten bewiesenen Lemma 2. Es ist bemerkenswert, daß der Satz 1 und Lemma 2 für Gruppen mit unendlich vielen Erzeugenden in dieser Allgemeinheit nicht gelten (siehe jedoch Lemma 3); als Gegenbeispiel wähle man etwa die spezielle lineare Gruppe $\mathrm{SL}(2,R)$ über dem Körper R der rationalen Zahlen, welche eine einfache Gruppe ist. (Ein Beispiel einer Gruppe mit überabzählbar vielen Erzeugenden ist etwa die Drehgruppe O_3^+.) Offenbar ist jede ihrer Faktorgruppen $F \neq 1$ unendlich, aber die Gruppe besitzt eine treue Darstellung durch 2×2-Matrizen. Eine (gruppentheoretische) Charakterisierung *aller* nichtlinearen Gruppen scheint schwierig. Folgendes Lemma ist ziemlich naheliegend.

LEMMA 1. *Die Gruppe G sei Produkt von Untergruppen G_α. Sind alle G_α nichtlinear, so ist auch G nichtlinear.*

BEWEIS. Jede endlich-dimensionale Darstellung D von G liefert eine endlich-dimensionale Darstellung ihrer Untergruppen G_α und nach Vor-

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aussetzung ist $D(G_\alpha)=1$ für jedes α. Insbesondere gilt daher für die Erzeugenden $a_\sigma \in G$: $D(a_\sigma)=1$ und somit ist $D(G)=1$.

KOROLLAR 1. *Das direkte und das freie Produkt nichtlinearer Gruppen ist wieder eine nichtlineare Gruppe.*

Die Nichtlinearität der Gruppe $G=\{a,b,c,d\}$, mit den Relationen

$$b^{-1}ab = a^2, \qquad c^{-1}bc = b^2, \qquad d^{-1}cd = c^2, \qquad a^{-1}da = d^2,$$

wurde in [3] auf direktem Wege nachgewiesen. Dasselbe Resultat läßt sich aber auch aus Satz 1 ableiten, da in [4] bewiesen wurde, daß jede Faktorgruppe $\neq 1$ dieser Gruppe unendlich ist. Um den Satz 1 zu beweisen, zeigen wir nun

LEMMA 2. *Jede endlich-erzeugbare Gruppe G, welche eine nichttriviale endlich-dimensionale Darstellung D besitzt, hat einen Normalteiler N mit endlicher Faktorgruppe $F=G/N \neq 1$.*

Eine unmittelbare Konsequenz dieses Lemmas ist das

KOROLLAR 2. *Eine endlich-erzeugbare, einfache Matrizengruppe ist endlich.*

BEWEIS VON LEMMA 2. Sei $a_1,\ldots,a_n$ ein Erzeugendensystem von G und die gegebene, nichttriviale Darstellung D etwa k-dimensional. Nach Voraussetzung ist $\bar{G}=D(G)\neq 1$. Die Bilder der Erzeugenden von G, welche ja ein Erzeugendensystem von $\bar{G}$ bilden, mögen

$$\bar{a}_1 = D(a_1), \ \ldots, \ \bar{a}_n = D(a_n)$$

heißen. Die Menge der Elemente $\bar{a}\in\bar{G}$ mit $\mathrm{Det}(\bar{a})=1$ ist ein Normalteiler $\bar{N}$, welcher die Kommutatorgruppe enthält und daher abelsche Faktorgruppe $\bar{A}=\bar{G}/\bar{N}$ besitzt. Wir unterscheiden nun zwei Fälle:

1) $\bar{A}\neq 1$. Dann besitzt aber $\bar{A}$, das wie $\bar{G}$ endlich erzeugbar ist, sicherlich eine endliche Faktorgruppe $\overline{\overline{A}}\neq 1$ (die man erhält, indem man die Ordnung eines jeden erzeugenden Elementes endlich macht!). Durch Zusammensetzen erhalten wir die homomorphe Abbildung $G\to\overline{\overline{A}}\neq 1$, welche das Lemma im Falle 1 beweist.

2) $\bar{A}=1$ d.h. $\bar{G}=\bar{N}$ oder $\mathrm{Det}(\bar{a})=1$ für jedes $\bar{a}\in\bar{G}$. Seien nun $x_1,\ldots,x_m$ die transzendenten, voneinander algebraisch unabhängigen und weiter $\alpha_1,\ldots,\alpha_s$ die algebraischen Zahlen $\neq 0$, welche als Elemente in den (erzeugenden) Matrizen $\bar{a}_1,\ldots,\bar{a}_n$ auftreten. Es ist $m+s\leqq k^2n$ und $\bar{G}$ eine Untegruppe von

$$\mathrm{SL}(k, \Gamma[\alpha_1,\ldots,\alpha_s,x_1,\ldots,x_m]) .$$

Es gibt gewiß algebraische Zahlen $\beta_1,\ldots,\beta_m$ so, daß beim Homomorphismus

$$\varphi\colon\ \mathrm{SL}(k, \Gamma[\alpha_1,\ldots,\alpha_s,x_1\ldots x_m]) \to \mathrm{SL}(k, \Gamma[\alpha_1\ldots\alpha_s,\beta_1\ldots\beta_m])$$

mindestens eine Matrix $\bar{a}_\mu \neq 1$ auf eine Matrix $\bar{\bar{a}}_\mu \neq 1$ abgebildet wird. Das Bild von $\bar{G}$ bei φ sei

$$\bar{\bar{G}} \subset \mathrm{SL}(k, \Gamma[\alpha_1\ldots\alpha_s,\beta_1\ldots\beta_m])\ .$$

Wir wählen nun ein Hauptideal $\mathfrak{a}=(p) \subset \Gamma[\alpha_1\ldots\beta_m]$, p eine rationale Primzahl, so daß $\mathfrak{a}$ teilerfremd zu allen Nennern der algebraischen Zahlen $\neq 0$, welche in den Matrizen $\bar{\bar{a}}_\mu \neq 1$ auftreten. Da die Menge dieser algebraischen Zahlen endlich ist, ist durch sie auch nur eine endliche Menge P von Primzahlen p ausgeschlossen. Mittels $\mathfrak{a}$ bilden wir den Homomorphismus

$$\eta_p\colon\ \mathrm{SL}(k, \Gamma[\alpha_1\ldots\beta_m]) \to \mathrm{SL}(k,R)\ ,$$

wobei R der endliche Restklassenring $\Gamma[\alpha_1\ldots\beta_m] \bmod \mathfrak{a}$ ist. Daher ist $\mathrm{SL}(k,R)$ endliche Gruppe, also auch $G_p=\eta_p(\bar{\bar{G}})$. Wir behaupten nun den

HILFSSATZ 1. *Ein erzeugendes Element* $\bar{\bar{a}}_\sigma(\neq 1) \in \bar{\bar{G}}$ *kann nicht bei jedem Homomorphismus* η_p, $p \notin P$, *auf* 1 *abgebildet werden*, d.h. G_p *ist für mindestens ein* $p \notin P$ *nicht* 1.

BEWEIS. Wenn $\eta_p(\bar{\bar{a}}_\sigma)=1$ für jedes $p \notin P$, so heißt dies $\bar{\bar{a}}_\sigma - 1 \equiv 0 \bmod \mathfrak{a}$ für alle $p \notin P$ und daraus folgt $\bar{\bar{a}}_\sigma - 1 = 0$, weil es unendlich viele $p \notin P$ gibt, eine ganzalgebraische Zahl aber nur durch endlich viele verschiedene Ideale $\mathfrak{a}$ teilbar sein kann. Wir wissen bereits, daß es mindestens ein $\bar{\bar{a}}_\sigma \neq 1$ gibt, also gibt es auch ein p mit $G_p \neq 1$.

Insgesamt erhalten wir wieder eine homomorphe Abbildung

$$\eta_p \varphi D\colon\ \bar{\bar{G}} \to G_p \neq 1$$

mit der gewünschten Faktorgruppe. Damit ist das Lemma vollständig bewiesen.

Wir können den Beweis von Lemma 2 auch im Falle unendlich erzeugter Gruppen durchführen, sofern über die Darstellung D einige zusätzliche Voraussetzungen bekannt sind. Das Lemma lautet dann folgendermaßen

LEMMA 3. *Eine Gruppe* G *besitze eine nichttriviale, endlich-dimensionale Darstellung derart, daß die Matrizen* $D(a_\sigma)$ *der erzeugenden Elemente* $a_\sigma \in G$ *die Eigenschaften haben, daß*

a) *nur endlich viele transzendente Zahlen als Elemente vorkommen*;

b) *die algebraischen Zahlen, welche als Elemente auftreten, aus ein und*

demselben Zahlkörper K endlichen Grades sind, und daß die Nenner dieser algebraischen Elemente sämtliche teilerfremd zu einem Primideal $\mathfrak{p} \subset K$ sind.

Dann hat G einen Normalteiler N mit endlicher Faktorgruppe $F = G/N \neq 1$.

BEWEIS. Man substituiere für die transzendenten Zahlen ganzalgebraische Zahlen β so, daß die entstehende Faktorgruppe $\bar{G}$ wieder $\neq 1$ wird. (Nun kann man sich wieder auf den Fall $\mathrm{Det}(a) = 1$, $a \in G$, beschränken, wie man mit Hilfe des durch die Normfunktion gegebenen Homomorphismus einsieht!) Mittels $\mathfrak{p}$, das nach Voraussetzung existiert, bilde man weiter die Homomorphismen

$$\eta_n\colon\ \mathrm{SL}(k, \Gamma[\ldots]) \to \mathrm{SL}(k, R)\,,$$

wobei $R \equiv \Gamma[\ldots] \bmod \mathfrak{p}^n$ ist (endlicher Restklassenring). Diese liefern also für G lauter endliche Faktorgruppen. Es gilt dabei der entsprechende Hilfssatz

HILFSSATZ 2. *Falls $\bar{a}_\mu \neq 1$ erzeugendes Element von $\bar{G}$, so gibt es ein genügend großes, natürliches n mit $\eta_n(\bar{a}_\mu) \neq 1$.*

Wäre nämlich $\eta_n(\bar{a}_\mu) = 1$ für alle natürlichen n, so hieße das gerade $\bar{a}_\mu - 1 = 0 \bmod \mathfrak{p}^n$ für alle n und daher $\bar{a}_\mu - 1 = 0$. Widerspruch!

Damit ist der Beweis von Lemma 3 vollendet.

Eine Gruppe nennt man darstellbar, falls sie eine nichttriviale Darstellung durch (endlich-dimensionale) Matrizen gestattet. Eine treu darstellbare Gruppe heißt entsprechend Matrizengruppe. Es gilt dann für beliebige Gruppen der Hilfssatz

HILFSSATZ 3. *Eine Gruppe G mit treu darstellbarem Normalteiler N von endlichem Index ist selbst Matrizengruppe.*

Zusammen mit Lemma 2 hat man daher folgendes

KRITERIUM. *Eine endlich-erzeugbare Gruppe ist genau dann Matrizengruppe, wenn sie einen treu darstellbaren Normalteiler mit endlichem Index besitzt.*

BEWEIS DES HILFSSATZES. Sei $D\colon n \to D(n)$ die gegebene treue Darstellung von N. Wir definieren

$$D_i(n) = D(a_i^{-1} n a_i), \qquad i = 1, 2, \ldots, r\,,$$

wobei $a_1, a_2, \ldots, a_r$ ein Repräsentantensystem von G/N ist. Mit $\bar{g}$ bezeichnen wir die Restklasse von $g \in G$; eine Permutationsdarstellung von $F = \{\bar{a}_1, \ldots, \bar{a}_r\}$ ist gegeben durch die Permutationen

$$P_g = \begin{pmatrix} \bar{a}_1 \dots \bar{a}_r \\ \bar{g}\bar{a}_1 \dots \bar{g}\bar{a}_r \end{pmatrix} = \begin{pmatrix} 1 \dots r \\ i_1 \dots i_r \end{pmatrix},$$

wofür wir kurz $P_g(k)=i_k$, $k=1,2,\dots,r$, schreiben. Nun gilt wegen $\overline{ga}_i=\bar{g}\bar{a}_i$ aber

$$ga_i = n_g^{(i)} a_{P_g(i)} \quad \text{mit} \quad n_g^{(i)} \in N ,$$

und daher können wir die Darstellung T von G durch

$$T_g = \bigoplus_{i=1}^{r} D_{P_g(i)}(n_g^{(i)})$$

für alle $g \in G$ definieren. Da P treu auf F und D treu auf N, so folgt offenbar, dass T treu auf G ist. Damit ist der Satz bewiesen.

Beispiele zu Hilfssatz 3 sind die freien Produkte von endlich vielen Gruppen endlicher Ordnung.

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A CHARACTERIZATION OF REFLEXIVITY OF BANACH SPACES

KJELD B. LAURSEN

1. Introduction.

A Banach-space X is reflexive if and only if the natural embedding of X into X^{**} is onto. A large number of necessary and sufficient conditions for reflexivity are known, one of them being that a Banach-space X is reflexive if and only if its unitball B is weakly compact. In this paper we prove an extension of this criterion. We shall construct a topology on a normed linear space X which is in general weaker than the weak topology and show that the space is reflexive if and only if the unitball is compact in this weaker topology.

The problem was originally related to the notion of a semi-inner-product space [5]. Our topology is defined in terms of a mapping $W: X \to X^*$ satisfying two simple conditions (see section 2). One can show that W satisfies these conditions if and only if $[x,y] = W_y(x)$, where W_y is the image of y under W, is a semi-inner-product consistent with the norm on X. We shall not make any use of this relationship.

The statement and proof of our theorem are contained in section 4. Section 2 lists the notation and terminology necessary for the remainder of the paper and section 3 mentions some known results that we use in section 4. We conclude with an example showing that the topology in question is generally weaker than the weak topology.

2. Notation and terminology.

If X is a vector space we denote the space of all linear functionals on X by X^+. If $F \subset X^+$ is a linear subspace with the property that if $f(x) = 0$ for all f in F then $x = 0$, we shall say that F is a total subspace of X^+ or that F is total over X.

Suppose that X is a vector space and that F is a total subspace over X. Following [2] we define the F-topology of X (or the weak topology in X induced by F) as the topology with base consisting of all sets of the form

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$$N(x_0, A, \varepsilon) = \{x \in X;\ |f(x) - f(x_0)| < \varepsilon \text{ for all } f \in A\}$$

where A is a finite subset of F and $\varepsilon > 0$.

Now, let X be a normed linear space (in the following abbreviated NLS) with dual X^*. Let $W: X \to X^*$ be a mapping satisfying the conditions (we write W_x for the image of x under W)

i) $\|W_x\| = \|x\|$,
ii) $W_x(x) = \|x\|^2$ for all x in X.

(It follows from the Hahn–Banach Theorem (cf. [2, p. 65] and the proof of Theorem 1) that such a mapping exists. In fact, there will in general be infinitely many such mappings on any given NLS X).

Using the notation $\mathrm{sp}(S)$ for the linear span (i.e. finite linear combinations of elements from S) of a subset S of a vector space we define the W-topology as the topology defined above with $F = \mathrm{sp}(W(X))$. This topology is Hausdorff since it is easily seen that $\mathrm{sp}(W(X))$ is a total subspace of X^+.

We end this section with a few definitions that will be needed in section 3:

Let X be a vector space and Y a subspace of X^+. If $S \subset X$ then S°, the polar of S in Y is defined as

$$S^\circ = \{f \in Y;\ |f(x)| \leqq 1 \text{ for all } x \text{ in } S\}.$$

S is convex if $x, y \in S$, $t \in [0,1]$ imply $tx + (1-t)\, y \in S$. S is radial at 0 if for every $x \in X$ there exists $t > 0$ such that $0 \leqq s < t$ implies $sx \in S$.

3. Some known results.

We list here two very powerful theorems which are crucial in our proof of the main theorem. They are Smulian's necessary and sufficient condition for compactness of a closed, circled convex subset of a linear space in the weak topology defined in section 2 by a total subspace of the algebraic dual (cf. [4, pp. 142–143]) and R. C. James' characterization of reflexivity of a Banach-space [3].

The Smulian Compactness Criterion:

Let X be a vector space and F a total subspace of X^+. Let B be a circled, convex subset of X such that B is closed in the weak topology induced by F. Then B is compact in this topology if and only if $B^\circ \subset F$ is radial at 0 and for each linear functional f on F which is bounded on B° there is an x in X such that $f(y) = y(x)$ for all $y \in F$.

James' characterization of reflexivity:

A Banach-space X is reflexive if and only if every $x^* \in X^*$ attains its norm on the unitball B of X, i.e. for every x^* in X^* there is an x in X of norm 1 such that $x^*(x) = \|x^*\|$.

The following results—found in [1]—are all needed in section 5:

Let S be an index set and let Y be a Banach-space of real-valued functions on S. If for each s in S a NLS N_s is given, let $P_Y N_s$ be the space of all functions x on S satisfying

i) x_s is in N_s for all s in S,
ii) if h is the real-valued function defined by $h(s) = \|x_s\|$ for all s in S then $h \in Y$.

LEMMA 1. *If Y satisfies the condition that whenever $h \in Y$ and $|k(s)| \leqq |h(s)|$ for all s then $k \in Y$ and $\|k\| \leqq \|h\|$, then $P_Y N_s$ is a normed linear space and is complete if all N_s are complete.*

LEMMA 2. *If $1 < p < \infty$ and $Y = \ell^p(S)$, then*

$$(P_Y N_s)^* = P_{Y^*} N_s^* .$$

From this follows directly

LEMMA 3. *If $1 < p < \infty$ and $Y = \ell^p(S)$ then $P_Y N_s$ is reflexive if and only if each N_s is reflexive.*

Since every finite dimensional NLS is reflexive ([2, p. 246]) it follows that with Y defined as in Lemma 3 $P_Y N_s$ is reflexive if every N_s is finite dimensional.

4. The main theorem.

THEOREM 1. *Let X be a Banach-space. Then X is reflexive if and only if there exists a mapping $W: X \to X^*$ satisfying the conditions*

$$\text{i)} \quad W_x(x) = \|x\|^2 ,$$
$$\text{ii)} \quad \|W_x\| = \|x\|$$

for all $x \in X$ such that the unitball B of X is compact in the weak topology induced by $\operatorname{sp}(W(X))$.

REMARK. Since all such W-topologies are weaker than the weak topology and since reflexivity implies weak compactness of B, it follows that if B is compact in one W-topology, it is compact in them all.

PROOF OF THEOREM 1. Suppose first that X is a reflexive Banach-space. By the Hahn–Banach Theorem for each x in X, $x \neq 0$, choose V_x in X^* such that $\|V_x\| = 1$ and $V_x(x) = \|x\|$. Define $W_x = \|x\| V_x$. Since each W_x is bounded, it follows that $W(X) \subset X^*$, i.e. $\mathrm{sp}(W(X)) \subset X^*$. From this it is clear that the W-topology induced by $\mathrm{sp}(W)$ is weaker than the weak topology. Since the unitball B is weakly compact, we conclude that B is compact in the topology induced by W.

Conversely, let $W: X \to X^*$ be a mapping satisfying the conditions i) and ii). Let $F = \mathrm{sp}(W(X)) \subset X^*$. F is a NLS with the norm inherited from X^*. The polar of B in F is exactly the unitball of F, i.e. $B^\circ = B^* \cap F$, where B^* is the unitball of X^*. It is trivial that B is convex and circled and since the W-topology is Hausdorff, W-compactness of B implies that B is W-closed. The Smulian compactness criterion therefore implies that every linear functional on F which is bounded on B° is evaluation at a point of X. But since F is a NLS, the set of linear functionals bounded on B° is precisely F^*, i.e. $f \in F^*$ implies that there is $x_f \in X$ such that $f(z) = z(x_f)$ for every $z \in F$.

Since every x in X by the natural embedding of X into X^{**} is a continuous linear functional on X^* and hence (by restriction) on F, we have a linear mapping Q of X into F^* such that $(Qx)(z) = z(x)$ for all $z \in F$. Given any $f \in F^*$, $(Qx_f)(z) = z(x_f) = f(z)$ for all z in F so $Qx_f = f$. This proves that Q is onto.

We shall show that Q is an isometry, i.e. $\|Qx\| = \|x\|$ for all $x \in X$.

$$\begin{aligned} \|Qx\| &= \sup\{|(Qx)(Z)|;\ Z \in B^\circ, \|Z\| = 1\} \\ &\leqq \sup\{|Z(x)|;\ Z \in B^*, \|Z\| = 1\} = \|x\|, \end{aligned}$$

since $B^\circ \subset B^*$. But

$$(Qx)(W_x/\|x\|) = W_x(x)/\|x\| = \|x\|$$

by condition i). This gives us the reverse inequality and thus that $\|Qx\| = \|x\|$ for all $x \in X$. The above computation also shows that every continuous functional on F attains its norm. The completion $\overline{F}$ of F in X^* has the same dual as F. Using the James characterization of reflexivity we conclude that $\overline{F}$ is reflexive. This implies that $\overline{F}^* = X$ is reflexive (cf. [2, p. 67]).

The following corollary is immediate.

COROLLARY. *If X is a reflexive Banach-space and W is a mapping as defined in Theorem* 1, *then the weak topology and the W-topology coincide on bounded sets.*

5. An example.

Theorem 1 is not particularly interesting unless it is known that there exists a reflexive Banach-space with a W-topology which is weaker than the weak topology. There are many examples of reflexive Banach-spaces in which the W-topology coincides with the weak topology. It can for instance be shown that if X has a smooth unitball (every point of the unit sphere has a unique supporting hyperplane) then W is uniquely determined and its induced topology equals the weak topology.

We shall not prove these facts here, but proceed immediately to our example.

It suffices to find a reflexive Banach-space X and a mapping W satisfying i) and ii) in Theorem 1 such that $\mathrm{sp}(W(X)) \neq X^*$ (cf. [2, p. 421 and p. 436]).

Let S be the set of natural numbers and $Y = \ell^2(S)$ be the space of square summable real sequences with the usual norm. For $s \in S$ let $N_s = \ell^\infty(s)$ be the space of real s-tuples with the max-norm and let $X = P_Y N_s$. We note that $x = (A_1, A_2, A_3, \dots)$ where $A_i = (a_{i1}, a_{i2}, \dots, a_{ii})$ for $i = 1, 2, \dots$ is an element of X if and only if

$$\sum_{i=1}^{\infty} \|A_i\|^2 < \infty, \qquad \text{where} \quad \|A_i\| = \max_{j \leq i} |a_{ij}| .$$

Furthermore

$$\|x\| = \left(\sum_{i=1}^{\infty} \|A_i\|^2 \right)^{\frac{1}{2}} .$$

It follows from the discussion in section 3 that X is reflexive and $X^* = P_Y N_s^*$ where $N_s^* = \ell^1(s)$; if $B = (b_1, b_2, \dots b_s) \in \ell^1(s)$, then $\|B\| = \sum_{k=1}^{s} |b_k|$. If $x^* = (B_1, B_2, \dots) \in X^*$ with $B_i = (b_{i1}, b_{i2}, \dots, b_{ii})$, then $\|x^*\| = \left(\sum_{i=1}^{\infty} (\sum_{j=1}^{i} |b_{ij}|^2)\right)^{\frac{1}{2}}$ and

$$x^*(x) = \sum_{i=1}^{\infty} B_i(A_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} b_{ij} a_{ij} .$$

To construct $W: X \to X^*$ we can proceed as follows:

Let $x = (A_1, A_2, \dots)$ with $A_i = (a_{i1}, a_{i2}, \dots, a_{ii})$ be an element of X. For each i let $j(i)$ be the smallest index such that $|a_{ij(i)}| = \|A_i\|$. Choose $x^* \in X^*$ such that $x^* = (B_1, B_2, \dots)$ where for each i, $B_i = (b_{i1}, b_{i2}, \dots, b_{ii})$ is defined by

$$b_{ij(i)} = a_{ij(i)} \qquad \text{and} \qquad b_{ij} = 0 \quad \text{if } j \neq j(i) .$$

Then

$$\|x^*\| = \left(\sum_{i=1}^{\infty} \|B_i\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} |b_{ij(i)}|^2 \right)^{\frac{1}{2}} = \|x\|$$

and

$$x^*(x) = \sum_{i=1}^{\infty} B_i(A_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} b_{ij} a_{ij} = \sum_{i=1}^{\infty} a_{ij(i)}^2 = \sum_{i=1}^{\infty} \|A_i\|^2 = \|x\|^2 .$$

Thus if for each x in X we let W_x be defined as indicated above we see that W satisfies the conditions i) and ii) of Theorem 1.

It remains to find an x_0^* in X^* which is not in $\mathrm{sp}(W(X))$. From the way we have constructed W we see that if $x^* \in W(X)$, $x^* = (B_1, B_2, \ldots)$ then each B_i has exactly one coordinate which is different from 0. This implies that if

$$y^* = \sum_{k=1}^{p} \delta_k x_k^* = (C_1, C_2, \ldots)$$

is an element of $\mathrm{sp}(W(X))$ then for $i > p$, C_i has at most p coordinates different from 0. Thus any element in X for which all coordinates are different from 0 is not in $\mathrm{sp}(W(X))$. One such $x_0^* = (B_1, B_2, \ldots)$ is defined by $B_i = (b_{i1}, b_{i2}, \ldots, b_{ii})$ with $b_{ij} = (i^2 2^i)^{-\frac{1}{2}}$ for each $j = 1, 2, \ldots, i$. We have

$$\|x_0^*\| = \left(\sum_{i=1}^{\infty} \|B_i\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} \left(\sum_{j=1}^{i} |b_{ij}| \right)^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} 2^{-i} \right)^{\frac{1}{2}} = 1 .$$

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A GENERALIZATION OF A THEOREM OF SYLVESTER ON THE LINES DETERMINED BY A FINITE POINT SET

STEN HANSEN

In 1893 Sylvester stated without proof: Given any finite set of non-collinear points in the real projective plane, there exists at least one line which contains exactly two of the given points.

This theorem remained unproved for nearly 40 years. In 1933 T. Gallai (Grünwald) gave a proof, and later on several other proofs have been found (for references cf. [3, p. 451] and [2, p. 65]). Here we mention R. Steinberg's (1944, cf. [1, p. 30]) because related ideas are used in the present paper.

In general it is not true that, given a finite point set in a projective space of dimension $d>2$ which is not contained in a hyperplane, there exists a hyperplane containing exactly d of the given points. A counter-example in 3-space is a set of 6 points, 3 on each of two skew lines. Another one is the Desargues configuration in 3-space [3, p. 452].

The following generalization of Sylvester's theorem has been proved by Th. Motzkin [3] for $d=3$:

Given a finite point set Γ_0 in the d-dimensional real projective space which is not contained in a hyperplane. Then among the hyperplanes determined by points of Γ_0 there is at least one with the property that the points of Γ_0 which it contains, with the exception of precisely one of them, lie in a $(d-2)$-dimensional projective subspace.

It is the aim of this paper to give a proof of this theorem for arbitrary dimension d.

Let Γ_p, $p=0,1,\ldots,d$, denote the set of the projective subspaces of dimension p which are spanned by the points of the given set Γ_0. Since Γ_0 spans the whole d-dimensional space P_d considered, none of these sets is empty, and Γ_d consists of P_d only. The elements of Γ_p will be denoted $A_p,\ldots,P_p,\ldots$. The union

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$$\Gamma = \bigcup_{p=0}^{d} \Gamma_p$$

will be called the *configuration* Γ.

A subspace $A_p \in \Gamma_p$ will be called *elementary* if it contains exactly $p+1$ points of Γ_0. If Γ_0 consists of $d+1$ points, then Γ consists of the subspaces spanned by the vertices of a simplex, and thus all elements of Γ are elementary. In this case the configuration Γ will be called elementary.

If a subspace A_p is spanned by a point B_0 and a subspace C_{p-1}, we shall write

$$A_p = B_0 C_{p-1} .$$

Obviously, $B_0 \in \Gamma_0$ and $C_{p-1} \in \Gamma_{p-1}$ imply $A_p \in \Gamma_p$. If in this case B_0 is the only point of $\Gamma_0 \cap A_p$ outside C_{p-1}, then the subspace A_p is called *ordinary*. (This makes also sense for $p=0$, if, as usual, the (-1)-dimensional projective space is defined to be the empty set.) It is clear that every elementary subspace is ordinary. In this terminology the theorem above states the existence of an ordinary hyperplane.

Every $A_p \in \Gamma$, $p>0$, is divided into polyhedral domains by the subspaces $A_{p-1} \in \Gamma_{p-1}$ contained in A_p. The closures of these domains will be called the *p-dimensional cells* of the configuration Γ. Each cell δ_p in A_p is obviously convex in the sense that, for any two points of it, that segment joining these points which does not intersect an $A_{p-1} \subset A_p$ belongs entirely to δ_p.

We shall prove the following theorem which is slightly stronger than the statement above:

THEOREM. *Suppose that the configuration Γ in the real projective space of dimension d is not elementary, and let δ_d be a d-dimensional cell of Γ. Then there exists an ordinary hyperplane $A_{d-1} = B_0 C_{d-2}$, where $B_0 \in \Gamma_0$ and*

$$A_{d-1} \cap \Gamma_0 \setminus \{B_0\} \subset C_{d-2} \in \Gamma_{d-2} ,$$

such that

$$A_{d-1} \cap \delta_d \subset C_{d-2} .$$

We start by proving two lemmas.

LEMMA 1. *Let σ_d, $d>0$, denote a closed d-dimensional simplex whose vertices belong to Γ_0, and suppose that the point $A_0 \in \Gamma_0$ is not contained in σ_d. Then there is a $(d-2)$-dimensional face σ_{d-2} of σ_d such that the hyperplane $B_{d-1} \in \Gamma_{d-1}$ spanned by A_0 and σ_{d-2} satisfies*

$$B_{d-1} \cap \sigma_d = \sigma_{d-2} .$$

PROOF. Consider the hyperplanes which contain the $(d-1)$-dimensional faces of σ_d. Every pair of distinct such hyperplanes divides the space P_d into two "wedges". The closure of one of these contains σ_d. Since σ_d is the intersection of such closed wedges, there must be at least one which does not contain A_0. The hyperplane spanned by A_0 and the intersection of the hyperplanes bounding such a wedge satisfies the requirement of the lemma.

LEMMA 2. *Let A_0 be a point of Γ_0, and let $C_{d-1} \in \Gamma_{d-1}$ be a hyperplane which does not contain A_0. Let further $\delta_{d-1} \subset C_{d-1}$ be a $(d-1)$-dimensional cell of Γ. If P_0 is a point such that the line A_0P_0 does not meet δ_{d-1} and Q_0 is an interior point of δ_{d-1}, then each of the two segments P_0Q_0 intersects at least one of the hyperplanes, belonging to Γ_{d-1}, which are spanned by A_0 and the $(d-2)$-dimensional faces of δ_{d-1}.*

PROOF. The union of the lines joining A_0 with the points of δ_{d-1} is a polyhedral convex cone. Since P_0 is an exterior point and Q_0 an interior point of this cone, each of the segments P_0Q_0 intersects its boundary. The statement now follows from the fact that every boundary point of the cone is contained in at least one of the hyperplanes spanned by A_0 and the $(d-2)$-dimensional faces of δ_{d-1}.

PROOF OF THE THEOREM. We proceed by induction on the dimension of the space. For $d=1$ the Theorem is obvious. Let $d>1$ be given. We assume the Theorem to be true for spaces of dimension $d-1$.

Let a d-dimensional cell δ_d of Γ be given. Obviously, δ_d is contained in some closed simplex with vertices belonging to Γ_0. If this simplex contains a point of Γ_0 different from its vertices, the hyperplanes spanned by this point and the vertices divide the simplex into smaller simplexes one of which contains δ_d. If this simplex contains a point of Γ_0 different from its vertices, the procedure can be repeated. After finitely many steps a simplex σ_d is obtained which contains δ_d but no point of Γ_0 other than its vertices.

Since Γ is not elementary, there is a point of Γ_0 outside σ_d. By Lemma 1, there exists a hyperplane $B_{d-1} \in \Gamma_{d-1}$ through this point for which

$$B_{d-1} \cap \delta_d \subset B_{d-1} \cap \sigma_d \subset S_{d-2},$$

where $S_{d-2} \in \Gamma_{d-2}$ is the $(d-2)$-dimensional subspace containing one of the $(d-2)$-dimensional faces of σ_d.

If B_{d-1} is elementary, it clearly satisfies the requirement of the Theorem. Hence, in the sequel we may assume that B_{d-1} is not elementary.

We consider a point $P_0 \in \Gamma_0$ which does not lie in B_{d-1}. We choose a

line L_1 which joins P_0 with an interior point of δ_d and which has no point different from P_0 in common with any of the $(d-2)$-dimensional subspaces in which two hyperplanes belonging to Γ_{d-1} intersect. (In particular, L_1 does then not meet any subspace belonging to Γ_{d-2} at a point different from P_0.)

By assumption Γ_{d-1} contains non-elementary hyperplanes, for instance B_{d-1}. Since these hyperplanes do not intersect the interior of δ_d, there is at least one among them, Q_{d-1} say, such that the point Q_0 at which it intersects L_1 satisfies the following condition: One of the open segments of L_1 determined by P_0 and Q_0 intersects neither the interior of δ_d nor any of the non-elementary hyperplanes. In other words, traversing L_1 from P_0 in that sense, or one of the senses, in which one meets non-elementary hyperplanes before meeting δ_d, Q_0 is the first point of intersection with a non-elementary hyperplane.

The point Q_0 belongs to the interior of exactly one of the $(d-1)$-dimensional cells of Γ into which Q_{d-1} is divided. The polyhedral cone γ_d consisting of the lines joining P_0 with the points of this cell δ_{d-1} contains δ_d because the hyperplanes which contribute to the boundary of γ_d belong to Γ_{d-1} and the line $L_1 \subset \gamma_d$ intersects δ_d. By the induction hypothesis, there exists in Q_{d-1} an ordinary $(d-2)$-dimensional subspace $C_0S_{d-3} \in \Gamma_{d-2}$, where

$$C_0 \in \Gamma_0, \qquad C_0S_{d-3} \cap \Gamma_0 \setminus \{C_0\} \subset S_{d-3} \in \Gamma_{d-3},$$

such that

$$C_0S_{d-3} \cap \delta_{d-1} \subset S_{d-3}.$$

Putting

$$S_{d-2} = P_0S_{d-3} \in \Gamma_{d-2},$$

we consider the hyperplane

$$C_0S_{d-2} \in \Gamma_{d-1}.$$

Obviously,

$$C_0S_{d-2} \cap \gamma_d \subset S_{d-2}, \tag{1}$$

hence

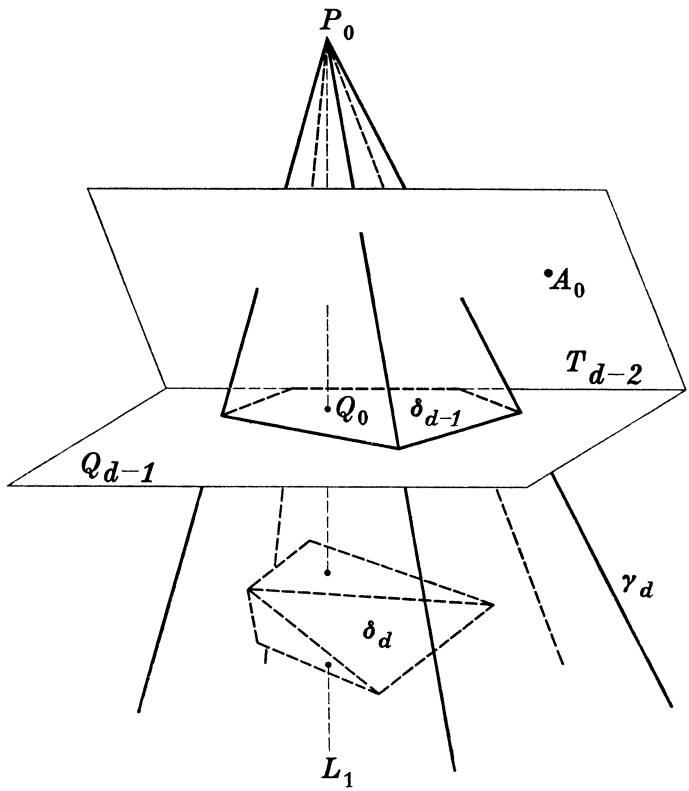
$$C_0S_{d-2} \cap \delta_d \subset S_{d-2} \tag{2}$$

since $\delta_d \subset \gamma_d$.

We distinguish now between two cases:

1°. In C_0S_{d-2} there is outside S_{d-2} no other point of Γ_0 than C_0. Then C_0S_{d-2} is ordinary, and because of (2) this hyperplane satisfies the requirements of the Theorem.

2°. In C_0S_{d-2} there is a point $A_0 \in \Gamma_0$ which does not lie in S_{d-2}. From the facts that



$$C_0 S_{d-2} \cap Q_{d-1} = C_0 S_{d-3},$$

$C_0 S_{d-3}$ is ordinary, and $S_{d-3} \subset S_{d-2}$ we can conclude that $A_0 \notin Q_{d-1}$. Further, from (1) and $A_0 \notin S_{d-2}$ it follows that $A_0 \notin \gamma_d$. Consequently, $P_0 A_0$ does not meet δ_{d-1}. By Lemma 2, there exists therefore a hyperplane $A_0 T_{d-2} \in \Gamma_{d-1}$ (see fig.) such that T_{d-2} contains a $(d-2)$-dimensional face of δ_{d-1}, and $A_0 T_{d-2}$ intersects that open segment $P_0 Q_0$ which does not meet δ_d. From the way in which Q_0 was determined it follows that $A_0 T_{d-2}$ is elementary, thus ordinary.

It remains to be shown that

$$A_0 T_{d-2} \cap \delta_d \subset T_{d-2}. \tag{3}$$

The hyperplanes Q_{d-1} and $P_0 T_{d-2}$ intersect in T_{d-2}. They belong to Γ_{d-1} and, thus, do not meet the interior of δ_d. Consequently, the closure of one of the two wedges into which they divide the space contains δ_d. Since the hyperplane $A_0 T_{d-2}$ intersects that open segment $P_0 Q_0$ which

does not meet δ_d, it can only have T_{d-2} in common with the wedge containing δ_d, and hence (3) holds.

This completes the proof of the Theorem.

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LINEAR RECURRING SEQUENCES OVER FINITE FIELDS

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In recent years, the interest in linear recurring sequences over finite fields has increased considerably because of their importance in the mathematical treatment of linear shift registers.

The aim of Part I of this paper is to show how most of the earlier general results concerning linear recurring sequences may be proved by a method due to W. W. Peterson.

Part II of the paper is devoted to the study of multigrams. This term (as used here) is due to E. S. Selmer. The object is to study the behaviour of the vector $(a_{n_1+i}, a_{n_2+i}, \ldots, a_{n_m+i})$ for $i = 0, 1, 2, \ldots$, where $a_0, a_1, a_2, \ldots$ is a solution of a linear recurrence relation, and $n_1, n_2, \ldots, n_m$ are arbitrary, fixed, non-negative integers.

The only previous result in this direction is a theorem by N. Zierler concerning bigrams ($m = 2$), and some of the results of this paper represent a generalisation of his investigations.

I wish to express my gratitude to Professor Ernst S. Selmer, whose lectures at the University of Bergen inspired me to work in this field, and who has helped me with the manuscript. Some of the results in Part II are also due to him.

PART I

1. Notation and definitions.

We consider the linear recurrence relation

$$\sum_{j=0}^{n} c_j a_{i-j} = 0, \quad i = n, n+1, \ldots, \qquad c_0 = 1,\ c_n \neq 0,$$

where all the elements are taken from a finite field GF[q]. Here $q = p^\tau$, p a prime and τ a positive integer.

The (shortest) period of a periodic sequence $(a_i) = a_0, a_1, a_2, \ldots$ will be denoted by $\mathrm{per}(a_i)$. If $f(x)$ is a polynomial, and r is the least positive

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integer such that $f(x) \mid (x^r - 1)$, we shall call r the period of $f(x)$ and write $\operatorname{per} f(x) = r$.

We frequently use the isomorphism

$$(a_i) = a_0, a_1, a_2, \ldots \rightleftarrows a(x) = a_0 x^{s-1} + a_1 x^{s-2} + \ldots + a_{s-1},$$

where $\operatorname{per}(a_i) = r$ and $r \mid s$, and we shall not distinguish between a sequence and the isomorphic polynomial. For instance, we write $\{a(x)\} \in G(f)$ whenever $(a_i) \in G(f)$, where the residue class $\{a(x)\}$ is defined below, and where $G(f)$ denotes the vector space consisting of all the solutions of the linear recurrence relation with characteristic polynomial

$$f(x) = c_0 x^n + c_1 x^{n-1} + \ldots + c_n, \qquad c_0 = 1, \ c_n \neq 0 .$$

We call $G(f)$ the *solution space* of $f(x)$.

The residue class determined by the polynomial $a(x)$ in the polynomial ring modulo a polynomial $F(x)$ will be denoted by $\{a(x)\}$, and the ideal generated by $\{a(x)\}$ in this ring will be denoted by $(\{a(x)\})$.

We will sometimes consider two sequences as identical if they are translates of each other. In this case, we shall speak of *unordered sequences.*

2. The theorem of Peterson, and comparisons with other methods.

The method used in this paper is based on the following theorem given by Peterson [3]:

THEOREM 1. *Let* $f(x) = c_0 x^n + c_1 x^{n-1} + \ldots + c_n$ *with* $\operatorname{per} f(x) = r$ *and* $f^*(x) = (x^r - 1)/f(x)$. *Then* r *is the shortest common period of the solutions of the recurrence relation*

$$\sum_{j=0}^{n} c_j a_{i-j} = 0, \qquad i = n, n+1, \ldots ,$$

and the solutions considered as polynomials by the isomorphism

$$(a_i) = a_0, a_1, a_2, \ldots \rightleftarrows a(x) = a_0 x^{r-1} + a_1 x^{r-2} + \ldots + a_{r-1}$$

make up the ideal generated by $\{f^*(x)\}$ *in the polynomial ring modulo* $x^r - 1$.

This theorem is closely connected with methods used by Zierler, Ward and Hall.

In Zierler [7], a sequence $a_0, a_1, a_2, \ldots$ is identified with the infinite series $a_0 + a_1 x + a_2 x^2 + \ldots$. His main tool is the result (Lemma 1) that

$$G(f) = \left\{ \frac{g(x)}{f_1(x)}; \deg g(x) < n \right\}, \qquad f_1(x) = x^n f(1/x) .$$

If we formally have

$$\frac{g(x)}{f_1(x)} = a_0+a_1x+\ldots+a_{r-1}x^{r-1}+a_0x^r+a_1x^{r+1}+\ldots,$$

then

$$(1-x^r)\frac{g(x)}{f_1(x)} = a_0+a_1x+\ldots+a_{r-1}x^{r-1},$$

which gives the connection between the two methods.

In papers by Morgan Ward and Marshall Hall (see particularly [6] and [2]), the isomorphism

$$(a_i) \rightleftarrows A(x),$$

where

$$\begin{aligned}(a_i) &= a_0, a_1, a_2, \ldots,\\ A(x) &= a_0x^{n-1}+(a_1+c_1a_0)x^{n-2}+\ldots+(a_{n-1}+c_1a_{n-2}+\ldots+c_{n-1}a_0),\end{aligned}$$

between the sequences satisfying the recurrence relation and the ring of polynomials modulo the characteristic polynomial $f(x)$, is fundamental.

If now $\mathrm{per} f(x)=r$, and $a(x)$ and $f^*(x)$ are defined as in Theorem 1, it is easily verified by direct computation that

$$a(x) = A(x)f^*(x).$$

3. Relations between the solution spaces of different polynomials.

We first prove Lemma 3 of Zierler [7]:

THEOREM 2. *$G(f)\subset G(g)$ if and only if $f(x)\,|\,g(x)$.*

PROOF. When $g(x)=0$, the theorem is trivial (if $G(0)$ denotes the set of all periodic sequences). When $g(x)\neq 0$, $\mathrm{per} f(x)=r_1$, $\mathrm{per} g(x)=r_2$, and l.c.m.$(r_1,r_2)=r$, it is easily verified that

$$G(f) = \left(\left\{f^*(x)\frac{x^r-1}{x^{r_1}-1}\right\}\right), \qquad G(g) = \left(\left\{g^*(x)\frac{x^r-1}{x^{r_2}-1}\right\}\right) \pmod{x^r-1},$$

where $f(x)f^*(x)=x^{r_1}-1$ and $g(x)g^*(x)=x^{r_2}-1$. Hence

$$G(f)\subset G(g) \Leftrightarrow g^*(x)\frac{x^r-1}{x^{r_2}-1}\,\Big|\,f^*(x)\frac{x^r-1}{x^{r_1}-1} \Leftrightarrow f(x)\,\Big|\,g(x).$$

The following easily proved lemmas will be needed:

LEMMA 1. *If u_i and v_i, $i=1,2,\ldots,k$, are polynomials satisfying $u_1v_1=u_2v_2=\ldots=u_kv_k$, then*

$$(u_1,u_2,\ldots,u_k)\,\mathrm{l.c.m.}(v_1,v_2,\ldots,v_k) = u_iv_i.$$

LEMMA 2. *If* $f_i(x) \mid f(x)$, $i = 1,2,\ldots,k$, *then* $(\mathrm{mod} f(x))$

$$(\{f_1(x)\}) + (\{f_2(x)\}) + \ldots + (\{f_k(x)\}) = (\{(f_1(x), f_2(x), \ldots, f_k(x))\})$$

$$(\{f_1(x)\}) \cap (\{f_2(x)\}) \cap \ldots \cap (\{f_k(x)\}) = (\{\text{l.c.m.}(f_1(x), f_2(x), \ldots, f_k(x))\}) .$$

We now prove two more theorems from Zierler [7]:

THEOREM 3. *If* $f(x) = \text{l.c.m.}(f_1(x), f_2(x), \ldots, f_k(x))$, *then*

$$G(f_1) + G(f_2) + \ldots + G(f_k) = G(f) .$$

PROOF. If $\mathrm{per} f_i(x) = r_i$ and

$$f_i(x) f_i^{**}(x) = x^r - 1 = f(x) f^*(x), \qquad i = 1,2,\ldots,k ,$$

where $r = \text{l.c.m.}(r_1, r_2, \ldots, r_k)$ and consequently $\mathrm{per} f(x) = r$, then we have by Lemmas 1 and 2:

$$\begin{aligned} G(f_1) + G(f_2) + \ldots + G(f_k) &= (\{f_1^{**}(x)\}) + (\{f_2^{**}(x)\}) + \ldots + (\{f_k^{**}(x)\}) \\ &= (\{(f_1^{**}(x), f_2^{**}(x), \ldots, f_k^{**}(x))\}) \\ &= (\{f^*(x)\}) = G(f) \pmod{x^r - 1} . \end{aligned}$$

THEOREM 4. *If* $f(x) = (f_1(x), f_2(x), \ldots, f_k(x))$, *then*

$$G(f_1) \cap G(f_2) \cap \ldots \cap G(f_k) = G(f) .$$

The proof is analogous to the previous one, using instead the second formula of Lemma 2.

4. Recurrence relations with a given finite set of periodic sequences as solutions.

DEFINITION. A polynomial $f(x)$ is called the *minimum polynomial* of a finite set A of periodic sequences if and only if $f(x)$ is monic and

$$A \subset G(g) \Leftrightarrow f(x) \mid g(x) .$$

LEMMA 3. *If*

$$(a_i) = a_0, a_1, a_2, \ldots \rightleftarrows a(x) = a_0 x^{r-1} + a_1 x^{r-2} + \ldots + a_{r-1}$$

is a sequence of period r, *then* $f(x)$ *is the minimum polynomial of* (a_i) *if and only if*

$$f(x) = \frac{x^r - 1}{(x^r - 1, a(x))} . \tag{1}$$

PROOF. If (1) holds, then $\mathrm{per} f(x) = r$, since $\mathrm{per} f(x) = s$ implies $s \mid r$ and accordingly

$$(x^r-1, a(x)) = \frac{x^s-1}{f(x)} \cdot \frac{x^r-1}{x^s-1}, \qquad \text{hence} \qquad \frac{x^r-1}{x^s-1} \Big| a(x) ,$$

which shows that $r=s$. We now have $f^*(x)=(x^r-1, a(x))$, so that $f^*(x) \mid a(x)$ and $\{a(x)\} \in (\{f^*(x)\}) = G(f)$, which together with Theorem 2 implies that

$$\text{if} \quad f(x) \mid g(x) \quad \text{then} \quad (a_i) \in G(f) \subset G(g) .$$

Conversely, given $(a_i) \in G(g)$ with $\operatorname{per} g(x) = t$, then $r \mid t$ and

$$\left\{a(x)\frac{x^t-1}{x^r-1}\right\} \in \left(\left\{\frac{x^t-1}{g(x)}\right\}\right) = G(g) \qquad (\text{mod } x^t-1) ,$$

and accordingly

$$\frac{x^t-1}{g(x)} \Big| a(x)\frac{x^t-1}{x^r-1} \Rightarrow \frac{x^r-1}{f^*(x)} = f(x) \Big| \frac{a(x)}{f^*(x)} g(x) \Rightarrow f(x) \mid g(x) ,$$

since

$$\left(\frac{x^r-1}{f^*(x)}, \frac{a(x)}{f^*(x)}\right) = \left(f(x), \frac{a(x)}{f^*(x)}\right) = 1 .$$

The „only if" of Lemma 3 follows because the minimum polynomial of a set is obviously unique.

COROLLARY 1. (Zierler [7], lemma 9.) *If $f(x)$ is the minimum polynomial of the sequence (a_i), then* $\operatorname{per}(a_i) = \operatorname{per} f(x)$.

COROLLARY 2. *If* $\operatorname{per} f(x) = r$ *and*

$$f^*(x) = \frac{x^r-1}{f(x)} = d_0 x^{r-n} + d_1 x^{r-n-1} + \ldots + d_{r-n} ,$$

then $f(x)$ is the minimum polynomial of the sequence

$$\{f^*(x)\} = \{0x^{r-1} + 0x^{r-2} + \ldots + 0x^{r-n+1} + d_0 x^{r-n} + d_1 x^{r-n-1} + \ldots + d_{r-n}\} .$$

PROOF. If $\operatorname{per}\{f^*(x)\} = s < r$, we would get

$$\frac{x^r-1}{x^s-1} \Big| f^*(x) = \frac{x^r-1}{f(x)} \Rightarrow f(x) \mid (x^s-1) ,$$

contradicting $\operatorname{per} f(x) = r$. With $a(x) = f^*(x)$ in Lemma 3, we have

$$\frac{x^r-1}{(x^r-1, f^*(x))} = \frac{x^r-1}{f^*(x)} = f(x) .$$

The following theorem represents an extension of Lemma 3:

THEOREM 5. *If A is a finite set of periodic sequences, then the minimum polynomial of A is $f(x)$ if and only if this is the least common multiple of the minimum polynomials of the sequences in A.*

PROOF. Let the sequences in A be denoted by $(a_i)_1, (a_i)_2, \ldots, (a_i)_k$, with the minimum polynomials $f_1(x), f_2(x), \ldots, f_k(x)$ respectively, and take

$$f(x) = \text{l.c.m.}\big(f_1(x), f_2(x), \ldots, f_k(x)\big) .$$

By Theorem 2, we have $(a_i)_j \in G(f_j) \subset G(f)$ and consequently $A \subset G(f) \subset G(g)$ when $f(x) \mid g(x)$.

Conversely, by the definition of the minimum polynomial, $A \subset G(g)$ implies $f_j(x) \mid g(x)$, $j = 1, 2, \ldots, k$, and so $f(x) \mid g(x)$.

The „only if" of Theorem 5 is again guaranteed by the fact that there is at most one minimum polynomial.

COROLLARY 1. *If $f(x)$ is the minimum polynomial of the set A, then* per$f(x)$ *is the least common multiple of the periods of the sequences in A.*

COROLLARY 2. (Zierler [7], Theorem 2.) *Every finite set of periodic sequences has a minimum polynomial.*

We shall now investigate the conditions that a set A must satisfy in order to be the solution space $G(f)$ of some $f(x)$. Theorem 1 immediately gives the following necessary conditions: A is closed under

1° addition of its sequences;

2° multiplication of the sequences by an element of GF$[q]$;

3° translations of the sequences.

That these conditions are also sufficient is easily seen from the fact that A then constitutes an ideal in the ring of polynomials modulo $x^r - 1$. where r is the least common multiple of the periods of the sequences in A, As this is a principal ideal ring, there exists a $g^{**}(x)$ such that

$$A = \big(\{g^{**}(x)\}\big) \pmod{x^r - 1} ,$$

and so

$$A = G(g) \quad \text{where} \quad g(x) = (x^r - 1)/g^{**}(x) .$$

(This is Theorem 3 in Zierler [7].)

5. Blocks and multipliers.

In this section, we shall see how the method of Peterson applies to the theory of blocks and multipliers of sequences generated by linear recurrence relations with *irreducible* characteristic polynomials.

DEFINITION. If a sequence, after multiplication of each of its terms by an element of GF[q], gives a translate of the original sequence, this element is called a *multiplier* of the sequence.

If the sequence is translated M steps forward then the multiplier is said to be of span M.

If $\{a(x)\} \in G(f)$ and c is a multiplier of $\{a(x)\} = \{A(x)f^*(x)\}$ of span M, then we have

$$\{c\}\{a(x)\} = \{ca(x)\} = \{x^M a(x)\} \pmod{x^r - 1},$$

and consequently $A(x)(x^M - c) \equiv 0 \pmod{f(x)}$. Here and in the following $f^*(x)$ is as usual given by $f^*(x)f(x) = x^r - 1$, where $r = \mathrm{per} f(x)$.

By the concluding remark of § 2, the polynomials $A(x)$ given by

$$\{a(x)\} = \{A(x)f^*(x)\} \in (\{f^*(x)\}) = G(f)$$

for every $\{a(x)\}$ in the ideal $(\{f^*(x)\})$ comprise exactly the polynomial ring modulo $f(x)$.

When (and only when) $f(x)$ is irreducible, this ring is in fact a field, and the residue class x is a root of the polynomial $f(y)$.

Taking $f(x)$ to be irreducible and $\mathrm{per} f(x) = r$, we can always find a polynomial $\eta(x)$ which is a primitive element of the finite field modulo $f(x)$ and which satisfies

$$\eta(x)^{(q^n-1)/r} \equiv x \pmod{f(x)}.$$

With this choice, we want to make

$$\{x^M A(x)f^*(x)\} = \{cA(x)f^*(x)\} \pmod{x^r - 1},$$

or

$$x^M \equiv \eta(x)^{(q^n-1)M/r} \equiv c \pmod{f(x)},$$

where $c \in \mathrm{GF}[q]$. But the elements $y \neq 0$ of the field modulo $f(x)$ which belong to GF[q] are just the roots of $y^{q-1} \equiv 1 \pmod{f(x)}$, which shows that M must be a multiple of r/e, where $e = (r, q-1)$. The number $\mu = r/e$ is called the restricted or reduced period of the sequences of $G(f)$.

Thus we have the result (cf. Ward [4, Theorems 9.2, 9.3, 9.4 and 9.6] and Hall [2, Lemma 1, p. 215]):

THEOREM 6. *The multipliers of the sequences of $G(f)$, where $f(x)$ is irreducible,* $\mathrm{per} f(x) = r$, $e = (r, q-1)$ *and* $\mu = r/e$, *are exactly the elements of* GF[q] *satisfying* $x^e = 1$, *and they are of* span $0, \mu, 2\mu, \ldots, (e-1)\mu$.

DEFINITION. The set of different unordered sequences obtained when a sequence is multiplied by all the non-zero elements of GF[q] is called a *block*.

We write

$$t = \frac{q-1}{e}, \quad \varkappa = \frac{1}{\mu}\frac{q^n-1}{q-1}.$$

These numbers are both seen to be integers. The following result is easily proved as in Ward [5, p. 169]:

If $f(x)$ is irreducible, then there are t unordered sequences in each block and there are $\varkappa$ blocks.

With $f(x)$ still irreducible and $\operatorname{per} f(x) = r$, we have

$$\{\eta(x)^{(q^n-1)i/(q-1)} A(x)f^*(x)\} = \{cA(x)f^*(x)\} \pmod{x^r-1},$$

where $c \in \mathrm{GF}[q]$, and

$$\{\eta(x)^{(q^n-1)j/r} A(x)f^*(x)\} = \{x^j A(x)f^*(x)\} \pmod{x^r-1},$$

which represents a translation of $\{A(x)f^*(x)\}$ j steps. We conclude (cf. the methods used in Hall [2], p. 216) that

I) All the non-zero polynomials of $(\{f^*(x)\})$ are represented by $\{\eta(x)^i f^*(x)\}$, where $i = 0,1,\ldots,q^n-2$.

II) All the translates of the sequence $\{a(x)\} \in G(f)$ are given by $\{\eta(x)^{(q^n-1)i/r} a(x)\}$, where $i = 0,1,\ldots,r-1$.

III) If $\{\eta(x)^{(q^n-1)i/(q-1)} a(x)\} = \{a_i(x)\}$, then $\{a_M(x)\}$ and $\{a_N(x)\}$ represent the same unordered sequence in the block containing $\{a(x)\}$ if and only if $M \equiv N \pmod{t}$.

IV) If $\{\eta(x)^i b(x)\} = \{b_i(x)\} \in G(f)$, then $\{b_M(x)\}$ and $\{b_N(x)\}$ belong to the same block if and only if $M \equiv N \pmod{\varkappa}$, and represent the same unordered sequence if and only if $M \equiv N \pmod{\varkappa t}$.

PART II

6. Some lemmas.

The following result is classical:

LEMMA 4. *If* $\deg f(x) = n$ *and* $\{a(x)\} \in G(f) = (\{f^*(x)\})$, *then the polynomials*

$$\{a(x)\}, \{xa(x)\}, \ldots, \{x^{n-1}a(x)\}$$

span the ideal $(\{f^*(x)\})$ *if and only if* $f(x)$ *is the minimum polynomial of* $\{a(x)\}$.

PROOF. Suppose that the polynomials $\operatorname{span}(\{f^*(x)\})$. Then evidently $\operatorname{per}\{a(x)\} = \operatorname{per} f(x) = r$, and we can find a polynomial $c(x)$ of degree $< n$ such that

$$\{c(x)a(x)\} = \{f^*(x)\} \pmod{x^r-1}.$$

This implies

$$c(x)a(x)+q(x)(x^r-1) = f^*(x),$$

and as $f^*(x)\,|\,a(x)$ and $f^*(x)\,|\,(x^r-1)$ we then have $f^*(x)=(x^r-1,a(x))$. Consequently $f(x)$ is the minimum polynomial of $\{a(x)\}$ by Lemma 3.

Conversely, take $f(x)$ as the minimum polynomial of $\{a(x)\}$, then Corollary 1 of Lemma 3 implies that $\operatorname{per}\{a(x)\}=r$. If $\{a(x)\},\{xa(x)\},\ldots,$ $\{x^{n-1}a(x)\}$ do not span the ideal $(\{f^*(x)\})$, there exists a relation $\{c(x)a(x)\}=\{0\}$ (mod x^r-1) with $c(x)\neq 0$ and $\deg c(x)<n$. Then

$$\frac{x^r-1}{f^*(x)}\,\Big|\,c(x)\frac{a(x)}{f^*(x)},$$

but since (by Lemma 3)

$$\left(\frac{x^r-1}{f^*(x)},\frac{a(x)}{f^*(x)}\right) = 1 \qquad \text{and} \qquad \deg\frac{x^r-1}{f^*(x)} = n,$$

this is a contradiction.

The following notation will be used in the rest of the paper: Given a polynomial $f(x)$ of degree n and period r, and

$$\{a(x)\} = \{a_0x^{r-1}+a_1x^{r-2}+\ldots+a_{r-1}\} \in G(f),$$

then we write

$$\alpha_a(n_1,n_2,\ldots,n_m) = (a_{n_1},a_{n_2},\ldots,a_{n_m}),$$

$$A_a(n_1,n_2,\ldots,n_m) = \begin{Vmatrix} a_{n_1} & a_{n_2} & \ldots & a_{n_m} \\ a_{n_1+1} & a_{n_2+1} & \ldots & a_{n_m+1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n_1+n-1} & a_{n_2+n-1} & \ldots & a_{n_m+n-1} \end{Vmatrix},$$

where it is assumed that $0\leqq n_1<n_2<\ldots<n_m<n_1+r$.

LEMMA 5. *Given $\{a(x)\}\in G(f)$ with $f(x)$ as the minimum polynomial, then the set of vectors*

$$\{\alpha_b(n_1,n_2,\ldots,n_m);\ (b_i)\in G(f)\}$$

is identical with the row space of $A_a(n_1,n_2,\ldots,n_m)$.

PROOF. By Lemma 4, the sequences $\{a(x)\},\{xa(x)\},\ldots,\{x^{n-1}a(x)\}$ will span $G(f)$. The rows of $A_a(n_1,n_2,\ldots,n_m)$ represent the elements at the positions $n_1,n_2,\ldots,n_m$ of these sequences, and consequently the row space of this matrix is the set of the lemma.

COROLLARY. (Hall [2, Theorem 13.3].) *For every linear recurrence relation of order n, there is a non-trivial solution with zero at $n-1$ arbitrarily given positions.*

PROOF. Suppose that the positions are $n_1, n_2, \ldots, n_{n-1}$, where we may assume $n_{n-1} - n_1 < r$. Then $A_a(n_1, n_2, \ldots, n_{n-1})$ is a $n \times (n-1)$ matrix with rank at most $n-1$, and the row space contains the zero vector non-trivially.

LEMMA 6. *When (b_i) runs through the q^n sequences of $G(f)$, the different vectors $\alpha_b(n_1, n_2, \ldots, n_m)$ will all occur with the same multiplicity.*

PROOF. Let $\{b(x)\}$ run through the polynomials of $(\{f^*(x)\})$. Those polynomials which have coefficients zero for a set of prescribed powers of x clearly constitute a subgroup of the ideal $(\{f^*(x)\})$. The lemma follows directly by decomposition of $(\{f^*(x)\})$ modulo a suitable subgroup of this kind, corresponding to the powers x^{r-1-n_i}, $i = 1, 2, \ldots, m$.

7. Multigrams.

DEFINITION. Given the integers $n_1, n_2, \ldots, n_m$ with

$$0 \leqq n_1 < n_2 < \ldots < n_m < n_1 + \operatorname{per} f(x) .$$

The *multigram* of order m, corresponding to $n_1, n_2, \ldots, n_m$, of a linear recurrence relation with characteristic polynomial $f(x)$ of degree n, is the family of vectors

$$M_f(n_1, n_2, \ldots, n_m) = \text{all } \alpha_a(n_1, n_2, \ldots, n_m), \qquad (a_i) \in G(f) ,$$

taken over the q^n sequences of $G(f)$.

In contrast to the set-theoretical formulation of Lemma 5, the term "family" of the multigram definition indicates that repetitions of vectors are counted.

It is clear from the definition that

$$M_f(n_1, n_2, \ldots, n_m) = M_f(0, n_2 - n_1, \ldots, n_m - n_1) .$$

We note that a multigram contains q^n vectors, where $n = \deg f(x)$. From Lemmas 5 and 6, the following conclusion is immediate:

We consider a linear recurrence relation with characteristic polynomial $f(x)$ of degree n, where the set

$$\{a(x)\}, \{xa(x)\}, \ldots, \{x^{n-1}a(x)\}$$

spans $(\{f^*(x)\})$. Then the vectors of the row space of $A_a(n_1, n_2, \ldots, n_m)$

and only these will appear in $M_f(n_1, n_2, \ldots, n_m)$, all with the same multiplicity.

If the matrix A_a has rank ϱ, its row space will contain q^ϱ different vectors, so the multiplicity of each vector is $q^{n-\varrho}$.

We shall say that a multigram belongs to the *class* k if it contains q^k different vectors. A multigram $M_f(n_1, n_2, \ldots, n_m)$ will be called *skew* if it contains less than q^m different vectors, that is, if its class $k < m$. The above argument then gives the useful criterion:

The multigram $M_f(n_1, n_2, \ldots, n_m)$ is skew if and only if the matrix $A_a(n_1, n_2, \ldots, n_m)$ has a rank less than m, when $(a_i) \in G(f)$ is a sequence which has $f(x)$ as its minimum polynomial.

In particular, the multigram is always skew if $m > n = \deg f(x)$.

8. The distribution of skew multigrams.

By Corollary 2 to Lemma 3, we can always find a sequence $(a_i) \in G(f)$ with $f(x)$ as its minimum polynomial. According to the above criterion, the multigram $M_f(n_1, n_2, \ldots, n_m)$ is skew if and only if there exists at least one linear relation between the m columns of the matrix $A_a(n_1, n_2, \ldots, n_m)$:

$$d_1 a_{n_1+i} + d_2 a_{n_2+i} + \ldots + d_m a_{n_m+i} = 0 ,$$

$i = 0, 1, \ldots, n-1$; not all $d_j = 0$.

Let $(b_i) = b_0, b_1, b_2, \ldots$ be a sequence of $G(f)$, and let k be any integer $\geqq -n_1$. Because of Lemma 5, we can find numbers $e_0, e_1, \ldots, e_{n-1}$, depending on the choice of k, such that

$$b_{n_i+k} = e_0 a_{n_i} + e_1 a_{n_i+1} + \ldots + e_{n-1} a_{n_i+n-1}, \qquad i = 1, 2, \ldots, m .$$

Using the above relation between the columns of A_a, we find easily that

$$d_1 b_{n_1+k} + d_2 b_{n_2+k} + \ldots + d_m b_{n_m+k} = 0 .$$

This holds for all k, and consequently represents a recurrence relation satisfied by (b_i), with characteristic polynomial

$$g(x) = d_1 x^{n_1} + d_2 x^{n_2} + \ldots + d_m x^{n_m} .$$

Since (b_i) is any sequence of $G(f)$, we have $G(f) \subset G(g)$ and so $f(x) \mid g(x)$ by Theorem 2. If conversely $f(x) \mid g(x)$, then $G(f) \subset G(g)$ by the same theorem, and all the sequences of $G(f)$ satisfy the recurrence relation corresponding to $g(x)$. In particular, this recurrence implies the above relation between the columns of the matrix $A_a(n_1, n_2, \ldots, n_m)$.

We have consequently proved the following

THEOREM 7. (Selmer). *The multigram $M_f(n_1,n_2,\ldots,n_m)$ of a linear recurrence relation with characteristic polynomial $f(x)$ is skew if and only if there exist elements $d_1,d_2,\ldots,d_m$ of* GF[q], *not all zero, satisfying*

$$f(x) \mid (d_1x^{n_1}+d_2x^{n_2}+\ldots+d_mx^{n_m}) .$$

9. The structure of multigrams.

We shall apply the results of sections 7 and 8 to a closer analysis of the structure of multigrams of order m and class $k \leqq m$.

We know that the different vectors of a multigram constitute a vector space over GF[q], that the dimension k of this space is $\leqq \min(m,n)$, and that every vector appears q^{n-k} times, when the given recurrence relation is of order n.

In order to investigate the structure of $M_f(n_1,n_2,\ldots,n_m)$ for given $n_1,n_2,\ldots,n_m$, we first find the polynomials which $f(x)$ divides. Knowing these, we have all the linear relations between the columns of $A_a(n_1,n_2,\ldots,n_m)$, where $(a_i) \in G(f)$ and has $f(x)$ as its minimum polynomial. The rank ϱ of this matrix, which equals the class k of the multigram, can then easily be found.

Moreover, we can find ϱ linearly independent columns of

$$A_a(n_1,n_2,\ldots,n_m) ,$$

say the columns $i_1,i_2,\ldots,i_\varrho$. Then the multigram $M_f(n_{i_1},\ldots,n_{i_\varrho})$ is not skew, and the components $i_1,i_2,\ldots,i_\varrho$ of the vectors of $M_f(n_1,n_2,\ldots,n_m)$ form all the ϱ-dimensional vectors over GF[q], each appearing the same number of times. The remaining $m-\varrho$ components are then obtained, using the linear relations found by the method described above.

Consequently we may, given $f(x)$, m and $n_1,n_2,\ldots,n_m$, completely determine $M_f(n_1,n_2,\ldots,n_m)$ without knowing the solutions of the linear recurrence relation with characteristic polynomial $f(x)$.

To illustrate the above remarks, we shall now give some results on special multigrams.

Obviously the multigram $M_f(n_1,n_1+1,\ldots,n_1+n-1)$ belongs to the class n, and this implies that $M_f(n_1,n_2,\ldots,n_m)$ belongs to the class m (is not skew) whenever $n_m-n_1<n$.

For arbitrary $n_1,n_2,\ldots,n_m$, we shall consider now the cases $m=1$ (single elements), $m=2$ (bigrams) and $m=3$ (trigrams). It is assumed throughout that the coefficients d are elements $\neq 0$ of GF[q].

In the case $m=1$, $M_f(n_1)$ trivially consists of all the elements of GF[q], each taken q^{n-1} times.

In the case $m=2$, there are two different classes to which the multigrams may belong.

I) If $f(x)\,|\,(x^{n_2-n_1}-d)$, then $M_f(n_1,n_2)$ consists of the vectors $a_j(1,d)$, where a_j runs through all the elements of GF$[q]$, and each vector occurs q^{n-1} times.

This case illustrates the fact that multigrams belonging to the same (skew) class are not necessarily equal, but may consist of different vector spaces.

II) If no $d\in$ GF$[q]$ exists for which $f(x)\,|\,(x^{n_2-n_1}-d)$, then $M_f(n_1,n_2)$ is not skew, and hence consists of all the q^2 different vectors over GF$[q]$ with two components, and each vector occurs q^{n-2} times.

If $f(x)$ is *irreducible* and $\mathrm{per} f(x)=r$ is known, it is easy to decide which of the two cases above will occur. As in the proof of Theorem 6, we see that we can find $d\in$ GF$[q]$ such that $f(x)\,|\,(x^{n_2-n_1}-d)$ if and only if $\mu\,|\,(n_2-n_1)$, where, as before, $e=(r,q-1)$ and $\mu=r/e$. This gives the following result, which is a generalization of Theorem 12 of Zierler [7]:

THEOREM 8. *Let $f(x)$ be irreducible of degree n and period r, and take $e=(q-1,r)$ and $\mu=r/e$.*

If $n_1\equiv n_2 \pmod{\mu}$, then $M_f(n_1,n_2)$ consists of all the q^2 different vectors over GF$[q]$ *with two components, each vector with multiplicity q^{n-2}.*

If $n_1\equiv n_2 \pmod{\mu}$, then $M_f(n_1,n_2)$ consists of the vectors $a_j(1,d)$, each with multiplicity q^{n-1}, where a_j runs through all the elements of GF$[q]$, *and $d\in$ GF$[q]$ is uniquely determined by $f(x)\,|\,(x^{n_2-n_1}-d)$.*

When $m=3$, we have 3 possible classes:

I) If $f(x)\,|\,(x^{n_2-n_1}-d)$ and $f(x)\,|\,(x^{n_3-n_1}-d')$, then $M_f(n_1,n_2,n_3)$ consists of the vectors $a_j(1,d,d')$, where a_j runs through all the elements of GF$[q]$, and each vector occurs q^{n-1} times.

IIa) If $f(x)\,|\,(x^{n_2-n_1}-d)$ but $f(x)\nmid(x^{n_3-n_1}-d')$ for every d', then $M_f(n_1,n_2,n_3)$ consists of the vectors (a_i,da_i,a_j), where a_i and a_j independently run through all the elements of GF$[q]$, and each vector occurs q^{n-2} times.

We get similar cases by permutations of n_1, n_2 and n_3.

IIb) If $f(x)\,|\,(d_1x^{n_1}+d_2x^{n_2}+d_3x^{n_3})$, and we do not have case I, then $M_f(n_1,n_2,n_3)$ consists of the vectors $\left(a_i,a_j,-(d_1a_i+d_2a_j)/d_3\right)$, where a_i and a_j independently run through all the elements of GF$[q]$, and each vector occurs q^{n-2} times.

III) If we have none of the above cases, then $M_f(n_1,n_2,n_3)$ consists of all the q^3 vectors over GF$[q]$ with 3 components, each vector with multiplicity q^{n-3}.

When $f(x)$ is irreducible, then as in Theorem 8 case I will occur if and only if $n_1 \equiv n_2 \equiv n_3 \pmod{\mu}$; case IIa if and only if $n_1 \equiv n_2 \not\equiv n_3 \pmod{\mu}$; and a necessary condition for IIb or III is that $n_1 \not\equiv n_2 \not\equiv n_3 \not\equiv n_1 \pmod{\mu}$.

10. Multigrams for primitive characteristic polynomials.

We shall now consider the following question: Given m and a primitive polynomial $f(x)$ of degree n, and let $n_1, n_2, \ldots, n_m$ vary under the restrictions $0 \leqq n_1 < n_2 < \ldots < n_m$ and $n_m - n_1 < \mathrm{per} f(x) = q^n - 1$. How many of the multigrams $M_f(n_1, n_2, \ldots, n_m)$ will then belong to each of the multigram classes?

This problem will, however, be simplified if one of the n_i's is kept fixed, since

$$M_f(n_1, n_2, \ldots, n_m) = M_f(n_1 + s, n_2 + s, \ldots, n_m + s)$$

for arbitrary s. Which n_i is kept fixed and at which value it is fixed is of no importance. In the following we keep n_1 fixed.

Having a primitive characteristic polynomial with a single non-zero unordered solution $(a_i) = a_0, a_1, a_2, \ldots$, the set

$$\{(a_i, a_{i+1}, \ldots, a_{i+n-1});\ i = 0, 1, \ldots, \mathrm{per} f(x) - 1\}$$

contains all the non-zero vectors of the n-dimensional vector space over GF$[q]$. Consequently the collection of all

$$A_a(n_1, n_2, \ldots, n_m); \quad 0 \leqq n_1 < n_2 < \ldots < n_m, \quad n_m - n_1 < \mathrm{per} f(x), \quad n_1 \text{ fixed},$$

is composed of all the matrices obtainable from the vector space of dimension n over GF$[q]$, by choosing in every possible way m different non-zero vectors from this vector space and taking these as the columns of $n \times m$ matrices, with the restrictions that a particular given vector shall occur as a column of every matrix, and that permutations of the columns within each matrix are not counted.

Among the matrices thus obtained, we now ask for the number of those with a given rank k. From the arguments of § 7, we see that this number is exactly the number of multigrams $M_f(n_1, n_2, \ldots, n_m)$, with the restrictions imposed above, belonging to the class k.

This number is given by the following theorem, to be proved in the next section:

THEOREM 9. *Let $f(x)$ be a primitive polynomial over* GF$[q]$, *of degree n, and let m be a positive integer. The number of different multigrams $M_f(n_1, n_2, \ldots, n_m)$ satisfying the conditions*

$$0 \leqq n_1 < n_2 < \ldots < n_m, \qquad n_m - n_1 < \operatorname{per} f(x), \qquad n_1 \text{ fixed},$$

and belonging to the class $m-l$, *is then given by*

$$\frac{(q^n-1)(q^n-q)\ldots(q^n-q^{m-l-1})}{(m-1)!} \cdot \frac{A_{m,l}}{q^n-1}.$$

Here

$$A_{m,l} = (q^{m-l}-m)A_{m-1,l-1} + A_{m-1,l},$$

with

$$A_{1,0} = 1; \qquad A_{m,l} = 0 \quad \text{for } l = m \text{ or } l < 0.$$

The fraction in Theorem 9 is not reduced by q^n-1, to allow also for the case $m-l=1$.

11. Proof of Theorem 9.

LEMMA 7. *In the expansion*

$$\binom{q^n-1}{m} = \frac{(q^n-1)(q^n-q)\ldots(q^n-q^{m-1})}{m!} A_{m,0} + \\ + \frac{(q^n-1)\ldots(q^n-q^{m-2})}{m!} A_{m,1} + \ldots + \frac{q^n-1}{m!} A_{m,m-1},$$

where $q_n - 1 \geqq m \geqq 1$ *and* $A_{m,l}$ *is independent of* n *for* $l = 0, 1, \ldots, m-1$, *the coefficients are uniquely determined as in Theorem* 9.

PROOF. By considering the expansion as a function of n and letting n increase, it is immediately seen that there is at most one set of coefficients (for given m) satisfying the expansion.

It is then easily seen, by induction with respect to m, that the relations given in Theorem 9 provide such a set of coefficients.

Suppose now $l+1 \leqq m \leqq q^n-1$. To prove Theorem 9, we saw in § 10 that we must determine the number of $n \times m$ matrices of rank $m-l$, subject to some further restrictions.

We first choose $m-l$ linearly independent n-dimensional vectors over GF[q]. It is well known (cf. Dickson [1, pp. 49–50]) that this can be done in

$$Q_{m-l}^{(n)} = (q^n-1)(q^n-q)\ldots(q^n-q^{m-l-1})$$

different ways, when permutations are counted.

The remaining l column vectors of each matrix must be chosen from the vector space spanned by the $m-l$ independent vectors. This implies that the number of these choices only depends on l and m, but not on n.

We next group together those matrices which contain the same (permuted) columns, and choose one representative from each set thus obtained. Again, this reduction in number will depend on l and m only. Consequently, the number of $n \times m$ matrices of rank $m-l$, when permutations of columns are not counted, must be of the form $Q_{m-l}^{(n)} B_{m,l}$, where $B_{m,l}$ is independent of n.

Summing these numbers, we get the total number of combinations (without repetitions) of m vectors out of $q^n - 1$:

$$\binom{q^n-1}{m} = \sum_{i=0}^{m-1} (q^n-1)(q^n-q)\dots(q^n-q^{m-i-1})B_{m,i}\,.$$

This is an expansion of the form in Lemma 7, and since this is unique, we have $A_{m,l} = m!\, B_{m,l}$. The number of $n \times m$ matrices of non-zero column vectors with rank $m-l$ is consequently

$$\frac{(q^n-1)(q^n-q)\dots(q^n-q^{m-l-1})}{m!} A_{m,l}\,,$$

when permutations of the columns are not counted. Here the coefficients $A_{m,l}$ are determined as in Theorem 9.

We must finally impose the condition that one column vector of each matrix shall be kept fixed. Because of the symmetry in the choice of columns, we get the corresponding reduced number of matrices if we multiply the above number by m (the number of columns in each matrix) and divide by $q^n - 1$ (the number of different column vectors at our disposal). The result is the number of matrices given in Theorem 9, which is thus proved.

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Preface * Elementary definitions 1–40 * Extensions of fields 41–81 * Galois theory 82–124 * Applications 125–173 * Reading list 174 * Index of notations 175–176 * Index 177–180.

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Translator's preface * Definition of the problem 1–14 * Solution of the absolute stability problem by the direct method. The resolving equations of Lur'e 15–44 * A frequency method for solving the problem of absolute stability (The V. M. Popov method) 45–88 * The connection between the Popov criterium and the existence of a Liapunov function. Further refinement of the frequency criteria for the absolute stability of the particular cases of (!) 89–136 * Review of the state of the problem. New problems 137–148 * Appendix: The connection between the method of Lur'e resolving equations and the Popov frequency method 149–162 * Bibliography 163–170 * Index 171–173.

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Introduction 1–3.

I, *La notion de groupe*: Examples simples de groupes 4–8 * Définition du groupe 9 * Théorèmes simples sur les groupes 10–18.

II, *Groupes de substitutions*: Définitions des groupes de substitutions 19–22 * La notion de sous-groupe 23–24 * Les substitutions considérés comme applications d'un ensemble fini sur lui-même. Substitutions paires et impaires 25–30.

III, *Quelques considérations générales sur les groupes. La notion d'isomorphisme*: La notation « additive » et la notation « multiplicative » dans la théorie des groupes 31–33 * Isomorphisme entre deux groupes 34–37 * Le théorème de Cayley 38–41.

IV, *Sous-groupes cycliques d'un groupe donné*: Etude du sous-groupes engendré par un élément d'un groupe 42 * Groupes cycliques finis et infinis 43–47 * Systèmes de générateurs 48–49.

V, *Groupes géometriques simples*: Exemples et définition des groupes d'isométrie de figures géométriques 50–51 * Les groupes d'isométries de la droite, du cercle, du plan 52–55 * Les groupes d'isométries d'une pyramide régulière et du polyèdre formé par deux pyramides régulières symétriques par rapport à la base commune 56–59 * Le groupe de rotations du tétraèdre 60–63 * Les groupes de rotations du cube et de l'octaèdre 64–67 * Les groupes de rorations de l'isocaèdre et du dodécaèdre 68–69 * Considérations générales sur les groupes de rotations des polyèdre réguliers 70.

VI, *Sous-groupes invariants*: Eléments conjugués et sous-groupes conjugués 71–79 * Sous-groupes invariants (diviseurs normaux) 80–86.

VII, *Homomorphisme*: Définition d'une application homomorphe et de son noyau. Propriétés simples 87–89 * Exemples d'applications homomorphes 90–95.

VIII, *Décomposition en classes d'un groupe par un sous-groupe donné. Le groupe quotient*: Classes à gauche et classes à droite 96–102 * Le groupe d'un groupe par un sous-groupe invariant donné 103–109.

Appendice 1: Notions élémentaires de la théorie des ensembles 110–123 * Appendice 2: Démonstration du théorème 124–126 * Index terminologique 127–128.

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Preface * Preliminary 1–28 * The trihedral angle 29–46 * The skew quadrilateral 47–53 * The tetrahedron 54–126 * Transversals 127–144 * The oblique cone with a circular base 145–156 * Spheres 157–246 * Inversion 247–265 * Recent geometry of the tetrahedron 266–336 * Bibliographical notes 337–346 * Index 347–353.

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II, *La fonction* $\Gamma(x)$: Définitions formelles de la fonction $\Gamma(x)$ 17–23 * La fonction $\mu(x)$ 24–26 * Définition fonctionnelle de $\Gamma(x)$. La proposition de Bohr–Mollerup 27–28 * Autre définition fonctionnelle de $\Gamma(x)$ 29–30 * Les valeurs de $\Gamma(x)$ pour les grandes valuers de x 31–32 * Propriétés de la fonction $\Gamma(x)$ 32–39 * Autre définition fonctionnelle de $\Gamma(x)$ 40–41 * Quelques généralisations 41–50.

III, *La fonction* $B(x, y)$: Définition fonctionnelle de la fonction $B(x, y)$ 51–59 * La fonction $G(x)$ 60–68.

Note 69–73 * Bibliographie 74–76.

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Preface * Formation of the equations: the main problems 1–25 * Mathieu's equation—general theory 26–51 * Mathieu functions of integral order 52–78 * Mathieu

functions of integral order—further properties 79–106 * Asymptotic expansions 107–120 * Mathieu's general equation 121–140 * Hill's equation 141–152 * The spheroidal wave equation 153–190 * Lamé's equation 191–236 * The ellipsoidal wave equation 237–250 * Appendix A: Bessel functions 251–255 * Appendix B: Legendre, Gegenbauer and Tchebycheff functions 256–260 * Appendix C: Elliptic functions 261–263 * References 264–268 * Additional notes 269–276 * Index 277–281.

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Preface * Symbols used in text * The Fourier transform on the real line for functions in L_1 1–18 * The Fourier transform on the real line for functions in L_2 19–33 * Regular points and spectrum 34–54 * More on the Gel'fand theory and an introduction to point set topology 55–72 * Further topological notions 73–90 * Compactness of the space of maximal ideals over a Banach algebra; an introduction to topological groups and star algebras 91–106 * The quotient group of a topological group and some further topological notions 107–124 * Right Haar measure and the Haar covering function 125–149 * The existence of a right invariant Haar integral over any locally compact a topological group 150–165 * The Daniel extension from a topological point of view, some general results from measure theory, and group algebras 166–187 * Characters and the dual group of a locally compact, abelian. topological group 188–206 * Generalization of the Fourier transform to $L_1(G)$ and $L_2(G)$ 207–250 * Bibliography 251–252 * Index 253–256.

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Preface * Symbols used in text * Valuations of rank one 1–23 * Complete fields and the field of p-adic numbers 24–64 * Valuation rings, places, and valuations 65–91 * Normed linear spaces 92–116 * Extensions of valuations 117–152 * Appendix 153–167 * Bibliography 168–170 * Subject index 171–173.

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Preface * Introduction and general orientation 1–4 * Generating functions 5–15 * Recurrent events 16–23 * Random walk models 24–37 * Markov chains 38–57 * Discrete branching processes 58–65 * Markov processes in continuous time 66–83 * Homogeneous birth and death processes 84–106 * Some non-homogeneous processes 107–116 * Multi-dimentional processes 117–135 * Queueing processes 136–161 * Epidemic processes 162–185 * Competition and predation 186–193 * Diffusion processes 194–206 * Approximations to stochastic processes 207–221 * Some non-markovian processes 222–233 * References 234–236 * Solutions to problems 237–242 * Author index 243–244 * Subject index 245–249.

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Vorbemerkung zur 4. Auflage 3 * Literatur 6 * Der geschichtliche Weg zur Nichteuklidischen Geometrie 7–19 * Axiomatik der absoluten Geometrie 20–49 * Die Euklidische Geometrie 50–53 * Axiomatik der hyperbolischen Geometrie im Einheitskreise 54–70 * Die hyperbolische Geometrie als selbständige Disziplin 71–143 * Schlussbetrachtungen 144–155 * Register 156–158.

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Preface * Introduction: A glimpse at set theory 1–26 * The red numbers 27–57 * The topology of Cartesian spaces 58–97 * Convergence 98–145 * Continuous functions 146–205 * Differentiation 206–273 * Integration 274–374 * Infinite series 375–421 * References 422–423 * Hints for selected exercises 424–440 * Index 441–447.

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Volume I: Translator's preface * Author's preface * Notation * Introductory material 1–42 * Basic concepts and theorems in the theory of trigonometric series 43–208 * Fourier coefficients 209–259 * The convergence of a Fourier series at a point 260–295 * Fourier series of continuous functions 296–361 * Convergence and divergence of a Fourier series in a set 362–487 * "Adjustment" of functions in a set of small measure 488–527 * Appendix 528–536 * Bibliography 537–548 * Index 549–553.

Volume II: Translator's preface * Notation * The summability of Fourier series. Additional investigation into problems of convergence 1–50 * Conjugate trigonometric series 51–152 * Absolute convergence of Fourier series 153–197 * Sine and cosine series with monotonically decreasing coefficients 198–231 * Lacunary series 232–276 * Convergence and divergence of general trigonometric series 277–306 * The absolute convergence of general trigonometric series 307–341 * Problems of uniqueness of the expansion of a function into a trigonometric series 342–417 * Representation of a function by a trigonometric series 418–445 * Appendix 446–488 * Bibliography 489–500 * Index to volumes I and II 501–508.

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Literaturverzeichnis 5 * Einleitung 7–9 * Bezeichnungen 10–11.

I, *Mass- und Integrationstheorie*: Masstheorie 12–41 * Integrationstheorie 42–70 * Produktmasse 71–85.

II, *Wahrscheinlichkeitstheorie*: Grundbegriffe der Theorie 86–105 * Unabhängigkeit 106–126 * Gesetz der grossen Zahlen 127–151.
Namen- und Sachregister 152–154.

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Vorwort * Einführung 13–16 * Das Arbeiten mit digitalen Rechenautomaten 17–33 * Die Problematik des numerischen Rechnens (numerische Grundbegriffe) 34–84 * Programmierung mit einer algorithmischen Sprache 85–146 * Inhomogene Speicher und periphere Geräte 147–165 * Übergang zu maschinenorientierten Programmen 166–231 * Schaltnetze und Schaltwerke in digitalen Rechenanlagen 232–297 * Elektronische Analogrechner 298–349 * Sachverzeichnis 350–357.

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Einführung in den Gruppenbegriff 7–17 * Gruppentheoretische Grundbegriffe und -methoden 18–32 * Über endliche Gruppen 33–46 * Vertauschbarkeit von Elementen und Untergruppen 47–70 * Die Faktorgruppe 71–77 * Die Homomorphie 78–82 * Die Automorphie 83–92 * Die Endomorphie; charakteristische und vollinvariante Untergruppen 93–100 * Abelsche Gruppen 101–117 * Gruppen mit Operatoren 118–132 * p-Gruppen und p-Sylow-Gruppen 133–150 * Freie Gruppen und Gruppen mit Beziehungen zwischen den Elementen 151–157 * Folgen und Reihen von Gruppen 158–170 * Abschnitt. Genaueres über die Gruppenpostulate 171–173 * Lösung der Aufgaben 174–187 * Sach- und Namenverzeichnis 188–190.

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Preface * Introduction 1–10 * Set theory 11–52 * Mathematical induction 53–81 * The natural numbers 82–98 * The integers 99–134 * Elementary number theory 135–199 * The rational numbers 200–223 * The real numbers 224–285 * The complex numbers 286–308 * The theory of algebraic equations 309–393 * Systems of equations and matrices 394–460 * Appendix I: The proof of Sturm's theorem 461–465 * Appendix II: The proof of the fundamental theorem of symmetric polynomials 466–473 * Appendix II: The proof of the fundamental theorem of algebra 474–480 * References 481–483 * Index 484–486.

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Author's preface * Virtual delay 1–12 * Delay formulas: A direct approach 13–30 * Delay formulas: An approach using transforms 31–45 * Weak stationarity: Preliminary results 46–64 * Weak stationarity: Convergence theorems 65–76 * Weak Markov assumptions 77–83 * References 84–86 * Index 87–88.

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Foreword * Preface.

I, *Hyperbolic and parabolic equations*: Equations of hyperbolic and parabolic types 1–3 * The wave operator 4–37 * Cauchy's problem, characteristic surfaces, and propagation of discontinuities 38–61 * Linear hyperbolic differential equations 62–93 * A parabolic equation: The equation of heat conduction 94–107 * Approximation of solutions of partial differential equations by the method of finite differences 108–130.

II. *Elliptic equations*: Elliptic equations and their solutions 133–149 * The maximum principle 150–163 * Hilbert space approach. Periodic solutions 164–189 * Hilbert space approach. Dirichlet problem 190–210 * Potential theoretical approach 211–253 * Function theoretical approach 254–281 * Quasi-linear equations 282–300.

Supplement I: Eigenvalue expansions 301–326 * Supplement II: Parabolic equations 327–340 * Index 341–343.

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Preface * Introduction 1–12 * Equations of the first order and degree 13–47 * Applications of first-order equations 48–78 * Implicit differential equations of the first order 79–97 * Linear differential equations 98–131 * Applications of linear equations with constant coefficients 132–168 * The Laplace transform 169–198 * Miscellaneous topics 199–223 * Series solutions of differential equations 224–244 * Graphical and numerical methods 245–261 * Partial differential equations 262–291 * Fourier series and boundary value problems 292–310 * Appendix A: Algebra 311–313 * Appendix B: The calculus 314–325 * Appendix C: Brief numerical tables 326–327 * Answers 328–350 * Index 351–354.

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Vorwort zur zweiten Auflage * Vorwort zur ersten Auflage * Die grundlegenden Existenzsätze 1–31 * Singuläre Stellen bei gewöhnlichen Differentialgleichungen erster Ordnung 32–43 * Das Verhalten der Lösungen von $dw/dz = (aw + bz)/(cw + dz)$ für konstante a, b, c, d im Punkte (0, 0) 44–50 * Ausserwesentlich singuläre Stellen

zweiter Art 51–85 * Differentialgleichungen erster Ordnung im Grossen 86–115 * Lineare Differentialgleichungen im Kleinen 116–201 * Differentialgleichungen der Fuchsschen Klasse 202–209 * Die hypergeometrische Differentialgleichung 210–279 * Die Besselsche Differentialgleichung 280–310 * Differentialgleichungen der Fuchsschen Klasse mit vier singulären Punkten 311–328 * Differentialgleichungen mit periodischen Koeffizienten 329–348 * Einige weitere Untersuchungen 349–385 * Namen- und Sachverzeichnis 386–389.

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Preface * Manifolds 1–24 * Lie groups 25–37 * Fibre bundles 38–52 * Differential forms 53–73 * Connexions 74–88 * Affine connexions 89–121 * Riemannian manifolds 122–144 * Geodesics and complete riemannian manifolds 145–160 * Riemannian curvature 161–184 * Immersions and the second fundamental form 185–212 * Second variation of arc length 213–257 * Appendix: Theorems on differential equations 258–259 * Bibliography 260–264 * Subject index 265–273.

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Vorwort * Grundbegriffe und Grundlehren 1–24 * Elementare mathematische Hilfsmittel 25–34 * Der zentrale Problemkreis der Wahrscheinlichkeitsrechnung 35–51 * Weitere Anwendungen in Wissenschaft und Praxis 52–82 * Rückblick und Ausblich 83–96 * Anhang 97–110.

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Translator's preface * Preface to the German edition * Preface to the Russian edition.

Introduction 1–8.

I, *Elementary problems of dynamic instability*: Determination of Regions of dynamic instability 9–32 * The influence of damping on the regions of dynamic instability 33–43 * Determination of nonlinear terms 44–62 * Free and forced vibrations of a nonlinear system 63–74 * Amplitudes of vibration at a parametric resonance 75–89 * Nonsteady-state vibration 90–100 * Secondary resonances 101–110 * Interaction of forced and parametrically excited vibrations 111–122 * Extension of the applicability of the theory 123–144.

II, *General theory of the dynamic stability of elastic systems*: Elements of matrix operations 145–161 * Elements of the theory of linear integral equations 162–177 * Differential equations for boundaries of the dynamic stability of rods 178–193 *

Differential equations for the dynamic stability of elastic systems 194–208 * Determination of the regions of dynamic instability 209–229 * Dynamic stability taking damping into account 230–248 * The fundamentals of the nonlinear theory of dynamics stability 249–280.

III, *Applications of the general theory*: Dynamic stability of straight rods 281–304 * Dynamic stability of curved rods 305–332 * Dynamic stability of plane bending modes 333–357 * The dynamic stability of statically indeterminate frames 358–381 * Dynamic stability of plates 382–416 * Dynamic stability of shells 417–432.

References 433–444 * Author index 445–446 * Subject index 447–451.

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Preface * The parabola of safety 1–14 * The hyperbola: Boundary of the zone of audibility 15–31 * The astroid and the cycloid 32–39 * Envelopes 40–76.

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Introduction à la première édition * Introduction à la deuxième édition * Notions algebriques 1–8 * Les complexes 9–22 * Le théorème fondamental 23–31 * Applications et compléments 32–43 * Les faisceaux 44–55 * L'algèbre spectrale 56–73 * Algèbre spectrale des espaces fibres 74–81 * Applications aux espaces fibres 82–91.

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Forord 9–10 * Matematikkens forhistorie 11–16 * Talldyrkelse 17–22 * Tallsystemer 23–26 * Egypternes matematikk 27–41 * Babylonernes matematikk 42–48 * Indernes matematikk 49–70 * Kinesernes matematikk 71–80 * Arabernes matematikk 81–89 * Grekernes matematikk 90–194 * Romernes matematikk 195–197 * Matematikken i Europa i middelalderen 198–218 * Renessansens matematikere 219–232 * Litteratur 233–234 * Navn- og sakregister 235–239.

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$2a_{12}u_{xy}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu=f(x, y)$ 1–2 (163–167) * The equation with constant coefficients for a function of n independent variables 3 (168–170).

II, *Equations of hyperbolic type*: Physical problems reducible to equations of hyperbolic type; statement of boundary-value problems 4–16 (170–207) * Method of travelling waves (D'Alembert's method) 17–27 (208–247) * Method of separation of variables 28–38 (248–295) * Method of integral representations 39–46 (296–316).

III, *Equations of parabolic type*: Physical problems leading to equations of parabolic type; statement of boundary-value problems 47–51 (317–330) * Method of separation of variables 52–58 (331–350) * Method of integral representations and source functions 59–70 (351–375).

IV, *Equations of elliptic type*: Physical problems leading to equations of elliptic type, and the statement of boundary-value problems 71–73 (381–391) * Simplest problems for Laplace's equations 74–76 (392–399) * The cource function 77–80 (400–424) * The method of separation of variables 81–91 (425–489) * Potentials and their application 92–95 (490–505).

V, *Equations of parabolic type*: Physical problems leading to equations of parabolic type: statement of boundary-value problems 96–97 (506–510) * The method of separation of variables 98–105 (511–545) * The method of integral representations 106–114 (557–564).

VI, *Equations of hyperbolic type*: Physical problems leading to equations of hyperbolic type: statement of boundary-value problems 115–119 (565–573) * The simplest problems; different methods of solution 120–125 (574–585) * The method of separation of variables 126–133 (586–620) * The method of integral representations 134–139 (621–643).

VII, *Equations of elliptic type* $\Delta u+cu=-f$: Problems for the equation $\Delta u-x^2u=-f$ 140–141 (644–648) * Some problems on natural vibrations 142–145 (649–675) * Propagation and radiation of sound 146–150 (676–702) * Steady-state electromagnetic vibrations 151–160 (703–740).

Supplement: Different orthogonal systems of coordinates 741–747 * Some formulae of vector analysis 748–749 * Special functions 749–754 * Tables of the error integral and roots of some characteristic equations 755–758 * References 759–762 * Index 763–768.

(The number of the pages referring to answers and solutions is given in the brackets).

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Preface.

I: A new mathematics syllabus: Without geometry? 3–25 * School mathematics: An introduction to what? 25–30 * Royaumont and Southampton 31–43 * The value of school geometry 44–55 * What is the value of abstract algebra in the context of school mathematics? 56–69 * What is the value of utilitarian mathematics in the context of school mathematics 70–78 * What are the factors which ought to determine the composition of the school syllabus? 79–85 * Mathematics in a liberal education 86–89 * The crisis in school mathematics 90–96.

II: Introduction of part two 97–98 * The difficulty of geometry 99–115 * The

role of geometry in unifying school mathematics 116–146 * *O*-level topics 147–167 * *A*-level topics 168–187 * Projective geometry 188–194 * Conclusion 195–198.

Appendixes: Importance of the algebra axioms 199–200 * Proof that the orthocentre of a triangle of three tangents to a parabola lies on the directrix 201–202 * Questions suitable for testing geometrical knowledge and thought 203–208 * Iteration process for finding square roots 209–212 * Comparison of cartesian and euclidian proofs 213–217 * On numbering the theorems 218–220 * A new notation for elementary deductive geometry 221–226.

Index 227–230.

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Préface * Avertissement.

I, *Principes de l'analyse booléenne*: Fondements de l'algèbre de Boole 1–4 * Ensembles. Fonctions caractéristiques. Graphes 5–9 * Logiques des propositions 10–13 * Logique symbolique 14–18.

II, *Algèbre de Boole*: Calcul binaire 19–33 * Équations de Boole 34–57 * Fonctions croissantes. Fonctions linéairement séparables 58–68.

III, *Applications*: Logique de décision majoritaire 69–74 * Réseaux 75–86 * Systèmes booléens 87–93 * Programmes booléens 94–106.

IV, *Problèmes et solutions*: Logique des propositions 107–110 * Codage. Additionneur binaire 111–115 * Équations fonctionnelles 116–119 * Calcul binaire. Systèmes booléens 120–124 * Programme booléen 125–178.

Bibliographie 129–131.

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Preface * Functions. Limits, continuity, differentiation 1–49 * Integration. Simple methods 50–76 * The convergence of infinite series 77–103 * The logarithmic and exponential functions 104–129 * Integration. Further series 130–162 * Further theorems concerning functions of one variable. Expansion in series 163–196 * Complex numbers 197–224 * Functions of more than one variable 225–283 * Vector algebra 284–327 * Applications of vector algebra to analytical geometry of straight lines and planes 328–358 * Vector functions. Differential geometry of curves. Line integrals 359–401 * Matrices, determinants and linear dependence 402–436 * Linear equations. Eigenvectors and eigenvalues 437–472 * Curvilinear coordinates and multiple integrals 473–508 * Vector analysis 509–537 * Ordinary differential equations. The Laplace transform 538–565 * Functions of a complex variable 566–601 * The Dirac δ-function; Fourier series and integrals 602–622 * Factorial, Legendre and Bessel functions 623–651 * Statistics and probability 652–684 * Answers to exercises 685–706 * References 707 * Bibliography 708–711 * Index 712–719.

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Preface * Acknowledgements.

I, *Basic concepts*: An overview of a theory of data 3–31 * On methods of collecting data 32–58.

II, *Preferential choice data*: Introduction to Q I 60–64 * Parallelogram analysis 65–79 * The unfolding technique in one dimension 80–121 * Applications of unfolding theory 122–139 * Multidimensional unfolding 140–180 * Factor analysis of preferences 181–192 * Some alternative theories of preferential choice and further problems 193–208.

III, *Single stimulus data*: Introduction to Q IIa 211–223 * Unidimensional models QIIa 224–243 * Q IIa multidimensional models-nonmetric factor analysis 245–283 * Comparing incomparables: Compression of partial orders to form decisions 284–291 * Introduction to Q IIb 292–302 * Q IIb models 303–325 * Some interpretations of Q IIb theory 326–342.

IV, *Stimuli comparison data*: Q IIa data and models 345–382 * On the problem of social utility and majority decision 382–402 * Q IIb data and models 403–430.

V, *Similarities data*: Introduction to Q IVa 433–443 * Nonmetric multidimensional psychological scaling 444–462 * Some applications of nonmetric multidimensional psychological scaling 463–496.

VI, *Interrelations*: Thurshone's comparative judgement model with laterality effects 499–510 * Some interrelations among data matrices and among methods 511–544 * The scale grid 545–560 * Relation to an earlier formulation of a theory of data 561–566.

Bibliography 567–576 * Author index 577–580 * Subject index 581–585.

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Vorwort * Abkürzungen.

I, *Grundlagen der Funktionalanalysis mit Anwendungen*: Typische Fragestellungen der numerischen Mathematik 1–11 * Einige Typen von Räumen 12–35 * Ordnungen 36–40 * Konvergenz und Vollständigkeit 41–60 * Kompaktheit 61–68 * Operatoren in pseudometrischen und spezielleren Räumen 69–87 * Operatoren in Hilberträumen 88–109 * Eigenwertaufgaben 110–131 * Vektornormen und Matrixnormen 132–143 * Weitere Sätze über Vektor- und Matrixnormen 132–143 * Weitere Sätze über Vektor- und Matrixnormen 144–159.

II, *Iterative Verfahren*: Der Fixpunktsatz für das allgemeine Iterationsverfahren in pseudometrischen Räumen 160–168 * Spezialfälle des Fixpunktsatzes und Abänderung des Operators 169–176 * Iterationsverfahren bei Gleichungssystemen 177–184 * Gleichungssysteme und Differenzenverfahren 185–199 * Iterationsverfahren bei Differential- und Integralgleichungen 200–212 * Ableitung von Operatoren in supermetrischen Räumen 213–223 * Aufstellung von Iterationsverfahren 224–238 * Regula falsi 239–250 * Newtonsches Verfahren mit Verschärfungen 251–263 * Monotonie und Extremalprinzipien beim Newtonschen Verfahren 264–274.

III, *Monotonie, Ungleichungen und weitere Gebiete*: Monotone Operatoren 275–285 * Weitere Anwendungen des Schauderschen Satzes 286–295 * Monotone Art bei Matrizen und Randwertaufgaben 296–308 * Anfangswertaufgaben und weitere Monotoniesätze 309–319 * Approximation von Funktionen 320–337 * Diskrete

Tschebyscheff-Approximation und Austauschverfahren 338–349 * Anhang: Zum Schauderschen Fixpunktsatz 350–355.

Literaturverzeichnis 356–362 * Namenverzeichnis 363–365 * Sachverzeichnis 366–371.

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Preface * Introduction 1–4 * Preliminaries 5–12 * Integration by parts 13–26 * The method of stationary phase 27–35 * The method of Laplace 36–47 * Watson's lemma 48–62 * The method of steepest descents 63–90 * The saddle-point method 91–98 * Airy's integral 99–106 * Uniform asymptotic expansions 107–117 * Bibliography 118–119 * Index 120.

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Preface * Introduction 1–3 * History, pland and current status of system 4–7 * General description and usage techniques 8–38 * Supervisor subroutine calls 39–44 * Use of the 1301 disk memory 45–54 * CTSS library 55–66 * Console commands 67–88 * Appendix A: Example of a session at a console 89–91 * Appendix B: Current restrictions 92–93 * References 94–96.

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Introduction and summary 1–3 * The forecasting method 4–11 * Seasonal variations 12–15 * Control of forecasts 16–19 * Forecasting errors 20–25 * Appendix A. Seasonal variations 26–27 * Appendix B. Derivation of control limits 28–32 * Appendix C. Computer programmes 33–34 * Appendix D 35–36.

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Preface * A note to the reader * Introduction 1–13 * Triangles and quadrangels 14–23 * The principle of duality 24–32 * The fundamental theorem and Pappus's theorem 33–40 * One-dimensional projectivities 41–48 * Two-dimensional projectivities 49–59 * Polarities 60–70 * The conic 71–80 * The conic, continued 81–90 * A finite projective plane 91–101 * Parallelism 102–110 * Coordinates 111–132 * Answers to exercises 133–157 * References 158 * Index 159–162.

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Preface * Linear spaces 1–30 * Orthogonal functions 31–86 * Fourier series 87–190 * Legendre polynomials and Bessel functions 191–246 * Heat and temperature 247–280 * Waves and vibrations; harmonic analysis 281–342 * Supplementary exercises 343–346 * Appendix: Functions on groups 347–364 * Answers and notes 365–396 * Index 397–403.

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Geleitwort * Über die Theorie der ganzen algebraischen Zahlen 2–222 * Über die Komposition der binären quadratischen Formen 223–263 * Sur la théorie des nombres entiers algébriques 263–296 * Über die Theorie der ganzen algebraischen Zahlen 297–312 * Erläuterungen zu den vorstehenden Abhandlungen 313–314.

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Author's forword to the translation * Introduction * The theory of multiplicative lattices 1–81 * Some calculations with numbers in cubic fields 82–146 * Geometry, tabulation and classification of algebraic fields of the third and fourth degree. A: Tabulation of fields of the third degree. B: Some geometrical theorems. C: Tabulation of fields of the fourth degree. D: The construction of cubic regions on quadratic regions. E: The construction of a quadratic region on a cubic region 147–240 * The algorithm of Voronoi. A: The case $D>0$. B: The case $D<0$. 246–304 * Thue's theorem 305–342 * On indeterminate equations of the third degree in two unknowns. A: Integer solutions. B: The solution of cubic indeterminate equations in two unknowns in rationals 343–452 * Supplement I: Introduction to Dirichlet's 453–490 * Supplement II. Investigations in the geometry of Galois theory 491–503 * Bibliography 504–509.

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Le savant, qu'est-il? 17–32 * Géomètres: leur esprit, leurs visages 33–188 * Epoques mathématiques 189–264 * Causerie académique 265–282.

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Introduction 7–20 * Nombres réels 21–28 * Les axiomes de la géométrie euclidienne 29–30 * Espaces vectoriels 31–52 * Géométrie affine plane 53–83 * Géométrie euclidienne plane 84–126 * Géométrie affine a trois dimensions 127–148 * Géométrie euclidienne a trois dimensions 149–160 * Annexe I: sur la « mesure »

des angles 161–163 * Annexe II: Géométrie d'une forme bilinéaire symétrique. Les langages « projectif » et « non euclidien » 165–188 * Annexe III: Inversions et groupe conforme 189–196 * Annexe IV: Quaternions et rotations 197–207 * Bibliographie 208 * Index des notations 208–211 * Index termonologique 212–223.

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Diskussionsbeiträge 23–30.

Claus Müller: *Mathematische Probleme der modernen Wellenoptik* 31–80.

Diskussionsbeiträge 81–87.

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Part I: Preface * Theory of sets 1–109 * Supplements: Relations 118–135 * Functions 136–154 * Solutions sets 155–163 * Graphs of solution sets 164–182 * Application of Boolean algebra to switching networks, by J. R. Sjoblom 183–208 * Answers to selected exercises 209–230 * Index of special symbols 231–232 * Index 233–237.

Part II: Preface * The statement calculus 1–53 * The restricted predicate calculus 54–108 * Answers to selected exercises 109–118 * Frequently used statements 119–120 * Index 121–122.

Part III: Preface * Introduction to groups 1–56 * Introduction to rings and fields 57–88 * Answers to selected exercises 89–94 * Index 95–97.

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Introduction.

I, *C*-algèbres*: Algèbres normées involutives 1–20 * Formes positives et représentations 21–58 * Spectre d'une C^*-algèbre 59–81 * C^*-algèbres liminaires 82–99 * Type d'une représentation 100–114 * Traces et représentations 115–132 * Quasi-spectre 133–141 * Intégration et désintégration des représentations 142–167 * C^*-algèbres de type I 168–186 * Champs continus de C^*-algèbres 187–222 * Extension aux C^*-algèbres du théorème de Stone–Weierstrass 223–234 * Algèbre de von Neumann enveloppante d'une C^*-algèbre 235–248.

II, *Applications aux représentations des groupes*: Représentations unitaires des groupes localement compacts 249–275 * Représentations irréductibles de carré intégrable 276–283 * Représentations des groupes compacts 284–295 * Fonctions presque-périodiques 296–304 * Caractères d'un groupe localement compact 305–313 * Dual d'un groupe localement compact 314–332.

Appendice A: 333–352 * Appendice B: Résultats divers 353–360 * Bibliographie 361–368 * Index des notations 369–371 * Index terminologique 372–378 * Table des matières 379–392.

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Preface * The behaviour of linear systems with constant elements on the positive time axis; classical approach 9–27 * Function theory-properties of a class of matrices 28–35 * The behaviour of linear systems with constant elements on the positive time axis-general approach 36–81 * The behaviour of linear systems with constant elements on the entire time axis 82–112 * The behaviour of linear systems with time-variable elements 113–187 * General properties of m-poles and n-poles 188–224.

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Foreword * Various systems of notation 1–9 * Some facts from the theory of numbers 10–14 * Congruences 15–19 * Continued fractions and indeterminate equations 20–30 * Pythagorean and heronic triples of numbers 31–32 * Artihmetical pastimes 33–36 * Numerical tricks 37–44 * Rapid calculations 45–56 * Numerical giants 57–60 * Games with piles of objects 61–70 * Meleda 71–74 * Lucas' game 75–76 * Solitaire 77–78 * "The game of fifteen" and similar games 79–85 * Problems on determining the number of ways of reaching a goal 86–96 * Magic squares 97–104 * Euler squares 105–106 * Pastimes with dominos 107–108 * Problems connected with the chess board 109–119 * Making up timetables 120–123 * "The problem of Josephus Flavius" and similar ones 124–126* Pastimes connected with objects changing places 127–135 * The simplest methods of constructing pleasing patterns 136–141 * Regular polygons from Rhombi 142–144 * The construction of figures from given parts 145–148 * Construction of parquets 149–157 * Re-cutting of figures 158–165 * The construction of curves 166–187 * Mathematical borders 188–192 * Models of polyhedra 193–201 * Pastimes with a sheet and a strip of paper 202–206 * The four-colour problem 207–210 * Drawing figures at one stroke of the pencil 211–214 * Hamilton's game 215–218 * Arranging points on a plane and in space 219–221 * Problems of a logical nature 222–231 * Rag-bag 232–245 * Notes and answers to problems 246–296 * Bibliography 297–298.

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Preface 7–11 * What operational research is 12–21 * Operational research tech-

niques 22–114 * The organization of operational research 115–120 * The use of operational research 121–129 * Conclusion 130–136 * Appendix I: Rapid statistical tests for use by managers 137–146 * Appendix II: Business models project rules of play 147–153.

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Preface to the second edition * Preface to the first edition * Vector algebra 1–31 * Motion of a particle 32–65 * Kinematical motion of a rigid body 66–93 * Theory of rotating axes 94–122 * Moments of inertia 123–145 * Equations of motion of a rigid body 146–187 * Euler's equations and applications 188–231 * Precessional motion 232–267 * Impulsive motion 268–306 * Lagrange's equations 307–344 * Theory of small oscillations 345–376 * The Routhian and stability of states of steady motion 377–393 * Appendix: Three dimensional statics 394–406 * Historical note 407 * Bibliography 408 * Index 409–412.

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Preface * The cube as a member of a larger family of polyhedrons 1–5 * The planes of symmetry and axes of symmetry of the cube 6–20 * The group of rotations of the cube 21–36 * Sections of the cube 37–45 * Coloured blocks 46–51 * Constructions from coloured blocks 52–66 * Blocks that can be cut out of a cube 67–76 * How a bigger cube can slide through a smaller one 77–83.

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Förord * Bakgrund och historik 1–5 * Grundläggande begrepp 6–13 * Beräkningssatser och sammensatta satser 14–17 * Villkorsuttryck och villkorssatser 18–25 * Typdeklarationer och fältdeklarationer 26–32 * Block och blockstruktur 33–39 * Hoppsatser 40–49 * Väljare 50–54 * Deklarationer markerade med "own" 55–57 * Repetitionssatser 58–68 * ALGOL-kommentarer 69–71 * Funktionsprocedurer 52–78 * Procedurer 79–91 * Stränger och procedurer i maskinkod 92–93 * Metaspråket och ALGOL-rapporten 94–96 * Om vissa egenheter i ALGOL 60 97–100 * In- och utmatning 101–103 * Realisering av ALGOL 60 på datamaskin 104–107 * Om kompilatorkonstruktion 108–111 * ALGOLs framtid 112–114 * Lösningar och svar till övningsexempel 115–122 * Litteratur 123 * Register 124–126.

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Foreword * Introduction * Qualitative methods in extremal problems 1–54 * Qualitative methods in the theory of functions of complex variables 55–65 * The

fixed point method 66–80 * Qualitative methods in the theory of differential equations 81–154 * Differential equations with deviating arguments 155–214 * Variational problems with retarded argument 215–229 * Bibliography 230–250.

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Preface * Acknowledgments * The fountainhead 1–62 * Modern elementary geometry 63–116 * Elementary transformations 117–182 * Euclidian constructions 183–228 * Dissection theory 229–280 * Projective geometry 281–328 * Non-euclidian geometry 329–370 * The foundations of geometry 371–436 * Appendix 1 437–443 * Appendix 2 444–446 * Suggestions for solutions 447–476 * Index 475–489.

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Preface.

I, *The logical background*: Introduction 1–3 * Logic 4–13.

II, *The set-theoretical background*: Sets 14–24 * An algebra of sets 25–35 * Relations and function 36–54 * Mathematical systems of relations and functions 55–63.

III, *The positive integers*: Basic properties 64–72 * The arithmetic of positive integers 73–81 * Order 82–90 * Sequences, sums and products 91–100.

IV, *The integers and integral domains*: Toward extending the positive integers 101–107 * Integral domains 108–112 * Construction and characterization of the integers 113–122 * The integers as an indexing system 123–130 * Mathematical properties of the integers 131–146 * Congruence relations in the integers 147–157.

V, *Polynomials*: Polynomials functions and polynomial forms 158–169 * Polynomials in several variables 170–182.

VI, *The rational numbers and fields*: Toward extending integral domains 183–191 * Fields of quotients 192–199 * Solutions of algebraic equations in fields 200–209 * Polynomials over a field 210–221.

VII, *The real number*: Toward extending the rationals 222–234 * Continuously ordered fields 235–255 * Infinite series and representations of real numbers 256–266 * Polynomials and continuous functions on the real numbers 267–287 * Algebraic and transcendental numbers 288–302.

VIII, *The complex numbers*: Basic properties 303–321 * Polynomials and continuous functions in the complex numbers 322–336 * Roots of complex polynomials 337–352.

IX, *Algebraic number fields and field extensions*: Generation of subfields 353–

364 * Algebraic extensions 365–373 * Applications to geometric construction problems 374–385 * Conclusion 386–390.

Appendix I: Some axioms for set theory 391–399 * Appendix II: The analytic basis of the trigonometric functions 400–407 * Bibliography 408–410 * Index 411–418.

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Literaturverzeichnis * Einleitung 7–16.

I, *Simplizialkomplexe*: Geometrie des Simplizialkomplexes 17–29 * Homologie- und Kohomologiegruppen 30–51.

II, *Kettenkomplexe und ihre Anwendungen*: Allgemeine Theorie 52–61 * Freie Kettenkomplexe 62–78.

III, *Zellkomplexe, Invarianz*: Zellkomplexe 79–90 * Invarianz der Homologiegruppen 91–112.

IV, *Ausbau der Theorie*: Produkte in Polyedern 113–125 * Mannigfaltigkeiten 126–140 * Der Homologiering einer Mannigfaltigkeit 141–151.

Index 152–153.

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Preface * General notation * Fundamental solutions and the Cauchy problem 1–32 * The maximum principle and some applications 33–57 * The first initial-boundary value problem 58–89 * Derivation of a priori estimates 90–132 * The second initial-boundary value problem 133–155 * Asymptotic behaviour of solutions 156–185 * Semi-linear equations; nonlinear boundary conditions 186–214 * Free boundary problems 215–237 * Fundamental solutions for parabolic systems 238–269 * Boundary value problems for elliptic and parabolic equations of any order 270–326 * Appendix: Nonlinear equations 327–332 * Appendix bibliography 333–334.

Bibliographical remarks 335–337 * Bibliography 338–344 * Index 345–347.

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Preface * Notation * Introduction to set theory 1–20 * Topological spaces 21–76 * Separation properties 77–126 * Compactness and uniformization 127–174 * Continuity 175–253 * Theory of convergence 254–308 * Author index 309–310 * Subject index 311–317.

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Preface.

I, *Functions of a real variable*: The real number system 3–19 * Functions and limits 20–34 * Differentiation 35–41 * Riemann integration 42–46 * Sequences

47–52 * Infinite series 53–75 * Uniform convergence 76–82 * Sets and measure on the real axis 83–114.

II, *Higher dimensions*: Functions of two variables 115–127 * Plane sets 128–146 * Area 147–153 * Metric and topological spaces 154–171 * Function spaces 172–179.

Bibliography 180–182 * Special symbols 183–186 * Index 187–194.

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Preface.

I, The general theory of commutative normed rings 15–65 * The general theory of commutative normed rings (continued) 66–99.

II, The ring of absolutely integrable functions and their discrete analogues 100–120 * Harmonic analysis on commutative locally compact groups 121–164 * The ring of functions of bounded variation on a line 165–196.

III, Regular rings 197–222 * Rings with uniform convergence 223–239 * Normed rings with an involution and their representations 240–274 * The decomposition of a commutative normed ring into a direct sum of ideals 275–290.

Historico-bibliographical notes 291–294 * Bibliography 295–302 * Index 303–306.

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Translator's note * Foreword.

I, *The kernel theorem. Nuclear spaces. Rigged Hilbert space*: Bilinear functionals on countably normed spaces. The kernel theorem 2–25 * Operators of Hilbert-Schmidt type and nuclear operators 26–55 * Nuclear spaces. The abstract kernel theorem 56–102 * Rigged Hilbert spaces. Spectral analysis of self-adjoint and unitary operators 103–126 * Appendix 127–134.

II, *Positive and positive-definite generalized functions*: Introduction 135–141 * Positive generalized functions 142–150 * Positive-definite generalized functions. Bochner's theorem 151–174 * Conditionally positive-definite generalized functions 175–195 * Evenly positive-definite generalized functions 196–215 * Evenly positive-definite generalized functions on the space of functions of one variable with bounded supports 216–228 * Multiplicatively positive linear functionals on topological algebras with involutions 229–236.

III, *Generalized random processes*: Basic concepts connected with generalized random processes 237–245 * Moments of generalized random processes. Gaussian processes. Characteristic processes 246–261 * Stationary generalized random processes. Generalized random processes with stationary nth-order increments 262–272 * Generalized random processes with independent values at every point 273–288 * Generalized random fields 289–302.

IV, *Measures in linear topological spaces*: Basic definitions 303–311 * The countable additivity of cylinder set measures in spaces adjoint to nuclear spaces 312–334 * Gaussian measures in linear topological spaces 335–344 * Fourier transforms

of measures in linear topological spaces 345–349 * Quasi-invariant measures in linear topoligical spaces 350–370.

Notes and references to the literature 371–376 * Bibliography 377–380 * Subject index 381–384.

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Vorwort * Funktionen 12–53 * Elementare Funktionen 54–116 * Die Ableitung einer Funktion 117–159 * Unbestimmte Integrale 160–188 * Lokale Eigenschaften von Funktionen 189–234 * Das bestimmte Integral 235–310 * Die Laplacetransformation 311–364 * Register 365–368.

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Preface * Introduction 1–3 * Elementary examples 4–30 * Approximations and stability 31–70 * Concepts of an approximation and stability for difference schemes for partial differential equations 71–135 * On the solution of difference equations 136–154 * Stability as the boundedness of the powers of a certain operator 155–187 * Elements of the spectral theory of difference operators 188–214 * Conclusion 215–222 * Comments on the literature 223–228 * Bibliography 229–231 * Appendix I: On difference schemes for the solution of the equation of thermal conductivity, by I. M. Gelfand and O. V. Lokutsievski 232–238 * Appendix II: The double sweep method for solution of difference equations, by I. M. Gelfand and O. V. Lokutsievski 239–262 * Appendix III: The scope of the energy method, by P. D. Lax 263–267 * Appendix IV: On the estimate of the amount of computational labour necessary in approximate solutions, by N. S. Bakhvalov 268–279 * Appendix V: Difference schemes for parabolic equations and continuous integrals, by V. Yu. Krylov 280–289.

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Introduction: Assumptions and notations 1–2 * Sets and functions 3–23 * Sequences of real numbers 24–66 * Series of real numbers 67–97 * Limits and metric spaces 98–112 * Continuous functions on metric spaces 113–132 * Connectedness, completeness, and compactness 133–155 * Calculus 156–201 * The elementary functions. Taylor series 202–230 * Sequences and series of functions 231–258 * Three famous theorems 259–270 * The Lebesgue integral 271–327 * Fourier series 328–354 * Index of special symbols 355–356 * Index 357–359.

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Vorwort zur deutschen Ausgabe * Mengen 1–46 * Wahrscheinlichkeit in endlichen Ereignisräumen 47–134 * Kombinatorik 135–162 * Zufallsveränderliche (Zufallsvariable) 163–258 * Binomialverteilung und einige Anwendungen 259–299 * Literatur 300–320 * Sachwortverzeichnis 321–324.

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Preface * Introduction 1–5 * Basic concepts of matrix algebra 6–48 * Basic concepts of statistical inference 49–155 * Classical linear regression 156–212 * Extensions of linear regression 213–265 * Linear regression with stochastic regressors 266–287 * Systems of simultaneous linear relationships 288–388 * Bibliography 389–392 * Index 393–399.

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Foreword * Preface.

Introduction to the analysis of time series 3–11 * Nature of economic time series 12–24.

Part A: *Stationary time series*: Spectral theory 25–51 * Spectral analysis of economic data 52–73 * Cross-spectral analysis 74–94 * Cross-spectral analysis of economic data 95–108 * Processes involving feedback 109–128.

Part B: *Non-stationary time series*: Series with trending means 129–146 * Series with spectrum changing with time 147–169 * Demodulation 170–189 * Non-stationary and economic series 190–206 * Application of cross-spectral analysis and complex demodulation: Business cycle indicators (by M. Hatanaka) 207–263 * Application of partial cross-spectral analysis: Tests of acceleration principle for inventory cycle (by M. Hatanaka) 264–293 * Problems remaining 294–296.

Index 297–299.

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Foreword * Introduction 1–2 * Choice of trend curve 3–17 * Fitting trend curves 18–41 * Appendix 1. Alternative methods for estimating the slopes 42 * Appendix 2. Gomes functions for the fitting of modified exponentials 43 * Appendix 3. The slopes characteristic 44–46 * Appendix 4. The general modified exponential 47–50 * Appendix 5. Mathematical basis for fitting modified exponentials 51–57 * Appen-

dix 6. Computer programmes available from imperial chemical industries limited 58–59 * Tables and graphs 60–99.

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Preface * General theory 1–8 * Shock propagation in non-uniform ducts 9–29 * The piston-driven shock wave 30–36 * Flows with heat addition 37–49 * Simple wave flows 50–67 * Formation and decay of shock waves 68–77 * The effects due to an oblique applied field 78–95 * References 96–98 * Appendices 99–119.

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Geometric introduction 1–98 * The foundations of tensor algebra 99–138 * Invariants and concomitants of tensors and their simplest properties 139–169 * The fundamental theorem of the theory of invariants and its consequences 170–245 * Binary forms 246– 307 * Ternary forms. Second order tensors 308–352 * Polyvectors 353–395 * Answers and hints to exercises 396–417 * Literature 418 * Index 419–429.

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Preface * Examples of similarity solutions 1–8 * Free parameter method 9–30 * Separation of variables method 31–45 * Group-theory methods 46–75 * Dimensional analysis 76–85 * Miscellaneous topics 86–98 * Appendix A: Specification of velocity transformations for flow problems 99–100 * Problems 101–108 * Bibliography 109–112 * Index 113–114.

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Preface * For the reader * Preliminaries 1–19 * Setting for the book 20–23 * Real sequences 24–61 * Real functions 62–78 * Continuous real functions 79–95 * Differentiable real functions 96–129 * Integration of functions 130–148 * Index 149–154.

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Vorwort zur zweiten Auflage * Vorwort zur ersten Auflage.

I, *Grundlagen*: Primzerlegung 1–12 * Grösster gemeinsamer Teiler 13–28 * Vollkommene Zahlen, Mersennesche und Fermatsche Primzahlen 29–37 * Kongruenz, Restklassen 38–70 * Die Struktur der primen Restklassengruppen 71–85.

II, *Quadratische Reste*: Definition, Reduktion, Kriterien 86–95 * Das quadratische Reziprozitätsgesetz: Elementarer Beweis 96–111 * Das quadratische Reziprozitätsgesetz: Beweis mit Gausschen Summen 112–122 * Die Jacobische Verallgemeinerung 123–144 * Verteilungsfragen über quadratische Reste nach einer Primzahl 145–175.

III, *Der Dirichletsche Primzahlsatz*: Elementare Sonderfälle 176–191 * Die Methode von Dirichlet 192–205 * Die Charaktere endlicher abelscher Gruppen, Restklassencharaktere 206–226 * Der Beweis von Dirichlet 227–237 * Das Nichtverschwinden der L-Reihen 238–282.

IV, *Quadratische Zahlkörper*: Elementare Teilbarkeitslehre 283–336 * Divisorentheorie 337–387 * Bestimmung der Klassenzahl 388–431 * Quadratische Zahlkörper und quadratisches Reziprozitätsgesetz 432–443 * Systematische Theorie der Gausschen Summen 444–494.

Namenverzeichnis 495–496 * Sachverzeichnis 497–504.

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Preface * Analytic functions 1–6 * Invariant subspaces 7–14 * Individual theorems 15–24 * Invariant subspaces in L^p 25–36 * Invariant subspaces on the line 37–50 * Analytic vector functions 51–64 * Vectorial function theory 65–72 * Inner functions 73–90 * Range functions 91–100 * Invariant subspaces (in general) 101–110 * Factoring of operator functions 111–128 * References 129–130.

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Author's preface * Translator's note.

Part I, *Examples*: Introduction 3–10 * The wave equation 11–27 * The potential equation 28–42 * The hear equation 43–58.

Part II, *Classification into types, theory of characteristics, and normal form*: Differential equations of the first order 59–68 * Systems of differential equations of the first order 69–80 * On the necessity of classification into types 81–88.

Part III, *Questions of uniqueness*: Elliptic and elliptic-parabolic type 89–98 * Parabolic type 99–101 * Hyperbolic type 102–114 * Mixed type 115–124.

Part IV, *Questions of existence*: Equations of hyperbolic type in two independent variables 125–150 * Boundary and initial-value problems for equations of hyperbolic and parabolic type in two independent variables 151–171 * Equations of elliptic type 172–194 * Weyl's lemma for equations of elliptic type 195–208.

Part V, *Simple tools from functional analysis applied to questions of existence*:

Auxiliary tools 209–217 * Schauder's technique of proof for existence problems in elliptic differential equations 218–221 * The regular eigenvalue problem 222–231 * Elliptic systems of differential equations 232–250 * Solutions 251–256 * Bibliography 257–258 * Index 259–263.

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Vorwort.

I, *Einführung*: Die Aufgabe der Logik 9–11 * Beispiele für mathematische Beweise 12–14 * Der Folgerungsbegriff 15–22 * Bemerkungen zum Folgerungsbegriff 23–27 * Logikkalküle 28–35 * Symbolisierung mathematischer Aussagen: Junktoren und Quantoren 36–43 * Symbolisierung mathematischer Aussagen: Subjekte, Prädikate und Funktoren 44–49.

II, *Die Sprache der Prädikatenlogik*: Terme und Ausdrücke 50–54 * Elementare Entscheidbarkeitsfragen 55–59 * Beweise und Definitionen durch Induktionen über den Aufbau der Ausdrücke 60–61 * Freie und gebundene Variablen 62–63 * Substitution 64–70.

III, *Semantik der Prädikatenlogik*: Einführung in die Semantik der Prädikatenlogik 71–75 * Definition der wichtigsten semantischen Begriffe 76–80 * Sätze über die Modellbeziehung 81–82.

IV, *Ein Prädikatenkalkül*: Vorbemerkungen zu den Regeln des Prädikatenkalküls. Der Ableitbarkeitsbegriff 83–85 * Die Regeln des Prädikatenkalküls 86–92 * Die Korrektheit des Prädikatenkalküls 93–94 * Ableitbare Regeln 95–103 * Einige Eigenschaften des Ableitbarkeitsbegriffs. Widerspruchsfreiheit 104–107 * Die Entscheidbarkeit der aussagenlogischen Ableitbarkeit 108–113.

V, *Der Gödelsche Vollständigkeitssatz*: Verallgemeinerte Substitution 114–120 * Isomorphismen von Ausdrücken 121–125 * Übersicht über den Beweis für den Vollständigkeitssatz 126–127 * Der Prozess der Maximalisierung 128–130 * Abschluss des Vollständigkeitsbeweisen 131–133 * Folgerungen aus dem Vollständigkeitssatz 134–139.

VI, *Das Peanosche Axiomsystem*: Sprache der Logik der zweiten Stufe 140–141 * Isomorphe Interpretationen, Kategorizität von Axiomensystemen 142–146 * Die Charakterisierbarkeit der natürlichen Zahlen in der Sprache der Logik der zweiten Stufe 147–151 * Die Nichtcharakterisierbarkeit der natürlichen Zahlen in der Sprache der Prädikatenlogik 152–156.

VII, *Ergänzungen*: Erweiterungen der Sprache der Prädikatenlogik 157–161 * Abgeleitete Regeln und Ableitbarkeitsbeziehungen mit den Verknüpfungen v, V 162–165 * Weitere Ableitbarkeitsbeziehungen in Verbindung mit der Generalisierung und Partikularisierung 166–171 * Konjunktive und alternative Normalform 172–173 * Pränexe Normalformen 174–178.

Weiterführende Literatur 179 * Verzeichnis der Kurzbezeichnungen für definierende und abgeleitete Regeln 180 * Bezeichnungen und Symbole 181 * Namen- und Sachverzeichnis 182–187.

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Preface * Acknowledgement * Roger T. Eddison and David B. Hertz: Applying operations research in industry and government 1–10 * R. G. Brown: Some problems in production management 11–32 * Jerome D. Herniter and Ronald A. Howard: Stochastic marketing models 33–96 * Roger T. Eddison: Warehousing, distribution, and finished goods management 97–122 * Van Court Hare, Jr.: Systems analysis 123–158 * Charles Christenson: Capital budgeting and long-range planning 159–190 * Jacques Antoine and Jaques Lesourne: Government systems planning 191–222 * Alex M. Lee: Transportation 223–244 * S. L. Cook: Applications in the steel industry 245–262 * G. W. Sears: Petroleum 263–280 * J. Stringer: Operations research in the mining industry 299–328 * Charles D. Flagle: Operations research in community services 329–344 * Kurt H. Schaffir: Operations research in the textile industry 345–364 * J. K. W. Slater: Agriculture 365–372 * S. W. Hess: Operations research in the chemical and pharmaceutical industries 373–386 * George W. Morgenthaler: Operations research applied to the national ballistic missile and space programs 387–438 * Author index 439–450 * Subject index 451–455.

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Preface * Introduction 1–50 * Plane analytic geometry 51–111 * Functions 112–171 * The differential operator 172–221 * Elements of integration 222–275 * Infinite series 276–321 * Complex analysis 322–336 * Taylor's series 337–388 * Some transcendental functions 389–445 * Applications of the differential calculus 446–490 * Plane curves 491–543 * Some basic concepts 544–574 * Solutions to odd-numbered problems 575–616 * Index 617–626.

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Foreword * Introduction—Logic, mathematics, and cybernetics 2–51 * The principles of symbolic logic 52–83 * Classes (sets) 84–119 * Logical reasoning 120–153 * The algebra of logic 154–209 * Computing and machines 210–299 * Logic and circuits 300–329 * Logic and the principles of programming 330–367 * Human beings and their machines 368–401 * Bibliography 402–411 * Index 412–427.

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sionalen Sphäre auf die Kugelfläche (Math. Ann. 104 (1931)) 38–63 * Über den Begriff der vollständigen differentialgeometrischen Fläche (mit W. Rinow) (Comment. Math. Helv. 3 (1931)) 64–79 * Die Klassen der Abbildungen der n-dimensionalen Polyeder auf die n-dimensionale Sphäre (Comment. Math. Helv. 5 (1933)) 80–94 * Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension (Fund. Math. 25 (1935)) 95–106 * Systeme symmetrischer Bilinearformen und euklidische Modelle der projektiven Räume (Viertelj.schr. Naturforsch. Ges. Zürich 85 (1940). Beiblatt Nr. 32 (Festschrift Rudolf Fueter)) 107–118 * Über die Topologie der Gruppen-Mannigfaltigkeiten und ihrer Verallgemeinerungen (Ann. of Math. 42 (1941)) 119–151 * Über den Rang geschlossener Liescher Gruppen (Comment. Math. Helv. 13 (1940/41)) 152–174 * Bericht über einige neue Ergebnisse in der Topologie (Rev. Mat. Hisp-Amer. (4) 6 (1946)) 175–185 * Fundamentalgruppe und zweite Bettische Gruppe (Comment. Math. Helv. 14 (1941/42)) 186–206 * Nachtrag zu der Arbeit: Fundamentalgruppe und zweite Bettische Gruppe (Comment. Math. Helv. 15 (1942/43)) 207–210 * Über die Bettischen Gruppen, die zu einer beliebigen Gruppe gehören (Comment. Math. Helv. 17 (1944/45)) 211–234 * Beiträge zur Homotopietheorie (Comment. Math. Helv. 17 (1944/45)) 235–243 * Enden offener Räume und unendliche diskontinuierliche Gruppen (Comment. Math. Helv. 16 (1943/44)) 244–262 * Über Flächen mit einer Relation zwischen den Hauptkrümmungen (Math. Nachr. 4 (1950)) 263–283 * Schlichte Abbildungen und lokale Modifikationen 4-dimensionaler komplexer Mannigfaltigkeiten (Comment. Math. Helv. 29 (1955)) 284–306 * Verzeichnis der Publikation von Heinz Hopf 307–310.

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I, *Allgemeine Theorie der Funktionen einer komplexen Veränderlichen*: Die komplexen Zahlen 1–21 * Die Potenzreihen 22–41 * Der Begriff der analytischen Funktion 42–66 * Untersuchung einiger spezieller analytischer Funktionen 67–79 * Die Integration analytischer Funktionen 80–107 * Die meromorphen Funktionen 108–134 * Die Umkehrung der analytischen Funktionen 135–146.

II, *Elliptische Funktionen*: Die doppeltperiodischen meromorphen Funktionen 147–187 * Die Theta-Funktionen 188–211 * Die elliptischen Funktionen Jacobis 212–218 * Die elliptischen Modulfunktionen 219–228 * Elliptische Gebilde 229–236 * Elliptische Integrale 237–244 * Die Transformation der elliptische Funktionen 245–256.

III, *Geometrische Funktionentheorie*: Vorbereitende Betrachtungen 257–278 * Die Grundlagen der Theorie der meromorphen Funktionen 279–303 * Folgerungen aus der Cauchyschen Integralformel 304–345 * Spezielle Funktionen und ihre Singularitäten 346–375 * Analytische Fortsetzung und Riemannsche Flächen 376–396 * Die Konforme Abbildung einfach zusammenhängender schlichter Gebiete 397–430 * Spezielle Abbildungsfunktionen 431–453 * Die Verallgemeinerung

des Riemannschen Abbildungssatz. Das Dirichletsche Prinzip 454–496 * Weitere Existenztheoreme der Funktionentheorie 497–550.

Anhang: Weitere Abbildungstheoreme der Funktionentheorie 558–636 * Holomorphe und meromorphe Funktionen auf Riemannschen Flächen 637–696.

Namen- und Sachverzeichnis 697–706.

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Vorwort zur deutschen Ausgabe * Einführung 1–8 * Die Mathematik in China 9–88 * Die Mathematik in Indien 89–174 * Die Mathematik in den Ländern des Islam 175–325 * Die Mathematik im Europa des Mittelalters 326–446 * Namenverzeichnis 447–454.

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Preface * Foreword * Categories * Fundamental concepts 1–12 * Fundamental constructions 13–35 * Function spaces 36–55 * Mappings into polyhedra 56–77 * Dimension (1) 78–96 * Compactifications 97–122 * Locally fine spaces 123–145 * Dimension (2) 146–158 * Appendix. Line and plane 159–162 * Bibliography 163–172 * Index 173–175.

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Preface * Introduction 1 * Addition in a finite number system 2–7 * Addition modulo m 8–9 * An application 10–11 * Closed subsets 11–13 * Multiplication in a finite number system 14–16 * Multiplication modulo m 17–20 * Classifications of numbers 21–23 * Operations with congruences 24–28 * Solution of congruences 29–35 * Congruences and finite number systems 36–37 * Tests for divisibility 38–40 * Repeating decimals 41–45 * Subgroups 46–55 * Linear diophantine equations 56–59 * Finite fields 60–64 * League scheduling 65–67 * Bibliography 68–69 * Answers 70–92 * Index 93.

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Foreword.

I, *Linear operations and functionals*: Metric spaces 1–42 * Normed spaces 43–98 * Linear operations and functionals 99–126 * The extension of linear operations and functionals 127–154 * Spaces of operations and functionals 155–199 * The analytic representation of functionals 200–248 * Sequences of linear operations 249–269 * The weak convergence of functionals and elements 270–291 * Completely continuous and adjoint operations 292–346 * The analytic representation of certain classes of linear operations 347–390 * Linear topological spaces 391–464.

II, *Functional equations*: Adjoint equations 465–487 * Functional equations of the second kind 488–541 * The general theory of approximation method 542–601 * The method of steepest descent 602–626 * The fixed point principle 627–656 * The differentiation of non-linear operations 657–694 * Newton's method 695–750 * Principal literature on functional analysis and associated subjects 751–754 * Bibliography 755–763 * Index 764–771.

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Preface * Things, symbols, and statements 1–9 * Not, and, or 10–16 * If ... then ... 17–21 * If and only if 22–24 * Entailment 25–28 * Contradictions, tautologies 29–31 * Existence, nonexistence 32–34 * Each, some 35–44 * Each, some (continued) 45–51 * Definition 52–54 * Truth, falsity 55–62 * Proofs, disproofs 63–70 * Equal, unequal 71–73 * Sets 74–78 * Singletons, the empty set, doubletons 79–85 * Subsets of a set 86–89 * Combining sets 90–95 * Ordered pairs 96–101 * Functions 102–110 * Functions (continued) 111–115 * The inverse of a one-one function 116–118 * Introduction to the real number system 119–121 * The addition function 122–127 * The multiplication function 128–135 * The Max function 136–139 * The minimum of two numbers 140–143 * The signum of a number 144–147 * Positive, negative 148–150 * Less than, greater than 151–157 * Precedes ($\leqq$), exceeds ($\geqq$) 158–164 * Absolute value 165–168 * Absolute value (continued) 169–171 * Number sets 172–178 * Lower bounds of a number set 179–182 * Upper bounds of a number set 183–185 * The maximum of a number set 186–189 * The minimum of a number set 190–193 * Integers 194–198 * Sets of integers 199–202 * Inductive proofs 203–206 * The Dedekind axiom 207–209 * Square root 210–213 * The supremum of a number set 214–218 * The infinum of a number set 219–221 * Completeness 222–225 * $[x]$ 226–228 * Odd, even 229–231 * Rational, irrationals 232–234 * Positive integrals powers 235–238 * Integral powers 239–244 * Positive integral roots 245–248 * Rational powers 249–252 * Real powers 253–259 * Logarithms 260–263 * Real functions 264–268 * Arithmetic of real functions 269–273 * The sum of a finite number-sequence 274–280 * Σ 281–287 * Permutations 288–290 * Σ (continued) 291–297 * The product of a finite number-sequence 298–304 * Traditional notations 305–310 * Factorization of $x^n - y^n$ and other matters 311–315 * Binomial theorem 316–320 * Some useful inequalities 321–324 * Selected bibliography 325–327 * Glossary of special symbols 326 * Index 327–336.

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Preface.

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II, *Changes in population number*: Thomas Park: Competition: An experimental and statistical study 175–196 * Chin Long Chiang: Competition and other interactions between species 197–216 * Francis J. Ryan: Analysis of populations of mutating bacteria 217–228.

III, *Estimation of populations*: R. J. Jessen: General methodology in sampling human populations 229–240 * Harold F. Dorn: Some problems in sampling human populations 241–250 * S. R. Gevorkiantz: Problems in forest inventory: From the forester's point of view 251–262 * Austan A. Hasel: Problems in forest inventory: From the statistical point of view 263–272 * Leslie W. Scattergood: Estimating fish and wildlife populations: A survey of methods 273–286 * D. B. DeLury: On the assumptions underlying estimates of mobile populations 287–294 * Geoffrey Beall: Data in binomial or near-binomial distribution: With particular application to problems in entomological Research 295–302 * Pascal K. Whelpton and Halvor Gille: The growth of human population 303–314.

IV, *Determination of biological response*: Lloyd C. Miller: The quantal response in toxicity tests 315–326 * Jerome Cornfield: Measurement and comparison of toxicities: The quantal response 327–344 * C. I. Bliss: Insecticidal assays 345–360 * Peter B. Dews and Joseph Berkson: On the error of bio-assay with quantal response 361–370 * Geoerge F. Stewart: Analytical sensory tests for food products 371–382 * J. W. Hopkins: Some statistical aspects of flavor and aroma testing 383–392 * J. K. Loosli: Determination of the nutritional requirements of dairy animals and swine 393–398 * Paul G. Homeyer: Some problems of technique and design in animal feeding experiments 399–406 * Clyde Stormont: Research with cattle twins 407–418 * J. P. Scott: Biomathematical problems in animal behaviour 419–428.

V, *Genetical analysis of populations*: Philip Levine: Gene frequencies in nonexperimental populations 429–448 * C. W. Cotterman: Estimation of gene frequencies in nonexperimental populations 449–466 * James V. Neel: A description of studies on the potential genetic effects of the atomic bombs in Japan. I, Organizational aspects 467–476 * William J. Schull: A description of studies on the potential genetic effects of the atomic bombs in Japan. II, Analytical problems 477–486 * Ernest C. Pollard: Radiation effects on the functioning of body cells 487–494 * John W. Gowen: Effects of X-rays of different wave lengths on viruses 495–510 * Herbert H. Kramer: Recombination in selfed chromosome interchange heterozygotes 511–522 * Bruce Griffing: Contributions of genotypic and environmental effects to earlines of tomato fruiting 523–536 * Jay L. Lush: Breeding structure of populations. I, General considerations 537–542 * James F. Crow: Breeding struc-

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Bibliography 573–616 * Index 617–632.

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Vorwort * Einleitung 1–16 * Das Axiomensystem 17–62 * Klassenalgebra 63–112 * Endliche Mengen 113–129 * Ordnungsrelationen 130–233 * Kardinalzahlen, Ordinalzahlen, Ordnungstypen 234–354 * Arithmetik der Ordinalzahlen 355–489 * Arithmetik der Kardinalzahlen 490–572 * Literaturverzeichnis 573–574 * Sachregister 575–581.

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Introduction 5 * Index des notations 6 * Organigramme 7–8 * Fonction exponentielle et équations différentielles linéaires 9–36 * Théorie élémentaire des fonctions d'une variable 37–74 * Intégrales simples 75–96 * Fonctions de plusieurs variables 97–122 * Séries 123–146 * Intégrales multiples 147–166 * Géométrie analytique 167–206 * Problèmes d'examen 207–216 * Vocabulaire 217–224.

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The kinematics of a fluid medium 1–38 * The fundamental hydrodynamic equations for an ideal fluid 39–80 * Hydrostatics 81–110 * The simplest cases of motion of an ideal fluid 111–146 * Vortex motion of an ideal fluid 147–245 * The plane problem of motion of a body in an ideal fluid 246–372 * The three-dimensional problem of the motion of a body in an ideal fluid 373–415 * The wave motion of an ideal fluid 416–564 * References 565–570 * Author index 571–572 * Subject index 573–577.

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Preface * Introduction 1–4 * Characteristics of digital computers 5–11 * Various number bases 12–23 * Getting started with a guess 24–27 * Interpolation, or filling in the gaps 28–34 * Approximations, plain and fancy 35–47 * Iteration: Repetition with a purpose 48–59 * Relaxation methods 60–68 * Monte Carlo methods 69–76 * Some special techniques 77–83 * Computer applications—present and future 84–91 * Bibliography for futher study 92–94 * Index 95–98.

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Foreword.

I, *Spaces with a cone*: Fundamental definitions 17–19 * Normal cones 20–24 * The space E_{u_0} 25–29 * Linear positive functionals 30–35 * Regular cones 36–39 * Criteria for the regularity of a cone 40–48 * Minihedral cones 49–52 * A space with two cones 53–58.

II, *Linear positive operators*: Linear u_0-positive operators 59–65 * Existence of a characteristic vector 66–75 * Simplicity of a positive characteristic value 76–79 * Comparison with other characteristic values 80–85 * Non-homogeneous linear equations 86–95.

III, *Differentiability with respect to a cone*: Derivatives with respect to a cone 96–103 * Derivatives at infinity 104–107 * Inequality for elements with small norms 109–118 * Inequalities for elements with large norms 119–121.

IV, *Existence of positive solutions*: Equations with monotone operators 122–129 * Equations with non-monotone operators 130–139 * Auxiliary statements 140–147 * Fixed points of operators which expand a cone 148–154.

V, *Continuous branches of positive solutions*: Positive solutions of equations with a parameter 155–162 * Some topological theorems 163–173 * Operators with monotone minorants 174–184.

VI, *Equations with concave operators*: Uniqueness of a positive solution and the convergence of successive approximations 185–195 * Existence of a continuum of characteristic vectors 196–204 * Equations with convex operators 205–206 * Applications to the problem about the points of bifurcation 207–229.

VII, *Applications*: Existence of positive solutions for integral equations 230–254 * First boundary value problem for elliptic equations of the second order with non-linearities 255–267 * Existence of positive periodic solutions for a system of ordinary differential equations of the first order 268–282 * Two point boundary value problem 283–292 * Periodic solutions of systems of the second order 293–303 * Dirichlet problem for a Monge–Ampère equation 304–321.

Summary of fundamental results 322–359 * Bibliographical notes 360–371 * Literature 372–378 * Index 379–381.

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Vorwort * Mehr oder weniger — eine Vorstufe zur Zahl 11–15 * Zauberer zürnen Zahlen 16–20 * Ein Übereinmaleins 21–24 * Irrlichter 25–28 * Euklids Kette 29–34 * Spröde Zahlen 35–40 * Geometrie der Zahlen 41–43 * Kettenbrüche 44–48 * Ganz reell 49–55 * Libussa verteilt Körbe 56–63 * Cardano greift ein 64–68 * Revolutionierte Algebra 69–72 * Rechnen mit Polynomen 73–76 * Kongruenzen 77–82 * Grenzen 83–87 * Folgen 88–91 * Wie die Welt nicht erschaffen wurde 92–101 * Stetigkeit verpflichtet 102–108 * Monotone Überraschungen 109–113 * Inhalt und Integral 114–125 * Gegenspilerein Ableitung 126–133 * Ungleichungen 134–139 * Zu neuen Ufern 140–146 * Nachwort 147.

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Preface * Notation * Introduction * Basic arithmetic lemmas 1–19 * Additive number-theoretic functions and random variables 20–29 * The law of large numbers 30–42 * One-dimensional integral and local asymptotic distribution laws 43–93 * Asymptotic laws for sums of additive functions 94–107 * An estimate of the remainder term of integral asymptotic laws 108–113 * Distribution of sequences of truncated functions 114–125 * Many-dimensional asymptotic laws 126–138 * The method of Dirichlet generating series 139–168 * Additive number-theoretic functions in the gaussian number field 169–176 * Bibliography 177–182.

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Vorwort * Relationen 1–16 * Gruppen und Ringe 17–76 * Universelle Algebren. Multioperatorgruppen 77–134 * Verbände 135–168 * Operatorgruppen und -ringe. Moduln. Lineare Algebren 169–228 * Geordnete und topologische Gruppen und Ringe. Normierte Ringe 229–291 * Literaturverzeichnis 292–295 * Sachregister 296–302.

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Preface.

I, *Basic mechanics and theory*: Mathematical expressions 3–11 * Properties of addition 12–16 * The sum of squares 17–27 * Expected values 28–38 * The sum of squares of several groups 39–47 * Linear model and least-square estimates 48–55 * The expectation of mean squares 56–63 * The variance-ratio test 64–74 * Hierarchical classifications 75–83 * Twoway classifications 84–101 * Sum of squares as an invariant 102–109 * Orthogonal contrasts 110–124 * Linear regression 125–141 * Regression and analysis of variance 142–160.

II, *Experimental designs*: Randomized experiments 161–173 * Randomized blocks 174–188 * Latin square 189–206 * Change-over designs 207–226 * Missing observations 227–243 * Concomitant observations 244–257 * Factors at two levels 258–278 * Factors at three levels 279–296 * Other types of factorials 297–312 * Simple confounding systems 313–339 * Split-plot design 340–349 * Incomplete latin square 350–361 * Balanced incomplete blocks 362–379 * Symmetrical pairs 380–388 * Residual effects 389–402.

III, *Some related topics*: Multiple measurements 403–417 * Multiple comparisons 418–429 * Unequal group variances 430–440 * Change of scale 441–452.

References 453–456 * Index 457–460.

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Introduction 2–10 * General results 11–19 * The $\mathcal{N}_\lambda$ spaces 20–29 * Intersection properties of cells 30–49 * The connection between intersection and extension properties 50–61 * The extension theorem for $\lambda = 1$ and its applications 62–83 * Norm preserving extensions 84–108 * References 109–112.

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Preface * Linear programming and management 1–20 * The graphical method 21–35 * Systematic trial-anderror method 36–55 * Matrices and vectors 56–94 * The vector method 95–109 * The simplex method. I 110–129 * The simplex method. II 130–142 * The dual 143–150 * Degeneracy 151–159 * The transportation model 160–213 * The assignment model 214–240 * The meaning of linearity 241–249 * A note on inequalities 250–252 * A system of linear equations having a unique solution 253–255 * A system of linear equations having no solution 256–258 * A system of linear equations having an infinite number of solutions 259–262 * Exercises 263–278 * Selected bibliography 279–280 * Index 281–284.

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Preliminaries 3–13 * Perturbation of a constant solution 14–46 * Perturbation of a nonconstant solution 47–106 * Autonomous perturbating terms 107–132 * Bibliography 133.

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(Volume I: See Math. Scand. 14 (1964), p. 253.)

Preface * Stochastic learning theory 1–120 * Stimulus sampling theory 121–268 * Introduction to the formal analysis of natural languages 269–322 * Formal properties of grammars 323–418 * Finitary models of languages users 419–492 * Mathematical models of social interaction 493–580 * Index 581–606.

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Introduction.

I: Shortest paths on simple surfaces 1–32 * Some properties of plane and space curves and related problems 33–43 * Geodesic paths 44–56.

II: Problems connected with the potential energy of a streched string 57–69 * The isoperimetric problem 70–79 * Fermat's principle and its consequences 80–102.

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Basic concepts 1–10 * Fundamental formulas 11–30 * Sets, algorithms, functions 31–46 * The problems of analytic geometry 47–57 * The straight line 58–79 * The circle 80–93 * The equations of the conics in standard form 94–119 * Transformation of axes 120–137 * Parametric equations 138–148 * Limits, tangents, normals 149–166 * The elements of curve tracing 167–184 * The graph of some algebraic equations 185–198 * The graphs of some transcendental loci 199–213 * Polar coordinates 214–222 * Empirical equations 223–244 * Index 245–247.

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Preface * The difficulties associated with the appearance of calculus 1–12 * Early attempts to make analysis rigorous 13–34 * Fourier series and the beginning of researches into point sets 35–65 * Arithmetization of analysis 66–96 * Development of point set theory 97–111 * The emergence of point set topology 112–132 * From Newton to Hausdorff 133–150 * Bibliography 151–160 * Subject index 161–164 * Name index 165–166.

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From the editors * Antoni Zygmund: Jósef Marcinkiewicz 1–27 * Bibliography 28–34 * A new proof of a theorem on Fourier series 35 * On a class of functions and their Fourier series 36–41 * Sur les nombres dérivés 42–44 * Interpolating polynomials for absolutely continuous functions (The translation of the Polish original: Wielomiany interpolacyjne funkcji bezwzględnie ciągłych) 45–70 * On the convergence of Fourier series 71–75 * On Riemann's two methods of summation 76–80 * Note on the differentiability of multiple integrals (with B. Jessen and A. Zygmund) 81–95 * Sur les séries de Fourier 96–124 * On the differentiability of functions and summability of trigonometrical series (with A. Zygmund) 125–163 * Sur la convergence des séries orthogonales 164–170 * Sur l'interpolation (I) 171–185 * Sur l'interpolation (II) 186–199 * Quelques remarques sur l'interpolation 200–203 * Sur la divergence des polynomes d'interpolation 204–208 * Some theorems on orthgonal systems (with A. Zygmund) 209–232 * Sur les fonctions indépendantes (with A. Zygmund) 233–259 * Mean values of trigonometrical polynomials (with A. Zygmund) 260–292 * Two theorems on trigonometrical series (with A. Zygmund) 293–298 * Remarque sur la loi du logarithme itéré (with A. Zygmund) 299–306 * Quelques théorèmes sur les séries orthogonales 307–318 * On the summability of orthogonal series (The translation of the Polish original: O sumowalności szeregów ortogonalnych) 319–327 * Sur les fonctions indépendantes I 328–340 * Sur les fonctions indépendantes II 341–354 * Sur les suites d'opérations linéaires 355–373 * Quelques théorèmes sur les fonctions indépendantes (with A. Zygmund) 374–388 *

Sur les multiplicateurs des séries orthogonales (with S. Kaczmarz) 389–396 * Sur les fonctions indépendantes III 397–412 * Quelques théorèmes sur les séries et les fonctions 413–417 * Quelques théorèmes sur les sèries orthogonales lacunaires 418–423 * Proof of a gap theorem (with A. Zygmund) 424–427 * A theorem of Lusin (with A. Zygmund) 428–443 * Sur quelques intégrales du type de Dini 444–451 * Un théorème sur l'interpolation 452–453 * Sur les séries de puissances (with A. Zygmund) 454–462 * Sur une propriété de la loi de Gauss 463–469 * Une remarque sur les espaces de M. Besicowitch 470–472 * Sur le problème des moments 473–474 * Sur la sommabilité H_k de séries de Fourier 475–477 * Sur les séries orthogonales 478–500 * Sur les multiplicateurs des séries de Fourier 501–512 * Sur une propriété du mouvement brownien 513–523 * Sur les valeurs limites des fonctions de deux variables complexes (with S. Bergmann) 524–526 * Sur une méthode remarquable de sommation des séries doubles de Fourier 527–538 * Sur l'interpolation d'opérations 539–540 * Quelques inégalités pour les opérations linéaires (with A. Zygmund) 541–546 * On the summability of double Fourier series (with A. Zygmund) 547–556 * Sur les variables aléatoires enroulées 557–558 * Sur la sommabilité forte de séries de Fourier 559–565 * Quelques théorèmes de la théorie des probabilités 566–579 * Sur une nouvelle condition pour la convergence presque partout des séries de Fourier 580–581 * Sur la dérivée seconde généralisée (with A. Zygmund) 582–586 * Sur la convergence absolue des séries de Fourier 587–594 * Sur les sommes riemanniennes (with R. Salem) 595–608 * On the behavior of trigonometric series and power series (with A. Zygmund) 609–654 * Sur les fonctions analytiques de deux variables complexes (with S. Bergmann) 655–673.

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Foreword * Why study systems of linear equations 1–10 * What kind of systems are there? 11–15 * The method of successive elimination 16–33 * Some general remarks on investigating systems determinants 34–49 * The approximate solution of systems by the method of successive approximation 50–64 * The approximate solution of inconsistent systems 65–75 * The graphical solution of systems of linear equations 76–85 * Answers to the exercises 86–88.

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Cotorsion modules 1–65 * Bibliography 66.

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Vorwort.

I, *Lineare Approximationen*: Das allgemeine lineare Approximationsproblem 1–4 * Dichte Systeme 5–12 * Allgemeine Theorie linearer Tschebyscheff-Approxi-

mationen 13–26 * Spezielle Tschebyscheff-Approximationen 27–42 * Abschätzungen der Grössenordnung der Fehlers bei trigonometrischer und bei polynomialer Approximation 43–69 * Polynomapproximationen 70–97 * Numerische Verfahren bei linearen Tschebyscheff-Approximationen 98–124.

II, *Nicht-lineare Approximationen*: Allgemeine Theorie nicht-linearer Tschebyscheff-Approximationen 125–146 * Rationale Approximationen 147–165 * Exponentialapproximationen 166–171.

Literaturverzeichnis 172–178 * Sachverzeichnis 179–180.

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Vorwort * Axiome 9–22 * Mengen 23–49 * Rationale Zahlen 50–72 * Gruppen 73–111 * Verbände 112–134 * Räume 135–156 * Reelle Zahlen 157–177 * Lösungen 178–184 * Literatur 185–186 * Namen- und Sachverzeichnis 187–189.

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Preface * On proofs and definitions 1–7 * Hilbert's system of axioms 8–20 * From the history of the parallel postulate 21–34 * Lemmas 35–46 * The Poincaré model 47–58 * Elementary theorems of hyperbolic geometry 59–70 * Constructions 71–76 * Trigonometry 77–88 * Elliptic geometry 89–96 * Epilog 97–98 * References 99–102 * Index 102–104.

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Preface * The Pythagoreans 2–13 * Archimedes 14–25 * Nicholas of Cusa 26–33 * Blaise Pascal 34–47 * Gottfried Wilhelm Leibniz 48–61 * Carl Friedrich Gauss 62–71 * George Boole 72–85 * Weierstrass and his school 86–91 * Georg Cantor 92–107 * Bibliography 108 * Index of names 109–110.

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(Volume I: See Math. Scand. 7 (1958), p. 270.)

Théorie générale des groupes continus différentiables 1–54 * Invariants dans les espaces abstraits comme solutions d'équations aux F-différentielles partielles 55–68 * Formes différentielles quadratiques et hermitiennes de courbure constante dans les espaces normés 69–80 * Analyse tensorielle générale 81–92 * Théorèmes

d'existence et d'unicité pour les équations différentielles complèment intégrables en F-différentielles à variables dans les espaces normés 93–106 * Appendice: Les vibrations des cordes vibrantes comme études de géodésiques 107–119.

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Preface to second English edition * Preface to first edition * Translator's preface * Equations of fredholm type 3–66 * Symmetric equations (theory of Hilbert–Schmidt) 67–112 * Singular integral equations 113–136 * Dirichlet's problem and its application 137–175 * The biharmonic equation (Application of Green's function) 176–219 * The generalized method of Schwarz 220–242 * Certain applications of integrals analogous to potentials 243–274 * Application of the theory of symmetric integral equations 275–297 * Certain applications of the theory of singular integral equations 298–334 * Bibliography 335–338 * Index 339–341.

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Editor's preface * Foreword * Introduction: Historical review * The operators of mathematical physics 1–39 * Energy convergence 40–73 * The energy method 74–125 * Major applications of the energy method 126–206 * The eigenvalue problem 207–277 * Generalization of the preceding results 278–331 * Estimates of the error of an approximate solution 332–394 * Numerical examples 395–447 * The Bubnov–Gelerkin method 448–490 * The method of least squares 491–543 * Finite difference methods 544–566 * Bibliography 567–576 * Index 577–582.

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Preface * Quadratic forms 1–19 * Multidimensional distributions 20–69 * Miscellaneous results 70–92 * Some applications to Gaussian noise 93–112 * Appendix 1: Identities involving special functions 113–116 * Appendix 2: Multiple integrals 117–120 * Bibliography 121–124 * Index 125–129.

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Preface * Introduction 1–5 * Real numbers, sets, variables, and functions 6–27 * Limits of sequences 28–40 * Limits of functions 41–67 * The role of limits in in-

finite series 68–88 * An important limit: The derivative 89–108 * Another important limit: The definite integral 109–127 * The set of rational numbers 128–131 * The limit of a bounded monotone sequence 132–133 * The limits of the Darboux sums 133–135 * Limits of functions of more than one variable 136–141 * Answers and suggestions 142–146 * Index 147–149.

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Foreword * Introduction 7 * Inequalities involving mean values 9–14 * Bernoulli's inequality 15 * Chebychev's inequality 16–17 * Abel's inequality 18 * The Cauchy–Schwarz–Buniakowski inequality 19 * Young's inequality 20–22 * Jensen's inequality 23–29 * The Fejér–Jackson inequality 30 * Jordan's inequality 30 * Some integral inequalities 31 * Some inequalities for symmetric functions 32–35 * Elementary inequalities 36–52 * Inequalities obtainable from functional considerations 53–76 * Inequalities involving powers and factorials 77–90 * Inequalities involving finite sums and products 91–100 * Inequalities involving trigonometric functions 101–110 * Geometric inequalities 111–119 * Inequalities involving mean values and symmetric functions 120–128 * The inequalities of Cauchy–Schwarz-Buniakowski, Hölder, Minkowski and Chebychev 129–130 * Inequalities involving integrals 131–137 * Inequalities in the complex domain 138–143 * Miscellaneous inequalities 144–159.

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Preface * The algebra of the real numbers 1–35 * Incidence geometry in planes and space 36–39 * Distance and congruence 40–60 * Separation in planes and space 61–73 * Angular measure 74–80 * Congruences between triangles 81–94 * Geometric inequalities 95–101 * The Euclidean program: Congruence without distance 102–113 * Three geometries 114–121 * Absolute plane geometry 122–131 * The parallel postulate and parallel projection 132–141 * Similarities between triangles 142–152 * Polygonal regions and their areas 153–167 * The construction of an area function 168–175 * Perpendicular lines and planes in space 176–185 * Circles and spheres 186–201 * Rigid motion 202–207 * Catesian coordinate systems 208–213 * Constructions with ruler and compass 214–241 * From Eudoxus to Dedekind 242–262 * Length and plane area 263–276 * Jordan measure in the plane 277–290 * Solid mensuration: The elementary theory 291–305 * Hyperbolic geometry 306–347 * The consistency of the hyperbolic postulates 348–366 * The consistency of euclidian geometry 367–381 * The postulational method 382-387 * An example of an ordered field which is not archimedean 388–393 * The theory of numbers 394–399 * The theory of equations 400–403 * Limits of sequences 404–407 * Countable and uncountable sets 408–414 * Index 415–419.

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Preface * Elementary set theory 1–20 * Topological spaces 21–57 * Mappings of topological spaces 58–75 * Compactness 76–94 * Product spaces 95–109 * Metric spaces 110–124 * More on product spaces and function spaces 125–147 * Nets and convergence 148–155 * Peano spaces 156–170 * Index 171–174.

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Preface.

I. *The three-dimensional rotation group and the Lorentz group*: The three-dimensional rotation group 1–17 * The Lorentz group 18–24.

II, *The representations of the three-dimensional rotation group*: The basic concepts of the theory of finite-dimensional representations 25–30 * Irreducible representations of the three-dimensional rotation group in infinitesimal form 31–49 * The realization of finite-dimensional irreducible representations of the three-dimensional rotation group 50–62 * The decomposition of a given representation of the three-dimensional rotation group into irreducible representations 63–88.

III, *Irreducible linear representations of the proper and complete Lorentz group*: The infinitesimal operators of a linear representation of the proper Lorentz group 89–102 * Determination of the infintesimal operators of a representation of the group $\mathscr{K}_+$ 103–119 * The finite-dimensional representations of the proper Lorentz group 120–137 * Principal series of representations of the group $\mathscr{U}$ 138–153 * Description of the representations of the principal series and of spinor representations by means of the unitary group 154–169 * Complementary series of representations of the group $\mathscr{U}$ 170–187 * The trace of a representation of the principal or complementary series 188–209 * An analogue of Plancherel's formula 210–233 * A description of all the completely irreducible representations of the proper Lorentz group 234–296 * Description of all the completely irreducible representations of the complete Lorentz group 297–326.

IV, *Invariant equations*: Equations invariant with respect to rotations of three-dimensional space 327–346 * Equations invariant with respect to proper Lorentz transformations 347–372 * Equations invariant with respect to transformations of the complete Lorentz group 373–381 * Equations derived from an invariant Lagrangian function 382–422.

Appendix 423–439 * References 440–444 * Index 445–447.

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Introduction * Preface * V. S. Nemshinov: The use of mathematical methods in economics 1–32 * V. V. Novozhilov: Cost-benefit comparisons in a socialist economy

33–190 * O. Lange: Some observations in input-output analysis 191–224 * L. V. Kantorovich: Mathematical methods of production planning and organisation 225–280 * L. V. Kantorovich: Further development of mathematical methods and the prospects of their application in economic planning 281–322 * Methods of establishing the shortest running distances for freights on setting up transportation systems 323–356 * Short bibliography on linear programming and related problems 357–368 * Postscript 369–377.

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Préface * Notations et abréviations * Introduction * Espaces de probabilité 1–29 * Intégration des variables aléatoires 30–65 * Espaces produits et fonctions aléatoires 66–98 * Espérances conditionnelles et martingales 99–143 * Théorie ergodique et processus de Markov 144–198 * Bibliographie 197–200 * Index alphabétique 201–203.

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(Volume I: See Math. Scand. 15 (1964), p. 220.)

A note for the reader * Finite differences and the approximate representation of functions 157–191 * Polynomial interpolation 192–228 * Numerical integration and differentiation 229–254 * Ordinary differential equations 255–335 * Partial differential equations 336–369 * Index 370–372.

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Vorwort * Abkürzungen * Grundeigenschaften der reellen Zahlen 9–10 * (135, 195) * Körpereigenschaften der reellen Zahlen 11–15 (135–136, 195–198) * Ungleichungen 16–23 (137–139, 199–207) * Der Funktionsbegriff 24–26 (140, 208–209) * Nullfolgen 27–28 (141, 210) * Grenzwerte von Zahlenfolgen 29–31 (141, 211–212) * Spezielle Sätze und Methoden in der Theorie der konvergenten Zahlenfolgen 32–34 (142, 213–216) * Unendliche Reihen 35–39 (143–144, 217–220) * Grenzwerte von Funktionen eines stetigen Arguments 40–43 (145–146, 221–224) * Stetige Funktionen 44–46 (147, 225–226) * Trigonometrische Funktionen 47–54 (147–149, 227–235) * Definition des bestimmten Integrals 55–56 (150, 236) * Elementare Eigenschaften des Integrals 57–58 (151, 239) * Die Ableitung 59 (151, 239) * Der Zusammenhang zwischen der Ableitung und dem Differenzquotienten einer Funktion 60–61 (151, 240) * Die Fundamentalsätze der Infinitesimalrechnung 62 (152–241) * Ableitungen rationaler Verbindungen gegebener Funktionen 63–66 (152–153, 242–243) * Umkehrung monotoner Funktionen 67–70 (154, 244–247) * Die Kettenregel und ihre Anwendungen 71–75 (155–156, 248–253) * Partielle Integration und Variabelsubstitution 76–83 (157–160, 254–264) * Der Logarithmus und die Exponentialfunktion. Allgemeine Ungleichungen 84–92 (161–164, 165–274) * Differential- und Integralformeln mit Logarithmus- und Exponentialfunktionen. Hyperbolische Funktionen 93–96 (165–166, 275–281) * Integration rationaler Funktionen 97–100 (167–172, 282–287) * Anwendung der ersten Ableitung. Monotonie 101–105 (173–175, 288–295) * Anwendung der ersten Ableitung auf die Bestimmung der Extrema und der Grenzwerte der unbestimmten Ausdrücke 106–114 (176–180, 296–311) * Höhere Ableitungen. Anwendungen der zweiten Ableitung 115–120 (181–186, 312–321) * Darstellungen von Kurven, Tangente und Normale 121–125 (187–188, 322–326) * Bogenlänge 126–128 (189, 327–330) * Potenzreihen für den Logarithmus und den Arcustangens 129 (190, 331) * Die Taylorsche Formel 130–134 (190–194, 332–336).

(Die in Klammern angeführte Seitenzahlen verweisen auf Hinweise und Lösungen).

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Preface * Preface to English edition * Definition and fundamental properties of groups 1–11 * Supplementary comments. Illustrations of the idea of a group 12–29 * The scalar law of a group 30–37 * Subgroups 38–54 * Modules or commutative groups 55–61 * The group $Z, +$ 62–89 * Homomorphisms and quotient groups 90–124 * Permutations 125–149 * Groups with operators 150–170 * Dimension 171–194 * Revision exercises on the book 195–210 * Appendix 211 * Translator's appendix 212–214 * Terminological index 215–218 * List of plates 219–220.

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Preface * Introduction * Fundamental theorems on the mechanics of a free point mass 1–3 * Attractive force of a material body having a spherical structure. Basic earth attractive force 4–8 * Classical two-body problem. Equation of the trajectory of a point moving under the effect of the fundamental force of earth's attraction 9–18 * Relation of the fundamental parameters of Keplerian trajectories to the initial motion conditions 19–23 * Magnitude of the flight velocity 24–25 * Flight altitude 25–26 * Angular flight range 27–36 * Direction of the flight velocity 37–39 * Flight time 40–59 * Identical velocity trajectories. Maximum range trajectory 60–75 * Identical range trajectories. Minimum velocity trajectory 76–87 * Fundamental relations between parameters of the maximum-range and minimum-velocity trajectories 88 * Deviations in the angular flight range caused by initial deviations 89–102 * Deviations in flight time caused by initial deviations 103–111 * References 112.

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Erster Band:

I, *Unendliche Reihen und Folgen*: Das Rechnen mit Potenzreihen 1–9 (152–163) * Reihentransformationen. Ein Satz von Cesàro 10–16 (164–170) * Die Struktur reeller Folgen und Reihen 17–25 (171–179) * Vermischte Aufgaben 26–33 (180–192).

II, *Integralrechnung*: Das Integral als Grenzwert von Rechtecksummen 34–48 (193–207) * Ungleichungen 49–57 (208–214) * Einiges über reelle Funktionen 58–66 (215–229) * Verschiedene Arten der Gleichverteilung 67–76 (230–241) * Funktionen grosser Zahlen 77–83 (242–250).

III, *Funktionen einer komplexen Veränderlichen. Allgemeiner Teil*: Komplexen Zahlen und Zahlenfolgen 84–92 (251–264) * Abbildungen und Vektorfelder 93–103 (265–275) * Geometrisches über den Funktionsverlauf 104–112 (276–282) * Cauchyscher Integralsatz. Prinzip vom Argument 113–123 (283–299) * Folgen analytischer Funktionen 124–135 (300–319) * Das Prinzip vom Maximum 136–150 (320–336).

Namenverzeichnis 337–338.

Zweiter Band:

IV, *Funktionen einer komplexen Veränderlichen. Spezieller Teil*: Maximalglied und Zentralindex, Maximalbetrag und Nullstellenanzahl 1–12 (168–184) * Schlichte Abbildungen 13–28 (185–204) * Vermischte Aufgaben 29–36 (205–220).

V, *Die Lage der Nullstellen*: Der Satz von Rolle und die Regel von Descartes 37–54 (221–241) * Geometrisches über die Nullstellen von Polynomen 55–65 (242–252) * Vermischte Aufgaben 66–74 (253–265).

VI, *Polynome und trigonometrische Polynome*: 75–97 (266–298).

VII, *Determinanten und Quadratische Formen*: 98–116 (299–322).

VIII, *Zahlentheorie*: Zahlentheoretische Funktionen 117–131 (323–338) * Ganzzahlige Polynome und ganzwertige Funktionen 132–137 (339–350) * Zahlentheore-

tisches über Potensreihen 138–147 (351–366) * Einiges über algebraische ganze Zahlen 148–155 (367–375) * Vermischte Aufgaben 156–161 (376–387).

IX, *Einige geometrische Aufgaben*: 162–167 (388–401).

Namenverzeichnis zum II. Band 402–403.

Sachverzeichnis zu beiden Bänden 404–406 * Berichtigungen 407.

(Die in Klammern angegebenen Seitenzahlen verweisen auf die Lösungen.)

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Prefatory note * Introduction 1–7 * The maximum principle 8–71 * Proof of the maximum principle 72–110 * Linear time-optimal processes 111–177 * Miscellaneous problems 178–224 * The maximum principle and the variational calculus 225–242 * Optimal processes with restricted phase coordinates 243–297 * A statistical optimal control problem 298–333 * References 334–336 * Index 337–338.

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Preface * Introduction 1–3 * The Dam process in discrete time 4–16 * The Dam process in continuous time 17–32 * Some further models 33–42 * References 43–48.

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Preface * Introduction.

I: Logical deficiencies in euclidean geometry 3–19 * Euclid's parallel postulate 20–36 * Neutral geometry 37–52 * Introduction to non-euclidean geometry 53–89 * The logical consistency of the non-euclidean geometries 90–104 * The empirical validity of the non-euclidean geometries 105–116.

II: Theory of incidence 119–135 * Incidence geometries. Models of the theory of incidence 136–166 * Theory of affine geometries 167–183 * Theory of order on the line 184–215 * Planar and spatial order properties 216–239 * Angles and order of rays 240–262 * Separation properties of angles and triangles 263–286 * Theory of congruence. Introduction to euclidean geometry 287–315 * Congruence without numbers.—An introduction 316–338.

List of postulates 339–342 * Index 343–350.

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Theoretical papers, 62 pp.: Kurt Spielberg: On the fixed charge transportation problem, 13 pp. * Herschel H. Loomis: The delay of synchronous logic nets, 10 pp. * Ralph L. London: A computer program for discovering and proving recognition rules for Backus normal form grammers, 7 pp. * W. H. Burge: The evaluation, classification, and interpretation of expressions 22 pp.

Real-time systems, 37 pp.: James J. Doengan, Calvin Packard, and Paul P. Pashby: Experiences with the Goddard computing system during manned spaceflight missions, 8 pp. * J. E. Hamlin: A general discription of the national aeronautics and space administration real time computing complex, 22 pp. * W. I. Adams and P. R. Federico: Cadfiss test system computation and data flow integrated subsystem tests, 7 pp.

Invited papers I, 20 pp.: Anthony G. Oettinger: A bull's-eye view of management and engineering information systems 14 pp. * Franz L. Alt: The standardization of programming 6 pp.

Language processors, 34 pp.: R. G. Franciotti and Marjorie P. Lietzke: The organization of the share ALGOL 60 translator, 10 pp. * Ned Chapin: Implementation of IPL-V on a small computer, 6 pp. * D. V. Schorre: META II—A syntax-directed compiler writing language, 11 pp. * Frederick W. Schneider and Glen D. Johnson: META 3—A syntax-directed compiler-writing compiler to generate efficient code, 7 pp.

Pattern recognition, 57 pp.: Richard V. Smith: Similarity and entropy, 7 pp. * Rebecca C. Prather and Leonard M. Uhr: Discovery and learning techniques for pattern recognition, 10 pp. * R. D. Joseph, S. S. Viglione, and H. F. Wolf: Cloud pattern recognition, 17 pp. * Charles M. Vossler and Neil M. Branston: The use of context for correcting garbled Englich text, 13 pp. * L. N. Kanal and N. C. Randall: Recognition system design by statistical analysis, 10 pp.

Numerical analysis and related topics, 22 pp.: Duane Pyle: Generalized inverse computations using the gradient projection method, 1 p. * Allan W. Dickinson: Nonlinear optimization: Some procedures and examples, 8 pp. * D. B. MacMillan: Iterative methods for solution of diffusion problems in $r-\Theta$ geometry, 5 pp. * Aviezri S. Fraenkel: On the size of modular numbers, 8 pp.

Operating systems, 46 pp.: Manfred Kory and Paul Berning: The STL integrated computed operating system, 25 pp. * John A. Gosden: The operations control center multi-computer operating system, 10 pp. * G. F. Leonard and J. R. Goodroe: An environment for an operating system, 11 pp.

Invited papers II, 5 pp.: M. V. Wilkes: Lists and why they are useful, 5 pp.

Multiple console systems, 13 pp.: D. Fife and R. Rosenberg: Queueing in a memory-shared computer, 13 pp. * F. J. Corbato, John McCarthy, A. J. Perlis, and Jules Schwartz: Panel discussion on time sharing.

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Formal manipulation of mathematical expressions, 19 pp.: E. Bond, M. Aus-

lander, S. Grisoff, R. Kenney, M. Myszewski, J. Sammet, R. Tobey, and S. Zilles: Formac—An experimental formula manipulation compiler, 19 pp.

Computer science curriculum, 5 pp.: E. J. Schweppe: A proposed academic program in the computer sciences, 2 pp. * Sam D. Conte: The computer sciences program at Purdue University, 1 p. * Richard S. Varga: Computer technology at Case, 2 pp.

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EDP standardization, 4 pp.: Richard E. Utman: The standardization process * Vico E. Henriques: Current position and future of EDP standards, 2 pp. * Thomas B. Steel, Jr.: The ACM and standardization, 2 pp.

Administration of university computing centers, 6 pp.: John Hamblen: Procedures and policies on use of and rates charged for computing center facilities, 1 p. * Bernard A. Galler: Procedures for submission and processing of jobs, 2 pp. * Saul Rosen: Compilers for student jobs, 2 pp. * George E. Forsythe: Automatic machine grading programs, 1 p.

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Citation Index, 10 pp. * Participant index, 1 p.

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Introduction * G. Adomian: Stochastic Green's functions 1–39 * A. T. Bharucha-Reid: On the theory of random equations 40–69 * G. C. Rota: Reynolds operators 70–83 * Victor Twersky: On propagation in random media of discrete scatters 84–116 * W. C. Hoffman: Wave propagation in a general random continuous medium 117–144 * J. B. Keller: Stochastic equations and wave propagation in random media 145–170 * Richard Bellman: Stochastic transformations and functional equations 171–177 * K. B. Gray: The application of stochastic approximation to the optimazation of random circuits 178–192 * E. W. Montroll: Random walks on lattices 193–220 * Emanuel Parzen: On statistical spectral analysis 221–246 * W. L. Root: Stability in single detection problems 247–263 * Eugene Wong: The construction of a class of stationary Markoff processes 264–276 * David Blackwell: Probability bounds via dynamic programming 277–280 * Cyrus Derman: Markovian sequential decision processes 281–289 * J. M. Richardson: Application of truncated hierarchy techniques 290–302 * Author index 303–306 * Subject index 307–318.

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List of contributors * List of symposia * Preface * Milton S. Plesset: Bubble dynamics 1–18 * Joseph B. Keller: Growth and decay of bubbles 19–29 * P. R. Garabedian: Flow around a bubble rising in a tube 30–33 * J. W. Westwater: Measurements of bubble growth during mass transfer 34–54 * Marshall P. Tulin: Supercavitating flows and practical applications 55–79 * G. I. Taylor: Cavitation in hydrodynamic lubrication 80–101 * Garrett Birkhoff: Free boundary problems for viscous flows in channels 102–121 * Donald F. Hays and James B. Feiten: Cavities between moving parallel plates 122–137 * Leif Floberg: Cavitation in lubricating oil films 138–146 * I. Prigogine: The molecular theory of surface tension 147–163 * T. Brooke Benjamin: Surface effects in non-spherical motions of small cavities 164–180 * Subject index 181–189.

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Preface * List of participants * The scientific programme * R. A. Beumont and R. S. Pierce: Quasi-isomorphism of p-groups 13–28 * B. Charles: Méthodes topologiques en théorie des groupes abéliens 29–42 * A. L. S. Corner: On a conjecture of Pierce concerning direct decompositions of Abelian groups 43–48 * V. Dlab: A generalization of dependence relations 49–50 * E. Fried: On the subgroups of an Abelian group that are ideals in every ring 51–56 * L. Fuchs: Some generalizations of the exact sequences concerning Hom and Ext 57–76 * J. de Groot: Additive groups of integer-valued functions over topological spaces 77–80 * Kin-ya Honda: On the structure of Abelian p-groups 81–86 * J. M. Irwingfand S. A. Khabbaz: On generating subgroups of Abelian groups 87–98 * H. Leptin: Einige Bemerkungen über die Automorphismen Abelscher p-Gruppen 99–104 * F. Loonstra: A-ordering of the group $\mathrm{Ext}(B, A)$ 105–114 * R. J. Nunke: On the structure of Tor 115–124 * R. S. Pierce: Endomorphism rings of primary Abelian groups 125–138 * A. D. Sands: Factorization of cyclic groups 139–146 * E. A. Walker: Quotient categories and quasi-isomorpisms of Abelian groups 147–162.

Symposium held at Brown University, Providence, R.I., April 3–5, 1963: *Stress waves in anelastic solids.* (Editors: Herbert Kolsky and William Prager.) (International Union of Theoretical and Applied Mechanics.) Berlin · Göttingen · Heidelberg, Springer-Verlag, 1964. 11 + 342 pp. DM 67.50.

Preface * N. Davids and H. H. Calvit: Some dynamical applications of shock waves propagation in solids 1–19 * G. E. Duvall: Propagation of plane shock waves in a stress-relaxing medium 20–32 * J. Jacquesson: Précurseurs élastoplastiques d'ondes de choc en milieu solide isotrope 33–42 * S. Kaliski and W. Nowacki: Propagation of magentoelastic distrubances in viscoelastic bodies 43–53 * J. D. Camp bell and D. B. Taylor: On the numerical solution of a wave propagation in the theory of dislocation motion 54–68 * R. Mazet: Vibrations d'une structure lamellaire sous fluage localisé 69–77 * G. N. Savin: Dynamics of an anelastic sting of variable length 78–86 * H. Zorski: Anelastic waves in thin plates 87–100 * G. Bianchi: Some ex-

perimental and theoretical studies on the propagation of longitudinal plastic waves in a strain-rate-dependent material 101–117 * N. Cristescu: Some problems of the mechanics of extensible strings 118–132 * H. G. Hopkins: Mechanical waves and strain-rate effects in metals 133–148 * S. C. Hunter and I. A. Johnson: The propagation of small amplitude elastic-plastic waves in pre-stressed cylindrical bars 149–165 * J. F. Bell: The initiation of finite amplitude waves in annealed metals 166–182 * E. David, R. Schall and H. Schardin: Visualisation of wave propagation in impulse-loaded bars 183–192 * J. S. Rinehart: Transient stress wave boundary interactions 193-206 * N. V. Zvolinsky and O. V. Rykov: Reflection of a plastic wave at an obstacle 207–221 * C. C. Chao and J. D. Achenbach: A simple voscoelastic analogy for stress waves 222–238 * E. H. Lee and D. T. Liu: An example of the influence of yield on high pressure wave propagation 239–254 * J. Miklowitz: Pulse propagation in a viscoelastic solid with geometric dispersion 255–276 * W. Goldsmith and C. F. Austin: Some dynamic characteristics of rocks 277–303 * S. S. Grigorian: On some simplifications in the description of the motion of a soft soil 304–313 * W. Heierli: One-dimensional inelastic wave propagation in soils: Experimental and theoretical investigations 314–330 * J. Mandel: Propagation des surfaces de discontinuité dans un milieu élastoplastique 331–340 * Subject index 341–342.

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Preface * Wolfgang Wasow: Asymptotic expansions for ordinary differential equations: Trends and problems 3–26 * Hugh L. Turrittin: Solvable related equations pertaining to turning point problems 27–52 * Robert M. Lewis: Asymptotic methods for the solution of dispersive hyperbolic equations 53–108 * Robert W. McKelvey: Asymptotic solutions and indefinite boundary value problems 109–128 * C. C. Lin: Some examples of asymptotic problems in mathematical physics 129–144 * Yasutaka Sibuya: On the problem of turning points for systems on linear ordinary differentiel equations of higher orders 145–162 * Frank W. J. Olver: Error bounds for asymptotic expansions, with an application to cylinder functions of large argument 163–184 * Robert A. Clark: Asymptotic solutions of elastic shell problems 185–210 * Arthur Erdélyi: The integral equations of asymptotic theory 211–230 * Nicholas D. Kazarinoff: Application of Langer's theory of turning points to diffraction problems 231–244 * Index 245–249.

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Foreword 7–8 * List of participants 9–12

Lectures:

K. Čulík: Applications of graph theory to mathematical logic and linguistics

13–20 * G. A. Dirac: Generalizations of the five colour theorem 21–28 * P. Erdös: Extremal problems in graph theory 29–36 * M. Fiedler: Some applications of the theory of graphs in matrix theory and geometry 37–42 * T. Gallai: Critical graphs 43–46 * F. Harary: On the reconstruction of a graph from a collection of subgraphs 47–52 * H. Izbicki: An edge colouring problem 53–62 * A. Kotzig: Hamilton graphs and Hamilton circuits 63–82 * C. St. J. A. Nash-Williams: On well-quasi-ordering trees 83–84 * G. Ringel: Extremal problems in the theory of graphs 85–90 * H. Sachs: On regular graphs with given girth 91–98 * A. A. Zykov: Recursively calculable functions of graphs 99–108.

Communications:

A. Ádám: Some open problems of the switching circuit theory 109–112 * J. Blažek, M. Koman: A minimal problem concerning complete plane graphs 113–118 * J. Bosák: The graphs of semigroups 119–126 * G. A. Dirac: Extensions of the theorems of Turán and Zarankiewicz 127–132 * P. Erdös: Some applications of probability to graphs theory and combinatorial problems 133–136 * B. Mišek: Pólya's fundamental formula and incidence matrices 137–142 * J. W. Moon: Simple paths on polyhedra 143–144 * J. Sedláček: Some properties of interchange graphs 145–150 * A. A. Zykov: Graphtheoretical results of Novosibirsk mathematicians 151–156.

Problems 157–170 * Bibliography 171–234.

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E. P. Wigner: The Role of invariance principles in natural philosophy * M. Froissart: The proof of dispersion relations 1–39 * E. P. Wigner: Causality, R-matrix, and collision matrix 40–67 * D. Y. Wong: Dispersion relations and applications 68–96 * R. G. Newton: The nonrelativistic angular-momentum plane 97–134 * P. V. Landshoff: Production amplitudes 135–166 * R. Oehme: High-energy scattering and dispersion theory 167–251 * Appendix: Some simple rules for the asymptotic behaviour of Feynman integrals 252–256.

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Contributors * Preface * P. Wegner: Introduction 1–9 * General survey 10–19 * Programming language 20–72 * Fortran implementation 73–100 * Algol implementation 101–165 * Commercial languages and compilers 166–213 * Programming systems 214–248 * Advanced programming techniques 249–312 * Appendix 313–316.

B. Randell and L. J. Russell: *ALGOL 60 implementation. The translation and use of ALGOL 60 programs on a computer.* (A.P.I.C. studies in

data processing 5.) London · New York, Academic Press, 1964. 14+ 418 pp. sh. 84/–.

Foreword * Preface.

I, *ALGOL compilers*: Introduction 3–7 * ALGOL translation techniques 8–21 * Translation of arithmetic expressions 22–33 * The Whetstone compiler 34–46.

II, *The object program*: Assignment statements 48–61 * Blocks and procedures 62–79 * Arrays 80–88 * Labels and switches 89–96 * Parameters 97-124 * For statements 125–134 * Code procedures 135–140.

III, *The translator*: Introduction 143–148 * Translator stack 149–172 * Name list 173–192 * Translation techniques 193–239 * Translator routines 240–247.

References 248–252 * Appendix I: A worked example 253–268 * Appendix II: Restrictions imposed on ALGOL 60 in KDF9 ALGOL 269–271 * Appendix III: The KDF9 computer 272 * Appendix IV: KDF9 ALGOL hardware representations 273–275 * Appendix V: Implementation of program testing facilities 276 * Appendix VI: Implementation of segmentation 277 * Appendix VII: Object program operations 278–280 * Appendix VIII: State variables 281 * Appendix IX: Details of the various implementations of the Whetstone compiler 282–283 * Appendix X: Control routine flow diagrams 284–324 * Appendix XI: Translator flow diagrams 325–414 * Index 415–418.

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Erster Teil: *Fernbilder.*

I, *Anschauliche Bilder*: Zentral- und Fernbilder 6–11 * Axonometrie 12–19 * Die Ellipse 20–27 * Perspektivität und Affinität 28–33.

II, *Zugeordnete Risse*: Risse und Spuren 34–39 * Schnittaufgaben 40–45 * Massaufgaben 46–51 * Kreisaufgaben 52–57.

III, *Anschauliche Risse*: Normale Axonometrie 58–63 * Spezialfälle der normalen Axonometrie 64–69 * Einfache Kurven und Flächen zweiter Ordnung 70–77 * Umrisse 78–83.

IV, *Einfache Flächen*: Drehflächen 84–89 * Kegelflächen 90–95 * Böschungsflächen 96–101 * Schraubenflächen 102–107.

V, *Durchdringungen*: Kugelverfahren für Drehflächen 108–113 * Pendelebenenverfahren für Kegelflächen 114–119 * Zerfallende Durchdringungskurven 120–125 * Pendelebenenverfahren für Kugeln und Regelflächen 126–133.

Zweiter Teil: *Zentralbilder.*

VI, *Distanzpunktperspektive*: Fluchtpunkte 134–139 * Fluchtlinien 140–145 * Distanzpunkte und Frontansicht 146–151 * Zylinder und Kugel 152–161.

VII, *Messpunktsperspektive*: Messpunkte 162–167 * Eckansicht 168–177 * Lotfluchtpunkt und Kippansicht 178–185 * Bildausmessung 186–193.

VIII, *Gebundene Perspektive*: Sehstrahlverfahren 194–201 * Drehsehnenverfahren 202–213 * Reliefperspektive 214–219 * Aus alten Büchern 220–225.

Literaturauswahl 226 * Anmerkungen 227 * Sach- und Namenverzeichnis 228–235.

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Preface * Definitions and elementary properties 1–22 * Duality and the Hahn–Banach theorem 23–43 * Topologies on dual spaces and the Mackey–Arens theorem 44–64 * Barrelled spaces and the Banach–Steinhaus theorem 65–75 * Inductive and projective limits 76–100 * Completeness and the closed graph theorem 101–126 * Some further topics 127–141 * Compact linear mappings 142–154 * Bibliography 155–156 * Index 157–158.

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Preface.

I, *Fundamental methods and applications*: Differential equations and their solutions 1–21 * First-order equations for which exact solutions are obtainable 22–56 * Applications of first-order euqations 57–81 * Explicit methods of solving higher order linear differential equations 82–133 * Applications of second-order linear differential equations with constant coefficients 134–176 * Series solutions of linear differential equations 177–219 * Systems of linear differential equations 220–258.

II, *Further methods, fundamental theory, and special topics*: Approximate methods of solving first-order equations 259–292 * The Laplace transform 293–325 * The theory of first-order equations 326–359 * The theory of linear differential equations 360–413 * Sturm–Liouville problems and Fourier series 414–457 * Nonlinear differential equations 458-515 * Partial differential equations 516–572.

Answers to odd-numbered exercises 573–588 * Bibliography 589–590 * Index 591–594.

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Einleitung * Der Gegenstand der Bedienungstheorie 4–38 * Grundbegriffe der Bedienungstheorie 39–77 * Einige Bedienungsaufgaben und ihre Lösung 78–168 * Schlussbemerkungen 169–178 * Anhang 179–183 * Literatur 184–185.

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Peface * Introduction * Partially ordered sets 1–2 * Further definitions 2–3 * First definition of a lattice 4–5 * Second definition of a lattice 6–7 * Elementary properties 7–11 * Modular lattices 12–16 * Applications to abstract algebra 17–19 * Metric lattices 20–21 * Distributive lattices 22–26 * Reducibility and independence 27–29 * Complemented lattices 30–31 * Boolean algebras 31–32 * Canonical form

of a boolean polynomial 33–37 * Boolean rings 38–39 * Morphisms 40–41 * Direct products 42–45 * Propositional calculus 46–52 * The truth ideal 53–54 * Power sets 55–57 * Switching circuits 58–63 * Bridge circuits 64–65 * Boolean matrices 65–67 * Boolean determinants 68–70 * Multi-terminal networks 71–73 * Brouwer algebras 74–76 * Pseudo-completements 77–78 * Intuitionist logic 78–79 * De Morgan's laws in Brouwer algebras 80–81 * Atomic lattices 82–85 * Semi-modular lattices 86–88 * Atoms in a semi-modular lattice 89–90 * Geometric lattices 91–94 * Algebraic geometry 95–96 * Closure 97–100 * Kuratowski spaces 101–103 * Neighbourhoods 104–105 * Metric spaces 106–109 * Exercises 110–115 * Index 116–117.

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Preface * Introduction * Kinematics 3–17 * The nature of force 18–45 * Dynamics of a particle 47–102 * Dynamics of rigid bodies 103–160 * Generalised coordinates 161–204 * Index 205–206.

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Preface * L. Nirenberg: Partial differential equations with applications in geometry 1–41 * Marshall Hall, Jr.: Generators and relations in groups—The Burnside problem 42–92 * R. H. Bing: Some aspects of the topology of 3-manifolds related to the Poincaré conjecture 93–128 * Lars Gårding: Partial differential equations: Problems and uniformization in Cauchy's problem 129–150 * L. Ahlfors: Quasiconformal mappings and their applications 151–164 * J. Milnor: Differential topology 165–183.

(Volume I: See Math. Scand. 13 (1963), p. 263.)

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Preface * Determinants. Preliminary illustrations and definitions 1–9 * Reductions and calculation of determinants 10–20 * Multiplication of determinants 21–28 * Minor and reciprocal determinants 29–33 * Symmetrical and skew symmetrical determinants 34–47 * Descriminating symmetrical determinants 48–56 * Symmetric functions 57–65 * Order and weight of eliminants 66–75 * Expression of eliminants ad determinants 76–90 * Determination of common roots 91–98 * Discriminants 99–107 * Linear transformations 108–123 * Formation of invariants and covariants 124–136 * Symbolical representation of concomitants 137–149 * Canonical forms 150–160 * Systems of quantics 161–174 * Applications to binary quantics 175–226 * Applications to higher binary quantics 227–283 * On the order of restricted systems of equations 284–313 * Applications of symbolic methods 314–337 * Notes 338–373 * Index 373–376.

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Einleitung 7–8 * Abbildung von Punkten, Geraden und Ebenen in senkrechter Parallelprojektion 9–32 * Die Ellipse 33–45 * Die Kegelschnitte 46–56 * Drehkörper. Ihre Schnitte, Durchdringungen und Umrisse 57–71 * Röhrenflächen 72–77 * Schraubenlinien und Schraubenflächen 78–85 * Axonometrie 86–94 * Abriss der Zentralperspektive 95–106 * Perspektive mit geneigter Bildebene (nach Dr. Stohler) 107–116 * Anhang: 129 Abbildungen.

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Preface.

I, *General theorems about solutions of differential systems*: Integral curves 1–10 * Lipschitzian and Carathéodory systems 11–15 * The solution (t, t_0, x^0) of the system (1.1.1) 16–24 * Periodic solutions 25–27 * Autonomous systems 28–31 * Complements 32–33 * Bibliography.

II, *Particular plane autonomous systems*: The linear case 37–48 * Homogeneous systems 49–65 * The analytic case 66–83 * The problem of the center 84–97 * Singular points at infinity 98–106 * Complements 107–109 * Bibliography 110.

III, *The singularities of Briot-Bouquet*: Theorem of Briot-Bouquet for the analytic case 111–117 * Reduction of differential equations with an isolated singular point to a typical form in the analytic case. The theorem of I. Bendixon on the behaviour of the trajectories of the reduced equations of the second type 118–126 * Equation of Briot-Bouquet in the nodal case in the real domain. Theorems of A. Wintner 127–133 * Complements 134–138 * Bibliography 139–140.

IV, *Plane autonomous systems*: Limiting sets 141–158 * Plane cycles 159–177 * Isolated singular points 178–187 * The index 188–196 * The cylinder as phase space 197–199 * The torus as phase space 200–203 * A short account on dynamical systems 204–205 * Bibliography 206–209.

V, *Autonomous plane systems with perturbations*: Homogeneous perturbed systems 210–225 * Isolated singular points of systems of class C^1. Elementary points 226–234 * An asymptotic study of a node with two tangents and a saddle point of H. Weyl 235–255 * Isolated singular points of systems of class C^1. Non-elementary points 256–268 * Structurally stable systems. Systems with a parameter 269–273 * Bibliography 274–276.

VI, *On some autonomous systems with one degree of freedom*: Trajectories of the equation of linear motion of a point under viscous resistance 277 * The equation $\ddot{\theta}+a\dot{\theta}+\sin\theta-\beta=0$, $a\geqq 0$, $\beta\geqq 0$ 278–302 * Equations of van der Pol and Liénard of the oscillations of relaxation 303–320 * Periodic solutions of the generalized equation of Liénard 321–333 * Periodic solutions of the equation $\ddot{x}+f(\dot{x})x+g(x)=0$ without the hypothesis $xg(x)>0$ for $|x|>0$ 334–340 * The equation of damped vibrations: $A\ddot{x}+f(\dot{x})\dot{x}+Cx=0$ 341–343 * On an equation of dynamics and aerodynamics of wires 344–349 * Complements 350–352 * Bibliography 353–358.

VII, *Non-autonomous systems with one degree of freedom*: The problem of forced

oscillations. Linear case 359–362 * The fixed point theorem of L. E. J. Brouwer and the theorems of M. L. Cartwright, J. E. Littlewood and J. L. Massera 363–370 * Theorems of T. Yoshizawa 371–388 * Harmonic solutions out of phase of the equation $\ddot{x} = F(x, \cos t)$. Theorem of F. John 389–398 * The equation $\ddot{x} + f(\dot{x})\dot{x} + g(x) = p(t)$ 399–407 * The equation $\ddot{x} + F(\dot{x}) + x = p(t)$ 408–412 * Theorems of G. E. H. Reuter on the equations $\ddot{x} + kf(x)\dot{x} + g(x) = kp(t)$, $\ddot{x} + kF(\dot{x}) + g(x) = kp(t)$ 413–417 * The equation $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = p(t)$ 418–422 * Non-linear systems with subharmonic solutions 423–427 * General remarks concerning periodic solutions 428 * Complements 429–434 * Bibliography 435–440.

VIII, *Linear systems*: The adjoint system. The inequalities of T. Wazewski 441–443 * Linear autonomous systems with constant coefficients 444–450 * Linear periodic systems 451—454 * Reducible systems 455–456 * Type numbers of a function. Relation of t-similitude 457–461 * Regular systems 462–465 * Periodic solutions 466–471 * Complements 472 * Bibliography 472–475.

IX, *Stability*: The method of V functions 476–487 * Stability of linear systems 488–501 * Stability in the first approximation 502–513 * Asymptotic equivalence 514–519 * Complements 520–522 * Bibliography 523–530.

Index 530–533.

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Editorial preface * Foreword * Author's preface.

I, *Construction of net equations*: Introduction 3–17 * Absolutely unstable net equations 18–21 * Six-point symmetric equation 22–28 * Asymmetric net equations 29–42 * Alternating method 43–51 * Method of mean arithmetic, and multi-nodal symmetric method 52–63 * Comparison between explicit and implicit equations, and the "implicity-explicit" methods 64–72 * Spherical and cylindrical regions 73–82 * Equations of increased accuracy 83–100 * Net equations with fictitious nodes 101–107 * On bilateral approximations 108–115 * Two-dimensional and three-dimensional equations 116–129 * Two-dimensional and multi-dimensional net equations of increased accuracy 130–146 * Non-uniform nets 147–152 * Multi-step equations 153–162 * General case of variable and discontinuous coefficients 162–169 * Parabolic equations of higher than the second order 170–183 * Non-linear equations 184–194 * Conclusions 195–202.

II, *Solution of net equations*: Introduction 203–205 * "One-dimensional" elliptic net equations 206–211 * Direct methods 212–223 * Ill-conditioned net matrices 224–229 * Simplest iterative method 230–240 * Variational methods 241–249 * Methods using Chebyshev polynomials 250–259 * Iterative methods of the second degree 260–267 * Iterative methods of the nth degree 268–279 * Methods of successive displacements 280–298 * Method of block iteration 299–314.

Appendix: On the application of Chebyshev polynomials to parabolic net equations 314–320.

References 321–338 * Index 338–344 * Volumes published in this series 245–246.

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Vorbemerkung * Vorwort zur Originalausgabe * Vorwort zur deutschen Ausgabe * Einführung 1–54.

I, *Theorie der Kurven*: Theorie der ebenen Kurven 55–71 * Raumkurven 72–89.

II, *Theorie der Flächen*: Vorbemerkungen 90–106 * Die zentroaffine Theorie der Flächen 107–119 * Die Flächentheorie in der Geometrie der äquiaffinen Gruppe 120–174 * Einige spezielle Flächenklassen 175–196 * Regelflächen 197–209 * Relative Flächentheorie 210–216 * Kongruenzen von Geraden im äquiaffinen Raum 217–227 * Anhang: Über die innere Geometrie zweiter Art auf einer Hyperfläche des affinen Raumes 228–242 * Literaturverzeichnis 243–268 * Sachregister 269–275.

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Preface * Introduction 1–23 * Isomorphism theorems 24–47 * Transformations and subgroups 48–63 * Direct sums 64–88 * Abelian groups 89–130 * p-groups and p-subgroups 131–149 * Supersolvable groups 150–173 * Free groups and free products 174–208 * Extension 209–254 * Permutation groups 255–297 * Symmetric and alternating groups 298–318 * Representations 319–371 * Products of subgroups 372–425 * The multiplicative group of a division ring 426–440 * Topics in infinite groups 441–458 * Bibliography 459–470 * Index of notation 471–474 * Index 475–479.

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Foreword * Preface * Introduction: Varieties of approaches * Quantum phenomenology 1–14 * Canonical quantization 15–30 * Quantization and relativistic wave equations 31–45 * General structure of Bose–Einstein fields 46–57 * The clothed linear field 58–72 * Representations of the free field 73–84 * Interacting fields: Quantum electrodynamics 85–96 * New approaches and problems 97–107 * Bibliography 108–112 * Appendix: Group representations in Hilbert space (by G. W. Machey) 113–130 * Index 131.

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Preface * Errata * Introduction 1–7 * The symplectic group 8–15 * The symplectic

metric 16–24 * Discontinuous groups 25–29 * Hermitian forms 30–35 * The fundamental domain of the modular group 36–43 * The fundamental domain of the group $\Delta(\mathfrak{G}, \mathfrak{H})$ 44–49 * The volume of the fundamental domain of the modular group 50–64 * Commensurable groups 65–73 * Unit groups of quinary quadratic forms 74–86.

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On the borders of geometry and arithmetric 7–24 * What we know and what we do not know about prime numbers 25–97 * One hundred elementary but difficult problems in arithmetic 98–122 * References 123–126.

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Preface * Divisibility and indeterminate equations of first degree 7–34 * Diophantine analysis of second higher degrees 35–109 * Prime numbers 110–155 * Number of divisors and their sum 156–185 * Congruences 186–227 * Euler's totient function and the theorem of Euler 228–263 * Representation of numbers by decimals in a given scale 264–281 * Continued fractions 282–314 * Legendre symbol and Jacobi's symbol 315–333 * Mersenne numbers and Fermat numbers 334–350 * Representations of natural numbers as sums of non-negative kth powers 351–399 * Some problems of the additive theory of numbers 400–415 * Complex integers 416–447 * Bibliography 448–468 * Author index 469–473 * Subject index 474–476.

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Preface * Preface to the second edition * Terminology and notation 1–2 * Finite joins and meets 3–53 * Infinite joins and meets 54–190 * Appendix 191–211 * Bibliography 212–230 * List of symbols 231 * Author index 232–234 * Subject index 235–237.

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Volume I, *Elementary calculus.*

Introduction * Prefaces to the eighth and sixteenth Russian editions.

I, *Functional relationship and the theory of limits*: Variables 1–42 * The theory of limits. Continuous functions 43–100.

II, *Differentiation: Theory and applications*: Derivatives and differentials of the first order 101–120 * Derivatives and differentials of higher orders 121–127 * Application of derivatives to the study of functions 128–157 * Functions of two variables 158–163 * Some geometrical applications of the differential calculus 164–200 * Exercises 201–212.

III, *Integration: Theory and applications*: Basic problems of the integral calculus. The indefinite integral 213–236 * Properties of the definite integral 237–254 * Applications of definite integrals 255–290 * Further remarks on definite integrals 291–304 * Exercises 305–314.

IV, *Series. Applications to approximate evaluations*: Basic theory of infinite series 315–331 * Taylor's formula and its application 332–358 * Further remarks on the theory of series 359–391 * Exercises 392–397.

V, *Functions of several variables*: Derivatives and differentials 398–418 * Taylor's formula. Maxima and minima of functions of several variables 419–439 * Exercises 440–443.

VI, *Complex numbers. The foundations of higher algebra. Integration of various functions*: Complex numbers 444–479 * Basic properties and evaluations of the zeros of integral polynomials 480–505 * Integration of various functions 506–519 * Answers 520–536.

Index 537–543.

Volume II, Advanced calculus.

Introduction * Preface to the sixth edition * Preface to the fourteenth edition.

I, *Ordinary differential equations*: Equations of the first order 1–39 * Differential equations of higher orders; systems of equations 40–76.

II, *Linear differential equations. Supplementary remarks on the theory of differential equations*: General theory; equations with constant coefficients 77–127 * Integration with the aid of power series 128–140 * Supplementary notes on the theory of differential equations 141–163.

III, *Multiple and line integrals. Improper integrals that depend on a parameter*: Multiple integrals 164–204 * Line integrals 205–241 * Improper integrals and integrals that depend on a parameter 242–278 * Supplementary remarks on the theory of multiple integrals 279–297.

IV, *Vector analysis and field theory*: Basic vector algebras 298–309 * Field theory 310–348.

V, *Foundations of differential geometry*: Curves on a plane and in space 349–366 * Elementary theory of surfaces 367–396.

VI, *Fourier series*: Harmonic analysis 397–428 * Supplementary remarks on the theory of Fourier series 429–463 * Fourier integrals and multiple Fourier series 464–475.

VII, *The partial differential equations of mathematical physics*: The wave equation 476–535 * The equation of telegraphy 536–558 * The vibrations of rods 559–567 * Laplace's equation 568–602 * The equation of thermal conduction 603–624.

Index 625–630.

Volume III_1, *Linear algebra.*

Introduction * Preface to the fourth Russian edition * Determinants. The solution of systems of equations 1–69 * Linear transformations and quadratic forms 70–187 * The basic theory of groups and linear representations of groups 188–322.

Index 323–325.

Volume III_2, *Complex variables. Special functions.*

Introduction * Foreword to the fourth edition.

I, *The bais of the theory of functions of a complex variable* 1–119.

II, *Conformal transformation and the two-dimensional field* 120–222.

III, *The application of the theory of residues, integral and fractional functions* 223–312.

IV, *Functions of several variables and matrix functions* 313–356.

V, *Linear differential equations* 357–492.

VI, *Special functions*: Spherical functions 493–536 * Bessel functions 537–583 * The Hermitian and Laguerre polynomials 584–603 * Elliptic functions 604–669.

Supplement: *The conversion of matrices into the canonical form* 670–696.

Index 697–700.

Volume IV, *Integral equations and partial differential equations.*

Introduction * Preface to the second edition * Preface to the third edition.

I, *Integral equations* 1–190.

II, *The calculus of variations* 191–301.

III, *Fundamental theory of partial differential equations*: First order equations 302–381 * Equations of higher orders 382–479 * Systems of equations 480–511.

V, *Boundary value problems*: Boundary value problems for an ordinary differential equation 512–567 * Equations of the elliptic type 568–725 * Equations of the parabolic and hyperbolic type 726–806.

Index 807–811.

Volume V, *Integration and functional analysis.*

Introduction * Preface.

I, *The Stieltjes integral* 1–80.

II, *Set functions and the Lebesgue integral*: Set functions and the theory of measure 80–112 * Measurable functions 113–126 * The Lebesgue integral 127–208.

III, *Set functions. Absolute continuity. Generalization of the integral* 209–256.

IV, *Metric and normed spaces* 257–366.

V, *Hilbert space*: The theory of bounded operators 367–482 * Space l_2 and L_2 483–532 * Unbounded operators 533–632.

Index 633–636.

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Translations editor's preface * Author's prefaces to the first and third editions * Derivations of the fundamental equations 1–21 * The formulation of problems of mathematical physics. Hadamard's example 22–32 * The classification of linear

equations of the second order 33–45 * The equation for a vibrating string and its solution by d'Alembert's method 46–57 * Riemann's method 58–71 * Multiple integrals: Lebesgue integration 72–125 * Integrals dependent on a parameter 126–132 * The equation of heat conduction 133–145 * Laplace's equation and Poisson's equation 146–154 * Some general consequences of Green's formula 155–165 * Poisson's equation in an unbounded medium. Newtonian potential 166–170 * The solution of the Dirichlet problem for a sphere 171–179 * The Dirichlet problem and the Neumann problem for a half-space 180–187 * The wave equation and the retarded potential 188–201 * Properties of the potentials of single and double layers 202–221 * Reduction of the Dirichlet problem and the Neumann problem to integral equations 222–227 * Laplace's equation and Poisson's equation in a plane 228–236 * The theory of integral equations 237–257 * Application of the theory of Fredholm equations to the solution of the Dirichlet and Neumann problems 258–264 * Green's function 265–290 * Green's function for the Laplace operator 291–300 * Correctness of formulation of the boundary-value problems of mathematical physics 301–326 * Fourier's method 327–341 * Integral equations with real, symmetric kernels 342–356 * The bilinear formula and the Hilbert–Schmidt theorem 357–375 * The inhomogeneous integral equation with a symmetric kernel 376–381 * Vibrations of a rectangular parallelepiped 382–387 * Laplace's equation in curvilinear coordinates. Examples of the use of Fourier method 388–400 * Harmonic polynomials and spherical functions 401–413 * Some elementary properties of spherical functions 414–420 * Index 421–428.

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Preface * Homographies in one dimension 1–20 * Geometry of points in a line 21–37 * Geometry and invariants 38–71 * Homogeneous coordinates in two dimensions 72–90 * Homography in two dimensions 91–123 * Geometry in the projective plane 124–155 * Collineations and correlations in the plane 156–181 * The conic 182–224 * Non-euclidean geometries 225–251 * Higher-dimensional geometry 252–296 * Index 297–299.

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Preface * Introduction 1–10 * An operator algebra for differential equations 11–19 * The analysis of linear time-variable feedback systems via signal-flow graph theory 20–29 * Synthesis techniques for linear time-variabel feedback systems 30–41 * The approximation problem 42–59 * Reducibility of linear systems 60–74 * Approximations of differential equations 75–88 * Appendixes 89–106 * References 107–108.

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Preface * Symbolizing sentences 1–42 * Logical inference 43–109 * Truth and validity 110–160 * Truth tables 161–179 * Terms, predicates, and universal quantifiers 180–210 * Universal specification and laws of identity 211–240 * A simple mathematical system: axioms for addition 241–262 * Universal generalization 263–272 * Index 273–274.

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Introduction 1–4 * Chaines de Markov 4–35 * Processus de Markov 36–61 * Processus non markoviens 62–85 * Solutions des problèmes 86–185 * Références 186–189 * Index alphabétique 190.

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Preface.

I, *Formulating business problems*: Formalizing business problems 3–23 * Problems involving progressions and series 24–52 * Mathematics of interest 53–88.

II, *One-decision variable optimization models*: Management science models 89–115 * Optimization of functions of one variable 116–155 * Models involving polynomials and rational functions 156–179 * Models involving algebraic and transcendental functions 180–207 * Models involving integration and differential equations 208–243.

III, *Two-decision variable optimization models*: Optimization of functions of two-decision variables 245–279 * Examples of optimization with two-decision variables 280–294 * Optimization subject to constraints 295–324.

IV, *Linear systems*: Introduction to matrices and linear systems 325–356 * Matrix inversion and linear systems 357–386 * Examples of the use of matrices in business problems 387–424 * Linear programming 425–468 * Examples of linear programming 469–502 * Special cases of linear programming 503–542.

V, *Multivariate optimization models*: Optimization models with many variables 543–568 * Examples of optimization models with many variables 569–591 * Quadratic forms 592–609 * Dynamic programming 610–636.

Appendices 637–694 * Answers to selected exercises 695–708 * Author index 709–710 * Subject index 711–713.

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Commemoration of Dr. Antonín Spacek 9–10 * List of papers by Antonín

Spacek 11–12 * Preface 13 * Contents 14–16 * O. Barndorff-Nielsen: Characteristic subsequences and limit laws for weighted means 17–28 * R. Bartoszyński: On a certain concept of value of information in games 29–34 * O. Bunke: Bedingte Strategien in der Spieltheorie: Existenzsätze und Anwendung auf statistische Entscheidungsfunktionen 35–44 * J. Cigler und L. Schmetterer: Über die Summe Markowscher Ketten auf endlichen Gruppen 45–54 * F. Csáki: Simplified derivation of optimum transfer functions in the Wiener–Newton sense 55–62 * I. Csiszár: Über topologische und metrische Eigenschaften der relativen Information der Ordnung α 63–74 * H. Dinges: Einige Verteilungen, die mit dem Wienerprozess zusammenhängen 75–84 * V. Fabian: A new one-dimensional stochastic approximation method for finding a local minimum of a function 85–106 * W. Fieger: Zwei Verallgemeinerungen der Palmschen Formeln 107–122 * L. A. Gardner, Jr.: Adaptive predictors 123–192 * T. L. Grettenberg: The ordering of finite experiments 193–206 * O. Hanš: On conditional bayes loss 207–222 * O. Hanš and J. Křepela: Producer-consumer aspects of acceptance sampling by attributes 223–264 * J. Havel: Devices for checking and further transformations of the output signal from the random process generator 265–276 * H. Heyer: Über neuere Ergebnisse aus der Theorie der Wahrscheinlichkeitsmasse auf lokalkompakten und Lieschen Gruppen 277–284 * Hu Guo Ding: On the Shannon theorem and its converse for sequences of communication schemes in the case of abstract random variables 285–332 * M. Jiřina: Branching processes with measure-valued states 333–358 * D. A. Kappos: Strukturtheorie der Räume von Zufallsvariablen 359–376 * J. Kerstan: Teilprozesse Poissonscher Prozesse 377–404 * H. Klinger: Ein Wahrscheinlichkeitsmodell für die symmetrische, lineare Irrfahrt mehrerer Teilchen 405–416 * E. S. Kočetkov: On estimates of the mean value of a stochastic process 417–428 * Z. Koutský: Periodische Verzweigungsfolgen 429–440 * K. Krickeberg: Wahrscheinlichkeitsoperatoren von Verteilungen in Vektorräumen 441–452 * L. Kubát: Note on the reliability of parallel redundant systems 453–460 * L. Kubik: On analogies between two classes of probability distributions: The class of infinitely decomposable distributions and the class of selfdecomposable distributions 461–466 * R. G. Laha and E. Lukacs: On identically distributed stochastic integrals 467–474 * P. Mandl: Über die asymptotischen Verteilungen der Erst Passage Zeiten 475–498 * G. Marsaglia: Random variables and computers 499–512 * K. Matthes: Zur Theorie der Bedienungsprozesse 513–528 * J. Nedoma: The synchronization for ergodic channels 529–540 * K. R. Parthasarathy: A note on McMillan's theorem for coutable alphabets 541–544 * A. Perez: Extensions of Shannon–McMillan's limit theorem to more general stochastic processes 545–574 * V. S. Pygačev: On the theory of detection and estimation represented by stochastic functions 575–582 * C. Rajski: On the normed information rate of discrete random variables 583–586 * Rosenblatt and M. Roth: Some questions connected with a non-homogeneous Markow chain of stochastic variables 587–598 * Ju. A. Rozanov: Some problems for stationary processes on a finite interval which reduce to integral equations of the Wiener–Hopf type 599–604 * Ju. A. Rozanov: On the equivalence of probability measures corresponding to Gaussian stationary processes 605–610 * O. Šefl: Some questions concerning the automatic control of processes with unknown characteristics 611–620 * I. Selin: The design of a sequential test for the detection of known signal in normal non-white noise 621–628 * D. W. C. Shen and Aaron Rosenberg: A nonlinear stochastic learning model 629–636 * Shen Shi Yi: The fundamental problem of a stationary channel 637–640 * Z. Šhidák:

Eigenvalues of operators in denumerable Markow chains 641–656 * A. H. Širjaer: On the theory of desicion functions and control process of an observation relative to preliminary data 657–682 * V. A. Statuljavičus: Limit theorems and their refinement for additive stochastic functions and sums of weakly dependent variables 683–688 * V. Strassen: Asymptotische Abschätzungen in Shannons Informationstheorie 689–724 * A. Tortrat: Répartition asymptotique des fonctions presque périodiques de Besicovitch $f \in B_p$ 725–742 * K. Urbanik: The principle of increase of entropy in quantum mechanics 743–764 * K. Winkelbauer: On discrete information sources 765–830 * W. Winkler: Stetigkeitseigenschaften der Realisierungen Gaussscher zufälliger Felder 831–840 * F. Zítek: Quelques remarques au sujet de l'entropie du tchèque 841–846.

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Preface * The explanation of the laws of thermodynamics 1–31 * Theory of the non-ideal gas 32–60 * Remarks on the condensation problem 61–76 * The Boltzmann equation 77–88 * The propagation of sound 89–101 * The Chapman–Enskog development 102–117 * The kinetic theory of dense gases 118–142 * Appendix: E. W. Montroll: Quantum statistics of interacting particles 143–178 * Index 179–181.

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Preface * Basic concepts 1–18 * Hyperplanes and the separation theorem 19–30 * The Minkowski metric 31–46 * Some characterizations of convex sets, and local convexity 47–56 * The support function and the dual cone 57–68 * Helly-type theorems 69–90 * Further characterizations of convex sets 91–104 * Properties of S determined by conditions on each set of r points of S 105–116 * Maximal convex subsets and upper semicontinuous decompositions 117–128 * Convex functions 129–134 * Special boundary points 135–140 * Some properties of convex sets in L_n and E_n 141–162 * Exercises, propositions, and problems 163–194 * Appendix 195–208 * Bibliography 209–232 * Index 233–238.

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Preface * Introduction 1–3 * Successive approximations 4–6 * Achilles and the tortoise 7–8 * Division on digital computers 9–11 * Extracting square roots by successive approximation 12–18 * Extracting kth roots by successive approximation 19–21 * The derivative of a polynomial 22–24 * Newton's method for the approximate solution of algebraic equations 25–28 * The geometric interpretation of the derivative 29–31 * The geometric interpretation of Newton's method 32–34 * The derivative of more general functions 35–36 * The calculus of derivatives 37–39

* Finding a first approximation 40–42 * The method of chords 43–47 * The method of iteration 48–50 * The geometric interpretation of the method of iteration 51–55 * The speed of convergence of a sequence of successive approximations 56–59 * The solution of a system of linear equations by successive approximation 60–63 * Successive approximations in geometry 64–68 * Exercises 69 * Answers 70.

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Vorwort * Bezeichnungen * Einleitung 1–10 * Volterra-Integralgleichungen 11–53 * Gewöhnliche Differentialgleichungen 54–107 * Volterra-Integralgleichungen in mehreren Veränderlichen. Hyperbolische Differentialgleichungen 108–157 * Parabolische Differentialgleichungen 158–250 * Literaturverzeichnis 251–264 * Namenverzeichnis 265–266 * Sachverzeichnis 267–269.

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Preface * Numeral systems 1–22 * Sets and numbers 23–33 * Operations and their properties 34–63 * Computations: The applications of properties 64–90 * Special methods. Division theorem 91–105 * Primes and divisibility 106–124 * Fractions 125–158 * Sentences 159–173 * Applications 174–188 * Negative numbers. Rational numbers 189–216 * Real numbers 217–235 * Measurement: A mathematical approach 236–258 * Measurement in practice 259–271 * Mathematical systems 272–289 * Mathematical method 290–302 * Answers to selected exercises 303–325 * Index 326–328.

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Vorwort * Einfürung 1–3 * Lebesguesche Zerlegung von Massen aus $A^{\perp}$ 4–8 * Die Räume $H^{P}(\lambda)$ 9–12 * Eine Formal für Masse in $A^{\perp}$ 13–15 * Die Algebren $P(X)$ 16–20 * Der Satz von Mergelyan 21–22 * Die Klassen für $P(X)$ 23–25 * Beschränkte analytische Funktionen 26–28 * Literatur 29–30.

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Preface 3 pp.

I, *Concept and topology of Riemann surfaces:* Weierstrass' concept of an analytic function 1–4 * The concept of an analytic form 4–12 * The relation between the concepts "analytic function" and "analytic form" 12–16 * The concept of a two-dimensional manifold 16–25 * Examples of surfaces 25–32 * Specialization; in par-

ticular differential and Riemann surfaces 32–44 * Orientation 44–50 * Covering surfaces 50–60 * Differentials and line integrals; homology 60–72 * The intersection number 79–92.

II, Functions on Riemann surfaces: The Dirichlet integral and harmonic differentials 93–106 * Scheme for the construction of the potential arising from a doublet source 106–112 * The proof 112–118 * The elementary differentials 118–126 * The symmetry laws 126–131 * The uniform functions F as a subspace of the additive and multiplicative functions on $\hat{F}$; the Riemann-Roch theorem 131–140 * Abels theorem; the inversion problem 140–153 * The algebraic function field 153–159 * Uniformization 159–167 * Riemann surfaces and non-Euclidean groups of motions; fundamental regions; Poincaré θ-series 167–179 * The conformal mapping of a Riemann surface onto itself 179–186.

Index 187–191.

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Vorwort * Mengenalgebra 1–26 * Boolesche Algebra 27–46 * Symbolische Logik und Aussagenalgebra 47–82 * Schaltalgebra 83–114 * Relaisschaltungen und Steuerprobleme 115–151 * Rechenschaltungen 152–172 * Einführung in die Wahrscheinlichkeitsrechnung in endlichen Stichprobenräumen 173–190 * Lösungen ausgewählter Übungsaufgaben 191–201 * Literaturverzeichnis 202 * Register 203–207.

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Preface * Fundamental concepts 1–18 * Multiplicity of transitive groups 19–43 * The transitive constituents of G 44–51 * The method of Schur 52–77 * Relationship with representation theory 78–103 * Bibliography 104–110 * Author index 111 * Notation used in text 112 * Subject index 113–114.

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Preface * Acknowledgments * Introduction 1–12 * Linear space 13–45 * Separation 46–51 * Paranorms and seminorms 52–71 * Semimetric space 72–80 * Linear semimetric space 81–97 * Banach space 98–119 * Hilbert space 120–135 * Topology and nets 136–166 * Linear topology 167–94 * The closed graph theorem 195–214 * Local convexity 215–232 * Duality 233–250 * Banach algebra 251–278 * Appendix 279–280 * Glossary 281–282 * Index 283–288 * Table 289–291.

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Preface * List of axioms, definitions and theorems * Logic 1–31 * Sets 32–49 * Arithmetic and the real numbers 50–87 * Groups and fields 88–104 * Algebra 105–167 * Geometry 168–228 * Analysis 229–295 * Answers to exercises 296–310 * Index 311–316.

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Introduction 1–2 * Cumulative plotting 3–9 * Quality control—decision procedures 10–27 * Quality control—corrective action 28–37 * "Post-mortem" investigations 38–45 * Other applications 46–52 * Appendix 1. Warning note procedure for control of finish content of staple fibre 53–56 * Appendix 2. Input/output schedule for computer programme "Cusum control" 57 * Appendix 3. Input/output schedule for computer programme "Autocorners" 58–59 * Appendix 4. Operating instructions for acceptance/rejection procedures 60–62 * Appendix 5. Operating instructions for production stream sequential procedure 63–64 * References 65 * Glossary of abbreviations and symbols 66.

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Preface to the second edition * Preface to the first edition * Heuristic introduction to the discrete memoryless channel 1–5 * Combinatorial preliminaries 6–13 * The discrete memoryless channel 14–32 * Compound channels 33–54 * The discrete finite-memory channel 55–63 * Discrete channels with a past history 64–89 * General discrete channels 90–108 * The semicontinuous memoryless channel 109–123 * Continuous channels with additive Gaussian noise 124–132 * Mathematical miscellanea 133–139 * Group codes. Sequential decoding 140–151 * References 152–154 * Index 155 * List of channels studies or mentioned 156.

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Preface * Translator's note * General theory of quadratic curves 1–27 * General theory of quadratic surfaces 28–63 * Linear transformations and matrices 64–144 * Appendix 1: Vectors. Operations on vectors and some of their properties 145–148 * Appendix 2: Elements of the theory of determinants 149–162 * Subject index 163–164.

NEW MATHEMATICAL JOURNALS

Foundations of language, International journal of language and philosophy, is published quarterly. Volume 1, no. 1 was issued in January 1965. The journal is intended to be a medium for information about different approaches to problems including those based on logic and mathematics. Board of Editors: Morris Halle, Peter Hartmann, K. Kunjunni Raja, Benson Mates, J. F. Staal, Pieter A. Verburg, John W. M. Verhaar. Subscription price: Dfl. 42.00. Orders should be sent to: D. Reidel Publishing Co., P. O. Box 17, Dordrecht, The Netherlands.

Kibernetika is published bimonthly by the Ukrainian Academy of Sciences, Kiev. The journal will publish original and survey papers, reports on meetings, book reviews, and information about progress in cybernetics in the Soviet Union and abroad. There will be sections as follows: General cybernetic questions, methodology; theory of automata and algorithms; theory of electronic numerical and analogue machines; algorithmic languages and programming theory; methods for investigating operations and systems; theory of optimization; simulation of reflective processes; information and languages systems; mathematical linguistics. Editor-in-chief: B. M. Glučkov. Address of the journal: Ul. Lusogorskaja 4, Kiev 28. Price per issue: 65 Kopek.

Kybernetika is published bimonthly by the Czechoslovak Academy of Sciences. Papers are printed in Czech, Russian and (many) in English language. Volume 1 (1965) contains 568 pages. Subscription price: US $ 8.40 per year. Editor-in-chief: Jaroslov Kožešnik. Address: Institute for Information Theory and Automation, ČSAV; Vyšehradská 49, Praha 2.

Približennye metody rišennija differenčial'nyh uravneniĭ [Approximation methods for solving differential equations] is published by the Mathematical Institute of the Ukrainian Academy of Sciences, Kiev. The journal addresses itself to a wide circle of research mathematicians, aspirants, students and engineers interested in the theory of differential equations and its applications. Volume 1 (1963) and volume 2 (1964) contain 156 and 176 pages, respectively. Editor-in-chief: Ju. A. Mitropolskiĭ. Address of the journal: Mathematical Institute, Academy of Sciences of the USSR, Kiev.

Publications du Département mathematiques, Faculté des Sciences, Lyons, publishes research papers in duplicated form. Année 1965, tome 2, fasc. 2 contains two articles (74 pp). Address: Département de mathématiques, Faculté des Sciences, 15 Quai Claude Bernard, Lyon, France.

MATH. SCAND. 16 (1965), 262

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MATH. SCAND. 16 (1965), 263–277

MATHEMATICIANS AT SCANDINAVIAN INSTITUTIONS OF HIGHER EDUCATION AND RESEARCH

JULY 1, 1965

For every university, institute of technology, etc., its mathematicians and their positions are listed, followed by those among its institutes or laboratories having mathematicians as staff members. A number in brackets after a person's name indicates the institute or laboratory in question.

In case a mathematician holds two positions under two headings of the list his name appears at one of them, and the other position is indicated in brackets. Appropriate indication is also given in case a mathematician holds a second scientific position not listed here.

The list of persons comprises both pure and applied mathematicians, appropriate specification being given only in the latter case.

Translations of names of positions are not given in the list. This is due to the fact that in most cases it is difficult to compare positions in Scandinavia with positions elsewhere. It is even difficult to compare positions in one Scandinavian country with those in another; in some cases a position in one Scandinavian country has the same name as a position of quite a different character in another.—Translations of other Scandinavian terms occurring in the list have been considered unnecessary.

The address of an institute or laboratory is given only when different from that of the university or institution to which it adheres.

Previous lists: Math. Scand. 1 (1953), pp. 176–183; 3 (1955), pp. 189–197; 5 (1957), pp. 175–183; 7 (1959), pp. 287–297; 8 (1960), pp. 257–267; 9 (1961), pp. 313–325; 10 (1962), pp. 286–299; 12 (1963), pp. 279–292; 14 (1964), pp. 280–294.

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University of Umeå

Umeå

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Lektor	H. Kulldorff (2)
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KUNGL. TEKNISKA HÖGSKOLAN

The Royal Institute of Technology

Stockholm 70

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	L. G. Borg (2)
	G. Dahlquist (2)
	H. A. S. Eriksson (5)
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Lektor	A. Ramgard (3)
	L. Ullemar (2)
	M. Essén (2)
	M. Giertz (2)
	Å. Björck (2)
	..... (2)
	..... (2)
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	..... (5)
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Docent	L. Ingelstam (2)
	L. E. Zachrisson (8)

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(2) INSTITUTIONEN FÖR MATEMATIK
(3) INSTITUTIONEN FÖR MATEMATISK FYSIK
(4) INSTITUTIONEN FÖR MEKANIK I

(5) INSTITUTIONEN FÖR MEKANIK II
(6) INSTITUTIONEN FÖR TEKNISK HYDROMEKANIK
(7) INSTITUTIONEN FÖR TEORETISK ELEKTROTEKNIK
(8) INSTITUTIONEN FÖR OPTIMERINGSLÄRA OCH SYSTEMTEORI

CHALMERS TEKNISKA HÖGSKOLA

Chalmers' Institute of Technology

Göteborg

Professor emeritus	G. Hössjer
Professor	B. E. H. Bergström (1) B. Holmberg (2) H. O. Kreiss (4) S. Lundqvist (2) N. F. V. Svartholm (2) H. Wallman (3) V. Thomée (1) ..... (1)
Laborator	A. E. Broman (1) J. Nilsson (2) A. Sjölander (2) ..... (1)
Lektor	O. Brander (2) F. Eriksson (1) R. Grahn (2) I. Hansson (2) B. Nyman (1) J. Petersson (1) L. Råde (1) G. Wahde (1) E. Yhland (2) ..... (4)

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LUNDS TEKNISKA HÖGSKOLA

Lund Institute of Technology

Lund

Professor	G. Blom (1) P. O. Brundell (4) S. G. Nilsson (2) J. Peetre (1) U. Uhlhorn (2)

Lektor E. Andersén (2)
I. Brinck (1)
T. Ekman (3)
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MATH. SCAND. 16 (1965), 278–285

NOTES

JANUARY 1–JUNE 30, 1965

INTERNATIONAL

The 1966 International Congress of Mathematicians will be held in Moscow from Thursday, August 16, to Friday, August 26. The scientific sessions of the Congress will take place in the auditoria of Moscow University on Lenin Hills.

The scientific programme of the Congress includes: *Plenary sessions*, at which about 20 one-hour reports will be read, by invitation. *Section sessions*, at which about 70 half-hour reports will be read, by invitation, most of the time allotted to these sessions being devoted, however, to brief scientific communications by the participants of the Congress. The following 15 sections will be working at the Congress:

1. *Mathematical logic and foundations of mathematics.*
2. *Algebra.*
3. *Theory of numbers.*
4. *Classical analysis.*
5. *Functional analysis.*
6. *Ordinary differential equations.*
7. *Partial differential equations.*
8. *Topology.*
9. *Geometry.*
10. *Algebraic geometry and complex manifolds.*
11. *Probability and statistics.*
12. *Applied mathematics and mathematical physics.*
13. *Mathematical problems of control systems.*
14. *Numerical mathematics.*
15. *History and pedagogical questions.*

The Chairman of the organizing Committee is I. G. Petrovsky. Secretary General: V. G. Karmanov. Address of the Congress: Moscow University, Moscow, V-234, SSSR.

SCANDINAVIAN

A Scandinavian Summer School in Mathematics, arranged by the Swedish National Committee for Mathematics, will be held June 10–30, 1966 at Sigtuna, Sweden.

Supported by the Scandinavian Cultural Foundation and the Mittag–Leffler Foundation the establishing of this school represents a joint Scandinavian effort to stimulate the education of young research mathematicians. A similar summer school is likely to be held each year in one of the Scandinavian Countries.

This year the School will concentrate on *Harmonic Analysis*. Its staff will include the following mathematicians: L. Carleson, Uppsala; E. Hewitt, Seattle; J.-P. Kahane, Paris; W. Rudin, Madison; N. Varopoulos, Paris. Lectures and seminars will be held in English. The number of participants will be limited to appr. 40.

A few non-Scandinavian research students will be admitted to the school. More detailed information can be obtained from the Secretary of the Summer School: Professor Y. Domar, Matematiska institutionen, Sysslomansgatan 8, Uppsala, Sweden.

From the beginning of the fall term 1965 Y. Domar, University of Uppsala, will substitute Å. Pleijel as editor of Mathematica Scandinavica.

DENMARK

Appointments at the University of Copenhagen: E. Sparre Andersen of the University of Aarhus to a professorship from January 1, 1966. Olaf Schmidt to a professorship in the history of the exact sciences. A. Brøndsted to a position of amanuensis. G. Kjærgaard Pedersen to a position of kandidatstipendiat.

Appointments at the University of Aarhus: L. Kristensen and E. Thue Poulsen to professorships in mathematics from August 1, 1965. Kiyosi Ito of the University of Kyoto to a long term guestprofessorship from August 1, 1966. K. Fenchel to a position of lektor. Dr. E. Balslev and A. Kock to positions of universitetsadjunkt. F. Abildgaard Jørgensen, J. Hoffmann Jørgensen and J. Munkholm temporarily to positions of amanuensis. O. Jørsboe, Ib Madsen, N. H. Stetkær-Hansen and J. Vesterstrøm to positions of kandidatstipendiat. K. Pedersen to a position of kandidatinstruktor.

Appointments at the Technical University of Denmark, Copenhagen: E. B. Hansen to a professorship in applied mathematics. O. Brændstrup, V. Güntelberg, R. Hartelius, O. Moltke-Leth, F. Nielsen, I. Thygesen and P. Wulff Pedersen to positions of lektor. F. Hjortgaard Christensen, Hanne Christiansen and R. Jackson to positions of amanuensis.

Appointments at the Royal Danish College of Education, Copenhagen: T. Heiede and J. Lichtenberg to positions of lektor.

Professor A. Aaboe, Yale University, will spend September 1965 at the University of Aarhus as a visiting staff member.

E. Chi, Columbia University, will spend the academic year 1965/66 at the University of Aarhus, supported by the U. S. National Science Foundation.

Professor Ph. C. Curtis, on leave of absence from the University of California, Los Angeles, will spend the academic year 1965/66 at the University of Copenhagen.

Professors B. Eckmann, Zürich, and P. Hilton, Cornell University, will visit the University of Aarhus two weeks of August 1965 in order to do joint research.

Dr. E. G. Effros, Columbia University, will spend the first half of the academic year 1965/66 at the University of Aarhus, supported by the U. S. National Science Foundation.

Professor J. Glimm, Massachusetts Institute of Technology, will spend the academic year 1965/66 at the University of Aarhus as a visiting staff member, also supported by the Danish National Science Foundation.

Professor R. Kadison, on sabbatical leave of absence from the University of

Pennsylvania with half support from the U. S. National Science Foundation, will spend the academic year 1965/66 at the University of Aarhus.

Lektor E. T. Kehlet, on leave of absence from the University of Copenhagen, will spend the fall term of 1965 at the University of Aarhus.

G. Kjærgaard Pedersen, kandidatstipendiat at the University of Copenhagen, will spend the academic year 1965/66 at the University of Aarhus.

Dr. R. Lavine, Massachusetts Institute of Technology, will spend the academic year 1965/66 at the University of Aarhus as a visiting staff member.

Professor N. Oler, on leave of absence from the University of Pennsylvania, will spend the academic year 1965/66 at the University of Aarhus, supported by the U. S. National Science Foundation.

Dr. E. Störmer, universitetsstipendiat at the University of Oslo, will spend the academic year 1965/66 at the University of Aarhus.

Dr. S. Thomeier, on leave of absence from Frankfurt University, will spend the academic year 1965/66 at the University of Aarhus as a visiting staff member.

Dr. G. Yeo, University of Sheffield, will spend July and September 1965 at the University of Aarhus as a visiting staff member.

A computing department equipped with a GIER computer has been established at the University of Aarhus. Dr. B. Mayoh has been appointed afdelingsleder from August 1, 1965.

Northern Europe University Computing Center (NEUCC) has been established at the Technical University of Denmark, Building 305, Lyngby. It is epuipped with an IBM 7090. H. J. A. Helms has been appointed managing director.

Guest Lectures.

University of Copenhagen:

March 26 — Z. Ciesielski, University of Poznań and University of Aarhus, *Capacitory potentials and Brownian motion.*

April 23 — J. Mandel, National Bureau of Standards, Washington, D. C., *Fitting families of curves, with applications in the physical sciences.*

April 30 — M. G. Arsove, University of Washington, Seattle, and University of Hamburg, *A conformal mapping technique for infinitely connected regions.*

The Technical University of Denmark:

May 24 — A. Bruce Clarke, University of Michigan, Ann Arbor, *Functions of Markov chains, alternating renewal processes, and semi-Markov processes.*

University of Aarhus:

January 14 — H. Brøns, University of Copenhagen, *Maximum-likelihood estimation.*

January 14 — I. E. Segal, Massachusetts Institute of Technology, *Holomorphic functionals.*

February 18 — W. Gaschütz, Universität Kiel, *Existenz äusserer Automorphismen bei p-Gruppen.*

February 25 — Z. Ciesielski, University of Poznań, *Capacitory potentials and Brownian motion.*

March 4	E. Størmer, University of Oslo, *Jordan algebras of selfadjoint operators in Hilbert space.*
March 25	E. T. Kehlet, University of Copenhagen, *Introductory survey of C*-algebras.*
April 13	K. Rao Gade, University of Chicago, *Unstable homotopy of O(n).*
May 6	J. Vesterstrøm, University of Copenhagen, *A primitive representation of locally compact groups.*
May 19	B. H. Mayoh, University of Oslo, *Simple structures defined on a transitive and reflexive graph.*
May 24, 25 and 26	O. Schmidt, University of Copenhagen, *Topics in the history of mathematics and astronomy.*
May–June	Z. Semadeni, University of Poznań, *Selected topics in functional analysis and categories.* A series of lectures.
June 14	D. Anderson, Massachusetts Institute of Technology, *SU-Cobordism.*

Lectures at meetings of the Danish Mathematical Society.

At the Mathematical Institute, University of Copenhagen:

February 1	L. Mejlbo, *Kommutatorrelationerne i kvanteteorien* [*The commutator relations in quantum theory*].
March 8	G. Goodman, Stanford University and University of Aarhus, *Schlicht functions and optimal control theory.*
April 8	A. L. Schmidt, *Farey-trekanter i den komplekse plan* [*Farey triangles in the complex plane*].
May 24	Z. Semadeni, University of Poznań and University of Aarhus, *Universal morphisms.*

At the Mathematical Institute, University of Aarhus:

January 7	W. K. Hayman, Imperial College of Science and Technology, London, *The spherical derivative.*

NORWAY

Professor E. B. Schieldrop died on the 12th of June, 1965.

Appointments at the University of Oslo: Jon Reed and Johan Aarnes to positions of lektor. John Johnsen and Eivind Stensholt to positions of research assistants. Appointments at the Technical University of Norway, Trondheim: Arnljot Höyland to a position of dosent in mathematical statistics. Sigbjörn Halvorsen and Anders Lödemel to positions of lektor.

Professor Karl E. Aubert has spent the spring term 1965 as a visiting professor at the University of Washington, Seattle, U.S.A. Professor Sigmund Selberg will spend the academic year 1965–1966 as a visiting professor at the University of Colorado, Boulder, U.S.A. Professor Erik Hemmingsen, Syracuse University, Syracuse, N.Y., U.S.A. will spend the academic year 1965–1966 as a visiting professor at the Technical University of Norway. Universitetsstipendiat Jens E. Fenstad will spend the academic year 1965–1966 as research associate at Stanford University, U.S.A. Ph. D. Brian Mayoh has spent the academic years 1963–65 as a research assistant at the University of Oslo.

Professor E. S. Selmer, University of Bergen, spent the academic year 1964–1965 in Cambridge, England.

Guest lectures.

University of Oslo:

March 15, 16, 17 — H. Bühlmann, Zürich, *Which is the false coin? A problem from information theory. Exchangeable random variables and the central limit theorem. Theory of games.*

March 23, 24, 30 — P. H. Randolph, Purdue, University, *Optimal selection.*

March 29 — P. H. Randolph, Purdue, University, *The role of statistics in operations research.*

March 31 — P. H. Randolph, Purdue University, *Theory of teams.*

May 5 — B. Epstein, New Mexico State University, *A characterization of the conformal moduls of a doubly-connected domain.*

University of Bergen:

June 9, 10 — A. Simon, San Diego, California, *On finite Larmor radius equations.*

The Technical University of Norway, Trondheim:

May 4, 5 — B. Epstein, University of New Mexico and U.S. Office of Naval Research, London, *A three-point difference equation. A characterization of the conformal modulus of a doubly connected domain.*

Lectures at meetings of the Norwegian Mathematical Society:

March 30 — B. Fuglede, University of Copenhagen, *En anvendelse av spillteorien på potentialteorien* [*An application of the theory of games to potential theory*].

June 10 — D. Amble and P. Ofstad, *Kan en maskin læres opp til å spille bondesjakk?* [*Can a machine be trained to play „Five in a row" ?*].

FINLAND

A new professorship has been established at the University of Turku.

Appointments: H. Haahti, biträdande professor, University of Oulu. B. Sjöberg, lektor and docent, Åbo Academy. P. Paatero, observator, and T. Kuusalo, J. Segercrantz, K. Vala, assistants, University of Helsinki. A. Tietäväinen and J. Tienari, assistants, Universities of Turku and Oulu, resp.

O. Jussila will spend the academic year 1965–66 at Harvard University. O. Tammi will be visiting the Institute of Technology, Haifa, from November 1965 to February 1966. G. Elfving will spend the Spring term 1966 at Stanford University.

Lectures at meetings of the Mathematical Society of Finland:

January 27 — S. Turski, Universität Warschau, *Über die Lösung des Riquierproblems für die biharmonische Gleichung* $\Delta^2 u = 0$.

February 10 — J. Väisälä, *Eräita monotonisten funktioiden yleistyksiä useampiulotteisissa avaruuksissa* [*Some generalizations of monotonic functions in many-dimensional spaces*].

March 10 — F. Nevanlinna, *Seulametodeista alkulukututkimuksessa* [*On sieve methods in prime number theory*].

April 6 P. Dedecker, Université de Lille, *Non abelian homological algebra.*

April 28 A. Dinghas, Freie Universität Berlin, *Zentralsymmetrische Lösungen der Differentialgleichung* $|\varphi_{z_i\bar{z}_k}| = g(\varphi)$.

May 5 E. Stiefel, Eidgenössische Technische Hochschule, Zürich, *Konforme Abbildungen zur Regularisierung des Dreikörperproblems.*

May 13 A. B. Clarke, University of Michigan, *Functions of Markov chains.*

SWEDEN

From the beginning of the academic year 1965/66 Ch. Lech will be professor in mathematics at the University of Stockholm, C.-E. Fröberg professor in numerical analysis at the University of Lund, V. Thomée professor in mathematics at Chalmers Institute of Technology. H. Kulldorff has been appointed professor in statistics at the University of Umeå; it is expected that he will be professor in mathematical statistics at the same university. Arne Nordlander holds the vacant professorship in mathematics at the University of Umeå, and his position of universitetslektor in Uppsala is held by K. Haliste during 1965/66. A new professorship in numerical analysis at the University is partially held by E. Lundqvist. The vacant laboratorship in mathematics at Lund is held by G. Bergendal. As an universitetslektor Bergendal is temporarily substituted by K. Björup. T. Herlestam is temporarily laborator at the Radio Institute of Defence. His position at Lund is held by A. Tengstrand. Vacant laboratorships at the Royal Institute of Technology, one in mathematics, two in numerical analysis, and one in mathematical physics, are held by Lars Ingelstam, Åke Björck, L.-E. Thorelli and Johan Rundgren. During C.-E. Fröberg's leave of absence, spring 1965, his laboratorship has been held by T. Ekman, and Ekman's position of universitetslektor was for the same period temporarily held by Bengt-Erik Bengtsson. The laboratorship in mathematics at Uppsala which is vacant after Chr. Lech will be held by L. I. Hedberg.

Appointments to positions of universitetslektor from the beginning of the academic year 1965/66: S. Lyttkens in mathematics at Uppsala, I. Wik in mathematics at Umeå. In Lund J. Friberg has been appointed universitetslektor in mathematics. Matts Essén and Magnus Giertz have been appointed universitetslektor in mathematics at the Royal Institute of Technology. So has Magnus Tideman, but with duties at the regional education to the degree of teknologie magister (new line existing at Linköping and Örebro) at Linköping. Gösta Nordlander holds a corresponding position at Örebro.

The names and the number of biträdande lärare at the universities are difficult to predict because of negotiations about their salaries.

L. I. Hedberg and I. Wik have been appointed docents in mathematics at the University of Uppsala. B. Rosén and J. L. Calais are docents in mathematical statistics and in quantum chemistry at Uppsala, respectively.

Theses defended for Ph. D:

H. Brolin, Uppsala, *Sets invariant under iteration of rational functions.*

K. Haliste, Uppsala, *Estimates of harmonic measures.*

I. Wik, Uppsala, *Studies on absolutely convergent Fourier series.*

L. I. Hedberg, Uppsala, *Generators in certain function algebras.*

J. L. Calais, Uppsala, *Different bands for different spins, a study of the alternant molecular orbital method as applied to crystals.*

N. Nilsson, Lund, *Some results in the spectral theory of partial differential operators.*
Arne Persson, Lund, *Summation methods on locally compact spaces.*

In May the Mathematical Institute in Lund was visited by B. Malgrange, Paris, and by L. Carleson, Uppsala, for a conference on the appointment to the vacant laboratorship in mathematics. L. Hörmander, Princeton, was invited to visit Sweden by the Mittag–Leffler Institute; he lectured in Stockholm and Lund.

Professor A. N. Kolmogorov, USSR, plans to visit Stockholm during the fall term 1965.

During the period January–March 1965 O. Frostman, Stockholm, had leave of absence to visit the Tata Institute of Fundamental Research in Bombay. U. Grenander, Stockholm, visited Thomas J. Watson Research Center, Yorktown Heights, New York, and Brown University, Providence, Rhode Island, during the spring term 1965. Lars Ingelstam, Royal Institute of Technology, has spent the spring term 1965 in Paris.

L. Carleson, Uppsala, will be on leave of absence during the greater part of the academic year 1965/66 to visit Stanford University. His professorship at Uppsala will be held by H. Wallin. With leave of absence J. Friberg, Lund, will spend the main part of the academic year 1965/66 at the University of Minnesota. Rolf Andersen, Lund, will spend the year 1965/66 at Lawrence, Kansas. Both Friberg and Andersson are supported by the Thorild Dahlgren Foundation. C. Kiselman, University of Stockholm, will visit the Institute for Advanced Study, Princeton, during the same academic year. Magnus Giertz, Royal Institute of Technology, will be at the University of California, Los Angeles, during the next academic year.

L. Gårding, Lund, will participate in the Symposium on analytic functions to be held in September 1965 at Erevan, USSR.

A computation center has started in Uppsala. A computor of type CDC 3600 will be in use from around July 1st 1965. The center will be situated at Sturegatan 4 A. Mailing address: Box 2103, Uppsala 2.

The Swedish Mathematical Society has been reorganized in various respects, and the number of officers has been restricted to five.

Various institutes have been united into bigger units at the universities and institutes of technology.

Guest lectures.

University of Lund and the Lund Institute of Technology:

January 27, 29, February 2, 4 — Chr. U. Jensen, University of Copenhagen, *Algebraisk Talteori og Klasselegemeteori* [*Algebraic number theory and class field theory*].

April 23 — J. Mandel, National Bureau of Standards, Washington, D.C., *Some practical problems in fitting straight lines.*

May 14 — B. Epstein, University of New Mexico and U.S. Office of Naval Research, London, *Difference methods and Dirichlet's problem.*

May 18, 19 — L. Hörmander, Princeton, *Pseudo-differential operatorer och randvärdesproblem* [*Pseudo-differential operators and boundary value problems*].

May 21 — G. Weiss, Washington University, St. Louis, *Harmonic functions of several variables.*

University of Stockholm:

February 26 — B. Fuglede, University of Copenhagen, *Om fortolkningen af de Heisenbergske kommutatorrelationer* [*On the interpretation of Heisenberg's commutation relations*].

April 1 — P. H. Randolph, Purdue University and University of Heidelberg, *A marketing team: An example of stochastic linear programming.*

April 21 — L. Lorch, University of Alberta and University of Aarhus, *The Sturm comparison and separation theorems revisited and applied to special functions.*

May 12 — B. Epstein, University of New Mexico and U.S. Office of Naval Research, London, *A three-point difference equation.*

May 13, 14, 25, 26 — L. Hörmander, Institute for Advanced Study, Princeton, *Pseudo-differential operatorer och randvärdesproblem* [*Pseudo-differential operators and boundary value problem*].

Royal Institute of Technology in Stockholm:

February 23, 25 — B. Fuglede, University of Copenhagen, *Nogle træk af teorien for generaliserede potentialer* [*Some aspects of the theory of generalized potentials*].

April 27 — L. Lorch, University of Alberta and University of Aarhus, *The asymptotics of Bessel functions, illustrated by application to a singular integral.*

Lectures at meetings of the Swedish Mathematical Society.

April 3, at the Royal Institute of Technology in Stockholm:

L. Lorch, University of Alberta and University of Aarhus, *The Sturm comparison and representation theorems revisited and applied to special functions.*

A. Hyllengren, Stockholm, *En subadditiv mängdfunktion för små nollmängder* [*A subadditi set function for small sets of zero measure*].

G. Aronsson, Uppsala, *Om variationsproblem av min-max typ* [*On variational problem of min-max type*].

L. Carleson, Uppsala, *Forskarhandledning och högre utbildning i matematik* [*Research guidance and higher education in mathematics*]; for discussion.

June 12, at the Royal Institute of Technology in Stockholm:

L. Gårding, Lund, *Slutna och öppna lineära relationer* [*Closed and open linear relations*].

T. Ganelius, Göteborg, *Slutna lineära avbildningar och Tauber-satser* [*Closed linear mappings and Tauberian theorems*].

MATH. SCAND. 16 (1965), 286

INDEX

PER ASISOFF: On the directions of almost periodicity of entire functions. 94– 96
E. R. DEAL: A quasi-spectral operator 29– 32
WERNER FIEGER: Eine für beliebige Call-Prozesse geltende Verallgemeinerung der Palmschen Formeln 121–147
W. FLUCH: Gruppen ohne endlich-dimensionale Darstellungen 164–168
W. FLUCH: Maximal-Fastperiodizität von Gruppen I 148–158
W. FLUCH: Maximal-Fastperiodizität von Gruppen II 159–163
STEN HANSEN: A generalization of a theorem of Sylvester on the lines determined by a finite point set 175–180
TAH-KAI HU: On the bases of free algebras 25– 28
ANTON JENSEN: Some results concerning a general set theoretical approach to logic 5– 24
LEIF KRISTENSEN: On a Cartan formula for secondary cohomology operations 97–115
DAN LAKSOV: Linear recurring sequences over finite fields 181–196
KJELD B. LAURSEN: A characterization of reflexivity of Banach spaces .. 169–174
JÖRGEN LÖFSTRÖM: On certain interpolation spaces related to generalized semi-groups 41– 54
VLASTIMIL PTÁK: Some open mapping theorems in LF-spaces and their application to existence theorems for convolution equations 75– 93
ÅGOT E. RIIBER: Über die α-Kapazität einer verallgemeinerten Cantorschen Punktmenge 33– 37
PETER SCHERK: On sets of vectors 38– 40
WARREN STENBERG: On weak separation of convex sets and on α-ubiquitous sets 55– 74
HANS TORNEHAVE: On entire functions almost periodic in several directions 116–120

New books and journals
Books received 197–260
New mathematical journals 261

The Scandinavian mathematical societies
List of officers as of July 1, 1965 262

Mathematicians at Scandinavian institutions of higher education and research, July 1, 1965 263–277

Notes
International 278
Scandinavian 278
Denmark 279
Finland 281
Norway 282
Sweden 283
Acknowledgements 287

MATH. SCAND. 16 (1965), 287

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