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EDITORIAL STATEMENT

We regret to inform the readers and our authors that the article published in Math. Scand. 65 (1989), 50-58, by Paul E Jambor: "*Radical of splitting ring extensions*" (published 15th May 1990) was already published in B.J. Gardner, ed. "*Ring, modules, and radicals*" (Proceedings of the Hobart Conference 1987), Longman, New York, 1989.

The Editors

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AN ADDITION THEOREM IN A FINITE ABELIAN GROUP

JØRGEN CHERLY

1. Introduction.

We say that a subset A of a finite abelian group G is an additive basis if there exists an integer h such that any element of G can be written as a sum of at most h elements of A. The set A is said to be an additive basis of order h in case h is minimal. We denote by $|A|$ the cardinality of the set A.

Let F_q be a finite field of $q = p^m$, $m \in \mathsf{N}$ elements and let $\mathsf{F}_q[X]$ denote its polynomialring. The degree of a polynomial $a \in \mathsf{F}_q[X]$ is denoted $\delta^0 a$.

A subset A of $\mathsf{F}_q[X]$ is said to be an additive basis if there exists an integer h such that any element of $\mathsf{F}_q[X]$ can be written as a sum of at most h elements of A. The set A is said to be an additive basis of order h in case h is minimal.

The Snirel'man density dA of A is given by

$$dA = \inf_{n \geqq 0} q^{-n-1} \operatorname{card} A_n,$$

where $A_n = \{a \in A \mid \delta^0 a \leqq n\}$, see [1] [2] [3].

For any integer n, let $G_n = \{f \in \mathsf{F}_q[X] \mid \delta^\circ f \leqq n\}$. It is clear that $(G_n, +)$ is a finite abelian group of order $|G_n| = q^{n+1}$.

We prove here the following results:

THEOREM 1.1. *Let A be a subset of G which contains* 0. *If A is an additive basis of order h for G then*

$$h \leqq 2[|G|/|A|]$$

This inequality is optimal.

Theorem 1.1 implies the following addition theorem in $\mathsf{F}_q[X]$.

THEOREM 1.2. *Let A be a subset of* $\mathsf{F}_q[X]$ *such that*

(i) $0 \in A$

(ii) $dA > 0$

(iii) *For any* $n \geqq 0: A_n$ *is an additive basis for* G_n

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Then A is an additive basis of order at most $2\,[1/dA]$.

In [1] we obtained the estimate $h \leqq 1/(dA)^2$. J. M. Deshouillers proved in [3] that A is an additive basis of order at most $4/dA$. Our proof is different and we improve the constant factor from 4 to 2.

2. Preliminary results.

Let $(G, +)$ be a finite abelian group. We shall denote by $A, B, C, \ldots$ non empty subsets of G. We define the addition of sets of group elements by $A + B = \{a + b \mid a \in A, b \in B\}$. We shall need the following two theorems from H. B. Mann's book [4] chapter 1.

THEOREM 2.1. *Either* $A + B = G$ *or* $|G| \geqq |A| + |B|$

THEOREM 2.2. *If* $C = A + B$ *then* $|C| \geqq |A| + |B| - |H|$ *where H is the subgroup*

$$H = \{g \in G \mid C + g = C\}$$

3. Some lemmas.

LEMMA 3.1. *Let* $C = A + B$, *where A and B are non empty subsets of G, and* $0 \in B$. *Then either* $C + B = C$ *or* $|C| \geqq |A| + |H|$ *where H is the subgroup*

$$H = \{g \in G \mid C + g = C\}$$

PROOF. Assume $C + B \neq C$, then $C + B \supset C$ since $0 \in B$. Hence we can find elements $b_0 \in B$ and $c_0 + b_0 \notin C$. Since $C + b_0$ is a union of complete cosets of H we have

$$C + b_0 = \bigcup_{c \in C} \{c + b_0 + H\} = \bigcup_{a \in A} \{a + b_0 + H\} \bigcup_{c \in C \setminus A} \{c + b_0 + H\}$$

Now for any $a \in A$ we have

$$\{a + b_0 + H\} \cap \{c_0 + b_0 + H\} = \emptyset$$

Indeed let $a_0 \in A$ be such that $a_0 + b_0 + h_1 = c_0 + b_0 + h_2$ with $h_1, h_2 \in H$. Then $c_0 + b_0 \in a_0 + b_0 + H \subseteq C$, contrary to the fact that $c_0 + b_0 \notin C$.

It follows that

$$C + b_0 \supset \{A + b_0\} \cup \{c_0 + b_0 + H\}$$

Hence

$$|C| = |C + b_0| \geqq |A + b_0| + |c_0 + b_0 + H| = |A| + |H|$$

LEMMA 3.2. *Let $C = A + B$, where A and B are non empty subsets of G, and $0 \in B$. Then either*
$C + B = C$ or

$$|C| \geqq |A| + \frac{1}{2}|B|$$

PROOF. (for the non abelian case see J. E. Olson [5]). Assume that $C + B \supset C$. Hence by Theorem 2.2 and lemma 3.1

$$2|C| \geqq 2|A| + |B|$$

LEMMA 3.3. *Let A be a non empty subset of $G, 0 \in A$ and $k \geqq 2$: Then either $kA = (k+1)A$ or $|kA| \geqq \frac{k+1}{2}|A|$.*

PROOF. (see also J. E. Olson [6] Theorem 2.2). Assume $kA \neq (k+1)A$. Then $mA \neq (m+1)A$ for all m such that $2 \leqq m \leqq k$.
By lemma 3.2 with $A \to (m-1)A$, $B \to A$ we obtain

$$|mA| \geqq |(m-1)A| + \frac{1}{2}|A| \text{ for all } m \text{ such that } 2 \leqq m \leqq k$$

Hence $\sum_{m=2}^{k} |mA| \geqq \sum_{m=2}^{k} |(m-1)A| + \frac{k-1}{2}|A|$
which implies $|kA| \geqq \frac{k+1}{2}|A|$

4. Proof of Theorem 1.1.

Define the integer k_0 by $k_0 = [|G|/|A|]$. Assume $k_0 A \neq G$. Then by lemma 3.3 and the definition of k_0

$$|k_0 A| \geqq \frac{k_0 + 1}{2}|A| > \frac{|G|}{2}.$$

Whence by theorem 2.1 we have $2k_0 A = G$.

The inequality in Theorem 1.1 is optimal. Indeed let A be any subset of G such that $|A| = [(|G| + 2)/2]$. Then by Theorem 2.1, A is an additive basis of order 2. Also $2k_0 = 2[|G|/|A|] = 2$ for $|G| \neq 2$.

5. Proof of Theorem 1.2.

By (i), (ii), (iii) and Theorem 1.1 we have: For any $n \geqq 0$: A_n is a basis of order h_n for G_n with

$$h_n \leqq 2\,[q^{n+1}/|A_n|] \leqq 2\,[1/dA]$$

It is then clear that any element of $\mathsf{F}_q[X]$ can be written as a sum of at most $2\,[1/dA]$ elements of A.

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MODULAR SUBSTRUCTURES IN PSEUDOMODULAR LATTICES

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Abstract.

Pseudomodular lattices have been used in [DL 86] in order to investigate combinatorial properties of algebraic matroids and were further analyzed in [BL 87]. The purpose of our paper is to present local conditions, characterizing modular sublattices of a pseudomodular lattice. As an application, we derive a result of [HK 89], implying that Lovasz' min – max formula for matchings in projective geometries remains valid for pseudomodular lattices, and we discuss a relation woth B. Lindströms construction of subgeometries of full algebraic combinatorial geometries which are isomorphic to projective geometries over skew fields.

1. Introduction.

All lattices we consider will be geometric, i.e. of finite length, relatively complemented, graded, and the rank function defined by the grading is semimodular. Each lattice L will be endowed with a strictly increasing semimodular rank function (which may or may not be identical to the one induced by the grading), i.e. $r: L \to \mathbf{N}_0$ is strictly increasing and satisfies

$$\forall x, y \in L: r(x) + r(y) \geqq r(x \vee y) + r(x \wedge y).$$

Recall that r is called modular if this inequality is satisfied by equality for every pair $x, y \in L$.

Recently, [DL 86] and [BL 87] introduced an interesting generalization of modular lattices, the class of so called pseudomodular lattices, which was shown in [DL 86] to contain all full algebraic combinatorial geometries and which may be defined as follows:

DEFINITION (cf. [BL 87]). A geometric lattice L endowed with a strictly increasing semimodular function $r: L \to \mathbf{N}$ is called *pseudomodular*, is for any $a, b, c \in L$ the following implications holds:
If

$$r(a \vee c) - r(a) = r(b \vee c) - r(b) = r(a \vee b \vee c) - r(a \vee b),$$

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then

$$r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) = r(a \vee c) - r(a).$$

It is easy to see that pseudomodularity actually is a generalization of modularity. However, as we will see in Section 2, there is somewhat more to say about the relation between pseudomodular lattices and modular ones. Essentially, it will turn out that modular sublattices of pseudomodular ones can be chaacterized by local conditions which can be checked easily. In Section 3 we will use this fact to derive a result of [HK 89], implying that Lovasz' min – max formula for matching in projective geometries extends to pseudomodular lattices. Finally, in the last section we comment on B. Lindströms construction of large projective geometries, defined over skewfields, which are contained in full algebraic combinatorial geometries.

2. Modular Substructures.

Let L be a geometric lattice with a strictly increasing semimodular rank function $r: L \to \mathsf{N}_0$ and let M be a geometric lattice with strictly increasing modular rank function $\rho: M \to \mathsf{N}_0$. Furthermore, let $\varphi: M \to L$ be a mapping. We say that φ defines a *modular substructure* of L if

- $\varphi: M \to L$ is a homomorphism, i.e. it is $\wedge$- and $\vee$-preserving and
- $\rho(x) = r(\varphi(x))\ \forall x \in M$.

The following result states that, if L is pseudomodular, then modular substructures can be characterized by local conditions which in concrete situations are easy to check (cf. Section 3):

THEOREM 2.1. *If L is pseudomodulair, then $\varphi: M \to L$ defines a modular substructure if and only if*

(i) *φ is $\wedge$-preserving,*
(ii) *$\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ whenever $x \vee y = 1_M$,*
(iii) *if $I = [x, 1_M]$ is an interval of length 2, then $\rho(y) = r(\varphi(y))$ for all $y \in I$.*

PROOF. We show that if φ satisfies conditions (i) – (iii) above, then for any upper interval $I = [x, 1_D]$, the induced map $\varphi: I \to L$ is a homomorphism satisfying $\rho(y) = r\varphi(y)$ for all $y \in I$. The proof is by induction on $k :=$ length of I.
If $k = 2$, the claim is immediate from conditions (i) – (iii). Thus suppose that $k \geqq 3$ and assume the claim holds for smaller values of k. We have to show that for any two $y_1, y_2 \in I$ we have

(1) $$\varphi(y_1 \vee y_2) = \varphi(y_1) \vee \varphi(y_2) \text{ and}$$

(2) $$\rho(y_1 \wedge y_2) = r\varphi(y_1 \wedge y_2).$$

We may assume that $y_1 \wedge y_2 = x$, for otherwise the claim follows from our inductive assumption by looking at the interval $[y_1 \wedge y_2, 1_M]$. Let us first show that (1) holds. If $y_1 \vee y_2 = 1_M$, then (1) is the content of condition (ii). Hence assume that $y_1 \vee y_2 < 1_M$. Let $y_3 \in M$ be a complement of $y_1 \vee y_2$ in the interval $[x, 1_M]$.

Applying φ, we are in a situation as shown in Figure 1 below, where $u := \varphi(y_1) \vee \varphi(y_2)$, $v := \varphi(y_1) \vee \varphi(y_3)$, and $w := \varphi(y_2) \vee \varphi(y_3)$:

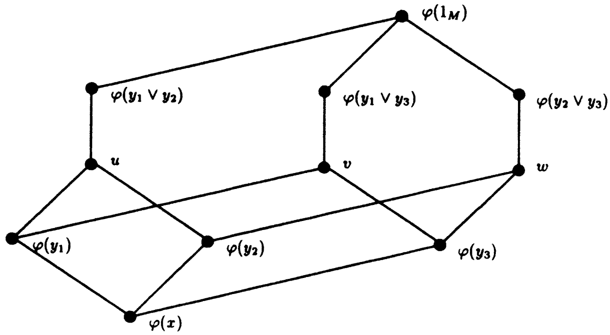

Figure 1.

Note that, due to condition (ii), we have

$$\varphi(y_1) \vee \varphi(y_2 \vee y_3) = \varphi(1_M),$$

hence

$$u \vee \varphi(y_2 \vee y_3) = \varphi(1_M),$$

as indicated in Figure 1 above. Furthermore, note that φ is order preserving since it was assumed to be $\wedge$-preserving. This implies that

$$\varphi(y_1 \vee y_2) \geqq u = \varphi(y_1) \vee \varphi(y_2)$$

etc. We are to show that actually equality holds. First note that modularity of ρ gives

$$\rho((y_1 \vee y_2) \wedge (y_2 \vee y_3)) = \rho(y_1 \vee y_2) + \rho(y_2 \vee y_3) - \rho(1_M) =$$
$$= \rho(y_1) + \rho(y_2) - \rho(x) + \rho(y_2) + \rho(y_3) - \rho(x) - \rho(1_M)$$
$$= \rho(y_1 \vee y_2) + \rho(y_3) - \rho(x) - \rho(1_M) + \rho(y_2) = \rho(y_2)$$

and therefore

$$(*) \qquad (y_1 \vee y_2) \wedge (y_2 \vee y_3) = y_2.$$

By induction on k, we know that $\varphi_{|[y_2, 1_M]}$ defines a modular substructure of L. Thus, in particular,

$$(**) \qquad r\varphi(y_1 \vee y_2) + r\varphi(y_2 \vee y_3) = r\varphi(1_M) + r\varphi(y_2).$$

Now, applying the semimodularity of L with respect to u and $\varphi(y_2 \vee y_3)$, we get – using $u \vee \varphi(y_2 \vee y_3) = \varphi(1_M)$ and $r(u \wedge \varphi(y_2 \vee y_3)) \geqq r\varphi(y_2)$ – the inequality

$$r(u) \geqq r\varphi(1_M) + r\varphi(y_2) - r\varphi(y_2 \vee y_3) \overset{(**)}{=} r\varphi(y_1 \vee y_2),$$

which together with $\varphi(y_1 \vee y_2) \geqq u$ implies (1), that is,

$$\varphi(y_1 \vee y_2) = u = \varphi(y_1) \vee \varphi(y_2)$$

in view of the strict monotonicity of r.

We are left to show that (2) holds, i.e.

$$\rho(x) = r\varphi(x).$$

Since the length k of the interval $I = [x, 1_M]$ is at least 3, we can find $y_1, y_2, y_3 \in I \setminus \{x\}$ such that each y_i is a relative complement to the join of the other two y_j's, i.e., with $y_1 \vee y_2 \vee y_3 = 1_M$ and $\rho(y_1) + \rho(y_2) + \rho(y_3) = \rho(1_M) + 2\rho(x)$. From our argument above, we conclude that, applying φ, we are in a situation as sketched in Figure 2 below:

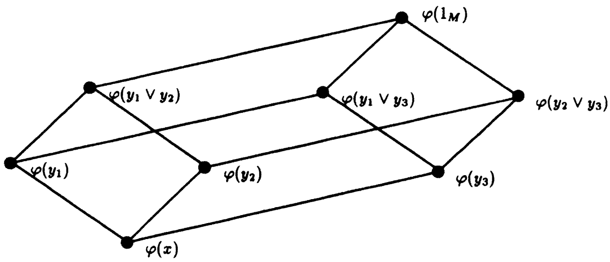

Figure 2.

Now let $a := \varphi(y_1)$, $b := \varphi(y_3)$ and $c := \varphi(y_2)$. Since, by induction, φ defines a modular substructure of L when restricted to each of the intervals $[y_i, 1_M]$, we get

$$r(a \vee c) - r(a) = r(a \vee b \vee c) - r(a \vee b) = r(b \vee c) - r(b) = \rho(y_2) - \rho(x).$$

Since L is pseudomodular, we conclude that

$r((a \vee c) \wedge (b \vee c)) - r(a \wedge b) = r(a \vee c) - r(a)$, i.e. (cf. (*)),

$$r\varphi(y_2) - r\varphi(x) = r\varphi(y_1 \vee y_2) - r\varphi(y_1)$$

and therefore

$$r\varphi(x) = \rho(y_2) + \rho(y_1) - \rho(y_1 \vee y_2) = \rho(x),$$

as claimed.

3. An application in Matching Theory.

In this section we assume the reader to be familiar with the basic theory of matroids and geometric lattices. Suppose L is a geometric lattice with point set E and let now r denote the rank function induced by the grading. A subset $A \subseteq E$ is called a *double circuit*, if $r(A) = r(A \setminus a) = |A| - 2$ for every $a \in A$ or, equivalently, if A is the complement of a coline, i.e., a flat of codimension 2, in the dual of L. These sets play a central role in the context of matching in geometric lattices (cf. [HK 89] for more details). It is easy to see (cf. [HK 89]) that the following holds:

LEMMA 3.1. *Let $A \subseteq E$ be a double circuit. Then there exists a partition*

$$A = A_1 \dot{\cup} \ldots \dot{\cup} A_d$$

such that $C_i := A \setminus A_i$ is a circuit for $i = 1, \ldots, d$ and these are all circuits contained in A.

PROOF. If in the dual of L the hyperplanes containing the coline $E \setminus A$ are $H_1, \ldots, H_d$, then $A_1 := H_1 \cap A, \ldots, A_d := H_d \cap A$ are precisely the subsets described above.

As can be seen from [HK 89], the crucial point in proving Lovasz' min – max formula for matchings conxists in showing that in case L is pseudomodular the closures $\bar{C}_i$ of the circuits in A induce a modular sublattice of L. More precisely, one has to show (cf. Theorem 3.1 of [HK 89]) that

$$r\left(\bigcap_{i=1}^{d} \bar{C}_i\right) \geqq d - 2.$$

As we will see next, this is a simple consequence of Theorem 2.1;

Indeed, let $M = M_d$ denote the Boolean algebra of (all) subsets of $\{1, \ldots, d\}$. Obviously, M is modular relative to the map $\rho \colon M \to \mathbf{N}_0$, defined by

$$\rho(I) := |A| - 2 - \sum_{i \notin I} (|A_i| - 1) = \sum_{i \in I} (|A_i| - 1) + d - 2$$

for all $I \subseteq \{1, \ldots, d\}$. We claim

PROPOSITION 3.2. *If – with the above notations – we define*

$$\varphi : M \to L$$

by

$$\varphi(I) := \bigcap_{i \notin I} \bar{C}_i$$

for all proper subsets $I \subsetneqq \{1, \ldots, d\}$ of $\{1, \ldots, d\}$ *and*

$$\varphi(\{1, \ldots, d\}) := \bar{A},$$

then φ *defines a modular substructure in* L.

PROOF. Obviously, φ is $\wedge$-preserving and for $I \cup J = \{1, \ldots, d\}$ we have

$$\begin{aligned} \bar{A} \geqq \varphi(I) \vee \varphi(J) &= \cap\{\overline{A \setminus A_i} \mid i \notin I\} \vee \cap\{\overline{A \setminus A_j} \mid j \notin J\} \\ &\supseteq \cap\{A \setminus A_i \mid i \notin I\} \vee \cap\{A \setminus A_j \mid j \notin J\} \\ &= \cup\{A_i \mid i \in I\} \vee \cup\{A_j \mid j \in J\} \supseteq A \end{aligned}$$

and therefore $\varphi(I) \vee \varphi(J) = \bar{A}$. Thus we are left to show that condition (iii) of Theorem 2.1 is satisfied, too. Too check this, observe that $\rho(\{1, \ldots, d\}) = |A| - 2 = r(A)$ and $\rho(\{1, \ldots, d\} \setminus \{i\}) = |A| - |A_i| - 1 = r(\bar{C}_i)$, since $C_i = A \setminus A_i$ is a circuit. Finally, for $i \neq j$, we have

$$r(\bar{C}_i \wedge \bar{C}_j) = \rho(\{1, \ldots, d\} \setminus \{i, j\}) = |A| - |A_i| - |A_j|$$

in view of the following sequence of inequalities

$$\begin{gathered} r(\bar{C}_i \wedge \bar{C}_j) \leqq r(\bar{C}_i) + r(\bar{C}_j) - r(\bar{C}_i \vee \bar{C}_j) = \\ (|A| - |A_i| - 1) + (|A| - |A_j| - 1) - (|A| - 2) = \\ |A| - |A_i| - |A_j| = |C_i \cap C_j| \leqq r(\bar{C}_i \wedge \bar{C}_j), \end{gathered}$$

the first one of which holds because L is semimodular, while the second one holds because $C_i \cap C_j$, being a proper subset of a circuit, must be independent.

COROLLARY 3.3. *Under the assumptions of Proposition* 3.2, *one has*

$$r\left(\bigcap_{i=1}^{d} \bar{C}_i\right) = d - 2.$$

PROOF. Since $\varphi : M \to L$ defines a modular substructure, we conclude that

$$r\left(\bigcap_{i=1}^{d} \bar{C}_i\right) = \rho(\emptyset) = |A| - 2 - \sum_{i=1}^{d} (|A_i| - 1) = d - 2,$$

as claimed.

4. Projective Geometries in Full Algebraic Matroids.

In [L 88] B. Lindström shows that for every field F of prime characteristic p the full algebraic matroid $L = L_n(F)$ of rank n, whose flats are the algebraically closed subfields of the algebraic closure $\overline{F(X_1,\ldots,X_n)}$ of the purely transcendental extension $F(X_1,\ldots,X_n)$ of F in n algebraically independent variables $X_1,\ldots,X_n$ contains as a subgeometry a full projective space of (projective) dimension $n-1$ over a certain skew-field, defined in terms of the "p-polynomials" in the polynomial ring $F[X]$ in one variable X over F, that is, the F-linear combinations of the monomials of type X^{p^h} $(h = 0,1,2,\ldots)$. More specifically, his surprising and beautiful result asserts that

(i) the set of all p-polynomials in $F[X]$ forms an Ore-domain $R(F)$ with quotient skew-field, say, Li(F), when for any two such polynomials $P[X], Q[X] \in R(F)$ one defines the sum $P[X] + Q[X]$ as usual and the product $P[X] \circ Q[X]$ by composition, i.e., by

$$P[X] \circ Q[X] := P[Q[X]],$$

and that

(ii) there exists a modular substructure $\varphi : L_n(F)$, where $M = M(Li(F)^n)$ denotes the relatively complemented modular lattice of (all) subspaces $U, V, \ldots,$ of the n-dimensional (right) vectorspace Li$(F)^n$ over Li(F) (with dimension as rank function) and where for $U \leqq \mathrm{Li}(F)^n$ the field $\varphi(U)$ is the algebraic closure in $\overline{F(X_1,\ldots,X_n)}$ of all polynomials

$$Q[X_1,\ldots,X_n] = Q_1[X_1] + \ldots + Q_n[X_n]$$

with $(Q_1[X],\ldots,Q_n[X]) \in U \cap R(F)^n$.

Unfortunately, to verify that $\varphi : M \to L$ satisfies the conditions in Theorem 2.1 requires almost all the work, Lindström has done to establish his result directly. So one saves not much by invoking Theorem 2.1 in this specific context. Still, Theorem 2.1 together with the fact from [DL 86], referred to already in the Introduction, that every full algebraic matroid is pseudomodular, seems to present a conceptual framework with regard to which Lindström's amazing construction can be understood and appreciated more easily and in a more systematic way and which may help to find similar results, e.g. for full algebraic matroids defined over fields of characteristic 0.

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APPROXIMATION BY NEAREST INTEGER CONTINUED FRACTIONS (II)

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Abstract.

In a paper with the same title recently published in this journal, a recurrence relation of a Diophantine inequality is established: $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2/(3 + \sqrt{5} - 2\alpha_k)$, where $\alpha_1 = 2/5$ and $\alpha_i = 1/(3 - \alpha_{i-1})$. In this note, we give the explicit form of this inequality: $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/\sqrt{5} + ((3 - \sqrt{5})/2)^{2k+3}/\sqrt{5}$.

Let x be an irrational number, $x = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n, \ldots]$ be its expansion in nearest integer continued fraction. Let $A_n/B_n = [\varepsilon_0 b_0; \varepsilon_1 b_1, \ldots, \varepsilon_n b_n]$ be the nth convergent and $\theta_n = B_n{}^2|x - A_n/B_n|$. It was proved in [2] that $\min(\theta_{n-1}, \theta_n, \theta_{n+1}) < 5(5\sqrt{5} - 11)/2$. The present author generalized this result. It is proved in [4] that $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2/(3 + \sqrt{5} - \alpha_k)$, where $\alpha_1 = 2/5$, $\alpha_i = 1/(3 - \alpha_{i-1})$. In this note, using the Fibonacci sequence, we give an explicit estimation of the value $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k})$ as a function of k directly.

THEOREM 1. $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/\sqrt{5} + ((3 - \sqrt{5})/2)^{2k+3}/\sqrt{5}$.

PROOF. Let $f_1 = 1$, $f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ be the Fibonacci sequence.

We first prove that the recurrence relation $\alpha_i = 1/(3 - \alpha_{i-1})$ and $\alpha_1 = 2/5$ imply that $\alpha_i = f_{2i+1}/f_{2i+3}$.

If $i = 1$, then $\alpha_1 = 2/5 = f_3/f_5$. Suppose $\alpha_k = f_{2k+1}/f_{2k+3}$. Then $\alpha_{k+1} = 1/(3 - \alpha_k) = 1/(3 - f_{2k+1}/f_{2k+3}) = f_{2k+3}/(3f_{2k+3} - f_{2k+1}) = f_{2k+3}/(2f_{2k+3} + f_{k+2}) = f_{2k+3}/(f_{2k+3} + f_{2k+4}) = f_{2k+3}/f_{2k+5}$. Therefore by induction we have $\alpha_i = f_{2i+1}/f_{2i+3}$.

Replacing α_k in the expression $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2/(3 + \sqrt{5} - \alpha_k)$ by f_{2k+1}/f_{2k+3}, we have

$$\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 2f_{2k+3}/((3 + \sqrt{5})f_{2k+3} - 2f_{2k+1}).$$

By Binet's formula for the Fibonacci sequence [1], we have

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$$f_n = (\phi^n - (-\phi^{-1})^n)/\sqrt{5},$$

where $\phi = (1 + \sqrt{5})/2$, $-\phi^{-1} = (1 - \sqrt{5})/2$. Now a direct calculation of $2((\phi/(-\phi^{-1}))^{2k+3} - 1)/((3 + \sqrt{5})((\phi/(-\phi^{-1}))^{2k+3} - 1) - 2((\phi/(-\phi^{-1}))^{2k+1} - 1)/(-\phi^{-1})^2)$ yields

$$\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/\sqrt{5} + ((3 - \sqrt{5})/2)^{2k+3}/\sqrt{5}.$$

Now we can have a comparison of the two approximations by simple continued fraction and by nearest integer continued fraction. Let x be an irrational number. Borel's theorem [3] asserts that among any three consecutive convergents p_i/q_i of simple continued fraction of x, there is at least one satisfies $|x - p_i/q_i| < 1/(\sqrt{5}\, q_i^2)$. As a much weaker corollary we know that there are infinitely many convergents p_i/q_i satisfying $|x - p_i/q_i| < 1/(\sqrt{5}\, q_i^2)$. For nearest integer continued fraction we only have the following even weaker form.

COROLLARY 1. *Let x be an irrational number. Then there are infinitely many convergents A_i/B_i of nearest integer continued fraction of x satisfying*

$$|x - A_i/B_i| < 1/((\sqrt{5} - \varepsilon) B_i^2).$$

PROOF. Since $((3 - \sqrt{5})/2)^{2k+3} \to 0$ when $k \to \infty$, we know that $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) \downarrow 1/\sqrt{5}$ for any fixed positive integer n. Therefore for any given small $\varepsilon > 0$ and any positive integer n, pick up k such that $\min(\theta_{n-1}, \theta_n, \ldots, \theta_{n+k}) < 1/(\sqrt{5} - \varepsilon)$ then we can have an integer $m(-1 < m < k)$ such that $\theta_{n+m} < 1/(\sqrt{5} - \varepsilon)$. Since there are infinitely many positive integer n, there are infinitely many i such that $\theta_i < 1/(\sqrt{5} - \varepsilon)$. Since $\theta_i = B_i^2|x - A_i/B_i|$, we have the conclusion.

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COUNTING MATRICES WITH COORDINATES IN FINITE FIELDS AND OF FIXED RANK

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Abstract.

Over a given finite field, we present a method for obtaining explicit expressions for the number of matrices of given rank satisfying certain conditions. As illustration of the method, we present a series of new formulas, and we obtain simple proofs of known formulas.

Introduction.

We present a method for obtaining explicit expressions for the number of matrices of fixed order and rank with entries in a finite field and satisfying certain additional conditions.

The method is simple and formal and makes it possible to derive many formulas using only standard linear algebra. Applying the method in various examples, we obtain simple proofs of known formulas, and we obtain a series of new formulas. Formulas of this kind have mostly been obtained through the use of recursion and exponential sums. In the references we have listed some articles that illustrate the differences in methods and that are not used elsewhere in the text.

On the other hand, our method explains the appearance of the recursion formulas. Moreover, the expressions obtained from our method are in many cases different from those obtained using exponential sums. As a consequence, we obtain new non-trivial identities between expressions that can be interpreted as special values of certain generalized hypergeometric series.

To illustrate our method, we consider the set of all $m \times t$ matrices and the four subsets defined by the following conditions on the matrix X:

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(1) The matrix X is arbitrary,
(2) The rows of the matrix X are non-zero and mutually different,
(3) The matrix X is a solution to the equation $X'SX = 0$ where S is a given regular symmetric $m \times m$ matrix,
(4) The matrix X is a solution to the equation $X'AX = 0$ where A is a given regular antisymmetric $m \times m$ matrix.

Explicit expressions and a recursion formula for the number of all matrices of rank r were given in [Lb]. A recursion formula for the number of matrices of rank r satisfying Condition (2) was obtained by [Ls] and an explicit expression was given in [C3]. Explicit expressions for the number of matrices satisfying Condition (3) or (4) with no condition on the rank were given in [C1, C2]. In the present work we obtain explicit expressions for the subsets of matrices of given rank r. The expressions we derive for the number of all matrices satisfying (3) or (4) are different from those obtained in [C1] and [C2]. Taken in connection with the previous expressions, the new expressions may be seen as a new family of nontrivial identities of hypergeometric series. In fact, these identities led us to the discovery of an error in the expressions obtained in [C2], see Note (6.3) below.

The explicit expressions involve the q-binomial coefficients or Gaussian polynomials,

$$\begin{bmatrix} t \\ r \end{bmatrix} = \frac{(q^t - 1)(q^{t-1} - 1)\ldots(q^{t-r+1} - 1)}{(q^r - 1)(q^{r-1} - 1)\ldots(q - 1)}.$$

When q is a variable, the expression on the right hand side is in fact a polynomial. In our formulas q will be the number of elements in a finite field. Our method explains the occurrence of these polynomials. In fact, as we show in Section 3, a number of well known properties of the Gaussian polynomials may be obtained on the basis of similar properties of numbers arising from finite dimensional vector spaces over finite fields.

1. Interpolation formulas.

In this section we prove an interpolation formula needed in Section 2. The formula is of Lagrange type, giving the transition between two different bases of the polynomial ring $R[x]$ over a ring R.

DEFINITON 1.1. Let R be a commutative ring with unity. Fix a sequence $\lambda_1, \lambda_2, \ldots$ of elements in R, and define polynomials for $r = 0, 1, \ldots$

$$Q_r(x) := (x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_r).$$

Then $Q_0(x) = 1$, and the sequence $Q_0(x)$, $Q_1(x), \ldots$ forms an R-basis for $R[x]$.

Hence every polynomial $f(x)$ of $R[x]$ uniquely determines coefficients $\begin{bmatrix} f \\ r \end{bmatrix}$ in R for $r = 0, 1, \ldots$ such that

$$f(x) = \sum_{r=0}^{\infty} \begin{bmatrix} f \\ r \end{bmatrix} Q_r(x) \tag{1.1.1}$$

where $\begin{bmatrix} f \\ r \end{bmatrix} = 0$ for $r > \deg f$.

Our notation for the coefficients resembles the usual notation for the q-binomial coefficients. The connection between the two notations will be explained in Remark 1.5 and Note 3.2.

Assume that the differences $\lambda_i - \lambda_j$ are invertible in R whenever $i \neq j$.

PROPOSITION 1.2. *For every polynomial $f(x)$, the following formulas hold,*

$$\begin{bmatrix} f \\ r \end{bmatrix} = \sum_{i=1}^{r+1} \frac{f(\lambda_i)}{Q'_{r+1}(\lambda_i)}, \quad \textit{for} \quad r = 0, 1, \ldots \tag{1.2.1}$$

where Q' denotes the formal derivative of the polynomial Q.

PROOF. Clearly, the two sides of (1.2.1) are R-linear maps $R[X] \to R$. Therefore, it suffices to prove that the equations (1.2.1) hold when $f = Q_n$ for $n = 0, 1, \ldots$ Clearly, the left hand side is equal to 1 for $r = n$, and zero otherwise. Consider the right hand side of (1.2.1) for $f = Q_n$. The numerator $Q_n(\lambda_i)$ vanishes when $i \leqq n$. In particular, the right hand side vanishes for $r < n$, and for $r = n$ the only non-vanishing term is equal to 1. Hence it remains to be shown that the right hand side is equal to 0 for $r > n$.

Clearly, it suffices to prove that the right hand side is zero for $r > n$ when the elements $\lambda_{n+1}, \ldots, \lambda_{r+1}$ are independent variables over Z. Set $p = r + 1 - n$, and $x_i := \lambda_{n+i}$ for $i = 1, \ldots, p$. Denote by G the right hand side. Then, after a change of indices,

$$G = \sum_{i=n+1}^{r+1} \frac{Q_n(\lambda_i)}{Q'_{r+1}(\lambda_i)} = \sum_{i=1}^{p} \frac{Q_n(x_i)}{Q'_{r+1}(x_i)}.$$

After a reduction of the fractions in the sum on the right hand side, we obtain the equation,

$$G = \sum_{i=1}^{p} \frac{1}{\prod_{1 \leqq j \leqq p, j \neq i}(x_i - x_j)}.$$

It follows that G is symmetric in the p variables $x_1, \ldots, x_p$. Moreover, every denominator on the right hand side divides the Vandermonde determinant $\Delta = \prod_{1 \leqq i < j \leqq p}(x_j - x_i)$. Therefore, the product $G\Delta$ is an alternating polynomial in $x_1, \ldots, x_p$, and of degree less than the degree of Δ. Consequently, $G = 0$ and we have proved the Proposition.

1.3. Assume for the rest of this section that the sequence of λ_i's consists of the powers of a single element q, that is, $\lambda_i = q^{i-1}$. Then

$$Q_i(x) = (x-1)(x-q)\ldots(x-q^{i-1}). \tag{1.3.1}$$

In this case, the coefficients of the interpolation formula can be expressed by the *q-binomial coefficients*,

$$\begin{bmatrix} r \\ i \end{bmatrix} := \frac{(q^r-1)(q^{r-1}-1)\ldots(q^{r-i+1}-1)}{(q^i-1)(q^{i-1}-1)\ldots(q-1)} \quad \text{for} \quad r, i \geqq 0.$$

For $0 \leqq i \leqq r$, it follows immediately from the definition that

$$\begin{bmatrix} r \\ i \end{bmatrix} = \begin{bmatrix} r \\ r-i \end{bmatrix}. \tag{1.3.2}$$

Moreover,

$$\begin{bmatrix} r \\ i \end{bmatrix} = \frac{Q_i(q^r)}{Q_i(q^i)} = (-1)^i q^{-\binom{i}{2}} \frac{Q_r(q^r)}{Q'_{r+1}(q^{r-i})}, \tag{1.3.3}$$

as follows by an extraction of powers of q and a simple reduction.

PROPOSITION 1.4. *Assume that* $\lambda_i = q^{i-1}$. *Then, for every polynomial* f *and* $r = 0, 1, 2, \ldots$, *the coefficient in the expansion* (1.1.1) *is given by the formulas,*

$$\begin{bmatrix} f \\ r \end{bmatrix} = \sum_{i=0}^{r} \frac{f(q^i)}{Q'_{r+1}(q^i)} = \frac{1}{Q_r(q^r)} \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} f(q^{r-i}).$$

PROOF. The first formula is just a rewriting of (1.2.1), and the second equation follows from Equation (1.3.3).

REMARK 1.5. The formulas apply whenever the element q and the differences $q^i - 1$ for $i > 0$ are invertible. When the latter elements are regular, the formulas can be interpreted in the total ring of fractions of the given ring. In particular, the formulas hold for an element q which is transcendental over a ground ring. Moreover, the formulas for an arbitrary element q follows by specialization from the transcendental case.

Consider the case where q is transcendental. Then the q-binomial coefficients are a priori rational functions of q. It is well known that they are in fact polynomials in q, called the *Gaussian polynomials*. Moreover, the following two formulas are well known,

$$\begin{bmatrix} x^m \\ r \end{bmatrix} = \begin{bmatrix} m \\ r \end{bmatrix}$$

(explaining our choice of notation), and

$$Q_r(x) = \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} x^{r-i}.$$

As we shall see in the Section 3 (Note 3.2 and Remark 3.3), these assertions follow easily from our results.

2. The method.

Let q be a power of a prime number and denote by F_q the field with q elements. Fix a vectorspace V of finite dimension m over F_q. For a given family $\mathscr{T}$ of subspaces of V, denote by $\mathscr{T}_r$ the set of subspaces of dimension r in the family $\mathscr{T}$. Moreover, for any $t \geqq 0$, denote by

$$\mathrm{Hom}_{\mathscr{T}}(\mathsf{F}_q^t, V) \subseteq \mathrm{Hom}(\mathsf{F}_q^t, V)$$

the set of those linear maps $\mathsf{F}_q^t \to V$ whose image belongs to $\mathscr{T}$. Note that the set $\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)$ consists of the linear maps φ: $\mathsf{F}_q^t \to V$ such that the rank of φ is equal to r and the image of φ belongs to $\mathscr{T}$. In particular, the set $\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^r, V)$ is the set of injective linear maps $\mathsf{F}_q^r \to V$ whose image belongs to $\mathscr{T}$.

Our method is to combine some simple relations between the numbers of elements in the sets $\mathscr{T}_r$, $\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)$ and $\mathrm{Hom}_{\mathscr{T}}(\mathsf{F}_q^t, V)$ with the interpolation formula obtained in Section 1 with respect to the polynomials

$$Q_i(x) = (x-1)(x-q)\dots(x-q^{i-1}).$$

When a basis for V is given, $V = \mathsf{F}_q^m$, then $\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)$ is a subset of the set of $m \times t$ matrices, and we obtain the explicit formulas mentioned in the introduction.

PROPOSITION 2.1. *Let $\mathscr{T}$ be a family of linear subspaces of V. Then the cardinalities of the sets $\mathscr{T}_r$, $\mathrm{Hom}_{\mathscr{T}}(\mathsf{F}_q^t, V)$ and $\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)$ are related by the following formulas,*

(2.1.1) $$|\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)| = |\mathscr{T}_r| \cdot Q_r(q^t),$$

(2.1.2) $$|\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)| = \begin{bmatrix} t \\ r \end{bmatrix} \cdot |\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^r, V)|,$$

(2.1.3) $$|\mathrm{Hom}_{\mathscr{T}}(\mathsf{F}_q^t, V)| = \sum_{r=0}^{\dim V} |\mathscr{T}_r| \cdot Q_r(q^t).$$

PROOF. Let U be a subspace of dimension r in V. Clearly, the number of surjective linear maps $\mathsf{F}_q^t \to U$ is equal to the number of injective linear maps $\mathsf{F}_q^r \to \mathsf{F}_q^t$; hence the number is equal to the product,

$$(q^t - 1)(q^t - q)\dots(q^t - q^{r-1}).$$

In other words, the number is equal to $Q_r(q^t)$, where $Q_r(x)$ is the polynomial of (1.3.1). Therefore, the number $|\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)|$ of linear maps $\mathsf{F}_q^t \to V$ with image in $\mathscr{T}_r$ is the product of $Q_r(q^t)$ and the number of possible images, $|\mathscr{T}_r|$. Hence the first equation holds. The second equation follows by using the first for t and for $t := r$, noting that the quotient $Q_r(q^t)/Q_r(q^r)$ is the q-binomial coefficient $\begin{bmatrix} t \\ r \end{bmatrix}$ by (1.3.3). Finally, the last equation follows from the first by summation over the possible ranks of the subspaces in $\mathscr{T}$.

Corollary 2.2. *Considered the polynomial $f = f_{\mathscr{T}}$ defined by*

$$f(x) := \sum_{r=0}^{\dim V} |\mathscr{T}_r|\, Q_r(x).$$

Then the following formulas hold:

$$|\mathscr{T}_r| = \begin{bmatrix} f \\ r \end{bmatrix} = \sum_{i=0}^{r} \frac{f(q^i)}{Q'_{r+1}(q^i)}, \tag{2.2.1}$$

$$|\mathrm{Hom}_{\mathscr{T}}(\mathsf{F}_q^t, V)| = f(q^t), \tag{2.2.2}$$

$$|\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^r, V)| = \begin{bmatrix} f \\ r \end{bmatrix} Q_r(q^r) = \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} f(q^{r-i}), \tag{2.2.3}$$

$$|\mathrm{Hom}_{\mathscr{T}_r}(\mathsf{F}_q^t, V)| = \begin{bmatrix} f \\ r \end{bmatrix} Q_r(q^t) = Q_r(q^t) \sum_{l=0}^{r} \frac{f(q^i)}{Q'_{r+1}(q^i)} \tag{2.2.4}$$

$$= \begin{bmatrix} t \\ r \end{bmatrix} \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} f(q^{r-i}). \tag{2.2.5}$$

Proof. By definition of f, the equation (2.2.2) follows from Equation (2.1.3). Moreover, it follows from the definition of the coefficient $\begin{bmatrix} f \\ r \end{bmatrix}$ that $|\mathscr{T}_r| = \begin{bmatrix} f \\ r \end{bmatrix}$. Therefore, the remaining formulas of the Corollary follow from the formulas of Proposition 2.1 and the general expressions for the coefficient $\begin{bmatrix} f \\ r \end{bmatrix}$ in Proposition 1.4.

Remark 2.3. The Proposition has an obvious dual form. Let $\mathscr{T}^*$ denote the family of polars to the family $\mathscr{T}$, that is, a subspace W of the dual space V^* belongs to $\mathscr{T}^*$, if and only if the intersection of the kernels of the linear forms in

W belongs to $\mathcal{T}$. Clearly, the set of all rank-r linear maps $V \to \mathsf{F}_q^t$ whose kernel belongs to $\mathcal{T}$ corresponds bijectively to the set of all rank-r linear maps $\mathsf{F}_q^t \to V^*$ whose image belongs to $\mathcal{T}^*$. Hence a dual form of the proposition is obtained by applying the proposition to the family $\mathcal{T}^*$ of subspaces in the dual vector space V^*.

3. General matrices.

In this and the following sections the method of Section 2 is illustrated by applying it to the cases mentioned in the introduction. As in Section 2, the vector space V is assumed to be of finite dimension m over the field F_q with q elements.

Let $\mathscr{F}$ be the set of all subspaces of V. Then $\mathrm{Hom}_{\mathscr{F}}(\mathsf{F}_q^t, V)$ is the set of all linear maps $\mathsf{F}_q^t \to \mathsf{F}_q^m$, and $\mathrm{Hom}_{\mathscr{F}_r}(\mathsf{F}_q^r, V)$ is the set $\mathrm{Hom}_r(\mathsf{F}_q^r, \mathsf{F}_q^m)$ of all injective linear maps $\mathsf{F}_q^r \to \mathsf{F}_q^m$. Clearly,

$$|\mathrm{Hom}_r(\mathsf{F}_q^r, \mathsf{F}_q^m)| = Q_r(q^m). \tag{3.0.1}$$

Hence, from Formula (2.1.2), we obtain the following result of Landsberg [Lb].

PROPOSITION 3.1. *The number of $\phi_r(t, m)$ of all $m \times t$ matrices of rank r with entries in F_q is given by the following expressions,*

$$\phi_r(t, m) = |\mathrm{Hom}_r(\mathsf{F}_q^t, \mathsf{F}_q^m)| = \begin{bmatrix} t \\ r \end{bmatrix} \cdot Q_r(q^m) = \frac{Q_r(q^t)Q_r(q^m)}{Q_r(q^r)}.$$

NOTE 3.2. It follows from Equation (2.1.1), or directly, that the number,

$$|\mathscr{F}_r| = \begin{bmatrix} m \\ r \end{bmatrix} = \frac{Q_r(q^m)}{Q_r(q^r)}, \tag{3.2.1}$$

is equal to the number of dimension-r subspaces of V. Clearly, the set $\mathrm{Hom}(\mathsf{F}_q^t, V)$ can be identified with the set of all $m \times t$ matrices, and consequently the cardinality of $\mathrm{Hom}(\mathsf{F}_q^t, V)$ is q^{tm}. Now, let $f = f_{\mathscr{F}} = \sum_{r=0}^{\dim V} \begin{bmatrix} m \\ r \end{bmatrix} Q_r(x)$ be the polynomial of Corollary 2.2. Then it follows from (2.2.2) that $q^{tm} = f(q^t)$. Consequently, since t is arbitrary, it follows that $f(x) = x^m$. From (3.2.1) and the first equation of (2.2.1), we obtain that

$$\begin{bmatrix} x^m \\ r \end{bmatrix} = \begin{bmatrix} m \\ r \end{bmatrix}. \tag{3.2.2}$$

Consider finally Equation (2.2.3). By (3.0.1), or directly, the left hand side of (2.2.3) is $Q_r(q^m)$. The polynomial f on the right hand side of (2.2.3) is $f = x^m$. Therefore, Equation (2.2.3) implies the following equation,

$$Q_r(q^m) = \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} (q^m)^{r-i}.$$

The latter equation holds for all m. Therefore it implies the following equation of polynomials,

$$Q_r(x) = \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} x^{r-i}. \tag{3.2.3}$$

REMARK 3.3. The results of 3.2 were obtained for the given prime power q. However, the results imply corresponding results for an element q which is transcendental over $\mathbf{Z}$. Indeed, when q is transcendental, the q-binomial coefficient $\begin{bmatrix} t \\ r \end{bmatrix}$ is a quotient of polynomials in q with integer coefficients and the denominator is a monic polynomial in q. It follows from the interpretation of (3.2.1) above, that the value of $\begin{bmatrix} t \\ r \end{bmatrix}$ is integral when evaluated on any prime power. Therefore, the q-binomial coefficients are polynomials in q with integer coefficients. Consider next the two sides of Equation (3.2.2) when q is transcendental. The two sides are polynomials in q, and equal when q is a prime power. Therefore, Equation (3.2.2) holds when q is transcendental. It follows similarly that Equation (3.2.3) holds in the transcendental case.

4. Matrices with different rows.

Consider the set of all $m \times t$ matrices whose rows are non-zero and mutually distinct. Let $V := \mathsf{F}_q^m$, and denote by $\xi_1, \ldots, \xi_m$ the dual of the canonical basis of V. Then, clearly, the latter set of matrices can be identified with the set of linear maps $\mathrm{Hom}_{\mathscr{D}}(\mathsf{F}_q^t, V)$ defined by the following family $\mathscr{D}$ of subspaces of V: A subspace U belongs to $\mathscr{D}$, if and only if the functionals ξ_i when restricted to U are non-zero and mutually distinct.

Clearly, the number of matrices in $\mathrm{Hom}_{\mathscr{D}}(\mathsf{F}_q^t, V)$ is $(q^t - 1)(q^t - 2)\ldots(q^t - m)$. Therefore it follows from Equation (2.2.2) that the polynomial of Corollary 2.2 is $f(x) = (x - 1)(x - 2)\ldots(x - m)$. Hence, from Formula (2.2.5), we obtain the following result of Carlitz [C3].

PROPOSITION 4.1. *The number $\delta_r(t, m)$ of $m \times t$ matrices of rank r with entries in F_q, whose rows are non-zero and mutually different, is given by the following expression,*

$$\delta_r(t,m) = |\mathrm{Hom}_{\mathscr{D}_r}(\mathsf{F}_q^t, \mathsf{F}_q^m)| = \begin{bmatrix} t \\ r \end{bmatrix} \sum_{i=0}^{r} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix} (q^{r-i} - 1)(q^{r-i} - 2)\ldots(q^{r-i} - m).$$

NOTE 4.2. The interest in the numbers $\delta_r(t,m)$ originally comes from their applications to coding theory. They appear in the investigations of the distribution of *multigrams* in solutions of maximal length of linear recurring sequences associated to linear codes. For more details see §7–10 of [Ls].

5. Matrices with quadratic symmetric conditions on the entries.

In this section the prime power q will be assumed to be odd. Let S be a regular symmetric bilinear form on V. In a given basis of V, identify S with a regular symmetric $m \times m$ matrix S. Consider the set of all $m \times t$ matrices X such that

$$X'SX = 0.$$

Clearly, the latter set of matrices can be identified with the set $\mathrm{Hom}_{\mathscr{S}}(\mathsf{F}_q^t, V)$ of linear maps defined by the set $\mathscr{S}$ of subspaces of V that are isotropic for S. Recall that a subspce $U \subseteq V$ is said to be *isotropic* for the bilinear form S if S is equal to zero on U.

5.1. Define for $\varepsilon = \pm 1$ a function $\sigma_r^\varepsilon(m)$ as follows: If m is odd, ignore the argument ε, and set

$$\sigma_r(m) := \prod_{i=1}^{r} \frac{q^{m+1-2i} - 1}{q^i - 1} = \prod_{i=1}^{r} \frac{(q^{\frac{m+1}{2}-i} + 1)(q^{\frac{m+1}{2}-i} - 1)}{q^i - 1}.$$

If m is even, set

$$\sigma_r^\varepsilon(m) := \prod_{i=1}^{r} \frac{q^{m+1-2i} + \varepsilon(q-1)q^{\frac{m}{2}-i} - 1}{q^i - 1}$$

$$= \prod_{i=1}^{r} \frac{(q^{\frac{m}{2}-i+1} - \varepsilon)(q^{\frac{m}{2}-i} + \varepsilon)}{q^i - 1} = \frac{q^{\frac{m}{2}-r} + \varepsilon}{q^{\frac{m}{2}} + \varepsilon} \prod_{i=1}^{r} \frac{q^{m+2-2i} - 1}{q^i - 1}$$

(where the last equation assumes $m > 0$ or $\varepsilon \neq -1$). Note that the numerator in the above products contains 0 as a factor when r is large. More precisely,

$$\sigma_r^\varepsilon(m) = 0 \Leftrightarrow \begin{cases} r > \dfrac{m-1}{2} & \text{when } m \text{ is odd,} \\[2ex] r > \dfrac{m}{2} & \text{when } m \text{ is even and } \varepsilon = 1. \\[2ex] r > \dfrac{m}{2} - 1 & \text{when } m \text{ is even and } \varepsilon = -1. \end{cases}$$

LEMMA 5.2. *Let S be a regular symmetric form on V. If the dimension m of V is* *even, define* $\varepsilon = \varepsilon(S)$ *as* $+1$ *or* -1 *according as* $(-1)^{\frac{m}{2}} \det S$ *is a square or a non-square in* F_q^*. *Then, the number of isotropic subspaces of dimension r for the form S is equal to the following expression,*

$$|\mathscr{S}_r| = \sigma_r^\varepsilon(m).$$

PROOF. The starting point of the proof is the following well known formula for the number of solutions in V to the equation $S(x, x) = 0$. The number of solutions is equal to q^{m-1} if m is odd, and equal to $q^{m-1} + \varepsilon(q-1)q^{\frac{m}{2}-1}$ if m is even, see [D, Theorems 65 and 66, pp. 47–48]. Hence the number of non-zero solutions to $S(x, x) = 0$ is equal to the expression,

$$\sigma_1^\varepsilon(m)(q-1) = \begin{cases} q^{m-1} - 1 & \text{when } m \text{ is odd,} \\ q^{m-1} + \varepsilon(q-1)q^{\frac{m}{2}-1} - 1 & \text{when } m \text{ is even.} \end{cases} \tag{5.2.1}$$

The number $|\mathscr{S}_1|$ of dimension-1 isotropic subspaces is obtained by dividing the latter expression by $q - 1$. Therefore, the formula of the Lemma holds for $r = 1$. Clearly, the formula holds for $r = 0$. In particular, the formula holds for $m = 1$ and $m = 2$. The formula is proved in the general case by induction on m.

Assume that $m > 2$. Then there are 1-dimensional isotropic subspaces of V, because the expression (5.2.1) is positive. Fix a 1-dimensional isotropic subspace L of V, and consider its "orthogonal complement" $L^\perp$. If v is a generator of L, then $L^\perp$ is the set of all vectors u such that $S(u, v) = 0$. The complement $L^\perp$ is of dimension $m - 1$, because S is regular. Moreover, $L^\perp$ contains L, because L is isotropic. The restriction, $S^\perp$, of S to $L^\perp$ is not regular, since L is contained in the null space of $S^\perp$. However, consider the form S_0 induced by $S^\perp$ on the quotient $V_0 := L^\perp/L$. Then, as will now be shown, the form S_0 is regular; moreover, $\dim V_0 = m - 2$ and, if m is even, then $\varepsilon(S) = \varepsilon(S_0)$.

To prove the latter assertions, choose a generator v for L, extend the vector v to a basis $(v, u_1, \ldots, u_{m-2})$ for $L^\perp$, and extend the latter set with a vector w to a basis $(v, w, u_1, \ldots, u_{m-2})$ for V. In the latter basis, the form S corresponds to a matrix of the following form,

$$\begin{pmatrix} 0 & a & 0 & \ldots & 0 \\ a & & & \ldots & \\ 0 & & & & \\ \vdots & \vdots & & S_0 & \\ 0 & & & & \end{pmatrix}$$

where $a = S(v, w)$ and S_0 is an $(m-2) \times (m-2)$ matrix. By Laplace development of the determinant of S we have that $\det S = -a^2 \det S_0$. The matrix S_0 is a matrix of the form S_0 defined above. Therefore, the form S_0 is regular and, for m even, $\varepsilon(S) = \varepsilon(S_0)$. Hence the assertions hold.

Consider the set of all isotropic subspaces containing L. Clearly, every isotropic subspace containing L is contained in $L^{\perp}$. Therefore, the isotropic subspaces containing L correspond bijectively to the isotropic subspaces in V_0 for the form S_0. Hence, by the induction hypothesis, the number of isotropic dimension-r subspaces containing L is equal to $\sigma_{r-1}^{\varepsilon}(m-2)$. In particular, the number is independent of L. Therefore, for $r \geqq 1$, the number $|\mathscr{S}_r|$ of all isotropic dimension-r subspaces is equal to $\sigma_{r-1}^{\varepsilon}(m-2)$ multiplied by the number $\sigma_1^{\varepsilon}(m)$ of L's and divided by the number $(q^r-1)/(q-1)$ of L's contained in a dimension-r subspace. Hence we obtain the equation,

$$|\mathscr{S}_r| = \frac{\sigma_1^{\varepsilon}(m)(q-1)}{q^r-1}\cdot\sigma_{r-1}^{\varepsilon}(m-2).$$

The numerator of the fraction is equal to the expression (5.2.1). The asserted formula follows from the definition of $\sigma_r^{\varepsilon}(m)$.

From Lemma 5.2 and Formula (2.1.1), we obtain the following result.

PROPOSITION 5.3. *Consider the $m \times t$ matrix solutions X to the equation,*

$$X'SX = 0.$$

The number $\sigma_r(t,m)$ of rank-r solutions is given by the expression,

$$\sigma_r(t,m) = |\mathrm{Hom}_{\mathscr{S}_r}(\mathsf{F}_q^t, \mathsf{F}_q^m)| = \sigma_r^{\varepsilon}(m)Q_r(q^t), \tag{5.3.1}$$

and the total number $\sigma(t,m)$ of solutions is given by the expression,

$$\sigma(t,m) = |\mathrm{Hom}_{\mathscr{S}}(\mathsf{F}_q^t, \mathsf{F}_q^m)| = \sum_r \sigma_r^{\varepsilon}(m)Q_r(q^t) \tag{5.3.2}$$

where the summation is from $r=0$ to the upper limit given by the inequalities following the definition of σ in 5.1.

EXAMPLE 5.4. The determinant of the form S is well defined modulo the subgroup of squares, $(\mathsf{F}_q^*)^2$. If $m \equiv 0 \pmod 4$, then $\varepsilon(S)=1$ if and only if $\det S \in (\mathsf{F}_q^*)^2$; if $m \equiv 2 \pmod 4$, then $\varepsilon(S)=1$ if and only if $-\det S \in (\mathsf{F}_q^*)^2$. For small values of m we obtain the following expressions for $\sigma(t,m)$:

$$\sigma(t,1),$$

$$\sigma(t,2) = \begin{cases} 1 & \text{if} \quad -\det S \notin (\mathsf{F}_q^*)^2, \\ 1 + 2(q^t-1) & \text{if} \quad -\det S \in (\mathsf{F}_q^*)^2, \end{cases}$$

$$\sigma(t,3) = 1 + (q+1)(q^t-1),$$

$$\sigma(t,4) = 1 + (q^2+1)(q^t-1) \quad \text{if} \quad \det S \notin (\mathsf{F}_q^*)^2.$$

Finally, when $m=4$ and $\det S \in (\mathsf{F}_q^*)^2$,

$$\sigma(t,4) = 1 + (q+1)^2(q^t - 1) + 2(q+1)(q^t-1)(q^t-q).$$

NOTE 5.5. The expression for the number $\sigma_r(t,m)$ in (5.3.1) seems to be new. The number $\sigma(t,m)$ in (5.3.2) was considered by Carlitz [C1, Theorem 4, p. 131], who obtained a different expression. The result of Carlitz is the following: The product $q^{\frac{1}{2}t(t+1)-mt}\sigma(t,m)$ is equal to the following expression when m is odd:

$$1 + \sum_{1<2r\leq t} q^{-(m+1)r} \frac{\prod_{i=0}^{2r-1}(1-q^{t-i})}{\prod_{i=1}^{r}(1-q^{-2i})},$$

and equal to the following expression when m is even:

$$1 + \sum_{1<2r\leq t} q^{-mr} \frac{\prod_{i=0}^{2r-1}(1-q^{t-i})}{\prod_{i=1}^{r}(1-q^{-2i})} - \varepsilon q^{\frac{1}{2}m} \sum_{1<2r\leq t+1} q^{-mr} \frac{\prod_{i=0}^{2r-2}(1-q^{t-i})}{\prod_{i=1}^{r-1}(1-q^{-2i})}.$$

Comparing the expressions (5.3.2) with the expressions of Carlitz we obtain a q-identity for each $m = 1, 2, \ldots$. The expressions of Carlitz can be interpreted as special values of certain generalized hypergeometric series, see [C1]. The identity obtained for $m = 1$ was observed by Carlitz [C1, Formula (4.9), p. 129]. It is the following:

$$q^{\frac{1}{2}m(m+1)} = \sum_{0\leq 2r\leq m+1} q^{-2r} \frac{\prod_{i=0}^{2r-1}(1-q^{m+1-i})}{\prod_{i=1}^{r}(1-q^{-2i})}.$$

6. Matrices with quadratic alternating conditions on the entries.

Let again q be an arbitrary prime power. Assume that the dimension m of V is even. Let A be a regular alternating bilinear form on V. In a given basis of V, identify A with a regular alternating $m \times m$ matrix A. Consider the set of all $m \times t$ matrices X such that

$$X'AX = 0.$$

Clearly, the latter set of matrices can be identified with the set of linear maps $\mathrm{Hom}_{\mathscr{A}}(\mathsf{F}_q^t, V)$ defined by the set $\mathscr{A}$ of subspaces of V that are isotropic for A. Recall that a subspace $U \subseteq V$ is said to be *isotropic* for the bilinear form A if A is equal to zero on U. It is easy to determine the number $|\mathscr{A}_r|$ of isotropic dimension-r subspaces. Indeed, consider a basis $(v_1, \ldots, v_i)$ for an i-dimensional isotropic subspace U. Let v be an arbitrary vector. Then $(v_1, \ldots, v_i, v)$ is a basis for an $(i+1)$-dimensional isotropic subspace, if and only if $v \notin U$ and $v \in U^\perp$. The complement $U^\perp$ has dimension $m - i$ because A is regular and $U \subseteq U^\perp$ because A is alternating. Hence the number of possible v's is equal to $q^{m-i} - q^i$. It follows by induction that the number of bases of isotropic dimension-r subspaces is equal to the product,

$$(q^m - 1)(q^{m-1} - q)(q^{m-2} - q^2)\dots(q^{m-r+1} - q^{r-1}).$$

Hence, the number of isotropic dimension-r subspaces is equal to the latter product divided by the number of bases for an r-dimensional subspace, that is, divided by $Q_r(q^r)$. Thus the number of isotropic dimension-r subspaces of V is equal to

$$\alpha_r(m) = \prod_{i=1}^{r} \frac{q^{m+2-2i} - 1}{q^i - 1}.$$

Therefore, from Formula (2.1.1) we obtain the following result.

PROPOSITION 6.1. *Consider the $m \times t$ matrix solutions X to the equation,*

$$X'AX = 0.$$

The number $\alpha_r(t, m)$ of rank-r solutions X is given by the expression,

(6.1.1) $$\alpha_r(t, m) = |\mathrm{Hom}_{\mathscr{A}}(\mathsf{F}_q^t, V)| = \prod_{i=1}^{r} \frac{q^{m+2-2i} - 1}{q^i - 1} Q_r(q^t),$$

and the total number $\alpha(t, m)$ of solutions is given by the expression,

(6.1.2) $$\alpha(t, m) = |\mathrm{Hom}_{\mathscr{A}}(\mathsf{F}_q^t, V)| = \sum_{r=0}^{m/2} \prod_{i=1}^{r} \frac{q^{m+2-2i} - 1}{q^i - 1} Q_r(q^t).$$

NOTE 6.2. It follows from the expressions for the number $\alpha_r(m)$ above and the number $\sigma_r(m)$ in Section 5 that the following equation holds,

$$\alpha_r(m) = \sigma_r(m + 1).$$

The latter equation has the following interpretation that seems to the authors to be a strange coincidence. Work over a finite field with an odd number q of elements. Assume that m is even. Let A be an alternating regular $m \times m$ matrix and let S be a symmetric regular $(m + 1) \times (m + 1)$ matrix. Then the number of dimension-r isotropic subspaces for A is equal to the number of dimension-r isotropic subspaces for S.

NOTE 6.3. An alternative expression for the number $\alpha(t, m)$ in (6.1.2) was obtained by Carlitz in [C2, Theorem 4, p. 25] using exponential sums. The result of Carlitz is the following:

$$q^{\frac{1}{2}t(t-1)}\alpha(t, m) = \sum_{0 \leq 2r \leq t} q^{m(t-r)-2r} \frac{\prod_{i=0}^{2r-1}(1 - q^{t-i})}{\prod_{i=1}^{r}(1 - q^{-2i})}.$$

In [C2], the exponent of q in the first term in the sum was erroneously given as $m(2t - r) - 2r$.

As in Section 5, the expressions of Carlitz can be interpreted as special values of hypergeometric series. Comparing the expression of Carlitz with the expression (6.1.2) we obtain for an even integer m the following q-identity:

$$\sum_{0 \leqq 2r \leqq t} q^{m(t-r)-2r} \frac{\prod_{i=0}^{2r-1}(1-q^{t-i})}{\prod_{i=1}^{r}(1-q^{-2i})} = q^{\frac{1}{2}t(t-1)} \sum_{r=0}^{m/2} \prod_{i=1}^{r} \frac{(q^{m+2-2i}-1)(q^t-q^{i-1})}{q^i-1}.$$

7. Recursion formulas.

In this section we return to the setup of Section 1. We prove a recursion formula for the coefficients in the interpolation formulas. When applied to the sequence $\lambda_i := q^{i-1}$, we recover the recursion formulas for the numbers $\phi_r(t,m)$ and $\delta_r(t,m)$ considered in Sections 4 and 5.

LEMMA 7.1. *The polynomial $f(x)$ can be factored in $R[x]$ as $f(x) = (x-\lambda)g(x)$ if and only if the recursion formulas,*

$$\begin{bmatrix} f \\ r \end{bmatrix} = (\lambda_{r+1} - \lambda)\begin{bmatrix} g \\ r \end{bmatrix} + \begin{bmatrix} g \\ r-1 \end{bmatrix},$$

or equivalently, the formulas

$$\begin{bmatrix} f \\ r \end{bmatrix} Q_r(x) = (\lambda_{r+1} - \lambda)\begin{bmatrix} g \\ r \end{bmatrix} Q_r(x) + (x-\lambda_r)\begin{bmatrix} g \\ r-1 \end{bmatrix} Q_{r-1}(x),$$

hold for $r = 1, 2, \ldots$.

PROOF. All the assertions of the Lemma follow immediately from the formulas

$$(x-\lambda)Q_r(x) = (\lambda_{r+1}-\lambda)Q_r(x) + Q_{r+1}(x) \quad \text{for} \quad r = 0, 1, \ldots.$$

7.2. As we saw in Section 3 and 4, the two conditions (1) and (2) considered in the Introduction correspond to the numbers $\phi_r(t,m)$ and $\delta_r(t,m)$ defined in Proposition 3.1 and Proposition 4.1. By Formula (2.2.4) and Note 3.2,

$$\phi_r(t,m) = \begin{bmatrix} f_m \\ r \end{bmatrix} Q_r(q^t),$$

where $f_m = x^m$. Hence, by applying Lemma 7.1 to the factorization $f_m(x) = x f_{m-1}(x)$, we obtain the following recursion formulas of Landsberg [Lb],

$$\phi_r(t,m) = q^r \phi_r(t,m-1) + (q^t - q^{r-1})\phi_{r-1}(t,m-1).$$

Similarly, by formula (2.2.4) and the analysis of Section 4,

$$\delta_r(t,m) = \begin{bmatrix} f_m \\ r \end{bmatrix} Q_r(q^t)$$

where $f_m(x) = (x-1)(x-2)\ldots(x-m)$. Here $f_m(x) = (x-m)f_{m-1}(x)$, and we obtain the recursion formulas of Laksov [Ls],

$$\delta_r(t,m) = (q^r - m)\delta_r(t,m-1) + (q^t - q^{r-1})\delta_{r-1}(t,m-1).$$

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FREE ARCHIMEDEAN l-GROUPS

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Abstract.

In this paper we discuss the existence and description of the free archimedean l-group $\mathscr{F}_{\mathscr{A}r}([G, P])$ generated by a po-group [G, P], and give some properties of the free abelian l-group (the free archimedean l-group) $\mathscr{A}_\alpha$ of rank α.

We use the standard terminologies and notations of [1, 5, 9]. We assume that all groups considered will be abelian. The group operation of an l-group is written by additive notation. Let G be an l-group and $S \subseteq G$. We denote by [S] the l-subgroup of G generated by S. The convex l-subgroup generated by an element $g \in G$ is denoted by $G(g)$. A po-group is a partially ordered group $[G, P]$ where $P = \{x \in G \mid x \geqq 0\}$ is the positive semigroup of G. P is said to be semi-group if $p \in P$ whenever $p \in G$ and $np \in P$ for some positive integer n. Let G and H be two po-groups. A map φ from G into H is called a po-group homomorphism, if φ is a group homomorphism and $x \geqq y$ implies $\varphi(x) \geqq \varphi(y)$ for any x, $y \in G$. A po-group homomorphism φ is called a po-group isomorphism if φ is an injection and φ^{-1} is also a po-group homomorphism. We use N and Z for the natural numbers and the integers, respectively.

1. Sub-product Radical Class of Archimedean l-groups.

A family $\mathscr{U}$ of l-groups is called a sub-product radical class, if it is closed under taking 1) l-subgroups, 2) joins of convex l-subgroups and 3) direct products. All our sub-product radical classes are always assumed to contain along with a given l-group all its l-isomorphic copies. Let $\mathscr{U}$ be a sub-product radical class and G be an l-group. Then the join of all convex l-subgroups of G belonging to $\mathscr{U}$ is the unique largest convex l-subgroup of G belonging to $\mathscr{U}$. It is denoted by $\mathscr{U}(G)$ and is called a sub-product radical of G. $\mathscr{U}(G)$ is a characteristic l-ideal of G.

An l-group G is said to be archimedean if it satisfies one of the following three equivalent conditions:

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1. For any $0 < a, b \in G$, there exists $n \in N$ such that $nb \nleqq a$.
2. For all $a, b \in G$, if $nb \leqq a$ for all $n \in Z$, then $b = 0$.
3. For all $a, b \in G$, if $nb \leqq a$ for all $n \in N$, then $b \leqq 0$.

Let G be an l-group. An element $a \in G$ is archimedean if $a \geqq 0$ and if for all $0 < b \leqq a$, there exists $n \in N$ such that $nb \nleqq a$ [12, 18]. Let $P(G)$ be the set of all archimedean elements of G. An element $a \in G$ is said to be generally archimedean if the positive part a^+ and the negative part a^- are both archimedean. The following lemma is easy to show using [18].

LEMMA 1.1. *Let G be an l-group and $g \in G$. Then the following are equivalent:*
(1) *g is generally archimedean.*
(2) *$|g|$ is archimedean.*
(3) *$G(g)$ is archimedean.*
(4) *$G(|g|)$ is archimedean.*

Let $\mathscr{A}r$ be the family of all archimedean l-groups. $\mathscr{A}r$ is a quasi-torsion class [13], that is, $\mathscr{A}r$ is closed under taking 1) convex l-subgroups, 2) joins of convex l-subgroups and 3) complete l-homomorphisms. It is clear that $\mathscr{A}r$ is closed under taking l-subgroups and direct products. So $\mathscr{A}r$ is a sub-product radical class. Let G be an l-group. Then there exists a unique largest archimedean l-subgroup of G, the $\mathscr{A}r$ radical $\mathscr{A}r(G)$. Clearly, G is archimedean if and only if $G = \mathscr{A}r(G)$. In [18] it was proved that the l-subgroup $A(G)$ of G is the unique largest archimedean convex l-subgroup of G. In [12] J. Jakubik also proved the existence of such $A(G)$. So we have $\mathscr{A}r(G) = [P(G)]$. By Theorem 1.3 of [5] $\mathscr{A}r(G)$ consists of the elements $g = x - y$ where $x, y \in P(G)$ and $x \wedge y = 0$. In fact, $x = g^+$ and $y = g^-$. And so such g are generally archimedean. Conversely, if $g \in G$ is a generally archimedean element, then $g \in \mathscr{A}r(G)$. Thus Lemma 1.1 infers

LEMMA 1.2.
$$\begin{aligned}\mathscr{A}r(G) &= [P(G)]\\ &= \{g \in G \mid g \text{ is generally archimedean}\}\\ &= \{g \in G \mid |g| \in P(G)\}\\ &= \{g \in G \mid G(g) \text{ is archimedean}\}\\ &= \{g \in G \mid G(|g|) \text{ is archimedean}\}.\end{aligned}$$

COROLLARY 1.3. *The set of all generally archimedean elements of an l-group G is closed under the addition, inverse, met and join.*

So we obtain a useful result.

PROPOSITION 1.4. *Suppose that an l-group G has a set of generators which consists of generally archimedean elements. Then G is archimedean.*

In what follows we will give an application of Proposition 1.4.

2. Free Archimedean l-group Generated by a po-group.

A partial l-group G is a set with partial operations corresponding to the usual l-group operations $.$, -1, 1, $\vee$ and $\wedge$ such that whenever the operations are defined for elements of G then the l-group laws are satisfied. Suppose $[G, P]$ is a po-group. Then G has implicit partial operations $\vee$ and $\wedge$ as determined by the partial order. That is,

$$x \vee y = y \vee x = y \text{ if and only if } x \leqq y \text{ and}$$

$$x \wedge y = y \wedge x = x \text{ if and only if } x \leqq y.$$

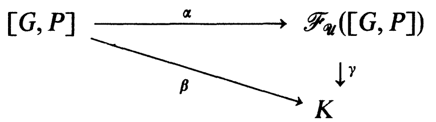
Using these two partial lattice operations together with the full group operations, G can be considered as a partial l-group. Thus we have the following definition as a special case of the $\mathscr{U}$-free algebra generated by a partial algebra.

DEFINITION 2.1. Let $\mathscr{U}$ be a class of l-groups and $[G, P]$ be a po-group. The l-group $\mathscr{F}_{\mathscr{U}}([G, P])$ is called the *$\mathscr{U}$-free l-group generated by* $[G, P]$ (or $\mathscr{U}$-free l-group over $[G, P]$) if the following conditions are satisfied:

(1) $\mathscr{F}_{\mathscr{U}}([G, P]) \in \mathscr{U}$;

(2) there exists an injective po-group isomorphism $\alpha: G \to \mathscr{F}_{\mathscr{U}}([G, P])$ such that $\alpha(G)$ generates $\mathscr{F}_{\mathscr{U}}([G, P])$ as an l-group;

(3) if $K \in \mathscr{U}$ and $\beta: G \to K$ is a po-group homomorphism, then there exists an l-homomorphism γ: $\mathscr{F}_{\mathscr{U}}([G, P]) \to K$ such that $\gamma\alpha = \beta$.



The classes of l-groups which will be refered to are $\mathscr{A}r$ and the following:

$\mathscr{L}$, the class of all l-groups,

$\mathscr{A}$, the class of all abelian l-groups.

$\mathscr{L}$, $\mathscr{A}$ and $\mathscr{A}r$ are all sub-product radical classes of l-groups.

In 1963 and 1965, E. C. Weinberg initially considered the $\mathscr{A}$-free l-group generated by a po-group $[G, P]$. He has given a necessary and sufficient condition for existence and a simple description of $\mathscr{F}_{\mathscr{A}}([G, P])$ as follows:

PROPOSITION 2.2. [17, 18]. *Let $[G, P]$ be a torsion-free abelian* po-*group.*

(1) *There exists an $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}([G, P])$ generated by $[G, P]$ if and only if there exists a* po-*group isomorphism of $[G, P]$ into an abelian l-group, if and only if P is semi-closed.*

(2) *Let $\mathscr{P}$ be the set of all total orders T of G such that $P \subseteq T$. Then $\mathscr{F}_{\mathscr{A}}([G, P])$ is*

the sublattice of the direct product $\prod_{T \in \mathscr{P}} [G, T]$ *which is generated by the long constants* $\langle g \rangle$ $(g \in G)$.

The elements of $\mathscr{F}_{\mathscr{A}}([G, P])$ *have the form*

$$x = \bigvee_{i \in I} \bigwedge_{j \in J} \langle x_{ij} \rangle$$

where I *and* J *are both finite and* $x_{ij} \in G$ $(i \in I, j \in J)$.

In 1970, P. Conrad generalized Weiberg's result.

PROPOSITION 2.3 [6]. *Let* $[G, P]$ *be a torsion-free* po-*group.*

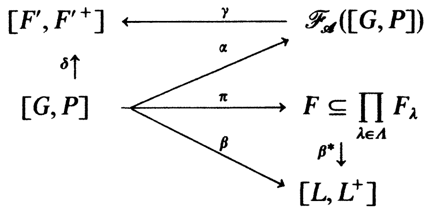
(1) *There exists an* $\mathscr{L}$*-free l-group* $\mathscr{F}_{\mathscr{L}}([G, P])$ *generated by* $[G, P]$ *if and only if there exists a* po-*group isomorphism of* $[G, P]$ *into an l-group, if and only if* P *is the intersection of right orders on* G.

(2) *Suppose that* $P = \bigcap_{\lambda \in \Lambda} P_\lambda$ *where* $\{P_\lambda \mid \lambda \in \Lambda\}$ *is the set of all right orders of* G *such that* $P_\lambda \supseteq P$. *If* G_λ *is* G *with one such right order, then denote by* $A(G_\lambda)$ *the l-group of order preserving permutations of* G_λ. *Each* $x \in G$ *corresponds to an element* ρ_x *of* $A(G_\lambda)$ *defined by* $\rho_x g = g + x$. *Then* $\mathscr{F}_{\mathscr{L}}([G, P])$ *is the sublattice of the direct product* $\prod_{\lambda \in \Lambda} A(G_\lambda)$ *which is generated by the long constants* $\langle g \rangle (g \in G)$.

In this section we will discuss the $\mathscr{A}r$-free l-group $\mathscr{F}_{\mathscr{A}r}([G, P])$ generated by a po-group $[G, P]$. Because $\mathscr{A}r$ is a sub-product radical class of l-groups, by Grätzer's existence theorem on a free algebra generated by a partial algebra (see Theorem 28.2 of [10]) we have

THEOREM 2.4. *There exists an* $\mathscr{A}r$*-free l-group* $\mathscr{F}_{\mathscr{A}r}([G, P])$ *generated by a* po-*group* $[G, P]$ *if and only if* $[G, P]$ *is* po-*group isomorphic into an archimedean l-group.*

Now suppose that a po-group $[G, P]$ is po-group isomorphic into an archimedean l-group $[F', F'^{+}]$ with the po-group isomorphism δ. Thus $[G, P]$ must be torsion-free abelian and semi-closed. By Proposition 2.2(1) there exists the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}([G, P])$ generated by $[G, P]$ with the po-group isomorphism α of $[G, P]$ into $\mathscr{F}_{\mathscr{A}}([G, P])$. By definition 2.1 there exists an l-homomorphism γ from $\mathscr{F}_{\mathscr{A}}([G, P])$ into F' such that $\gamma\alpha = \beta$. Let $D = \{F_\lambda \mid \lambda \in \Lambda\}$ be the set of all archimedean l-homomorphism images of $\mathscr{F}_{\mathscr{A}}([G, P])$ with the l-homomorphism β_λ. Thus $\gamma\mathscr{F}_{\mathscr{A}}([G, P]) \in D$ and D is not empty. For each $\lambda \in \Lambda$, $\gamma_\lambda \alpha$ is a po-group homomorphism of $[G, P]$ into F_λ. The direct product $\prod_{\lambda \in \Lambda} F_\lambda$ is an archimedean l-group. Let π be the natural map of the po-group G



onto the subgroup G' of long constants of $\prod_{\lambda \in \Lambda} F_\lambda$. That is, $\pi(g) = (\cdots, \gamma_\lambda \alpha(g), \cdots)$ for $g \in G$. Because $\gamma\alpha = \delta$ is a po-group isomorphism, π is a po-group isomorphism of G onto G'. Let F be the sublattice of $\prod_{\lambda \in \Lambda} F_\lambda$ generated by G'. For each $g \in G$, let $g' = \pi(g)$ denote the long constant of G'. Since $\prod_{\lambda \in \Lambda} F_\lambda$ is a distributive lattice, the sublattice generated by all g' is

$$F = \left\{ \bigvee_{i \in I} \bigwedge_{j \in J} g'_{ij} \,|\, g_{ij} \in G, I \text{ and } J \text{ finite} \right\}.$$

Suppose that β is a po-group homomorphism of $[G, P]$ into an archimedean l-group $[L, L^{+}]$. Then there exists an l-homomorphism γ' of $\mathscr{F}_{\mathscr{A}}([G,P])$ into $[L, L^{+}]$ such that $\gamma'\alpha = \beta$. So $\gamma' \mathscr{F}_{\mathscr{A}}([G,P]) \in D$. Now we extend β to F as follows:

$$\beta^*\left(\bigvee_{i \in I} \bigwedge_{j \in J} g'_{ij} \right) = \bigvee_{i \in I} \bigwedge_{j \in J} \beta(g_{ij}).$$

To see that β^* is well defined, suppose that

$$\bigvee_{i \in I} \bigwedge_{j \in J} \beta(g_{ij}) \neq \bigvee_{m \in M} \bigwedge_{n \in N} \beta(h_{mn}).$$

Then we have

$$\bigvee_{i \in I} \bigwedge_{j \in J} \bigvee_{m \in M} \bigvee_{f \in N^M} \beta(g_{ij} - h_{mf(m)}) \neq 0$$

in $[L, L^{+}]$. Because $\gamma' \mathscr{F}_{\mathscr{A}}([G,P]) \in D$,

$$\bigvee_{i \in I} \bigwedge_{j \in J} \bigvee_{m \in M} \bigvee_{f \in N^M} \beta(g'_{ij} - h'_{mf(m)}) \neq 0$$

in F. That is, we have

$$\bigvee_{i \in I} \bigwedge_{j \in J} g'_{ij} \neq \bigvee_{m \in M} \bigwedge_{n \in N} h'_{mn}$$

in F. Therefore β^* is single valued.

That β^* is a lattice homomorphism is an immediate consequence of the fact that L is a distributive lattice. Now consider $g = \bigvee_{i\in I}\bigwedge_{j\in J} g'_{ij}$ and $h = \bigvee_{m\in M}\bigwedge_{n\in N} h'_{mn}$ in F.

$$\begin{aligned}\beta^*(g-h) &= \bigvee_{i\in I}\bigwedge_{j\in J}\bigwedge_{m\in M}\bigvee_{f\in N^M}(g_{ij}-h_{mf(m)})' \\ &= \bigvee_{i\in I}\bigwedge_{j\in J}\bigwedge_{m\in M}\bigvee_{f\in N^M}\beta(g_{ij}-h_{mf(m)}) \\ &= \bigvee_{i\in I}\bigwedge_{j\in J}\beta(g_{ij}) - \bigvee_{m\in M}\bigwedge_{n\in N}\beta(h_{mn}) \\ &= \beta^*(g)-\beta^*(h).\end{aligned}$$

Hence β^* is an l-homomorphism of F into L and $\beta^*\pi = \beta$.

The above discussion proves the following theorem.

THEOREM 2.5. *Suppose that a* po-*group* $[G, P]$ *is* po-*group isomorphic into an archimedean l-group. Then the* $\mathscr{A}r$*-free l-group* $\mathscr{F}_{\mathscr{A}r}([G, P])$ *generated by* $[G, P]$ *is the sublattice F of the direct product* $\prod_{\lambda\in\Lambda} F_\lambda$ *which is generated by the long constants* g' $(g \in G)$ *where* $\{F_\lambda \mid \lambda \in \Lambda\}$ *are all archimedean l-homomorphic images of the* $\mathscr{A}$*-free l-group* $\mathscr{F}_{\mathscr{A}}([G, P])$ *generated by* $[G, P]$.

NOTE. Suppose that a po-group $[G, P]$ is po-group isomorphic into an archimedean l-group. By Proposition 2.3 there exists an $\mathscr{L}$-free l-group $\mathscr{F}_{\mathscr{L}}([G, P])$ generated by $[G, P]$. If we take $\mathscr{F}_{\mathscr{L}}([G, P])$ instead of $\mathscr{F}_{\mathscr{A}}([G, P])$ in the above discussion, we obtain another description of $\mathscr{F}_{\mathscr{A}r}([G, P])$.

Let $\mathscr{U}$ be a class of algebras and X be a nonempty set. The algebra $\mathscr{F}_{\mathscr{U}}(X)$ is called the $\mathscr{U}$-free algebra on X if X generates $\mathscr{F}_{\mathscr{U}}(X)$ as an algebra, and whenever $L \in \mathscr{U}$ and λ: $X \to L$ is a map, then there exists a homomorphism σ: $\mathscr{F}_{\mathscr{U}}(X) \to L$ which extends λ. By Birkhoff's Theorem ([4]) there exists a $\mathscr{U}$-free algebra $\mathscr{F}_{\mathscr{U}}(X)$ on any nonempty set X if $\mathscr{U}$ is closed under subalgebras and direct products. Let $\mathscr{U}$ be a class of l-groups and X be a nonempty set with $|X| = \alpha$. Then the $\mathscr{U}$-free l-group $\mathscr{F}_{\mathscr{U}}(X)$ on X is said to be of rank α. We can construct the $\mathscr{U}$-free l-group $\mathscr{F}_{\mathscr{U}}(X)$ on X using the $\mathscr{U}$-free l-group generated by a trivially ordered group. Let $\mathscr{U}$ be a class of l-groups which is closed under l-subgroups and direct products. We denote by $\mathscr{G}(\mathscr{U})$ the class of all groups that can be embedded (as subgroups) into the members of $\mathscr{U}$. It is clear that $\mathscr{G}(\mathscr{U})$ is closed under subgroups and direct products.

PROPOSITION 2.6. *Let* $\mathscr{U}$ *be a class of l-groups which is closed under l-subgroups and direct products and X be a nonempty set. The* $\mathscr{U}$*-free l-group* $\mathscr{F}_{\mathscr{U}}(X)$ *on X is the* $\mathscr{U}$*-free l-group generated by the* $\mathscr{G}(\mathscr{U})$*free group on X with trivial order.*

PROOF. By Birkhoff's Theorem there exists the $\mathscr{G}(\mathscr{U})$-free group $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ on X. $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X) \in \mathscr{G}(\mathscr{U})$ means $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ can be embedded (as a subgroup) into a member of $\mathscr{U}$. By Theorem 28.2 of [10] there exists a $\mathscr{U}$-free l-group $\mathscr{F}_{\mathscr{U}}([\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X), \{0\}])$ generated by the trivially ordered $\mathscr{G}(\mathscr{U})$-free group $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$. Now any map from X into an l-group $L \in \mathscr{U}$ can be extended to a group homomorphism of $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ into L and hence to an l-homomorphism of $\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X)$ into L and hence to an l-homomorphism of $\mathscr{F}_{\mathscr{U}}([\mathscr{F}_{\mathscr{G}(\mathscr{U})}(X), \{0\}])$ into L.

Theorem 2.7 of [14] is a special case of the above Proposition 2.6. The following theorem is a consequence of Proposition 2.6.

THEOREM 2.7. *Let X be a nonempty set. The $\mathscr{A}r$-free l-group $\mathscr{F}_{\mathscr{A}r}(X)$ on X is the $\mathscr{A}r$-free l-group generated by the $\mathscr{G}(\mathscr{A}r)$-free group $\mathscr{F}_{\mathscr{G}(\mathscr{A}r)}(X)$ with trivial order.*

$$X \to \mathscr{F}_{\mathscr{G}(\mathscr{A}r)}(X) \to \mathscr{F}_{\mathscr{A}r}(X).$$

PROPOSITION 2.8. *Suppose that $\mathscr{F}_{\mathscr{A}r}([G, P_1])$ and $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ are the $\mathscr{A}r$-free l-groups generated by* po-*group* $[G, P_1]$) *and* $[G, P_2]$, *respectively. If* $P_1 \subseteq P_1$. *Then* $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ *is an l-homomorphic image of* $\mathscr{F}_{\mathscr{A}r}([G, P_1])$.

PROOF. $[G, P_2]$ can be embedded into $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ as a po-group and G generates $\mathscr{F}_{\mathscr{A}r}([G, P_2])$. So $[G, P_1]$ is also embedded into $\mathscr{F}_{\mathscr{A}r}([G, P_2])$ as a po-group. Hence there exists an l-homomorphism φ from $\mathscr{F}_{\mathscr{A}r}([G, P_1])$ into $\mathscr{F}_{\mathscr{A}r}([G, P_2])$. But $[G, P_1]$ can be embedded into $\mathscr{F}_{\mathscr{A}r}([G, P_1])$ as a po-group and G generates $\mathscr{F}_{\mathscr{A}r}([G, P_1])$. Therefore φ is onto $\mathscr{F}_{\mathscr{A}r}([G, P_2])$.

3. The Relation between $\mathscr{F}_{\mathscr{A}}([G, P])$ and $\mathscr{F}_{\mathscr{A}r}([G, P])$.

In [6] P. Conrad has given the relation between the $\mathscr{L}$-free l-group $\mathscr{F}_{\mathscr{L}}(X)$ and the $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}(X)$ on a nonempty set X. Let Y be the l-ideal generated by the commutator subgroup $[\mathscr{F}_{\mathscr{L}}(X), \mathscr{F}_{\mathscr{L}}(X)]$. Then $\mathscr{F}_{\mathscr{A}}(X) \cong \mathscr{F}_{\mathscr{L}}(X)/Y$.

In this section we will give the relation between $\mathscr{F}_{\mathscr{A}}([G, P])$ and $\mathscr{F}_{\mathscr{A}r}([G, P])$. Clearly, if $\mathscr{F}_{\mathscr{A}}([G, P])$ is archimedean then $\mathscr{F}_{\mathscr{A}}([G, P]) \cong \mathscr{F}_{\mathscr{A}r}([G, P])$. We will give a necessary and sufficient condition in which $\mathscr{F}_{\mathscr{A}}([G, P])$ is archimedean. First we need some concepts. Let $[G, P]$ be a torsion free abelian po-group and S be a nonempty subset of G. S is said to be positively independent if for any finite subset $\{x_1, \cdots, x_k\}$ of S and non-negative integers $\{\lambda_1, \ldots, \lambda_k\}$, $\sum_{i=1}^{k} \lambda_i x_i \in -P$ only if $\lambda_i = 0 (i = 1, \ldots, k)$. There exists a total order P_1 of G such that $P_1 \supseteq P \cup S$ if and only if S is positively independent. Let $x = \bigvee_{i \in I} \bigwedge_{j \in J} \langle x_{ij} \rangle \in \mathscr{F}_{\mathscr{A}}([G, P])$. Then $x \nleq 0$ if and only if for some $i \in I$ the set $\{x_{ij} \mid j \in J\}$ is positively independent [3].

A po-group $[G, P]$ is said to be strong uniformly archimedean if, given $u \in G$ and a positively independent subset $\{v_1, \ldots, v_k\}$ of G, there exists $n \in N$ such that if

$\lambda_1,\cdots,\lambda_k$ are non-negative integers and $\sum_{i=1}^{k} \lambda_i \geqq mn$ with $m \in N$, then $\sum_{i=1}^{k} \lambda_i v_i \nleq mu$.

THEOREM 3.1. *The $\mathscr{A}$-free l-group $\mathscr{F}_{\mathscr{A}}([G, P])$ generated by a* po-*group* $[G, P]$ *is archimedean if and only if* $[G, P]$ *is strong uniformly archimedean.*

PROOF. Necessity. Suppose that $u \in G$ and $\{v_1, \ldots, v_k\}$ is a positively independent subset of G. Then, $\langle v_1 \rangle^+ \wedge \cdots \wedge \langle v_k \rangle^+ \neq 0$ in $\mathscr{F}_{\mathscr{A}}([G, P])$. Since $\mathscr{F}_{\mathscr{A}}([G, P])$ is archimedean, there exists $n \in N$ such that

$$n(\langle v_1 \rangle \wedge \cdots \wedge \langle v_k \rangle)^+ = n(\langle v_1 \rangle^+ \wedge \cdots \wedge \langle v_k \rangle^+) \nleq \langle u \rangle^+.$$

It follows that if $\lambda \geqq n$, $\lambda(\langle v_1 \rangle \wedge \cdots \wedge \langle v_k \rangle) \nleq \langle u \rangle$. Now if $\lambda_1, \cdots, \lambda_k$ are non-negative integers and $\sum_{i=1}^{k} \lambda_i \geqq mn$ with $m \in N$, then we have

$$\sum_{i=1}^{k} \lambda_i \langle v_i \rangle \geqq \left(\sum_{i=1}^{k} \lambda_i\right)(\langle v_1 \rangle \wedge \cdots \wedge \langle v_k \rangle) \nleq m\langle u \rangle,$$

because P is semi-closed and $mn(\langle v_1 \rangle \wedge \cdots \wedge \langle v_k \rangle) \leqq \left(\sum_{i=1}^{k} \lambda_i\right)(\langle v_1 \rangle \wedge \cdots \wedge \langle v_k \rangle) \leqq m\langle u \rangle$ would imply $n(\langle v_1 \rangle \wedge \cdots \wedge \langle v_k \rangle) \leqq \langle u \rangle$, a contradiction. Hence we have $\sum_{i=1}^{k} \lambda_i v_i \nleq mu$ in $[G, P]$.

Sufficiency. It follows from Proposition 1.4 that it suffices to show that $\langle g \rangle$ is generally archimedean in $\mathscr{F}_{\mathscr{A}}([G, P])$ for each $g \in G$. And because G is a group and $g^- = (-g) \vee 0$, it suffices to show that g^+ is archimedean in $\mathscr{F}_{\mathscr{A}}([G, P])$ for each $g \in G$. Let $g \in G$ and $0 < x = \bigvee_{i \in I} \bigwedge_{j \in J} \langle x_{ij} \rangle \in \mathscr{F}_{\mathscr{A}}([G, P])$ where $x_{ij} \in G$. We must show there exists $n \in N$ such that $nx \nleq g^+$. Since $x > 0$, the set $\{x_{ij} \mid j \in J\}$ is positively independent for some i. It suffices to show that there exists $n \in N$ such that $n(\bigwedge_{j \in J} \langle x_{ij} \rangle) \nleq \langle g \rangle \vee 0$. And so it suffices to show that if $\{v_1, \cdots, v_k\}$ is a positively independent subset of G and $g \in G$, then there exists $n \in N$ and a total order T of G such that $T \supseteq P$, $v_i \in T$ and $nv_i - g \in T$ $(i = 1, \cdots, k)$. Then, lifting the identity map of $[G, P]$ onto $[G, T]$ to an l-homomorphism of $\mathscr{F}_{\mathscr{A}}([G, P])$ onto $[G, T]$ we would have $\bigwedge_{i=1}^{k} [(nv_i - g) \wedge nv_i] \nleq 0$, and so $n\left(\bigwedge_{i=1}^{k} v_i\right) \nleq g \vee 0$.

It therefore suffices to show that there exists $n \in N$ so that the set

$$\{v_i \mid i = 1, \ldots, k\} \cup \{nv_i - g \mid i = 1, \ldots, k\}$$

is positively independent. Because $[G, P]$ is strong uniformly archimedean, there exists $n \in N$ such that if $\lambda_1, \ldots, \lambda_k$ are non-negative integers and $\sum_{i=1}^{k} \lambda_i \geqq mn$ with $m \in N$, then $\sum_{i=1}^{k} \lambda_i v_i \nleqq mg$. Suppose that $\mu_1, \ldots, \mu_k$ and $v_1, \ldots, v_k$ are all non-negative integers and

$$\sum_{i=1}^{k} \mu_i v_i + \sum_{i=1}^{k} v_i(nv_i - g) \in -P.$$

Then $\sum_{i=1}^{k} (\mu_i + nv_i)v_i \leqq \left(\sum_{i=1}^{k} v_i\right) g$ which contradicts the choice of n unless all v_i are zero and then contradicts positive independence of the v_i unless all μ_i are zero. Thus $\{v_i \mid i = 1, \ldots, k\} \cup \{nv_i - g \mid i = 1, \ldots, k\}$ is positively independent.

COROLLARY 3.2. *Suppose that a* po-*group* $[G, P]$ *is* po-*group isomorphic into an archimedean l-group. Then* $\mathcal{F}_{\mathcal{A}}([G, P]) \cong \mathcal{F}_{\mathcal{A}r}([G, P])$ *if and only if* $[G, P]$ *is strong uniformly archimedean.*

Let G be a group. A nonempty subset S of G is said to be indepedent if for any finite subset $\{x_1, \ldots, x_k\}$ of S and non-negative integers $\{\lambda_1, \ldots, \lambda_k\}$, $\sum_{i=1}^{k} \lambda_i x_i = 0$ only if $\lambda_i = 0$ $(i = 1, \cdots, k)$. Clearly, S is independent in G if and only if S is positively independent in the po-group $[G, \{0]\}]$ with the trivial order. Let G be a torsion-free and abelian group. Weinberg has proved that the $\mathcal{A}$-free l-group $\mathcal{F}_{\mathcal{A}}([G, \{0\}])$ is archimedean (Corollary 3.4 of [17]). From this we get

COROLLARY 3.3. *Suppose that G is a torsion-free and abelian group. Given* $u \in G$ *and an independent subset* $\{v_1, \ldots, v_k\}$ *of G, then there exists* $n \in N$ *such that if* $\lambda_1, \ldots, \lambda_k$ *are non-negative integers and* $\sum_{i=1}^{k} \lambda_i \geqq mn$ *with* $m \in N$, $\sum_{i=1}^{k} \lambda_i v_i \neq mu$.

4. Some properties of an archimedean l-group.

In order to discuss properties of $\mathcal{F}_{\mathcal{A}r}([G, P])$ we need to know some properties of an Archimedean l-group. First we introduce some concepts. Let $\{G_\alpha \mid \alpha \in A\}$ be a system of l-groups. For $g \in \prod_{\alpha \in A} G_\alpha$, we denote by g_α the α component of g. An l-group G is said to be an ideal subdirect sum of l-groups G_α, in symbol $G \subseteq^* \prod_{\alpha \in A} G_\alpha$, if G is a subdirect sum of G_α and G is an l-ideal of $\prod_{\alpha \in A} G_\alpha$. An l-group G is said to be a completely subdirect sum, if G is an l-subgroup of $\prod_{\alpha \in A} G_\alpha$ and $\sum_{\alpha \in A} G_\alpha \subseteq G$. We use the symbol $\subseteq'$ to denote subdirect sum. Let G be an l-group. We denote by vG the least cardinal α such that $|A| \leqq \alpha$ for each bounded disjoint subset A of G. G is said to be v-homogeneous if $vH = vG$ for any convex l-subgroup $H \neq 0$ of G. A v-homogeneous l-group G is said to be v-homogeneous of α type if $vG = \alpha$. By Theorem 3.7 of [11] it is easy to verify the following lemma. The proof is left to the reader.

LEMMA 4.1. *Any complete l-group is l-isomorphic to an ideal subdirect sum of complete v-homogeneous l-groups.*

By using 4.3 of [11] it is easy to verify that if an l-group G is *v-homogeneous and non-totally ordered, then* $vG \geqq \aleph_0$. It is well known that any non-zero complete totally ordered group is l-isomorphic to a real group R or an integer group Z. So from Lemma 4.1 we obtain the structure theorem of a complete l-group.

THEOREM 4.2. *Any complete l-group G is l-isomorphic to an ideal subdirect sum of real groups, integer groups and complete v-homogeneous l-groups of* $\aleph_i$ *type* ($i \geqq 0$).

THEOREM 4.3. *Let G be an archimedean v-homogenous l-group of* $\aleph_i$ *type. Then G has the following properties:*

(1) *G has no basic element.*
(2) *G has no basis.*
(3) *The radical* $R(G) = G$.
(4) *G is not completely distributive.*
(5) *The distributive radical* $D(G) = G$.

Moreover, every non-trivial convex l-subgroup of G enjoys these same five properties.

PROOF. By Theorems 5.4 and 5.10 of [5] we need only to show (1). For any $0 < g \in G$, $vG(g) = \aleph_i > 1$. So $G(g)$ is not totally ordered, and $[0, g]$ is also not totally ordered by 4.3 of [11].

An l-group G is said to be continuous, if for any $0 < x \in G$ we have $x = x_1 + x_2$ and $x_1 \wedge x_2 = 0$ where $x_1 \neq 0$, $x_2 \neq 0$.

LEMMA 4.4 (Lemma 2.4 of [20]). *A complete l-group G is continuous if and only if G has no basic element.*

An l-group G is said to be projectable if each of its principal polars is a cardinal summand. The following lema is clear.

LEMMA 4.5. *Let G be a projectable (in particular, complete) and non-totally ordered l-group. Then G is directly decomposable.*

An l-group G is said to be ideal subdirectly irreducible if G cannot be expressed as an ideal subdirect sum of l-groups.

LEMMA 4.6 (Lemma 2.6 of [20]). *A complete l-group G is directly indecomposable if and only if G is ideal subdirectly irreducible.*

LEMMA 4.7 (Lemma 2.7 of [20]). *An archimedean l-group G is subdirectly irreducible if and only if the Dedekind completion $G^\wedge$ of G is ideal subdirectly irreducible.*

Now from Lemma 4.4, Lemma 4.5 and Lemma 4.6 we have

THEOREM 4.8. *Let G be a complete v-homogeneous l-groups of $\aleph_i$ type. Then*
(1) *G is continuous.*
(2) *G is directly decomposable.*
(3) *G is not ideal subdirectly irreducible.*
Moreover, every nontrivial convex l-subgroup of G enjoys these same three properties.

From Lemma 4.7 and Theorem 4.8 we obtain

COROLLARY 4.9. *An archimedean v-homogeneous l-group of $\aleph_i$ type is not subdirectly irreducible.*

A subset D in a lattice L is called a d-set if there exists $x \in L$ such that $d_1 \wedge d_2 = x$ for any pair of distinct elements of D and $d > x$ for each $d \in D$. We denote by $w[a, b]$ the least cardinal α such that $|D| \leqq \alpha$ for each d-set D of $[a, b]$.

LEMMA 4.10. *Let G be a v-homogeneous l-group of $\aleph_i$ type and be a dense l-subgroup of an l-group G′. Then G′ is also a v-homogeneous l-group of $\aleph_i$ type.*

PROOF. Suppose that H is an arbitrary convex l-subgroup of G. Let $H = H' \cap G$. Then H is dense in H' and H is a convex l-subgroup of G. We will prove that $vH' = vH$. It is clear that $vH' \geqq vH$. Let $\{x'_\alpha \in H'^+ \mid \alpha \in A\}$ be a disjoint of H' with an upper bound x'. Then there exists $x_\alpha \in H$ such that $0 < x_\alpha \leqq x'_\alpha$ for each $\alpha \in A$ and there exists $x \in H$ such that $0 < x \leqq x'$. Hence $\{x_\alpha \wedge x \mid \alpha \in A\}$ is a disjoint subset of H with an upper bound x. Hence $|A| \leqq \aleph_i$ and $vH' \leqq vH$. Therefore

$$vH' = vH = vG = \aleph_i$$

and so G' is a v-homogeneous l-group of $\aleph_i$ type.

From Theorem 2.6 and Theorem 5.2 of [8] and the above Lemma 4.10 we get

THEOREM 4.11. *Let G be an archimedean v-homogeneous l-group of $\aleph_i$ type. Then the Dedekind completion $G^\wedge$ of G and the lateral completion G^L of G are also v-homogeneous l-groups of $\aleph_i$ type.*

LEMMA 4.12. *Let G be a v-homogeneous l-group of $\aleph_i$ type and $\{x_\alpha \mid \alpha \in A\}$ be a disjoint subset in G. Then $|A| \leqq \aleph_i$.*

PROOF. Let G^L be the lateral completion of G. By Theorem 4.11 G^L is also v-homogeneous of $\aleph_i$ type. Let x be the least upper bound of a disjoint subset $\{x_\alpha \mid \alpha \in A\}$ of G in G^L. So $\{x_\alpha \mid \alpha \in A\}$ is a bounded disjoint subset in G^L. Therefore $|A| \leqq \aleph_i$.

THEOREM 4.13. *Let G be a v-homogeneous l-group of $\aleph_i$ type. Then the divisible hull G^d of G is also a v-homogeneous l-group of $\aleph_i$ type.*

PROOF. Let P be any nontrivial convex l-subgroup of G^d. For any $0 \neq x \in P$ there exists $n \in N$ such that $0 \neq nx \in P \cap G$. So $P \cap G$ is also a nontrivial convex l-subgroup of G. It is clear that $vP \geqq v(P \cap G) = \aleph_i$. On the other hand, $P = \left\{\frac{1}{n_g} \middle| g \in G \cap P, n \in N\right\}$. So if $\{c_j \mid j \in J\}$ is a bounded disjoint subset in P, let $c_j = \frac{1}{n_j} g_j$ $(j \in J,\ g_j \in G \cap P,\ n_j \in N)$. By the Bernau representation of an archimedean l-group [2] we see that $c_j \wedge c_{j'} = 0$ if and only if $g_j \wedge g_{j'} = 0\ (j \neq j')$. So $\{g_j \mid j \in J\}$ is a disjoint subset in $G \cap P$. By Lemma 4.12, $|J| \leqq \aleph_i$. Hence $vP \leqq \aleph_i$. Therefore $vP = \aleph_i$.

Now we turn to an archimedean l-group. In [19] we proved the following result.

LEMMA 4.14. *An l-group G is archimedean if and only if G is l-isomorphic to a subdirect sum of subgroups of reals and archimedean v-homogeneous l-groups of $\aleph_i$ type.*

Suppose that G is a subdirect sum of subgroups of reals and v-homogeneous l-groups of $\aleph_i$ type, $G \subseteq' \prod_{\delta \in \Delta} T_\delta$. Let $\Delta_1 = \{\delta \in \Delta \mid T_\delta$ is a subgroup of reals$\}$. If $\sum_{\delta \in \Delta_1} T_\delta \subseteq G$, G is said to be a semicomplete subdirect sum of subgroups of reals and v-homogeneous l-groups of $\aleph_i$ type, in symbols $\sum_{\delta \in \Delta_1 \subseteq \Delta} T_\delta \subseteq G \subseteq' \prod_{\delta \in \Delta} T_\delta$.

THEOREM 4.15. (Theorem 4.7 of [19]). *An l-group G is archimedean if and only if G is l-isomorphic to a semicomplete subdirect sum of subgroups of reals and archimedean v-homogeneous l-groups of $\aleph_i$ type.*

5. Properties of $\mathscr{A}_\alpha$.

We denote by $\mathscr{A}_\alpha$ the $\mathscr{A}$-free l-group $\mathscr{F}_\mathscr{A}(X)$ of rank α. By Proposition 2.6 $\mathscr{A}_\alpha$ is the $\mathscr{A}$-free l-group $\mathscr{F}_\mathscr{A}([G, \{0\}])$ generated by $\mathscr{G}(\mathscr{A})$-free group G with trivial order. It follows from Corollary 3.4 of [17] that $\mathscr{A}_\alpha$ is archimedean. Hence $\mathscr{A}_\alpha \cong \mathscr{A}r_\alpha$. We have already known some properties of $\mathscr{A}_\alpha$. For example, $\mathscr{A}_\alpha$ is a subdirect sum of integers (Theorem 2.5 of [3]); $\mathscr{A}_\alpha(\alpha > 1)$ has a countably infinite disjoint subset but no uncountable disjoint subset (Theorem 6.2 of [16]); every infinite chain in $\mathscr{A}_\alpha$ must be countable (Theorem 5.1 of [15]); the word problem for $\mathscr{A}_\alpha$ is solvable (Theorem 2.11 of [14]); $\mathscr{A}_\alpha(\alpha > 1)$ has no singular elements (Theorem 2.8 of [3]). In this section we will give further properties of $\mathscr{A}_\alpha$ using the structure theorem of an archimedean l-group.

THEOREM 5.1. *$\mathscr{A}_\alpha(\alpha > 1)$ is an archimedean v-homogeneous l-group of $\aleph_0$ type.*

PROOF. Since $\mathscr{A}_\alpha$ is archimedean, by Theorem 4.15, without loss of generality, we have

$$\sum_{\delta \in \Delta_1 \subseteq \Delta} T_\delta \subseteq \mathscr{A}_\alpha \subseteq' \prod_{\delta \in \Delta} T_\delta,$$

where each T_δ $(\delta \in \Delta_1)$ is a subgroup of reals and each T_δ $(\delta \in \Delta \backslash \Delta_1)$ is an archimedean v-homogeneous l-group of $\aleph_i$ type. By Theorem 3.5 of [3] (or Theorem 1 of [18]) $\mathscr{A}_\alpha(\alpha > 1)$ has no nontrivial direct summands. Hence $\Delta_1 = \emptyset$ and $\mathscr{A}_\alpha(\alpha > 1)$ is a subdirect sum of archimedean v-homogeneous l-groups of $\aleph_i$ type. Let

$$\text{(1)} \qquad \sum_{\delta \in \Delta'} T_\delta \subseteq \mathscr{A}_\alpha^{\wedge} \subseteq^* \prod_{\delta \in \Delta'} T_\delta^{\wedge}$$

$$\mathscr{A}_\alpha \subseteq' \prod_{\delta \in \Delta'} T_\delta,$$

where $\Delta' = \Delta \backslash \Delta_1$ and each $T_\delta\,(\delta \in \Delta')$ is an archimedean v-homogeneous l-groups of $\aleph_i$ type. For any $0 < x \in \mathscr{A}_\alpha$. We denote by $\mathscr{A}_\alpha(x)$ the convex l-subgroup in $\mathscr{A}_\alpha$ generated by x and $\mathscr{A}_\alpha^{\wedge}(x)$ the convex l-subgroup in $\mathscr{A}_\alpha^{\wedge}$ generated by x. By Theorem 2 of [18] we have

$$\text{(2)} \qquad v\mathscr{A}_\alpha(x) \leqq v\mathscr{A}_\alpha \leqq \aleph_0.$$

On the other hand, $\mathscr{A}_\alpha(x)$ is dense in $\mathscr{A}_\alpha^{\wedge}(x)$. If $\{x_\alpha \mid \alpha \in A\}$ is a disjoint subset with an upper bound x_0 in $\mathscr{A}_\alpha^{\wedge}(x)$. Then there exists $x_0' \in \mathscr{A}_\alpha(x)$ such that $0 < x_0' < x_0$. Put $x_\alpha' = x_\alpha \wedge x_0'$. Then $\{x_\alpha' \mid \alpha \in A\}$ is a disjoint subset with an upper bound x_0' in $\mathscr{A}_\alpha(x)$. Hence $v\mathscr{A}_\alpha^{\wedge}(x) \leqq v\mathscr{A}_\alpha(x)$. And it is clear that $v\mathscr{A}_\alpha(x) \leqq v\mathscr{A}_\alpha^{\wedge}(x)$. Thus,

$$\text{(3)} \qquad v\mathscr{A}_\alpha(x) = v\mathscr{A}_\alpha^{\wedge}(x).$$

For any $\delta_0 \in \Delta'$, put $\bar{x}_{\delta_0} = (\ldots 0 \ldots, x_{\delta_0}, \ldots 0 \ldots)$. Then $\bar{x}_{\delta_0} \leqq x$. Since $vT_{\delta_0}(x_{\delta_0}) = vT_{\delta_0} \geqq \aleph_0$ where $T_{\delta_0}(x_{\delta_0})$ is the convex l-subgroup of T_{δ_0} generated by x_{δ_0}, there exists a disjoint subset $\{x^\beta \mid \beta \in B\}$ in $T_{\delta_0}(x_{\delta_0})$ such that $x^\beta \leqq x_{\delta_0}$ and $|B| \geqq \aleph_0$. Then $\bar{x}^\beta = (\ldots 0 \ldots, x^\beta, \ldots 0 \ldots) \in \mathscr{A}_\alpha^{\wedge}$ by (1) and $\{\bar{x}^\beta \mid \beta \in B\}$ is a disjoint subset with an upper bound $\bar{x}_{\delta_0}$ in $\mathscr{A}_\alpha^{\wedge}(\bar{x}_{\delta_0})$. Hence

(4) $$v\mathscr{A}_\alpha^\wedge(x) = v\mathscr{A}_\alpha^\wedge(\bar{x}_{\delta_0}) \geqq \aleph_0.$$

Combining (2), (3) and (4) we get $v\mathscr{A}_\alpha(x) = \aleph_0$ for any $0 < x \in \mathscr{A}_\alpha$. Now for any nontrivial convex l-subgroup K in $\mathscr{A}_\alpha$. Let $0 < x \in K$. Then

$$\aleph_0 = v\mathscr{A}_\alpha(x) \leqq vK \leqq v\mathscr{A}_\alpha \leqq \aleph_0.$$

Therefore $vK = \aleph_0$ and $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.

By Theorem 4.3 and Theorem 5.1 we obtain

THEOREM 5.2. *$\mathscr{A}_\alpha(\alpha > 1)$ has the following properties:*
(1) $\mathscr{A}_\alpha$ has no basic element.
(2) $\mathscr{A}_\alpha$ has no basis.
(3) The radical $R(\mathscr{A}_\alpha) = \mathscr{A}_\alpha$.
(4) $\mathscr{A}_\alpha$ is not completely distributive.
(5) The distributive radical $D(\mathscr{A}_\alpha) = \mathscr{A}_\alpha$.
Moreover, every nontrivial convex l-subgroup of $\mathscr{A}_\alpha$ enjoys these same five properties.

By Theorem 3.6 of [7] and the above Theorem 4.11, Theorem 4.13 and Theorem 5.1 we have

THEOREM 5.3. (1) *The Dedekind completion $\mathscr{A}_\alpha^\wedge$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.*
(2) The lateral completion $\mathscr{A}_\alpha^L$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.
(3) The divisible hull $\mathscr{A}_\alpha^d$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.
(4) The essential closure $\mathscr{A}_\alpha^e$ of $\mathscr{A}_\alpha$ is a v-homogeneous l-group of $\aleph_0$ type.

From Theorem 4.8 we get

THEOREM 5.4. *The Dedekind completion $\mathscr{A}_\alpha^\wedge$ of $\mathscr{A}_\alpha$ has the following properties:*
(1) $\mathscr{A}_\alpha^\wedge$ is continuous.
(2) $\mathscr{A}_\alpha^\wedge$ is directly decomposable.
(3) $\mathscr{A}_\alpha^\wedge$ is not ideal subdirectly irreducible.
(4) $\mathscr{A}_\alpha^\wedge$ has a closed l-ideal.
Moreover, each nontrivial convex l-subgroup of $\mathscr{A}_\alpha^\wedge$ enjoys these same four properties.

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ON PRINCIPAL GRAPHS AND WEAK DUALITY

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Abstract.

The main result is that if a finite tree occurs as a principal graph of a subfactor N of a II_1 factor M of index greater than 4, if the contragredient maps on the principal graphs are trivial, and if no vertex has degree greater than 3, then the tree must contain a subgraph isomorphic to what is denoted by $E_6^{(1)}$ in [GHJ] (and by T in this paper). The proof uses a notion that we call "weak duality" of graphs. The result needed is that certain kinds of graphs can be weakly dual only to themselves. This paper also contains a proof of the assertion that the distinguished vertex $*$ of a principal graph is essentially determined up to a parity-preserving automorphism of the bipartite graph.

1. Introduction.

This paper is devoted to a study of the properties of a pair $(\mathscr{G}, \mathscr{H})$ of pointed finite bipartite graphs wich arise as a part of Ocneanu's paragroup invariant of a finite index subfactor. After setting up some notation and recalling some basic facts about subfactors, we proceed to analyse the role of the distinguished vertex of the graph $\mathscr{G}$; in particular, we describe the extent of uniqueness of the vertex $*$ up to a graph automorphism. In the last section, we single out a property possessed by a pair of principal graphs of a subfactor for which the contragredient maps are trivial, which we term 'weak duality'. The main result here is that if a pair of graphs $\mathscr{G}$ and $\mathscr{H}$ are weakly dual, then $\mathscr{G}$ is necessarily isomorphic to $\mathscr{H}$ if $\mathscr{G}$ satisfies some conditions – at most triple points, no double bonds, and the absence of two specific kinds of subgraphs. This is the technically complicated part of the paper, although, when suitably combined with an observation by Ocneanu, it leads fairly easily to a proof of what has been referred to, in the Abstract, as the main result.

2. Notation and other preliminaries.

In this paper, we will be dealing with bipartite graphs. If $\mathscr{G}$ is such a graph, its vertex set $V(\mathscr{G})$ admits a partition $V(\mathscr{G}) = \mathscr{G}^0 \coprod \mathscr{G}^1$, the vertices of the former

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(resp., latter) being referred to as even (resp. odd) vertices. (We use the symbol $\coprod$ here and elsewhere to indicate a disjoint union, and the even vertices of $\mathscr{G}$ are written with a superscript 0 and odd vertices with superscript 1.) We shall also find it convenient to encode the data of the graph $\mathscr{G}$ by the matrix G with rows (resp., columns) indexed by $\mathscr{G}^0$ (resp., $\mathscr{G}^1$), with the (β^0, ξ^1) entry equal to the number of bonds joining the even vertex β^0 to the odd vertex ξ^1. It is clear that the adjacency matrix $A(\mathscr{G})$, whose rows and columns are indexed by $V(\mathscr{G})$ is given in block form by

$$\begin{bmatrix} 0 & G \\ G^t & 0 \end{bmatrix}.$$

It follows that the non-zero eigenvalues of $A(\mathscr{G})$ are precisely the numbers $\pm\lambda$, where λ^2 is a non-zero eigenvalue of G^tG; further, the restrictions of the Perron-Frobenius eigenvector of $A(\mathscr{G})$ to $\mathscr{G}^1$ and $\mathscr{G}^0$ are the Perron-Frobenius eigenvector of G^tG and a suitable $(= \|\mathscr{G}\|^{-1})$ multiple of its image under G respectively.

We now recall some basic facts concerning the principal graphs associated to a finite-index inclusion $N \subset M$ of II_1 factors (usually assumed to be hyperfinite). Most of these facts stem from one of three sources – the work of Jones, Ocneanu and Popa – and we will not take the trouble of painstakingly ascribing each to a specific author.

Let $N \subset M$ be a pair of hyperfinite II_1 factors, with finite index. Then the *basic construction* of Jones yields a canonical tower $N \subset M \subset M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$ and a semi-canonical tunnel $M_0 = M \supset M_{-1} = N \supset M_{-2} \supset M_{-3} \supset \ldots \supset M_{-n} \supset \ldots$ In any case, the doubly indexed grid $\{A_{k,n} = M'_{-n} \cap M_k : k, n \geq 0\}$ is canonical. It is well-known that the inclusion data of this grid is encoded, in a precise manner, by a pair $\{\mathscr{G}, \mathscr{H}\}$ of bipartite graphs which satisfy several conditions. (Recall that the subfactor N is said to have finite depth precisely when either (equivalently both) of the principal graphs is (are) finite.)

3. The distinguished vertex $*$.

Any graph $\mathscr{G}$ that arises as a principal graph is, in addition, a pointed connected graph; i.e., it contains an even vertex – always denoted by $*_{\mathscr{G}}$ – which has the distinguished feature of being an even vertex at which the Perron-Frobenius eigenvector of the adjacency matrix $A(\mathscr{G})$ is minimal. The content of the next proposition is to note that the above "feature" determines the vertex essentially uniquely. Although fairly straightforward, this proposition has been included here because the authors have not seen a proof of this in the literature.

PROPOSITION 1. *Let $\mathscr{G}$ be a principal grph of a subfactor $N \subset M$, of finite depth.*

Then the vertex $*_{\mathscr{G}}$ is uniquely determined up to a "parity-preserving" automorphism of the bipartite graph $\mathscr{G}$.

PROOF. (In this proof and elsewhere, when we have to use the language of hypergroups, we shall use the results as well as the terminology of [S] and [SV].) If $\mathscr{G}$ is a hypergroup, the set $\mathscr{G}_{(0)} = \{\beta \in \mathscr{G}: d_\beta = 1\}$ is a *sub-group* of the hypergroup $\mathscr{G}$, where of course, $\alpha \mapsto d_\alpha$ denotes the dimension function of $\mathscr{G}$ (which is the unique "positive homomorphism" from the hypergroup ring $\mathbf{Z}\mathscr{G}$ to $\mathbf{R}$). Also since the uniqueness of the dimension function implies that $d_{\bar{\alpha}} = d_\alpha$, it follows that

$$\alpha \in \mathscr{G}_{(0)} \Rightarrow 1 = d_\alpha^2 = d_\alpha \cdot d_\alpha = \sum_{\gamma \in \mathscr{G}} \langle \alpha \cdot \bar{\alpha}, \gamma \rangle d_\gamma = 1 + \sum_{1 \neq \gamma \in \mathscr{G}} \langle \alpha \cdot \bar{\alpha}, \gamma \rangle d_\gamma$$

and hence that $\langle \alpha \cdot \bar{\alpha}, \gamma \rangle = 0$ for $1 \neq \gamma \in \mathscr{G}$. In other words, $\alpha \cdot \bar{\alpha} = 1$. It is immediate that $\mathscr{G}_{(0)}$ is a group.

On the other hand, if the graph $\mathscr{G}$ is a principal graph, then in the terminology of [SV], the set $\mathscr{G}^0$ has the structure of a hypergroup which acts on the set $\mathscr{G}^1$. If $\alpha \in \mathscr{G}^0_{(0)}$, and if L_0 and L_1, respectively, denote the matrices (with respect to the natural bases) of left multiplication by α on $\mathbf{Z}\mathscr{G}^0$ and $\mathbf{Z}\mathscr{G}^1$, the already established equation $\alpha \cdot \bar{\alpha} = 1$ implies that the matrices L_0 and L_1 are orthogonal matrices; as these are non-negative integral matrices, they must be permutation matrices. Hence, the map $\lambda \mapsto f(\lambda) = \alpha \cdot \lambda$ defines a parity-preserving permutation of $V(\mathscr{G})$ such that $f(1) = \alpha$, where 1 denotes the (unique) identity element of the hypergroup $\mathscr{G}^0$.

To complete the proof, we need to verify that the map f defines an automorphism of the graph $\mathscr{G}$, i.e., $G(\beta, \xi) = G(f(\beta), f(\xi))$, for all $\beta \in \mathscr{G}^0$, $\xi \in \mathscr{G}^1$. However, we have $G(\beta, \xi) = \langle \beta \cdot \lambda, \xi \rangle$, where λ denotes the sum (with multiplicities taken into account in case of multiple bonds) of the neighbours in $\mathscr{G}^1$ of $*_{\mathscr{G}}$ (where we think of λ as an element of $\mathbf{Z}\mathscr{G}^1$). Hence,

$$G(f(\beta), f(\xi)) = \langle (\alpha \cdot \beta) \cdot \lambda, (\alpha \cdot \xi) \rangle = \langle (\bar{\alpha} \cdot \alpha) \cdot \beta \cdot \lambda, \xi \rangle = G(\beta, \xi)$$

as desired.

REMARK 2. Note that we have *not* proved that the distinguished vertex $*$ is determined uniquely up to an automorphism of the graph; what we have shown is that once we have pre-determined the parity, then among the even vertices, the vertex $*$ is uniquely determined up to an automorphism. This prompts the question: is it possible for a graph $\mathscr{G}$ that arises as a principal graph to admit vertices α and ξ of different parity at both of which the Perron-Frobenius eigenvector of the adjacency matrix $A(\mathscr{G})$ assumes the minimal value, and yet such that there is no (necessarily, parity reversing) automorphism which maps α to ξ? (The graph A_{2n} is a non-example.)

EXAMPLE 3. We are grateful to Bhaskar Bagchi for this combinatorial example which, besides being pretty, illustrates two different phenomena; viz., (i) there is a bipartite graph $\mathscr{G}$ at all of whose even vertices the Perron-Frobenius eigenvector takes on the same value and yet the automorphism group of $\mathscr{G}$ does not act transitively on the set of even vertices; (hence by the above proposition $\mathscr{G}$ cannot arise as a principal graph;) (ii) there is a second pointed bipartite graph $\mathscr{H}$ which is not isomorphic to $\mathscr{G}$ although the two graphs are "weakly dual" – cf. Definition (4).

Both the graphs $\mathscr{G}$, $\mathscr{H}$ have $15 = \binom{6}{2}$ odd vertices indexed by the 15 edges of the complete graph K_6. Each graph has 10 even vertices indexed by certain subgraphs of K_6 (isomorphic to $C_3 \coprod C_3$ or C_6, where C_k denotes a k-cycle – thus C_6 is a hexagon, etc.). In both graphs, an odd vertex is adjacent to an even vertex precisely when the relevant edge belongs to the relevant subgraph.

The even vertices of $\mathscr{H}$ correspond to all the $10 = \left(\binom{6}{3}\right)/2$ subgraphs of K_6 isomorphic to $C_3 \coprod C_3$.

The graph $\mathscr{G}$ also has ten even vertices; these correspond to six subgraphs isomorphic to $C_3 \coprod C_3$ and four subgraphs isomorphic to C_6. They are: $\{(124)\coprod(356),\ (125)\coprod(346),\ (136)\coprod(245),\ (145)\coprod(236),\ (134)\coprod(256),\ (146)\coprod(235)\}$ and $\{(123456), (126453), (156423), (153426)\}$ – where we have used the obvious notation $(v_1 \ldots v_k)$ to denote the k-cycle that successively passes through the vertices $v_1, \ldots, v_k$.

Both graphs share the following properties: (i) each odd vertex has degree 4 and each even vertex has degree 6 (and hence the value of the Perron-Frobenius eigenvector at a vertex depends only on the parity of the vertex); (ii) given any two distinct odd vertices, the number of paths of length two which join them is 1 or 2 according as the corresponding edges (in K_6) share a common vertex or not. These facts ensure that the graphs are weakly dual as asserted, provided the distinguished vertices of both $\mathscr{G}$ and $\mathscr{H}$ are taken to be the same graph isomorphic to $C_3 \coprod C_3$.

Note now that the graph $\mathscr{G}$ has two kinds of even vertices. We assert that any automorphism of $\mathscr{G}$ leaves invariant the set of all even vertices of either kind. Since the Perron-Frobenius eigenvector is constant on the set of even vertices of $\mathscr{G}$, this example does indeed illustrate the claimed features. (In fact $\mathrm{Aut}(\mathscr{H})$ is isomorphic to S_6 while $\mathrm{Aut}(\mathscr{G})$ can be seen to be a group of order 48.)

To prove the assertion, first observe that $\mathscr{G}^1$ is identified with the edges of K_6, which in turn constitute the vertices of $L(K_6)$-the so-called line graph of K_6. (Recall that two vertices are adjacent in the line graph $L(K)$ precisely when the corresponding edges in K have a vertex in common.) Suppose now that we are given an automorphism σ of $\mathscr{G}$; by property (i) above of G, the automorphism σ must preserve parity; so $\sigma|_{\mathscr{G}^1}$ yields a self-map of the vertices of $L(K_6)$; the fact that $\mathscr{G}$ has the property (ii) of the previous paragraph is seen to imply that this map must preserve adjacency in the graph $L(K_6)$, and is hence an automorphism of $L(K_6)$. It is not hard to see that every automorphism of $L(K_6)$ is induced by an automorphism of K_6. Thus, $\sigma \mapsto \sigma|_{\mathscr{G}^1}$ yields a map from $\mathrm{Aut}(\mathscr{G})$ to $\mathrm{Aut}(K_6) = S_6$, which is clearly a monomorphism. Since no automorphism of K_6 can map a subgraph of the form $C_3 \coprod C_3$ onto a subgraph of the form C_6, the proof of the assertion is complete.

4. Weak duality.

If two pointed bipartite graphs $\mathscr{G}, \mathscr{H}$ arise as the two principal graphs corresponding to a finite-index subfactor, then the sets $\mathscr{G}^0$, $\mathscr{H}^0$ of even vertices of the two graphs are naturally equipped with involutions corresponding to the contragredient mapping at the level of bimodules. We shall be concerned with subfactors for which both these involutions are trivial; for brevity, we shall simply say that the subfactor has trivial contragredient maps when this happens. For such a subfactor – i.e., one with trivial contragredient maps – it follows from the description of the grid $\{A_{k,n}\}$ discussed earlier, that the graphs $\mathscr{G}$ and $\mathscr{H}$ are "weakly dual" in the sense of the next definition.

DEFINITION 4. Two pointed finite connected bipartite graphs $(\mathscr{G}, *_{\mathscr{G}})$ and $(\mathscr{H}, *_{\mathscr{H}})$ are said to be "weakly dual" if the following conditions are satisfied:

(1) $\mathscr{G}^1 = \mathscr{H}^1$.

(2) $G^t(*_{\mathscr{G}}) = H^t(*_{\mathscr{H}})$ (i.e. the neighbours of $*$ *in* $\mathscr{G}$ and $\mathscr{H}$ are the same).

(3) $G^tG(\xi^1, \eta^1) = H^tH(\xi^1, \eta^1)$ for all ξ^1, $\eta^1 \in \mathscr{G}^1$, (i.e. the number of paths, of length 2, between ξ^1 and η^1 is the same in $\mathscr{G}$ and $\mathscr{H}$).

(This would be the appropriate place to acknowledge our gratitude to Uffe Haagerup for leading us to think along these lines; he had, in oral communication, pointed out that if the graph $\mathscr{G}$ "looks like an A_n up to a certain distance from $*_{\mathscr{G}}$", so also must $\mathscr{H}$ – cf. Remark 7 (2).)

REMARK. Note that when $\mathscr{G}$ and $\mathscr{H}$ are a pair of principal graphs, the identification of the odd vertices in (1) is via the contragredient map τ. When the contragredient map on the even vertices is nontrivial the graphs do not satisfy condition (3), but they satisfy $G^t\tau_{\mathscr{G}^0}G = H^t\tau_{\mathscr{H}^0}H$.

We turn now to pairs of graphs which are not isomorphic but which are weakly dual. The simplest known example comes from the principal graphs for the inclusion $N \subset M$, when M is the crossed-product of N with a non-abelian group of outer automorphisms of N. In this example, as is well-known, the graph $\mathscr{H}$ has multiple bonds while $\mathcal{G}$ does not. An example of graphs without multiple bonds is furnished by Example 3. Another such, but smaller, example is given below.

Example 5. We begin by discussing a pair of graphs which are "almost" weakly dual, but just fail to be so; nevertheless they have near relatives which do furnish an example of a pair of graphs which are weakly dual but not isomorphic. The non-example is discussed here mainly because of the key role these two graphs play in Proposition 6.

(a) Consider the following pair of graphs, with even and odd vertices labelled as indicated (fig. 1).

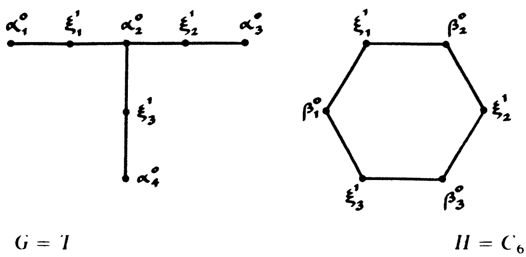

Figure 1.

(Here and in the sequel, we write T to denote the "T-graph" each of whose arms is two edges long. This graph is denoted by $E_6^{(1)}$ in [GHJ], but we use the notation T because it is more suggestive.)

It is easily verified that the condition $G'G = H'H$ is satisfied. Since every even vertex in $\mathcal{H}$ has degree 2 while none in $\mathcal{G}$ does, clearly these two graphs cannot be wealy dual (by condition (2)).

(b) Consider these graphs with labelling as indicated (fig. 2).

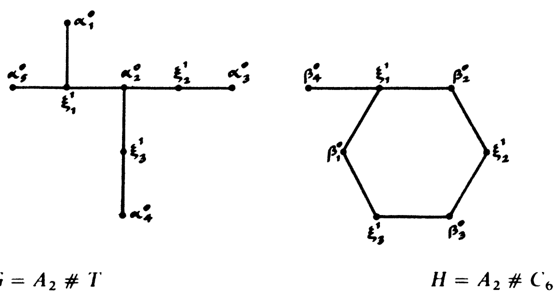

Figure 2.

(Here and elsewhere, we use the symbol # to denote "connected sum", whereby we mean that a pair of vertices, one from each of the graphs in question, has been identified; to be sure, there are several ways of forming such a connected sum.) Again, the condition $G^tG = H^tH$ is satisfied. While the vertices α_5^0 and β_4^0 have the same degree, what fails now is that the minimum value of the Perron-Frobenius eigenvector of $A(\mathscr{G})$ occurs not at α_5^0 but at the vertices α_3^0 and α_4^0.

(c) Finally, the desired example comes from the following graphs, with labelling as indicated (fig. 3).

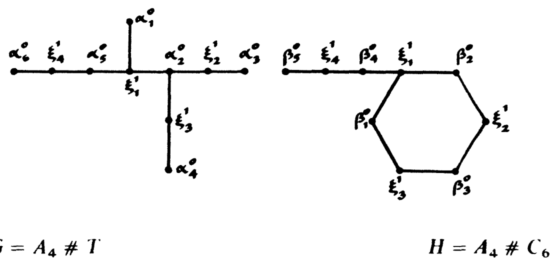

Figure 3.

Here, it is the case that $*_{\mathscr{G}} = \alpha_6^0$ and $*_{\mathscr{H}} = \beta_5^0$.

(d) It goes without saying that by extending the A-part of the graphs more and more, the graphs $\mathscr{G}$ and $\mathscr{H}$ generate a whole sequence of pairs of non-isomorphic graphs – namely $A_{2n} \# T$ and $A_{2n} \# C_6$ – which are weakly dual.

We are now ready to prove the following proposition which gives some criteria on a bipartite graph $\mathscr{G}$ which ensure that the only graph, up to isomorphism, which is weakly dual to $\mathscr{G}$ is $\mathscr{G}$ itself. (Observe that in view of Example 5 (c), (d), the conditions (3) and (4) in the proposition are almost necessary.)

PROPOSITION 6. *Suppose $\mathscr{G}$ is a finite connected bipartite graph satisfying the following conditions:*

(1) *no vertex of $\mathscr{G}$ has degree greater then* 3;

(2) *$\mathscr{G}$ does not have double bonds;*

(3) *$\mathscr{G}$ has no* 6-*cycles; and*

(4) *$\mathscr{G}$ has no subgraph isomorphic to T such that each of the vertices of degree* 1 *in T is an even vertex in $\mathscr{G}$ whose degree in $\mathscr{G}$ is still* 1.

Then the identification $\mathscr{G}^1 = \mathscr{H}^1$ extends to a graph isomorphism of $\mathscr{G}$ to $\mathscr{H}$.

Before proceeding to the proof proper, we set up some notation. We shall use the notation $(\xi_0 - \xi_1 - \ldots - \xi_n) \in \mathscr{G}$ to signify that $\xi_0, \xi_1, \ldots, \xi_n$ are vertices of the graph $\mathscr{G}$ such that ξ_{i-1} is adjacent to ξ_i in $\mathscr{G}$ for $1 \leq i \leq n$.

The set of neighbours of α in $\mathscr{G}$ will be denoted by $\mathscr{N}_\alpha^{\mathscr{G}}$. In the following, since we shall be dealing with a pair of weakly dual graphs $\mathscr{G}$ and $\mathscr{H}$, which have the with a pair of weakly dual graphs $\mathscr{G}$ and $\mathscr{H}$, which have the y $\mathscr{N}_\alpha$ when α is an even vertex of either $\mathscr{G}$ or $\mathscr{H}$.

We shall also employ the following notation: for vertices ξ, η in $\mathscr{G}$:

(i) the symbol $\mathscr{G}(\xi, \eta)$ will denote the set of common neighbours in $\mathscr{G}$ of ξ and η, i.e. $\mathscr{G}(\xi,\eta) = \mathscr{N}_\xi^{\mathscr{G}} \cap \mathscr{N}_\eta^{\mathscr{G}}$; (note that, in the absence of double bonds, $|\mathscr{G}(\xi^1,\eta^1)| = G^tG(\xi^1,\eta^1)$, whence $|\mathscr{G}(\xi^1,\xi^1)|$ is the degree of ξ^1 in $\mathscr{G}$;)

(ii) the symbol $\mathscr{I}^{\mathscr{G}}(\xi)$ will denote the set of degree one neighbours of ξ in $\mathscr{G}$; i.e., $\mathscr{I}^{\mathscr{G}}(\xi) = \{\beta \in \mathscr{N}_\xi^{\mathscr{G}}: \deg_{\mathscr{G}}(\beta) = 1\}$;

(iii) the symbol Λ will denote the set of triple points (i.e., vertices of degree 3) in $\mathscr{G}^0$; suppose $\Lambda = \{\lambda_1^0, \lambda_2^0, \ldots, \lambda_l^0\}$, $l \geqq 0$.

Proof. It is not hard to see that the above conditions (1), (2) and (4) of the proposition imply the conditions (1′), (2′) and (4′) below. (To be precise, conditions (1) and (2) are together equivalent to conditions (1′) and (2′); while condition (4) is equivalent to (4′).) What we shall prove is that conditions (1′), (2′), (3) and (4′) suffice to ensure the validity of the conclusion of the Proposition. (We have, however, chosen to state the proposition as we have, since we fell that this formulation is more "visual" and easier to verify.)

(1′) $(G^tG)(\xi^1,\xi^1) \leqq 3$, for all ξ^1 in $\mathscr{G}^1$;

(2′) for all $\beta^0 \in \mathscr{G}^0$ $\deg(\beta^0) \leqq 3$; and

(4′) for all $\lambda^0 \in \Lambda$ there exists $\xi_{\lambda^0}^1 \in \mathscr{N}_{\lambda^0}$ such that $\mathscr{I}^{\mathscr{G}}(\xi_{\lambda^0}^1) = \emptyset$.

In the proof we would have occasion to use the following condition (3′) which can be seen to be implied by (3).

(3′) If $\Omega = \{\omega_1^1, \omega_2^1, \omega_3^1\}$ is a subset of $\mathscr{G}^1$ such that between any two vertices in Ω there is a path of length 2 in $\mathscr{G}$, i.e., $(G^tG)(\omega_i^1, \omega_j^1) \neq 0$ for $i \neq j$, then, Ω is the set of neighbours of some triple point in $\mathscr{G}$, i.e., $\Omega = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$.

We break the proof, which is somewhat involved, into the following steps.

Step 1. $\mathscr{H}$ has no double bonds.

Reason: $H^tH(\xi^1,\xi^1) = G^tG(\xi^1,\xi^1) \leqq 3$ for all $\xi^1 \in \mathscr{G}^1$.

Step 2. Each vertex in $\mathscr{H}$ has degree at most 3.

Reason: For the same reason as in Step 1, this is clear for the odd vertices. Suppose, now, that there is an even vertex δ^0 in $\mathscr{H}^0$ such that $\deg(\delta^0) \geqq 4$.

Case (1). There is an even vertex δ^0 in $\mathscr{H}^0$ such that $\deg(\delta^0) > 4$. Then δ^0 has at least five neighbours, ξ_1^1, ξ_2^1, ξ_3^1, ξ_4^1, and ξ_5^1. The path $(\xi_i^1 - \delta^0 - \xi_j^1)$ in $\mathscr{H}$ ensures that $G^tG({\xi_1}^i, \xi_j^1) = H^tH(\xi_i^1, \xi_j^1) \neq 0$ for all i and j. By (3′), for any choice of distinct i, j and k, $\{\xi_i^1, \xi_j^1, \xi_k^1\} = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$, i.e. each ξ_i^1 is adjacent to $\left(\binom{4}{2} = \right)$ 6 distinct triple points in $\mathscr{G}$, which contradicts (1).

Case (2). Suppose there is an even vertex δ^0 in $\mathscr{H}^0$ such that $\deg(\delta^0) = 4$. Then δ^0 has four neighbours ξ_1^1, ξ_2^1, ξ_3^1, and ξ_4^1. By the same reasoning as above, for distinct i, j and k, $\{\xi_i^1, \xi_j^1, \xi_k^1\} = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$. So there are 4 triple points λ_1^0, $\lambda_2^0, \lambda_0^3, \lambda_4^0$ in $\mathscr{G}$ such that $\mathscr{N}_{\lambda_i^0} = \{\xi_j^1 : j \neq i\}$. Let $\mathscr{G}'$ be the induced subgraph of $\mathscr{G}$ on the vertices $\{\xi_j^1 : 1 \leqq j \leqq 4\} \cup \{\lambda_i^0 : 1 \leqq i \leqq 4\}$. Since each λ_i^0 and ξ_j^1 has degree 3 in $\mathscr{G}'$, the conditions (1) and (2) imply that $\mathscr{G}'$ is a connected component of $\mathscr{G}$, and hence $\mathscr{G}' = \mathscr{G}$ by the assumed connectedness of $\mathscr{G}$.

Since $\mathscr{G}$ and $\mathscr{H}$ are weakly dual, we have the following:

(i) $\mathscr{H}^1 = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1\}$.

(ii) $H^tH(\xi_i^1, \xi_j^1) = 2$ for $1 \leqq i \neq j \leqq 4$.

(iii) $H^tH(\xi_i^1, \xi_i^1) = 3$ for $1 \leqq \mathrm{i} \leqq 4$.

We proceed to deduce that there must exist another even vertex $\delta_1^0 \neq \delta^0$ of degree 4 in $\mathscr{H}$ such that $\mathscr{N}_{\delta_1^0} = \mathscr{N}_{\delta^0} = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1\}$.

By (ii) there are unique even vertices κ_{ij}^0, distinct from δ^0, such that $(\xi_i^1 - \kappa_{ij}^0 - \xi_j^1)$ are in $\mathscr{H}$. Then for any ξ_i^1 and $j \neq i$, we have $(\xi_i^1 - \delta^0)$, and $(\xi_i^1 - \kappa_{ij}^0)$, are in $\mathscr{G}$. But $\deg(\xi_i^1) \leq 3$. Therefore for each i, $\kappa_{ij}^0 = \kappa_{ik}^0$ for some $j \neq k$. Now $(\xi_j^1 - \kappa_{ij}^0 = \kappa_{ik}^0 - \xi_k^1)$ is in $\mathscr{H}$. But κ_{jk}^0 is the unique vertex other than δ^0 such that $(\xi_j^1 - \kappa_{jk}^0 - \xi_k^1)$ is in $\mathscr{H}$. So $\kappa_{ij}^0 = \kappa_{ik}^0 = \kappa_{jk}^0$, which is then a vertex of degree at least three. Hence each ξ_i^1 is connected to a $\kappa_i^0 \neq \delta^0$ such that $\deg(\kappa_i^0) \geqq 3$. We now show that all the κ_i^0 are the same.

Now, for $1 \leqq i, j \leqq 4$, we see that,

$$|\mathscr{N}_{\kappa_i^0} \cap \mathscr{N}_{\kappa_j^0}| = |\mathscr{N}_{\kappa_i^0}| + |\mathscr{N}_{\kappa_j^0}| - |\mathscr{N}_{\kappa_i^0} \cup \mathscr{N}_{\kappa_j^0}|$$
$$\geqq 3 + 3 - |\mathscr{H}^1| = 2.$$

Let $1 \leqq i \neq j \leqq 4$. Then there exist $1 \leqq k \neq l \leqq 4$ such that $\xi_k^1, \xi_l^1 \in \mathscr{N}_{\kappa_i^0} \cap \mathscr{N}_{\kappa_j^0}$. Then since $\mathscr{H}(\xi_k^1, \xi_l^1) \supset \{\delta^0, \kappa_i^0, \kappa_j^0\}$ and $\delta^0 \neq \kappa_i^0, \kappa_j^0$, the property (ii), stated above, implies that $\kappa_i^0 = \kappa_j^0$.

So there does indeed exist $\delta_1^0 \neq \delta^0 \in \mathscr{H}^0$, such that $\deg(\delta_1^0) = 4$, and $\mathscr{N}_{\delta_1^0} = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1\}$.

By (iii) there must exist even vertices β_1^0, β_2^0, β_3^0, β_4^0 in $\mathscr{H}^0$, such that $({\xi_1}^i - \beta_i^0)$ are in $\mathscr{H}$ and $\deg \beta_i^0 = 1$.

Thus the graphs $\mathscr{G}$ and $\mathscr{H}$ are fully determined. Observe that all the even vertices of $\mathscr{G}$ $(\lambda_i^0, 1 \leqq i \leqq 4)$ have degree 3, while the even vertices of $\mathscr{H}$ have degree either 4 ($\deg(\delta^0) = \deg(\delta_1^0) = 4$) or 1 ($\deg(\beta_i^0) = 1$ for all i). So there can be no choice of $*_{\mathscr{G}}$ and $*_{\mathscr{H}}$ such that $\mathscr{G}^t(*_{\mathscr{G}}) = \mathscr{H}^t(*_{\mathscr{H}})$.

This completes the proof of Step 2.

Let $\mathscr{M} = \{\mu_1^0, \mu_2^0, \ldots, \mu_m^0\}$, $m \geqq 0$, be the set of triple points in $\mathscr{H}^0$.

Consider the following partition of the sets of even vertices of $\mathscr{G}$ and $\mathscr{H}$ respectively, obtained by considering the degrees of the even vertices:

$$\mathscr{G}^0 = \coprod_{\xi^1 \in \mathscr{G}^1} \mathscr{I}^{\mathscr{G}}(\xi^1) \coprod_{\substack{\xi^1, \eta^1 \in \mathscr{G}^1 \\ \xi^1 \neq \eta^1}} ((\mathscr{G}(\xi^1, \eta^1) \backslash \Lambda) \coprod \Lambda.$$

$$\mathscr{H}^0 = \coprod_{\xi^1 \in \mathscr{H}^1} \mathscr{I}^{\mathscr{H}}(\xi^1) \coprod_{\substack{\xi^1, \eta^1 \in \mathscr{H}^1 \\ \xi^1 \neq \eta^1}} ((\mathscr{H}(\xi^1, \eta^1) \backslash \mathscr{M}) \coprod \mathscr{M}.$$

To establish that $\mathscr{G}$ is isomorphic to $\mathscr{H}$, it is enough to set up bijections between the corresponding components of the above partition for $\mathscr{G}^0$ and $\mathscr{H}^0$, which preserve neighbours – i.e., if $f\colon \mathscr{H}^0 \to \mathscr{G}^0$ is the resulting "grand bijection", then $\mathscr{N}_{\alpha^0} = \mathscr{N}_{f(\alpha)^0}$ for all α^0 in $\mathscr{H}^0$.

Step 3. In order that there exist a bijection between $\mathscr{G}^0$ and $\mathscr{H}^0$ as in the preceding sentence, it is necessary and sufficient that the following conditions (A) – equivalently (A′) – and (B) are satisfied:

(A) There is a bijection $f\colon \mathscr{M} \mapsto \Lambda$ so that $\mathscr{N}_\mu = \mathscr{N}_{f(\mu)}$ for all μ in $\mathscr{M}$.

(A′) For any three element subset $\mathscr{N}$ of $\mathscr{G}^1$, $|\{\lambda_i^0 \in \Lambda\colon \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}| = |\{\mu_i^0 \in \mathscr{M}\colon \mathscr{N}_{\mu_i^0} = \mathscr{N}\}|$.

(B) $|\mathscr{I}^{\mathscr{G}}(\xi^1)| = |\mathscr{I}^{\mathscr{H}}(\xi^1)|$ for all ξ^1 in $\mathscr{G}^1$.

Reason: The necessity of the conditions (A) and (B) is easy to see, as is the equivalence of the conditions (A) and (A′). (One way of seeing that (A) $\Leftrightarrow$ (A′) is by appealing to the "marriage lemma".)

As for sufficiency, suppose the conditions (A′) and (B) are met. Note that $|\mathscr{G}(\xi^1, \eta^1)| = G^t G(\xi^1, \eta^1) = H^t H(\xi^1, \eta^1) = |\mathscr{H}(\xi^1, \eta^1)|$; on the other hand, the condition (A′) implies that, for all ξ^1, η^1 in $\mathscr{G}^1$, we have the equality

$$|\{\lambda_i^0 \in \Lambda\colon (\xi^1 - \lambda_i^0 - \eta^1) \in \mathscr{G}\}| = |\{\mu_i^0 \in \mathscr{M}\colon (\xi^1 - \mu_i^0 - \eta^1) \in \mathscr{H}\}|.$$

Therefore, for all ξ^1, η^1 in $\mathscr{G}^1$, we have

$$|\mathscr{G}(\xi^1, \eta^1) \backslash \Lambda| = |\mathscr{H}(\xi^1, \eta^1) \backslash \mathscr{M}|,$$

which establishes a bijection between the vertices of degree two connecting ξ^1 and η^1 in $\mathscr{G}$ and $\mathscr{H}$. This completes the proof of Step 3.

Hence, in order to complete the proof of the proposition, we only need to verify the validity of (A′) and (B). The proof of (A′) will be achieved in Steps 4 and 5, while Step 6 will prove (B).

Step 4. For $\mathscr{N} = \{\xi_1^1, \xi_2^1, \xi_3^1\} \subseteq \mathscr{G}^1$, $|\{\lambda^0 \in \Lambda\colon \mathscr{N}_{\lambda^0} = \mathscr{N}\}| \geqq |\{\mu^0 \in \mathscr{M}\colon \mathscr{N}_{\mu^0} = \mathscr{N}\}|$.

Reason : We consider three cases according to the number of triple points $\mu^0 \in \mathscr{M}$ such that $\mathscr{N} = \mathscr{N}_{\mu^0}$, (which cannot exceed 3 since the odd vertices can have degree at most 3, in either graph).

Case (*i*). Let $\mathscr{N} = \mathscr{N}_{\mu^0}$ for some $\mu^0 \in \mathscr{M}$. Then $G^t G(\xi_i^1, \xi_j^1) = H^t H(\xi_i^1, \xi_j^1) \neq 0$ for all i, j. By (3′) $\mathscr{N}_{\mu^0} = \mathscr{N}_{\lambda^0}$ for some $\lambda^0 \in \Lambda$.

Case (ii). Suppose there exist $\mu_1^0, \mu_2^0 \in \mathcal{M}$ such that $\mu_1^0 \neq \mu_2^0$ and $\mathcal{N}_{\mu_1^0} = \mathcal{N}_{\mu_2^0} = \mathcal{N}$. By (i) above, there exists $\lambda_1^0 \in \Lambda$ such that $\mathcal{N}_{\lambda_1^0} = \mathcal{N}_{\mu_1^0} = \mathcal{N}$. Since there are at least two triple points in $\mathcal{H}$ each of whose set of neighbours equals $\mathcal{N}$, $G^t G(\xi_i^1, \xi_j^1) = H^t H(\xi_i^1, \xi_j^1) \geqq 2$. Therefore, there exist $\kappa_{ij}^0 \neq \lambda_1^0$ in $\mathcal{G}^0$, such that $(\xi_i^1 - \kappa_{ij}^0 - \xi_j^1)$ are in $\mathcal{G}$ for all distinct i and j.

If all the κ_{ij}^0's were distinct, $(\xi_1^1 - \kappa_{12}^0 - \xi_2^1 - \kappa_{23}^0 - \xi_3^1 - \kappa_{31}^0 - \xi_1^1)$ would from a 6-cycle in $\mathcal{G}$. Therefore for some $j \neq k$, $\kappa_{ij}^0 = \kappa_{ik}^0$, which then is a triple point, λ_2^0, in $\mathcal{G}^0$, such that $\mathcal{N}_{\lambda_2^0} = \mathcal{N}$.

Case (iii). Suppose there exist three distinct points $\mu_1^0, \mu_2^0, \mu_3^0 \in \mathcal{M}$ such that $\mathcal{N}_{\mu_i} = \mathcal{N}$ for all i. By (i) above there exists $\lambda_1^0 \in \Lambda$ such that $\mathcal{N}_{\lambda_1^0} = \mathcal{N}_{\mu_1^0} = \mathcal{N}$. Since there are three triple points in $\mathcal{H}$ each of whose set of neighbours equals $\mathcal{N}$, we have $H^t H(\xi_1^1, \xi_2^1) = 3$. So there exist distinct κ_1^0, κ_2^0, distinct from λ^0, such that $(\xi_1^1 - \kappa_1^0 - \xi_2^1)$, $(\xi_1^1 - \kappa_2^0 - \xi_2^1)$ are in $\mathcal{G}$. Now $G^t G(\xi_1{}^1, \xi_3^1) = 3$ and $\mathrm{Deg}(\xi_1^1) \leqq 3$. Therefore $(\xi_1^1 - \kappa_1^0 - \xi_3^1)$ and $(\xi_1^1 - \kappa_2^0 - \xi_3^1)$ are in $\mathcal{G}$. So we have $\{\lambda_1^0, \lambda_2^0 = \kappa_1^0, \lambda_3^0 = \kappa_2^0\} \in \Lambda$ such that $\mathcal{N}_{\lambda_i^0} = \mathcal{N}$.

Step 5. End of proof of (A′).

For all $\xi^1 \in \mathcal{G}^1$

$$(1) \qquad |\mathcal{G}(\xi^1, \xi^1)| = \sum_{\xi^1 \neq \eta^1} |\mathcal{G}(\xi^1, \eta^1)| - |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| + |\mathcal{I}^{\mathcal{G}}(\xi^1)|.$$

and,

$$(2) \quad |\mathcal{H}(\xi^1, \xi^1)| = \sum_{\xi^1 \neq \eta^1} |\mathcal{H}(\xi^1, \eta^1)| - |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H}\}| + |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

(Reason: While $|\mathcal{G}(\xi^1, \eta^1)|$ counts the number of even vertices β^0 such that $(\xi^1 - \beta^0 - \eta^1)$ is in $\mathcal{G}$, the first summation on the right side counts such vertices of degree two precisely once and vertices of degree three twice.)

If ξ^1 is such that $|\mathcal{I}^{\mathcal{G}}\xi^1)| = 0$, then

$$|\mathcal{G}(\xi^1, \xi^1)| = \sum_{\eta^1 \neq \xi^1} |\mathcal{G}(\xi^1, \eta^1)| - |\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| \text{ and}$$

$$|\mathcal{H}(\xi^1, \xi^1)| = \sum_{\eta^1 \neq \xi^1} |\mathcal{H}(\xi^1, \eta^1)| - |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H}\}| + |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

Now $|\mathcal{G}(\xi^1, \xi^1)| = |\mathcal{H}(\xi^1, \xi^1)|$, and $|\mathcal{G}(\xi^1, \eta^1)| = |\mathcal{H}(\xi^1, \eta^1)|$.
Therefore,

$$|\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| = |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H})\}| - |\mathcal{I}^{\mathcal{H}}(\xi^1)|.$$

And hence,

$$|\{\lambda_i^0 \in \Lambda : (\lambda_i^0 - \xi^1) \in \mathcal{G}\}| \leqq |\{\mu_i^0 \in \mathcal{M} : (\mu_i^0 - \xi^1) \in \mathcal{H}\}|.$$

So, we have,

$$\begin{aligned}0 &\geqq |\{\lambda_i^0 \in \Lambda: (\lambda_i^0 - \xi^1) \in \mathscr{G}\}| - |\{\mu_i^0 \in \mathscr{M}: (\mu_i^0 - \xi^1) \in \mathscr{H}\}| \\ &= |\{\lambda_i^0 \in \Lambda: \xi^1 \in \mathscr{N}_{\lambda_i^0}\}| - |\{\mu_i^0 \in \mathscr{M}: \xi^1 \in \mathscr{N}_{\mu_i^0}\} \\ &= \sum_{\mathscr{N} \ni \xi^1} (|\{\lambda_i^0 \in \Lambda: \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}| - |\{\mu_i^0 \in \mathscr{M}: \mathscr{N}_{\mu^{0i}} = \mathscr{N}\}| \\ &\geqq 0 \text{ (since each term in the sum is positive by Step 4 above)}\end{aligned}$$

Hence each term in the sum is zero, i.e.,

$$|\{\lambda_i^0 \in \Lambda: \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}| = |\{\mu_i^0 \in \mathscr{M}: \mathscr{N}_{\mu_i^0} = \mathscr{N}\}$$

for all $\mathscr{N} \subseteq \mathscr{G}^1$, containing an element ξ^1 such that $|\mathscr{I}^{\mathscr{G}}(\xi^1)| = 0$.

But, by (4'), every $\mathscr{N}_\lambda$ has an element ξ_λ^1 with $|\mathscr{I}^{\mathscr{G}}(\xi_\lambda^1)| = 0$. If $\mathscr{N} \neq \mathscr{N}_\lambda$ for any λ, there is no triple point in $\mathscr{G}$ whose set of neighbours equals $\mathscr{N}$ and so, by Step 4, there is no such triple point in $\mathscr{H}$ either. So we have (A').

Step 6. Proof of (B).

By (A') we know that for any ξ^1 in $\mathscr{G}^1$

$$\begin{aligned}|\{\lambda_i^0 \in \Lambda: (\lambda_i^0 - \xi^1) \in \mathscr{G}\}| &= \sum_{\mathscr{N} \ni \xi^1} |\{\lambda_i^0 \in \Lambda: \mathscr{N}_{\lambda_i^0} = \mathscr{N}\}| \\ &= \sum_{\mathscr{N} \ni \xi^1} |\{\mu_i^0 \in \mathscr{M}: \mathscr{N}_{\mu_i^0} = \mathscr{N}\}| \\ &= |\{\mu_i^0 \in \mathscr{M}: (\mu_i^0 - \xi^1) \in \mathscr{H}\}|\end{aligned}$$

Therefore by comparing (1) and (2) we have

$$|\mathscr{I}^{\mathscr{G}}(\xi^1)| = |\mathscr{I}^{\mathscr{H}}(\xi^1)|$$

The proof of the proposition is finally complete.

REMARK 7. (1) Each of the graphs A_n, D_n and E_n satisfies the four hypotheses of the last proposition. (For $n > 8$, we write $E_n = A_{n-8} \# E_8$, where the $*$'s of the two graphs are identified.)

(2) For a bipartite graph $\mathscr{G}$, there is a natural induced metric on $V(\mathscr{G})$. For each integer $n \geqq 0$, write $\mathscr{G}_{0]} = \{*\}$, $\mathscr{G}_{1]} = \mathscr{N}_*^{\mathscr{G}} \cup \{*\}$, etc.) It is clear that if $\mathscr{G}$ and $\mathscr{H}$ are weakly dual, so are $\mathscr{G}_{2n]}$ and $\mathscr{H}_{2n]}$. In particular, we recapture Haagerup's observation: if $\mathscr{G}$ and $\mathscr{H}$ are the principal graphs of a finite-index subfactor and if $\mathscr{G}_{n]} = A_{n+1}$, then $\mathscr{H}_{n]} = A_{n+1}$ for even n. (To be sure, it must be verified that the contragredient map is trivial on the even vertices; but for this it is enough to note that for all n the set $\mathscr{G}_{2n+2]}^0 - \mathscr{G}_{2n]}^0$ is invariant under the involution of the even vertices.) The above statement is also valid for odd n; this follows from the case of even n and the connectedness of the principal graphs.

(3) It is tempting to call the subgraph conditions – cf. (3) and (4) of Proposition

6 – a "double of a star-triangle" relation; more precisely, is there more than just a superficial similarity between the two notions?

We now recall the following observation made by Ocneanu (see [K] for the statement and [OK] for a proof).

OBSERVATION 8. Suppose a graph $\mathcal{G}$ satisfies the following conditions:
(1) $\mathcal{G}$ does not contain a subgraph isomorphic to C_4;
(2) $\mathcal{G}$ contains a triple point; and
(3) $\|G\| > 2$.
Then it is not possible to construct a commuting square of the following form:

$$\begin{array}{ccc} C & \overset{G^t}{\subset} & D \\ {}_{G}\cup & & \cup_{G^t} \\ A & \overset{G}{\subset} & B \end{array}$$

In particular, there does not exist a finite depth subfactor $N \subset M$ with trivial contragredient maps, both of whose principal graphs are $\mathcal{G}$.

The above observation, in conjunction with the preceding proposition, has the following interesting consequence.

THEOREM 9. *Let $N \subset M$ be a pair of II_1 factors such that $[M: N] > 4$. Assume that the subfactor N has trivial contragredient maps. If one of the associated principal graphs $\mathcal{G}$ is a finite tree, each of whose vertices has degree at most three, then $\mathcal{G}$ contains a subgraph isomorphic to T such that the vertices of degree one in T are even vertices of degree one in $\mathcal{G}$.*

PROOF. Suppose now that a graph $\mathcal{G}$ arises as in the statement of the theorem. The hypothesis ensures that $\mathcal{G}$ satisfies conditions (1), (2) and (3) of Proposition 6, as well as conditions (1) and (3) of Observation 8. Since $\|G\| > 2$, the graph $\mathcal{G}$ is not A_n for any n. Since $\mathcal{G}$ is assumed to have at most triple points, it follows that $\mathcal{G}$ also satisfies condition (2) of Observation 8.

On the other hand, it must be obvious that a finite graph cannot arise as a principal graph of a subfactor with trivial contragredient maps, if it satisfies conditions (1)–(4) of Proposition 6 as well as conditions (1)–(3) of Observation 8.

Hence it must be the case that $\mathcal{G}$ violates condition (4) of Proposition 6, and the proof is complete.

REMARK 10. The hypothesis about trivial contragredient maps is essential. It has been shown by Haagerup (in an unpublished manuscript) that there exists a subfactor whose principal graphs are as shown below, where the non-trivial contragredient mapping in the graph $\mathcal{G}$ is indicated by the dotted line (fig. 4).

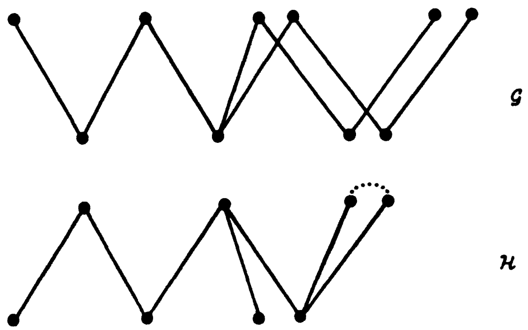

Figure 4.

The above theorem shows that no T-graph – i.e., a graph with a unique vertex of degree 3, with all other vertices of degree at most two – with norm greater than 2 can arise as a principal graph of a subfactor with trivial contragredient maps.

We conclude by describing some more graphs which cannot arise as a principal graph of a subfactor with trivial contragredient maps, as these are trees which satisfy condition (4) of Proposition 6 (and which have norm greater than 2 and have no vertex of degree greater than 3):

(i) any version of a connected sum of A_n, $n \geqq 2$ and E_8 in which a vertex of degree one from A_n has been identified with one of the vertices of degree one in E_8 or one of their degree two neighbours (fig. 5);

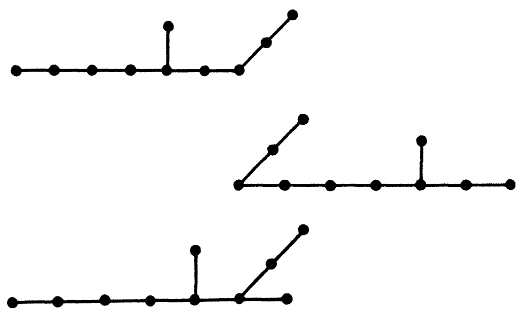

Figure 5.

(ii) the Cayley tree and many other subgraphs of the Bethe lattice (fig. 6).

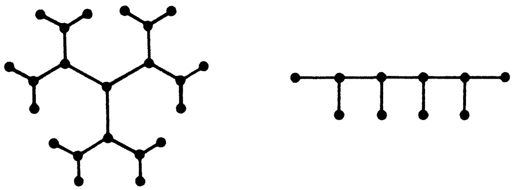

Figure 6.

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A VARIATIONAL PRINCIPLE FOR THE HAUSDORFF DIMENSION OF FRACTAL SETS

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Abstract.

Let $\mathscr{P}(E)$ denote the set of probability measures on a Borel set $E \subseteq \mathsf{R}^n$, and let $\underline{R}(\mu)$, $\bar{R}(\mu)$ denote respectively the lower and upper Rényi dimensions associated with a measure $\mu \in \mathscr{P}(E)$. We prove that the Hausdorff dimension $\dim(E)$ satisfies

$$\dim(E) \leqq \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu)$$

while, if E is additionally bounded, the packing dimension $\operatorname{Dim}(E)$ satisfies

$$\operatorname{Dim}(E) \geqq \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu).$$

As a consequence, for any bounded Borel set E satisfying Taylor's definition of a fractal (i.e. $\dim(E) = \operatorname{Dim}(E)$) we obtain the variational principle

$$\dim(E) = \operatorname{Dim}(E) = \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu) = \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu).$$

In addition we provide an example showing that the hypothesis "bounded" cannot be eliminated.

1. Introduction.

In recent papers on fractals attention has shifted from sets to measure, cf. [1, 2, 3, 4, 5, 6, 8, 9, 10, 12]. Thus it seems reasonable to make an attempt at finding a relation between the dimension of a fractal E and parameters connected with measures supported by E. Such relations have already been investigated, cf. in particular [14, Theorem 1 p. 62] and Young [18]. Our principal result states that if $E \subseteq \mathsf{R}^n$ is a bounded Borel set satisfying Taylor's definition of a fractal, i.e. the Hausdorff dimension $\dim(E)$ of E is equal to the packing dimension $\operatorname{Dim}(E)$ of E, cf. [15] and [16], then

$$\dim(E) = \operatorname{Dim}(E) = \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu) = \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu) \tag{1}$$

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where $\underline{R}(\mu)$ and $\bar{R}(\mu)$ denote, respectively, the lower and upper Rényi dimensions and $\mathscr{P}(E)$ is the family of all Borel probability measures on E.

Formula (1) is a variational principle – i.e. it establishes an equality between a number naturally connected with a space or a map (in this case dim E) and the supremum of certain numbers connected to a class of probability measures supported by E. It is well-known that variational principles play a major role in ergodic theory (cf. e.g. [17, Chapter 8-9]) since these principles yield a canonical way of choosing measures. Formula (1) yields in a similar way a canonical way of choosing measures – namely measures $\mu \in \mathscr{P}(E)$ such that $\underline{R}(\mu)$ and $\bar{R}(\mu)$ are close to dim (E) and Dim (E). It is interesting to note that our variational principle is formulated in terms of the Rényi dimension since generalised Rényi dimensions play an important part in so-called multifractal analysis, cf. e.g. Rand [13] and the references therein.

We begin in section 2 by collecting the relevant facts and setting the notation. Then in section 3 we derive some auxiliary inequalities and prove the variational principle contained in formula (1).

2. Preliminaries.

This section contains a survey of the fractal dimensions which we will consider.

Let (X, d) be a separable metric space, $E \subseteq X$ and $s \geqq 0$. Then the s-dimensional Hausdorff measure $\mathscr{H}^s(E)$ of E is defined by

$$\mathscr{H}^s(E) = \sup_{\delta>0} \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s \mid E \subseteq \cup_{i=1}^{\infty} E_i, \operatorname{diam} E_i < \delta \text{ for all } i \in \mathbb{N} \right\}.$$

The Hausdorff dimension dim E of E is defined by

$$\dim E = \inf\{s \geqq 0 \mid \mathscr{H}^s(E) < \infty\} = \sup\{s \geqq 0 \mid \mathscr{H}^s(E) > 0\}.$$

The s-dimensional packing measure $\mathscr{P}^s(E)$ of E is defined in two stages. First put

$$\mathscr{P}^s_0(E) = \inf_{\delta>0} \sup\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} B_i)^s \mid B_i \cap B_j = \varnothing \text{ for } i \neq j \right.$$
$$\text{and } B_i \text{ is a closed ball of radius at most } \delta$$
$$\left. \text{with center in } E \text{ for all } i \in \mathbb{N} \right\}.$$

Then

$$\mathscr{P}^s(E) = \inf\left\{ \sum_{i=1}^{\infty} \mathscr{P}^s_0(E_i) \mid E \subseteq \cup_{i=1}^{\infty} E_i \right\}.$$

The packing dimension Dim E of E is defined by

$$\text{Dim}\, E = \inf\{s \geqq 0 \mid \mathscr{P}^s(E) < \infty\} = \sup\{s \geqq 0 \mid \mathscr{P}^s(E) > 0\}.$$

It is a well-known fact that $\dim E \leqq \text{Dim}\, E$ for all $E \subseteq \mathsf{R}^n$, cf. [14].

Two other useful dimensions of a bounded set E are the upper and lower box dimensions. For each $\delta > 0$ let $N_\delta(E)$ be the least number of sets of diameter at most δ that cover E. Then the upper and lower box dimensions of E are defined by

$$\bar{C}(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}$$

and

$$\underline{C}(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}$$

respectively.

Let us introduce the Rényi dimension. Fix $\mu \in \mathscr{P}(X)$ and write

$$h_r(\mu) = \inf\left\{ -\sum_{i=1}^{\infty} \mu(E_i) \log \mu(E_i) \,\middle|\, (E_i)_i \text{ is a countable Borel partition of } X \text{ and } \operatorname{diam} E_i \leqq r \right\}$$

for $r > 0$. Then the upper and lower Rényi dimensions of μ are defined by

$$\bar{R}(\mu) = \limsup_{r \to 0} -\frac{h_r(\mu)}{\log r}$$

and

$$\underline{R}(\mu) = \liminf_{r \to 0} -\frac{h_r(\mu)}{\log r}$$

respectively, (cf. [18]).

3. Inequalities and the Variational Principle.

We want to prove that

$$\dim(E) \leqq \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu) \tag{2}$$

for a Borel subset E of R^n, and

$$\text{Dim}(E) \geqq \sup_{\mu \in \mathscr{P}(E)} \bar{R}(\mu) \tag{3}$$

for a bounded Borel subset E of R^n. Both proofs are based on the following result:

THEOREM 1. *Let $E \subseteq \mathsf{R}^n$ be a Borel set. Then the following assertions hold:*
i)

$$\dim(E) = \sup_{\mu \in \mathscr{P}(E)} \left(\inf_{x \in E} \liminf_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} \right).$$

ii) *If*

$$E \subseteq \left\{ x \mid \limsup_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} \geqq \alpha \right\} \text{ and } \mu(E) > 0,$$

then

$$\operatorname{Dim}(E) \geqq \alpha.$$

PROOF. i) Follows easily from [14, Theorem 1]. ii) Follows from [14, Theorem 1], however see also Theorem 3.2 of [5].

We begin with three small technical lemmas

LEMMA 2. *Let μ be a Borel probability measure on R^n. Let E be a Borel set, $t \geqq 0$ and $\delta \in]0, 1[$. Suppose*

$$\log \mu(B(x,r)) \leqq t \log r$$

for all $x \in E$ and $r \in]0, \delta[$. Then

$$\underline{R}(\mu) \geqq \mu(E)t.$$

PROOF. Let $r \in]0, \delta[$ and $(E_i)_i$ be a partition of R^n such that $\operatorname{diam}(E_i) \leqq r$. Let $I = \{i \mid E_i \cap E \neq \emptyset\}$. If $i \in I$ then we can choose a point $x_i \in E_i \cap E$ such that $E_i \subseteq B(x_i, r)$, whence

$$\log \mu(E_i) \leqq \log \mu(B(x_i, r)) \leqq t \log r \text{ for } i \in I. \tag{4}$$

By (4) we have

$$- \sum_i \mu(E_i) \log \mu(E_i) \geqq - \sum_{i \in I} \mu(E_i) \log \mu(E_i) \geqq - \sum_{i \in I} \mu(E_i) t \log r$$
$$= -\mu\left(\bigcup_{i \in I} E_i \right) t \log r \geqq - \mu(E) t \log r.$$

Since the partition $(E_i)_i$ was arbitrary this inequality implies that

$$h_r(\mu) \geqq - \mu(E) t \log r \text{ for } r \in]0, \delta[$$

whence

$$\underline{R}(\mu) = \liminf_{r \to 0} - \frac{h_r(\mu)}{\log r} \geqq t\mu(E).$$

LEMMA 3. *Let $F \subseteq \mathbb{R}^n$ be a bounded Borel set and $r > 0$. Then there exists a finite collection $F_1, \ldots, F_m$ of disjoint Borel sets with* $\operatorname{diam}(F_i) \leqq r$ *such that $F \subseteq \cup_i F_i$ and such that for each i, there exists an $x_i \in F$ satisfying*

$$B(x_i, \tfrac{1}{4}r) \subseteq F_i.$$

PROOF. Construct a sequence of balls $B(x_1, \frac{1}{2}r)$, $B(x_2, \frac{1}{2}r), \ldots$ such that $x_i \in F$ and $d(x_i, x_j) > \frac{1}{2}r$ for $i \neq j$. Because F is totally bounded this process must terminate at some finite stage, giving balls $B(x_1, \frac{1}{2}r), \ldots, B(x_m, \frac{1}{2}r)$ such that any $x \in F$ must satisfy $\min_i d(x, x_i) \leqq \frac{1}{2}r$ (consequently $F \subseteq \cup_{i=1}^m B(x_i, \frac{1}{2}r)$). Note that the smaller balls $B(x_1, \frac{1}{4}r), \ldots, B(x_m, \frac{1}{4}r)$ are disjoint. Set

$$F_1 = B(x_1, \tfrac{1}{2}r) \setminus \bigcup_{j=2}^{m} B(x_j, \tfrac{1}{4}r)$$

$$F_i = B(x_i, \tfrac{1}{2}r) \setminus \left(\bigcup_{j=1}^{i-1} F_j \cup \bigcup_{j=i+1}^{m} B(x_j, \tfrac{1}{4}r) \right) \text{ for } i = 2, \ldots, m-1$$

$$F_m = B(x_m, \tfrac{1}{2}r) \setminus \bigcup_{j=1}^{m-1} F_i.$$

It is clear that the F_i's are disjoint, and since $B(x_1, \frac{1}{4}r), \ldots, B(x_m, \frac{1}{4}r)$ are disjoint we can conclude that $B(x_i, \frac{1}{4}r) \subseteq F_i$ and $F \subseteq \cup_i F_i$.

LEMMA 4. *Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set and $\mu \in \mathscr{P}(E)$. Let $F \subseteq E$ be a Borel set, $t \geqq 0$ and $\delta \in]0, 1[$. Assume*

$$\log \mu(B(x, r)) \geqq t \log r$$

for all $x \in F$ and $0 < r < \delta$. Then

$$\bar{R}(\mu) \leqq t + \mu(E \setminus F)\bar{C}(E \setminus F).$$

PROOF. Let $r \in]0, \delta[$ and choose by Lemma 3 a finite pairwise disjoint covering $(F_1, \ldots, F_m)$ of F with $\operatorname{diam} F_i \leqq r$ and such that there exists points $x_i \in F$ for all i satisfying

$$B(x_i, \tfrac{1}{4}r) \subseteq F_i.$$

The set $E \setminus F$ can be covered by $N = N_r(E \setminus F)$ closed balls $B_1, \ldots, B_N$ of diameter at most r. Define $Q_1, \ldots, Q_N$ by

$$Q_1 = (B_1 \cap (E \setminus F)) \setminus \cup_j F_j$$
$$Q_i = (B_i \cap (E \setminus F)) \setminus (\cup_j F_j \cup \cup_{j=1}^{i-1} Q_j) \text{ for } i = 2, \ldots, N.$$

Then $F_1, \ldots, F_m, Q_1, \ldots, Q_N$ are disjoint sets of diameter not exceeding r, and

$$E = \cup_i (F_i \cap E) \cup \cup_i Q_i, \quad \cup_i Q_i \subseteq E \setminus F.$$

Hence

$$\begin{aligned}
h_r(\mu) &\leqq -\sum_{i=1}^{m} \mu(F_i \cap E)\log\mu(F_i \cap E) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&= -\sum_{i=1}^{m} \mu(F_i)\log\mu(F_i) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&\leqq -\sum_{i=1}^{m} \mu(F_i)\log\mu(B(x_i, \tfrac{1}{4}r)) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&\leqq -\sum_{i=1}^{m} \mu(F_i)t\log(\tfrac{1}{4}r) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i) \\
&\leqq -t\log(\tfrac{1}{4}r) - \sum_{i=1}^{N} \mu(Q_i)\log\mu(Q_i).
\end{aligned}$$

We know that if $p_1, \ldots, p_k \geqq 0$ and $\sum_{i=1}^{k} p_i = s \in [0, 1]$ then in fact $-\sum_{i=1}^{k} p_i \log p_i \leqq s\log k - s\log s \leqq s\log k + \dfrac{1}{e}$. Therefore

$$\begin{aligned}
h_r(\mu) &\leqq -t\log(\tfrac{1}{4}r) + \sum_{i=1}^{N} \mu(Q_i)\log N + \frac{1}{e} \\
&\leqq -t\log(\tfrac{1}{4}r) + \mu\left(\bigcup_{i=1}^{N} Q_i\right)\log N_r(E\setminus F) + \frac{1}{e} \\
&\leqq -t\log(\tfrac{1}{4}r) + \mu(E\setminus F)\log N_r(E\setminus F) + \frac{1}{e}
\end{aligned}$$

for $r < \delta$, whence

$$\begin{aligned}
\bar{R}(\mu) = \limsup_{r \searrow 0} \frac{h_r(\mu)}{-\log r} &\leqq \limsup_{r \searrow 0} \left(\frac{t\log(\frac{1}{4}r)}{\log r} + \mu(E\setminus F)\frac{\log N_r(E\setminus F)}{-\log r} - \frac{1}{e\log r}\right) \\
&\leqq t + \mu(E\setminus F)\bar{C}(E\setminus F).
\end{aligned}$$

We are now ready to prove (2) and (3).

PROPOSITION 5. *Let $E \subseteq \mathrm{R}^n$. Then the following assertions hold:*
i) *If E is a Borel set then*

$$\dim E \leqq \sup_{\mu \in \mathscr{P}(E)} \underline{R}(\mu).$$

ii) *If E is a bounded Borel set then*

$$\sup_{\mu\in\mathscr{P}(E)} \bar{R}(\mu) \leqq \operatorname{Dim} E.$$

PROOF. i) Let $t < \dim E$. Then Theorem 1 part i) implies that there exists a measure $\mu\in\mathscr{P}(E)$ such that

$$t < \liminf_{r\downarrow 0} \frac{\log\mu(B(x,r))}{\log r} \text{ for all } x\in E. \tag{5}$$

Now put

$$E_m = \left\{x\in E \mid \frac{\log\mu(B(x,r))}{\log r} > t \text{ for } 0 < r < \frac{1}{m}\right\},\quad m\in\mathsf{N}.$$

Let $\varepsilon > 0$ and observe that (5) implies that $E_m \uparrow E$. We can thus choose an integer $N\in\mathsf{N}$ so $\mu(E_N) \geqq \mu(E) - \varepsilon = 1 - \varepsilon$. An application of Lemma 2 then yields

$$\sup_{\lambda\in\mathscr{P}(E)} \underline{R}(\lambda) \geqq \underline{R}(\mu) \geqq \mu(E_N)t \geqq (1-\varepsilon)t$$

which proves the first part of the proposition since $t < \dim E$ and $\varepsilon > 0$ were arbitrary.

ii) Let $\mu\in\mathscr{P}(E)$ and $t > \operatorname{Dim}(E)$. Then Theorem 1 part ii) implies that

$$\limsup_{r\downarrow 0} \frac{\log\mu(B(x,r))}{\log r} \leqq \operatorname{Dim}(E)\quad \mu\text{-a.s.}$$

and we can thus choose a subset F of E with $\mu(F) = 1$ such that $\limsup_{r\downarrow 0} \dfrac{\log\mu(B(x,r))}{\log r} < t$ for all $x\in F$. Now put

$$F_m = \left\{x\in F \mid \frac{\log\mu(B(x,r))}{\log r} < t \text{ for } 0 < r < \frac{1}{m}\right\},\quad m\in\mathsf{N}.$$

An application of Lemma 4 then yields

$$\bar{R}(\mu) \leqq t + \mu(E\setminus F_m)\bar{C}(E) = t + \mu(F\setminus F_m)\bar{C}(E).$$

Since $F_m \uparrow F$ we conclude that $\bar{R}(\mu) \leqq t$. This completes the proof since both $\mu\in\mathscr{P}(E)$ and $t > \operatorname{Dim}(E)$ were arbitrary.

Proposition 5 immediately yields the following variational principle

PROPOSITION 6. *If $E \subseteq \mathsf{R}^n$ is a bounded Borel set satisfying* $\dim(E) = \operatorname{Dim}(E)$, *then*

$$\dim(E) = \operatorname{Dim}(E) = \sup_{\mu\in\mathscr{P}(E)} \underline{R}(\mu) = \sup_{\mu\in\mathscr{P}(E)} \bar{R}(\mu).$$

It is easily seen that the inequality in Proposition 5) ii) may not hold if the assumption "bounded" is omitted. Indeed put $E = \mathbf{N}$ and $q_n = c((n+1)(\log(n+1))^2)^{-1}$ for $n \in \mathbf{N}$ where $c = 1/\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, and define $\mu \in \mathscr{P}(E)$ by $\mu = \sum_n q_n \delta_n$ (here δ_x denotes the Dirac measure concentrated at x). If $0 < r < 1$ and $(E_i)_i$ is a countable partition of $E = \mathbf{N}$ then $(E_i \cap E)_i = (\{n\})_{n \in \mathbf{N}}$, whence

$$\frac{h_r(\mu)}{-\log r} = \frac{-\sum_n \mu(\{n\}) \log(\mu\{n\})}{-\log r} = \frac{-\sum_n q_n \log q_n}{-\log r} = \infty$$

which implies that $\bar{R}(\mu) = \underline{R}(\mu) = \infty > 0 = \mathrm{Dim}(E)$.

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EXTENSION OF ENTIRE FUNCTIONS WITH CONTROLLED GROWTH

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Contents.

1. Introduction.

The order ρ of an entire function $F \in \mathcal{O}(\mathbf{C}^n)$ is classically defined as the infimum of all numbers $a > 0$ such that $|F(z)| \leqq C_a e^{|z|^a}$ for some constant C_a. Given the order the type σ is then defined as the infimum of all numbers $b > 0$ such that $|F(z)| \leqq C_b e^{b|z|^\rho}$ for some constant C_b. To make these notions dual in the sense of convex analysis C. O. Kiselman has introduced the concept of relative order and type, generalizing the classical order and type. This was first done in [1], but the idea is more developed in [2]. There among others he considers an extension problem for entire functions. Given two entire functions F and G in $\mathbf{C}^n$, what size can a disk $D_r = \{w \in \mathbf{C};\ |w| < r\}$ have such that there exists a holomorphic function H in $\mathbf{C}^n \times D_r$ satisfying certain growth conditions determined by F and G on the sets $C_1 = \{(z,e) \in \mathbf{C}^n \times \mathbf{C};\ |w| = 1\}$ and $C_e = \{(z,w) \in \mathbf{C}^n \times \mathbf{C};\ |w| = e\}$ respectively? It turns out that the size of the disk is determined by the relative order of G with respect to F. However using these disks we cannot let the growth of H be controlled from both above and below on both of the sets C_1 and C_e, nor can we specify that both $H(z,1) = F(z)$ and $H(z,e) = G(z)$. This paper is a continuation of that work. Instead of disks we consider annuli. In this way we can extend F and G to H with H satisfying mainly the same growth conditions as

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above but now from both above and below on both of the sets C_1 and C_e. (The difference in the growth condition is an arbitrary $\varepsilon > 0$). It is equivalent also to assume that $H(z, 1) = F(z)$ and $H(z, e) = G(z)$. The size of the annulus turns out to be determined by both the relative order of F with respect to G and the relative order of G with respect to F. More generally we consider extensions to logarithmically convex Reinhardt domains in $\mathbf{C}^n \times \mathbf{C}^m$ and we see that all domains to which we can extend the functions can be written in an explicit way.

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2. Relative order and type.

The notion of relative order and type is introduced in Kiselman [1] and studied in detail in [2]. We give here the definitions and some simple facts.

DEFINITION 2.1. Let f, g: $E \to [-\infty, +\infty]$ be two functions defined on a real vector space E. We define *the order of f relative to g* as

$$\mathrm{order}(f:g) = \inf[a > 0; \exists c_a \in \mathbf{R}, \forall x \in E, f(x) \leqq \frac{1}{a} g(ax) + c_a].$$

If g is convex and $g(0) < +\infty$ then the set above is an interval $]\rho, +\infty[$ or $[\rho, +\infty[$, where $0 \leqq \rho \leqq +\infty$.

DEFINITION 2.2. Let f, g be two functions as above. We then define *the type of f relative to g* as

$$\mathrm{type}(f:g) = \inf[b > 0; \exists c_b \in \mathbf{R}, \forall x \in E, f(x) \leqq bg(x) + c_b].$$

The set above is an interval $]\sigma, +\infty[$ or $[\sigma, +\infty[$, where $0 \leqq \sigma \leqq +\infty$, if g is bounded from below.

Let E^* be the algebraic dual of the real vector space E and E' a fixed linear subspace of E^*. We define the spaces $\mathscr{F}(E, E')$ and $\mathscr{F}(E', E)$ in the following way: $\mathscr{F}(E, E')$ is the space of all functions from E to $[-\infty, +\infty]$ which are convex, lower semicontinuous for the weak topology $\sigma(E, E')$ and takes the value $-\infty$ only for the constant function $-\infty$. $\mathscr{F}(E', E)$ is defined similarly for functions from E' to $[-\infty, +\infty]$ but with the weak star topology $\sigma(E', E)$ instead.

Let f: $E \to [-\infty, +\infty]$ be a function on the real vector space E. We define the *Fenchel transform of f* by

$$\tilde{f}(\xi) = \sup_{x \in E} (\xi \cdot x - f(x)), \qquad \xi \in E' \tag{2.1}$$

We can apply the transformation twice getting

$$\tilde{\tilde{f}}(x) = \sup_{\xi \in E'} (\xi \cdot x - \tilde{f}(\xi)), \qquad x \in E. \tag{2.2}$$

A direct consequence of the definition is that we have $\tilde{f} \in \mathscr{F}(E', E)$ and $\tilde{\tilde{f}} \in \mathscr{F}(E, E')$. Obviously the transform is dependent on the subspace E' chosen. Usually in a topological vector space one takes E' as the topological dual of E. For instance in R^n, where the topological dual is isomorphic to R^n, one takes E' as R^n. Some general properties of the Fenchel transform are $\tilde{\tilde{f}} \leqq f$, $\tilde{\tilde{\tilde{f}}} = \tilde{f}$ and

$$\tilde{\tilde{f}} = \sup[v \in \mathscr{F}(E, E');\ v \leqq f]. \tag{2.3}$$

Thus $\tilde{\tilde{f}} = f$ if and only if $f \in \mathscr{F}(E, E')$. For more information see Rockafellar [3].

There is a duality theorem connecting the relative order and type via the Fenchel transform.

THEOREM 2.3 (Kiselman [2], Theorem 4.3). *Let E be a real vector space and E' a linear subspace of E^*. Assume that $f, g \in \mathscr{F}(E, E')$. Then*

$$\operatorname{order}(\tilde{g} : \tilde{f}) = \operatorname{type}(f : g) \text{ and } \operatorname{type}(\tilde{g} : \tilde{f}) = \operatorname{order}(f : g).$$

3. Growth functions.

Let F be an entire function in C^n. We then define its *growth function* as

$$f(t) = \sup[\log |F(z)|;\ z \in \mathrm{C}^n,\ |z| \leqq e^t], \qquad t \in \mathrm{R}. \tag{3.1}$$

We shall also have use for holomorphic functions $H \in \mathcal{O}(\mathrm{C}^n \times \Omega')$, where $\Omega' \subset \mathrm{C}^m$ is a logarithmically convex Reinhardt domain. We will then define *the partial growth function of H* by

$$h_w(t) = \sup_z [\log |H(z, w)|;\ z \in \mathrm{C}^n,\ |z| \leqq e^t], \qquad t \in \mathrm{R},\ w \in \Omega'. \tag{3.2}$$

and also

$$\begin{aligned} h(t, s) &= \sup_{z, w} [\log |H(z, w)|;\ (z, w) \in \mathrm{C}^n \times \Omega',\ |z| \leqq e^t,\ |w_i| = e^s,\ \forall i] \\ &= \sup_w [h_w(t);\ |w_i| = e^s,\ \forall i = 1, \ldots, m], \qquad t, s \in \mathrm{R}. \end{aligned} \tag{3.3}$$

In view of Hadamard's three-circle-theorem, all functions here considered are convex in the variables t and s.

An open set in C^n is a *Reinhardt domain* if $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n)$ belongs to the set for all real $\theta_1, \ldots, \theta_n$ if $z = (z_1, \ldots, z_n)$ does. A set $S \in \mathrm{C}^n$ is *logarithmically convex* if the set $\{x \in \mathrm{R}^n;\ x = (\log |z_1|, \ldots, \log |z_n|)$, for some $z \in S\}$ is convex.

If F and G are two entire functions, we define *the order of F with respect to G* as

$$\operatorname{order}(F : G) = \operatorname{order}(f : g), \tag{3.4}$$

where f and g are defined by (3.1). The order so defined is independent of the norm, since if B_j denotes the closed ball with respect to the norm j, then B_i is included in $e^{k_{ij}}B_j$, for some constant k_{ij}. Thus we have the estimate $f_i(t) \leqq f_j(t + k_{ij})$, where f_i denotes the growth function with respect to the norm i. The independence now follows from the following lemma.

LEMMA 3.1. (Kiselman [2] Lemma 3.2). *Let f_y denote the translate of f: $E \to [-\infty, +\infty]$ by the vector y: $f_y(x) = f(x - y)$. If one of f and g is convex and real valued, then*

$$\operatorname{order}(f_y : g) = \operatorname{order}(f : g_y) = \operatorname{order}(f : g).$$

We can also define what we will call the *refined growth function of F*, as

$$f_r(t) = \sup[\log|F(z)|;\ z \in \mathbf{C}^n,\ |z_i| \leqq e^{t_i}], \qquad t \in \mathbf{R}^n. \tag{3.5}$$

Also this function is convex by Hadamard.

If F and G are two entire functions in $\mathbf{C}^n$, then $\operatorname{order}(f_r : g_r)$ with f_r, g_r defined by (3.5) is in general larger than or equal to $\operatorname{order}(f : g)$, with f and g defined by (3.1), since we can always take all $t_i = t \in \mathbf{R}$. On the other hand if for example $F(z_1, z_2) = F_1(z_1)F_2(z_2)$ and $G(z_1, z_2) = F_2(z_1)F_1(z_2)$, with $\operatorname{order}(F_1 : F_2) > 1$, we get $\operatorname{order}(F : G) = 1$, but $\operatorname{order}(f_r : g_r) > 1$. But with the change of variables

$$z_1' = \frac{z_1 + z_2}{\sqrt{2}}, \qquad z_2' = \frac{z_1 - z_2}{\sqrt{2}},$$

we get $\operatorname{order}(f_r : g_r) = 1$ also for f_r, g_r. Thus we see that the relative order between two refined growth functions is coordinate dependent.

If we take g as the exponential function $g(t) = e^t$ and f as defined by (3.1), we get the order of f relative to g as the classical order of F. We now take g_r as the convex function defined by $g_r(t) = e^{\max_i t_i}$, $t \in \mathbf{R}^n$ and f_r as defined by (3.5). Then we have $\operatorname{order}(f : g) = \operatorname{order}(f_r : g_r)$, since $f_r(t_1, \ldots, t_n) \leqq f(\max)_i\, t_i)$ if we use the norm $|z| = \max_i |z_i|$ in (3.1). Similarly we can define $g(t) = e^{\rho t}$ and $g_r(t) = e^{\rho \max_i t_i}$ to retain the classical type.

4. Coefficient functions.

Let F be an entire function in $\mathbf{C}^n$. We can then expand F in homogeneous polynomials

$$\sum_{j=0}^{\infty} P_j(z), \tag{4.1}$$

where P_j is homogeneous of degree j. We define the norm of the polynomials P_j as

$$\|P_j\| = \sup_{|z| \leqq 1} |P_j(z)|. \tag{4.2}$$

With this norm we define *the coefficient function of F* as

$$p(j) = \begin{cases} -\log \|P_j\|, & j \in \mathbf{N}; \\ +\infty & j \in \mathbf{R} \setminus \mathbf{N}; \end{cases} \tag{4.3}$$

Note that in this definition we set $-\log 0 = +\infty$.

5. Properties of coefficient functions.

Let $f, g\colon E \to [-\infty, +\infty]$ be two functions on a real vector space E. We then define the *infimal convolution of f and g* by

$$f \,\square\, g(x) = \inf_y [f(y) \dot{+} g(x - y)], \qquad x \in E;$$

where $\dot{+}$ is upper addition extending the usual addition to hold from $[-\infty, +\infty]^2$ to $[-\infty, +\infty]$, so that $(+\infty) \dot{+} (-\infty) = +\infty$. In the same manner lower addition $\underset{\cdot}{+}$ is defined, so that $(+\infty) \underset{\cdot}{+} (-\infty) = -\infty$. If we apply the Fenchel transformation to an infimal convolution we get

$$(f \,\square\, g)^{\sim} = \tilde{f} \underset{\cdot}{+} \tilde{g}. \tag{5.1}$$

See also Rockafellar [3] concerning the infimal convolution.

Define the function K as

$$K(t) = \begin{cases} -\log(1 - e^t), & t < 0; \\ +\infty & t \geqq 0. \end{cases} \tag{5.2}$$

Then we have the following theorem connecting the growth and coefficient functions of an entire function.

THEOREM 5.1 (Kiselman [2], Theorem 6.1). *Let $F \in \mathcal{O}(\mathbf{C}^n)$ be an entire function. Let f and p be defined by* (3.1) *and* (4.3) *respectively. Then*

$$\tilde{p} \leqq f \leqq \tilde{p} \,\square\, K \qquad \textit{on } \mathbf{R}. \tag{5.3}$$

COROLLARY 5.2 (Kiselman [2], Corollary 6.5). *Let F, G be two entire functions in $\mathbf{C}^n$. Let f, g be their growth functions defined by* (3.1) *and p, q be their coefficient functions defined by* (4.3). *Then*

$$\operatorname{order}(f : g) = \operatorname{order}(\tilde{p} : \tilde{q}) = \operatorname{type}(\tilde{\tilde{q}} : \tilde{\tilde{p}}).$$

PROOF. This follows from Lemma 3.1 and Theorem 2.3. See Kiselman [2] for the details.

LEMMA 5.3. *Let $F \in \mathcal{O}(\mathbf{C}^n)$ and let p be its coefficient function defined by* (4.3). *Then $p(j)$, $j \in \mathbf{N}$, has faster growth than any linear function, or equivalently*

$$\frac{p(j)}{j} \to +\infty \qquad \text{as } j \to +\infty.$$

PROOF. Since F is entire, $\|P_j\| R^j \to 0$ as $j \to +\infty$, for all $R > 0$. Taking logarithms we get

$$j \log R - p(j) \to -\infty \qquad \text{as } j \to +\infty.$$

Since this holds for all positive R, the lemma follows.

It actually follows that

$$\frac{p(j)}{|j|} \to +\infty \qquad \text{as } |j| \to +\infty \text{ for } j \in \mathbf{R},$$

since $p(j) = +\infty$ for all $j \in \mathbf{R} \setminus \mathbf{N}$.

LEMMA 5.4. *Let $a\colon E \to [-\infty, +\infty]$ be a function on a finite-dimensional real vector space E which grows faster than any linear function:*

$$\frac{a(x)}{|x|} \to +\infty \qquad \text{as } |x| \to +\infty.$$

Then also

$$\frac{\tilde{\tilde{a}}(x)}{|x|} \to +\infty \qquad \text{as } |x| \to +\infty.$$

Moreover if a is lower semicontinuous, then $\tilde{a}$ and $\tilde{\tilde{a}}$ are determined by the set $M = \{(x, a(x)); a(x) = \tilde{\tilde{a}}(x)\}$. That is, if a is redefined to $+\infty$ at all other points, then $\tilde{a}$ and $\tilde{\tilde{a}}$ are unchanged.

PROOF. To prove the first part of the lemma we just observe that the affine functions are among the functions we take supremum of in (2.3). To prove the second part let a_M be the function obtained from a by setting

$$(5.4) \qquad a_M(x) = \begin{cases} a(x) & \text{if } (x, a(x)) \in M; \\ +\infty & \text{if } (x, a(x)) \notin M. \end{cases}$$

We will show that $\tilde{a}_M = \tilde{a}$. Thus also $\tilde{\tilde{a}}_M = \tilde{\tilde{a}}$. Obviously $\tilde{a}_M \leqq \tilde{a}$. Pick some $\xi \in E'$. By the growth condition and lower semicontinuity of a and the definition of the Fenchel transform there exists an $x \in E$ such that $\tilde{a}(\xi) = \xi \cdot x - a(x)$. Since $\tilde{\tilde{a}}(x) \geqq \xi \cdot x - \tilde{a}(x) = a(x)$ and $\tilde{\tilde{a}} \leqq a$, we conclude that x is such that $(x, a(x)) \in M$, which implies $a_M(x) = a(x)$. We get $\tilde{a}_M(\xi) \geqq \xi \cdot x - a_M(x) = \xi \cdot x - a(x) = \tilde{a}(\xi)$. Thus actually $\tilde{a}_M(\xi) = \tilde{a}(\xi)$. But ξ was arbitrary so if follows that $\tilde{a}_M = \tilde{a}$.

By Lemma 5.3, this holds in particular for a coefficient function, since by definition a coefficient function is obviously lower semicontinuous.

6. Extension of entire functions.

Before we proceed with the main theorem we will need the following lemma.

LEMMA 6.1 (Kiselman [2], Part of Theorem 7.2). *If the problem*

$$\begin{cases} \textit{Find } F\colon E \times \mathsf{R} \to]-\infty, +\infty], & \textit{such that} \\ F(x,j) = f_j(x), & x \in E,\ j = 0, 1, \textit{ where} \\ f_j\colon E \to]-\infty, +\infty], & \textit{are convex and lower} \\ & \textit{semicontinuous for } \sigma(E, E'), \end{cases}$$

has a convex solution F which is finite at a point $(0, t)$, $1 < t < +\infty$, *then*

$$\operatorname{order}(f_1 : f_0) \leqq \frac{t}{t-1}.$$

Now let $\alpha \in \mathsf{R}^m$ satisfy $\sum_{j=1}^m \alpha_i = 1$ and $p \in \mathsf{R}\setminus\{0\}$ be a nonzero real number. We consider hyperplanes H_p^α, defined by

$$H_p^\alpha = \{x \in \mathsf{R}^m;\ x \cdot \alpha = p\}. \tag{6.1}$$

The set $\{H_p^\alpha\}_\alpha$, consists of all hyperplanes containing $\mathbf{p} = (p, \ldots, p)$ but not the origin. The set $\{H_p^\alpha\}_p$ is a set of parallel hyperplanes. In R, H_p^α is just the point p.

Let S_p^α denote the closed halfspace which is bounded by H_p^α and contains the origin. More explicitly

$$S_p^\alpha = \begin{cases} \{x \in \mathsf{R}^m;\ x \cdot \alpha \leqq p\} & \text{if } p > 0; \\ \{x \in \mathsf{R}^m;\ x \cdot \alpha \geqq p\} & \text{if } p < 0. \end{cases} \tag{6.2}$$

Now define the logarithmically convex Reinhardt domain ω_p^α by

$$\omega_p^\alpha = \{w \in \mathbf{C}^m;\ (\log|w_1|, \log|w_2|, \ldots, \log|w_m|) \in \operatorname{int} S_p^\alpha\}. \tag{6.3}$$

This is the smallest open set Ω' which satisfies

$$\operatorname{int} S_p^\alpha = \{x \in \mathsf{R}^m;\ x = (\log|w_1|, \ldots, \log|w_m|) \text{ for some } w \in \Omega'\}.$$

There is also a largest open set which satisfies this property. We will denote this set by Ω_p^α. We then have

$$\Omega_p^\alpha = \omega_p^\alpha \cup B_J \tag{6.3'}$$

where $J = \{i\colon p\alpha_i \geqq 0\}$ and B_J the union of all sets $\Pi\omega_p^\alpha$, where Π is a projection which takes some of the components with index in the set J to zero. That is, Ω_p^α is the interior of the closure of ω_p^α. The definition of Ω_p^α just says that if $m = 1$ and

$p > 0$ then we add the origin to ω_p^1 and if $m \geqq 2$ that if some component w_i of $w \in \omega_p^\alpha$ can be arbitrarily small with the other components having fixed values then we add to ω_p^α images of all points in ω_p^α with w_i projected to zero. If there are more such components we go on recursively. For instance if $p = 1$ and $\alpha = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$ then Ω_p^α contains all points $w \in \mathbf{C}^m$ with some component $w_i = 0$, whereas ω_p^α contains none. This is why we introduce the extension (6.3′). We want to fill in unnecessary boundaries.

Define $S_{+\infty}^\alpha = S_{-\infty}^\alpha = \mathbf{R}^m$. Then (6.3) and (6.3′) can be used to define ω_p^α and Ω_p^α also for $p = -\infty, +\infty$, if we let $0 \cdot \infty = 0$ in the definition of J. We have $\omega_{-\infty}^\alpha = \omega_{+\infty}^\alpha = (\mathbf{C}\backslash\{0\})^m$, for all α. If $\alpha_i \geqq 0$ for all i then $\Omega_{+\infty}^\alpha = \mathbf{C}^m$, but there is no α such that $\Omega_{-\infty}^\alpha = \mathbf{C}^m$ since some α_i must be positive. For instance $\Omega_{+\infty}^{(1,0)} = \mathbf{C}^2$ but $\Omega_{-\infty}^{(1,0)} = (\mathbf{C}\backslash\{0\}) \times \mathbf{C}$.

THEOREM 6.2. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ be two transcendental entire functions. Define their growth functions f, g by* (3.1) *respectively. Let λ, $\rho \in \mathbf{R}$, α, $\beta \in \mathbf{R}^m$ satisfy $0 < \lambda \leqq 1 \leqq \rho < +\infty$, and $\sum \alpha_i = \sum \beta_i = 1$. Define the domain*

$$\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m;\ w \in \Omega_{\lambda'}^\alpha \cap \Omega_{\rho'}^\beta\}$$

where $\lambda' = \dfrac{\lambda}{\lambda - 1}, (-\infty \leqq \lambda' < 0), \rho' = \dfrac{\rho}{\rho - 1}, (1 < \rho' \leqq +\infty)$ *and* $\Omega_{\lambda'}^\alpha, \Omega_{\rho'}^\beta$ *are defined by* (6.3′). *Then the following conditions are equivalent:*

(a) $$\operatorname{order}(G:F) \leqq \rho;$$

$$\operatorname{order}(F:G) \leqq \frac{1}{\lambda};$$

(b) *For each $\varepsilon > 0$ (or equivalently some $\varepsilon > 0$) there exists an $H \in \mathcal{O}(\Omega)$ such that*

$$\begin{cases} f \leqq h_w \,\square\, K + \varepsilon, & h_w \leqq f \,\square\, K + \varepsilon, \quad |w_i| = 1,\ \forall i = 1, \ldots, m; \\ g \leqq h_w \,\square\, K + \varepsilon, & h_w \leqq g \,\square\, K + \varepsilon, \quad |w_i| = e,\ \forall i = 1, \ldots, m, \end{cases}$$

where h_w is defined by (3.2) *and K by* (5.2),

(b′) *The condition* (b) *holds together with the extra assumption*

$$H(z, \mathbf{1}) = F(z), \quad H(z, \boldsymbol{e}) = G(z),$$

where $\mathbf{1} = (1, \ldots, 1)$ and $\boldsymbol{e} = (e, \ldots, e)$ (m times).

(c) *There exists an $H \in \mathcal{O}(\Omega)$ such that*

$$\operatorname{order}(f : h(\cdot, 0)) = \operatorname{order}(h(\cdot, 0) : f) = 1,$$

$$\operatorname{order}(g : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g) = 1.$$

where $h(\cdot, \cdot)$ is defined by (3.3).

(c′) *The condition* (c) *holds together with the extra assumption*

$$H(z,\mathbf{1}) = F(z), \qquad H(z,e) = G(z).$$

(*The conditions are equivalent also for polynomials if condition* (c) *is altered to*

$$\operatorname{order}(f : h(\cdot,0)) = \operatorname{order}(h(\cdot,0) : f) = 0,$$

$$\operatorname{order}(g : h(\cdot,1)) = \operatorname{order}(h(\cdot,1) : g) = 0.$$

In this case condition (a) *is equivalent with* $\operatorname{order}(G:F) = \operatorname{order}(F:G) = 0$.)

If $H \in \mathcal{O}(\Omega)$ *satisfies condition* (b) *or* (b′) *for some* $\varepsilon > 0$ *then it also satisfies condition* (c) *or* (c′) *respectively.*

PROOF. Note first that sometimes $\Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'} = \omega^{\alpha}_{\lambda'} \cap \omega^{\beta}_{\rho'}$. That is, if $\operatorname{sign}(\alpha_i) = 1$ when $\operatorname{sign}(\beta_i) \geqq 0$. For instance when $m = 1$ this is always the case:

$$\Omega = \{(z,w) \in \mathbf{C}^n \times \mathbf{C};\, e^{\lambda'} < |w| < e^{\rho'}\}$$

and w is never zero even if $\lambda' = -\infty$.

(a) implies (b′). Make the expansions $F(z) = \sum_{j=0}^{\infty} P_j(z)$ and $G(z) = \sum_{j=0}^{\infty} Q_j(z)$, as in (4.1). Let p and q be the coefficient functions of F and G defined by (4.3) respectively. Put

$$H(z,w) = \frac{1}{E_N\left(\dfrac{1}{e}\right)}\left[E_N\left(\frac{w^{\gamma}}{e}\right)\sum_{j=0}^{\infty} P_j(z) w^{\mu_j} + E_N\left(\frac{1}{w^{\gamma'}}\right)\sum_{j=0}^{\infty} Q_j(z)\left(\frac{w}{e}\right)^{\nu_j}\right], \tag{6.4}$$

where $\gamma, \gamma' \in \mathbf{Z}^m$ are multi-indices satisfying $\sum_{i=1}^{m} \gamma_i = \sum_{i=1}^{m} \gamma'_i = 1$. Moreover, we choose all γ_i nonnegative. That is $\gamma_i = \delta_{ik}$ for some $k = 1,\dots,m$, with δ_{ik} as the Kronecker delta. Some α_i must be positive and we choose γ'_i positive for this i and all the others nonpositive. In this way the functions $w \mapsto w^{\gamma}$ and $w \mapsto \dfrac{1}{w^{\gamma'}}$ are holomorphic in $\Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$. We have as usual $w^{\gamma} = w_1^{\gamma_1} \dots w_m^{\gamma_m}$. The multi-indices $\mu_j, \nu_j \in \mathbf{Z}^m$ are chosen such that $\mu_{ji} = \nu_{ji} = 0$, if $\tilde{\tilde{p}}(j) = +\infty$, which occurs if and only if $\tilde{\tilde{q}}(j) = +\infty$, since we are demanding finite order. (Recall Corollary 5.2). Otherwise we take μ_{ji} as the integer part of $\alpha_i \min(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) - N, 0)$ and ν_{ji} as the integer part of $\beta_i \max(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}((j) + N + m, 0)$. For clarity:

$$\left(\frac{w}{e}\right)^{\nu_j} = \prod_{i=1}^{m}\left(\frac{w_i}{e}\right)^{\nu_{ji}}.$$

The function $E_N(\xi) = (1-\xi)\exp\left(\xi + \dfrac{\xi^2}{2} + \dots + \dfrac{\xi^N}{N}\right)$, $N = 1,2,\dots$, is a so called Weierstrass function. We have $E_N(1) = 0$, so that $H(z,\mathbf{1}) = F(z)$, $H(z,e) = G(z)$. It can be shown (Rudin [4]), that for $|\xi| \leqq 1$

$$|1 - E_N(\xi)| \leqq |\xi|^{N+1}. \tag{6.5}$$

The integer N is chosen such that $e^{2-N} < \varepsilon$.

When $|w_i| = 1, i = 1, \ldots, m$, we make the estimates

$$\left\| P_j w^{\mu_j} - E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| = \left| 1 - E_N\left(\frac{w^\gamma}{e}\right) \right| \|P_j\| \leqq e^{-N-1} \|P_j\| \quad \text{and} \tag{6.6}$$

$$\left\| E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| = \left| E_N\left(\frac{1}{w^{\gamma'}}\right) \right| \exp\left(-q(j) - \sum_i \nu_{ji}\right) \tag{6.7}$$

$$\leqq 2\exp(-q(j) + \tilde{\tilde{q}}(j) - \tilde{\tilde{p}}(j) - N) \leqq 2\exp(-N - \tilde{\tilde{p}}(j)),$$

using $\sum_i \nu_{ji} \geqq \tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) + N$, $q(j) \geqq \tilde{\tilde{q}}(j)$ and (6.5).

When $|w_i| = e, i = 1, \ldots, m$, we make the estimates

$$\left\| E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| = \left| E_N\left(\frac{w^\gamma}{e}\right) \right| \exp\left(-p(j) + \sum_i \mu_{ji}\right) \tag{6.8}$$

$$\leqq 2\exp(-p(j) + \tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) - N) \leqq 2\exp(-N - \tilde{\tilde{q}}(j)),$$

using $\sum_i \mu_{ji} \leqq \tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) - N$, $p \geqq \tilde{\tilde{p}}$ and (6.5). We also have

$$\left\| Q_j \left(\frac{w}{e}\right)^{\nu_j} - E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| = \left| 1 - E_N\left(\frac{1}{w^{\gamma'}}\right) \right| \|Q_j\| \leqq e^{-N-1} \|Q_j\|. \tag{6.9}$$

The partial coefficient function r_w of H is

$$r_w(j) = -\log \left\| \frac{1}{E_N(1/e)} \left[E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} + E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right] \right\|.$$

By the triangle inequality and (6.5), we have

$$-\log(1 + e^{-N-1}) - \log\left(\left\| E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| + \left\| E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| \right) \leqq r_w(j) \tag{6.10}$$

and

$$r_w(j) \leqq -\log(1 - e^{-N-1}) - \log \left| \left\| E_N\left(\frac{w^\gamma}{e}\right) P_j w^{\mu_j} \right\| - \left\| E_N\left(\frac{1}{w^{\gamma'}}\right) Q_j \left(\frac{w}{e}\right)^{\nu_j} \right\| \right|. \tag{6.11}$$

When $|w_i| = 1, i = 1, \ldots, m$, we have by our estimates (6.6), (6.7) and since $p \geqq \tilde{\tilde{p}}$

$$(6.12)\qquad -\log(1+e^{-N-1})-\log(1+e^{-N-1}+2e^{-N})+\tilde{\tilde{p}}(j)\leqq r_w(j),$$

which implies $\tilde{\tilde{p}}(j)-e^{2-N}\leqq r_w(j)$, hence taking the Fenchel transformation $\tilde{r}_w\leqq\tilde{p}+e^{2-N}$. (Recall the general rule $\tilde{\tilde{\tilde{p}}}=\tilde{p}$.)

For j such that $p(j)=\tilde{\tilde{p}}(j)$, we also have

$$(6.13)\qquad r_w(j)\leqq -\log(1-e^{-N-1})-\log(1-e^{-N-1}-2e^{-N})+\tilde{\tilde{p}}(j),$$

which implies $r_w(j)\leqq\tilde{\tilde{p}}(j)+e^{2-N}$, for j satisfying $p(j)=\tilde{\tilde{p}}(j)$. Define p_M from p by (5.4). Then $r_w\leqq p_M+e^{2-N}$, hence $\tilde{p}_M\leqq\tilde{r}_w+e^{2-N}$. But by Lemma 5.4 $\tilde{p}_M=\tilde{p}$, so it follows that $\tilde{p}-e^{2-N}\leqq\tilde{r}_w$. So far, we have $\tilde{p}-e^{2-N}\leqq\tilde{r}_w\leqq\tilde{p}+e^{2-N}$, thus finally by Theorem 5.1

$$(6.14)\qquad \begin{aligned} &f\leqq\tilde{p}\,\square\,K\leqq(\tilde{r}_w+\varepsilon)\,\square\,K=\tilde{r}_w\,\square\,K+\varepsilon\leqq h_w\,\square\,K+\varepsilon,\\ &h_w\leqq\tilde{r}_w\,\square\,K\leqq(\tilde{p}+\varepsilon)\,\square\,K=\tilde{p}\,\square\,K+\varepsilon\leqq f\,\square\,K+\varepsilon,\end{aligned}$$

when $|w_i|=1$, for all $i=1,\dots,m$, since we chose $e^{2-N}<\varepsilon$.

When $|w_i|=e, i=1,\dots,m$, we have similarily using (6.10), (6.8), (6.9) and $q\geqq\tilde{\tilde{q}}$

$$(6.15)\qquad -\log(1+e^{-N+1})-\log(2e^{-N}+1+e^{-N-1})+\tilde{\tilde{q}}(j)\leqq r_w(j),$$

which implies $\tilde{\tilde{q}}(j)-e^{2-N}\leqq r_w(j)$, hence taking the Fenchel transformation $\tilde{r}_w\leqq\tilde{q}+e^{2-N}$. For j such that $q(j)=\tilde{\tilde{q}}(j)$, we get from (6.11), (6.8) and (6.9)

$$(6.16)\qquad r_w(j)\leqq -\log(1-e^{-N-1})-\log(1-e^{-N-1}-2e^{-N})+\tilde{\tilde{q}}(j),$$

which implies $r_w(j)\leqq\tilde{\tilde{q}}(j)+e^{2-N}$, for j satisfying $q(j)=\tilde{\tilde{q}}(j)$. Define q_M from q by (5.4). Then $r_w\leqq q_M+e^{2-N}$, hence $\tilde{q}_M\leqq\tilde{r}_w+e^{2-N}$. But by Lemma 5.4 $\tilde{q}_M=\tilde{q}$, so it follows that $\tilde{q}-e^{2-N}\leqq\tilde{r}_w$. So far, we have $\tilde{q}-e^{2-N}\leqq\tilde{r}_w\leqq\tilde{q}+e^{2-N}$, thus finally by Theorem 5.1

$$(6.17)\qquad g\leqq h_w\,\square\,K+\varepsilon,\quad h_w\leqq g\,\square\,K+\varepsilon,\quad |w_i|=e,\quad i=1,\dots,m.$$

We now have to show that H is holomorphic in Ω. Directly after the definition of H in (6.4) we have chosen γ and γ' so that $w\mapsto w^\gamma$ and $w\mapsto 1/w^{\gamma'}$ becomes holomorphic in Ω. Since the Weierstrass functions are entire these do not cause any trouble. Apart from the Weierstrass functions we show that the first part of the series defining H converges locally uniformly in $\mathbf{C}^n\times\Omega^\alpha_{\lambda'}$ and that the second part converges locally uniformly in $\mathbf{C}^n\times\Omega^\beta_{\rho'}$. Then we can conclude that the whole of the series defining H actually converges locally uniformly in Ω.

It simplifies the argument to show the convergence just in $\mathbf{C}^n\times\omega^\alpha_{\lambda'}$ and $\mathbf{C}^n\times\omega^\beta_{\rho'}$ respectively. By our choice of μ and v it is then clear that we have locally uniform convergence also in $\mathbf{C}^n\times\Omega^\alpha_{\lambda'}$ and $\mathbf{C}^n\times\Omega^\beta_{\rho'}$. This is because we have no negative exponents on components of w which may be zero. For instance if $\alpha_i<0$ then $\sup_{j,w}|w^{\mu_j}|<+\infty$ on compact subsets of $(\mathbf{C}\backslash\{0\})^{i-1}\times\{0\}\times(\mathbf{C}\backslash\{0\})^{m-i}$

$\subset B_J$ since then w^{μ_j} becomes zero for $\mu_{ji} > 0$. If $\alpha_i = 0$ then w^{μ_j} does not depend on w_i.

The first part of the series converges locally uniformly in $\mathbf{C}^n \times \omega^{\alpha}_{\lambda'}$, if

$$\|P_j\|\, R^j r^{\mu_j} = \|P_j\|\, R^j \prod_{i=1}^{m} r_i^{\mu_{ji}} \to 0 \quad \text{as}\, j \to +\infty, \tag{6.18}$$

uniformly for all R, $0 \leqq R \leqq R_1 < +\infty$ and for all $r \in \mathrm{R}^m_+$ such that

$$(\log r_1, \ldots, \log r_m) \in S,\ S \text{ is compact and } S \subset \operatorname{int} S^{\alpha}_{\lambda'}, \tag{6.19}$$

since every r_i is bounded from below on compact subsets of $\omega^{\alpha}_{\lambda'}$. Taking logarithms, this is equivalent to

$$\frac{\sum_{i=1}^{m} \mu_{ji} \log r_i - p(j)}{j} \to -\infty \quad \text{as}\, j \to +\infty, \tag{6.20}$$

with r_i as above. From the definition of μ_j, we have $\tilde{\tilde{p}}(j) - N < \tilde{\tilde{q}}(j)$, when μ_j is nonzero. When $\mu_j = 0$, we have convergence independently of $r \in \mathrm{R}^m_+$, by Lemma 5.3. For μ_j not equal to zero

$$\frac{\sum_{i=1}^{m} \mu_{ji} \log r_i - p(j)}{j} \leqq \frac{\sum_{i=1}^{m} \alpha_i(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) + O(1)) \log r_i - \tilde{\tilde{p}}(j)}{j}, \tag{6.21}$$

since $p \geqq \tilde{\tilde{p}}$. We will now use Corollary 5.2 to make estimates. We have $\tilde{\tilde{q}} \leqq b\tilde{\tilde{p}} + c_b$, for $b > 1/\lambda$, since $\operatorname{type}(\tilde{\tilde{q}} : \tilde{\tilde{p}}) = \operatorname{order}(f : g) \leqq 1/\lambda$. The expression above is linear in $\tilde{\tilde{q}}(j)$. Thus, we only need to estimate it on the endpoints of the possible values of $\tilde{\tilde{q}}(j)$. Since $\tilde{\tilde{q}}(j)$ has a bound from above and since μ_j is zero when $\tilde{\tilde{q}}(j)$ is less than $\tilde{\tilde{p}}(j) - N$, the expression cannot be larger than the value for $\tilde{\tilde{q}}(j) \leqq \tilde{\tilde{p}}(j) - N$, or $\tilde{\tilde{q}}(j) = b\tilde{\tilde{p}}(j) + c_b$. On the bound when $\tilde{\tilde{q}}(j) \leqq \tilde{\tilde{p}}(j) - N$ we have $\mu_j = 0$ and this case we have already considered. On the bound when $\tilde{\tilde{q}}(j) = b\tilde{\tilde{p}}(j) + c_b$ the expression above is equal to

$$\frac{\sum_{i=1}^{m} \alpha_i(\tilde{\tilde{p}}(j) - b\tilde{\tilde{p}}(j) + O(1)) \log r_i - \tilde{\tilde{p}}(j)}{j} \tag{6.22}$$

$$= \frac{[(1-b)\sum_{i=1}^{m} \alpha_i \log r_i - 1]\tilde{\tilde{p}}(j) + O(\max|\log r_i|)}{j} \to -\infty \quad \text{as}\, j \to +\infty,$$

locally uniformly for $(\log r_1, \ldots, \log r_m) \in S^{\alpha}_{b'}$, where $b' > \dfrac{1}{1-b} = \dfrac{1/b}{1/b - 1} > \dfrac{\lambda}{\lambda - 1} = \lambda'$, by Lemma 5.3 and 5.4. (Recall that $b > 1/\lambda \geqq 1$.) For each compact subset S of the interior of $S^{\alpha}_{\lambda'}$, there exist b, b', such that $S \subset S^{\alpha}_{b'} \subset \operatorname{int} S^{\alpha}_{\lambda'}$. Thus we see that the first part of the series converges locally uniformly in $\mathbf{C}^n \times \omega^{\alpha}_{\lambda'}$ and by our discussion above we can conclude that we have locally uniform convergence in $\mathbf{C}^n \times \Omega^\alpha_{\lambda'}$.

In the same way we see that the second part of (6.4) converges locally uniformly in $\mathbf{C}^n \times \omega^\beta_{\rho'}$. We want

$$\|Q_j\| R^j \left(\frac{r}{e}\right)^{\nu_j} = \|Q_j\| R^j \prod_{i=1}^{m} \left(\frac{r_i}{e}\right)^{\nu_{ji}} \to 0 \quad \text{as } j \to +\infty; \tag{6.23}$$

Taking logarithms, this is equivalent to

$$\frac{\sum_{i=1}^m \nu_{ji} \log\left(\frac{r_i}{e}\right) - q(j)}{j} \to -\infty \quad \text{as } j \to +\infty; \tag{6.24}$$

We have $\tilde{\tilde{p}} \leqq a\tilde{\tilde{q}} + c_a$, for $a > \rho$, since type$(\tilde{\tilde{p}}:\tilde{\tilde{q}}) = $ order$(g:f) \leqq \rho$. For ν_j not to be equal to zero, we must have $\tilde{\tilde{p}}(j) > \tilde{\tilde{q}}(j) - N - m$. In this case

$$\frac{\sum_{i=1}^m \nu_{ji} \log\left(\frac{r_i}{e}\right) - q(j)}{j} \leqq \frac{\sum_{i=1}^m \beta_i(\tilde{\tilde{p}}(j) - \tilde{\tilde{q}}(j) + O(1)) \log\left(\frac{r_i}{e}\right) - \tilde{\tilde{q}}(j)}{j}. \tag{6.25}$$

since $q \geqq \tilde{\tilde{q}}$. This is an expression linear in $\tilde{\tilde{p}}(j)$. We have convergence independently of $r \in \mathbf{R}^m_+$ on the bound putting $\tilde{\tilde{p}}(j) = \tilde{\tilde{q}}(j) - N - m$, when $\nu_j = 0$, by Lemma 5.3. On the upper bound for $\tilde{\tilde{p}}(j)$, that is $\tilde{\tilde{p}}(j) = a\tilde{\tilde{q}}(j) + c_a$, the expression above is equal to

$$\frac{\sum_{i=1}^m \beta_i(a\tilde{\tilde{q}}(j) - \tilde{\tilde{q}}(j) + O(1)) \log r_i - a\tilde{\tilde{q}}(j)}{j} \tag{6.26}$$

$$= \frac{[(a-1)\sum_{i=1}^m \beta_i \log r_i - a]\tilde{\tilde{q}}(j) + O(\max|\log r_i|)}{j} \to -\infty \quad \text{as } j \to +\infty,$$

locally uniformly for $(\log r_1, \ldots, \log r_m) \in S^\beta_{a'}$, where $a' < \dfrac{a}{a-1} < \dfrac{\rho}{\rho-1} = \rho'$. But $S^\beta_{a'} \subset \operatorname{int} S^\beta_{\rho'}$. By a discussion similar to that for the first part we can conclude that the second part of the series defining H converges locally uniformly in $\mathbf{C}^n \times \Omega^\beta_{\rho'}$. (The Weierstrass function is an exception. It is holomorphic in $\Omega^\alpha_{\lambda'}$).

REMARK. It can be shown that if ν_j had not been put to zero for $\tilde{\tilde{p}}(j)$ less than $\tilde{\tilde{q}} - N - m$ then the domain of convergence of the second part of the series defining H had been only $\mathbf{C}^n \times \Omega^\beta_{\rho'} \cap \Omega^\beta_{\lambda'}$. A similar statement holds for the first part.

(b) implies (c) and (b′) implies (c′). Pick some $\varepsilon > 0$. By condition (b) there exists a holomorphic function $H \in \mathcal{O}(\Omega)$ satisfying

$$(6.27)\qquad \begin{aligned} h_w(t) &\leqq f \,\square\, K(t) + \varepsilon \leqq f(t+1) + K(-1) + \varepsilon, \\ f(t) &\leqq h_w \,\square\, K(t) + \varepsilon \leqq h_w(t+1) + K(-1) + \varepsilon, \end{aligned}$$

when $|w_i| = 1$ for all i. If also condition (b′) holds one can take a function satisfying

$$(6.28)\qquad H(z, 1) = F(z), H(z, e) = G(z).$$

The estimates still hold if we take the supremum over w. Using Lemma 3.1, we get $\operatorname{order}(h(\cdot, 0): f) \leqq 1$ and $\operatorname{order}(f: h(\cdot, 0)) \leqq 1$. In general the following property holds

$$(6.29)\qquad \operatorname{order}(f:f) = \begin{cases} 1, & \text{if } F \text{ is not a polynomial;} \\ 0, & \text{if } F \text{ is a polynomial,} \end{cases}$$

as is perhaps easiest seen by Corollary 5.2 and Lemma 5.3. By submultiplicativity when F is not a polynomial

$$(6.30)\qquad 1 = \operatorname{order}(f:f) \leqq \operatorname{order}(f: h(\cdot, 0)) \operatorname{order}(h(\cdot, 0): f) \leqq 1,$$

so that all orders are 1. If F is a polynomial the only possibilities for the order are 0 or $+\infty$ and the latter is excluded by the estimates above.

In a similar manner, we get $\operatorname{order}(g: h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1): g) = 1$, when G is not a polynomial. Otherwise 0.

(c) implies (a). We use Lemma 6.1 on the convex function $h(\cdot, \cdot)$. The lemma implies, since $h(0, \rho' - \delta) < +\infty$ for all $\delta > 0$, that

$$(6.31)\qquad \operatorname{order}(h(\cdot, 1): h(\cdot, 0)) \leqq \frac{\rho'}{\rho' - 1} = \rho.$$

With a change of variables $s \mapsto 1 - s$, we get from the other side

$$(6.32)\qquad \operatorname{order}(h(\cdot, 0): h(\cdot, 1)) \leqq \frac{1 - \lambda'}{(1 - \lambda') - 1} = 1/\lambda.$$

By submultiplicativity

$$\operatorname{order}(g:f) \leqq \operatorname{order}(g: h(\cdot, 1)) \operatorname{order}(h(\cdot, 1): h(\cdot, 0)) \operatorname{order}(h(\cdot, 0): f) \leqq \rho;$$

$$\operatorname{order}(f:g) \leqq \operatorname{order}(f: h(\cdot, 0)) \operatorname{order}(h(\cdot, 0): h(\cdot, 1)) \operatorname{order}(h(\cdot, 1): g) \leqq \frac{1}{\lambda}.$$

If F and G are polynomials both orders will be zero.

(c′) implies (c) and (b′) implies (b) obviously, so we are done.

Remark. We see that condition (b) actually is two conditions. We will refer to these as condition (b) holds for every $\varepsilon > 0$ or condition (b) holds for some $\varepsilon > 0$

respectively. Note also that the conditions in Theorem 6.2 are independent of α and β, for if some of the conditions in Theorem 6.2 holds for $\alpha, \beta \in \mathbf{R}^m$, with $\sum \alpha_i = \sum \beta_i = 1$, then condition (a) holds. Since condition (a) is independent of α, β the condition also holds for some other α', β', $\sum \alpha'_i = \sum \beta'_i = 1$.

THEOREM 6.3. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ be two entire functions and $\Omega' \subset \mathbf{C}^m$ a logarithmically convex Reinhardt domain such that condition* (c) *holds in Theorem* 6.2 *for $\Omega = \mathbf{C}^n \times \Omega'$. If* $\operatorname{order}(F:G) > 1$, *then $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ for some $\alpha, \beta \in \mathbf{R}^m$ and for some $\lambda' = \dfrac{\lambda}{\lambda - 1}$ and $\rho' = \dfrac{\rho}{\rho - 1}$ such that ρ and λ satisfies condition* (a). *If* $\operatorname{order}(F:G) \leq 1$ *then $\Omega' \subset \Omega^{\beta}_{\rho'}$.*

Proof. Put

$$S' = \{x \in \mathbf{R}^m;\ x = (\log|w_1|, \ldots, \log|w_m|) \text{ for some } w \in \Omega'\}. \tag{6.33}$$

Then S' is an open convex set and its intersection with the line $t \mapsto (t, \ldots, t)$ is the line segment $(t, \ldots, t)$, $\lambda' < t < \rho'$ for some λ' and ρ' satisfying $-\infty \leq \lambda' < 0$ and $1 < \rho' \leq +\infty$. Thus by convexity and since S' contains the origin $S' \subset \operatorname{int} S^{\alpha}_{\lambda'} \cap S^{\beta}_{\rho'}$ for some $\alpha, \beta \in \mathbf{R}^m$ satisfying $\sum \alpha_i = \sum \beta_i = 1$, with $S^{\alpha}_{\lambda'}$, $S^{\beta}_{\rho'}$ defined by (6.2). Moreover if $H \in \mathcal{O}(\Omega) = \mathcal{O}(\mathbf{C}^n \times \Omega')$ is a holomorphic function, then the function $h(\cdot,\cdot)$ defined by (3.3) satisfies

$$\operatorname{order}(h(\cdot, 1) : h(\cdot, 0)) \leq \frac{\rho'}{\rho' - 1} \tag{6.34}$$

and

$$\operatorname{order}(h(\cdot, 0) : h(\cdot, 1)) \leq \frac{\lambda' - 1}{\lambda'}. \tag{6.35}$$

If H also satisfies condition (c) then by submultiplicativity

$$\operatorname{order}(g:f) \leq \operatorname{order}(g : h(\cdot, 1)) \operatorname{order}(h(\cdot, 1) : h(\cdot, 0)) \operatorname{order}(h(\cdot, 0) : f) \leq \frac{\rho'}{\rho' - 1};$$

$$\operatorname{order}(f:g) \leq \operatorname{order}(f : h(\cdot, 0)) \operatorname{order}(h(\cdot, 0) : h(\cdot, 1)) \operatorname{order}(h(\cdot, 1) : g) \leq \frac{1 - \lambda'}{\lambda'}.$$

Thus

$$\operatorname{order}(G:F) \leq \frac{\rho'}{\rho' - 1} = \rho, \tag{6.36}$$

$$\operatorname{order}(F:G) \leq \frac{\lambda' - 1}{\lambda'} = \frac{1}{\lambda},$$

for some ρ and λ satisfying $0 < \lambda \leqq 1$, $1 \leqq \rho < +\infty$, $\rho' = \frac{\rho}{\rho - 1}$, $\lambda' = \frac{\lambda}{\lambda - 1}$ and also as we see condition (a) in Theorem 6.2. But if condition (a) holds for ρ and λ we can extend F and G to $\mathbf{C}^n \times \Omega^\alpha_{\lambda'} \cap \Omega^\beta_{\rho'}$. Thus since S' is contained in $S^\alpha_{\lambda'} \cap S^\beta_{\rho'}$ all we have to do now is to check the points in Ω' for which some component $w_i = 0$. The set of i for which this can happen is the set of i for which the component x_i in $x \in S^\alpha_{\lambda'} \cap S^\beta_{\rho'}$ with the other components given fixed values is unbounded from below. If λ' and ρ' are finite this is exactly the set of i for which $\lambda'\alpha_i \geqq 0$ and $\rho'\beta_i \geqq 0$. By the definition of $\Omega^\alpha_{\lambda'}$ and $\Omega^\beta_{\rho'}$ we conclude that $\Omega' \subset \Omega^\alpha_{\lambda'} \cap \Omega^\beta_{\rho'}$ if λ' and ρ' are finite. If $\rho' = +\infty$ we just choose all β_i positive so that $\Omega^\beta_{+\infty} = \mathbf{C}^m$, but if $\lambda' = -\infty$ there is no α such that $\Omega^\alpha_{-\infty} = \mathbf{C}^m$. The best we can do is to choose $\alpha_i = \delta_{ij}$ for some j. Then $\Omega^\alpha_{-\infty} = \{w \in \mathbf{C}^m; w_j \neq 0\}$. This is why we need the condition order$(F\colon G) > 1$, hence $\lambda' > -\infty$, to conclude that there exist λ', ρ', α and β such that $\Omega' \subset \Omega^\alpha_{\lambda'} \cap \Omega^\beta_{\rho'}$ for $\lambda' = \frac{\lambda}{\lambda - 1}$, $\rho' = \frac{\rho}{\rho - 1}$ with λ and ρ satisfying condition (a).

The obvious counterexample of Theorem 6.3 for order$(F\colon G) \leqq 1$ is when $F = G$. However if $f(t) > g \,\square\, K(t)$ for some t then condition (b′) cannot be satisfied for all $\varepsilon > 0$ using a function H holomorphic in $\mathbf{C}^n \times \Omega^\beta_{\rho'}$ with all $\beta_i \geqq 0$ by the maximum modulus principle. This follows since then the polydisk with radii $e^{\rho'}$ is contained in $\Omega^\beta_{\rho'}$, so that $(t, s) \mapsto h(t, s)$ is increasing in $s \in]-\infty, \rho'[$. If condition (b′) is fulfilled for some $\varepsilon > 0$ then $h(t, 0) \geqq f(t)$ and $h(t, 1) \leqq g \,\square\, K(t) + \varepsilon$, which leads to a contradiction if $0 < \varepsilon < f(t) - g \,\square\, K(t)$. Thus even if order $(F\colon G) \leqq 1$ it might happen that the function H must have a pole for some $w_i = 0$. To be precise either Ω' is contained in $\Omega^\alpha_{\lambda'} \cap \Omega^\beta_{\rho'}$ or there is a constant A such that AF, G cannot be extended to $\mathbf{C}^n \times \Omega'$ fulfilling condition (b) or (b′).

PROPOSITION 6.4. *Let F and G be two entire functions and let p, q be their coefficient functions respectively. For each $\varepsilon > 0$ in condition* (b) *or* (b′) *in Theorem* 6.2, *we can choose a holomorphic function H in $\Omega \subset \mathbf{C}^n \times \mathbf{C}^m$ which is rational in the $w \in \mathbf{C}^m$* variable if *$\tilde{\tilde{p}}$ and $\tilde{\tilde{q}}$ are finite in the same set and $\tilde{\tilde{p}} - \tilde{\tilde{q}}$ is bounded there. This is not possible iff $f - g \,\square\, K$ or $g - f \,\square\, K$ is unbounded from above.*

PROOF. Recall the definition of H in (6.4). We can approximate the Weierstrass functions by polynomials, since we made estimates only on compact sets. Recalling the definitions of μ and ν in (6.4) we conclude that we can choose H rational in the w variable if $\tilde{\tilde{p}}$ and $\tilde{\tilde{q}}$ are finite in the same set and $\tilde{\tilde{p}} - \tilde{\tilde{q}}$ is bounded there. We will see later in Proposition 6.6 that they must be finite in the same set if condition (b) is to hold. If on the other hand H is rational in the $w \in \mathbf{C}^m$ variable then there exists a polynomial P *in* $\mathbf{C}^m$ such that PH can be extended to an entire function in $\mathbf{C}^n \times \mathbf{C}^m$. (We regard P as constant in the first n variables). But $|P|$ is bounded on the set of $w \in \mathbf{C}^m$ such that $|w_i| = e$ and we can choose $P(1)$ nonzero. Thus if H satisfies condition (b′) for some $\varepsilon > 0$ and $f - g \square K$ is unbounded from above we get a contradiction by the maximum modulus theorem in the same manner as in the paragraph proceeding Theorem 6.3. After a change of variables $(z_1,\ldots,z_n,w_1,\ldots,w_m) \mapsto \left(z_1,\ldots,z_n,\frac{1}{w_1},\ldots,\frac{1}{w_m}\right)$ we can apply the result on the case when $g - f \square K$ is unbounded.

REMARK. The first part of the proposition is also true for condition (c) and (c′) since if H satisfies condition (b) or (b′) for some $\varepsilon > 0$ then it also satisfies condition (c) or (c′) respectively.

We can get bounds for the coefficient functions from the growth functions. Assume that $f \square K \leqq g + C$ or $g \square K \leqq f + D$. Then the following estimates hold by Theorem 5.1.

$$\tilde{q} \square K \geqq g \geqq f \square K - C \geqq \tilde{p} \square K - C \tag{6.37}$$

or

$$\tilde{p} \square K \geqq f \geqq g \square K - D \geqq \tilde{q} \square K - D \tag{6.38}$$

Using (5.1) and since none of the functions $\tilde{\tilde{p}}, \tilde{\tilde{q}}$ or $\tilde{K}$ attains $-\infty$ we get $\tilde{\tilde{p}} + C \geqq \tilde{\tilde{q}}$ or $\tilde{\tilde{q}} + D \geqq \tilde{\tilde{p}}$. That is, we can estimate $\tilde{\tilde{p}} - \tilde{\tilde{q}}$ from above or below. However we cannot have both $f \square K \leqq g + C$ and $g \square K \leqq f + D$ at the same time if F and G are transcendental. On the other hand if $|f - g| \leqq C$ then we can make estimates as above to get $|\tilde{\tilde{p}} - \tilde{\tilde{q}}| \leqq C - \tilde{K}$. It is easy to calculate $\tilde{K}$ and it is done in Kiselman [2], where also the estimates $\log(\tau + 1) \leqq -\tilde{K}(\tau) \leqq \log(\tau + 1) + 1$ are obtained. We see that in this case the function H can have logarithmic growth in the powers of the $w \in \mathbf{C}^m$ variable. Taking $C = 0$ we see that this should be possible to improve.

Let us call a logarithmically convex Reinhardt domain $\Omega' \subset \mathbf{C}^m$ a λ', ρ'*-domain* if $\inf[t;(e^t,\ldots,e^t) \in \Omega'] = \lambda' < 0$, $\sup[t;(e^t,\ldots,e^t) \in \Omega'] = \rho' > 0$ and the complement of Ω' contains $\{w \in \mathbf{C}^m; w_i = 0\}$ for some $i = 1,\ldots,m$, which is always the case when $\lambda' > -\infty$. Then $\Omega' \subset \Omega^\alpha_{\lambda'} \cap \Omega^\beta_{\rho'}$ for some $\alpha, \beta \in \mathbf{R}^m$ and the following proposition holds.

PROPOSITION 6.5. *Each condition in Theorem 6.2 is equivalent with the same condition using* $\omega = \mathbf{C}^n \times \Omega'$, $\Omega' \subset \mathbf{C}^m$ *a* (λ', ρ')*-domain instead of* $\Omega = \{(z,w) \in \mathbf{C}^n \times \mathbf{C}^m; w \in \Omega^\alpha_{\lambda'} \cap \Omega^\beta_{\rho'}\}$.

PROOF. If we examine the proof that (b), (b′) implies (c), (c′) implies (a) in Theorem 6.2 we see that this works also for a $(\lambda' \rho')$-domain. Thus if some condition holds for ω then condition (a) will hold and from this we can conclude that the same condition holds for Ω. The other implication is trivial.

Thus we have found all finite-dimensional logarithmically convex Reinhardt domains to which we can extend in the way of Theorem 6.2 without putting any extra conditions on F and G.

REMARK. It is clear that it is enough for ω to contain a (λ', ρ')-domain and be contained in one to satisfy Proposition 6.5.

A theorem like Theorem 6.2 can not hold if F is a polynomial when G is not, or vice versa, as seen by the following proposition.

PROPOSITION 6.6. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ satisfy condition* (a) *or* (b) *in Theorem* 6.2. *Expand F and G in homogeneous polynomials. Then $F(z) = \sum_{j=M}^{N} P_j(z)$, where $P_M, P_N \neq 0$, if and only if $G(z) = \sum_{j=M}^{N} Q_j(z)$, $Q_M, Q_N \neq 0$. Also $F(z) = \sum_{j=M}^{\infty} P_j(z)$, where $P_M \neq 0$, if and only if $G(z) = \sum_{j=M}^{\infty} Q_j(z)$, $Q_M \neq 0$.*

PROOF. We have already noted in the proof of Theorem 6.2 that condition (a) would be violated otherwise. If we examine the proof that (b′) implies (c′) implies (a), we see that (b) implies (a) regardless of if F and G are both polynomials or not.

7. Extension of entire functions, refined case.

If we instead expand $F \in \mathcal{O}(\mathbf{C}^n)$ in a Taylor series

$$F(z) = \sum_k A_k z^k, \qquad z \in \mathbf{C}^n,\ k \in \mathbf{N}^n, \tag{7.1}$$

where k is a multi-index, we define *the refined coefficient function of F* as

$$a(k) = \begin{cases} -\log |A_k|, & k \in \mathbf{N}^n; \\ +\infty & k \in \mathbf{R}^n \setminus \mathbf{N}^n; \end{cases} \tag{7.2}$$

Similar connections hold between the refined growth and coefficient functions, as between the ordinary growth and coefficient functions. Define K_n by

$$K_n(\xi) = K(\xi_1) + \ldots + K(\xi_n), \qquad \xi \in \mathbf{R}^n, \tag{7.3}$$

with K defined by (5.2). Then we have the following theorem.

THEOREM 7.1 (Kiselman [2], Theorem 6.6). *Let F be an entire function in $\mathbf{C}^n$. Define a, f_r by* (7.2), (3.5) *respectively and K_n by* (7.3). *Then*

$$\tilde{a} \leqq f_r \leqq \tilde{a} \,\square\, K_n \qquad \text{on } \mathbf{R}^n.$$

COROLLARY 7.2. *Let F, G be two entire functions in $\mathbf{C}^n$. Let f_r, g_r be defined by* (3.5) *and a, b by* (7.2), *with F, G respectively. Then*

$$\text{order}(f_r : g_r) = \text{order}(\tilde{a} : \tilde{b}) = \text{type}(\tilde{\tilde{b}} : \tilde{\tilde{a}}). \tag{7.4}$$

PROOF. This follows from Lemma 3.1 and Theorem 2.3 in the same way as in Corollary 5.2.

LEMMA 7.3. *Let $F \in \mathcal{O}(\mathbf{C}^n)$ be an entire function and a be its refined coefficient function defined by* (7.2). *Then*

$$\frac{a(k)}{|k|} \to +\infty \qquad \text{as } |k| \to +\infty,$$

where $|k| = \sum_{j=1}^n |k_j| = \sum_{j=1}^n k_j$.

PROOF. Since F is entire

$$|A_k| \prod_{l=1}^n R_l^{k_l} \to 0 \qquad \text{as } |k| \to +\infty,$$

for all $R \in \mathbf{R}_+^n$. Hence taking logarithms

$$\sum_{l=1}^n k_l \log R_l - a(k) \to -\infty \qquad \text{as } |k| \to +\infty.$$

From the definition of a this actually holds for $k \in \mathbf{R}^n$. Thus we can apply Lemma 5.4 on a refined coefficient function since by definition it is obviously lower semicontinuous.

Define the convex functions

$$h_w(t) = \sup_z [\log |H(z, w)|; z \in \mathbf{C}^n, |z_i| \leqq e^{t_i}], \qquad t \in \mathbf{R}^n, w \in \Omega' \subset \mathbf{C}^m; \tag{7.5}$$

and

$$\begin{aligned} h(t,s) &= \sup_{z,w} [\log |H(z,w)|; z \in \mathbf{C}^n, w \in \Omega', |z_i| \leqq e^{t_i}, |w_i| = e^s, \forall i] \\ &= \sup_w [h_w(t); |w_i| = e^s, \forall i = 1, \ldots, m], \qquad t \in \mathbf{R}^n, s \in \mathbf{R}; \end{aligned} \tag{7.6}$$

Then in complete analogy with Theorem 6.2, we have

THEOREM 7.4. *Let $F, G \in \mathcal{O}(\mathbf{C}^n)$ be two transcendental entire functions. Define their refined growth functions f_r, g_r by* (3.5) *respectively. Let $\lambda, \rho \in \mathbf{R}, \alpha, \beta \in \mathbf{R}^m$ satisfy $0 < \lambda \leqq 1 \leqq \rho < +\infty$ and $\sum \alpha_i = \sum \beta_i = 1$. Define the domain*

$$\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m; w \in \Omega_{\lambda'}^{\alpha} \cap \Omega_{\rho'}^{\beta}\}$$

where $\lambda' = \dfrac{\lambda}{\lambda - 1}, (-\infty \leqq \lambda' < 0), \rho' = \dfrac{\rho}{\rho - 1}, (1 < \rho' \leqq +\infty)$ *and* $\Omega^{\alpha}_{\lambda'}, \Omega^{\beta}_{\rho'}$ *are defined by* (6.3′). *Then the following conditions are equivalent:*

(a)
$$\operatorname{order}(g_r : f_r) \leqq \rho;$$
$$\operatorname{order}(f_r : g_r) \leqq \frac{1}{\lambda};$$

(b) *For each* $\varepsilon > 0$ *(or equivalently some* $\varepsilon > 0$*) there exists an* $H \in \mathcal{O}(\Omega)$, *such that*
$$\begin{cases} f_r \leqq h_w \,\square\, K_n + \varepsilon, \ h_w \leqq f_r \,\square\, K_n + \varepsilon, & |w_i| = 1, \ \forall i = 1, \ldots, m; \\ g_r \leqq h_w \,\square\, K_n + \varepsilon, \ h_w \leqq g_r \,\square\, K_n + \varepsilon, & |w_i| = e, \ \forall i = 1, \ldots, m, \end{cases}$$
where h_w *is defined by* (7.5) *and* K_n *by* (7.3),

(b′) *The condition* (*b*) *holds with the extra assumption*
$$H(z, \mathbf{1}) = F(z) \text{ and } H(z, \mathbf{e}) = G(z);$$
where $\mathbf{1} = (1, \ldots, 1)$ *and* $\mathbf{e} = (e, \ldots, e)$. (*m times*).

(c) *There exists an* $H \in \mathcal{O}(\Omega)$, *such that*
$$\operatorname{order}(f_r : h(\cdot, 0)) = \operatorname{order}(h(\cdot, 0) : f_r) = 1;$$
$$\operatorname{order}(g_r : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g_r) = 1,$$
where $h(\cdot, \cdot)$ *is defined by* (7.6).

(c′) *The condition* (c) *holds with the extra assumption*
$$H(z, \mathbf{1}) = F(z) \text{ and } H(z, \mathbf{e}) = G(z).$$

(If F and G are polynomials the theorem still holds if condition (c) *is altered to*
$$\operatorname{order}(f_r : h(\cdot, 0)) = \operatorname{order}(h(\cdot, 0) : f_r) = 0;$$
$$\operatorname{order}(g_r : h(\cdot, 1)) = \operatorname{order}(h(\cdot, 1) : g_r) = 0.$$
In this case condition (a) *is equivalent with* $\operatorname{order}(f_r : g_r) = \operatorname{order}(g_r : f_r) = 0$.)

If a holomorphic function $H \in \mathcal{O}(\Omega)$ *satisfies condition* (b) *or* (b′) *for some* $\varepsilon > 0$ *then it also satisfies condition* (c) *or* (c′) *respectively.*

PROOF. Make the expansions
$$F(z) = \sum_{k \in N^n} A_k z^k, \qquad G(z) = \sum_{k \in N^n} B_k z^k.$$
Let a and b be the refined coefficient functions of F and G respectively. This time in the proof that (a) implies (b′) we use the function H defined by

$$(7.7)\quad H(z,w) = \frac{1}{E_N\left(\frac{1}{e}\right)}\left[E_N\left(\frac{w^\gamma}{e}\right)\sum_{k\in N^n} A_k z^k w^{\mu_k} + E_N\left(\frac{1}{w^{\gamma'}}\right)\sum_{k\in N^n} B_k z^k\left(\frac{w}{e}\right)^{\nu_k}\right],$$

with the same definitions of N, E_N, γ and γ' as in the proof of Theorem 6.2. The multi-indices $\mu_k, \nu_k \in \mathbf{Z}^m$ are chosen such that $\mu_{ki} = \nu_{ki} = 0$, if $\tilde{\tilde{a}}(k) = +\infty$, which occurs if and only if $\tilde{\tilde{b}}(k) = +\infty$, since we are demanding finite order. (Recall Corollary 7.2). Otherwise we take μ_{ki} as the integer part of $\alpha_i \min(\tilde{\tilde{a}}(k) - \tilde{\tilde{b}}(k) - N, 0)$ and ν_{ki} as the integer part of $\beta_i \max(\tilde{\tilde{a}}(k) - \tilde{\tilde{b}}(k) + N + m, 0)$. We get the partial refined coefficient function c_w of H as

$$(7.8)\quad c_w(k) = -\log\left|\frac{1}{E_N(1/e)} E_N\left(\frac{w^\gamma}{e}\right)A_k w^{\mu_k} + E_N\left(\frac{1}{w^{\gamma'}}\right)B_k\left(\frac{w}{e}\right)^{\nu_k}\right]\right|.$$

By the triangle inequality and (6.5), we have

$$(7.9)\quad -\log(1 + e^{-N-1}) - \log\left(\left|E_N\left(\frac{w^\gamma}{e}\right)A_k w^{\mu_k}\right| + \left|E_N\left(\frac{1}{w^{\gamma'}}\right)B_k\left(\frac{w}{e}\right)^{\nu_k}\right|\right) \leqq c_w(k)$$

and

$$(7.10)\quad c_w(k) \leqq -\log(1 - e^{-N-1}) - \log\left|\left|E_N\left(\frac{w^\gamma}{e}\right)A_k w^{\mu_k}\right| - \left|E_N\left(\frac{1}{w^{\gamma'}}\right)B_k\left(\frac{w}{e}\right)^{\nu_k}\right|\right|.$$

The estimates are now the same as those in Theorem 6.2. When $|w_i| = 1$, we get $\tilde{a} - e^{2-N} \leqq c_w$ by (7.9) and estimates like (6.6) and (6.7), which implies $\tilde{c}_w \leqq \tilde{a} + e^{2-N}$. For $k \in \mathbf{N}^n$ such that $a(k) = \tilde{\tilde{a}}(k)$, we get by (7.10) $c_w(k) \leqq \tilde{\tilde{a}}(k) + e^{2-N}$. Hence using Lemma 5.4 we get $\tilde{a} \leqq \tilde{c}_w + e^{2-N}$, so we have $\tilde{a} - \varepsilon \leqq \tilde{c}_w \leqq \tilde{a} + \varepsilon$ using $e^{2-N} < \varepsilon$. Finally by Theorem 7.1

$$f_r \leqq \tilde{a} \,\square\, K_n \leqq \tilde{c}_w \,\square\, K_n + \varepsilon \leqq h_w \,\square\, K_n + \varepsilon, \qquad |w_i| = 1;$$

$$h_w \leqq \tilde{c}_w \,\square\, K_n \leqq \tilde{a} \,\square\, K_n + \varepsilon \leqq f_r \,\square\, K_n + \varepsilon, \qquad |w_i| = 1;$$

When $|w_i| = e$ we get using (7.9), (7.10) and similar estimates as in (6.8), (6.9), $\tilde{b} - \varepsilon \leqq \tilde{c}_w \leqq \tilde{b} + \varepsilon$. Hence

$$g_r \leqq \tilde{b} \,\square\, K_n \leqq \tilde{c}_w \,\square\, K_n + \varepsilon \leqq h_w \,\square\, K_n + \varepsilon, \qquad |w_i| = e;$$

$$h_w \leqq \tilde{c}_w \,\square\, K_n \leqq \tilde{b} \,\square\, K_n + \varepsilon \leqq g_r \,\square\, K_n + \varepsilon, \qquad |w_i| = e;$$

To show that H is holomorphic in Ω, we now show that

$$|A_k| R^k r^{\mu_k} = |A_k| \prod_{l=1}^{n} R_l^{k_l} \prod_{i=1}^{m} r_i^{\mu_{ki}} \to 0 \qquad \text{as } |k| \to +\infty,$$

uniformly for all $R \in \mathbf{R}^n_+$, $0 \leqq R_l \leqq R' < +\infty$ and for all $r \in \mathbf{R}^m_+$ such that

$$(\log r_1, \ldots, \log r_m) \in S,\ S \text{ is compact and } S \subset \operatorname{int} S^{\alpha}_{\lambda'}.$$

Taking logarithms, this is equivalent to

$$\text{(7.11)} \qquad \frac{\sum_{i=1}^m \mu_{ki} \log r_i - a(k)}{|k|} \to -\infty \qquad \text{as } |k| \to +\infty.$$

By Corollary 7.2 we have $\tilde{\tilde{b}} \leqq d\tilde{\tilde{a}} + c_d$, for $d > 1/\lambda$, so as in the proof of Theorem 6.2 we can make estimates

$$\text{(7.12)} \qquad \frac{\sum_{i=1}^m \mu_{ki} \log r_i - a(k)}{|k|}$$

$$\leqq \max\left(-\frac{a(k)}{|k|}, \frac{\sum_{i=1}^m \alpha_i(\tilde{\tilde{a}}(k) - d\tilde{\tilde{a}}(k) + O(1)) \log r_i - \tilde{\tilde{a}}(k)}{|k|} \right)$$

$$= \max\left(-\frac{a(k)}{|k|}, \frac{[(1-d)\sum_{i=1}^m \alpha_i \log r_i - 1]\tilde{\tilde{a}}(k) + O(\max|\log r_i|)}{|k|} \right)$$

and the last expression tends to $-\infty$ locally uniformly on $S^{\alpha}_{d'}$ as $|k| \to +\infty$ for $d' > \dfrac{1}{1-d} > \lambda'$ by Lemma 7.3 and Lemma 5.4.

Also we can show that

$$|B_k| R^k \left(\frac{r}{e}\right)^{\nu_k} = |B_k| \prod_{l=1}^n R_l^{k_l} \prod_{i=1}^m \left(\frac{r_i}{e}\right)^{\nu_{ki}} \to 0 \qquad \text{as } |k| \to +\infty,$$

uniformly for all $R \in \mathbf{R}^n_+$, $0 \leqq R_l \leqq R' < +\infty$ and for all $r \in \mathbf{R}^m_+$ such that

$$(\log r_1, \ldots, \log r_m) \in S,\ S \text{ is compact and } S \subset \operatorname{int} S^{\beta}_{\rho'}.$$

The proofs that (b) implies (c) and (b′) implies (c′) are similar to those of Theorem 6.2 and omitted. Also the proof that (c) implies (a) is similar and omitted. We have the trivial implications (c′) implies (c) and (b′) implies (c′), so we are done.

We have some similar statements as those following Theorem 6.2.

COROLLARY 7.5. *The conditions in Theorem* 7.4 *are independent of* α *and* β.

THEOREM 7.6. *Let* $F, G \in \mathcal{O}(\mathbf{C}^n)$ *be two entire functions and* $\Omega' \subset \mathbf{C}^m$ *be a logarithmically convex Reinhardt domain such that condition* (c) *holds in Theorem* 7.4 *for* $\Omega = \mathbf{C}^n \times \Omega'$. *If* $\operatorname{order}(f_r : g_r) \geqq 1$, *then* $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ *for some* $\alpha, \beta \in \mathbf{R}^m$ *and* *for some* $\lambda' = \dfrac{\lambda}{\lambda - 1}$ *and* $\rho' = \dfrac{\rho}{\rho - 1}$ *such that* ρ *and* λ *satisfies condition* (a). *If* $\operatorname{order}(f_r : g_r) \leqq 1$ *then* $\Omega' \subset \Omega^{\beta}_{\rho'}$.

Also in the refined case it is necessary to have $\Omega' \subset \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}$ to ensure the possibility to extend all functions with specified orders to $\mathbf{C}^n \times \Omega'$.

PROPOSITION 7.7. *Let F and G be two entire functions and let a, b be their refined coefficient functions respectively. For each* $\varepsilon > 0$ *in condition* (b) *or* (b′) *in Theorem* 7.4, *we can choose a holomorphic function H in* $\Omega \subset \mathbf{C}^n \times \mathbf{C}^m$ *which is rational in the* $w \in \mathbf{C}^m$ *variable if* $\tilde{\tilde{a}}$ *and* $\tilde{\tilde{b}}$ *are finite in the same set and* $\tilde{\tilde{a}} - \tilde{\tilde{b}}$ *is bounded there. This is not possible if* $f_r - g_r \,\square\, K_n$ *or* $g_r - f_r \,\square\, K_n$ *is unbounded from above.*

PROPOSITION 7.8. *Each condition in Theorem* 7.4 *is equivalent with the same condition using* $\omega = \mathbf{C}^m \times \Omega'$, $\Omega' \subset \mathbf{C}^m$ *a* (λ', ρ')*-domain instead of* $\Omega = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m; w \in \Omega^{\alpha}_{\lambda'} \cap \Omega^{\beta}_{\rho'}\}$.

PROPOSITION 7.9. *Let* $F, G \in \mathcal{O}(\mathbf{C}^n)$ *satisfy condition* (a) *or* (b) *in Theorem* 7.4. *Expand F, G in Taylor series*

$$F(z) = \sum_{k \in N^n} A_k z^k, \qquad G(z) = \sum_{k \in N^n} B_k z^k.$$

Let $C(F)$ *denote the convex hull of those* $k \in \mathbf{N}^n$ *for which* $A_k \neq 0$ *and define* $C(G)$ *similarly. Then* $C(F) = C(G)$.

PROOF. Let $\operatorname{dom}(\tilde{\tilde{a}}) = \{x \in \mathbf{R}^n; \tilde{\tilde{a}}(x) < +\infty\}$ be the *effective domain* of $\tilde{\tilde{a}}$, where a is the refined coefficient function of F. If b is the refined coefficient function of G then by Corollary 7.2 we must have $\operatorname{dom}(\tilde{\tilde{a}}) = \operatorname{dom}(\tilde{\tilde{b}})$, if condition (a) is to hold. Now $C(F) \subset \operatorname{dom}(\tilde{\tilde{a}}) \subset \operatorname{cl} C(F)$ holds in general and we will see that actually $\operatorname{dom}(\tilde{\tilde{a}}) = C(F)$. Define for $j \in \mathbf{N}$

$$a_j(k) = \begin{cases} a(k), & k_i \leqq j, \forall i; \\ +\infty & \text{otherwise}; \end{cases} \tag{7.13}$$

and let $C(a_j)$ denote the convex hull of $\operatorname{dom}(a_j)$. Then $C(a_j)$ is a subset of $C(F)$ and is closed since it is finitely generated. Now take a point $x \in \operatorname{cl} C(F) \setminus C(F)$ and an arbitrary number $M \in \mathbf{R}$. Then $x \notin C(a_j)$. Since $C(a_j)$ is closed and a_j is bounded from below there exists a non-vertical hyperplane $P_j = \{(y, z) \in \mathbf{R}^n \times \mathbf{R}; z = \xi_j \cdot y + c_j\}$ passing through (x, M) and also satisfying $\xi_j \cdot k + c_j \leqq a_j(k)$ for each k. Since by Lemma 7.3 a has faster growth than any linear function we can take $P_{j+1} = P_j = P$ for all $j \geqq N$, for some number N and some hyperplane P. Now $a_j \to a$ pointwise, so by (2.3) and since M was arbitrary we conclude that

$\tilde{\tilde{a}}(x) = +\infty$. This shows that condition (a) implies $C(F) = C(G)$. Also (b) implies (a) regardless of the expansions of F and G.

8. Transformation of the relative order.

In Theorem 6.2 the correspondence between the maximum size of the domain Ω and the relative orders is complete when $\operatorname{order}(F:G)$, $\operatorname{order}(G:F) > 1$ or if F, G are polynomials. This is not always the case, but we will see that it is possible to transform the orders. Assume that $F, G \in \mathcal{O}(\mathbf{C}^n)$ are transcendental and that l, k are integers. We define the mapping σ_l: $\mathbf{C}^n \to \mathbf{C}^n$ by $\sigma_l(z) = (z_1^l, \ldots, z_n^l)$ and make the transformations

$$(8.1) \qquad \begin{aligned} F_{lk}(z) &= F(\sigma_l(z))^k; \\ G_{kl}(z) &= G(\sigma_k(z))^l. \end{aligned}$$

We have $|\sigma_l(z)| = |z|^l$, with the norm $|z| = \max_i |z_i|$, which is the norm we will now use to determine the growth function of F_{lk}

$$(8.2) \qquad \begin{aligned} f_{lk}(t) &= \sup[\log|F_{lk}(z)|;\ z \in \mathbf{C}^n,\ |z| \leqq e^t] \\ &= \sup[\log|F(\sigma_l(z))^k|;\ z \in \mathbf{C}^n,\ |z| \leqq e^t] \\ &= \sup[k\log|F(z')|;\ z' \in \mathbf{C}^n,\ |z'| \leqq e^{lt}] \\ &= kf(lt), \end{aligned}$$

f being the growth function of F. Similarily $g_{kl}(t) = lg(kt)$. We can now see the effect of the transformation on the relative order:

$$f_{lk}(t) \leqq \frac{1}{a} g_{kl}(at) + c_a,$$

$$kf(lt) \leqq \frac{l}{a} g(akt) + c_a,$$

$$f(s) \leqq \frac{l}{ka} g\left(\frac{ka}{l} s\right) + c_a.$$

Thus

$$(8.3) \qquad \operatorname{order}(F_{lk} : G_{kl}) = \frac{l}{k} \operatorname{order}(F:G)$$

and similarly

$$(8.4) \qquad \operatorname{order}(G_{kl} : F_{lk}) = \frac{k}{l} \operatorname{order}(G:F).$$

This gives the invariance

$$\text{(8.5)} \qquad \operatorname{order}(F_{lk} : G_{kl}) \operatorname{order}(G_{kl} : F_{lk}) = \operatorname{order}(F : G) \operatorname{order}(G : F) \geqq 1.$$

If we have strict inequality there are numbers $k, l \in \mathbb{N}$ such that both

$$\operatorname{order}(F_{lk} : G_{kl}), \operatorname{order}(G_{kl} : F_{lk}) > 1.$$

Otherwise we can get the orders arbitrarily close to one.

These transformations also work on the refined growth functions.

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ABOUT CERTAIN SINGULAR KERNELS $K(x, y) = K_1(x - y)K_2(x + y)$

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§ 1. Introduction.

In this paper we give a solution to a problem about the L^p-boundedness, $1 < p < \infty$, and the weak type 1-1 of certain singular integral operators. Here we study operators of the form

$$(1.1)\qquad Tf(\xi) = \int_{R^n} k_1(\xi - y)k_2(\xi + y)f(y)\,dy$$

for a wide class of functions k_1 and k_2.

The case $n = 1, p = 2$, has been solved in [Ri-S] when k_1 is the Hilbert kernel and k_2 satisfies

$$(1.2)\qquad |k_2(x)| \leqq c \quad \text{and} \quad |k_2'(x)| \leqq \frac{c}{|x|}, \quad \text{for some} \quad c > 0$$

The authors used strongly the L^2-boundedness of the Hilbert transform and the local Lipschitz condition (1.2).

Following this approach, we take $k_1(x) = \sum_{j \in Z} 2^{jn}\varphi_j(2^j x)$ where $\{\varphi_j\}_{j \in Z}$ is a family of functions in $L^1(R^n)$ satisfying

$$(1.3)\qquad \int \varphi_j(x)\,dx = 0$$

and for some $0 < \varepsilon < 1$

$$(1.4)\qquad \int |\varphi_j(x + h) - \varphi_j(x)|\,dx \leqq c\,|h|^{\varepsilon}$$

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$$\int (1+|x|^{\varepsilon})\,|\varphi_j(x)|\,dx \leqq c \tag{1.5}$$

with c independent of j. It is known that $k_1(x)$ is a tempered distribution and that the operator of convolution by k_1 is bounded on $L^p(R^n)$, $1 < p < \infty$. See for example [Sa-U]. So we ask for suitable conditions about k_2 in order to obtain the boundedness of the operator given by (1.1), for this kind of kernels k_1.

Condition (1.2) leads us to consider functions k_2 satisfying

$$\|k_2\|_\infty \leqq c \tag{1.6}$$

and for some $0 < \delta < 1$, for all $|h| < \dfrac{|x|}{2}$,

$$|k_2(x+h) - k_2(x)| \leqq c\left(\left|\frac{h}{x}\right|\right)^{\delta} \tag{1.7}$$

The main result we obtain is the following.

THEOREM A. *Let $\{\varphi_j\}_{j\in Z}$ be a family of functions in $L^1(R^n)$ with compact support contained in $\{x \in R^n\colon 2^{-1} \leqq |x| \leqq 2\}$ satisfying* (1.3) *and* (1.4). *Let k_2 be a function satisfying* (1.6) *and* (1.7). *Then for $f \in L^p(R^n)$, $1 < p < \infty$,*

$$Tf(\xi) = \lim_{(N,M)\to(-\infty,\infty)} \int \sum_{N\leqq j\leqq M} 2^{jn}\varphi_j(2^j(\xi - y))k_2(\xi+y)f(y)\,dy$$

exists almost everywhere in R^n and $\|Tf\|_p \leqq c_p\|f\|_p$. Moreover, if $f \in S(R^n)$ $|\{x\colon |Tf(x)| > \lambda\}| \leqq c\lambda^{-1}\|f\|_1$ for all $\lambda > 0$ (weak type 1–1).

In § 2 we give some preliminaries, in § 3 we prove Theorem A, in § 4 we obtain the same result replacing the hypothesis of compact support of φ_j by (1.5), and in § 5 we show some examples of kernels k_1 and k_2 that give rise to operators Tf as in theorem A.

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§ 2. Preliminaries.

In this section we state some properties about the L^p-boundedness and the weak type 1–1 of certain singular integral operators and the maximal operators associated. We set, for $g\colon R^n \to C$ and $j \in Z$, $g^{(j)}(x) = 2^{jn}_g(2^j x)$.

Let us consider a family of functions $\{\varphi_j\}_{j\in Z}$ in $L^1(R^n)$ satisfying (1.3), (1.4) and (1.5). It is not hard to see that if we take $\{\sigma_k\}_{k\in Z}$ and $\{\mu_k\}_{k\in Z}$ the Borel measures with density $\varphi^{(-k)}_{-k}$ and $|\varphi^{(-k)}_{-k})|$ respectively, then

$$|\hat{\mu}_k(\xi) - \hat{\mu}_k(0)| \leqq c(2^k|\xi|)^\varepsilon, \quad k \in Z$$

$$|\hat{\mu}_k(\xi)| \leqq c(2^k|\xi|)^{-\varepsilon}, \quad k \in Z$$

with ε as in (1.4) and (1.5). Moreover, the same conditions are satisfied by $\{\sigma_k\}_{k\in Z}$. Applying straightforward Theorem F in [D-R] we obtain the following results: For $1 < p < \infty$

$$Mf(x) = \sup_k (|\varphi_k^{(k)}| * |f|)(x) \quad \text{is bounded on} \quad L^p(R^n) \tag{2.1}$$

$$K_1 f(x) = \sum_k (\varphi_k^{(k)} * f)(x) \quad \text{is bounded on} \quad L^p(R^n) \tag{2.2}$$

Moreover if supp $\varphi_k^{(k)} \subseteq \{x: |x| < 2^{k+1}\}$, then

$$K_1^* f(x) = \sup_j \left| \sum_{k \leqq j} (\varphi_k^{(k)} * f)(x) \right| \quad \text{is bounded on} \quad L^p(R^n) \tag{2.3}$$

REMARK 2.4. We also observe that, for $f \in S(R^n)$, the operators M, K_1 and K_1^* above defined, are of weak type 1–1. Indeed, with standard techniques we can see that, for some $c > 0$

$$\sum_{k \in Z} \int_{|x| > |2y|} |\varphi_k^{(k)}(x+y) - \varphi_k^{(k)}(x)|\, dx \leqq c \quad \text{for all } y \in R^n \tag{2.5}$$

See for example [G-St], [Sa-U]. So, the boundedness on $L^2(R^n)$, (2.5) and theorem 2.4 in [C–W] imply the weak type 1–1 of K_1. The proofs of the weak type (1-1) of M and K_1^* follow the same lines than those in the theorem last mentioned.

§3. The Main Result.

Before beginning with the proof of Theorem A, we make the following

REMARK 3.1. Let $\{\varphi_j\}_{j\in Z}$ be a family of functions satisfying (1.3), (1.4) and supp $\varphi_j \subseteq \{x: 2^{-1} \leqq |x| \leqq 2\}$. Then, for $r \in Z$, $f \in L^p(R^n)$, $1 < p < \infty$, there exists

$$S_r f(\xi) = \lim_{(N,M)\to(-\infty,\infty)} \sum_{N \leqq j \leqq M} \int_{|x| > 2^r} \varphi_j^{(j)}(x) f(\xi - x)\, dx \tag{3.2}$$

for almost every $\xi \in R^n$

Indeed, since supp $\varphi_j^{(j)} \subseteq \{x: 2^{-j-1} \leqq |x| \leqq 2^{-j+1}\}$, we have that

$$S_r f(\xi) = \int_{|x| > 2^r} \varphi_{-r}^{(-r)}(x) f(\xi - x)\, dx + \sum_{j \leqq -r-1} \int \varphi_j^{(j)}(x) f(\xi - x)\, dx$$

Thus, for all $r \in Z$,

$$|S_r f(\xi)| \leqq Mf(\xi) + K_1^* f(\xi)$$

where M and K_1^* are defined by (2.1) and (2.3) respectively.

REMARK 3.3. The last inequality of the previous remark, (2.1), (2.3) and remark (2.4) imply that $\sup_{r \in Z} |S_r f(\xi)|$ is bounded on $L^p(R^n)$, $1 < p < \infty$, and for $f \in S(R^n)$ it is of weak type 1–1.

The same results hold for

$$\tilde{S}_r f(\xi) = \lim_{(N,M) \to (-\infty, \infty)} \sum_{N \leqq j \leqq M} \int_{|x| \leqq 2^r} \varphi_j^{(j)}(x) f(\xi - x)\, dx \tag{3.4}$$

Indeed, $\tilde{S}_r f(\xi) = K_1 f(\xi) - S_r f(\xi)$ and we apply (2.2)

PROOF OF THEOREM A. For $M, N \in Z$, $N < M$, $f \in S(R^n)$ and $\xi \in R^n$, we set

$$T_{NM} f(\xi) = \sum_{N \leqq j \leqq M} \int \varphi_j^{(j)}(\xi - y) k_2(\xi + y) f(y)\, dy$$

With a change of variables, we obtain

$$T_{NM} f(\xi) = \sum_{N \leqq j \leqq M} \int \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx.$$

We fix $l = l(\xi) \in Z$ such that $2^l \leqq |\xi| < 2^{l+1}$ and we decompose

$$T_{NM} f(\xi) = \sum_{N \leqq j \leqq M} \int_{|x| \leqq 2^l} + \sum_{N \leqq j \leqq M} \int_{2^l < |x| \leqq 2^{l+3}} + \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \tag{3.5}$$

Since $\operatorname{supp} \varphi_j^{(j)} \subseteq \{x \colon 2^{-j-1} \leqq |x| \leqq 2^{-j+1}\}$, the central sum is independent of N and M, for $|N|$ and $|M|$ large enough, and

$$\left| \sum_{N \leqq j \leqq M} \int_{2^l < |x| \leqq 2^{l+3}} \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx \right| \leqq 6\, \|k_2\|_\infty\, Mf(\xi)$$

where Mf is the maximal operator defined in (2.1). Furthermore (2.1) and remark (2.4) imply

$$\lim_{(N,M) \to (-\infty, \infty)} \sum_{N \leqq j \leqq M} \int_{2^l < |x| \leqq 2^{l+3}} \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx$$

is bounded on $L^p(R^n)$ and of weak type 1–1.

We now study the first sum of (3.5)

$$\sum_{N \leq j \leq M} \int_{|x| \leq 2^l} \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx$$

$$= \sum_{N \leq j \leq M} \int_{|x| \leq 2^l} \varphi_j^{(j)}(x)[k_2(2\xi - x) - k_2(2\xi)] f(\xi - x)\, dx$$

$$+ k_2(2\xi) \sum_{N \leq j \leq M} \int_{|x| \leq 2^l} \varphi_j^{(j)}(x) f(\xi - x)\, dx$$

Now

$$\sum_{N \leq j \leq M} \int_{|x| \leq 2^l} |\varphi_j^{(j)}(x)|\, |k_2(2\xi - x) - k_2(2\xi)|\, |f(\xi - x)|\, dx \leqq c\, Mf(x)$$

Indeed, since $|x| \leqq 2^l \leqq |\xi|$ we apply (1.7) to obtain

$$\sum_{N \leq j \leq M} \int_{|x| \leq 2^l} |\varphi_j^{(j)}(x)|\, |k_2(2\xi - x) - k_2(2\xi)|\, |f(\xi - x)|\, dx$$

$$\leqq c \sum_{N \leq j \leq M} \int_{|x| \leq 2^l} |\varphi_j^{(j)}(x)| \left(\frac{|x|}{|\xi|}\right)^{\delta} |f(\xi - x)|\, dx$$

$$\leqq c \sum_{j \geq -l-1} \int_{|x| \leq 2^l} |\varphi_j^{(j)}(x)|\, 2^{-j\delta}\, |f(\xi - x)|\, dx \leqq c\, Mf(\xi)$$

So $\lim_{(N,M) \to (-\infty, \infty)} \sum_{N \leq j \leq M} \int_{|x| \leq 2^l} \varphi_j^{(j)}(x)[k_2(2\xi - x) - k_2(2\xi)] f(\xi - x)\, dx$ exists for all $\xi \in R^n$, moreover (2.1) and remark (2.4) imply that it is bounded on $L^p(R^n)$, $1 < p < \infty$, and of weak type 1–1.

Now, if $\tilde{S}_l$ is as in (3.4),

$$\lim_{(N,M) \to (-\infty, \infty)} k_2(2\xi) \sum_{N \leq j \leq M} \int_{|x| \leq 2^l} \varphi_j^{(j)}(x) f(\xi - x)\, dx = k_2(2\xi) \tilde{S}_l f(\xi)$$

and it is absolutely bounded by $\|k_2\|_\infty \sup_{r \in Z} |\tilde{S}_r f(\xi)|$. From this and Remark 3.3 we obtain the L^p-boundedness, $1 < p < \infty$, and the weak type 1–1 of the above limit. So the study of the first sum in (3.5) is completed.

We now perform an analogous decomposition for the last sum in (3.5)

$$\sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x) k_2(2\xi - x) f(\xi - x)\, dx$$

$$= \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x)[k_2(2\xi - x) - k_2(\xi - x)] f(\xi - x)\, dx$$

$$+ \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx$$

But $|x| > 2^{l+3}$ implies $|x - \xi| \geqq |x| - |\xi| \geqq 2|\xi|$ and by (1.7),

$$\sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} |\varphi_j^{(j)}(x)|\, |k_2(2\xi - x) - k_2(\xi - x)|\, |f(\xi - x)|\, dx$$

$$\leqq c \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} |\varphi_j^{(j)}(x)| \frac{|\xi|^\delta}{|\xi - x|^\delta} |f(\xi - x)|\, dx$$

$$\leqq c \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} |\varphi_j^{(j)}(x)| \left(\frac{|\xi|}{|x|}\right)^\delta |f(\xi - x)|\, dx$$

In the last inequality we use that $|\xi - x| \geqq \frac{3}{4}|x|$ if $|x| > 2^{l+3}$.

As before, this sum is bounded by

$$c \sum_{m \geqq 0} 2^{-m\delta} \int |\varphi_{-l-m-2}^{(-l-m-2)}(x)|\, |f(\xi - x)|\, dx$$

which, in turn, is bounded by $c\, Mf(\xi)$.

Finally

$$\lim_{(N,M) \to (-\infty, \infty)} \sum_{N \leqq j \leqq M} \int_{|x| > 2^{l+3}} \varphi_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx = S_{l+3}(k_2 f)(\xi)$$

with S_{l+3} as in (3.2). If $f \in L^p(R^n)$ so does $k_2 f$ and thus the L^p-boundedness, $1 < p < \infty$, and the weak type 1–1 of the above limit follow from Remark 3.3.

§4.

In this paragraph we extend the result obtained in § 3, asking the family $\{\varphi_j\}_{j \in Z}$ to satisfy (1.3), (1.4) and (1.5). We need the following .

LEMMA 4.1. *Let* $\{\varphi_j\}_{j \in Z}$ *be a family of functions in* $L^1(R^n)$ *satisfying* (1.3), (1.4) *and* (1.5). *Then, for* $0 < a < b$, *there exists a finite constant* $c(b/a)$, *depending only on* b/a *and* n *such that*

$$\sum_{j \in Z} \int_{a < |x| < b} |\varphi_j^{(j)}(x)|\, dx \leqq c(b/a)$$

PROOF. As an easy consequence of the Theorem 4', pag. 153 [St], we note that there exist $q > 1$ and $c > 0$ such that for all $j \in Z$, $\|\varphi_j\|_q \leqq c$. Since $\|\varphi_j^{(j)}\|_q = 2^{jn(1-1/q)}\|\varphi_j\|_q$ we have, by Hölder's inequality that

$$\sum_{2^j < a^{-1}} \int_{a < |x| < b} |\varphi_j^{(j)}(x)|\, dx \leqq c \sum_{2^j < a^{-1}} 2^{jn(1-1/q)}(b^n - a^n)^{1-1/q} = c(b/a).$$

On the other hand,

$$\sum_{2^j > a^{-1}} \int_{a < |x| < b} |\varphi_j^{(j)}(x)|\, dx = \sum_{2^j > a^{-1}} \int_{a < |x| < b} 2^{jn} |\varphi_j(2^j x)|\, (2^j |x|)^\delta (2^j |x|)^{-\delta}\, dx$$

$$\leqq a^{-\delta} \sum_{2^j > a - 1} 2^{-j\delta} \int |\varphi_j(y)|\, |y|^\delta\, dy$$

being the last term bounded independently of a and b.

By personal communication, F. Ricci told us the following result.

LEMMA 4.2. *Let* K_1 *be the tempered distribution given by* $K_1(f) = \sum_{j \in Z} \langle \varphi_j^{(j)}, f \rangle$ *with* φ_j *satisfying* (1.3), (1.4) *and* (1.5), *where, as usual,* $\langle \varphi_j^{(j)}, f \rangle = \int_{R^n} \varphi_j^{(j)}(x) f(x)\, dx$. *Then* K_1 *can be decomposed as* $\sum \langle \psi_j^{(j)}, f \rangle$ *where* $\{\psi_j\}_{j \in Z}$ *is a family of functions with compact support contained in* $\{x: 2^{-1} \leqq |x| \leqq 2\}$, *and satisfying* (1.3) *and* (1.4).

A slight modification of the proof of 4.2, gives us the following.

LEMMA 4.3. *Let* $\{\varphi_j\}_{j \in Z}$ *be a family of functions satisfying* (1.3), (1.4) *and* (1.5). *Let* k_2 *be a function satisfying* (1.6) *and* (1.7). *Then there is a family of functions* $\{\beta_j\}_{j \in Z}$ *with compact support contained in* $\{x: 2^{-1} \leqq |x| \leqq 2\}$ *satisfying* (1.3) *and* (1.4) *such that for each* $f \in S(R^n)$ *and* $\xi \in R^n - \{0\}$,

$$\sum_{j \in Z} \int \varphi_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx = \sum_{j \in Z} \int \beta_j^{(j)}(x) k_2(\xi - x) f(\xi - x)\, dx$$

PROOF. Before dealing with the proof, we introduce some additional notation.

We set for $k \in Z$, $E_k = \{x \in R^n: 2^k < |x| \leqq 2^{k+1}\}$, also for $g \in L^{1,\mathrm{loc}}(R^n)$ we write $m_k(g) = |E_k|^{-1} \int_{E_k} g$ *and we define, for* $x \in R^n$, $\Phi(x) = k_2(2\xi - x)f(\xi - x)$.

We give the proof in several steps.

Step 1. $\sum_{j,1 \in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|\Phi(x) - m_1(\Phi)|\,dx < \infty.$

Indeed, we pick $r \in R$ such that $2^r = |\xi|/8$. Then

$$\sum_{1 \geqq r}\sum_{j \in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|\Phi(x) - m_1(\Phi)|\,dx \leqq \sum_{1 \geqq r}\sum_{j \in Z} 2\,\|\Phi\|_{L^\infty(E_1)} \int_{E_1} |\varphi_j^{(j)}(x)|\,dx$$

Now, since $f \in S(R^n)$, we have $\|\Phi\|_{L^\infty(E_1)} \leqq c\,2^{-1}$ for some positive constant c. Then by lemma 4.1 the above sum converges.

On the other hand

$$\sum_{1 < r}\sum_{j \in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|\Phi(x) - m_1(\Phi)|\,dx$$

$$\leqq \sum_{1 < r}\sum_{j \in Z} \int_{E_1} |\varphi_j^{(j)}(x)|\,|E_1|^{-1} \int_{E_1} |\Phi(x) - \Phi(t)|\,dt\,dx$$

$$\leqq \sum_{1 < r}\sum_{j \in Z} \sup_{s,t \in E_1} |\Phi(s) - \phi(t)| \int_{E_1} |\varphi_j^{(j)}(x)|\,dx$$

Since $1 < r$ for $s, t \in E_1$ we have $|2\xi - s| \geqq \max\{|\xi|, 2\,|s - t|\}$. So we can apply (1.7) to obtain

$$|k_2(2\xi - s) - k_2(2\xi - t)| \leqq c\frac{|s - t|^\delta}{|2\xi - s|^\delta} \leqq c\,2^{1\delta}|\xi|^{-\delta} \text{ for some positive constant } c.$$

Then we can write

$$|\Phi(s) - \Phi(t)| = |k_2(2\xi - s)f(\xi - x) - k_2(2\xi - t)f(\xi - t)|$$
$$\leqq |k_2(2\xi - s) - k_2(2\xi - t)|\,|f(\xi - t)| + |f(\xi - s) - f(\xi - t)|\,|k_2(2\xi - t)|$$
$$\leqq c\,2^{1\delta}|\xi|^{-\delta}\,\|f\|_\infty + 2^{1+2}\,\|\nabla f\|_\infty\,\|k_2\|_\infty$$

and we can apply again lemma 4.1 to obtain the statement of step 1.

Step 2. For $j, k \in Z$, let $\varphi_{j,k}$ be the function defined by $\varphi_{j,k} = \varphi_j\chi_{E_k} - |E_k|^{-1}\chi_{E_k} \int_{E_k} \varphi_j$ where χ_{E_k} is the characteristic function of the set E_k. Then there is a family of functions $\{\vartheta_j\}_{j\in Z}$ with compact support contained in $\{x: 2^{-1} \leqq |x| \leqq 2\}$ and satisfying (1.3) and (1.4) such that $\sum_{j,k\in Z} \langle \varphi_{j,k}^{(j)}, \Phi\rangle = \sum_{j\in Z} \langle \vartheta_j^{(j)}, \Phi\rangle$.

Indeed, since

$$\langle \varphi_{j,k}^{(j)}, \Phi\rangle = \int_{E_{k-j}} \varphi_j^{(j)}(x)(\Phi(x) - m_{k-j}(\Phi))\,dx \tag{4.4}$$

the double sum $\sum_{j,k\in Z} \langle \varphi_{j,k}^{(j)}, \Phi\rangle$ is absolutely convergent by step 1 and we can rearrange it to obtain

$$\sum_{j,k\in Z} \langle \varphi_{j,k}^{(j)}, \Phi\rangle = \sum_{1} \sum_{j-k=1} \langle \varphi_{j,k}^{(j)}, \Phi\rangle = \sum_{1} \langle \vartheta_1^{(1)}, \Phi\rangle$$

where $\vartheta_1(x) = \sum_{j-k=1} \varphi_j^{(j-1)}(x)\chi_{E_0}(x)$.

It is not hard to see that the family $\{\vartheta_1\}_{1\in Z}$ satisfies (1.3) and (1.4). This completes the proof of step 2.

Step 3. We define for $j\in Z$ $\lambda_j(x) = \sum_{k\in Z} |E_k|^{-1} \int_{E_k} \varphi_j(t)\,dt\, \chi_{E_k}(x)$ then there is a family of functions $\{\eta_1\}_{1\in Z}$ with compact support contained in $\{x: 2^{-1} \leqq |x| \leqq 2\}$ satisfying (1.3) and (1.4) such that

$$\sum_{j\in Z} \langle \lambda_j^{(j)}, \Phi\rangle = \sum_{j\in Z} \langle \eta_j^{(j)}, \Phi\rangle$$

Indeed, we set $\sigma_k(x) = |E_k|^{-1}\chi_{E_k}(x)$ and $c_{jk} = \int_{E_k} \varphi_j$. Then for each $j\in Z$

$$\left\langle \sum_{N\leqq k\leqq M} c_{jk}\sigma_k^{(j)}, \Phi\right\rangle = \sum_{N+1\leqq k\leqq M} \int_{|x|\leqq 2^k} \varphi_j(x)\,dx\,\langle \sigma_{k-1} - \sigma_k, \Phi\rangle$$

$$+ \int_{|x|\leqq 2^{M+1}} \varphi_j(x)\,dx\,\langle \sigma_M^{(j)}, \Phi\rangle - \int_{|x|\leqq 2^N} \varphi_j(x)\,dx\,\langle \sigma_N^{(j)}, \Phi\rangle$$

Since $\lim_{s\to\infty} \langle \sigma_s, \Phi\rangle = 0$ and $\varphi_j \in L^1(R^n)$, the last two terms go to zero as $M \to +\infty$ and $N \to -\infty$. Now $\sigma_{k-1} - \sigma_k = (\sigma_{-1} - \sigma_0)^{(-k)}$.

So $\langle \lambda_j^{(j)}, \Phi\rangle = \sum_{k\in Z} \int_{|x|\leqq 2^k} \varphi_j(x)\,dx\,\langle (\sigma_{-1} - \sigma_0)^{(j-k)}, \Phi\rangle$.

We observe that $\sum_{k,l}\left|\int_{|x|\leqq 2^k} \varphi_{l+k}(x)\,dx\,\langle(\sigma_{-1} - \sigma_0)^{(l)}, \Phi\rangle\right| < \infty$. Indeed

$$\sum_{k\geqq 0}\left|\int_{|x|\leqq 2^k} \varphi_{l+k}(x)\,dx\right| = \sum_{k\geqq 0}\left|\int_{|x|\geqq 2^k} \varphi_{l+k}(x)\,dx\right|$$

$$\leqq \sum_{k\geqq 0} 2^{-k\delta} \int_{|x|\leqq 2^k} |x|^{\delta}\,|\varphi_{l+k}(x)|\,dx < \infty$$

by (1.5). And, by Hölder's inequality,

$$\sum_{k<0} \int_{|x|\leqq 2^k} |\varphi_{l+k}(x)|\,dx \leqq \sum_{k<0} 2^{kn(1-1/q)}\omega_n^{1-1/q}\,\|\varphi_{l+k}\|_q < \infty$$

where ω_n denotes the measure of the n-dimensional unit sphere. Then

$$\sum_{k,l}\left|\int_{|x|\leqq 2^k} \varphi_{l+k}(x)\,dx\,\langle(\sigma_{-1} - \sigma_0)^{(l)}, \Phi\rangle\right| \leqq c\sum_{l}|\langle(\sigma_{-1} - \sigma_0)^{(l)}, \Phi\rangle|$$

$$= \sum_{l}\left|\left\langle\left(\sum_{k}\sigma_{-1} - \sigma_0)\chi_{E_k}\right)^{(l)}, \Phi\right\rangle\right| \leqq \sum_{k,l}|\langle(\sigma_{-1} - \sigma_0)\chi_{E_k})^{(l)}, \Phi\rangle| < \infty.$$

The convergence of the last sum is a consequence of (4.4) and of the statement in Step 1. Then we can write

$$\sum_{j\in Z}\langle\lambda_j^{(j)}, \Phi\rangle = \sum_{j}\sum_{k}\int_{|x|\leqq 2^k}\varphi_j(x)\,dx\,\langle(\sigma_{-1} - \sigma_0)^{(j-k)}, \Phi\rangle = \sum_{j\in Z}\langle\eta_j^{(j)}, \Phi\rangle$$

where $\eta_l = \sum_{-k+j=l}\int_{|x|\leqq 2^k}\varphi_j(x)\,dx\,(\sigma_{-1} - \sigma_0)$. A computation shows that $\{\eta_1\}_{1\in Z}$ is a family of functions, with compact support contained in $\{x\colon 2^{-1} \leqq |x| \leqq 2\}$, satisfying (1.3) and (1.4). Two complete the proof we write $\sum_{j\in Z}\langle\varphi_j^{(j)}, \Phi\rangle = \sum_{j\in Z}\langle\vartheta_j^{(j)}, \Phi\rangle + \sum_{j\in Z}\langle\eta_j^{(j)}, \Phi\rangle$.

Then we have the following.

THEOREM B. *Let* $\{\varphi_j\}_{j\in Z}$ *be a family of functions in* $L^1(R^n)$ *satisfying* (1.3), (1.4) *and* (1.5). *Let* k_2 *be a function satisfying* (1.6) *and* (1.7). *Then, for* $f \in L^p(R^n)$, $1 < p < \infty$,

$$Tf(\xi) = \lim_{(N,M)\to(-\infty,\infty)} \int \sum_{N \leqq j \leqq M} 2^{jn}\varphi_j(2^j(\xi - y))k_2(\xi + y)f(y)\,dy$$

exists almost everywhere in R^n *and* $\|Tf\|_p \leqq c_p \|f\|_p$. *Moreover, if* $f \in S(R^n)$ $|\{x: |Tf(x)| > \lambda\}| \leqq c\,\lambda^{-1}\|f\|_1$ *for all* $\lambda > 0$ *(weak type* 1–1).

§5.

In this paragraph we give some applications of the results before obtained.

REMARK 5.1. Let $k_1 = \Omega(x)/|x|^n$, with Ω a homogeneous function of degree zero satisfying $\int_{S^{n-1}} \Omega(x)\,dx = 0$, and, for some $\varepsilon > 0$,

$$\int_{S^{n-1}} |\Omega(gx) - \Omega(x)|\,dx \leqq c\,|g|^{\varepsilon}$$

for all g in So(n). Here $|\ |$ denotes a smooth distance to the identity. Let k_2 be a function satisfying (1.6) and (1.7). Then the operator given by

$$Tf(\xi) = \text{p.v.} \int_{R^n} k_1(\xi - y)k_2(\xi + y)f(y)\,dy$$

is bounded on $L^p(R^n)$, $1 < p < \infty$, and of weak type 1–1.

Indeed, if we define $\varphi_0(x) = k_1(x)X_{E_0}(x)$, then $k_1(x) = \sum 2^{jn}\varphi_0(2^j x)$, $\operatorname{supp}\varphi_0 \subseteq \{x \in R^n: 2^{-1} \leqq |x| \leqq 2\}$ and it satisfies (1.3). In order to apply Theorem A it only remains to check the L^1-Hölder condition (1.4). We must estimate

$$\int_{\substack{2^{-1} \leqq |x-h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x-h)/|x-h|^n - \Omega(x)/|x|^n|\,dx$$

$$+ \int_{\substack{2^{-1} \leqq |x-h| \leqq 2 \\ \{x: |x| \leqq 2^{-1}\} \cup \{x: |x| \geqq 2\}}} |\Omega(x-h)/|x-h|^n|\,dx$$

$$+ \int_{\substack{2^{-1} \leqq |x| \leqq 2 \\ \{x: |x-h| \leqq 2^{-1}\} \cup \{x: |x-h| \geqq 2\}}} + |\Omega(x)/|x|^n|\,dx$$

The second and third integrals are similar. We study the last one. We can assume $|h| < 1/4$, since for $|h| \geqq 1/4$

$$\int |\varphi_0(x + h) - \varphi_0(x)|\, dx \leqq 2\, \|\varphi_0\|_1 \leqq c\, |h|^{\varepsilon}\, \|\varphi_0\|_1$$

Now, for $|x - h| \leqq 2^{-1}$, we have $|x| \leqq 2^{-1} + |h|$ and for $|x - h| \geqq 2$ we have $|x| \geqq 2 - |h|$. Then

$$\int\limits_{\substack{2^{-1} \leqq |x| \leqq 2 \\ \{x:\, |x - h| \leqq 2^{-1}\} \cup \{x:\, |x - h| \geqq 2\}}} |\Omega(x)| / |x|^n\, dx \leqq \int\limits_{\substack{2^{-1} \leqq |x| \leqq 2 \\ |x - h| \leqq 2}} |\Omega(x)| / |x|^n\, dx$$

$$+ \int\limits_{\substack{2^{-1} \leqq |x| \leqq 2 \\ |x - h| \geqq 2}} |\Omega(x)| / |x|^n\, dx \leqq c\, \|\Omega\|_1\, |h|^{\varepsilon}.$$

A change to polar coordinates gives us the last bound. It remains to treat the first integral.

$$\int\limits_{\substack{2^{-1} \leqq |x - h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x - h)/ |x - h|^n - \Omega(x)/ |x|^n|\, dx$$

$$\leqq \int\limits_{\substack{2^{-1} \leqq |x - h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x - h) - \Omega(x)|\, |x - h|^{-n}\, dx$$

$$+ \int\limits_{\substack{2^{-1} \leqq |x - h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x)|\, ||x - h|^{-n} - |x|^{-n}|\, dx$$

We note that $||x - h|^{-n} - |x|^{-n}| \leqq |h| \sum\limits_{0 \leqq k \leqq n-1} \binom{n}{k} |x|^{k-n} |h|^{n-k-1} |x - h|^{-n}$

then $\int\limits_{\substack{2^{-1} \leqq |x - h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x)|\, ||x - h|^{-n} - |x|^{-n}|\, dx \leqq c\, |h|\, \|\Omega\|_1$. On the other hand, the change of variable $z = x - h$ gives us

$$\int\limits_{\substack{2^{-1} \leqq |x - h| \leqq 2 \\ 2^{-1} \leqq |x| \leqq 2}} |\Omega(x - h) - \Omega(x)|\, |x - h|^{-n}\, dx \leqq \int\limits_{2^{-1} \leqq |z| \leqq 2} |\Omega(z + h) - \Omega(z)|\, |z|^{-n}\, dz$$

For $z \in R^n$ we set $z = z'r$ with $z' \in S^{n-1}$ and $r \geqq 0$, we also set $\alpha = h/r$. Then the last integral can be written $\int\limits_{2^{-1} \leqq r \leqq 2} r^{-1} \left(\int\limits_{S^{n-1}} |\Omega(z' + \alpha) - \Omega(z')|\, dz' \right) dr.$

Now we apply lemma 5 of [C-W-Z] to obtain, for α small enough,

$$\int_{S^{n-1}} |\Omega(z'+\alpha) - \Omega(z')|\, dz' \leqq \sup_{|g| \leqq |\alpha|} \int_{S^{n-1}} |\Omega(gu) - \Omega(u)|\, dz' \leqq c\, |\alpha|^{\varepsilon} = c\, |h|^{\varepsilon} r^{-\varepsilon}$$

Remark 5.1 follows from this last inequality.

REMARK 5.2. Let $k_2(x)$ be a $C^1(R^n - \{0\})$ function such that, for some constant $c > 0$ and for all $x \in R^n$ $|k_2(x)| \leqq c$ and $|\nabla k_2(x)| \leqq c\,|x|^{-1}$. Then it is easy to see that k_2 satisfies (1.7) for all $\delta \leqq 1$. For example k_2 being homogeneous of degree 0 and smooth out of the origin. A less restrictive condition for K_2 is given by the following remark.

REMARK 5.3. Let $\{\psi_j\}_{j \in Z}$ be a family of measurable functions on R^2 satisfying

(i) Supp $\psi_j \subseteq \{x \in R^n\colon 2^{-1} \leqq |x| \leqq 2\}$.

(ii) There exist $c > 0$ and $0 < \delta < 1$ such that $|\psi_j(x+h) - \psi_j(x)| \leqq c\,|h|^{\delta}$ for almost all $x \in R^n$.

By (i) and (ii) $\psi_j \in L^{\infty}(R^n)$ and $\|\psi_j\|_{\infty} \leqq c$, so if we define $k_2(x) = \sum_{j \in Z} \psi_j(2^j x)$, we have that $k_2 \in L^{\infty}(R^n)$ and satisfies (1.7). Indeed $|k_2(x+h) - k_2(x)| \leqq c \sum_{j \in Z} 2^{j\delta}\,|h|^{\delta}$. If $|h| \leqq |x|/2$ and either 2^j or $2^j(x+h)$ belongs to supp ψ_j, then $2^j \leqq c/|x|$. The result follows since for each h and x fixed, at most six terms are involved.

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ON APPROXIMATION IN WEIGHTED SOBOLEV SPACES AND SELF-ADJOINTNESS

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Abstract.

Necessary and sufficient conditions for approximation by test functions in a type of weighted Sobolev spaces are given. As an application a necessary condition for essential self-adjointness of a perturbed Laplacian is proved. A lemma on the equivalence of two capacities is proved and used to obtain criteria for closability, continuity and compactness of certain embeddings.

1. Introduction.

Let $\rho \in L^1_{\text{loc}}(\mathsf{R}^N)$ be a positive function, locally bounded away from zero. In [6] the spectral properties of the operator $A = -\frac{1}{\rho}\Delta$ on $L^2(\rho)$ with domain $C_0^\infty(\mathsf{R}^N)$ are investigated. In particular it is proved that for $N \geqq 3$ a necessary condition for A to be essentially self-adjoint is that $\int \rho(x)dx = \infty$. In the present paper we sharpen this result. This is done by giving necessary and sufficient conditions for the density of test functions in a weighted Sobolev space.

For $p \geqq 1$ we define $L^{m,p}(\mathsf{R}^N)$ as the set of distributions u on R^N such that

$$\|u\|_{m,p} = \left(\sum_{k=1}^{m} \int |\nabla^k u(x)|^p dx\right)^{1/p} < \infty;$$

here $\nabla^k u$ denotes the vector $(D^\alpha u)_{|\alpha|=k}$. $L_0^{m,p}(\mathsf{R}^N)$ is the completion of $C_0^\infty(\mathsf{R}^N)$ with respect to the $L^{m,p}$ norm. Now let $\Omega \subset \mathsf{R}^N$ be an open set and let μ be a nontrivial positive Radon measure on Ω. We will study the space $H_\mu^{m,p}(\Omega)$, defined as the completion of $L^p(\mu) \cap L^{m,p}(\mathsf{R}^N) \cap C_\Omega^\infty$ with respect to the norm

$$\|u\|_{m,p;\mu} = \|u\|_{L^p(\mu)} + \|u\|_{m,p};$$

here $C_\Omega^\infty = \{u \in C^\infty(\mathsf{R}^N) : \operatorname{supp} u \subset \Omega\}$. The closure of $C_0^\infty(\Omega)$ in $H_\mu^{m,p}(\Omega)$ is denoted $\mathring{H}_\mu^{m,p}(\Omega)$. Note that if $p < N$ then by Sobolev's inequality the elements in

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$L_0^{m,p}$ can be identified with functions in L^{p^*}, where $p^* = \dfrac{Np}{N-p}$. The elements in $\mathring{H}_\mu^{m,p}(\Omega)$ are naturally identified with elements in $L_0^{m,p}$. Note also that $L^{m,p} \subset L^p_{\text{loc}}$.

The theorem to be proved is the following. (See Section 2 for definitions of the capacitites $B_{m,p}$ and H_1^{N-m} used.)

THEOREM 1. *Let μ be a nontrivial positive Radon measure concentrated on $\Omega \subset \mathsf{R}^N$ and let C denote $B_{m,p}$ for $p > 1$ and H_1^{N-m} for $p = 1$. Then*

(i) *$\mathring{H}_\mu^{m,p}(\Omega) = H_\mu^{m,p}(\Omega)$ if either $p \geqq N$ or $p < N$ and $C(\Omega^c) = \infty$.*

Suppose now that $p < N$ and $C(\Omega^c) < \infty$. Then $\mathring{H}_\mu^{m,p}(\Omega) = H_\mu^{m,p}(\Omega)$ if and only if

(ii) *$\mu(\Omega) = \infty$, when either $m \geqq N$, $p = 1$ or $mp > N$, $p > 1$.*

(iii) *$\mu(F^c) = \infty$ for every closed set $F \subset \mathsf{R}^N$ satisfying $C(F) < \infty$, when either $1 < p \leqq \dfrac{N}{m}$ or $m < N, p = 1$.*

REMARK. When $1 < p < N$, $mp > N$ or $1 = p < N$, $m \geqq N$ the condition $C(\Omega^c) < \infty$ is just a complicated way of saying that Ω^c should be bounded; see [2].

Letting A be the operator above we can now prove the following theorem.

THEOREM 2. *A necessary condition for the operator A to be essentially self-adjoint is that $\int_{F^c} \rho(x)dx = \infty$ whenever F is a closed set such that $B_{1,2}(F) < \infty$.*

PROOF. Suppose ρ does not satisfy the condition in the theorem. Then by Theorem 1 and the Hahn-Banach theorem there is a function $0 \neq u \in H_\rho^{1,2}(\mathsf{R}^N)$ such that

$$\int u(x)v(x)\rho(x)dx + \int \nabla u(x)\cdot\nabla v(x)\,dx = 0$$

for all $v \in \mathring{H}_\rho^{1,2}(\mathsf{R}^N) \supset D(\bar{A})$, where $\bar{A}$ is the closure of A. Thus

$$(u, v + \bar{A}v)_{L^2(\rho)} = 0,$$

so $u \in D((I + A)^*) = D(A^*)$ and $u + A^*u = 0$. Now suppose $\bar{A} = A^*$. Then $u \in D(\bar{A})$ and hence

$$\int |u(x)|^2 \rho(x)\,dx + \int |\nabla u(x)|^2\,dx = 0$$

which implies that $u = 0$. This contradiction shows that $\bar{A}$ is not self-adjoint.

2. Capacities.

We will denote different constants, not depending on the essential functions or variables considered, by A. The ball with radius r and centered at x will be denoted by $B(x,r)$. If $x = 0$ we will write only $B(r)$. The annulus $B(R)\setminus B(r)$ is denoted by $A(R,r)$.

We start by defining some convolution kernels needed.

DEFINITION 1. The *Bessel kernels* G_α, the *Riesz kernels* I_α and the *truncated Bessel kernels* $G_{\alpha;1}$ are defined by

$$\hat{G}_\alpha(\xi) = (1 + |\xi|^2)^{-\alpha/2},$$

$$I_\alpha(x) = |x|^{\alpha-N}, \ 0 < \alpha < N$$

and

$$G_{\alpha;1}(x) = \theta(x)G_\alpha(x)$$

where $\theta \in C_0^\infty(B(1))^+$ is an arbitrary but fixed function such that $\theta = 1$ on $B(\frac{1}{2})$. For $\alpha \geqq 1$ we *define*

$$K_\alpha = I_1 * G_{\alpha-1}.$$

It is easy to see, analogously to the cases with Riesz or Bessel potentials, that, for $1 < p < N$, $L_0^{m,p} = \{K_m * f : f \in L^p(\mathbb{R}^N)\}$.

Each of these kernels gives rise to corresponding capacities as follows.

DEFINITION 2. For $E \subset \mathbb{R}^N$ and $p > 1$ we *define*

$$B_{\alpha,p}(E) = \inf\{\|f\|_p^p : f \in L_+^p, G_\alpha * f \geqq 1 \text{ on } E\},$$

$$R_{\alpha,p}(E) = \inf\{\|f\|_p^p : f \in L_+^p, I_\alpha * f \geqq 1 \text{ on } E\}$$

$$C_{\alpha,p}(E) = \inf\{\|f\|_p^p : f \in L_+^p, K_\alpha * f \geqq 1 \text{ on } E\}$$

and

$$B_{\alpha,p;1}(E) = \inf\{\|f\|_p^p : f \in L_+^p, G_{\alpha;1} * f \geqq 1 \text{ on } E\}.$$

It is proved in [2] that $B_{\alpha,p}$ and $R_{\alpha,p}$ are finite simultaneously, although not comparable, for $1 < p < \dfrac{N}{\alpha}$. It is not hard to see from that proof that $C_{\alpha,p}$ and $B_{\alpha,p}$ are finite at the same time, for $1 < p < N$. The set functions $B_{\alpha,p}$ and $B_{\alpha,p;1}$ are comparable. See [4] for a proof of this fact. We will need the following lemma, which, for technical convenience, is the reason for introducing the truncated Bessel kernel.

LEMMA 1. *Let $p > 1$. If $F \subset \mathbb{R}^N$ is closed then*

$$B_{\alpha,p;1}(F) = \inf\{\|f\|_p^p : f \in L_+^p, C^\infty \ni G_{\alpha;1} * f \geqq 1 \text{ on } F\}.$$

PROOF. We may assume $B_{\alpha,p;1}(F)$ finite. Let $A_0 = \overline{B(1)}$ and $A_j = \overline{B(j+1)}\setminus B(j)$ for $j \geqq 1$. It is proved in [2] that $\sum_{j=1}^{\infty} B_{\alpha,p}(F \cap A_j) \leqq AB_{\alpha,p}(F)$. Hence

$$\sum_{j=1}^{\infty} B_{\alpha,p;1}(F \cap A_j) < \infty. \tag{1}$$

Now choose $f_j \in L^p_+$ such that $\|f_j\|_p^p \leqq 2^{-j} + B_{\alpha,p;1}(F \cap A_j)$ and $G_{\alpha;1} * f_j \geqq 1$ on a neighbourhood of $F \cap A_j$. This can be done since $G_{\alpha;1} * f_j$ is lower semicontinuous. It is seen from the definition of $B_{\alpha,p;1}$ that we may assume that $\operatorname{supp} f_j \subset A_j'$ where we denote $E' = \{x: \operatorname{dist}(x, E) \leqq 1\}$. By a standard regularization we obtain functions $g_j \in C_0^\infty(A_j'')^+$ such that

$$\|g_j\|_p^p \leqq A(2^{-j} + B_{\alpha,p;1}(F \cap A_j))$$

and $G_{\alpha;1} * g_j \geqq 1$ on $F \cap A_j$. Consequently, if we set $g = \sum_{j=M}^{\infty} g_j$ for some M to be specified later we get $G_{\alpha;1} * g \geqq 1$ on $F \setminus B(M)$. Also, since the sum defining g is uniformly locally finite, $G_{\alpha;1} * g \in C^\infty$ and

$$\|g\|_p^p \leqq A \sum_{j=M}^{\infty} \|g_j\|_p^p \leqq A \sum_{j=M}^{\infty} (2^{-j} + B_{\alpha,p;1}(F \cap A_j)). \tag{2}$$

Now let $\varepsilon > 0$. By (1) and (2) we get $\|g\|_p < \varepsilon$ if M is large enough. By the same argument as before with lower semicontinuity and regularization we can find a function $h \in C_0^\infty(\mathsf{R}^N)^+$ such that $\|h\|_p \leqq B_{\alpha,p;1}(F \cap \overline{B(M)})^{1/p} + \varepsilon$ and $G_{\alpha;1} * h \geqq 1$ on $F \cap \overline{B(M)}$. Setting $f = g + h$ we obtain $\|f\|_p \leqq B_{\alpha,p;1}(F)^{1/p} + 2\varepsilon$ and $C^\infty \ni G_{\alpha;1} * f \geqq 1$ on F. Since ε was arbitrary the lemma follows.

For $p = 1$ the appropriate capacities are Hausdorff capacities.

DEFINITION 3. Let $0 \leqq d < N$. Then for subsets E of R^N we *define*

$$H_\rho^d(E) = \inf \sum_{i=1}^{\infty} r_i^d$$

where the infimum is taken over all countable coverings $\bigcup_{i=1}^{\infty} B(x_i, r_i) \supset E$, with $r_i \leqq \rho$. For $d < 0$ we *define* $H_\rho^d(E) = H_\rho^0(E)$.

The following lemma is immediate except for (iii) which is (a variant of) the well-known Frostman lemma.

LEMMA 2. *Let* $0 < d < N$. *Then*

(i) $H_\infty^d(E) \leqq H_1^d(E) \leqq AH_\infty^d(E) + A(H_\infty^d(E))^{N/d}$. *In particular* H_∞^d *and* H_1^d *are finite at the same time.*

(ii) $\sum_{j=0}^{\infty} H_1^d(E \cap A_j) \leqq AH_1^d(E)$, *where* A_j *is as in the proof of Lemma 1.*

(iii) $H_1^d(E)$ *is comparable to* $\sup\{\mu(E): |\mu|(B(x,r)) \leqq r^d, r \leqq 1, x \in \mathsf{R}^N\}$, *for Borel sets* E.

The following lemma, partly proved by Adams [3], will give a substitute when $p = 1$ for the potentials used when $p > 1$.

LEMMA 3. *Let m be an integer, $0 < m < N$. Then for closed sets $F \subset \mathrm{R}^N$, $H_1^{N-m}(F)$ is comparable to*

$$\inf\{\|\varphi\|_1 + \|\nabla^m \varphi\|_1 : \varphi \in C^\infty, 0 \leqq \varphi \leqq 1, \varphi = 1 \text{ on a neighbourhood of } F\}.$$

PROOF. Suppose $\varphi \in C^\infty$, $\varphi = 1$ on a neighbourhood of F and $\|\varphi\|_1 + \|\nabla^m \varphi\|_1 < \infty$. If μ is a positive measure supported by F then by [9, Sec. 1.4] we have

$$\mu(F) \leqq \int \varphi \, d\mu \leqq A \sup_{x; 0 < r \leqq 1} \frac{\mu(B(x,r))}{r^{N-m}} (\|\varphi\|_1 + \|\nabla^m \varphi\|_1).$$

Taking the supremum over μ with

$$\sup_{x; 0 < r \leqq 1} \frac{\mu(B(x,r))}{r^{N-m}} \leqq 1$$

we get by Lemma 2 (iii) that $H_1^{M-m}(F) \leqq A(\|\varphi\|_1 + \|\nabla^m \varphi\|_1)$ which proves one direction of the lemma.

To prove the other direction suppose first that F is compact. Cover F by balls $B(x_i, r_i)$, $r_i \leqq 1$, $1 \leqq i \leqq s$, such that

$$\sum_{i=1}^{s} r_i^{N-m} \leqq H_1^{N-m}(F) + \varepsilon,$$

where $\varepsilon > 0$. By Lemma 3.1 of [7] there are functions $\psi_i \in C_0^\infty(B(x_i, 2r_i))$, $1 \leqq i \leqq s$, such that $|D^\alpha \psi_i| \leqq A_\alpha r_i^{-|\alpha|}$ and such that $\varphi = \sum_{i=1}^{s} \psi_i$ satisfies $\varphi = 1$ on a neighbourhood of F. We get

$$\|\varphi\|_1 + \|\nabla^m \varphi\|_1 \leqq \sum_{i=1}^{s} \int_{B(x_i, 2r_i)} (|\psi_i(x)| + |\nabla^m \psi_i(x)|) \, dx$$

$$\leqq A \sum_{i=1}^{s} (r_i^N + r_i^{N-m}) \leqq A H_1^{N-m}(F) + A\varepsilon.$$

Since ε was arbitrary we are done in case F is compact.

For the general case we introduce a partition of unity $1 = \sum_{n=0}^{\infty} \zeta_n$, where $0 \leqq \zeta_n \leqq 1$, $\zeta_n = 1$ on a neighbourhood of A_{2n}, $\operatorname{supp} \zeta_n \subset A'_{2n}$ and $|\nabla^k \zeta_n| \leqq A$ for $1 \leqq k \leqq m$. Here A_j and A'_j are as in the proof of Lemma 1. Now choose functions φ_n corresponding to the sets $F \cap A'_{2n}$ according to the construction for compact sets in a way that

$$\|\varphi_n\|_1 + \|\nabla^m \varphi_n\|_1 \leqq A H_1^{N-m}(F \cap A'_{2n}) + 2^{-n}\varepsilon,$$

where $\varepsilon > 0$. Letting $\varphi = \sum_{n=0}^{\infty} \zeta_n \varphi_n$ we obtain, using Leibniz' rule, interpolation and Lemma 2(ii),

$$\begin{aligned}
\|\varphi\|_1 + \|\nabla^m \varphi\|_1 &\leqq A \sum_{n=0}^{\infty} \sum_{k=0}^{m} \int_{A'_{2n}} |\nabla^k \varphi(x)|\, dx \\
&\leqq A \sum_{n=0}^{\infty} \int (|\varphi_n(x)| + |\nabla^m \varphi_n(x)|)\, dx \\
&\leqq \sum_{n=0}^{\infty} (H_1^{N-m}(F \cap A'_{2n}) + 2^{-n}\varepsilon) \\
&\leqq 2\varepsilon + A \sum_{n=0}^{\infty} H_1^{N-m}(F \cap A_n) \leqq 2\varepsilon + A H_1^{N-m}(F).
\end{aligned}$$

Since ε was arbitrary the lemma follows.

3. Some Applications of Lemma 3.

We record here some generalizations, depending on Lemma 3, to the case $p = 1$ of some results in [9, Ch. 12]. As before let μ be a positive Radon measure concentrated on $\Omega \subset \mathbb{R}^N$ and let W, X and Y be the completions of $C_0^\infty(\Omega)$ with respect to the norms

$$\|u\|_W = \int |u(x)|\, dx + \int |\nabla^m u(x)|\, dx$$

$$\|u\|_X = \int |u|\, d\mu + \int |\nabla^m u(x)|\, dx,$$

and

$$\|u\|_Y = \int |\nabla^m u(x)|\, dx,$$

respectively. Then we have the following theorems.

THEOREM 3. *The identity operator defined on $C_0^\infty(\Omega)$ and mapping $L^1(\Omega)$ into X is closable if and only if μ is absolutely continuous with respect to H_1^{N-m}.*

THEOREM 4. *The identity operator defined on $C_0^\infty(\Omega)$ and mapping W into $L^1(\mu)$ is closable if and only if μ is absolutely continuous with respect to H_1^{N-m}.*

THEOREM 5. *Let $m \leqq N$. Then the identity operator defined on $C_0^\infty(\Omega)$ and mapping Y into $L^1(\mu)$ is closable if and only if μ is absolutely continuous with respect to H_1^{N-m}.*

REMARK. Note that for $m \geqq N$ and for $m = N$ respectively the above condition on absolute continuity is empty so the operators are always closable. In the proof of Theorem 3 below "quasieverywhere" can be read "everywhere" in this case.

For the proofs we will need two lemmas, proved in [5]. We give the proofs here for the convenience of the reader. Recall that a function u is called H_1^d-quasicontinuous if it is defined H_1^d-quasieverywhere and if for every $\varepsilon > 0$ there is an open set G such that $u|_{G^c}$ is continuous and $H_1^d(G) < \varepsilon$.

LEMMA 4. *Suppose that $u_n \in C_0^\infty(\Omega)$ and that $\|u_{n+1} - u_n\|_W < 4^{-n}$. Then u_n converges H_1^{N-m}-quasieverywhere to an H_1^{N-m}-quasicontinuous function.*

PROOF. Let μ be a positive Radon measure such that $\mu(B(x,r)) \leqq r^{N-m}$ for all $x \in \mathbf{R}^N$ and all $r \leqq 1$. Then by [9, Sec. 1.4] we have

$$\int |u_{n+1} - u_n| \, d\mu \leqq A 4^{-n}.$$

By monotone convergence, $\tilde{u}(x) = \lim_{n\to\infty} u_n(x)$ exists a.e. $[\mu]$. Now, let

$$F = \{x : \tilde{u}(x) \text{ is defined}\}$$

and

$$E_n = \{x \in F : |\tilde{u}(x) - u_n(x)| \geqq 2^{-n}\}.$$

Then, by part (iii) of Lemma 2, $H_1^{N-m}(F^c) = 0$. Also,

$$\mu(E_n) \leqq A\, 2^n \int |\tilde{u} - u_n| \, d\mu \leqq A\, 2^{-n},$$

so $H_1^{N-m}(E_n) \leqq A\, 2^{-n}$. Let $F_k = \cup_{n=k}^{\infty} E_n$. Then $H_1^{N-m}(F_k) \leqq A\, 2^{-k}$. Thus, given $\varepsilon > 0$, we may choose k and an open set G_k with $H_1^{N-m}(G_k) < \varepsilon$ such that $F_k \cup F^c \subset G_k$. Since $u_n \to \tilde{u}$ uniformly on G_k^c, the lemma is proved.

LEMMA 5. *Suppose that u is H_1^d-quasicontinuous and that $E = \{x : u(x) \neq 0\}$ is a Borel set with $|E| = 0$. Then $H_1^d(E) = 0$.*

PROOF. Suppose that $H_1^d(E) = c > 0$. There is an open set G with $H_1^d(G) < \varepsilon$ such that $u|_{G^c}$ is continuous. We do not specify ε here because the choice of it depends on a certain constant, appearing later in the proof. However, ε is a fixed positive number less than c.

Let $K \subset E \setminus G$ be a compact set such that $H_1^d(K) > c - \varepsilon$ and set $K_n = \left\{x : \operatorname{dist}(x, K) \leqq \frac{1}{n}\right\}$. By Lemma 2, part (iii), we can choose measures μ_n supported by K_n such that

$$\sup_{x;0<r\leqq 1} \frac{\mu_n(B(x,r))}{r^d} \leqq A$$

and $\mu_n(K_n) = H_1^d(K_n)$.
Define $\phi_n(x) = n^N \phi(nx)$, where $\phi \in C_0^\infty(B(1))$ is a function such that $0 \leqq \phi \leqq A$ and $\int \phi = 1$. Set $\nu_n = \phi_n * \mu_n$. Then we have

$$\phi_n * \mu_n(y) = \int \phi_n(y-t)d\mu_n(t) \leqq An^N \mu_n\left(B\left(y, \frac{1}{n}\right)\right) \leqq An^{N-d}.$$

Thus, for $r \leqq \dfrac{1}{n}$

$$\frac{\nu_n(B_r(x))}{r^d} \leqq A(nr)^{N-d} \leqq A.$$

For $\dfrac{1}{n} \leqq r \leqq 1$ we have

$$\begin{aligned}
\frac{\nu_n(B(x,r))}{r^d} &\leqq \frac{1}{r^d}\int \chi_{B(x,r)}(y)\int \phi_n(y-t)\,d\mu_n(t)\,dy \\
&= \frac{1}{r^d}\int \chi_{B(x,r)} * \phi_n(t)\,d\mu_n(t) \\
&\leqq \frac{1}{r^d}\int \chi_{B(x,r+\frac{1}{n})}(t)\,d\mu_n(t) \\
&\leqq A\frac{\mu_n\left(B\left(x, r+\dfrac{1}{n}\right)\right)}{\left(r+\dfrac{1}{n}\right)^d} \leqq A.
\end{aligned}$$

Thus we obtain

$$\sup_{x;0<r\leqq 1} \frac{\nu_n(B(x,r))}{r^d} \leqq A.$$

Also,

$$\begin{aligned}
\nu_n(\mathsf{R}^N) &= \int\int \phi_n(x-y)\,d\mu_n(y)\,dx \\
&= \int\int \phi_n(x-y)\,dx\,d\mu_n(y) \\
&= \mu_n(K_n) = H_1^d(K_n) \\
&\geqq H_1^d(K) \geqq c - \varepsilon.
\end{aligned}$$

Let $K_n^* = \operatorname{supp} \nu_n$. Then, since $|E| = 0$, we get

$$H_1^d(K_n^* \setminus E) \geqq A^{-1} \nu_n(E^c) = A^{-1} \nu_n(\mathrm{R}^N) \geqq A^{-1}(c - \varepsilon).$$

Now we are in the position to specify ε: take any positive ε satisfying $A^{-1}(c - \varepsilon) > \varepsilon$. Then we obtain

$$H_1^d(K_n^* \setminus E) > H_1^d(G).$$

Hence there are points $x_n \in K_n^* \cap E^c \cap G^c$. We may assume that x_n converges to some point x_0. Since $x_n \in K_n^*$ there are points $y_n \in K$ such that $|x_n - y_n| < \dfrac{2}{n}$. Then $y_n \to x_0$, so in particular $x_0 \in K$. By the continuity of u on G^c we obtain that $0 = u(x_n) \to u(x_0) \neq 0$. From this contradiction we conclude that $H_1^d(E) = 0$ and the lemma is proved.

Proof of Theorem 3. We start with the sufficiency part. Suppose $\{u_n\} \subset C_0^\infty(\Omega)$ is a Cauchy sequence in X, converging to zero in $L^1(\Omega)$. Then $D^\alpha u_n$ converges in $L^1(\Omega)$ for $|\alpha| = m$ and since obviously $D^\alpha u_n \to 0$ as distributions we get $D^\alpha u_n \to 0$ in $L^1(\Omega)$. Hence, passing to a subsequence, we may assume that u_n converges H_1^{N-m}-quasieverywhere by Lemma 4. Also, by Lemma 5 we get that, H_1^{N-m}-quasieverywhere, $u_n \to 0$. Thus $u_n \to 0$ a.e $[\mu]$, and since $\{u_n\}$ is a Cauchy sequence in $L^1(\mu)$ we obtain $u_n \to 0$ in X.

For the necessity part suppose $F \subset \Omega$ is a compact set satisfying $H_1^{N-m}(F) = 0$ and $\mu(F) > 0$. Let $G_n \supset F$ be shrinking open sets such that $H_1^{N-m}(\bar{G}_n) \to 0$ and $\mu(G_n \setminus F) \to 0$ as $n \to \infty$. Then, by Lemma 3, we can find $\varphi_n \in C_0^\infty(\Omega)$ such that $\varphi_n = 1$ on G_n and

$$\|\varphi_n\|_1 + \|\nabla^m \varphi_n\|_1 \to 0.$$

Moreover, by construction, $\varphi_n \to 0$ uniformly outside every neighbourhood of F and there is a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_n \subset K$ for all n. Now let $\varepsilon > 0$ and choose n so that $\mu(G_n \setminus F) < \varepsilon/4$. Then, for j and k large enough,

$$\int_\Omega |\varphi_j - \varphi_k| \, d\mu \leqq \int_{K \setminus G_n} |\varphi_j - \varphi_k| \, d\mu + 2\mu(G_n \setminus F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{\varphi_n\}$ is a Cauchy sequence in X, converging to zero in $L^1(\Omega)$. However, since

$$\|\varphi_n\|_{L^1(\mu)} \geqq \mu(F) > 0,$$

we cannot have $\varphi_n \to 0$ in X. This proves the necessity part.

The proofs of Theorems 4 and 5 follow the same lines as in [9, Sec. 12.4], making use of the proof of Theorem 3.

Using Lemma 3 we can also obtain necessary and sufficient conditions for continuity and compactness of the embedding of X into the Sobolev space $W^{k,q}(\mathsf{R}^N)$. To state these theorems we need first some definitions. Let $m < N$. Then a set $F \subset B(x,r)$ is called $(m,1)$-unessential if

$$H_\infty^{N-m}(F) \leqq \gamma r^{N-m},$$

where γ is a sufficiently small constant, depending only on m and N. For $m \geqq N$ only the empty set is called $(m,1)$-unessential.

Let $\mathscr{F}(\Omega)$ be the family of all balls $B(x,r)$ such that $B(x,r)\backslash\Omega$ is $(m,1)$-unessential. Then we define

$$D_{m,1}(\mu,\Omega) = \sup\{r; B(x,r)\in\mathscr{F}(\Omega), \inf \mu(B(x,r)\backslash F) \leqq r^{N-m}\},$$

where the infimum is taken over all $(m,1)$-unessential closed sets $F \subset B(x,r)$. We then have the following theorems.

THEOREM 6. *Let $0 \leqq k \leqq m$, $1 \leqq q < \infty$ and $m - k > N(1 - 1/q)$. Then*

$$\|u\|_q + \|\nabla^k u\|_q \leqq A\|u\|_X$$

for all $u \in C_0^\infty(\Omega)$ if and only if there are positive constants r and a such that

$$\mu(B(x,r)\backslash F) \geqq a$$

for all balls $B(x,r)\in\mathscr{F}(\Omega)$ and all $(m,1)$-unessential sets $F \subset B(x,r)$. The best constant A is comparable to

$$D^{m-N(1-1/q)} \max\{D^{-k}, 1\},$$

where $D = D_{m,1}(\mu,\Omega)$.

THEOREM 7. *Let $0 \leqq k \leqq m$, $1 \leqq q < \infty$ and $m - k > N(1 - 1/q)$. Then X is compactly embedded into $W^{k,q}(\mathsf{R}^N)$ if and only if $D_{m,1}(\mu,\Omega) < \infty$ and*

$$\lim_{R\to\infty} D_{m,1}(\mu, \Omega\backslash\overline{B(R)}) = 0.$$

The proofs of these theorems are the same as those of the corresponding theorems for $p > 1$ in [9, Sec. 12.2–12.3], relying now on Lemma 3 of the present paper.

4. Proof of Theorem 1.

We will need some basic results on the function spaces $L^{m,p}$ and $L_0^{m,p}$. We state first a well-known lemma of Hardy type.

LEMMA 6. (i) *If* $1 \leqq p < N$ *and* $u \in L_0^{1,p}(\mathsf{R}^N)$ *then*

$$\int \frac{|u(x)|^p}{|x|^p}\,dx \leqq A \int |\nabla u(x)|^p\,dx.$$

(ii) *If* $p > N$ *and* $u \in L^{1,p}(\mathsf{R}^N)$ *then*

$$\int_{B(1)^c} \frac{|u(x)|^p}{|x|^p}\,dx \leqq A\left(\int_{B(1)} |u(x)|^p\,dx + \int |\nabla u(x)|^p\,dx\right).$$

(iii) *If* $p = N$ *and* $u \in L^{1,p}(\mathsf{R}^N)$ *then*

$$\int_{B(2)^c} \frac{|u(x)|^p}{(|x|\log|x|)^p}\,dx \leqq A\left(\int_{B(2)} |u(x)|^p\,dx + \int |\nabla u(x)|^p\,dx\right).$$

A proof of essentially the following decomposition lemma can be found in Lizorkin [8].

LEMMA 7. *Let* $1 \leqq p < N$. *Then for each* $u \in L^{m,p}(\mathsf{R}^N)$ *there is a unique constant* c *such that* $u - c \in L_0^{m,p}(\mathsf{R}^N)$.

We turn now to the proof of Theorem 1, divided into four cases starting with the main one.

The case $1 < p \leqq \dfrac{N}{m}$ *or* $p = 1, m < N$. We start by proving the sufficiency part. Suppose that μ satisfies the condition in the theorem. We will show then that $C_\Omega^\infty \cap H_\mu^{m,p} \subset L_0^{m,p}$. Let $u \in C_\Omega^\infty \cap H_\mu^{m,p}$ and suppose that $c \neq 0$, where $u - c \in L_0^{m,p}$. We can assume that $c > 0$. Let

$$F = \left\{x : |u(x) - c| \geqq \frac{c}{2}\right\}$$

and suppose first that $p > 1$. Then $u - c$ can be written $u - c = K_m * f$ where $f \in L^p$ and we get

$$C_{m,p}(F) \leqq C_{m,p}\left(\left\{x : K_m * |f| \geqq \frac{c}{2}\right\}\right) \leqq \frac{2^p}{c^p}\,\|f\|_p^p < \infty.$$

If $p = 1$ then in the same way as in the proof of Lemma 3

$$H_\infty^{N-m}(F) \leqq A \int |\nabla^m(u(x) - c)|\,dx < \infty.$$

On the other hand, since $|u| \geqq \dfrac{c}{2}$ on F^c, we have

$$\mu(F^c) \leqq \left(\frac{2}{c}\right)^p \int |u|^p \, d\mu < \infty.$$

This contradicts the condition on μ so $c = 0$. It follows that $u \in \mathring{L}_0^{m,p}$. Note that since $\Omega^c \subset F$ we must have $c = 0$ if $C(\Omega^c) = \infty$, without using any condition on μ.

Now let $\eta_R \in C_0^\infty(B(2R))$ satisfy $0 \leqq \eta_R \leqq 1, \eta_R = 1$ on $B(R)$ and $|\nabla^k \eta_R| \leqq AR^{-k}$ for $k \leqq m$. Then for $R \geqq 1$ we get

$$\|u - u\eta_R\|_{m,p}^p \leqq A \sum_{1 \leqq l+k \leqq m} \int |\nabla^k u(x)|^p |\nabla^l (1 - \eta_R)(x)|^p \, dx$$

$$\leqq A \sum_{k=1}^m \int_{B(R)^c} |\nabla^k u(x)|^p dx + A \sum_{\substack{1 \leqq l+k \leqq m \\ l \geqq 1}} R^{-lp} \int_{A(2R,R)} |\nabla^k u(x)|^p dx$$

$$\leqq A \sum_{k=1}^m \int_{B(R)^c} |\nabla^k u(x)|^p dx + AR^{-p} \int_{A(2R,R)} |u(x)|^p dx.$$

Thus $\eta_R u \to u$ in $L^{m,p}$ as $R \to \infty$ by Lemma 6 (i). Also

$$\int |\eta_R u - u|^p \, d\mu \leqq \int_{B(R)^c} |u|^p \, d\mu \to 0$$

as $R \to \infty$ since $u \in H_\mu^{m,p}(\Omega)$. It follows that $u \in \mathring{H}_\mu^{m,p}(\Omega)$, i.e. $C_\Omega^\infty \cap H_\mu^{m,p}(\Omega) \subset \mathring{H}_\mu^{m,p}(\Omega)$ and hence $H_\mu^{m,p}(\Omega) = \mathring{H}_\mu^{m,p}(\Omega)$.

We now turn to the necessity part. Suppose that $H_\mu^{m,p}(\Omega) = \mathring{H}_\mu^{m,p}(\Omega)$ and that F is a closed set such that $B_{m,p}(F) < \infty$ and $\mu(F^c) < \infty$ where now $p > 1$. Assume also that $B_{m,p}(\Omega^c) < \infty$. Then there is an open set $G \supset \Omega^c$ such that $B_{m,p}(\bar{G}) < \infty$. By Lemma 1 we can find $f \in L_+^p$ such that $G_{m;1} * f \in C^\infty$ and $G_{m;1} * f \geqq 1$ on $F \cup \bar{G}$. Now let T be a smooth function on $\mathbb{R}_+$ such that $T(t) = 1$ if $t \geqq 1$ and $\sup_{t>0} |t^{k-1} T^{(k)}(t)| < \infty$ for $0 \leqq k \leqq m$. Then by the truncation theorem in [1] (it works also for the truncated kernel) there is a function $g \in L^p$ such that $T \circ (G_{m;1} * f) = G_{m;1} * g$ and $\|g\|_p \leqq A \|f\|_p$. We set $u = 1 - G_{m;1} * g$. Then $u \in C_\Omega^\infty \cap L^{m,p}$ but, by Lemma 7, $u \notin \mathring{L}_0^{m,p}$ since $G_{m;1} * g \in \mathring{L}_0^{m,p}$. Moreover

$$\int |u|^p \, d\mu = \int_{F^c} |u|^p \, d\mu \leqq \mu(F^c) < \infty$$

by the assumption on F, so $u \in H_\mu^{m,p}(\Omega) \setminus \mathring{H}_\mu^{m,p}(\Omega)$. This contradiction shows that μ must satisfy the condition in the theorem.

For $p = 1$ we use instead $u = 1 - \varphi$ where φ is the function constructed in

Lemma 3, satisfying $\varphi = 1$ on a neighbourhood of $F \cup \bar{G}$. Since $\int|\varphi(x)|dx + \int|\nabla^m \varphi(x)|\,dx < \infty$ it follows easily, using the multiplier η_R above, that $\varphi \in L_0^{m,1}$. Hence $u \notin \mathring{H}_\mu^{m,1}(\Omega)$. But $\int |u|\,d\mu \leqq \mu(F^c) < \infty$ so $u \in H_\mu^{m,1}$ and we have again obtained a contradiction.

The case $1 < p < N$, $mp > N$ *or* $1 = p < N$, $m \geqq N$. For the sufficiency we again decompose $u = v + c$ where $v \in L_0^{m,p}$. Using Sobolev's inequality

$$\sup_{B(x,1)} |v| \leqq A\left(\int_{B(x,1)} |v(y)|^{p^*}\,dy\right)^{1/p^*} + A\left(\int_{B(x,1)} |\nabla^m v(y)|^p\,dy\right)^{1/p}$$

for every $x \in \mathbb{R}^N$, we see that $v(x) \to 0$ uniformly as $|x| \to \infty$. Thus, if $c \neq 0$, we can find R such that $|v| \leqq \frac{|c|}{2}$ on $B(R)^c$. Then, if μ is not finite,

$$\int |u|^p\,d\mu \geqq \left(\frac{|c|}{2}\right)^p \mu(B(R)^c) = \infty$$

and we have a contradiction. Thus $c = 0$ and we can proceed as in the first case, again using Lemma 6(i). Note that if Ω^c is unbounded it follows immediately that $c = 0$, without any condition on μ, since $v = c$ on Ω^c.

For the necessity we observe that if Ω^c is bounded then there is $u \in C_\Omega^\infty$ such that $u(x) = 1$ for large $|x|$. Then $u \in L^{m,p} \setminus L_0^{m,p}$ since $u \notin L^{p^*}$. If μ is finite we get $u \in H_\mu^{m,p}(\Omega) \setminus \mathring{H}_\mu^{m,p}(\Omega)$ and we are done.

The case $p > N$ is proved as above, now invoking Lemma 6 (ii).

The case $p = N$ is proved by Lemma 6 (iii), this time using the multiplier

$$\eta_R(x) = \chi\left(\frac{1}{\log R} \log \frac{R^2}{|x|}\right),$$

where χ is a smooth function satisfying $\chi(t) = 0$ for $t \leqq \frac{1}{4}$ and $\chi(t) = 1$ for $t \geqq \frac{3}{4}$. This completes the proof.

REMARK. The same question of density of test functions can be asked about the more general norm

$$\|u\| = \|u\|_{L^p(\mu)} + \sum_{k=l}^{m} \|\nabla^k u\|_p,$$

where $m \geqq l \geqq 2$. In case $lp \geqq N$ approximation is always possible. This is proved in the same way as in this paper, using only somewhat different Hardy inequalitites. In the general case it is easy to give an implicit necessary and sufficient condition. Namely, with obvious notation, $H_\mu^{l,m,p} = \mathring{H}_\mu^{l,m,p}$ if and only if

$$u \in \mathring{H}_{\mu}^{l,m,p},\ P \in \mathsf{P}_{l-1},\ \int |u - P|^p \, d\mu < \infty \Rightarrow P = 0.$$

However, it is not clear whether this condition can be stated in a more transparent way in the spirit of Theorem 1, for example in terms of polynomial capacities; cfr [9, Ch. 10].

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AN EXISTENCE RESULT FOR SIMPLE INDUCTIVE LIMITS OF INTERVAL ALGEBRAS

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Given a C^*-algebra A with unit 1 we define the Elliot triple of A to be

$$(K_0(A), T(A), r_A)$$

where $K_0(A)$ has the usual ordering and $[1]$ as order-unit, $T(A)$ is the tracial state space of A, and with S the state space functor, $r_A: T(A) \to S(K_0(A))$ is given by $r_A(\tau)([p] - [q]) = \tau(p - q)$ for all $\tau \in T(A)$ and projections $p, q \in M_\infty(A)$ where τ is extended to $M_\infty(A)$ by $(a_{ij}) \mapsto \sum_i \tau(a_{ii})$. We identify two such triples (G_i, Δ_i, f_i), $i = 1, 2$ if there are isomorphisms $\phi_0: G_2 \to G_1$, $\phi_T: \Delta_1 \to \Delta_2$ such that the diagram

$$\begin{array}{ccc} \Delta_1 & \xrightarrow{\phi_T} & \Delta_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ S(G_1) & \xrightarrow[S(\phi_0)]{} & S(G_2) \end{array}$$

commutes. Elliott [3] proved that this triple is a complete invariant for the simple unital C^*-algebras which arise as inductive limits of finite direct sums of matrix algebras over $C([0,1])$ – AI algebras for short. The project of determining the range of the Elliott triple when applied to simple unital AI algebras was initiated by Thomsen [6]. When A is a simple unital AI algebra $K_0(A)$ is a simple dimension group, $S(K_0(A))$ is a metrizable Choquet simplex, $T(A)$ is another metrizable Choquet simplex and the map r_A is an affine continuous surjection. Furthermore, we know from [6] that the map r_A preserves extreme points i.e. $r_A(\partial_e T(A)) = \partial_e S(K_0(A))$.

In this paper we show that, whenever G is a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected, Δ is a metrizable Choquet simplex and $f: \Delta \to S(G)$ an affine continuous map with $f(\partial_e \Delta) = \partial_e S(G)$, then (G, Δ, f) is the Elliott triple of some simple unital AI algebra.

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NOTATION 1. For K a compact convex subset of a linear topological space, we denote by $\mathrm{Aff}(K)$ the complete order-unit space of affine continuous real-valued functions on K with pointwise ordering and 1 as the order-unit. For $\lambda: K \to L$ a continuous affine map between compact convex sets, let $\mathrm{Aff}(\lambda): \mathrm{Aff}(L) \to \mathrm{Aff}(K)$ be the positive order-unit-preserving homomorphism given by $\mathrm{Aff}(\lambda)(h) = h \circ \lambda$ for all $h \in \mathrm{Aff}(L)$. It is well-known that for K a compact convex subset of a locally convex Hausdorff space, the state space of $\mathrm{Aff}(K)$ is naturally isomorphic to K via evaluation.

For convenience we write $s^{\#}$ for $\mathrm{Aff}S(s)$ when s is a homomorphism of ordered groups, and write "homomorphism" instead of "positive order-unit-preserving homomorphism" when dealing with homomorphisms of order-unit spaces.

DEFINITIONS 2. Let Δ be a Choquet simplex. A partition of unity $v_1, \dots, v_k$ in $\mathrm{Aff}(\Delta)$ is said to be *extreme* if there are closed non-empty faces $\Delta_1, \dots, \Delta_k$ in Δ with $\Delta = \mathrm{hull}\{\Delta_1, \dots, \Delta_k\}$ such that $v_i|_{\Delta_j} \equiv \delta_{ij}$ for $1 \leqq i, j \leqq k$. A partition of unity $v_1, \dots, v_k$ in $\mathrm{Aff}(\Delta)$ is *peaked* if $\|v_j\| = 1, j = 1, \dots, k$. Note that in this case $\mathrm{span}\{v_1, \dots, v_k\} \cong l_k^\infty$ with $v_1, \dots, v_k$ as the standard basis.

For $v, 1 - v$ an extreme partition of unity in $\mathrm{Aff}(\Delta)$ with corresponding faces E, E^c we let $\mathrm{Aff}(\Delta)_v = \{f \in \mathrm{Aff}(\Delta): f|_{E^c} \equiv 0\}$, which is an order-unit space with order-unit v, and define a homomorphism $\pi_v: \mathrm{Aff}(\Delta) \to \mathrm{Aff}(\Delta)_v$ by $\pi_v(f)|_E = f|_E$ and $\pi_v(f)|_{E^c} \equiv 0$ for $f \in \mathrm{Aff}(\Delta)$.

LEMMA 3. *Suppose that Δ is a Choquet simplex and $(l_{n_i}^\infty, v_i)$ an inductive system with $\varinjlim (l_{n_i}^\infty, v_i) = \mathrm{Aff}(\Delta)$. Let $v_1, \dots, v_k$ be a peaked partition of unity in $\mathrm{Aff}(\Delta)$ and let $\varepsilon > 0$. There is an $i \in \mathbb{N}$ and a homomorphism $\rho: \mathrm{span}\{v_1, \dots v_k\} \to l_{n_i}^\infty$ such that $\|v_{\infty i} \circ \rho - id\| < \varepsilon$.*

PROOF. Since $\cup_{i=1}^\infty \nu_{\infty i}((l_{n_i}^\infty)^+)$ is dense in $\mathrm{Aff}(\Delta)^+$ there are $x_1, \dots, x_{k-1} \in (l_{n_{i_0}}^\infty)^+$ with $\left\|\left(1 - \frac{\varepsilon}{2k}\right) v_j - \nu_{\infty i_0}(x_j)\right\| < \frac{\varepsilon}{2k^2}, j = 1, \dots, k-1$. Now

$$\nu_{\infty i_0}\left(\sum_{j=1}^{k-1} x_j\right) < \left(1 - \frac{\varepsilon}{2k}\right) \sum_{j=1}^{k-1} v_j + \frac{\varepsilon}{2k} \leqq 1$$

and there is an $i \geqq i_0$ so that $v_{ii_0}(\sum_{j=1}^{k-1} x_j) < 1$. Let $y_j = v_{ii_0}(x_j), j = 1, \dots, k-1$ and $y_k = 1 - \sum_{j=1}^{k-1} y_j$. Then $(y_j)_{j=1}^k$ is a partition of unity in $l_{n_i}^\infty$. Since

$$1 - \frac{\varepsilon}{k} - \left(1 - \frac{\varepsilon}{2k}\right) v_k < \sum_{j=1}^{k-1} \nu_{\infty i}(y_j) < 1 - \left(1 - \frac{\varepsilon}{2k}\right) v_k$$

we have that

$$\left(1 - \frac{\varepsilon}{2k}\right)v_k < v_{\infty i}(y_k) < \frac{\varepsilon}{k} + \left(1 - \frac{\varepsilon}{2k}\right)v_k.$$

Hence, $\|v_j - v_{\infty i}(y_j)\| < \frac{\varepsilon}{k}, j = 1, \ldots, k.$

Let $\rho : \operatorname{span}\{v_1, \ldots, v_k\} \to l^{\infty}_{n_i}$ be the homomorphism given by $\rho(v_j) = y_j$ for $j = 1, \ldots, k$. Clearly, $\|v_{\infty i} \circ \rho - id\| < \varepsilon$.

LEMMA 4. *Let Δ be a metrizable Choquet simplex and $w, 1 - w$ an extreme partition of unity in* Aff(Δ). *Let V be a subspace of* Aff(Δ) *with $1 \in V$ and $V \cong l^{\infty}_m$ for some $m \in \mathbb{N}$. Let $F \subseteq \operatorname{Aff}(\Delta)_w$ be a finite subset and let $\varepsilon > 0$. There is a subspace W of* Aff$(\Delta)_w$ *with $w \in W \cong l^{\infty}_n$ for some $n \in \mathbb{N}$ such that* dist$(f, W) < \varepsilon$ *for all $f \in F$ and a homomorphism $\eta : V \to W$ with $\|\eta - \pi_w|_V\| < \varepsilon$.*

PROOF. There is a $\delta > 0$ such that if $x_1, \ldots, x_l \in l^{\infty +}_k$ with $\|\sum_{i=1}^{l} x_i - 1\| < \delta$ then there are $y_1, \ldots, y_l \in l^{\infty +}_k$ with $\sum_{i=1}^{l} y_i = 1$ and $\|x_i - y_i\| < \frac{\varepsilon}{2m}$ for $1 \leqq i \leqq l$. It follows from Theorem 2.7.2 of [1] that there is a subspace W of Aff$(\Delta)_w$ with $w \in W$ and $W \cong l^{\infty}_n$ for some $n \in \mathbb{N}$ such that dist$(f, W) < \varepsilon$ for all $f \in F$ and dist$(\pi_w(e_i), W) < \frac{\delta \wedge \varepsilon}{4m}$ for $1 \leqq i \leqq m$ where $e_1, \ldots, e_m$ is the standard basis for $V \cong l^{\infty}_m$. Since dist$(\pi_w(e_i), W^+) \leqq 2$dist$(\pi_w(e_i), W)$ there are $x_1, \ldots, x_m \in W^+$ with $\|x_i - \pi_w(e_i)\| < \frac{\delta \wedge \varepsilon}{2m}$ for $1 \leqq i \leqq m$. So $\|\sum_{i=1}^{m} x_i - w\| < \delta$ and there are $y_1, \ldots, y_m \in W^+$ with $\sum_{i=1}^{m} y_i = w$ and $\|x_i - y_i\| < \frac{\varepsilon}{2m}$ for $1 \leqq i \leqq m$. Let η: $V \to W$ be the homomorphism given by $\eta(e_i) = y_i$ for $1 \leqq i \leqq m$. Let $x \in V$ with $\|x\| \leqq 1$. Then $x = \sum_{i=1}^{m} \alpha_i e_i$ for some $\alpha_1, \ldots, \alpha_m \in [-1, 1]$ and

$$\|\eta(x) - \pi_w(x)\| \leqq \sum_{i=1}^{m} |\alpha_i| \, \|y_i - \pi_w(e_i)\| \leqq \sum_{i=1}^{m} (\|y_i - x_i\| + \|x_i - \pi_w(e_i)\|) < \varepsilon.$$

DEFINITION 5. A *tree* is a triple $(X, \leqq, x_0)$ where $(X, \leqq)$ is a partially ordered set with a maximal element x_0 such that $s(x) = \{y \in X : y < x, \forall z < x : y \nless z\}$ is finite for every $x \in X$, $s(x) \cap s(y) = \emptyset$ when $x \neq y$ and X is the union of the level sets $\mathscr{L}^i$ given by $\mathscr{L}^1 = \{x_0\}$ and $\mathscr{L}^{i+1} = \cup_{y \in \mathscr{L}^i} s(y)$ for $i \in \mathbb{N}$. For $i \in \mathbb{N}$ we let $L^i = \mathscr{L}^i \cup \{y \in \cup_{j=1}^{i} \mathscr{L}^j : \forall x \in X : x \nless y\}$ – *the leaves* of $\cup_{j=1}^{i} \mathscr{L}^j$ – and $c(x) = \{y \in L^{i+1} : y \leqq x\}$ for $x \in L^i$.

REMARK 6. Let Δ be a metrizable Choquet simplex with compact and totally disconnected extreme boundary. Since every locally compact, totally disconnected topological space has a basis of sets being both open and closed, there is a basis Y for the compact, totally disconnected metric space $\partial_e \Delta$ such that $\partial_e \Delta \in Y$ and $(Y, \subseteq, \partial_e \Delta)$ is a tree, where $s(y)$ is either empty or consists of mutually disjoint sets with union y for every $y \in Y$. Now put $X = \{g \in \mathrm{Aff}(\Delta) : \exists y \in Y : g|_{\partial_e \Delta} = 1_y\}$. Then $(X, \leqq, 1)$ is a tree where each L^i is an extreme partition of unity in $\mathrm{Aff}(\Delta)$ and the span of X is dense in $\mathrm{Aff}(\Delta)$.

PROPOSITION 7. *Let G be a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected. Let Δ be a metrizable Choquet simplex and let $f : \Delta \to S(G)$ be a continuous affine map with $f(\partial_e \Delta) = \partial_e S(G)$. There is a system $(\mathbf{Z}^{n_i}, s_i)$ with positive order-unit-preserving connecting homomorphisms and inductive limit $\varinjlim(\mathbf{Z}^{n_i}, s_i) = G$, a collection $1 \in X \subseteq \mathrm{AffS}(G)$ with dense span such that $(X, \leqq, 1)$ is a tree where each L^i is an extreme partition of unity in $\mathrm{AffS}(G)$ and homomorphisms $\rho_i : W_i \to \mathrm{AffS}(\mathbf{Z}^{n_i})$, δ_i: $\mathrm{AffS}(\mathbf{Z}^{n_i}) \to W_{i+1}$ where $W_i = \mathrm{span}(L^i)$ such that*

$$\|\delta_i \circ \rho_i - \mathrm{id}\| < 2^{-i},$$

$$\|\rho_{i+1} \circ \delta_i - s_i^{\#}\| < 2^{-i} n_i^{-1},$$

$$\|s_{\infty i}^{\#} - \delta_i\| < 2^{-i}$$

for all $i \in \mathbf{N}$ and moreover there are Markov operators θ_h: $C_{\mathbf{R}}([0,1]) \to C_{\mathbf{R}}([0,1])$ of the form $\theta_h = N_i^{-1}(\theta_h^1 + \ldots + \theta_h^{N_i})$ where $\theta_h^1, \ldots, \theta_h^{N_i}$ are restrictions of unital $*$-endomorphisms of $C([0,1])$ for $h \in L^i$, $i \geqq 2$ such that

$$\mathrm{Aff}(\Delta) = \varinjlim(W_i \otimes C_{\mathbf{R}}([0,1]), \theta_i),$$

$$\theta_{\infty i}(v \otimes 1) = \mathrm{Aff}(f)(v), \; v \in W_i, \; i \in \mathbf{N}$$

where $\theta_i : W_i \otimes C_{\mathbf{R}}([0,1]) \to W_{i+1} \otimes C_{\mathbf{R}}([0,1]$ is the homomorphism given by

$$\theta_i\left(\sum_{g \in L^i} g \otimes x_g\right) = \sum_{\substack{g \in L^i \\ h \in c(g)}} h \otimes \theta_h(x_g)$$

and

$$\mathrm{mult}(s_i) \geqq 2^i N_i \# L^{i+1}.$$

PROOF. Let $(\mathbf{Z}^{n_i}, s_i)$ be a system with positive order-unit-preserving connecting homomorphisms and inductive limit $\varinjlim(\mathbf{Z}^{n_i}, s_i) = G$. Since G is simple and noncyclic we may assume that $\mathrm{mult}(s_i) \to \infty$ as $i \to \infty$ where mult denotes the smallest entry of a given matrix. We may therefore also assume that $\mathrm{mult}(s_i) \geqq 1$ for all $i \in \mathbf{N}$.

By the above remark there is a collection $1 \in X \subseteq \mathrm{AffS}(G)$ with dense span such that $(X, \leqq, 1)$ is a tree where each L^i is an extreme partition of unity in $\mathrm{AffS}(G)$.

Note that $\mathrm{Aff}(f)$ takes extreme partitions of unity in $\mathrm{Aff}S(G)$ to extreme partitions of unity in $\mathrm{Aff}(\Delta)$ because $f(\partial_e\Delta) = \partial_e S(G)$.

For convenience we write $\mathrm{Aff}(\Delta)_g$ and π_g for $\mathrm{Aff}(\Delta)_{\mathrm{Aff}(f)(g)}$ and $\pi_{\mathrm{Aff}(f)(g)}$ respectively for $g \in X$.

For $m \in \mathbb{N}$ a partition of unity $\zeta_1, \ldots, \zeta_m$ in $C_{\mathbb{R}}([0,1])$ is chosen such that $\zeta_i\left(\frac{i-1}{m}\right) = 1$, $1 \leqq i \leqq m$, and we let $\iota_m : l_m^\infty \to C_{\mathbb{R}}([0,1])$, $\kappa_m : C_{\mathbb{R}}([0,1]) \to l_m^\infty$ be the homomorphisms given by $\iota_m(e_i) = \zeta_i$, $1 \leqq i \leqq m$ and $\kappa_m(x) = \sum_{i=1}^m x\left(\frac{i-1}{m}\right)e_i$ for $x \in C_{\mathbb{R}}([0,1])$. Note that $\kappa_m \circ \iota_m = \mathrm{id}$.

Let $(d_i)_{i=1}^\infty$ and $(a_i)_{i=1}^\infty$ be dense sequences in $C_{\mathbb{R}}([0,1])$ and $\mathrm{Aff}(\Delta)$ respectively. We show that there are increasing sequences $(i_p)_{p=1}^\infty$, $(j_p)_{p=1}^\infty$ in $\mathbb{N}$ ($j_1 = 1$) and

- subspaces $\mathrm{Aff}(f)(g) \in Z_g \subseteq \mathrm{Aff}(\Delta)_g$ such that $Z_g \cong l_{m_g}^\infty$ for some $m_g \in \mathbb{N}$ and $\mathrm{dist}(\pi_g(a_q), Z_g) < 2^{-p}$ for all $1 \leqq q \leqq p$, $g \in L^{j_p}$, $p \in \mathbb{N}$,
- homomorphisms $\eta_h : Z_g \to Z_h$ such that $\|\eta_h - \pi_h|_{Z_g}\| < 2^{-p}$ for all $g \in L^{j_p}$, $h \in L^{j_{p+1}}$, $h \leqq g$, $p \in \mathbb{N}$,
- Markov operators $\theta_h : C_{\mathbb{R}}([0,1]) \to C_{\mathbb{R}}([0,1])$ which are of the form $\theta_h = N_p^{-1}(\theta_h^1 + \ldots + \theta_h^{N_p})$ where $\theta_h^1, \ldots, \theta_h^{N_p}$ are restrictions of unital $*$-endomorphisms of $C([0,1])$ and $\|\theta_h(f) - \iota_h \circ \eta_h \circ \kappa_g(f)\| < 2^{-p}$ for all $f \in F_g$, $g \in L^{j_p}$, $h \in L^{j_{p+1}}$, $h \leqq g$, $p \in \mathbb{N}$ where $\iota_g = \iota_{m_g} \circ \phi_g : Z_g \to C_{\mathbb{R}}([0,1])$, $\kappa_g = \phi_g^{-1} \circ \kappa_{m_g}$: $C_{\mathbb{R}}([0,1]) \to Z_g$ for some isomorphism $\phi_g : Z_g \to l_{m_g}^\infty$ and

$$F_g = \bigcup_{q=1}^{p} (\theta_{gg_q}(\{d_1, \ldots, d_p\}) \cup \iota_g \circ \eta_{gg_q} \circ \kappa_{g_q}(\{d_1, \ldots, d_p\}))$$

  where $g_q \in L^{j_q}$ is the unique function $g_q \geqq g$,
- homomorphisms $\rho_p : W_p \to \mathrm{Aff}S(\mathbb{Z}^{n_{i_p}})$ and $\delta_p : \mathrm{Aff}S(\mathbb{Z}^{n_{i_p}}) \to W_{p+1}$ where $W_p = \mathrm{span}(L^{j_p})$ such that

$$\|s_{\infty i_p}^\# \circ \rho_p - \mathrm{id}\| < 2^{-p-1},$$

$$\|s_{\infty i_p}^\# - \delta_p\| < 2^{-p-1} n_{i_p}^{-1},$$

$$\|\rho_{p+1} \circ \delta_p - s_{i_{p+1} i_p}^\#\| < 2^{-p} n_{i_p}^{-1}$$

  for all $p \in \mathbb{N}$ and such that
- $\mathrm{mult}(s_{i_{p+1} i_p}) \geqq 2^p N_p \,\#\, L^{j_{p+1}}$ for all $p \in \mathbb{N}$.

By Lemma 3, there is an $i_1 \in \mathbb{N}$ and a homomorphism $\rho_1 : W_1 \to \mathrm{Aff}S(\mathbb{Z}^{n_{i_1}})$ such that $\|s_{\infty i_1}^\# \circ \rho_1 - \mathrm{id}\| < 2^{-2}$. Since the span of X is dense in $\mathrm{Aff}S(G)$ there is a $j_2 > 1$ and a homomorphism $\delta_1 : \mathrm{Aff}S(\mathbb{Z}^{n_{i_1}}) \to W_2$ such that $\|s_{\infty i_1}^\# - \delta_1\| < 2^{-2} n_{i_1}^{-1}$. It follows from Theorem 2.7.2 of [1] that there is a subspace $1 \in Z_1 \subseteq \mathrm{Aff}(\Delta)$ such that $Z_1 \cong l_{m_1}^\infty \in \mathbb{N}$ and $\mathrm{dist}(a_1, Z_1) < 2^{-1}$.

Suppose that $(i_p)_{p=1}^P$, $(j_p)_{p=1}^{P+1}$ are increasing sequences in $\mathbb{N}$ ($j_1 = 1$) and that Z_g,

$g \in L^{j_P}$, $1 \leqq p \leqq P$, η_h, θ_h, $h \in L^{j_p}$, $2 \leqq p \leqq P$, δ_p, ρ_p, $1 \leqq p \leqq P$ and $\mathrm{mult}(s_{i_{p+1}i_p})$, $1 \leqq p \leqq P-1$ satisfies the above conditions. It follows from Lemma 4 that for every $g \in L^{j_P}$, $h \in L^{j_{P+1}}$, $h \leqq g$ there is a subspace $\mathrm{Aff}(f)(h) \in Z_h \subseteq \mathrm{Aff}(\Delta)_h$ such that $Z_h \cong l^{\infty}_{m_h}$ for some $m_h \in \mathbb{N}$ and a homomorphism $\eta_h: Z_g \to Z_h$ such that

$$\mathrm{dist}(\pi_h(a_q), Z_h) < 2^{-(P+1)},\ 1 \leqq q \leqq P+1,$$

$$\|\eta_h - \pi_h|_{Z_g}\| < 2^{-P}.$$

It follows from the Krein-Milman theorem for Markov operators of [5] that there is a Markov operator $\theta_h: C_{\mathbb{R}}([0,1])$ of the form $\theta_h = N_{Ph}^{-1}(\theta_h^1 + \ldots + \theta_h^{N_{Ph}})$ where $\theta_h^1, \ldots, \theta_h^{N_{Ph}}$ are restrictions of unital $*$-endomorphisms of $C([0,1])$ such that $\|\theta_h(f) - \iota_h \circ \eta_h \circ \kappa_g(f)\| < 2^{-P}$ for all $f \in F_g$. We may assume that $N_{Ph} = N_P$ for all $h \in L^{j_{P+1}}$. There is a $k_1 > i_P$ such that $\mathrm{mult}(s_{k_1 i_P}) \geqq 2^P N_P \,\#\, L^{j_{P+1}}$. It follows from Lemma 3 that there is a $k_2 > k_1$ and a homomorphism $\rho: W_{P+1} \to \mathrm{AffS}(\mathbb{Z}^{n_{k_2}})$ such that $\|s^{\#}_{\infty k_2} \circ \rho - \mathrm{id}_{W_{P+1}}\| < 2^{-P-2} n_{i_P}^{-1}$. Since

$$\|s^{\#}_{\infty k_2} \circ \rho \circ \delta_P - s^{\#}_{\infty i_P}\| \leqq \|s^{\#}_{\infty k_2} \circ \rho \circ \delta_P - \delta_P\| + \|\delta_P - s^{\#}_{\infty i_P}\| < 2^{-P} n_{i_P}^{-1}$$

and the unit ball in $\mathrm{AffS}(\mathbb{Z}^{n_{i_P}})$ is compact, it follows that there is an $i_{P+1} \geqq k_2$ such that $\|s^{\#}_{i_{P+1} i_P} \circ \rho \circ \delta_P - s^{\#}_{i_{P+1} i_P}\| < 2^{-P} n_{i_P}^{-1}$. Note that $\mathrm{mult}(s_{i_{P+1} i_P}) \geqq 2^P N_P \,\#\, L^{j_{P+1}}$ and put $\rho_{P+1} = s^{\#}_{i_{P+1} i_P} \circ \rho$. Since the span of X is dense in $\mathrm{AffS}(G)$ there is a $j_{P+2} > j_{P+1}$ and a homomorphism $\delta_{P+1}: \mathrm{AffS}(\mathbb{Z}^{n_{i_{P+1}}}) \to W_{P+2}$ such that $\|s^{\#}_{\infty i_{P+1}} - \delta_{P+1}\| < 2^{-(P+1)-1} n_{i_{P+1}}^{-1}$. The result follows from Zorn's lemma.

For $p \in \mathbb{N}$ let Z_p be the subspace of $\mathrm{Aff}(\Delta)$ spanned by $(Z_g)_{g \in L^{j_p}}$. Note that Z_p is isomorphic to the order-unit-space direct sum of $(Z_g)_{g \in L^{j_p}}$. Moreover $\mathrm{dist}(a_q, Z_p) < 2^{-p}$ for all $1 \leqq q \leqq p$. Let $\eta_p: Z_p \to Z_{p+1}$ be the homomorphism given by

$$\eta_p\left(\sum_{g \in L^{j_p}} v_g\right) = \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} \eta_h(v_g)$$

for $v_g \in Z_g$, $g \in L^{j_p}$. Then η_p is a homomorphism with $\eta_p(\mathrm{Aff}(f)(g)) = \mathrm{Aff}(f)(g)$ for all $g \in W_p$. Let $v \in Z_p$ with $\|v\| \leqq 1$. Then v is of the form $v = \sum_{g \in L^{j_p}} v_g$ where $v_g \in Z_g$, $\|v_g\| \leqq 1$ and

$$\|\eta_p(v) - v\| = \left\| \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} (\eta_h(v_g) - \pi_h(v_g)) \right\| = \max_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} \|\eta_h(v_g) - \pi_h(v_g)\| < 2^{-p}.$$

Therefore the sequence $(\eta_{qp}(v))_{q=p}^{\infty}$ in $\mathrm{Aff}(\Delta)$ is Cauchy for every $v \in Z_p$ – let $\alpha_p(v)$ denote the limit. Then $\alpha_p: Z_p \to \mathrm{Aff}(\Delta)$ is a homomorphism with $\alpha_{p+1} \circ \eta_p = \alpha_p$ and $\alpha_p(\mathrm{Aff}(f)(g)) = \mathrm{Aff}(f)(g)$ for all $g \in W_p$. Thus there is a homomorphism $\alpha: \varinjlim(Z_p, \eta_p) \to \mathrm{Aff}(\Delta)$ with $\alpha \circ \eta_{\infty p} = \alpha_p$. By $\|\eta_p(v) - v\| < 2^{-p}\|v\|$ the homomorphism α is seen to be isometric. Using that $(a_i)_{i=1}^{\infty}$ is dense in $\mathrm{Aff}(\Delta)$, $\mathrm{dist}(a_q, Z_p) < 2^{-p}$ for all $1 \leqq q \leqq p$ and $1 \in Z_p$ for all $p \in \mathbb{N}$ and the fact that α is isometric, one shows that α is an isomorphism.

Define $I_p: Z_p \to W_p \otimes C_{\mathbb{R}}([0,1])$ and $K_p: W_p \otimes C_{\mathbb{R}}([0,1]) \to Z_p$ by

$$I_p\left(\sum_{g \in L^{j_p}} v_g\right) = \sum_{g \in L^{j_p}} g \otimes \iota_g(v_g),$$

$$K_p\left(\sum_{g \in L^{j_p}} g \otimes x_g\right) = \sum_{g \in L^{j_p}} \kappa_g(x_g)$$

for $v_g \in Z_g$, $x_g \in C_{\mathbb{R}}([0,1])$, $g \in L^{j_p}$. The maps I_p, K_p are homomorphisms and $K_p \circ I_p = \mathrm{id}$. Letting $\omega_p = I_{p+1} \circ \eta_p \circ K_p$ and $\beta_p = \eta_{\infty p} \circ K_p$ we have $\beta_{p+1} \circ \omega_p = \beta_p$ and so there is a homomorphism $\beta: \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \omega_p) \to \varinjlim (Z_p, \eta_p)$ with $\beta \circ \omega_{\infty p} = \beta_p$, $p \in \mathbb{N}$. It is easy to see that this is an isomophism. In addition $\alpha \circ \beta_p(g \otimes 1) = \alpha \circ \eta_{\infty p} \circ K_p(g \otimes 1) = \alpha \circ \eta_{\infty p}(\mathrm{Aff}(f)(g)) = \alpha_p(\mathrm{Aff}(f)(g)) = \mathrm{Aff}(f)(g)$ for all $g \in W_p$.

Let $\theta_p: W_p \otimes C_{\mathbb{R}}([0,1]) \to W_{p+1} \otimes C_{\mathbb{R}}([0,1])$ be the homomorphism given by

$$\theta_p\left(\sum_{g \in L^{j_p}} g \otimes x_g\right) = \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} h \otimes \theta_h(x_g).$$

The set $\{\sum_{g \in L^{j_p}} g \otimes x_g : x_g \in \{d_i : i \in \mathbb{N}\}\}$ is dense in $W_p \otimes C_{\mathbb{R}}([0,1])$ and

$$\bigcup_{q=1}^{p}\left(\theta_{pq}\left\{\sum_{h \in L^{j_q}} h \otimes x_h : x_h \in \{d_1, \ldots, d_p\}\right\} \cup \omega_{pq}\left\{\sum_{h \in L^{j_q}} h \otimes x_h : x_h \in \{d_1, \ldots, d_p\}\right\}\right)$$

is equal to $\{\sum_{g \in L^{j_p}} g \otimes x_g : x_g \in F_g\}$. For $z = \sum_{g \in L^{j_p}} g \otimes x_g$ where $x_g \in F_g$ we have

$$\|\theta_p(z) - \omega_p(z)\| = \sum_{\substack{g \in L^{j_p} \\ h \in L^{j_{p+1}} \\ h \leqq g}} \|\theta_h(x_g) - \iota_h \circ \eta_h \circ \kappa_g(x_g)\| \leqq 2^{-p}$$

and it follows from (a slight modification of) Lemma 3.4 of [5] that there is an isomorphism $\gamma: \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \theta_p) \to \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \omega_p)$ such that $\gamma \circ \theta_{\infty p} = \gamma_p$ where $\gamma_p: W_p \otimes C_{\mathbb{R}}([0,1]) \to \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \omega_p)$ is the homomorphism given by $\gamma_p(x) = \lim_{q \to \infty} \omega_{\infty q} \circ \theta_{qp}(x)$ for $x \in W_p \otimes C_{\mathbb{R}}([0,1])$. Note that $\gamma_p(g \otimes 1) = \omega_{\infty p}(g \otimes 1)$ for all $g \in W_p$.

To sum up, there is an isomorphism $\alpha \circ \beta \circ \gamma: \varinjlim (W_p \otimes C_{\mathbb{R}}([0,1]), \theta_p) \to \mathrm{Aff}(\Delta)$ with $\alpha \circ \beta \circ \gamma \circ \theta_{\infty i}(g \otimes 1) = \mathrm{Aff}(f)(g)$ for all $g \in W_i$ and $i \in \mathbb{N}$.

LEMMA 8. *Let $M, N \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \geqq 0$ with $\sum_{j=1}^{m} \lambda_j > 0$. There are $k_1, \ldots, k_m \in \mathbb{N}_0$ such that*

$$M - N < N \sum_{j=1}^{m} k_j \leqq M,$$

$$\sum_{j=1}^{m} \left| \frac{Nk_j}{M} - \lambda_j \right| < \frac{2Nm}{M} + \left| 1 - \sum_{j=1}^{m} \lambda_j \right|.$$

PROOF. Put $\lambda = \sum_{j=1}^{m} \lambda_j$ and $\mu_j = \frac{M}{\lambda N} \sum_{l=1}^{j} \lambda_l$ for $1 \leqq j \leqq m$. There is an $h_j \in \mathbf{N}_0$ with $\mu_j - 1 < h_j \leqq \mu_j$ for $1 \leqq j \leqq m$. Put $k_1 = h_1$ and $k_{j+1} = h_{j+1} - h_j$ for $1 \leqq j \leqq m - 1$. We have

$$M - N = N(\mu_m - 1) < Nh_m = N \sum_{j=1}^{m} k_j \leqq N\mu_m = M,$$

$$\left| \frac{Nk_1}{M} - \frac{\lambda_1}{\lambda} \right| = \frac{N}{M} \left| k_1 - \frac{M\lambda_1}{N\lambda} \right| = \frac{N}{M} |k_1 - \mu_1| < \frac{N}{M}$$

and

$$\left| \frac{Nk_j}{M} - \frac{\lambda_j}{\lambda} \right| = \frac{N}{M} \left| k_j - \frac{M\lambda_j}{N\lambda} \right| = \frac{N}{M} |(h_j - h_{j-1}) - (\mu_j - \mu_{j-1})| < 2\frac{N}{M}$$

for $2 \leqq j \leqq m$. Therefore

$$\sum_{j=1}^{m} \left| \frac{Nk_j}{M} - \lambda_j \right| < \frac{2Nm}{M} + |1 - \lambda|.$$

THEOREM 9. *Let G be a simple noncyclic dimension group, such that the extreme boundary of the state space is compact and totally disconnected. Let Δ be a metrizable Choquet simplex and let $f: \Delta \to S(G)$ be a continuous affine map with $f(\partial_e \Delta) = \partial_e S(G)$. Then (G, Δ, f) is the Elliott triple of some simple unital* AI *algebra.*

PROOF. Let $(\mathbf{Z}^{n_i}, s_i)$, X, $(\rho_i)_{i=1}^{\infty}$, $(\delta_i)_{i=1}^{\infty}$, $(\theta_h)_{h \in X - \{1\}}$ and mult (s_i) be as in Proposition 7. Let (a_{pq}^i), (λ_{hq}^i), (μ_{pq}^i) and (v_{ph}^i) be the matrices for $s_i, \delta_i, s_i^{\#}$ and ρ_i respectively and put $Z_i = \{(p,q): 1 \leqq p \leqq n_{i+1}, 1 \leqq q \leqq n_i, \sum_{h \in L^{i+1}} v_{ph}^{i+1} \lambda_{hq}^i = 0\}$ – the zero entries of the matrix for $\rho_{i+1} \circ \delta_i$ for $i \in \mathbf{N}$.

It follows from Lemma 8 that for $(p,q) \notin Z_i$ there are $(k_{pq}^h)_{h \in L^{i+1}}$ in $\mathbf{N}_0$ such that

$$a_{pq}^i - N_i < N_i \sum_{h \in L^{i+1}} k_{pq}^h \leqq a_{pq}^i,$$

$$\sum_{h \in L^{i+1}} \left| \frac{N_i k_{pq}^h}{a_{pq}^i} - \frac{v_{ph}^{i+1} \lambda_{hq}^i}{\mu_{pq}^i} \right| < \frac{2N_i \# L^{i+1}}{a_{pq}^i} + \left| 1 - \sum_{h \in L^{i+1}} \frac{v_{ph}^{i+1} \lambda_{hq}^i}{\mu_{pq}^i} \right|.$$

For $(p,q) \in Z_i$ choose $(k_{pq}^h)_{h \in L^{i+1}} \subseteq \mathbf{N}_0$ such that $a_{pq}^i - N_i < N_i \sum_{h \in L^{i+1}} k_{pq}^h \leqq a_{pq}^i$.

Let $r_{pq}^i = a_{pq}^i - N_i \sum_{h \in L^{i+1}} k_{pq}^h$ for $1 \leqq p \leqq n_{i+1}$, $1 \leqq q \leqq n_i$ and $i \in \mathbf{N}$. Let $m_i =$

$(m_1^i, \ldots, m_{n_i}^i)$ denote the order-unit in $\mathbf{Z}^{n_i}$. Then $A_i = \oplus_{p=1}^{n_i} M_{m_p^i} \otimes C([0,1])$ is the interval algebra with $K_0(A_i) = \mathbf{Z}^{n_i}$. For all $h \in L^{i+1}$, $1 \leqq r \leqq N_i$ and $i \in \mathbf{N}$ there is a continuous function $\omega_h^r: [0,1] \to [0,1]$ such that $\theta_h^r(x) = x \circ \omega_h^r$ for all $x \in C([0,1])$. Let $\psi_i: A_i \to A_{i+1}$ be the $*$-homomorphism with characteristic functions ω_h^r repeated k_{pq}^h times, $h \in L^{i+1}$, $1 \leqq r \leqq N_i$ and $\mathrm{id}_{[0,1]}$ repeated r_{pq}^i times from the qth summand of A_i to the pth summand of A_{i+1} for $1 \leqq p \leqq n_{i+1}$, $1 \leqq q \leqq n_i$ and $i \in \mathbf{N}$. Let $B = \varinjlim (A_i, \psi_i)$ and note that $K_0(B) = \varinjlim (\mathbf{Z}^{n_i}, s_i) = G$.

Let $\iota_i: W_i \hookrightarrow W_{i+1}$ be inclusion and let $\vartheta_i: W_{i+1} \otimes C_R([0,1]) \to W_{i+1} \otimes C_R([0,1])$ be the homomorphism given by

$$\vartheta_i\left(\sum_{h \in L^{i+1}} h \otimes x_h\right) = \sum_{h \in L^{i+1}} h \otimes \theta_h(x_h).$$

Observe that $\theta_i \circ (\iota_i \otimes \mathrm{id}_{C_R([0,1])})$. Define

$$\zeta_i = \vartheta_i \circ (\delta_i \otimes \mathrm{id}_{C_R([0,1])}): \mathrm{Aff}\,\mathrm{T}(A_i) \to W_{i+1} \otimes C_R([0,1]),$$

$$\eta_i = \rho_i \otimes \mathrm{id}_{C_R([0,1])}: W_i \otimes C_R([0,1]) \to \mathrm{Aff}\,\mathrm{T}(A_i)$$

for $i \in \mathbf{N}$.

We show that the triangles of the following diagram commutes up to an error which is summable.

$$\begin{array}{ccc}
W_i \otimes C_R([0,1]) & \xrightarrow{\ \theta_i\ } & W_{i+1} \otimes C_R([0,1]) \\
\Big\downarrow{\scriptstyle \eta_i} & \nearrow{\scriptstyle \zeta_i} & \Big\downarrow{\scriptstyle \eta_{i+1}} \\
\mathrm{Aff}\,\mathrm{T}(A_i) & \xrightarrow[\mathrm{Aff}\,\mathrm{T}(\psi_i)]{} & \mathrm{Aff}\,\mathrm{T}(A_{i+1})
\end{array}$$

As for the upper triangle we have

$$\begin{aligned}
\|\theta_i - \zeta_i \circ \eta_i\| &= \|\vartheta_i \circ (\iota_i \otimes \mathrm{id}_{C_R([0,1])}) - \vartheta_i \circ (\delta_i \otimes \mathrm{id}_{C_R([0,1])}) \circ (\rho_i \otimes \mathrm{id}_{C_R([0,1])})\| \\
&\leqq \|\iota_i - \delta_i \circ \rho_i\| \\
&< 2^{-i}.
\end{aligned}$$

Let $x = (x_1, \ldots, x_{n_i}) \in \mathrm{Aff}\,\mathrm{T}(A_i)$ and $1 \leqq p \leqq n_{i+1}$. The pth coordinate of $\mathrm{Aff}\,\mathrm{T}(\psi_i)(x)$ is

$$\sum_{q=1}^{n_i} \frac{m_q^i}{m_p^{i+1}} \left(\sum_{h \in L^{i+1}} k_{pq}^h \sum_{r=1}^{N_i} x_q \circ \omega_h^r + r_{pq}^i x_q \right)$$

and the pth coordinate of $\eta_{i+1} \circ \zeta_i(x)$ is

$$\sum_{q=1}^{n_i} \sum_{h \in L^{i+1}} v_{ph}^{i+1} \lambda_{hq}^i \theta_h(x_q).$$

Assume that $\|x\| \leqq 1$. Then

$$\|\mathrm{AffT}(\psi_i)(x) - \eta_{i+1}\circ\zeta_i(x)\|$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{q=1}^{n_i} \sum_{h\in L^{i+1}} \left| \frac{m_q^i k_{pq}^h}{m_p^{i+1}} - v_{ph}^{i+1}\lambda_{hq}^i \frac{1}{N_i} \right| \left\| \sum_{r=1}^{N_i} x_q \circ \omega_h^r \right\| + \sum_{q=1}^{n_i} \frac{m_q^i r_{pq}^i}{m_p^{i+1}} \|x_q\| \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{q=1}^{n_i} \sum_{h\in L^{i+1}} \left| \frac{m_q^i k_{pq}^h N_i}{m_p^{i+1}} - v_{ph}^{i+1}\lambda_{hq}^i \right| + \sum_{q=1}^{n_i} \frac{m_q^i a_{pq}^i 2^{-i}}{m_p^{i+1}} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{q=1}^{n_i} \mu_{pq}^i \sum_{h\in L^{i+1}} \left| \frac{k_{pq}^h N_i}{a_{pq}^i} - \frac{v_{ph}^{i+1}\lambda_{hq}^i}{\mu_{pq}^i} \right| + 2^{-i} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{\substack{1\leqq q\leqq n_i\\ (p,q)\in Z_i}} \mu_{pq}^i \sum_{h\in L^{i+1}} \frac{k_{pq}^h N_i}{a_{pq}^i} + \sum_{\substack{1\leqq q\leqq n_i\\ (p,q)\notin Z_i}} \mu_{pq}^i \right.$$
$$\left. \times \left(\frac{2N_i \# L^{i+1}}{a_{pq}^i} + \left| 1 - \sum_{h\in L^{i+1}} \frac{v_{ph}^{i+1}\lambda_{hq}^i}{\mu_{pq}^i} \right| \right) + 2^{-i} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{\substack{1\leqq q\leqq n_i\\ (p,q)\in Z_i}} \mu_{pq}^i + 2^{1-i} + \sum_{\substack{1\leqq q\leqq n_i\\ (p,q)\notin Z_i}} \left| \mu_{pq}^i - \sum_{h\in L^{i+1}} v_{ph}^{i+1}\lambda_{hq}^i \right| + 2^{-i} \right\}$$

$$\leqq \max_{1\leqq p\leqq n_{i+1}} \left\{ \sum_{\substack{1\leqq q\leqq n_i\\ (p,q)\in Z_i}} \|\rho_{i+1}\circ\delta_i - s_i^{\#}\| + 2^{1-i} + \sum_{\substack{1\leqq q\leqq n_i\\ (p,q)\notin Z_i}} \|\rho_{i+1}\circ\delta_i - s_i^{\#}\| + 2^{-i} \right\}$$

$$= n_i\|\rho_{i+1}\circ\delta_i - s_i^{\#}\| + 3\cdot 2^{-i}$$

$$\leqq 4\cdot 2^{-i}.$$

Since

$$\|\theta_{i+1}\circ\zeta_i - \zeta_{i+1}\circ\mathrm{AffT}(\psi_i)\|$$
$$\leqq \|\theta_{i+1}\circ\zeta_i - \zeta_{i+1}\circ\eta_{i+1}\circ\zeta_i\| + \|\zeta_{i+1}\circ\eta_{i+1}\circ\zeta_i - \zeta_{i+1}\circ\mathrm{AffT}(\psi_i)\|$$
$$\leqq \|\theta_{i+1} - \zeta_{i+1}\circ\eta_{i+1}\| + \|\eta_{i+1}\circ\zeta_i - \mathrm{AffT}(\psi_i)\|$$
$$\leqq 5\cdot 2^{-i}$$

the sequence $(\theta_{\infty k+1}\circ\zeta_k\circ\mathrm{AffT}(\psi_{ki})(x))_{k=i}^{\infty}$ is Cauchy for all $x\in\mathrm{AffT}(A_i)$ – let $\alpha_i(x)$ denote the limit. Then $\alpha_i:\mathrm{AffT}(A_i)\to\mathrm{Aff}(\Delta)$ is a homomorphism with $\alpha_{i+1}\circ\mathrm{AffT}(\psi_i) = \alpha_i$. Thus there is a homomorphism $\alpha:\mathrm{AffT}(B)\to\mathrm{Aff}(\Delta)$ with $\alpha\circ\mathrm{AffT}(\psi_{\infty i}) = \alpha_i$. Using that $(\|\mathrm{AffT}(\psi_i) - \eta_{i+1}\circ\zeta_i\|)_{i=1}^{\infty}$ and $(\|\zeta_i\circ\eta_i - \theta_i\|)_{i=1}^{\infty}$ are summable one shows that α is an isomorphism.

Note that $T(B)\cong\Delta$ via $S(\alpha):\Delta\to T(B)$. We now show that $r_B\circ S(\alpha) = f$. Let κ_i and χ_i be the inclusions

$$\kappa_i : W_i \hookrightarrow W_i \otimes C_{\mathsf{R}}([0,1]),\ g \mapsto g \otimes 1,$$

$$\chi_i : \mathrm{AffS}(\mathsf{Z}^{n_i}) \hookrightarrow \mathrm{AffS}(\mathsf{Z}^{n_i}) \otimes C_{\mathsf{R}}([0,1]) = \mathrm{AffT}(A_i), g \mapsto g \otimes 1.$$

With these definitions we have that

$$\theta_{\infty i} \circ \kappa_i = \mathrm{Aff}(f)|_{W_i},$$

$$\zeta_i \circ \chi_i = \kappa_{i+1} \circ \delta_i$$

and $p = \chi_i[p]$ for every projection $p \in A_i \subset \mathrm{AffT}(A_i)$. Let $w \in \Delta$ and $p \in A_i$ be a projection.

$$\begin{aligned}
&r_B \circ S(\alpha)(w)[\psi_{\infty i}(p)] \\
&= S(\alpha)(w)(\psi_{\infty i}(p)) \\
&= w \circ \alpha \circ \psi_{\infty i}(p) \\
&= w \circ \alpha_i(p) \\
&= \lim_{k \to \infty} w \circ \theta_{\infty k+1} \circ \zeta_k \circ \psi_{ki}(p) \\
&= \lim_{k \to \infty} w \circ \theta_{\infty k+1} \circ \zeta_k \circ \chi_k[\psi_{ki}(p)] \\
&= \lim_{k \to \infty} w \circ \theta_{\infty k+1} \circ \kappa_{k+1} \circ \delta_k[\psi_{ki}(p)] \\
&= f(w)\left(\lim_{k \to \infty} \delta_k[\psi_{ki}(p)]\right) \\
&= f(w)[\psi_{\infty i}(p)].
\end{aligned}$$

Hence (G, Δ, f) is the Elliott triple of B.

The final step of the proof consists of replacing B by a simple AI algebra with the same Elliott triple. Let ϕ_i be the $*$-homomorphism obtained from ψ_i by replacing two of the characteristic functions in each entry of ψ_i by h_0 and h_1 where $h_0(t) = \frac{t}{2}$ and $h_1(t) = \frac{t+1}{2}$ for $t \in [0,1]$ and all $i \in \mathsf{N}$. It follows from [2] that the C^*-algebra $A = \varinjlim (A_i, \phi_i)$ is simple. Note that $K_0(\phi_i) = K_0(\psi_i) = s_i$. Since

$$\|\mathrm{AffT}(\phi_i) - \mathrm{AffT}(\psi_i)\| < 2\mathrm{mult}(s_i)^{-1} \leqq 2^{1-i}$$

the sequence $(\mathrm{AffT}(\psi_{\infty k} \circ \phi_{ki})(x))_{k=i}^{\infty}$ is Cauchy for every $x \in \mathrm{AffT}(A_i)$, $i \in \mathsf{N}$ – let $\gamma_i(x)$ denote the limit. Then $\gamma_i : \mathrm{AffT}(A_i) \to \mathrm{AffT}(B)$ is a homomorphism with $\gamma_{i+1} \circ \mathrm{AffT}(\phi_i) = \gamma_i$ for all $i \in \mathsf{N}$. There is an isomorphism $\gamma : \mathrm{AffT}(A) \to \mathrm{AffT}(B)$ such that $\gamma \circ \mathrm{AffT}(\phi_{\infty i}) = \gamma_i$ for all $i \in \mathsf{N}$. Let $\tau \in T(B)$ and $p \in A_i$ be a projection.

Then

$$\begin{aligned} r_A \circ S(\gamma)(\tau)[\phi_{\infty i}(p)] &= S(\gamma)(\tau)(\phi_{\infty i}(p)) \\ &= \gamma \circ \mathrm{AffT}\,(\phi_{\infty i})(p)(\tau) \\ &= \gamma_i(p)(\tau) \\ &= \lim_{k\to\infty} \mathrm{AffT}\,(\psi_{\infty k} \circ \phi_{ki})(p)(\tau) \\ &= \lim_{k\to\infty} \tau(\psi_{\infty k} \circ \phi_{ki}(p)) \\ &= \tau(\psi_{\infty i}(p)) \\ &= r_B(\tau)[\psi_{\infty i}(p)]. \end{aligned}$$

We conclude that $r_A \circ S(\gamma) \circ S(\alpha) = f$.

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ISOMETRIES OF BANACH ALGEBRAS SATISFYING THE VON NEUMANN INEQUALITY

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Introduction.

A famous classical result of R. Kadison [K] says that every isometry of one C^*-algebra onto another is given by a Jordan isomorphism followed by a unitary multiplication. This result is generalized in the recent works [AS], [MT1] and [MT2] on isometries of certain non self-adjoint algebras of operators on Hilbert space. In the context of these works it is even possible to describe explicitly the Jordan isomorphisms. In this paper we generalize Kadison's theorem even further to the context of Banach algebras satisfying the von Neumann inequality.

In what follows Z denotes a complex, unital Banach algebra (i.e. Z has a unit e and $\|e\| = 1$) with an open unit ball D. The term "Banach algebra" will always mean "complex, unital Banach algebra". We say that the *von Neumann inequality* hold in Z if

$$\|f(z)\| \leqq \|f\|_\infty := \max\{|f(\lambda)|; |\lambda| = 1\}$$

for every $z \in D$ and every polynomial f. It is well known that the von Neumann inequality holds in $B(H)$, the algebra of all bounded operators on the Hilbert space H, see [N], [FSN, Chapter I.8]. Therefore, it holds in every subalgebra of a C^*-algebra. The following generalization of Kadison's theorem is our main result.

THEOREM. *Let Z be a Banach algebra satisfying the von Neumann inequality, and let φ be a surjective isometry of Z. Then there exists a Jordan automorphism ψ of Z and a unitary element u of Z so that $\varphi(z) = u\psi(z)$ for all $z \in Z$.*

The notions of Jordan automorphisms and unitary elements in general Banach algebras will be explained latter.

Our approach is similar to that of [AS] and uses the notion of the partial Jordan triple product and its connection to the holomorphic structure. We show

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that under the validity of the von Neumann inequality, the symmetric part Z_s of Z (in the sense of holomorphy) is the space $H + iH$ of decomposable elements (where H is the space of the Hermitean elements of Z), and that it is a C^*-algebra. Moreover, the abstract partial Jordan triple product (defined in terms of the holomorphy) coincides with the algebraic one: $\{x, y, z\} = (xy^{\#}z + zy^{\#}x)/2$ for all $x, z \in Z$ and $y \in Z_s$. A key result (Proposition 2.2) is the fact that the validity of the von Neumann inequality in Z is equivalent to the complete integrability of the vector field $h(z) = e - z^2$ in D. We also use the well-known fact that isometries preserve the partial Jordan triple product in general complex Banach spaces.

The paper is organized as follows. Section 1 below contains the background material on Hermitean elements of Banach algebras and on the partial Jordan triple product in Banach spaces and its connection to the holomorphic structure. Section 2 contains the connection between the von Neumann inequality and the vector field $h(z) = e - z^2$, as well as the proof that $Z_s = H + iH$. Section 3 contains the result on isometries generalizing Kadison's theorem mentioned above, as well as its Lie algebraic counterpart which describe the bounded Hermitean operators. Finally, in section 4 we collect some known facts and raise some questions on Banach algebras satisfying the von Neumann inequality.

1. Background.

Hermitean elements of a Banach algebra.

We begin with a quick review of some basic facts concerning Hermitean elements in Banach algebras which will be needed later. See [BD3], [BD1], [BD2] and [Do] for more details.

Let Z be a Banach algebra with a dual space Z^*. Let $S := \{\varphi \in Z^*; \varphi(e) = \|\varphi\| = 1\}$ be the set of *states*. The *numerical range* of $z \in Z$ is $V(z) := \{\varphi(z); \varphi \in S\}$. An element $z \in Z$ is *Hermitean* if $V(z) \subset \mathbb{R}$. It is known that z is Hermitean if and only if $\|\exp(itz)\| = 1$ for every $t \in \mathbb{R}$. Let H denote the set of all Hermitean elements in Z. H is a real-linear, closed subspace of Z containing e, with the property that $a, b \in H$ implies $i(ab - ba) \in H$. The subspace $H + iH$ of *decomposable* elements is also closed, and $H \cap iH = \{0\}$. The *involution* on $H + iH$ is defined by $(a + ib)^{\#} := a - ib$, $a, b \in H$. It is continuous, but need not be isometric. In general not much more can be said about H and $H + iH$; there are many examples where $H = \mathbb{R}e$ (this hold for instance in the disk algebra and in H^∞), and in general, H and $H + iH$ are not algebras. A fundamental result concerning $H + iH$ is the *Vidav-Palmer Theorem*, see [BD1, Chap. 2, Th. 9, p. 65], [BD3] and the references therein.

1.1 THEOREM. *Let Z be a Banach algebra so that $Z = H + iH$, namely every element in Z is decomposable. Then Z is a C^*-algebra with respect to the involution $\#$ and the given algebraic operations and norm.*

The partial Jordan triple product and the symmetric part of Banach space. We survey here the notions of the *partial Jordan triple product* and the *symmetric part* of a general complex Banach space. For more details see the survey article [A] or the original papers [Vi] and [U1]. The books [L] and [U2] are general references to Jordan triples and the associated bounded symmetric domains.

Let Z be a complex Banach space with an open unit ball D. $\mathrm{Aut}(D)$ denotes the real Banach Lie group of all biholomorphic automorphisms of D. Its Lie algebra, $\mathrm{aut}(D)$, is a real Banach Lie algebra, and it is identified with the completely integrable holomorphic vector fields on D. Namely, a holomorphic function $h := D \to Z$ belongs to $\mathrm{aut}(D)$ if and only if there exists a one-parameter subgroup $\{\varphi_t\}_{t\in \mathbb{R}}$ of $\mathrm{Aut}(D)$ so that $\frac{\partial}{\partial t}\varphi_t(z) = h(\varphi_t(z))$ for every $z \in D$ and $t \in \mathbb{R}$. One denotes $\varphi_t = \exp(th)$. The Lie brackets in $\mathrm{aut}(D)$ are given by $[h,k](z) = h'(z)k(z) - k'(z)h(z)$, $z \in D$. The *symmetric part* of Z is

$$Z_s := \mathrm{aut}(D)(0) = \{h(0); h \in \mathrm{aut}(D)\}.$$

It is known that Z_s is closed *complex* linear subspace of Z whose open unit ball is $D_s := \mathrm{Aut}(D)(0) = D \cap Z_s$, the *symmetric part of* D. $\mathrm{Aut}(D)$ admits a *Cartan decomposition* $\mathrm{aut}(D) = k \oplus p$, where

$$k = \mathrm{aut}(D) \cap B(Z) = \{h \in \mathrm{aut}(D); h(0) = 0\}$$

is the subspace of *skew-Hermitean* bounded operators on Z, and

$$p = \{h \in \mathrm{aut}(D); h'(0) = 0\}$$

is a subspace of even polynomials of degree $\leqq 2$. Precisely, $p = \{h_a; a \in Z_s\}$, where $h_a(z) := a - q_a(z)$, and q_a is a continuous, homogeneous polynomial of degree 2. q_a extends to a continuous, symmetric bilinear form by polarization: $q_a(z,w) = (q_a(z+w) - q_a(z) - q_a(w))/2$. The partial *Jordan triple product* is the map $\{\ldots\}: Z \times Z_s \times Z \to Z$, defined by $\{z,a,w\} := q_a(z,w)$. Z is a JB*-triple if $Z_s = Z$, i.e. the Jordan triple product is defined everywhere. This is the case precisely when $D_s = D$, i.e. when $\mathrm{Aut}(D)$ acts transitively on D, and D is a *bounded symmetric domain.* For example, every C^*-algebra is a JB*-triple with respect to the triple product $\{z,a,w\} := (za^*w + wa^*z)/2$.

Let $s_0(z) := -z$ be the symmetry at 0. The corresponding *Cartan involution* on $\mathrm{aut}(D)$, $\theta(h) := s_0 h s_0$, is a Lie-automorphism and $\vartheta^2 = I$. It is easy to check that $k = \{h \in \mathrm{aut}(D); \vartheta(h) = h\}$ and $p = \{h \in \mathrm{aut}(D); \vartheta(h) = -h\}$. From these facts it follows that $[p,p] \subset k$, $[k,p] \subset p$, and $[k,k] \subset k$. Explicitly, for $a, b \in Z_s$ and $u \in k$,

$$[h_a, h_b](z) = 2\{a,b,z\} - 2\{b,a,z\}, [u,h_a] = h_{u(a)}.$$

In particular, for any $a \in Z_s$ the operator $D(a,a) := \{a, a, \cdot\}$ is Hermitean. Notice also that the previous formula implies that $D(a,a)b = \{a,a,b\} \in Z_s$. It follows that *$Z_s$ is closed under the triple product and it is therefore a* JB*-*triple.* Basic facts in JB*-triples yield that *for every $a \in Z_s$, the restriction of $D(a,a)$ to Z_s has a positive spectrum* and the following identity (called the "*C*-axiom*") holds: $\|D(a,a)\|_{B(Z_s)} = \|a\|^2$.

The group of *linear isometries* of Z is identified naturally with

$$K := \{\varphi \in \mathrm{Aut}(D); \varphi(0) = 0\} = \mathrm{Aut}(D) \cap \mathrm{GL}(Z).$$

The Lie algebra of K is clearly k. The Isometries are automorphisms and the skew-Hermitean operators are derivations of the partial Jordan triple product. Precisely,

1.2. PROPOSITION. (i) *Let Z_1, Z_2 be complex Banach spaces with open unit balls D_1, D_2 respectively, and let φ be an isometry of Z_1 onto Z_2. Let $\mathrm{aut}(D_j) = k_j \oplus p_j$, $(j = 1,2)$, be the Cartan decompositions. Then $\varphi k_1 \varphi^{-1} = k_2$, $\varphi p_1 \varphi^{-1} = p_2$, and $\varphi h_a \varphi^{-1} = h_{\varphi(a)}$ for every $a \in (Z_1)_s$. In particular $\varphi((Z_1)_s) = (Z_2)_s$, and*

$$\varphi\{z,a,w\} = \{\varphi(z), \varphi(a), \varphi(w)\}, \quad a \in (Z_1)_s, \quad z, w \in Z_1.$$

(ii) *Let Z be a complex Banach space with an open unit ball D and let $h \in k$. Then $h(Z_s) \subseteq Z_s$, and for every $a \in Z_s$ and $z, w \in Z$,*

$$h\{z,a,w\} = \{h(z),a,w\} + \{z,h(a),w\} + \{z,a,h(w)\}.$$

PROOF. (i) The fact that $\varphi(\mathrm{aut}(D_1))\varphi^{-1} = \mathrm{aut}(D_2)$ is obvious. $\varphi k_1 \varphi^{-1} = k_2$ follows from this and from $\varphi(0) = 0$. Let $a \in (Z_1)_s$, and let $h := \varphi h_a \varphi^{-1}$. Then $h \in \mathrm{aut}(D_2)$, $h'(0) = 0$ and $h(0) = \varphi(a)$. Thus, $\varphi(a) \in (Z_2)_s$ and $h = h_{\varphi(a)} \in p_2$. This completes the proof of the first three identities as well as the inclusion $\varphi((Z_1)_s) \subseteq (Z_2)_s$. The reverse inclusion follows by using φ^{-1} instead of φ; this yields the fourth identity. Applying both sides of the third identity on the element $\varphi(z)$, we get $\varphi\{z,a,z\} = \{\varphi(z), \varphi(a), \varphi(z)\}$. By polarization, φ preserves the partial triple product.

(ii) Let $\varphi_t = \exp(th)$, $t \in \mathrm{R}$. Then $\varphi_t \in K$ and by (i) $\varphi_t((Z_1)_s) = (Z_2)_s$. Differentiating with respect to t at 0, we get $h((Z_1)_s) \subseteq (Z_2)_s$. Let $a \in (Z_1)_s$, and let $z, w \in Z_1$. Then by the last formula in (i) $\varphi_t\{z,a,w\} = \{\varphi_t(z), \varphi_t(a), \varphi_t(w)\}$. Differentiating this at $t = 0$, we get the desired result.

Linear operators h on Z which satisfy the identity in (ii) are called derivations of the partial Jordan triple product, or *triple derivations* for short. Notice that Proposition 1.2 implies that for any $a \in Z_s$ the operator $iD(a,a)$ is a triple derivation.

The completely integrable holomorphic vector fields are characterized by tangency to the unit sphere.

1.3. PROPOSITION. *Let Z be a complex Banach space with an open unit ball D. Let $h\colon Z \to Z$ be a holomorphic polynomial. Then the following conditions are equivalent:*

(i) $h_{|D} \in \operatorname{aut}(D)$;
(ii) *h is tangent to the unit sphere ∂D. Namely, if $z \in Z$, $f \in Z^*$ are so that* $\|z\| = \|f\| = f(z) = 1$ *then* $\operatorname{Re}(f(h(z))) = 0$.

See [Ka] and [U2, Lemma 4.4] for a proof.

2. The symmetric part of Banach algebras which satisfy the von Neumann inequality.

Let Z be a Banach algebra with a unit e, an open unit ball D and a symmetric part $Z_s = \operatorname{aut}(D)(0)$. Let H denote the space of all Hermitean elements in Z.

2.1. PROPOSITION. (i) $Z_s \subseteq H + iH$. (ii) *If $a \in Z_s$, then* $a^\# = \{e, a, e\}$.

PROOF. Let $a \in Z_s$, and let $h_a := a - q_a \in p \subset \operatorname{aut}(D)$ be the corresponding vector field on D. Set $b := q_a(e) = \{e, a, e\}$. By Proposition 1.3 we have for every state $\varphi \in S, 0 = \operatorname{Re}(\varphi(h_a(e)) = \operatorname{Re}(\varphi(a - b))$. It follows that $a - b \in iH$. Since Z_s is C-linear, we get $ia \in Z_s$ and $q_{ia}(e) = ib$. Thus by the same arguments, $ia - q_{ia}(e) = i(a + b) \in iH$. Set $a_1 := (a + b)/2$ and $a_2 := (a - b)/2i$. Then $a_1, a_2 \in H$ and $a = a_1 + ia_2 \in H + iH$. Also, $b = a^\# = a_1 - ia_2$, and so the restriction of the involution of $H + iH$ to Z_s is given by $a^\# = \{e, a, e\}$.

2.2. PROPOSITION. (i) *The integral curves of the vector field $h(z) = h_e(z) := e - z^2$ on D are given by*

$$\varphi_t(z) = \exp(th)(z) := (r(t) + z)(e + r(t)z)^{-1}, \quad t \in \mathrm{R}, \quad r(t) := \tanh(t);$$

(ii) *h is completely integrable in D if and only if Z satisfies the von Neumann inequality.*

PROOF. For any $z \in Z$ let J_z be the maximal open interval containing 0 in which the initial value problem: $\partial\varphi_t(z)/\partial t = h(\varphi_t(z))$, $t \in J_z$; $\varphi_0(z) = z$; has a solution $\varphi_t(z) \in D$. Let I_z be the maximal open interval containing 0 of those $t \in \mathrm{R}$ for which $(e + r(t)z)^{-1}$ exists and $\sigma_t := (r(t) + z)(e + r(t)z)^{-1}$ belongs to D. The meaning of (i) is that $J_z = I_z$ and $\varphi_t(z) = \sigma_t(z)$, $t \in J_z$. Indeed, since $(\partial/\partial t)r(t) = 1/\cosh^2(t) = 1 - r(t)^2$, and $(\partial/\partial r)(e + rz)^{-1} = -(e + rz)^{-2}z$, we get by a direct differentiation $(\partial/\partial t)\sigma_t(z) = e - \sigma_t(z)^2$. This establishes (i). To prove (ii), we observe first that the completeness of h is equivalent to the fact that the Mobius transformations

$$\psi_r(z) := (r + z)(e + rz)^{-1}, \quad -1 < r < 1, \quad z \in D,$$

map D into itself. Let $\mathsf{D} := \{\lambda \in \mathsf{C}; |\lambda| < 1\}$. Since D is circular and Aut(D) is generated by the ψ_r, $-1 < r < 1$, and the rotations $\rho_t(z) := e^{it}z$, we get that every $\psi \in$ Aut(D) maps D into itself. The submultiplicativity of the norm of Z implies that D is a multiplicative semigroup. Thus, the finite products of members of Aut(D) (namely, the finite Blaschke products) map D into itself. By [F], the convex combinations of finite Blaschke products are norm-dense in the closed unit ball of the disk algebra A. It follows that for every $f \in A$ with $\|f\|_\infty \leqq 1$ and $z \in D$ we have $\|f(z)\| \leqq 1$. This is equivalent to the validity of the von Neumann inequality in Z. Conversely, if the von Neumann inequality holds in Z then ψ_r maps D into itself for all $-1 < r < 1$. Since $\psi_r = \exp(th)$, $r = \tanh(t)$, we see that h is completely integrable in D.

2.3. Remark. Proposition 2.2 yields the von Neumann inequality in C^*-algebras. Indeed, it is well known that if Z is a C^*-algebra, then for every $a \in D$ the Potapov-Mobius transformation

$$\Phi_a(z) := (e - aa^*)^{-1/2}(a + z)(e + a^*z)^{-1}(e - a^*a)^{1/2}$$

belongs to Aut(D), see [IS]. Applying this with $a = re$, $-1 < r < 1$, we see that $\psi_r \in$ Aut(D) becomes a member of Aut(D) in the natural way. By the proof of Proposition 2.2, this implies the validity of the von Neumann inequality in Z. The original proof of the von Neumann inequality (see [N] and [RSN, Section 135]) uses also similar analytic tools. Another proof is given in [FSN], and is based on much heavier tools (unitary dilations of contractions and the spectral theorem for unitary operators).

2.4. Proposition. (i) *If* $a \in H + iH$ *and* $b \in Z_s$ *then* $ab, ba \in Z_s$. *In particular,* Z_s *is an algebra,* $H + iH$ *is a module over* Z_s, *and* $(H + iH)Z_s = Z_s(H + iH) = Z_s$.
(ii) $H + iH = Z_s$ *if and only if* $e \in Z_s$.

Proof. For $a \in Z$ consider the multiplication operators $L_a(z) := az$ and $R_a(z) := za$. Then

$$\exp(itL_a) = L_{\exp(ita)} \quad \text{and} \quad \exp(itR_a) = R_{\exp(ita)}, \quad t \in \mathsf{R}.$$

Since $\|L_z\| = \|R_z\| = \|z\|$, it follows that $a \in H \Leftrightarrow L_a \in H(B(Z)) \Leftrightarrow R_a \in H(B(Z))$. Here $H(B(Z))$ denotes the space of Hermitean elements of the Banach algebra $B(Z)$ of all bounded operators on Z. Let $a \in H$ and $b \in Z_s$. Then, by Proposition 1.3, $\exp(itL_a)(b) = \exp(ita)\cdot b$, and $\exp(itR_a)(b) = b \cdot \exp(ita)$ belong to Z_s for all $t \in \mathsf{R}$. Differentiating with respect to t at 0, we get $iab, iba \in Z_s$. Since Z_s is C-linear, this implies (i). Next, if $H + iH = Z_s$ then $e \in H \subseteq iH = Z_s$. Conversely, assume that $e \in Z_s$ and let $a \in H$. By (i) with $b = e$, we get $a \in Z_s$. Thus $H \subseteq Z_s$. Since Z_s is C-linear, this implies $H + iH \subseteq Z_s$. Using Proposition 2.1 we get $H + iH = Z_s$.

2.5. REMARK. If $e \in Z_s$, and in particular – if the von Neumann inequality holds in Z (Proposition 2.2), then $Z_s = H + iH$ is a Banach algebra (Proposition 2.3) in which every element is decomposable. By Theorem 1.1, Z_s is a C^*-algebra with respect to the involution $\#$ and the given algebraic operations and norm. We do not know whether $e \in Z_s$ by itself implies the von Neumann inequality in Z.

The following theorem generalizes the result discussed in Remark 2.5. We prefer to avoid Theorem 1.1 and to give an almost self contained proof, in order to illustrate the power of the Jordan theoretic techniques.

2.6. THEOREM. *Let Z be a Banach algebra with a unit e, satisfying the von Neumann inequality. Then $Z_s = H + iH$ is a C^*-algebra with respect to the given multiplication, norm, and the involution $(a_1 + ia_2)^\# = a_1 - ia_2, a_1, a_2 \in H$. Moreover, the partial Jordan triple product $\{\ldots\}: Z \times Z_s \times Z \to Z$ constructed via the holomorphy coincides with the algebraic partial triple product, namely:*

$$\{x, y, z\} = (xy^\# z + zy^\# x)/2, \quad x, z \in Z, \quad y \in Z_s.$$

PROOF. By Proposition 2.4, $Z_s = H + iH$ is a closed subalgebra of Z with a unit e and involution $\#$ (which at the moment is known only to be an anti-linear homeomorphism of Z_s). By Proposition 2.2 we have $\{z, e, z\} = z^2$ for all $z \in Z$. Polarizing, we get $\{z, e, w\} = (zw + wz)/2, z, w \in Z$. Let $a \in H$, then as in the proof of Proposition 2.3, $\exp(itL_a) = L_{\exp(ita)}$, $t \in \mathrm{R}$, belong to the group $K = \mathrm{Aut}(D) \cap \mathrm{GL}(Z)$ of linear isometries of Z. By Proposition 1.2 the members of K are automorphisms of the partial triple product. Hence, for any $z \in Z$ and $t \in \mathrm{R}$,

$$\begin{aligned}\exp(ita)z^2 &= \exp(itL_a)\{z, e, z\} \\ &= \{\exp(itL_a)(z), exp(itL_a)(e), \exp(itL_a)(z)\} \\ &= \{\exp(ita)z, \exp(ita), \exp(ita)z\}.\end{aligned}$$

Differentiating with respect to t at 0, we get

$$az^2 = 2\{az, e, z\} - \{z, a, z\} = (az)z + z(az) - \{z, a, z\} = az^2 + zaz - \{z, a, z\}.$$

Thus, $\{z, a, z\} = zaz$. It follows that for $a = a_1 + ia_2 \in Z_s$ with $a_1, a_2 \in H$ and $z \in Z$,

$$\{z, a, z\} = \{z, a_1, z\} - i\{z, a_2, z\} = za_1z - iza_2z = za^\# z.$$

Polarizing, we get $\{z, a, w\} = (za^\# w + wa^\# z)/2$ for every $z, w \in Z$ and $a \in Z_s$.

Next, we show that $(ab)^\# = b^\# a^\#$ for all $a, b \in Z_s$. Since $Z_s = H + iH$, it is certainly enough to show that $(ab)^\# = ba$ for $a, b \in H$. By the above arguments we have for all $t \in \mathrm{R}$

$$\exp(ita)b = \{\exp(ita), \exp(ita)b, \exp(ita)\}.$$

Differentiating with respect to t at 0, we get $iab = i(ab + ba) - i\{e, ab, e\}$. However, by Proposition 2.1, $\{e, z, e\} = z^{\#}$ for all $z \in Z_s$. Applying this with $z = ab$ (and using the fact that Z_s is an algebra), we get $(ab)^{\#} = ba$ as desired.

The C^*-axiom $\|a^{\#}a\| = \|a\|^2$ in Z_s, follows from the C^*-axiom for JB*-triples, see Section 1 above. Indeed, in our case $D(a, a)z = (aa^{\#}z + za^{\#}a)/2, z \in Z$. Hence,

$$\|a\|^2 = \|D(a, a)\|_{B(Z_s)} \leqq (\|a^{\#}a\| + \|aa^{\#}\|)/2 \leqq \|a\| \, \|a^{\#}\|,$$

and thus, $\|a\| \leqq \|a^{\#}\|$ for all $a \in Z_s$. Replacing a by $a^{\#}$ we get $\|a\| = \|a^{\#}\|, a \in Z_s$. It follows that equality holds in the above inequality, and so $\|a^{\#}a\| = \|aa^{\#}\| = \|a\|^2, a \in Z_s$. This completes the proof.

2.7. Remark. We have a direct argument yielding $\|z^{\#}\| = \|z\|$ for every $z \in Z_s$. Indeed, let $\alpha_n \in \mathrm{R}$ be so that $\tanh(t) = \sum_{n=0}^{\infty} \alpha_n t^{2n+1}$, $t \in \mathrm{R}$, and let $a \in Z_s$. It is easy to check (from the initial value problem defining $\exp(th_a)$) that $\exp(th_a)(0) = \tanh(ta) := \sum_{n=0}^{\infty} \alpha_n t^{2n+1} a(a^{\#}a)^n$, $t \in \mathrm{R}$. It follows from the anti multiplicativity of $\#$ that $(\exp(th_a)(0))^{\#} = \exp(th_{a^{\#}}) = \tanh(ta^{\#}) \in D_s$. Since $D_s = \mathrm{Aut}(D)(0)$ and $K(0) = \{0\}$, we get $D_s = \exp(p)(0)$, where $\exp(p)$ is the subgroup of $\mathrm{Aut}(D)$ generated by $\{\exp(h_a); a \in Z_s\}$. It follows that $(D_s)^{\#} = D_s$, and this is equivalent to $\|z^{\#}\| = \|z\|$ for all $z \in Z_s$.

2.8. Problem. Does $e \in Z_s$ imply the von Neumann inequality in Z? In particular, does $e \in Z_s$ imply that $h_e(z) = e - z^2$, i.e. that $\{z, e, z\} = z^2$ for all $z \in Z$? Notice that, by the proof of Theorem 2.6, the last identity is equivalent to the identity $\{z, a, z\} = za^{\#}z$ for all $z \in Z$ and $a \in Z_s$.

3. The isometries and the Hermitean operators.

3.1. Theorem. *Let Z, W be unital Banach algebras which satisfy the von-Neumann inequality, and let $\varphi: Z \to W$ be a surjective isometry. Then,*

(i) $u := \varphi(e)$ is a unitary element of W_s;

(ii) $\varphi(z) = u\psi(z), z \in Z$, where ψ is an isometric Jordan isomorphism of Z onto W. Namely, ψ is an isometry satisfying $\psi(e) = e$ and

$$\psi(ab + ba) = \psi(a)\psi(b) + \psi(b)\psi(a), \text{ for } a, b \in Z;$$

$$\psi(Z_s) = W_s \text{ and } \psi(z)^{\#} = \psi(z^{\#}) \text{ for } z \in Z_s.$$

3.2. Remarks. (i) Theorem 3.1 reduces the study of a geometrical problem (the description of the isometries of Banach algebras satisfying the von Neumann inequality) to that of an algebraic one (namely, the description of the Jordan isomorphisms of these algebras). It is our generalization of Kadison's theorem discussed in the introduction.

(ii) The partial Jordan triple product is expressed in terms of the (binary) Jordan product $x \circ y := (xy + yx)/2$ and the involution $\#$ on Z_s:

$$\{x, y, z\} = x \circ (y^{\#} \circ z) + z \circ (y^{\#} \circ x) - (x \circ z) \circ y^{\#}, \quad x, z \in Z, \quad y \in Z_s.$$

It follows that a linear map which preserves the Jordan product and the partial involution preserves also the partial Jordan triple product.

(iii) We do not know whether the converse of Theorem 3.1 is true, namely whether a Jordan isomorphism of Banach algebras satisfying the von Neumann inequality must be an isometry. This is true in C^*-algebras. More generally, an automorphism of a JB*-triple must be an isometry.

We need the characterization of tripotents as partial isometries.

3.3. PROPOSITION. *Let Z be as Theorem 3.1 and let $u \in Z_s$.*

(i) *u is a tripotent (namely, $\{u, u, u\} = u$) if and only if it is a partial isometry (namely, $u^{\#} u$ is an idempotent);*

(ii) *u is a unitary tripotent (namely, $\{u, u, z\} = z$ for every $z \in Z$) if and only if $u^{\#} u = e = uu^{\#}$, i.e. u is a unitary element of the C^*-algebra Z_s.*

PROOF. Part (i) is well known (see, for instance, [H]), If u is a unitary element of Z_s, then for every $z \in Z$, $2\{u, u, z\} = uu^{\#}z + zu^{\#}u = 2z$. On the other hand, if $\{u, u, z\} = z$ for every $z \in Z$, then $e = \{u, u, e\} = (u^{\#}u + uu^{\#})/2$ and u is a partial isometry. Since e is an extreme point of the unit ball of Z_s (this holds in any unital Banach algebra, see [BD1, Chap. 1, Th. 5, p. 38]), we get $u^{\#}u = uu^{\#} = e$.

PROOF OF THEOREM 3.1. By Proposition 1.2, $\varphi(Z_s) = W_s$ and φ preserves the partial triple product. Using Theorem 2.6 we get

$$\varphi(xy^{\#}z + zy^{\#}x) = \varphi(x)\varphi(y)^{\#}\varphi(z) + \varphi(z)\varphi(y)^{\#}\varphi(x)$$

for all $x, z \in Z$ and $y \in Z_s$. Set $u = \varphi(e)$ and notice that for every $z \in Z$, $\varphi(z) = \varphi(\{e, e, z\}) = \{\varphi(e), \varphi(e), \varphi(z)\} = \{u, u, \varphi(z)\}$. It follows from Proposition 3.3 that u is a unitary element of W_s. Let $\psi(z) := u^{\#}\varphi(z)$. Then ψ is an isometry of Z onto W and $\psi(e) = e$. By proposition 1.2, ψ preserves the partial triple product. Hence, for every $z \in Z_s$

$$\psi(z^{\#}) = \psi(\{e, z, e\}) = \{\psi(e), \psi(z), \psi(e)\} = \{e, \psi(z), e\} = \psi(z)^{\#}.$$

Thus ψ is self adjoint. Finally, for every $z \in Z$,

$$\psi(z^2) = \psi(\{z, e, z\}) = \{\psi(z), \psi(e), \psi(z)\} = \psi(z)^2$$

Polarizing this identity we see that ψ preserves the Jordan product. This completes the proof since $\varphi(z) = u\psi(z)$.

The next result deals with Hermitean operators on Z and it is the Lie-algebraic analog of Theorem 3.1. By Proposition 1.2 we know that every Hermitean operator $T\colon Z \to Z$ satisfies $T(Z_s) \subseteq Z_s$. Moreover, iT is a derivation of the partial triple product, namely the identity in Proposition 1.2 (ii) holds with

$h = iT$. An operator $A: Z \to Z$ is a *derivation of the (binary) Jordan product* $x \circ y := (xy + yx)/2$ if $A(x \circ y) = Ax \circ y + x \circ Ay$ for all $x, y \in Z$. It is obvious that in this case $A(e) = 0$. An operator $T: Z \to Z$ is *self-adjoint* if $T(H) \subseteq H$. This is equivalent to $T(H + iH) \subseteq H + iH$ and $(T(z))^{\#} = (T(z^{\#})$ for every $z \in H + iH$. T is *skew-adjoint* if iT is self-adjoint, and this is equivalent to $T(H + iH) \subseteq H + iH$ and $(T(z))^{\#} = -T(z^{\#})$ for every $z \in Z_s$.

3.4. PROPOSITION. *Let Z be a unital Banach algebra satisfying the von Neumann inequality. Then every Hermitean operator $T: Z \to Z$ has a unique decomposition $T = A + L_a$, where $a := T(e) \in H$ and A is a skew-adjoint derivation of the Jordan product.*

PROOF. Set $a := T(e)$. Then $a \in Z_s = H + iH$ and

$$a = T(e) = T\{e, e, e\} = 2\{a, e, e\} - \{e, a, e\} = 2a - a^{\#}.$$

Thus $a = a^{\#}$, and so $a \in H$. Set $A := T - L_a$. Then A is Hermitean and $A(e) = 0$. Since $Z_s = H + iH$, we get $A(H + iH) \subseteq H + iH$ and for $z \in Z_s$,

$$A(z^{\#}) = A\{e, z, e\} = 2\{A(e), z, e\} - \{e, Az, e\} = -(Az)^{\#}.$$

Thus A is skew adjoint. Next, for any $z \in Z$

$$A(z^2) = A\{z, e, z\} = 2\{Az, e, z\} - \{z, A(e), z\} = (Az)z + z(Az).$$

Polarizing, we get $A(z \circ w) = (Az) \circ w + z \circ (\mathrm{Aw})$ for every $z, w \in Z$. Thus A is also a derivation of the Jordan product. The uniqueness of the decomposition $T = A + L_a$ is obvious.

3.5. REMARKS. (i) Proposition 3.4 reduces a geometrical problem (the description of the Hermitean operators) to an algebraic one (the description of the skew-adjoint derivations of the Jordan product).

(ii) In the situation described in Proposition 3.3, T admits also the decomposition $T = B + R_a$, where $a = T(e) \in H$ and B is a skew-adjoint derivation of the Jordan product. Of course, $B = A + L_a - R_a$.

(iii) Let $T: Z \to Z$ be of one of the forms $T = A + L_a$ or $T = B + R_a$, where $a \in H$ and A, B are skew-adjoint derivations of the (binary) Jordan product. Then iT is a derivation of the partial Jordan triple product. This follows from Remark 3.2 (ii).

(iv) We do not know whether a bounded operator $T: Z \to Z$ for which iT is a derivation of the partial Jordan triple product must be Hermitean. It is easy to see that in this case $\exp(i\lambda T)$ is an automorphism of the triple product for all $\lambda \in \mathbb{R}$. But we do not know whether $\exp(i\lambda T)$ is in fact an isometry. See Remark 3.2 (iii).

4. Banach algebras which satisfy the von Neumann inequality.

The results obtained above lead naturally to the problem of the description of the class of Banach algebras satisfying the von Neumann inequality. This problem is interesting from many point of views, but it is difficult and far from being solved. We collect bellow some known facts relevant to this problem, and raise some questions.

The first point we would like to make is that *the von Neumann inequality is an inequality in the category of Jordan Banach algebras.* A *Jordan algebra* is a commutative algebra with a product $x \circ y$ (called a Jordan product), in which the associative law is replaced by the weaker law (the *Jordan algebra identity*): $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$, where $x^2 := x \circ x$.

It is easy to see that for every element x the powers x^n (which are defined inductively via $x^n = x \circ x^{n-1}$) satisfy $x^n \circ x^m = x^k \circ x^l$ whenever $n + m = k + l$. Thus the subalgebra generated by x is associative. A *Jordan Banach algebra* is a Jordan algebra which is also a Banach space, so that $\|x \circ y\| \leqq \|x\| \, \|y\|$ for all x and y. We will assume that the field of scalars is $\mathbf{C}$. Clearly, it is possible to consider the von Neumann inequality in the category of Jordan Banach algebras, and it is interesting to characterize those Jordan Banach algebras in which this inequality holds. Notice that the von Neumann inequality has a local nature, namely it is a statement on all the singly generated subalgebras of Z.

For any Banach algebra Z, let $Z^J := \langle Z, +, \circ, \|\cdot\| \rangle$ denote the associated Jordan Banach algebra with the Jordan product $x \circ y := (xy + yx)/2$ and the same addition and norm. Notice that for every $z \in Z$, the powers z^n are the same in Z and in Z^J, thus the singly generated subalgebras are the same for Z and Z^J. Therefore we get,

4.1. Corollary. *The von Neumann inequality holds in Z if and only if it holds in Z^J.*

We remark that not all Jordan Banch algebras hae the form Z^J for some Banach algebra Z.

A JB*-*algebra* is a unital, complex Jordan Banach algebra Z with an involution $z \mapsto z^*$ (i.e. an anti-linear, multiplicative map of period 2), so that $\|\{z, z^*, z\}\| = \|z\|^3$ for every $z \in Z$, where the triple product is defined via the binary product and the involution by

$$\{x, y, z\} := x \circ (y^* \circ z) + z \circ (y^* \circ x) - y^* \circ (x \circ y).$$

In this case, $X := \{x \in Z;\, x^* = x\}$ is a real JB-*algebra* (i.e. a real Banach Jordan algebra satisfying $\|x\|^2 \leqq \|x^2 + y^2\|$ for all x and y) and $Z = X \otimes \mathbf{C}$. Conversely, the complexification of every real, unital JB-algebra has a unique norm with respect to which it is a JB*-algebra. An equivalent description is that a JB*-algebra is a JB*-triple having a unitary tripotent e, with the binary product $z \circ w := \{z, e, w\}$ and the involution $z^* := \{e, z, e\}$. Clearly, every C^*-algebra is a JB*-algebra.

4.2. PROPOSITION. *The von Neumann inequality holds in any* JB*-*algebra Z.*

Indeed, the vector field $h_e(z) := e - \{z, e, z\} = e - z^2$ is completely integrable since Z is a JB*-triple. The rest follows from Proposition 2.2 and the fact that the singly generated subalgebras of Z are associative.

It is obvious that if the von Neumann inequality holds in the Jordan Banach algebra Z then it holds in any subalgebra. The same is true for quotient algebras, but this require some explanation. A *quotient map* in the category of Banach spaces is a linear operator $T: X \to Y$ which maps the open unit ball of X onto the open unit bal of Y. A *multiplicative quotient map* in the category of Banach algebras (or, Jordan Banach algebras) is a quotient map which preserves the product (respectively, the Jordan product). Notice that a multiplicative quotient map must preserve the unit element. Also, a multiplicative quotient map $Q: Z \to W$ in the category of Banach algebras is also a multiplicative quotient map $Q: Z^J \to W^J$ in the category of Jordan Banach algebras. The converse is false.

4.3. PROPOSITION. *Assume that the von Neumann inequality holds in the Jordan Banach algebra Z and let Q be a multiplicative quotient map of Z onto a* JB-*algebra W. Then the von Neumann inequality holds in W. The same is true in the category of Banach algebras.*

PROOF. The first statement implies the second via Proposition 4.1. To prove the first, let f be a polynomial and let $w \in W$, $\|w\| < 1$. Let $z \in Z$ be so that $Qz = w$ and $\|z\| < 1$. Since $Q(z^n) = (Q(z))^n$ for all $n \geqq 0$, we get $Q(f(z)) = f(Q(z))$. Thus,

$$\|f(w)\| = \|f(Qz)\| = \|Q(f(z))\| \leqq \|f(z)\| \leqq \|f\|_\infty.$$

4.4. COROLLARY. *The von Neumann inequality holds in every subalgebra and in very quotient algebra of a* JB*-*algebra.*

There is an extensive literature on the generalizations of von Neumann inequality in the context of *commutative* Banach algebras. These generalizations deal mainly with the multi-variable (isometric and isomorphic) analogs of the von Neumann inequality. See, for instance, [Da], [MT], [DD] and the references therein. Nevertheless, to our knowledge, there is no complete characterization of the commutative Banach algebras which satisfy the von Neumann inequality. However, there is a characterization, due to I. G. Craw (see [Da, Lemma 3.1]), of the commutative Banach algebras satisfying a multi variable generalized version of the von Neumann inequality.

4.5. THEOREM. *Let Z be a commutative Banach algebra. Then*

$$\|f(z_1, z_2, \ldots, z_n)\| \leqq \|f\|_\infty := \sup\{|f(\zeta_1, \zeta_2, \ldots, \zeta_n)|;\ \zeta_j \in \mathbf{C}, |\zeta_j| = 1\}$$

for every finite sequence $z_1, z_2, \ldots, z_n \in Z$ with $\|z_j\| \leqq 1$ and every polynomial $f(\zeta_1, \zeta_2, \ldots, \zeta_n)$, if and only if Z is isometrically isomorphic to a quotient of a uniform algebra.

We would like to mention also the result of T. Ando, saying that if T_1, T_2 are commuting contractions on Hilbert space and $f(\zeta_1, \zeta_2)$ is a polynomial, then $\|f(T_1, T_2)\| \leqq \|f\|_\infty$. N. Th. Varopoulos [Va1], [Va2] showed that this result cannot be extended to three or more commuting contractions.

Let $\mathscr{P}$ be the algebra of all polynomials in the non-commuting variables $x_1, x_2, \ldots, x_n, \ldots$. For every Banach algebra Z consider the norm

$$\|f\|_{\mathscr{P}Z} := \sup\{\|f(a_1, a_2, \ldots, a_n, \ldots)\|;\ a_j \in Z,\ \|a_j\| \leqq 1, j = 1, 2, \ldots\}$$

on $\mathscr{P}$. Notice that $\|\cdot\|_{\mathscr{P}Z}$ is submultiplicative.

4.6. DEFINITION. ([Di1]). A class $\mathscr{C}$ of Banach algebras is called a *variety* if there exists a submultiplicative norm $\|\cdot\|_{\mathscr{C}}$ on $\mathscr{P}$ so that $\mathscr{C}$ is the class of all Banach algebras Z for which $\|f\|_{\mathscr{P}Z} \leqq \|f\|_{\mathscr{C}}$ for every $f \in \mathscr{P}$, namely the inclusion map $\langle \mathscr{P}, \|\cdot\|_{\mathscr{C}} \rangle \to \langle \mathscr{P}, \|\cdot\|_{\mathscr{P}Z} \rangle$ is contractive for all $Z \in \mathscr{C}$.

4.7. THEOREM ([Di1]). *A class $\mathscr{C}$ of Banach algebras is a variety if and only if it is closed under taking closed subalgebras, quotient algebras, l_∞-direct products and isometric isomorphisms.*

4.8. COROLLARY. *The class $\mathscr{VN}$ of all Banach algebras satisfying the von Neumann inequality is a variety.*

Indeed, it is easy to see that $\mathscr{VN}$ is closed under l_∞-direct products. Thus, Corollary 4.8 follows from Proposition 4.3 and Theorem 4.7.

4.9. PROBLEM. Compute the norm $\|f\|_{\mathscr{VN}}$ on $\mathscr{P}$.

It is plain that $\|f\|_\infty := \|f\|_{\mathscr{P}\mathbf{C}} \leqq \|f\|_{\mathscr{VN}}$ for every $f \in \mathscr{P}$, but the two norms are inequivalent. Notice that it f depends on one variable then the above inequality becomes an equality.

The following characterization of subalgebras of $B(H)$ is due to Bernard (see [Be], [Di2]).

4.10. THEOREM. *A Banach algebra Z is isometrically isomorphic to a subalgebra of $B(H)$ for some Hilbert space H if and only if $\|f\|_{\mathscr{P}Z} \leqq \|f\|_{\mathscr{P}B(l_2)}$ for every $f \in \mathscr{P}$, where l_2 is the separable, infinite dimensional Hilbert space.*

Thus, the Banach subalgebras of C^*-algebras form a variety, which is clearly contained in $\mathscr{VN}$, and so $\|f\|_{\mathscr{P}B(l_2)} \geqq \|f\|_{\mathscr{VN}}$ for every $f \in P$.

Algebras of the form $B(X)$, the bounded operators on a Banach space X, do not satisfy the von Neumann inequality in general:

4.11. THEOREM. ([Fo]). *Let X be a complex Banach space. Then $B(X)$ satisfies the von Neumann inequality if and only if X is isometric to a Hilbert space.*

Let D be the open unit ball of the Banach algebra Z. We denote by $A(D, Z)$ the Banach algebra of all bounded continuous functions on the closure of D which are analytic in D, with the pointwise multiplication and norm $\|f\|_{A(D,Z)} := \sup\{\|f(z)\|;\ z \in D\}$. Let A be the disk algebra, namely $A = A(\mathrm{D}, \mathrm{C})$. For every $f \in A$ let $Jf \in \mathscr{H}(D, Z)$ (= the space of all holomorphic functions from D into Z) be defined by the Cauchy integral

$$Jf(z) = f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta)(\zeta - z)^{-1} d\zeta.$$

In general, Jf need not be bounded on D.

4.12. PROPOSITION. *The following conditions are equivalent:*

(i) *Z satisfies the von Neumann inequality;*

(ii) *J maps A isometrically into $A(D, Z)$.*

(iii) *J maps A into $A(D, Z)$, and J: $A \to A(D, Z)$ is a contraction.*

PROOF. By definition, (i) $\leftrightarrow$ (iii). Let R: $A(D, Z) \to A$ be the restriction map, defined by $(Rf)(\zeta)e := f(\zeta e)$, $\zeta \in \mathrm{D}$. Then $\|Rf\|_\infty \leqq \|f\|_{A(D,Z)}$ and $RJf = f$ for every $f \in A$. It follows that for every $f \in A$, $\|f\|_\infty = \|RJf\|_\infty \leqq \|Jf\|_{A(D,Z)}$. Thus (ii) $\leftrightarrow$ (iii).

4.13. PROPOSITION. (i) *The map C: $Z \to A(D, Z)$ defined by $C_z(w) = z$ is an isometric homomorphism;*

(ii) *The map Q_0: $A(D, Z) \to Z$ defined by $Q_0 f = f(0)$ is a multiplicative quotient map.*

(iii) *CQ_0 is a projection of norm* 1;

(iv) *The von Neumann inequality holds in Z if and only if it holds in $A(D, Z)$.*

PROOF. Clearly, (iv) follows from (i) and (ii) via Proposition 4.3. Parts (i), (ii) and (iii) are easily checked.

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THE NEVANLINNA MATRIX OF ENTIRE FUNCTIONS ASSOCIATED WITH A SHIFTED INDETERMINATE HAMBURGER MOMENT PROBLEM

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Abstract.

The shifted moment problem is introduced and its associated polynomials of the first and second kind are calculated. The Nevanlinna matrix associated with the shifted problem is found and a one-to-one correspondence between the solutions to the shifted problem and those to the original is given. It is proved that any of the four functions in the Nevanlinna matrix associated with an indeterminate moment problem belongs to a certain class of functions, introduced by Hamburger. A two-variable analogue of this result is established and is applied to the r times shifted moment problem.

0. Introduction.

We consider a normalized Hamburger moment sequence $(s_n)_{n \geqq 0}$ and the associated polynomials of the first and second kind, $(P_k)_{k \geqq 0}$ and $(Q_k)_{k \geqq 0}$, following the notation of Akhiezer, [1]. The sequence $(P_k)_{k \geqq 0}$ forms an orthonormal system with respect to the inner product given by $\langle x^n, x^m \rangle = \int_{\mathsf{R}} x^{n+m} d\mu(x)$, where μ is any measure from the set $V = \{\mu \geqq 0 \mid s_n = \int x^n d\mu(x) \, \forall n \geqq 0\}$ of solutions to the moment problem. The P_k's are uniquely determined by the additional condition that their leading coefficients are positive. The sequence $(Q_k)_{k \geqq 0}$ is given by

$$Q_k(x) = \int_{\mathsf{R}} \frac{P_k(x) - P_k(t)}{x - t} d\mu(t),$$

where μ is any measure from V.

The sequences $(a_k)_{k \geqq 0}$ and $(b_k)_{k \geqq 0}$ given by the formulas $a_k = \int x P_k(x)^2 d\mu(x)$, $b_k = \int x P_k(x) P_{k+1}(x) d\mu(x)$ define the Jacobi-matrix

$$J = \begin{pmatrix} a_0 & b_0 & 0 & 0 & \dots \\ b_0 & a_1 & b_1 & 0 & \dots \\ 0 & b_1 & a_2 & b_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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associated with the moment problem. The matrix obtained from J by deleting the first row and column is in fact still a Jacobi-matrix associated with a certain normalized moment sequence $(\tilde{s}_n)_{n \geqq 0}$. This is because of Favard's theorem, cf. [1] p. 5. The sequence $(\tilde{s}_n)_{n \geqq 0}$ is called the shifted moment sequence.

The investigation of the shifted moment problem goes back to Sherman, [7], using continued fractions.

In case of an indeterminate moment sequence, the series

$$p(z) \equiv \left(\sum_{k=0}^{\infty} |P_k(z)|^2 \right)^{1/2} \tag{1}$$

$$q(z) \equiv \left(\sum_{k=0}^{\infty} |Q_k(z)|^2 \right)^{1/2} \tag{2}$$

converge uniformly on compact subsets of the complex plane.

The so-called Nevanlinna matrix of entire functions, cf. [1] p. 55,

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

plays an important role in the parametrization of all solutions to the moment problem. The functions are defined as follows:

$$\begin{aligned} A(z) &= z \sum_{k=0}^{\infty} Q_k(0)Q_k(z) \\ B(z) &= -1 + z \sum_{k=0}^{\infty} Q_k(0)P_k(z) \\ C(z) &= 1 + z \sum_{k=0}^{\infty} P_k(0)Q_k(z) \\ D(z) &= z \sum_{k=0}^{\infty} P_k(0)P_k(z). \end{aligned} \tag{3}$$

The objective of this paper is to express the Nevanlinna matrix associated with the shifted moment problem in terms of the functions (3) in the Nevanlinna matrix associated with the original moment problem. The functions A and C of the original problem are given in terms of the functions B and D of the shifted problem. This is applied to the interrelation between Nevanlinna matrices of indeterminate moment problems and a certain class $\mathscr{A}$ of entire functions, introduced by Hamburger [5]. A two-variable analogue of this interrelation is established and applied to the r times shifted moment problem. A one-to-one correspondence between the set of solutions to the shifted problem and V is derived.

The above description of A and C in terms of B and D of the shifted problem can be used to obtain results on the growth of the functions (1), (2) and (3) associated with an indeterminate moment problem, see Berg & Pedersen [3].

1. The shifted moment problem and its associated polynomials.

Suppose that $(s_n)_{n \geqq 0}$ is a normalized Hamburger moment sequence with associated polynomials $(P_k)_{k \geqq 0}$ and $(Q_k)_{k \geqq 0}$. Put

$$\tilde{a}_k \equiv a_{k+1}$$
$$\tilde{b}_k \equiv b_{k+1} \tag{4}$$
$$\tilde{P}_k(x) \equiv b_0 Q_{k+1}(x), \quad k \geqq 0$$

and consider the three-term recurrence relation

$$xY_k = \tilde{b}_k Y_{k+1} + \tilde{a}_k Y_k + \tilde{b}_{k-1} Y_{k-1}, \quad k \geqq 1. \tag{5}$$

The sequence $(P_k)_{k \geqq 0}$ satisfies this recurrence relation with initial values $\tilde{P}_0(x) = 1, \tilde{P}_1(x) = (x - \tilde{a}_0)/\tilde{b}_0$. Since the b_k's are all strictly positive so are the $\tilde{b}_k$'s. Favard's theorem then ensures the existence of a uniquely determined normalized moment sequence $(\tilde{s}_n)_{n \geqq 0}$ having the sequence $(\tilde{P}_k)_{k \geqq 0}$ as associated polynomials of the first kind. We denote by $(\tilde{Q}_k)_{k \geqq 0}$ the associated polynomials of the second kind. These can actually be determined in terms of the P_k's and Q_k's and we have

LEMMA 1.1. *The associated polynomials of the shifted moment problem* $(\tilde{P}_k)_{k \geqq 0}$ *and* $(Q_k)_{k \geqq 0}$ *are given by*

$$\tilde{P}_k(x) = b_0 Q_{k+1}(x)$$
$$\tilde{Q}_k(x) = P_1(x) Q_{k+1}(x) - \frac{1}{b_0} P_{k+1}(x). \tag{6}$$

PROOF. The polynomials $(\tilde{Q}_k)_{k \geqq 0}$ also satisfy the recurrence relation (5) with initial values $\tilde{Q}_0(x) = 0, \tilde{Q}_1(x) = 1/b_0$. The two sequences $(\tilde{P}_k)_{k \geqq 0}$ and $(Q_k)_{k \geqq 0}$ are clearly linearly independent and so they form a basis of the space of solutions to (5).

Now, the sequence $(P_{k+1})_{k \geqq 0}$ is in fact a solution to (5) and therefore there exist real numbers $\alpha(x), \beta(x)$ such that $P_{k+1}(x) = \alpha(x)\tilde{P}_k(x) + \beta(x)\tilde{Q}_k(x), k = 0, 1, 2, \ldots$. In particular $P_1(x) = \alpha(x)$ and $P_2(x) = P_1(x) b_0 Q_2(x) + \beta(x)/b_1$, giving us $\beta(x) = -b_0$ so that

$$\tilde{Q}_k(x) = P_1(x) Q_{k+1}(x) - \frac{1}{b_0} P_{k+1}(x).$$

REMARK. The above lemma is a special case of [2], Lemma 2.5, where the associated polynomials of the "r times" shifted moment problem are considered.

2. The Nevanlinna matrix of the shifted problem.

Using the lemma above we are now able to compute the entire functions of the shifted problem.

PROPOSITION 2.1. *The shifted moment sequence* $(\tilde{s}_n)_{n \geq 0}$ *is indeterminate exactly when* $(s_n)_{n \geq 0}$ *is indeterminate, and the entire functions in the Nevanlinna matrix associated with the shifted problem, as well as the function in* (1) *can be expressed as follows:*

$$
\begin{aligned}
\tilde{p}(z) &= b_0 q(z) \\
\tilde{A}(z) &= b_0^{-2}(D(z) - a_0(z - a_0)A(z) - (z - a_0)C(z) + a_0 B(z)) \\
\tilde{B}(z) &= -C(z) - a_0 A(z) \\
\tilde{C}(z) &= (z - a_0)A(z) - B(z) \\
\tilde{D}(z) &= b_0^2 A(z).
\end{aligned}
\tag{7}
$$

PROOF. Since $\tilde{P}_k(x) = b_0 Q_{k+1}(x)$, the two moment problems are indeterminate simultaneously, cf. [1] p. 16 and p. 19. Using Lemma 1.1 and the formulas (1) and (3) we get

$$
\tilde{p}(z) = \left(\sum_{k=0}^{\infty} |\tilde{P}_k(z)|^2 \right)^{1/2} = b_0 q(z),
$$

$$
\tilde{D}(z) = b_0^2 \sum_{k=0}^{\infty} Q_{k+1}(0) Q_{k+1}(z) = b_0^2 A(z),
$$

$$
\begin{aligned}
\tilde{B}(z) &= -1 + z \sum_{k=0}^{\infty} \tilde{Q}_k(0) \tilde{P}_k(z) \\
&= -1 + z \sum_{k=0}^{\infty} (-a_0 Q_{k+1}(0) - P_{k+1}(0)) Q_{k+1}(z) \\
&= -C(z) - a_0 A(z).
\end{aligned}
$$

The functions $\tilde{C}$ and $\tilde{A}$ may be calculated similarly.

REMARK. The function $\tilde{q}(z)$ can also be calculated. By definition $\tilde{q}(z) = (\sum_{k=0}^{\infty} |\tilde{Q}_k(z)|^2)^{1/2}$ so that

$$b_0^2\tilde{q}(z)^2 = \sum_{k=0}^{\infty} |(z-a_0)Q_{k+1}(z) - P_{k+1}(z)|^2$$

$$= |z-a_0|^2 q(z)^2 + p(z)^2 - 1 - (z-a_0)\sum_{k=0}^{\infty} Q_k(z)P_k(\bar{z})$$

$$-(\bar{z}-a_0)\sum_{k=0}^{\infty} Q_k(\bar{z})P_k(z).$$

Following the notation of Buchwalter & Cassier [4], p. 175 we obtain ($z \notin \mathsf{R}$)

$$b_0^2\tilde{q}(z)^2 = |z-a_0|^2 q(z)^2 + p(z)^2 - 1 + \frac{(z-a_0)(B(z,\bar{z})+1)}{z-\bar{z}}$$

$$-\frac{(\bar{z}-a_0)(B(\bar{z},z)+1)}{\bar{z}-z}$$

$$= |z-a_0|^2 q(z)^2 + p(z)^2 + \frac{(z-a_0)B(z,\bar{z}) - (\bar{z}-a_0)B(\bar{z},z)}{z-\bar{z}}.$$

Using the equality $B(z,w) = B(w)C(z) - A(z)D(w)$, see [4], p. 177 we get

$$b_0^2\tilde{q}(z)^2 = |z-a_0|^2 q(z)^2 + p(z)^2 + \frac{\operatorname{Im}\{(z-a_0)(B(\bar{z})C(z) - A(z)D(\bar{z}))\}}{\operatorname{Im} z}.$$

For $z \neq w$ we have

$$\sum_{k=0}^{\infty} Q_k(z)P_k(w) = \frac{B(z,w)+1}{w-z} = \frac{B(w)C(z) - A(z)D(w) + 1}{w-z},$$

and making $w \to z$ we get

$$\sum_{k=0}^{\infty} Q_k(z)P_k(z) = B'(z)C(z) - A(z)D'(z).$$

This implies the equality

$$b_0^2\tilde{q}(x)^2 = (x-a_0)^2 q(x)^2 + p(x)^2 - 1 - 2(x-a_0)(B'(x)C(x) - A(x)D'(x))$$

for $x \in \mathsf{R}$.

3. The Nevanlinna parametrization of the shifted problem.

When the sequence $(s_n)_{n \geqq 0}$ is indeterminate, we have the so-called Nevanlinna parametrization of the set V of representing measures by means of the class $\mathscr{P}$ of Pick-functions in the upper half-plane H^+:

$$\mathscr{P} \equiv \{f \in \mathscr{H}(H^+) \mid \operatorname{Im} f \geqq 0\}.$$

There is a one-to-one correspondence $\varphi \leftrightarrow \mu_\varphi$ between $\mathscr{P} \cup \{\infty\}$ and the set V, given by

$$\int_{\mathsf{R}} \frac{d\mu_\varphi(t)}{z-t} = \frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathsf{C}\backslash\mathsf{R}. \tag{8}$$

The solutions to the shifted problem can be described in terms of those to the original problem; we have:

PROPOSITION 3.1. *The transformation $\varphi \mapsto -b_0^2/\varphi - a_0 = \varphi^*$ is a homeomorphism of $\mathbb{P} \cup \{\infty\}$ onto itself, and we have the following one-to-one correspondence between the sets $\tilde{V}$ and V:*

$$b_0^2 \int \frac{d\tilde{\mu}_\varphi(t)}{z-t} = z - a_0 - \left(\int \frac{d\mu_{\varphi^*}(t)}{z-t}\right)^{-1}. \tag{9}$$

PROOF. Since $-1/\varphi \in \mathscr{P}$ whenever $\varphi \in \mathscr{P}\backslash\{0\}$, $\varphi \mapsto \varphi^*$ is easily seen to be a homeomorphism. Using (8) and Proposition 2.1 we get

$$\begin{aligned} b_0^2 \int \frac{d\tilde{\mu}_\varphi(t)}{z-t} &= b_0^2 \frac{\tilde{A}(z)\varphi(z) - \tilde{C}(z)}{\tilde{B}(z)\varphi(z) - \tilde{D}(z)} \\ &= \frac{(z-a_0)(A(z)(a_0\varphi(z) + b_0^2) + C(z)\varphi(z)) - D(z)\varphi(z) - B(z)(a_0\varphi(z) + b_0^2)}{A(z)(a_0\varphi(z) + b_0^2) + C(z)\varphi(z)} \\ &= z - a_0 - \left(\frac{A(z)(b_0^2\varphi(z)^{-1} + a_0) + C(z)}{B(z)(b_0^2\varphi(z)^{-1} + a_0) + D(z)}\right)^{-1} \\ &= z - a_0 - \left(\int \frac{d\mu_{\varphi^*}(t)}{z-t}\right)^{-1}. \end{aligned}$$

4. On Hamburger's class $\mathscr{A}$ of entire functions.

The Nevanlinna matrix and the so-called Nevanlinna-extremal solutions (i.e. the subset $\{\mu_t \mid t \in \mathsf{R} \cup \{\infty\}\}$ of V) are closely interrelated. There has been made attempts to characterize these solutions in terms of entire functions, see [1], p. 161 ff. and [5] p. 515 (the result is partly wrong, see [6]).

Hamburger introduced a certain class $\mathscr{A}$ of entire functions. An entire function f belongs to this class if it is real, if it has infinitely many zeros $(\lambda_n)_{n \geq 1}$ all of which are real and simple, and if the following relations hold

$$\sum_{n=1}^{\infty} \left|\frac{\lambda_n^k}{f'(\lambda_n)}\right| < \infty \quad \text{for each} \quad k \in \mathsf{N},$$

$$\frac{1}{f(z)} = \sum_{n=1}^{\infty} \frac{1}{f'(\lambda_n)(z-\lambda_n)} \quad \text{for} \quad z \in \mathsf{C}\backslash\{\lambda_n \mid n \geqq 1\}.$$

It is proved in [1] p. 165 that any combination $tB - D$, $t \in \mathrm{R} \cup \{\infty\}$ ($\equiv B$ if $t = \infty$) belongs to the class $\mathscr{A}$, but nothing is stated about the natural question whether the "numerator" functions $tA - C$ occuring in the Nevanlinna parametrization (8) also belong to this class $\mathscr{A}$.

PROPOSITION 4.1. *For any $t \in \mathrm{R} \cup \{\infty\}$ the function $tA - C$ belongs to $\mathscr{A}$. In particular A, B, C, D belong to $\mathscr{A}$.*

PROOF. Using the formulas for $\tilde{B}$ and $\tilde{D}$ in Proposition 2.1 we get that

$$tA(z) - C(t) = \frac{t + a_0}{b_0^2}\tilde{D}(z) + \tilde{B}(z). \tag{10}$$

Applying the theorem in [1] p. 165 to the shifted moment problem we get that the righthand side of (10) belongs to the class $\mathscr{A}$, and so does the "numerator" function $tA - C$.

5. A class of entire functions of two variables.

We denote by $\mathscr{A}_2$ the class of entire functions F of two complex variables such that

$$\begin{aligned} &\forall u \in \mathrm{R}\colon (v \mapsto F(u, v)) \in \mathscr{A} \\ &\forall v \in \mathrm{R}\colon (u \mapsto F(u, v)) \in \mathscr{A} \end{aligned} \tag{11}$$

It is possible to define four entire functions of two complex variables associated with an indeterminate moment problem, see [4], p. 175. These functions generalize the functions in the Nevanlinna matrix and in terms of those they may be written as follows, see [4], p. 177.

$$\begin{aligned} A(u, v) &= A(v)C(u) - A(u)C(v) \\ B(u, v) &= B(v)C(u) - A(u)D(v) \\ C(u, v) &= A(v)D(u) - B(u)C(v) \\ D(u, v) &= B(v)D(u) - B(u)D(v) \end{aligned} \tag{12}$$

PROPOSITION 5.1. *The two-variable entire functions A, B, C and D belong to the class $\mathscr{A}_2$.*

PROOF. The above equations (12), the fact that $A(z)D(z) - B(z)C(z) \equiv 1$ and Proposition 4.1 yield the result.

We shall now compute the two-variable functions associated with the r times shifted moment problem. To do this we need some notation. We define $A_1 \equiv \tilde{A}$,

$A_2 \equiv (A_1)^{\sim}, \ldots, A_r \equiv (A_{r-1})^{\sim}$ and similarly for the other functions B, C and D. We also put, for $k \geqq 0$,

$$M_k(u,v) = \begin{pmatrix} b_k^{-2}(u-a_k)(v-a_k) & -b_k^{-2}(u-a_k) & -b_k^{-2}(v-a_k) & b_k^{-2} \\ u-a_k & 0 & -1 & 0 \\ v-a_k & -1 & 0 & 0 \\ b_k^2 & 0 & 0 & 0 \end{pmatrix}.$$

With this notation we can formulate:

PROPOSITION 5.2. *Let $r \geqq 1$. Then,*

$$\begin{pmatrix} A_r \\ B_r \\ C_r \\ D_r \end{pmatrix} = M_{r-1}M_{r-2} \cdot \ldots \cdot M_1 M_0 \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \tag{13}$$

PROOF. A computation as in the proof of Proposition 2.1 (or simply (12) and Proposition 2.1) gives

$$\tilde{A}(u,v) = b_0^{-2}((u-a_0)(v-a_0)A(u,v) - (u-a_0)B(u,v) - (v-a_0)C(u,v) + D(u,v))$$

$$\tilde{B}(u,v) = (u-a_0)A(u,v) - C(u,v)$$

$$\tilde{C}(u,v) = (v-a_0)A(u,v) - B(u,v)$$

$$\tilde{D}(u,v) = b_0^2 A(u,v).$$

This can be written more compactly as

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix} = M_0 \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

Then (13) follows by induction.

By Proposition 5.1 we see:

COROLLARY 5.3. *The four coordinates of the vector*

$$M_{r-1}M_{r-2} \cdot \ldots \cdot M_1 M_0 \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

belong to the class $\mathscr{A}_2$.

REMARK. Proposition 5.2 generalizes Proposition 2.1 and Corollary 5.3 generalizes Proposition 4.1. I wish to thank the referee for suggesting the class $\mathscr{A}_2$ and these generalizations.

REMARK. Most of the results of this paper were presented in my Masters Thesis written under the guidence of Christian Berg, University of Copenhagen, January 1991.

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A THEOREM OF GROTHENDIECK USING PICARD GROUPS FOR THE ALGEBRAIST*

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Abstract.

This is an application of a new algebraic reformulation of the Picard group pic(G) of a quasi-compact subset $G \subseteq X = \operatorname{Spec} A$ for a commutative ring A. A torsion theoretic (algebraic) proof is given of A. Grothendieck's theorem that a complete intersection that is factorial in co-dimension 3 is factorial. Our proof is along the lines of Grothendieck's SGA2 proof, but eliminates the need for spectral sequences and (formal) sheaf theory.

All torsion theoretic details are given in a lengthy appendix, including the proof of the long standing conjecture that $Q_G = \varinjlim Q_U$ for G quasi-compact and U open. This yields the interesting result that restriction of an O_X-Module to a quasi-compact, generically closed subset does not require the sheafification process.

One of A. Grothendieck's theorems is that a complete intersection ring which is locally a unique factorization domain (UFD) in codimension 3, is a UFD. This conjecture of P. Samuel was proved about thirty years ago [9, Corollaire XI 3.14]. Grothendieck's proof uses the "most sophisticated techniques of algebraic geometry" [8, p. 3]. We provide in this paper a purely algebraic proof of this purely algebraic theorem.

Such a proof, accessible to the pure algebraist is sorely needed. Our technique is "merely" to translate Grothendieck's proof into commutative algebra, using torsion theory to finesse the material on formal schemes, and to avoid the occasional spectral sequence. We have used this technique in [5] to find a simple proof of the local Lichtenbaum-Hartshorne theorem. This illustrates again how torsion theory can handle difficult algebraic geometry with relative ease. Of course, some simplifications come by narrowing the focus of the highly complex and general theory that [9] provides.

Many of the ideas of our proof parallel those in [9], and for comparison, references to [9] are provided for those familiar with the algebraic geometry language. Algebraists should find most interesting the three methods (Steps 1, 2,

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and 3) of "lifting" (modulo nilpotents or a regular element) or "extending" a finitely generated module that is locally free on just a portion of spec A.

The difficult part of Grothendieck's proof is the theorem that a complete intersection (A, m) of dimension $\geqq 4$ is parafactorial, i.e., depth $A \geqq 2$ and the Picard group $\operatorname{Pic}(U)$ of the punctured spectrum $U = \operatorname{Spec} A - m$ is trivial. This leads to a study of a purely algebraic formulation, $\operatorname{pic}(G)$, where $G \subseteq X = \operatorname{Spec} A$. Our object is to define an abelian group $\operatorname{pic}(G)$ that agrees with the algebraic geometers' $\operatorname{Pic}(G, \mathcal{O}_G)$, the group of isomorphism classes of invertible $\mathcal{O}_G$-Modules where $\mathcal{O}_G = \mathcal{O}_X|_G$, yet is easy to manipulate and is analogous in definition to the ring theorists' $\operatorname{Pic}(A)$ ($\cong \operatorname{Pic}(X)$), the group of isomorphism classes of finitely generated, locally free A-modules of constant rank 1 (these are just the (finitely generated) rank 1 projectives).

Formulations of $\operatorname{pic}(G)$ are being developed in the literature, e.g. [20, 19, 18, 6], and we have provided a reworking in an extensive appendix (see Definition A-6 and the variations that follow) when G is quasi-compact. This includes the three cases (1) when Spec A is noeetherian and G is arbitrary, (2) when A is arbitrary and G is an intersection of quasi-compact opens, and (3) when A is a Krull domain and G is the set of height 1 primes. We only show that $\operatorname{pic}(G) \cong \operatorname{Pic}(G)$ for case (2), and though our application is for A noetherian and G open, our $\operatorname{pic}(G)$ is useful in the other cases as well. Part of our work shows that, in case (2), the definition of $\tilde{M}|_G$ does not require the sheafification process even though G may not be open.

The proofs of the results in the Appendix and of the Grothendieck-Samuel theorem will use the torsion functor $\mathcal{T}_G$ and the localization functor Q_G, as well as some techniques from torsion theory, so we have included this needed material in order to make the paper easier to read.

We hope this presentation will encourage other commutative ring theorists to use torsion theory to explore algebraic geometry from the algebraic point of view.

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We need to discuss when a noetherian local ring is a unique factorization domain (UFD). If a noetherian ring is already a domain, then it is a UFD if and only if each height 1 prime is principal [15, Theorem 13.1]. Any localization of a UFD is a UFD, and a noetherian UFD is a normal domain, i.e., noetherian and integrally closed [15, 13.2 and 13.3]. We refer the reader to the Appendix (A-6, A-7, A-11) for our module theoretic formulation of pic.

The following definition should be compared with [9, XI Proposition 3.5].

DEFINITION 1. A noetherian local ring (A, m) of dimension $\geqslant 2$ is called *parafactorial* if depth $A \geqq 2$ and $\mathrm{pic}(U) = 0$ where $U = \operatorname{spec} A - m$.

The next item is useful in an induction proof.

PROPOSITION 2 [9, XI Corollaire 3.10]. *A noetherian local ring (A, m) of dimension $\geqq 2$ is a* UFD *if and only if A is parafactorial and A_p is a* UFD *for all $p \neq m$.*

PROOF. ($\Leftarrow$) Since A_p is a normal domain for each $p \neq m$, Serre's criterion for normality $(R_1) + (S_2)$ is satisfied by A for the cases $p \neq m$ [13, Theorem 23.8]. Dim $A \geqq 2$, and depth $A \geqq 2$ from the parafactorial property, covers the case $p = m$. The normal ring A is then a finite direct product of normal domains [13, Exercise 9.11] and, since it is local, A must be a domain. The result (A-13) is applicable as a normal domain is Krull [13, Theorem 12.4].

($\Rightarrow$) Use (A-13) again.

LEMMA 3. *Let $\varphi: A \to B$ be a flat homomorphism of local rings such that ${}_AmB$ is primary to the maximal ideal ${}_Bm$ of B. Then* $\dim A = \dim B$ *and* depth $A =$ depth B. *If* $\dim B \geqq 2$ *and B is parafactorial, so is A.*

PROOF. For flat local homomorphisms we have dimension and depth formulas $\dim B = \dim A + \dim B/{}_AmB$ and depth $B =$ depth $A +$ depth $B/{}_AmB$ [13, Theorem 15.1 and Corollary to Theorem 23.3]. The first parts follow from $\dim B/{}_AmB = 0 =$ depth $B/{}_AmB$.

Let $U = \operatorname{spec} A - {}_Am$ and $U' = \operatorname{spec} B - {}_Bm$. We use (A-11) for our description of $\mathrm{pic}(U)$: the group of isomorphism classes $[[M]]_U$ of finitely generated reflexive A-modules M, locally free rank 1 on U. Then by the flatness of $B, M \otimes_A B$ is reflexive, $[[M \otimes B]]_{U'} \in \mathrm{pic}(U') = 0$, so $M \otimes B \cong B$. Faithful flatness (M is finitely generated) and the Ext^1 condition will show M is projective, hence free over the local ring A, of rank 1 (cf., [9, XI Lemme 3.6]).

Recall that a regular local ring R is a noetherian local ring (R, m) such that the number of generators of a minimal generating set for m is equal to the dimension of R. Regular local rings are UFD's [13, Theorem 20.3]. If R is regular local, so is R_p for each $p \in \operatorname{spec} R$ [13, Theorem 19.3]. A noetherian local ring is regular if and only if its m-adic completion is regular [2, Proposition 11.24].

The next definition has been updated to include completions.

DEFINITION 4. A (noetherian) local ring (A, m) is a *complete intersection* (c.i.) if its m-adic completion $\hat{A} = R/(x_1, \ldots, x_s)$, where R is a (complete) regular local ring and $x_1, \ldots, x_s$ is an R-sequence.

A (c.i.) is Gorenstein, hence Cohen-Macaulay [13, Theorems 21.3 and 18.1], so depth $A = \dim A$.

THEOREM 5 [9, Théorème XI 3.13 (ii)]. *Any complete intersection of dimension $\geqq 4$ is parafactorial.*

PROOF. We induct on s. If $s = 0$, we mean $\hat{A}$ is regular. Then A is regular, hence parafactorial by Proposition 2.

For $s > 0$, it is sufficient to consider the complete case, Lemma 3. We now change notation. For the induction hypothesis, for a fixed $s \geqq 0$, we suppose every (c.i.) of dimension $\geqq 4$ whose completion can be written as a regular local ring mod a regular sequence of length s is parafactorial. Let A_1 be a complete (c.i.) of dimension $\geqq 4$, $A_1 = A/tA$ where $A = R/(x_1, \ldots, x_s)$ is complete of dimension $\geqq 5$, parafactorial by the induction hypothesis, and t is a non-zero-divisor of A in the maximal ideal m of A. We need to show $\operatorname{pic}(U_1) = 0$ where we denote by U_i the punctured spectrum of $A_i = A/t^iA$ (A_i is also Cohen-Macaulay, depth $A_i = \dim A_i \geqq 4$, so the depth condition in the definition of parafactorial is automatically satisfied for every i). We accomplish this in three steps.

Let $U = \operatorname{spec} A - m$, $G = g(V(tA) - m) = \{p \in \operatorname{spec} A \mid p \subseteq \text{some } q \in \operatorname{spec} A, t \in q \neq m\} \subseteq U$. We have natural group homomorphisms

$$0 = \operatorname{pic}(U) \xrightarrow{\alpha_1} \operatorname{pic}(G) \xrightarrow{\alpha_2} \lim_{\leftarrow i} \operatorname{pic}(U_i) \xrightarrow{\alpha_3} \operatorname{pic}(U_1)$$

which we will discuss each in turn.

Step (1). First note that $U \backslash G = \{p \mid \dim A/p = 1 \text{ and } t \notin p\}$. This is because if $p \in U \backslash G$ then the principal ideal generated by t in A/p is primary to the maximal ideal. By Krull's principal ideal theorem [13, Theorem 13.5], $\dim A/p \leqq 1$. Secondly, since A_p is Cohen-Macaulay [13, Theorem 17.3 (iii)], $U \supseteq G \supseteq \{p \in \operatorname{spec} A \mid \operatorname{depth} A_p \leqq 1\}$. Thus, we may use (A-11) for our description of pic, and define the group homomorphism α_1 by $\alpha_1([[M]]_U) = [[M_p]]_G \in \operatorname{pic}(G)$ where M is a finitely generated reflexive A-module, locally free of rank 1 on U.

To show α_1 is surjective, let $[[M]]_G \in \operatorname{pic}(G)$ where M is a finitely generated reflexive A-mdule, locally free rank 1 on G. M is actually free on all of U, since $[[M]]_{U_p} \in \operatorname{pic}(U_p)$ where U_p is the punctured spectrum of A_p, $p \in U \backslash G$. This is true since $U_p \subseteq G$ and M is locally free on G (of course, A_p and U_p satisfy the hypothesis of (A-11) since $\dim A_p \geqq 4 > 1$). Since A is a regular local ring modulo a regular sequence of length s, the same is true for A_p by earlier remarks. The dimension of $A_p \geqq 4$ and thus, by the induction hypothesis, A_p is parafactorial, and $\operatorname{pic}(U_p) = 0$. We conclude that $M_p \cong A_p$, $[[M]]_U \in \operatorname{pic}(U)$, and it follows that α_1 is surjective (compare with [9, XI Proposition 3.12 and X Exemple 2.1]).

Step (2). For the groups $\operatorname{pic}(G)$ and $\operatorname{pic}(U_i)$ we use (A-6). The natural group homomorphisms, for $j \geqq i$, π_{ij}: $\mathrm{pic}(U_j) \to \mathrm{pic}(U_i)$ are defined by $\pi_{ij}([\mathrm{M}_j]_{U_j}) = [M_j \otimes A_i]_{U_i}$. These maps from an inverse system and the natural maps $\mathrm{pic}(G) \to \mathrm{pic}(U_i)$ factor through the π_{ij}, hence we have a homomorphism α_2: $\mathrm{pic}(G) \to \varprojlim_{i} \mathrm{pic}(U_i)$ given by $\alpha_2([M]_G) = ([M \otimes A_i]_{U_i})_{i \geqq 1}$. To show α_2 is surjective, let $([M_i]_{U_i})_{i \geqq 1} \in \varprojlim_{i} \mathrm{pic}(U_i)$. By the remarks at the end of (A-11) we can assume the representative M_i is a finitely generated A_i-module and $M_i = Q_{U_i}(M_i)$ $(= M_i^{**})$. Furthermore, we fix isomorphisms, for $j = i + 1, i \geqq 1, Q_U(M_j \otimes A_i) \cong Q_{U_i}(M_j \otimes A_i) \cong M_i$ where the first equality comes from (A-1j) and the second from compatibility relations in the inverse limit. Set $Q = Q_U$ for ease of notation, and define, for $j = i + 1$, A-module maps φ_{ij}: $M_j \to M_j \otimes A_i \to Q(M_j \otimes A_i) \cong M_i$. The obvious compositions give us an inverse system of maps between any pair of the M_i's. $M = \varprojlim M_i$ is the module we want. To show this, we first claim that the φ's satisfy the Mittag-Leffler condition which we proceed to prove (cf., [9, IX Théorème 2.2]). These φ's fit into a large commutative diagram with many maps.

$$
(*) \qquad
\begin{array}{ccccccccc}
 & & & & & & & & \leftrightarrows \\
 & & & & & & M_4 & & \cong \;\cdots \\
 & & & & & \leftrightarrows & \downarrow\uparrow & & \\
 & & & & M_3 & \cong & Q(M_4 \otimes A_3) & \cong & \cdots \\
 & & & \varphi \leftrightarrows \psi & \downarrow\uparrow & & \mathrm{nat}\downarrow\uparrow t & & \\
 & & M_2 & \cong & Q(M_3 \otimes A_2) & \cong & Q(M_4 \otimes A_2) & \cong & \cdots \\
 & \leftrightarrows & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \\
M_1 & \cong & Q(M_2 \otimes A_1) & \cong & Q(M_3 \otimes A_1) & \cong & Q(M_4 \otimes A_1) & \cong & \cdots
\end{array}
$$

In each column of isomorphisms, the top one is the chosen fixed one mentioned earlier (thus the small triangles with the φ's commute by definition). The remaining ones in the column are induced from these by tensoring with A_i and applying Q. The downward maps are the natural ones induced from the surjections $M_k \otimes A_{i+1} \xrightarrow{\mathrm{nat}} M_k \otimes A_i$. The upward maps are induced from $M_k \otimes A_i \xrightarrow{t} M_k \otimes A_{i+1}$, and these are used to construct, for $j > i$, maps ψ_{ji}: $M_i \to M_j$, i.e., the small triangles commute by the definition of the ψ's. If we can show that the squares commute (both kinds) then we can work along any column to study the M_i's. But this is easy to establish; set $N = M_k$, $N^+ = M_{k+1}$, and let $j = i + 1$. We have a diagram

$$
(**) \qquad
\begin{array}{ccccc}
N \otimes A_j & \cong & Q(N^+ \otimes A_k) \otimes A_j & \leftarrow & N^+ \otimes A_j \\
\downarrow\uparrow\ 1 \otimes t & & \downarrow\uparrow\ 1 \otimes t & & \downarrow\uparrow\ 1 \otimes t \\
N \otimes A_i & \cong & Q(N^+ \otimes A_k) \otimes A_i & \leftarrow & N^+ \otimes A_i
\end{array}
$$

The squares commute, the horizontal arrows are locally isomorphisms on U, so when Q is applied to (**) we obtain the small squares in (*), by (A-1c).

We are ready to extract information about the M_i's and φ's from (*) by working along a column. First, the φ's and ψ's yield, for any i and j, a long exact sequence $0 \to M_i \xrightarrow{\psi} M_{i+j} \xrightarrow{\varphi} M_j \to R^1Q(M_i) \to R^1Q(M_{i+j}) \to \ldots$, where we have deleted the subscripts on φ and ψ for clarity. To obtain this sequence, let $k > i + j$ and set $N = M_k$. The *right* exact sequence $0 \to N \otimes A_i \xrightarrow{t^j} N \otimes A_{i+j} \to N \otimes A_j \to 0$ becomes exact locally at each $p \in U$ since $N = M_k$ is locally either zero or free, and t is not a zero-divisor on A. By (A-1c), the kernel of the map marked t^j is $\mathscr{T}_U$-torsion (see the introduction to the Appendix for terminology). Replace $N \otimes A_i$ by its image K_i and apply Q to the resulting short exact sequence to obtain a long exact sequence. Since the class of $\mathscr{T}_U$-torsion modules is closed under the formation of injective envelopes when A is noetherian [6, Proposition 6.3 (6)] and Q kills this torsion class by (A-1c), it follows that $R^nQ(N \otimes A_i) \xrightarrow{\sim} R^nQ(K_i), n \geqq 0$. Substituting the M_i's (and φ's and ψ's) yields the quoted long exact sequence.

Now for the claimed Mittag-Leffler condition. Fix $d \geqq 1$. We must show that the images of the M_i's ($i \geqq d$) in M_d stabilize for all i sufficiently large. Use the preceding long exact sequences to construct, for each $j \geqq 2$, the commutative diagrams

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & M_d & = & M_d & & \\
 & & \psi\downarrow & & \downarrow & & \\
0 & \to & M_{jd} & \xrightarrow{\varphi} & M_{(j+1)d} & \xrightarrow{\varphi} & M_d \\
 & & \varphi\downarrow & t^d & \downarrow & & \| \\
0 & \to & M_{(j-1)d} & \to & M_{jd} & \to & M_d \\
 & & \downarrow & & \downarrow & & \\
 & & R^1Q(M_d) & = & R^1Q(M_d) & &
\end{array} \qquad (j \geqq 2)$$

Denote by $c_j = c_j(d)$ the image of M_{jd} in M_d and by $l_j = l_j(d)$ the image of M_{jd} in $R^1Q(M_d)$. Conclude that $c_{j+1} \subseteq c_j \subseteq M_d$, $l_{j-1} \subseteq l_j \subseteq R^1Q(M_d)$, and $l_j/l_{j-1} \cong c_j/c_{j+1}$ by the snake lemma. It is clear that $R^1Q(M_d) = \lim\limits_{\to j} \operatorname{Ext}^1_A(m^j, M_d) = H^2_m(M_d)$, from (A-1i) and the fact that the powers of the maximal ideal m of A are cofinal in the torsion filter F_U (see the introduction to the Appendix). By a standard change of rings formula [6, Theorem 7.2 (9)], the second local cohomology $H^2_m(M_d) = H^2_{mA_d}(M_d)$. By duality for the Gorenstein ring A_d [6, Proposition 7.10] or [9, V Proposition 3.5], $H^2_{mA_d}(M_d) = \operatorname{Ext}^{n-3}_{A_d}(M_d, A_d)^*$ where $*$ is the Matlis dual $\operatorname{Hom}_{A_d}(\ , E(A_d/mA_d))$ and $n = \dim A = \dim A_d + 1 \geqq 5$. A quick check locally gives that the finitely generated A_d-module $\operatorname{Ext}^{n-3}_{A_d}(M_d, A_d)$ vanishes on U_d since

$n - 3 \geqq 1$, hence is of finite length. Thus $R^1Q(M_d)$ is also of finite length and the l_j's stabilize, hence so do the c_j's, say to $c(d)$, $d \geqq 1$. This establishes the Mittag-Leffler claim.

We have yet to prove that $M = \varprojlim_i M_i = \varprojlim_d c(d)$ has the required properties: M is finitely generated as an A-module, locally free rank one on G, and $Q(M \otimes A_i) \cong M_i$. For the last item, apply $\varprojlim_j$ to the family of short exact sequences $0 \to M_{(j-1)d} \to M_{jd} \to c(d) \to 0$ (for $j >> 0$) to obtain the exact sequence $0 \to M \to M \to c(d) \to 0$ (Mittag-Leffler is used here). The endomorphism of M is just multiplication by t^d, a fact deduced from the diagonal map in the diagrams ($j \geqq 2$) above. All the φ's are locally surjective at each $p \in U$ so we have $Q(M_d) = Q(c(d))$. Thus $M_d = Q(M_d) = Q(c(d)) \cong Q(M/t^dM) = Q_{U_d}(M/t^dM)$ for each d. We now use [13, Theorem 8.4] to prove M is finitely generated. It is clear from the definition of $M = \varprojlim c(d)$ that $0 = \cap \ker(M \to c(d)) = \cap t^dM$ so that M is t-adically separated. Also, $M/tM \cong c(1) \subseteq M_1$ is finitely generated over the (t-adically) complete ring A. Thus M is finitely generated. For local freeness on G, let $p \in V(tA) - m$. Then $M_p/t^dM_p \cong c(d)_p \cong (M_d)_p \cong A_p/t^dA_p$, $d \geqq 1$. Nakayama's lemma and Krull's intersection theorem $\cap t^dA_p = 0$ imply $M_p \cong A_p$. We have completed the proof that α_2 is surjective.

Step (3). Grothendieck quotes some (now) well-known theorems from algebraic geometry to prove the natural maps $\mathrm{Pic}(U_{i+1}) \to \mathrm{Pic}(U_i)$ are isomorphisms [9, XI Proposition 1.1], hence so is the projection α_3 from the inverse limit to the first coordinate. His argument is that the Picard group can be written as the first Čech cohomology of a sheaf of units and that this is the same as the first (Grothendieck) sheaf cohomology [10, Exercises III 4.4 and 4.5]. Then $\mathrm{Pic}(U_{i+1})$ and $\mathrm{Pic}(U_i)$ fit into part of a long exact sequence obtained by modding out a nilpotent ideal sheaf of order 2 [10, Exercise III 4.6]. Thus the kernel and cokernel of $\mathrm{Pic}(U_{i+1}) \to \mathrm{Pic}(U_i)$ lie in first (respectively, second) cohomology groups, which can be shifted to the local cohomology modules $H^2_m(t^iA/t^{i+1}A)$ and $H^3_m(t^iA/t^{i+1}A)$, respectively. Since $t^iA/t^{i+1}A \cong A/tA$ and $\operatorname{depth} A/tA = \dim A/tA \geqq 4$, these modules are zero, and $\mathrm{Pic}(U_{i+1}) \cong \mathrm{Pic}(U_i)$, for all i. This is Grothendieck's proof that α_3 is surjective. If the reader accepts these ideas, then he or she may continue to the main theorem.

However, for anyone who would like a "purely algebraic" proof of this theorem, we provide directions.

First we need that a U-invertible module M can be determined by local data, where U is the punctured spectrum of, say, any noetherian local ring (A, m). We assume $\operatorname{depth} A \geqq 2$, $M = Q_U(M) = M^{**}$, and that M is U-invertible. From (A-3), choose a finite covering of $U = \cup D(f_i)$ such that $M_{f_i} = Q_{D(f_i)}(M) \cong Q_{D(f_i)}(A) = A_{f_i}$, and let $\sigma_i\colon M_{f_i} \xrightarrow{\sim} A_{f_i}$ be these isomorphisms. For each i,j set

$\sigma_{ij} = \sigma_i \otimes A_{f_j}: M_{f_i f_j} \xrightarrow{\simeq} A_{f_i f_j}$. We identify $(M_{f_i})_{f_j} = M_{f_i f_j} = M_{f_j f_i} = (M_{f_j})_{f_i}$. Then for each $x_i \in M_{f_i}$ we have $\sigma_i(x_i)' = \sigma_{ij}(x_i')$, where we use "$'$" to denote an image in a further localization. Also, for $i < j$, $r_{ij} := \sigma_{ij}\sigma_{ji}^{-1}$ is an automorphism of $A_{f_i f_j}$, so corresponds to a unit in $A_{f_i f_j}$, i.e., $r_{ij} \in A^{\times}_{f_i f_j}$. These r_{ij}, $i < j$, satisfy in $A_{f_i f_j f_k}$ the relations $r'_{jk} r'^{-1}_{ik} r'_{ij} = 1$. The point is that M can be reconstructed (up to isomorphism) from the local data $r_{ij} \in A^{\times}_{f_i f_j}$, $i < j$. We know $M = Q_U(M)$ can be written [6, Theorem 5.1] as a kernel of a map which is part of the commutative diagram

$$\begin{array}{ccccccc} 0 & \to & M & \to & \oplus_i M_{f_i} & \to & \oplus_{i<j} M_{f_i f_j} \\ & & & & \simeq \downarrow \oplus \sigma_i & & \simeq \downarrow \oplus \sigma_{ij} \\ 0 & \to & K & \to & \oplus A_{f_i} & \to & \oplus A_{f_i f_j} \end{array}$$

where M is (isomorphic to) the kernel of the (standard) homomorphism that sends $(x_i) \in \oplus M_{f_i}$ to the element of $\oplus M_{f_i f_j}$ whose (i, j)-coordinate is $x'_j - x'_i$. K is the kernel of the twist map that sends $(a_i) \in \oplus A_{f_i}$ to $(r_{ij} a'_j - a'_i) \in \oplus A_{f_i f_j}$. However, if for this M we have another set of σ_i's, i.e., of the form $\sigma_i c_i$ where $c_i \in A^{\times}_{f_i}$ corresponds to an automorphism, then the old local data $(r_{ij}) \in \Pi A^{\times}_{f_i f_j}$ is related to the new local data (s_{ij}) by the formula $(c'_j c'^{-1}_i)(s_{ij}) = (r_{ij})$. Conversely, given any element $(r_{ij}) \in \Pi_{i<j} A^{\times}_{f_i f_j}$ such that $r'_{ij} r'_{jk} = r'_{ik}$ in $A_{f_i f_j f_k}$, the kernel K of the twist map will be trivial on each $D(f_i)$ (we mean K_{f_i} is a free A_{f_i}-module; for example K_{f_1} is generated by $(1, r_{12}^{-1}, \ldots, r_{1n}^{-1})$). By (A-3), K is U-invertible. Moreover, $K = Q_U(K)$ since each A_{f_i} and $A_{f_i f_j}$ is stable under Q_U and Q_U is left exact, (A-1e) and (A-1a). Furthermore, K is finitely generated, by the observation in (A-11). If we also have any other element $(s_{ij}) \in \Pi_{i<j} A^{\times}_{f_i f_j}$ and $(c_i) \in \Pi_i A^{\times}_{f_i}$ such that $c'_j c'^{-1}_i s_{ij} = r_{ij}$ then $s'_{ij} s'_{jk} = s'_{ik}$ and the constructed kernel L of the twist map will be isomorphic to K, as the next commutative diagram shows

$$\begin{array}{ccccccc} 0 & \to & \mathrm{K} & \to & \oplus A_{f_i} & \to & \oplus A_{f_i f_j} \\ & & & & \simeq \downarrow \oplus c_i & & \simeq \downarrow \oplus_{i<j} c'_i \\ 0 & \to & L & \to & \oplus A_{f_i} & \to & \oplus A_{f_i f_j} \end{array}$$

All the above may be nicely summarized in terms of the first cohomology group of a Čech complex. Consider the complex of multiplicative abelian groups with (standard) coboundary homomorphisms [14, p. 72]:

$$1 \to \Pi_i A^{\times}_{f_i} \xrightarrow{\delta_1} \Pi_{i<j} A^{\times}_{f_i f_j} \xrightarrow{\delta_2} \Pi_{i<j<k} A^{\times}_{f_i f_j f_k} \to \cdots$$

where $\delta_1((a_i)) = (a'_j a'^{-1}_i)$, $\delta_2((a_{ij})) = (a'_{jk} a'^{-1}_{ik} a'_{ij})$. Then for each (finite) cover of $U = \cup D(f_i)$ there is a one-to-one correspondence between the elements of the cohomology group $C := \operatorname{Ker} \delta_2 / \operatorname{Im} \delta_1$ and those isomorphism classes $[[M]] \in \operatorname{pic}(U)$ such that M_{f_i} is a free A_{f_i}-module for all i.

Next we show how to lift this data (cohomology) mod nilpotents. Note the rather astonishing fact that if I is an ideal of a commutative ring A, and $I^2 = 0$

then there is an exact sequence $0 \to I \to A^{\times} \to (A/I)^{\times} \to 1$ of *abelian groups*, in which I has as its group structure its natural additive structure, and the other two groups are multiplicative. The first map sends $a \in I$ to the unit $1 + a$. For ease of notation, we consider lifting a module from U_1 to U_2 where U_i is the punctured spectrum of $A_i = A/I^i$, $i = 1, 2$. Let $[M] \in \mathrm{pic}(U_1)$ and pick suitable elements $f_i \in A$ such that σ_i: $M_{f_i} \cong (A_1)_{f_i}$ and $U_1 = \cup D(f_i A_1)$ (hence $U_2 = \cup D(f_i A_2)$). Then the σ's determine an element in the first cohomology group C_1 as described above. We have a commutative diagram of abelian groups (and standard maps) with columns that are exact and rows that are complexes.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\oplus_i (I/I^2)_{f_i} & \to & \oplus_{i<j} (I/I^2)_{f_i f_j} & \to & \oplus_{i<j<k} (I/I^2)_{f_i f_j f_k} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\Pi (A/I^2)^{\times}_{f_i} & \to & \Pi (A/I^2)^{\times}_{f_i f_j} & \to & \Pi (A/I^2)^{\times}_{f_i f_j f_k} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\Pi (A/I)^{\times}_{f_i} & \to & \Pi (A/I)^{\times}_{f_i f_j} & \to & \Pi (A/I)^{\times}_{f_i f_j f_k} & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & &
\end{array}$$

A diagram chase shows that the cohomology C_2 of the second row at the second term maps onto our cohomology C_1 (of the third row at the second term), if the first row is exact (as abelian groups or as A-modules) at the third term. Exactness occurs precisely when the local cohomology module $H^3_m(I/I^2) = 0$.

An explanation of this last remark is given in [17, Proposition 2.3, p. 78], or we may see this in the following way. Let N be any module over a noetherian ring A and first suppose the f_i's form a regular sequence. Then the complex $D^{\cdot}_A(N) = D^{\cdot}_A(A) \otimes_A N$: $0 \to N \to \oplus_i N_{f_i} \to \oplus_{i<j} N_{f_i f_j} \to \cdots \to N_{f_1 f_2 \dots f_n} \to 0$ has ith cohomology $H^i(D^{\cdot}(N)) = \lim_{\to j} \mathrm{Ext}^i(A/(f_1^j, \dots, f_n^j), N) = H^i_m(N)$, since (1) the Koszul complex $K.(f_1^j, \dots, f_n^j)$ is a resolution of $A/(f_1^j, \dots, f_n^j)$ by free modules of finite rank (the f_i^j's also form a regular sequence [13, Theorem 16.1]), (2) the complex $\mathrm{Hom}(K.(f_1^j, \dots, f_n^j), N)$ is chain isomorphic to $K.(f_1^j, \dots, f_n^j) \otimes N$, and (3) $\lim_{\to j} K.(f_1^j, \dots, f_n^j)$ is $D^{\cdot}(A)$ indexed in the reverse order. In case the f_i's do not form an A-sequence, map the polynomial ring $B = \mathbf{Z}[X_1, \dots, X_n]$ to A by sending the indeterminate X_i to f_i. Then we have that $H^i_m(N) = H^i_{(X_1, \dots, X_n)}(N) = H^i(D^{\cdot}_B(B) \otimes_B N) = H^i(D^{\cdot}_A(A) \otimes_A N) = H^i(D^{\cdot}_A(N))$, even as modules.

Return to our application. In our case, $N = I/I^2 = tA/t^2A \cong A/tA$ (in general $I^i/I^{i+1} \cong A/tA$) and we know $H^3_m(A/tA) = 0$, so we can lift. Thus, given $[[M]] = [[K]] \in \mathrm{pic}(U_1)$ with K the kernel of a twist map defined by data $(r_{ij}) \in \Pi(A/tA)^{\times}_{f_i f_j}$ induced from M, we can construct, by the above, a kernel K'

from some lifting $(s_{ij}) \in \Pi(A/t^2A)^{\times}_{f_if_j}$ of (r_{ij}), with $[[K']] \in \mathrm{pic}(U_2)$, It remains to show $Q(K'/tK') \cong Q(K) = K$. This is easy: localizing the natural map $K'/tK' \to K$ at any of the f_i's will yield a surjection of rank one free modules, which must then be an isomorphism. The conclusion follows from (A1-c). So we have that π_{12} is surjective, and in general each of the maps $\mathrm{pic}(U_{i+1}) \to \mathrm{pic}(U_i)$ is surjective, hence so is α_3.

This completes the proof that $\mathrm{pic}(U_1) = 0$ and the induction step.

REMARKS 6. (i) Grothendieck uses $\varinjlim \mathrm{Pic}(U_\lambda)$, the direct limit taken over all open $U_\lambda \supseteq G$, instead of our $\mathrm{pic}(G)$.

(ii) His proof uses the Picard group of a formal scheme where we have used $\varprojlim \mathrm{pic}(U_i)$. These are frequently equal, in general [10, Exercise II 9.6]. The methods of Step 2 are based on those of [9, IX §2].

(iii) Grothendieck actually shows when α_1, α_2, and α_3 are injective or bijective using minimal hypothesis (see, for example, [8, Lemma 18.14]). This can be translated to some extent as well using Q and methods similar to the above. For example, in proving that $\alpha_2 \circ \alpha_1$ is injective, one uses the long exact sequence $0 \to Q(M) \to Q(M) \to Q(M/t^iM) \to R^1Q(M) \to \cdots$, where the first map is multiplication by t^i, to show $Q(M)/t^iQ(M) \cong Q(M/t^iM)$ for $i >> 0$. These ideas are left to the reader to explore, as more complicated techniques need be developed first (e.g., we found [1, §2] to be helpful).

Now for the main result [9, Corollaire XI 3.14].

THEOREM 7 (Grothendieck-Samuel). *If A is a complete intersection and A_p is a* UFD *for all primes $p \in \mathrm{spec}\, A$ of height $\leqq 3$, then A is a* UFD.

PROOF. We prove that each A_p is a UFD for all $p \in \mathrm{spec}\, A$ by inducting on the height of p. The cases $ht(p) \leqq 3$ being trivial by hypothesis, assume $ht(p) \geqq 4$, and for all $p' \subsetneq p$ that $A_{p'}$ is a UFD. Let $q \in \mathrm{spec}\, \hat{A}$ be minimal over $p\hat{A}$. Then $p = q \cap \hat{A}$, $A_p \to \hat{A}_q$ is flat, $p\hat{A}_q$ is primary to $q\hat{A}_q$. By Lemma 3, $\dim \hat{A}_q \geqq 4$, and it is clear that $\hat{A}_q$ is also a complete intersection. Theorem 5 gives that $\hat{A}_q$ is parafactorial, hence so is A_p by Lemma 3. Thus A_p is a UFD from Proposition 2.

EXAMPLES 8. (1) The height condition in Theorem 5 and Theorem 7 cannot be lowered. If $A = k[X, Y, U, V]_m/(XY - UV)$, where $m = (X, Y, U, V)$ and k is a field, then A is a complete intersection of dimension 3, and A_p is regular, hence a UFD, for $\mathrm{ht}(p) \leqq 2$ (at least one of the elements $x, y, u, v \in A$ becomes a unit in A_p). However, A is not a UFD since $xy = uv$ (or the ideal of A generated by x and u is a height one prime that is not principal), hence not parafactorial.

(2) In general, the complete intersection hypothesis cannot be replaced by Gorenstein in Theorem 5 and Theorem 7. Let $A = k[X_{ij}]_m/I$, $1 \leqq i \leqq 3$ and

$1 \leqq j \leqq 3$, where m is generated by all the indeterminates X_{ij} and I is generated by all the 2×2 minors of the (generic) 3×3 matrix $[X_{ij}]$. A is Gorenstein and $\mathrm{ht}(I) = \mathrm{gd}(I) = (3 - 2 + 1)^2 = 4$ [4, Corollary 8.9 and Theorem 2.5, respectively]. Thus, $\dim A = 9 - 4 = 5$. For $p \neq mA$, A_p is a regular local ring (same argument as in (1)), hence a UFD. Let J be the ideal of A generated by x_{11}, x_{21}, x_{31}. Then $A/J \cong k[Y_{ij}]_{m'}/I'$, $1 \leqq i \leqq 3$ and $1 \leqq j \leqq 2$, where m' is generated by all the (indeterminates) Y_{ij} and I' is generated by all the 2×2 minors of the 3×2 matrix $[Y_{ij}]$. Thus, $\mathrm{ht}(I') = \mathrm{gd}(I') = (3 - 2 + 1)(2 - 2 + 1) = 2$ and $\dim A/J = 6 - 2 = 4$, i.e., $\mathrm{ht}(J) = 5 - 4 = 1$. A/J is Cohen-Macaulay [11, Theorem 1] means depth $A/J = \dim A/J = 4 > 2$, and A/J is locally regular on the punctured spectrum, so by Serre's criterion for normality A/J is a normal domain and J is a height one prime of the normal domain A that is not principal. Hence A is not a UFD and not parafactorial.

(3) In general, $\mathcal{O}(G)$ and $Q_G(A)$ may not agree when G is a quasi-compact (cf. Theorem A-15). Let $A = S^{-1}\mathbf{Z}$ where $S = \mathbf{Z}\backslash(p) \cup (q)$, p and q distinct prime numbers. Set $m_1 = pA$ and $m_2 = qA$, $G = \{m_1, m_2\}$. Since G is discrete in the induced topology, $\tilde{M}(G) = M_{m_1} \times M_{m_2}$ for any A-module M. But $g(G) = X = \operatorname{spec} A$, so $Q_G(M) = Q_X(M) = M$. None-the-less, $\mathrm{Pic}(G) = \mathrm{pic}(G) = 0$.

Appendix.

It is in this appendix that we recall some facts from torsion theory and describe, for any commutative ring A, a module-theoretic formulation of the Picard group of a generically closed, quasi-compact subset of $X = \operatorname{Spec} A$ (the structure sheaf is that induced from the natural one $\mathcal{O}_X = \hat{A}$ on X by restriction). This formulation uses no sheaf theory and is reasonably easy to use in practice, so we consider a more general case as well.

For the torsion theory preliminaries, let A be any commutative ring, $G \subseteq X = \operatorname{spec} A$, with *generic closure* $g(G) = \{q \in X \mid q \subseteq p \text{ for some } p \in G\}$. Denote by $F_G = \{I \subseteq A \mid I \nsubseteq \text{any } p \in G\}$ the *torsion filter corresponding to* G (we may assume that G is generically closed, i.e., $G = g(G)$). G is quasi-compact if and only if F_G contains a cofinal subset of finitely generated ideals [6, Theorem 3.3].

F_G is a directed set under reverse inclusion so we can form, for any A-module M, the A-module $P_G(M) := \varinjlim_{I \in F_G} \mathrm{Hom}(I, M)$. Let $Q_G(M) := P_G(P_G(M))$ and $\mathcal{T}_G(M) := \{x \in M \mid \mathrm{Ann}_A x \in F_G\} \cong \varinjlim_{I \in F_G} \mathrm{Hom}(A/I, M)$, the *$\mathcal{T}_G$-torsion submodule of M*. M is *$\mathcal{T}_G$-torsion* if $\mathcal{T}_G(M) = M$ and *$\mathcal{T}_G$-torsionfree* if $\mathcal{T}_G(M) = 0$. Note that $\bar{M} := M/\mathcal{T}_G(M)$ is $\mathcal{T}_G$-torsionfree. We summarize some facts about $\mathcal{T}$ and Q in the following.

PROPOSITION A-1. (a) *$\mathcal{T}_G$ and Q_G are left exact idempotent functors* [6, *Theorem* 1.3 (1)].

(b) *There is a canonical map* φ_M^G: $M \to Q_G(M)$ *whose kernel is* $\mathcal{T}_G(M)$ *and cokernel is the first cohomology* $R^1\mathcal{T}_G(\bar{M})$. $Q_G(M) = \{x \in E(\bar{M}) \mid \exists I \in F_G, Ix \subseteq \bar{M}\}$ *where* $E(\bar{M})$ *is the injective envelope of* $\bar{M}$ [6, *Theorem* 1.3 (6′) *and* (3)].

(c) *If* α: $M \to M'$ *is a homomorphism of A-modules, then* $\alpha \otimes_A A_p$ *is an isomorphism for each* $p \in G$ *if and only if* $\ker \alpha$ *and* $\operatorname{coker} \alpha$ *are* $\mathcal{T}_G$*-torsion, if and only if* $Q_G(\alpha)$: $Q_G(M) \xrightarrow{\cong} Q_G(M')$ *is an isomorphism. In particular,* $M_p \cong Q_G(M)_p$ *for any* $p \in G$ [6, *Theorem* 1.3 (4)].

(d) $Q(A)$ *is a commutative ring,* $Q(M)$ *is a* $Q(A)$*-module, and* $\operatorname{Hom}_A(M, Q(A)) = \operatorname{Hom}_A(Q(M), Q(A)) = \operatorname{Hom}_{Q(A)}(Q(M), Q(A))$ *for any* $Q = Q_G$ [6, p. 10–11, *Proposition* 4.1 (1) *and* (2)].

(e) *If* $G \supseteq H$ *then there is a unique map* $Q_G(M) \to Q_H(M)$ *compatible with* φ_M^G *and* φ_M^H. *Thus* $Q_H Q_G = Q_H = Q_G Q_H$ [6, *Proposition* 4.1 (2) *and* 4.5].

(f) *Given* α: $M \to N$ *then* $N = Q_G(M)$ *if and only if* $\ker \alpha$ *and* $\operatorname{coker} \alpha$ *are* $\mathcal{T}_G$*-torsion,* N *is* $\mathcal{T}_G$*-torsionfree, and the natural map* $N = \operatorname{Hom}(A, N) \to \operatorname{Hom}(I, N)$ *is surjective for all* $I \in F_G$ [6, *Theorem* 1.3 (5), *Remarks* (i) *on* p. 7, *and Proposition* 1.5].

(g) *If* G *and* H *are generically closed, quasi-compact subsets of* $\operatorname{spec} A$ *then* $Q_G Q_H(M) = Q_{G \cap H}(M)$. *In particular,* $Q_G(M)_p = Q_{G \cap \operatorname{spec} A_p}(M_p)$ *for any* $p \in \operatorname{spec} A$ [6, *Theorem* A-6; *or* 19, *Corollary* 5.23].

(h) *If A is a domain then* $Q_G(A) = \bigcap_{p \in G} A_p$. *This follows easily from the description of* $Q_G(A)$ *in terms of* $E(A) = K$*, the quotient field of A, given in* (b) *above.*

(i) *If A is a noetherian ring then* $Q_G = P_G$ *for any subset* $G \subseteq \operatorname{spec} A$ [6, *Proposition* 6.3 (8)].

(j) *If* φ: $A \to B$ *is a homomorphism of noetherian rings,* $G \subseteq \operatorname{spec} A$, $G' = {}^a\varphi^{-1}(G) \subseteq \operatorname{spec} B$, *and M is a B-module, then* $Q_G(M) = Q_{G'}(M)$. *This is deduced by applying* [6, *Lemma* 7.3 *and Example* 10.4] *to part* (b) *above.*

We first investigate the notion of invertibility on a subset of $\operatorname{spec} A$ for an arbitrary commutative ring A.

DEFINITION A-2. Let $G \subseteq \operatorname{spec} A$. An A-module M is called *G-invertible* if there is a finitely generated submodule $N \subseteq M$ such that for all $p \in G$ we have $A_p \cong N_p = M_p$.

To show the needed equivalent conditions of G-invertible we use an idea of G. Picavet [16]. Let G be a quasi-compact subset of $X = \operatorname{spec} A$, M an A-module, t an indeterminate over A and M. Set $S_G = \{h \in A[t] \mid c(h) \in F_G\}$ where $c(h)$ is the ideal of A generated by the coefficients of the polynomial h. Let $\mathbf{G}(M) := S_G^{-1} M[t]$. Then the natural composition $A \to A[t] \to \mathbf{G}(A)$ is a flat ring homomorphism and $\mathbf{G}(M) = \mathbf{G}(A) \otimes_A M$ so that $\mathbf{G}$ is an exact functor. If $G = X$, write $M(t)$ in place of $\mathbf{X}(M)$ (cf. [15, p. 17–18]). Each maximal ideal of $\mathbf{G}(A)$ is of the form $p\mathbf{G}(A)$ where p is a (maximal) element of G [16, Lemma IV 11]. Then

$\mathbf{G}(M)_{p\mathbf{G}(A)} = M_p(t)$, and $\mathbf{G}$ vanishes precisely on the class of $\mathscr{T}_G$-torsion modules since each maximal element of the quasi-compact set G is the contraction of a maximal ideal of $\mathbf{G}(A)$ [16, Proposition IV 3].

We are ready for the key lemma, a generalization of [6, Theorem 8.2]. Recall that a topological space is *quasi-noetherian* [12] if it is quasi-compact and has a quasi-compact open basis (closed under finite intersections).

LEMMA A-3. *Let G be a quasi-compact subset of* spec A. *Then the following two conditions on the A-module M are equivalent.*

(1) *M is G-invertible.*

(2) a) *There is a complex $A^r \to A^s \to M \to 0$ which is exact at each $p \in G$, and*

b) *$M_p \cong A_p$ for all $p \in G$.*

These imply:

(3) *There is a finite covering of $G = \cup G_i$ by relatively open subsets G_i with $Q_{G_i}(M) \cong Q_{G_i}(A)$.*

Conversely, if the G_i can be chosen to be quasi-compact (e.g., *if G is quasi-noetherian*) *then* (3) *is equivalent to* (1) *and* (2).

PROOF. (1) $\Leftrightarrow$ (2) G-invertible means that 2 (b) holds and there is a complex $A^s \to M \to 0$ that is exact at each $p \in G$. So let $K = \ker(A^s \to M)$ and apply the exact functor $\mathbf{G}$ to obtain an *exact* sequence $0 \to \mathbf{G}(K) \to \mathbf{G}(A)^s \to \mathbf{G}(M) \to 0$. From the above remarks, $\mathbf{G}(M)$ is a locally free rank one $\mathbf{G}(A)$-module. It follows from the flatness of $\mathbf{G}(A)$ that there is a finitely generated A-submodule $L \subseteq K$ such that $\mathbf{G}(K/L) = \mathbf{G}(A) \otimes (K/L) = \mathbf{G}(A) \otimes K/\mathbf{G}(A) \otimes L = 0$, i.e., $K_p = L_p$ for all $p \in G$. Use this L to construct the desired complex 2 (a).

(1) $\Rightarrow$ (3) If $p \in G$, pick $x \in N$ that generates M at p, and map a rank one free onto $Ax \subseteq N \subseteq M$. Since N is finitely generated and M is locally free on G, there is a sufficiently small neighborhood $G_f = G \cap D(f)$ of p such that the composition $A \to N \subseteq M$ is locally an isomorphism at every $q \in G_f$. This composition induces the homomorphism $Q_{G_f}(A) \xrightarrow{\cong} Q_{G_f}(M)$ by (A-1c).

(3) $\Rightarrow$ (1) If $p \in G_i \subseteq G$ then $M_p = Q_i(M)_p \cong Q_i(A)_p = A_p$ by (A-1c) where we use the notation Q_i in place of Q_{G_i}. So we have that M is locally free on G. To construct a suitable $N \subseteq M$, let x be a generator of the cyclic $Q_i(A)$-module $Q_i(M)$. The hypothesis that G_i is quasi-compact implies that there is a finitely generated ideal $I \in F_{G_i}$ with $Ix \subseteq \bar{M} = \operatorname{Im}(M \to Q_i(M))$. Select a finitely generated A-submodule $N_i \subseteq M$ that maps onto Ix, (A-1b). This gives the diagram

$$\begin{array}{ccccc} N_i & \subseteq & M & & \\ \downarrow & & \downarrow & & \\ Ix & \subseteq & \bar{M} & \subseteq & Q_i(M). \end{array}$$

At each $p \in G_i$, all these maps become bijective so that N_i generates M locally at each such p. The finitely generated submodule $N = \Sigma N_i$, for the finite number of modules N_i so constructed, is the desired submodule.

COROLLARY A-4. *If G is quasi-compact and M is G-invertible then, for each $p \in G$, we have* $\operatorname{Hom}_A(M, Q_G(A))_p = \operatorname{Hom}_{A_p}(M_p, Q_G(A)_p) \cong A_p$.

PROOF. As in the end of the proof of [6, Theorem 8.2, p. 59–60] or [19, Lemma 5.28].

REMARK A-5. More generally, M can be locally of any (finite) constant rank on G in (A-3) and (A-4).

We now come to our definition of the Picard group of a quasi-compact subset pf the prime spectrum.

DEFINITION A-6. Let G be a quasi-compact subset of spec A. We shall say that two A-modules M and M' are *G-equivalent* ($M \sim_G M'$) if there is another module L, and maps $\varphi: M \to L$ and $\varphi': M' \to L$ such that for all $p \in G$, $\varphi \otimes A_p: M_p \to L_p$ and $\varphi' \otimes A_p: M'_p \to L_p$ are isomorphisms. G-equivalence is indeed an equivalence relation since $M \sim_G M'$ if and only if $Q_G(M) \cong Q_G(M')$, by (A-1c) (or, to check transitivity, use the pushout $L_1 \oplus L_2/\{\varphi'_1(x) - \varphi'_2(x) \mid x \in M'\}$). We use $[M]_G$ or just $[M]$ to denote the equivalence class.

Let $\operatorname{pic}(G) = \{[M]_G \mid M$ is a G-invertible A-module$\}$. That $\operatorname{pic}(G)$ is a set follows from the mapping $M \to \Pi_{p \in G} M_p \cong \Pi A_p$ since M is G-equivalent to its image in the last module. The *picard group of* G is $\operatorname{pic}(G)$ with addition $[M_1]_G + [M_2]_G = [M_1 \otimes_A M_2]_G$, identity $[A]_G$, and inverse $-[M]_G = [\operatorname{Hom}(M, Q_G(A))]_G$. The verification of inverse requires special handling, but the other properties are straightforward and left to the reader.

First $\operatorname{Hom}(M, Q_G(A))$ is locally free rank one on G, by (A-4). We can replace G by its generic closure without loss of generality. Since M is assumed G-invertible, there is a (finite) cover of $G = \cup G_i$ by sets G_i such that $Q_i(M) \cong Q_i(A)$ (see (A-3)). We may assume that the G_i are quasi-compact since G is now quasi-noetherian. If $M \to Q_G(A)$ is any A-linear homomorphism then there is induced a homomorphism $Q_i(A) \cong Q_i(M) \to Q_i(Q_G(A)) = Q_i(A)$, by (A-1e). This mapping yields the commutative diagram, for each $p \in G_i$,

$$\begin{array}{ccc} \operatorname{Hom}(M, Q_G(A))_p & \to & \operatorname{Hom}(Q_i(M), Q_i(A))_p \\ \simeq\downarrow & & \simeq\downarrow \\ \operatorname{Hom}(M_p, Q_G(A)_p) & \xrightarrow{\simeq} & \operatorname{Hom}(Q_i(A)_p, Q_i(A)_p) \end{array}$$

The right hand vertical map is an isomorphism since $Q_i(M)$ is a free $Q_i(A)$-module and from (A–1d); the left hand from (A-4); the bottom from $p \in G_i \subseteq G$, $Q_i(M) \cong Q_i(A)$, and (A-1c and e). This proves $Q_i(\operatorname{Hom}(M, Q_G(A)) \xrightarrow{\simeq} Q_i(Q_i(A)) = Q_i(A)$ is an isomorphism. Since G is quasi-noetherian, it follows that $\mathrm{Hom}(M, Q_G(A))$ is G-invertible, (A-3). The natural homomorphism $M \otimes_A \mathrm{Hom}(M, Q_G(A)) \to Q_G(A) \sim_G A$ is locally an isomorphism at each $p \in G$, thus $[\mathrm{Hom}(M, Q_G(A))]_G$ is an inverse of $[M]_G$.

We should remark that an alternative formulation of $Q_G(A)$ is $\{g/h \in \mathbf{G}(A) \mid g = \Sigma a_i t^i,\ h = \Sigma b_i t^i,\ b_i(g/h) = a_i/1 \text{ for all } i\}$ for any quasi-compact G [6, Theorem A-7], but we do not exploit this representation here.

Observe that pic is natural in that given a ring homomorphism $\varphi\colon A \to B$ and quasi-compact subsets $G \subseteq \operatorname{spec} A$, $G' \subseteq \operatorname{spec} B$, $G \supseteq {}^a\varphi(G')$ then there exists a natural group homomorphism $\mathrm{pic}(G) \to \mathrm{pic}(G')$ given by sending $[M]_G$ to $[M \otimes_A B]_{G'}$.

Here are some variations of the above definition of pic that under certain hypotheses may be more suitable.

VARIATION A-7. The representative $Q_G(M) \in [M]_G$ is uniquely determined up to isomorphism. Thus it is clear we can use *isomorphism* classes $[[M]]$ where M is G-invertible and $M = Q_G(M)$, (A-1a). The identity is then $[[Q_G(A)]]$, $[[M_1]] + [[M_2]] = [[Q_G(M_1 \otimes M_2)]]$, but the inverse remains $[[\mathrm{Hom}(M, Q_G(A))]]$, by [6, Proposition 4.1 (3)]. In this case, the group homomorphism $\mathrm{pic}(G) \to \mathrm{pic}(G')$ mentioned above sends $[[M]]$ to $[[Q_{G'}(M \otimes_A B)]]$.

VARIATION A-8. In the inverse, we could have replaced $Q_G(A)$ by $\bar{A} = A/\mathscr{T}_G(A)$ by doing the following complicated maneuver: first choose a finitely generated $N \subseteq M$ that demonstrates M is G-invertible; then show $\mathrm{Hom}(N, \bar{A})$ is G-invertible and G-equivalent to $\mathrm{Hom}(M, Q(A))$, where we denote Q_G by Q. For the latter, set $T = Q(A)/\bar{A}$, a $\mathscr{T}_G$-torsion module, (A-1b). In the exact sequence $0 \to \mathrm{Hom}(N, \bar{A}) \to \mathrm{Hom}(N, Q(A)) \to \mathrm{Hom}(N, T)$, the last term is $\mathscr{T}_G$-torsion since N is finitely generated, so the first two terms are G-equivalent. But the middle term is isomorphic to $\mathrm{Hom}(Q(N), Q(A)) = \mathrm{Hom}(Q(M), Q(A)) = \mathrm{Hom}(M, Q(A))$, (A-1c and d). Demonstrating $\mathrm{Hom}(N, \bar{A})$ is G-invertible is easy now, for it amounts to proving the following: if $\varphi_M\colon M \to Q_G(M)$ is the canonical map and $Q_G(M)$ is G-invertible, so is M. But from the quasi-compactness of G, there is a finitely generated $I \in F_G$ with $IN \subseteq \bar{M}$, from which we can construct the necessary submodule of M.

VARIATION A-9. If $T = \mathscr{T}_G(A)$ is *bounded*, i.e., if there is an ideal $I \in F_G$ such that $IT = 0$, then we may use $[\mathrm{Hom}(N, A)]_G$ for the inverse. For then we have an exact sequence $0 \to \mathrm{Hom}(N, T) \to \mathrm{Hom}(N, A) \to \mathrm{Hom}(N, \bar{A}) \to \mathrm{Ext}^1(N, T)$, with $\mathrm{Ext}^i(N, T)$ being $\mathscr{T}_G$-torsion (it's killed by I) for $i = 0, 1$. Thus, $\mathrm{Hom}(N, A)$ and $\mathrm{Hom}(N, \bar{A})$ are G-equivalent.

While this variation includes the case when A is a domain, it also is true that we may use $[\mathrm{Hom}(M, Q_G(A))]_G$ where $Q_G(A) = \cap_{p \in G} A_p$, by (A-1h). A Krull domain, for example, is particularly nice if $G = \{p \in \operatorname{spec} A \mid \operatorname{ht}(p) = 1\}$, for then $Q_G(A) = A$.

VARIATION A-10. If A is noetherian, then we need only use the finitely generated M in our equivalence classes, with inverse $-[M]_G = [\operatorname{Hom}(M, A)]_G$. This is because the natural homomorphism $M \otimes \operatorname{Hom}_A(M, A) \to A$ is an isomorphism locally at each $p \in G$ (for *any* $G \subseteq \operatorname{spec} A$) when M is finitely generated and locally free rank one on G. Of course, $\operatorname{Hom}(M, A)$ is finitely generated locally free rank one, too. Our pic is then defined without any reference to Q, for any subset $G \subseteq \operatorname{spec} A$.

VARIATION A-11. If A is noetherian *and* G contains those primes p such that depth $A_p \leqq 1$ (but otherwise arbitrary) then we have the classical situation: isomorphism classes $[[M]]$ of finitely generated, reflexive A-modules which are locally free rank one on G. This happens because each representative of the isomorphism class in (A-7) is of this form, due to the relations $\operatorname{Hom}(N, Q_G(A)) = \operatorname{Hom}(M, Q_G(A))$, $Q_G(M) \cong \operatorname{Hom}(\operatorname{Hom}(M, Q_G(A)), Q_G(A))$ (checked locally on G and use [6, Proposition 4.1 (3)]), and $A = Q_G(A)$, which we prove in the next proposition. The addition is then $[[M_1]] + [[M_2]] = [[(M_1 \otimes M_2)^{**}]]$, the identity is $[[A]]$, and inverse is $-[[M]] = [[M^*]]$ where the dual is $\operatorname{Hom}_A(\ , A)$. Also, the natural group homomorphism $\operatorname{pic}(G) \to \operatorname{pic}(G')$ (with both G and G' satisfying the depth hypothesis) sends $[[M]]$ to $[[(M \otimes B)^{**}]]$ where the dual is $\operatorname{Hom}_B(\ , B)$.

If the double dual proves awkward in an application, we note that (A-6) could also be used, with the additional knowledge that every element in the equivalence class $[M]_G$ is finitely generated and that $Q_G(M) = M^{**}$ whenever M is G-invertible.

PROPOSITION A-12. *Let A be a noetherian ring, M an A-module. Let $H \subseteq G$ be two generically closed subsets of* spec A *such that for all $p \in G \backslash H$ either $M_p = 0$ or $0 \neq M_p$ is finitely generated and* depth $M_p \geqq 2$. *Then $Q_G(M) = Q_H(M)$.*

PROOF. If we do not have an isomorphism locally at each $p \in G$, choose a $p \in G$ of smallest height where $M_p = Q_G(M)_p \neq Q_H(M)_p = Q_{H'}(M_p)$, $H' = H \cap \operatorname{spec} A_p$, by (A-1g). By (A-1g and c), we can enlarge H' to be the punctured spectrum of A_p, by the choice of p, so we are reduced to the case of a local ring (A, m), $U = \operatorname{spec} A - m$, $0 \neq M$ finitely generated of depth $\geqq 2$, and $M \neq Q_U(M)$. But depth $M > 0$ implies $\operatorname{Hom}(A/m, M) = 0$, so $M = \bar{M}$. In regards to (A-1b), we see, for a non-zero-divisor $a \in m$ on M, that $\operatorname{Hom}(A/m, E(M)/M) \cong \operatorname{Ext}^1(A/m, M) \cong \operatorname{Hom}(A/m, M/aM) = 0$, since depth $M \geqq 2$. Thus $Q_U(M)/M = \mathcal{T}_U(E(M)/M) = 0$, a contradiction.

We try our formulation of pic on a known result for Krull domains.

Let $H_1 = \{p \in \operatorname{spec} A \mid \operatorname{ht}(p) = 1\}$. Recall that a domain A is *Krull* if (i) $A = \cap_{p \in H_1} A_p$, (ii) for each $p \in H_1$, A_p is a principal ideal domain (i.e., A_p is a DVR), and (iii) each $0 \neq a \in A$ is contained in only finitely many elements of H_1. It is clear that H_1 is a (quasi-)noetherian space.

PROPOSITION A-13. *Let A be a Krull domain, $X = \operatorname{spec} A$. Suppose $G \subseteq X$ is quasi-compact and contains H_1. Then*

(1) $\operatorname{pic}(X) \subseteq \operatorname{pic}(G) \subseteq \operatorname{pic}(H_1)$.

(2) *A is a* UFD *if and only if* $\operatorname{pic}(H_1) = 0$.

(3) *If each element in H_1 is finitely generated and if A_p is a* UFD *for each $p \in G$ then* $\operatorname{pic}(G) = \operatorname{pic}(H_1)$.

PROOF. (1) Set $H = H_1$. There are natural group homomorphisms $\operatorname{pic}(X) \to \operatorname{pic}(G) \to \operatorname{pic}(H)$ given by restricting the equivalence relations to smaller sets. So to prove injectivity, let $[M]_G \in \operatorname{pic}(G)$. If $[M]_H = [A]_H \in \operatorname{pic}(H)$ then $Q_H(M) \cong Q_H(A) = \cap_{p \in H} A_p = A$ since A is Krull, (A-1b). We check that the natural map $M \to Q_H(M)$ is locally an isomorphism at each $p \in G \supseteq H$, using $M_p \cong A_p$ in the commutative diagram

$$\begin{array}{ccccccc} & & M_p & \cong & A_p & & \\ & \swarrow & \downarrow & & \downarrow & \searrow^{\cong} & \\ Q_H(M)_p & = & Q_H(M_p) & \cong & Q_H(A_p) & = & Q_{H'}(A_p), \end{array}$$

where we have used (A-1g), and have put $H' := H \cap \operatorname{spec} A_p$, the set of height one primes of the Krull domain A_p. Then $M \sim_G Q_H(M) \cong A$ and our maps are injective.

(2) Now suppose $\operatorname{pic}(H) = 0$ and p is any height one prime. For A to be a UFD we need to show [8, Proposition 6.1] that p is principal (i.e., free rank one). It is easy to see from the above two defining properties (ii) and (iii) of a Krull domain that p is H-invertible, so that $\operatorname{pic}(H) = 0$ implies $Q_H(p) \cong Q_H(A) = A$. But [6, Lemma 8.9 (2)] says for any $I \subseteq A$ that $Q_H(I)$ is the intersection of all the symbolic powers (containing I) of height one primes. In particular $Q_H(p) = p$, hence p is principal for each $p \in H$ and A is a UFD.

Conversely, if A is a UFD, let $[[M]] \in \operatorname{pic}(H)$ (we use (A-7) here) with $M = Q_H(M)$. Then M embeds in $\Pi_{p \in H} M_p \cong \Pi A_p$, a torsionfree (usual) module over the domain A. So M is isomorphic to a non-zero A-submodule $I \subseteq M_{(0)} \cong K$, the quotient field of A. $I \neq 0$ implies $I^{-1} \cong I^* \neq 0$, (A-4). If $0 \neq x \in I^{-1}$ then $I \cong xI \subseteq A$ so we may assume M is isomorphic to an ideal I of A with $I = Q_H(I) = \cap p_i^{(n_i)}$. Now it is true in any domain that a symbolic power of a principal prime p_i is an ordinary power, and since there are no containment relations among the p_i's, their intersections will agree with their products, so I is principal, i.e., rank one free, and $\operatorname{pic}(H) = 0$.

(3) If A_q is a UFD for all $q \in G$, and $p \in H$ is finitely generated then p is

G-invertible and $[p]_G \in \mathrm{pic}(G)$ maps to $[p]_H \in \mathrm{pic}(H)$. On the other hand, we have seen earlier that each equivalence class in $\mathrm{pic}(H)$ has a representative $I = Q_H(I) = \cap p_i^{(n_i)}$, $p_i \in H$. But $Q_H(\otimes_i p_i^{\otimes n_i}) = Q_H(\Pi p_i^{n_i}) = \cap p_i^{(n_i)} = I$ (check locally on H using A-1c)) so that $\mathrm{pic}(H)$ is generated by the classes of height one primes and our map is surjective.

The reader may prove that indeed $\mathrm{pic}(H_1)$ is just the *divisor class group of* A, the group of divisorial fractional ideals modulo the subgroup of principal fractional ideals. Also, $\mathrm{pic}(X)$ $(= \mathrm{Pic}(A))$ is just the *ideal class group of* A, the invertible fractional ideals modulo the principal's. For these results the following are useful: if $I^* \neq 0$ then $I^{**} \cong I^{-1-1}$; if I is not the quotient field K of A then $I \xrightarrow{\simeq} Q_H(I) \cong I^{**}$ if and only if $I = I^{-1-1}$; $I \cong J$ if and only if $I = xJ$ for some $0 \neq x \in K$; Q_X is the identity functor so $M \otimes M^* \xrightarrow{\simeq} A$ whenever M is X-invertible (= rank one projective).

Our next goal is to show that, with sufficient hypothesis on $G \subseteq X = \mathrm{spec}\, A$, $\mathrm{pic}(G)$ is isomorphic to the algebraic geometers' $\mathrm{Pic}(G)$, the group of isomorphism classes of invertible (= locally free rank one) $\mathcal{O}_G$-Modules, where $\mathcal{O}_G$ is the natural structure sheaf $\mathcal{O}_X$ on X restricted down to G. There is a discussion of this in [19, Proposition 7.10], but a different approach (sheaf theoretic) to $\mathrm{pic}(G)$ is used. Some of our ideas are from [19], but our method of proof is different in that we first prove an important conjectured result, a generalization of [19, 5.25].

LEMMA A-14. *If G is a quasi-compact subset of $X = \mathrm{spec}\, A$ then for each A-module M we have $Q_G(M) = \varinjlim Q_U(M)$, where the direct limit* (*with the natural restriction maps of* (A-1e)) *is taken over all open subsets U of X containing G. Furthermore, these isomorphisms are compatible with the restriction maps $Q_G(M) \to Q_{G'}(M)$ for all quasi-compact G and G' with $G \supseteq G'$.*

PROOF. We need only use the quasi-compact open $U \supseteq G$ in the direct limit as they form a cofinal subset. The natural maps $\varphi_M^U: M \to Q_U(M)$ induce $\varphi: M \to \varinjlim Q_U(M)$. We need, by (A-1f), that $\ker \varphi$ and $\mathrm{coker}\, \varphi$ are $\mathcal{T}_G$-torsion. This follows from (A-1c) since for each $p \in G \subseteq U$, $M_p \to Q_U(M)_p$ is an isomorphism, so that $\varphi \otimes A_p$ is an isomorphism, too.

We also need that this direct limit is $\mathcal{T}_G$-torsionfree. Let $[x] \in \varinjlim Q_U(M)$, and $I = (a_1, \ldots, a_n) \in F_G$ a finitely generated ideal such that $I[x] = 0$ (G is quasi-compact). Then each $a_i x$ maps to zero in the direct limit, hence to zero in some $Q_{U_i}(M)$, $U_i \supseteq G$. Thus $Ix = 0$ in $Q_V(M)$, $V = \cap U_i$. Since $I \in F_U \subseteq F_V$ and $Q_V(M)$ is always $\mathcal{T}_V$-torsionfree, $x = 0$ and we have proved this part.

It remains to prove that for any $I \in F_G$, a homomorphism $f: I \to \varinjlim Q_U(M)$ lifts to A. We can assume $I = (a_1, \ldots, a_n)$ is finitely generated. Map a free module A^n onto I with kernel $K \subseteq A^n$. Since I is U-invertible, $U = D(I)$, apply (A-3) to find a finitely generated $L \subseteq K$ such that K/L is $\mathscr{T}_U$-torsion (or prove this directly). Now lift the $f(a_i)$'s to x_i's in a common $Q_V(M)$, V open and $G \subseteq V \subseteq U$. Then the images of a finite generating set for L in $Q_V(M)$ will map to zero in the direct limit. Replacing V by a sufficiently smaller open set containing G, we can assume L maps to zero in $Q_V(M)$, hence the images of K and K/L are the same in $Q_V(M)$. However, K/L is $\mathscr{T}_U$-torsion, hence $\mathscr{T}_V$-torsion, while $Q_V(M)$ is $\mathscr{T}_V$-torsionfree. Thus K/L (hence K) maps to zero in $Q_V(M)$. Deduce from this that f factors through some $f'\colon I \to Q_V(M)$. But $I \in F_U \subseteq F_V$, so by the lifting property of $Q_V(M)$ there is a lifting of f' to A from which is obtained the desired lifting f'' of f.

$$\begin{array}{ccccc}
K & \subseteq & A^n & \twoheadrightarrow & I \subseteq A \\
\downarrow & & \downarrow & \swarrow^{f'} \; f\downarrow & \swarrow^{f''} \\
K/L & \xrightarrow{0} & Q_V(M) & \to & \varinjlim Q_U(M)
\end{array}$$

To show compatibility of our isomorphisms with the restriction maps, for $G \supseteq G'$ quasi-compact, look at the diagram

$$\begin{array}{ccc}
Q_G(M) & \xrightarrow{\simeq} & \varinjlim\limits_{U \supseteq G} Q_U(M) \\
 & \nwarrow \quad \nearrow & \\
\downarrow & M & \downarrow \\
 & \swarrow \quad \searrow & \\
Q_{G'}(M) & \xrightarrow{\simeq} & \varinjlim\limits_{V \supseteq G'} Q_V(M)
\end{array}$$

which has commutative triangles. Since $\operatorname{Hom}(M, Q_{G'}(M)) = \operatorname{Hom}(Q_G(M), Q_{G'}(M))$ by (A-1d), the square commutes.

Recall that a quasi-compact, generically closed subset of spec A is always the image of the (reverse) spec map ${}^a\varphi$ of some flat ring homomorphism $\varphi\colon A \to B$, and conversely. Since these subsets form a basis of the closed sets of the flat topology [7, Theorem 2.2], we shall refer to them as the *flat subsets of spec A*. A flat subset G of spec A is quasi-noetherian since subsets of the form $G \cap U$ where U is a quasi-compact open subset of spec A is again quasi-compact.

For an arbitrary commutative ring A, let $\mathcal{O} = \mathcal{O}_X = \tilde{A}$ denote the natural structure sheaf of rings on $X = \operatorname{Spec} A$, and, for an A-module M, let $\tilde{M}$ be the $\mathcal{O}$-Module canonically associated to M. Let $G \subseteq \operatorname{Spec} A$. Recall that the definition of the restriction sheaf $\mathcal{O}_G := \mathcal{O}|_G$ and the $\mathcal{O}_G$-Module $\tilde{M}|_G$ involve direct limits of sections over open $U \supseteq G$ to define a presheaf, followed by the sheafification process [10, p. 65 and Proposition-Definition II 1.2].

THEOREM A-15. *Let G be a flat subset of X = Spec A, M an A-module. Then, in the definition of $\mathcal{O}_G$ and $\tilde{M}|_G$, the sheafification process is not needed. More* *precisely, the presheaf assignment* $G_0 \mapsto \varinjlim \{\tilde{M}(U) \mid G_0 \subseteq U \text{ open } \subseteq X\}$ *defined on the basis of all flat relatively open subsets* G_0 *of* G *is, in fact, a sheaf on this basis, with* $\tilde{M}|_G(G \cap U) \cong \varprojlim \{Q_{G \cap U_0}(M) \mid G \cap U_0 \subseteq G \cap U,\ U_0 \text{ quasi-compact open } \subseteq X\}$ *for any open* $U \subseteq X$. *In particular,* $\tilde{M}(G) := \tilde{M}|_G(G) \cong Q_G(M)$ *and* $\mathcal{O}(G) := \mathcal{O}_G(G) \cong Q_G(A)$, *and these isomorphisms commute with restrictions.*

PROOF. For a discussion of (pre)sheaves on a basis, see [14, p. 32–34]. If U_0 is a quasi-compact open subset of X then, if G is flat, $G_0 = G \cap U_0$ is a quasi-compact relatively open subset of G. By [6, Theorem 5.1; or 19, Proposition 5.16] and (A-14), the assignment $G_0 \mapsto Q_{G_0}(M)$ defines our above stated presheaf (on the basis of quasi-compact relatively open subsets G_0 of G) for the sheaf $\tilde{M}|_G$. But this presheaf is actually a sheaf on this basis, as we now show.

(1) If $G_0 = \cup G_i$, $G_0 \subseteq G$, and $x \in Q_{G_0}(M)$ is such that x is zero in $Q_{G_i}(M)$ for each i, then $x \in \cap_i \mathcal{T}_{G_i}(Q_{G_0}(M)) = \mathcal{T}(Q_{G_0}(M)) = 0$, (A-1b and e). This establishes uniqueness.

(2) Let $G_0 = \cup G_i$, where G_0 and the G_i are flat relatively open subsets of G, and let $\{x_i\}$, $x_i \in Q_{G_i}(M)$, be a family of elements that agree on overlaps $G_i \cap G_j$. We want to find an $x \in Q_{G_0}(M)$ that will agree with the x_i's (x is unique by part (1)). We can assume the G_i are of the form $G \cap D(f_i)$, $f_i \in A$, and argue as in [6, Theorem A-2, p. 105] using this refined cover of G_0 (just as in that proof, one needs to use part (1) again to show that the chosen x agrees with the x_i's on the original cover $\{G_i\}$).

The compatibility follows from the facts (a) the presheaf defines the sheaf, (b) it is true for all quasi-compact open $U \subseteq X$, and (c) the last item in (A-14).

REMARKS A-16. It is not known if $\tilde{M}|_G \cong Q_G(M)$ when G is *not* quasi-compact, even if G is open in X. Nor has the case where G is not generically closed been studied in the literature (see Example 8.3 before the Appendix). Also, there still remains the question of whether $\tilde{M}|_G(G \cap U) = Q_{G \cap U}(M)$ when G is flat and U is *any* open subset of X.

PROPOSITION A-17. *Let* G *be a flat subset of* Spec A, *and* $\mathcal{L}$ *an invertible* $\mathcal{O}_G$*-module on* G. *Then* $\mathcal{L} \cong \widetilde{\mathcal{L}(G)}|_G$, *and for each* $x = p \in G$ *we have the stalk* $\mathcal{L}_x \cong \mathcal{L}(G)_p$.

PROOF. Our aim is to define the local maps of sections over a typical flat relatively open subset $G_0 \subseteq G$. First choose a cover of G of flat relatively open subsets G_i such that $\mathcal{L}|_{G_i} \cong \mathcal{O}_{G_i}$ (from the definition of an invertible sheaf). We have a commutative diagram, since $\mathcal{L}$ is a sheaf,

$$(*) \qquad \begin{array}{ccccccc} 0 & \to & \mathcal{L}(G) & \to & \oplus \mathcal{L}(G_i) & \to & \oplus \mathcal{L}(G_i \cap G_j) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathcal{L}(G_0) & \to & \oplus \mathcal{L}(G_0 \cap G_i) & \to & \oplus \mathcal{L}(G_0 \cap G_i \cap G_j) \end{array}$$

Now the commutative diagram (use (A-15))

$$\begin{array}{ccccc} \mathscr{L}(G_i) & \cong & \mathcal{O}(G_i) & \cong & Q_{G_i}(A) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{L}(G_0 \cap G_i) & \cong & \mathcal{O}(G_0 \cap G_i) & \cong & Q_{G_0 \cap G_i}(A) \end{array}$$

yields upon application of Q_{G_0} that $Q_{G_0}(\mathscr{L}(G_i)) \cong \mathscr{L}(G_0 \cap G_i)$, by (A-1g). Similarly, $Q_{G_0}(\mathscr{L}(G_i \cap G_j)) \cong \mathscr{L}(G_0 \cap G_i \cap G_j)$. From (*), (A-15), and these isomorphisms, we conclude that $\widetilde{\mathscr{L}(G)}(G_0) \cong Q_{G_0}(\mathscr{L}(G)) \xrightarrow{\cong} \mathscr{L}(G_0)$. These maps will define an isomorphism $\widetilde{\mathscr{L}(G)} \xrightarrow{\cong} \mathscr{L}$ of sheaves (on a basis) if we show they are compatible with restrictions. To see this, let $G_0 \supseteq G_0'$. Then the commutative diagram

$$\begin{array}{ccc} \mathscr{L}(G) & \rightarrow & \mathscr{L}(G_0) \\ \| & & \downarrow \\ \mathscr{L}(G) & \rightarrow & \mathscr{L}(G_0') \end{array}$$

induces

$$\begin{array}{ccccc} \mathscr{L}(G) & & \rightarrow & & \mathscr{L}(G_0) \\ & \searrow & & \nearrow & \\ & & Q_{G_0}(\mathscr{L}(G)) & & \\ \| & & \downarrow & & \downarrow \\ & & Q_{G_0'}(\mathscr{L}(G)) & & \\ & \nearrow & & \searrow & \\ \mathscr{L}(G) & & \rightarrow & & \mathscr{L}(G_0') \end{array}$$

We have just shown the upper and lower triangles are commutative, and the left side is (A-1e), so the right hand trapezoid is commutative, by (A-1d), whence the sheaf isomorphism is established.

Now take stalks at $x = p \in G$ to show $\mathscr{L}(G)_p \cong \mathscr{L}_x$.

THEOREM A-18. *Let G be a flat subset of $X = \operatorname{Spec} A$. Then* $\operatorname{pic}(G) \cong \operatorname{Pic}(G, \mathcal{O}_G)$.

PROOF. If $\mathscr{L}$ is an invertible $\mathcal{O}_G$-Module then (A-15) and (A-17) tell us that $Q_G(\mathscr{L}(G)) = \mathscr{L}(G)$. It is also clear that if $\mathscr{L} \cong \mathscr{L}'$ then $\mathscr{L}(G) \cong \mathscr{L}'(G)$. Thus, if we use (A-7), the map $\operatorname{Pic}(G) \to \operatorname{pic}(G) = \{[[M]] \mid M = Q_G(M), M \text{ is } G\text{-invertible}\}$ given by $[[\mathscr{L}]]$ goes to $[[\mathscr{L}(G)]]$ is well defined provided we show that $\mathscr{L}(G)$ is G-invertible. Cover $G = \cup G_i$, $G_i = G \cap D(f_i)$, $f_i \in A$, so that $\mathscr{L}|_{G_i} \cong \mathcal{O}_{G_i}$. Then $Q_{G_i}(\mathscr{L}(G)) \cong \mathscr{L}(G_i) \cong \mathcal{O}(G_i) \cong Q_{G_i}(A)$, by (A-15) and (A-17). From (A-3), we have that $\mathscr{L}(G)$ is G-invertible.

To see that the map is bijective, define a map in the reverse direction by sending the isomorphism class $[[M]]$ to $[[\tilde{M}|_G]] \in \operatorname{Pic}(G)$; of course, M is G-invertible and $M = Q_G(M)$. $\tilde{M}|_G$ is, indeed, an invertible sheaf since, for a suitable covering of $G = \cup G_i$, we have $Q_{G_i}(M) \cong Q_{G_i}(A)$, by (A-3). Hence, the canonical homomorphism $M \to Q_{G_i}(M)$ induces the isomorphism $\tilde{M}|_{G_i} \xrightarrow{\cong} \widetilde{Q_{G_i}(M)}|_{G_i} \cong \widetilde{Q_{G_i}(A)}|_{G_i} \xleftarrow{\cong} \tilde{A}|_{G_i} \cong \mathcal{O}_{G_i}$, and $\tilde{M}|_G$ is an invertible sheaf. These maps are inverses of each other since $\mathcal{L} \cong \widetilde{\mathcal{L}(G)}|_G$ by (A-17), and $\tilde{M}|_G(G) \cong Q_G(M) = M$, by (A-15).

Our mappings are group homomorphisms. To see this, let $[[\mathcal{L}]]$, $[[\mathcal{L}']] \in \mathrm{Pic}(G)$ and consider the morphism of preschemes to the sheafification $\mathcal{L} \otimes_{\mathcal{O}_G} \mathcal{L}'$. The homomorphism over the set G is then $\mathcal{L}(G) \otimes_{\mathcal{O}G)} \mathcal{L}'(G) \to (\mathcal{L} \otimes \mathcal{L}')(G)$. We claim this homomorphism of A-modulus is locally an isomorphism at each $p = x \in G$. It induces the commutative diagram

$$\begin{array}{ccccc} \mathcal{L}(G)_p \otimes_{\mathcal{O}(G)_p} \mathcal{L}'(G)_p & \xrightarrow{\cong} & (\mathcal{L}(G) \otimes_{\mathcal{O}(G)} \mathcal{L}'(G))_p & \to & (\mathcal{L} \otimes \mathcal{L}')(G)_p \\ \downarrow{\cong} & & & & \cong\downarrow \\ \mathcal{L}_x \otimes_{\mathcal{O}_x} \mathcal{L}'_x & & \xrightarrow{\cong} & & (\mathcal{L} \otimes \mathcal{L}')_x \end{array}$$

The vertical maps are isomorphisms from (A-17), the lower one an isomorphism since a presheaf and its sheafification have the same stalks [10, p. 64]. From these isomorphisms, for each $p \in G$, we conclude $Q_G(\mathcal{L}(G) \otimes_A \mathcal{L}'(G)) \cong Q_G((\mathcal{L} \otimes \mathcal{L}')(G)) = (\mathcal{L} \otimes \mathcal{L}')(G)$ since $\mathcal{L} \otimes \mathcal{L}'$ is an invertible sheaf on G.

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ON A NON-ABELIAN VARIETY OF GROUPS WHICH ARE SYMMETRIC ALGEBRAS

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I.

It is known that symmetric operations have nice properties and there are many types of algebraic systems in which commutative systems play very special role. This is the case, for example, of groups, rings and modules. From the algebraic point of view two algebras $(A; \mathbf{F}_1)$ and $(A; \mathbf{F}_2)$ are equal if the sets $A(\mathbf{F}_1)$ and $A(\mathbf{F}_2)$ of all their algebraic operations (= all superpositions of fundamental operations and the projections) coincide (cf. [2]). It may happen that not all fundamental operations $\mathbf{F}$ of the algebra $(A; \mathbf{F})$ are symmetric, i.e. do admit of all permutations of their variables but $(A; \mathbf{F})$ is a symmetric algebra. This means that all non-symmetric operations from $\mathbf{F}$ can be presented as a superpositions of symmetric algebraic operations of the algebra $(A; \mathbf{F})$ and the projections $e_n^k(x_1, x_2, \ldots, x_n) = x_k$, $1 \leqq k \leqq n$, $n = 1, 2, 3, \ldots$.

It is clear that the Abelian group $(G; ., ^{-1}, 1)$ is symmetric algebra in the sense. In [4] E. Marczewski has asked whether there are non-Abelian groups which are symmetric algebras. Since 0-ary and 1-ary operations are symmetric the question is whether the group operation $\cdot$ can be expressed as a superposition of projections and of symmetric algebraic operations, which, in the case of groups, are symmetric words. It turned out [5] that such group exists.

In this note we find a non-Abelian variety (= equationally definable class) of groups which have the same property.

We prove the following

THEOREM. *If the group G satisfies the following identities*

$$x^6 = 1 \tag{1}$$

$$[x, y]^3 = 1 \tag{2}$$

$$[x^2 y^2] = 1 \tag{3}$$

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then G is symmetric algebra. Namely we have

$$xy = w(q(q(x, y), y^4), w(w^4(x, y), s(x, y, x))), \tag{4}$$

where

$$w(x, y) = xy[x, y] \tag{5}$$

$$q(x, y) = xy[x, y]x^2y^2 \tag{6}$$

$$s(x, y, z) = [x, y, z][z, y, x] \tag{7}$$

are symmetric operations in the group G.

II.

Let us begin with notations and auxiliary results which enable us to prove the theorem. As usual, $[x, y] = x^{-1}y^{-1}xy$, and

$$[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n] \quad \text{for} \quad n > 2.$$

The following relations

$$xy = yx[x, y] \tag{8}$$

$$[x, y]^{-1} = [y, x] \tag{9}$$

$$[xy, z] = [x, z][x, z, y][y, z] \tag{10}$$

$$[x, yz] = [x, z][x, y][x, y, z] \tag{11}$$

are identities in any group [cf. [1]).

Lemma 1. *If* (1)–(3) *are identities in G, then the following equations*

$$[[x, y], z^2] = 1 \tag{12}$$

$$[[x, y], [z, u]] = 1 \tag{13}$$

$$[y, x, z] = [x, y, z]^2 \tag{14}$$

$$[y, x, y, x] = [x, y, y, x]^2 \tag{15}$$

$$[y, x, y, x] = [y, x, x, y] \tag{16}$$

$$[y, x, x, y, y] = [y, x, x, y] \tag{17}$$

$$[y, x, y, x, x] = [y, x, y, x] \tag{18}$$

$$[y, x, x, x] = [y, x, x] \tag{19}$$

$$[x, y^{2k}, y] = 1 \tag{20}$$

are identities in G for $k = 1, 2, 3, \ldots$

PROOF. If we take $[x,z]^2$ instead of x into (3) and apply (2) we get (12). This implies (13) by putting $[z,u]^2$ instead of z. Using (2), (10) and (13) we have

$$[y,x,z] = [[x,y]^{-1},z] = [[x,y]^2,z]$$
$$= [x,y,z][[x,y,z],[x,y]][x,y,z] = [x,y,z]^2,$$

which yields (14).

From (3), (10), (12) and (14) it follows

$$[y,x,y,x] = [[x,y,y]^2,x] = [x,y,y,x]^2$$

which gives (15).

It is known (cf. e.g. [1]) that in metabelian groups, i.e. in the groups with identity (12), the following Jacobi identity

$$[x,y,z][y,z,x][z,x,y] = 1 \tag{21}$$

holds. Thus we have

$$1 = [x,y,[x,y]][y,[x,y],x][x,y,x,y]$$

which together with (2), (3), (9), (10), (12) and (14) gives

$$[x,y,x,y] = [y,[x,y],x]^{-1} = [[x,y,y]^2,x]^2 = [x,y,y,x]^4 = [x,y,y,x]$$

i.e. equality (16) is fulfiled.

The equalities (17), (18) and (19) follow from (2), (11) and (12), because we have

$$1 = [y,x,x,y^2] = [y,x,x,y]^2[y,x,x,y,y]$$
$$1 = [y,x,y,x^2] = [y,x,y,x]^2[y,x,y,x,x]$$
$$1 = [y,x,x^2] = [y,x,x]^2[y,x,x,x].$$

Now the Jacoby identity yields

$$1 = [x,y^{2k},y][y^{2k},y,x][y,x,y^{2k}],$$

which together with (12) gives (20) and lemma 1 follows.

LEMMA 2. *If the equations* (1)–(3) *are identities in the group G, then the words w, q and s defined by the formulas* (5), (6) *and* (7), *respectively, are symmetric operations in G.*

PROOF. By (2), (8) and (9) we have

$$w(y,x) = yx[y,x] = xy[y,x]^2 = xy[x,y] = w(x,y)$$

Using this and (12) we get

$$q(y,x) = yx[y,x]y^2x^2 = w(y,x)y^2x^2 = w(x,y)x^2y^2 = q(x,y).$$

To prove s is ternary symmetric operation in G observe that the cycles (2, 3, 1) and (1, 3) generate the symmeetric group S_3 of all permutations on three leters x, y and z. Now from (13) we have

$$s(z, y, x) = [z, y, x][x, y, z] = s(x, y, z)$$

We have also by (2), (13), (14) and (21)

$$s(y, z, x) = [y, z, x][x, z, y] = [z, y, x][z, y, x][x, z, y]$$
$$= [z, y, x][y, x, z]^{-1} = [x, y, z][z, y, x] = [x, y, z],$$

which completes the proof.

III. Proof of the theorem.

First of all we calculate $q^2(x, y)$. Using (8) we get

$$q^2(x, y) = xy[x, y]x^2y^2xy[x, y]x^2y^2$$
$$= x^2y[x, y, x]x^2y^2[y^2, x]y[x, y]x^2y^2$$

Observe that $[y^2, x] = [x, y^2]^2$, because of (2) and (9). This together with (20) gives $[y^2, x]y = y[y^2, x]$. Hence, once more from (8), we get

$$q^2(x, y) = x^2y^2[x, y, x][x, y, x, y]x^2[x^2, y]y^2[y^2, x][x, y]x^2y^2$$

Now the equality (13), (9) and (10) yield

$$[x^2, y][y^2, x] = [x, y][x, y, x][x, y][y, x][y, x, y][y, x] = [x, y, x][y, x, y].$$

Therefore, in view of (1), (2), (3), (12) and (13) we get

$$q^2(x, y) = [x, y][x, y, x]^2[y, x, y][x, y, x, y]$$

Now we are going to calculate $a = q(q(x, y), y^4)$. Taking into account (1), (3), (6), (10), (12) and (13) we get

$$a = xy[x, y]x^2y^2y^4[xy[x, y]x^2y^2, y^4]q^2(x, y)y^8$$
$$= xyx^2y^2[x, y][xy, y^4]q^2(x, y)$$

It follows from (10) and (13) that

$$[x, y^2] = [x, y]^2[x, y, y] \tag{22}$$

This together with (2), (10) and (13) gives

$$[xy, y^4] = [x, y^4][x, y^4, y] = [x, y^4] = [x, y^2]^2 = [x, y][x, y, y]^2$$

Therefore we have

$$a = x^3y[y,x^2]y^2[x,y]^2[x,y,y]^2q^2(x,y) = x^3y^3[y,x,x][x,y,y]^2q^2(x,y),$$

as a consequence of (8), (9) and (13). Now using (13) and (14) we obtain

$$a = x^3y^3[x,y][x,y,y][x,y,x][x,y,x,y] = x^3y^3c_1$$

for a suitable product c_1 of commutators. It follows from (13) and the definitions of the operations s and w that

$$b = w(w^4(x,y), s(x,y,x)) = (xy[x,y])^4[x,y,x]^2$$

Thus, in viewof (2), (3), (6) and (9), we get

$$(xy[x,y])^4 = (xy[x,y]xy[x,y])^2 = (x^2y[y,x][x,y][x,y,x]y[x,y])^2$$
$$= (x^2y^2[x,y][x,y,x][x,y,x,y])^2 = x^4y^4[y,x][x,y,x]^2[x,y,x,y]^2$$

and consequently

$$b = x^4y^4[y,x][x,y,x][x,y,x,y]^2 = x^4y^4c_2$$

for a suitable product c_2 of commutators.

In order to calculate $[a,b]$ let us consider the commutator $[x^3y^3c_1,\alpha]$, α being x or y or else c_2. In view of (3), (6) and (10) we have

$$[x^3y^3c_1,\alpha] = [xyc_1,\alpha][xy,\alpha] = [x,\alpha][x,\alpha,y][y,\alpha]$$

This together with (11) and (20) yields

$$[a,b] = [x^3y^3c_1, x^4y^4c_2] = [x^3y^3c_1,x^4][x^3y^3c_1,y^4][x^3y^3c_1,c_2]$$
$$= [y,x^4][x,y^4][x,c_2][x,c_2,y][y,c_2]$$

Now by (3) and (22) we have

$$[y,x^4] = [y,x^2]^2 = ([y,x]^2[y,x,x])^2 = [y,x]^4[y,x,x]^2$$

and similarly

$$[x,y^4] = [x,y]^4[x,y,y]^2$$

which together with (13) and (14) gives

$$[a,b] = [x,y,x][y,x,y][x,c_2][x,c_2,y][y,c_2]$$

It follows from (2), (11) and (13) that

$$[x,c_2] = [y,x,x]^2 = [x,y,x,x]^2[[x,y,x,y]^2,x]^2,$$

which, in view of (14), (15), (16) and (18), can be rewrite as

$$[x,c_2] = [x,y,x][y,x,x][y,x,x,y,x]^2 = [x,y,y,x]$$

Thus we get

$$[x, c_2, y] = [x, y, y, x, y] = [x, y, x, y, y] = [x, y, x, y]$$

as it follows from (16) and (18).

Using the same arguments we have also

$$[y, c_2] = [y, x, y]^2[x, y, x, y]^2[[x, y, x, y]^2, y]^2$$
$$= [x, y, y][y, x, x, y][y, x, x, y]^2 = [x, y, y].$$

Thus

$$[a, b] = [x, y, x][y, x, y][x, y, y, x][x, y, x, y][x, y, y]$$
$$= [x, y, x][y, x, y, x]$$

as it follows from (2), (3), (14), (15) and (16).

Now we are able to calculate $w(a, b)$. The identities (1)–(3) and (8) give

$$w(a, b) = ab[a, b]$$
$$= x^3y^3[x, y][x, y, x][x, y, y][x, y, x, y]x^4y^4[y, x][x, y, x][x, y, x, y]^2ab$$
$$= x^3y^3[x, y, x]^2x^4y^4[x, y, x][x, y, y][y, x, y, x] = x^3yx^4[x, y, y][y, x, y, x]$$
$$= xy[x^2, y][x, y, y][y, x, y, x]$$

Since $[x^2, y] = [x, y]^2[x, y, x]$ we have, by (16),

$$w(a, b) = xy[y, x][x, y, x][x, y, y][y, x, x, y]$$

Thus, it is enough to prove that the last product of commutators equals 1. To do this we use (2), (3), (10) and (11). We have

$$1 = [y^2, x^2] = [y, x^2]^2[y, x^2, y] = ([y, x]^2[y, x, x])^2[[x, y][y, x, x], y]$$
$$= [y, x][x, y, x][x, y, y][y, x, x, y],$$

which completes the proof of the theorem.

COROLLARY. *In the normal product Z_3Z_2 of the cyclic group Z_3 by the group Z_2 of all its automorphisms* (i.e. *the group S_3*) *all equations* (1)–(3) *are fulfilled. This gives another proof of a result from* [5].

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COHOMOLOGIES ET EXTENSIONS DE CATEGORIES

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0. Introduction.

Dans des travaux précédents (voir [10] et [11]), nous avons défini une notion d'extension de catégories qui généralise celle, classique, d'extension de groupes (exposée dans [12]). Elle fut introduite dans le but d'interpréter la cohomologie (abélienne) des petites catégories en dimension 2. Elle a été utilisée par T. Porter (dans [14]) pour l'étude des modules croisés dans $\mathscr{C}at$ qu'il relie aux catégories internes. Elle intervient aussi en cohomologie de dimension supérieure, comme l'a montré M. Golasinski (dans [7]), en considérant les extensions n-uples. Des cas particuliers sont utilisés dans divers ouvrages ([5] et [16] par exemple). On verra ici qu'elle contribue à interpréter des cohomologies non nécessairement abéliennes et à coefficients non nécessairement fonctoriels comme, par exemple, celle définie par Z. Wojtkowiak (dans [17]) qui apparait naturellement en théorie de l'obstruction concernant les limites homotopiques.

Considérant ses relations avec les catégories fibrées de Grothendieck (dans [9]), on peut aussi élargir la notion d'extension de catégories et les cohomologies ainsi décrites.

Baues définit, dans [3], une cohomologie abélienne des petites catégories qui généralise la cohomologie à coefficient dans un module (i.e. un foncteur à valeurs dans les groupes abéliens) étudiée dans [10] et [11]. Des extensions définies dans ce cadre contribuent, entre autre, à l'étude de la classification des types d'homotopie d'espaces et des classes d'homotopie d'applications (voir [1] et [2]).

Nous commencerons (section 1) par un exposé des principales propriétés des extensions de catégories définies en [10] et [11]. Cette notion est ensuite élargie (section 2) et l'on voit apparaître naturellemant les cocycles. On a alors une cohomologie (section 3) qui classifie les extensions larges.

La cohomologie est souvent associée à un coefficient, cette notion est ici précisée (section 4) et confrontée aux extensions de catégories. Nous terminons (section 5) par l'étude des cohomologies à coefficients en groupes.

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Les catégories considérées C, H, K,... sont des petites catégories à l'exception de la catégorie des groupes $\mathcal{G}r$, de la catégorie des groupes abéliens $\mathcal{A}b$, de la catégorie des petites catégories $\mathcal{C}at$ ou de catégories dont les objets sont les groupes, $\mathcal{C}gr$ ou $\mathcal{H}gr$, ou les petites catégories $\mathcal{C}cat$.

Si C est une catégorie, on note C_0 l'ensemble de ses objets, on note C_1 l'ensemble de ses morphismes. On considère C_0 comme une partie de C_1 en identifiant chaque objet et son morphisme identique. Pour deux objets C et C' de C, on note $C(C, C')$ l'ensemble des morphismes $c \in C_1$ de source $\alpha(c) = C$ et de but $\beta(c) = C'$. On note respectivement C_2 et C_3 les ensembles des couples $(c, c') \in C_1 \times C_1$ et des triples $(c, c', c'') \in C_1 \times C_1 \times C_1$ composables, i.e. tels que $\alpha(c) = \beta(c')$ et $\alpha(c') = \beta(c'')$.

1. Extensions de categories.

Rappelons la notion que nous avons définie en [10] et [11]. On la retrouve, pour les groupoïdes dans [5].

DÉFINITION 1.1. Une *extension de catégories* (de la catégorie C par la catégorie K) est une suite

$$E\colon \mathsf{K} \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$$

où i est un foncteur fidèle et C une catégorie quotient de H, le foncteur projection étant p. De plus, pour h et $h' \in H_1$ on a

$$(*) \qquad p(h) = p(h') \Leftrightarrow \exists!\ k \in K_1 \text{ tel que } h' = i(k)h.$$

La définition de catégorie quotient est celle de [13], c'est à dire que p est un foncteur plein et bijectif sur les objets.

PROPOSITION 1.2. (a) *Le foncteur composé* pi *envoie* K_1 *sur* C_0.

(b) *Pour* $h \in H_1$ *et* $k \in K_1$ *tels que* $hi(k)$ *est défini, il existe un unique* ${}^h k \in K_1$ *tel que* $hi(k) = i({}^h k)h$.

(c) *Le foncteur* i *identifie* K *à une sous-catégorie de* H.

PREUVE. (a) et (b) sont des conséquences de la condition (*) de la définition 1.1. tandis que (c) est issu de la définition d'un foncteur fidèle.

PROPOSITION 1.3. *La catégorie* K *est réunion disjointe de groupes* K_C *indexés par* C_0.

PREUVE. La proposition 1.2.(a) nous a dit que K est réunion disjointe de monoïdes. La condition (*) de la définition 1.1. dit alors que chacun de ceux-ci est un groupe.

PROPOSITION 1.4. *Pour* k *et* $k' \in K_1$ *et* $h \in H_1$, *on a*

$$kh = k'h \Rightarrow k = k'.$$

PREUVE. Comme k est élément d'une groupe, on a $h = k^{-1}k'h$ et, de par la condition (∗) de la définition 1.1., on a $k^{-1}k' = 1$ et donc $k = k'$.

COROLLAIRE 1.5. *On a une opération de* H *sur* K, i.e.

$$\forall (h, h') \in \mathsf{H}_2 \textit{ et } \forall k \in \mathsf{K}_1 \textit{ avec } \beta(k) = \alpha(h'),$$
$$\textit{on a } \ {}^{hh'}k = {}^{h}({}^{h'}k).$$

PREUVE. On a les relations:

$$\begin{aligned}(hh')k &= {}^{hh'}k(hh') \\ &= h(h'k) = h({}^{h'}kh') = (h^{h'}k)h' = {}^{h}({}^{h'}k)hh',\end{aligned}$$

ce qui implique, de par la proposition 1.4., la relation annoncée.

DÉFINITION 1.6. Une *section* de l'extension E est la donnée, pour chaque $c \in \mathsf{C}_1$, d'un représentant $s(c)$ tel que $ps(c) = c$ de sorte que $s(c)$ soit une identité quand c en est une.

PROPOSITION 1.7. *La donnée d'une section de l'extension E définit, pour chaque morphisme $c\colon C \to C'$ de* C, *un homomorphisme de groupes $\mathsf{K}_C \to \mathsf{K}_{C'}$ qui à un k associe ${}^{s(c)}k$. Cet homomorphisme est une identité quand c en est une.*

PREUVE. On a les relations

$${}^{s(c)}(kk')s(c) = s(c)kk' = {}^{s(c)}ks(c)k' = {}^{s(c)}k^{s(c)}k's(c),$$

et, de par la proposition 1.4., il vient

$${}^{s(c)}(kk') = {}^{s(c)}k^{s(c)}k'.$$

On n'a pas nécessairement un foncteur $\mathsf{C} \to \mathcal{G}r$ car les sections s ne sont pas nécessairement des foncteurs $\mathsf{C} \to \mathsf{H}$. Lorsqu'il existe un foncteur $s\colon \mathsf{C} \to \mathsf{H}$ alors [14] dit que l'extension est scindée et que K a une C-structure.

Nous sommes ici sur le chemin de la définition d'une cohomologie de C à coefficients dans K. Les techniques y seraient semblables à celles de [11], mais nous la considèrerons ci-après dans un cadre plus général.

Mettons maintenant en évidence une propriété des extensions de catégories qui permettra d'élargir cette notion. Pour cela, il nous faut rappeler des définitions dues à [9] et souvent exposées (par exemple dans [8] et [15]).

DEFINITION 1.8. Soit $p\colon \mathsf{H} \to \mathsf{C}$ un foncteur. Un morphisme h de H est dit *cocartésien* au dessus de $c = p(h)$ si

$$\forall h' \in \mathsf{H}_1 \text{ tel que } p(h') = c \text{ et } \alpha(h') = \alpha(h)$$

$$\exists! \mathrm{k} \text{ tel que } p(k) = \beta(c) \text{ et } kh = h'.$$

On dit que *p possède assez de morphismes cocartésiens* si pour tout morphisme c: $C \to C'$ et pour tout objet H tel que $p(H) = C$, il existe un morphisme de source H cocartésien au dessus de c.

DÉFINITION 1.9. Un foncteur p qui possède assez de morphismes cocartésiens est ce qu'on appelle une *précofibration* (selon [9] et [15]) ou une préopfibration (dans le langage de [3]) au sens de Grothendieck. Les *fibres* de p sont alors les catégories $\mathsf{H}_C = p^{-1}(C)$ avec $C \in \mathsf{C}_0$.

THÉORÈME 1.10. *Les extensions de catégories sont les précofibrations de Grothendieck dont les fibres sont des groupes.*

PREUVE. Soit $\mathsf{K} \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$ une extension de catégories. La propriété $(*)$ de 1.1. nous dit que tous les morphismes de H sont cocartésiens. Comme p est plein, donc surjectif sur les morphismes et n'ayant qu'un seul objet dans chaque fibre K_C, il est clair que p possède assez de morphismes cocartésiens. Réciproquement, soit p: $\mathsf{H} \to \mathsf{C}$ une précofibration de Grothendieck dont les fibres H_C sont des groupes. Considérons alors $\coprod_{C \in \mathsf{C}_0} \mathsf{H}_C \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$. L'inclusion i est un foncteur fidèle. Les fibres étant des groupes, elles n'ont qu'un objet et, par ailleurs, ayant assez de morphismes cocartésiens, le foncteur p est bijectif sur les objets et surjectif sur les morphismes: c'est un quotient. Si on remarque alors que tout morphisme est cocartésien, car on a assez de morphismes cocartésiens et car les fibres sont des groupes, on a la propriété $(*)$ de 1.1.: nous sommes bien en présence d'une extension de catégories.

DÉFINITION 1.11. Un *morphisme d'extensions* de E à E' est un triplet de foncteurs $\Gamma = (\eta, \mu, \nu)$ tel que le diagramme suivant soit commutatif

$$\begin{array}{cccccc} E: & \mathsf{K} & \xrightarrow{i} & \mathsf{H} & \xrightarrow{p} & \mathsf{C} \\ {\scriptstyle \Gamma}\downarrow & {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} \\ E': & K' & \xrightarrow{i'} & \mathsf{H}' & \xrightarrow{p'} & \mathsf{C}'. \end{array}$$

La composition des morphismes d'extensions est définie de manière évidente. Le morphisme Γ est appelé *isomorphisme d'extensions* si les foncteurs η, μ et ν sont inversibles.

Les isomorphismes d'extensions Γ tels que η et ν sont des foncteurs identiques fournissent une congruence entre les extensions de C par K. Ceci permet de décrire, en termes d'extensions, la cohomologie évoquée à la suite de la proposition 1.7.

2. Extensions larges et cocycles.

On peut être amené à souhaîter que les extensions de catégories aient des fibres possédant plus d'un objet. Du point de vue de la cohomologie, il s'agit d'avoir des foncteurs "surjectifs" p: $\mathsf{H} \to \mathsf{C}$ tels qu'une bonne notion de section permette de définir des cocycles sur C_2 à valeurs dans une catégorie "noyau" K. Le théorème 1.10. nous suggère la généralisation suivante.

DÉFINITION 2.1. Une *extension large de catégories* (de la catégorie C par la catégorie K) est une suite

$$E\colon \mathsf{K} \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$$

telle que

(1) le foncteur p possède assez de morphismes cocartésiens,
(2) le foncteur i est fidèle (on pourra identifier K à la souscatégorie $i(\mathsf{K})$ de H),
(3) $\mathsf{K} = \coprod_{C \in \mathsf{C}_0} \mathsf{K}_C$ et $i(\mathsf{K}_C) = p^{-1}(C)$ $\forall C \in \mathsf{C}_0$.

En vertu du théorème 1.10., on retrouve les extensions de catégories dans le cas où les catégories K_C sont des groupes.

DÉFINITION 2.2. Soit E: $\mathsf{K} \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$ une extension large de catégories. On appelle *section* de E la donnée s, pour chaque morphisme c: $C \to C'$ de C et chaque objet H tel que $p(H) = C$ d'un morphisme $s_H(c)$ de source H et cocartésien au dessus de c de sorte que $s_H(c)$ soit une identité quand c en est une.

Selon la terminologie issue de [9], une section de l'extension E ust un clivage de la précofibration p.

PROPOSITION 2.3. *La donnée d'une section s de E définit*

(a) *pour chaque morphisme* c: $C \to C'$ *de* C, *un foncteur* $S(c)$: $\mathsf{K}_C \to \mathsf{K}_{C'}$, *qui est le foncteur identique quand c est une identité,*

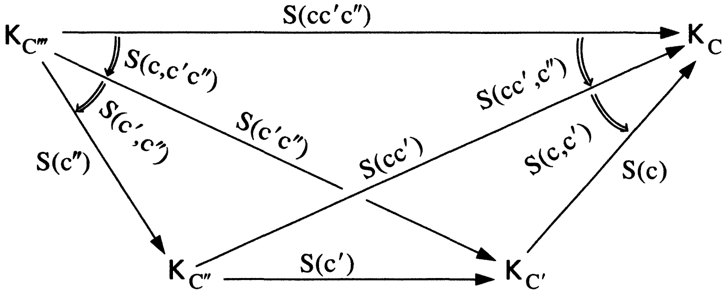
(b) *pour chaque* $(c, c') \in \mathsf{C}_2$, *une transformation naturelle* $S(c, c')$: $S(cc') \Rightarrow S(c)S(c')$ *qui est la transformation identique quand c ou c' est une identité,*

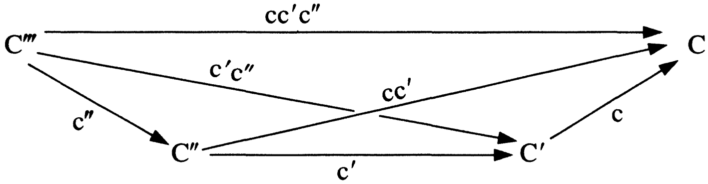
(c) *et pour chaque* $(c, c', c'') \in \mathsf{C}_3$, *on a*

$$S(c, c')S(c'')_0 S(cc', c'') = S(c)S(c', c'')_0 S(c, c'c'')\colon S(cc'c'') \Rightarrow S(c)S(c')S(c''),$$

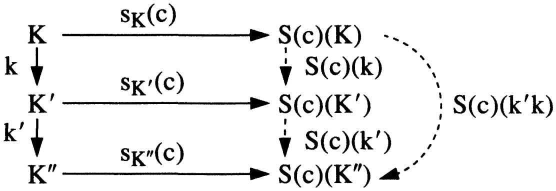
ce qui se traduit par la commutativité du 2-diagramme ci-dessous.

La composition $_0$ *des transformations naturelles est la composition "verticale" définie dans* [13].



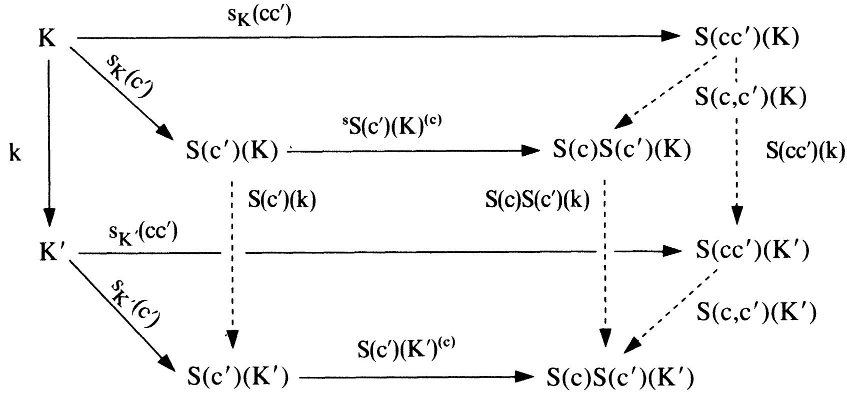


PREUVE. (a) Si K est un objet de $\mathsf{K}_C \subset \mathsf{H}$, l'objet $S(c)(K)$ de K_C est le but de $s_K(c)$. Si k: $K \to K'$ est un morphisme de K_C, le morphisme $S(c)(k)$ est l'unique morphisme de $\mathsf{K}_{C'}$ tel que $S(c)(k)s_K(c) = s_{K'}(c)k$. Depar la cocartésianité de $s_K(c)$, on a aussi $S(c)(k')S(c)(k) = S(c)(k'k)$ quand $k'k$ est défini dans K_C.

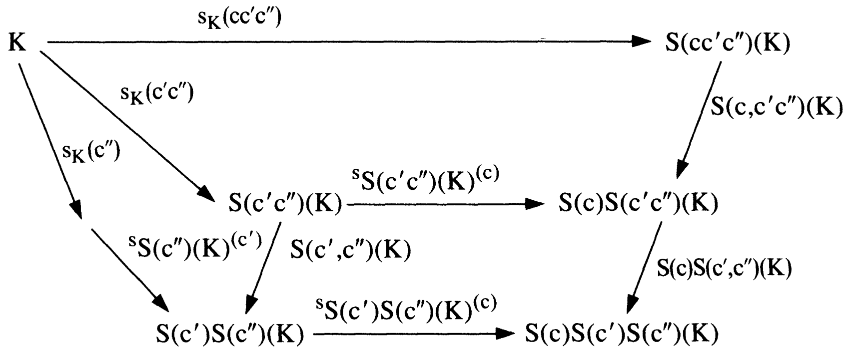


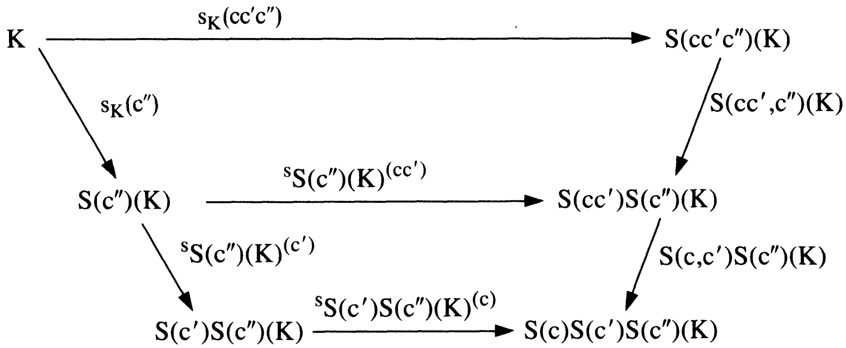
(b) Pour chaque K, la cocartésianité de $s_K(cc')$ donne l'existence d'un unique $S(c, c')(K)$ tel que $S(c, c')(K)s_K(cc') = s_{S(c')(K)}(c)s_K(c')$. Pour chaque k: $K \to K'$, elle donne aussi

$$S(c)[S(c')(k)]S(c, c')(K) = S(c, c')(K')S(cc')(k).$$



(c) Pour chaque objet K, on a les diagrammes commutatifs suivants:





De par la cocartésianité de $s_K(cc'c'')$, on a alors

$$S(c,c')S(c'')(K)S(cc',c'')(K) = S(c)S(c',c'')(K)S(c,c'c'')(K),$$

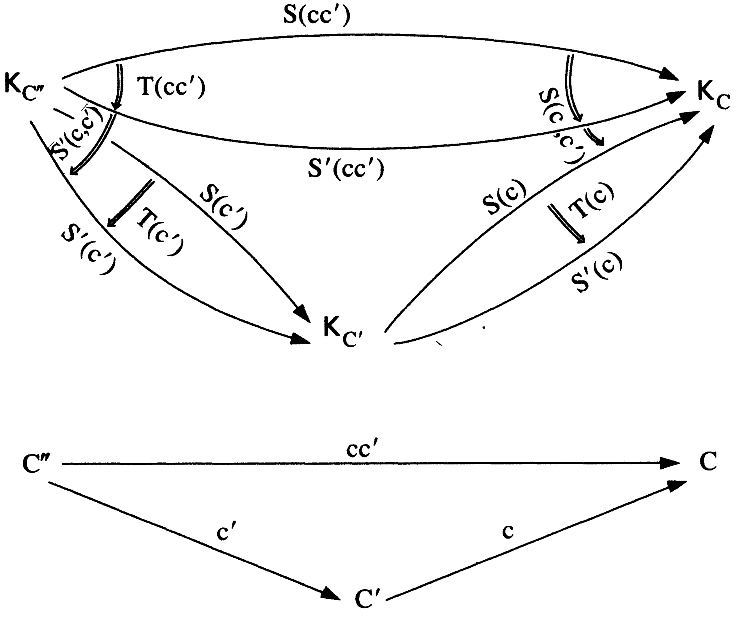
d'où le résultat annoncé.

DÉFINITION 2.4. Un tel "pseudofoncteur" S de $\mathbf{C}$ dans $\mathscr{C}at$ est appelé 2-*cocycle associé à l'extension E* par la section s.

PROPOSITION 2.5. *Soient S et S' des 2-cocycles associés à l'extension E par des sections s et s' respectivement. Pour chaque* $c \in \mathbf{C}_1$, *on a une équivalence naturelle* $T(c)$: $S(c) \Rightarrow S'(c)$ *telle que pour chaque* $(c,c') \in \mathbf{C}_2$, *on a*

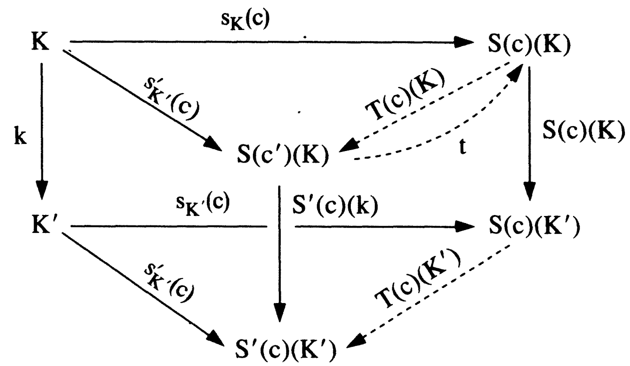
$$S'(c,c')_\circ T(cc') = [T(c)_* T(c')]_\circ S(c,c'): S(cc') \Rightarrow S'(c)S'(c'),$$

ce qui se traduit par la commutativité du 2-diagramme suivant.

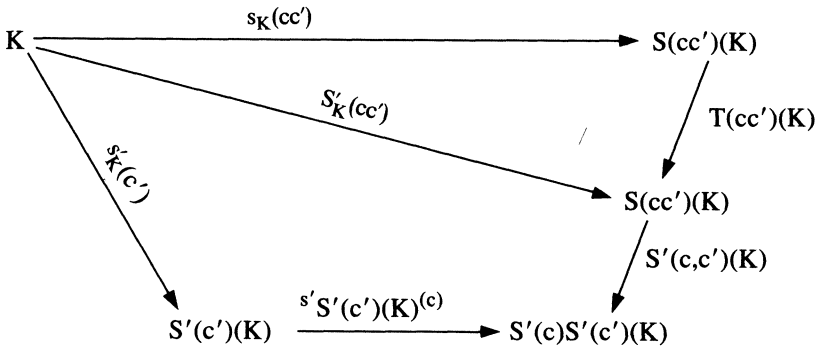


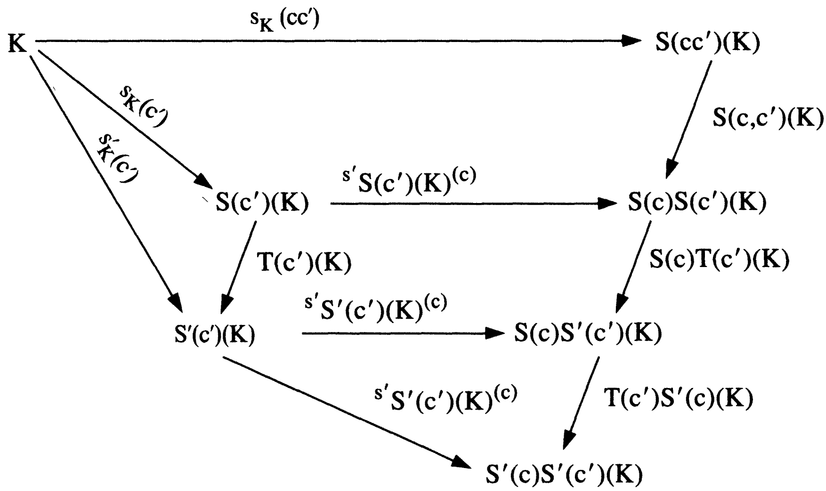
*La composition * des transformations naturelles est la composition "horizontale" définie dans* [13].

PREUVE. (a) Pour chaque K, le morphisme $T(c)(K)$ est l'unique tel que $T(c)(K)s_K(c) = s'_K(c)$. L'unique morphisme t tel que $ts'_K(c) = s_K(c)$ est évidemment inverse de $T(c)(K)$. Pour k: $K \to K'$, on a aussi $T(c)(K')S(c)(k) = S'(c)(k)\,T(c)(K)$ de par la cocartésianité de $s_K(c)$.



(b) Pour chaque objet K, on a les diagrammes commutatifs suivants:





De par la cocartésianité de $s_K(cc')$, on a alors

$$S'(c,c')(K)\,T(cc')(K) = T(c')S'(c)(K)S(c)T(c')(K)S(c,c')(K)$$

d'où le résultat annoncé, compte tenu de la définition de

$$[T(c) * T(c')](K) = T(c')S'(c)(K)S(c)T(c')(K).$$

Comme pour les extensions de catégories en 1.11., on peut parler de morphisme d'extensions larges dont on ne considère ici que le cas particulier suivant.

DÉFINITION 2.6. Deux extensions larges de catégories, E: $\mathsf{K} \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$ et E': $\mathsf{K} \xrightarrow{i'} \mathsf{H}' \xrightarrow{p'} \mathsf{C}$, sont dites *congrues* s'il existe un foncteur inversible μ: $\mathsf{H} \to \mathsf{H}'$ tel que

$$\mu i = i' \text{ et } p'\mu = p.$$

On notera Ext (C, K) l'ensemble des classes de congruence d'extensions de C par K.

PROPOSITION 2.7. *Soient E et E' deux extensions de* C *par* K.

(a) *Si E et E' sont congrues et si S est un* 2-*cocycle associé à E, alors S est aussi un* 2-*cocycle associé à E'.*

(b) *Réciproquement, si un même* 2-*cocycle S est associé à E et E', alors E et E' sont congrues.*

Preuve. (a) Soit s la section qui associe S à E. En posant $s'_H(c) = \mu(s_{\mu^{-1}(H)}(c))$, on définit une section s' qui associe S à E'. Notons que $H = \mu^{-1}(H)$ car $H \in \mathsf{H}_0 = \mathsf{K}_0$.

(b) Soient s et s' les sections qui associent S à E et E' respectivement. Un morphisme h: $K \to H$ de H s'écrit de manière unique sous la forme $ks_K(p(h))$ avec $k \in \mathsf{K}_1 \subset \mathsf{H}_1$. En posant $\mu(h) = ks'_K(p(h))$, on définit un foncteur μ: $\mathsf{H} \to \mathsf{H}'$ qui fait de E et E' des extensions congrues.

3. Cohomologie.

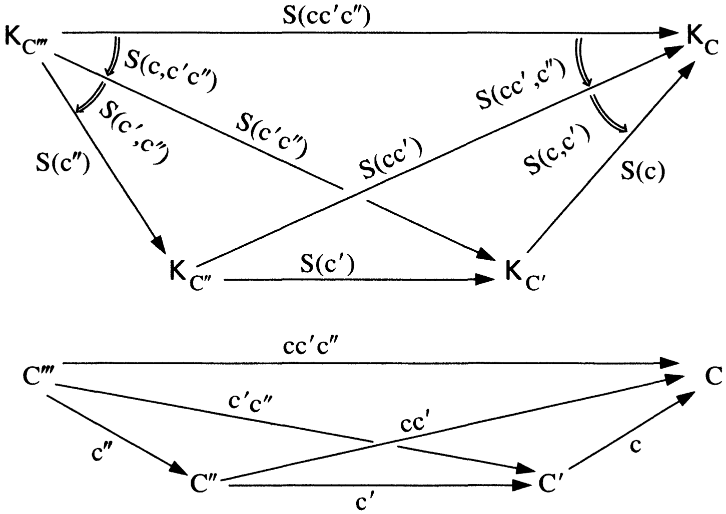
Soient C et $\mathsf{K} = \coprod_{C \in \mathsf{C}_0} \mathsf{K}_C$ deux catégories.

DÉFINITION 3.1. Un 2-*cocycle* de C vers K est la donnée S:

(a) pour chaque morphisme c: $C \to C'$ de C, d'un foncteur $S(c)$: $\mathsf{K}_C \to \mathsf{K}_{C'}$ qui est le foncteur identique quand c est une identité,

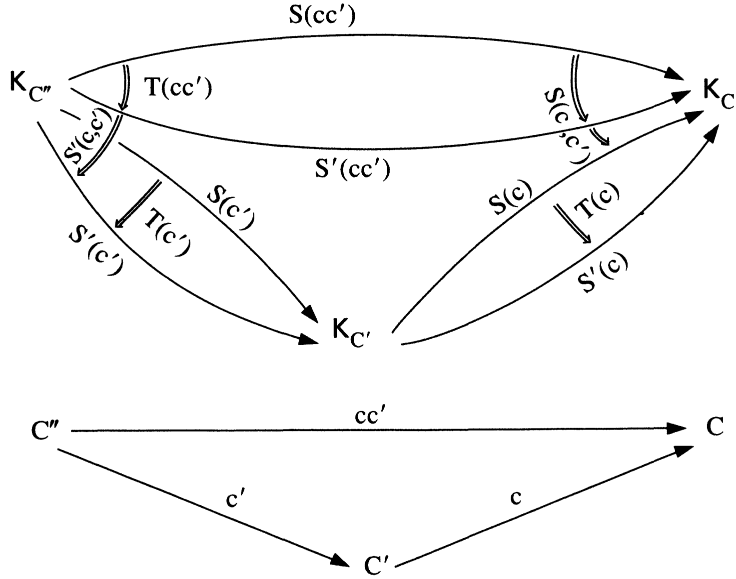
(b) pour chaque couple composable $(c, c') \in \mathsf{C}_2$, d'une transformation naturelle $S(c,c')$: $S(cc') \Rightarrow S(c)S(c')$ qui est la transformation identique quand c ou c' est une identité, de sorte que pour chaque triple composable $(c, c', c'') \in \mathsf{C}_3$, on a

$$S(c,c')S(c'')_\circ T(cc',c'') = S(c)S(c',c'')_\circ S(c,c'c''): S(cc'c'') \Rightarrow S(c)S(c')S(c'').$$



Définition 3.2. Des 2-cocycles S et S' de C vers K sont dits *cohomologues* si pour chaque morphisme $c \in \mathsf{C}_1$ on a une équivalence naturelle $T(c)$: $S(c) \Rightarrow S'(c)$ de sorte que pour chaque couple composable $(c, c') \in \mathsf{C}_2$ on a

$$S(c,c')_\circ T(cc') = [T(c)_* T(c')]_\circ S(c,c'): S(cc') \Rightarrow S'(c)S'(c').$$



On retrouve le même type de relations dans la 2-cohomologie non-abélienne étudiée par les gerbes, comme en [6], ou par les bitorseurs, comme en [4]. Comme on y travaille dans ses groupoïdes, il se peut que l'on ait alors des 2-flèches allant dans le sens inverse.

DÉFINITION 3.3. L'ensemble des classes de cohomologie de 2-cocycles de C vers K est noté $H^2(\mathsf{C}, \mathsf{K})$ et est appelé ensemble de *cohomologie de dimension* 2 de C vers K.

THÉORÈME 3.4. *L'application qui à chaque extension de* C *par* K *associe la classe de cohomologie d'un de ses 2-cocycles associés définit une bijection*

$$\omega\colon \operatorname{Ext}(\mathsf{C}, \mathsf{K}) \to H^2(\mathsf{C}, \mathsf{K}).$$

PREUVE. (a) La proposition 2.3. montre comment à une extension E on associe un 2-cocycle et la proposition 2.5. que la classe d'icelui dans $H^2(\mathsf{C}, \mathsf{K})$ ne dépend que de E. La proposition 2.7. nous dit qu'à deux extensions congrues on associe un même élément de $H^2(\mathsf{C}, \mathsf{K})$. L'application ω est donc bien définie.

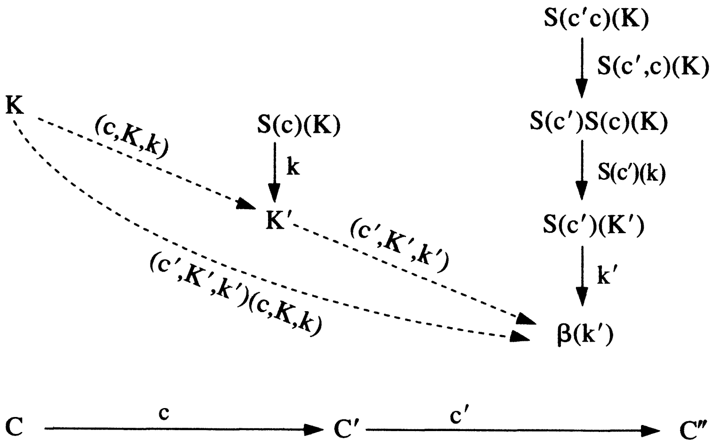
(b) Si à deux extensions E et E' on associe des 2-cocycles S et S' cohomologues, comme dans la démonstration de la proposition 2.7. (b), on construit une congruence entre E et E'. L'application ω est donc injective.

(c) Il reste à montrer la surjectivité de ω. Soit S un 2-cocycle de C vers K, nous allons construire une extension à la quelle S est associée. Notre modèle est l'écriture unique des morphismes de H vue dans la démonstration de la proposition 2.7. (b). Soit H la catégorie dont les morphismes sont les triples (c, K, k) tels que

$$c \in \mathsf{C}_1,\ K \in (\mathsf{K}_{\alpha(c)})_0,\ k \in (\mathsf{K}_{\beta(c)})_1 \text{ avec } \alpha(k) = S(c)(K);$$

la source de (c, K, k) est K, son but est $\beta(k)$; la composition dans H est définie par

$$(c', K', k')(c, K, k) = (c'c, K, k'S(c')(k)S(c', c)(K)).$$



La définition d'un 2-cocycle permet de vérifier que cette composition est bien associative et donc que H est bien une catégorie. Les foncteurs i: $\mathsf{K} \to \mathsf{H}$: $k \mapsto (1_C, \alpha(k), k)$ pour $k \in (\mathsf{K}_C)_1$ et p: $\mathsf{H} \to \mathsf{C}$: $(c, K, k) \mapsto c$ font de $\mathsf{K} \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$ une extension large à laquelle le 2-cocycle S est associé par la section définie par

$$s_K(c) = (c, K, 1_{S(c)(K)}).$$

4. Coefficients.

Précisons cette notion qui, souvent, définit la cohomologie.

DÉFINITION 4.1. Un *coefficient* M: $\mathsf{C} \nrightarrow \mathscr{C}at$ sur une catégorie C est la donnée d'application $\mathsf{C}_0 \to \mathscr{C}at_0$: $C \mapsto M(C)$ et $\mathsf{C}(C, C') \to \mathscr{C}at(M(C), M(C'))$: $c \mapsto M(c)$ de sorte que $M(c)$ est une identité quand c en est une.

Un coefficient $\mathsf{C} \nrightarrow \mathscr{C}at$ n'est pas en général un foncteur $\mathsf{C} \to \mathscr{C}at$ ni même un pseudofoncteur. Dans le cas où M est un foncteur à valeurs dans $\mathscr{A}b$, c'est un C-module au sens de [11], le cas particulier des modules sur un groupoïde est considéré dans [16].

DÉFINITION 4.2. Une *extension de* C *par le coefficient* M: $\mathsf{C} \nrightarrow \mathscr{C}at$ est une extension de categories

$$E: \coprod_{C \in \mathsf{C}_0} M(C) \xrightarrow{i} \mathsf{H} \xrightarrow{p} \mathsf{C}$$

telle que $x \in M(C)$ si et seulement si $pi(x) = C$ et il existe une section s telle que le 2-cocycle S associé à E par s vérifie

$$S(c) = M(c) \;\; \forall c \in \mathsf{C}_1.$$

On dit alors que S est *compatible* à M.

Si l'on reprend la terminologie de 1.5., étendue aux extensions larges à l'aide de 1.8., on peut dire qu'alors les "opérations" de C (par M) et de H sur K sont compatibles.

DÉFINITION 4.3. Suivant 2.5., on peut parler de *classes de congruence d'extensions de* C *par* M. On notera $\operatorname{Ext}(\mathsf{C}, M)$ l'ensemble de ces classes.

DÉFINITION 4.4. Un *2-cocycle de* C *de coefficient* M est un 2-cocycle de C vers $\coprod_{C \in \mathsf{C}_0} M(C)$ vérifiant

$$S(c) = M(c) \;\; \forall c \in \mathsf{C}_1.$$

DÉFINITION 4.5. L'ensemble des classes de cohomologie de 2-cocycles de coefficient M est noté $H^2(\mathsf{C}, M)$ et est appelé ensemble de *cohomologie de dimension* 2 *de coefficient* M.

THÉORÈME 4.6. *L'application qui à chaque extension de* C *par le coefficient* M *associe la classe de cohomologie d'un de ses* 2*-cocycles compatibles à* M *définit une bijection*

$$\omega \colon \operatorname{Ext}(\mathsf{C}, M) \to H^2(\mathsf{C}, M).$$

PREUVE. Ceci découle directement de la démonstration du théorème 3.4.

EXEMPLES 4.8. On peut voir, à la lecture de [17] par exemple, que ceci, avec éventuellement des coefficients contravariants (on remplace alors dans ce qui précède la catégorie C par la catégorie opposée C^{op}), recouvre nombre de cohomologies que l'on peut trouver dans la littérature. En particulier, on retrouve la cohomologie à coefficients dans un module de [11].

Notre notion de coefficient peut encore être élargie. Montrons comment dans une définition que nous étudierons dans la suite.

Soit $\mathscr{C}cat$ une catégorie dont les objets sont les petites catégories munie d'un foncteur $H\colon \mathscr{C}at \to \mathscr{C}cat$ identique sur les objets et soit $\tilde{M}\colon \mathsf{C} \to \mathscr{C}cat$ un foncteur.

DÉFINITION 4.9. Un *coefficient* $M\colon \mathsf{C} \nrightarrow \mathscr{C}at$ est dit *compatible* avec le foncteur $\tilde{M}$ si pour chaque $C \in \mathsf{C}_0$ on a $M(C) = \tilde{M}(C)$ et si les diagrammes du type suivant sont tous commutatifs.

$$\begin{array}{ccc}
\mathsf{C}(C, C') & \xrightarrow{\quad \tilde{M} \quad} & \mathscr{C}cat(M(C), M(C')) \\
& {\scriptstyle M} \searrow \qquad \nearrow {\scriptstyle H} & \\
& \mathscr{C}at(M(C), M(C')) &
\end{array}$$

Une *extension de* C *par* $\tilde{M}$ est une extension de C par un coefficient M compatible avec $\tilde{M}$.

Il est utile de revenir maintenant au cas particulier des extensions de catégories initiales, celles de 1.1., car elles nous permettent de donner une expression intéressante de ce que nous avons décrit. On verra ainsi concrètement que nous avons obtenu une généralisation de ce que l'on rencontre souvent (par exemple dans [12], [11] ou [17]) en cohomologie.

5. Le cas des coefficients en groupes.

La catégories $\mathcal{G}r$ étant une sous-catégorie de $\mathcal{C}at$, soit M: $\mathsf{C} \nrightarrow \mathcal{G}r$ un coefficient en groupes sur la catégorie C.

REMARQUES 5.1. (1) Il est clair que les extensions de C par M sont des extensions de catégories au sens de 1.1.

(2) Dans le cas où M est un foncteur à valeurs dans la catégorie des groupes abéliens $\mathcal{A}b$, on retrouve exactement les extensions de C par le C-module M définies en [11].

Quand il n'y aura pas de risque de confusion, on notera $c \cdot x = M(c)(x)$ pour $c \in \mathsf{C}_1$ et $x \in M(\alpha(c))$.

THÉORÈME 5.2. *L'ensemble des 2-cocycles de* C *de coefficient M est en bijection avec l'ensemble des applications*

$$\Psi\colon \mathsf{C}_2 \to \coprod_{C \in \mathsf{C}_0} M(C)$$

telles que

(a) $\psi(c, c') \in M(\beta(c))$ est une identité si c ou c' en est une,

(b) $c,c')(c' \cdot x) = \psi(c, c')(cc' \cdot x)\psi(c, c')^{-1}$ $\forall x \in M(\alpha(c'))$,

(c) $(c \cdot \psi(c', c''))\psi(c, c'c'') = \psi(c, c')\psi(cc', c'')$ $\forall (c, c', c'') \in \mathsf{C}_3$.

PREUVE. Les 2-cocycles S de C de coefficient M sont tous tels que $S(c) = M(c)$ par définition. Les groupes $M(C)$ ayant un seul objet, la donnée d'une transformation naturelle $S(c, c')$ de $M(cc')$ vers $M(c)M(c')$ est exactement la donnée d'un morphisme de $M(\beta(c))$, i.e. d'un élément $\psi(c, c')$ de ce groupe, qui vérifie (b). On a bien sûr la propriété (a) quand S est un 2-cocycle. La relation (c) n'est autre que la traduction de la relation de cocycle de 3.1.

THÉORÈME 5.3. *Des 2-cocycles S et S' de* C *de coefficient M sont cohomologues si, et seulement si, les applications correspondantes* ψ *et* ψ' *sont telles qu'il existe une application*

$$\tau\colon \mathsf{C}_1 \to \coprod_{C \in \mathsf{C}_0} M(C)$$

telle que

(a) $\tau(c) \in M(\beta(c))$,
(b) $c \cdot x = \tau(c)(c \cdot x)\tau(c)^{-1} \quad \forall x \in M(\alpha(c))$,
(c) $\psi'(c, c') = \tau(c)(c \cdot \tau(c'))\psi(c, c')\tau(cc')^{-1} \quad \forall (c, c') \in \mathsf{C}_2$.

PREUVE. On a encore ici $S(c) = S'(c) = M(c)$. La donnée d'une transformation naturelle $T(c)$: $M(c) \to M(c)$ est exactement la donnée d'un élément $\tau(c)$ du groupe $M(\beta(c))$ tel que

$$M(c)(x) = \tau(c)M(c)(x)\tau(c)^{-1} \quad \forall x \in M(\alpha(c)).$$

La relation de cohomologie entre S et S' de 3.2. se traduit alors par la relation (c).

Considérons maintenant une catégorie $\mathcal{C}gr$ dont les objets sont les groupes, munie d'un foncteur H: $\mathcal{G}r \to \mathcal{C}gr$ identique sur les objets, et un foncteur $\tilde{M}$: $\mathsf{C} \to \mathcal{C}gr$.

EXEMPLE 5.4. Dans [17], on considère la catégorie $\mathcal{C}gr = \mathcal{H}gr$ dont les morphismes sont définis par

$$\mathcal{H}gr(G, G') = [K(G, 1); K(G', 1)]$$

les classes d'homotopie libre d'applications entre espaces d'Eilenberg-MacLane définis par les groupes G et G'. Le foncteur H est alors le foncteur naturel de $\mathcal{G}r$ vers $\mathcal{H}gr$.

DÉFINITION 5.5. La *cohomologie* $H^2(\mathsf{C}, \tilde{M})$ est définie comme suit. On considère tous les coefficients M compatibles avec $\tilde{M}$ et les applications de 5.2. correspondantes. La relation d'équivalence sur ces dernières est alors définie par les applications

$$\tau\colon \mathsf{C}_1 \to \coprod_{C \in \mathsf{C}_0} M(C)$$

telles que
(a) $\tau(c) \in \tilde{M}(\beta(c))$,
(b) $c_o x = \tau(c)(c \cdot x)\tau(c)^{-1} \quad \forall x \in \tilde{M}(\alpha(c))$,
(c) $\psi'(c, c') = \tau(c)(c \cdot \tau(c'))\psi(c, c')\tau(cc')^{-1} \quad \forall (c, c') \in \mathsf{C}_2$,
quand M et M' sont deux coefficients compatibles avec $\tilde{M}$ et où l'on note:

$$c \cdot x = M(c)(x) \text{ et } c_o x = M'(c)(x).$$

THÉORÈME 5.6. *La relation précédente détermine une congruence entre extensions de* C *par* $\tilde{M}$ *et on a une interprétation de* $H^2(\mathsf{C}, \tilde{M})$ *en termes d'extensions de catégories.*

PREUVE. La définition 5.5. étend la relation de cohomologie aux 2-cocycles associés à toutes les extensions de C par M avec M compatible avec $\tilde{M}$. On reprend, mutatis mutandis, les démonstrations des théorèmes 3.4. et 4.6. adaptées aux extensions à coefficients en groupes par les théorèmes 5.2 et 5.3.

EXEMPLE 5.7. Dans l'exemple 5.4., avec des coefficients contravariants, on retrouve la cohomologie de dimension 2 de [17] et donc une interpretation de celle-ci en termes d'extensions, où les coefficients M compatibles avec $\tilde{M}$ interviennent dans la recherche de l'obstruction à un problème posé par les limites homotopiques.

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ON EMBEDDINGS OF PROPER SMOOTH G-MANIFOLDS

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By a linear Lie group we mean a Lie group isomorphic to a closed subgroup of a general linear group. A euclidean space R^n equipped with the linear action of G via some representation $\varrho\colon G \to \mathrm{Gl}(n, \mathsf{R})$ is denoted by $\mathsf{R}^n(\varrho)$ and called a linear G-space. If M is a smooth, i.e., a C^∞-differentiable manifold and the action $G \times M \to M$ is smooth, we call M a smooth G-manifold. In case the mapping $G \times M \to M \times M$, $(g, x) \mapsto (gx, x)$, is proper, i.e., if the inverse image of every compact set is compact, we call M a proper smooth G-manifold. This definition of properness is equivalent to the definition used in [Pa3]. The purpose of this paper is to prove the following result:

THEOREM. *Let G be a linear Lie group and M a proper smooth G-manifold having only finitely many orbit types. Then there exists a G-equivariant, closed smooth embedding of M into some linear G-space.*

Suppose for a moment G is an arbitrary Lie group. Let G act on itself via multiplication on the left. This action makes G a proper smooth G-manifold having only one orbit type. Assume there exists a G-equivariant topological embedding f of the G-manifold G into a linear G-space $\mathsf{R}^n(\varrho)$. Then $f(g) = \varrho(g)f(e)$ for every $g \in G$ where e is the identity element of G. Since f is injective it follows that also $\varrho\colon G \to \mathrm{Gl}(n, \mathsf{R})$ is injective. It now follows from Proposition 5.1.2 in [Pr] that ϱ is a smooth immersion. Since the mappings $\varrho(G) \to \mathsf{R}^n(\varrho)$, $\varrho(g) \mapsto \varrho(g)f(e)$, and $f(G) \to G$, $f(g) \mapsto g$, are continuous, it follows that their composition $\varrho(G) \to G$, $\varrho(g) \mapsto g$, is continuous. Theorem II 2.10 in [He] finally implies that $\varrho(G)$ is closed in $\mathrm{Gl}(n, \mathsf{R})$, i.e., that G is a linear Lie group. Thus we see that in the previous theorem it is necessary to assume G is a linear Lie group.

The smooth embedding is constructed essentially in the same way as the topological embedding in [Pa3] and the subanalytic embedding in [Ka]. It will

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be used in a paper to appear where we will prove the real analytic version of the theorem by using G-equivariant real analytic approximations.

If $x \in M$ we denote its isotropy subgroup by G_x. We note that if G acts properly on M, then G_x is compact for every $x \in M$. For a subgroup H of G we denote $(H) = \{gHg^{-1} \mid g \in G\}$ and $M_{(H)} = \{x \in M \mid (G_x) = (H)\}$. The set (G_x) is called the orbit type of x. Let S be a G_x-invariant smooth submanifold of M containing x. If GS is an open subset of M and there exists a G-equivariant smooth mapping $f\colon GS \to G/G_x$ such that $S = f^{-1}(eG_x)$, then we call S a slice at x and GS a tube at x. It has been proven in [Pa3] that if M is a proper smooth G-manifold, then there exists a slice at every $x \in M$.

Suppose S is a slice at x. Let $g_0 \in G$, U be an open neighbourhood of $g_0 G_x$ in G/G_x and $\sigma\colon U \to G$ be a local cross section. It follows from Proposition 2.1.2 in [Pa3] that the mapping $F\colon U \times S \to V$, $(u, s) \mapsto \sigma(u)s$, is a homeomorphism onto an open neighbourhood V of $g_0 S$. In fact, F is a diffeomorphism with the inverse mapping $F^{-1}\colon V \to U \times S$, $gs \mapsto (gG_x, \sigma(gG_x)^{-1}gs)$.

1. Lemma. *Assume G is a Lie group, M a smooth G-manifold, $x \in M$ such that G_x is compact and S a slice at x. Let A and B be disjoint, closed G-invariant subsets of GS. Then there exists a G-invariant smooth mapping $f\colon GS \to [0, 1]$ such that $f \mid A = 0$ and $f \mid B = 1$.*

Proof. We first remark that $A \cap S$ and $B \cap S$ are disjoint, closed G_x-invariant subsets of S. Thus there is a smooth mapping $f_1\colon S \to \mathsf{R}$ such that $f_1(y) = 0$ when $y \in A \cap S$ and $f_1(y) = 1$ when $y \in B \cap S$. By Theorem 0.3.3 in [Br] the mapping $f_2\colon S \to \mathsf{R}$, $y \mapsto \int_{G_x} f_1(gy)dg$, is smooth. Obviously, $f_2(y) = 0$ when $y \in A \cap S$ and $f_2(y) = 1$ when $y \in B \cap S$. Let $f\colon GS \to \mathsf{R}$, $gs \mapsto f_2(s)$. Let $g_0 \in G$ and U, V, σ and F be as in the previous paragraph. Let $p\colon U \times S \to S$ be the projection. Then $f \mid V = f_2 \circ p \circ F^{-1}$ is smooth as composite of smooth mappings. Since g_0 was arbitrary, it follows that f is smooth. Clearly, f is G-invariant.

2. Proposition. *Let G be a Lie group, M a proper smooth G-manifold and A and B disjoint, closed G-invariant subsets of M. Then there exists a G-invariant smooth mapping $f\colon M \to [0, 1]$ such that $f \mid A = 0$ and $f \mid B = 1$.*

Proof. Let $\{GS_i\}_{i=1}^{\infty}$ be a cover of M by tubes. Since M/G is paracompact by Theorem 4.3.4 in [Pa3], $\{GS_i\}_{i=1}^{\infty}$ has locally finite refinements $\{W_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ by open G-invariant sets W_i and V_i, respectively, such that $\overline{V}_i \subset W_i$ and $\overline{W}_i \subset GS_i$ for every i. Then $B \cap \overline{V}_i$ and $W_i \backslash A$ are G-invariant subsets of GS_i, $W_i \backslash A$ is open, $B \cap \overline{V}_i \subset W_i \backslash A$ and the closure of $W_i \backslash A$ is a subset of GS_i for every i. Thus, by Lemma 1 there exists for every i a G-invariant smooth mapping $f_i' \mid\colon GS_i \to [0, 1]$ such that $f_i' \mid (GS_i \backslash (W_i \backslash A)) = 0$ and $f_i' \mid (B \cap \overline{V}_i) = 1$. We extend f_i' to $f_i\colon M \to [0, 1]$ by setting $f_i(y) = 0$ when $y \in M \backslash GS_i$ and $f_i(y) = f_i'(y)$ when

$y \in GS_i$. Then f_i is G-invariant, and since $\overline{W}_i \subset GS_i$, it follows that f_i is smooth. Since $\{\operatorname{supp} f_i\}_{i=1}^{\infty}$ is locally finite, it follows that $f_B\colon M \to \mathsf{R}$, $x \mapsto \sum_{i=1}^{\infty} f_i(x)$, is smooth. Clearly, f_B is G-invariant and non-negative, $f_B \mid A = 0$ and $f_B(x) > 0$ for every $x \in B$.

Let A' and B' be closed G-invariant neighbourhoods of A and B, respectively, such that $A' \cup B' = M$, $B \cap A' = \emptyset$ and $A \cap B' = \emptyset$. Then there exist non-negative G-invariant smooth mappings $f_{B'}$, $f_{A'}\colon M \to \mathsf{R}$ such that $f_{B'} \mid A = 0$, $f_{B'}(x) > 0$ for every $x \in B'$, $f_{A'} \mid B = 0$ and $f_{A'}(x) > 0$ for every $x \in A'$. Since $f_{A'}(x) + f_{B'}(x) \neq 0$ for every $x \in M$, the mapping

$$f\colon M \to [0,1], \qquad x \mapsto \frac{f_{B'}(x)}{f_{A'}(x) + f_{B'}(x)},$$

is well-defined. Since f is smooth and G-invariant, $f \mid A = 0$ and $f \mid B = 1$, the proposition follows.

3. PROPOSITION. *Assume G is a linear Lie group, M a proper smooth G-manifold and $x \in M$. Then there exists a slice S at x such that the tube GS admits a G-equivariant smooth embedding in a linear G-space.*

PROOF. Let S_0 be a relatively compact slice at x. Then, by Proposition IV 1.2 in [Br], S_0 only has finitely many orbit types when regarded as a G_x-space by restriction. It has been proven in [Mo] and in [Pa1] that there exists a representation $\varrho_0\colon G_x \to \mathrm{Gl}(n, \mathsf{R})$ for some $n \in \mathsf{N}$ and a G_x-equivariant smooth embedding $j_0\colon S_0 \to \mathsf{R}^n(\varrho_0)$. According to Theorem 3.1 in [Pa3], there exists a representation $\varrho\colon G \to \mathrm{Gl}(p, \mathsf{R})$ for some $p \geqq n$ and a linear G-space $\mathsf{R}^p(\varrho)$ which, considered as a linear G_x-space by restriction, contains $\mathsf{R}^n(\varrho_0)$ as an invariant linear subspace. Therefore we can regard j_0 as an embedding in $\mathsf{R}^p(\varrho)$.

Since G is a linear Lie group, Theorem 3.2 in [Ka] implies that there exists a representation $\psi\colon G \to \mathrm{Gl}(q, \mathsf{R})$ for some $q \in \mathsf{N}$ and a point $v \in \mathsf{R}^q(\psi)$ such that $G_v = G_x$ and the mapping $G/G_x \to \mathsf{R}^q(\psi)$, $gG_x \mapsto \psi(g)v$, is a closed smooth, in fact a real analytic, embedding. We define

$$j\colon GS_0 \to \mathsf{R}^{p+q}(\varrho \oplus \psi), \qquad gs \mapsto (\varrho(g)j_0(s), \psi(g)v).$$

Since j_0 is G_x-equivariant and injective, it immediately follows that j is G-equivariant and injective.

Let $g_0 \in G$ and $\sigma\colon U \to G$ be a local cross section at $g_0 G_x$. The mapping $F_0\colon U \times S_0 \to V_0$, $(u,s) \mapsto \sigma(u)s$, is a diffeomorphism onto an open neighbourhood V_0 of $g_0 S_0$. Also $h\colon U \times S_0 \to U \times j(S_0)$, $(u,s) \mapsto (u, j(s))$, is a diffeomorphism. Since easily $j(S_0)$ is a topological slice at $j(x)$ in the G-space $j(GS_0)$ the mapping $F\colon U \times j(S_0) \to V$, $(u, j(s)) \mapsto \sigma(u)j(s)$, is a homeomorphism onto an open neighbourhood V of $j(g_0 S_0)$ in $j(GS_0)$. Clearly F is smooth. Then $j \mid V_0 =$

$F \circ h \circ F_0^{-1}$ is a smooth homeomorphism onto V. Since g_0 was chosen arbitrarily it follows that j is smooth and $j^{-1}: j(GS_0) \to GS_0, j(gs) \mapsto gs$, is continuous.

The restriction $j \mid S_0$ is a smooth embedding. Since the mapping $Gx \to G/G_x$, $gx \mapsto gG_x$, is a smooth diffeomorphism (see Proposition 1.1.5 in [Pa3] and Theorem VI 1.2 in [Br]) and the mapping $G/G_x \to \mathsf{R}^{p+q}(\varrho \oplus \psi), gG_x \mapsto (\varrho(g)j_0(x), \psi(g)v)$, is a smooth embedding it follows that the restriction $j \mid Gx$ is a smooth embedding. Let $y = (y_1, y_2) \in T_xGS_0 = T_xS_0 \oplus T_xGx$ and let $dj_x(y) = 0$. Let $j^1: GS_0 \to \mathsf{R}^p(\varrho)$, $gs \mapsto \varrho(g)j_0(s)$, and $j^2: GS_0 \to \mathsf{R}^q(\psi)$, $gs \mapsto \psi(g)v$. Then $dj_x^1(y_1) + dj_x^1(y_2) = 0$ and $dj_x^2(y_1) + dj_x^2(y_2) = 0$. Since $j^2 \mid S_0$ is a constant mapping, it follows that $dj_x^2(y_1) = 0$. Thus also $dj_x^2(y_2) = 0$. Since $dj_x^2 \mid T_xS_0 \oplus T_xGx$ is injective, it follows that $y_2 = 0$. Since $dj_x^1 \mid T_xS_0$ is injective, it follows that also $y_1 = 0$. Thus $y = 0$, which implies that dj_x is injective. Therefore x has an open neighbourhood W in GS_0 such that $j \mid W$ is an immersion. Now, $S = GW \cap S_0$ is a slice at x and $GS = GW$. Obviously, $j \mid GS$: $GS \to \mathsf{R}^{p+q}(\varrho \oplus \psi)$ is a G-equivariant smooth embedding.

We next show that for each orbit type (H_i), $i = 1, \ldots, m$, in M there exists a representation $\varrho_i: G \to \mathrm{Gl}(q_i, \mathsf{R})$ such that every $x \in M_{(H_i)}$ has a tube which admits a G-equivariant smooth embedding in $\mathsf{R}^{q_i}(\varrho_i)$. The representations ϱ_i are constructed in Lemma 4. In Lemma 7 they are used in showing that there exists a representation $\varrho: G \to \mathrm{Gl}(q, \mathsf{R})$ for some $q \in \mathsf{N}$, such that M can be covered with finitely many open sets each of which admits a G-equivariant smooth embedding in $\mathsf{R}^q(\varrho)$. Finally, the embedding of M is constructed by using Lemma 7 and Proposition 2. Lemma 5 and Corollary 6 are needed to make the embedding of M closed.

4. LEMMA. *Suppose G is a linear Lie group and M a proper smooth G-manifold with only finitely many orbit types. Suppose H is a compact subgroup of G. Then there exists a representation v: $G \to \mathrm{Gl}(n, \mathsf{R})$ of G for some $n \in \mathsf{N}$ with the following property: If $x \in M_{(H)}$, there is a slice S_x at x such that the tube GS_x has a G-equivariant smooth embedding in $\mathsf{R}^n(v)$.*

PROOF. Proposition 4.4.2 in [Pa3] yields that M only has finitely many orbit types when regarded as an H-space by restriction. Let φ: $H \to O(m)$ be a representation for some $m \in \mathsf{N}$ such that there exists an H-equivariant smooth embedding f: $M \to \mathsf{R}^m(\varphi)$. The existence of f follows from [Mo] and [Pa1]. As in Proposition 3 we can consider f as an embedding in some linear G-space $\mathsf{R}^p(\varrho)$.

Let $x \in M$ be such that $G_x = H$ and let S'_x be a relatively compact slice at x. Let $\psi: G \to \mathrm{Gl}(q, \mathsf{R})$ and $v \in \mathsf{R}^q(\psi)$, where $q \in \mathsf{N}$, be such that the mapping $G/H \to \mathsf{R}^q(\psi)$, $gH \mapsto \psi(g)v$, is a closed smooth embedding. Proposition 3 implies that there exists a slice $S_x \subset S'_x$ at x such that $j_x: GS_x \to \mathsf{R}^{p+q}(\varrho \oplus \psi)$, $gs \mapsto (\varrho(g)f(s), \psi(g)v)$, is a G-equivariant smooth embedding. For every $g \in G$, gS_x

is a slice at gx and $G(gS_x) = GS_x$. Thus j_x embeds also $G(gS_x)$ and the lemma follows.

5. LEMMA. *Let G be a linear Lie group, H a compact subgroup of G and M a proper smooth G-manifold. Then there exists a representation $\psi\colon G \to \mathrm{Gl}(k, \mathbb{R})$ for some $k \in \mathbb{N}$ with the following property: If $x \in M_{(H)}$, S_x is a slice at x and K_x is a compact subset of S_x, then there exists a G-equivariant smooth mapping $h_x\colon GS_x \to \mathbb{R}^k(\psi)$ whose restriction to GK_x is proper.*

PROOF. Let $x \in M$ be such that $G_x = H$. The mapping $f_x\colon GS_x \to G/H$, $gs \mapsto gH$, is smooth. Let $f_x|$ be the restriction of f_x to GK_x and ϕ_x the restriction of the group action mapping to $G \times K_x$. Since the projection $p_x\colon G \times K_x \to G$ and the natural projection $\pi\colon G \to G/H$ are proper mappings, it follows that $f_x| \circ \phi_x = \pi \circ p_x$ is proper. Since $\phi_x(G \times K_x) = GK_x$ it follows that $f_x|$ is proper. Let $f\colon G/H \to \mathbb{R}^k(\psi)$ be a G-equivariant, closed smooth embedding in some linear G-space $\mathbb{R}^k(\psi)$. Then $h_x = f \circ f_x\colon GS_x \to \mathbb{R}^k(\psi)$ is a G-equivariant smooth mapping whose restriction to GK_x is proper.

Let $g \in G$, S_{gx} be a slice at gx and K_{gx} be a compact subset of S_{gx}. Then $g^{-1}S_{gx}$ is a slice at x and $g^{-1}K_{gx}$ is a compact subset of $g^{-1}S_{gx}$. Since $GS_{gx} = G(g^{-1}S_{gx})$ and $GK_{gx} = G(g^{-1}K_{gx})$ we can choose $h_{gx} = h_x$.

6. COROLLARY. *Assume G is a linear Lie group and M a proper smooth G-manifold having only finitely many orbit types. Let H be a compact subgroup of G. Then there exists a representation $\varrho\colon G \to \mathrm{Gl}(m, \mathbb{R})$ for some $m \in \mathbb{N}$ with the following property: If $x \in M_{(H)}$, then there is a slice S_x at x such that if K_x is a compact subset of S_x, the tube GS_x has a G-equivariant smooth embedding f_x in $\mathbb{R}^m(\varrho)$ where the restriction $f_x \mid GK_x$ is proper.*

PROOF. Let $v\colon G \to \mathrm{Gl}(n, \mathbb{R})$ and $\psi\colon G \to \mathrm{Gl}(k, \mathbb{R})$ be as in Lemmas 4 and 5, respectively. Let $x \in M_{(H)}$, S_x be a slice at x as in Lemma 4 and K_x be a compact subset of S_x. Then, obviously, $(h_x, j_x)\colon GS_x \to \mathbb{R}^{k+n}(\psi \oplus v)$ is the desired mapping.

7. LEMMA. *Let G be a linear Lie group and M a proper smooth G-manifold having only finitely many orbit types. Then M has covers $\{O'_k\}_{k=1}^n$ and $\{O_k\}_{k=1}^n$ for some $n \in \mathbb{N}$, satisfying the following three conditions:*

1) *Every O'_k and O_k is open and G-invariant.*

2) *$\bar{O}_k \subset O'_k$ for every k.*

3) *There exists a representation $\varrho\colon G \to \mathrm{Gl}(q, \mathbb{R})$ for some $q \in \mathbb{N}$ such that for every k there is a G-equivariant smooth embedding $j_k\colon O'_k \to \mathbb{R}^q(\varrho)$ whose restriction to $\bar{O}_k$ is proper.*

PROOF. Let $(H_1), \ldots, (H_m)$ be the orbit types of M. Let $\{GS_{x_i}\}_{i=1}^{\infty}$ be a cover of M by such tubes that every S_{x_i} has the same properties as the slice in Corollary 6. The orbit space M/G is a paracompact space with finite covering dimension.

Thus, by Theorem 1.8.2 in [Pa2], there is an open cover $\{O'_{k\beta} \mid \beta \in B_k, k = 1,\ldots,n\}$ refining $\{GS_{x_i}\}_{i=1}^{\infty}$ such that each $O'_{k\beta}$ is G-invariant and $O'_{k\beta} \cap O'_{k\beta'} = \emptyset$ if $\beta \neq \beta'$. Here we can assume that each $B_k \subset \mathsf{N}$ and that $\{O'_{k\beta} \mid \beta \in B_k, k = 1,\ldots,n\}$ is locally finite and has an open G-invariant refinement $\{O_{k\beta} \mid \beta \in B_k,\ k = 1,\ldots,n\}$, where $\bar{O}_{k\beta} \subset O'_{k\beta}$ for every k and β.

We next choose for every k and β a tube GS_i such that $O'_{k\beta} \subset GS_i$ and denote this tube by $GS_{k\beta}$. We divide the family $\{GS_{k\beta} \mid \beta \in B_k,\ k = 1,\ldots,n\}$ into m subfamilies $\{GS^1_{k\beta}\},\ldots,\{GS^m_{k\beta}\}$ in such a way that exactly those tubes $GS_{k\beta}$ for which $(G_{x_{k\beta}}) = (H_l)$ belong to the family $\{GS^l_{k\beta}\}$. By Corollary 6, there exists for each $l \in \{1,\ldots,m\}$ a representation $\varrho_l\colon G \to \mathrm{Gl}(n_l, \mathsf{R})$ for some $n_l \in \mathsf{N}$, such that every tube $GS^l_{k\beta}$ admits a G-equivariant smooth embedding $j^l_{k\beta}$ in $\mathsf{R}^{n_l}(\varrho_l)$. Since $\bar{O}_{k\beta} \cap S_{k\beta}$ is compact and $\bar{O}_{k\beta} = G(\bar{O}_{k\beta} \cap S_{k\beta})$ we can assume that the restriction $j^l_{k\beta} \mid \bar{O}_{k\beta}$ is proper.

The representation $\tilde{\varrho} = \varrho_1 \oplus \cdots \oplus \varrho_m$ makes $\mathsf{R}^p(\tilde{\varrho}) = \mathsf{R}^{n_1 + \cdots + n_m}(\tilde{\varrho})$ a linear G-space. Then $j_{k\beta}\colon GS^l_{k\beta} \to \mathsf{R}^p(\tilde{\varrho})$, $y \mapsto (0,\ldots,0, j^l_{k\beta}(y), 0,\ldots,0)$, is a G-equivariant smooth embedding whose restriction to $\bar{O}_{k\beta}$ is proper. Finally, let

$$\varrho\colon G \to \mathrm{Gl}(p+1, \mathsf{R}), \qquad g \mapsto \begin{pmatrix} \tilde{\varrho}(g) & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $O'_{k\beta} \cap O'_{k\beta'} = \emptyset$ when $\beta \neq \beta'$, it follows that $j_k\colon \bigcup_{\beta \in B_k} O'_{k\beta} \to \mathsf{R}^{p+1}(\varrho)$, $y \mapsto (j_{k\beta}(y), \beta)$ when $y \in O'_{k\beta}$, is a G-equivariant smooth embedding. Since only finitely many values of β can occur in any compact subset of R it follows that the restriction $j_k \mid \bigcup_{\beta \in B_k} \bar{O}_{k\beta}$ is proper. Thus we can choose $O'_k = \bigcup_{\beta \in B_k} O'_{k\beta}$ and $O_k = \bigcup_{\beta \in B_k} O_{k\beta}$.

Proof of the theorem. Let $\{O'_k\}_{k=1}^n$ and $\{O_k\}_{k=1}^n$ be the covers of M as in Lemma 7. Let $\{W_k\}_{k=1}^n$ be a refinement of $\{O_k\}_{k=1}^n$ by open G-invariant sets W_k, where $\overline{W}_k \subset O_k$ for every k. According to Proposition 2 there exists for every k a G-invariant smooth mapping $h_k\colon M \to [0, 1]$, which is identically one on $\overline{W}_k$ and zero outside O_k. Let $\varrho\colon G \to \mathrm{Gl}(q, \mathsf{R})$ be a representation such that for every k there is a G-equivariant smooth embedding $j_k\colon O'_k \to \mathsf{R}^q(\varrho)$ whose restriction to $\bar{O}_k$ is proper. Next, for every k let $j^*_k\colon M \to \mathsf{R}^q(\varrho)$ be a mapping defined by $j^*_k(x) = h_k(x) j_k(x)$ if $x \in O_k$ and $j^*_k(x) = 0$ if $x \in M \backslash O_k$. Then each j^*_k is smooth and G-equivariant. Let R^n be a euclidean space where G acts trivially. Then the mapping

$$j\colon M \to \mathsf{R}^n \oplus \mathsf{R}^q(\varrho) \oplus \cdots \oplus \mathsf{R}^q(\varrho), \qquad x \mapsto (h_1(x),\ldots,h_n(x), j^*_1(x),\ldots,j^*_n(x)),$$

is G-equivariant and smooth. It is an immersion since each j^*_k is immersive in W_k.

Let $x \in M$ and let $(x_d)_{d=1}^{\infty}$ be a sequence in M such that $j(x_d) \to j(x)$. We know that $x \in W_k$ for some k. Thus $h_k(x) = 1$. Since $h_k(x_d) \to h_k(x)$, it follows that $h_k(x_d) > 0$ for sufficiently large d. Thus $x_d \in O_k$ for sufficiently large d. Since

$h_k(x_d)j_k(x_d) \to h_k(x)j_k(x)$, it now follows that $j_k(x_d) \to j_k(x)$. Since the restriction $j_k \mid O_k$ is an embedding, it follows that $x_d \to x$. Therefore j is injective and j^{-1} is continuous.

Since all the restrictions $j_k^* \mid \overline{W}_k$ are proper also the restrictions $j \mid \overline{W}_k$ are proper for every k. Since $\{\overline{W}_k\}_{k=1}^n$ is a closed cover of M it follows that j is proper. This completes the proof.

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HOLOMORPHIC FUNCTIONS AND THE (BB)-PROPERTY

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§ 1. Introduction.

A holomorphic function on a balanced domain in a locally convex space may be regarded as a sequence of polynomials which satisfies certain growth conditions. Locally convex topologies on the space of all holomorphic functions are a quantification of these conditions on aggregates of functions. This quantification is frequently obtained by combining, in an appropriate fashion, estimates on spaces of homogeneous polynomials. In this introduction we give an intuitive view of the process and in doing so, reformulate a number of known results.

To be more specific we require some definitions. E will denote a locally convex space over the complex numbers $\mathbf{C}$, $\mathscr{P}(^nE)$ is the space of continuous n-homogeneous polynomials on E and $\mathscr{H}(U)$ will denote the space of $\mathbf{C}$-valued holomorphic functions on the open subset U of E. The three most frequently studied topologies in infinite dimensional holomorphy are τ_0, the compact open topology, τ_ω and τ_δ. A seminorm p on $\mathscr{H}(U)$ is said to be τ_ω continuous if there exists a compact subset K of U such that for every V open, $K \subset V \subset U$, there exists $c(V) > 0$ such that

$$(1.1) \qquad p(f) \leqq c(V) \, \| f \|_V \quad \text{for all } f \in \mathscr{H}(U).$$

The τ_ω topology on $\mathscr{H}(U)$ is the locally convex topology generated by the τ_ω continuous semi-norms. A semi-norm p on $\mathscr{H}(U)$ is said to be τ_δ continuous if for every increasing countable open cover of U, $\mathscr{V} = (V_n)_n$, there exists $c > 0$ and a positive integer n_0 such that

$$p(f) \leqq c \, \| f \|_{V_{n_0}} \quad \text{for all } f \text{ in } \mathscr{H}(U).$$

We always have $\tau_0 \leqq \tau_\omega \leqq \tau_\delta$ and conditions for equality have been investigated by various authors [1, 2, 3, 12, 13, 16]. An important special case is obtained by taking U balanced since this leads, via the Taylor series expansion, to a Schauder decomposition of $\mathscr{H}(U)$. As $\mathscr{P}(^nE)$, n arbitrary, is a complemented subspace of

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$\mathscr{H}(U)$ for any of the above topologies we see that any equality of topologies on $\mathscr{H}(U)$ will lead to the same equality on each space of homogeneous polynomials. When we restrict to $\mathscr{P}(^nE)$ simplifications and refinements are possible. On the one hand $\tau_\omega = \tau_\delta$ on $\mathscr{P}(^nE)$ for all n. On the other hand $\mathscr{P}(^nE)$ may be identified with the dual of $\left(\widehat{\bigotimes_{n,\pi,s}} E\right)$ – the completion of the space of symmetric n-tensors on E endowed with the projective π topology. This duality can be used to describe τ_0 and τ_ω as well as two other natural topologies in $\mathscr{P}(^nE)$ (we use the notation $\bigotimes_n x_i$ for $x_i \otimes \ldots \otimes x_i$ (n times) and $\bigotimes_{i,n} x_i$ to denote $x_1 \otimes x_2 \ldots \otimes x_n$);

τ_b – the topology of uniform convergence on the bounded subsets of E,
β – the strong topology inherited from $\widehat{\bigotimes_{n,\pi,s}} E$.

The duality between $\widehat{\bigotimes_{n,\pi,s}} E$ and $\mathscr{P}(^nE)$ is given by $\left\langle P, \bigotimes_n x \right\rangle = P(x)$. The τ_0, τ_b and β topologies on $\mathscr{P}(^nE)$, E metrizable, are then identified as uniform convergence on the following subsets of $\widehat{\bigotimes_{n,\pi,s}} E$;

$$\bar{\Gamma}\left(\bigotimes_{n,s} K\right), \bar{\Gamma}\left(\bigotimes_{n,s} B\right), \bigcap_m \bar{\Gamma}\left(\bigotimes_{n,s} r_m U_m\right),$$

respectively, where K is compact in E, B is bounded in E, $(U_m)_m$ denotes a fundamental sequence of convex balanced neighbourhoods of zero in E and $(r_m)_m$ is a sequence of positive numbers, Γ is the convex balanced hull and $\bar{\Gamma}$ the closed convex balanced hull.

For each positive integer m let $\mathscr{P}(^nE)_m$ denote the set of all $P \in \mathscr{P}(^nE)$ such that $\|P\|_{U_m} < \infty$ and we endow this space with the topology of uniform convergence on U_m. $\mathscr{P}(^nE)_m$ is a Banach space and $\mathscr{P}(^nE) = \bigcup_m \mathscr{P}(^nE)_m$. With the above duality $\mathscr{P}(^nE)_m$ has the topology of uniform convergence on $\bar{\Gamma}\left(\bigotimes_{n,s} U_m\right)$ and

$$(\mathscr{P}(^nE), \tau_\omega) = \lim_{\overrightarrow{m}} \mathscr{P}(^nE)_m.$$

From the above description it is clear that $\tau_0 \leqq \tau_b \leqq \beta \leqq \tau_\omega$ on $\mathscr{P}(^nE)$ for all n.

By the Hahn-Banach theorem we have $\tau_0 = \tau_b$ on $\mathscr{P}(^nE)$ for some (and hence for all n) if and only if E is semi-Montel (i.e. the closed bounded subsets of E are compact).

We have $\tau_b = \beta$ on $\mathscr{P}(^nE)$ if and only if the sets $\bar{\Gamma}\left(\bigotimes_{n,s} B\right)$, B bounded in E, form a fundamental system of bounded subsets of $\widehat{\bigotimes_{n,\pi,s}} E$. This is the n-fold symmetric version of Grothendieck's "Problème des topologies" which we now state.

For locally convex spaces E and F is every bounded subset of E $\hat{\bigotimes}_{\pi}$ *F contained in the closed absolutely convex hull of a set of the form* $B_1 \otimes B_2$ *where* B_1 *is a bounded subset of E and* B_2 *is a bounded subset of F?*

If this is the case then the pair $\{E, F\}$ is said to have the (BB)-property. We shall say that the locally convex space E has the $(\mathrm{BB})_n$-property if each bounded subset of $\hat{\bigotimes}_{n,\pi,s} E$ is contained in $\bar{\Gamma}\left(\bigotimes_{n,s} B\right)$ for some bounded subset B of E. If E has $(\mathrm{BB})_n$ for all n we say that E has $(\mathrm{BB})_\infty$. With our new notation we have that $\tau_b = \beta$ on $\mathscr{P}({}^nE)$ if and only if E has the $(\mathrm{BB})_n$-property. If $\{E, E\}$ has the (BB)-property then E has the $(\mathrm{BB})_2$-property and $\tau_b = \beta$ on $\mathscr{P}({}^2E)$. The history of Grothendieck's problem can be divided into two distinct phases: the positive solutions of Grothendieck ([20]), circa 1955, and the recent developments, all of which follows from Taskinen's fundamental 1986 paper [20]. It is no coincidence that the Fréchet spaces for which the most interesting results have been obtained in infinite dimensional holomorphy – Banach spaces, nuclear and Schwartz spaces – are included in the classes for which Grothendieck obtained positive solutions to the "problème des topologies" and that the class of spaces which often appeared as the critical case in infinite dimensional holomorphy – the Fréchet-Montel spaces – should yield mixed results (i.e. both positive results and counterexamples) to the "problème des topologies". It is our belief that the $(\mathrm{BB})_\infty$-property will frequently appear as an essential hypothesis in topological problems of infinite dimensional holomorphy. Taskinen and his followers have shown by example and counterexample, that the collection of pairs of spaces with the (BB)-property will probably not coincide with any of the usual linear collections but will contain large subcollections of interesting spaces. We shall consider the (BB)-property more closely in the next section and confine ourselves here to a general presentation.

If E is a Fréchet-Montel space and E has the $(\mathrm{BB})_n$-property then each bounded subset of $\hat{\bigotimes}_{n,\pi,s} E$ is contained in a subset of the form $\bar{\Gamma}\left(\bigotimes_{n,s} K\right)$. By our previous remarks this implies that $\tau_0 = \beta$ on $\mathscr{P}({}^nE)$ and, moreover, since $\bar{\Gamma}\left(\bigotimes_{n,s} K\right)$ is a compact subset of $\hat{\bigotimes}_{n,\pi,s} E$ it follows that $\hat{\bigotimes}_{n,\pi,s} E$ is itself Fréchet-Montel. Hence $(\mathscr{P}({}^nE), \tau_0)$ is reflexive and $\tau_0 = \tau_\omega$ on $\mathscr{P}({}^nE)$. To summarize we have the following result.

PROPOSITION 1. *If E is a Fréchet-Montel space then E has the* $(\mathrm{BB})_n$*-property if and only if* $\tau_0 = \tau_\omega$ *on* $\mathscr{P}(^nE)$.

Taskinen [22] constructed a Fréchet-Montel space which did not have the (BB)-property and a suitable modification by Ansemil-Taskinen [3] yielded the first example of a Fréchet-Montel space for which $\tau_0 \neq \tau_\omega$ on $\mathscr{P}(^2E)$. For positive results arising from proposition 1 we refer to proposition 4 and § 2.

Since $\hat{\bigotimes}_\pi$ is an associative functor the following is immediate.

PROPOSITION 2. [8, *proposition* 8]. *If* $\mathscr{E}$ *is a collection of Fréchet spaces which is stable under the formation of completed projective tensor products and* $\{E, F\}$ *has the* (BB)*-property for any* $E, F \in \mathscr{E}$ *then each* $E \in \mathscr{E}$ *has the* $(\mathrm{BB})_\infty$*-property.*

The collection of tensor-(FG) spaces introduced in [8] satisfies the hypothesis of proposition 2. Remarks regarding the definition of this collection and examples are given in the next section.

The collection of separable Banach spaces is stable under completed projective tensor products. Using this fact, associativity of $\bigotimes_\pi$ and [21, proposition 2.13] modified for separable Banach spaces we can easily show the following.

PROPOSITION 3. *If E is a separable Fréchet space,* $\{E, F\}$ *has the* (BB)*-property for every separable Banach space F, and E contains a fundamental system of absolutely convex bounded sets* $\mathscr{B}$ *such that* E_B *has the approximation property for all* $B \in \mathscr{B}$ *then E has the* $(\mathrm{BB})_\infty$*-property.*

Separable Hilbertizable Fréchet spaces satisfy the hypothesis of proposition 3. The hypothesis of proposition 3 may be weakened by using recent results from [6].

Finally we consider the equality $\beta = \tau_\omega$ on $\mathscr{P}(^nE)$. Since τ_ω is the barrelled topology associated with τ_0 on $\mathscr{P}(^nE)$ and $\tau_\omega \geqq \beta \geqq \tau_0$ it follows that τ_ω is also the barrelled topology associated with β. A locally convex space is said to be distinguished if its strong dual is barrelled. Hence for arbitrary Fréchet spaces we have $\beta = \tau_\omega$ on $\mathscr{P}(^nE)$ if and only if $\hat{\bigotimes}_{n,\pi,s} E$ is distinguished. Since we require this property for all n it is more convenient to consider the density condition [5]. A Fréchet space E is said to have the density condition if the bounded subsets of $E'_\beta (= (\mathscr{P}(^1E), \beta))$ are metrizable. If E has $(\mathrm{BB})_n$ and the density condition then $\hat{\bigotimes}_{n,\pi,s} E$ also has the density condition ([5]) and hence is distinguished. Quasinormable spaces and Fréchet-Montel spaces have the density condition.

This concludes our first examination of locally convex topologies on spaces of homogeneous polynomials. We now consider the problem of how these locally convex spaces of homogeneous polynomials are combined to provide locally convex structures on $\mathscr{H}(U)$, U a balanced domain in a locally convex space E. In later sections we shall see how to lift estimates to obtain more precise information about $\mathscr{H}(U)$.

The τ_0 topology on $\mathscr{H}(U)$, U a balanced domain in a Fréchet space, is generated by all seminorms of the form

$$p_K(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_K \tag{1.2}$$

where $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$ and K is a compact subset of U.

The τ_ω topology on $\mathscr{H}(U)$, U a balanced domain in a Fréchet space (or indeed in any locally convex space) is generated by all seminorms which have the following three properties

$$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) \tag{1.3}$$

for all $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$,

$$p \mid \mathscr{P}(^nE) \text{ is } \tau_\omega \text{ continuous for all } n \tag{1.4}$$

there exists a compact subset K of U such that for every V open, $K \subset V \subset U$, there exists $c(V) > 0$ such that

$$p(P) \leqq c(V) \, \|P\|_V \tag{1.5}$$

for all $P \in \mathscr{P}(^nE)$ and all n.

The τ_δ topology on $\mathscr{H}(U)$ is generated by all seminorms which satisfy (1.3) and (1.4).

Ansemil-Ponte [2] used (1.2), holomorphic germs and a result of Mujica to show that if $\tau_0 = \tau_\omega$ on $\mathscr{P}(^nE)$ for all n where E is a Fréchet-Montel space then $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$, U an arbitrary balanced open subset of U. In our terminology and using proposition 1 we may rephrase this as follows.

PROPOSITION 4. *If E is a Fréchet-Montel space then $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$ for any balanced open subset U of E if and only if E has the* (BB)$_\infty$*-property.*

If $\tau_0 = \tau_\omega$ on $\mathscr{P}(^nE)$ then condition (1.4) is equivalent to

(1.4)′ *there exists a compact subset K_n of E such that*

$$p(P) \leqq \|P\|_{K_n} \text{ for all } P \in \mathscr{P}(^nE)$$

We may thus regard (1.4)′ and (1.5) as our estimates on spaces of homogeneous polynomials and (1.3) plays a role in putting these estimates together to obtain $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$. Proposition 4 solves the $\tau_0 = \tau_\omega$ problem as a holomorphic problem on balanced domains and what remains is now a polynomial problem. The combined conditions (1.3), (1.4) and (1.5) may be regarded as a refinement of (1.1).

More precise descriptions are available for certain classes of Fréchet spaces. By proposition 4, if E is a Fréchet-Schwartz space then the τ_ω topology on $\mathscr{H}(U)$, U balanced on E, is generated by all seminorms which satisfy (1.2).

If E is a Banach space with unit ball B and U is a balanced open subset of E then the τ_ω topology on $\mathscr{H}(U)$ is generated by all semi-norms of the form

$$p(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+\alpha_n B} \tag{1.6}$$

where $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$, K is a compact subset of U and $(\alpha_n)_n \in c_0$ ([4, 12]).

We now consider the τ_b topology on $\mathscr{H}(U)$. A subset B of U is called a bounded subset of U if it is a bounded subset of E and there exists a neighbourhood V of 0 in E such that $B + V \subset U$. The natural analogue of τ_b on $\mathscr{H}(U)$ is the topology of uniform convergence on the bounded subsets of U. However, it is rarely the case that each holomorphic function on U is bounded on all the bounded subsets of U. In this situation attention is often restricted to those holomorphic functions which are bounded on the bounded subsets of U and one obtains a subspace of $\mathscr{H}(U)$ which is denoted by $\mathscr{H}_b(U)$ (see example 10). We are interested in $\mathscr{H}(U)$ and so motivated by (1.2) for Fréchet-Schwartz spaces and (1.6) for Banach spaces we define a new topology on $\mathscr{H}(U)$. We shall say that a sequence of subsets $(B_n)_n$, of a locally convex space E, converges to a subset B if for every neighbourhood V of 0 in E there exists a positive integer n_0 such that $B_n \subset B + V$ for all $n \geqq n_0$.

Definition 5. If U is a balanced open subset of a locally convex space E then the τ_b *topology on* $\mathscr{H}(U)$ is generated by all seminorms on $\mathscr{H}(U)$ which have the form

$$p(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{B_n} \tag{1.7}$$

for all $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(U)$, where $(B_n)_n$ is a sequence of bounded subsets of E which converges to some compact subset of U.

If U is an arbitrary open subset of U we let

$$(\mathscr{H}(U), \tau_b) = \varprojlim_{\xi, V} (\mathscr{H}(\xi + V), \tau_b)$$

where the projective limit is taken over all pairs (ξ, V) with $\xi \in U$, V balanced in E and $\xi + V \subset U$ (we use the canonical identification between $\mathscr{H}(V)$ and $\mathscr{H}(\xi + V)$ to define τ_b on $\mathscr{H}(\xi + V)$). Many of the properties of $\mathscr{H}(U)$, U balanced, shared by τ_0, τ_ω and τ_δ remain true for τ_b and are proved in the same way. For instance $\{(\{\mathscr{P}(^nE), \tau_b)\}_{n=0}^{\infty}$ is an $\mathscr{S}$-absolute decomposition for $(\mathscr{H}(U), \tau_b)$ ([13]).

Clearly we have $\tau_0 \leqq \tau_b \leqq \tau_\omega$ on $\mathscr{H}(U)$, and $\tau_0 = \tau_b$, for U balanced, if and only if E is a semi-Montel space. We have already noted that $\tau_b = \tau_\omega$ for balanced domains in Banach spaces and Fréchet-Schwartz spaces.

Our aim in this paper is to obtain a similar result for a large class of Fréchet spaces.

For unexplained terminology, general definitions and results mentioned without proof or reference we refer to [13].

§ 2. *T*-Schauder decompositions.

In § 1 we saw that a pair of locally convex spaces has the (BB)-property if every bounded subset of their completed π-projective tensor products splits (modulo taking the absolute convex hull). For this reason it is only to be expected that good splittings of the component spaces should lead to examples of pairs with the (BB)-property. A splitting of the whole space is a projection. The mere existence of projections is not sufficient, however, as bounded sets in Fréchet spaces are defined by estimates involving a sequence of semi-norms and it is necessary to have further interactions between the projections and the semi-norms. The first step in this direction was taken by Taskinen [20] who defined a class of Fréchet spaces with an unconditional basis and a property allowing the extension of norm estimates on subsets of the basis to their linear span in a uniform fashion. The projections, given by the basis, led to a partition of the basis into disjoint subsets on each of which tensor norms could be estimated and the results from the different sections combined to obtain the (BB)-property. The technique of using good estimates between different norms on sufficiently many projections appears to be fundamental as in the same paper (see also [22]) the author obtained counterexamples to the general problem by using projections. Taskinen's method has been developed to more general situations but in all cases it is possible to see implicitly or explicitly the presence of projections.

Bonet-Diaz [7] introduced *T*-spaces by replacing the basis by projections and Bonet-Diaz-Taskinen [8] went a step further and replaced projections with a partition of the identity. Diaz-Metafune [10] (see also [9]) characterized the standard quojections of Moscatelli type E such that $\{E, F\}$ has the (BB)-property for every Banach space F as those spaces for which E'' is a product of Banach spaces (in other words the spaces with a single twist or spaces whose second dual contains sufficiently many projections onto spaces with the (BB)-property (i.e. Banach spaces)). The other major class for which there is a positive solution are the Hilbertizable Fréchet space and here again projections are implicitly available as every closed subspace of a Hilbert space is complemented.

In infinite dimensional holomorphy the methods of Taskinen [20] and Bonet-Diaz [7] was extended to the n-fold symmetric case in [14] and [16] to obtain further examples of Fréchet-Montel spaces E for which $\tau_0 = \tau_\omega$ on $\mathscr{H}(U)$, U balanced in E.

In this article we define a collection of spaces which are very similar to the T-spaces of Bonet-Diaz [7] and adjust the method of [14] and [16] to obtain more precise estimates for the $(BB)_n$-property. These estimates are then combined to obtain examples of spaces of holomorphic functions with $\tau_b = \tau_\omega$.

Let $\{E_n\}_n$ denote an unconditional Schauder decomposition of the locally convex space E. For each n let P_n denote the canonical projection, defined by the decomposition, from E onto E_n and for any subset J of N let $P_J = \sum_{j \in J} P_j$.

DEFINITION 6. An *unconditional Schauder decomposition*, $\{E_n\}_n$ of a Fréchet space E is a T-Schauder decomposition if there exists a fundamental system of semi-norms for E, $(\|\cdot\|_k)_{k \in N}$ such that

(2.1) $\|P_J(x)\|_k \leqq \|x\|_k$ for all $J \subset N$, $k \in N$ and $x \in E$

(2.2) for every sequence $\alpha = (\alpha_k)_k$, $0 < \alpha_k \leqq 1$, there exists a partition

$$J_\alpha = (J_{\alpha,k})_k \text{ of } N \text{ such that if } P_{\alpha,k} := P_{J_{\alpha,k}} \text{ then}$$

$$\|P_{\alpha,k}(x)\|_{k-1} \leqq \alpha_k \|P_{\alpha,k}(x)\|_k \text{ for all } x \in E \text{ and all } k \geqq 2.$$

(2.3) $\|\cdot\|_k$ defines the topology induced by E on $P_{\alpha,k}(E)$ for all α and all k.

A basic sequence of semi-norms satisfying (2.1), (2.2) and (2.3) is called T-adaptable and a subset A of E is said to be T-invariant if $P_J(A) \subset A$ for all $J \subset N$.

Definition 6 is very similar to the definition of T-decomposable space given in [7]. The extra condition, required in order to prove lemma 7, is that each $P_{\alpha,k}$ is obtained by adding together projections from the original Schauder decomposition and condition (2.3) shows that it may be difficult to construct an example of a T-decomposable space which does not have a T-Schauder decomposition. The FG-spaces of [8] are obtained by replacing the decomposition and (2.1) by a partition of the identity $\sum_{j=1}^{\infty} P_j$ and tensor (FG) spaces are FG-spaces for which $\left\| \sum_{j=1}^{l} P_j \right\|_k \leqq 1$ for all $l \leqq k$ (see the remarks following proposition 2).

LEMMA 7. *If the Fréchet space E has a T-Schauder decomposition and U is a T-invariant convex balanced open subset of E then each compact subset of U is contained in an absolutely convex T-invariant compact subset of U.*

PROOF. Let K denote a convex balanced compact subset of U. Let $\tilde{K} = \bigcup_{J \subset N} P_J(K)$. Clearly $\tilde{K} \subset U$ and we show that $\tilde{K}$ is relatively compact. Let $(y_n)_n$ denote a sequence in $\tilde{K}$ and for each n choose x_n in K and $J_n \subset N$ such that $y_n = P_{J_n}(x_n)$. Since K is compact, $(x_n)_n$ contains a convergent subsequence (which we may suppose is the original sequence) which converges to $x \in K$. We identify subsets of N with points in 2^N in the usual way. If the set consisting of 2 elements is given its discrete topology and 2^N is given the product topology then 2^N is a compact metric space. It follows that $\{J_n\}_n$ contains a subsequence, which we again assume to be the original sequence, which converges to an element J of N.

If k is any positive integer then

$$\begin{aligned}\|P_{J_n}(x_n) - P_J(x)\|_k &\leqq \|P_{J_n}(x_n) - P_{J_n}(x)\|_k + \|P_{J_n}(x) - P_J(x)\|_k \\ &\leqq \|x_n - x\|_k + \|P_{J_n}(x) - P_J(x)\|_k \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $\bar{\Gamma}(\tilde{K})$ is a compact subset of E.

For any $J_1 \subset N$ we have

$$P_{J_1}(\tilde{K}) = \bigcup_{J \subset N} P_{J_1} P_J(K) = \bigcup_{j \subset N} P_{J_1 \cap J}(K) \subset \tilde{K}$$

Hence

$$P_{J_1}(\bar{\Gamma}(\tilde{K})) = \bar{\Gamma}(P_{J_1}(\tilde{K})) \subset \bar{\Gamma}(\tilde{K})$$

and $\bar{\Gamma}(\tilde{K})$ is a convex balanced compact T-invariant subset of E which contains K. This completes the proof.

PROPOSITION 8. *Let $(\|\cdot\|_k)_k$ denote a T-adaptable set of seminorms for the T-Schauder decomposition $\{E_n\}_n$ of the Fréchet space E. Let K denote an absolutely convex compact T-invariant subset of E and let $U_1 = \{x \in E;\ \|x\|_1 \leqq 1\}$. Let q_1 denote the seminorm on E with closed unit ball $U_1 + K$ and let $q_k = \|\cdot\|_k$ for $k \geqq 2$. Then $(q_k)_k$ is a T-adaptable set of seminorms for the decomposition $\{E_n\}_n$.*

PROOF. We retain the same set of projections $(P_{\alpha,k})_k$ for the new set of semi-norms. Since K is compact the norms $\|\cdot\|_1$ and q_1 are equivalent and hence (2.3) is satisfied by $(q_k)_{k \geqq 1}$. To complete the proof we must check (2.1) for q_1 and (2.2) for $k = 2$. For (2.1) it suffices to show that $q_1(P_J x) \leqq q_1(x)$ for all $J \subset N$ and all $x \in E$. This is equivalent to showing $P_J(U_1 + K) \subset U_1 + K$. By (2.1) for $\|\cdot\|_1$, we have $P_J(U_1) \subset U_1$ and since K is T-invariant we have $P_J(K) \subset K$.

Hence $P_J(U_1 + K) = P_J(U_1) + P_J(K) \subset U_1 + K$.

For (2.2) we are required to show that

$$q_1(P_{\alpha,1}(x)) \leqq \alpha_2 q_2(P_{\alpha,1}(x)) \quad \text{for all } x \in E.$$

Since $U_1 \subset U_1 + K$ it follows that $q_1 \leqq \|\cdot\|_1$.

Hence

$$q_1(P_{\alpha,1}(x)) \leqq \|P_{\alpha,1}(x)\|_1 \leqq \alpha_2 \|P_{\alpha,1}(x)\|_2 = \alpha_2 q_2(P_{\alpha,1}(x))$$

and this completes the proof.

EXAMPLE 9 (of Fréchet spaces with T-Schauder decompositions).

(a) Fréchet spaces with unconditional basis of type (T), l^p and X valued Köthe sequence spaces ([7, 8, 22]). In [11] the authors show that a Köthe echelon space, $\lambda_p(I, A)$, $1 \leqq p \leqq \infty$ or $p = 0$ and I of arbitrary cardinality, has a T-Schauder decomposition if and only if it has a total bounded set or equivalently if and only if $\lambda_p(I, A)'_\beta$ admits a continuous norm.

(b) Banach spaces [7].

(c) Fréchet-Schwartz spaces which admit a continuous norm and a finite dimensional decomposition ([7]).

(d) Fréchet-Montel spaces with unconditional basis $(e_n)_n$ which satisfy

$(c\Omega)$ for all $k, t \in N$, $k > t$, there exists $M_{k,t} > 0$ such that if

$\|e_i\|_k \leqq C \|e_i\|_t$, $i \in J$, for some $C > 0$ and $J \subset N$, then

$\|x\|_k \leqq CM_{k,t} \|x\|_t$ for every $x \in \overline{sp(e_i, i \in J)}$ ([7, 9]).

The above are all examples of T-decomposable spaces and the proofs given in [7] show that they all have T-Schauder decompositions. Furthermore, the proof of Observation 2 in [7] shows that the countable product of Fréchet spaces with T-Schauder decompositions also has a T-Schauder decomposition.

Condition (2.3) implies that all except a finite number of semi-norms agree on $P_{\alpha,k}(E)$. For this reason we find (see for instance the proof of proposition 3 in [7]) that in many examples the spaces $P_{\alpha,k}(E)$ are finite dimensional. Finite dimensionality also plays a key role in the final part of Taskinen's proof [20, theorem 3.1 and 3.3] and something similar is often necessary in the general case. Our next example gives a natural situation in which all of the spaces $P_{\alpha,k}(E)$ are infinite dimensional Banach spaces.

EXAMPLE 10. Let E denote an infinite dimensional Banach space. The Taylor series decomposition of $\mathcal{H}_b(E)$ (the entire holomorphic functions of bounded type endowed with the topology of uniform convergence on the bounded subsets of E) is a T-Schauder decomposition of the Fréchet space $\mathcal{H}_b(E)$. We have $E_n = \mathcal{P}(^nE)$

and $P_n(f) = \dfrac{\hat{d}^n f(0)}{n!}$ for all n and all $f \in \mathscr{H}_b(E)$. The topology on $\mathscr{H}_b(E)$ is generated by the seminorms

$$p_k(f) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{kB}, \quad k = 1, 2 \ldots$$

where B is the unit ball of E.

Clearly (2.1) is satisfied. Let $\alpha = (\alpha_n)_n$ be given. Since

$$p_k\left(\frac{\hat{d}^n f(0)}{n!}\right) = \left(\frac{k}{l}\right)^n p_l\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

for all k, l and $n \in N$ we can choose an increasing sequence of positive integers $(n_k)_{k \geqq 2}$ such that $p_{k-1}(P) \leqq \alpha_k p_k(P)$ for all $P \in \mathscr{P}(^nE)$ and all $n \geqq n_k$. Let $J_1 = \{0, 1, \ldots, n_2 - 1\}$ and $J_k = \{n_k, \ldots, n_{k+1} - 1\}$ for $k \geqq 2$. Let $P_{\alpha,k}(f) = \dfrac{\hat{d}^j f(0)}{j!}$. Then

$$\begin{aligned} \|P_{\alpha,k}(f)\|_{k-1} &= \sum_{j \in J_k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{(k-1)B} \\ &\leqq \alpha_k \sum_{j \in J_k} \left\| \frac{\hat{d}^j f(0)}{j!} \right\|_{kB} \\ &= \alpha_k \|P_{\alpha,k}(f)\|_k \end{aligned}$$

and (2.2) is satisfied. Since $\|\cdot\|_k \sim \|\cdot\|_l$ on $\mathscr{P}(^nE)$ for all n, k and l and the projections $P_{\alpha,k}$ only involve a finite number of derivatives it follows that (2.3) is also satisfied. Hence $\mathscr{H}_b(E)$ has a T-Schauder decomposition. We note in passing that the norms given above for $\mathscr{H}_b(E)$ satisfy condition $(c\Omega)$ of example 9 (d) without any restriction on k and l. A similar proof works for U balanced.

The proof of the following proposition was motivated by the proofs of [20, theorems 3.1 and 3.3], [14, theorem 1] and [16, proposition 3]. The crucial point in proposition 11 is to obtain *symmetric tensor representations* and not just *tensor representations* which have previously been given in [14] and [16].

We let S denote the symmetrization (projection) from $\hat{\bigotimes}_{d,\pi} E$ onto $\hat{\bigotimes}_{d,\pi,s} E$ obtained by extending

$$S(x_1 \otimes x_2 \ldots \otimes x_d) := \frac{1}{d!} \sum_{\pi \in S_d} x_{\pi(1)} \otimes \ldots \otimes x_{\pi(d)}$$

where S_d is the set of all permutations of $\{1, \ldots, d\}$. If B is a convex balanced subset of E then the polarization formula [13, p. 4] and duality theory show that for $\theta \in \widehat{\bigotimes}_{d,\pi,s} E$ we have

$$\text{(2.4)} \quad \|\theta\|_{B^d} := \inf\left\{\sum_{j=1}^{\infty} \|x_{j,1}\|_B \|x_{j,2}\|_B \cdots \|x_{j,d}\|_B ; \theta = \sum_{j=1}^{\infty} x_{j,1} \otimes x_{j,2} \cdots \otimes x_{j,d}\right\}$$

$$\leqq \|\theta\|_B := \inf\left\{\sum_{j=1}^{\infty} \|x_j\|_B^d ; \theta = \sum_{j=1}^{\infty} \bigotimes_d x_j\right\}$$

$$\leqq \frac{d^d}{d!} \|\theta\|_{B^d}$$

Proposition 11. *Let E denote a Fréchet space with T-Schauder decomposition $\{E_n\}$ and T-adaptable set of semi-norms $(\|\cdot\|_k)_{k \geqq 1}$. Let d be a positive integer, $U_n = \{x \in E ; \|x_n\| \leqq 1\}$, B be a bounded subset of $\widehat{\bigotimes}_{d,\pi,s} E$ and $\varepsilon > 0$ be arbitrary. If $B \subset \bar{\Gamma}\left(\bigotimes_{d,s} U_1\right)$ then there exists a bounded subset A in $(1+\varepsilon)U_1$ such that $B \subset \bar{\Gamma}\left(\bigotimes_{d,s} A\right)$.*

Proof. We may suppose without loss of generality that $d > 1$ and $0 < \varepsilon < 1$. Since B is bounded there exists an increasing sequence of positive numbers, $(r_n)_{n=1}^{\infty}$, such that $B \subset \bigcap_n \bar{\Gamma}\left(\bigotimes_{d,s} r_n U_n\right)$. By our hypothesis we may take $r_1 = 1$ and, without loss generality, we may suppose $r_2 > \max\left((2d)^{d^2}; \frac{d^{2d}}{\varepsilon}\right)$.

For each positive integer k and each z in B we have a representation

$$z = \sum_{i=1}^{\infty} \lambda_{i,k}(z) \bigotimes_d x_{i,k}(z)$$

where

$$\sum_{i=1}^{\infty} |\lambda_{i,k}(z)| \leqq 1 + \varepsilon \text{ for all } k \in N \text{ and } z \in B$$

and $x_{i,k}(z) \in r_k U_k$ for all $i, k \in N$ and $z \in B$.

Let $\alpha_k = 2^{-k} r_k^{-(d+2)}$ for $k \in N$ and let $(P_{\alpha,k})_k$ be the decomposition for $\alpha := (\alpha_k)_k$ given by (2.2). For the d-tuple $k = (k_1, \ldots, k_d)$ of positive integers let

$$A_k = \left\{i, 1 \leqq i \leqq d, \hat{k} := \sup_{1 \leqq j \leqq d} k_j = k_i\right\} \quad \text{and let } B_k = \{i; 1 \leqq i \leqq d, k_i = 1\}$$

Let $|A_k| = e$ and $|B_k| = f$. For $1 \leqq j \leqq d$ let

$$\Phi_j(k) = \begin{cases} r_{\hat{k}}^{(d+1-e)/e} & \text{if } j \in A_k \text{ and } f > 0 \\ r_{\hat{k}}^{(d-e)/e} & \text{if } j \in A_k \text{ and } f = 0 \\ r_{\hat{k}}^{-1} & \text{if } j \notin A_k \text{ and } j \notin B_k \\ r_{\hat{k}}^{-1-\frac{1}{f}} & \text{if } j \in B_k. \end{cases}$$

If $|B_k| = 0$ then

$$\prod_{j=1}^{d} \Phi_j(k) = (r_{\hat{k}}^{(d-e)/e})^e (r_{\hat{k}}^{-1})^{d-e} = r_{\hat{k}}^{d-e-d+e} = 1.$$

If $|B_k| = f > 0$ then

$$\prod_{j=1}^{d} \Phi_j(k) = (r_{\hat{k}}^{(d+1-e)/e})^e (r_{\hat{k}}^{-1})^{d-e-f} (r_{\hat{k}}^{-1-\frac{1}{f}})^f$$
$$= r_{\hat{k}}^{d+1-e-d+e+f-f-1} = 1$$

so that $\prod_{j=1}^{d} \Phi_j(\mathrm{k}) = 1$ for all $k \in N^d$.

We now consider the formal sum

$$\sum_{i=1}^{\infty} \lambda_{i,1}(z) \bigotimes_d P_{\alpha,1}(x_{i,1}(z))$$
$$+ \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} \lambda_{i,k}(z) 2^{-kd} r_2^{-d} \bigotimes_d (r_2 2^k P_{\alpha,k}(x_{i,k}(z)))$$
$$+ \sum_{i=1}^{\infty} \sum_{\substack{k=(k_1,\dots k_d) \\ |A_k|<d}} \lambda_{i,\hat{k}}(z) 2^{-\Sigma_{j=1}^d k_j} S\left(\bigotimes_{d,j} (r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z)))\right)$$

We claim that the elements of E which appear in the tensor part of the above sum form a bounded subset of E. We consider the different cases that may arise and adopt the following notation.

If $j \leqq q$ we may use (2.3) to find $C_{q,j} > 0$ such that $\|\cdot\|_q \leqq C_{q,j} \|\cdot\|_j$ on $P_{\alpha,j}(E)$ and we let $c_q = \sup_{1 \leqq j \leqq q} C_{q,j}$.

In cases 3, 4 and 5, $k \in N$, $k \geqq 2$, and let $y_{k,i}(z) = r_2 2^k P_{\alpha,k}(x_{i,k}(z))$.

In cases 6 to 12, $k = (k_1, \dots, k_d) \in N^d$ and $|A_k| < d$ and we let

$$w_{i,k,j}(z) = d r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z)).$$

We let q denote a positive integer.

Case 1. $q = 1$.

$$\|P_{\alpha,1}(x_{i,1}(z))\|_1 \leqq \|x_{i,1}(z)\|_1 \leqq 1$$

Case 2. $q > 1$.

$$\|P_{\alpha,1}(x_{i,1}(z))\|_q \leqq C_{q,1}\,\|P_{\alpha,1}(x_{i,1}(z))\|_1 \leqq C_{q,1}$$

Case 3. $q < k$.

$$\begin{aligned}\|y_{k,i}(z)\|_q &\leqq r_2 2^k\,\|P_{\alpha,k}(x_{i,k}(z))\|_{k-1}\\ &\leqq r_2 2^k 2^{-k} r_k^{-(d+2)}\,\|P_{\alpha,k}(x_{i,k}(z))\|_k\\ &\leqq r_2 r_k^{-(d+2)} r_k\\ &< r_2^{-d}\end{aligned}$$

Case 4. $q = k$.

$$\begin{aligned}\|y_{k,i}(z)\|_q &= r_2 2^q\,\|P_{\alpha,q}(x_{i,q}(z))\|_q\\ &\leqq r_2 r_q 2^q\end{aligned}$$

Case 5. $q > k$.

$$\begin{aligned}\|y_{k,i}(z)\|_q &\leqq r_2 2^k C_{q,k}\|\,P_{\alpha,q}(x_{i,k}(z))\|_k\\ &\leqq r_2 2^q c_q r_q\end{aligned}$$

Case 6. $q < k_j = \hat{k}$. We have $\Phi_j(k) \leqq r_{\hat{k}}^{\frac{d+1-e}{e}}$

$$\begin{aligned}\|w_{i,k,j}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^{\hat{k}}\,\|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_{\hat{k}-1}\\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^{\hat{k}} 2^{-\hat{k}} r_{\hat{k}}^{-(d+2)}\,\|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_{\hat{k}}\\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}-(d+2)+1}\\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1}\\ &\leqq d r_2^{\frac{1}{2}} r_2^{-1}\\ &\leqq 1 \quad \text{since } r_2 > (2d)^{d^2} > d^2\end{aligned}$$

Case 7. $q < k_j < \hat{k}$. We have $\Phi_j(k) \leqq r_{\hat{k}}^{-1}$

$$
\begin{aligned}
\|w_{i,k,j}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{k_j} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_{k_j-1} \\
&\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{-k_j} r_{k_j}^{-(d+2)} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_{k_j} \\
&\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} r_{k_j}^{-(d+2)} \|(x_{i,\hat{k}})(z)\|_{\hat{k}} \\
&\leqq d r_2^{\frac{1}{2d}} r_{k_j}^{-(d+2)} \\
&\leqq d r_2^{-(d+1)} \\
&\leqq r_2^{-d}
\end{aligned}
$$

Case 8. If $q = k_j = \hat{k}$. We have $\Phi_j(k) \leqq r_q^{\frac{d+1-e}{e}}$

$$
\begin{aligned}
\|w_{i,j,k}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_q^{\frac{d+1-e}{e}} 2^q \|P_{\alpha,q}(x_{i,q}(z))\|_q \\
&\leqq d 2^q r_q^{\frac{1}{2d}+d+1}
\end{aligned}
$$

(Note that in this case we cannot have $q = 1$ since this would imply $\hat{k} = 1$ and $k = (1, \ldots, 1)$. Hence $|A_k| = d$ and this is not possible).

Case 9. If $1 = q = k_j < \hat{k}$. In this case $|B_k| \neq 0$ and $\Phi_j(k) = r_{\hat{k}}^{-1-\frac{1}{f}}$

$$
\begin{aligned}
\|w_{i,j,k}(z)\|_1 &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1-\frac{1}{f}} 2 \|P_{\alpha,1}(x_{i,\hat{k}}(z))\|_1 \\
&\leqq 2 d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1-\frac{1}{f}} r_{\hat{k}} \\
&= 2 d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-\frac{1}{f}} \\
&\leqq 2 d r_2^{\frac{1}{2d}-\frac{1}{d}} \quad \text{since } 1 \leqq f \leqq d \\
&\leqq 2 d r_2^{-\frac{1}{2d}} \\
&\leqq 1 \quad \text{since } r_2 \leqq (2d)^{2d}.
\end{aligned}
$$

Case 10. *If* $1 < k_j \leqq q < \hat{k}$. We have $\Phi_j(k) = r_{\hat{k}}^{-1}$.

$$
\begin{aligned}
\|w_{i,j,k}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{k_j} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_q \\
&\leqq d 2^q r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} r_{\hat{k}} \\
&= d 2^q r_2^{\frac{1}{2d}}
\end{aligned}
$$

Case 11. If $k_j = \hat{k} < q$. We have $\Phi_j(k) \leqq r_{\hat{k}}^{\frac{d+1-e}{e}}$.

$$\begin{aligned}\|w_{i,j,k}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^{\hat{k}} \|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_q \\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} 2^q C_{q,\hat{k}} \|P_{\alpha,\hat{k}}(x_{i,\hat{k}}(z))\|_{\hat{k}} \\ &\leqq d 2^q r_2^{\frac{1}{2d}} r_{\hat{k}}^{\frac{d+1-e}{e}} r_{\hat{k}} C_{q,\hat{k}} \\ &= d 2^q r_2^{\frac{1}{2d}} r_q^{\frac{d+1}{e}} c_q\end{aligned}$$

Case 12. If $k_j < \hat{k} < q$. Then $\Phi_j(\mathrm{k}) \leqq \mathrm{r}_{\hat{k}}^{-1}$.

$$\begin{aligned}\|w_{i,j,k}(z)\|_q &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^{k_j} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_q \\ &\leqq d r_2^{\frac{1}{2d}} r_{\hat{k}}^{-1} 2^q C_{q,\hat{k}} \|P_{\alpha,k_j}(x_{i,\hat{k}}(z))\|_k \\ &\leqq d r_2^{\frac{1}{2d}} 2^q c_q\end{aligned}$$

Hence the set consisting of

$$\{P_{\alpha,1}(x_{i,1}(z))\}_{\substack{i=1\\ z\in B}}^{\infty} \bigcup \{r_2 2^k P_{\alpha,k}(x_{i,k}(z))\}_{i=1,k\geqq 2, z\in B}^{\infty}$$
$$\bigcup \{d r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z))\}_{\substack{i=1,k=(k_1,\dots,k_d)\in N^d\\ z\in B, 1\leqq j\leqq d, |A_k|<d}}^{\infty}$$

is a bounded subsetof E and we denote by B_2 its closed convex balanced hull. By cases 1, 3, 6, 7 and 9 it follows that $B_2 \subset U_1$.

By (2.4)

$$\begin{aligned}&\left\| S\left(\bigotimes_{d,j} \Phi_m{}^j(k) r_2^{\frac{1}{2d}} 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z))\right)\right\|_{B_2} \\ &\leqq \frac{d^d}{d!} \prod_{j=1}^{d} \left\| r_2^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,\hat{k}}(z))\right\|_{B_2} \\ &\leqq \frac{d^d}{d!}\cdot\frac{1}{d^d} < 1 \quad \text{(by case (6) to (12)).}\end{aligned}$$

Hence

$$S\left(\bigotimes_{d,j} r^{\frac{1}{2d}} \Phi_j(k) 2^{k_j} P_{\alpha,k_j}(x_{i,k}(z))\right)$$

$$= \sum_{i=1}^{\infty} \bigotimes_{d} \theta_{l,k,i}^d$$

where $\sum_{l=1}^{\infty} \|\theta_{l,k,i}\|_{B_2}^d \leqq 1$.

We now have the formal sum of symmetric tensors

$$\sum_{i=1}^{\infty} \lambda_{i,1}(z) \bigotimes_{d} P_{\alpha,1}(x_{i,1}(z)) + \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} \lambda_{i,k}(z) 2^{-kd} r_2^{-d} \bigotimes_{d} (r_2 2^k P_{\alpha,k}(x_{i,k}(z))$$

$$+ \sum_{i=1}^{\infty} \sum_{\substack{k=(k_1,\dots,k_d) \\ |A_k|<d}}^{\infty} \lambda_{i,\hat{k}}(z) 2^{-\Sigma_{j=1}^{d} k_j} r_2^{-\frac{1}{2}} \left(\sum_{l=1}^{\infty} \|\theta_{l,k,i}\|_{B_2}^{d} \bigotimes_{d} \left(\frac{\theta_{l,k,i}}{\|\theta_{l,k,i}\|_{B_2}} \right) \right)$$

Now

$$\sum_{i=1}^{\infty} |\lambda_{i,1}(z)| \leqq 1 + \varepsilon, \quad \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} |\lambda_{i,k}(z)| 2^{-kd} r_2^{-d} \leqq \frac{1+\varepsilon}{r_2^d} \leqq \varepsilon$$

and

$$\sum_{i=1}^{\infty} \sum_{\substack{k=(k_1,\dots,k_d) \\ |A_k|<d}} |\lambda_{i,\hat{k}}(z)| 2^{-\Sigma_{j=1}^{d} k_j} r_2^{-\frac{1}{2}} \left(\sum_{l=1}^{\infty} \|\theta_{l,k,i}\|_{B_2}^{d} \right)$$

$$\leqq \frac{1+\varepsilon}{r_2^{\frac{1}{2}}} \leqq \varepsilon$$

and

$$\|P_{\alpha,1}(x_{i,1}(z))\|_{B_2} \leqq 1, \quad \|r_2 2^k P_{\alpha,k}(x_{i,k}(z))\|_{B_2} \leqq 1$$

so that this formal sum is absolutely convergent to some z_0 in $\widehat{\bigotimes}_{d,\pi,s} E$.

For any $k = (k_1, \dots, k_d) \in N^d$ we have

$$P_{\alpha,k_1} \otimes P_{\alpha,k_2} \cdots \otimes P_{\alpha,k_d}(z_0) = \sum_{i=1}^{\infty} \lambda_{i,\hat{k}}(z) \bigotimes_{j,d} P_{\alpha,k_j}(x_{i,\hat{k}}(z))$$

$$= P_{\alpha,k_1} \otimes P_{\alpha,k_2} \cdots \otimes P_{\alpha,k_d}(z).$$

Hence $z = z_0$ and

$$z = \sum_{j=1}^{\infty} \beta_j \bigotimes_{d} x_j \quad \text{where} \sum_{j=1}^{\infty} |\beta_j| \leqq 1 + 3\varepsilon$$

and $\|x_j\|_{B_2} \leqq 1$ for all j.

This shows that

$$B \subset (1+3\varepsilon)\bar{\Gamma}\left(\bigotimes_{d,s} B_2 \right) = \bar{\Gamma}\left(\bigotimes_{d,s} (1+3\varepsilon)^{\frac{1}{d}} B_2 \right)$$

Since ε was arbitrary this completes the proof.

§ 3. Applications to Spaces of Holomorphic Functions.

To apply propositon 11 we first need to improve inequality (1.5). This we do in the following proposition.

PROPOSITION 12. *Let U denote a balanced open subset of a Fréchet space E and let $(V_j)_j$ denote a neighbourhood basis at the origin. If p is a τ_ω continuous semi-norm on $\mathscr{H}(U)$ then there exists a compact subset K of U and a non-decreasing surjective mapping Φ: $N \cup \{0\} \to N$ and $c > 0$ such that*

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq c \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K + V_{\Phi(n)}}$$

for all $f \in \mathscr{H}(U)$ and all n.

PROOF. By (1.5) there exists a compact balanced subset K of U such that for every neighbourhood V of 0 there exists $c(V) > 0$ such that

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq c(V) \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K+V}$$

for all $f \in \mathscr{H}(U)$ and all n.

Choose $\lambda > 1$ such that λK is also a compact subset of U. For each positive integer j we can choose a positive integer n_j such that $c(\lambda^{-1} V_j)/\lambda^n \leqq 1$ for all $n \geqq n_j$. Then

$$\begin{aligned} p\left(\frac{\hat{d}^n f(0)}{n!}\right) &\leqq c(\lambda^{-1} V_j) \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K + \frac{1}{\lambda} V_j} \\ &\leqq \frac{c(\lambda^{-1} V_j)}{\lambda^n} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\lambda K + V_j} \\ &\leqq \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\lambda K + V_j} \end{aligned}$$

for all $f \in \mathscr{H}(U)$ and all $n \geqq n_j$.

We may suppose, without loss of generality, that the sequence $(n_j)_j$ is strictly increasing.

Let

$$\Phi(n) = \begin{cases} 1 & \text{for} \quad n < n_1 \\ j & \text{for} \ \ n_j \leqq n < n_{j+1}, j \geqq 1 \end{cases}$$

Then

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\lambda K + V_{\Phi(n)}}$$

for all $f \in \mathscr{H}(U)$ and $n \geqq n_1$.

For $n < n_1$ we have

$$p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq \frac{c(\lambda^{-1}V_1)}{\lambda^n}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\lambda K + V_1}$$

for all $f \in \mathscr{H}(U)$.

If $c = 1 + c(\lambda^{-1}V_1)$ this completes the proof.

THEOREM 13. *Let E denote a Fréchet space with a T-Schauder decomposition and the density condition and let U denote a convex balanced T-invariant open subset of E. Then*

$$(\mathscr{H}(U), \tau_\omega) = (\mathscr{H}(U), \tau_b).$$

PROOF. Let p denote a τ_ω continuous semi-norm on $\mathscr{H}(U)$ and let $(\|\cdot\|_k)_{k \geqq 1}$ denote a T-adaptable set of seminorms on E. We may suppose, by proposition 12 and Lemma 7, that

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)$$

and

$$(3.1) \qquad p\left(\frac{\hat{d}^n f(0)}{n!}\right) \leqq \left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{K + U_{\phi(n)}}$$

for all $f \in \mathscr{H}(U)$ and all n where K is an absolutely convex balanced T-invariant compact subset of U, $U_j = \{x \in E; \|x\|_j \leqq 1\}$ and $\phi: N \cup \{0\} \to N$ is a non-decreasing surjective mapping.

We now consider the semi-norm $p_n := p\,|_{\mathscr{P}(^nE)}$ restricted to $\mathscr{P}(^nE)$ for some fixed n. By proposition 11, $\tau_b = \beta$ on $\mathscr{P}(^nE)$ and since E has the density condition $\beta = \tau_\omega$ on $\mathscr{P}(^nE)$.

Let q_1 denote the semi-norm on E with unit ball $V := K + U_{\phi(n)}$, and let $q_k = \|\cdot\|_{\phi(n)+k-1}$ for $k \geqq 2$. Since $(\|\cdot\|_k)_{k \geqq \phi(n)}$ is a T-adaptable set of semi-norms on E proposition 8 implies that $(q_k)_{k \geqq 1}$ is also a T-adaptable set of seminorms on E.

Let $B = \{\phi \in \mathscr{P}(^nE)'; |\phi| \leqq p_n\}$. We have $p_n(P) = \|P\|_B$ for all $P \in \mathscr{P}(^nE)$.

By (3.1)

$$p_n(P) \leqq \|P\|_{\left(\bigotimes_{s,n} V\right)^{00}}$$

where we are considering P as an element of $\left(\widehat{\bigotimes}_{n,\pi,s} E\right)'$.

By the Hahn-Banach theorem $B \subset \left(\bigotimes_{s,n} V\right)^{00}$. Since E has the density condition $\left(\widehat{\bigotimes}_{n,\pi,s} E\right)$ is distinguished and hence, as B is a bounded subset of $\left(\left(\widehat{\bigotimes}_{n,\pi,s} E\right)'_\beta\right)'_\beta$ we have, by remark 1.4 of [17] (see also [19, lemma 1]), that there exists a bounded subset A *of* $\overline{\bigotimes_{n,s} V}$ such that $B \subset A^{00}$.

Hence $p_n(P) \leqq \|P\|_A$ for all $P \in \mathscr{P}(^nE)$.

By proposition 11 there exists, for any $\varepsilon > 0$, a bounded subset C of $(1 + \varepsilon)\bar{V}$ such that $A \subset \bar{\Gamma}\left(\bigotimes_{s,n} C\right)$. Hence $p_n(P) \leq \sup_{x \in C} |P(x)|$ for all $P \in \mathscr{P}(^nE)$.

Now let $C = C_n$ and $\varepsilon = \varepsilon_n$.

We have thus shown that for all n there exists a bounded subset C_n of $(1 + \varepsilon_n)(K + U_{\phi(n)})$ such that

$$p(P) \leqq \|P\|_{C_n} \quad \text{for all } P \in \mathscr{P}(^nE).$$

By choosing the ε_n's sufficiently small and by noting that $\phi(n) \to \infty$ as $n \to \infty$ we see that the sequence $(C_n)_n$ converges to the compact set K. By (3.1) we have

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) \leqq \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{C_n}$$

and hence p is a τ_β continuous seminorm on $\mathscr{H}(U)$. This completes the proof.

COROLLARY 14. *If K is a T-invariant compact convex balanced subset of a Fréchet space E with a T-Schauder decomposition and E has the density condition then the τ_ω topology on $\mathscr{H}(K)$ is generated by semi-norms of the form*

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{B_n}$$

where $(B_n)_n$ is a sequence of bounded subsets of E which converges to K as $n \to \infty$.

Theorem 13 and corollary 14 are also true for entire functions and germs at the origin when E is a complemented subspace of a Fréchet space with the density condition and a T-Schauder decomposition.

By proposition 3.6 of [15] we also have the following corollary.

COROLLARY 15. *If U is a balanced open subset of a Fréchet space E with a T-Schauder decomposition and the density condition then $(\mathscr{H}_b(U), \beta)$ is quasinormable.*

(β is the topology of uniform convergence on the bounded subsets of E which lie strictly inside U).

By example 10 we have the following result.

COROLLARY 16. *If U is a balanced open subset of a Banach space then $(\mathscr{H}_b(\mathscr{H}_b(U)), \beta)$ is a quasinormable Fréchet space.*

The method used in the proof of proposition 12 can also be used to prove the following result.

PROPOSITION 17. *If K is a compact balanced subset of a Fréchet space then the τ_0 topology on $\mathscr{H}(K)$ is generated by all seminorms of the form*

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{K_n}$$

where $(K_n)_n$ is a sequence of compact subsets of E which converges to K.

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LA SOLUTION UNIQUE DE L'ÉQUATION DIFFÉRENTIELLE DE LIOUVILLE

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Résumé.

Soit Ω un domaine hyperbolique du plan complexe $\mathbf{C}$ avec l'élément de la métrique de Poincaré: $P_\Omega(z)\,|dz|$. Alors, $\phi_\Omega = \log P_\Omega$ est une solution de l'équation de Liouville: $\Delta u = 4e^{2u}$ dans Ω. Nous connaissons que si u satisfait l'inégalité différentielle: $\Delta u \geqq 4e^{2u}$ dans Ω, alors, $u \leqq \phi_\Omega$ dans Ω (le résultat dû à L. V. Ahlfors). Nous démontrons que $u \geqq \phi_\Omega$ dans Ω si $\Delta u \leqq 4e^{2u}$ dans Ω et de plus, $u - \phi_\Omega$ est bornée inférieurement dans Ω. Ces faits, avec leur corollaire, étendent le théorème de B. Gustafsson vrai pour Ω simplement connexe. Supposons que Ω est borné et $\partial\Omega = \partial(\Omega \cup \partial\Omega)$ dans $\mathbf{C}$, où ∂X est la frontière de $X \subset \mathbf{C}$. Supposons que u satisfait l'inégalité: $\Delta u \leqq 4e^{2u}$ dans Ω, et encore que $u(z) \to +\infty$ quand z tend vers chaque point de $\partial\Omega$. Alors $u \geqq \phi_\Omega$ dans Ω. Comme une conséquence nous obtenons le théorème de L. A. Caffarelli et A. Friedman qui supposent une régularité de $\partial\Omega$.

1. Introduction et les résultats.

Un domaine Ω du plan complexe $\mathbf{C} = \{|z| < +\infty\}$ s'appelle hyperbolique si $\partial\Omega$ contient au moins deux points, ∂X étant la frontière de $X \subset \mathbf{C}$ dans $\mathbf{C}$. Nous supposerons une fois pour toutes que Ω est un domaine hyperbolique dans $\mathbf{C}$. Chaque Ω a l'élément de la métrique de Poincaré: $P_\Omega(z)\,|dz|$, $z \in \Omega$. C'est-à-dire, si f est une projection analytique sur Ω du disque unité ouvert $D = \{|z| < 1\}$ regardé comme le revêtement universel de Ω, en notation: $f \in \mathrm{Proj}(\Omega)$, alors,

$$1/P_\Omega(z) = (1 - |w|^2)\,|f'(w)| \tag{1.1}$$

pour la densité de Poincaré P_Ω en $z = f(w)$, $w \in D$; le côté droit de (1.1) est indépendant du choix particulier de f et w, pour autant que $z = f(w)$ soit satisfaite. Il est bien connu que $P_\Omega(z) \to +\infty$ quand z tend vers un point de $\partial\Omega$;

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voir [J. p. 116]. La densité P_Ω satisfait l'identité de la courbure de Gauss: $P_\Omega^{-2}\Delta \log P_\Omega = 4$ dans Ω, ou, la fonction

$$\phi_\Omega = \log P_\Omega$$

satisfait l'équation différentielle de J. Liouville, citée dans le titre du mémoire présent:

$$\text{(EDL)} \qquad \Delta u = 4e^{2u}$$

dans Ω, où $\Delta = 4\partial^2/\partial z\partial\bar{z} = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $z = x + iy$, est l'opérateur laplacien. Notre but est la démonstration de ce fait que ϕ_Ω est la solution unique de (EDL) dans Ω sous une certaine condition frontière de u en termes de ϕ_Ω (le théorème 1), ou directement, celle de u (le théorème 2 avec le théorème 1). Nous désignons par $\mathrm{RC}^2(\Omega)$ la famille des fonctions à valeurs réelles et deux fois continûment différentiables dans Ω.

THÉORÈME 1. *Soit Ω un domaine hyperbolique de* **C**, *et supposons que $u \in \mathrm{RC}^2(\Omega)$ satisfait l'inégalité différentielle: $\Delta u \geqq 4e^{2u}$ dans Ω. Alors, $u \leqq \phi_\Omega$ dans Ω. Encore, supposons que $u \in \mathrm{RC}^2(\Omega)$ satisfait l'inégalité différentielle: $\Delta u \leqq 4e^{2u}$ dans Ω et qu'il y a une constante réelle A telle que*

$$\text{(1.2)} \qquad u \geqq \phi_\Omega + A$$

dans Ω. Alors $u \geqq \phi_\Omega$ dans Ω.

La première part du théorème 1 est due essentiellement à L. V. Ahlfors [A1, Theorem A]. En particulier, $u \leqq \phi_\Omega$ dans Ω si u satisfait (EDL) dans Ω; il y a deux cas: $u \equiv \phi_\Omega$ ou $u < \phi_\Omega$ toujours dans Ω d'après le principe de la comparaison des solutions de (EDL) dû à V. Jørgensen (voir [J, p. 115]).

On ne peut pas conclure que: $u \geqq \phi_\Omega$ dans Ω de la seule condition: $\Delta u \leqq 4e^{2u}$ (ou plus fortement: $\Delta u = 4e^{2u}$) sans (1.2) dans Ω. Si c'était vrai, au contraire, alors ϕ_Ω serait la seule solution de (EDL) dans Ω. Mais, il y a solutions en nombre infini de (EDL) dans chaque Ω, fait indiqué dans [J, p. 116]; pour les détails, voir la section 2 du mémoire présent qui se compose des huit sections.

Nous allons démontrer, dans la section 2, que le théorème de B. Gustafsson [G1, p. 105, Theorem 7.3] pour Ω simplement connexe est obtenu par le théorème 1.

Un domaine $\Omega \subset$ **C** est dit admissible si Ω est borné et $\partial\Omega = \partial(\Omega \cup \partial\Omega)$ dans **C**. En particulier, pour Ω admissible $\partial\Omega$ n'a aucun point isolé.

THÉORÈME 2. *Supposons que $u \in \mathrm{RC}^2(\Omega)$ satisfait l'inégalité différentielle: $\Delta u \leqq 4e^{2u}$ dans un domaine admissible $\Omega \subset$* **C**. *Supposons encore que*

$$\text{(1.3)} \qquad u(z) \to +\infty$$

quand z tend vers chaque point de $\partial\Omega$. Alors, $u \geqq \phi_\Omega$ dans Ω.

Nous ne pouvons pas enlever l'admissibilité de Ω dans le théorème 2: voir Σ et $D_{\#}$ dans la section 5.

Une conséquence immédiate des théorèmes 1 et 2 est la proposition suivante:

Supposons que $u \in \mathrm{RC}^2(\Omega)$ *satisfait* (EDL) *dans* Ω *admissible. Supposons encore que* $u(z) \to +\infty$ *quand* z *tend vers chaque point de* $\partial\Omega$. *Alors,* $u = \phi_\Omega$ *dans* Ω.

Le cas spécial de cette proposition sous une certaine régularité de $\partial\Omega$ est le théorème de L. A. Caffarelli et A. Friedman [CF, p. 448, Theorem 3.3]. Le résultat de S. Richardson [R, p. 331, Theorem 2] pour Ω borné et limité par une courbe de Jordan se suit aussi.

2. Domaines du type fini, les corollaires du théorème 1 et un théorème de Gustafsson.

Étant donné des constantes $a > 0$ et $b > 0$, on peut réduire l'inégalité ($\Delta v \leqq ae^{bv}$, respectivement) dans Ω à $\Delta u \geqq 4e^{2u}$ ($\Delta u \leqq 4e^{2u}$, respectivement) dans Ω par la transformation: $u = (b/2)v + (1/2)\log(ab/8)$. En particulier, (EDL) est équivalente à l'équation: $\Delta v = e^v$ par la transformation: $v = 2u + \log 8$.

Le théorème dû à É. Picard [P1, P2], en la forme spéciale, est, alors, suivant: *Fixons des points distincts:* $a_1, \ldots, a_n$ *de* $\mathbf{C}$ ($n \geqq 2$). *Soient* $\beta_j > -1$ ($1 \leqq j \leqq n$) *et* $\gamma > 1$ *des constantes telles que* $\gamma + \beta_1 + \ldots + \beta_n < 0$ (*d'où*, $\gamma < n$). *Alors, il y a une et seulement une solution* u *de* (EDL) *dans* $\Omega(a_1, \ldots, a_n) = \mathbf{C}\backslash\{a_1, \ldots, a_n\}$ *telle que des limites* $\neq \infty$ *existent:*

$$\lim_{z \to a_j} (u(z) - \beta_j \log|z - a_j|), \ 1 \leqq j \leqq n; \ \lim_{z \to \infty} (u(z) + \gamma \log|z|).$$

Soient $a_1, a_2 \in \partial\Omega$ ($a_1 \neq a_2$) et soient u_j les solutions de (EDL) dans $\Omega(a_1, a_2)$ correspondantes aux trios des constantes donnés: $(\beta_1^{(j)}, \beta_2^{(j)}, \gamma^{(j)})$, j = 1, 2, avec $\beta_1^{(1)} \neq \beta_1^{(2)}$. Alors, la différence $u_1 - u_2$ n'est pas bornée dans un voisinage de a_1 relatif à Ω. En particulier, $u_1 \neq u_2$ considerée dans Ω. Cette observation montre qu'il y a solutions en nombre infini de (EDL) dans chaque Ω.

Soit $\partial^*\Omega$ la frontière de Ω dans la sphère de Riemann: $\mathbf{C} \cup \{\infty\}$. Évidemment, $\partial\Omega = \partial^*\Omega$ si et seulement si Ω est borné dans $\mathbf{C}$. La condition (1.2) est donc équivalente à l'inégalité:

$$u(z) \geqq \phi_\Omega(z) + O(1)$$

quand z tendvers chaque point $\zeta \in \partial^*\Omega$ en ce sens qu'il y a deux constantes $r(\zeta) > 0$ et $A(\zeta)$ dépendantes de ξ telles qu'on ait

$$u(z) \geqq \phi_\Omega(z) + A(\xi)$$

pour chaque $z \in \Omega$ sujet à $|z - \zeta| < r(\zeta)$ ou $|z| > r(\zeta)$ selon que $\zeta \neq \infty$ ou $\zeta = \infty$ (si $\infty \in \partial^* \Omega$).

Soit $\delta_\Omega(z)$ la distance entre $z \in \Omega$ et $\partial \Omega$:

$$\delta_\Omega(z) = \inf_{w \in \partial \Omega} |z - w|.$$

Il est bien connu que $\delta_\Omega P_\Omega \leqq 1$ dans Ω. Suivant [Y1, Y2] nous appelons que Ω est du type fini, si $A(\Omega) > 0$, où

$$A(\Omega) \equiv \inf_{z \in \Omega} \delta_\Omega(z) P_\Omega(z).$$

Sinon, ou en d'autres termes, si $A(\Omega) = 0$, nous appelons que Ω est du type infini. Un critère typique pour Ω soit du type fini est que la dérivée $(\partial/\partial z)(1/P_\Omega(z)) = -(\exp(-\phi_\Omega(z)))(\partial/\partial z)\phi_\Omega(z)$ est bornée dans Ω [Y1, Theorem 1]. Plus précisément, en posant

$$\gamma(\Omega) = \sup_{z \in \Omega} |(\partial/\partial z)(1/P_\Omega(z))|.$$

on a $\gamma(\Omega) > 0$ et les inégalités: $1/(2\gamma(\Omega)) \leqq A(\Omega) \leqq 2/(2 + \gamma(\Omega))$; voir [Y2, p. 116] ($\omega(\Omega) = A(\Omega)$ dans [Y1]). On connait que $\gamma(\Omega) \leqq 2$ pour Ω simplement connexe, et $\gamma(\Omega) = 1$ si et seulement si Ω est convexe; voir [Y1, Proposition 2] pour le fait dernier. En particulier, $A(\Omega) \geqq 1/4$ pour Ω simplement connexe et $A(\Omega) \geqq 1/2$ pour Ω convexe. Voir [G1, p. 37, (64) et (65)] aussi; notons que la fonction c_0 pour Ω définie dans [G1] est exactement $-\log P_\Omega = -\phi_\Omega$ si Ω est simplement connexe, mais c_0 est strictement plus grande que $-\log P_\Omega$ dans tout Ω non simplement connexe; voir [G1, p. 66, (180)]. Plus généralement, si les composantes connexes de $(\mathbf{C} \cup \{\infty\}) \setminus \Omega$ sont en nombre fini et chaqune est un continu non dégénéré dans $\mathbf{C} \cup \{\infty\}$, ou, un ensemble fermé et connexe contenant au moins deux points dans $\mathbf{C} \cup \{\infty\}$, alors Ω est du type fini. De plus, il y a quelques exemples de Ω du type fini avec les composantes en nombre infini de $\partial \Omega$; voir, par exemple, [BP] et [Y2, § 8]. Spécialement, pour des domaines Ω du type fini, il n'y a aucun point isolé de $\partial \Omega$. Des domaines du type fini ont beaucoup de propriétés importantes dans la théorie des fonctions; voir, par exemple. [BP] et [Y1, Y2]. Si Ω est non borné et du type fini, alors Ω n'est pas admissible. Si Ω est D coupé le long du intervalle fermé $[0, 1/2]$ sur l'axe réel, alors Ω est borné et du type fini, mais non admissible. Nous proposerons un exemple de Ω admissible mais du type infini dans la section 6. Maintenant, si Ω est du type fini, alors

$$u(z) \geqq -\log \delta_\Omega(z) + O(1) \tag{2.1}$$

quand z tend vers chaque point de $\partial^* \Omega$ si et seulement si

$$u(z) \geqq \phi_\Omega(z) + O(1)$$

quand z tend vers chaque point de $\partial^*\Omega$; en d'autres termes, on a (1.2) dans Ω. Visiblement (2.1) implique (1.3) quand z tend vers chaque point de $\partial\Omega$. Mais la converse n'est pas vraie; voir l'exemple (ii) dans la section 5. Il y a donc une différence délicate entre (2.1) et (1.3).

Or, le théorème [G1, p. 105, Theorem 7.3] est exactement le suivant:

Si $u \in RC^2(\Omega)$ *satisfait* (EDL) *dans* Ω *simplement connexe, alors* $u \leqq \phi_\Omega$ *dans* Ω. *Encore si* (2.1) *est vraie quand* z *tend vers chaque point de* $\partial^*\Omega$, *alors* $u = \phi_\Omega$.

Gustafsson a dit de plus que son théorème, dans la forme ci-dessus, se maintient aussi pour Ω ayant des continus non-dégénérés dans $\mathbf{C} \cup \{\infty\}$, en nombre fini, comme les composantes connexes de $(\mathbf{C} \cup \{\infty\})\backslash\Omega$ [G1, p. 106]. Nous avons ici une extension de son théorème à Ω du type fini à l'aide du théorème 1 et du corollaire 1 du théorème 1 suivant.

COROLLAIRE 1. *Supposons que* $u \in RC^2(\Omega)$ *satisfait* (EDL) *dans* Ω *du type fini et* (2.1) *est vraie quand* z *tend vers chaque point de* $\partial^*\Omega$. *Alors* $u = \phi_\Omega$.

Voir aussi [R, p. 330, Theorem 1] pour Ω borné et simplement connexe, et [G2]. L'équation $\Delta u = e^u$ s'appelle celle de Liouville dans [R].

Comme une autre conséquence du théorème 1 nous avons le

COROLLAIRE 2. *Supposons qu'une fonction* $f\colon \Omega \to D$ *est analytique et qu'il y a une constante* $B > 0$ *telle que*

$$|f'(z)|/(1 - |f(z)|^2) \geqq BP_\Omega(z) \tag{2.2}$$

en tout $z \in \Omega$. *Alors* Ω *est simplement connexe et* f *est un homéomorphisme conforme de* Ω *sur* D.

N.B. Il y a une estimation de grad ϕ_Ω dans un sous-domaine Ω_0 d'un domaine Ω convexe [CF, p. 440, Lemma 2.2]; mais, elle est incomplète. Nous pouvons démontrer que

$$|\text{grad}\, \phi_\Omega(z)| \leqq 4(1 - A(\Omega))/\delta_\Omega(z), \quad z \in \Omega; \tag{2.3}$$

pour Ω du type fini. En effet, par [Y2, p. 116, (7.3)] on a

$$(1/2)\,|\text{grad}\, \phi_\Omega(z)| = P_\Omega(z)\,|(\partial/\partial z)(1/P_\Omega(z))| \leqq 2/\delta_\Omega(z) - 2P_\Omega(z), \quad z \in \Omega.$$

Parce que $P_\Omega(z) \geqq A(\Omega)/\delta_\Omega(z)$, on a (2.3). On peut remplacer le côté droit de (2.3) par $3/\delta_\Omega(z)$ si Ω est simplement connexe, et encore par $2/\delta_\Omega(z)$ si Ω est convexe.

3. Preuve du théorème 1.

Nous commençons par une petite modification du lemme d'Ahlfors (voir [A2, p. 13, Lemma 1-1]; notons que $K(\rho) \leqq -1$ là; voir aussi [A1, Theorem A]); pour la démonstration, en réalité, nous suivrons l'argument d'Ahlfors et/ou Gustafsson.

LEMME. *Supposons que $\Phi \in \mathrm{RC}^2(\Omega)$ satisfait $\Delta\Phi \leqq 4e^{2\Phi}$ dans Ω avec $\Phi \geqq \phi_\Omega + B$ dans Ω où B est une constante. Supposons que $\Psi \in \mathrm{RC}^2(\Omega)$* satisfait $\Delta\Psi \geqq 4e^{2\Psi}$ *dans Ω. Alors $\Phi \geqq \Psi$ dans Ω.*

En particulier, si $\Phi = \phi_\Omega$ et $\Psi = u$ avec $\Delta u \geqq 4e^{2u}$, alors, $\phi_\Omega \geqq u$ dans Ω: c'est le résultat d'Ahlfors, cité ci-dessus.

PREUVE. Considérons $f \in \mathrm{Proj}(\Omega)$ avec (1.1). Fixons r, $0 < r < 1$, et posons

(3.1) $$g(w) = \Phi(f(w)) + \log|f'(w)| - [\Psi(f(rw)) + \log(r|f'(rw)|)]$$

pour $w \in D$. Par l'inégalité: $\Phi(f(w)) \geqq -\log[(1-|w|^2)|f'(w)|] + B$ on a alors:

$$g(w) \geqq -\log(1-|w|^2) - [\Psi(f(rw)) + \log(r|f'(rw)|)] + \beta \quad \text{dans} \quad D,$$

d'où $g(w) \to +\infty$ quand $|w| \to 1$. Par conséquent, il y a $a = \alpha + i\beta \in D$ tel que $g(\alpha, \beta) = g(a) \leqq g(w)$ pour tout $w = \xi + i\eta \in D$. Ensuite,

$$\Delta g(a) = g_{\xi\xi}(\alpha, \beta) + g_{\eta\eta}(\alpha, \beta) \geqq 0$$

parce que $g_{\xi\xi}(\alpha, \beta) \geqq 0$ et $g_{\eta\eta}(\alpha, \beta) \geqq 0$.

D'autre part, on a

(3.2) $$4^{-1}\Delta g(w) = 4^{-1}(\Delta_z\Phi(f(w)))|f'(w)|^2 - 4^{-1}(\Delta_\zeta\Psi(f(rw)))(r|f'(rw)|)^2$$
$$\leqq \exp[2\Phi(f(w)) + 2\log|f'(w)|] - \exp[2\Psi(f(rw)) + 2\log(r|f'(rw)|)];$$

ici, les opérateurs laplaciens Δ, Δ_z et Δ_ζ sont tenus par rapport à w, $z = f(w)$, et $\zeta = f(rw)$, respectivement. Ainsi, inégalité (3.2) avec $\Delta g(a) \geqq 0$ donne que $0 \leqq g(a) \leqq g(w)$ pour chaque $w \in D$. Fixons $w \in D$ et faisons $r \to 1$ dans (3.1). Alors, $\Phi(f(w)) - \Psi(f(w)) \geqq 0$ dans D, ou, $\Phi \geqq \Psi$ dans Ω.

PREUVE DU THÉORÈME 1. Toujours, $\phi_\Omega \geqq u$ si $\Delta u \geqq 4e^{2u}$ dans Ω par la remarque après notre lemme. Ensuite, sous la condition (1.2), on peut appliquer notre lemme à $\Phi = u$ avec $\Delta u \leqq 4e^{2u}$, et $\Psi = \phi_\Omega$. Alors, $u \geqq \phi_\Omega$.

N.B. (I) Il est facile d'étendre le théorème 1 aux surfaces S de Riemann qui sont hyperboliques, ou, plus précisément, qui admettent D comme leur revêtement universel. Dans ce cas, $\Delta u(z)\,dxdy \geqq$ (ou $\leqq$) $4e^{u(z)}\,dxdy$ est tenue en termes des paramètres locaux $z = x + iy \in \mathbf{C}$.

(II) Si Ω est simplement connexe, alors u là satisfait (EDL) si et seulement si $u = \log\{|\psi'|/(1-|\psi|^2)\}$, où ψ: $\Omega \to D$ est une fonction analytique dans Ω dont la dérivée n'annule pas dans Ω. C'est un résultat célèbre de Liouville [L]. En effet, $v = 2u$ satisfait $\Delta v = 8e^v$ dans Ω. En posant $K = -4$ dans [B, p. 27, Proposition 1.6] on a $v = 2\log\{|\psi'|/(1-|\psi|^2)\}$ dans Ω. Pour Ω simplement connexe, on a $u \leqq \phi_\Omega$ dans Ω; c'est exactement:

$$(1-|w|^2)|g'(w)|/(1-|g(w)|^2) = P_\Omega(z)^{-1}|\psi'(z)|/(1-|\psi(z)|^2) \leqq 1$$

pour chaque $z = f(w)$, $f \in \mathrm{Proj}(\Omega)$, et $g = \psi \circ f\colon D \to D$. C'est aussi un résultat direct d'une inégalité due à H. A. Schwarz et G. Pick, appliquée a g; voir [A2, p. 3].

4. Preuve du corollaire 2.

La fonction $u = \log[|f'|/(1 - |f|^2)]$ satisfait (EDL) dans Ω. Il résulte donc du théorème 1 avec (2.2) que $u = \phi_\Omega$, d'où

$$|f'|/(1 - |f|^2) = P_\Omega \quad \text{dans} \quad \Omega.$$

Soit $g \in \mathrm{Proj}(\Omega)$ et considérons la fonction composée $h(w) = f(g(w))$ dans E. Alors,

$$(1 - |w|^2)|h'(w)|/(1 - |h(w)|^2) = P_\Omega(z)^{-1}|f'(z)|/(1 - |f(z)|^2) = 1$$

pour chaque $w \in D$, avec $z = g(w)$. Donc $h(w) \equiv \varepsilon(w - a)/(1 - \bar{a}w)$, $|\varepsilon| = 1 > |a|$. Il n'est pas difficile d'observer que f est une transformation biunivoque entre Ω et D.

N.B. Encore, on peut étendre le corollaire 2 aux surfaces de Riemann hyperboliques.

Dans le cas où Ω est du type fini dans le corollaire 2, et pour $f\colon \Omega \to D$ dont la derivée ne s'annule pas dans Ω, nous pouvons remplacer (2.2) par:

$$\liminf \delta_\Omega(z)|f'(z)|/(1 - |f(z)|^2) > 0$$

quand z tend vers chaque point de $\partial^*\Omega$. De même, dans le cas où Ω est admissible dans le corollaire 2, et pour $f\colon \Omega \to D$ dont la derivée ne s'annule pas dans Ω, nous pouvons remplacer (2.2) par

$$|f'(z)|/(1 - |f(z)|^2) \to +\infty$$

quand z tend vers chaque point de $\partial\Omega$. Dans ce cas on doit employer le théorème 2 démontré dans la prochaine section.

5. Preuve du théorème 2.

Posons $\Omega(n) = \{z \in \mathbf{C};\ \mathrm{dis}(z, \Omega \cup \partial\Omega) < 1/n\}$ pour les nombres naturels n. Alors, $\Omega(n) \subset \Omega(m)$ si $n \geqq m$. Notons que tous les domaines $\Omega(n)$ sont bornées, d'où, hyperboliques. Notons encore que $\Omega \cup \partial\Omega = \bigcap_{n \geqq 1} \Omega(n)$. Nous observerons que $G \subset \Omega$ pour chaque domaine $G \subset \mathbf{C}$ tel qu'on ait $K \subset \Omega \cup \partial\Omega$ pour chaque ensemble K fermé (dans $\mathbf{C}$), contenu dans G. Pour le voir, nous supposons qu'il y a $z \in G \backslash \Omega$. Alors, nous pouvons trouver un disque fermé $k \subset G$, $z \in k$. Donc, $z \in k \subset \Omega \cup \partial\Omega$, d'où $z \in \partial\Omega = \partial(\Omega \cup \partial\Omega) = \partial(\mathbf{C} \backslash (\Omega \cup \partial\Omega))$. Ensuite, Il y a un point de $\mathbf{C} \backslash (\Omega \cup \partial\Omega)$ dans $k \subset \Omega \cup \partial\Omega$. C'est une contradiction.

En remplaçant ∞ par un point fixé de Ω, on sait que Ω est le noyau de la suite $\{\Omega(n)\}$ au sens de D. A. Hejhal [Hj, p. 7, et p. 10, Remark]. Il est facile d'observer que Ω est aussi le noyau de chaque sous-suite $\{\Omega(n_j)\}$ de $\{\Omega(n)\}$. Donc, $\Omega(n) \to \Omega$ au sens de Hejhal. Il résulte alors du théorème profond [Hj, p. 8, Theorem 1, et p. 10, Remark], en termes de la densité de Poincaré, que $P_{\Omega(n)}$ converge à P_Ω uniformément quand $n \to \infty$ sur chaque ensemble fermé contenu dans Ω.

En posant $\phi_n = \phi_{\Omega(n)}$ et en considérant la différence $u - \phi_n$ dans Ω on a $u(z) - \phi_n(z) \to +\infty$ quand z tend vers chaque point de $\partial\Omega$. Il y a donc un point $b \in \Omega$ où $u - \phi_n$ attend son minimum. Comme on vu dans la preuve de notre lemme dans la section 3, on sait que les inégalités: $0 \leqq 4^{-1}\Delta(u - \phi_n)(b) \leqq e^{2u(b)} - \exp(2\phi_n(b))$ donnet l'inégalité: $u \geqq \phi_n$ dans Ω. En faisant $n \to \infty$ dans Ω nous obtenons que $u \geqq \phi_\Omega$ dans Ω.

EXEMPLES. (i) Pour les contre-exemples, nous considérons, d'abord, le demi-plan $\Sigma = \{z; \operatorname{Re} z > 0\}$, étant non borné (donc, non admissible), pour que

$$\phi_\Sigma(z) = -\log(2x), \quad z = x + iy \in \Sigma.$$

Posons $F(z) = (e^z + 1)/(e^z - 1)$, de sorte que $F(\Sigma) = \Sigma \backslash \{1\}$, et encore, posons:

$$u(z) = \phi_\Sigma(F(z)) + \log|F'(z)| = \log[e^x/(e^{2x} - 1)], \quad z = x + iy \in \Sigma.$$

Alors, u est une solution de (EDL) dans Σ, et u(z) $\to +\infty$ quand z tend vers chaque point de $\partial\Sigma$, axe imaginaire. Mais,

$$u(z) - \phi_\Sigma(z) = \log[2xe^x/(e^{2x} - 1)] \to -\infty$$

quand $x \to +\infty$ dans Σ. Alors, l'inégalité: $u \geqq \phi_\Omega$ n'est pas vraie dans Σ.

(ii) Le domaine $D_\# = D \backslash \{0\}$ n'est pas admissible (et encore, est du type infini) et

$$\phi_{D_\#}(z) = -\log[2|z|\log(1/|z|)], \quad z \in D_\#.$$

Pour une constante α, $0 < \alpha < 1$, la fonction

$$u(z) = \phi_D(|z|^\alpha) + \log(\alpha|z|^{\alpha-1}) = \log[\alpha|z|^{\alpha-1}/(1 - |z|^{2\alpha})]$$

satisfait l'équation (EDL) dans $D_\#$ et $u(z) \to +\infty$ quand z tend vers chaque point de $\partial D_\#$. Mais,

$$u(z) - \phi_{D_\#}(z) = \log\frac{2\alpha|z|^\alpha \log(1/|z|)}{1 - |x|^{2\alpha}} \to -\infty$$

quand $z \to 0 \in \partial D_\#$. Alors, $u \geqq \phi_{D_\#}$ dans $D_\#$ est une faute. Encore on a

$$u(z) + \log\delta_{D_\#}(z) = \log[\alpha|z|^\alpha/(1 - |z|^{2\alpha})] \to -\infty$$

pour $z \to 0 \in \partial D_\#$.

6. Un exemple de Ω.

Toujours, $2^{n+3} > 3n(n+1)(n+2)$ pour tout $n \geqq 9$. Pour $n \geqq 9$, $a_n = 1 - n^{-1}$ et $r_n = 2^{-n}$, posons

$$\Delta_n = \{z; |z - a_n| \leqq r_n\} \text{ et encore } \Omega = D \Big\backslash \left(\bigcup_{n=9}^{\infty} \Delta_n \right);$$

les disques Δ_n et Δ_m sont disjoints pour $n \neq m$. Alors, Ω est admissible. Considérons les domaines annulaires:

$$A_n = \{z; r_n < |z - a_n| < a_{n+1} - a_n - r_{n+1}\}, \quad n \geqq 9.$$

Alors, $A_n \subset \Omega$ pour $n \geqq 9$, et de plus,

$$\operatorname{mod} A_n \equiv (2\pi)^{-1} \log((a_{n+1} - a_n - r_{n+1})/r_n) \to +\infty$$

quand $n \to +\infty$. Donc, il résulte de [BP, p. 478, Corollary 1] que Ω est du type infini.

7. Une candidate de la densité P_Ω pour $\Omega \subset \mathsf{R}^n$, $n \geqq 3$.

Soient $ax = (ax_1, \ldots, ax_n)$ et $|x| = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2}$ pour $a \in \mathsf{R}$ et pour un point $x = (x_1, \ldots, x_n)$ de l'espace euclidien R^n ($n \geqq 3$). L'élément de la métrique de Poincaré de la balle $B = \{|x| < 1\}$ est $P_B(x)\,ds$, où $P_B(x) = (1 - |x|^2)^{-1}$, et $ds = \left(\sum_{j=1}^{n} dx_j^2 \right)^{1/2}$. Alors, la fonction

$$P_B(n^{-1/2}(n-2)^{-1/2}x)^{(n-2)/2}$$

satisfait l'équation:

$$\Delta u = u^{(n+2)/(n-2)} \tag{7.1}$$

dans la balle $\{|x| < n^{1/2}(n-2)^{1/2}\}$, où $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ est laplacien.

Pour chaque domaine $\Omega \subset \mathsf{R}^n$ ayant des hypersurfaces compactes, connexes, lisses (de C^∞, par exemple), et en nombre fini, comme sa frontière $\partial\Omega$, C. Loewner et L. Nirenberg [LN, p. 253, Theorem 4, (a) et (b)] ont démontré qu'il y a une solution $\phi_\Omega > 0$ de C^∞ et unique de (7.1) dans Ω telle que $\phi_\Omega(y) \to +\infty$ quand y tend vers chaque point de $\partial\Omega$. De plus, en désignant la distance de $y \in \Omega$ et $\partial\Omega$ encore par $\delta_\Omega(y)$, on a

$$\delta_\Omega(y)^{(n-2)/2} \phi_\Omega(y) \to (n(n-2)/4)^{(n-2)/4}$$

quand y tend vers chaque point de $\partial\Omega$. De plus, si Ω n'est pas borné, alors

$$|y|^{n-2}\phi_\Omega(y) \to c(\Omega) > 0$$

quand $|y| \to +\infty$ dans Ω; [LN, Theorem 4, (4.3)].

Soit Ω un domaine du type spécifié par Loewner et Nirenberg et soit $\Omega_n = \{n^{1/2}(n-2)^{1/2}x; x \in \Omega\}$ qui, sans aucun doute, encore satisfait la régularité de leur sens. Posons

$$P_\Omega(x) = [\phi_{\Omega_n}(n^{1/2}(n-2)^{1/2}x)]^{2/(n-2)}, \quad x \in \Omega.$$

La fonction P_Ω est alors une candidate naturelle de la densité de Poincaré de Ω. En effet, elle est exactement celle de Poincaré de B si $\Omega = B$. Encore, il y a bien une raison pour P_Ω être une candidate en termes de la courbure scalaire $K(x, P_\Omega)$ de la métrique $P_\Omega(x)\,ds$ en $x \in \Omega$. La fonction $u = P_\Omega^{(n-1)/2}$ satisfait, en effet, l'équation $\Delta u = n(n-2)u^{(n+2)/(n-2)}$ dans Ω. Donc, $K(x, P_\Omega)$ de $P_\Omega(x)\,ds = u(x)^{2/(n-2)}\,ds$ en chaque point $x \in \Omega$ est toujours -4; voir [LN, p. 247, Remark]. De plus, $P_\Omega(x)\,ds$ est complète en ce sens que toute courbe a sa longueur infinie si elle tend vers $\partial\Omega$. La métrique $P_\Omega(x)\,ds$ est essentiellement celle de [LN, p. 246, (7)]. Il est intéressant donc que

$$\delta_\Omega(x)P_\Omega(x) \to 1/2 \tag{7.2}$$

quand x tend vers chaque point de $\partial\Omega$, avec la constante absolue 1/2 indépandante de la dimension, car,

$$\delta_{\Omega_n}(n^{1/2}(1-2)^{1/2}x) = n^{1/2}(n-2)^{1/2}\delta_\Omega(x), \quad x \in \Omega.$$

Et encore, si Ω n'est pas borné, alors,

$$|x|^2 P_\Omega(x) \to n^{-1}(n-2)^{-1}c(\Omega_n)^{2/(n-2)} \tag{7.3}$$

quand $|x| \to +\infty$ dans Ω. En particulier, si Ω est borné, on a

$$0 < \inf_{x\in\Omega} \delta_\Omega(x)P_\Omega(x) \quad \text{et} \quad \sup_{x\in\Omega} \delta_\Omega(x)P_\Omega(x) < +\infty.$$

Retournons au cas $\Omega \subset \mathbf{C}$ pour un moment. Si Ω est simplement connexe, on a $A(\Omega) \geqq 1/4$ et, de plus, si Ω est convexe, on a $A(\Omega) \geqq 1/2$; voir la section 2. Il est intéressant de comparer ces faits avec (7.2). Il est connu que si $\partial\Omega$ de $\Omega \subset \mathbf{C}$ est bornée mais Ω n'est pas borné (donc ∞ est un point isolé de $\partial^*\Omega$ et, d'où Ω est du type infini) on a $2|z|(\log|z|)P_\Omega(z) \to 1$ quand $|z| \to +\infty$ dans Ω (voir la section 8). Donc, $|z|^2 P_\Omega(z) \to +\infty$ quand $|z| \to +\infty$ dans Ω. Dans ce sens (7.3) montre une différence entre les cas $n = 2$ et $n > 2$ pour $\mathbf{R}^n$.

Notons finalemant que H. Yamabe [Yb] a considéré l'équation du type analogue à (7.1) dans une variété riemannienne, de C^∞, mais compacte de dimension au moins 3.

8. Une allure de P_Ω.

Soit ∞ est un point isolé de $\partial^*\Omega$ de $\Omega \subset \mathbf{C}$. Pour démontrer:

$$\lim_{|z| \to +\infty, z \in \Omega} (2|z| \log |z|) P_\Omega(z) = 1, \tag{8.1}$$

nous tenons a, b de $\partial\Omega$, $a \neq b$, et nous considérons $T(z) = (b - a)/(z - a)$. En posant

$$\sigma = \{w; 0 < |w| < d\} \quad \text{et} \quad R = \mathbf{C}\backslash\{0, 1\},$$

où d est la distance de 0 et $\partial T(\Omega)\backslash\{0\}$, on a: $\sigma \subset T(\Omega) \subset R$, d'où

$$1/P_\sigma(w) \leqq 1/P_{T(\Omega)}(w) \leqq 1/P_R(w) \tag{8.2}$$

en chaque $w \in \sigma$. D'abord,

$$2|w| \log(d/|w|) = 1/P_\sigma(w), \quad w \in \sigma. \tag{8.3}$$

D'autre part, l'inégalité de J. A. Hempel [Hm, p. 443, (4.1)] se lit:

$$1/P_R(w) \leqq 2|w| (|\log |w|| + c_H), \quad w \in R, \tag{8.4}$$

où $c_H = \Gamma(1/4)^4/(4\pi^2) = 4.376\ldots$. En combinant (8.2), (8.3), et (8.4) avec l'identité:

$$1/P_\Omega(z) = |z - a|^2/\{|b - a| \, P_{T(\Omega)}(w)\}$$

pour $w = T(z)$, $z \in \Omega$, on a

$$2|z - a| \log(d|z - a|/|b - a|) \leqq 1/P_\Omega((z)$$
$$\leqq 2|z - a| (|\log(|b - a|/|z - a|)| + c_H)$$

en chaque point z de $T^{-1}(\sigma) = \{|z - a| > |b - a|/d\}$. En conséquence, pour tout $a \in \partial\Omega$ fixé, on en conclut que

$$\lim_{|z| \to +\infty, z \in \Omega} (2|z - a| \log |z - a|) P_\Omega(z) = 1.$$

Il est facile d'observer (8.1).

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APPROXIMATION BY SOLUTIONS OF ELLIPTIC EQUATIONS ON CLOSED SUBSETS OF EUCLIDEAN SPACE

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0. Introduction.

In the last twenty years the question of how to reduce problems of qualitative approximation by analytic or harmonic functions on unbounded closed sets to the compact case has been often considered and different answers have been given in a variety of particular cases. In this paper we provide a general method of localization which applies to all instances previously dealt with and which considerably simplyfies the available proofs.

As an example of the kind of results we are envisaging we mention Nersesjan's Theorem on uniform approximation by analytic functions of one complex variable. Denoting by α continuous analytic capacity [17] we have

THEOREM 1. (Nersesjan [11]). *Let F be a closed subset of the complex plane. Then the following are equivalent:*

(i) *Each continuous function on F which is analytic on F^0 can be uniformly approximated on F by functions which are analytic on some neighbourhood (depending on the approximating function) of F.*

(ii) $\alpha(D\backslash F) = \alpha(D\backslash F^0)$, *for each disc D.*

For compact F the above statement is just the well known Vitushkin Theorem [17, p. 183] on rational approximation. To settle the case of unbounded closed sets Nersesjan had to find a suitable new localization argument. Later on Hadjiisky [6] discovered another way of localizing the problem which gave a direct proof of (ii) $\Rightarrow$ (i) without appealing to the compact case. This idea was further exploited in [2] to deal with the analytic approximation problem on general closed sets in L^p, Lipschitz and BMO norms. Another interesting reference is [15].

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In Section 1 we give a short proof of Theorem 1 which already contains some of the elements of our general method. To get a better insight into it we prove in Section 2 the harmonic analog of Theorem 1 (see Section 2 for a precise statement). In that proof the reader will find in action all the ingredients of our idea. In Section 3 we present a fairly general context in which our localization method works: we are able to deal with a homogeneous elliptic operator on $\mathbf{R}^n$ with constant complex coefficient, the approximation taking place in the norm of a Banach space satisfying certain conditions.

In Section 4 we study weighted uniform analytic approximation in the plane showing that the conditions required in Section 3 are, in some sense, sharp.

Our notational conventions will be standard. For example, C will denote a constant, independent of the relevant variables under consideration and which might be different in different occurencies. The open ball with center a and radius δ is denoted by $B(a, \delta)$. If B is a ball, kB is the ball with the same center and radius k times the radius of B.

1. Proof of Nersesjan's Theorem.

The important part in Theorem 1 is (ii) $\Rightarrow$ (i). Let us then assume that F is a closed subset of the complex plane and that f is a continuous function on F which is analytic on F^0. Extend f continuously to the whole of $\mathbf{C}$. We wish now to localize the singularities of f by means of Vitushkin's method. To this end take a covering of $\mathbf{C}$ by discs (D_j) of radius 1, which is almost disjoint, in the sense that each point in $\mathbf{C}$ belongs to at most a fixed number of discs D_j. Let (φ_j) be a C^∞ partition of unity subordinated to (D_j). Set $f_j = V_{\varphi_j}(f) \equiv \dfrac{1}{\pi z} * (\varphi_j \bar{\partial} f)$. Then f_j is continuous on $\mathbf{C}$, analytic on F^0 and outside D_j, and [17, p. 150]

$$\|f\| \leqq CN_j \|f\|_{D_j} \tag{1}$$

where C is an absolute constant and $N_j = \|\varphi_j\| + \|\nabla \varphi_j\|$. In this and in the next section we will denote by $\| \ \|_E$ the supremum norm on the set E and we will let $\| \ \|$ stand for $\| \ \|_{\mathbf{C}}$.

Condition (ii) implies that there exists $C > 0$ such that, setting $F_j = F \cap \bar{D}_j$, we have $\alpha(D \setminus F_j^0) \leqq C\alpha(2D \setminus F_j)$ for all discs D. Then Vitushkin's Theorem gives (i) with F replaced by F_j. Therefore, for fixed j, there exists h_j, analytic on a neighbourhood of F_j such that $\|f - h_j\|_{F_j} < \eta = \varepsilon/(2^j N_j)$, where ε is a given positive number. A well known modification argument, which we reproduce below for the reader's convenience, gives that in fact we can assume $\|f - h_j\| < \eta$. To see this set $d_j = f - h_j$. For some open neighbourhood U of F_j on which h_j is analytic we still have $\|d_j\|_{\bar{U}} < \eta$. Extend d_j from $\bar{U}$ to a continuous function on $\mathbf{C}$, still denoted by d_j, satisfying $\|d_j\| < \eta$. Modify h_j outside U in such a way that the identity

$h_j = f - d_j$ holds everywhere. Then h_j is analytic on a neighbourhood of F_j and $\|f - h_j\| < \eta$.

Define $g_j = V_{\varphi_j}(h_j)$, so that for some absolute constant C we have by (1) $\|f_j - g_j\| = \|V_{\varphi_j}(f - h_j)\| \leqq CN_j \|f - h_j\| < C\varepsilon/2^j$. It is not difficult to see that $g = f - \sum_j (f_j - g_j)$ is analytic on some neighbourhood of F. Since $\|f - g\| < C\varepsilon$, g is the desired approximant.

2. Uniform harmonic approximation.

In this section we will prove the following harmonic analog of Nersesjan's Theorem [5], [9].

THEOREM 2. *Let F be a closed subset of R^n. Then the following are equivalent.*

(i) *Each continuous function on F which is harmonic on F^0 can be uniformly approximated on F by functions which are harmonic on neighbourhoods of F.*

(ii) $\operatorname{Cap}(B \backslash F) = \operatorname{Cap}(B \backslash F^0)$, *for each ball B.*

Here Cap stands for the classical Wiener capacity of potential theory. For F compact the result goes back to Deny and Keldysh [3], [8] in slightly different formulations. For $n \geqq 3$ the proof proceeds along the lines of the preceeding section but for $n = 2$ a new difficulty arises owing to the fact that the fundamental solution $\frac{1}{2\pi}\log|z|$ of the Laplacean Δ is unbounded at ∞. We shall therefore concentrate on the proof of (ii) $\Rightarrow$ (i) in the plane. Before starting with the details some remarks are in order concerning Vitushkin's localization operator for Δ in dimension 2.

Let D be a disc of radius δ, $\varphi \in C_0^2(D)$ and set $N(\varphi) = \sum_{j=0}^{2} \delta^j \|\nabla^j \varphi\|$. Given a continuous function f in C define $V_\varphi f = \frac{1}{2\pi}\log|z| * (\varphi \Delta f)$, so that $\Delta(V_\varphi f) = \varphi \Delta f$ in the distributional sense. A simple computation [1] gives

$$V_\varphi f(z) = \varphi(z)f(z) + \frac{1}{2\pi}(\log|\zeta| * f\Delta\varphi)(z) - \frac{1}{\pi}\left(\frac{1}{\zeta} * f\bar{\partial}\varphi\right)(z) - \frac{1}{\pi}\left(\frac{1}{\bar{\zeta}} * f\partial\varphi\right)(z),$$

and so

$$\|V_\varphi f\|_D \leqq CN(\varphi)\|f\|_D, \tag{2}$$

the constant C depending only on δ. It is important to realize that we cannot replace the left hand side of (2) by $\|V_\varphi f\|$ because $V_\varphi f$ has a logarithmic singularity at ∞, and this fact is the only obstruction to the argument used in Section 1.

We proceed now to the proof of Theorem 2. Let F be a closed subset of the plane and f a continuous function on C which is harmonic on F^0. Let (D_j) be an almost disjoint covering of C by open discs of radius 1 and (φ_j) a C^∞ partition of the unity subordinated to (D_j). For fixed j and given $\eta_j > 0$ (to be specified later) choose h_j harmonic on a neighbourhood of $F_j = \bar{D}_j \cap F$ such that $\|f - h_j\|_{F_j} < \eta_j$. This is possible, arguing as in the preceding section, because we know that Theorem 2 holds for the compact sets F_j. Using the modification argument of Section 1 we can furthermore suppose that h_j is continuous on C and $\|f - h_j\| < \eta_j$. Set $g_j = V_{\varphi_j}(h_j)$. The function $f_j - g_j$ is harmonic outside D_j and has a logarithmic singularity at ∞. Thus, assuming that D_j is centered at the origin,

$$f_j(z) - g_j(z) = a_j \log|z| + H_j(z), \quad |z| > 1,$$

where H_j is harmonic outside D_j and at ∞, and $a_j = \dfrac{1}{2\pi}\displaystyle\int \Delta\varphi_j(f - h_j)\,dx\,dy$.

Hence, for some constant C_j depending only on j, $|a_j| \leqq C_j\|f - h_j\|_{D_j} \leqq C_j\eta_j$. If $D_j \subset F$ then $f_j \equiv 0$, so we can take for granted that D_j contains a disc $D \subset \mathrm{C}\backslash F$. Let ψ be a C^∞ function such that $\psi = 1$ outside D and $\psi = 0$ on $\frac{1}{2}D$. Set $L_j(z) = a_j\psi(z)\log|z - c|$, where c is the center of D. Then L_j is harmonic on a neighbourhood of F, $\|L_j\|_{D_j} \leqq C_j\eta_j$, and $f_j - g_j - L_j$ is harmonic outside D_j and at ∞. Therefore, using (2),

$$\|f_j - g_j - L_j\| \leqq \|f_j - g_j - L_j\|_{D_j} \leqq \|V_{\varphi_j}(f - h_j)\|_{D_j} + \|L_j\|_{D_j} \leqq C_j\eta_j.$$

Choose now η_j so that $C_j\eta_j = \varepsilon/2^j$, where ε has been given in advance. Set $g = f - \sum_j (f_j - g_j - L_j)$. It is easy to check that g is harmonic on a neighbourhood of F. Since $\|f - g\| < \varepsilon$ the proof is complete.

3. The main result.

The goal of this section is to describe a general setting in which our localization method works.

We do not wish to restrict our attention to uniform approximation, so we start by introducing a certain class of Banach spaces in whose norm the approximation will take place.

Following [12] we let V stand for a Banach space, whose norm is denoted by $\|\ \|$, which contains C_0^∞, the set of test functions in R^n, and is contained in $(C_0^\infty)^*$, the space of distributions. We assume that V is a topological C_0^∞-submodule of $(C_0^\infty)^*$, which means that for $f \in V$ and $\varphi \in C_0^\infty$

$$|\langle f, \varphi\rangle| \leqq C(\varphi)\|f\|, \tag{3}$$

$\langle f, \varphi\rangle$ denoting the action of the distribution f on the test function φ, and

$$\|\varphi f\| \leqq C(\varphi)\|f\|, \tag{4}$$

where $C(\varphi)$ is a constant independent of f.

Given a closed subset F of R^n let $I(F)$ be the closure in V of those $f \in V$ whose support (in the sense of distributions) is disjoint from F. The Banach space $V(F) = V/I(F)$, endowed with the quotient norm, should be viewed as the natural version of V on F. We will write $\|f\|_F$ for the norm of the equivalence class in $V(F)$ of the distribution $f \in V$.

We need also to introduce local versions of V and of $V(F)$. Let V_{loc} be the set of distributions f such that $\varphi f \in V$, $\varphi \in C_0^\infty$. There is a natural Frechet topology in V_{loc} given by the seminorms $\|f\|_m = \|\varphi_m f\|_{B_m}$, where $B_m = \{|x| \leqq m\}$ and $\varphi_m \in C_0^\infty$ is a fixed function taking the value 1 on some neighbourhood of B_m. Define $V_{\text{loc}}(F) = V_{\text{loc}}/J(F)$, where $J(F)$ is the closure in V_{loc} of those distributions in V_{loc} whose support is disjoint from F.

We present now some examples (see also Section 4 in which a non-translation invariant example of V is considered).

EXAMPLE 1. $V = L^p(\mathsf{R}^n)$, $1 \leqq p \leqq \infty$. Clearly $L^p(F)$, L^p_{loc} and $L^p_{\text{loc}}(F)$ are the standard spaces denotes by these symbols.

EXAMPLE 2. $V = \text{VMO}(\mathsf{R}^n)$, the space of functions of vanishing mean oscillation. In [7] one finds an intrinsic characterization of VMO(F) involving only the values taken by functions on F.

EXAMPLE 3. $V = C^m(\mathsf{R}^n)$, m being a non-negative integer. This is the space of functions with bounded continuous derivatives up to order m endowed with any of the standard norms associated to it. For example,

$$\|f\| = \sup_{|\alpha| \leqq m} \|\partial^\alpha f\|_\infty.$$

In this case $C^m(\mathsf{R}^n)_{\text{loc}}$ is just the space of functions with continuous partial derivatives up to order m. Notice that, for $m \geqq 1$ and $F \neq \mathsf{R}^n$, $C^m(F)$ is not a space of functions. With the help of the Whitney extension theorem, it can be identified with a space of jets (see [14, Chapter VI]).

EXAMPLE 4. $V = \lambda^s(\mathsf{R}^n)$, s a non-integer positive real number. Writing $s = m + \sigma$, with m an integer and $0 < \sigma < 1$, we have that $f \in \lambda^s(\mathsf{R}^n)$ if and only if $f \in C^m(\mathsf{R}^n)$, $\sup_{\delta > 0} \omega(\delta)\delta^{-\sigma} < \infty$ and $\omega(\delta)\delta^{-\sigma} \to 0$ as $\delta \to 0$, where $\omega(\delta) = \sup_{|\alpha| = m} \sup_{|x-y| < \delta} |\partial^\alpha f(x) - \partial^\alpha f(y)|$. We set $\|f\| = \|f\|_{C^m(\mathsf{R}^n)} + \sup_{\delta > 0} \omega(\delta)\delta^{-\sigma}$.

For $0 < s < 1$, $\lambda^s_{\text{loc}}(F)$ turns out to be the set of functions on F which locally satisfy a little "o" Lipschitz condition of order s. For $1 < s$, the remark concerning jets made in the previous example still applies.

The approximating functions in our abstract theorem will not be necessarily analytic or harmonic, but instead they will be annihillated by a complex constant coefficients homogeneous elliptic operator L of order r, as in [16].

We describe now the two basic assumptions on V which make our localization argument work.

First, the Vitushkin localization operator associated to L satisfies adequate estimates. Let B be an open ball of radius δ, $\varphi \in C_0^\infty(B)$, and set $V_\varphi f = \Phi * (\varphi L f)$, where Φ is a fundamental solution of L and f a distribution on R^n. We recall that Φ can be taken of the form $\Phi(x) = \Phi_0(x) + P(x)\log|x|$, where $\Phi_0(x)$ is a C^∞ function in $\mathsf{R}^n \setminus \{0\}$, homogeneous of degree $r - n$, and $P(x)$ is a polynomial which is either zero (this is the case if $r < n$) or homogeneous of degree $r - n$. We will require that our Banach space V satisfies the estimate

$$\|V_\varphi f)\|_{\bar{B}} \leqq N(\varphi, B)\,\|f\|, \tag{5}$$

where $N(\varphi, B)$ is independent of f. The definition of $\|f\|_K$, $K =$ support of φ, immediately gives that (5) can be improved to

$$\|V_\varphi f\|_{\bar{B}} \leqq N(\varphi, B)\,\|f\|_K. \tag{6}$$

Notice that (6) (or equivalently (5)) means exactly that V_φ sends continuously V_{loc} into V_{loc}.

The proof of (5) for the examples considered above can be found in [1], [2], [10], [12], [13], [16]. In [12] the reader will even find a proof of (5) for a wide class of abstract Banach spaces.

Our second assumption on V is more technical. We require that for some non-negative integer p one has

$$\|\partial^\alpha \Phi\|_{\mathsf{R}^n \setminus B(0,R)} \leqq \alpha!\,\varepsilon(R)^{|\alpha|}, \quad |\alpha| \geqq p, \tag{7}$$

where $\varepsilon(R) \to 0$ as $R \to \infty$.

That (7) holds in the examples 1–4 follows from

$$|\partial^\alpha \Phi(x)| \leqq \alpha!\,C^{|\alpha|}\,|x|^{-(n-r+|\alpha|)}(\log|x|| + 1), \quad x \neq 0, \tag{8}$$

which is essentially equivalent to the real analyticity of Φ outside the origin. For instance, p is 0 for $L = \bar{\partial}$ and $V = C^m(\mathsf{C})$, and p is 1 for $L = \bar{\partial}$ and $V = L^2(\mathsf{C})$. For $L = \Delta$, $V = C^m(\mathsf{C})$, p is 1 in the plane and p is 0 for all dimensions larger than 2.

Our next task will be to prove that (7) gives a sort of maximum principle for the exterior of a ball and the norm of V. We start by discussing expansions of potentials at ∞.

From (8) it follows that there exists $k > 1$ such that given any $x \neq 0$ we have an expansion $\Phi(z) = \sum_{|\alpha| \geqq 0} \frac{\partial^\alpha \Phi(x)}{\alpha!}(z - x)^\alpha$, in the ball $|z - x| < k^{-1}|x|$, the series being absolutely convergent there. Consequently, given points a and x such that $|x - a| > k\delta$ for some $\delta > 0$, we have an expansion

$$\Phi(x-y) = \sum_{|\alpha| \geqq 0} (-1)^{|\alpha|} \frac{\partial^\alpha \Phi(x-a)}{\alpha!} (y-a)^\alpha, \quad |y-a| < \delta,$$

the series converging in $C^\infty(B(a, \delta))$. Let T be a distribution with compact support contained in $B(a, \delta)$ and set $f = \Phi * T$. Then, for $|x - a| > k\delta$

$$f(x) = \langle T, \Phi(x-y) \rangle = \sum_{|\alpha| \geqq 0} c_\alpha \partial^\alpha \Phi(x-a),$$

where $c_\alpha = (-1)^{|\alpha|}(\alpha!)^{-1}\langle T, (y-a)^\alpha \rangle$, the series converging in $C^\infty(|x-a| > k\delta)$.

We would like to point out here that the above statement is not true for $k = 1$ as was claimed in [10] and [16]. The results proved there are not affected by this missing dilation factor.

LEMMA 1. *Let f be a distribution such that Lf has compact support and $f(x) \to 0$ as $|x| \to \infty$. Then $f = \Phi * Lf$.*

PROOF. $f - (\Phi * Lf)$ is a tempered distribution annihillated by L. Thus $f = \Phi * Lf + P$ for some polynomial P. To show that $P \equiv 0$ set, for $|x| \to \infty$,

$$(\Phi * Lf)(x) = \sum_{|\alpha| \geqq k} c_\alpha \partial^\alpha \Phi(x),$$

where $c_\alpha \neq 0$ for some α with $|\alpha| = k$. One can write [16, p. 161]

$$\sum_{|\alpha| = k} c_\alpha \partial^\alpha \Phi(x) = H(x) + Q(x) \log |x|$$

for some polynomial Q of degree $r - n - k$ and some function H, C^∞ outside the origin and homogeneous of degree $r - n - k$. Clearly

$$f = H + Q \log |x| + g + P, \tag{9}$$

where $g = \sum_{|\alpha| > k} c_\alpha \partial^\alpha \Phi$. Arguing from the homogeneities of the different terms in (9), it is not difficult to conclude that $P \equiv 0$.

LEMMA 2. *Let B be an open ball and f a distribution in V_{loc} such that $Lf = 0$ on $\mathbb{R}^n \setminus B$ and $f(x) = O(|x|^{-d})$ as $x \to \infty$, where $d = \max\{p + n - r, 1\}$, p being the integer appearing in condition (7). Then $f \in V$ and $\|f\| \leqq C \|f\|_{3\bar{B}}$.*

PROOF. By Lemma 1 $f = \Phi * Lf$. Thus, for $q = d - n + r \geqq p$,

$$f(x) = \sum_{|\alpha| \geqq q} c_\alpha \partial^\alpha \Phi(x), \quad x \notin k\bar{B}.$$

Choose $\varphi \in C_0^\infty(2B)$, $\varphi = 1$ on B and $\psi \in C_0^\infty(3B)$, $\psi = 1$ on $2B$. Let us suppose first that B is centered at the origin. Then, using the Leibnitz formula,

$$\alpha!\,|c_\alpha| = |\langle Lf, \varphi(x)x^\alpha\rangle| = |\langle f, L(\varphi(x)x^\alpha)\rangle| \leqq \sum_{|\beta|=0}^{r} |\alpha|^r\,|\langle f, L^\beta(\varphi)x^{\alpha-\beta}\rangle| =$$

$$= \sum_{|\beta|=0}^{r} |\alpha|^r\,|\langle(\psi(x)x)^{\alpha-\beta} f, L^\beta(\varphi)\rangle|,$$

where L^β is a differential operator of order $r - |\beta|$ and in the sums above only indexes β with $\alpha - \beta \in \mathbf{Z}^n_+$ are taken. Observe now that (3) and (4) hold with $\|f\|$ replaced by $\|f\|_K$, K being the support of φ. Applying (3) and (4) in this sharper form, we get

$$\alpha!\,|c_\alpha| \leqq A\,|\alpha|^r C_1^{|\alpha|}\,\|f\|_{3\bar{B}},$$

where A depends only on r and

$$C_1 = \max_{|\beta|\leqq r} C(L^\beta\varphi) + \max_{1\leqq j\leqq n} C(\psi(x)x_j),$$

the constants in the right hand side being those appearing in (3) and (4) for the indicated functions.

If R satisfies $B(0,R) \supset kB$, we get

$$\|f\|_{\mathbf{R}^n\setminus B(0,R)} = \left\|\sum_{|\alpha|\geqq q} c_\alpha \partial^\alpha \Phi\right\|_{\mathbf{R}^n\setminus B(0,R)} \leqq$$

$$\leqq \sum_{|\alpha|\geqq q} (\alpha!)^{-1} A\,|\alpha|^r C_1^{|\alpha|}\,\|f\|_{3\bar{B}}\,\alpha!\,\varepsilon(R)^{|\alpha|} \leqq C\,\|f\|_{3\bar{B}},$$

provided R is large enough so that $C_1\varepsilon(R) < 1$.

It is easily proved that

$$\|f\| \leqq C(\|f\|_{2\bar{B}(0,R)} + \|f\|_{\mathbf{R}^n\setminus B(0,R)}).$$

On the other hand

$$\|f\|_{2\bar{B}(0,R)} = \|V_\varphi f\|_{2\bar{B}(0,R)} \leqq C\,\|f\|_{2\bar{B}},$$

because of (6) applied to $2\bar{B}(0,R)$, and so the desired estimate follows.

We are left with the task of removing the assumption that B is centered at the origin. Let $B = B(a,\delta)$ and take $\varphi \in C_0^\infty(3B)$, $\varphi = 1$ on $2B$. Then $f = V_\varphi(f)$ and so we get from (6)

$$\|f\|_{3\bar{B}(0,|a|+\delta)} \leqq C\,\|f\|_{3\bar{B}}. \tag{10}$$

Since Lf vanishes outside $\bar{B}(0,|a|+\delta)$,

$$\|f\| \leqq C\,\|f\|_{3\bar{B}(0,|a|+\delta)}. \tag{11}$$

Combining (10) and (11) one completes the proof of the Lemma.

We are now ready to prove our main result.

THEOREM 3. *Let V be a Banach space satisfying* (3), (4), (5) *and* (7), *F a closed subset of* $\mathbb{R}^n$ *and $f \in V_{loc}$. Then the following statements are equivalent.*

(i) *Given a positive number ε there exists $g \in V_{loc}$ such that $Lg = 0$ on some neighbourhood of F and $\|f - g\|_F < \varepsilon$.*

(ii) *Given a ball B and a positive number ε there exists $g \in V_{loc}$ such that $Lg = 0$ on some neighbourhood of $F \cap \bar{B}$ and $\|f - g\|_{F \cap \bar{B}} < \varepsilon$.*

PROOF. We only need to show that (ii) implies (i). Let (B_j) be an almost disjoint covering of $\mathbb{R}^n$ by open balls B_j of radius 1 and let (φ_j) be a partition of unity subordinated to (B_j). Set $N_j = N(\varphi_j, 3\bar{B}_j)$. For fixed j and given $\eta > 0$ (to be specified later) choose $h_j \in V_{loc}$ such that $Lh_j = 0$ on some neighbourhood of $F_j = F \cap \bar{B}_j$ and $\|f - h_j\|_{F_j} < \eta$.

We claim now that h_j can be modified to $H_j \in V_{loc}$ so that $LH_j = 0$ on some neighbourhood of F_j and $\|f - H_j\| < \eta$. Using the definition of the norm in $V(F_j)$ we find an open neighbourhood U of F_j on which $Lh_j = 0$ and $\|f - h_j\|_{\bar{U}} < \eta$. Let $g_j \in V$ be such that $f - h_j = g_j$ on U and $\|g_j\| < \eta$. The distribution $H_j = f - g_j$ fulfills all requirements in the claim.

Set $f_j = V_{\varphi_j}(f)$ and $G_j = V_{\varphi_j}(H_j)$. If $B_j \subset F$ then $f_j = 0$ and if $B_j \subset \mathbb{R}^n \setminus F$ then $L(f_j) = 0$ on a neighbourhood of F. Hence, in what follows we will consider only indexes j such that B_j intersects ∂F. For such indexes B_j contains a ball $B = B(a, \delta) \subset \mathbb{R}^n \setminus F$. Let $\psi \in C^\infty(\mathbb{R}^n)$ be 1 outside B and 0 on $\frac{1}{2}B$. Set $\chi_j = \psi(x)\Phi(x - a)$ and $K_j = \sum_{|\alpha| < q} c_\alpha \partial^\alpha \chi_j$, where the coefficients c_α are defined by the expansion $f_j(x) - G_j(x) = \sum_{|\alpha| \geqq 0} c_\alpha \partial^\alpha \Phi(x - a)$ and $q = \max\{p, r - n + 1\}$, p being the integer appearing in (7).

Since $\alpha! \, |c_\alpha| = |\langle f - H_j, L(\varphi_j(x)(x - \alpha)^\alpha)\rangle|$, applying (3) we get the estimate

$$|c_\alpha| \leqq C(\alpha, j) \, \|f - H_j\| \leqq C(\alpha, j)\eta,$$

where $C(\alpha, j)$ is a constant depending only on α and j. Hence

$$\|K_j\|_{3\bar{B}_j} \leqq \sum_{|\alpha| < q} C(\alpha, j)\eta \, \|\partial^\alpha \chi_j\|_{3\bar{B}_j} = C(q, j)\eta,$$

where now $C(q, j)$ stands for a constant depending only on q and j.

On the other hand, the function $f_j - G_j - K_j$ satisfies the hypothesis of Lemma 2. Applying Lemma 2 and (6) we obtain

$$\|f_j - G_j - K_j\| \leqq C_j \, \|f_j - G_j - K_j\|_{3\bar{B}_j} \leqq$$
$$\leqq C_j \, \|V_{\varphi_j}(f - H_j)\|_{3\bar{B}_j} + C(q, j)\eta \leqq C_j N_j \eta + C(q, j)\eta = C(q, j)\eta.$$

Choose η so that $C(q, j)\eta = \varepsilon/2^j$, where ε has been given in advance, and define

$g = f - \sum_j (f_j - G_j - K_j)$. Then $\|f - f\| < \varepsilon$ and $Lg = 0$ on some neighbourhood of F. This shows (i).

REMARK. It is clear that Theorem 3 also gives the corresponding approximation results for classes of functions in the spirit of theorems 1 and 2. We also would like to mention that small modifications of our arguments would prove analogous theorems for Banach spaces V defined on subdomains of R^n.

4. Weighted uniform approximation.

Let ω be a positive radial continuous function on the plane. Let V be the set of continuous functions on C such that

$$\|f\|_\omega \equiv \sup_{z \in \mathsf{C}} |f(z)|\,\omega(z) < \infty.$$

If $F \subset \mathsf{C}$ is closed then, as it is easily seen, $f \in V(F)$ if and only if $\|f\|_{\omega,F} \equiv \sup_{z \in F} |f(z)|\,\omega(z) < \infty$, and the norm of $V(F)$ is exactly $\| \ \|_{\omega,F}$. Conditions (3), (4) and (5) clearly hold for any ω. Our last result states that condition (7) with Φ replaced by $1/\pi z$ is equivalent to the fact that local analytic approximation implies global analytic approximation.

THEOREM 4. *The following statements are equivalent.*

(i) *Let f be a continuous function on a closed subset F of the plane. If for each disc D and $\varepsilon > 0$ there exists a function g, analytic on some neighbourhood of $F \cap \bar{D}$ such that $\|f - g\|_{\omega, F\cap\bar{D}} < \varepsilon$, then for each $\varepsilon > 0$ there exists a function g, analytic on some neighbourhood of F such that $\|f - g\|_{\omega,F} < \varepsilon$.*

(ii) *Condition* (7), *with Φ replaced by $1/\pi z$, holds.*

(iii) *There exists a positive integer d such that if $f \in C(\mathsf{C})$ is analytic on $\mathsf{C}\backslash\bar{D}(0,\delta)$ and $f(z) = O(|z|^{-d})$ as $z \to \infty$ then $\|f\|_\omega \leqq C\,\|f\|_{\omega,\bar{D}(0,3\delta)}$.*

(iv) *There exists a positive integer q such that $\lim_{z\to\infty} \omega(z)\,|z|^{-q} = 0$.*

PROOF. That (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) follows from Theorem 3. A simple computation shows that (iv) $\Rightarrow$ (ii). Thus we only need to prove that (i) $\Rightarrow$ (iv).

If (iv) is not true then $\limsup_{z\to\infty} \omega(z)\,|z|^{-q} = \infty$ for all $q > 0$. Set $F = \{z \in \mathsf{C}: |z| \geqq 1\}$. Then the only function in $V(F)$ which is analytic on $\mathring{F}$ is the zero function, as one can easily check using the maximum principle and the fact that ω is radial. Let now f be a continuous function on F, analytic on F^0 and such that it can not be continued analytically on a neighbourhood of F. If g is analytic on a neighbourhood of F and $\|f - g\|_{\omega,F} < 1$ then $f = g$, which is impossible. Therefore f can not be globally approximated on F, but a local approximation is possible because $\| \ \|_\omega$ is locally equivalent to the uniform norm. Thus (i) fails.

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SERIES REPRESENTATIONS OF LINEAR FUNCTIONALS

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F. John [1] has given a series representation of the form $\int f(x)d\alpha(x) = \sum a_n f(x_n)$ for Stieltjes integrals. The purpose of this paper is to prove the following generalisation:

THEOREM. *If X is a compact Hausdorff space with a countable base then for any continuous linear functional Λ on $C(X)$ there exist sequences $a_n \in \mathsf{R}$, $x_n \in X$ such that*

$$\Lambda(f) = \sum a_n f(x_n) \quad \textit{for any } f \in C(X)$$

PROOF. By the Riesz representation theorem this is equivalent to proving

$$\int f d\mu = \sum a_n f(x_n) \tag{1}$$

where μ is a bounded regular Borel measure on X, which we may assume to be non-atomic. It is not difficult to see that there exist closed subsets $E_{k,n}$, $1 \leqq k \leqq n$ of X such that

$$|\mu|(E_{k,n} \cap E_{j,n}) = 0 \quad \text{if} \quad k \neq j, \tag{2}$$

$$\bigcup_{k \leqq n} E_{k,n} = X, \ \lim_{k \to \infty} \max_{k \leqq n} |\mu|(E_{k,n}) = 0 \quad \text{and} \quad E_{k,n} \subseteq E_{j,n-1} \quad \text{for some } j,$$

and

$$\int f d\mu = \lim_{n \to \infty} \sum_{k \leqq n} \mu(E_{k,n}) f(y_{k,n}). \tag{3}$$

where $y_{k,n}$ is any sequence such that $y_{k,n} \in E_{k,n}$. One can for example choose a sequence $E_{k,n}$ such that (2) holds and (3) holds for every f_n, where f_n is a dense sequence in $C(X)$, from which it follows that (3) holds for every $f \in C(X)$.

We may suppose the $E_{k,n}$ ordered so that the $E_{k,n}$ contained in $E_{1,n-1}$ come first, then those contained in $E_{2,n-1}$ etc. We now construct the sequence a_n in blocks as follows: the first block is $\mu(E_{1,1})$, for $n > 1$ the nth block consists of

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$\mu(E_{k,n})$ for those $E_{k,n}$ that are contained in $E_{1,n-1}$ followed by $-\mu(E_{1,n-1})$ and then $\mu(E_{k,n})$ for those $E_{k,n}$ that are contained in $E_{2,n-1}$ followed by $-\mu(E_{2,n-1})$ etc. Choose x_n such that $x_n \in E_{j,k}$ where $a_n = +-\mu(E_{j,k})$, and the same x is chosen whenever $E_{j,k}$ occurs.

Then (1) holds. For if we sum to the end of the n first blocks we get

$$\sum_{k \leqq n^2} a_k f(x_k) = \sum_{k \leqq n} \mu(E_{k,n}) f(y_{k,n}) \to \int f d\mu$$

From this it follows that if (1) did not hold we would have for some sequences $n(k) \to \infty$, $c(k)$, $d(k)$ and $\varepsilon > 0$

$$\left| \sum_{j \leqq c(k)} \mu(E_{j,n(k)+1}) f(y_{j,n(k)+1}) - \sum_{j \leqq d(k)} \mu(E_{j,n(k)}) f(y_{j,n(k)}) \right| \geqq \varepsilon \quad \text{for all } k,$$

where $c(k)$, $d(k)$ are such that the expression inside the absolute value sign is the sum of the first $c(k) + d(k)$ terms of the $(n(k) + 1)$th block.

Let $M = \sup |f(x)|$ and $\delta = \varepsilon/(8M + 7)$. Take r so large that $\max_{k \leqq n} |\mu|(E_{k,n}) \leqq \delta$ if $n \geqq r$. Let A be the set of all elements that are contained in $\bigcup_{j \leqq d(k)} E_{j,n(k)}$ for infinitely many k. Since μ is regular there is an open set B containing A and a closed set C contained in A such that $|\mu|(B - C) < \delta$, where C can be chosen to be $\bigcup_{j \leqq m} E_{j,n(k)}$ for some m, k. By Urysohn's lemma there is a continuous g such that $|g| \leqq |f|$, $g = f$ on C and $g = 0$ on the complement of B.

Choose $h \geqq r$ such that $|\mu|(A - \cup E_{j,n(h)})$ and $|\mu|(\cup E_{j,n(h)} - A)$, where the union is taken for all $j \leqq d(h)$, are both less than δ and

$$\left| \int g d\mu - \sum \mu(E_{j,r}) g(y_{j,r}) \right| < \delta \quad \text{for all } r \geqq n(h)$$

We then have

$$\begin{aligned}
(5) \qquad & \left| \sum_{c(h)} \mu(E_{j,n(h)+1}) f(y_{j,n(h)+1}) - \sum_{d(h)} \mu(E_{j,n(h)}) f(y_{j,n(h)}) \right| \\
\leqq & \left| \sum_{c(h)} \mu(E_{j,n(h)+1}) f(y_{j,n(h)+1}) - \sum_{c(h)} \mu(E_{j,n(h)+1}) g(y_{j,n(h)+1}) \right| \\
& + \left| \sum_{c(h)} \mu(E_{j,n(h)+1}) g(y_{j,n(h)+1}) - \sum_{d(h)} \mu(E_{j,n(h)}) g(y_{j,n(h)}) \right| \\
& + \left| \sum_{d(h)} \mu(E_{j,n(h)}) f(y_{j,n(h)}) - \sum_{d(h)} \mu(E_{j,n(h)}) g(y_{j,n(h)}) \right|
\end{aligned}$$

It follows from the way the $E_{k,n}$ were ordered and the choice of h that $|\mu|(S) < 3\delta$ where S is the union of those $E_{j,n(h)+1}$ with $j \leqq c(h)$ that are not contained in C. Hence $\left|\sum_{c(h)} \mu(E_{j,n(h)+1}) f(y_{j,n(h)+1}) - \sum_{c(h)} \mu(E_{j,n(h)+1}) g(y_{j,n(h)+1})\right| =$

$$\left|\sum_{E_{j,n(h)+1} \subseteq S} \mu(E_{j,n(h)+1}) f(y_{j,n(h)+1}) - \sum_{E_{j,n(h)+1} \subseteq S} \mu(E_{j,n(h)+1}) g(y_{j,n(h)+1})\right| \leqq 6M\delta.$$

A similar estimation of the other terms of (5) gives $|\sum \mu(E_{j,n(h)+1}) f(y_{j,n(h)+1}) - \sum \mu(E_{j,n(h)}) f(y_{j,n(h)})| < \delta(8M + 7) = \varepsilon$, a contradiction which proves the theorem.

If we let $X = [1, 2]$, $\mu =$ Lebesgue measure $E_{k,n} = [(k-1)/2^n, k/2^n]$, $2^n + 1 \leqq k \leqq 2^{n+1}$, $x_{k,n} = (k-1)/2^n$ we obtain after a change of variables F. John's formula

$$\sum (-1)^{n+1}/n f(\{\log n/\log 2\}) = \log 2 \int f(x)\, dx,$$

for all bounded Riemann integrable f, where $\{x\}$ is the fractional part of x.

(It is clear that the above proof applies also to bounded Riemann integrable functions.)

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AUTOMORPHISMS OF INDUCTIVE LIMIT C^*-ALGEBRAS

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Introduction.

After the remarkable results obtained in [2], [7] and especially in [8], the study of C^*-algebra inductive limits of finite direct sums of matrix algebras over commutative C^*-algebras (suggested by E. G. Effros in [5]), and also of their automorphism groups, become obviously more attractive and important.

In this note we prove some results concerning mainly the automorphisms of a certain class of C^*-algebras which we call *almost constant* (see Definition 1). These are some C^*-algebra inductive limits of matrix algebras over commutative C^*-algebras, including many Goodearl algebras [10] of real rank zero [3] and, in particular, all the Bunce-Deddens algebras [4].

The main purpose of this paper is to show that a similar type of results with those given in [13] can be obtained for a large class of algebras. (Compare the very recent work in a similar direction in [9]).

Let $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ be an almost constant C^*-algebra and consider the UHF algebra $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}}) \subset A$. If moreover $K_0(A)$ is *weakly torsion free* (see Definiton 3) and A has cancellation it is shown that any endomorphism of A is approximately inner with respect to the trace seminorm (see Theorem 1 for a much more complete and general result). Necessary and sufficient conditions for an automorphism of B to be extended to an automorphism of A are given, provided that the (unique) trace of A is faithful (see Theorem 2). A key fact in proving these results is that B is dense in A with respect to the trace seminorm (see Proposition 2b)). Also it is shown that the centralizer of $\{\Phi \in \mathrm{Aut}(B): \Phi = \tilde{\Phi}_{|B}$ for some $\tilde{\Phi} \in \mathrm{Aut}(A)\}$ in $\mathrm{Aut}(B)$ is trivial and if moreover $A \cap A' = \mathbf{C} \cdot 1_A$ then the centralizer of $\{\Phi \in \mathrm{Aut}(A): \Phi(B) = B\}$ in $\mathrm{Aut}(A)$ is also trivial (see Propositions 3 and 4 for much more general situations.)

We shall present now some notations used in this paper. We shall work only with *unital C^*-algebras*. For a compact space X and a C^*-algebra A we shall

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consider the embedding $A \subset C(X, A)$, where each element in A is seen as a constant map on X and also the embedding $C(X)$ $f \to f \otimes 1_A \in C(X) \otimes A = C(X, A)$. We denote by $U(A)$ the unitary group of the C^*-algebra A. By a homomorphism of C^*-algebras we shall mean a unital $*$-homomorphism and by an automorphism of a C^*-algebra, a $*$-automorphism. We denote by $\mathrm{Hom}(A, B)$ the homomorphisms $A \to B$ and by $\mathrm{Aut}(A)$ the automorphisms of A. By M_n we mean the $n \times n$ complex matrices. $K_0(A)$ will denote the K_0-group of the C^*-algebra A and by a trace on A we shall mean a tracial state on A. Let A be a C^*-algebra and let τ be a trace on A. We shall denote by $\|\cdot\|_\tau$ the seminorm on A given by $\|a\|_\tau = \tau(a^*a)^{\frac{1}{2}}$, $a \in A$. When $(x_n)_{n \geqq 1}$ is a sequence in A and $\|x_n - x\|_\tau \to 0$ for some $x \in A$, we shall write $\tau - \lim x_n = x$.

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Results.

We begin with some definitions:

DEFINITION 1. We shall say that a C^*-algebra A is *almost constant* if there is an inductive system $(C(X_n, M_{p(n)}), \Phi_{n,m})$ such that $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ and:

1) each X_n is a compact space

2) for any $n < m$, the homomorphism $\Phi_{n,m}\colon C(X_n, M_{p(n)}) \to C(X_m, M_{p(m)})$ is given by:

$$\Phi_{n,m}(f) = \mathrm{diag}(f \circ \phi_{n,m}^{(1)}, f \circ \phi_{n,m}^{(2)}, \dots, f \circ \phi_{n,m}^{(p(m)/p(n))})$$

for any $f \in C(X_n, M_{p(n)})$, where $\phi_{n,m}^{(i)}\colon X_m \to X_n$ are some continuous maps

(3) for any $n \in N$ we have:

$$\lim_{n \leqq m \to \infty} \frac{\mathrm{card}\{i \mid \phi_{n,m}^{(i)} \text{ is constant}\} \cdot p(n)}{p(m)} = 1$$

EXAMPLES. Let $A = \varinjlim (C(X, M_{p(n)}), \Phi_n)$ be a Goodearl algebra [10] of real rank zero, where X is a compact separable space which is not totally disconnected. Then $A = \varinjlim (C(X, M_{p(n)}), \Phi_n)$ is almost constant (see [10, Theorem 9]). Note also that by Elliott's remarkable classification results from [7], any Bunce-Deddens algebra A [4] can be written as a Goodearl algebra $A = \varinjlim (C(S^1, M_{p(n)}), \Phi_n)$ of real rank zero, where the multiplicity of the identity map in each Φ_n is 1 (use also [10]). Now it is obvious that $A = \varinjlim (C(S^1, M_{p(n)}), \Phi_n)$ is an almost constant C^*-algebra.

DEFINITION 2. We shall say that (A, B) is an *inductive pair* if $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ is an almost constant C^*-algebra and $B \subset A$ is the UHF algebra $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}})$.

DEFINITION 3. Let A be a C^*-algebra. We shall say that $K_0(A)$ is *weakly torsion free* if $nx = ny = [1_A]$ in $K_0(A)$ for some $n \in N$ and $x, y \in K_0(A)_+$ implies $x = y$.

The following Theorem shows that for certain C^*-algebras, any two homomorphisms between them are approximately inner equivalent in a weak sense.

THEOREM 1. *Let (A, B) be an inductive pair and et C be a C^*-algebra with a unique trace τ and such that C has cancellation and $K_0(C)$ is weakly torsion free. If $\Phi, \Psi \in \mathrm{Hom}(A, C)$, then there exists a sequence $(u_n)_{n \geqq 1}$ in $U(C)$ such that:*

$$\Phi(a) = \tau - \lim u_n \Psi(a) u_n^*, \quad a \in A$$

and

$$\Phi(b) = \lim u_n \Psi(b) u_n^*, \quad b \in B.$$

To prove this Theorem we shall need the following two Propositions:

PROPOSITION 1. *Let A be a* UHF *algebra and B a C^*-algebra with cancellation and such that $K_0(B)$ is weakly torsion free. If $\Phi, \Psi \in \mathrm{Hom}(A, B)$, then there is a sequence $(u_n)_{n \geqq 1}$ in $U(B)$ such that:*

$$\Phi(a) = \lim u_n \Psi(a) u_n^*, \quad a \in A.$$

PROOF. Denote $A = \varinjlim (M_{p(n)}, \Phi_n)$, where each homomorphism $\Phi_n \colon M_{p(n)} \to M_{p(n+1)}$ is unital. Let $(e_{ij}^{(n)})_{1 \leqq i,j \leqq p(n)}$ be a system of matrix units for $A_n := M_{p(n)}$. Since:

$$p(n)[\Phi(e_{11}^{(n)})] = p(n)[\Psi(e_{11}^{(n)})] = [1_B]$$

in $K_0(B)$, by hypothesis it follows that:

$$\Phi(e_{11}^{(n)}) = v_n \Psi(e_{11}^{(n)}) v_n^*$$

for some v_n in $U(B)$, $n \geqq 1$. Define $u_n := \sum_{i=1}^{p(n)} \Phi(e_{i1}^{(n)}) v_n \Psi(e_{1i}^{(n)})$, $n \geqq 1$. It is easy to see that each u_n is a unitary in B and:

$$\Phi(x) = u_n \Psi(x) u_n^*, \quad x \in A_n.$$

It follows that:

$$\Phi(a) = \lim u_n \Psi(a) u_n^*, \quad a \in A.$$

REMARK 1. The statement and proof of the above Proposition is a slight variation of a result of Effros and J. Rosenberg ([6, Theorem 3.8]); the argument is apparently due to Elliott. This argument has been used many times since by various authors.

PROPOSITION 2. *Let (A, B) be an inductive pair. Then:*
a) *A has a unique trace τ.*
b) *B is dense in A with respect to the seminorm $\|\cdot\|_\tau$.*
c) *If A has trivial center then $B' \cap A = \mathbf{C} \cdot 1_A$.*

PROOF. a) and b). Assume that $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ and $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}}) (\subset A)$ where:

$$\Phi_{n,m}(f) = \operatorname{diag}(f \circ \phi_{n,m}^{(1)}, \ldots, f \circ \phi_{n,m}^{(p(m)/p(n))}), f \in C(X_n, M_{p(n)})$$

for some continuous maps $\phi_{n,m}^{(i)}: X_m \to X_n$, and:

$$\lim_{n \leqq m \to \infty} \frac{\operatorname{card}\{i \mid \phi_{n,m}^{(i)} \text{ is constant}\}\, p(n)}{p(m)} = 1$$

for any $n \in N$.

Denote $A_n := C(X_n, M_{p(n)}), n \geqq 1$. Fix $0 \neq a \in A_n$ for some n. Then, for any $k \in N$ there is $m(k) = m(k, n) > n$ such that:

$$\frac{\operatorname{card}\{i \mid \phi_{n,m(k)}^{(i)} \text{ is not constant}\}\, p(n)}{p(m(k))} \leqq 2^{-k} \|a\|^{-2}.$$

Let $I_k := \{i \mid \phi_{n,m(k)}^{(i)}$ is not constant$\}$. Define $b_k \in A_{m(k)}$ to be the element of $A_{m(k)}$ obtained replacing in $\Phi_{n,m(k)}(a)$ the block $a \circ \phi_{n,m(k)}^{(i)}$ with 0 only for $i \in I_k$. Denote by $\mu_k: A_k \to A$ the canonical homomorphisms. It is obvious that $(\mu_{m(k)}(b_k))_{k \geqq 1}$ is a sequence in B.

Now, let σ be an *arbitrary* trace of A. Denote $\sigma_{m(k)} = \sigma \circ \mu_{m(k)}$. We have:

$$\begin{aligned}
\|\mu_n(a) - \mu_{m(k)}(b_k)\|_\sigma^2 &= \|\Phi_{n,m(k)}(a) - b_k\|_{\sigma_{m(k)}}^2 \\
&\leqq \sum_{i \in I_k} \|a \circ \phi_{n,m(k)}^{(i)}\|^2 \cdot \frac{p(n)}{p(m(k))} \\
&\leqq (\operatorname{card} I_k) \cdot \|a\|^2 \frac{p(n)}{p(m(k))} \leqq \frac{p(m(k))}{p(n)} 2^{-k} \cdot \|a\|^{-2} \cdot \|a\|^2 \cdot \frac{p(n)}{p(m(k))} \\
&= 2^{-k}, \quad k \geqq 1.
\end{aligned}$$

Hence, we have:

$$\sigma(\mu_n(a)) = \lim_{k \to \infty} \sigma(\mu_{m(k)}(b_k)) = \lim_{k \to \infty} \lambda(\mu_{m(k)}(b_k))$$

where λ is the unique trace of the UHF algebra B. Since $\bigcup_{k \geqq 1} \mu_k(A_k)$ is dense in A in the C^*-algebra norm, it follows that A has a unique trace and B is dense in A with respect to the trace seminorm.

c) follows from the following more general fact: Let C be a C^*-algebra with trivial center and let σ be a trace of C. Consider a unital C^*-subalgebra D of C which is dense in C with respect to $\|\cdot\|_\sigma$. Then $D' \cap C$ is trivial.

The proof is similar with that of [13, Corollary 3] and will be not given.

Proof of Theorem 1. By Proposition 1 it follows that there is a sequence $(u_n)_{n\geqq 1}$ in $U(C)$ such that

$$\Phi(b) = \lim u_n \Psi(b) u_n^*, \quad b \in B.$$

Since A has a unique trace denoted σ (see Proposition 2a)), we have:

$$\|\Phi(a)\|_\tau = \|a\|_\sigma = \|\Psi(a)\|_\tau, \quad a \in A.$$

Define $\Psi_n: A \to C$ by $\Psi_n(a) := u_n \Psi(a) u_n^*$, $a \in A (n \geqq 1)$. Then $\Psi_n, \Phi: (A, \|\cdot\|_\sigma) \to (C, \|\cdot\|_\tau)$ are linear and continuous. Since $\|\Psi_n(b) - \Phi(b)\|_\tau \to 0$, $b \in B$ and $\|\Psi_n(a)\|_\tau = \|a\|_\sigma$, $a \in A$, by Proposition 2b) we deduce that $\|\Psi_n(a) - \Phi(a)\|_\tau \to 0$, $a \in A$.

Remark 2. Let (A, B) be an inductive pair and suppose that A has cancellation and $K_0(A)$ is weakly torsion free. Denote by τ the trace of A. Then, if Φ is an endomorphism of A, the above Theorem implies that there is a sequence $(u_n)_{n\geqq 1}$ in $U(A)$ such that:

$$\Phi(a) = \tau - \lim u_n a u_n^*, \quad a \in A$$

and

$$\Phi(b) = \lim u_n b u_n^*, \quad b \in B.$$

Note that Φ isn't approximately inner in general; indeed, e.g. by [13, Proposition 3] there are automorphisms of Bunce-Deddens algebras which don't induce the identity of the K_1-group (see also [1, problem 10.11.5 (b)] and [12]]).

Now we are interested to find necessary and sufficient conditions under which an automorphism of the "canonical" UHF subalgebra of an almost constant C^*-algebra can be extended to an automorphism of the whole C^*-algebra.

Notation. Let A be a C^*-algebra with a unique trace τ, which is faithful. We shall denote by $L^2(A)$ the completion of A with respect to the norm $\|\cdot\|_\tau$. The induced norm on $L^2(A)$ will be also denoted by $\|\cdot\|_\tau$. If $(x_n)_{n\geqq 1}$ is a sequence in $(L^2(A), \|\cdot\|_\tau)$ we shall denote by $\tau - \lim x_n \in L^2(A)$ the corresponding limit.

Theorem 2. *Let (A, B) be an inductive pair, where $A = \varinjlim (C(X_n, M_{p(n)}), \Phi_{n,m})$ as in Definition 1, $B = \varinjlim (M_{p(n)}, \Phi_{n,m|M_{p(n)}})$ and suppose that the trace τ of A is* *faithful. Consider* $\Phi \in \mathrm{Aut}(B)$ *and let* $(u_n)_{n \geq 1} \subset U(B)$ *such that* $\Phi(x) = \lim u_n x u_n^*$, $x \in B$. *Denote by* $\mu_n \colon C(X_n, \mathrm{M}_{p(n)}) \to A$ *the canonical homomorphisms. Then the following two conditions are equivalent:*

a) Φ *extends to an automorphism of* A

b) $\tau - \lim u_n f u_n^*$ *and* $\tau - \lim u_n^* f u_n$ *exist in* A *for any* $f \in \bigcup_{m=1}^{\infty} \mu_m(C(X_m))$.

Moreover, when Φ *extends, it has a unique extension* $\tilde{\Phi} \in \mathrm{Aut}(A)$, *where*

$$\tilde{\Phi}(x) = \tau - \lim u_n x u_n^*$$

and

$$\tilde{\Phi}^{-1}(x) = \tau - \lim u_n^* x u_n$$

for any $x \in A$.

The following rigidity result will be needed to prove the above Theorem:

LEMMA 1. *Let* (A, B) *be an inductive pair and* C *a* C^**-algebra with a unique trace. If* $\Phi, \Psi \in \mathrm{Hom}(A, C)$ *and* $\Phi_{|B} = \Psi_{|B}$ *then* $\Phi = \Psi$.

PROOF. In fact the folowing more general result is true:

Let M be a C^*-algebra with a unique trace τ and N a unital C^*-subalgebra of M such that N is dense in M with respect to $\|\cdot\|_\tau$. If P is a C^*-algebra with a unique trace, $\Phi, \Psi \in \mathrm{Hom}(M, P)$ and $\Phi_{|N} = \Psi_{|N}$ then $\Phi = \Psi$.

The proof of this fact is immediate and similar with that of [13, Lemma 2] and therefore will not be given.

PROOF OF THEOREM 2. This proof is inspired by that of [13, Theorem 2].

First of all observe that the unicity of the extension (when it exists), follows from the above Lemma.

a) $\Rightarrow$ b). Consider $\tilde{\Phi} \in \mathrm{Aut}(A)$ such that $\tilde{\Phi}_{|B} = \Phi$. By the proof of Theorem 1 and the above observation, we obtain:

$$\tilde{\Phi}(x) = \tau - \lim u_n x u_n^*, \quad x \in A.$$

It follows that $\tau - \lim u_n f u_n^* = \tilde{\Phi}(f) \in A$ for any $f \in \bigcup_{m=1}^{\infty} \mu_m(C(X_m))$. Working with Φ^{-1} we obtain the other relations.

b) $\Rightarrow$ a). Recall that by $L^2(A)$ (resp. $L^2(B)$) we mean the completion of A (resp. B) with respect to the trace norm $\|\cdot\|_\tau$ (resp. $\|\cdot\|_\sigma$), where $\sigma = \tau_{|B}$ is the trace of B. Observe that by Proposition 2b) we have $L^2(A) = L^2(B)$.

If B is seen in its GNS representation in $\mathscr{B}(L^2(B))$ associated with σ, we have:

$$\Phi(x) = UxU^*, \quad x \in B.$$

Here $U \in U(\mathscr{B}(L^2(B)))$ and $U(b) := \Phi(b), \quad b \in B$.

Since by a previous observation we have also $U \in U(\mathscr{B}(L^2(A)))$, one can define $\tilde{\Phi} \in \mathrm{Hom}(A, \mathscr{B}(L^2(A)))$ by:

$$\tilde{\Phi}(x) = UxU^*, \quad x \in A,$$

where A is seen in its GNS representation in $\mathscr{B}(L^2(A))$ associated with τ. It is clear that $\tilde{\Phi}_{|B} = \Phi$.

Consider an arbitrary element f in $\bigcup_{m=1}^{\infty} \mu_m(C(X_m))$. By a version of Kaplansky's Density Theorem (use a slight modification of the proof of [11, Lemma 3.11]) there is a sequence $(b_k)_{k \geqq 1}$ in B such that $\|b_k - f\|_\tau \to 0$ and $\|b_k\| \leqq \|f\|$. Since $x_n \xrightarrow{\|\cdot\|_\tau} 0$ in A means $x_n \xrightarrow{\text{so}} 0$ in $\mathscr{B}(L^2(A))$ when $\{\|x_n\|\}$ is bounded, we have:

$$\tilde{\Phi}(f) = UfU^* = \text{so-lim}\, Ub_kU^* = \text{so-lim}\, \Phi(b_k)$$

$$= \text{so-lim}_k\, (\text{so-lim}_n\, u_n b_k u_n^*)$$

It is not difficult to see that $\tau - \lim u_n x u_n^*$ exists in $L^2(A)$ for any $x \in A$ (indeed, the limit exists for all $x \in B$, by Proposition 2b) B is dense in A with respect to $\|\cdot\|_\tau$ and $\|u_n x u_n^*\|_\tau = \|x\|_\tau$ for any $x \in A$ and $n \in N$). Hence:

$$\|\tau - \lim_n u_n b_k u_n^* - \tau - \lim_n u_n f u_n^*\|_\tau = \lim_n \|u_n(b_k - f)u_n^*\|_\tau = \|b_k - f\|_\tau$$

which implies that in $L^2(A)$ one has:

$$\tau - \lim_k (\tau - \lim_n u_n b_k u_n^*) = \tau - \lim_n u_n f u_n^*.$$

Since $\tau - \lim u_n f u_n^* \in A$ by hypothesis and $\|x_n\|_\tau \to 0$ in A means $x_n \xrightarrow{\text{so}} 0$ in $\mathscr{B}(L^2(A))$ if $\{\|x_n\|\}$ is bounded, we can write:

$$\tilde{\Phi}(f) = \text{so-lim}_k\, (\text{so-lim}_n\, u_n b_k u_n^*)$$

$$= \text{so-lim}_n\, u_n f u_n^* \in A$$

But A is the C^*-algebra generated by B and $\bigcup_{m=1}^{\infty} \mu_m(C(X_m))$ and we already knew that $\tilde{\Phi}$ belongs to $\mathrm{Hom}(A, \mathscr{B}(L^2(A))$ and $\tilde{\Phi}(B) = B$. It follows that $\tilde{\Phi}(A) \subset A$ and as in the proof of a) $\Rightarrow$ b) one obtains:

$$\tilde{\Phi}(x) = \tau - \lim_n u_n x u_n^*, \quad x \in A.$$

Repeating the above argument for Φ^{-1}, where $\Phi^{-1}(x) = \lim u_n^* x u_n$, $x \in B$, finally we get $\tilde{\Phi}^{-1} \in \mathrm{Aut}(A)$.

The following two Propositions give, in particular, additional informations about $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$, where (A, B) is an inductive pair. Their proofs are similar with those of [13, Proposition 4 and Proposition 5] and therefore will be not given.

PROPOSITION 3. *Let A be a C^*-algebra and let B be a* UHF *algebra which is a unital C^*-subalgebra of A. Then the centralizer of $\{\Phi \in \mathrm{Aut}(B)\colon \Phi = \tilde{\Phi}_{|B}$ for some $\tilde{\Phi} \in \mathrm{Aut}(A)\}$ in $\mathrm{Aut}(B)$ is trivial.*

PROPOSITION 4. *Let A and B be as in the above Proposition. Suppose moreover that:*

a) *the center of A is trivial.*

b) *A has a unique trace τ.*

c) *B is dense in A with respect to the trace seminorm $\|\cdot\|_\tau$.*

Then the centralizer of $\{\Phi \in \mathrm{Aut}(A)\colon \Phi(B) = B\}$ in $\mathrm{Aut}(A)$ is trivial.

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WHEN DOES bvca(Σ, X) CONTAIN A COPY OF l_∞?

JUAN CARLOS FERRANDO*

Abstract.

Let X be a Banach space and let Σ be a σ-algebra of subsets of a set Ω. Denoting by bvca(E, X) the Banach space of all X-valued countably additive measures of bounded variation defined on Σ endowed with the variation norm, we show that if X has the Radon-Nikodym property with respect to each element of ca$^+(\Sigma)$, bvca(Σ, X) has acopy of l_∞ if and only if X does.

Our notation is standard (see [1] and [2]). In what follows X will be a Banach space, Ω a set and Σ a σ-algebra of subsets of Ω. We denote by ca(Σ, X) the Banach space of all X-valued countable additive measures equipped with the semivariation norm. By ca(Σ) we denote the Banach space of all scalar countably additive measures defined on Σ provided with the variation norm, whose positive members we denote by ca$^+(\Sigma)$. If $\mu \in$ ca$^+(\Sigma)$ and $1 \leqq p < \infty$. $L_p(\mu, X)$ will stand for the Banach space of all (classes of) X-valued Bochner p-integrable functions defined on Ω equipped with their usual norms. By bvca(Σ, X) we denote the Banach space of all X-valued countably additive measures F of bounded variation defined on Σ with the variation norm $|F| = |F|(\Omega)$.

If $\mu \in$ ca$^+(\Sigma)$ we denote by bvca(Σ, μ, X) the linear subspace of bvca(Σ, X) formed by all those measures that are μ-continuous. The linear operator $T: L_1(\mu, X) \to$ bvca(Σ, μ, X) defined by $(Tf)(E) = \int_E f\,d\mu$ (integral of Bochner) for all $E \in \Sigma$, is an isometry onto if and only if X has the Radon-Nikodym property with respect to μ.

The main aim of this note is to demonstrate the theorem below. Our proof is based upon the proofs of Lemma 4 and Theorem 2 of [4].

THEOREM. *If X has the Radon-Nikodym property with respect to each $\mu \in$ ca$^+(\Sigma)$, then the space* bvca(Σ, X) *contains an isomorphic copy of l_∞ if and only if X does.*

This result can be aligned with those given in [3] and [4], where conditions are

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imposed on the spaces $\mathrm{cca}(\Sigma, X)$ and $\mathrm{ca}(\Sigma, X)$ in order to ensure that they contain copies of l_∞ and c_0. We shall need the following lemmas.

LEMMA 1 (Rosenthal, [7]). *Let T be a bounded linear operator from l_∞ into X. If X does not contain a copy of l_∞, T is weakly compact.*

LEMMA 2 (Mendoza, [6]). *If $1 \leqq p < \infty$, then $L_p(\mu, X)$ contains an isomorphic copy of l_∞ if and only if X does.*

PROOF OF THE THEOREM. The *if* part of the theorem is trivial. So we concentrate on the converse. Suppose that $\mathrm{bvca}(\Sigma, X)$ contains a copy of l_∞. Let J denote a canonical isomorphism from l_∞ into $\mathrm{bvca}(\Sigma, X)$ and let (e_n) be the unit vector basis of c_0. We are going to define some useful linear operators (these definitions have been taken from [4, Lemma 4]).

If X does not contain any copy of l_∞, then for each E in Σ the operator $J_E\colon l_\infty \to X$, defined by $J_E(\xi) = (J\xi)(E)\ \forall \xi \in l_\infty$, is weakly compact by Lemma 1, and therefore the series $\sum_n J_E e_n$ is unconditionally convergent. Hence the operator $T\colon l_\infty \to \mathrm{ba}(\Sigma, X)$ defined by $T\xi(E) = \sum_n \xi_n J_E e_n$ for each $\xi \in l_\infty$ and each $E \in \Sigma$ is well-defined. It is bounded too, since if $\{E_i, 1 \leqq i \leqq n\}$ is a partition of Ω by elements of Σ and we write $\xi^n := (\xi_1, \ldots, \xi_n, 0, 0, \ldots, 0, \ldots)$, then

$$\sum_{i=1}^{n} \|T\xi(E_i)\| = \sum_{i=1}^{n} \|\textstyle\sum_j \xi_j J_{E_i} e_j\| = \displaystyle\sum_{i=1}^{n} \lim_k \|J_{E_i}\xi^k\| = \lim_k \sum_{i=1}^{n} \|J\xi^k(E_i)\|$$

$$\leqq \sup_k \sum_{i=1}^{n} \|J\xi^k(E_i)\| \leqq \sup_k |J\xi^k| \leqq \sup_k \|J\|\,\|\xi^k\|_\infty \leqq \|J\|\,\|\xi\|_\infty$$

Thus $T\xi$ is of bounded variation for each $\xi \in l_\infty$, and $\|T\| \leqq \|J\|$. So setting $v := \sum_n 2^{-n} |Je_n|$, clearly $Je_n \in \mathrm{bvca}(\Sigma, v, X)$ for each $n \in \mathsf{N}$, and hence $J\xi^n \ll v$ for each $\xi \in l_\infty$ and each $n \in \mathsf{N}$. Now, since $\lim_n J\xi^n(E)\,(= T\xi(E) \in X)$ exists $\forall \xi \in l_\infty$ and $\forall E \in \Sigma$, the Vitali-Hahn-Saks theorem guarantees that $T\xi \ll v$ for each $\xi \in l_\infty$. Therefore $T(l_\infty) \subseteq \mathrm{bvca}(\Sigma, v, X)$.

As $Te_n(E) = J_E e_n = Je_n(E)$ for each $E \in \Sigma$ and each $n \in \mathsf{N}$, then $Te_n = Je_n$ for each $n \in \mathsf{N}$ and hence $\inf_n \|Te_n\| > 0$. Now a well-known theorem of Rosenthal ([17]) assures that there is some $\mathsf{M} \subseteq \mathsf{N}$ with card $\mathsf{M} = \aleph_0$ such that the restriction of T to $l_\infty(\mathsf{M})$ is an isomorphism. So the space $\mathrm{bvca}(\Sigma, v, X)$ has a copy of l_∞. But $\mathrm{bvca}(\Sigma, v, X)$ is isometric to $L_1(v, X)$, since X has the Radon-Nikodym property with respect to the positive measure v. Hence Lemma 2 applies to get the contradiction.

COROLLARY 1. *If Σ is such that each $\mu \in \mathrm{ca}^+(\Sigma)$ is purely atomic, then* $\mathrm{bvca}(\Sigma, X)$ *has a copy of l_∞ if and only if X does.*

REMARK. If X has the Radon-Nikodym property with respect to each μ in $ca^+(\Sigma)$ and bvca(Σ, X) has a copy L of c_0, then there clearly exists a $\mu \in ca^+(\Sigma)$ so that $L \subseteq$ bvca(Σ, μ, X). According to a well-known theorem of Kwapien, [5], X must contain a copy of c_0.

COROLLARY 2. *If X is a reflexive Banach space, then* bvca(Σ, X) *does not contain any copy of* c_0.

PROOF. Since X is reflexive, X has the Radon-Nikodym property and hence does not contain any copy of c_0. Therefore bvca(Σ, X) cannot have a copy of c_0.

PROPOSITION. *Suppose that the σ-algebra Σ is countably generated and that* bvca(Σ, μ, Y) *is separable whenever Y is a closed separable subspace of X and $\mu \in ca^+(\Sigma)$. Then* bvca(Σ, X) *contains a copy of l_∞ if and only if X does.*

PROOF. Let J be an isomorphism from l_∞ into bvca(Σ, X) and assume X has not any copy of l_∞. Then define for each $E \in \Sigma$ the weakly compact linear operator J_E as in the Theorem and denote by Z the closed linear hull of $\{J_E e_n, n \in \mathrm{N}, E \in \mathscr{A}\}$, where $\mathscr{A}$ denotes a sequence of elements of Σ containing Ω which generates Σ. Obviously, Z is a separable Banach space.

Next we shall see that $J_E e_n \in Z \; \forall E \in \Sigma$. In fact, given a family $\mathscr{B}$ of elements of Σ, denote by $\mathscr{B}^*$ the family of all countable unions of sets of $\mathscr{B}$ and all the complementary sets of sets of $\mathscr{B}$. Let ω be the first ordinal of uncountable cardinal. Set $\Sigma_0 = \mathscr{A}$ and for each ordinal α with $0 < \alpha < \omega$ define $\Sigma_\alpha = \{\cup\{\Sigma_\beta, \beta < \alpha\}\}^*$. Note that $\Sigma_\beta \subseteq \Sigma_\alpha \; \forall \beta \leqq \alpha$ and $\Sigma = \cup\{\Sigma_\alpha, \alpha < \omega\}$. We shall proceed by transfinite induction.

We know that $J_E e_n \in Z$ for each $E \in \Sigma_0$. Suppose that, if $0 < \alpha < \omega$, $J_E e_n \in Z$ for each $E \in \Sigma_\beta$ with $\beta < \alpha$. As $\Sigma_\alpha = \{\cup\{\Sigma_\beta, \beta < \alpha\}\}^*$, choosing $E = \cup\{E_k, k \in \mathrm{N}\}$ with $E_k \in \Sigma_{\beta_k}$ and $\beta_k < \alpha$ for each k, then one has that $J_E e_n = J e_n(E) = \lim_k J e_n\left(\bigcup_{j=1}^{k} E_j\right) \in Z$. On the other hand, if $E \in \Sigma_\beta$ with $\beta < \alpha$, then $J_{\Omega \setminus E} e_n = J e_n(\Omega \setminus E) = J e_n(\Omega) - J e_n(E) \in Z$.

Using the same notation of the Theorem, since $T\xi(E) = \sum_n \xi_n J_E e_n$ for each $\xi \in l_\infty$ and $E \in \Sigma$, and since all $J_E e_n \in Z$ as we have just seen, we have $T\xi(E) \in Z$ for each $\xi \in l_\infty$ and each $E \in \Sigma$. Besides, reasoning as in the last part of the proof of the Theorem, there exists a $\mu \in ca^+(\Sigma)$ such that $T\xi << \mu$ for each $\xi \in l_\infty$. This shows that T is a bounded linear operator of l_∞ into bvca(Σ, μ, Z). Thus Rosenthal's theorem implies that bvca(Σ, μ, Z) has a copy of l_∞. But we suppose that bvca(Σ, μ, Z) is separable, a contradiction.

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AMENABILITY IN GROUP ALGEBRAS AND BANACH ALGEBRAS

ANDREW KEPERT

0. Introduction.

In this paper, we will examine some aspects of the relationship between amenability in groups and amenability in Banach algebras. While leaving specific characterizations of these properties for the next section, we have the following results from [13].

0.1. PROPOSITION. *A locally compact group G is amenable if and only if $L'(G)$ is amenable.*

0.2. PROPOSITION. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras and $v\colon \mathfrak{A} \to \mathfrak{B}$ is a continuous homomorphism with range dense in $\mathfrak{B}$. If $\mathfrak{B}$ is amenable then $\mathfrak{B}$ is amenable.*

0.2.1. COROLLARY. *If X is a locally compact Hausdorff topological space, then $C_0(X)$ is amenable.*

0.2.2. COROLLARY. *If H is a Hilbert space then $\mathscr{K}(H)$ is amenable.*

Each of these Corollaries is proven by constructing an amenable locally compact group G and a dense-ranged continuous homomorphism $L^1(G) \to \mathfrak{A}$. It is natural to ask whether *any* amenable Banach algebra $\mathfrak{A}$ can be shown to be amenable by a similar construction. It will be shown in Sections 2 and 3 that this is not the case, and for certain classes of Banach algebras we will develop some necessary and sufficient conditions for there to exist such a homomorphism.

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leading to the results in Section 3, and to Dr N. Grønbæk for the suggestion that lead to the results in Section 4.

1. Notation and Preliminary Results.

Throughout, $G, G_1, \ldots$ will denote locally compact groups, each represented multiplicatively with unit denoted e. If H is a closed normal subgroup of G, T_H will denote the epimorphism $L^1(G) \to L^1(G/H)$ as described by Reiter in [20, 3.3.2–3.5.3]. When G is Abelian, its dual group will be denoted Γ, and this will be identified with $\Phi_{L^1(G)}$, the maximal ideal space of the group algebra $L^1(G)$. If $\mathfrak{A}$ is a commutative semisimple Banach algebra, the *hull* of a set $X \subseteq \mathfrak{A}$ is $Z(X) = \{\varphi \in \Phi_{\mathfrak{A}}: \varphi(X) = 0\}$ and the *kernel* of a set $S \subseteq \Phi_{\mathfrak{A}}$ is $\mathscr{I}(S) = \{a \in \mathfrak{A}: \varphi(a) = 0, (\varphi \in S)\}$.

Let $C_b(G)$ be the space of continuous bounded functions on G, then $M \in C_b(G)^*$ is called a *mean* when $\inf f(G) \leqq M(f) \leqq \sup f(G)$ for each $f \in C_b(G)$ with rng $f \subseteq \mathsf{R}$. A group G is *amenable* if there exists a mean M on $C_b(G)$ that is left-invariant, that is, $M(f) = M({}_xf)$, $(f \in C_b(G), x \in G)$. Locally compact groups that are Abelian or compact are amenable. There are many equivalent characterizations to the above, including the existence of left and/or right-invariant means on other function spaces on G, such as $L^\infty(G)$, $UC(G)$, $UC_r(G), \ldots$ and a variety of structural conditions on G, referred to as Følner conditions. See [10] or [17] for more on invariant means and Følner conditions.

Let $\mathfrak{A}$ be a Banach algebra, and let $\mathfrak{A} \hat{\otimes} \mathfrak{A}$ (defined as in [3, Chapter 6]) be endowed with its canonical structure as a Banach $\mathfrak{A}$-bimodule, given by $(a \otimes b) \cdot c = a \otimes (bc)$ and $a \cdot (b \otimes c) = (ab) \otimes c$. Let π be the mapping $\mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A}$ given by extending $a \otimes b \mapsto ab$ by linearity and continuity. An *approximate diagonal* for $\mathfrak{A}$ is a bounded net $\{d_n\}_{n \in \Delta} \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$ such that for each $a \in A$, $\pi(d_n)a \to a$, and $d_n \cdot a - a \cdot d_n \to 0$. If an approximate diagonal exists, then we say that $\mathfrak{A}$ is *amenable*. Again, there are other characterizations equivalent to this, such as the condition that every derivation from $\mathfrak{A}$ into a dual Banach $\mathfrak{A}$-bimodule is inner (see [3, Theorem 43.9]), or that $\mathfrak{A}$ has bounded approximate identity and $\ker \pi$, when considered as an ideal of the algebra $\mathfrak{A} \hat{\otimes} \mathfrak{A}^{\mathrm{op}}$, has a bounded approximate identity (see [8]).

We say a Banach algebra $\mathfrak{A}$ has *property* (**G**) if there exists an amenable locally compact group G and a continuous homomorphism $\nu: L^1(G) \to \mathfrak{A}$ with range dense in $\mathfrak{A}$. Then as noted in the introduction, a Banach algebra with property (**G**) is amenable.

We present some basic results on property (**G**) which will aid us in later sections, when we will characterize property (**G**) for certain Banach algebras $\mathfrak{A}$.

1.1. PROPOSITION. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are Banach algebras with property* (**G**), *then $\mathfrak{A} \oplus \mathfrak{B}$ and $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ have property* (**G**).

PROOF. By hypothesis, there exist amenable locally compact groups G_1 and G_2 and continuous homomorphisms $\nu_1: L^1(G_1) \to \mathfrak{A}$ and $\nu_2: L^1(G_2) \to \mathfrak{B}$ with $\overline{\operatorname{rng} \nu_1} = \mathfrak{A}$ and $\overline{\operatorname{rng} \nu_2} = \mathfrak{B}$. Then the continuous homomorphisms $\nu_1 \oplus \nu_2$: $L^1(G_1) \oplus L^1(G_2) \to \mathfrak{A} \oplus \mathfrak{B}$ and $\nu_1 \otimes \nu_2: L^1(G_1) \hat{\otimes} L^1(G_2) \to \mathfrak{A} \hat{\otimes} \mathfrak{B}$ have dense range, so it suffices to show that $L^1(G_1) \oplus L^1(G_2)$ and $L^1(G_1) \hat{\otimes} L^1(G_2)$ have property (G). For this, note that the groups $G_1 \times G_2$ and $G_1 \times G_2 \times \mathbf{Z}_2$ are amenable with

$$L^1(G_1 \times G_2) \cong L^1(G_1) \hat{\otimes} L^1(G_2),$$

$$L^1(G_1 \times G_2 \times \mathbf{Z}_2) \cong L^1(G_1 \times G_2) \hat{\otimes} \mathbf{C}^2$$

$$\cong L^1(G_1 \times G_2) \oplus L^1(G_1 \times G_2),$$

and $T_{G_2} \oplus T_{G_1}: L^1(G_1 \times G_2) \oplus L^1(G_1 \times G_2) \to L^1(G_1) \oplus L^1(G_2)$ is an epimorphism.

1.2. PROPOSITION. *Suppose G is a locally compact group and $\nu: L^1(G) \to \mathfrak{A}$ is a continuous homomorphism into a commutative Banach algebra, then there is a locally compact Abelian group G' and a continuous homomorphism $\nu': L^1(G') \to$ with* $\operatorname{rng} \nu = \operatorname{rng} \nu'$.

PROOF. Let C be the closure of the commutator subgroup of G, then by [12, Theorem 23.8, G/C is Abelian. We show firstly that the kernel of $T_C: L^1(G) \to L^1(G/C)$ is $\mathscr{J}$, the commutator ideal of $L^1(G)$. The inclusion $\mathscr{J} \subseteq \ker T_C$ follows from the observation that T_C is a continuous homomorphism into a commutative Banach algebra.

Conversely, by [20, 3.6.4], we have $\ker T_C = \overline{\operatorname{span}}\{f - {}_x f: x \in C, f \in L^1(G)\}$. Put $H = \{x \in G: f - {}_x f \in \mathscr{J}, (f \in L^1(G))\}$, a closed subgroup of G, and let $\{e_n\}_{n \in \Delta}$ be a bounded approximate identity for $L^1(G)$. For each $x, y \in G$ and each $f \in L^1(G)$,

$$\|{}_{yx}f - {}_y e_n * {}_x e_n * f\| \leqq \|{}_y({}_x f - e_n * {}_x f)\| + \|{}_y e_n * {}_x(f - e_n * f)\|$$

$$\leqq \|{}_x f - e_n * {}_x f\| + \|e_n\| \, \|f - e_n * f\|$$

$$\to 0,$$

so that $({}_y e_n * {}_x e_n - {}_x e_n * {}_y e_n) * f \to {}_{yx}f - {}_{xy}f$. Furthermore, $({}_y e_n * {}_x e_n - {}_x e_n * {}_y e_n) * f \in \mathscr{J}$, which is closed and translation-invariant, so that ${}_{x^{-1}y^{-1}xy}f - f \in \mathscr{J}$ and $x^{-1}y^{-1}xy \in H$. But C is the closed subgroup generated by $\{x^{-1}y^{-1}xy: x, y \in G\}$, so $C \subseteq H$. Hence $\ker T_C \subseteq \mathscr{J}$.

Now, since $\mathfrak{A}$ is commutative, $\ker T_C \subseteq \ker \nu$, and so $\nu \circ T_C^{-1}: L^1(G/C) \to \mathfrak{A}$ defines a continuous algebra homomorphism, as required.

2. Subalgebras of Commutative Group Algebras.

In this section we will examine closed subalgebras of commutative group algebras, which we call *group subalgebras*, and for certain classes of these, develop necessary and sufficient conditions for property (**G**).

By Proposition 1.2, a subalgebra $\mathfrak{A}$ of $L^1(G)$ has property (**G**) if and only if there is a locally compact Abelian group G' and a continuous homomorphism $\nu: L^1(G') \to L^1(G)$ with $\mathfrak{A} = \overline{\text{rng}\,\nu}$. Thus we can use the Theorem of Cohen, [6, Theorem 1], which characterizes homomorphisms between commutative group algebras. For this we define the terms *coset ring*, *affine*, *piecewise affine* and *proper*. The *coset ring* of a locally compact Abelian group Γ, denoted $\mathscr{R}(\Gamma)$, is the Boolean ring generated by the open cosets in Γ. If $E \subseteq \Gamma$, a map $\psi: E \to \Gamma'$ is *affine* if for any $\gamma_1, \gamma_2, \gamma_3 \in E$, $\psi(\gamma_1\gamma_2^{-1}\gamma_3) = \psi(\gamma_1)\psi(\gamma_2)^{-1}\psi(\gamma_3)$. If $E \subseteq \Gamma$, a map $\psi: S \to \Gamma'$ is *piecewise affine* if there exist disjoint $S_1, \ldots, S_n \in \mathscr{R}(\Gamma)$ such that $S = S_1 \cup \ldots \cup S_n$ and for each $1 \leqq k \leqq n, \alpha|_{S_k}$ has a continuous affine extension $\alpha_k: E_k \to \Gamma'$. (Here it is understood that E_k is a coset containing S_k.) If X and Y are locally compact topological spaces then a map $\psi: X \to Y$ is *proper* if for any compact $C \subseteq Y$, $\psi^{-1}(C)$ is compact.

Then with $\nu: L^1(G') \to L^1(G)$ as above, it follows from [6, Theorem 1] that $Y = \{\gamma \in \Gamma: \nu^*(\gamma) \neq 0\} \in \mathscr{R}(\Gamma)$ and $\alpha = \nu^*|_Y$ is a proper piecewise affine map. Furthermore, any such α uniquely determines a homomorphism ν by the relations $\widehat{\nu(f)}(\gamma) = \hat{f} \circ \alpha(\gamma)$ if $\gamma \in Y$ and $\widehat{\nu(f)}(\gamma) = 0$ if $\gamma \notin Y$.

In the paper [14], it was shown that for such a homomorphism, $Y = \Gamma \backslash Z(\text{rng}\,\nu)$ and $\text{rng}\,\nu = \kappa(\alpha)$, where

$$\kappa(\alpha) = \{f \in L^1(G): \hat{f} = 0 \text{ off } Y \text{ and } \hat{f}(\gamma_1) = \hat{f}(\gamma_2) \text{ whenever } \alpha(\gamma_1) = \alpha(\gamma_2)\},$$

which is closed. Thus we can classify the group subalgebras with property (**G**) as follows.

2.1. PROPOSITION. *If G is a locally compact Abelian group and $\mathfrak{A}$ is a closed subalgebra of $L^1(G)$, then $\mathfrak{A}$ has property* (**G**) *if and only if*

(i) *$Y = \Gamma \backslash Z(\mathfrak{A}) \in \mathscr{R}(\Gamma)$, and*

(ii) *there is a locally compact Abelian group Γ' and a proper piecewise affine map $\alpha: Y \to \Gamma'$ with $\mathfrak{A} = \kappa(\alpha)$.*

We now consider specific classes of group subalgebras and develop necessary and sufficient conditions for amenability and property (**G**). The simplest such class consists of the closed ideals of commutative group algebras. For this, define the *discrete coset ring* of a locally compact Abelian group Γ to be $\mathscr{R}(\Gamma_d)$, the coset ring of Γ with its discrete topology. We denote this $\mathscr{R}_d(\Gamma)$, and it is the Boolean ring generated by *all* cosets in Γ.

We will, in fact, mainly be interested in sets in

$$\mathscr{R}_c(\Gamma) = \{X \in \mathscr{R}_d(\Gamma) \colon X \text{ is a closed subset of } \Gamma\},$$

as these are the only sets in $\mathscr{R}_d(\Gamma)$ that can be hulls of ideals. The fact that an ideal has hull in $\mathscr{R}_c(\Gamma)$ if and only if it has bounded approximate identity is vital in the next theorem and subsequent results.

2.2. THEOREM. *Let $\mathscr{I}$ is a closed ideal of $L^1(G)$, and put $E = Z(\mathscr{I})$. Then $\mathscr{I}$ is amenable if and only if $E \in \mathscr{R}_c(\Gamma)$, whereas $\mathscr{I}$ has property* (**G**) *if and only if $E \in \mathscr{R}(\Gamma)$. In either case, $\mathscr{I} = \mathscr{I}(E)$.*

PROOF. The first part of this is [16, Theorem 1]. For the second, we have by Proposition 2.1 that if $\mathscr{I}$ is an ideal with property (**G**), then $Y = \Gamma \backslash E \in \mathscr{R}(\Gamma)$, and so $E \in \mathscr{R}(\Gamma)$. Conversely, if $E = Z(\mathscr{I}) \in \mathscr{R}(\Gamma)$, then E is clopen, and consequently of synthesis, so that $\mathscr{I} = \mathscr{I}(E)$. Moreover, if we define $\alpha\colon Y \to \Gamma$ to be the inclusion mapping, then α is a proper piecewise affine map with $\kappa(\alpha) = \mathscr{I}(E)$, so by the above discussion, $\mathscr{I}$ has property (**G**).

REMARK. In the above proof, the epimorphism $\nu\colon L^1(G) \to \mathscr{I}$ determined by α has $\nu(f) = \chi_{\Gamma\backslash E} \cdot \hat{f}$. This is clearly a multiplicative projection. Then by [5, Theorem 1], there is an idempotent measure $\mu \in M(G)$ with $\hat{\mu} = \chi_{\Gamma\backslash E}$, so that ν is given by $f \mapsto f * \mu$. This is a demonstration of the fact that $M(G)$ is the multiplier algebra of $L^1(G)$.

We now turn to another construction of closed subalgebras of $L^1(G)$ that are amenable and yet lack property (**G**). Suppose $\mathfrak{A}$ is a commutative Banach algebra and H is a group of automorphisms of $\mathfrak{A}$. Put

$$\mathfrak{A}_H = \{a \in \mathfrak{A} \colon h(a) = a, (h \in H)\}$$

certainly $\mathfrak{A}_H$ is a closed subalgebra of $\mathfrak{A}$. We then have the following result, whose proof in this generality was kindly suggested by Professor B.E. Johnson.

2.3. PROPOSITION. *If $\mathfrak{A}$ is a commutative amenable Banach algebra and H is a finite group of automorphisms of $\mathfrak{A}$, then $\mathfrak{A}_H$ is an amenable Banach algebra.*

PROOF. Let $\{d_n\}_{n\in\Delta} \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$ be an approximate diagonal for $\mathfrak{A}$, and let H have identity $\imath$ and cardinality N. Put $K = \max_{h\in H} \|h\|$.

The group $H \times H$ can be made into a group of automorphisms on $\mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H$ via $(h_1, h_2)(a_1 \otimes a_2) = h_1(a_1) \otimes h_2(a_2)$ and then $\mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H = (\mathfrak{A} \hat{\otimes} \mathfrak{A})_{(H\times H)}$. For each $n \in \Delta$, put $d_n' = \dfrac{1}{N} \sum_{h\in H} (h, h)(d_n)$. Then $\{d_n'\}_{n\in\Delta}$ is an approximate diagonal for $\mathfrak{A}$ with $(h, h)(d_n') = d_n'$, for each $h \in H$; let $M = \sup_{n\in\Delta} \|d_n'\|$. Now put

$$d_n'' = e \otimes e - \prod_{h\in H} (e \otimes e - (h, \imath)(d_n')),$$

where this product is in the algebra $\mathfrak{A} \hat{\otimes} \mathfrak{A}$, and the term $e \otimes e$ plays a purely formal rôle as a multiplicative identity. It is clear that $\{d_n''\}_{n\in\Delta}$ is a bounded net in $\mathfrak{A} \hat{\otimes} \mathfrak{A}$. Moreover, if $(h_1, h_2) \in H \times H$, then

$$\begin{aligned}(h_1, h_2)(d_n'') &= e \otimes e - \prod_{h\in H} (e \otimes e - (h_1 h, h_2)(d_n')) \\ &= e \otimes e - \prod_{h\in H} (e \otimes e - (h_1 h h_2^{-1}, \iota)(d_n')) = d_n'',\end{aligned}$$

so that $d_n'' \in \mathfrak{A}_H \hat{\otimes} \mathfrak{A}_H$. Also, if $a \in \mathfrak{A}_H$ then

$$\begin{aligned}\|a - a\pi(d_n'')\| &= \left\| a \prod_{h\in H} \pi(e \otimes e - (h, \iota)(d_n')) \right\| \\ &\leqq \|a - a\pi(d_n')\| \prod_{h\in H\setminus\{\iota\}} \|e \otimes e - (h, \iota)(d_n')\| \\ &\leqq \|a - a\pi(d_n')\| (1 + KM)^{N-1} \\ &\to 0,\end{aligned}$$

so that $\{\pi(d_n'')\}_{n\in\Delta}$ is an approximate left identity for $\mathfrak{A}_H$. Finally, we have

$$d_n'' = \sum_{\emptyset \neq S \subseteq H} (-1)^{|S|} \prod_{h\in S} (h, \iota)(d_n'),$$

so if we let $S \mapsto h_S \in S$ be a choice function, then for each $a \in \mathfrak{A}_H$,

$$\begin{aligned}\|d_n'' \cdot a - a \cdot d_n''\| &\leqq \sum_{\emptyset \neq S \subseteq H} \|(h_S, \iota)d_n' \cdot a - a \cdot (h_S, \iota)d_n'\| \prod_{h\in S\setminus\{h_S\}} \|(h, \iota)\| \, \|d_n'\| \\ &\leqq \sum_{\emptyset \neq S \subseteq H} \|(h_S, \iota)\| \, \|d_n' \cdot a - h_S^{-1}(a) \cdot d_n'\| \, (KM)^{|S|-1} \\ &\leqq (2^N - 1)K \, \|d_n' \cdot a - a \cdot d_n'\| \, (KM)^{N-1} \\ &\to 0.\end{aligned}$$

Hence $\{d_n''\}_{n\in\Delta}$ is an approximate diagonal for $\mathfrak{A}_H$, and so $\mathfrak{A}_H$ is amenable.

So we see that if G is a locally compact Abelian group and H is a finite group of automorphisms of $L^1(G)$, then $L_H^1(G) = L^1(G)_H$ is amenable. To determine when $L_H^1(G)$ has property **(G)**, note that, by [6, Theorem 1], the automorphisms of $L^1(G)$ are characterized by the piecewise affine homeomorphisms of Γ. Hence we can consider H as a finite group of piecewise affine homeomorphisms $\Gamma \to \Gamma$. Then

$$\begin{aligned}L_H^1(G) &= \{f \in L^1(G): \hat{f} \circ h = \hat{f} \quad (h \in H)\} \\ &= \{f \in L^1(G): \hat{f} \text{ is constant on each orbit } H(\gamma)\}.\end{aligned}$$

So, applying Proposition 2.1, we see that $L_H^1(G)$ has property **(G)** if and only if there is a proper piecewise map α from $Y = \Gamma\backslash Z(L^1_H(G))$ into some other locally compact Abelian group such that $L^1_H(G) = \kappa(\alpha)$. However,

$$\kappa(\alpha) = \{f \in L^1(G): \hat{f}(\Gamma\backslash Y) = 0 \text{ and } \hat{f} \text{ is constant on each set } \alpha^{-1}\{\alpha(\gamma)\}\},$$

so it would seem that the partition of Γ into orbits $H(\gamma)$ is identical to the partition of Γ into sets on which α is constant. The following lemma delivers precisely this result.

2.4. Lemma. *Let v: $L^1(G') \to L^1(G)$ be a homomorphism between commutative group algebras with $Y \in \mathscr{R}(\Gamma)$ and α: $Y \to \Gamma'$ as above, and let H be a finite group of piecewise affine homeomorphisms of Γ. If* rng $v = L^1_H(G)$, *then $Y = \Gamma$ and for γ_1, $\gamma_2 \in \Gamma$, $\alpha(\gamma_1) = \alpha(\gamma_2) \Leftrightarrow H(\gamma_1) = H(\gamma_2)$.*

Proof. For each $\gamma \in \Gamma$, $H(\gamma)$ is finite, and since $L^1(G)^\wedge$ separates points of Γ, there exists $f \in L^1(G)$ with $\hat{f}(H(\gamma)) = \{1\}$. Put $\tilde{f} = \frac{1}{|H|}\sum_h \hat{f} \circ h$, then $\tilde{f} \in (L^1_H(G))^\wedge$ and $\tilde{f}(\gamma) = 1$. Hence $\gamma \in \Gamma\backslash Z(L^1_H(G)) = Y$, so $Y = \Gamma$.

Now suppose $\gamma_1, \gamma_2 \in \Gamma$ are such that $H(\gamma_1) = H(\gamma_2)$. For each $f \in L^1(G')$, $v(f) \in L^1_H(G)$, so $v(f)(\gamma_1) = v(f)(\gamma_2)$. Thus $\hat{f}(\alpha(\gamma_1)) = \hat{f}(\alpha(\gamma_2))$, and since $A(\Gamma')$ separates points, $\alpha(\gamma_1) = \alpha(\gamma_2)$.

On the other hand, if $H(\gamma_1) \neq H(\gamma_2)$, then $H(\gamma_1)$ and $H(\gamma_2)$ are finite disjoint sets, so there exists $f \in L^1(G)$ with $\hat{f}(H(\gamma_1)) = \{0\}$ and $\hat{f}(H(\gamma_2)) = \{1\}$. Then $\tilde{f} = \frac{1}{|H|}\sum_h \hat{f} \circ h \in (L^1_H(G))^\wedge = (\kappa(\alpha))^\wedge$ and $\tilde{f}(\gamma_1) \neq \tilde{f}(\gamma_1)$, so $\alpha(\gamma_1) \neq \alpha(\gamma_2)$.

We now use the above to characterize property (**G**) in algebras $L^1_H(G)$ in the case where H is a finite group of automorphisms of Γ. This is a natural situation to consider, as we then have a finite group of automorphisms on G, given by H^*, the group of adjoints of elements of H. Then $L^1_H(G) = \{f \in L^1(G): f \circ h^* = f, (h^* \in H^*)\}$, which is $L^1(G^{H^*})$, a convolution algebra on the *orbit hypergroup* $G^{H^*} = \{H^*(g): g \in G\}$. The amenability of hypergroups and hypergroup algebras is studied further in [23].

We will need a stronger characterization of the terms "coset ring" and "piecewise affine". For more details of this, see [14, Section 2]. Define $\mathscr{R}_0(\Gamma)$ to be the subset of $\mathscr{R}(\Gamma)$ of sets of the form $S = E_0\backslash(\bigcup_1^m E_k)$, where $E_0, \ldots, E_m$ are clopen cosets in Γ and each of $E_1, \ldots, E_m$ is a subcoset of infinite index in E_0. Then E_0 is the coset generated by S, which we denote $E_0 = E_0(S)$. Also, any member of $\mathscr{R}(\Gamma)$ can be represented as a finite disjoint union of elements of $\mathscr{R}_0(\Gamma)$, and so we can suppose that in the definition of *piecewise affine*, each S_k is in $\mathscr{R}_0(\Gamma)$ and each α_k has domain $E_0(S_k)$.

In the situation where $\kappa(\alpha) = L^1_H(G)$, we have seen that we have $Y = \Gamma$. The following lemmas allow us to obtain further special properties of such a piecewise affine map.

2.5. LEMMA. *Suppose G is an Abelian group and $E_1,\dots,E_n$ are cosets in G such that $G = \bigcup_1^n E_k$, then for some $1 \leqq k \leqq n$, E_k is a subgroup of finite index in G.*

PROOF. Without loss, we have that for some $0 \leqq m \leqq n$, $E_1,\dots,E_m$ are subcosets of finite index in G and $E_{m+1},\dots,E_n$ are subcosets of infinite index in G. For $1 \leqq k \leqq m$, let H_k be the subgroup $E_k E_k^{-1} \subseteq G$, then H_k is of finite index in G, and so $H = \bigcap_1^m H_k$ is of finite index in G. (If $m = 0$, put $H = G$.) For each $k > m$, $H \cap E_k$ is empty or a coset of infinite index in H, so by [22, Theorem 4.3.3], $\bigcup_{m+1}^n (H \cap E_k)$ is a proper subset of H. However, $H = \bigcup_1^n (H \cap E_k)$, so for some $k \leqq m$, $H \cap E_k \neq \emptyset$, so that $H_k \cap E_k \neq \emptyset$. Hence $H_k = E_k$, and we are done.

2.5.1. COROLLARY. *Suppose Γ_1, Γ_2 are locally compact Abelian groups and $\alpha\colon \Gamma_2 \to \Gamma_1$ is a piecewise affine map. Then there is a set $S \in \mathscr{R}_0(\Gamma_2)$ such that $E_0(S)$ is a subgroup of finite index in Γ_2 and $\alpha|_S$ has a continuous affine extension $\alpha_0\colon E_0(S) \to \Gamma_1$. Further, if α is proper, then so is α_0.*

PROOF. By the discussion above, that is, [14, Lemmas 2.1 & 3.1].

2.6. LEMMA. *Suppose $S \in \mathscr{R}_0(\Gamma)$ is such that $E_0(S)$ a subgroup of finite index in Γ, and H is a finite group of automorphisms of Γ. Then $\tilde{S} = \bigcap_{h \in H} h(S) \in \mathscr{R}_0(\Gamma)$ and $E_0(\tilde{S}) = \bigcap_{h \in H} h(E_0(S))$ is a subgroup of finite index in Γ.*

PROOF. Suppose $S = E_0 \backslash (\bigcup_1^m E_k)$, as in the definition of $\mathscr{R}_0(\Gamma)$, and put $\tilde{E}_0 = \bigcap_{h \in H} h(E_0)$. Each of $\{h(E_0)\colon h \in H\}$ is a subgroup of finite index in Γ, so $\tilde{E}_0$ is a subgroup of finite index in Γ. Also, $\tilde{S} = \tilde{E}_0 \backslash (\bigcup_{h \in H} \bigcup_1^m (h(E_k) \cap \tilde{E}_0))$, with each $h(E_k) \cap \tilde{E}_0$ being empty or of infinite index in $\tilde{E}_0$. Hence $\tilde{S} \in \mathscr{R}_0(\Gamma)$ and $E_0(\tilde{S}) = \tilde{E}_0$.

2.6.1. COROLLARY. *With Γ_1, Γ_2 and $\alpha\colon \Gamma_2 \to \Gamma_1$ as in Corollary 2.5.1, if H is a finite group of automorphisms of Γ_2, then we can obtain S with the additional properties that $h(S) = S$ and $h(E_0(S)) = E_0(S)$, for each $h \in H$.*

In the following theorem, we will use the natural generalizations of [6, Theorem 1] and [14, Theorem A] to the situation where we have a homomorphism between two algebras, each a finite direct sum of commutative group algebras.

Suppose we have $\mathfrak{A} = L^1(G_1) \oplus \cdots \oplus L^1(G_n)$, where $G_1,\dots,G_n$ are locally compact Abelian groups. We can naturally identify $\Phi_{\mathfrak{A}}$ with $\Gamma_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \Gamma_n$, the disjoint union of the duals of $G_1,\dots,G_n$. We can also define the coset ring of $\Gamma_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \Gamma_n$, denoted $\mathscr{R}(\Gamma_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \Gamma_n)$, to be

$$\{Y \subseteq \Gamma_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} \Gamma_n\colon Y \cap \Gamma_k \in \mathscr{R}(\Gamma_k) \quad (1 \leqq k \leqq n)\},$$

which happens to be the Boolean ring generated by all the open cosets of each of

$\Gamma_1, \ldots, \Gamma_n$. Similarly, for $G'_1, \ldots, G'_m$ locally compact Abelian groups, and $Y \in \mathscr{R}(\Gamma'_1 \cup \cdots \cup \Gamma'_m)$, we can define a map $\alpha: Y \to \Gamma_1 \cup \cdots \cup \Gamma_n$ to be piecewise affine if we can partition Y into sets $\{Y_{jk}: 1 \leqq j \leqq m, 1 \leqq k \leqq n\}$ such that for each j, k, $Y_{jk} \in \mathscr{R}(\Gamma'_j)$, $\alpha(Y_{jk}) \subseteq \Gamma_k$, and $\alpha_{jk} = \alpha|_{Y_{jk}}: Y_{jk} \to \Gamma_k$ is piecewise affine.

With such notation, it is elementary to show that a homomorphism v from $\mathfrak{A} = L^1(G_1) \oplus \cdots \oplus L^1(G_n)$ into $\mathfrak{B} = L^1(G'_1) \oplus \cdots \oplus L^1(G'_m)$ is uniquely determined by the proper piecewise affine map $v^*|_Y$, where $Y = \Phi_{\mathfrak{B}} \backslash Z(\operatorname{rng} v)$. Moreover, the proof of [14, Theorem A] generalizes naturally to considering such homomorphisms, giving the conclusion $\operatorname{rng} v = \kappa(\alpha)$. This is merely an extension of the observation made in Section 4 of [14] regarding homomorphisms $L^1(G) \to L^1(G_1) \oplus \cdots \oplus L^1(G_n)$.

2.7. THEOREM. *Suppose H is a finite group of automorphisms of a locally compact Abelian group Γ. Then the following are equivalent:*

(i) *$L^1_H(G)$ has property* (**G**),

(ii) *the subgroup $\Lambda = \{\gamma \in \Gamma: H(\gamma) = \{\gamma\}\}$ is of finite index in Γ, and*

(iii) *$L^1_H(G)$ is isomorphic to a finite direct sum of group algebras.*

PROOF. Supposing (i), then by Proposition 2.1 and Lemma 2.4, there is a locally compact Abelian group G' and a proper piecewise affine map $\alpha: \Gamma \to \Gamma'$ such that the level sets of α are precisely the orbits of the action of H on Γ. By Corollary 2.6.1, there exists $S \in \mathscr{R}_0(\Gamma)$ such that $E_0(S)$ is a subgroup of finite index in Γ, $\alpha|_S$ has a proper continuous affine extension $\alpha_0: E_0(S) \to \Gamma'$, and for each $h \in H$, $h(S) = S$ and $h(E_0(S)) = E_0(S)$.

Now, $\alpha \circ h = \alpha$, for each $h \in H$, so $\Lambda_0 = \{\gamma \in E_0(S): \alpha_0 \circ h(\gamma) = \alpha_0(\gamma), (h \in H)\}$ is a subgroup of $E_0(S)$ with $S \subseteq \Lambda_0$. Since $E_0(S)$ is the coset generated by S, we have that $\Lambda_0 = E_0(S)$, and so $\alpha_0 \circ h = \alpha_0$, $(h \in H)$. Put $\Xi = \{\gamma \in E_0(S): \alpha_0(\gamma) = \alpha_0(e)\} = \alpha_0^{-1}\{\alpha_0(e)\}$, a subgroup of $E_0(S)$. For each $\gamma \in S$, $H(\gamma) \subseteq S$, so

$$\begin{aligned} \gamma' \in H(\gamma) &\Leftrightarrow \alpha(\gamma') = \alpha(\gamma) \\ &\Leftrightarrow \alpha_0(\gamma') = \alpha_0(\gamma) \text{ and } \gamma' \in S, \end{aligned}$$

so $H(\gamma) = \gamma\Xi \cap S$. Thus $\{\gamma \in E_0(S): H(\gamma) \subseteq \gamma\Xi\}$, a subgroup of $E_0(S)$, contains S. It follows that $H(\gamma) \subseteq \gamma\Xi$ for all $\gamma \in E_0(S)$. For each $h \in H$, let $\tilde{h}: E_0(S) \to \Xi$ be the homomorphism defined by $\tilde{h}(\gamma) = h(\gamma)\gamma^{-1}$, so that $\Lambda = \bigcap_{h \in H} \tilde{h}^{-1}\{e\}$. It remains to be proven that Ξ is finite, for then each $\tilde{h}^{-1}\{e\}$ is of finite index in $E_0(S)$, which is in turn of finite index in Γ.

By [14, Lemma 2.2], there exists $\gamma_1, \ldots, \gamma_N \in E_0(S)$ such that $E_0(S) = \bigcup_1^N \gamma_k S$, giving

$$\Xi = \Xi \cap E_0(S) = \bigcup_{1 \leqq k \leqq N} \gamma_k(\gamma_k^{-1}\Xi) \cap S) = \bigcup_{1 \leqq k \leqq N} \gamma_k H(\gamma_k^{-1}),$$

which is evidently finite.

Now assume (ii). For each coset $\gamma\Lambda$ of Λ, and each $h \in H$, $h(\gamma\Lambda)$ is the coset $h(\gamma)\Lambda$, so that H acts on Γ/Λ. Let $H(\gamma_1\Lambda), \ldots, H(\gamma_N\Lambda)$ be the orbits of this action, and for each $1 \leqq k \leqq N$, let $h_{k,1}, \ldots, h_{k,n_k} \in H$ be such that the cosets of Λ that make up $H(\gamma_k\Lambda)$ are $\{h_{k,j}(\gamma_k\Lambda): 1 \leqq j \leqq n_k\}$.

For each $1 \leqq k \leqq N$, $H_k = \{h \in H: h(\gamma_k) \in \gamma_k\Lambda\}$ is a subgroup of H, and $\Lambda_k = \{h(\gamma_k)\gamma_k^{-1}: h \in H_k\}$ is a subgroup of Λ. Furthermore, H_k acts on $\gamma_k\Lambda$ by $h(\gamma_k\lambda) = (\gamma_k\lambda)\cdot(h(\gamma_k)\gamma_k^{-1})$, that is, by translations by elements of Λ_k. For $1 \leqq j \leqq n_k$, define $\alpha_{kj}: h_{kj}(\gamma_k)\Lambda \to \Lambda/\Lambda_k$ by $\alpha_{kj}(h_{kj}(\gamma_k)\lambda) = \lambda\Lambda_k$. This is continuous and affine, and since $\alpha_{kj}^{-1}(\lambda\Lambda_k) = h_{kj}(\gamma_k)\lambda\Lambda_k$ is finite, α_{kj} is also proper.

Each coset of Λ in Γ is of the form $h_{kj}(\gamma_k)\Lambda$, for some unique k and j, so we can define a proper piecewise affine map $\alpha: \Gamma \to \Lambda/\Lambda_1 \cup \cdots \cup \Lambda/\Lambda_N$ by "piecing together" all the α_{kj}. For each $\gamma \in \Gamma$, say $\gamma = h_{kj}(\gamma_k)\lambda$, we have $H(\gamma) = H(h_{kj}(\gamma_k\lambda)) = H(\gamma_k\lambda)$. Also $\alpha_{kj}^{-1}(\lambda\Lambda_k) = h_{kj}(\gamma_k\lambda\Lambda_k) = h_{kj}(H_k(\gamma_k\lambda))$. Hence

$$\alpha^{-1}\{\alpha(\gamma)\} = \bigcup_{1 \leqq j \leqq n_k} \alpha_{kj}^{-1}(\lambda\Lambda_k) = \bigcup_{1 \leqq j \leqq n_k} h_{kj}(H_k(\gamma_k\lambda)) = H(\gamma_k\lambda) = H(\gamma),$$

and as this holds for each $\gamma \in \Gamma$, $\kappa(\alpha) = L_H^1(G)$. Now, by the extension of Cohen's characterization of group algebra homomorphisms, as outlined above, α determines a homomorphism $v: A(\Lambda/\Lambda_1) \oplus \cdots \oplus A(\Lambda/\Lambda_N) \to A(\Gamma)$ with range $\kappa(\alpha)$. Also, $\ker v = \mathscr{I}(\mathrm{rng}(\alpha))$ and since α is surjective, we have that v is a monomorphism. Hence $A_H(\Gamma) = \kappa(\alpha) \cong A(\Lambda/\Lambda_1) \oplus \cdots \oplus A(\Lambda/\Lambda_N)$.

The last implication (iii) $\Rightarrow$ (i) follows from Proposition 1.1.

So we see that the amenable algebras of the form $L_H^1(G)$ will usually fail to have property (**G**). For instance, if Γ is connected, then for $L_H^1(G)$ to have property (**G**), we must have $\Lambda = \Gamma$, and so $H = \{\iota\}$ and $L_H^1(G) = L^1(G)$.

If G is a locally compact Abelian group, we always have the automorphism η on G given by $x \mapsto x^{-1}$. (Although occasionally we have $\eta = \iota$, as we will see.) Then $H = \{\iota, \eta\}$ is a finite group of automorphisms of G and $L_H^1(G) = L_{\mathrm{sym}}^1(G)$, the subalgebra of symmetric (or even) functions in $L^1(G)$. We now apply the preceding theorem to this case

2.8. Theorem. *If G is a locally compact Abelian group, the following are equivalent:*

(i′) $L_{\mathrm{sym}}^1(G)$ *has property* (**G**),

(ii′) $G \cong \sum_{\mathfrak{a}} \mathbf{Z}_2 \times \prod_{\mathfrak{b}} \mathbf{Z}_2 \times F$, *for some cardinals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *and some finite group* F, *and*

(iii′) $L_{\mathrm{sym}}^1(G)$ *is isomorphic to a group algebra.*

Proof. Suppose (i′), then we have by Theorem 2.7 that $\Lambda = \{\gamma \in \Gamma: H(\gamma) = \{\gamma\}\}$ is of finite index in Γ, say $|\Gamma/\Lambda| = N$. Then $\gamma \in \Gamma \Rightarrow \gamma^N \in \Lambda \Rightarrow \gamma^N = \gamma^{-N} \Rightarrow \gamma^{2N} = e$, so that Γ is of bounded order. Thus, by [12, Theorem A.25], there is an *algebraic* isomorphism $\psi\colon \Gamma_{2,d} \to \sum_{i\in \mathbf{I}} \mathbf{Z}_{n_i}$, where $\mathbf{I}$ is an index set and $\{n_i: i \in \mathbf{I}\}$ is a bounded set of integers greater than 2. Then $\{\gamma \in \Gamma: \gamma^2 = e\} = \Lambda$ is of finite index, so $F = \psi^{-1}(\sum_{n_i > 2} \mathbf{Z}_{n_i})$ is a finite subgroup of Γ with $(\Gamma/F)_d \cong \sum_{n_i = 2} \mathbf{Z}_2$.

Let Λ_0 be a compact open subgroup of Γ, which we can assume to contain F. If we now apply the argument of the above paragraph to $\hat{\Lambda}_0$, we obtain that $\Lambda_0 \cong F \times \prod_{\mathfrak{a}} \mathbf{Z}_2$, for some cardinal $\mathfrak{a}$. By continuing with an argument similar to that used in [12, 25.29], or by a straightforward application of Zorn's Lemma, we can obtain a complement to Λ_0, which will be isomorphic to $\sum_{\mathfrak{b}} \mathbf{Z}_2$ for some cardinal $\mathfrak{b}$, giving $G \cong F \times \sum_{\mathfrak{a}} \mathbf{Z}_2 \times \prod_{\mathfrak{b}} \mathbf{Z}_2$.

For the implication (ii′) ⇒ (iii′), we show that for $G = \sum_{\mathfrak{a}} \mathbf{Z}_2 \times \prod_{\mathfrak{b}} \mathbf{Z}_2 \times F$, $L^1_{\mathrm{sym}}(G)$ is isomorphic to a group algebra. Let $H = \sum_{\mathfrak{a}} \mathbf{Z}_2 \times \prod_{\mathfrak{b}} \mathbf{Z}_2$, so that $H^{(2)} = \{e\}$ and $G = H \times F$. Let $\Psi\colon L^1(G) \to L^1(H) \hat{\otimes} l^1(F)$ be the natural isomorphism. It is easily verified that $\Psi(L^1_{\mathrm{sym}}(G)) = L^1(H) \hat{\otimes} l^1_{\mathrm{sym}}(F)$. Now, $l^1_{\mathrm{sym}}(F)$ is a finite-dimensional commutative semisimple Banach algebra, so $l^1_{\mathrm{sym}}(F) \cong \mathbf{C}^m \cong l^1(\mathbf{Z}_m)$, where $m = \dim(l^1_{\mathrm{sym}}(F))$. Consequently $L^1_{\mathrm{sym}}(G) \cong L^1(H) \hat{\otimes} l^1(\mathbf{Z}_m) \cong L^1(H \times \mathbf{Z}_m)$.

The final implication (iii′) ⇒ (i′) is trivial.

In light of the conclusion (iii′) in Theorem 2.8, it is natural to ask whether we can reach the same conclusion in Theorem 2.7. We will give an example of an Abelian group G with a finite group of automorphisms H such that $l^1_H(G)$ has property **(G)**, but is not isomorphic to a group algebra.

Let U and V be as constructed in [15, p. 616–7]. That is, U is a countably infinite torsion-free Abelian group and V is a non-isomorphic subgroup that is of index 2 in U. Let $\Upsilon = \hat{U}$, a connected compact Abelian group, then $\Xi = \mathrm{Ann}_\Upsilon(V)$ is a two-element group, say $\Xi = \{e, \xi\}$, and $\Upsilon/\Xi = \hat{V}$ is also compact and connected. Put $G = U \times \mathbf{Z}_2$, so that $\Gamma = \Upsilon \times \mathbf{Z}_2$, and define $\eta \in \mathrm{Aut}(\Gamma)$ by $\eta(v, 0) = (v, 0)$ and $\eta(v, 1) = (v\xi, 1)$. Then $\eta^2 = \iota$, so $H = \{\iota, \eta\}$ is a finite group of automorphisms of G which clearly satisfies the criterion (ii) in Theorem 2.7. It then follows that $l^1_H(G)$ is isomorphic to a finite direct sum of groups algebras, and by applying the construction in the proof of Theorem 2.7, we obtain $l^1_H(G) \cong l^1(U) \oplus l^1(V)$, which has maximal ideal space $\Upsilon \cup \Upsilon/\Xi$.

Suppose $l^1(U) \oplus l^1(V)$ is isomorphic to a group algebra $L^1(G')$, so that there exists a piecewise affine homeomorphism $\alpha\colon \Upsilon \cup \Upsilon/\Xi \to \Gamma'$. Thus Γ' has two connected components, which are necessarily affinely homeomorphic. It follows that Υ and Υ/Ξ are topologically isomorphic, and so U and V are isomorphic. (Contradiction.)

Many thanks to Dr Laci Kovács for suggesting the group U used in this example.

3. A Non-commutative Amenable Banach Algebra Without Property (G).

In this section we examine some amenable Banach algebras which we show to lack property (G) by methods entirely different to those in Section 2. For the results presented in this section, I am indebted to a suggestion of U. Haagerup, and its communication through P.C. Curtis and George Willis.

3.1. LEMMA. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are unital Banach algebras and $\mathscr{I}$ is a closed left ideal of $\mathfrak{A}$ with a left approximate identity $\{e_n\}_{n\in\Delta}$, bounded by $M > 0$. If $v\colon \mathscr{I} \to \mathfrak{B}$ is a continuous homomorphism with* $\operatorname{rng} v \cap \mathfrak{B}^{-1} \neq \emptyset$, *then there is a unique homomorphism $\tilde{v}\colon \mathfrak{A} \to \mathfrak{B}$ extending v. Moreover, $\tilde{v}(e) = e = \lim_{n\in\Delta} v(e_n)$, $\overline{v(\mathscr{I})} = \overline{\tilde{v}(\mathfrak{A})}$, and $\|\tilde{v}\| \leqq M\,\|v\|$.*

PROOF. Suppose $a\in\mathscr{I}$ is such that $v(a)\in\mathfrak{B}^{-1}$, then any homomorphism $\tilde{v}\colon \mathfrak{A} \to \mathfrak{B}$ extending v must satisfy $\tilde{v}(x) = v(xa)[v(a)]^{-1}$, $(x\in\mathfrak{A})$. Define $\tilde{v}$ to be exactly this. Then $\tilde{v}$ is a continuous linear extension of v with $\tilde{v}(e) = e$.

For each $n\in\Delta$, and each $x\in\mathfrak{A}$,

$$v(xe_n) - \tilde{v}(x) = v(x(e_n a - a))[v(a)]^{-1} \to 0$$

so $\tilde{v}(x) = \lim_{n\in\Delta} v(xe_n)$. Hence $\tilde{v}(\mathfrak{A}) \subseteq \overline{v(\mathscr{I})}$, $\|\tilde{v}\| \leqq M\,\|v\|$, and $e = \tilde{v}(e)$ $\lim_{n\in\Delta} v(e_n)$. Also, if $x, y\in\mathfrak{A}$, then $ya\in\mathscr{I}$, so $v(xya) = \lim_{n\in\Delta} v(xe_n ya)$. However, $v(xya) = \tilde{v}(xy)v(a)$ and $\lim_{n\in\Delta} v(xe_n ya) = [\lim_{n\in\Delta} v(xe_n)]v(ya) = \tilde{v}(x)\tilde{v}(y)v(a)$, and since $v(a)\in\mathfrak{A}^{-1}$, we have that $\tilde{v}(xy) = \tilde{v}(x)\tilde{v}(y)$, as desired.

3.2. PROPOSITION. *Suppose $\mathfrak{A}$ is a Banach algebra with unit e, G is a locally compact group, and $v\colon L^1(G) \to \mathfrak{A}$ is a continuous homomorphism. If* $\operatorname{rng} v \cap \mathfrak{A}^{-1} \neq \emptyset$, *then v has a unique extension to a homomorphism $\tilde{v}\colon M(G) \to \mathfrak{A}$. Further, $\|\tilde{v}\| = \|v\|$ and $\overline{v(L^1(G))} = \overline{\tilde{v}(l^1(G))} = \overline{\tilde{v}(M(G))}$.*

PROOF. Let Δ be the set of compact neighbourhoods of $e\in G$, and order Δ by $\supseteq$. For each $U\in\Delta$, take $e_U\in C_{00}^{+}(G)$ with support within U and $\|e_U\| = 1$. Then $\{e_U\}_{U\in\Delta}$ is a bounded approximate identity for $L^1(G)$, a closed ideal of $M(G)$. Hence, by Lemma 3.1, v has a unique extension to a homomorphism $\tilde{v}\colon M(G) \to \mathfrak{A}$, with $e = \tilde{v}(\delta_e) = \lim_{U\in\Delta} v(e_U)$, $\|\tilde{v}\| = \|v\|$, and $\overline{v(L^1(G))} = \overline{\tilde{v}(M(G))}$.

Since $\overline{\tilde{v}(l^1(G))} \subseteq \overline{\tilde{v}(M(G))}$, it remains to be proven that $v(L^1(G)) \subseteq \overline{\tilde{v}(l^1(G))}$. For this it suffices to prove that $v(C_{00}^{+}(G)) \subseteq \overline{\tilde{v}(l^1(G))}$.

For this we can use a portion of the proof of existence and uniqueness of Haar measure, as given in [12, 15.5–6]. (As is done in [24, Lemma 2.1].) This states that for $f\in C_{00}^{+}(G)$ and $\varepsilon > 0$, there exists $U\in\Delta$ such that if $g\in C_{00}^{+}(G)$ is zero off U with $\|g\| = 1$, then there exists $h\in l^1(G)$ with $\|h\| \leqq \|f\|$ and $\|f - h * g\| < \varepsilon$. Take $V\in\Delta$ with $V \subseteq U$, and $\|v(e_V) - e\| < \varepsilon$. Then $e_V\in C_{00}^{+}(G)$ is zero off U, so we can take $h\in l^1(G)$ with $\|f - h * e_V\| < \varepsilon$. Then

$$\begin{aligned}\|v(f) - \tilde{v}(h)\| &\leqq \|v(f - h * e_V)\| + \|\tilde{v}(h)(v(e_V) - e)\| \\ &\leqq \|v\| \, \|f - h * e_V\| + \|\tilde{v}\| \, \|h\| \, \|v(e_V) - e\| \\ &< \|v\| \, \varepsilon + \|v\| \, \|f\| \, \varepsilon.\end{aligned}$$

Hence $v(f) \in \overline{\tilde{v}(l^1(G))}$.

In the following, $\mathscr{Z}(\mathfrak{A})$ is the *centre* of $\mathfrak{A}$, that is, $\mathscr{Z}(\mathfrak{A}) = \{a \in \mathfrak{A}: ab = ba, (b \in \mathfrak{A})\}$.

3.3. THEOREM. *Suppose* $\mathfrak{A}$ *is a unital Banach algebra with property* (**G**). *Then*

$$\overline{\text{span}}\{ab - ba: a, b \in \mathfrak{A}\} \cap \mathscr{Z}(\mathfrak{A}) = \{0\}.$$

PROOF. Let G be an amenable locally compact group and $v: L^1(G) \to \mathfrak{A}$ be a dense-ranged homomorphism. Then $\mathfrak{A}^{-1}$ is open, so $\operatorname{rng} v \cap \mathfrak{A}^{-1} \neq \varnothing$, and we can apply Lemma 3.2 to obtain an extension $\tilde{v}: M(G) \to \mathfrak{A}$ with $\|\tilde{v}\| = \|v\|$, $\mathfrak{A} = \overline{\tilde{v}(l^1(G))}$ and $\tilde{v}(\delta_e) = e$. Then

$$\begin{aligned}\overline{\text{span}}\{ab - ba: a, b \in \mathfrak{A}\} &= \overline{\text{span}}\{a\tilde{v}(f) - \tilde{v}(f)a: a \in \mathfrak{A}, f \in l^1(G)\} \\ &= \overline{\text{span}}\{a\tilde{v}(\delta_x) - \tilde{v}(\delta_x)a: a \in \mathfrak{A}, ax \in G\} \\ &= \overline{\text{span}}\{\tilde{v}(\delta_{x^{-1}})a\tilde{v}(\delta_x) - a: a \in \mathfrak{A}, x \in G\}.\end{aligned}$$

Thus it suffices to show that for each $z \in \mathscr{Z}(\mathfrak{A})$, there is an element of $\mathfrak{A}^*$ that annihilates each $\tilde{v}(\delta_{x^{-1}})a\tilde{v}(\delta_x) - a$, but not z.

Take $z \in \mathscr{Z}(\mathfrak{A})$. Let $\psi \in \mathfrak{A}^*$ be such that $\psi(z) \neq 0$ and $\|\psi\| < 1$. For each $a \in \mathfrak{A}$, define the function ψ_a on G by $\psi_a(x) = \psi(\tilde{v}(\delta_{x^{-1}})a\tilde{v}(\delta_x))$, $(x \in G)$. Then $\sup_{x \in G} |\psi_a(x)| \leqq \|v\|^2 \, \|a\|$, so $\psi_a \in l^\infty(G)$. Define $\Psi: \mathfrak{A} \to l^\infty(G)$ by $\Psi(a) = \psi_a$. Then Ψ is linear with $\|\Psi\| \leqq \|v\|^2$. If $a \in \operatorname{rng} v$, say $a = v(f)$, then for each $x \in G$, $\psi_a(x) = \psi \circ v(\delta_{x^{-1}} * f * \delta_x)$, so $\psi_a \in C_b(G)$. Hence $\Psi(\operatorname{rng} v) \subseteq C_b(G)$, a closed subalgebra of $l^\infty(G)$, and since Ψ is continuous, $\Psi(\mathfrak{A}) \subseteq C_b(G)$.

Now, if $a \in \mathfrak{A}$ and $x, y \in G$, then

$${}_y(\Psi(a))(x) = \psi_a(yx) = \psi(\tilde{v}(\delta_{x^{-1}})\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y)\tilde{v}(\delta_x)) = \Psi(\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y))(x),$$

so that if M is a left-invariant mean on $C_b(G)$, then

$$M \circ \Psi(a) = M({}_y\Psi(a)) = M \circ \Psi(\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y)).$$

Hence $M \circ \Psi \in \mathfrak{A}^*$ annihilates each $\tilde{v}(\delta_{y^{-1}})a\tilde{v}(\delta_y) - a$. But $\psi_z(x) = \psi(\tilde{v}(\delta_{x^{-1}}) z\tilde{v}(\delta_x)) = \psi(\tilde{v}(\delta_{x^{-1}})\tilde{v}(\delta_x)z) = \psi(z)$ is the constant function $\psi(z)$. Hence $M \circ \Psi(z) = \psi(z) \neq 0$.

Suppose H is a separable Hilbert space, n is an integer greater than 2, and $H_1, \ldots, H_n$ are orthogonal closed infinite-dimensional subspaces with $H_1 + \cdots +$

$H_n = H$. For each $1 \leqq k \leqq n$, let S_k be a linear isometry $H \to H_k$. Then $S_1, \ldots, S_n \in \mathfrak{B}(H)$ and $I = S_1^* S_1 = \cdots = S_n^* S_n = S_1 S_1^* + \cdots + S_n S_n^*$. Let $\mathcal{O}_n$ be the C^*-algebra generated by $S_1, \ldots, S_n$, which we call the *Cuntz algebra on n generators.* This algebra was introduced in [7], where it is shown not to depend on the actual isometries $S_1, \ldots, S_n$ chosen, but only on n. In [21], it is shown that the Cuntz algebras are amenable. However,

$$(S_1^* S_1 - S_1 S_1^*) + \cdots + (S_n^* S_n - S_n S_n^*) = (n-1)I \in \mathscr{L}(\mathcal{O}_n),$$

so we see that $\mathcal{O}_n$ cannot have property (**G**).

This seems related to other properties of the Cuntz algebras related to amenability. In particularly, the Cuntz algebras are amenable, but not strongly amenable. (Strong amenability is a property of C^*-algebras defined in [13]. The Cuntz algebras were shown to not be strongly amenable in [21].)

Suppose $\mathfrak{A}$ is a C^*-subalgebra of $\mathfrak{B}(H)$, and $v: L^1(G) \to \mathfrak{A}$ is a homomorphism with $\operatorname{rng} v \cap \mathfrak{A}^{-1} \neq \emptyset$. By Proposition 3.2, we have a homomorphism $l^1(G) \to \mathfrak{A}$, which gives a continuous representation $\pi: G \to \mathfrak{B}(H)$ with $\pi(x) \leqq \|v\|$, for each $x \in G$. (cf. [18, p. 77].) Then by [18, Corollary 17.6], π is equivalent to a unitary representation, that is, there is an isomorphism $\Psi: H \to H$ such that $\pi': x \mapsto \Psi^{-1}\pi(x)\Psi$ is a continuous representation of G such that each $\pi'(x)$ is unitary. Then, by [13, Proposition 7.8], $\mathfrak{A}' = \overline{\pi(l^1(G))}$ is a strongly amenable Banach algebra. Moreover $\overline{\pi(l^1(G))} = \overline{\Psi^{-1}\pi(l^1(G))\Psi} = \Psi^{-1}\mathfrak{A}\Psi$. Now, if strong amenability was preserved by such a transformation, then we could conclude that $\mathfrak{A}$ is strongly amenable. Unfortunately, this avenue is not open to us.

4. Other Constructions Preserving Amenability.

Having demonstrated that property (**G**) falls short of providing a characterization of amenability, it is natural to ask whether other stability properties of amenability can be used to provide a "constructive" characterization of amenability in Banach algebras.

For this, define a Banach algebra $\mathfrak{A}$ to have *property* (**G**$'$) if there are closed subalgebras $\{0\} = \mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots \subset \mathfrak{A}_n = \mathfrak{A}$ such that for each $1 \leqq k \leqq n$, $\mathfrak{A}_{k-1}$ is a closed ideal of $\mathfrak{A}_k$ and $\mathfrak{A}_k/\mathfrak{A}_{k-1}$ has property (**G**). A repeated application of [13, Proposition 5.1] demonstrates the amenability of such $\mathfrak{A}$. It is also a simple matter to show, using the fact that each $\mathfrak{A}_k$ factorizes, that each $\mathfrak{A}_k$ is an ideal of $\mathfrak{A}$.

Furthermore, algebras such as $L^1(G)^\#$, where G is a nondiscrete locally compact Abelian group, are easily shown to have property (**G**$'$) while lacking property (**G**). (To show the latter, consider $L^1(G)^\#$ as the closed subalgebra $L^1(G) + \mathbf{C}\delta_e$ of $M(G)$. This can be shown to lack property (**G**) by a simple application of Cohen's characterization of homomorphisms $L^1(G_1) \to M(G_2)$, [6, Theorem 1].)

Unfortunately, examples we have already seen are sufficient to show that property (**G**′) is not necessary for amenability. For instance, if $n \geqq 2$, then the Cuntz algebra $\mathcal{O}_n$ is simple – it has no nontrivial ideals, closed or otherwise. Hence the above chain of ideals could only be $\{0\} = \mathfrak{A}_0 \subset \mathfrak{A}_1 = \mathcal{O}_n$, and since $\mathcal{O}_n$ lacks property (**G**), it lacks property (**G**′). Also, if H is a group of automorphisms of R^n with $2 \leqq |H| < \infty$, then by [14, Corollary 1.6.2], $L^1_H(\mathsf{R}^n)$ has no subalgebra with property (**G**), and so $L^1_H(\mathsf{R}^n)$ cannot have property (**G**′).

This last example can also be used to show that similar attempts to use other constructions that preserve amenability will also fail. In particular, it is possible to show that if $\{\mathfrak{A}_n\}_{n \in \Delta}$ is a net of amenable closed subalgebras of $\mathfrak{A}$ with union dense in $\mathfrak{A}$, and the approximate diagonals of the $\mathfrak{A}_n$ have a common bound, then $\mathfrak{A}$ is amenable. (This is similar to the construction in [19, Proposition 1.12]. It can be shown to be equivalent.) Thus we can define a property ($\mathbf{G}^\infty$) to be that of having such a net of closed subalgebras, each having property (**G**). As already noted, this cannot occur in $L^1_H(\mathsf{R}^n)$. It can also be shown, by quite different methods, that the Cuntz algebras and many of the closed ideals of commutative group algebras also lack this property. (In fact, the author's PhD thesis presented a characterization of property ($\mathbf{G}^\infty$) in such ideals: $\mathscr{I} \subseteq L^1(G)$ has property ($\mathbf{G}^\infty$) if and only if $X = Z(\mathscr{I}) \in \mathscr{R}_c(\Gamma)$ and $X\Xi \subseteq X$, where Ξ is the component of the identity in Γ.)

Given the examples $L^1_H(G)$, it seems that we will need to consider other constructions, if we are to achieve the goal of obtaining such a characterization of amenability. An obvious place to start is to consider allowing the use of Proposition 2.3, as this is the result that gives us the amenability of the algebras $L^1_H(G)$. However, this is of little use in the non-commutative case, as it provides no guarantee of the amenability of $\mathfrak{A}_H$, when $\mathfrak{A}$ is not commutative. It is not known to the author whether a noncommutative version of Proposition 2.3 does hold.

5. Dense-ranged Homomorphisms of Amenable Banach Algebras.

We are left with the prospect that we cannot characterize amenability of Banach algebras in terms of amenable group algebras. The question arises as to whether there is some other "canonical" class $\mathscr{A}$ of amenable Banach algebras which we could use in place of the amenable group algebras in the definition of property (**G**), to arrive at a characterization of property (**G**). That is:

5.1. Question. Is there some class $\mathscr{A}$ of amenable Banach algebras such that for each amenable Banach algebra $\mathfrak{A}$, there is a $\mathfrak{B} \in \mathscr{A}$ and a dense-ranged continuous homomorphism $\nu: \mathfrak{B} \to \mathfrak{A}$?

Evidently, setting $\mathscr{A}$ to be the class of *all* amenable Banach algebras will suffice, but we seek a class considerably smaller. By enlarging $\mathscr{A}$ slightly, we can ensure that $\mathscr{A}$ is closed under taking quotients by closed ideals. Then the above question is equivalent to the one where "homomorphism" is replaced by "monomorphism".

5.2. DEFINITION. Let $\mathfrak{A}$ be a Banach algebra. A *Banach subalgebra* of $\mathfrak{A}$ is a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, with its own norm by which it is a Banach algebra, such that the injection $\mathfrak{B} \hookrightarrow \mathfrak{A}$ is continuous.

With this definition, Question 5.1 is asking for a class of amenable Banach algebras $\mathscr{A}$ such that each amenable Banach algebra $\mathfrak{A}$ has a dense Banach subalgebra $\mathfrak{B}$ that is (isomorphic to) a member of $\mathscr{A}$. In any such class $\mathscr{A}$, we must include each amenable Banach algebra $\mathfrak{A}$ which has no dense amenable Banach subalgebras. Define a Banach algebra $\mathfrak{A}$ to be *minimal-amenable* if it has this property, or equivalently, if every dense-ranged homomorphism from an amenable Banach algebra into $\mathfrak{A}$ is onto.

5.3. QUESTION. Which amenable Banach algebras are minimal-amenable?

The only examples known to the author of minimal-amenable Banach algebras are those which are finite-dimensional, and those of the form $C(X)$, where X is a compact F-space. (See [9] for definitions, and [1, Theorem A] for the relevant result.) These examples are not particularly illuminating, in that they are also *minimal*, in that they have no proper dense Banach subalgebras. Also, for such X, $C(X)$ is either finite-dimensional or nonseparable, and so we ask:

5.4. QUESTION. Are there minimal-amenable Banach algebras that are not minimal?

5.5. QUESTION. Are there infinite-dimensional separable minimal-amenable Banach algebras?

A possible answer to each of these questions would be that commutative group algebras are minimal-amenable. The result of [14] can be used to show that if $\mathfrak{A}$ has property (**G**), then any dense-ranged homomorphism $\mathfrak{A} \to L^1(G)$ is onto, so that any proper dense amenable Banach subalgebra of $L^1(G)$ must lack property (**G**). It is interesting to note that two standard sources of proper dense Banach subalgebras of group algebras can never yield an amenable algebra. The first of these, Segal algebras, defined as in [20, Section 6.2] lack bounded approximate identities, due to [4, Theorem 1.2]. The second construction is that of Beurling algebras, defined to be $L^1(G, \omega)$, for some submultiplicative weight $\omega: G \to \mathbb{R}^+$, as in [20, Section 6.3]. By [11, Theorem 0], such an algebra is amenable if and only if $x \mapsto \omega(x)\omega(x^{-1})$ is bounded. However, since $L^1(G, \omega)$ is assumed to be contained within $L^1(G)$, ω is bounded below, and hence ω is also bounded above. However, this implies $L^1(G,\omega) = L^1(G)$.

We should note that the term "minimal" (or "minimal-amenable", etc) is only supposed to indicate the lack of a certain type of dense subalgebra, and as such, only refers to an ordering (by inclusion) of such dense subalgebras. It is tempting to lift this to an order on the category of Banach algebras (or the category of amenable Banach algebras, etc). Such an order would be defined by $\mathfrak{A} \preceq \mathfrak{B}$ if there is a dense-ranged monomorphism $\mathfrak{A} \to \mathfrak{B}$. However, it is possible to have non-isomorphic Banach algebras $\mathfrak{A}$, $\mathfrak{B}$ with $\mathfrak{A} \preceq \mathfrak{B} \preceq \mathfrak{A}$. We give an example where both $\mathfrak{A}$ and $\mathfrak{B}$ have property (G).

Define dense Banach subalgebras $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{B}_1, \mathfrak{B}_2$ of $C_0(\mathsf{Z} \times \mathsf{R})$ by

$$\mathfrak{A}_1 = \mathfrak{B}_1 = A(\mathsf{Z} \times \mathsf{R})$$

$$\mathfrak{A}_2 = \{f \in C_0(\mathsf{Z} \times \mathsf{R}) \colon f(n,\cdot) \in A(\mathsf{R})\ (n \in \mathsf{Z})\}$$

$$\mathfrak{A}_3 = \mathfrak{B}_2 = C_0(\mathsf{Z} \times \mathsf{R}).$$

Each of these has carrier space $\mathsf{Z} \times \mathsf{R}$ and $\mathfrak{A}_1 \preceq \mathfrak{A}_2 \preceq \mathfrak{A}_3$. Also, $\mathfrak{B}_1 \cong \mathfrak{B}_1 \oplus \mathfrak{B}_1$ and $\mathfrak{B}_2 \cong \mathfrak{B}_2 \oplus \mathfrak{B}_2$, so that if we define

$$\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \mathfrak{A}_3 \qquad \text{and} \qquad \mathfrak{B} = \mathfrak{B}_1 \oplus \mathfrak{B}_2$$

$$\text{then} \qquad \mathfrak{B} \cong \mathfrak{B}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \preceq \mathfrak{A} \preceq \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_2 \cong \mathfrak{B}.$$

Suppose $\mathfrak{A} \cong \mathfrak{B}$, so that there is an isomorphism $v\colon \mathfrak{A} \to \mathfrak{B}$. Now, each of $\mathfrak{A}_1$, $\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{B}_1, \mathfrak{B}_2$ has carrier space $\mathsf{Z} \times \mathsf{R}$, and so $v^*|_{\Phi_\mathfrak{B}}$ is a homeomorphism

$$\alpha\colon (\mathsf{Z} \times \mathsf{R}) \cup (\mathsf{Z} \times \mathsf{R}) \to (\mathsf{Z} \times \mathsf{R}) \cup (\mathsf{Z} \times \mathsf{R}) \cup (\mathsf{Z} \times \mathsf{R}).$$

Consider a coset $E_1 = \{n\} \times \mathsf{R} \subseteq \Phi_{\mathfrak{B}_1}$, then $\mathfrak{B}|_{E_1} \cong A(\mathsf{R})$, and so $\mathfrak{A}|_{\alpha(E_1)} \cong A(\mathsf{R})$. However, if $\alpha(E_1) \subseteq \Phi_{\mathfrak{A}_3}$, then $\mathfrak{A}|_{\alpha(E_1)} \cong C_0(\mathsf{R})$. Hence $\alpha(E)$ is either one of the lines in $\Phi_{\mathfrak{A}_1}$ or one of the lines in $\Phi_{\mathfrak{A}_2}$. Similarly, if $E_2 = \{m\} \times \mathsf{R} \subseteq \Phi_{\mathfrak{B}_2}$, then $\alpha(E_2) \subseteq \Phi_{\mathfrak{A}_3}$. Hence $v(\mathfrak{A}_1 \oplus \mathfrak{A}_2) = \mathfrak{B}_1$ and $v(\mathfrak{A}_3) = \mathfrak{B}_2$. For $r = 1, 2$, put $Y_r = \alpha^{-1}(\Phi_{\mathfrak{A}_r}) \subseteq \Phi_{\mathfrak{B}_1}$. Then since the monomorphism $v|_{\mathfrak{A}_1}\colon \mathfrak{A}_1 \to \mathfrak{B}_1$ is a homomorphism of group algebras, $\alpha|_{Y_1}$ is piecewise affine. Thus $Y_1 \in \mathscr{R}(\mathsf{Z} \times \mathsf{R})$ is piecewise-affinely homeomorphic to $\mathsf{Z} \times \mathsf{R}$. By considering the structure of an element of $\mathscr{R}(\mathsf{Z} \times \mathsf{R})$, it is easily shown that $Y_2 = (\mathsf{Z} \times \mathsf{R}) \setminus Y_1 \in \mathscr{R}(\mathsf{Z} \times \mathsf{R})$ is also piecewise-affinely homeomorphic to $\mathsf{Z} \times \mathsf{R}$. Thus $\mathfrak{B}_1|_{Y_2} \cong A(\mathsf{Z} \times \mathsf{R})$, and so $\mathfrak{A}_2 \cong A(\mathsf{Z} \times \mathsf{R})$. This is clearly not the case.

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ON THE LOCATION OF ZEROS OF SOLUTIONS OF $f'' + A(z)f = 0$ WHERE $A(z)$ IS ENTIRE

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Abstract.

We investigate the distribution of zero-sequence of solutions of $f'' + Af = 0$, where A is polynomial or transcendental entire, near some rays. Results are obtained concerning the rays near which the exponent of convergence of zeros of the solutions attains its maximal value.

1. Introduction and main results.

Since 1982 there have been many papers on the oscillation theory of the solutions of the differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function. In this paper we shall investigate the distribution of zeros of solutions of (1.1). We first consider the case where $A(z)$ in (1.1) is a polynomial of degree $n \geqq 1$. It follows from the Wiman-Valiron theory that any nontrivial solution of (1.1) is an entire function of order $\dfrac{n+2}{2}$ [2, Th. 1]. The first general reslt on the exponent of convergence of the zero-sequence of the solutions is the following theorem which was due to Bank and Laine.

THEOREM A [2, Th. 1). *Let $A(z)$ be a polynomial of degree $n \geqq 1$. If f_1 and f_2 are two linearly independent solutions of* (1.1), *then at least one of f_1,f_2 has the property that the exponent of convergence of its zero-sequence is* $\dfrac{n+2}{2}$.

By generalizing a result of Hellerstein, Shen and Williamson [10], Gundersen [5, Th. 1] proved a stronger result that the conclusion of Theorem A still holds if the zero-sequence is replaced by the nonreal one.

In order to state our results, we need give some definitions.

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Let $g(z)$ be an entire function in the plane and let $\arg z = \theta \in \mathrm{R}$ be a ray. We denote, for each $\varepsilon > 0$, the exponent of convergence of zero-sequence of $g(z)$ in the angular region $\Omega(\theta - \varepsilon, \theta + \varepsilon) = \{z \mid \theta - \varepsilon \leqq \arg z \leqq \theta + \varepsilon, |z| > 0\}$ by $\lambda_{\theta,\varepsilon}(g)$, and by $\lambda_\theta(g) = \lim_{\varepsilon \to 0} \lambda_{\theta,\varepsilon}(g)$. We also denote the order of growth of $g(z)$ by $\sigma(g)$. We are interested in the distribution of those rays for which $\lambda_\theta(g) = \sigma(g)$. Our first result that concerns the case where A in (1.1) is a polynomial is the following:

THEOREM 1. *Let $A(z)$ be a polynomial of degree $n \geqq 1$ and let f_1 and f_2 be two linearly independent solutions of* (1.1). *If for some real number θ_0*

$$\overline{\lim_{r \to \infty}} \frac{\log\log|E(re^{i\theta_0})|}{\log r} = \frac{n+2}{2}, \tag{1.2}$$

where $E = f_1 f_2$, then there exist θ_1 and θ_2 with $\theta_1 \leqq \theta_0 \leqq \theta_2$ such that $\theta_2 - \theta_1 = \dfrac{2\pi}{n+2}$ and $\lambda_{\theta_1}(E) = \lambda_{\theta_2}(E) = \dfrac{n+2}{2}$.

Since E is of order $\dfrac{n+2}{2}$ [2, Le., A], a routine application of the Phragmen-Lindelöf principle implies that there certainly exists θ such that (1.2) holds. Thus we have the following:

COROLLARY 1. *Let $A(z)$ be a polynomial of degree $n \geqq 1$, and let f_1 and f_2 be two linearly independent solutions of* (1.1). *Then there exist two rays* $\arg z = \theta_1, \theta_2$ *with $\theta_2 - \theta_1 = \dfrac{2\pi}{n+2}$ such that* $\max(\lambda_{\theta_1}(f_1), \lambda_{\theta_1}(f_2)) = \max(\lambda_{\theta_2}(f_1), \lambda_{\theta_2}(f_2)) = \dfrac{n+2}{2}$.

Since $\dfrac{2\pi}{n+2} < \pi$ for $n \geqq 1$, Corollary 1 implies Gundersen's result (Theorem 1 in [5]).

We next turn to the case where A in (1.1) is a transcendental entire function of finite order. It is well known that any non-trivial solution of (1.1) is an entire function of infinite order. Let f_1 and f_2 be two linearly independent solutions of (1.1) and let $E = f_1 f_2$. Then $\lambda(E) = +\infty$ is equivalent to $\sigma(E) = +\infty$ [2, Le. B], where $\lambda(E)$ denotes the exponent of convergence of zero-sequence of E. Unlike the case of polynomial, when A is transcendental, the distribution of the rays $\arg z = \theta$ for which $\lambda_\theta(E) = +\infty$ largely depends on the growth of E itself along the rays. If we denote for any $\alpha < \beta$,

$$\Omega(\alpha, \beta) = \{z \mid \alpha \leqq \arg z \leqq \beta, |z| > 0\};$$

$$\Omega(\alpha, \beta, r) = \{z \mid z \in \Omega(\alpha, \beta), |z| < r\};$$

and for an entire function $g(z)$ in the plane,

$$M(r, \Omega(\alpha, \beta), g) = \sup_{\alpha \leqq \theta \leqq \beta} |g(re^{i\theta})|,$$

we may state our next result in the following form.

THEOREM 2. *Let $A(z)$ be a transcendental entire function of finite order in the plane and let f_1, f_2 be two linearly independent solutions of* (1.1). *Set $E = f_1 f_2$. Then $\lambda_\theta(E) = +\infty$, if and only if*

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), E)}{\log r} = +\infty \tag{1.3}$$

for any $\varepsilon > 0$.

Especially, if

$$\overline{\lim_{r\to\infty}} \frac{\log\log |E(re^{i\theta})|}{\log r} = +\infty, \tag{1.4}$$

then we have $\lambda_\theta(E) = +\infty$.

Although Theorem 2 provides no information concerning the distribution of zeros of solutions of (1.1) in terms of the entire function $A(z)$, it is possible to get some further results in some cases. We next consider some applications of Theorem 2. Our starting point is the following theorem due to Bank, Laine and Langley.

THEOREM B [3, Th. 1]. *Let $A(z)$ be a transcendental entire function of finite order ρ with the following property: there exists a set $H \subseteq \mathrm{R}$ of measure zero, such that for each real number $\theta \in \mathrm{R}\backslash H$, either*

(1.5) (i) $$r^{-N} |A(re^{i\theta})| \to \infty \text{ as } r \to +\infty, \text{ for each } N > 0,$$

or

(1.6) (ii) $$\int_0^\infty r |A(r)e^{i\theta}| \, dr < +\infty,$$

or

(iii) *there exist positive real numbers K and b, and a nonnegative real number n (all possibly depending on θ), such that $(n + 2)/2 < \rho$, and*

$$|A(re^{i\theta})| \leqq Kr^n \quad \text{for all} \quad r \geqq b. \tag{1.7}$$

Then if f_1 and f_2 are linearly independent solutions of

$$f'' + Af = 0,$$

we have

$$\max(\lambda(f_1), \lambda(f_2)) = +\infty.$$

By using Theorem 2, we can prove

THEOREM 3. *Suppose that* $A(z)$ *satisfies the conditions of Theorem* B. *If* f_1 *and* f_2 *are linearly independent solutions of* $f'' + Af = 0$ *and* $\Omega(\alpha, \beta)$ *is an angular region with* $\beta - \alpha > \dfrac{\pi}{\rho}$ *such that there exists a ray* $\arg z = \theta \in (\alpha, \beta)$ *with*

$$\overline{\lim_{r\to\infty}} \frac{\log\log|A(re^{i\theta})|}{\log r} = \rho, \tag{1.8}$$

then there exists at least one ray $\arg z = \theta_0 \in (\alpha, \beta)$ *such that*

$$\max(\lambda_{\theta_0}(f_1), \lambda_{\theta_0}(f_2)) = +\infty.$$

From the Phragment-Lindelöf principle, those θ's such that (1.8) holds always form a union of intervals. Thus we have the following:

COROLLARY 2. *Under the assumption of Theorem* B, *if* $\rho = \sigma(A) > \frac{1}{2}$ *and* f_1 *and* f_2 *are two linearly independent solutions of* $f'' + Af = 0$, *then there exist at least two rays* $\arg z = \theta_1, \theta_2$ *with* $0 < \theta_2 - \theta_1 \leqq \dfrac{\pi}{\rho}$ *such that*

$$\max\{\lambda_{\theta_1}(f_1), \lambda_{\theta_1}(f_2)\} = \max\{\lambda_{\theta_2}(f_1), \lambda_{\theta_2}(f_2)\} = +\infty$$

Especially, if $\rho > 1$, *then at least one of* f_1 *and* f_2 *has the property that the exponent of convergence of its nonreal zero-sequence is infinite.*

Recently there have also been some results concerning (1.1) with $A = \sum_{j=q}^{m} Q_j \exp(jP)(-\infty < q \leqq m < +\infty)$, where Q_j and P are polynomials (see [1] and [9]). In this direction, we have the following result.

COROLLARY 3. *Let* $J \geqq 1$, *and let* $P_1, \ldots, P_J$ *be nonconstant polynomials whose degrees are* $d_1, \ldots, d_J$ *respectively, and suppose that for* $i \neq j$,

$$\deg(P_i - P_j) = \max(d_i, d_j).$$

Set

$$A(z) = \sum_{j=1}^{J} B_j(z) e^{P_j(z)}$$

where, each j, $B_j(z)$ *is an entire function, not identically zero, of order strictly less than* d_j. *If* f_1 *and* f_2 *are linearly independent solutions of* $f'' + (A + Q)f = 0$, *where* $Q(z)$ *is a polynomial whose degree* m *satisfies* $\dfrac{m+2}{2} \leqq \sigma(A) = \max(d_j)$, *then there* *exists at least one ray* $\arg z = \theta$ *in every angular region* $\Omega(\alpha, \beta)$ *of opening larger than* $\dfrac{\pi}{\sigma(A)}$ *such that*

$$\max(\lambda_\theta(f_1), \lambda_\theta(f_2)) = +\infty.$$

If $J = 1$ in Corollary 3, we can prove a stronger result. For a polynomial

$$P(z) = (x + iy)z^n + \ldots + a_0$$

with x, y real, we define, for each real θ,

$$\delta(P, \theta) = x \cos n\theta - y \sin n\theta.$$

Then we can state our result as follows.

THEOREM 4. *Suppose that* $A(z) = B(z)e^{P(z)} \not\equiv 0$, *where* $B(z)$ *is an entire function of order strictly less than the degree of the polynomial* $P(z)$. *If* f_1 *and* f_2 *are two linearly independent solutions of* $f'' + (A + Q)f = 0$, *where* $Q(z)$ *is a polynomial with degree m satisfies* $\dfrac{m+2}{2} < \sigma(A)$, *then for any* θ *satisfying* $\delta(P, \theta) = 0$ *we have*

$$\max(\lambda_\theta(f_1), \lambda_\theta(f_2)) = +\infty.$$

2. Preliminaries.

We shall assume that the reader is familiar with the standard notation of Nevenlinna theory (see [4] or [6]). Our proofs require the Nevanlinna characteristic for an angle (see [4], [14]): If $0 < \beta - \alpha \leqq 2\pi$ and $k = \dfrac{\pi}{\beta - \alpha}$ and $g(z)$ is meromorphic on the angular domain $\Omega(\alpha, \beta)$, we denote

$$A_{\alpha\beta}(r, g) = \frac{k}{\pi} \int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}} \right) \left\{ \log^+ |g(te^{i\alpha})| + \log^+ g(te^{i\beta})| \right\} \frac{dt}{t};$$

$$B_{\alpha\beta}(r, g) = \frac{2k}{\pi r^k} \int_\alpha^\beta \log^+ |g(re^{i\theta})| \sin k(\theta - \alpha)\, d\theta;$$

$$C_{\alpha\beta}(r, g) = 2 \sum_{1 < |b_v| < r} \left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{r^{2k}} \right) \sin k(\beta_v - \alpha);$$

$$D_{\alpha\beta}(r, g) = A_{\alpha\beta}(r, g) + B_{\alpha\beta}(r, g);$$

$$S_{\alpha\beta}(r, g) = A_{\alpha\beta}(r, f) + B_{\alpha\beta}(r, g) + C_{\alpha\beta}(r, g),$$

where $|b_v| = |b_v| e^{i\beta_v}$ $(v = 1, 2, \ldots,)$ are the poles of $g(z)$ in $\Omega(\alpha, \beta)$, counting multiplicities. If we only consider the distinct poles of g, we denote the corresponding angular counting function by $\bar{C}_{\alpha\beta}(r, g)$.

For a positive function $\varphi(r)$, $r \in (0, \infty)$, the order of $\varphi(r)$ is defined by

$$\varlimsup_{r \to \infty} \frac{\log \varphi(r)}{\log r}.$$

Especially the order of $S_{\alpha\beta}(r, g)$ is denoted by $\sigma_{\alpha\beta}(g)$.

3. Lemmas required for the proof of Theorem 1.

LEMMA 1. *Suppose that $g(z)$ ($\not\equiv$ const) is meromorphic in the plane and that $\Omega(\alpha, \beta)$ is an angular domain, where $0 < \beta - \alpha \leqq 2\pi$. Then*

(i) [4, Chap. 1] *for any complex number $a \neq \infty$*

$$S_{\alpha\beta}\left(r, \frac{1}{g-a}\right) = S_{\alpha\beta}(r, g) + \varepsilon(r, a), \tag{3.1}$$

where $\varepsilon(r, a) = O(1)$ $(r \to \infty)$;

(ii) [4, P. 138] *for any $r < R$*

$$A_{\alpha\beta}\left(r, \frac{g'}{g}\right) \leqq K\left\{\left(\frac{R}{r}\right)^k \int_1^R \frac{\log^+ T(t, g)}{t^{1+k}} dt + \log^+ \frac{r}{R-r} + \log \frac{R}{r} + 1\right\}, \tag{3.2}$$

and

$$B_{\alpha\beta}\left(r, \frac{g'}{g}\right) \leqq \frac{4k}{r^k} m\left(r, \frac{g'}{g}\right), \tag{3.3}$$

where $k = \dfrac{\pi}{\beta - \alpha}$ and K is a positive constant not depending on r and R.

LEMMA 2 [13, 7, P. 193]. *Suppose that $\Omega(\alpha, \beta)$ and $\Omega(\alpha', \beta')$ are two angular domains such that $\alpha < \alpha' < \beta' < \beta$ and that $g(z)$ is analytic on $\Omega(\alpha, \beta)$. If*

$$\varlimsup_{r \to \infty} \frac{\log\log M(r, \Omega(\alpha', \beta'), g)}{\log r} \equiv \rho(\Omega(\alpha', \beta'), g) > \frac{\pi}{\beta - \alpha}, \tag{3.4}$$

then we have for every a with at most one exception

$$\varlimsup_{r \to \infty} \frac{\log n(\Omega(\alpha, \beta, r), g = a)}{\log r} \geqq \rho(\Omega(\alpha', \beta'), g), \tag{3.5}$$

where $n(\Omega(\alpha, \beta, r), g = a)$ denotes the roots of the equation $g(z) = a$, counting multiplicities, in the sector $\Omega(\alpha, \beta, r)$.

LEMMA 3 [11, *Chap.* 7.4]. *Let $A(z) = a_n z^n + \ldots + a_0$ be a polynomial with $a_n = |a_n| e^{i\alpha_n} \neq 0$ $(0 \leqq \alpha_n < 2\pi)$. Define $\theta_k = \dfrac{\alpha_n + 2k\pi}{n+2}$ for $k = 0, 1, \ldots, n+1$, and* *fix* $\varepsilon > 0$. *If* f *is a solution to* $f'' + Af = 0$, *only finitely many of the zeros of* f *lie outside* $\bigcup_{k=0}^{n+1} \Omega(\theta_k - \varepsilon, \theta_k + \varepsilon)$. *If for some* k, f *has infinitely many zeros in* $\Omega(\theta_k - \varepsilon, \theta_k + \varepsilon)$, *then*

$$n(\Omega(\theta_k - \varepsilon, \theta_k + \varepsilon, r), f = 0) = (1 + o(1))\sqrt{|a_n|}\, r^{\frac{n+2}{2}} \Big/ \frac{\pi(n+2)}{2}. \tag{3.6}$$

4. Proof of Theorem 1.

Let f_1 and f_2 be two linearly independent solutions of $f'' + Af = 0$, where A is a polynomial of degree $n \geqq 1$. Suppose that

$$\overline{\lim_{r \to \infty}} \frac{\log \log |E(re^{i\theta_0})|}{\log r} = \frac{n+2}{2}, \tag{4.1}$$

where $\theta_0 \in \mathrm{R}$ and $E = f_1 f_2$. It follows from the Phragmen-Lindelöf principle that there exists an interval $[\theta_1', \theta_2']$ containing θ_0 such that for all $\theta \in [\theta_1', \theta_2']$ we have

$$\overline{\lim_{r \to \infty}} \frac{\log \log |E(re^{i\theta})|}{\log r} = \frac{n+2}{2}. \tag{4.2}$$

By using Lemma 3, we need only to prove that there exist two rays $\arg z = \theta_1$, θ_2 with $\theta_0 \in (\theta_1, \theta_2)$ such that $\lambda_{\theta_1}(E) = \lambda_{\theta_2}(E) = \dfrac{n+2}{2}$ and $\theta_2 - \theta_1 \leqq \dfrac{2\pi}{n+2}$. If this is not true, then there must exist an angular domain $\Omega(\theta_1, \theta_2)$ satisfying the following properties:

(a) $\theta_2 - \theta_1 > \dfrac{2\pi}{n+2}$;

(b) there exists a ray $\arg z = \theta_3 \in (\theta_1, \theta_2)$ such that (4.2) holds for θ_3;

(c) $\lambda_\theta(E) < \dfrac{n+2}{2}$ for all $\theta \in [\theta_1, \theta_2]$.

From (c), the definition of $\lambda_\theta(E)$ and the fact that finitely many zeros of f only lie outside of the critical sectors described in Lemma 3, we deduce that

$$\overline{\lim_{r \to \infty}} \frac{\log n(\Omega(\theta_1 + \varepsilon, \theta_2 - \varepsilon, r), E = 0)}{\log r} < \frac{n+2}{2}, \tag{4.3}$$

for every $\varepsilon > 0$.

In order to obtain a contradiction, we choose a fixed $\varepsilon_0 > 0$ such that $\theta_2 - \theta_1 - 6\varepsilon_0 > \dfrac{2\pi}{n+2}$ and $\theta_3 \in (\theta_1 + 3\varepsilon_0, \theta_2 - 3\varepsilon_0)$. From this choice of ε_0 we have

$$\begin{aligned}(4.4)\qquad &\varlimsup_{r\to\infty}\frac{\log\log M(r,\Omega(\theta_1+3\varepsilon_0,\theta_2-3\varepsilon_0),E)}{\log r}\\ &\geqq \varlimsup_{r\to\infty}\frac{\log^+\log^+|E(re^{i\theta_3})|}{\log r}\\ &=\frac{n+2}{2}>\frac{\pi}{\theta_2-\theta_1-4\varepsilon_0}.\end{aligned}$$

By using Lemma 2, we have for all $a\in C$ with at most one exception

$$(4.5)\qquad \varlimsup_{r\to\infty}\frac{\log n(\Omega(\theta_1+2\varepsilon_0,\theta_2-2\varepsilon_0,r),E=a)}{\log r}=\frac{n+2}{2}.$$

Taking a fixed $a\in \mathrm{C}$ such that (4.5) holds, we deduce from (4.5) that there exists a sequence (r_n) of real numbers with $r_n\to+\infty(n\to\infty)$ such that for every $\varepsilon>0$ we have

$$n(\Omega(\theta_1+2\varepsilon_0,\theta_2-2\varepsilon_0,r_n),E=a)\geqq r_n^{\frac{n+2}{2}-\varepsilon}$$

for all sufficiently large n.

Suppose that $a_v=|a_v|\,e^{i\alpha_v}(v=1,2,\ldots)$ are the roots of $E=a$, counting multiplicities, in $\Omega(\theta_1+\varepsilon_0,\theta_2-\varepsilon_0)$. To compute $\sigma_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(E)$, we first observe that $\theta_1+2\varepsilon_0<\alpha_v<\theta_2-2\varepsilon_0$ implies for $k=\dfrac{\pi}{\theta_2-\theta_1-2\varepsilon_0}$ the inequalities

$$k\varepsilon_0<k(\alpha_v-\theta_1-\varepsilon_0)<\pi-k\varepsilon_0,$$

hence

$$(4.6)\qquad \sin k(\alpha_v-\theta_1-\varepsilon_0)\geqq\sin(k\varepsilon_0).$$

Moreover, we write a sum below as a Stieltjes-integral:

$$\begin{aligned}\sum\left(\frac{1}{|a_v|^k}-\frac{|a_v|^k}{(2r_n)^{2k}}\right)&=\sum\frac{1}{|a_v|^k}-\sum\frac{|a_v|^k}{(2r_n)^{2k}}\\ &=\int_1^{r_n}\frac{dn(t)}{t^k}-\frac{1}{(2r_n)^{2k}}\int_1^{r_n}t^k\,dn(t),\end{aligned}$$

where a short-hand notation $n(t)=n(\Omega(\theta_1+2\varepsilon_0,\theta_2-2\varepsilon_0,t),E=a)$ will be used. Application of Lemma 1 (i), the formula (4.6) and the partial integration of the above Stieltjes-integrals now results in

$$(4.7)\quad S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(2r_n, E) = S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(2r_n, \frac{1}{E-a}\right) + O(1)$$

$$\geqq C_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(2r_n, \frac{1}{E-a}\right) + O(1)$$

$$= 2\sum_{1<|a_v|<2r_n}\left(\frac{1}{|a_v|^k} - \frac{|a_v|^k}{(2r_n)^{2k}}\right)\sin k(\alpha_v - \theta_1 - \varepsilon_0) + O(1)$$

$$\geqq 2\sum_{\substack{1<|a_v|<r_n\\ \theta_1+2\varepsilon_0<a_v<\theta_2-2\varepsilon_0}}\left(\frac{1}{|a_v|^k} - \frac{||a_v|^k}{(2r_n)^{2k}}\right)\sin(k\varepsilon_0) + O(1)$$

$$\geqq 2\sin(k\varepsilon_0)\left\{k\int_1^{r_n}\frac{n(t)}{t^{1+k}}\,dt + \frac{n(r_n)}{r_n^k}\right.$$

$$\left. - \frac{(r_n)^k(r_n)}{(2r_n)^{2k}} + \frac{k}{(2r_n)^{2k}}\int_1^{r_n} t^{k-1}n(t)\,dt\right\} + O(1)$$

$$\geqq \left(1 - \frac{1}{2^{2k}}\right)\frac{\sin(k\varepsilon_0)}{r_n^k}n(r_n) + O(1)$$

$$\geqq r_n^{\frac{n+2}{2}-k-2\varepsilon}.$$

Therefore we have

$$\overline{\lim_{r\to\infty}}\frac{\log S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(r, E)}{\log r} \geqq \frac{n+2}{2} - k - 2\varepsilon.$$

As ε can be arbitrary small, $\sigma_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(E)$ is at least $\dfrac{n+2}{2} - k > 0$.

On the other hand, in [2, P. 354], Bank and Laine proved that

$$(4.8)\qquad E^2 = c^2\left(\left(\frac{E'}{E}\right)^2 + 2\left(\frac{E''}{E}\right) - 4A\right)^{-1},$$

where $c \neq 0$ is the Wronskian of f_1 and f_2.

By using Lemma 1 (ii) in which we set $R = 2r$ and the fact that E is of finite order, we deduce that

$$D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E'}{E}\right) = O(1),$$

$$D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E''}{E}\right) \leqq D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E'}{E}\right) + D_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r, \frac{E''}{E'}\right) + O(1)$$

$$= O(1)$$

and

$$S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(r,A)=O(1).$$

Thus we have

$$S_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}(r,E)=O\left(\bar{C}_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r,\frac{1}{E}\right)+O(1)\right). \tag{4.9}$$

(4.7) and (4.9) show that the order of $\bar{C}_{\theta_1+\varepsilon_0,\theta_2-\varepsilon_0}\left(r,\frac{1}{E}\right)$ is at least $\frac{n+2}{2}-k>0$.

Therefore, by Lemma 3, there must be a critical ray $\arg z=\theta_k\in(\theta_1+\varepsilon_0,\theta_2-\varepsilon_0)$ with infinitely many zeros around that ray. Hence, by Lemma 3 again, we know that the order of $n(\Omega(\theta_1+\varepsilon_0,\theta_2-\varepsilon_0,r),E=0)$ is $\frac{n+2}{2}$. This contradicts (4.3), proving theorem 1.

5. Discussion of Theorem 1.

REMARK 1. Theorem 1 is sharp. In fact, consider the equation

$$f''-zf=0. \tag{5.1}$$

According to a result of Hille [11, chap. 7.4], there exist three pairwise independent solutions $f_k(z)$ $(k=1,2,3)$ to (5.1) such that for $z\notin\Omega\left(\frac{\pi+2k\pi}{3}-\varepsilon,\frac{\pi+2k\pi}{3}+\varepsilon\right)$ and $|z|$ sufficiently large

$$f_k(z)=(1+o(1))(-z)^{-\frac{1}{4}}\exp\left(\tfrac{2}{3}e^{\frac{i\pi}{2}}(-1)^{k+1}iz^{\frac{3}{2}}(1+o(1))\right).$$

It is seen that $f_k(z)\to 0$ *in* $\Omega\left(\frac{\pi+2(k+1)\pi}{3}+\varepsilon,\frac{\pi+2(k+2)\pi}{3}-\varepsilon\right)$ as $|z|$ tends infinity and $\lambda(f_k)=\frac{3}{2}$. Thus, as Hille observed, $\lambda_\theta(f_k)=\frac{3}{2}$ only when $\theta=\frac{\pi+2k\pi}{3}$. Therefore $\lambda_\theta(f_kf_{k+1})=\frac{3}{2}$ only for $\theta=\frac{\pi+2k\pi}{3}$ and $\frac{\pi+2(k+1)\pi}{3}$.

REMARK 2. Let θ_k be as defined in Lemma 3 and let f_1 and f_2 be two linearly independent solutions of (1.1). It follows from Lemma 3 that we have in fact proved that there exists an integer k such that $\lambda_{\theta_k}(f_1f_2)=\lambda_{\theta_{k+1}}(f_1f_2)=\frac{n+2}{2}$.

It is also easily seen from the proof of Theorem 1 that if $\lambda_{\theta_1}(f_1 f_2) = \frac{n+2}{2}$ for some $\theta_1 \in \mathrm{R}$, then we can find $\theta_2 \in \mathrm{R}$ such that $\lambda_{\theta_2}(f_1 f_2) = \frac{n+2}{2}$ and the magnitude of $\Omega(\theta_1, \theta_2)$ or $\Omega(\theta_2, \theta_1)$ is $\frac{2\pi}{n+2}$.

REMARK 3. When $A(z) = \frac{P(z)}{Q(z)}$ is rational with $n = \mathrm{di}(A) = \text{degree } P - \text{degree } Q \geqq 1$ in (1.1), using our methods with obvious modifications we can prove that the conclusion of Theorem 1 remains true provided that $E = f_1 f_2$ is transcendental [8, Th. 1].

6. Lemmas required for the proofs of Theorem 2–4.

To prove Theorem 2 and Theorem 3, we need some estimates, restricted in an angle, for the logarithmic derivative of an entire function. The first lemma in this section is due to A. Mokhon'ko.

LEMMA 4 [12]. *Let $z = r\exp(i\varphi)$, $r_0 + 1 < r$ and $\alpha < \varphi < \beta$, where $0 < \beta < \alpha \leqq 2\pi$. If $g(z)$ is meromorphic in the angular region $\Omega(\alpha, \beta)$ and $\sigma_{\alpha\beta}(g)$ is finite, then there exist $K_1 > 0$ and $M_1 > 0$ depending only on g and $\Omega(\alpha, \beta)$, and not depending on z, such that*

$$\left|\frac{g'(z)}{g(z)}\right| \leqq K_1 r^{M_1}(\sin k(\varphi - \alpha))^{-2} \tag{6.1}$$

for all $z \notin D_1$, where $k = \frac{\pi}{\beta - \alpha}$ and D_1 is an R-set, that is, a countable union of discs whose radii have finite sum.

As an application we may estimate the growth of $\frac{g''}{g}$ where g is regular in an angle.

LEMMA 5. *Let $z = r\exp(i\varphi)$, $r_0 + 1 < r$ and $\alpha \leqq \varphi \leqq \beta$, where $0 < \beta - \alpha \leqq 2\pi$. If $g(z)$ is regular in $\Omega(\alpha, \beta) \cap (|z| \geqq r_0)$ and $\sigma_{\alpha\beta}(g)$ is finite, then for every $\varepsilon \in \left(0, \frac{\beta - \alpha}{2}\right)$ except for a set of ε with linear measure zero, there exist $K > 0$ and $M > 0$ depending only on g, ε and $\Omega(\alpha, \beta)$, and not depending on z such that*

$$\left|\frac{g''(z)}{g(z)}\right| \leqq K r^{M}(\sin k(\varphi - \alpha)\sin k_\varepsilon(\varphi - \alpha - \varepsilon))^{-2} \tag{6.2}$$

for all $z \in \Omega(\alpha + \varepsilon, \beta - \varepsilon)$ *outside an R-set D, where* $k = \dfrac{\pi}{\beta - \alpha}$ *and* $k_\varepsilon = \dfrac{\pi}{\beta - \alpha - 2\varepsilon}$.

PROOF. Since $\sigma_{\alpha\beta}(g)$ is finite, it follows from Lemma 2 that for every $\varepsilon > 0$ there exists $K_\varepsilon (0 < K_\varepsilon < +\infty)$ such that

$$\varlimsup_{r \to \infty} \frac{\log \log M(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), g)}{\log r} < K_\varepsilon. \tag{6.3}$$

From (6.3) we have

$$\log |g(re^{i(\alpha+\varepsilon)})| < r^{K_\varepsilon + 1},$$
$$\log |g(re^{i(\beta-\varepsilon)})| < r^{K_\varepsilon + 1}$$

and

$$\log |g(re^{i\theta})| < r^{K_\varepsilon + 1}$$

for all large r and all $\theta \in [\alpha + \varepsilon, \beta - \varepsilon]$. Noting that $g(z)$ is regular in $\Omega(\alpha, \beta)$, we deduce from the definition of the Nevanlinna angular characteristic that $\sigma_{\alpha+\varepsilon, \beta-\varepsilon}(g)$ is finite.

Let D_1 be the R-set in Lemma 4. Then the set of ε for which the rays $\arg z = \alpha + \varepsilon$ or $\beta - \varepsilon$ meet D_1 infinitely often (i.e., meet infinitely many discs in D_1) has measure zero. Suppose that ε is a number such that $0 < \varepsilon < \dfrac{\beta - \alpha}{2}$ and $\arg z = \alpha + \varepsilon$ and $\beta - \varepsilon$ meet D_1 at most finitely many times. For such ε, we have

$$\begin{aligned} S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g') &= D_{\alpha+\varepsilon, \beta-\varepsilon}(r, g') \\ &\leqq D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \frac{g'}{g}\right) + D_{\alpha+\varepsilon, \beta-\varepsilon}(r, g) + O(1) \\ &= D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \frac{g'}{g}\right) + S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g) + O(1). \end{aligned} \tag{6.4}$$

If $|z| = r$ does not meet D_1, by using Lemma 4 and from the definition of $D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \dfrac{g'}{g}\right)$, we have

$$D_{\alpha+\varepsilon, \beta-\varepsilon}\left(r, \frac{g'}{g}\right) = O(1). \tag{6.5}$$

Combining (6.4) and (6.5), we deduce that

(6.6) $$S_{\alpha+\varepsilon,\beta-\varepsilon}(r, g') \leqq S_{\alpha+\varepsilon,\beta-\varepsilon}(r, g) + O(1)$$

for all r except for a set of r with finite linear measure. This implies that $\sigma_{\alpha+\varepsilon,\beta-\varepsilon}(g')$ is finite.

Applying Lemm 4 to $g'(z)$ and $\Omega(\alpha + \varepsilon, \beta - \varepsilon)$, if $z = re^{i\varphi}$, $r_0 + 1 < r$ and $\alpha + \varepsilon < \varphi < \beta - \varepsilon$, then there exists $K_2 > 0$ and $M_2 > 0$ depending only on $g'(z)$ and $\Omega(\alpha + \varepsilon, \beta - \varepsilon)$ such that

$$\left|\frac{g''(z)}{g'(z)}\right| \leqq K_2 r^{M_2}(\sin k_\varepsilon(\varphi - \alpha - \varepsilon))^{-2}$$

for all $z \notin D_2$, where D_2 is an R-set. Thus if $z \notin D_1 \cup D_2$, we have

$$\left|\frac{g''(z)}{g(z)}\right| \leqq \left|\frac{g''(z)}{g'(z)}\right|\left|\frac{g'(z)}{g(z)}\right|$$
$$\leqq K_1 K_2 r^{M_1+M_2}(\sin k(\varphi - \alpha)\sin k_\varepsilon(\varphi - \alpha - \varepsilon))^{-2}.$$

Using K, M, D instead of K_1K_2, $M_1 + M_2$, $D_1 \cup D_2$, we obtain (6.2).

7. Proof of Theorem 2.

Suppose that $f(z)$ is a nontrivial solution to $f'' + Af = 0$. Then

(7.1) $$\frac{f''}{f} \equiv -A.$$

We apply Wiman-Valiron theory to (7.1). Hence there exists a set $D \subset [1, \infty)$ of finite logarithmic measure such that if $r \notin D$ and z is a point on $|z| = r$ at which $|f(z)| = M(r, f)$, then

(7.2) $$\left|\frac{f''(z)}{f(z)}\right| = \left(\frac{v(r)}{r}\right)^2 (1 + \eta(z)) = |A(z)| \leqq M(r, A),$$

where $\eta(z) \to 0$ (as $|z| \to \infty$) and $v(r)$ denotes the central index of f. Thus we have [4, pp. 360–361]

(7.3) $$v(r) \leqq 4r(M(2r, A))^{\frac{1}{2}}$$

for all sufficiently large r. (7.3) implies that the order of $\log T(r, f)$ is at most $\sigma(A)$.

Let f_1 and f_2 be two linearly independent solutions of $f'' + Af = 0$ and $E = f_1 f_2$. The above argument implies that $\sigma(\log T(r, E)) \leqq \sigma(A)$, since

$$T(r, E') = m(r, E')$$
$$\leqq 2m(r, f_1) + 2m(r, f_2) + m\left(r, \frac{f_1'}{f_1}\right) + m\left(r, \frac{f_2'}{f_2}\right) + O(1),$$

we deduce that $\sigma(\log T(r, E')) \leqq \sigma(A)$. Thus if ε is sufficiently small, we deduce from Lemma 1 (ii) in which we set $R = 2r$ that

$$A_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) = O\left(\int_1^{2r} \frac{\log^+ T(t, E)}{t^{1+\frac{\pi}{2\varepsilon}}dt}\right)$$
$$= O\left(\int_1^{2r} \frac{t^{\sigma(A)+1}}{t^{1+\frac{\pi}{2\varepsilon}}}dt\right)$$
$$= O(1).$$

Since [6, P. 36]

$$m\left(r, \frac{E'}{E}\right) = O(\log^+ T(2r, E) + \log r)$$
$$= O(r^{\sigma(A)+1}),$$

we deduce from (3.3) that

$$B_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) = O\left(r^{-\frac{\pi}{2\varepsilon}} m\left(r, \frac{E'}{E}\right)\right)$$
$$= O(r^{\sigma(A)+1-\frac{\pi}{2\varepsilon}})$$
$$= O(1).$$

Therefore we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) = O(1). \tag{7.4}$$

Similarily we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E''}{E}\right) \leqq D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E'}{E}\right) + D_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{E''}{E'}\right) + O(1) \tag{7.5}$$
$$= O(1)$$

and

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r, A) = O(1), \tag{7.6}$$

for any $\theta \in \mathsf{R}$.

From the Nevanlinna theory it follows from (7.4), (7.5), (7.6) and (4.8) that

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r, E) = O\left(\bar{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r, \frac{1}{E}\right) + O(1)\right) \tag{7.7}$$

for all sufficiently small $\varepsilon > 0$.

Now suppose that $\theta_0 \in \mathrm{R}$ such that for any sufficiently small $\varepsilon > 0$

$$\overline{\lim_{r\to\infty}} \frac{\log\log M\left(r, \Omega\left(\theta_0 - \frac{\varepsilon}{3}, \theta_0 + \frac{\varepsilon}{3}\right), E\right)}{\log r} = +\infty,$$

by using Lemma 2, we can find a complex number a such that

$$\overline{\lim_{r\to\infty}} \frac{\log n(\Omega(\theta_0 - \frac{2}{3}\varepsilon, \theta_0 + \frac{2}{3}\varepsilon, r), E = a)}{\log r} = +\infty.$$

As we did in the proof of (4.7), we deduce that $\sigma_{\theta_0 - \varepsilon, \theta_0 + \varepsilon}(r, E)$ is infinite. It follows from (7.7) that $\lambda_{\theta_0}(E) = +\infty$.

On the other hand, if there exists $\varepsilon_0 > 0$ such that

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0), E)}{\log r} \leqq K < +\infty,$$

as we did in the proof of Lemma 5, we know that $\sigma_{\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0}(E)$ must be finite. As in the proof of (4.7) we deduce that the order of $n(\Omega(\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0, r), E = 0)$ is finite. Since $\lambda_{\theta_0, \varepsilon}(E) \leqq \lambda_{\theta_0, \varepsilon_0}(E)$ for any $\varepsilon (0 < \varepsilon \leqq \varepsilon_0)$, therefore $\lambda_{\theta_0}(E) < +\infty$. The proof of Theorem 2 is completed.

8. Proof of Theorem 3.

Observe first that if $\rho \leqq \frac{1}{2}$, then $\beta - \alpha > 2\pi$. Since $\lambda(E) = +\infty$ [15], we see easily from the definition of $\lambda_\theta(E)$ that there exists at least one ray $\arg z = \theta_0$ such that $\lambda_{\theta_0}(E) = +\infty$. In the following we assume that $\rho > \frac{1}{2}$.

Suppose that $\Omega(\alpha, \beta)$ is an arbitrary angular domain with $\beta - \alpha > \dfrac{\pi}{\rho}$ and that there exists a ray $\arg z = \theta_0$ such that $\alpha < \theta_0 < \beta$ and (1.8) holds. It follows from the Phragmen-Lindelöf principle that there exists an interval $[\theta_1, \theta_2]$ containing θ_0 such that (1.8) holds for all $\theta \in [\theta_1, \theta_2]$. So we may suppose $[\theta_1, \theta_2] \subset (\alpha, \beta)$. Let f_1 and f_2 be two linearly independent solutions of $f'' + Af = 0$. If there is no ray $\arg z = \theta$ with $\alpha < \theta < \beta$ such that $\lambda_\theta(E) = +\infty$, where $E = f_1 f_2$, we shall derive a contradiction.

We choose a fixed $\varepsilon_0 > 0$ such that $\beta - \alpha - 4\varepsilon_0 > \dfrac{\pi}{\rho}$ and $4\varepsilon_0 < \theta_2 - \theta_1$. From Theorem 2 we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\alpha + \varepsilon_0, \beta - \varepsilon_0), E)}{\log r} < +\infty. \tag{8.1}$$

Consequently we deduce from (8.1) that $\sigma_{\alpha+\varepsilon_0,\beta-\varepsilon_0}(E)$ is finite. Now we claim that there exists θ'_1, θ'_2 with $\alpha + \frac{3}{2}\varepsilon_0 < \theta'_1 < \alpha + 2\varepsilon_0$ and $\beta - 2\varepsilon_0 < \theta'_2 < \beta - \frac{3}{2}\varepsilon_0$ such that $\sigma_{\theta'_1,\theta'_2}(E)$ is at least $\rho - \dfrac{\pi}{\theta'_2 - \theta'_1}$.

In fact, by using Lemma 4 and 5, we may choose a fixed $\varepsilon \in (\varepsilon_0, \frac{5}{4}\varepsilon_0)$ such that

$$\left|\frac{E'}{E}(re^{i\varphi})\right| \leqq Kr^M(\sin k(\varphi - \alpha - \varepsilon_0))^{-2} \tag{8.2}$$

$$\left|\frac{E''}{E}(re^{i\varphi})\right| \leqq Kr^M(\sin k_\varepsilon(\varphi - \alpha - \varepsilon)\sin k(\varphi - \alpha - \varepsilon_0))^{-2} \tag{8.3}$$

for all $z = re^{i\varphi} \in \Omega(\alpha + \varepsilon, \beta - \varepsilon)$ outside an R-set D, where $k = \dfrac{\pi}{\beta - \alpha - 2\varepsilon_0}$, $k_\varepsilon = \dfrac{\pi}{\beta - \alpha - 2\varepsilon}$, and K and M are constants depending only on $\Omega(\alpha + \varepsilon_0, \beta - \varepsilon_0)$, E and ε, and not depending on r and φ. Since the set of θ for which the ray $\arg z = \theta$ meets D infinitely often (i.e., meets infinitely many discs of D) has measure zero, we can find two rays $\arg z = \theta'_1, \theta'_2$ with $\alpha + \frac{3}{2}\varepsilon_0 < \theta'_1 < \alpha + 2\varepsilon_0$ and $\beta - 2\varepsilon_0 < \theta'_2 < \beta - \frac{3}{2}\varepsilon_0$ such that they only meet finitely many discs in D. So if r is sufficiently large,

$$\{z = re^{i\theta} \mid \theta = \theta'_1 \text{ or } \theta'_2\} \cap D = \phi. \tag{8.4}$$

From

$$4A = \left(\frac{E'}{E}\right)^2 - 2\left(\frac{E''}{E}\right) - \frac{c^2}{E^2},$$

where $c = W(f_1, f_2) \neq 0$, we have

$$\begin{aligned} S_{\theta'_1\theta'_2}(r, E) &= S_{\theta'_1\theta'_2}\left(r, \frac{1}{E}\right) + O(1) \\ &\geqq \tfrac{1}{2}S_{\theta'_1\theta'_2}\left(r, \frac{c^2}{E^2}\right) + O(1) \\ &\geqq \tfrac{1}{2}D_{\theta'_1\theta'_2}\left(r, \frac{c^2}{E^2}\right) + O(1) \\ &= \tfrac{1}{2}D_{\theta'_1\theta'_2}\left(r, 4A + 2\left(\frac{E''}{E}\right) - \left(\frac{E'}{E}\right)^2\right) + O(1) \\ &\geqq \tfrac{1}{2}\left(D_{\theta'_1\theta'_2}(r, A) - 2D_{\theta'_1\theta'_2}\left(r, \frac{E'}{E}\right) - D_{\theta'_1\theta'_2}\left(r, \frac{E''}{E}\right)\right) + O(1) \end{aligned} \tag{8.5}$$

$$= \tfrac{1}{2}S_{\theta'_1\theta'_2}(r, A) - D_{\theta'_1\theta'_2}\left(r, \frac{E'}{E}\right) - \tfrac{1}{2}D_{\theta'_1\theta'_2}\left(r, \frac{E''}{E}\right) + O(1).$$

From the choice of θ'_1 and θ'_2, as we did in the proof of (4.6), we deduce that $\sin k(\varphi - \alpha - \varepsilon_0) \geqq \sin \dfrac{k\varepsilon_0}{4}$ for all $\varphi \in [\theta'_1, \theta'_2]$. If $\varphi \in [\theta'_1, \theta'_2]$ and $z = re^{i\varphi}$ lie outside D, we deduce from (8.2) and (8.3) that

$$\left|\frac{E'}{E}(re^{i\varphi})\right| \leqq K\left[\sin\left(\frac{k\varepsilon_0}{4}\right)\right]^{-2} r^M \tag{8.6}$$

and

$$\left|\frac{E''}{E}(re^{i\varphi})\right| \leqq K\left[\sin\left(\frac{k\varepsilon_0}{4}\right)\sin\left(\frac{k_\varepsilon\varepsilon_0}{4}\right)\right]^{-2} r^M. \tag{8.7}$$

From (8.4), (8.6), (8.7) and definition of $D_{\theta'_1,\theta'_2}$, we have

$$D_{\theta'_1\theta'_2}\left(r, \frac{E'}{E}\right) + D_{\theta'_1\theta'_2}\left(r, \frac{E''}{E}\right) = O\left(\int_1^r \frac{\log t}{t^{1+\frac{\pi}{\theta'_1\theta'_2}}}\,dt + \frac{\log r}{r^{1+\frac{\pi}{\theta'_1\theta'_2}}}\right) = O(1) \tag{8.8}$$

for all r except for a set of r with finite linear measure. It follows from (8.5) and (8.8) that

$$S_{\theta'_1\theta'_2}(2r, E) \geqq \tfrac{1}{6}S_{\theta'_1\theta'_2}(r, A), \tag{8.9}$$

for all larger r.

We next show that the order of $S_{\theta'_1\theta'_2}(r, A)$ is at least $\rho - \dfrac{\pi}{\theta'_2 - \theta'_1}$. In fact, since there exists a ray $\arg z = \theta \in [\alpha + 3\varepsilon_0, \beta - 3\varepsilon_0] \subset (\theta'_1, \theta'_2)$ such that (1.8) holds, we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\alpha + 3\varepsilon_0, \beta - 3\varepsilon_0), A)}{\log r} = \rho > \frac{\pi}{\beta - \alpha - 4\varepsilon_0}. \tag{8.10}$$

By using Lemma 2, we deduce that there exists a complex number a such that the order of $n\{\Omega(\alpha + 2\varepsilon_0, \beta - 2\varepsilon_0, r), A = \alpha\}$ is ρ. As in the proof of (4.7), we deduce that $\sigma_{\theta'_1\theta'_2}(A)$ is at least $\rho - \dfrac{\pi}{\theta'_2 - \theta'_1}$. From (8.9) we know that $\sigma_{\theta'_1\theta'_2}(E)$ is at least $\rho - \dfrac{\pi}{\theta'_2 - \theta'_1}$. The claim is proved.

From the claim we must have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\theta_1', \theta_2'), E)}{\log r} \geqq \rho, \tag{8.11}$$

otherwise by direct calculation we deduce that $\sigma_{\theta_1'\theta_2'}(r, E)$ is less than $\rho - \dfrac{\pi}{\theta_2' - \theta_1'}$.

On the other hand, since $A(z)$ satisfies the condition (i), (ii) and (iii) of the Theorem B and $\sigma_{\alpha+\varepsilon_0, \beta-\varepsilon_0}(E)$ is finite, by using Lemma 4 and 5 and in the similarity to the proof of Theorem 1 in [3], we deduce that for every $\theta \in [\alpha + \frac{5}{4}\varepsilon_0, \beta - \frac{5}{4}\varepsilon_0]$ except for a set of θ with linear measure zero there exists $r(\theta) > 0$ such that

$$\log^+ |E(re^{i\theta})| < 0(r^{\rho-\varepsilon}) \quad r > r(\theta), \varepsilon = \varepsilon(\theta) > 0.$$

It is from (8.1) that the Phragmen-Lindelöf principle is applicable in $\Omega(\alpha + \varepsilon_0, \beta - \varepsilon_0)$. Therefore we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r, \Omega(\theta_1 + \frac{3}{2}\varepsilon_0, \theta_2 - \frac{3}{2}\varepsilon_0), E)}{\log r} \leqq \rho - \varepsilon$$

for some $\varepsilon > 0$. This contradicts (8.11). Theorem 3 is completely proved.

9. Proof of Corollary 3.

It was shown in [3, Le. 5] that $A(z)$ satisfies the assumption of Theorem 3 and that for some j, $A(re^{i\theta}) = B_j(re^{i\theta})e^{P_j(re^{i\theta})}(1 + o(1))$ as $r \to \infty$ with $z = re^{i\theta}$ outside a fixed R-set. Thus every angular domain $\Omega(\alpha, \beta)$ with $\beta - \alpha > \dfrac{\pi}{\rho}$ where $\rho = \sigma(A) =$ degree P_j, must contain a ray $\arg z = \theta \in (\alpha, \beta)$ such that (1.8) holds. The corollary follows.

10. Proof of Theorem 4.

Suppose that degree $P = n$ and that $\delta(\theta_0, P) = 0$ for $\theta_0 \in R$. Then $\delta(\theta, P) < 0$ for all $\theta \in \left(\theta_0 - \dfrac{\pi}{n}, \theta_0\right)$ or $\left(\theta_0, \theta_0 + \dfrac{\pi}{n}\right)$. We assume $\delta(\theta, P) < 0$ for $\theta \in \left(\theta_0 - \dfrac{\pi}{n}, \theta_0\right)$ (the case $\delta(\theta, P) < 0$ for $\theta \in \left(\theta_0, \theta_0 + \dfrac{\pi}{n}\right)$ can be similarly treated). When $z = re^{i\theta} \in \Omega\left(\theta_0, \theta_0 + \dfrac{\pi}{n}\right)$ and lies outside an R-set D, we have [3, Le. 3]

$$|A(z)| \geqq \exp(\tfrac{1}{2}\delta(P,\theta)r^n). \tag{10.1}$$

Thus for any $\varepsilon > 0$, there exists $\theta \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 + \varepsilon\right)$ such that

$$\varlimsup_{r \to \infty} \frac{\log\log(|A(re^{i\theta}) + Q(re^{i\theta})|)}{\log r} = n.$$

It follows from Theorem 3 that $\lambda_{\theta_1}(E) = +\infty$ for some $\theta_1 \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 + \varepsilon\right)$, where $E = f_1 f_2$.

Since the order of $B(z)$ is strictly less than n, there exists $\sigma < n$ such that

$$M(r, B) = \max_{0 \leqq \theta \leqq 2\pi} |B(re^{i\theta})| < \exp(r^\sigma).$$

If $\theta \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right)$,

$$\begin{aligned} |A(re^{i\theta}) + Q(re^{i\theta})| &\leqq M(r,B)\exp(\tfrac{1}{2}\delta(P,\theta)r^n) + r^m \\ &\leqq M(r,B)\exp(-Kr^n) + r^m \\ &\leqq \exp(-Kr^n) + r^m \end{aligned}$$

where $K > 0$ is a constant depending only on ε.

It follows from [3, Lemma 2] that there exists $b > 0$ such that every solution f of $f'' + (A + Q)f = 0$ satisfies

$$\log^+ |f(re^{i\theta})| \leqq Kr^{\frac{m}{2}+1}$$

for all $\theta \in \left[\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right]$ and for all $r > b$, where K is a constant depending only on ε.

Thus we have

$$\log^+ |E(re^{i\theta})| \leqq 2Kr^{\frac{m}{2}+1} \leqq r^{n-\varepsilon},$$

for all $\theta \in \left[\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right]$. This implies that

$$\varlimsup_{r \to \infty} \frac{\log\log M\left(r, \Omega\left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right), E\right)}{\log r} \leqq n - \frac{\varepsilon}{2}.$$

It follows from Theorem 2 that there is no ray $\arg z = \theta \in \left(\theta_0 - \frac{\pi}{n} + \frac{\varepsilon}{2}, \theta_0 - \varepsilon\right)$ such that $\lambda_\theta(E) = +\infty$. So we must have $\lambda_{\theta_0-\varepsilon,\theta_0+\varepsilon}(E) = +\infty$ for any $\varepsilon > 0$. Therefore $\lambda_{\theta_0}(E) = +\infty$. The proof of Theorem 4 is completed.

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CARLEMAN APPROXIMATION ON TOTALLY REAL SUBSETS OF CLASS C^k

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Introduction.

Let X be a complex manifold and $S \subset X$ a totally real submanifold of class C^k. In [10] we showed that there is a Stein neighborhood Ω of S in X such that $\mathcal{O}(\Omega)$ is dense in $C^k(S)$ in the Whitney C^k-topology on $C^k(S)$ (or equivalently, that Carleman approximation of class C^k is possible). In this paper we extend these results to the case where $S \subset X$ is a totally real subset of class C^k.

This type of approximation was first introduced by Carleman in [2]. Papers which deal with Carleman approximation in several complex variables are [1], [4], [10], [11], [13], and [14].

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Notation.

We will use standard multiindex notation

$$\nu = (\nu_1, \ldots, \nu_n) \in \mathsf{N}_0^n,$$

$$|\nu| = \nu_1 + \ldots + \nu_n.$$

Differentiation in R^n is denoted by

$$D^\nu = D_t^\nu = \frac{\partial^{|\nu|}}{\partial t^\nu} = \frac{\partial^{|\nu|}}{\partial t_1^{\nu_1} \ldots \partial t_n^{\nu_n}}.$$

All manifolds are assumed to be second countable. We will refer to both the "usual" C^k-topology on a manifold, and a stronger topology which we will call the Whitney C^k-topology. If Whitney's name is not mentioned explicitly, we always mean the "usual" C^k-topology.

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Jet Bundles and Whitney Functions.

We give a description of the jet bundle and the various subbundles that will be used. For more details and proofs we refer to [5] and [9]. Let X and Y be smooth manifolds. If $f\colon X \to Y$ is a map of class C^k, then $df\colon TX \to TY$ is a map of class C^{k-1}. We say that two maps $f, g\colon X \to Y$ have 0th order contact at $p \in X$ if $f(p) = g(p)$. Inductively, we say that two maps $f, g\colon X \to Y$ of class C^k have k-th order contact at p if $df, dg\colon TX \to TY$ have $(k-1)$th order contact at every point of T_pX. The notion of kth order contact at p is an equivalence relation on $C^k(X, Y)$, and the equivalence classes are called k-jets at p. The set of such equivalence classes will be denoted by $J_p^k(X, Y)$. The disjoint union

$$J^k(X, Y) = \cup_{p \in X} J_p^k(X, Y)$$

is called the bundle of k-jets (or simply the jet bundle), it is a fiber bundle over X in a natural manner. We will only be concerned with the case where $Y = \mathbf{C}$, in this case $J^k(X, \mathbf{C})$ is a complex vector bundle over X.

Let $\Gamma_k(X)$ be the set of continuous sections of the jet bundle $J^k(X, \mathbf{C})$. We introduce a topology on $\Gamma_k(X)$ in the following manner. If $E \subset J^k(X, \mathbf{C})$ is an open subset, then let $M(E) = \{\sigma \in \Gamma_k(X) : \sigma(p) \in E \text{ for all } p \in X\}$. If $\sigma_0 \in \Gamma_k(X)$, then a neighborhood system at σ_0 is given by $\{M(E)\}$ where E runs over all open sets in $J^k(X, \mathbf{C})$ which contain $\sigma_0(X)$. The topology on $\Gamma_k(X)$ defined in this manner is called the Whitney C^k-topology.

We give a convenient alternative description of the Whitney C^k-topology on $\Gamma_k(X)$. Choose a norm $\|\cdot\|_p$ on each $J_p^k(X, \mathbf{C})$ which varies continuously with respect to p. Let $\mathscr{A} \subset \Gamma_k(X)$ be a set of sections of the jet bundle, and let $\phi \in \Gamma_k(X)$ be given. Then ϕ lies in the closure of $\mathscr{A}$ in the Whitney C^k-topology on $\Gamma_k(X)$ iff for each positive continuous function $\varepsilon\colon X \to \mathbf{R}$ there exists $\phi_\varepsilon \in \mathscr{A}$ such that

$$\|\phi_\varepsilon(p) - \phi(p)\|_p < \varepsilon(p)$$

for all $p \in X$.

Let $S \subset X$ be a closed subset. A continuous section over S of the jet bundle $J^k(X, \mathbf{C})$ is a continuous map $\phi\colon S \to J^k(X, \mathbf{C})$ such that $\phi(p) \in J_p^k(X, \mathbf{C})$ for all $p \in S$. The set of all continuous sections over S is denoted by $\Gamma_k(S)$. The map Θ: $\Gamma_k(X) \to \Gamma_k(S)$ given by restricting the domain of a section is surjective, hence we can define the Whitney C^k-topology on $\Gamma_k(S)$ by letting $U \subset \Gamma_k(S)$ be open iff $\Theta^{-1}(U)$ is open in $\Gamma_k(X)$.

Any function $f\colon X \to \mathbf{C}$ of class C^k induces a continuous section $j_k(f)$ in the jet bundle $J^k(X, \mathbf{C})$. The question of which sections are induced by functions is answered by Whitney's extension theorem (see [15]). Let $S \subset X$ be a closed subset, and let ϕ be a continuous section over S. For each $p \in S$, let f_p be a representative for $\phi(p)$. Let (x, U) be some choice of local coordinates on X, and let $K \subset x(S \cap U)$ be a compact set. Then ϕ induces a family of functions on K (i.e. a jet in the sense of [9]) by

$$F(t) = \{D^\nu(f_{x^{-1}(t)} \circ x^{-1})(t)\}_{|\nu| \leqq k} = \{g_\nu(t)\}_{|\nu| \leqq k},$$

where $t \in K$. Whitney's condition is that

$$g_\nu(t) - D_t^\nu \sum_{|\alpha| \leqq k} \frac{g_\alpha(s)}{\alpha!}(t-s)^\alpha = o(|t-s|^{k-|\nu|})$$

uniformly for $s, t \in K$ and for all ν with $|\nu| \leqq k$. If Whitney's condition is satisfied for all choices (x, U) of local coordinates on X and for all compacts $K \subset x(S \cap U)$, then there is some $f \in C^k(X)$ such that the restriction of $j_k(f)$ to S is equal to ϕ. In that case we will call ϕ a Whitney function of class C^k. The set of Whitney functions of class C^k on S will be denoted by $W^k(S)$. We give $W^k(S)$ the induced topology from $\Gamma_k(S)$. Clearly, $W^k(S)$ is closed in $\Gamma_k(S)$.

From now on, if f is a function of class C^k in a neighborhood of $S \subset X$, then $j_k(f)$ will denote the section over S induced by f.

Let X be a complex n-dimensional manifold. Let $S \subset X$ be a closed subset, and let $\phi \in W^k(S)$ be given. Choose some function $f \in C^k(X)$ such that $j_k(f) = \phi$, and let (z, U) be some choice of holomorphic coordinates such that $S \cap U \neq \emptyset$. Consider the condition

$$(*) \qquad \bar{\partial}\frac{\partial^{|\nu|} f}{\partial z^\nu} = 0 \quad \text{on } z(S \cap U)$$

for all multiindices $\nu = (\nu_1, \ldots, \nu_n)$ of order $\leqq k-1$. This condition is independent of the choice of representative f. If (ζ, V) is another choice of local coordinates with $S \cap U \cap V \neq \emptyset$, then $(*)$ implies that

$$\bar{\partial}\frac{\partial^{|\nu|} f}{\partial \zeta^\nu} = 0 \quad \text{on } \zeta(S \cap U \cap V)$$

for all multiindices of order $\leqq k-1$. Hence we can define the closed subspace $H^k(S) \subset W^k(S)$ by $\phi \in H^k(S)$ iff $(*)$ is satisfied for all choices of representatives f for ϕ and all choices of local coordinates (z, U) with $S \cap U \neq \emptyset$. We give $H^k(S)$ the induced topology from $W^k(S)$. We will interpret $H^k(S)$ as those Whitney functions of class C^k which satisfy the Cauchy-Riemann equations up to order k on S.

We observe that if f is holomorphic in a neighborhood of S, then necessarily $j_k(f) \in H^k(S)$.

Totally Real Subsets.

Let X be a complex n-dimensional manifold. We say that a closed subset $S \subset X$ is a totally real subset of class C^k ($k \geqq 1$) if there exists a non-negative function

$\rho \in C^{k+1}(X)$ which is strictly plurisubharmonic on a neighborhood of S and such that $S = \rho^{-1}(0)$. It is shown in [7] that if $S \subset X$ satisfies the condition above then for each $p \in S$ there are a neighborhood U of p and a totally real submanifold $M \subset U$ of class C^k such that $S \cap U \subset M$. (In [7] only the case $k = 1$ is considered, but the same proof works without change for all positive integers k.) In [8] it is shown that a totally real submanifold of class C^1 is also a totally real subset of class C^1, and this is generalized in [12] to totally real submanifolds and subsets of class C^k, $k \geqq 1$. The argument given in the Note added in proof of [6] shows that any closed subset of a totally real submanifold of class C^k is a totally real subset of class C^k. Hence $S \subset X$ is a totally real subset of class C^k iff S can locally be embedded as a closed subset of a totally real submanifold of class C^k. We note that in [3] an example is given of a totally real subset which cannot be globally embedded in any totally real submanifold.

Let $M \subset X$ be a totally real submanifold of class C^k and real dimension n (i.e. the maximal possible). Let $S \subset M$ be a closed subset, and let $\phi \in H^k(S)$. Let $\tilde{f} \in C^k(X)$ be a function such that $j_k(\tilde{f}) = \phi$, and let f be the restriction of $\tilde{f}$ to M. Then it is possible to recover ϕ from f, since the partial derivatives of $\tilde{f}$ in the non-tangential directions are determined by the partial derivatives in the tangential directions together with the Cauchy-Riemann equations.

Again, let $M \subset X$ be as in the preceeding paragraph, and let $f \in C^k(M)$ be given. In [8, Lemma 4.3] it is shown that there exists an extension $\tilde{f}$ of f which is C^k on a neighborhood of M and which satisfies the Cauchy-Riemann equations up to order k on M. Hence f determines an element of $H^k(M)$, and since $\dim_R M = n$ we see that this is a one-to-one correspondence between $C^k(M)$ and $H^k(M)$.

We can now state the theorem that will prove in this paper.

THEOREM. *Let X be a complex n-dimensional manifold and let $S \subset X$ be a totally real subset of class C^k. Then there is a Stein neighborhood Ω of S in X such that the set $\{j_k(h)\colon h \in \mathcal{O}(\Omega)\}$ is dense in $H^k(S)$ in the Whitney C^k-topology.*

Approximation.

Proposition 1 and Proposition 2 below are both taken from [10].

PROPOSITION 1. *Let X be a complex manifold and let $M \subset X$ be a totally real submanifold of class C^k, $k \geqq 1$. For each $p \in M$ there are neighborhoods*

$$U' \subset\subset U'' \subset\subset U \subset\subset X$$

around p and a neigborhood $W \subset U$ around $M \cap \partial U''$ such that if $f \in C^k(M)$ has compact support contained in $M \cap U'$, then there are holomorphic functions $h_t \in \mathcal{O}(U)$, $t > 0$, such that $h_t \to f$ in the C^k-topology on $M \cap U$ and $h_t \to 0$ in the

C^k-topology on W as $t \to 0$. It is possible to choose U such that if $V \subset\subset U$ is an open subset, then U' and U'' can be chosen such that $V \subset\subset U'$.

The last assertion of Proposition 1 is not stated explicitly in [10], but it follows immediately from the proof, since U', U'', and U are images of polydiscs which may be chosen arbitrarily close to each other.

Let $\{U_j\}$ be a locally finite cover of S by open sets $U_j \subset\subset X$ with the following properties:

(1) For each j there is a totally real submanifold $M_j \subset U_j$ of class C^k and of real dimension n such that $S \cap U_j \subset M_j$.

(2) For each j there is an open set $V_j \subset\subset U_j$, and $\{V_j\}$ is also a locally finite cover of S.

(3) For each j there are open sets $U_j' \subset\subset U_j'' \subset\subset U_j$ and $W_j \subset U_j$ such that $V_j \subset\subset U_j'$ and the conditions in Proposition 1 are satisfied for these sets.

Since S has a fundamental system of Stein neighborhoods (see[6]), we can choose a Stein neighborhood Ω of S such that $\Omega \cap \partial U_j'' \subset W_j$ for all j. For each j, choose $\eta_j \in C^k(X)$ such that $0 \leqq \eta \leqq 1$, $\eta_j \equiv 1$ on V_j, $\operatorname{supp} \eta_j \subset U_j'$, and the k-jet induced by η_j lies in $H^k(M_j)$.

PROPOSITION 2. *Under the assumptions above, if $f \in C^k(M_j)$ has compact support contained in $M_j \cap U_j'$, then there are functions $h_t \in \mathcal{O}(\Omega)$ such that $h_t \to f$ in the C^k-topology on $\Omega \cap M_j \cap U_j$ and $h_t \to 0$ on the C^k-topology on $\Omega \backslash U_j''$ as $t \to 0$.*

Let $\phi \in H^k(S)$ be given. For each $p \in S$, choose a norm $\|\cdot\|_p$ on $J_p^k(X, \mathrm{C})$ such that $\|\cdot\|$ varies continuously with respect to $p \in S$. Let $A\colon S \to \mathrm{R}$ be a continuous function such that if g_1, g_2 are C^k-functions then

$$\|j_k(g_1 g_2)(p)\|_p \leqq A(p) \|j_k(g_1)(p)\|_p \|j_k(g_2)(p)\|_p$$

for all $p \in S$. Let $\varepsilon\colon S \to \mathrm{R}$ be a positive, continuous function. We will show that there is $h \in \mathcal{O}(\Omega)$ such that $\|j_k(h)(p) - \phi(p)\|_p < \varepsilon(p)$ for all $p \in S$. Let $\tilde{f} \in C^k(X)$ be such that $j_k(\tilde{f}) = \phi$ at all points of S. Let $\tilde{f}_j = \eta_j \tilde{f}$, and let f_j be the restriction of $\tilde{f}_j$ to M_j. By Proposition 1, there are $\tilde{h}_j^{(t)} \in \mathcal{O}(\Omega \cap U_j)$ such that $\tilde{h}_j^{(t)} \to f_j$ in the C^k-topology on $\Omega \cap M_j \cap U_j$ and $\tilde{h}_j^{(t)} \to 0$ in the C^k-topology on $\Omega \cap W_j$ as $t \to 0$. By Proposition 2, there are $h_j^{(t)} \in \mathcal{O}(\Omega)$ such that $h_j^{(t)} \to f_j$ in the C^k-topology on $\Omega \cap M_j \cap U_j$ and $h_j^{(t)} \to 0$ in the C^k-topology on $\Omega \backslash U_j''$.

Let $\{K_m\}$ be a sequence of compact sets in Ω with $K_m \subset K_{m+1}$ such that

$$\Omega = \bigcup_{m=1}^{\infty} K_m \quad \text{and} \quad \bigcup_{j=1}^{m} S \cap U_j \subset S \cap K_m$$

for all positive integers m. Let

$$k(m) = \max\{j : K_m \cap U_j \neq \emptyset\},$$

$$\alpha_m = \max\{A(p)\,\|j_k(1 - \eta_j)(p)\|_p : p \in K_m, j \leqq k(m)\},$$

and let $\{C_m\}$ be an increasing sequence such that $C_m \geqq \alpha_{m+1} \dots \alpha_{k(m)}$ for all m. By Proposition 2, we can choose $h_1 \in \mathcal{O}(\Omega)$ such that

$$\|j_k(h_1 - \eta_1 \tilde{f})\|_{S \cap \bar{U}_1} < \inf\left\{\frac{\varepsilon(p)}{2C_1} : p \in S \cap \bar{U}_1\right\},$$

$$\|j_k(h_1)\|_{K_1 \setminus U_1} < \inf\left\{\frac{\varepsilon(p)}{2C_1} : p \in S \cap K_1\right\}.$$

Inductively, choose $h_m \in \mathcal{O}(\Omega)$ such that

$$(*) \qquad \left\|j_k\left(h_m - \eta_m\left(\tilde{f} - \sum^{m-1} h_j\right)\right)\right\|_{S \cap \bar{U}_m} < \inf\left\{\frac{\varepsilon(p)}{2^m C_m} : p \in S \cap \bar{U}_m\right\},$$

$$(**) \qquad \|j_k(h_m)\|_{K_m \setminus U_m} < \inf\left\{\frac{\varepsilon(p)}{2^m C_m} : p \in S \cap K_m\right\}.$$

Let $h = \sum h_m$. From (**) we easily get that the series converges uniformly on compacts in Ω, and hence that $h \in \mathcal{O}(\Omega)$. We claim that $\|j_k(h)(p) - \phi(p)\|_p < \varepsilon(p)$ for all $p \in S$. So let $p \in S$ be given and let $m_0 = \max\{j : p \in V_j\}$, $m_1 = \max\{j : p \in U_j\}$. The norms below are all the norm $\|\cdot\|_p$ on $J_p^k(X, \mathbf{C})$. From (**) we get that

$$\left\|\sum_{j > m_1} j_k(h_j)(p)\right\| < \frac{\varepsilon(p)}{2^{m_1} C_{m_1+1}},$$

and from (*) we get that

$$\left\|\sum^{m_0} j_k(h_j)(p) - \phi(p)\right\| < \frac{\varepsilon(p)}{2^{m_0} C_{m_0}},$$

Let $m_0 < m \leqq m_1$, then $p \in K_m$. If $p \notin U_m$ then

$$\left\|\sum^{m} j_k(h_j)(p) - \phi(p)\right\| \leqq \|j_k(h_m)(p)\| + \left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\|$$
$$\leqq \frac{\varepsilon(p)}{2^m C_m} + \left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\|.$$

If $p \in U_m$ then

$$\left\|\sum^{m} j_k(h_j)(p) - \phi(p)\right\| = \left\|j_k\left(h_m - (\eta_m + 1 - \eta_m)\left(\tilde{f} - \sum^{m-1} h_j\right)\right)(p)\right\|$$
$$\leqq \left\|j_k\left(h_m - \eta_m\right)\left(\tilde{f} - \sum^{m-1} h_j\right)\right)(p)\right\| + A(p)\|j_k(1 - \eta_m)(p)\| \left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\|$$
$$\leqq \frac{\varepsilon(p)}{2^m C_m} + \alpha_m \left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\|.$$

Putting these results together, we get that

$$\|j_k(h)(p) - \phi(p)\| \leqq \left\|\sum^{m_1} j_k(h_j)(p) - \phi(p)\right\| + \left\|\sum_{j>m_1} j_k(h_j)(p)\right\|$$

$$\leqq \sum_{j=m_0+1}^{m_1} \alpha_j \dots \alpha_{m_1} \frac{\varepsilon(p)}{2^{j-1}C_{j-1}} + \frac{\varepsilon(p)}{2^{m_1}C_{m_1}} + \frac{\varepsilon(p)}{2^{m_1}C_{m_1+1}}$$

$$< \varepsilon(p).$$

This ends the proof of the Theorem.

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