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Integral Formulae Associated with the Euler-Lagrange Operators of Multiple Integral Problems in the Calculus of Variations

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Introduction

It is well known that, in the calculus of variations of $m$-fold integrals, there exists certain classes of integrands, to be denoted collectively by $\Phi$, which are such that the corresponding Euler-Lagrange equations are satisfied by all field functions. In particular, it is possible to construct special types of such functions, to be collectively denoted by $\Psi$, which are explicitly derivable from a given set of $m$ scalar functions $S^x (x=1,\ldots, m)$, by means of which all known field theories, including those of Carathéodory [2] and Weyl [11], [12], may be obtained. This is feasible because each of these $\Psi$ defines an independent integral.

This state of affairs immediately gives rise to the question as to whether all functions $\Phi$ possess this property, and if so, whether the associated independent integrals are applicable to new field-theoretic constructions. Accordingly the first section of this paper is devoted to an exhaustive characterization of all functions $\Phi$ for which the Euler-Lagrange equations are identically satisfied; the explicit representation of these $\Phi$ is found to be identical with an expression given by Edelen [4] (who did not, however, present the somewhat complicated analysis of all relevant integrability conditions on which this derivation is based).

The functions $\Phi$ thus obtained may be expressed as divergences, and therefore they give rise directly to certain integral formulae. However, the integrands of the corresponding $(m-1)$-fold integrals over the boundary $\partial G$ of the $m$-dimensional region $G$ of integration depend not only on the field functions, but also on their derivatives. Thus it is not immediately obvious that these integrals assume the same value for all field functions if it is merely assumed that the latter (but not necessarily their derivatives) coincide on $\partial G$. Unless our integral formulae possess this property, they are essentially trivial. In section 2 it is, however, shown that corresponding to each function $\Phi$ for which the Euler-Lagrange equations are identically satisfied, there exists an integral formula which is non-trivial in this sense. In other words, all such functions $\Phi$ give rise to independent integrals.

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This conclusion suggests – *inter alia* – that the aforementioned functions $\Psi$ can be characterized explicitly as a subclass of the functions $\Phi$, and it is briefly shown in section 3 how this can be done. In particular, it is found that the $\Psi$ constitute a proper subclass of the $\Phi$, and accordingly it is possible – at least in theory – to construct multiple integral field theories more general than those considered in the past. This is not done here; instead, the general integral formula is applied in section 4 to the derivation of a sufficiency condition for a general multiple integral optimal control problem whose state equation is specified by a system of first order partial differential equations. The sufficiency condition thus obtained represents a substantial generalization of a sufficiency criterion given recently by Butkovskii [1] for control problems of this kind.

1. Complete Specification of Integrands for which the Euler-Lagrange Equations are satisfied identically

We shall begin by specifying our notation. The $m$ independent variables are denoted by $t^z$, $(z=1, \ldots, m)$, while the dependent variables (i.e., the ‘field’ functions) are represented by $x^j = x^j (t^z)$, $(j=1, \ldots, n)$, these being assumed to be of class $C^2$ on a given simply connected closed region $G$ in the domain $R_m$ of the variables $t^z$. [Here, and in the sequel, Greek and Latin indices range from 1 to $m$ and from 1 to $n$ respectively; the summation convention is operative in respect of both sets of indices.] We shall write $\dot{x}_a^j = \partial x^j / \partial t^a$, and for any differentiable function $F=F(t^z, x^j, \dot{x}_a^j)$ the operator $d/dt^a$ is defined by

$$\frac{dF}{dt^a} = \frac{\partial F}{\partial t^a} + \frac{\partial F}{\partial x^j} \dot{x}_a^j + \frac{\partial F}{\partial \dot{x}_a^j} \ddot{x}_a^j.$$  

In the configuration space $R_{m+n}$ of the variables $(t^z, x^j)$ a system of equations of the type

$$x^j = x^j (t^z),$$  

(1.2)

gives the parametric representation of an $m$-dimensional subspace $C_m$, it being assumed that the derivatives $\dot{x}_a^j$ are linearly independent.

Now let us suppose that we are given a Lagrange function $L(t^z, x^j, \dot{x}_a^j)$ which is of class $C^1$ in its $m+n+mn$ arguments. This gives rise to a variational problem associated with the $m$-fold integral

$$I(C_m) = \int_G L(t^z, x^j (t^z), \dot{x}_a^j (t^z)) d(t),$$  

(1.3)

where we have used the notation

$$d(t) = dt^1 \ldots dt^m.$$  

(1.4)
In (1.3) the arguments of $L$ refer explicitly to the functions (1.2) which define the subspace $C_m$; clearly the value of (1.3) will in general depend on the choice of $C_m$. One is usually concerned with the values of (1.3) as obtained by substitution in $L$ from the set of all class $C^2$ functions $x^j = \tilde{x}^j(t^a)$ which coincide with (1.2) on the boundary $\partial G$ of $G$:

$$\tilde{x}^j(t^a) = x^j(t^a) \quad \text{on} \quad \partial G.$$  

(1.5)

In order that $x^j(t^a)$ afford an extreme value to the integral (1.3) from amongst this class of admissible functions, it is necessary that the $x^j(t^a)$ be solutions of the Euler-Lagrange equations

$$E_j(L) = 0,$$

(1.6)

in which the Euler-Lagrange operator $E_j$ is defined by

$$E_j(L) = \frac{d}{dt^a} \left( \frac{\partial L}{\partial \dot{x}^j} \right) - \frac{\partial L}{\partial x^j}.$$  

(1.7)

It is known, however, that there exist certain class $C^2$ functions $\Phi(t^a, x^j, \dot{x}^j_z)$ for which

$$E_j(\Phi) = 0$$

(1.8)

identically, and it is the object of this section to give a complete specification of all functions of this type. Indeed, it can be shown ([4], p. 119; [8], p.253) that if $\Phi$ is to be of this kind it is necessary and sufficient that it be a polynomial in $\dot{x}^j_z$ of the form

$$\Phi(\tau) = \sum_{s=1}^{r} A_{j_1...j_s}(t^a, x^h) \dot{x}^{j_1}_{z_1} ... \dot{x}^{j_s}_{z_s} + A(t^a, x^h),$$

(1.9)

provided that the following conditions are satisfied: (i) The integer $r$ is restricted according to

$$1 \leq r \leq M, \quad M = \min(m, n);$$

(1.10)

(ii) the coefficients $A_{j_1...j_s}$, $A$ are class $C^2$ functions of $(t^a, x^h)$, the former being completely skew-symmetric in pairs of superscripts and in pairs of subscripts, which also implies that the symmetry conditions

$$A_{j_1...j_p...j_q...j_s} = A_{j_1...j_q...j_p...j_s},$$

(1.11)

hold; (iii) the following system of first order partial differential equations must be satisfied:
\[(s + 1) \frac{\partial}{\partial t^s} \left( A_{j_1 \ldots j_s} \right) - \sum_{\mu=1}^{s} \frac{\partial}{\partial x^\mu} \left( A_{j_1 \ldots j_s} \right) = 0, \quad (s = 1, \ldots, r), \]
\[(1.12)\]

together with
\[\frac{\partial}{\partial t^s} \left( A_{j} \right) - \frac{\partial A}{\partial x^j} = 0. \quad (1.13)\]

It will now be shown that it is in fact possible to integrate this system in such a manner as to obtain an expression for \( \Phi_{(r)} \) which clearly displays \( \Phi_{(r)} \) as a divergence. To this end we introduce the following notation: For any set of quantities \( T^{j_1 \ldots j_p} \) endowed with \( p \) indices we shall write
\[\sum_{(j_1 \ldots j_p)} T^{j_1 j_2 \ldots j_p} = T^{j_1 j_2 \ldots j_p} + T^{j_p j_1 \ldots j_{p-1}} + \ldots + T^{j_{p-1} j_p j_1} \quad (1.14)\]
and
\[\sum_{(j_1 \ldots j_p)}^* T^{j_1 j_2 \ldots j_p} = T^{j_1 j_2 \ldots j_p} - T^{j_p j_1 \ldots j_{p-1}} + \ldots + (-1)^{p+1} T^{j_{p-1} j_p j_1} \quad (1.15)\]
The system (1.12) can then be written in the form
\[(s + 1) \frac{\partial}{\partial t^s} \left( A_{j_1 \ldots j_s} \right) - \sum_{(j_1 \ldots j_s)} \frac{\partial}{\partial x^j} \left( A_{j_1 \ldots j_s} \right) = 0, \quad \text{if } s \text{ is even}, \quad (1.16)\]
and
\[(s + 1) \frac{\partial}{\partial t^s} \left( A_{j_1 \ldots j_s} \right) - \sum_{(j_1 \ldots j_s)}^* \frac{\partial}{\partial x^j} \left( A_{j_1 \ldots j_s} \right) = 0, \quad \text{if } s \text{ is odd} \quad (1.17)\]

Let us assume that \( r \) is odd. Putting \( s = r \) in (1.17), and noting that the polynomial (1.9) is at most of order \( r \), we obtain
\[\sum_{(j_1 \ldots j_r)}^* \frac{\partial}{\partial x^j} \left( A_{j_1 \ldots j_r} \right) = 0. \quad (1.18)\]

However, this represents a set of integrability conditions (\cite{5}, p. 105 et seq.) which ensure the existence of a system of class \( C^3 \) functions \( V_{j_2 \ldots j_r} \) \((t^a, x^b)\) possessing symmetry and skew-symmetry properties identical with those of the coefficients in (1.9) such that
\[A_{j_1 \ldots j_r}^{(r) a_1 \ldots a_r} = \sum_{(j_1 \ldots j_r)} \frac{\partial}{\partial x^{j_1}} \left( V_{j_2 \ldots j_r}^{(r) a_1 a_2 \ldots a_r} \right). \quad (1.19)\]
For future reference we note that, as a result of these symmetry properties,
\[ A_{j_1 \ldots j_r} \dot{x}_{j_1}^{j_1} \ldots \dot{x}_{j_r}^{j_r} = r \frac{\partial}{\partial x_{j_1}^{j_1}} \left( A_{j_2 \ldots j_r} \dot{x}_{j_2}^{j_1} \ldots \dot{x}_{j_r}^{j_r} \right). \tag{1.20} \]

Let us now put \( s = r - 1 \) (which is even) in (1.16), which gives
\[ r \frac{\partial}{\partial t^2} \left( A_{j_{j_2 \ldots j_r}} \right) - \sum_{(j_{j_2 \ldots j_r})} \frac{\partial}{\partial x^j} \left( A_{j_2 \ldots j_r} \right) \dot{x}_{j_2}^{j_2} \ldots \dot{x}_{j_r}^{j_r} = 0, \]

in which we substitute from (1.19):
\[ \sum_{(j_{j_2 \ldots j_r})} \frac{\partial}{\partial x^j} \left[ \left( A_{j_{j_2 \ldots j_r}} - r \frac{\partial}{\partial t^2} \left( A_{j_2 \ldots j_r} \right) \right) \dot{x}_{j_2}^{j_2} \ldots \dot{x}_{j_r}^{j_r} \right] = 0. \tag{1.21} \]

This is once more a set of integrability conditions, which imply the existence of class \( C^3 \) functions \( V_{j_3 \ldots j_r} \) with suitable skew-symmetry properties such that
\[ A_{j_2 \ldots j_r} - r \frac{\partial}{\partial t^2} \left( A_{j_2 \ldots j_r} \right) = \sum_{(j_{j_2 \ldots j_r})} \frac{\partial}{\partial x^j} \left( V_{j_3 \ldots j_r} \right). \tag{1.22} \]

As a result of this skew-symmetry we have
\[ \frac{\partial^2 V_{j_2 \ldots j_r}}{\partial t^2 \partial t^{j_2}} = 0, \tag{1.23} \]

so that differentiation of (1.22) with respect to \( t^{j_2} \) yields
\[ \frac{\partial}{\partial t^{j_2}} \left( A_{j_2 \ldots j_r} \right) = \sum_{(j_{j_2 \ldots j_r})} \frac{\partial}{\partial x^j} \left( \frac{\partial V_{j_3 \ldots j_r}}{\partial t^{j_2}} \right). \tag{1.24} \]

It should also be noted that (1.22) gives rise to
\[ A_{j_2 \ldots j_r} \dot{x}_{j_2}^{j_2} \ldots \dot{x}_{j_r}^{j_r} = r \frac{\partial}{\partial t^{j_2}} \left( A_{j_3 \ldots j_r} \dot{x}_{j_2}^{j_2} \ldots \dot{x}_{j_r}^{j_r} \right) + \]
\[ + (r - 1) \frac{\partial}{\partial x^j} \left( V_{j_3 \ldots j_r} \right) \dot{x}_{j_2}^{j_2} \ldots \dot{x}_{j_r}^{j_r}. \tag{1.25} \]
We now put $s=r-2$ (which is odd) in (1.17), after which we substitute from (1.24), and thus further integrability conditions analogous to (1.21) are obtained. Proceeding in this manner we obtain, for $s=r-t$ with $1 \leq t \leq r-1$, on the assumption that $t$ is odd,

$$(r-t+1) \frac{\partial}{\partial t^x} \left( \begin{array}{c}
A \\
\mathcal{A}_{jt+1...jr} \\
\end{array} \right) \left( \begin{array}{c}
(r-t+1) \mathcal{A}_{jt+1...jr} \\
\end{array} \right) - \sum_{(jjt+1...jr)} \frac{\partial}{\partial \lambda^j} \left( \begin{array}{c}
A \\
\mathcal{A}_{jt+1...jr} \\
\end{array} \right) = 0, \quad (1.26)$$

while, according to the discussion of the previous case ($s=r-t+1$), the existence of a system of functions $V_{jt+1...jr}$ is assured such that

$$\frac{\partial}{\partial t^x} \left( \begin{array}{c}
A \\
\mathcal{A}_{jt+1...jr} \\
\end{array} \right) = \sum_{(jjt+1...jr)} \frac{\partial}{\partial \lambda^j} \left( \begin{array}{c}
\frac{\partial}{\partial t^x} V_{jt+1...jr} \\
\end{array} \right).$$

This is substituted in (1.26), which again gives rise to a set of integrability conditions according to which we may write

$$(r-t) \mathcal{A}_{jt+1...jr} - (r-t+1) \frac{\partial}{\partial t^x} \left( \begin{array}{c}
\mathcal{A}_{jt+1...jr} \\
V_{jt+1...jr} \\
\end{array} \right) = \sum_{(jt+1...jr)}^* \frac{\partial}{\partial \lambda^j} \left( \begin{array}{c}
\mathcal{A}_{jt+1...jr} \\
V_{jt+1...jr} \\
\end{array} \right). \quad (1.27)$$

Similar results are obtained when $t$ is even, in which case one merely has to inter-change $\sum$ with $\sum^*$ in each of these relations. In particular, with $t=r-1$, equation (1.27) reduces to

$$(1)^{a_x} \mathcal{A}_{j} = 2 \frac{\partial}{\partial t^x} \left( \begin{array}{c}
\mathcal{A}_{j} \\
V_{j} \\
\end{array} \right) + \frac{\partial}{\partial \lambda^j} \left( \begin{array}{c}
\mathcal{A}_{j} \\
V_{j} \\
\end{array} \right), \quad (1.28)$$

so that, by virtue of the skew-symmetry of $V_j$ in $\varepsilon$ and $\alpha$,

$$\frac{\partial}{\partial t^x} \mathcal{A}_{j} = \frac{\partial^2}{\partial t^x \partial \lambda^j} V_j.$$

When this result is substituted in (1.13), the latter can be integrated immediately to yield

$$A = \frac{\partial}{\partial t^x} V^{(0)} + V^{(t^x)}, \quad (1.29)$$
where \( V(t^a) \) is an arbitrary function of \( t^a \) only. Since terms of this kind merely contribute a constant to the integral of a variational problem, we shall henceforth discard the function \( V \).

We now substitute from (1.20), (1.25), ..., (1.27), ..., (1.28), and (1.29) in (1.9), obtaining

\[
\Phi_r = r \left[ \frac{\partial}{\partial x^i} \left( V^{(r)} \frac{\partial^{(r)} x^{j_1}}{\partial x^{j_2}}} + \frac{\partial}{\partial t^a} \left( V^{(r)} \frac{\partial^{(r)} x^{j_1}}{\partial t^a} \right) \right] \frac{\partial x^{j_2}}{\partial x^{j_1}} + \frac{\partial}{\partial t^a} \left( V^{(r-1)} \frac{\partial^{(r-1)} x^{j_2}}{\partial x^{j_2}}} \right) \frac{\partial x^{j_2}}{\partial x^{j_1}} \\
+ (r - 1) \left[ \frac{\partial}{\partial x^{j_2}} \left( V^{(r-1)} \frac{\partial^{(r-1)} x^{j_2}}{\partial x^{j_2}}} \right) \frac{\partial x^{j_2}}{\partial x^{j_1}} + \frac{\partial}{\partial t^a} \left( V^{(r-1)} \frac{\partial^{(r-1)} x^{j_2}}{\partial t^a} \right) \frac{\partial x^{j_2}}{\partial t^a} \right] \\
+ \cdots + \left[ \frac{\partial}{\partial x^{j_2}} \left( V^{(1)} \frac{\partial^{(1)} x^{j_2}}{\partial x^{j_2}}} \right) \frac{\partial x^{j_2}}{\partial x^{j_1}} + \frac{\partial}{\partial t^a} \left( V^{(1)} \frac{\partial^{(1)} x^{j_2}}{\partial t^a} \right) \frac{\partial x^{j_2}}{\partial t^a} \right].
\]

(1.30)

Here it should be noted that each of the expressions in square brackets can be written more succinctly in terms of the operator \( d/dt^a \) as defined by (1.1). For, in view of the symmetry of the second derivatives \( x^{j_2}_{\alpha \beta} = \partial x^{j_2}_\beta / \partial t^\beta \) one has

\[
V^{(s)} x^{j_2}_{\alpha_1 \alpha_2} = 0, \text{ etc.,}
\]

(1.31)

which in turn allows us to express (1.30) in the form

\[
\Phi_r = r \frac{d}{dt^a} \left[ V^{(r)} x^{j_2}_\alpha x^{j_2}_{\alpha_2} \right] + (r - 1) \frac{d}{dt^a} \left[ V^{(r-1)} x^{j_2}_{\alpha_2} x^{j_2}_{\alpha_3} \right] \\
+ \cdots + \frac{d}{dt^a} \\
+ \left[ \frac{\partial}{\partial x^{j_2}} \left( V^{(1)} \frac{\partial^{(1)} x^{j_2}}{\partial x^{j_2}}} \right) \frac{\partial x^{j_2}}{\partial x^{j_1}} + \frac{\partial}{\partial t^a} \left( V^{(1)} \frac{\partial^{(1)} x^{j_2}}{\partial t^a} \right) \frac{\partial x^{j_2}}{\partial t^a} \right].
\]

(1.32)

This result suggests that we define the following functions:

\[
W = V^{(s)} x^{j_2}_\alpha x^{j_2}_{\alpha_2} \text{, } 2 \leq s \leq r, \quad 1 \leq r \leq M,
\]

(1.33)

with

\[
W = V^{(1)} x^{j_2}_\alpha x^{j_2}_{\alpha_2} \text{, }
\]

(1.34)

so that (1.32) assumes the following final form

\[
\Phi_r = \sum_{s=1}^{r} s \frac{dW}{dt^a}.
\]
The above derivation of formula (1.32) is subject to the assumption that the integer \( r \) is odd; a very similar argument may be applied to the case when \( r \) is even, and a formally identical final result is thus obtained. We may therefore summarize our conclusions in the form of the following

**THEOREM.** In order that a function \( \Phi(t^e, x^h, \dot{x}_r^h) \) be such that \( E_j(\Phi) = 0 \) identically, where the Euler-Lagrange operator \( E_j \) is defined by (1.7), it is necessary and sufficient that \( \Phi \) be of the form

\[
\Phi = \sum_{r=1}^{M} c_r \Phi_{(r)}, \quad M = \min(m, n),
\]

(1.35)

in which the \( c_r \) denote arbitrary constants, and the \( \Phi_{(r)} \) are defined by (1.34) and (1.33), the class \( C^3 \) functions \( V_{j_2, \ldots, j_s}(t^e, x^h) \) being subject solely to the requirement that they be skew-symmetric in pairs of subscripts and in pairs of superscripts.

Remark 1. This result is essentially equivalent to a theorem stated by Edelen [4], whose discussion, however, omits an explicit analysis of the integrability conditions upon which the derivation is crucially dependent.

Remark 2. It is obvious from the theorem that any function \( \Phi \) which identically satisfies the condition \( E_j(\Phi) = 0 \) is a divergence; however, because of the skew-symmetry properties stipulated above, this divergence is not entirely arbitrary. Moreover, as a result of this skew-symmetry, it follows from (1.31) that \( \Phi_{(r)} \) does not depend on the second derivatives \( \ddot{x}^j_{\alpha \beta} \). In this connection it should nevertheless be pointed out that one may prove the following somewhat different result (Lovelock [7]): if \( \phi^a(t^e, x^h, \dot{x}_r^h) \) denotes any set of \( m \) class \( C^2 \) functions, the divergence

\[
\phi(t^e, x^h, \dot{x}_r^h, \ddot{x}_r^h) = \frac{d}{dt^a} \left[ \phi^a(t^e, x^h, \dot{x}_r^h) \right]
\]

(1.36)

satisfies the conditions

\[
\frac{d}{dt^a} \left[ \frac{\partial \phi}{\partial \dot{x}^i} - \frac{d}{dt^\theta} \left( \frac{\partial \phi}{\partial \dot{x}^i_{\alpha \beta}} \right) \right] - \frac{\partial \phi}{\partial x^i} = 0
\]

(1.37)

identically, in which the left-hand side is obviously derived from the Euler-Lagrange operator of second order problems in the calculus of variations.

2. The Integral Formulae

Since the functions (1.34) are divergences, one may obtain certain integral formulae, whose derivation we shall now consider in some detail. To this end we shall suppose that the boundary \( \partial G \) of the domain \( G \) in \( \mathbb{R}^m \) can be represented para-
metrically in the form
\[ t^a = t^a(u^a), \tag{2.1} \]
in which the functions on the right-hand side are assumed to be of class \( C^2 \) in the \( m - 1 \) parameters \( u^a \) and such that the rank of the matrix \( (\partial t^a/\partial u^b) \) is \( m - 1 \). [Here, and in the sequel, the indices \( a, b \) range from 1 to \( m - 1 \), the summation convention being operative in respect of these indices also.] At each point of \( \partial G \) a normal vector to \( \partial G \) in \( R_m \) is defined by the components
\[ \pi_a = (-1)^{a+1} \frac{\partial (t^1, \ldots, t^{a-1}, t^{a+1}, \ldots, t^m)}{\partial (u^1, \ldots, u^{m-1})}, \tag{2.2} \]
which appear in the formulation of the divergence theorem.

In fact, when the latter is applied to (1.34), it immediately follows that
\[ \int_G \Phi_{(r)} d(t) = \sum_{s=1}^r s \int_{\partial G} \pi_a W^{(s)} d(u), \tag{2.3} \]
where we have written \( d(u) = du^1 \cdots du^{m-1} \) in accordance with the notation (1.4).
However, it should be noted immediately that the functions \( W \) as defined by (1.33) contain the derivatives \( \dot{x}_a^j \) explicitly. Therefore, if the integral (2.3) is to be evaluated relative to two distinct sets of functions \( x^j(t^a) \), \( \dot{x}_a^j(t^a) \), of which it is merely assumed that they coincide on the boundary \( \partial G \) of \( G \) in the sense of condition (1.5), it is not directly evident that the same values are obtained, since (1.5) does not necessarily imply that \( \dot{x}_a^j = \dot{x}_a^j \) on \( \partial G \). Consequently we cannot, at this stage, infer that the functions \( \Phi_{(r)} \) represent the integrands of independent integrals, this term being used in its customary sense in the calculus of variations. Thus, unless the left-hand side of (2.3) is, in fact, an independent integral, the conclusion (2.3) is essentially trivial. It is therefore the object of this section to show that the right-hand side of (2.3) indeed assumes the same values for all functions \( x^j(t^a) \) which coincide on \( \partial G \), even when their derivatives are not identical on \( \partial G \).

To this end we note that, when (2.1) is substituted in (1.5), we obtain
\[ \dot{x}_a^j(t^a(u^a)) = x^j(t^a(u^a)). \tag{2.4} \]

When this is differentiated with respect to \( u^a \), it is seen that
\[ \frac{\partial \dot{x}_a^j}{\partial t^a} \frac{\partial t^a}{\partial u^a} = \frac{\partial x^j}{\partial t^a} \frac{\partial t^a}{\partial u^a}, \]
from which it follows that
\[
\frac{\partial \hat{x}^i}{\partial u^a} = \frac{\partial x^j}{\partial u^a}
\]
(2.5)
on \partial G, and consequently our object is achieved if we can replace all arguments \( \hat{x}^j_\alpha \) in the integrand on the right-hand side of (2.3) by functions which merely depend on the derivatives \( \partial x^j/\partial u^a \). The resulting integrand will then yield a non-trivial integral theorem.

It has been shown elsewhere ([9]) that the components (2.2) can be expressed in the form
\[
(m - 1)! \pi_\alpha = \varepsilon_{\alpha_2 \ldots \alpha_m} J^{\beta_2 \ldots \beta_m},
\]
(2.6)
in which \( \varepsilon_{\alpha_1 \ldots \alpha_m} \) denotes the permutation symbol of Levi-Civita, while
\[
J^{\beta_2 \ldots \beta_m} = \frac{\partial \left(t^{\beta_2}, \ldots, t^{\beta_m}\right)}{\partial \left(u^{\alpha_2}, \ldots, u^{\alpha_{m-1}}\right)}
\]
(2.7)
for any selection of \((m - 1)\) distinct integers \( \beta_2, \ldots, \beta_m \) from the set \( 1, \ldots, m \). From (1.33) and (2.6) we therefore conclude that
\[
\pi_\alpha W^{(s)_m} = \frac{1}{(m - 1)!} \varepsilon_{\alpha_1 \beta_2 \ldots \beta_m} J^{\beta_2 \ldots \beta_m} (s)_{\alpha_2 \ldots \alpha_s}^{} \hat{x}^i_{\alpha_2} \ldots \hat{x}^i_{\alpha_s}.
\]
(2.8)
This expression can be simplified with the aid of the following general formula:
\[
\varepsilon_{\alpha_1 \beta_2 \ldots \beta_m} J^{\beta_2 \ldots \beta_m} (s)_{\alpha_2 \ldots \alpha_s}^{} \hat{x}^i_{\alpha_2} \ldots \hat{x}^i_{\alpha_s} = \frac{(m - 1) \cdots (m - s + 1)}{s!} \varepsilon_{\alpha_1 \beta_2 \ldots \beta_m} J^{\beta_2 \ldots \beta_m} (s)_{\alpha_2 \beta_2 \ldots \beta_s}^{} \hat{x}^i_{\alpha_2} \ldots \hat{x}^i_{\alpha_s}.
\]
(2.9)
The validity of this formula is established by a comparison of its two sides. To this end we recall the identities
\[
\varepsilon^{\alpha_1 \ldots \alpha_s \beta_2 + 1 \ldots \beta_m} \varepsilon_{\beta_1 \ldots \beta_s \beta_2 + 1 \ldots \beta_m} = (m - s)! \delta^{\alpha_1 \ldots \alpha_s}_{\beta_1 \ldots \beta_s},
\]
together with
\[
\delta_{\alpha_1 \ldots \alpha_s} T^{\alpha_1 \ldots \alpha_s} = s! T^{\beta_1 \ldots \beta_s}
\]
for any set of quantities \( T^{\alpha_1 \ldots \alpha_s} \) which are completely skew-symmetric. Thus multiplication of (2.6) by \( \varepsilon_{\alpha_2 \ldots \alpha_m} \) yields
\[
\pi_\alpha \varepsilon_{\alpha_2 \ldots \alpha_m} = J^{\alpha_2 \ldots \alpha_m}
\]
(2.10)
by virtue of the complete skew-symmetry of $J^{x_2...x_m}$, and hence the left-hand side of (2.9) can be written as

$$
\varepsilon_{a_1b_2...b_mp_1p_2...p_m}\pi^a_{p_1p_2...p_m} V^{b_2...b_m} x_{a_2}^{j_2} ... x_{a_s}^{j_s} = (m-1)! \pi_a V^{b_2...b_s} x_{a_2}^{j_2} ... x_{a_s}^{j_s}.
$$

(2.11)

On the other hand, substitution of (2.10) on the right-hand side of (2.9) yields, in the same manner,

$$
\frac{(m-1)...(m-s+1)}{s!} \varepsilon_{a_1b_2...b_s b_{s+1}...b_m} e^{a_2...a_s b_{s+1}...b_m} \pi_a V^{b_2...b_s} x_{a_2}^{j_2} ... x_{a_s}^{j_s} = (m-1)! \pi_a \delta^{a_2...a_s}_{b_2...b_s} V^{j_2...j_s} x_{a_2}^{j_2} ... x_{a_s}^{j_s}.
$$

$$
= (m-1)! \pi_a V^{j_2...j_s} x_{a_2}^{j_2} ... x_{a_s}^{j_s}.
$$

(2.11)

where, in the last step, we have made use of the complete skew-symmetry of $V^{j_2...j_s}$.

But this expression is identical with (2.11), which proves the assertion (2.9).

When (2.9) is substituted in (2.8), the latter becomes

$$
\pi^a W = \frac{1}{s!(m-s)!} \varepsilon_{a_1b_2...b_s b_{s+1}...b_m} V^{b_2...b_s} x_{a_2}^{j_2} ... x_{a_s}^{j_s}.
$$

(2.12)

However, since $\partial x^l/\partial u^a = \dot{x}_a^l (\partial t^a/\partial u^a)$, it follows directly from the definition (2.7) that

$$
J^{a_2...a_s b_{s+1}...b_m} \dot{x}_{a_2}^{j_2} ... \dot{x}_{a_s}^{j_s} = \frac{\partial (x^{j_2}, ..., x^{j_s}, t^{b_{s+1}}, ..., t^{b_m})}{\partial (u^1, ..., u^{m-1})},
$$

(2.13)

and accordingly (2.12) becomes

$$
\pi^a W = \frac{1}{s!(m-s)!} \varepsilon_{a_1b_2...b_s b_{s+1}...b_m} V^{j_2...j_s} \frac{\partial (x^{j_2}, ..., x^{j_s}, t^{b_{s+1}}, ..., t^{b_m})}{\partial (u^1, ..., u^{m-1})}.
$$

(2.14)

By means of the divergence theorem we therefore conclude that

$$
\int_\mathcal{G} \frac{dW}{dt^a} d(t) = \int_\mathcal{G} \pi^a W d(u)
$$

$$
= \frac{1}{s!(m-s)!} \int_\mathcal{G} \varepsilon_{a_1a_2...a_s a_{s+1}...a_m} V^{j_2...j_s} \frac{\partial (x^{j_2}, ..., x^{j_s}, t^{a_{s+1}}, ..., t^{a_m})}{\partial (u^1, ..., u^{m-1})} d(u).
$$

(2.15)
Because $V_{j_2...j_s}$ is a function of $(t^e, x^h)$ only, it is clear that the integrand on the right-hand side does not contain the derivatives $x^j_t$, despite the explicit dependence of $W$ on these quantities. Instead, only the derivatives $\partial x^j/\partial u^a$ occur, namely in the Jacobian which appears in the integrand, and according to (2.5) these derivatives are identical for all functions $x^j(t^a)$ which coincide on $\partial G$. Thus (2.15) represents an independent integral of the type which we have been seeking.

The required expression for the integral (2.3) is now obtained by substitution of (2.15) on the right-hand side of (2.3), which yields

$$
\int_G \Phi_{(r)}(t^e, x^h, x^h_t) \, d(t) = \sum_{s=1}^r \frac{1}{(s-1)!(m-s)!} \times \int_{\partial G} e_{a_1a_2...a_s} \frac{\partial (x^j_{j_2}, ..., x^j_{j_s}, t^{a_{s+1}}, ..., t^{a_m})}{\partial (u^1, ..., u^{m-1})} \, d(u). \tag{2.16}
$$

When this result is combined with the theorem of section 1, one obtains the following

**THEOREM.** Corresponding to any function $\Phi(t^e, x^h, x^h_t)$ for which the condition $E_j(\Phi)=0$ is identically satisfied, there exists an independent integral of the type

$$
\int_G \Phi(t^e, x^h, x^h_t) \, d(t) = \sum_{r=1}^M c_r \int_G \Phi_{(r)}(t^e, x^h, x^h_t) \, d(t), \tag{2.17}
$$

where the integrals in the sum on the right-hand side are given by (2.16). These integral formulae are non-trivial in the sense that the integrals assume the same values for all subspaces $C_m$ which coincide on the boundary $\partial G$ of $G$.

3. A Significant Special Case

It is known that there exists a class of certain relatively simple functions $\Psi_{(r)}(t^e, x^h, x^h_t)$ for which the condition $E_j(\Psi_{(r)})=0$ is satisfied identically. This class is of fundamental importance in the general field theory of multiple integral problems in the calculus of variations: indeed, it gives rise to all the known field theories. In the present section we shall briefly discuss the relation of these special functions to the general solution (1.35) of the system $E_j(\Phi)=0$.

In order to be able to define the functions $\Psi_{(r)}$ explicitly ([9], [10]) it is necessary to assume that one is given a set of $m$ functions $S^x(t^e, x^j)$ of class $C^2$. We shall write

$$
\frac{\partial S^x}{\partial t^e} = S^x_{t^e}, \quad \frac{\partial S^x}{\partial x^j} = S^x_{x^j}, \tag{3.1}
$$
and, relative to any subspace \( C_m \) as defined by (1.2), we construct the quantities

\[
e^a_\beta = S^a_\beta + S^a_j x^j_\beta.
\]  

(3.2)

For each value of the integer \( r \), with \( 1 \leq r \leq m \), we then define the functions \( \Psi(r)(t^i, x^h, \dot{x}^h_i) \) by writing

\[
r! \Psi(r) = \delta_{\beta_1 \ldots \beta_r} \epsilon_{\alpha_1 \ldots \alpha_r},
\]  

(3.3)

in which \( \delta_{\ldots} \) denotes the generalized Kronecker delta. It should be noted that \( \Psi(r) \) is identical with the sum of all principal \( r \times r \) minors of \( \det(c^a_\beta) = \Psi(m) \). It can be shown that

\[
E_j(\Psi(r)) = 0 \quad (r = 1, \ldots, m),
\]  

(3.4)

identically, and that each \( \Psi(r) \) gives rise to an independent integral [9]. [Thus each \( \Psi(r) \) can be used to construct an equivalent integral in the sense of Carathéodory [2]; the case \( r = m \) yields the field theory of the latter, whereas the case \( r = 1 \) leads to the field theory of Weyl [11], [12]. Moreover, it should be observed that, although \( \Psi(r) \) formally appears to be a polynomial of order \( r \), it follows from the definition (3.3) that the sums of all terms in \( \Psi(r) \) involving products of the type \( \dot{x}^i_{\alpha_1} \ldots \dot{x}^i_{\alpha_s} \) vanish identically when \( s > M = \min(m, n) \), which occurs whenever \( m > n \). Thus the order of the polynomials \( \Psi(r) \) actually never exceeds \( M \), which is in accordance with (1.35)].

Clearly the functions (3.3) represent a subclass of the functions defined by (1.35); this immediately gives rise to the question as to whether or not it is possible to represent all functions of the type (1.35) as linear combinations of the \( \Psi(r) \). We shall see that this cannot be true in general, and, because of the negative nature of this result, the relevant calculations are outlined very briefly.

For a given set of \( m \) class \( C^2 \) functions \( S^a(t^i, x^j) \) and a pair of integers \( (r, s) \), with \( 1 \leq r, s \leq m \), \( 0 \leq r \leq r - 2 \), we can define the following quantities:

\[
\begin{align*}
r! & \quad U_{j_1 + 2 \ldots j_r}^{(r-s) \beta_1 + 2 \ldots \beta_r} = \left( \frac{r - 1}{s} \right) \delta_{\sigma_1 \sigma_2 \ldots \sigma_s + 1 \sigma_{s+1} + 2 \ldots \sigma_{r-s+1}} \sigma_{s+1} \sigma_{s+2} \ldots \sigma_r S^\sigma_{\beta_1} S^\sigma_{\beta_2} \ldots S^\sigma_{\beta_s+1} S^\sigma_{\beta_{s+2}} \ldots S^\sigma_{\beta_r}, \end{align*}
\]  

(3.5)

together with

\[
\begin{align*}
U^{(1) \beta} = \frac{1}{2} \delta^{\beta \sigma} S^\sigma S^\lambda.
\end{align*}
\]  

(3.6)

Clearly the functions on the left-hand side of (3.5) possess skew-symmetry properties which are identical with those of the \( V \) which enter (1.33), and hence these function can serve as candidates for the construction of solutions of the type (1.35). Indeed, a
fairly long but straight-forward calculation yields the following identity:

\[
\sum_{s=0}^{r-1} \frac{d}{dt^s} \left[ \frac{(r-s)\beta_{\alpha_2+2...\alpha_r}}{U_{j_{\alpha_2+2...\alpha_r}}} \frac{\delta_{\alpha_1...\alpha_r}}{j_1} \right] = \frac{1}{r!} \delta_{\alpha_1...\alpha_r} \left( S_{\alpha_1}^1 + S_{j_1}^1 \frac{\delta_{\alpha_1}}{\alpha_1} \right) ... \left( S_{\alpha_r}^{j_r} + S_{j_r}^{j_r} \frac{\delta_{\alpha_r}}{\alpha_r} \right) = \Psi_{(r)} \tag{3.7}
\]

where the last step is a direct consequence of (3.2) and (3.3). This suggests that, in analogy with (1.33), we define the functions \( W^* \) by writing

\[
(r-s)_{\beta_{\alpha_2+2...\alpha_r}} W^* = U_{j_{\alpha_2+2...\alpha_r}} \frac{(r-s)\alpha_{\alpha_2+2...\alpha_r}}{j_1 \alpha_1+2 ... \alpha_r} \tag{3.8}
\]

which allows us to write (3.7) in the form

\[
\Psi_{(r)} = \sum_{s=1}^{r} \frac{dW^*}{dt^s} \tag{3.9}
\]

If this is compared with (1.34), it follows that \( \Psi_{(r)} = \Phi_{(s)} \), so that under these circumstances the general solution (1.3) can be expressed as a linear combination of the functions defined by (1.35). However, this construction depends crucially on the relation (3.5), and its general validity would be assured only if it were possible to assert that, given any system of skew-symmetric functions \( U_{j_{\alpha+2...\alpha_r}} \) one can find a set of \( m \) class \( C^2 \) function \( S^x \) such that a counterpart of (3.5) would be satisfied. It is not difficult to show that this is not generally true. To this end we merely put \( r=2 \), \( s=0 \) in (3.5), which then reduces to

\[
(2)_{\beta_\alpha} \frac{\delta S^x}{\partial \alpha} = \frac{1}{2} \left( S^x \frac{\partial S^\beta}{\partial x^j} - S^\beta \frac{\partial S^x}{\partial x^j} \right),
\]

or

\[
2U_{j} = S^x S^\beta \frac{\partial}{\partial x^j} \left( \ln S^x - \ln S^\beta \right), \text{(no summation over } \alpha, \beta) \tag{3.10}
\]

However, not every function \( V_{j}^{\beta_\alpha} \) which is skew-symmetric in \( \alpha \) and \( \beta \) admits a representation of this kind. For (3.10) indicates that such a function must be proportional to a gradient, for which it is necessary and sufficient ([3], p. 97) that it satisfies the very stringent conditions
\[
\begin{align*}
V_k \left( \frac{\partial V_j}{\partial x^h} - \frac{\partial V_h}{\partial x^j} \right) + V_j \left( \frac{\partial V_h}{\partial x^k} - \frac{\partial V_k}{\partial x^h} \right) \\
+ V_h \left( \frac{\partial V_k}{\partial x^j} - \frac{\partial V_j}{\partial x^k} \right) = 0, \text{ (no summation over } \alpha, \beta). \tag{3.11}
\end{align*}
\]

Similar, but considerably more complicated requirements must be imposed on the \( V_{(r)} \) in order that a counterpart of (3.5) be valid. Clearly such conditions need not be satisfied in general, and accordingly we conclude that there exist functions other than linear combinations of the \( \Psi_{(r)} \) which give rise to independent integrals (and hence to field theories for multiple integral variational problems other than those mentioned above).

Nevertheless, it is of interest to observe that our general integral formula (2.16) reduces to a much simpler form when the functions \( V_{(r)} \) which appear in the integrand on the right-hand side of (2.16) are of the special form (3.5). In fact, when the latter is substituted in (2.16), certain simplifications may be effected with the aid of the following result. For any integer \( p \), with \( 1 \leq p \leq s - 1 \), where \( 2 \leq s \leq m \), the relation

\[
p! S_{h_1}^p \ldots S_{h_p}^p \frac{\partial (x^{h_1}, \ldots, x^{h_p}, x^{h_{p+1}}, \ldots, x^{h_s-1}, t^{a_{s+1}}, \ldots, t^{a_m})}{\partial (u^1, \ldots, u^{m-1})} \\
= \delta_{\lambda_1\ldots\lambda_p}^{h_1\ldots h_p} \sum_{q=0}^p (-1)^q \binom{p}{q} S_{\epsilon_1}^{\lambda_1} \ldots S_{\epsilon_q}^{\lambda_q} \\
\times \frac{\partial (t^{\epsilon_1}, \ldots, t^{\epsilon_q}, S_{\lambda_{q+1}}^{\lambda_{q+1}}, \ldots, S_{\lambda_p}^{\lambda_p}, x^{h_{p+1}}, \ldots, x^{h_s-1}, t^{a_{s+1}}, \ldots, t^{a_m})}{\partial (u^1, \ldots, u^{m-1})} \tag{3.12}
\]

holds identically. Again, since the proof of this identity is straight-forward but somewhat lengthy, it will be omitted here. The application of (3.12), together with (3.9), reduces (2.16) to the form

\[
\int_G \Psi_{(r)}(t^i, x^l(t^i), \dot{x}^l_2(t^i)) \, d(t) = \frac{1}{r!(m-r)!} \\
\times \int_{\delta G} e_{\beta_1\ldots\beta_{r-1}\beta_r\ldots\beta_{m-1}} S^\theta \frac{\partial (S_1^\beta, \ldots, S_1^{\beta_{r-1}}, t^{\beta_r}, \ldots, t^{\beta_{m-1}})}{\partial (u^1, \ldots, u^{m-1})} \, d(u). \tag{3.13}
\]

This is identical with an integral formula derived from first principles elsewhere [9].
4. Sufficiency Conditions for an Optimal Control Problem

We shall consider an optimal control problem whose state is represented by the functions \( x^j(t^z) \), while the control functions, to be denoted by \( v^A(t^z) \), \( (A = 1, \ldots, p) \), are supposed to belong to a given set of admissible controls, which consists of piece-wise continuous functions of \( t^z \) on \( G \) whose values range in a certain subset of \( R^p \). The state equation is assumed to be of the form

\[
\dot{x}_z^j = f_z^j(t^z, x^h(t^z), v^A(t^z)), \quad (4.1)
\]

where the given functions \( f_z^j \) are of class \( C^1 \) and such that the integrability conditions of (4.1) are satisfied. A solution \( x^j(t^z) \) of (4.1) corresponding to an admissible control \( v^A(t^z) \) gives rise to a so-called admissible pair \( (x^j, v^A) \). Moreover, let \( f(t^z, x^h, v^A) \) be a given class \( C^1 \) function by means of which the performance index is defined as an \( m \)-fold integral, namely

\[
J(x^h, v^A) = \int_G f(t^z, x^h(t^z), v^A(t^z)) \, d(u). \quad (4.2)
\]

The optimal control problem consists of the determination of an admissible pair \( (x^j, v^A) \) which minimizes the functional (4.2) relative to all other admissible pairs; a pair \( (\cdot, \cdot) \) by means of which this is achieved is said to be optimal.

Now, for a given function \( \Phi(t^z, x^h, \dot{x}_z^h) \) of the type (1.35) let us write the corresponding integral formula (2.17) in the form

\[
\int_G \Phi \, d(t) = \int_{\partial G} \Phi \, d(u), \quad (4.3)
\]

where, for the sake of brevity, we have written in accordance with (2.16):

\[
\phi = \sum_{s=1}^{r} \frac{c_r}{(s-1)!(m-s)!} e_{a_1 a_2 \ldots a_s a_{s+1} \ldots a_m}^{(s)} \frac{\partial}{\partial (u^1, \ldots, u^{m-1})} \left( x^{j_1}, \ldots, x^{j_s}, t^{a_{s+1}}, \ldots, t^{a_m} \right). \quad (4.4)
\]

in which the constants \( c_r \) are determined by the given function \( \Phi \). In order to apply (4.3) to our optimal control problem we now proceed as follows. Suppose that, corresponding to a given admissible pair \( (x^j, v^A) \), there exists a function \( \Phi \) of the type
(1.35) which is such that

\[ f(t^e, x^h(t^e), v^A(t^e)) + \Phi[t^e, x^h(t^e), f^A_x(t^e, x^h(t^e), v^A(t^e))] \leq f(t^e, x^h(t^e), v^A(t^e)) + \Phi[t^e, x^h(t^e), f^A_x(t^e, x^h(t^e), v^A(t^e))] , \]  

(4.5)

for all \( t^e \in G \), while the associated function \( \phi \) as defined by (4.4) is such that the following maximum condition relative to \( \partial G \), namely

\[ \int_{\partial G} \phi(t^e, x^h, \partial x^h/\partial u^a) \, d(u) \geq \int_{\partial G} \phi(t^e, x^h, \partial x^h/\partial u^a) \, d(u), \]  

(4.6)

is satisfied, these inequalities being valid for all admissible pairs \((x^j, v^A)\).

By virtue of (4.3) it then follows from (4.5) that

\[ \int_G f(t^e, x^h(t^e), v^A(t^e)) \, d(t) - \int_G f(t^e, x^h(t^e), v^A(t^e)) \, d(t) \leq \int_{\partial G} \phi(t^e, x^h, \partial x^h/\partial u^a) \, d(u) - \int_{\partial G} \phi(t^e, x^h, \partial x^h/\partial u^a) \, d(u), \]  

(4.7)

and hence, because of (4.6),

\[ \int_G f(t^e, x^h(t^e), v^A(t^e)) \, d(t) \leq \int_G f(t^e, x^h(t^e), v^A(t^e)) \, d(t), \]  

(4.8)

which implies that the pair \((x^j, v^A)\) is optimal. This conclusion establishes the following

**Sufficiency condition for optimal control:** If, for an admissible pair \((x^j, v^A)\) there exists a function \( \Phi \) of the type (1.35) such that the inequalities (4.5) and (4.6) are satisfied for all other admissible pairs \((x^j, v^A)\), then the pair \((x^j, v^A)\) is optimal.

**Remark 1.** Frequently the definition of ‘admissibility’ of the pair \((x^j, v^A)\) is strengthened by the inclusion of the requirement that all \( x^j(t^e) \) coincide on \( \partial G \), in which case the condition (4.6) can be omitted from the statement of the above sufficiency condition. For, under these circumstances it follows from (2.5) and the structure of (4.4) that

\[ \phi(t^e, x^h, \partial x^h/\partial u^a) = \phi(t^e, x^h, \partial x^h/\partial u^a) \]  

(4.9)

for all pairs \((x^j, v^A)\) which are admissible in the stronger sense, so that (4.8) is a direct consequence of (4.7).

**Remark 2.** Although the requirements of our sufficiency condition appear to be unduly stringent, it should be borne in mind that the construction of \( \Phi \), carried
out in accordance with (1.33), (1.34), and (1.35), affords a considerable measure of freedom as a result of the arbitrary nature of the functions $V_{j_2 \ldots j_s}$ and the constants $c_r$ on which this construction is based. In this respect, therefore, the above criterion is substantially more flexible than the sufficiency condition given previously [9], which is based on the more specialized functions $\Psi_{(r)}$ as defined by (3.2) and (3.3). In particular, when $\Phi = \Psi_{(1)}$, one simply has $\Phi = \partial S^2 / \partial t^a + (\partial S^2 / \partial x^j) \dot{x}_j$, in which case our sufficiency condition reduces to that given by Butkovskiy [1] which in turn represents a generalization of the sufficiency criterion given by Leitmann [6] for the case when the performance index (4.2) is represented by a single integral.

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