aequationes mathematicae

Volume 13, 1975

University of Waterloo, Ontario, Canada
Birkhäuser Verlag, Basel, Switzerland

aequationes mathematicae

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Single copy:
sFr. 24.–
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PUBLISHER: Birkhäuser Verlag, 4010 Basel, Switzerland
EDITORIAL OFFICE: University of Waterloo, Faculty of Mathematics, Waterloo, Ontario, Canada N2L3G1

ISSN 0001-9054

Contents Vol. 13 1975

aequationes mathematicae

CODEN: AEMABN
Vol. 13
No. 1/2
Pag. 1–200
1975

University of Waterloo, Ontario, Canada
Birkhäuser Verlag, Basel/Switzerland

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Aequationes Math. 13 (1975), 1–7. University of Waterloo Birkhäuser Verlag, Basel

Expository papers

Aequationes Mathematicae launches a systematic program of expository papers. We will endeavour to publish at least one in every volume.

The existence of Room squares

R. C. Mullin and W. D. Wallis

Abstract

The authors give a condensed proof of the existence of Room squares for positive odd sides except 3 and 5. Some areas of current research on Room squares are also discussed.

1. Introduction

A Room square of side $2n+1$ is a $2n+1$ by $2n+1$ array of cells and a set S of $2n+2$ objects called symbols which satisfy the following conditions:

(1) Every cell of the array is either empty or contains an unordered pair of distinct symbols from S.

(ii) Each symbol occurs in every row and column of the array.

(iii) Every unordered pair of symbols occurs precisely once in the array.

At the Fourth Southeastern Conference on Combinatorics, Graph Theory and Computing [56], one of the authors, W. D. Wallis, announced the existence of a Room square of side 257. This was sufficient to establish the existence of Room squares of side v for all odd positive v with the exception of $v=3$ and $v=5$, for which values no such squares exist. The fact that this square alone was needed to complete the solution to the existence problem was the culmination of the work of several researchers.

The search for Room squares has assumed many guises. Mathematicians have tackled the problem algebraically (in terms of Room quasigroups, see [5]), graph theoretically (in terms of orthogonal one-factorizations, see [47]), as well as combinatorially.

In this article, using hindsight, the authors extract a condensed version of the proof of the existence of Room squares from the large volume of literature on the subject. (See bibliography.)

AMS (1970) subject classification: Primary 05B30. Secondary 20N05, 05C99.

Received September 19, 1974.

2. The starter-adder approach

Let G be a finite Abelian group of odd order $2n+1$. By a starter in G we mean a set

$$X = \{(x_1, y_1)\ (x_2, y_2), \ldots, (x_n, y_n)\},$$

of unordered pairs of elements of G such that:

(i) the elements $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ comprise all the non-zero elements of G; and

(ii) the differences $\pm(x_i - y_i)$, $i = 1, 2, \ldots, n$, comprise all the non-zero elements of G (generating each precisely once).

By an adder for X we mean a set $A(X)$ of n distinct non-zero elements $a_1, a_2, \ldots, a_n$ from G such that the elements $\{x_i + a_i, y_i + a_i\}$, $i = 1, 2, \ldots, n$, are all distinct and comprise all the non-zero elements of G.

The following theorem is implicit in [43] and explicit in [28].

THEOREM 1. *If a finite Abelian group of order* $2n+1$ *admits a starter and adder, then there exists a Room square of order* $2n+1$.

3. Fermat primes

The members of the sequence of integers $F_n = 2^{2^n} + 1$; $n = 0, 1, 2, \ldots$, are called Fermat numbers and any primes of the sequence are Fermat primes. The first five members of the sequence, 3, 5, 17, 257, and 65537, are known to be prime but no other Fermat primes are yet known. Fermat primes are of interest in the theory of Room squares because of their appearance as exceptional cases in the Mullin-Nemeth construction [29], given in the following:

THEOREM 2. *If* $v = p^n$ *is a prime power not equal to* 9 *or a Fermat prime, then the additive group of the Galois field* $GF(p^n)$ *admits a starter and adder.*

Although this theorem produces Room squares for most prime power sides, the fact that it fails for Fermat primes caused complications in the solution of the general existence problem for a considerable time. The fact that there existed a starter and adder in Z_9, the cyclic group of order 9 was established, see [43], [59], before Theorem 2, and although 15 is clearly not a prime power it is important to note that a Room square of side 15 based on Z_{15} was found in [37] and [43].

Although chronologically the following result appeared rather late, we cite it here as the most logical place in our development.

THEOREM 3. (Chong and Chan [7].) *If* p *is a Fermat prime greater than or equal to* 17, *then* Z_p *admits a starter and adder.*

In view of this result we have established the existence of Room squares of side v

for all prime powers v except 3 and 5. At this point it is evident that a 'multiplication' theorem for Room squares is desirable. Although an early construction [5] given for such a theorem proved incorrect [27], a proper construction was supplied by Stanton and Horton [41].

THEOREM 4. *If there exist Room squares of side* v_1 *and* v_2, *then there exists a Room square of side* $v_1 v_2$.

Note that this theorem would be sufficient to complete the existence problem if squares of side 3 and 5 had existed. At this point we have established the existence of Room squares for all positive odd sides v except possibly some of those divisible by 3 or 5. This difficulty is overcome by W. D. Wallis [47] in the following:

THEOREM 5. *Given any odd positive integer* n *and a Room square of side* $v>n$, *then there exists a Room square of side* nv.

This theorem will be used in the special cases of $n=3$ and $n=5$. It is probably evident that the above theorems are sufficient to cover all cases in the existence problem, but the authors have encountered 'proofs' suggested by others to cover all cases which in fact left gaps. Therefore we prove

THEOREM 6. *Let* v *be an odd positive integer different from* 3 *or* 5. *Then there exists a Room square of side* v.

Proof. It is trivial that there is a Room square of side 1. Any positive integer v may be written in the form $3^a 5^b n$ where $(15, n)=1$ and a and b are uniquely determined. Let $m(v)=a+b$. Our proof proceeds by induction on $m(v)$. Clearly by the initial remarks of this proof and Theorems 2, 3, and 4 establish the fact that if $m(v)=0$ there is a square of side v. If $m(v)=1$ and $v>5$ then there exists a square of side v by Theorem 5. Note that the restriction $v>5$ excludes the two values of v such that $m(v)=1$ for which no square of side v exists, namely 3 or 5. Now consider those values of v for which $m(v)=2$. Such a value can be written in the form sn where $m(n)=0$ and $s=3^a 5^b$ and $a+b=2$. But as pointed out squares of side 9 and 15 were shown to exist in [43] and a square of side 25 exists by Theorem 2. Hence a square of side v exists for all odd positive v for which $m(v)=2$. The rest follows easily by induction since if $m(v)\geqslant 3$, then $v=nt$ where n is 3 or 5 and $m(t)\geqslant 2$, and therefore $t\geqslant 9$. Hence we can apply Theorem 5 to obtain a square of side v. □

4. Other aspects of the Room square problem

A Room square on the set of symbols $\{0, 1, 2, \ldots, v\}$ is *standardized* with respect to 0) if cell (i, i) of the array contains the unordered pair $\{0, i\}$. (Since the property of being a Room square is invariant under row and column permutations, any Room square can in effect be standardized with respect to any of its symbols.)

We can also associate with any standardized Room square of side $2n+1$ a matrix of order $2n+1$, which is obtained by defining $A=(a_{i,j})$ where

$$a_{i,j}=1 \quad \text{if cell } (i, j) \text{ is not empty,}$$
$$a_{i,j}=0 \quad \text{otherwise.}$$

A standardized square is *skew* if the associated matrix A satisfies

$$A+A^T=J+I,$$

where A^T is the transpose of A and I and J are the identity and all ones matrices respectively of order $2n+1$.

Although the existence problem for Room squares is solved, the corresponding problem for skew squares is far from complete although it is known [55] that skew squares exist for all but a finite number of positive odd orders. It is not yet known whether a skew Room square of side 9 exists. The results of Theorems 2 and 3 are skew squares, and the product of two skew squares as in Theorem 4 is skew. Apart from side 9, the first side for which the problem lies open is 39.

Another aspect of the problem subject to much current activity is that of sets of pairwise orthogonal symmetric Latin squares. A pair (R, C) of Latin squares on the symbols $1, 2, \ldots, v$ is an orthogonal symmetric pair if it satisfies:

(i) R and C are both symmetric;
(ii) R and C both have the ith diagonal entry i;
(iii) If R and C have (i, j) entries ϱ and σ respectively where $i<j$, then there are no numbers k and l for which $k<l$ and R and C have (k, l) entries ϱ and σ respectively except for $i=k$ and $j=l$.

It is known that a standardized Room square is equivalent to a pair of orthogonal symmetric Latin squares [11]. The problem here is for any v to find as large a set as possible of pairwise orthogonal symmetric Latin squares of side v. In many instances $(v-1)/2$ such squares have been found but there is no proof yet known to the authors that this is best possible. Gross (see for example [15]) has recently obtained many results concerning pairwise orthogonal symmetric squares, yet the problem remains open.

Another aspect of the Room square problem worthy of consideration is the balanced Room square problem. If the unordered pairs in a Room square are assigned an order, then one can form $2v$ sets of symbols, namely the first elements of each pair in a given row can constitute a set of elements, as can the second elements. If these subsets constitute a balanced incomplete block design the ordered Room square is said to be balanced. Balanced squares are particularly important for applications. It is known that balanced squares can only exist for sides congruent to 3 mod 4. The reader is referred to [4] and [39] for more on this interesting topic.

We close with a historical comment. Although Room squares were brought to the attention of modern researchers in a brief note by T. G. Room [38] in 1955, they were first introduced by Howell in 1897 for use in the bridge tournaments under the name Howell rotations. This fact was first pointed out to the authors by B. Wolk and N. S. Mendelsohn of the University of Manitoba.

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Aequationes Math. *13* (1975), 9–13. University of Waterloo Birkhäuser Verlag, Basel

Research papers

Über das Carnotsche Skalarprodukt in schwach normierten Vektorräumen

S. Gołąb

Das Problem, wann ein normierter linearer Raum (ein Banachscher Raum) ein Skalarprodukt zuläßt, entstand im J. 1934 [1] und seit dieser Zeit erschien eine lange Reihe von Arbeiten die mit diesem Problem verbunden sind (J. von Neumann, B. D. Roberts, S. Kakutani, G. Birkhoff, P. Jordan-J. von Neumann, M. Fréchet, R. S. Phillips, F. A. Ficken, R. C. James, M. M. Day, E. R. Lorch, I. J. Schoenberg, H. Rubin-M. H. Stone, S. Kurepa, J. Aczél, D. A. Senechalle, S. Gołąb-H. Światak).

Die vorliegende Note soll einen weiteren Beitrag zu diesem Problem bilden.

Die Mehrheit von Autoren bedient sich der Benennung des linearen normierten Raumes (linear normed space) im Sinne eines Vektorraumes (über dem Körper **R** bzw **C**) der mit einer Norm $|x|$ ausgestattet ist, welche den folgenden drei Axiomen genügt

I $x \neq \theta \Rightarrow |x| > 0$

II $|\alpha x| = |\alpha|\,|x|$

III $|x+y| \leqslant |x| + |y|$.

In dieser Note betrachten wir einen Vektorraum V über dem Körper **R**, wobei drei folgende Axiome über die Norm $|x|$ vorausgesetzt werden:

1) $|x| \geqslant 0$

2) $|x| = 0 \Rightarrow x = \theta$

3) $|\alpha x| = \alpha |x|$ für alle $\alpha \geqslant 0$.

In diesem Falle werden wir sagen, daß *V schwach normiert ist*. Wir setzen weder die Symmetrie der Norm $|x|$ (die aus II folgt), noch die Konvexität III, voraus.

In einem Vektorraum, der a priori mit keinem Skalarprodukt ausgestattet ist, kann das letzte mit Hilfe der folgenden (weiter Carnotschen genannten) Formel definiert werden

$$\varphi(x, y) \overset{\text{def}}{=} \tfrac{1}{2}\{|x|^2 + |y|^2 - |y-x| \cdot |x-y|\} \tag{1}$$

AMS subject classifications: Primary 46C05, 46C10. Secondary 39A30.

Eingegangen am 19.12.71, und in veränderter Form am. 29.3.1973

Da die Norm $|x|$ im allgemeinen nicht symmetrisch ist, könnte man auch eine andere (nicht äquivalente) Definition annehmen und zwar

$$\varphi^*(x, y) \overset{\text{def}}{=} \tfrac{1}{2}\{|x|^2 + |y|^2 - |y-x|^2\}. \tag{1*}$$

Bemerken wir, daß φ immer symmetrisch ist, während by φ^* dies nicht der Fall ist.

Es entsteht die Frage für welche zusätzliche Voraussetzung über φ (und mittelbar über die Norm) der Raum V zu einem euklidischen wird. Es gibt in dieser Richtung mehrere Möglichkeiten.

Eine von diesen Möglichkeiten ist eben Gegenstand der vorliegenden Note.

Bekanntlich ist die Homogenität in den abstrakten euklidischen Räumen (nach der Terminologie von R. C. James [2]) eine der Eigenschaften des Skalarproduktes:

$$\varphi(\alpha x, y) = \alpha \varphi(x, y). \tag{2}$$

Im Anschluß auf die Homogenität (2) kann der folgende Satz ausgesprochen werden.

SATZ 1. *Ist der Vektorraum V schwach normiert und ist das Carnotsche Skalarprodukt* (1) *in bezug auf die erste Veränderliche x homogen, d.h. ist* (2) *für jedes $\alpha \in \mathbf{R}$ und für jedes Paar von Vektoren x, y, die zu einer V_2 gehören, erfüllt, so ist die Norm der V_2 eine euklidische.*

Im folgenden wollen wir aber die Forderung (2) abschwächen. Die Abschwächung kann auf zweierlei Weise erreicht werden. Entweder kann man

$$(\exists \alpha)\,(\forall x, y) : \varphi(\alpha x, y) = \alpha \varphi(x, y), \tag{3a}$$

fordern, oder aber

$$(\exists x, y)\,(\forall \alpha) : \varphi(\alpha x, y) = \alpha \varphi(x, y). \tag{3b}$$

Je nachdem (3a) oder (3b) vorausgesetzt wird, bekommen wir unten zwei verschiedene Sätze. Es zeigt sich aber, daß bei den Abschwächungen (3a) bzw (3b) der Homogenität etwas anderes zusätzlich über die Norm $|x|$ vorausgesetzt werden muß.

SATZ 2. *Besitzt das Carnotsche Skalarprodukt* (1) *die Eigenschaft* (2) *für $\alpha = -1$, d.h.*

$$\varphi(-x, y) = -\varphi(x, y) \quad \textit{für alle} \quad x, y \in V_2 \tag{4}$$

und ist außerdem die Norm $|x|$ stetig in V_2, so diktiert die Norm $|x|$ in V_2 eine euklidische Metrik.

In einem endlich-dimensionalen schwach normierten Raum kann man sinnvoll über die Stetigkeit der Norm sprechen.

Beweis. Aus (1) bekommen wir

$$\varphi(-x, y)=\tfrac{1}{2}\{|-x|^2+|y|^2-|y+x|\cdot|-y-x|\}$$

was mit (1) verglichen

$$|-x|^2+|y|^2-|y+x|\cdot|-y-x|=-|x|^2-|y|^2+|y-x|\cdot|x-y|$$

ergibt, woraus, falls $y=x$ hier eingesetzt wird,

$$|-x|^2+|x|^2-|2x|\cdot|-2x|=-|x|^2-|x|^2+|\theta|\cdot|\theta|$$

oder

$$|-x|^2+3|x|^2-|2x|\,|-2x|=0$$

folgt. Das Axiom 3) ergibt ferner

$$|-x|^2-4|x|\,|-x|+3|x|^2=0$$

oder

$$[|-x|-3|x|]\cdot[|-x|-|x|]=0.$$

$|-x|=3|x|$ führt aber zu einem Widerspruch (für $x\neq\theta$) und so bleibt $|-x|=|x|$, was besagt, daß die Norm symmetrisch sein muß. In diesem Falle kann man schreiben

$$\begin{cases} \varphi(x, y)=\tfrac{1}{2}\{|x|^2+|y|^2-|y-x|^2\} \\ \varphi(-x, y)=\tfrac{1}{2}\{|x|^2+|y|^2-|y+x|^2\} \end{cases}$$

und beide obigen Relationen ergeben wegen (4) die Identität

$$|y+x|^2+|y-x|^2=2\{|x|^2+|y|^2\}, \tag{5}$$

d.h. die bekannte Jordan-v. Neumann Relation. Die allgemeine Lösung der Gleichung (5) bekommt man nach [3] in der folgenden Form

$$|x|=\sqrt{\tfrac{1}{2}F(x, y)}, \tag{6}$$

wo $F(x, y)$ eine biadditive und symmetrische Funktion bedeutet. Bei der Voraussetzung der Stetigkeit der Norm $|x|$ erhält man also die Euklidizität der Norm, womit der Satz 2 bewiesen ist.

SATZ 3. *Wenn es ein Paar* (x_0, y_0) *von zwei linear unabhängigen Vektoren gibt mit der Eigenschaft*

$$\varphi(\alpha x_0, \beta y_0)=\alpha\varphi(x_0, \beta y_0) \quad \textit{für alle} \quad \alpha, \beta\in R \tag{7}$$

und falls

$$\varphi(\theta, x)=0 \tag{8}$$

für jedes $x \in V_2$, wo V_2 den zweidimensionalen Vektorraum bedeutet, der durch x_0 und y_0 aufgespannt ist, so ist die Norm $|x|$ in V_2 eine euklidische Norm.

Beweis. Aus (1) bekommen wir

$$\varphi(\theta, x) = \tfrac{1}{2}\{|\theta|^2 + |x|^2 - |x-\theta| \cdot |\theta - x|\} = \tfrac{1}{2}[|x|^2 - |x|\,|-x|]$$

da $|\theta|$ nach 3) gleich Null ist. Laut Voraussetzung (8) haben wir also

$$|x|^2 - |x|\,|-x| = 0 \quad \text{für jedes} \quad x \in V_2$$

oder $|x| = |-x|$, was besagt, daß die Norm $|x|$ symmetrisch ist. Statt (1) können wir also die Formel (1*) benutzen. (7) mit $\beta = 1$ nimmt dann die folgende Gestalt an, wenn das Axiom 3) berücksichtigt wird

$$\alpha^2 |x_0|^2 + |y_0|^2 - |\alpha x_0 - y_0|^2 = \alpha\{|x_0|^2 + |y_0|^2 - |y_0 - x_0|^2\} \tag{9}$$

(9) kann nun infolge der folgenden kurzen Bezeichnungen

$$\lambda \overset{\text{def}}{=} |x_0|^2, \qquad \mu \overset{\text{def}}{=} |y_0|^2, \qquad \nu \overset{\text{def}}{=} |y_0 - x_0|^2 \quad (\lambda > 0, \mu > 0) \tag{10}$$

folgendermaßen umgeschrieben werden

$$|\alpha x_0 - y_0|^2 = \lambda\alpha^2 + (\nu - \mu - \lambda)\,\alpha + \mu. \tag{11}$$

Setzen wir jetzt in (9) $-\beta y_0$ statt y_0 ein (S. (7)) und nützen noch einmal das Axiom 3) aus, das jetzt auf Grund der bewiesenen Symmetrie der Norm folgendermaßen geschrieben werden kann

$$|\alpha x| = |\alpha|\,|x|, \tag{12}$$

so ergibt sich

$$\alpha^2\lambda + \beta^2\mu - |\alpha x_0 + \beta y_0|^2 = \alpha\lambda + \alpha\beta^2\mu - \alpha|\beta y_0 + x_0|^2$$

oder

$$|\alpha x_0 + \beta y_0|^2 = \lambda\alpha^2 + \mu\beta^2 - \alpha\lambda - \alpha\beta^2\mu + \alpha|x_0 + \beta y_0|^2. \tag{13}$$

Der Reihe nach setzen wir in (11) β statt $(-1/\alpha)$ ein und wenden noch einmal (12) an. Wir erhalten dann

$$\frac{1}{\beta^2}|x_0 + \beta y_0|^2 = \frac{\lambda}{\beta^2} + \frac{\lambda + \mu - \nu}{\beta} + \mu$$

oder

$$|x_0 + \beta y_0|^2 = \lambda + \mu\beta^2 + (\lambda + \mu - \nu)\,\beta. \tag{14}$$

Einsetzen von (14) in (13) ergibt endlich

$$|\alpha x_0+\beta y_0|^2=\lambda\alpha^2+\mu\beta^2-\alpha\lambda-\alpha\beta^2\mu+\alpha\lambda+\alpha\mu\beta^2+\alpha\beta(\lambda+\mu-v)$$

oder

$$|\alpha x_0+\beta y_0|^2=\lambda\alpha^2+(\lambda+\mu-v)\,\alpha\beta+\mu\beta^2. \tag{15}$$

In der Formel (15) ist zunächst $\alpha\neq 0$ und $\beta\neq 0$ vorauszusetzen, aber man bestätigt ohne Schwierigkeit, daß die Formel (15) ihre Gültigkeit auch für $\alpha=0$ bzw $\beta=0$ behält. In dieser Weise erhalten wir endlich

$$|\alpha x_0+\beta y_0|=\sqrt{\lambda\alpha^2+(\lambda+\mu-v)\,\alpha\beta+\mu\beta^2}\qquad(\lambda,\mu>0) \tag{16}$$

für *alle* α, β, was besagt, daß für alle Vektoren x der V_2 die von der Norm $|x|$ aufgeprägte Metrik eine euklidische ist, w.z.b.w.

Bemerkung. Man könnte sich im Satz 3 von der Voraussetzung (8) losmachen, aber man müßte statt (7) eine stärkere Annahme machen und zwar, daß es eine endliche Anzahl von Paaren (x_i, y_i) linear unabhängigen Vektorer in V_2 gibt, für welche

$$\varphi(\alpha x_i, y_i)=\alpha\varphi(x_i, y_i)\qquad(i=1,2,\ldots,N) \tag{7*}$$

erfüllt ist.

Der Verfasser ist den Referenten für Literaturangaben und einem von ihnen für den kurzen Beweis van Satz 1 und für den Hinweis dankbar, dass in diesem Satze die Voraussetzung (4) statt (2) genügt.

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Note on the differentiability of solutions of class C^r of a functional equation with respect to a parameter

S. Czerwik

1. In this paper we are concerned with the functional equation

$$\varphi(x, t) = H(x, \varphi[f(x), t], t), \tag{1}$$

where $\varphi(x, t)$ is an unknown function and $f(x)$, $H(x, y, t)$ are known real functions of real variables and t is a real parameter.

In paper [3] J. Matkowski has proved under assumptions which guarantee the existence and uniqueness of solutions $\varphi_n \in C^r$ of equations

$$\varphi_n(x) = H_n(x, \varphi_n[f_n(x)]), \qquad n = 0, 1, 2, \ldots, \tag{2}$$

that φ_n tends to φ_0 with derivatives up to order r, uniformly on every compact set.

We shall prove that under some assumptions, solution $\varphi(x, t)$ of equation (1) is of class C^p with respect to the parameter t. For the linear equation

$$\varphi[f(x), t] = g(x, t)\,\varphi(x, t) + F(x, t)$$

this problem has been investigated in [1].

2. Let I be an interval and let $0 \in I$. We will assume the following hypotheses:

(I) $f \in C^r[I]$ and $0 < f(x)/x < 1$ for $x \in I$, $x \neq 0$.

We denote by Ω a domain fulfilling the conditions:

(i) $\Omega \subset R^2$, $(0, 0) \in \Omega$; for every $x \in I$ the set $\Omega_x = \{y : (x, y) \in \Omega\}$ is a nonempty open interval;

(ii) for every $x \in I$ and $t \in T$, where T is an open interval, we have

$$H(f(x), \Omega_{f(x)}, t) \subset \Omega_x.$$

(II) H is of class C^r in Ω and $\partial^{i+j}H/\partial x^i \partial y^j$, $0 \leqslant i+j \leqslant r$ are continuous in $\Omega \times T$ and $H(0, 0, t) = 0$ for $t \in T$.

(III) $H \in C^{r+p}[\Omega \times T]$, $p > 0$.

AMS Primary Subject Classification: 39A15

Received December 13, 1972 and, in revised form, December 12, 1973.

(IV) We assume

$$|f'(x)|\leqslant 1, \quad |x|\leqslant \varepsilon, \quad \varepsilon>0, \tag{3}$$

$$\left|[f'(0)]^r \frac{\partial H}{\partial y}(0, 0, t)\right|<1, \quad t\in T, \tag{4}$$

$$[f'(0)]^k \frac{\partial H}{\partial y}(0, 0, t)\neq 1, \quad k=1, 2, \dots, r-1, \quad t\in T. \tag{5}$$

By $\Lambda[I, T]$ we denote the class of functions $\varphi: I\times T\to R$ such that for $x\in I$, $t\in T$ $\varphi[f(x), t]\in\Omega_x$ and $\varphi(0, t)=0$.

THEOREM 1. *Suppose that hypotheses* (I), (II), (IV) *are fulfilled. Then equation* (1) *for every* $t\in T$ *has exactly one solution* $\varphi\in\Lambda[I, T]\cap C^r[I]$. *Moreover, there exists an interval* $J\subset I$ *containing* 0 *such that* $(\partial^i\varphi/\partial x^i)(x, t)$, $i=0, \dots, r$ *are continuous in* $J\times T$.

Here $(\partial^i\varphi/\partial x^i)(x, t)$ denotes the ith derivative of the function φ at the point (x, t) and $(\partial^i/\partial x^i)(\varphi[f(x), t])$ denotes the ith derivative of the function $\varphi[f(x), t]$.

This theorem is an immediate consequence of Theorem 1 in [3] and the details of calculations are omitted here.

3. We shall prove the following

THEOREM 2. *Let hypotheses* (I), (III), (IV) *be fulfilled. Then equation* (1) *has for every* $t\in T$ *exactly one solution* $\varphi\in\Lambda[I, T]\cap C^r[I]$. *Moreover, there exist the derivatives* $(\partial^{r+i}\varphi/\partial x^r\partial t^i)(x, t)$, $i=1, 2, \dots, p$ *and they are continuous in* $J\times T$, *where* J *is an interval containing* 0 *and* $J\subset I$.

Proof. The proof will be by induction with respect to i. On account of Theorem 1, there exists exactly one function $\varphi(x, t)$ satisfying equation (1) and such that $(\partial^i\varphi/\partial x^i)(x, t)$, $i=0, 1, \dots, r$ are continuous in $J\times T$.

a) First we shall show that the derivative $(\partial^{r+1}\varphi/\partial x^r\partial t)(x, t)$ exists and is continuous in $J\times T$.

Let us write (t is fixed)

$$\psi(x, t, h)\overset{\text{df}}{=}\frac{1}{h}[\varphi(x, t+h)-\varphi(x, h)].$$

Since φ is the solution of equation (1), we obtain

$$\psi(x, t, h)=\frac{1}{h}[H(x, \varphi[f(x), t+h], t+h)-H(x, \varphi[f(x), t], t)]. \tag{6}$$

We easily obtain the relation (cf. also [4], p. 66, Hadamard's Lemma)

$$H(x, \varphi[f(x), t+h], t+h) - H(x, \varphi[f(x), t], t) \\ = H_1 \cdot (\varphi[f(x), t+h] - \varphi[f(x), t]) + H_2 \cdot h$$

where

$$H_1 = H_1(x, t, h) = \\ = \int_0^1 \frac{\partial H}{\partial y} \{x, \varphi[f(x), t] + s(\varphi[f(x), t+h] - \varphi[f(x), t]), t+sh\}\, ds,$$

$$H_2 = H_2(x, t, h) = \\ = \int_0^1 \frac{\partial H}{\partial t} \{x, \varphi[f(x), t] + s(\varphi[f(x), t+h] - \varphi[f(x), t]), t+sh\}\, ds.$$

Consequently, (6) may also be written in the form

$$\psi(x, t, h) = H_1 \cdot \psi[f(x), t, h] + H_2. \tag{7}$$

The function ψ fulfils the condition

$$\psi(0, t, h) = \frac{1}{h}[\varphi(0, t+h) - \varphi(0, h)] = \frac{1}{h}(0-0) = 0. \tag{8}$$

Now we shall prove that $\lim_{h\to 0} (\partial^r \psi/\partial x^r)(x, t, h)$ exists. The function $(\partial^r \psi/\partial x^r)(x, t, h)$ fulfils the equation

$$\frac{\partial^r \psi}{\partial x^r}(x, t, h) = [f'(x)]^r \cdot H_1 \cdot \frac{\partial^r \psi}{\partial x^r}[f(x), t, h] + H_1 \cdot W + \\ + \sum_{i=1}^{r} \binom{r}{i} \frac{\partial^i H_1}{\partial x^i}(x, t, h) \frac{\partial^{r-i}}{\partial x^{r-i}}(\psi[f(x), t, h]) + \frac{\partial^r H_2}{\partial x^r}(x, t, h), \tag{9}$$

where

$$W = W\left(\frac{df}{dx}, \ldots, \frac{d^r f}{dx^r}, \frac{\partial \psi}{\partial x}, \ldots, \frac{\partial^{r-1}\psi}{\partial x^{r-1}}\right)$$

is a polynomial in the variables $df/dx, \ldots, d^r f/dx^r$, $\partial\psi/\partial x, \ldots, \partial^{r-1}\psi/\partial x^{r-1}$ whose coefficients are real constants.

We put

$$H_3(x, t, h)=\sum_{i=1}^{r}\binom{r}{i}\frac{\partial^i H_1}{\partial x^i}(x, t, h)\frac{\partial^{r-i}}{\partial x^{r-i}}(\psi[f(x), t, h])+H_1\cdot W,$$

$$H_4(x, t, h)=\frac{\partial^r H_2}{\partial x^r}(x, t, h), \quad \alpha(x, t, h)=\frac{\partial^r\psi}{\partial x^r}(x, t, h).$$

Hence by (9) we obtain

$$\alpha(x, t, h)=[f'(x)]^r\cdot H_1\cdot\alpha[f(x), t, h]+H_3(x, t, h)+H_4(x, t, h). \tag{10}$$

In view of (III), (4) and (5) there exists an interval $T_1=\langle t+t_1, t+t_2\rangle\subset T$ such that $t\in T_1$ and

$$|[f'(0)]^r H_1(0, t, h)|=|[f'(0)]^r\int_0^1\frac{\partial H}{\partial y}(0, 0, t+sh)\,ds|<1, \qquad h\in\langle t_1, t_2\rangle. \tag{11}$$

The functions H_3 and H_4 are continuous in $\Omega\times T_1$. If $h=0$, let $\bar\alpha(x, t)$ be the continuous solution of the equation

$$\bar\alpha(x, t)=[f'(x)]^r H_1(x, t, 0)\,\bar\alpha[f(x), t]+H_3(x, t, 0)+H_4(x, t, 0). \tag{12}$$

According to the Theorem 2.7 in [2] this solution exists and is unique. We shall prove that the function $\alpha(x, t, h)$, defined for $h=0$ being equal $\bar\alpha(x, t)$, is continuous also for $h=0$. We see that equation (12) is the limit case of (10). Theorem 2 in [1] applied to equation (10) yields the continuity of $\alpha(x, t, h)$ also for $h=0$, i.e. continuity of $(\partial^r\psi/\partial x^r)(x, t, h)$ for $h=0$, and consequently

$$\lim_{h\to 0}\frac{\partial^r\psi}{\partial x^r}(x, t, h)=\bar\alpha(x, t).$$

Hence the existence of the derivative $(\partial^{r+1}\varphi/\partial x^r\partial t)(x, t)$ follows. Next, from (10), passing to the limit as $h\to 0$, we obtain

$$\frac{\partial^{r+1}\varphi}{\partial x^r\partial t}(x, t)=[f'(x)]^r\frac{\partial H}{\partial y}(x, \varphi[f(x), t], t)\frac{\partial^{r+1}\varphi}{\partial x^r\partial t}(f(x), t)+ +H_3(x, t, 0)+H_4(x, t, 0) \tag{13}$$

whence, again in view of Theorem 2 in [1], it follows that the function $(\partial^{r+1}\varphi/\partial x^r\partial t)(x, t)$ is continuous in $J\times T$, which completes the proof of the first case.

b) Now we assume that for k, $1 \leqslant k < p$, the derivative $(\partial^{r+k}\varphi/\partial x^r \partial t^k)(x, t)$ exists and is continuous in $J \times T$. Then differentiating (13) $k-1$ times we obtain

$$\frac{\partial^{r+k}\varphi}{\partial x^r \partial t^k}(x, t) = [f'(x)]^r \frac{\partial H}{\partial y}(x, \varphi[f(x), t], t) \frac{\partial^{r+k}\varphi}{\partial x^r \partial t^k}(f(x), t) + H_5(x, t), \quad (14)$$

where

$$H_5(x, t) = \left[\sum_{i=1}^{k-1} \binom{k-1}{i} \frac{\partial^{i+1}}{\partial y \partial t^i}(H(x, \varphi[f(x), t], t)) \frac{\partial^{r+k-i}\varphi}{\partial x^r \partial t^{k-1}}(f(x), t)\right] [f'(x)]^r + \\ + \frac{\partial^{k-1} H_3}{\partial t^{k-1}}(x, t, 0) + \frac{\partial^{k-1} H_4}{\partial t^{k-1}}(x, t, 0).$$

Equation (14) is a linear equation with respect to the function $\alpha(x, t) = (\partial^{r+k}\varphi/\partial x^r \partial t^k)(x, t)$ and coefficients occurring in it are of class C^{p-k}. From Theorem 6 in [1] follows that α has the derivatives $(\partial^i \alpha/\partial t^i)(x, t)$, $i = 1, 2, \ldots, p-k$, continuous in $J \times T$, and consequently the derivatives $(\partial^{r+i}\varphi/\partial x^r \partial t^i)(x, t)$, $i = 1, 2, \ldots, p$ exist and are continuous in $J \times T$, which completes the proof.

Remark. Theorems 1 and 2 are also true for the equation

$$\varphi(x, t_1, \ldots, t_n) = H(x, \varphi[f(x), t_1, \ldots, t_n], t_1, \ldots, t_n).$$

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Funktionalungleichungen und Iterationsverfahren

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Abstract

This paper is concerned with a class of iterative processes of the form $u_{k+1} = Tu_k$ $(k = 0, 1, \ldots)$ for solving nonlinear operator equations $u = Tu$ or $Fu = 0$. By studying the relationship between a linear functional inequality $\varphi(Ah)\,\beta(h) + \gamma(h) \leqslant \varphi(h)$ and estimates for the iteration operator T a general semilocal convergence theorem is obtained. The theorem contains as special cases theorems for various iterative methods. Numerical examples illustrate the accuracy of the error estimates for the approximation u_k.

1. Einleitung

Zur Lösung von nichtlinearen Operatorgleichungen $u = Tu$ oder $Fu = 0$ im metrischen Raum werden häufig iterative Verfahren der Gestalt $u_{k+1} = Tu_k$, $u_0 \in D(T)$ verwendet. Für solche Verfahren wird ein allgemeiner semilokaler Konvergenzsatz hergeleitet (Abschnitt 2): Durch Bedingungen an die Ausgangsnäherung u_0 werden die Konvergenz des Verfahrens und die Existenz einer Lösung u^* der Gleichung $u = Tu$ gesichert. Gleichzeitig werden Schranken für die Fehler $\varrho(u^*, u_k)$ $(k = 0, 1, 2, \ldots)$ angegeben. Dabei spielen Lösungen einer linearen Funktionalungleichung vom Typ

$$\varphi(Ah)\,\beta(h) + \gamma(h) \leqslant \varphi(h)$$

in einer geeigneten Parametermenge H eine wesentliche Rolle. Solche Funktionalungleichungen werden in Abschnitt 3 auf ihre für die Konvergenztheorie dieser Arbeit wichtigen Eigenschaften hin untersucht. Für $\beta(h) \equiv 1$ enthält der allgemeine Konvergenzsatz das bekannte auf Kantorowitsch zurückgehende Majorantenprinzip und insbesondere einen allgemeinen Satz von Rheinboldt. Für den Fall $\beta(h) \not\equiv 1$ werden in den folgenden Abschnitten einerseits eine Reihe von bekannten Konvergenzsätzen (u.a. der Fixpunktsatz für kontrahierende Abbildungen und der Satz von Kantorowitsch für das Newton-Verfahren) in die allgemeine Theorie des Abschnitts 2 eingeordnet, und andererseits werden durch Anwendung des allgemeinen Satzes neue Konvergenzsätze gewonnen oder auch bekannte Ergebnisse durch Abschwächung der Konvergenzbedingungen bzw. durch Verschärfung der Fehlerschranken verbessert.

Im folgenden bezeichne $\mathbf{R}_n$ $(n \geqslant 1)$ stets die Menge der reellen n-tupel mit der gewöhnlichen (komponentenweise) Halbordung und $\mathbf{R}_n^+$ die Menge der nichtnegativen

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AMS Primary Subject Classification: 65J05. Secondary Subject Classification: 39A25

Eingegangen am 15.1.1973, und in veränderter Form am 25.7.1973

n-tupel, speziell sei $\mathbf{R}=\mathbf{R}_1$ und $\mathbf{R}^+=\mathbf{R}_1^+$. Von einer mit $\{u_k\}$ bezeichneten Folge werde stets angenommen, daß k die Menge $\mathbf{N}=\{0, 1, 2,\dots\}$ durchläuft.

2. Ein allgemeiner Konvergenzsatz

Es sei (R, ϱ) ein vollständiger metrischer Raum. Mit $S(u, r)=\{v\in R:\varrho(u, v)\leqslant r\}$ werde eine Kugel in R bezeichnet.

LEMMA 2.1. *Zu einer Folge* $\{u_k\}\subset R$ *gebe es Folgen* $\{r_k\}$, $\{s_k\}\subset\mathbf{R}^+$ *mit*

$$\varrho(u_{k+1}, u_k)\leqslant s_k, \quad r_{k+1}+s_k\leqslant r_k \qquad (k=0, 1, 2,\dots) \tag{2.1}$$

und $\lim_{k\to\infty} r_k=0$. *Dann existiert* $\lim_{k\to\infty} u_k=u^*$ *und mit* $S_k=S(u_k, r_k)$ *gilt:*

$$u^*\in S_{k+1}\subset S_k\subset\cdots\subset S_1\subset S_0 \qquad (k=0, 1, 2,\dots). \tag{2.2}$$

Beweis. 1.) Aus $u\in S_{k+1}$ folgt $\varrho(u, u_k)\leqslant\varrho(u, u_{k+1})+\varrho(u_{k+1}, u_k)\leqslant r_{k+1}+s_k\leqslant r_k$, d.h. $u\in S_k$ $(k=0, 1, 2,\dots)$. 2.) Wegen $u_{k+l}\in S_{k+l}\subset S_k$ für alle $k, l\in\mathbf{N}$ und $\lim r_k=0$ ist $\{u_k\}$ Cauchyfolge in R. Es existiert also $\lim u_k=u^*$, und da S_k für jedes k abgeschlossen ist, so gilt auch $u^*\in S_k$.

Bemerkungen. 1. Setzt man im Lemma $s_k=t_{k+1}-t_k$ und, falls $\lim t_k=t^*<+\infty$ existiert, $r_k=t^*-t_k$ $(k=0, 1, 2,\dots)$, so erhält man als Spezialfall das, in dieser Form von Rheinboldt [20], [22] angegebene Majorantenprinzip.

2. Im Fall $R=\mathbf{R}^+$, $u_k=\sum_{l=0}^{k-1} s_l$ $(k=1, 2,\dots)$ mit $s_l>0$ für alle $l\in\mathbf{N}$, $u_0=0$, $r_k=1/p_k$ $(k=0, 1, 2,\dots)$ enthält das Lemma das allgemeine Konvergenzkriterium von Pringsheim für unendliche Reihen mit positiven Gliedern (vergl. [16], S. 317).

Zur Konstruktion von Folgen $\{s_k\}$ und $\{r_k\}$ mit den geforderten Eigenschaften wird mit einer Parameterfolge $\{h_k\}$ einer geeigneten Menge H der lineare Ansatz

$$s_k=\gamma(h_k)\xi_k \quad \text{und} \quad r_k=\varphi(h_k)\,\xi_k \tag{2.3}$$

gemacht.

SATZ 2.2. (V1) *Zu einer Folge* $\{u_k\}\subset R$ *gebe es Folgen* $\{\xi_k\}\subset\mathbf{R}^+$, $\{h_k\}\subset H$, *ferner Abbildungen* $A: H\to H$ *und* $\beta, \gamma: H\to\mathbf{R}^+$ *mit*

$$\varrho(u_{k+1}, u_k)\leqslant\gamma(h_k)\,\xi_k, \quad \xi_{k+1}\leqslant\beta(h_k)\,\xi_k, \quad h_{k+1}=Ah_k \qquad (k=0, 1, 2,\dots). \tag{2.4}$$

(V2) $\varphi: H\to\mathbf{R}^+$ *sei Lösung der Funktionalungleichung*

$$\varphi(Ah)\,\beta(h)+\gamma(h)\leqslant\varphi(h) \quad \text{in} \quad H. \tag{2.5}$$

(V3) *Für jedes* $h\in H$ *sei* $\lim_{k\to\infty}\varphi(A^k h)\prod_{\mu=0}^{k-1}\beta(A^\mu h)=0$.
Dann existiert $\lim_{k\to\infty} u_k=u^*$ *und es gelten die Fehlerabschätzungen*

$$\varrho(u^*, u_k)\leqslant\varphi(h_k)\,\xi_k \tag{2.6}$$

$$\varrho(u^*, u_{k+1})\leqslant\varphi(A^{k+1}h_0)\,\xi_0\prod_{\mu=0}^{k}\beta(A^\mu h_0) \qquad (k=0, 1, 2,\dots). \tag{2.7}$$

Beweis. Die durch (2.3) definierten Folgen $\{r_k\}$ und $\{s_k\}$ erfüllen die Voraussetzungen des Lemmas.

Zur Berechnung der Schranke $r_k = \varphi(h_k)\,\xi_k$ ist wegen (2.4) häufig die Kenntnis von u_{k+1}, d.h. die Ausführung eines weiteren Iterationsschrittes (oder doch die Berechnung von wesentlichen Größen dazu) nötig, so daß die folgenden Abschätzungen, die sich in einfacher Weise mit Hilfe von (2.4) und (2.5) aus (2.6) ergeben, von Nutzen sind.

ZUSATZ 2.2. $\{u_k\}$ *erfülle* (V1), (V2) *und* (V3). *Dann gelten die Abschätzungen*

$$\left.\begin{aligned} \varrho(u^*, u_{k+1}) &\leqslant \varphi(Ah_k)\,\beta(h_k)\,\xi_k \\ \varrho(u^*, u_{k+1}) &\leqslant [\varphi(h_k) - \gamma(h_k)]\,\xi_k \end{aligned}\right\} \quad (k=0, 1, 2, \ldots). \qquad \begin{aligned}(2.8)\\(2.9)\end{aligned}$$

Falls φ *die Funktional*gleichungen (2.5) *erfüllt, stimmen die beiden Schranken überein.*

Bemerkungen. 1. Es sei H halbgeordnet; A, β und φ seien isoton auf H. Dann dürfen in (V1) anstelle der Gleichungen $h_{k+1} = Ah_k$ die Ungleichungen $h_{k+1} \leqslant Ah_k$ $(k=0, 1, 2, \ldots)$ gesetzt werden.

2. Es sei H halbgeordnet; A, β, γ und φ seien isoton auf H. Dann bleiben alle Fehlerschranken (2.6)–(2.9) gültig, wenn man $\{\xi_k\}$ durch $\{\tilde{\xi}_k\}$ mit $\xi_k \leqslant \tilde{\xi}_k$ und $\{h_k\}$ durch $\{\tilde{h}_k\} \subset H$ mit $h_k \leqslant \tilde{h}_k$ ersetzt.

SPEZIALFALL 2.2.1. (V1.1) Zu $\{u_k\} \subset R$ gebe es eine Folge $\{h_k\} \subset [0, h_0] \subset \mathbf{R}^+$ und Abbildungen b, c: $[0, h_0] \to \mathbf{R}^+$ mit

$$\varrho(u_{k+1}, u_k) \leqslant c(h_k)\,h_k, \quad h_{k+1} \leqslant b(h_k)\,h_k^p \;(p \geqslant 1) \quad (k=0, 1, 2, \ldots) \tag{2.10}$$

(V2.1) $\varphi: [0, h_0] \to \mathbf{R}^+$ sei eine beschränkte, isotone Lösung der Funktionalungleichung

$$\varphi(b(h)\,h^p)\,b(h)\,h^{p-1} + c(h) \leqslant \varphi(h) \tag{2.11}$$

(V3.1) Es sei $b(h) \leqslant K$ für alle $h \in [0, h_0]$ und $Kh_0^{p-1} = L < 1$.
Dann existiert $\lim_{k\to\infty} u_k = u^*$ und es gelten die Fehlerabschätzungen (2.6)–(2.9), insbesondere

$$\varrho(u^*, u_{k+1}) \leqslant \varphi(\tilde{h}_{k+1})\,\tilde{h}_{k+1} \quad \text{mit} \quad \tilde{h}_{k+1} = h_0 L^{1+p+p^2+\cdots+p^k}. \tag{2.12}$$

Beweis. Die Aussagen ergeben sich aus Satz 2.2 durch die folgende Spezialisierung: $H = [0, h_0]$, $\xi_k = h_k$ $(k=0, 1, 2, \ldots)$, $Ah = b(h)\,h^p$, $\beta(h) = b(h)\,h^{p-1}$, $\gamma(h) = c(h)$. Wegen $\varphi(A^k h) \leqslant \varphi(h_0)$ und $\beta(A^k h) \leqslant L < 1$ für alle k ist (V3) erfüllt und mit $A^{k+1}h_0 \leqslant h_0 L^{1+p+p^2+\cdots+p^k}$ und $\beta(A^k h_0) \leqslant L^{p^k}$ $(k \in \mathbf{N})$ folgt aus (2.7) schließlich (2.12).

Der spezielle Ansatz (2.3) enthält auch das Majorantenprinzip, falls die Majorantenfolge durch ein m-stufiges Verfahren erzeugt wird:

SPEZIALFALL 2.2.2. (V1.2) Zu $\{u_k\} \subset R$ gebe es eine Menge $J \subset \mathbf{R}^+$, eine Folge

$\{t_k\} \subset J$ und eine Abbildung $\psi: J^m \to J\,(m \geqslant 1$, fest) mit

$$\varrho(u_{k+1}, u_k) \leqslant t_{k+1} - t_k \quad (k=0, 1, 2, \ldots) \quad \text{und}$$
$$t_{k+1} = \psi(t_k, t_{k-1}, \ldots, t_{k-m+1}) \quad (k = m-1, m, \ldots). \tag{2.13}$$

(V2.2) Es existiere $\lim t_k = t^* < +\infty$.
Dann existiert $\lim u_k = u^*$ mit

$$\varrho(u^*, u_k) \leqslant t^* - t_k \quad (k=0, 1, 2, \ldots). \tag{2.14}$$

Beweis. Die Aussagen ergeben sich aus Satz 2.2 durch die folgenden Spezialisierungen: $\xi_k = 1$, $h_k = (t_{k+m-1}, t_{k+m-2}, \ldots, t_k)'$ $(k=0, 1, 2, \ldots)$, $H = \{h_k\} \subset J^m$, $Ah = (\psi(h), h^{(1)}, h^{(2)}, \ldots, h^{(m-1)})$, $\beta(h) = 1$, $\gamma(h) = \delta(Ah) - \delta(h)$, $\delta(h) = h^{(m)}$ mit $h = (h^{(1)}, \ldots, h^{(m)})$. $\varphi(h) = K - \delta(h)$, $K \in \mathbf{R}$ löst die Funktionalgleichung $\varphi(Ah) + \delta(Ah) - \delta(h) = \varphi(h)$ in H. Nach (V2.2) existiert $\lim \delta(A^k h) = \delta^*(h) = t^*$ für alle $h \in H$. $\varphi(h) = \delta^*(h) - \delta(h)$ erfüllt (V3) und es gilt $\varphi(h_k) = t^* - t_k$.

Bemerkungen. 1. Zur Majorisierung eines Iterationsverfahrens (im Banachraum)

$$u_{k+1} = Tu_k \quad (k=0, 1, 2, \ldots) \quad \text{mit} \quad T: D \subset R \to R, \quad u_0 \in D_0 \subset D \tag{2.15}$$

benutzt z.B. Kantorowitsch [14] eine Funktion $\psi: [t_0, t'] \to \mathbf{R}^+$ $(m=1)$ mit

$$\|T'(u)\| \leqslant \psi'(t) \quad \text{für alle} \quad u \in D_0, t \in [t_0, t'] \quad \text{mit} \quad \|u - u_0\| \leqslant t - t_0 \tag{2.16}$$

und der Anfangsbedingung $\|Tu_0 - u_0\| \leqslant \psi(t_0) - t_0$.

2. Rheinboldt [22] dagegen kann (2.16) durch eine ableitungsfreie Bedingung ersetzen und verwendet $(m=2)$ $\psi(s, t) = s + \hat{\psi}(s-t, s, t)$ mit

$$\varrho(TTu, Tu) \leqslant \psi(\varrho(Tu, u), \varrho(Tu, u_0), \varrho(u, u_0)) \text{ für alle } u \in D_0 \text{ mit } Tu \in D_0 \tag{2.17}$$

und den Anfangsbedingungen $t_0 = 0$, $t_1 \geqslant \varrho(Tu_0, u_0)$.

3. Von Dennis [6] wird das Majorantenprinzip $(m=1)$ mit Hilfe von Abschätzungen der Form (2.16) auf eine große Klasse von „Newton-ähnlichen" Verfahren angewendet.

4. Cavanagh (vergl. [20], S. 429) konstruiert Majorantenfolgen $\{t_k\}$ mit Hilfe eines Systems von Folgen $\{\xi_k\} \subset \mathbf{R}^+$, $\{h_k\} \subset H \subset \mathbf{R}_n$ mit $\varrho(u_{k+1}, u_k) \leqslant \xi_k$, $\xi_{k+1} = \psi(\xi_k, h_k)$: $t_{k+1} = t_k + \xi_k$ $(k=1, 2, \ldots)$, $t_1 = \xi_0$, $t_0 = 0$. Für $\xi_{k+1} = \beta(h_k)\xi_k$, $h_{k+1} = Ah_k$ erhält man das Verfahren direkt als Spezialfall von Satz 2.2.

Im folgenden Satz werden die Bedingungen an die Folge $\{u_k\}$ auf einen die Folge erzeugenden Operator T und die Ausgangsnäherung u_0 übertragen.

SATZ 2.3. *Es sei $T: D \subset R \to R$ ein Operator im vollständigen metrischen Raum (R, ϱ).* (V1′) *Es gebe $D_0 \subset D$, $u_0 \in D_0$, eine Menge H und Abbildungen $A: H \to H$,*

$\beta, \gamma: H \to \mathbf{R}^+$, $h: R \times D_0 \times \{u_0\} \to H$ und $\xi: R \times D_0 \times \{u_0\} \to \mathbf{R}^+$, so daß mit

$$h(TTu, Tu, u_0) = Ah(Tu, u, u_0) \tag{2.18}$$

die Abschätzungen

$$\varrho(Tu, u) \leqslant \gamma(h(Tu, u, u_0))\, \xi(Tu, u, u_0), \tag{2.19}$$

$$\xi(TTu, Tu, u_0) \leqslant \beta(h(Tu, u, u_0))\, \xi(Tu, u, u_0) \tag{2.20}$$

für alle $u \in D_0$ mit $Tu \in D_0$ gelten.

(V2), (V3) *seien erfüllt.*

(V4) *Das Iterationsverfahren $u_{k+1} = Tu_k$, $u_0 \in D_0$ sei unbeschränkt ausführbar mit $\{u_k\} \subset D_0$.*

Dann existiert $\lim u_k = u^*$ *und mit $h_k = h(Tu_k, u_k, u_0)$ und $\xi_k = \xi(Tu_k, u_k, u_0)$ gelten die Fehlerabschätzungen* (2.6)–(2.9). *Falls T stetig ist, $u^* \in D$, so ist $u^* = Tu^*$.*

Beweis. Da $\{u_k\}$ in D_0 verläuft, erfüllen $\{\xi_k\}$ und $\{h_k\}$ die Voraussetzungen des Satzes 2.2. Aus der Stetigkeit von T folgt die Fixpunkteigenschaft von u^*.

Bemerkungen. 1. Durch Induktion läßt sich leicht zeigen, daß (V4) erfüllt ist, wenn neben (V1'), (V2), (V3) eine der folgenden Bedingungen gilt:

(V4') Es sei $TD_0 \subset D_0$.

(V4'') Es sei $S(Tu_0, \varphi(Ah_0)\, \beta(h_0)\, \xi_0) \subset U$ für eine Menge $U \subset R$ mit der Eigenschaft, daß aus $u \in D_0$ und $Tu \in U$ stets $Tu \in D_0$ folgt.

2. Alle Bemerkungen zu Satz 2.2 gelten sinngemäß auch für Satz 2.3. Insbesondere enthält Satz 2.3 den Satz von Rheinboldt [22].

3. Lösungen der Funktionalungleichung $\varphi(Ah)\, \beta(h) + \gamma(h) \leqslant \varphi(h)$

Aufgrund des linearen Ansatzes (2.3) sind in Konvergenzsatz 2.2 nur *lineare* Funktionalungleichungen des Typs

$$\varphi(Ah)\, \beta(h) + \gamma(h) \leqslant \varphi(h) \tag{3.1}$$

zu lösen. H ist dabei eine (beliebige) Menge, und $A: H \to H$, $\beta, \gamma: H \to \mathbf{R}^+$ sind gegebene Abbildungen. Unter anderen, z.T. ähnlichen Voraussetzungen sind solche Funktional*gleichungen* bereits von einer Reihe von Autoren untersucht worden. Anstelle eines ausführlichen Literaturverzeichnisses sei auf die umfassende Monographie von Kuczma [18], Chap. II hingewiesen. In diesem Abschnitt werden einige Ergebnisse zusammengestellt, die für die in dieser Arbeit entwickelte Konvergenztheorie von Interesse sind. Φ bzw. Φ^+ bezeichne die Menge aller Funktionen $\varphi: H \to \mathbf{R}$ bzw. $\mathbf{R}^+$. Durch

$$\varphi \leqslant \psi \Leftrightarrow \varphi(h) \leqslant \psi(h) \quad \text{für alle} \quad h \in H \tag{3.2}$$

wird in Φ eine Halbordnung definiert. Der durch $G\varphi(h) = \varphi(Ah)\, \beta(h)$ definierte

Operator $G:\Phi\to\Phi$ ist linear und isoton. Weiter bezeichnen $\mathfrak{U}=\{\varphi\in\Phi^+:G\varphi+\gamma\leqslant\varphi\}$ bzw. $\mathfrak{G}=\{\varphi\in\Phi^+:G\varphi+\gamma=\varphi\}$ die Lösungsmengen der Funktionalungleichung bzw. der Funktionalgleichung und $\mathfrak{U}^0=\{\varphi\in\Phi^+:G\varphi\leqslant\varphi\}$ bzw. $\mathfrak{G}^0=\{\varphi\in\Phi^+:G\varphi=\varphi\}$ die Lösungsmengen der zugehörigen homogenen Gleichungen. Schließlich erfüllt $\varphi\in\Phi$ genau dann die Konvergenzbedingung (V3), wenn $\varphi\in\mathfrak{K}=\{\varphi\in\Phi:\lim_{k\to\infty}G^k\varphi(h)=0$ für alle $h\in H\}$ gilt.

Aus der Linearität ergeben sich die folgenden einfachen Eigenschaften von $\mathfrak{U}$ und $\mathfrak{G}$:

(1) $\varphi\in\mathfrak{U},\ \varphi^0\in\mathfrak{U}^0,\ \lambda\in\mathbf{R}^+\Rightarrow\varphi+\lambda\varphi^0\in\mathfrak{U}$

(2) $\varphi_i\in\mathfrak{U},\ \lambda_i\in\mathbf{R}^+\ (i=1,2),\ \lambda_1+\lambda_2\geqslant1\Rightarrow\lambda_1\varphi_1+\lambda_2\varphi_2\in\mathfrak{U}$

(3) $\varphi\in\mathfrak{G},\ \varphi^0\in\mathfrak{G}^0,\ \lambda\in\mathbf{R},\ \psi=\varphi+\lambda\varphi^0\geqslant0\Rightarrow\psi\in\mathfrak{G}$

(4) $\varphi_i\in\mathfrak{G},\ \lambda_i\in\mathbf{R}\ (i=1,2),\ \lambda_1+\lambda_2=1,\ \psi=\lambda_1\varphi_1+\lambda_2\varphi_2\geqslant0\Rightarrow\psi\in\mathfrak{G}$

Die Aussagen bleiben richtig, wenn man $\mathfrak{U}$ bzw. $\mathfrak{G}$ durch $\mathfrak{U}\cap\mathfrak{K}$ bzw. $\mathfrak{G}\cap\mathfrak{K}$ ersetzt.

Aus $\psi\in\mathfrak{U}$ bzw. $\psi\in\mathfrak{G}$ lassen sich mit Hilfe des folgenden Satzes Lösungen $\varphi\in\mathfrak{U}\cap\mathfrak{K}$ bzw. $\mathfrak{G}\cap\mathfrak{K}$ konstruieren.

SATZ 3.1. (1) *Es sei* $\psi\in\mathfrak{U}$, *und es existiere* $\lim_{k\to\infty}G^k\psi=\psi^*\leqslant\psi$. *Dann ist* $\varphi=\psi-\psi^*\in\mathfrak{U}\cap\mathfrak{K}$.

(2) *Es sei* $\psi\in\mathfrak{G}$, *und es existiere* $\lim_{k\to\infty}G^k\psi=\psi^*\underset{(\geqslant)}{\leqslant}\psi$. *Dann ist* $\varphi=\underset{(-)}{+}(\psi-\psi^*)\in\mathfrak{G}\cap\mathfrak{K}$.

Beweis. In beiden Fällen ist $\psi^*\in\mathfrak{G}^0$, so daß φ die Funktionalungleichung bzw. -gleichung erfüllt. Es gilt $\lim_{k\to\infty}G^k(\psi-\psi^*)=\psi^*-\psi^*=0$.

SPEZIALFÄLLE. (3.1.1) Es sei $\beta(h)\equiv1$, $\gamma(h)=\delta(Ah)-\delta(h)\geqslant0$ mit $\delta:H\to\mathbf{R}^+$ und es existiere $\lim_{k\to\infty}\delta(A^kh)=\delta^*(h)\ (\geqslant\delta(h))$. Dann ist $\psi=\delta\in\mathfrak{G}$ und $\lim_{k\to\infty}G^k\delta=\delta^*$, so daß $\varphi=\delta^*-\delta\in\mathfrak{G}\cap\mathfrak{K}$ ist.

(3.1.2) Es sei $\beta(h)\leqslant L<1$, $\gamma(h)\leqslant K$ und $\varphi(h)=K/(1-L)\ (h\in H)$. Dann ist $\varphi\in\mathfrak{U}$ und

$$\lim_{k\to\infty}G^k\varphi(h)\leqslant\frac{K}{1-L}\lim_{k\to\infty}\prod_{\mu=0}^{k-1}L^\mu=0,\quad\text{für jedes } h\in H,$$

also $\varphi\in\mathfrak{U}\cap\mathfrak{K}$.

(3.1.3) Es sei $\beta(h)<1$, $\beta(Ah)\leqslant\beta(h)$, $\gamma(Ah)\leqslant\gamma(h)$ und $\varphi(h)=\gamma(h)/[1-\beta(h)]\ (h\in H)$. Dann ist $\varphi\in\mathfrak{U}$ und

$$\lim_{k\to\infty}G^k\varphi(h)\leqslant\frac{\gamma(h)}{1-\beta(h)}\lim_{k\to\infty}\prod_{\mu=0}^{k-1}\beta(h)=0\quad\text{für jedes}\quad h\in H,$$

also $\varphi\in\mathfrak{U}\cap\mathfrak{K}$.

Für die Fehlerschranken (2.6)–(2.9) ist $\varphi\in\mathfrak{U}$ *optimal*, wenn $\varphi\leqslant\psi$ für alle $\psi\in\mathfrak{U}$ gilt. Notwendige Bedingung dafür ist $\varphi\in\mathfrak{G}$, denn es gilt

SATZ 3.2. *Es sei* $\varphi\in\mathfrak{U}(\cap\mathfrak{K})$ *und* $\psi=G\varphi+\gamma$. *Dann ist* $\psi\in\mathfrak{U}(\cap\mathfrak{K})$ *mit* $\psi\leqslant\varphi$. $\psi=\varphi$ *gilt genau dann, wenn* $\varphi\in\mathfrak{G}(\cap\mathfrak{K})$.

Beweis. Es ist $\varphi\in\mathfrak{U}$, d.h. $\psi=G\varphi+\gamma\leqslant\varphi$. Da G isoton ist, folgt $G\psi\leqslant G\varphi$, also auch $G\psi+\gamma\leqslant G\varphi+\gamma$, also $\psi\in\mathfrak{U}$ und $\psi=\varphi$ gilt nur, wenn bereits $\varphi\in\mathfrak{G}$. Die Aussagen bleiben richtig, wenn man $\mathfrak{U}$ bzw. $\mathfrak{G}$ durch $\mathfrak{U}\cap\mathfrak{K}$ bzw. $\mathfrak{G}\cap\mathfrak{K}$ ersetzt.

Stellt man an φ neben $\varphi\in\mathfrak{G}$ noch die Bedingung (V3), so ist φ sogar eindeutig bestimmt:

SATZ 3.3. $\mathfrak{G}\cap\mathfrak{K}$ *enthält höchstens ein Element. Für* $\varphi\in\mathfrak{G}\cap\mathfrak{K}$ *gilt die Darstellung*

$$\varphi(h)=\sum_{k=0}^{\infty}\gamma(A^k h)\prod_{\mu=0}^{k-1}\beta(A^\mu h)\qquad(h\in H). \tag{3.3}$$

Beweis. (1) Es seien ψ, $\varphi\in\mathfrak{G}\cap\mathfrak{K}$. Dann gilt $\psi-\varphi=G(\psi-\varphi)$ und es folgt

$$\psi(h)-\varphi(h)=\lim_{k\to\infty}G^k\psi(h)-\lim_{k\to\infty}G^k\varphi(h)=0\qquad(h\in H).$$

(2) Aus der Funktionalgleichung (3.1) ergibt sich

$$\varphi(A^{k+1}h)\,\beta(A^k h)+\gamma(A^k h)=\varphi(A^k h)\qquad(k=0,1,2,\ldots)$$

und daher gilt (nach Multiplikation mit $b_k(h)=\prod_{\mu=0}^{k-1}\beta(A^\mu h)$)

$$\sum_{k=0}^{m}\gamma(A^k h)\,b_k(h)+\varphi(A^{m+1}h)\,b_{m+1}(h)=\varphi(h)\qquad(m=0,1,2,\ldots)$$

mit

$$\varphi(A^{m+1}h)\,b_{m+1}(h)=G^{m+1}\varphi(h)\to 0\quad\text{für}\quad m\to\infty\qquad(h\in H).$$

Bemerkung. Unter den Voraussetzungen der Spezialfälle (3.1.1)–(3.1.3) können Existenzaussagen direkt durch Abschätzen der Reihe (3.3) gewonnen werden.

Hinreichende Bedingungen für die Existenz und Eindeutigkeit von Lösungen der Funktional*gleichung* (3.1) erhält man auch mit dem Fixpunktsatz für kontrahierende Abbildungen:

SATZ 3.4. *Es sei* $H\subset\mathbf{R}^n$ *kompakt, und* A, β, γ *seien stetig auf* H *mit* $\beta(h)\leqslant L<1$ *für alle* $h\in H$. *Dann gilt für jede Funktion* $\varphi_0\in\mathfrak{U}$: *Das Intervall* $\langle 0,\varphi_0\rangle=\{\psi\in C(H):0\leqslant\psi(h)\leqslant\varphi_0(h),\,(h\in H)\}$ *enthält genau eine Lösung der Funktionalgleichung* (3.1).

Beweis. Mit $\|\varphi\|=\max_{h\in H}|\varphi(h)|$ wird Φ zu einem Halbordnungs-Banachraum. Der Operator $\hat{G}$, definiert durch $\hat{G}\varphi=G\varphi+\gamma$ ist stetig und bildet das Intervall $\langle 0,\varphi_0\rangle$ in sich ab: $\hat{G}\langle 0,\varphi_0\rangle=\langle\gamma,G\varphi_0+\gamma\rangle\subset\langle 0,\varphi_0\rangle$. Schließlich ist $\hat{G}$ lipschitzbeschränkt durch $L<1$:

$$\|\hat{G}\varphi-\hat{G}\psi\|=\max_{h\in H}\beta(h)\,|\varphi(Ah)-\psi(Ah)|\leqslant L\max_{h\in AH}|\varphi(h)-\psi(h)|$$
$$\leqslant L\max_{h\in H}|\varphi(h)-\psi(h)|=L\,\|\varphi-\psi\|\,.$$

Bemerkungen. 1. Die Bedingung $\mathfrak{U}\neq\emptyset$ ist z.B. in den Fällen (3.1.1)–(3.1.3) erfüllt.

2. Falls keine Lösung der Funktionalgleichung (3.1) bekannt ist, kann (für möglichst scharfe Fehlerschranken) die Auswertung der Reihe (3.3) (einschließlich einer Restgliedabschätzung) durch das Iterationsverfahren

$$\varphi_{k+1}=G\varphi_k+\gamma \qquad (k=0,1,2,\ldots),\ \varphi_0\in\mathfrak{U} \tag{3.4}$$

ersetzt werden, welches nach Satz 3.2 eine Folge $\{\varphi_k\}\subset\mathfrak{U}$ mit

$$\varphi_{k+1}\leqslant\varphi_k\leqslant\cdots\varphi_1\leqslant\varphi_0 \qquad (k=0,1,\ldots) \tag{3.5}$$

liefert.

Die folgende, in geschlossener Form lösbare Funktionalgleichung (3.6) tritt in Konvergenzsätzen für das Newton-Verfahren und dessen Verschärfungen mehrfach auf.

SATZ 3.5. *Es sei $H=[0,\frac{1}{4}]$ und $\gamma:H\to\mathbf{R}$ eine beliebige Funktion mit $\gamma(h)<2$ für alle $h\in H$. Dann hat die Funktionalgleichung*

$$\varphi\left(\frac{[1-\gamma(h)+h\gamma^2(h)]\,h}{[1-2h\gamma(h)]^2}\right)\frac{1-\gamma(h)+h\gamma^2(h)}{1-2h\gamma(h)}+\gamma(h)=\varphi(h) \tag{3.6}$$

die Lösungen

$$\varphi(h)=\frac{2}{1\pm\sqrt{1-4h}}.$$

Beweis. Unter Beachtung von $[1-2h\gamma(h)]^2-4[1-\gamma(h)+h\gamma^2(h)]\,h=1-4h$ bestätigt man die Aussage durch Einsetzen.

4. Der Fixpunktsatz für kontrahierende Abbildungen

In dem einfachsten Spezialfall des Satzes 2.3, nämlich $H\subset\mathbf{R}^+$, $Ah=h$, $\beta(h)=L<1$, $\gamma(h)=1$ und $h(v,u,u_0)=\text{const.}$, $\xi(v,u,u_0)=\varrho(v,u)$ hat die Funktionalgleichung (2.5) die Gestalt $\varphi(h)\,L+1=\varphi(h)$. Ihre Lösung $\varphi(h)=1/(1-L)$ erfüllt (V3) und man erhält (vergl. Rheinboldt [22]):

SATZ 4.1. *Es sei $T:D\subset R\to R$ ein Operator im vollständigen, metrischen Raum (R,ϱ). Es gebe eine Teilmenge $D_0\subset D$ und $u_0\in D_0$, so daß*

$$\varrho(TTu,Tu)\leqslant L\varrho(Tu,u) \quad \textit{mit} \quad L<1 \quad \textit{für alle} \quad u\in D_0 \quad \textit{mit} \quad Tu\in D_0 \tag{4.1}$$

erfüllt ist und $S=S\,(Tu_0,\ L/[1-L]\ \varrho(Tu_0,\ u_0))\subset D_0$ *gilt. Dann ist das Iterationsverfahren* $u_{k+1}=Tu_k,\ u_0\in D_0\ (k=0,1,2,\ldots)$ *unbeschränkt ausführbar. Es existiert* $\lim u_k=u^*\in S$ *und es gelten die Fehlerabschätzungen*

$$\|u^*-u_{k+1}\|\leqslant\frac{L}{1-L}\|u_{k+1}-u_k\| \quad \text{und} \quad \|u^*-u_{k+1}\|\leqslant\frac{L^k}{1-L}\|u_1-u_0\|\,. \tag{4.2}$$

Bemerkung. Ersetzt man (4.1) durch die Bedingung

$$\varrho(Tv,\ Tu)\leqslant L\varrho(v,\ u) \quad \text{mit} \quad L<1 \quad \text{für alle} \quad v,\ u\in D_0, \tag{4.3}$$

so ergibt sich aus Satz 4.1 der Kontraktionssatz, denn es ist T stetig, also gilt $u^*=Tu^*$. Darüberhinaus ist u^* in D_0 eindeutig bestimmt.

5. Konvergenzsätze für das Newton-Verfahren mit Verschärfungen

Der Operator $F\colon D(F)\subset R\to R$ im Banachraum $(R,\ \|\cdot\|)$ besitze in einer konvexen Menge $U\subset D(F)$ beschränkte Fréchet-Ableitungen $F^{(\nu)}(u)$ $(u\in U,\ \nu=1,2,\ldots,n+1)$. Es sei $\Gamma(u)=\{F'(u)\}^{-1}$, falls die Inverse existiert. Zur Lösung der Operatorgleichung $Fu=0$ wird für *festes* $n\geqslant 1$ ein Iterationsverfahren

$$u_{k+1}=Tu_k=g(u_k,\ \Gamma_k,\ F_k,\ F'_k,\ldots,\ F_k^{(n)}),\ u_0\in D \quad (k=0,1,2,\ldots) \tag{5.1}$$

mit $F_k=Fu_k$, $F_k^{(\nu)}=F^{(\nu)}(u_k)$ $(\nu=1,2,\ldots,n)$ und $\Gamma_k=\Gamma(u_k)$ betrachtet. Dabei ist g ein gegebenes Polynom in den angegebenen Argumenten und $D=\{u\in U\colon \Gamma(u)$ existiert und ist beschränkt$\}$.

5.1. *Ein allgemeiner Satz von Collatz*

Als Spezialfall von Satz 2.2.1 ergibt sich

SATZ 5.1 (Collatz). 1. *Für das Verfahren* (5.1) *gebe es Abschätzfunktionen* $\tilde b,\ \tilde c\colon \mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}^+$, *die in beiden Argumenten isoton sind, und eine Konstante* $p\geqslant 1$, *so daß mit* $h_k=\|F_k\|$ *und* $B_k=\|\Gamma_k\|$ *gilt:*

$$\|u_{k+1}-u_k\|\leqslant\tilde c(B_k,\ h_k)\,h_k \quad \textit{und} \quad h_{k+1}\leqslant\tilde b(B_k,\ h_k)\,h_k^p \quad (k=0,1,2,\ldots) \tag{5.2}$$

2. *Zu* $u_0\in D$ *existiere eine Zahl* $B>B_0$, *so daß mit* $c(h)=\tilde c(B,\ h)$, $b(h)=\tilde b(B,\ h)$, $K=b(h_0)$, $L=Kh_0^{p-1}$ *und* $\eta=2M_2B_0c(h_0)\,h_0$ *mit* $M_2\geqslant\frac{1}{2}\|F''(u)\|$ *für alle* $u\in U$ *gilt:*

$$L<1,\quad 0\leqslant\eta\leqslant 1-L \quad \text{und} \quad L+\frac{\eta}{1-\eta}\cdot\frac{B^2}{B_0(B-B_0)}\leqslant 1\,. \tag{5.3}$$

3. *Mit einer isotonen Lösung* φ *der Funktionalungleichung* (2.11) *in* $H=[0,\ h_0]$ *sei* $S=S\,(v_1,\ \varphi(Lh_0)\,Lh_0)\subset U$.

Dann ist das Verfahren (5.1) *unbeschränkt in S ausführbar, die Folge* $\{u_k\}$ *konvergiert gegen eine Lösung* u^* *der Gleichung* $Fu=0$ *und es gilt die Fehlerabschätzung* (2.12).

Beweis. Auf Einzelheiten des Beweises sei hier nicht eingegangen, da er im wesentlichen wie bei Collatz [4], [5] verläuft: Es wird bewiesen, daß die Folge $\{u_k\}$ existiert und U nicht verläßt. Dabei gilt stets $B_k \leqslant B$ und $h_k \leqslant h_0$. $\{u_k\}$ erfüllt daher (V1.1)-(V3.1).

Bemerkung. Die Collatz'sche Formulierung des Satzes entsteht, wenn man in (2.12) [unter Berücksichtigung von $c(\tilde{h}_{k+1}) \leqslant c(h_0)$] als Lösung der Funktionalungleichung (2.11) speziell $\varphi_0(h) = c(h)/[1-b(h)\,h^{p-1}]$ einsetzt:

$$\|u^* - u_{k+1}\| \leqslant c(h_0)\, h_0 \frac{L^{1+p+p^2+\cdots+p^k}}{1-L^{p^{k+1}}} \quad (k=0, 1, 2, \ldots). \tag{5.4}$$

Die hier gewählte Form hat den Vorteil, daß auch Lösungen $\varphi \leqslant \varphi_0$ von (2.11) und damit Verschärfungen der Fehlerabschätzungen zugelassen werden, wie Iterierte des Verfahrens (3.4) oder etwa in geschlossener Form angebbare Lösungen der Funktionalgleichung.

5.2. *Sätze für das Verfahren von E. Schröder*

Eine spezielle Klasse von Verfahren (5.1), die auf E. Schröder zurückgehen, wird z.B. bei Collatz [4], [5] und Ehrmann [8], [9] ausführlich beschrieben:

$$\left.\begin{aligned}
&u_{k+1} = Tu_k, \quad u_0 \in D \quad (k=0, 1, 2, \ldots) \quad \text{mit} \\
&\quad Tu = u + d(u) \quad \text{und} \quad d(u) = d_n(u) \quad \text{definiert durch} \\
&d_\nu(u) = -\Gamma(u)\, Fu - \sum_{\mu=2}^{\nu} \frac{1}{\mu!} \Gamma(u)\, F^{(\mu)}(u)\, d_{\nu-1}^{\mu}(u), \qquad (\nu=2, 3, \ldots, n) \\
&d_1(u) = -\Gamma(u)\, Fu
\end{aligned}\right\} \tag{5.5}$$

Obwohl diese Verfahren einerseits in (5.1) enthalten sind, andererseits in dieser speziellen Gestalt schon in einer Reihe von Arbeiten [1], [4], [8], [9], für $n=2$ z.B. in [7], [13], [19], [23] untersucht worden sind, lassen sich doch durch Anwendung des allgemeinen Konvergenzsatzes für $n \geqslant 2$ Abschwächungen bzw. Vereinfachungen der Voraussetzungen und Verschärfungen der Fehlerabschätzungen erreichen.

Aufgrund der rekursiven Definition des Iterationsoperators T empfiehlt es sich die folgenden isotonen Funktionen $p, q, r_\mu : \mathbf{R}_n^+ \to \mathbf{R}^+$ und $P, Q, R_\mu : \mathbf{R}_n^+ \to \mathbf{R}_n^+$ zu definieren: Für $h = (h^{(2)}, h^{(3)}, \ldots, h^{(n+1)}) \in \mathbf{R}_n^+$ sei

$$\left.\begin{aligned}
&p(h) = p_n(h) \quad \text{mit} \\
&p_\nu(h) = 1 + \sum_{\mu=2}^{\nu} p_{\nu-1}^{\mu}(h)\, h^{(\mu)}, \qquad p_1(h) = 1
\end{aligned}\right\} \tag{5.6}$$

$$\left.\begin{aligned} &q(h)=q_{n+1}(h) \quad \text{mit} \\ &q_\nu(h)=q_{\nu-1}(h)\, s_{\nu-1}(h)+p_{\nu-1}^{\nu}(h)\, h^{(\nu)}, \quad q_1(h)=0, \quad \text{und} \\ &s_\nu(h)=\sum_{\mu=2}^{\nu} h^{(\mu)} \sum_{\sigma=1}^{\mu} p_\nu^{\sigma-1}(h)\, p_{\nu-1}^{\mu-\sigma}(h), \quad s_1(h)=0 \end{aligned}\right\} \tag{5.7}$$

$$r_\nu(h)=\sum_{\mu=2}^{\nu} \mu h^{(\mu)} p^{\mu-1}(h) \tag{5.8}$$

$$P(h)=(p^{(\nu,\mu)}(h)) \quad \text{mit} \quad p^{(\nu,\mu)}(h)=\begin{cases} \binom{\mu}{\mu-\nu} p^{\mu-\nu}(h) & \text{für} \quad \mu\geqslant\nu \\ 0 & \text{für} \quad \mu<\nu \end{cases} \tag{5.9}$$

$$Q(h)=\operatorname{diag}(q^{\nu-1}(h)) \tag{5.10}$$

$$R_\nu(h)=\operatorname{diag}\left\{\frac{1}{[1-r_\nu(h)]^\mu}\right\} \quad (\nu,\mu=2,3,\ldots,n,n+1) \tag{5.11}$$

SATZ 5.2. *Mit den Definitionen der Tabellen* I.1, I.2, II.1 *oder* II.2 *gelte für ein* $u_0\in D$:

Es sei $h_0\in H$ *und mit einer isotonen Lösung* φ *der Funktionalungleichung* $\varphi(Ah)\,\beta(h)+\gamma(h)\leqslant\varphi(h)$ *in* H *sei* $S=S(v_1, \varphi(Ah_0)\,\beta(h_0)\,\xi_0)\subset U$.

Dann ist das Iterationsverfahren (5.5), *ausgehend von* u_0, *unbeschränkt in* S *ausführbar, die Folge* $\{u_k\}$ *konvergiert gegen eine Lösung* u^* *der Gleichung* $Fu=0$ *und es gelten die Fehlerabschätzungen* (2.6)–(2.9) *mit* h_k, ξ_k *aus der entsprechenden Tabelle.*

Tabellen:

	I.1	I.2
H =	$\left\{h\in\mathbf{R}_n^+ : \begin{array}{l} r_2(h)<1 \quad \text{und} \\ q(h)\leqslant[1-r_2(h)]^2 \end{array}\right\}$	$\left\{h\in\mathbf{R}_n^+ : \begin{array}{l} r_{n+1}(h)<1 \quad \text{und} \\ q(h)P(h)h\leqslant[1-r_{n+1}(h)]^2\,h \end{array}\right\}$
Ah =	$R_2(h)\,Q(h)\,h$	$R_{n+1}(h)\,Q(h)\,P(h)\,h$
$\beta(h)$=	$q(h)/[1-r_2(h)]$	$q(h)/[1-r_{n+1}(h)]$
$\gamma(h)$=	$p(h)$	$p(h)$
ξ_k =	$\lVert\Gamma_k F_k\rVert$	$\lVert\Gamma_k F_k\rVert$
h_k =	$(M_{\nu,k}\xi_k^{\nu-1}) \quad (\nu=2,3,\ldots,n+1)$	$Ah_{k-1} \quad (k=1,2,\ldots)$ $h_0\geqslant\left\{\begin{array}{l} m_{\nu,0}\xi_0^{\nu-1} \\ M_{n+1,0}\xi_0^n \end{array}\right\} \quad (\nu=2,3,\ldots,n)$
	mit $M_{\nu,k}=1/\nu!\,\sup_{u\in U}\lVert\Gamma_k F^{(\nu)}(u)\rVert$	und $m_{\nu,0}=1/\nu!\,\lVert\Gamma_0 F_0^{(\nu)}\rVert$

Tabellen (Fortsetzung)

	II.1	II.2
$H =$	$\{h \in \mathbf{R}_n^+ : q(h) < 1\}$	$\{h \in \mathbf{R}_n^+ : q(h)\, P(h)\, h < h\}$
$Ah =$ $\beta(h) =$ $\gamma(h) =$	$Q(h)\, h$ $q(h)$ $p(h)$	$Q(h)\, P(h)\, h$ $q(h)$ $p(h)$
$\xi_k =$	$B\|F_k\|$	$B\|F_k\|$
$h_k =$	$(BM_\nu \xi_k^{\nu-1}) \quad (\nu=2, 3, \ldots, n+1)$	$Ah_{k-1} \quad (k=1, 2, \ldots)$ $h_0 \geqslant \begin{Bmatrix} Bm_\nu \xi_0^{\nu-1} \\ BM_{n+1} \xi_0^n \end{Bmatrix} \quad (\nu=2, 3, \ldots, n)$
	mit $\left.\begin{matrix} M_\nu \geqslant 1/\nu!\ \|F^{(\nu)}(u)\| \\ B \geqslant \|\Gamma(u)\| \end{matrix}\right\}$ für alle $u \in U \subset D$ und $m_\nu = 1/\nu!\ \|F_0^{(\nu)}\|$	

Beweis. Der Satz ergibt sich als Spezialfall von Satz 2.3 (I.1, II.1) bzw. Satz 2.2 (I.2, II.2):

In allen 4 Fällen sind A, β und γ isoton auf H und es gilt $Ah \leqslant h$ für alle $h \in H$, insbesondere gilt also $\beta(Ah) \leqslant \beta(h)$ und $\gamma(Ah) \leqslant \gamma(h)$. Jede isotone Lösung der Funktionalungleichung $\varphi(Ah)\, \beta(h) + \gamma(h) \leqslant \varphi(h)$ erfüllt wegen $\varphi(A^k h) \leqslant \varphi(h)\ (k=0, 1, 2, \ldots)$ und $\beta(h) < 1$ für alle $h \in H$ auch (V3).

Der weitere Beweis soll hier nur im Fall I.1 angedeutet werden. Einzelheiten (und auch die Beweise der übrigen Fälle) können [17] entnommen werden. Es werden Funktionen

$$M_\nu : D \to \mathbf{R}^+ \quad (\nu=2, 3, \ldots, n+1), \qquad \xi : D \to \mathbf{R}^+, \qquad h : D \to \mathbf{R}_n^+$$

durch

$$M_\nu(u) = \frac{1}{\nu!} \sup_{v \in U} \|\Gamma(u)\, F^{(\nu)}(v)\|, \qquad \xi(u) = \|\Gamma(u)\, Fu\|,$$

$$h(u) = (M_\nu(u)\, \xi^{\nu-1}(u)) \qquad (\nu=2, 3, \ldots, n+1)$$

definiert.

Mit Hilfe der Taylorformel und dem bekannten Satz von Banach über die Existenz einer Inversen zeigt man, daß für alle $u \in D_0 = \{u \in D : h(u) \in H\}$ mit $Tu \in U$ die Ungleichungen $h(Tu) \leqslant R_2(h(u))\, Q(h(u))\, h(u)$ und $\xi(Tu) \leqslant q(h(u))/[1 - r_2(h(u))]\, \xi(u)$ richtig sind und $Tu \in D_0$ gilt. Damit ist (V1′) erfüllt, wenn man Bemerkung 1 zu Satz 2.2 berücksichtigt und wegen $S \subset U$ gilt schließlich auch (V4″).

Bemerkungen. 1. Da A, β, γ und φ auf H isoton sind, dürfen in der Bedingung

$S \subset U$ und in den Fehlerschranken (2.6)–(2.9) die Folgen $\{\xi_k\}$, $\{h_k\}$ durch Folgen $\{\tilde{\xi}_k\}$ und $\{\tilde{h}_k\} \subset H$ mit $\xi_k \leqslant \tilde{\xi}_k$, $h_k \leqslant \tilde{h}_k$ ersetzt werden.

2. Für $n \geqslant 2$ scheinen Lösungen der Funktional*gleichungen* (2.5) nicht „in geschlossener Form“ bekannt zu sein. Daher wird man (in allen 4 Fällen) die (isotone) Lösung $\varphi_0(h) = \gamma(h)/[1-\beta(h)]$ der Funktionalungleichung oder auch, falls h_k die Ungleichung $\beta(h_k) < 1$ nur schwach erfüllt, die ersten Iterierten φ_k aus dem Verfahren (3.4) verwenden.

3. Für $n \geqslant 2$ kann die Abschätzung (2.8) verschärft werden, wenn man

$$\|u^* - u_{k+1}\| \leqslant \varphi(h_{k+1})\, \xi_{k+1} \leqslant \varphi(Ah_k)\, \tilde{\xi}_{k+1} \qquad (k=0, 1, 2, \ldots) \tag{5.12}$$

benutzt und $\tilde{\xi}_{k+1}$ in den 4 betrachteten Fällen wie folgt definiert ($k=0$):

$$\text{I.1:} \quad \tilde{\xi}_1 = \frac{1}{1-2M_{2,0}\,\|d_n\|}\, \tilde{q}(M_{2,0}, M_{3,0}, \ldots, M_{n+1,0}) \tag{5.13}$$

$$\text{I.2:} \quad \tilde{\xi}_1 = \frac{1}{1-\sum\limits_{\nu=2}^{n+1} \nu m_{\nu,0}\, \|d_n\|^{\nu-1}}\, \tilde{q}(m_{2,0}, m_{3,0}, \ldots, m_{n,0}, M_{n+1,0}) \tag{5.14}$$

$$\text{II.1:} \quad \tilde{\xi}_1 = B\tilde{q}(M_2, M_3, \ldots, M_{n+1}) \tag{5.15}$$

$$\text{II.2:} \quad \tilde{\xi}_1 = B\tilde{q}(m_2, m_3, \ldots, m_n, M_{n+1}) \tag{5.16}$$

Dabei ist analog zu (5.7)

$$\tilde{q}(K_2, K_3, \ldots, K_{n+1}) = K_{n+1}\, \|d_n\|^{n+1} + \|d_n - d_{n-1}\| \sum_{\nu=2}^{n} K_\nu \sum_{\sigma=1}^{\nu} \|d_n\|^{\sigma-1}\, \|d_{n-1}\|^{\nu-\sigma} \tag{5.17}$$

mit $d_\nu = d_\nu(u_0) \quad (\nu = n, n-1)$.

4. I.2 und II.2 haben gegenüber I.1, II.1 für $n \geqslant 2$ den Vorteil, daß die Normen der Frechetschen Ableitungen $F^{(\nu)}(u)\,(\nu=2, 3, \ldots, n)$ nur an *einer* Stelle $u=u_0$ abgeschätzt zu werden brauchen. Dies führt häufig zu schärferen Fehlerschranken. Die Konvergenzbedingungen dagegen fallen i.a. schärfer aus.

SPEZIALFÄLLE. (5.2.1) Für $n=1$ geht (5.5) in das Newton-Verfahren über. Es ist dann $p(h) = P(h) = 1$, $q(h) = Q(h) = h$, $r_2(h) = 2h$ und $R_2(h) = 1/[1-2h]^2$.

(5.2.1.1) Im Fall I.1 ist Satz 5.2 mit dem bekannten Satz von Kantorowitsch [14] identisch: Es ist $H = [0, \frac{1}{4}]$ und die Funktionalgleichung (2.5) geht über in

$$\varphi\left(\left(\frac{h}{1-2h}\right)^2\right) \frac{h}{1-2h} + 1 = \varphi(h) \tag{5.18}$$

mit der isotonen Lösung $\varphi(h) = 2/[1+\sqrt{1-4h}]$ (Satz 3.5 mit $\gamma(h)=1$).

Für die „a-priori"-Abschätzung (2.7) ergibt sich (vergl. Ostrowski [21]):

$$\|u^* - u_{k+1}\| \leqslant \varphi(A^{k+1} h_0)\, \xi_0 \prod_{\mu=0}^{k} \beta(A^\mu h_0) = e^{-2^k \tau} \frac{\sinh \tau}{\sinh 2^k \tau}\, \xi_0 \qquad (k=0, 1, 2, \ldots) \quad (5.19)$$

mit

$$2h_0 = \frac{1}{1+\cosh \tau}, \qquad \tau \geqslant 0.$$

(5.2.1.2) Im Fall II.1 ($U \subset D$) ist Satz 5.2 mit dem Satz von Mysowskich (siehe [14]) identisch. Es ist $H=[0, 1)$ und die Funktionalgleichung (2.5) geht über in

$$\varphi(h^2)\, h + 1 = \varphi(h) \qquad (5.20)$$

mit der isotonen Lösung $\varphi(h) = \sum_{\nu=0}^{\infty} h^{2^\nu - 1}$ (vergl. (3.3)).

Mit der Lösung $\varphi(h) = 1/[1-h]$ der Funktional*un*gleichung folgt aus (2.7) die „a-priori"-Abschätzung:

$$\|u^* - u_{k+1}\| \leqslant \frac{h_0^{2^{k+1}-1}}{1-h_0^{2^{k+1}}}\, \xi_0 \qquad (k=0, 1, 2, \ldots). \qquad (5.21)$$

(5.2.2) Für $n=2$ ist (5.5) auch unter dem Namen Tschebyscheff-Verfahren bekannt. Mit $h=(f, g)' \in \mathbf{R}_2^+$ ist

$$p(h) = 1+f, \qquad q(h) = (2+f) f^2 + (1+f)^3 g,$$
$$r_2(h) = 2(1+f) f, \qquad r_3(h) = (2+f) f^2 + 3(1+f)^2 g,$$
$$P(h) = \begin{pmatrix} 1 & 3(1+f) \\ 0 & 1 \end{pmatrix}, \qquad Q(h) = \begin{pmatrix} q(h) & 0 \\ 0 & q^2(h) \end{pmatrix},$$
$$R_\nu(h) = \begin{pmatrix} \dfrac{1}{[1-r_\nu(h)]^2} & 0 \\ 0 & \dfrac{1}{[1-r_\nu(h)]^3} \end{pmatrix} \qquad (\nu=2, 3)$$

Die Grenzlinien der „Konvergenzbereiche" H sind in Abb. 1 dargestellt.

(5.2.2.1) Der Fall I.1 enthält unter den einschränkenden Voraussetzungen $U=S(u_0, \frac{14}{9}\xi_0)$, $h_0 \leqslant (\frac{1}{6}, \frac{1}{18})'$ Ergebnisse von Döring [7], wenn man die (grobe) Lösung $\varphi(h) = \frac{14}{9}$ der Funktionalungleichung (2.5) benutzt.

(5.2.2.2) Der Fall I.2 enthält mit $\varphi(h) = \gamma(h)/[1-\beta(h)]$ und $U=S(u_0, \varphi(h_0)\,\xi_0)$ Ergebnisse von Shafiyev [23].

(5.2.3) Im Fall $F^{(3)}(u) = F^{(4)}(u) = \cdots = F^{(n+1)}(u) = 0$ für alle $u \in U$, der z.B. bei der nichtlinearen Behandlung von Matrizen-Eigenwertaufgaben auftritt (vergl. z.B. [5]), erhält man mit $\hat{h} = (h, 0, \ldots, 0)' \in \mathbf{R}_n^+$:

$$p(\hat{h}) = \hat{p}(h) = \hat{p}_n(h) \quad \text{mit} \quad \hat{p}_\nu(h) = 1 + h\hat{p}_{\nu-1}^2(h) \quad (\nu=2, 3, \ldots, n), \quad \hat{p}_1(h) = 1$$

und analog

$$q(\hat{h})=\hat{q}(h)=1-\hat{p}(h)+\hat{p}^2(h)\,h, \qquad r_2(\hat{h})=\hat{r}_2(h)=2h\hat{p}(h).$$

Im Fall I.1 wird $H=[0,\frac{1}{4}]$, es gilt $\hat{p}(h)<2$ für alle $h\in H$ und die Funktionalgleichung (2.5) reduziert sich auf

$$\varphi\left(\frac{\hat{q}(h)\,h}{[1-\hat{r}_2(h)]^2}\right)\frac{\hat{q}(h)}{1-\hat{r}_2(h)}+\hat{p}(h)=\varphi(h). \tag{5.22}$$

Sie besitzt nach Satz 3.5 ($\gamma(h)=\hat{p}(h)$) die isotone Lösung $\varphi(h)=2/[1+\sqrt{1-4h}]$. (5.13) schließlich geht über in

$$\tilde{\xi}_1=\frac{M_{2,0}}{1-2M_{2,0}\,\|d_n\|}\,\|d_n-d_{n-1}\|\,(\|d_n\|+\|d_{n-1}\|). \tag{5.23}$$

Falls man *zusätzlich* F_1 berechnet, kann man statt (5.23) auch

$$\tilde{\tilde{\xi}}_1=\frac{\|\Gamma_0\|}{1-2M_{2,0}\,\|d_n\|}\,\|F_1\| \tag{5.24}$$

benutzen.

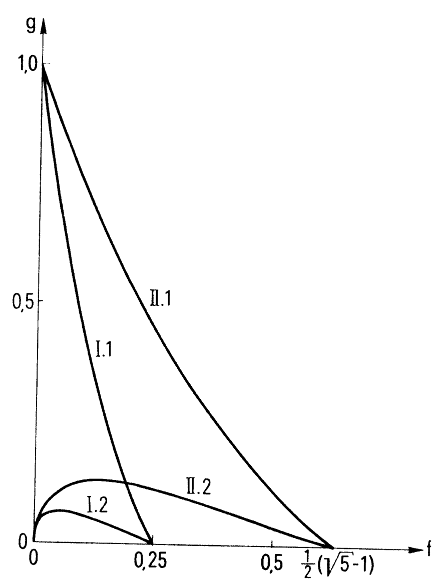

Abb. 1

5.3. *Sätze für ein Verfahren von Kleinmichel*

Mit den allgemeinen Bezeichnungen dieses Abschnitts für $n=2$ werde anstelle des

Verfahrens (5.1) das von Kleinmichel [15] vorgeschlagene Iterationsverfahren

$$\left.\begin{aligned} u_{k+1}=Tu_k,\ u_0\in\tilde{D} \quad (k=0,1,2,\dots) \\ \text{mit}\quad Tu=u-\Gamma(u-\tfrac{1}{2}\Gamma(u)\,Fu)\,Fu \\ \text{und}\quad \tilde{D}=\{u\in D:\Gamma(u-\tfrac{1}{2}\Gamma(u)\,Fu)\ \text{existiert und ist beschränkt}\} \end{aligned}\right\} \tag{5.25}$$

betrachtet. Die Anwendung des allgemeinen Satzes 2.3 führt auf

SATZ 5.3. *Es gilt Satz* 5.2 *für das Verfahren* (5.25) *mit den Definitionen der Tabellen* III *oder* IV (*anstelle von* I.1 I.2, II.1 *oder* II.2), *wenn man* D *durch* $\tilde{D}$ *ersetzt.*

Tabelle III

$H=\left\{h=(f,g)'\in\mathbf{R}_2^+:f\leqslant\frac{1}{4},\quad g<\frac{4(1-f)}{4+3(1-f)^2}(1-6f+8f^2)\right\}$
$Ah=\frac{1-f}{1-3f}(\beta(h)f,\beta^2(h)g)'$ $\beta(h)=\frac{1}{1-3f}(f^2\gamma(h)+[\frac{3}{4}+\gamma^2(h)]g)$ $\gamma(h)=\frac{1}{1-f}$
$\xi_k=\lVert\Gamma_k F_k\rVert, \quad h_k=(M_{2,k}\xi_k, M_{3,k}\xi_k^2)'$ mit $M_{\nu,k}=\frac{1}{\nu!}\sup_{u\in U}\lVert\Gamma_k F^{(\nu)}(u)\rVert \quad (\nu=2,3)$

Tabelle IV

$H=\{h\in\mathbf{R}^+:Kh^2<1\}$ $Ah=Kh^3,\quad \beta(h)=Kh^2,\quad \gamma(h)=1$ $\xi_k=h_k=B\lVert F_k\rVert$
mit $\left.\begin{aligned} B\geqslant\lVert\Gamma(u)\rVert \\ M_\nu\geqslant\frac{1}{\nu!}\lVert F^{(\nu)}(u)\rVert\end{aligned}\right\}$ für alle $u\in U\subset\tilde{D}$ und $K=(M_2B)^2+\frac{7}{4}M_3B$

Beweis. Der Beweis verläuft mit den Abschätzfunktionen A, β, γ im Prinzip wie der des Satzes 5.2. Für Einzelheiten sei wieder auf [17] verwiesen.

Bemerkungen. 1. Wie in den Fällen I.1, I.2, II.1 und II.2 sind A, β, γ isoton auf H und es gilt $Ah \leqslant h$ und $\beta(h) < 1$ für alle $h \in H$.

2. Im Fall III werden von Kleinmichel [15] mit $U = S(u_0, r)$, $q(h) = = \frac{1}{4}(3 + \sqrt{1 + 25g/12f^2}) \geqslant 1$ und $\varphi_K(h) = 2q(h)/[1 + \sqrt{1 - 4fq(h)}]$ die schärferen Konvergenzbedingungen

$$h_0 \in H_K = \{h = (f, g)' \in \mathbf{R}_2^+ : f \leqslant \tfrac{1}{4}, g < \tfrac{12}{25}(1 - 6f + 8f^2)\} \quad \text{und} \quad r \geqslant \varphi_K(h_0)\,\xi_0$$

angegeben.

$\varphi_K(h)$ löst die Funktional*un*gleichung (2.5) in H_K. Sie stellt aber nur in der „Nähe" von $g \equiv 0$ (vergl. Bemerkung 4) eine gute Näherung für die Lösung der Funktional*gleichung* dar. Sonst ist etwa $\varphi(h) = \gamma(h)/[1 - \beta(h)]$ eine wesentlich bessere Näherung, wie ein Vergleich der Funktion bestätigt. Die Grenzlinien der „Konvergenzbereiche" H und H_K sind in Abb. 2 dargestellt.

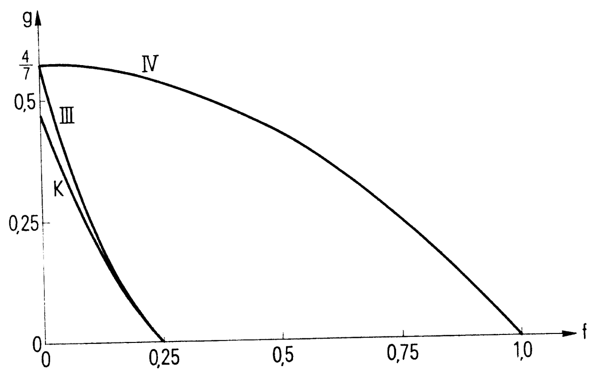

Abb. 2

3. Der Fall IV ist ein Spezialfall des Satzes 2.2.1 mit $b(h) = K$ und $L = Kh_0^2$. Die zugehörige Funktionalgleichung $\varphi(Kh^3)\,Kh^2 + 1 = \varphi(h)$ hat die Lösung

$$\varphi(h) = \sum_{\nu=0}^{\infty} (Kh^2)^{(3^\nu - 1)/2}.$$

4. Falls $F^{(3)}(u) = 0$ für alle $u \in U$ gilt, (vergl. Spezialfall (5.2.3)) ist $g \equiv 0$ und die Funktionalgleichung (2.5) im Fall III reduziert sich auf

$$\varphi\left(\frac{h^3}{(1-3h)^2}\right)\frac{h^2}{1-3h}\cdot\frac{1}{1-h} + \frac{1}{1-h} = \varphi(h) \quad \text{in} \quad H = [0, \tfrac{1}{4}], \tag{5.26}$$

wenn man $\hat{h} = \begin{pmatrix} h \\ 0 \end{pmatrix}$ mit h identifiziert. Sie besitzt nach Satz 3.5 ($\gamma(h) = 1/[1 - h]$) die isotone Lösung $\varphi(h) = 2/[1 + \sqrt{1 - 4h}]$.

5.4. *Beispiele*

Die Anwendung der Fehlerabschätzungen dieses Abschnitts soll an einfachen Beispielen demonstriert werden.

5.4.1. Es werde das nichtlineare Gleichungssystem

$$Fu=F\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3x^2y+y^2-1\\x^4+xy^3-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

in $R=\mathbf{R}_2$ mit $\|u\|=\max(|x|,|y|)$ für $u=\begin{pmatrix}x\\y\end{pmatrix}\in\mathbf{R}_2$ betrachtet (vergl. [14]).

5.4.1.1. Für $u_0=\begin{pmatrix}1\\\frac{1}{3}\end{pmatrix}$ existiert $\Gamma_0=\frac{1}{1145}\begin{pmatrix}-27 & 297\\327 & -162\end{pmatrix}$ und mit dem Verfahren (5.5) ($n=1, 2, 3$) ergeben sich die Näherungen $u_{1,n}=u_0+d_n(u_0)$:

$$u_{1,1}=\begin{pmatrix}0.993\,013\,101\\0.306\,841\,339\end{pmatrix},\quad u_{1,2}=\begin{pmatrix}0.992\,782\,965\\0.306\,459\,261\end{pmatrix},\quad u_{1,3}=\begin{pmatrix}0.992\,780\,311\\0.306\,440\,912\end{pmatrix}.$$

Die Existenz einer Lösung u^* und die Konvergenz der Folgen $\{u_{k,n}\}$ ($n=1, 2, 3$) folgen aus Satz 5.2:

Für $n=3$ ist der Fall I.2 am einfachsten anwendbar, denn es darf $U=\mathbf{R}_2$ gesetzt werden. Mit

$$\xi_0=\frac{91}{3435},\qquad m_{2,0}=\frac{2718}{1145},\qquad m_{3,0}=\frac{1953}{1145},\qquad M_{4,0}=\frac{594}{1145},$$

$$h_0=\begin{pmatrix}m_{2,0}\xi_0\\m_{3,0}\xi_0^2\\M_{4,0}\xi_0^3\end{pmatrix}=\begin{pmatrix}6.288\,667\,27\cdot10^{-2}\\1.197\,087\,95\cdot10^{-3}\\9.645\,503\,68\cdot10^{-6}\end{pmatrix},$$

$p(h_0)=1.072\,482\,271$, $q(h_0)=1.340\,601\,189\cdot10^{-3}$, $r_4(h_0)=0.139\,068\,015$,

$$P(h_0)=\begin{pmatrix}1 & 3p(h_0) & 6p^2(h_0)\\0 & 1 & 4p(h_0)\\0 & 0 & 1\end{pmatrix}$$

gilt $r_4(h_0)<1$ und $q(h_0)\,P(h_0)\,h_0<(1-r_4(h_0))^2\,h_0$, also $h_0\in H$. Damit sind alle Voraussetzungen von Satz 5.2 (I.2) erfüllt.

Man erhält mit $\varphi(h)=p(h)/[1-\beta(h)]$, $\beta(h)=q(h)/[1-r_4(h)]$ die Fehlerschranken:

$$\|u^*-u_{1,3}\|\leqslant[\varphi(h_0)-p(h_0)]\,\xi_0\leqslant4.44\cdot10^{-5}$$

(nach (2.9)) und

$$\|u^*-u_{1,3}\|\leqslant\varphi(Ah_0)\,\tilde{\xi}_1\leqslant\varphi(h_0)\,\tilde{\xi}_1\leqslant3.32\cdot10^{-6}$$

(nach (5.12), (5.14)).

Der exakte Fehler ist $\|u^* - u_{1,3}\| < 4.7 \cdot 10^{-7}$.

Für $n=2$ wählt man $U=\{(x, y): 0.9 \leqslant x \leqslant 1,\ 0.3 \leqslant y \leqslant \frac{1}{3}\}$. Mit

$$h_0 = \begin{pmatrix} m_{2,0}\xi_0 \\ M_{3,0}\xi_0^2 \end{pmatrix}, \qquad \varphi(h) = p(h)/[1-\beta(h)], \qquad \beta(h) = q(h)/[1-r_3(h)]$$

sind die Voraussetzungen $h_0 \in H$, $S \subset U$ des Satzes 5.2 (I.2) erfüllt.

Man erhält die Schranken

$$\|u^* - u_{1,2}\| \leqslant \begin{cases} 3.17 \cdot 10^{-4} & \text{(nach (2.9))} \\ 1.01 \cdot 10^{-4} & \text{(nach (5.12), (5.14))} \\ 1.89 \cdot 10^{-5} & \text{(exakter Fehler)} \end{cases}$$

Für $n=1$ liefert der Spezialfall (5.2.1.1) die Schranke $\|u^* - u_{1,1}\| \leqslant 1.92 \cdot 10^{-3}$ (nach (2.9)) gegenüber dem exakten Fehler $\|u^* - u_{1,1}\| \leqslant 4.01 \cdot 10^{-4}$.

5.4.1.2. Das Verfahren (5.25) führt ausgehend von $u_0 = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix} \in \tilde{D}$ auf

$$u_1 = \begin{pmatrix} 0.992\,783\,127 \\ 0.306\,449\,493 \end{pmatrix}.$$

Mit $U=\{(x, y) \in \mathbf{R}_2 : 0.9 \leqslant x \leqslant 1,\ 0.3 \leqslant y \leqslant \frac{1}{3}\}$,

$$\varphi(h) = \gamma(h)/[1-\beta(h)], \qquad \xi_0 = 0.026\,491\,9942, \qquad M_{2,0} = \frac{2718}{1145}, \qquad M_{3,0} = \frac{1953}{1145}$$

sind alle Voraussetzungen des Satzes 5.3 (III) erfüllt. Es existiert daher eine Lösung $u^* \in U$ mit $\|u^* - u_0\| \leqslant 0.0285$ (nach (2.6)) und $\|u^* - u_1\| \leqslant 2.277 \cdot 10^{-4}$ (nach (2.9)).

Mit $U = S(u_0, 0.0306)$ erhält man nach dem Satz von Kleinmichel die Schranken

$$\|u^* - u_0\| \leqslant 0.03056 \quad \text{und} \quad \|u^* - u_1\| \leqslant 1.657 \cdot 10^{-3}.$$

Die exakten Fehler sind

$$\|u^* - u_0\| \leqslant 0.0269 \quad \text{und} \quad \|u^* - u_1\| \leqslant 0.905 \cdot 10^{-5}.$$

5.4.2. Die Matrix-Eigenwertaufgabe $Ax = \lambda Bx$ ($A = (a_{\sigma\tau})$, $B = (b_{\sigma\tau})$, $x = (x_{(\sigma)})$ Eigenvektor, λ Eigenwert) werde in bekannter Weise als Operatorgleichung

$$Fu = \begin{pmatrix} (A - xB)\, y \\ y_{(r)} - 1 \end{pmatrix} \qquad (0 \leqslant r \leqslant s, \quad r \text{ fest})$$

im Banachraum $\mathbf{R}_{s+1}$ der Elemente $u = \begin{pmatrix} y \\ x \end{pmatrix}$, $y \in \mathbf{R}_s$, $x \in \mathbf{R}$ mit $\|u\| = \max_\sigma(|y_{(\sigma)}|, |x|)$ geschrieben (vergl. [4], [5]). Es tritt dann der Spezialfall (5.2.3) ein und das Verfahren

(5.5) ist von der besonders einfachen Gestalt $u_{k+1,n}=u_{k,n}+d_{k,n}$ mit

$$d_{k,1}=\begin{pmatrix} z_{k,1} \\ \zeta_{k,1} \end{pmatrix}=-\Gamma_k F_k, \qquad d_{k,\nu}=\begin{pmatrix} z_{k,\nu} \\ \zeta_{k,\nu} \end{pmatrix}$$

$$=d_{k,1}+\Gamma_k\begin{pmatrix} \zeta_{k,\nu-1}Bz_{k,\nu-1} \\ 0 \end{pmatrix} \quad (\nu=2,3,\dots,n). \tag{5.27}$$

Es sei

$$s=r=10,\ B=E,\ \alpha_{\sigma\tau}=\begin{cases} 1 & \text{für } \sigma=1, \\ 1/(\sigma+\tau-1) & \text{für } \sigma=2,3,\dots,10, \end{cases} \tau=1,2,\dots,10\Big\}.$$

Für $u_0=(11.78, 3.84, 2.76, 2.18, 1.81, 1.55, 1.36, 1.21, 1.10, 1, 2.43)'$ existiert Γ_0 und es ist $\xi_0=4.809\,397\,191\,072\,82\ldots\cdot 10^{-3}$, $M_{2,0}\leqslant 5.401\,832$, also $h_0<\frac{1}{4}$. Daher ist nach Satz 5.2 (Spezialfall (5.2.3)) die Konvergenz des Verfahrens (5.27) für jedes $n\geqslant 1$ gesichert. Man erhält für $\|u^*-u_{1,n}\|$ die Schranken:

n	nach (2.9)	nach (5.23)	nach (5.24)	exakt
1	$1.4\cdot 10^{-4}$	–	$1.3\cdot 10^{-4}$	$2.9\cdot 10^{-6}$
2	$7.0\cdot 10^{-6}$	$1.7\cdot 10^{-7}$	$1.5\cdot 10^{-8}$	$3.6\cdot 10^{-10}$
3	$3.8\cdot 10^{-7}$	$2.1\cdot 10^{-11}$	$1.1\cdot 10^{-11}$	$2.2\cdot 10^{-13}$
4	$2.0\cdot 10^{-8}$	$1.3\cdot 10^{-14}$	$5.9\cdot 10^{-15}$	$1.3\cdot 10^{-16}$

Bemerkungen. Die Konvergenzbedingungen des Satzes 5.1 sind erst nach „rekursiver Rechnung“ analog zu (5.12), (5.23) erfüllbar. Für $s=2$, und $s=3$ finden sich Beispiele mit günstigeren Schranken in [17].

5.4.3. Die Betrachtung der Gleichung $Fu=u^3-10=0$ im Banachraum $(R, |\cdot|)$ ermöglicht mit geringem Aufwand einen Vergleich der Fehlerabschätzungen des Satzes 5.2 für $n=2$ mit einigen anderen Schranken. Ausgehend von $u_0=2$ führt das Verfahren (5.5) für $n=2$ auf $u_1=2.152\bar{7}$ und $u_2=2.154\,434\,688\,394\ldots$. Wenn man nur Größen berücksichtigt, die bis zur Berechnung von u_1 aufgetreten sind, erhält man die folgenden Abschätzungen:

$10^3\cdot\lvert u^*-u_1\rvert\leqslant$	$10^9\cdot\lvert u^*-u_2\rvert\leqslant$	Lit. quelle	U
3.25 / 2.52 (nach (5.12))	73.62 / 16.60 nach (2.7)	Satz 5.2 I.1 / II.2	[2, 13/6]
3.65	29.3	Collatz [4] (Ade [1])	[2, 13/6]
3.96	71.2	Shafiyev [23]	$S(0, 0.1846)$
4.28	1344	Döring [7]	$S(2, 0.26)$

Dabei ist in Satz 5.2 $\varphi(h)=\gamma(h)/[1-\beta(h)]$ verwendet worden. Läßt man auch Größen zu, die bis zur Berechnung von u_2 aufgetreten sind, so ergeben sich z.B. nach Satz 5.2 (I.1) die Schranken $|u^*-u_1|\leqslant 1.660\cdot 10^{-3}$ (2.6) und $|u^*-u_2|\leqslant 2.33\cdot 10^{-9}$ (2.9). Döring [7] gibt als beste Schranke $2.92\cdot 10^{-9}$ an. Die exakten Fehler sind

$$|u^*-u_1|\leqslant 1.657\cdot 10^{-3} \quad \text{und} \quad |u^*-u_2|\leqslant 1.64\cdot 10^{-9}.$$

6. Ein Satz für eine Verallgemeinerung des Iterationsverfahrens von Schulz

Es sei $(R, \|\cdot\|)$ ein Banachraum von linearen Abbildungen eines linearen Raumes in sich. Zur Bestimmung der Inversen X eines Operators $P\in R$ führt die Anwendung des Verfahrens (5.5) auf die Gleichung $P-X^{-1}=0$ auf die Iterationsvorschrift

$$X_{k+1}=X_k+X_k\sum_{\nu=1}^{n}(E-PX_k)^\nu,\ X_0\in R \qquad (k=0, 1, 2, \dots) \tag{6.1}$$

(vergl. [10], [25]). Für $n=1$ erhält man daraus das Verfahren von Schulz [26]. Schmidt und Leder [25] haben gezeigt, daß dieses Verfahren unter der Voraussetzung $\|E-PX_0\|<1$ konvergiert und von der Ordnung $n+1$ ist. Eine Fehlerabschätzung wird im folgenden durch Anwendung von Satz 2.3 hergeleitet. Diese stimmt für $n=1$ bzw. $n=2$ mit den von Albrecht [2] bzw. Forster [10] angegebenen Abschätzungen überein.

SATZ 6.1. *Für $X_0\in R$ mit $\|E-PX_0\|<1$ konvergiert das Verfahren* (6.1) *für jede feste ganze Zahl $n\geqslant 1$ gegen den inversen Operator P^{-1} des gegebenen Operators $P\in R$. Es gelten die Fehlerabschätzungen*

$$\|P^{-1}-X_{k+1}\|\leqslant\frac{\|E-PX_k\|^n}{1-\|E-PX_k\|^n}\|X_{k+1}-X_k\| \qquad (k=0, 1, 2, \dots) \tag{6.2}$$

mit $\|E-PX_k\|\leqslant\|E-PX_0\|^{(n+1)^k}$.

Beweis. Für den durch

$$TX=X+X\sum_{\nu=1}^{n}(E-PX)^\nu$$

definierten Iterationsoperator $T:R\to R$ gelten, wie man durch elementare Umformungen nachweist, die Identitäten

$$E-PTX=(E-TX)^{n+1} \quad \text{und} \quad TTX-TX=(TX-X)\sum_{\nu=1}^{n+1}(E-PX)^{\nu n}$$

für alle $X\in R$. Mit $D_0=\{X\in R: \|E-PX\|<1\}$ und $H=[0, 1)$,

$$A:H\to H, \qquad \beta,\ \gamma:H\to\mathbf{R}^+, \qquad \xi:D_0\to\mathbf{R}^+, \qquad h:D_0\to H$$

definiert durch

$$Ah=h^{n+1}, \quad \beta(h)=\sum_{\nu=1}^{n+1} h^{\nu n}, \quad \gamma(h)=1, \quad \xi(X)=\|TX-X\|, \quad h(X)=\|E-PX\|$$

folgen daraus die Abschätzungen

$$\xi(TX)\leqslant\beta(h(X))\,\xi(X) \quad \text{und} \quad h(TX)\leqslant Ah(X) \quad \text{für alle} \quad X\in D_0,$$

d.h. (V1′) ist erfüllt. $\varphi(h)=1/(1-h^n)$ löst die Funktionalgleichung

$$\varphi(h^{n+1})\sum_{\nu=1}^{n+1} h^{\nu n}+1=\varphi(h) \quad \text{in} \quad H. \tag{6.3}$$

φ erfüllt außerdem (V3), denn es ist $\lim_{k\to\infty} A^k h=0$ für alle $h\in H$ und daher $\beta(A^k h)<1$ für alle $k\geqslant k_0$. Mit $TD_0\subset D_0$ gilt schließlich auch (V4′).

7. Ein Satz von Ulm für die Regula falsi

Es sei $(R, \|\cdot\|)$ ein Banachraum. Der Operator $F: D(F)\subset R\to R$ besitze in $U\subset D(F)$ eine Steigung, d.h. zu jedem Paar $(u, v)\in U\times U$ existiere ein linearer Operator $\delta F(u, v): R\to R$ mit $\delta F(u, v)(u-v)=Fu-Fv$.

Mit $D=\{(u, v)\in U\times U: \{\delta F(u, v)\}^{-1}$ existiert und ist beschränkt$\}$ lautet die Regula falsi zur Lösung der Gleichung $Fu=0$

$$u_{k+1}=u_k-\{\delta F(u_{k-1}, u_k)\}^{-1} Fu_k, \quad (u_0, u_{-1})\in D \quad (k=0, 1, 2, \ldots) \tag{7.1}$$

(vergl. Schmidt [24]).

Als Spezialfall von Satz 2.2 ergibt sich mit

Tabelle V

$H=\{h=(f, g)'\in \mathbf{R}_2^+: g<1, f\leqslant(1-\sqrt{g})^2\}$

$$Ah=\frac{1}{(1-g)^2}(fg, f)', \quad \beta(h)=\frac{1}{1-g}, \quad \gamma(h)=1$$

$$\varphi(h)=\frac{2}{1+f-g+\sqrt{(1+f-g)^2-4f}} \tag{7.2}$$

$$\xi_k=\|u_{k+1}-u_k\|$$

$$h_k=\begin{pmatrix} K_k\,\|u_{k+1}-u_k\| \\ K_k\,\|u_{k+1}-u_{k-1}\| \end{pmatrix} \quad \text{mit} \quad K_{k+1}=\frac{K_k}{1-g_k} \quad (k=0, 1, 2, \ldots) \tag{7.3}$$

und $\|\{\delta F(u_{-1}, u_0\}^{-1}[\delta F(u, v)-\delta F(v, w)]\|\leqslant K_0\,\|u-w\|$ für alle $u, v, w\in U$.

SATZ 7.1 (Ulm). *Mit den Definitionen der Tabelle* V *gelte für* $(u_0, u_{-1}) \in D$: *Es sei* $h_0 \in H$ *und* $S = S(u_1, [\varphi(h_0) - 1]\, \xi_0) \subset U$. *Dann ist das Verfahren, ausgehend von* u_0, u_{-1}, *unbeschränkt ausführbar und konvergiert gegen eine Lösung* u^* *der Gleichung* $Fu = 0$ *und es gelten die Fehlerabschätzungen* (2.6)–(2.9).

Beweis. Die wesentlichen Abschätzungen für den Beweis findet man bei Ulm [27] oder auch bei Hofmann [12], welche beide das Majorantenprinzip benutzen. $\varphi(h)$ ist Lösung der Funktionalgleichung (2.5). Es gilt $AH \subset H$ und $A^2 H \subset H_2 = \{(f, g)' \in H : f < g \leqslant \frac{4}{9}\}$ mit $AH_2 \subset H_2$.

Da φ auf H_2 beschränkt ist und $\beta(h) \leqslant \frac{4}{5}$ für alle $h \in H_2$ gilt, ist auch (V3) erfüllt.

8. Sätze von Burmeister für inversionsfreie Verfahren

Es sei $(R, \|\cdot\|)$ ein Banachraum. Der Operator $F: D(F) \subset R \to R$ sei in einer konvexen Menge $U \subset D(F)$ Fréchet-differenzierbar. Mit $L(R)$ werde die Menge der linearen Operatoren von R in sich bezeichnet. Um die Bestimmung des inversen Operators zu vermeiden, kann das Newton-Verfahren (5.5) $(n=1)$ bzw. die Regula falsi (7.1) mit dem Verfahren von Schulz (6.1) $(n=1)$ kombiniert werden (vergl. Helfrich [11], Ulm [28], Schmidt-Leder [25]):

$$\left.\begin{array}{ll} u_{k+1} = u_k - X_{k+1} F_k & X_{k+1} = X_k + (E - X_k F'_k)\, X_k \quad (k = 0, 1, 2, \ldots) \\ & u_0 \in U,\ X_0 \in L(R) \end{array}\right\} \tag{8.1}$$

$$\left.\begin{array}{ll} u_{k+1} = u_k - X_{k+1} F_k, & X_{k+1} = X_k + (E - X_k \delta F(u_{k-1}, u_k))\, X_k \\ & \qquad\qquad (k = 0, 1, 2, \ldots) \\ & u_{-1}, u_0 \in U,\ X_0 \in L(R) \end{array}\right\} \tag{8.2}$$

Für das Verfahren (8.2) wird zusätzlich vorausgesetzt, daß F in U eine Steigung besitzt.

Die folgenden Sätze von Burmeister [3] für diese Verfahren ordnen sich ebenfalls in die Konvergenztheorie dieser Arbeit ein.

Tabelle VI

$H = \{h = (e, f)' \in \mathbf{R}_2^+ : e < 1,\ 2f \leqslant (1 - e^2)^2\}$
$Ah = \begin{pmatrix} e^2 + f \\ (1 + e^2 + f)^2\, (\frac{1}{2} f + e^2)\, f \end{pmatrix}$, $\beta(h) = (1 + e^2 + f)\, (\frac{1}{2} f + e^2)$, $\gamma(h) = 1$
$\varphi(h) = \dfrac{2}{1 - e^2 + \sqrt{(1 - e^2)^2 - f}}$
$\xi_k = \|u_{k+1} - u_k\|$, $h_k = \begin{pmatrix} \|E - X_k F'_k\| \\ K\, \|X_{k+1}\|\, \xi_k \end{pmatrix}$ mit $\|F'(u) - F'(v)\| \leqslant K\, \|u - v\|$ für alle $u, v \in U$.

SATZ 8.1. *Mit den Definitionen der Tabelle* VI *gelte für* $u_0 \in U$, $X_0 \in L(R)$: *Es sei* $h_0 \in H$ *und* $S(u_1, [\varphi(h_0)-1]\,\xi_0) \subset U$. *Dann ist das Verfahren* (8.1) *unbeschränkt ausführbar. Es konvergiert gegen eine Lösung* u^* *der Gleichung* $X_0 Fu = 0$ *und für* u^* *gelten die Fehlerabschätzungen* (2.6)–(2.9).

SATZ 8.2. *Es gilt Satz* 8.1 *für das Verfahren* (8.2) (*ausgehend von* u_{-1}, $u_0 \in U$, $X_0 \in L(R)$), *wenn man Tabelle* VI *durch Tabelle* VII *ersetzt.*

Tabelle VII

$$H = \{h = (e, f, g)' \in \mathbf{R}_3^+ : e^2 + 2\sqrt{f} + g \leqslant 1\}$$

$$Ah = \begin{pmatrix} d \\ f d(1+d)^2 \\ f(1+d) \end{pmatrix}, \quad \beta(h) = d(1+d), \quad \gamma(h) = 1$$

mit $d = e^2 + f + g$

$$\varphi(h) = \frac{2}{1 - e^2 - g + \sqrt{(1 - e^2 - g)^2 - 4f}}$$

$$\xi_k = \|u_{k+1} - u_k\|, \qquad h_k = \begin{pmatrix} \|E - X_k \delta F(u_{k-1}, u_k)\| \\ K\,\|X_{k+1}\|\;\|u_{k+1} - u_k\| \\ K\,\|X_{k+1}\|\;\|u_k - u_{k-1}\| \end{pmatrix}$$

$$\text{mit} \quad \left.\begin{matrix} \|\delta F(u, v) - F'(u)\| \\ \|\delta F(u, v) - F'(v)\| \end{matrix}\right\} \leqslant K\,\|u - v\| \quad \text{für alle} \quad u, v \in U.$$

Beweise. Siehe Burmeister [3]. Es gilt $AH \subset H$ und φ ist in beiden Fällen Lösung der zugehörigen Funktionalgleichung (2.5) und erfüllt (V3).

Diese Arbeit enthält in wesentlich verallgemeinerter Fassung Teile der von der Fakultät für Natur- und Geisteswissenschaften der Technischen Universität Clausthal genehmigten Dissertation [17] des Verfassers.

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Aequationes Math. *13* (1975), 47–59. University of Waterloo Birkhäuser Verlag, Basel

Continuous solutions of a homogeneous functional equation

Morris Newman and Mark Sheingorn

0. Introduction

In this paper we consider the problem of finding continuous solutions of the homogeneous functional equation

$$\sum_{k=1}^{n} f(a_k x)=0, \quad 0<a_1<\cdots<a_n.$$

Here $f:R\to R$, R the real numbers. This problem is a generalization of a problem suggested by Erwin Just; namely, to find a real-valued continuous f such that $f(x)+f(2x)+f(3x)=0$ for all real x.

In the case when $\log a_1, \log a_2,\dots, \log a_n$ are all rational multiples of a fixed real number this equation reduces to one of a type considered by J. Kordylewski and M. Kuczma, and may be found in Kuczma's book [2], p. 266. However this is a trivial case both for us and for Kordylewski and Kuczma, and their results do not apply to the general case.

It is not hard to see that finding continuous solutions of this equation is equivalent to finding continuous solutions of the finite difference equation

$$\sum_{k=1}^{n} g(y+\alpha_k)=0, \quad y\in R,$$

such that

$$\lim_{y\to-\infty} g(y)=0.$$

Here $\alpha_k=\log a_k$. If the α_k are all rational multiples of a fixed real number, this may be reduced to a classical linear difference equation with constant coefficients. These have been considered in detail (see Gelfond, [1] for example) but usually for values of y of the form $\alpha z+\beta$, where α, β are fixed reals and z is an integer, rather than for all real y. A more general non-homogeneous equation of this type was also considered by S. Pincherle in 1888, but without the boundary condition [5, 6].

AMS Primary Subject Classifications: 30A20, 30A08; Secondary Subject Classification: 39A15.

Received April 5, 1973 and, in revised form, January 18, 1974.

In section 1 we give a sufficient condition for the first equation to have a continuous solution (Theorem 2). This allows us to treat the case $n=3$ completely. In section 2 we prove a partial converse to Theorem 2. In section 3 we use results concerning the density of zeros of certain sums of exponentials to analyse the case $n=4$ completely. In section 4 these techniques are applied to yield a wide class of equations with $n>4$ which have continuous solutions. We conclude by indicating that if $n>4$, no simple complete solution (in the sense of Corollary 1 and Theorem 4) seems possible.

We remark that our techniques are entirely applicable with minor changes to the more general equation

$$\sum_{k=1}^{n} c_k f(a_k x)=0,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary reals, and $a_1, a_2, \ldots, a_n$ are as before.

The authors wish to thank the referees for many useful comments, including the difference equation formulation.

1. The problem under consideration is whether the functional equation

$$\sum_{k=1}^{n} f(a_k x)=0, \qquad 0<a_1<a_2<\cdots<a_n, \tag{1}$$

has solutions which are continuous for all real values of x. There is no loss of generality in assuming that

$$a_1=1,$$

since $f(x)$ may be replaced by $f(x/a_1)$.

When $n=2$, (1) becomes

$$f(x)+f(ax)=0, \qquad a=a_2>1, \tag{2}$$

which has only the solution $f(x)=0$, since

$$f(x)=-f\left(\frac{x}{a}\right)=f\left(\frac{x}{a^2}\right)=\cdots=(-1)^t f\left(\frac{x}{a^t}\right).$$

Replace t by $2t$, and take the limit as t goes to ∞. This gives $f(x)=f(0)$, which in turn implies $f(x)=0$.

The example

$$f(x)=\cos\left(\pi\frac{\log|x|}{\log a}\right)$$

is most revealing. $f(x)$ is continuous for all real values of x except $x=0$, and satisfies the functional equation (2).

When $n>2$, (1) may or may not have solutions. Thus the equation

$$f(x)+f(ax)+\cdots+f(a^{n-1}x)=0, \quad a>1,$$

implies that

$$f(ax)+f(a^2x)+\cdots+f(a^nx)=0,$$

so that by subtraction we find that

$$f(x)=f(a^nx),$$

which has only the solution $f(x)=\text{constant}$, so that $f(x)\equiv 0$.

We set

$$\alpha_k=\log a_k, \quad 1\leqslant i\leqslant n$$

so that

$$0=\alpha_1<\alpha_2<\cdots<\alpha_n,$$

There is no loss of generality in assuming that

$$\alpha_2=1,$$

which may be accomplished by choosing the logarithms to the base a_2.

We now prove

THEOREM 1. *Suppose that the equation*

$$\sum_{k=1}^{n} w^{\alpha_k}=0 \tag{3}$$

has a root w such that $|w|>1$. *Then* (1) *has a solution which is continuous for all real* x.

Proof. We seek a solution of (1) of the form

$$f(x)=e^{c\log|x|}.$$

Then $f(x)$ is continuous for all real x, except possibly at $x=0$. If $\operatorname{Re}(c)=a$, then $f(x)$ is continuous at 0 (and has the value 0 there) if and only if $a>0$. Put $w=e^c$. Then $a>0$ if and only if $|w|>1$. Furthermore, $f(x)$ satisfies (1) if and only if w satisfies (3). This completes the proof.

The real part (or the imaginary part) of the solution described above is a real solution which is continuous for all real x.

In order to apply Theorem 1, we need the following:

LEMMA 1. *Let α_k be as above, and put*

$$g(w)=\sum_{k=1}^{n} w^{\alpha_k}.$$

Then a w_0 exists such that

$$g(w_0)=0.$$

Proof. $g(w)$ is an entire function of w only if the α_k are all integers; otherwise, it is not even single-valued. Lemma 1 means that there is a branch of g and a complex number w_0 such that $g(w_0)=0$. We begin by choosing a branch which is analytic in the complex plane C with $(-\infty, 0]$ deleted, which we denote by K. The branch is defined by setting $w^\alpha = e^{\alpha \log w}$, and choosing the principal branch of $\log w$, which is 0 when $w=1$, and may be defined by the condition that

$$-\pi<\operatorname{Im}(\log w)\leqslant\pi.$$

Assume this done and call the branch $g(w)$, $w\in K$. Since $\alpha_1=0$, we may find an $\varepsilon>0$ such that $g(w)$ does not vanish in the set

$$\{z:|z|<\varepsilon\}\cap K$$

(in fact, $\varepsilon=1/n$ will suffice).

g may be approximated uniformly on compact subsets by (proper branches of) the functions

$$f_m(w)=\sum_{i=1}^{n} w^{p_{im}/q_{im}}, \tag{4}$$

where p_{im}, q_{im} are non-negative integers, $p_{1m}/q_{1m}=0$, $p_{2m}/q_{2m}=1$, and p_{im}/q_{im} converges monotonically to α_i from below as $m\to\infty$, for each i. Furthermore we may assume that

$$\frac{\alpha_{i-1}+\alpha_i}{2}\leqslant\frac{p_{im}}{q_{im}}\leqslant\alpha_i, \qquad \text{for all } i>2 \tag{5}$$

and also that

$$q_{im}>1.$$

To uniformize f_m, set

$$w=t^{q_m}, \qquad q_m=q_{1m}q_{2m}\cdots q_{nm},$$

and put

$$\varphi_m(t)=f_m(w)=\sum_{i=1}^{n} t^{p_{im}q_m/q_{im}}=\sum_{i=1}^{n} t^{\beta_{im}},$$

where

$$\beta_{im}=p_{im}q_m/q_{im} \quad \text{is a non-negative integer}.$$

Since $\varphi_m(t)$ is a polynomial it has zeros; and therefore so does $f_m(w)$. In fact the zeros of f_m are the zeros of φ_m raised to the power q_m.

Now the zeros of $f_m(w)$ are bounded (independently of m): for, if w is a zero of f_m, with $|w|>1$ then, by (5),

$$|w|^{p_{nm}/q_{nm}}=\left|\sum_{i=1}^{n-1} w^{p_{im}/q_{im}}\right|\leqslant(n-1)\,|w|^{p_{n-1,m}/q_{n-1,m}},$$

so that

$$|w|^{(p_{n,m}/q_{n,m})-(p_{n-1,m}/q_{n-1,m})}\leqslant n-1.$$

Then (5) implies that

$$|w|^{(\alpha_n-\alpha_{n-1})/2}\leqslant n-1,$$

or

$$|w|\leqslant(n-1)^{2(\alpha_n-\alpha_{n-1})^{-1}}$$

Let w_0 be a limit point of these zeros of $f_m(w)$. If $w_0\in K$ then by uniform convergence on compact subsets this limit point must be a zero of g. If all limit points lie on $(-\infty, -\varepsilon]$ we proceed as follows. Select one of these limit points w_0. Without loss of generality assume that w_0 is the limit of a sequence of zeros of the f_m all of which lie in the upper half plane. Consider the branch of the logarithm given by

$$\frac{-\pi}{2}<\operatorname{Im}(\log w)\leqslant\frac{3\pi}{2}.$$

Accordingly adjust the branches of g and the f_m. These new branches agree with the old ones in the upper half plane. So the new f_m's have a limit point of zeros $w_0\in(-\infty, -\varepsilon]$, but now w_0 is in the interior of the domain of analyticity of (the new branch of) g, and so $g(w_0)=0$ by uniform convergence on compact subsets. This completes the proof of Lemma 1.

Theorem 1 requires that a zero of g be located outside the unit circle. In this connection we prove

LEMMA 2. *Suppose that g has a zero in* $\{z: |z|<1\}$. *Then it has a zero in* $\{z: |z|>1\}$.

Remark. This lemma shows that either all the zeros of g lie on the unit circle, or there is one outside the unit circle.

Proof of Lemma 2. To prove this lemma we must analyze the zeros of the functions f_m given by (4) in some detail. Suppose then that g has a zero w_0 such that $|w_0|<1$. Then Hurwitz's theorem implies that w_0 is a limit point of the zeros of the f_m. Let $\{h_m\}$ be a subsequence of $\{f_m\}$ such that z_m is a zero of h_m and the sequence $\{z_m\}$ converges to w_0. To simplify the notation and avoid double subscripts, let us assume that $h_m=f_m$. Since $|w_0|<1$, there is an $r_0<1$ such that

$$|z_m|<r_0, \quad \text{if} \quad m>m_0=m_0(r_0). \tag{6}$$

Because of (6), φ_m must have q_m zeros $t_1, t_2, \ldots, t_{q_m}$ such that

$$|t_j|<r_0^{1/q_m}, \qquad 1\leqslant j\leqslant q_m.$$

But the product of the roots of φ_m is ± 1. Thus if $t_1, t_2, \ldots, t_{\beta_{nm}}$ are all the roots of φ_m, we have

$$1=\prod_{1\leqslant k\leqslant q_m}|t_k|\prod_{q_m<k\leqslant \beta_{nm}}|t_k|$$

$$<r_0\prod_{q_m<k\leqslant\beta_{nm}}|t_k|.$$

So if

$$|t_0|=\max_{q_m<k\leqslant\beta_{nm}}|t_k|,$$

t_0 must satisfy

$$|t_0|>r_0^{-1/l}, \qquad l=\beta_{nm}-q_m.$$

But that means that f_m has a root τ_m satisfying

$$|\tau_m|>r_0^{-q_m/l}.$$

Since

$$\beta_{nm}=q_m p_{nm}/q_{nm},$$

we have

$$q_m/l=\left(\frac{p_{nm}}{q_{nm}}-1\right)^{-1}.$$

Hence for sufficiently large m, there is a fixed positive ε such that

$$|\tau_m|>1+\varepsilon,$$

since $r_0<1$ and $p_{nm}/q_{nm}\to\alpha_n>1$.

Thus for sufficiently large m each f_m has a zero in the annulus

$$\{z:1+\varepsilon\leqslant|z|\leqslant n\}.$$

This implies there is a limit point there which will be a zero of g. This completes the proof of Lemma 2.

Lemmas 1 and 2 now imply

THEOREM 2. *The functional equation* (1) *has continuous solutions whenever the roots of the function* $\sum_{k=1}^{n} w^{\alpha_k}$ *do not all lie on the unit circle.*

This is sufficient to solve the case $n=3$ completely. In fact, we have

COROLLARY 1. *The equation*

$$f(x)+f(ax)+f(bx)=0, \qquad 1<a<b, \tag{7}$$

has a non-trivial solution which is continuous for all real values of x if and only if $b\neq a^2$.

Proof. Suppose first that $b\neq a^2$. The associated function in this case is

$$g(w)=1+w+w^{\alpha}, \qquad \alpha=\log_a b.$$

Suppose that $g(w)$ has a root ζ on the unit circle. Then $|1+\zeta|=|\zeta^{\alpha}|=1$, which implies that ζ must be a primitive cube root of 1. Since $1+\zeta+\zeta^2=0$ and $1+\zeta+\zeta^{\alpha}=0$, $\zeta^2=\zeta^{\alpha}$, $\zeta^{\alpha-2}=1$, and so α must be an integer. In fact, $\alpha\equiv 2 \pmod 3$.

Now suppose that $g(w)$ has all its roots on the unit circle. Then the roots of $g(w)$ coincide with the roots of

$$w^{\alpha}g(1/w)=1+w^{\alpha-1}+w^{\alpha}$$

whose roots are the reciprocals, and hence the conjugates, of the roots of $g(w)$. This is only possible if the polynomials $g(w)$ and $w^{\alpha}g(1/w)$ are identical, which implies that

$\alpha=2$, $b=a^2$. It follows that not all the roots of $g(w)$ lie on the unit circle and so Theorem 2 implies that (7) has a non-trivial solution which is continuous for all real x.

Now suppose that $b=a^2$. We have already seen that in this case no non-trivial solution exists. This completes the proof.

COROLLARY 2. *If the a_k appearing in* (1) *satisfy*

$$a_k=a_2^{e_k}, \quad 3\leqslant k\leqslant n,$$

where the e_k are integers, then (1) *will have non-trivial solutions if and only if the polynomial*

$$g(w)=1+w+w^{e_3}+\cdots+w^{e_n}$$

has a root which is not a root of unity: equivalently, if and only if $g(w)$ has some factor irreducible over the rationals which is not a cyclotomic polynomial.

Proof. This follows from a theorem of Kronecker to the effect that an algebraic integer which with all its conjugates lies on the unit circle must be a root of unity.

2. Is the sufficiency condition of Theorem 2 also necessary? In general we do not know. However the following partial converse is true.

THEOREM 3. *Let the equation* (1) *have $g(w)=\sum_{k=1}^{n} w^{\alpha_k}$ as its associated function. If the α_k are rational, and $g(w)$ has all its roots on the unit circle, then no non-trivial continuous solution of* (1) *exists.*

Proof. Let q be the least positive integer such that $q\alpha_k$ is an integer, $1\leqslant k\leqslant n$. We see by the uniformizing transformation

$$w=t^q$$

and Kronecker's theorem that the roots of $g(w)=\varphi(t)$ are roots of unity. Furthermore, $\varphi(t)$ is a monic integral polynomial in t. Thus there is a monic integral polynomial $p(t)$ such that for appropriate positive integers m, k,

$$\varphi(t)\,p(t)=(t^m-1)^k,$$

and hence

$$g(w)\,p(w^{1/q})=(w^{m/q}-1)^k.$$

Now multiplying $\varphi(t)$ by t^s (s an integer) corresponds to substituting $a_2^s x$ for x in equation (1). Thus multiplying $\varphi(t)$ by $b\,w^s$ (b, s integers) corresponds to substituting $a_2^s x$ for x in (1) and multiplying the result by b. Multiplying by the sum of a number of such terms corresponds to adding the resulting equations above. This means that after a sequence of such substitutions, multiplications by integers and additions, equation (1) may be transformed into the equation associated with $(t^m-1)^k = (w^{m/q}-1)^k$. But that equation is

$$\sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} f(a_2^{rm}x)=0,$$

which has only the trivial solution $f \equiv 0$, as may be seen by induction on k. This concludes the proof.

We include an example which is a small piece of evidence in favor of necessity in general.

EXAMPLE. Let $\alpha > 1$. The function

$$1+z+z^\alpha+z^{\alpha+1}=(1+z)(1+z^\alpha)$$

has all its roots on the unit circle. This function arises from the equation

$$f(x)+f(a_2x)+f(a_3x)+f(a_2a_3x)=0, \tag{8}$$

where we have $1<a_2<a_3$. Put

$$g(x)=f(x)+f(a_2x).$$

Then (8) becomes

$$g(x)+g(a_3x)=0,$$

which has only the continuous solution $g \equiv 0$. Thus

$$f(x)+f(a_2x)=0,$$

which has only the continuous solution $f \equiv 0$.

3. The purpose of this section is to prove the following result:

THEOREM 4. *Let $1<a_2<a_3<a_4$. Then there exists a continuous function $f: R \to R$ satisfying*

$$f(x)+f(a_2x)+f(a_3x)+f(a_4x) \equiv 0 \tag{8}$$

if and only if $a_4 \neq a_2a_3$.

At the end of section 1 we showed that if $a_4=a_2a_3$ then (8) cannot have a continuous solution. Thus by Theorem 2 it will suffice to show that if $P(w)=1+e^w+e^{\alpha_3 w}+e^{\alpha_4 w}$ has all roots on the imaginary axis, then $a_4=a_3a_2$. (Here $\alpha_i=\log_{a_2} a_i$, $i=3, 4$.) To see this, put $e^w=z$ in Theorem 2.

Assume all roots of $P(w)$ are on the imaginary axis. We must show that $\alpha_4=\alpha_3+1$. Let $w_0=i\theta_0$ be a root of $P(w)$. Then $w_0=-i\theta_0$ is also a root:

$$0=P(-i\theta_0)=1+e^{-i\theta_0}+e^{-i\alpha_3\theta_0}+e^{-i\alpha_4\theta_0}.$$

So,

$$0=e^{i\alpha_4\theta_0}+e^{i(\alpha_4-1)\theta_0}+e^{i(\alpha_4-\alpha_3)\theta_0}+1.$$

But of course,

$$0=e^{i\alpha_4\theta_0}+e^{i\alpha_3\theta_0}+e^{i\theta_0}+1. \tag{9}$$

Therefore,

$$e^{i(\alpha_4-1)\theta_0}+e^{i(\alpha_4-\alpha_3)\theta_0}=e^{i\alpha_3\theta_0}+e^{i\theta_0}.$$

This simplifies to:

$$(\xi^{\alpha_4-\alpha_3-1}-1)(\xi^{\alpha_3}+\xi)=0,$$

where we have put $\xi=e^{w_0}$. Put

$$Q(w)=(e^{(\alpha_4-\alpha_3-1)w}-1)(e^{\alpha_3 w}+e^w).$$

Then every root of $P(w)$ is also a root of $Q(w)$. We shall show this is impossible unless $Q(w)$ is identically 0, which can only happen if $\alpha_4-\alpha_3-1=0$.

Assume that $Q(w)$ is not identically 0. All the roots of $Q(w)$ are imaginary. Let

$$P^*(w)=e^{i\alpha_4 w}+e^{i\alpha_3 w}+e^{iw}+1,$$
$$Q^*(w)=(e^{i(\alpha_4-\alpha_3-1)w}-1)(e^{i\alpha_3 w}+e^w).$$

The roots of $P^*(w)$ and $Q^*(w)$ are all real, and each root of $P^*(w)$ is a root of $Q^*(w)$. $P^*(w)$ is of class $[\alpha_4/2]$ and $Q^*(w)$ is of class $[(\alpha_{4-1})/2]$, as defined in [4], p. 479. M. Krein and B. Ja. Levin [3] have shown the following:

THEOREM. *Let f be a function of class $[\Delta]$. Let $n(t)=$ the number of roots of f in the rectangle $|\mathrm{Im}\,z|\leqslant h$, $\mathrm{Re}\,z\leqslant t$. Then $n(t)=(\Delta/\pi)\,t+w(t)$ where $w(t)$ is bounded as $t\to\infty$. (Here h is a positive number such that all the roots of f lie in $\{z:|\mathrm{Im}\,z|<h\}$. Such a number exists for functions of class $[\Delta]$ ([3])).*

Letting $n_{P^*}(t)$=number of zeros of P^* in the interval $(0, t)$ and defining $n_{Q^*}(t)$ similarly we get:

$$\lim_{t\to\infty}\frac{n_{P^*}(t)}{t}=\frac{\alpha_4}{2\pi}$$

and

$$\lim_{t\to\infty}\frac{n_{Q^*}(t)}{t}=\frac{\alpha_4-1}{2\pi}<\frac{\alpha_4}{2\pi}.$$

But we have already shown that $n_{Q^*}(t)\geqslant n_{P^*}(t)$. This contradiction forces $Q(w)$ to be identically 0, so that $\alpha_4-\alpha_3-1=0$.

4. In this section we discuss the cases $n>4$. The structure of the proof of Theorem 4 is as follows: Starting with the functional equation, we obtain an associated function P (of the variable ξ, for convenience). If the functional equation fails to have a continuous solution, $P(\xi)$ has all its roots on the unit circle. But then $\xi^{\alpha_4}P(1/\xi)$ has the same roots as $P(\xi)$, and thus $Q(\xi)=P(\xi)-\xi^{\alpha_4}P(1/\xi)$, which is of lesser class than $P(\xi)$, has zeros where $P(\xi)$ does. But this is impossible unless $Q(\xi)\equiv 0$. This last fact implies the condition $\alpha_4=\alpha_3+1$; i.e. $P(\xi)=1+\xi+\xi^{\alpha_3}+\xi^{1+\alpha_3}$, which *does* have all its roots on the unit circle. We were able to show previously that the functional equation that this P arises from,

$$f(x)+f(a_2x)+f(a_3x)+f(a_2a_3x)=0,$$

cannot have a continuous solution; i.e. this particular $P(\xi)$ is really an exceptional case.

Except for the last statement, everything in the proof for $n=4$ goes through for $n>4$. As examples we give two theorems (we continue to assume that $a_1=1$).

THEOREM 5 ($n=5$). *The functional equation*

$$\sum_{i=1}^{5} f(a_ix)=0$$

will have a non-trivial continuous solution except possibly when it has the form

$$f(x)+f(a_2x)+f\left(\frac{a_3^2}{a_2}\right)+f(a_3^2x)=0.$$

Here the associated function is

$$P_5(\xi)=1+\xi^{\alpha_3}+\xi^{2\alpha_3-1}+\xi^{2\alpha_3}.$$

THEOREM 6 ($n=6$). *The functional equation*

$$\sum_{i=1}^{6} f(a_i x)=0$$

will have a non-trivial continuous solution except possibly when it has the form

$$f(x)+f(a_2x)+f(a_3x)+f\left(\frac{a_4}{a_3}x\right)+f\left(\frac{a_4}{a_2}x\right)+f(a_4x)=0.$$

Here the associated function is

$$P_6(\xi)=1+\xi+\xi^{\alpha_3}+\xi^{\alpha_4-\alpha_3}+\xi^{\alpha_4-1}+\xi^{\alpha_4}.$$

Now in the case $n=4$ the exceptional P did arise from a functional equation with no non-trivial continuous solutions, but this is not necessarily true for $n>4$, even if we assume that the exponents are integers. For brevity, put

$$P_5(\xi)=1+\xi+\xi^{\alpha}+\xi^{2\alpha-1}+\xi^{2\alpha},$$
$$P_6(\xi)=1+\xi+\xi^{\beta}+\xi^{\gamma-\beta}+\xi^{\gamma-1}+\xi^{\gamma},$$

where α, β, γ are integers such that $\alpha>1$, $\beta>1$, $\gamma>2\beta$. Then the following may be shown to be true:

(i) If α is odd, $P_5(\xi)$ has a root strictly between -1 and 0.
(ii) If α is even, $P_5(\xi)$ has all its roots on the unit circle if and only if $\alpha=2$, 4.
(iii) If β is odd and γ is even, $P_6(\xi)$ has a root strictly between -1 and 0.
(iv) If $\beta=2$, or $\gamma=2\beta+1$, then all the roots of $P_6(\xi)$ are on the unit circle.
(v) If $\beta=3$, $\gamma=9$ then $P_6(\xi)$ has a root not on the unit circle.

Thus the pattern with respect to the exceptional P is quite complicated.

The authors would like to thank D. J. Newman and M. Knopp for many helpful comments.

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Aequationes Math. *13* (1975), 61–76. University of Waterloo Birkhäuser Verlag, Basel

Nonlinear eigenvalue problems with monotonically compact operators

Theodore Laetsch

1. Introduction

In this paper we discuss the positive solutions of the equation

$$u=\lambda Au, \tag{1.1}$$

where $\lambda>0$ and A is an isotonic, forced (that is, $A0>0$) operator on a subset D of the positive cone K of a partially ordered real normed linear space E [14]. We are principally interested in the existence and dependence on λ of minimum positive fixed points of λA; that is, solutions $u^0(\lambda)$ of (1.1) such that, for any other solution $u=\lambda Au\in D$, we have $u^0(\lambda)\leqslant u$.

Our discussion of the nonlinear eigenvalue problem (1.1) differs from most others (e.g., [4; 7; 9; 14; 22; 23; 25] and the references given in these works) in that we do not assume that A is compact, and we explicitly assume that A is forced rather than that A is unforced ($A0=0$) as is often done. Of course, once one knows that (1.1) has a minimum positive solution $u^0(\lambda)$ for a given λ, then the search for other positive fixed points of λA can be accomplished by seeking the positive fixed points of the unforced operator A^0_λ defined by $A^0_\lambda h=\lambda A[u^0(\lambda)+h]-u^0(\lambda)$; however, this introduces a nonlinear dependence on λ. The forced case of (1.1) for nonlinear elliptic boundary value problems has been studied in [12; 13; 27], where other references are given; a nonlinear integral equation which can be reduced to (1.1) with a forced operator A has been studied by Pimbley [20]; see Example 5-3.4.

In Section 2 we introduce the basic definitions and notation. In particular, we introduce the concept of monotonic compactness, or m-compactness, and the related concept of om-compactness. A continuous, compact, isotonic operator is m-compact, but the class of m-compact, isotonic operators is much broader than the class of continuous, compact, isotonic operators. For example, in a partially ordered reflexive Banach space with a normal cone, every continuous operator is both m-compact and om-compact.

Section 3 describes the minimum positive fixed points for the more general equa-

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AMS Subject Classifications: Primary 47H15. Secondary 46A40, 35J65.

Received May 25, 1973 and, in revised form, January 29, 1974.

tion $u=A_\lambda u$, where the operators A_λ are *om*-compact, forced, and isotonic on their domain D, and increase as the parameter λ increases. Most of the results are analogous to the results of Keller and Cohen [13] for the minimum positive eigenfunctions of nonlinear elliptic boundary value problems.

In a future paper [17], we will consider operators which satisfy an algebraic convexity property instead of the analytic property of monotonic compactness, and we show that many of the results of this paper can be extended to such operators.

In Section 4 of this paper, we discuss briefly some generalizations of the theory of Section 3. Examples which illustrate applications of the results are given in Section 5; further examples are presented in [17].

Similar problems have recently been treated by Amann [3].

This paper was completed in part while the author held a National Research Council Postdoctoral Resident Research Associateship sponsored by the Air Force Systems Command at the Aerospace Research Laboratories of Wright-Patterson Air Force Base, and supported in part by NSF Grant GP-33652.

2. Definitions and notation

Let E be a partially ordered real normed space ordered by a positive cone K[14; 19; 26]; K is assumed to have the following properties: $K+K\subseteq K$, $\alpha K\subseteq K$ for all scalars $\alpha\geqslant 0$, $K\cap(-K)=\{0\}$, and K is closed. We use the usual notation of linear algebra for the linear sum S_1+S_2, difference S_1-S_2, and scalar multiple αS_1 of *subsets* S_1 and S_2 of E; the set theoretic difference is denoted by $S_1\backslash S_2=\{x\in S_1 : x\notin S_2\}$. If x and y belong to E and $y-x\in K$, then we write $x\leqslant y$ or $y\geqslant x$. The cone K is called *normal* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in K such that $x_n\leqslant y_n$ for all n and $\lim_{n\to\infty} y_n=0$, we have $\lim_{n\to\infty} x_n=0$ (for equivalent conditions, see [14, §1.2; 19, §2.1; 26, §V.3]).

A set S is *o-convex* (this is called *full* in [19] and *K-saturated in* [26]) if, whenever $u\in S$, $v\in S$, and $u\leqslant x\leqslant v$, then $x\in S$. For any set S, the *o-convex hull* $[S]$ of S is the set

$$\begin{aligned}[S]&=(S+K)\cap(S-K)\\ &=\{z\in E : \exists u,\ v\in S \ni u\leqslant z\leqslant v\};\end{aligned}$$

S is *o*-convex if and only if $S=[S]$. For two points u, v in E, we set $[u, v]=[\{u, v\}]$. Thus, if $u\leqslant v$, then

$$[u, v]=\{z\in E : u\leqslant z\leqslant v\}.$$

Let $S_1\subseteq S_2\subseteq E$. We say that S_1 is *o-bounded* (*in* S_2) if there exist u, v (u, $v\in S_2$) such that $S_1\subseteq[u, v]$. The set S_1 is *bounded in* S_2 if it is contained in a closed bounded subset of S_2. If K is normal, then every *o*-bounded set in K is bounded [19, Prop. 2.1.4]; if in addition S_2 is *o*-convex and S_1 is *o*-bounded in S_2, then S_1 is bounded in S_2.

Let E_1 and E_2 be partially ordered normed spaces with positive cones K_1 and K_2, respectively. We consider an operator $A: D \to E_2$ on a subset $D \subseteq E_1$. If $A(D \cap K_1) \subseteq K_2$, then A is *positive* on D.

The operator A is *isotonic from above* (or *below*) *relative to D at* $u \in D$ if $v \in D$ and $v \geqslant u$ (or $v \leqslant u$) imply $Av \geqslant Au$ (or $Av \leqslant Au$). It is *isotonic* (or *monotonic* or *increasing*) *on D* if it is isotonic from above relative to D at each point of D. *Strict isotonicity* is defined in an obvious way by replacing $Av \geqslant Au$ by $Av > Au$.

We say that A is *uniformly* (*o*-)*bounded on D* if $A(D)$ is (*o*-)bounded; (*o*-)*bounded in D* if A is uniformly (*o*-)bounded on every subset of D which is (*o*-)bounded in D; *compact* (or *completely continuous*) *on D* if A is continuous on D and the image of every closed bounded subset of D is relatively compact; (*o*)*m-compact on D* if, for every monotonic (i.e., increasing or decreasing) sequence $\{u_n\}$ in D which is (*o*-)bounded in D, the sequence $\{Au_n\}$ is convergent, and if the monotonic sequence $\{u_n\}$ converges to $v \in D$, then $\{Au_n\}$ converges to Av.

If $0 \in D$, then A is *forced* if $A0 > 0$ and *unforced* if $A0 = 0$.

Since any relatively compact monotonic sequence is convergent [14, p. 40], if A is isotonic and compact on D, then it is *m*-compact on D. If K_2 is normal and A is isotonic on D, then A is (*o*)*m*-compact on D if and only if, for every monotonic sequence $\{u_n\}$ which is (*o*-)bounded in D, the sequence $\{Au_n\}$ is weakly convergent, and if $\{u_n\}$ is strongly convergent to $v \in D$, then $\{Au_n\}$ converges weakly to Av; this is true since weakly convergent monotonic sequences in a partially ordered space with a normal cone are convergent [26, Theorem V.4.3].

Let $D \subseteq K_1$. If K_1 is normal, if D is *o*-convex, and A is *m*-compact on D, then A is *om*-compact on D. If every monotonic (*o*-)bounded sequence in K_1 is convergent, then A is (*o*)*m*-compact on D if and only if for every convergent monotonic sequence $\{u_n\}$ which has a limit in D, $A(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} Au_n$. The same conclusion holds if A is isotonic and either every *o*-bounded monotonic sequence in K_2 converges or every bounded monotonic sequence in K_2 converges and A is bounded in D. These conditions on the convergence of (*o*-)bounded monotonic sequences hold in partially ordered spaces E which are reflexive and have a normal positive cone [26, p. 224]. Thus, for example, any continuous isotonic operator on the cone of nonnegative functions on the Lebesgue spaces $L^p(\Omega)$ $(1 < p < +\infty)$ is *m*-compact and *om*-compact. Since on these spaces every positive linear operator is continuous [14, p. 64], we see that *every positive linear operator on* $L^p(\Omega)$ $(1 < p < +\infty)$ *is m-compact and om-compact.*

Example 5-4 exhibits an operator which is *om*-compact but not *m*-compact.

If T is a linear operator on $K_1 - K_1$ which is positive (that is, $T(K_1) \subseteq K_2$), we define (cf. [6], [26, p. 266], and [24, § 10]):

$$\|T\| = \sup\{\|Tu\|/\|u\| : u \in \bar{K}_1^1\}, \quad K_1^1 = \{u \in K_1 : \|u\| < 1\},$$

and, if $\|T\| < +\infty$, we say that T is *bounded in* K_1 or K_1*-bounded* and define the spectral radius (relative to K_1)

$$\operatorname{spr}[T] = \lim_{n\to\infty} \|T^n\|^{1/n}$$

and the reciprocal spectral radius $\mu_0[T] = \{\operatorname{spr}[T]\}^{-1}$.

The norm $\|T\|$ defined above coincides with the usual definition of the norm of T considered as a mapping of $K_1 - K_1$ into E_2 if the norm on $K_1 - K_1$ is taken to be the gauge of the set $K_1^1 - K_1^1$. If $E_1 = K_1 - K_1$, and K_1 is complete, then E_1 is complete (that is, a Banach space) [24, (10.1)], and the original topology on E_1 coincides with the topology induced by the gauge of $K_1^1 - K_1^1$ [19, p. 194], so that in this case $\operatorname{spr}[T]$ as defined above coincides with the usual definition of spectral radius.

If $E = E_1 = E_2 = \overline{K - K}$, and $T: E \to E$ is compact, then again $\operatorname{spr}[T]$ as defined above coincides with the usual spectral radius of T [6, pp. 64–65; 24, p. 281].

Now let $E = E_1 = E_2$, $K = K_1 = K_2$, and let K be complete. For $0 < \lambda < \mu_0[T]$, the resolvent of a K-bounded operator T,

$$R(\lambda) = [I - \lambda T]^{-1} = \sum_{n=0}^{\infty} \lambda^n T^n,$$

exists on $K - K$ (for more details, see [24, p. 278]), is a positive linear operator, and satisfies the resolvent equations

$$R(\lambda_2) - R(\lambda_1) = (\lambda_2 - \lambda_1)\, TR(\lambda_2)\, R(\lambda_1) \geqslant 0 \tag{2.3}$$

and

$$\lambda_2 R(\lambda_2) - \lambda_1 R(\lambda_1) = (\lambda_2 - \lambda_1)\, R(\lambda_2)\, R(\lambda_1) \geqslant 0 \tag{2.4}$$

for $0 < \lambda_1 \leqslant \lambda_2 < \mu_0[T]$.

3. Fixed points of *om*-compact operators

The following is a fundamental result on the existence of a fixed point of an isotonic, *om*-compact operator (cf. [16, Lemma 4-3]). The proof is immediate from the definition of an *om*-compact operator. By the remarks of Section 2, this lemma includes parts (b), (c), and (d) of Theorem 4.1 of [14]. Amann [2] has proved a corresponding result for semilinear second order elliptic partial differential equations; see Example 5-1.1 below.

LEMMA 3-1. *Let $A: D \to E$ be isotonic and om-compact on a set $D \subseteq E$. Let D satisfy the following condition: If $\{u_n\}$ is a convergent monotonic sequence in D which is o-bounded in D, and if $Au_n \in D$ for all n, then* $\lim_{n\to\infty} Au_n$ *(which exists since A is om-compact on D) belongs to D. Suppose there exist u_0 and v_0 in D such that $u_0 \leqslant Au_0 \leqslant Av_0 \leqslant v_0$ and*

$$A([u_0, v_0] \cap D) \subseteq D.$$

Then A has a fixed point u in D and each of the sequences $\{A^n u_0\}$ and $\{A^n v_0\}$ converges monotonically to fixed points u^0 and v^0 of A in D such that for all $n \geqslant 1$,

$$u_0 \leqslant A^n u_0 \leqslant u^0 \leqslant u \leqslant v^0 \leqslant A^n v_0 \leqslant v_0 .$$

Remark 3-1.1. This lemma and most of the results which follow remain valid if A is assumed to be the sum of an isotonic, *om*-compact operator and an isotonic, condensing operator [3]. This is a genuine generalization, for any contraction operator is condensing but not necessarily *om*-compact, while on a reflexive space with a normal positive cone any continuous isotonic operator is *om*-compact but not necessarily condensing. However, we are mainly interested in applications to families of operators of the form $\{c+\lambda A : \lambda \geqslant 0\}$ (see Theorem 3-4 below), and for $\lambda > 1$, λA may not be condensing, although A is. Thus we continue to consider only order-monotonically compact operators.

We now consider a family $\{A_\lambda : \lambda \in J\}$ of forced, isotonic, *om*-compact operators having a common domain $D \subseteq K$, where K is a complete positive cone in a partially ordered normed space E; J denotes a subinterval of $[0, +\infty)$. We assume that A_λ and D satisfy the following conditions: $0 \in D$; for all $\lambda \in J$,

$$A_\lambda([0, u] \cap D) \subseteq D \tag{3.1}$$

whenever both u and $A_\lambda u$ belong to D; and if $\{u_n\} \subseteq D$ and $\{\lambda_n\} \subseteq J$ are both convergent increasing sequences or both convergent decreasing sequences such that $\{u_n\}$ is o-bounded in D, $v_n = A_{\lambda_n} u_n \in D$ for all n, and $v = \lim_{n\to\infty} v_n$ exists, then $v \in D$. The assumption (3.1) is satisfied if $A_\lambda 0 \in D$ and either D or $A(D)$ is o-convex; however, in Section 5, we shall see several examples for which (3.1) is satisfied although neither D nor $A_\lambda(D)$ is o-convex. By the minimum positive fixed point of A_λ we mean a fixed point $u^0(\lambda) \in D$ of A_λ such that, for any other fixed point $u = A_\lambda u \in D$ of A_λ, $u \geqslant u^0(\lambda)$. We denote by Λ the set of $\lambda \in J$ such that A_λ has a positive fixed point in D; in Theorems 3-2 to 3-5, we assume that $\Lambda \neq \emptyset$.

We omit the proofs of the following theorems; the proofs are not difficult, they are usually immediate consequences of Lemma 3-1, and they are given in detail in [16, Chapter I.4]. (See [13] for similar results for nonlinear elliptic partial differential equations.)

THEOREM 3-1. *For each $\lambda \in \Lambda$, the sequence $\{A_\lambda^n 0 : n = 1, 2, \ldots\}$ converges to the minimum positive fixed point $u^0(\lambda)$ of A_λ. If there exists a family $\{B_\lambda : \lambda \in J\}$ of operators on D such that $B_\lambda u \geqslant A_\lambda u$ for all $\lambda \in J$ and $u \in D$, then*

$$\Lambda \supseteq \{\lambda \in J : B_\lambda \text{ has a fixed point in } D\}.$$

We say that a family $\{u_\lambda : \lambda \in J\}$ of elements of K is a (strictly) increasing family if the mapping $\lambda \to u_\lambda$ is (strictly) isotonic on J; the family of operators $\{A_\lambda : \lambda \in J\}$ is a (strictly) increasing family if, for each $u \in D$, $u > 0$, the family $\{A_\lambda u : \lambda \in J\}$ is a (strictly) increasing family.

THEOREM 3-2. *Let $\{A_\lambda : \lambda \in J\}$ be a (strictly) increasing family. Then the set Λ is an interval, with $\inf \Lambda = \inf J$, and the minimum positive fixed points $\{u^0(\lambda) : \lambda \in \Lambda\}$ form a (strictly) increasing family. If $\lambda_0 \in J$ and $\lambda_0 < \lambda \in \Lambda$, then the sequence $\{A_\lambda^n u^0(\lambda_0) : n = 1, 2, \ldots\}$ is an increasing sequence which converges to $u^0(\lambda)$. If the mapping $\lambda \to A_\lambda u$ is continuous from the left at $\lambda_0 \in \Lambda \cap (\inf J, \sup J]$ uniformly with respect to $u \in [0, u^0(\lambda_0)] \cap D$, then the mapping $\lambda \to u^0(\lambda)$ is continuous from the left at λ_0.*

The mapping $\lambda \to u^0(\lambda)$ may not be continuous from the right at all points of Λ, as is shown by the infinitely differentiable operator $A_\lambda : [0, +\infty)$ given by $A_\lambda u = \lambda \exp[-5/(1+u)]$; this example also shows that for $\lambda_0 < \lambda \in \Lambda$, $\{A_{\lambda_0}^n u^0(\lambda)\}$ does not necessarily converge to $u^0(\lambda_0)$. See also [18].

THEOREM 3-3. *Let $\{A_\lambda : \lambda \in J\}$ be as in Theorem 3-2, and in addition, for all $\lambda_0 \in J$ and all $w \in D$, let the mapping $\lambda \to A_\lambda u$ be continuous from the left at λ_0 uniformly with respect to $u \in [0, w] \cap D$. If Λ contains the nonempty interval $(\inf(\Lambda), \lambda_0)$ and either the family $\{u^0(\lambda) : \lambda \in (\inf(\Lambda), \lambda_0)\}$ is o-bounded in D or this family is bounded in D and A_{λ_0} is m-compact, then $\lambda_0 \in \Lambda$ and $\lim_{\lambda \to \lambda_0 -} u^0(\lambda)$ exists and equals $u^0(\lambda_0)$. Either $\sup(\Lambda) = \sup(J)$, or $\sup(\Lambda) \in \Lambda$, or the family $\{u^0(\lambda) : \lambda \in \Lambda\}$ is not o-bounded in D. If A_λ is m-compact on D for all $\lambda \in J$, and D is contained in and closed relative to K^ρ for some $\rho \in (0, +\infty]$, then either $\lambda^* = \sup(\Lambda) \in \Lambda$, or $\lambda^* = \sup(J)$, or $\lim_{\lambda \to \lambda^* -} \|u^0(\lambda)\| = \rho$.*

We comment briefly on the proof of the last assertion of Theorem 3-3: Suppose $\lambda^* < \sup(J)$ and $\lim_{\lambda \to \lambda^* -} \|u^0(\lambda)\|$ either does not exist or is less than ρ. Then there exists an increasing sequence $\{\lambda_n\}$ converging to λ^* such that, for some number $\sigma > 0$, $\sup\{\|u^0(\lambda_n)\|\} = \sigma < \rho$. Since D is closed relative to K^ρ, $D \cap \overline{K^\sigma}$ is a closed, bounded subset of D, and, since A_{λ^*} is m-compact, $\{A_{\lambda^*} u^0(\lambda_n)\}$ converges to, say, w. Because of the uniform continuity of the mapping $\lambda \to A_\lambda u$ with respect to u, $\{u^0(\lambda_n)\}$ converges to w, and thus $A_{\lambda^*} w = w$ and $\lambda^* \in \Lambda$. (The uniform continuity of the map $\lambda \to A_\lambda u$ on o-bounded subsets of D implies the om-compactness of the map $(\lambda, u) \to \to A_\lambda u$ on $J \times D$ with the order defined by $(\lambda, u) \leqslant (\mu, v)$ if $\lambda \leqslant \mu$ and $u \leqslant v$.)

If A_λ has the form $A_\lambda u = c + \lambda A u$, where $c \in K$, A is isotonic and om-compact on D, and either $c > 0$, $\lambda \geqslant 0$, and A is positive on D, or $c = 0$, $\lambda > 0$, and A is forced, then the uniform continuity assumptions on the mappings $\lambda \to A_\lambda u$ of Theorems 3-2 and 3-3 are satisfied. If $\lambda_0 \in \Lambda$ and $0 < \lambda < \lambda_0$, then

$$0 \leqslant u^0(\lambda) - c = \lambda A u^0(\lambda) \leqslant \lambda A u^0(\lambda_0),$$

and thus we have the following description of the fixed points of A_λ as $\lambda \to 0+$:

THEOREM 3-4. *Let* $A_\lambda = c + \lambda A$, $\lambda \geqslant 0$, *as just described. Then* Λ *is an interval (possibly unbounded) with* $\inf(\Lambda)=0$, *and* $\inf\{u^0(\lambda):0<\lambda\in\Lambda\}=c$ *(that is,* $\{u^0(\lambda):0<\lambda\in\Lambda\}$ *order-converges to c as* $\lambda\to 0+$ [5, *page* 244]). *If every o-bounded set in K is bounded, then* $\lim_{\lambda\to 0+} u^0(\lambda)=c$.

THEOREM 3-5. *Let* $\{u_n\}$ *be a sequence of fixed points of* $A_\lambda = c+\lambda A$ *in D corresponding to values* $\lambda=\lambda_n>0$ *converging to* 0. *Then either* $\{u_n\}$ *order-converges to c or* $\{u_n\}$ *is not o-bounded in D. Let D be contained in and closed relative to* K^ρ *for some* $\rho\in(0, +\infty]$, *let A be bounded in D, and suppose that there exist positive numbers* ε *and* δ *such that, for every* $\lambda\in(0,\delta)$, $A_\lambda=c+\lambda A$ *has at most one fixed point in* $(c+K^\varepsilon)\cap D$; *let K be normal. If none of the* u_n *is the minimum positive fixed point of* $c+\lambda_n A$, *then* $\lim_{n\to\infty}\|u_n\|=\rho$.

Remark 3-5.1. The *om*-compactness of A is used in Theorems 3-4 and 3-5 only to assure that for all sufficiently small $\lambda\in\Lambda$, the operators $c+\lambda A$ have the minimum positive fixed points referred to in the theorems, and that the mapping $\lambda\to u^0(\lambda)$ is isotonic (Theorems 3-1 and 3-2).

We conclude this section with the following existence and uniqueness theorem (cf. [11; 30, §4.6; 3]):

THEOREM 3-6. *Let* $A_\lambda=c+\lambda A$ *as above, and suppose that there exists a K-bounded positive linear operator T on* $K-K$ *such that for all* $u, v\in D$ *with* $v\geqslant u$, *we have*

$$Av-Au\leqslant T(v-u). \tag{3.2}$$

Let $R(\lambda)=[I-\lambda T]^{-1}$ *be the resolvent of T for* $0<\lambda<\mu_0[T]$. *If* λ *is such that* $0<\lambda<\mu_0[T]$, $R(\lambda)[c+\lambda A0]\in D$, *and* $c+\lambda AR(\lambda)[c+\lambda A0]\in D$, *then* $A_\lambda=c+\lambda A$ *has a unique positive fixed point in D. In fact,* $u^0(\lambda)$ *is the only fixed point of* A_λ *in D for all* $\lambda\in(0,\mu_0[T])\cap\Lambda$.

Remark 3-6.1. The conditions $R(\lambda)[c+\lambda A0]\in D$ and $c+\lambda AR(\lambda)[c+\lambda A0]\in D$ are satisfied if $D=K$.

Remark 3-6.2. The uniqueness assertion of the last sentence of Theorem 3-6 remains valid without the assumption that A is *om*-compact if A_λ has a minimum positive fixed point $u^0(\lambda)$ for the values of λ in question.

The following example of an operator $A:[0,+\infty)\to(0,+\infty)$ shows that, without further restrictions on A, we cannot improve the upper limit $\mu_0[T]$ of Theorem 3-6 for the values of λ for which A_λ has a unique positive fixed point. Choose a positive number r, and define $Au=e^u$ for $0\leqslant u\leqslant r$ and $Au=e^r(u+1-r)$ for $u\geqslant r$. Let $Tu=e^r u$. Then (3.2) holds for all $v\geqslant u\geqslant 0$, and $\mu_0[T]=e^{-r}$. If $r=1$, $A_\lambda=\lambda A$ has infinitely many positive fixed points corresponding to $\lambda=\mu_0[T]=e^{-1}$, and if $r>1$, λA has two positive fixed points for all $\lambda\in(\mu_0[T], e^{-1})$. See also [11].

4. Generalizations

We now indicate some ways in which the preceding results can be generalized; these generalizations were omitted from Section 3 in the interest of simplicity.

Most of the results of Section 3 remain valid if we assume merely that E is a partially ordered linear space (not necessarily provided with a norm or a topology) and we modify the definition of *om*-compactness of an (isotonic) operator A on D to read: If $\{u_n\}$ is a monotonic sequence which is *o*-bounded in D, then $\sup_n \{Au_n\}$ and $\inf_n \{Au_n\}$ exist, and if $\sup_n \{u_n\}$ and $\inf_n \{u_n\}$ exist in D, then $A(\sup_n \{u_n\}) = \sup_n \{Au_n\}$ and $A(\inf_n \{u_n\}) = \inf_n \{Au_n\}$. In this case, the assertions concerning the existence of limits and continuity made in Section 3 should be interpreted in terms of order convergence [5, page 244] or the order topology on E [19, Section 3.1].

A closely related fact is that many of the results of Sections 3 can be carried over to ordered topological vector spaces [19; 26, Chapter V] in place of normed spaces. (For the appropriate definition of the spectral radius of a linear operator, see [24, pages 277–278].)

5. Examples

The following examples present illustrations, possible applications, and counterexamples for the theory developed above. In all examples, the space E is a function space (including the special case $E = R^n$). We use the following standard notations: $C^m(\bar{\Omega})$ denotes the set of uniformly continuous functions on a bounded open connected set $\Omega \subseteq R^n$ which have uniformly continuous derivatives of all orders less than or equal to m, with the norm

$$\|u\|_m = \sum_{|\beta| \leqslant m} \sup \{|D^\beta u(x)| : x \in \Omega\} ;$$

for $0 < \alpha < 1$, $C^{m,\alpha}(\bar{\Omega})$ is the set of functions in $C^m(\bar{\Omega})$ which have uniformly Hölder continuous (exponent α) derivatives of all orders less than or equal to m, with the norm

$$\|u\|_{m,\alpha} = \|u\|_m + \max_{|\beta| \leqslant m} \sup \left\{ \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} : x, y \in \Omega \right\}.$$

We will usually omit the subscripts on the symbols $\|u\|_m$ and $\|u\|_{m,\alpha}$. $L^p(\Omega)$ is the Lebesgue space of p-th power integrable functions ($1 \leqslant p < +\infty$) on a measurable set Ω, with the usual norm.

The letter K will always denote the cone of never negative functions in the space E under consideration (or, in the case of $L^p(\Omega)$, the functions which are almost everywhere nonnegative). If Ω is an interval of the real line, $K_\uparrow$ is the subset of K consisting of never decreasing functions.

EXAMPLES 5-1. An immediate application of our results is to well-known problems which can be formulated in terms of compact operators. We describe two such examples:

Example 5-1.1. Nonlinear second order elliptic boundary value problems can be expressed in terms of compact operators (see, for example, Amann's papers [1; 2]). It follows that the results of Section 3 of this paper contain, with somewhat more precision and greater generality, all the corresponding results of Keller and Cohen [13]. For convenience, we describe the situation precisely here, referring to the papers of Amann cited above for proofs (see especially the proof of [2, Proposition 3.3]).

Let Ω be a bounded open connected set in n-dimensional space with boundary $\partial\Omega$, let L be a uniformly elliptic operator defined on Ω:

$$Lu(x) = -a_{ij}(x)\, D_i D_j u(x) + b_i(x)\, D_i u(x) + c(x)\, u(x),$$

where repeated subscripts are to be summed from 1 to n, and let B_ι ($\iota = 0$ or 1) be the boundary operators defined on $\partial\Omega$ by

$$\begin{aligned} B_0 u(x) &= u(x) \\ B_1 u(x) &= \beta_0(x)\, u(x) + \beta_i(x)\, D_i u(x). \end{aligned}$$

For a fixed choice of ι, we seek a solution $u \in C^2(\bar{\Omega})$ of the boundary value problem

$$\begin{aligned} Lu(x) &= f(u(x), x, \lambda), \qquad x \in \Omega, \\ B_\iota u(x) &= \phi_\iota(x, \lambda), \qquad x \in \partial\Omega, \end{aligned} \tag{5.1}$$

under the following assumptions:

For $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$, the coefficients a_{ij}, b_i, and c belong to $C^\alpha(\bar{\Omega})$, $c \geqslant 0$ on $\bar{\Omega}$, $\partial\Omega \in C^{2,\alpha}$, $\beta_i \in C^{1,\alpha}(\partial\Omega)$ for $0 \leqslant i \leqslant n$, c and β_0 are not both identically zero, $\beta_0 \geqslant 0$ on $\partial\Omega$, and $\beta_i(x)\, v_i(x) > 0$ for all $x \in \Omega$, where $\{v_1(x), \ldots, v_n(x)\}$ is the normal vector to $\partial\Omega$ at $x \in \partial\Omega$.

We assume given a positive number $r \leqslant +\infty$, a nonempty interval $J \subseteq [0, +\infty]$, a function $f \in C([0, r) \times \bar{\Omega} \times J)$, and functions $\phi_\iota \in C(\partial\Omega \times J)$, and define $f_\lambda(w, x) = f(w, x, \lambda)$ and $\phi_{\iota,\lambda}(x) = \phi_\iota(x, \lambda)$; for each $\lambda \in J$, $f_\lambda \in C^\alpha([0, r'] \times \bar{\Omega})$ for every $r' \in (0, r)$ and $\phi_{\iota,\lambda} \in C^{2-\iota,\alpha}(\partial\Omega)$. For every $x \in \bar{\Omega}$ and $y \in \partial\Omega$,

$$0 \leqslant \phi_\iota(y, \lambda) \leqslant \phi_\iota(y, \lambda')$$

and

$$0 \leqslant f(w, x, \lambda) \leqslant f(w', x, \lambda')$$

whenever $0 \leqslant w \leqslant w' < r$, $\lambda \in J$, $\lambda' \in J$, and $\lambda \leqslant \lambda'$; furthermore, for each $\lambda \in J$, either $f_\lambda(0, x)$ is not identically zero for $x \in \bar{\Omega}$, or $\phi_{\iota,\lambda}(x)$ is not identically zero for $x \in \partial\Omega$.

Under these conditions, the boundary value problem (5.1) can be formulated as an operator equation in $C(\bar{\Omega})$,

$$u = A_\lambda u, \tag{5.2}$$

with an increasing family $\{A_\lambda\}$ of compact, isotonic, forced operators on the set $D=C^\alpha(\bar{\Omega})\cap K^r\subseteq K\subseteq C(\bar{\Omega})$.

The operator A_λ is constructed as follows: Let Γ_λ denote the operator which assigns to each $v\in C^{\alpha^2}(\bar{\Omega})$ the unique solution $u=\Gamma_\lambda v$ of the linear boundary value problem

$$Lu=v \quad \text{in} \quad \bar{\Omega}, \qquad B_\iota u=\phi_{\iota,\lambda} \quad \text{on} \quad \partial\Omega.$$

Then Γ_λ is well-defined and compact as a mapping from $C^{\alpha^2}(\bar{\Omega})$ with the $C(\bar{\Omega})$ topology to $C^{2,\alpha^2}(\bar{\Omega})$ with the $C(\bar{\Omega})$ topology. For every $u\in K^r\subseteq C(\bar{\Omega})$, let $f_\lambda(u)$ be the function defined by $f_\lambda(u)(x)=f(u(x), x, \lambda)$; f_λ is a bounded, continuous operator (in the $C(\bar{\Omega})$ topology) from $D=C^\alpha(\bar{\Omega})\cap K$ to $C^{\alpha^2}(\bar{\Omega})$. Thus the composition $A_\lambda=\Gamma_\lambda f_\lambda: D\to K$ is on isotonic, forced, compact operator on D, and the family $\{A_\lambda:\lambda\in J\}$ is increasing by the maximum principle. The set D satisfies the assumptions laid down following Lemma 3.1.

Using the $C(\bar{\Omega})$ topology is convenient if we wish to use arguments in which the normality of the positive cone K is important. If this is not required, it may be more convenient to consider A_λ in a space of smoother functions; for example, Rabinowitz [22] uses the spaces $C^1(\bar{\Omega})$.

The assumption that $f_\lambda(w, x)$ is an increasing function of w can be weakened (cf. [2]): It suffices to assume that there exists a sufficiently smooth function $k\geqslant 0$ on $\bar{\Omega}$ such that for all $\lambda\in J$ and $x\in\bar{\Omega}$,

$$f_\lambda(w', x)-f_\lambda(w, x)\geqslant -k(x)(w'-w) \tag{5.3}$$

whenever $0\leqslant w\leqslant w'<r$. Then the function

$$g_\lambda(w, x)=f_\lambda(w, x)+k(x)w$$

is an increasing function of w with the same smoothness properties as f_λ, the operator $\tilde{L}u(x)=Lu(x)+k(x)u(x)$ has the same smoothness properties as L, and the problem (5.1) is equivalent to

$$\begin{aligned} \tilde{L}u(x)&=g_\lambda(u(x), x), & x&\in\bar{\Omega},\\ B_\iota u(x)&=\phi_{\iota,\lambda}(x), & x&\in\partial\Omega, \end{aligned}$$

to which the preceding arguments can be applied.

For the Neumann or mixed problems ($\iota=1$), one may also consider nonlinear boundary conditions

$$B_1u(x)=\phi_{1,\lambda}(u(x), x), \qquad x\in\partial\Omega,$$

where $\phi_{1,\lambda}\in C^{1,\alpha}([0, r']\times\partial\Omega)$ for all $r'\in(0, r)$. For isotonicity, we assume that the first derivative of $\phi_{1,\lambda}(w, x)$ with respect to w is uniformly bounded below (for all $\lambda\in J$) by a constant $-\kappa$, with $\kappa\geqslant 0$. Then $\psi_\lambda(w, x)=\phi_{1,\lambda}(w, x)+\kappa w$ is an increasing function of w, and the desired solutions u are solutions of the problem

$$\begin{aligned} \tilde{L}u(x) &= g_\lambda(u(x), x), \quad x \in \Omega, \\ \tilde{B}u(x) &= \psi_\lambda(u(x), x), \quad x \in \partial\Omega, \end{aligned} \tag{5.4}$$

where $\tilde{B}u = B_1 u + \kappa u$. If one assumes $a_{ij} \in C^{2,\alpha}(\bar{\Omega})$ and $b_i \in C^{1,\alpha}(\bar{\Omega})$, one may define compact operators A_λ on D such that (5.2) and (5.4) are equivalent (cf. [2, Proposition 3.3]).

It follows from the preceding remarks that Amann's Theorem 1 [2] is essentially a special case of Lemma 3-1 above (but his Theorem 3, which does not assume the one-sided Lipschitz condition (5.3), is, for this problem, more general).

Example 5-1.2. Integral equations with weakly singular kernels,

$$u(x) = \int_\Omega K(x, y) f_\lambda(u(y), y)\, dy,$$

where Ω is a bounded open subset of R^n and

$$0 \leqslant K(x, y) \leqslant \frac{\kappa}{|x-y|^\alpha}$$

for some constants $\kappa > 0$ and $\alpha \in (0, n)$, and all $x \in \bar{\Omega}$, $y \in \bar{\Omega}$, have an obvious formulation as operator equations in $C(\bar{\Omega})$. If $f \in C([0, r) \times \bar{\Omega})$, the operator is compact on $K^r \subseteq \subseteq C(\bar{\Omega})$ (e.g., see [21, §X.3]).

EXAMPLE 5-2. (Cf. Chandra and Fleishman [8].) Let τ be a given positive number. Consider the integral operator

$$A_\tau u(x) = \int_0^\tau G(x, y) f(u(y), y)\, dy, \quad 0 \leqslant x \leqslant \tau,$$

where $G: [0, \tau] \times [0, \tau] \to [0, +\infty)$ is a measurable bounded kernel which is continuous and isotonic in its first variable, and $f: [0, r) \times [0, \tau] \to [0, +\infty)$ is piecewise continuous and isotonic in its first variable and integrable in its second variable (that is, $\int_0^\tau f(w, t)\, dt < +\infty$ for each $w \in [0, r)$). Suppose that $A0(x) = \int_0^\tau G(x, y) f(0, y)\, dy$ is strictly isotonic in $x \in [0, \tau]$. Then A_τ is a well-defined operator on $K_\uparrow^r \subseteq C[0, \tau]$; it is forced and isotonic relative to the order induced by the normal cone $K_\uparrow$ and maps $K_\uparrow^r$ into the set $D_0 \subseteq K_\uparrow \subseteq C[0, \tau]$ of strictly increasing functions in $K_\uparrow$. Moreover, it is easily verified, using the Lebesgue monotone convergence theorem, that A_τ is m-compact on $D = K_\uparrow^r \cap (\{0\} \cup D_0)$. Thus the theory of Section 3 may be applied to the family $\{\lambda A_\tau : \lambda > 0\}$. If f and G are defined for all $\tau \in (0, \tau_0]$, then we may also consider the family $\{A_\tau : \tau \in (0, \tau_0]\}$ as τ varies.

In particular, if there exist numbers $\lambda>0$, $\rho\in(0, r)$, and $\tau\in(0, \tau_0]$ such that

$$\varrho \geqslant \lambda \int_0^\tau G(1, y) f(\varrho, y)\, dy,$$

then it follows from Lemma 3-1 that the integral equation $u=\lambda A_\tau u$ has a solution $u\in[\lambda A_\tau 0, \lambda A_\tau \rho]\cap D$; the iterations of Lemma 3-1 yield a solution of this equation [8, Theorem 1].

Similarly, Lemma 3-1 may be applied to Examples 1 and 2 of [8], which give functions $u_0\in D$ and $v_0=A_\tau\rho\in D$ such that $u_0\leqslant A_\tau u_0\leqslant A_\tau v_0\leqslant v_0$, and hence proves the existence of more than one nonnegative solution of certain differential equations.

EXAMPLE 5-3. The theory of Section 3, which requires only monotonic compactness rather than compactness, covers more general situations than considered in Example 5-1. For example, whenever differential or integral equations can be formulated in terms of continuous, bounded, isotonic operators in $L_p(\Omega)$, the operators are m-compact and Section 3 is applicable. We consider some examples of integral equations in $C(\bar{\Omega})$.

Examples 5-3.1. Compactness is often lost when the region of integration in an integral equation becomes unbounded. Let us consider, for example, the case of one real variable. Let $N(x, y, w)$ be measurable and never negative on $R\times R\times[0, r)$, for some positive number $r\leqslant+\infty$, and increasing and continuous in w for all $(x, y)\in R\times R$. Suppose that for every $x\in R$ and any function $u\in L^\infty(R)$, with $0\leqslant u\leqslant |u|_\infty<r$, $N(x, y, u(y))$ is an integrable function of y and $\int_R N(x, y, u(y))\, dy$ is a continuous function of x. Then the operator A defined in the space $C_b(R)=C(R)\cap L^\infty(R)$ of bounded continuous functions on R by

$$Au(x)=\int_R N(x, y, u(y))\, dy$$

is easily seen to be m-compact on $K^r\subseteq C_b(R)$ by the Lebesgue monotone convergence theorem and Dini's theorem. The m-compactness condition will be satisfied, for example, if $N(x, y, w)=G(x, y) f(w, y)$, where $0\leqslant f\in C([0, r)\times R)$, $f\in C_b([0, r']\times R)$ for every $r'\in(0, r)$, $f(w, y)$ is increasing in w for every $y\in R$, $0\leqslant G\in C(R\times R)$, and $\int_{-\infty}^{\infty} G(x, y)\, dy$ is uniformly convergent.

Example 5-3.2. An example with a 'discontinuous' kernel is

$$u(x)=\lambda\int_x^\infty e^{x-t} f(u(t), t)\, dt. \tag{5.5}$$

This operator is in general not compact on $C_b(R)$, since the linear equation

$$\phi(x)=\lambda e^x \int_x^\infty e^{-t}\phi(t)\,dt=\lambda\Gamma\phi(x)$$

has the solutions $\phi_\lambda(x)=e^{(1-\lambda)x}$, and $\phi_\lambda \in C_b(R)$ for $\lambda=1+i\eta$, where η is an arbitrary real number. Thus the eigenvalues of the linear operator Γ form a continuum, and Γ is therefore not compact.

If we take $f(w, y)$ to be the linear function (of w) $w+e^{-|y|}$, the equation can be solved explicitly. The integral Equation (5.5) can be reduced by the substitution $u(x)=v(x)-e^{|x|}$ to

$$v(x)=e^{-|x|}+\lambda e^x \int_x^\infty e^{-t}v(t)\,dt;$$

thus the bounded solutions of (5.5) are [28, §57, Ex. 1]

$$u(x;\lambda)=\begin{cases}\dfrac{\lambda}{2-\lambda}e^{-x}, & x\geqslant 0,\\ \dfrac{2}{2-\lambda}e^{(1-\lambda)x}-e^x, & x\leqslant 0,\end{cases}$$

for $0<\lambda<1$, and, for $\lambda=1$,

$$u(x;1)=\begin{cases}e^{-x}+c, & x\geqslant 0,\\ 2+c-e^x, & x\leqslant 0,\end{cases}$$

where c is an arbitrary nonnegative constant.

Note that there are an infinite number of solutions corresponding to $\lambda=1$ which have the form $u^0(x;1)+\phi(x)$, where u^0 is the minimum positive solution and ϕ is a positive eigenfunction of the homogeneous linear equation.

If in the latter example we limit ourselves to the space $C_{b0}(R)$ of functions in $C_b(R)$ satisfying $\lim_{|x|\to\infty}u(x)=0$, the solution for $\lambda=1$ is unique.

Example 5-3.3. As another example, our results can be applied to the problem of finding bounded positive solutions of the differential equation

$$u''(x)-a^2u(x)+\lambda f(u(x),x)=0,\quad -\infty<x<\infty$$

which can be formulated as the problem of solving the operator equation (for $a>0$)

$$u(x)=\lambda Au(x)=\frac{\lambda}{2a}\int_{-\infty}^{\infty} e^{-a|x-y|}f(u(y),y)\,dy.$$

See [28, §57, Ex. 2] for a discussion of a special case of this equation.

Example 5-3.4. The operator

$$Au(x) = \int_x^1 u(y)\, u(y-x)\, dy$$

has been considered by Pimbley [20]. As an operator on $K \subseteq C[0, 1]$, it is m-compact and om-compact, but not compact; it is positively convex [17] but not convex. To see that A is not compact on K, consider the bounded sequence $\{u_n\}$ given by $u_n(x) = \cos^2(n\pi x)$. We calculate

$$Au_n(x) = \tfrac{1}{4}(1-x) + \tfrac{1}{8}v_n(x) + w_n(x),$$

where $v_n(x) = (1-x)\cos(2\pi n x)$ and $\lim_{n\to\infty} w_n(x) = 0$ uniformly for $x \in [0, 1]$. Thus $\{Au_n\}$ contains a convergent subsequence if and only if $\{v_n\}$ contains a convergent subsequence. However, for $n > m$,

$$\|v_n - v_m\|^2 \geqslant \int_0^1 |v_n - v_m|^2 \geqslant \frac{\pi^2 - 9}{12\pi^2} > 0,$$

so $\{v_n\}$ does not contain a convergent subsequence, and therefore A is not compact on K. The same argument shows that A is not compact on $K \subseteq L^p(0, 1)$ for $1 \leqslant p \leqslant 2$. The m-compactness of A in $C[0, 1]$ follows from the Lebesgue monotone convergence theorem and the continuity in the mean of the Lebesgue integral [29, page 397] (cf. [10, p. 159]).

The results of Section 3 apply to the equation $u = 1 + \lambda A u$ considered by Pimbley.

EXAMPLE 5-4. We give an example of an operator (in a complete space with a normal cone) which is om-compact but not m-compact. The example shows that *the last conclusion of Theorem* 3-3 *is not necessarily true if* A_λ *is om-compact but not m-compact.*

For $N = 1$ or 2, and $u \in C[0, 1]$, define

$$A^{(N)}u(x) = \begin{cases} x\left[1 + \displaystyle\int_x^1 \frac{[u(t)]^N}{t^2}\, dt\right], & 0 < x \leqslant 1, \\ [u(0)]^N, & x = 0. \end{cases}$$

It is easily verified that

$$\lim_{x \to 0+} x\left[1 + \int_x^1 \frac{[u(t)]^N}{t^2}\, dt\right] = [u(0)]^N,$$

and hence $A^{(N)}$ is a forced, isotonic, convex, positive operator mapping $K \subseteq C[0, 1]$ into K. We consider $C[0, 1]$ with the ordering induced by the positive cone

$$K_0 = \{u \in K : u(0) = 0\}.$$

With this ordering, $A^{(N)}$ is *om*-compact on K_0:

If $\{u_k\}$ is a monotonic sequence in K_0 which is order-bounded above by $w \in K_0$, then $\{u_k\}$ converges monotonically to a function $u \leqslant w$ on $[0, 1]$ with $u(0) = 0$. For $0 < x \leqslant 1$, $Au_k(x)$ converges (by the Lebesgue monotone convergence theorem) to

$$v(x) \equiv x\left[1 + \int_x^1 \frac{[u(t)]^N}{t^2}\, dt\right],$$

which is a positive, continuous function of x for $x > 0$, bounded above by $Aw(x)$. Since $\lim_{x \to 0+} Aw(x) = [w(0)]^N = 0$, we have $\lim_{x \to 0+} v(x) = 0 = \lim_{k \to \infty} Au_k(x) \equiv v(0)$. Thus Au_k converges to $v \in K_0$, and if $u_k \to u \in K_0$, then $Au_k \to Au = v \in K_0$ uniformly on $[0, 1]$.

To see that $A^{(N)}$ is not *m*-compact on K_0, we investigate the nonlinear eigenvalue problem for the operator family $A_\lambda^{(N)} = \lambda A^{(N)}$. The equation $u = \lambda A^{(N)} u$, which is equivalent to the differential equation

$$xu' - u + \lambda u^N = 0,$$
$$u(1) = \lambda,$$

has solutions for each $\lambda \in \Lambda = (0, 1)$:

$$u^0(\lambda; x) = \frac{\lambda x}{1 - \lambda^2 + \lambda^2 x}, \quad 0 \leqslant x \leqslant 1,$$

for $N = 2$, and

$$u^0(\lambda; x) = \lambda x^{\lambda - 1}, \quad 0 \leqslant x \leqslant 1,$$

for $N = 1$. For $\lambda = 1$, the only solution of $u = \lambda A^{(N)} u$ is the constant 1, which is not in K_0. Thus: Λ is the open interval $(0, 1)$; the family $\{u^0(\lambda) : \lambda \in \Lambda\}$ is bounded (but not *o*-bounded) in K_0; and, with $\lambda^* = \sup(\Lambda) = 1$, $\lim_{\lambda \to \lambda^*-} u^0(\lambda)$ does not exist in $C[0, 1]$. It follows from the last sentence of Theorem 3-3 that the operator A is not *m*-compact on K_0; this can be seen directly by considering the bounded sequence $\{u_k(x)\} = \{u^0(1 - k^{-1}; x)\}$, for which $\{Au_k\}$ does not converge in $C[0, 1]$.

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Aequationes Math. *13* (1975), 77–87. University of Waterloo Birkhäuser Verlag, Basel

Uniqueness theorems for the representation $\phi(f(x)g(y)+h(y))$

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1. Introduction

The functional equation

$$\phi(f(x)g(y)+h(y))=\psi(l(x)m(y)+n(y)) \tag{1.1}$$

occurs in various contexts. For instance, in the extension and uniqueness problem for solutions of the generalized distributivity equation (cf. [6], Section 7). It also occurs in the theory of measurement as a uniqueness problem for certain scales. In this context, it is already solved by Aczél, Djoković, Pfanzagl [3], but under more restrictive conditions than we will assume here. We will give the solutions of (1.1) under the conditions (i) to (vi) below:

(i) $\phi, f, g, h, \psi, l, m$, and n are real valued and continuous,
(ii) (x, y) ranges over a rectangle $I\times J$, where I and J are real, proper intervals,
(iii) ϕ and ψ are philandering,
(iv) f and l are non-constant,
(v) (g, h) and (m, n) are non-constant,
(vi) g and m have no zeroes in J.

'*Philandering*' is here used in the sense 'nowhere constant'; i.e. not constant on any proper interval (cf. [7], Definition 4.2 or MR 45: 761).

We will use the abbreviation *c.s.m.* to denote that a function is continuous and strictly monotonic.

The sentence 'F is a function of G' will mean: 'There exists a function H such that $F=H\circ G$'.

Our condition (vi) is not as strong as it seems to be, since a zero of g must also be a zero of m and vice versa. In fact, if $g(y_0)=0$, then the left member of (1.1) is independent of x, hence also the right member. This implies $m(y_0)=0$, since ψ is philandering and l non-constant. So (vi) merely means that we restrict our study to any of the common non-zero intervals of g and m.

It seems natural to divide the treatment of (1.1) into six cases, depending on the relationship between g and h and between m and n.

AMS Subject Classifications: Primary 39A15, 39A20, 39A30. Secondary 26A72.
This work was supported in part by National Research Council of Canada grants A2972 and A8212.

Received August 8, 1973 and, in revised form, March 11, 1974.

DEFINITION 1.1. An n-tuple of real valued functions $(f_1, \ldots, f_n)$ is said to be *affinely dependent* if $(f_1, \ldots, f_n, 1)$ is linearly dependent. Otherwise, $(f_1, \ldots, f_n)$ is said to be *affinely independent*.

The six cases are described as follows:

Case	*Condition*
1	(g, h) as well as (m, n) is affinely independent,
2	g is constant; (m, n) is affinely independent,
3	h is a linear function of g; (m, n) is affinely independent,
4	g and m are both constant,
5	g is constant; n is a linear function of m,
6	h is a linear function of g and n a linear function of m.

Remark: Notations and symbols used in different theorems are independent.

2. A basic equation

In cases 1 to 3, we will reduce (1.1) to the equation

$$\Phi(u+v) = F(u)\, G(v) + H(v). \tag{2.1}$$

This equation can be reduced to the Cauchy equation, which is shown in [1], pp. 148–150. More details are discussed in [2], pp. 20–22. We only need the solution for continuous functions. The treatments in [1] and [2] give rise to the following

LEMMA 2.1. *If* (2.1) *is satisfied by real valued, continuous functions* Φ, F, G, *and* H *when* u *and* v *range over real, proper intervals, then the solution* (Φ, F, G, H) *must be representable in one of the following four forms, where* α, a, b, c, *and* d *are appropriate constants:*

Solution	$\Phi(t)$	$F(u)$	$G(v)$	$H(v)$
I	$abt+c$	$au+d$	b	$abv-bd+c$
II	$abe^{\alpha t}+c$	$ae^{\alpha u}+d$	$be^{\alpha v}$	$-bde^{\alpha v}+c$
III	a	arbitrary	0	a
IV	a	b	arbitrary	$a-bG(v)$

3. Treatment of case 1

In this section we will prove the following

THEOREM 3.1. *Let $(\phi, f, g, h, \psi, l, m, n)$ satisfy Equation (1.1) under assumptions (i) to (vi). Assume also that (g, h) as well as (m, n) is affinely independent. Then, there are constants a, b, c, and d such that*

$$\left.\begin{aligned} \psi(at+b)&=\phi(t)\\ l&=cf+d\\ m&=\frac{a}{c}g\\ n&=ah-\frac{ad}{c}g+b.\end{aligned}\right\} \tag{3.1}$$

The proof is based on reduction to monotonic functions. We will use two lemmas from [5]. The first one is

LEMMA 3.2. *Let $\{g_{1n}\}_1^\infty$ and $\{g_{2n}\}_1^\infty$ be two sequences of continuous functions on a real interval I, and suppose $g_{1n}\to g$, $g_{2n}\to g$, uniformly as $n\to\infty$. Assume further that $g_{1n}(s)<g_{2n}(s)$ and $f\circ g_{1n}=f\circ g_{2n}$, where f is continuous. Then, f is constant on $g(I)$.*

The second one is Lemma A2 of the Appendix.

Proof of Theorem 3.1. We first show that f and l must be c.s.m. functions of each other. They must have the same constancy intervals, since ϕ and ψ are philandering and g and m non-zero. Hence, it is sufficient to treat the case when Huneke's mountain climbing theorem (cf. [4]) can be applied in accordance with Lemma 4.4 of [8]. Suppose for instance, that $f(x_1)=f(x_2)$ and $l(x_1)<l(x_2)$. Applying the lemma mentioned, we choose two continuous functions ξ_1 and ξ_2 such that $\xi_1(0)=\xi_2(0)$, $f\circ\xi_1=f\circ\xi_2$, and $l\circ\xi_1(t)<l\circ\xi_2(t)$ when $0<t<1$. Suppose for instance that $m(y)>0$, which is no essential restriction, since m does not change sign. Then, we have

$$\left.\begin{aligned} l(\xi_1(t))\,m(y)+n(y)&<l(\xi_2(t))\,m(y)+n(y)\\ l(\xi_1(0))\,m(y)+n(y)&=l(\xi_2(0))\,m(y)+n(y)\\ \psi(l(\xi_1(t))\,m(y)+n(y))&=\psi(l(\xi_2(t))\,m(y)+n(y))\end{aligned}\right\}\quad 0<t<1,\ y\in J.$$

Applying Lemma 3.2 to arbitrary compact subintervals of J, we see that ψ is constant in $(l\circ\xi_1(0)\cdot m+n)(J)$, which is a proper interval, since (m, n) is affinely independent. This contradicts the philandering of ψ. Thus, $f(x_1)=f(x_2)$ must imply $l(x_1)=l(x_2)$. The converse is proved similarly, and hence f and l are c.s.m. functions of each other. Now, put

$$f(x)=t,\qquad l(x)=k(t), \tag{3.2}$$

where k is c.s.m. From

$$\phi(tg(y)+h(y))=\psi(k(t)\,m(y)+n(y)), \tag{3.3}$$

we can prove the existence of a c.s.m. function χ such that

$$\chi(tg(y)+h(y))=k(t)\,m(y)+n(y). \tag{3.4}$$

This proof is given in the appendix. Since χ is differentiable a.e. and since, for every t, $tg(y)+h(y)$ ranges over a proper interval ((g, h) being affinely independent), k must be differentiable. Hence, χ is also differentiable. Differentiating (3.4) with respect to t gives

$$\chi'(tg(y)+h(y))\cdot g(y)=k'(t)\cdot m(y). \tag{3.5}$$

Since g and m are continuous and nowhere zero, and since $y\mapsto tg(y)+h(y)$ is non-constant for every t, χ' and k' must be continuous. We will show that they are both either constant or c.s.m.

If any of χ' and k' has a constancy interval, so has also the other. Let K be a maximal constancy interval of k'. Then χ' is constant in the interval

$$\{tg(y)+h(y)\mid t\in K,\ y\in J\},$$

again because $y\mapsto tg(y)+h(y)$ is non-constant. By this fact, we easily see that K could be extended if K were a proper subset of $f(I)$. Hence $K=f(I)$. Thus, χ' and k' are either constant or philandering.

The case when χ' and k' are constant, i.e. χ and k linear functions, gives immediately (3.1) if we use $\chi(t)=at+b$ and $k(t)=ct+d$ together with (3.2), (3.3), and (3.4).

Now, suppose that χ' and k' are philandering. We will show that they are even c.s.m. It is sufficient to show that k' is strictly monotonic. If $k'(t_1)=k'(t_2)$, we get

$$\chi'(t_1g(y)+h(y))=\chi'(t_2g(y)+h(y)).$$

The mountain climbing technique again, together with Lemma 3.2, implies that χ' has a constancy interval if $t_1\neq t_2$. Hence k' must be strictly monotonic and so also χ'. We may suppose that k' and χ' are positive.

Writing (3.5) in logarithmic form, we obtain

$$\ln k'(t)+\ln\frac{m(y)}{g(y)}=\ln\chi'(tg(y)+h(y)). \tag{3.6}$$

From this, we easily see that $m(y)/g(y)$ is non-constant and that $g(y)$ and $h(y)$ are both functions of $m(y)/g(y)$. Thus from (3.6) we obtain (2.1) if we put

$$\ln k'(t)=u, \qquad \ln\frac{m(y)}{g(y)}=v,$$

$\Phi=$ the inverse of $\ln\chi'$, $F=$ the inverse of $\ln k'$, $G(v)=g(y)$, and $H(v)=h(y)$. Consider the solutions of (2.1), given in Lemma 2.1. Solutions I and III imply that g is constant, which contradicts our assumptions. IV implies that Φ is constant, which contradicts the fact that Φ is the inverse of a c.s.m. function. Finally, solution II implies an affine dependency between g and h (and between m and n), which also violates the assumption of Theorem 3.1. Hence, χ' and k' cannot be strictly monotonic, and so the solutions (3.1) of (1.1) are the only possible. This completes the proof.

4. Treatment of cases 2, 3

In this section we shall prove the following non-existence theorems.

THEOREM 4.1. *There is no solution* $(\phi, f, g, h, \psi, l, m, n)$ *of equation* (1.1) *under the assumptions* (i) *to* (vi) *such that g is constant and* (m, n) *is affinely independent.*

Proof. We let

$$g=\text{constant}=\beta \tag{4.1}$$

and follow the argument used in the proof of Theorem 3.1 up to Equation (3.4). Hence we may let

$$f(x)=t, \quad l=k\circ f, \tag{4.2}$$

where k is c.s.m. and obtain from (1.1)

$$\chi(\beta t+h(y))=k(t)\,m(y)+n(y), \tag{4.3}$$

where χ is c.s.m. The non-constancy of k in (4.3) implies that m, n are functions of h, and hence we may let

$$\left.\begin{aligned}\beta t=u,\quad & k(u/\beta)=K(u)\\ h(y)=v,\quad & m=M\circ h,\quad n=G\circ h\end{aligned}\right\} \tag{4.4}$$

and obtain from (4.3)

$$\chi(u+v)=K(u)\,M(v)+G(v) \tag{4.5}$$

which is our basic equation studied in Section 2. We get from Lemma 2.1 that (M, N) can only be affinely dependent, which in turn shows that (m, n) is affinely dependent.

THEOREM 4.2. *There is no solution* $(\phi, f, g, h, \psi, l, m, n)$ *of Equation* (1.1) *under the assumptions* (i) *to* (vi) *such that h is a linear function of g,* (m, n) *is affinely independent.*

Proof. We again follow the argument used in the proof of Theorem 3.1 up to Equation (3.4) and let

$$f(x)=t, \quad l=k\circ f \tag{4.6}$$

to get

$$\chi(tg(y)+h(y))=k(t)\,m(y)+n(y), \tag{4.7}$$

where k and χ are c.s.m. Since h is a linear function of g we may let

$$\left.\begin{aligned} h=\lambda g+\mu, \quad & s=t+\lambda \\ \chi(z+\mu)=\theta(z), \quad & k(s-\lambda)=K(s), \end{aligned}\right\} \tag{4.8}$$

where λ, μ are constants and rewrite (4.7) as

$$\theta(sg(y))=K(s)\, m(y)+n(y). \tag{4.9}$$

It follows from (4.9) that m, n are functions of g and we may put

$$m=M\circ g, \quad n=G\circ g, \quad g(y)=u \tag{4.10}$$

to write (4.9) as

$$\theta(su)=K(s)\, M(u)+G(u). \tag{4.11}$$

The variable s cannot be zero, otherwise (M, N) will be affinely dependent and so is (m, n) which contradicts our hypothesis. The variable u cannot be zero as g never vanishes. We may confine ourselves to the case where both s and u are always positive; other cases can be treated similarly. We let

$$\ln s=p, \quad \ln u=q \tag{4.12}$$

and rewrite (4.11) as

$$\theta(\mathrm{Exp}(p+q))=K(\mathrm{Exp}(p))\, M(\mathrm{Exp}(q))+N(\mathrm{Exp}(q)). \tag{4.13}$$

It follows from Lemma 2.1 that $(M\circ\mathrm{Exp}, N\circ\mathrm{Exp})$ is affinely dependent, and so is (m, n). This is again a contradiction to our hypothesis.

5. Treatment of cases 4, 5, 6

THEOREM 5.1. *Let* $(\phi, f, g, h, \psi, l, m, n)$ *satisfy Equation* (1.1) *under the assumptions* (i) *to* (vi). *Assume that g and m are both constant. Then there exist constants a, b, c, d, e such that*

$$\left.\begin{aligned} g=a, \quad & \phi(t)=\psi(ct+d+e) \\ l=(ca/b)f+(d/b), \quad & m=b, \quad n=ch+e \end{aligned}\right\} \tag{5.1}$$

Proof. This special case is the subject of [8].

THEOREM 5.2. *Let* $(\phi, f, g, h, \psi, l, m, n)$ *satisfy Equation* (1.1) *under the assumptions* (i) *to* (vi). *Assume also that g is constant and n is a linear function of m. Then there exist constants a, b, c,* α, β, γ *such that*

$$\left.\begin{aligned} g=\alpha, \quad & \phi(t)=\psi(ab\,\mathrm{Exp}(ct)+\gamma) \\ l=a\,\mathrm{Exp}\circ(c\alpha f)-\beta & \\ m=b\,\mathrm{Exp}\circ(ch), \quad & n=b\beta\,\mathrm{Exp}\circ(ch)+\gamma. \end{aligned}\right\} \tag{5.2}$$

Proof. We let

$$\left.\begin{array}{lll} g=\alpha, & \alpha f=F, & n=\beta m+\gamma \\ \psi(z+\gamma)=\Phi(z), & & l+\beta=L \end{array}\right\} \tag{5.3}$$

and rewrite (1.1) as

$$\phi(F(x)+h(y))=\Phi(L(x)\,m(y)). \tag{5.4}$$

The functions appearing in (5.4) are continuous and non-constant with ϕ and Φ philandering. The function L never vanishes, otherwise Φ will not be philandering. We consider the case where L and m are both positive valued; other cases can be treated in a similar way. Hence we can rewrite (5.4) as

$$\phi(F(x)+h(y))=\Phi\circ\mathrm{Exp}(\ln L(x)+\ln\circ m(y))$$

and apply Theorem 5.1 to get

$$\left.\begin{array}{l} \phi(t)=\Phi\circ\mathrm{Exp}(ct+A+B) \\ \ln\circ L=cF+A \\ \ln\circ m=ch+B, \end{array}\right\} \tag{5.5}$$

where A, B, c are constants. This proves that in the case $L>0$, $m>0$ the solution is described by (5.2) with $a>0$, $b>0$.

THEOREM 5.3. *Let $(\phi, f, g, h, \psi, l, m, n)$ satisfy Equation* (1.1) *under the assumptions* (i) *to* (vi). *Assume that h is a linear function of g, n is a linear function of m and write*

$$\left.\begin{array}{l} h=\alpha g+\beta \\ n=\gamma m+\delta, \end{array}\right\} \tag{5.6}$$

where α, β, γ, δ are constants. Then

(1) *$f+\alpha$ and $l+\gamma$ are continuous maps of I into R having common zeroes. We denote by Z the zeroes of them, and let $\{I_i\}$ be the component intervals of $I\backslash Z$, which is open in I.*

(2) *There exist constants $c\neq 0$, $a_i>0$, $b>0$ such that*

$$\left.\begin{array}{l} \phi(\mathrm{sgn}(f|_{I_i}+\alpha)\cdot\mathrm{sgn}(g)\cdot a_i b t^c+\beta)=\psi(\mathrm{sgn}(l|_{I_i}+\gamma)\cdot\mathrm{sgn}(m)\cdot t+\delta) \\ |f|_{I_i}+\alpha|=a_i|l|_{I_i}+\gamma|^c, \qquad |g|=b|m|^c. \end{array}\right\} \tag{5.7}$$

(3) *The constants c, a_i satisfy the following constraints.*

(3a) *If $Z\neq\emptyset$, then $c>0$ and $\phi(\beta)=\psi(\delta)$.*

(3b) *We classify $\{i\}$ into four families*

$$\begin{array}{l} C_{++}=\{i\,|\,f|_{I_i}+\alpha>0,\ l|_{I_i}+\gamma>0\} \\ C_{+-}=\{i\,|\,f|_{I_i}+\alpha>0,\ l|_{I_i}+\gamma<0\} \\ C_{-+}=\{i\,|\,f|_{I_i}+\alpha<0,\ l|_{I_i}+\gamma>0\} \\ C_{--}=\{i\,|\,f|_{I_i}+\alpha<0,\ l|_{I_i}+\gamma<0\}. \end{array}$$

The constants a_i are the same when the i's belong to the same class C. Let us write

$$\begin{aligned} A_{++} &= a_i \quad \textit{for} \quad i \in C_{++} \\ A_{+-} &= a_i \quad \textit{for} \quad i \in C_{+-} \\ A_{-+} &= a_i \quad \textit{for} \quad i \in C_{-+} \\ A_{--} &= a_i \quad \textit{for} \quad i \in C_{--} \end{aligned}$$

Furthermore, if none of the four classes of the C's is empty so that the four A's all exist, then they satisfy

$$A_{++} \cdot A_{--} = A_{+-} \cdot A_{-+}.$$

Proof. We set

$$\left.\begin{aligned} &f+\alpha=F, \quad l+\gamma=L \\ &\phi(z+\beta)=\Phi(z), \quad \psi(\omega+\delta)=\Psi(\omega) \end{aligned}\right\} \tag{5.8}$$

and rewrite (1.1) as

$$\Phi(F(x)\,g(y))=\Psi(L(x)\,m(y)). \tag{5.9}$$

The functions appearing in (5.9) are continuous and non-constant with Φ, Ψ philandering and g, m non-vanishing on J. Hence F and L must have the same zeroes on I, and this proves (1). We may now confine our discussion on $I_i \times J$ for (5.9), where F and L have no zero; and we may assume for convenience that $F|_{I_i}$, $L|_{I_i}$, g, m are all positive valued. We can rewrite (5.9) as

$$\Phi \circ \operatorname{Exp}(\ln \circ F|_{I_i}(x) + \ln \circ g(y)) = \Psi \circ \operatorname{Exp}(\ln \circ L|_{I_i}(x) + \ln \circ m(y))$$

and apply Theorem 5.1 to get

$$\left.\begin{aligned} &\Phi \circ \operatorname{Exp}(cs + A_i + B) = \Psi \circ \operatorname{Exp}(s) \\ &\ln \circ F|_{I_i} = c \ln \circ L|_{I_i} + A_i \\ &\ln \circ g = c \ln \circ m + B \end{aligned}\right\}$$

which will in turn give (5.7) with $a_i = \operatorname{Exp} A_i$, $b = \operatorname{Exp} B$. This proves (2).

If $Z \neq \emptyset$ it follows from the second equation of (5.7) that $c > 0$; and from the first equation of (5.7) that $\phi(\beta) = \psi(\delta)$. The consistency of the first equation of (5.7) for different i's and the philandering of ϕ give the asserted dependence of the constants a_i's as in (3). This completes the proof of our theorem.

EXAMPLE 5.4. An example to Theorem 5.3 is given as follows:

$$\begin{aligned} &I = J = \text{the reals } R \\ &\phi(t) = |t| && t \in R \\ &f(x) = \sin^2 x && x \geqslant 0 \\ &\qquad\;\; -\sin^2 x && x < 0 \\ &g(y) = \exp(2y) && y \in R \\ &h(y) = 0 && y \in R \end{aligned}$$

$$\begin{aligned} \psi(t)&=t^2 && t\in R\\ l(x)&=\sin x && x\in R\\ m(y)&=\exp(y) && y\in R\\ n(y)&=0 && y\in R \end{aligned}$$

Then $(\phi, f, g, h, \psi, l, m, n)$ satisfy Equation (1.1) and the assumptions (i) to (vi). Equation (5.6) is satisfied with $\alpha=\beta=\gamma=\delta=0$. The set Z appearing in Theorem 5.3.1 consists of integral multiples of π; and $I_i=(i\pi, (i+1)\pi)$ for each integer i. The remaining parts (2) and (3) are fulfilled with $c=2$, $a_i=1$ all i, $b=1$, $C_{++}=\{0, 2, 4, \ldots\}$, $C_{+-}=\{1, 3, 5, \ldots\}$, $C_{-+}=\{-2, -4, -6, \ldots\}$, $C_{--}=\{-1, -3, -5, \ldots\}$, $A_{++}=A_{+-}=A_{-+}=A_{--}=1$.

Appendix

Under the assumptions of Theorem 3.1, we will now show that G and H are functions of each other, where

$$G(t, y)=tg(y)+h(y)$$
$$H(t, y)=k(t)\,m(y)+n(y)$$

from which the existence of the c.s.m. function χ, satisfying (3.4), follows. Because of symmetry, it is sufficient to show that H is a function of G. We first show

LEMMA A1. *In the interior of $f(I)\times J$, H is locally a c.s.m. function of G.*

Proof. Since ψ is philandering, H is constant on every connected component of a level curve of G. These level curves are continuous functions: $y\mapsto t$ (i.e. $(y, t)\in L$, $(y, t')\in L \Rightarrow t=t'$, where L is a level curve). Therefore, every point in $\operatorname{Int}(f(I)\times J)$ has a neighbourhood where all level curves are connected. In such a neighbourhood, H is obviously a function of G, which proves the lemma.

We also need the following

LEMMA A2. *Let N be a one-sided neighbourhood (punctured or not) of a real number α, let two real valued functions g_1 and g_2 be continuous in N, and suppose $g_1(s)<g_2(s)$ when $s\in N$, $s\neq\alpha$. Assume that*

$$a=\limsup g_1(s)=\limsup g_2(s) \tag{A1}$$

as s tends to α in N, and that $g_1(N)$ and $g_2(N)$ are left neighbourhoods of a. If a function f is left continuous at a, and

$$f(g_1(s))=f(g_2(s)); \quad s\in N \tag{A2}$$

then

$$f(u)=f(a) \quad \text{for every} \quad u\in g_1(N)\cup g_2(N). \tag{A3}$$

If $g_1(N)$ *and* $g_2(N)$ *are right neighbourhoods of a, and f is right continuous at a, then* (A3) *holds if* lim sup *is replaced by* lim inf *in* (A1).

Proof. Given in [5] (Lemma 4.3).

Now, to prove our assertion, suppose P, $Q \in \operatorname{Int}(f(I)\times J)$ and $G(P)=G(Q)$. We will show that $H(P)=H(Q)$. For points on the boundary $\partial(f(I)\times J)$, this equality then follows from the subsequent Lemma A4.

Join P and Q by a continuous curve Γ, running entirely in the interior of $f(I)\times J$. Since G and H are locally functions of each other $G\circ\Gamma$ and $H\circ\Gamma$ must have the same constancy intervals. Therefore, we may assume that they are philandering, for otherwise we can identify all points of every constancy interval, according to Lemma 3.1 of [8]. This identification does not destroy the property of G and H being locally continuous functions of each other. Suppose, in order to obtain a contradiction, that $H(P)\neq H(Q)$. Using the mountain climbing technique again, we may suppose that there are two continuous functions ξ_1 and ξ_2 such that

$$G\circ\Gamma\circ\xi_1=G\circ\Gamma\circ\xi_2 \tag{A4}$$

$$H\circ\Gamma\circ\xi_1(0)=H\circ\Gamma\circ\xi_2(0) \tag{A5}$$

and

$$H\circ\Gamma\circ\xi_1(t)<H\circ\Gamma\circ\xi_2(t);\; 0<t<\delta.$$

Since H, as a local function of G, is c.s.m., we may suppose, for instance, that

$$H\circ\Gamma\circ\xi_1(0)\leqslant H\circ\Gamma\circ\xi_1(t)<H\circ\Gamma\circ\xi_2(t);\quad 0<t<\delta. \tag{A6}$$

From (3.3), i.e. $\phi\circ G=\psi\circ H$, and (A4), we obtain

$$\psi\circ H\circ\Gamma\circ\xi_1=\psi\circ H\circ\Gamma\circ\xi_2. \tag{A7}$$

Now, (A5), (A6), and (A7), together with Lemma A2, imply the existence of a constancy interval of ψ. This contradicts the assumption that ψ is philandering. Thus $H(P)=H(Q)$, which was to be proved.

DEFINITION A3. A locally connected subset S of R^n is said to be a *body* if $\operatorname{Int} S$ is nonempty, connected and $S\subset\overline{\operatorname{Int} S}$.

LEMMA A4. *Let S be a body in* R^n *and let* $G, H: S\to R$ *be continuous. If* $H\mid_{\operatorname{Int} S}$ *is a function of* $G\mid_{\operatorname{Int} S}$, *then H is a function of G.*

Proof. Suppose

$$G(\mathbf{x})=G(\mathbf{y}) \quad \text{implies} \quad H(\mathbf{x})=H(\mathbf{y}), \tag{A8}$$

holds for every $\mathbf{x}$, $\mathbf{y}\in\operatorname{Int} S$. We will show that (A8) is true if $\mathbf{x}$ or $\mathbf{y}$ is a boundary point. Let $\mathbf{a}\in\partial S$ and $\mathbf{b}\in\operatorname{Int} S$ be arbitrary such that $G(\mathbf{a})=G(\mathbf{b})$. Suppose first that every neighbourhood of $\mathbf{a}$ contains a point $\mathbf{x}$ such that $G(\mathbf{x})>G(\mathbf{a})$ and every neighbourhood of $\mathbf{b}$ contains a point y such that $G(\mathbf{y})>G(\mathbf{b})$. Then there exist sequences $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ in Int$S$ such that $\mathbf{x}_n\to\mathbf{a}$, $\mathbf{y}_n\to\mathbf{b}$ as $n\to\infty$ and that $G(\mathbf{x}_n)=G(\mathbf{y}_n)$ for all n. Hence by (A8) we get $H(\mathbf{x}_n)=H(\mathbf{y}_n)$ for all n, and by continuity of H that $H(\mathbf{a})= =H(\mathbf{b})$.

We obtain the same conclusion if in the above paragraph $>$ is replaced by $<$. We also get the same conclusion if $G(\mathbf{x})=G(\mathbf{a})$ for values of $\mathbf{x}\in\text{Int}\,S$ arbitrarily close to $\mathbf{a}$.

Now, suppose $G(\mathbf{x})>G(\mathbf{a})$ in some punctured neighbourhood of $\mathbf{a}$ and $G(\mathbf{y}) \leqslant G(\mathbf{b})$ in some neighbourhood of $\mathbf{b}$. Then, since Int$S$ is connected, there is a point $\mathbf{b}'\in\text{Int}\,S$ such that $G(\mathbf{b}')=G(\mathbf{b})=G(\mathbf{a})$ and $G(\mathbf{y})>G(\mathbf{b}')$ for values of $\mathbf{y}\in\text{Int}\,S$ arbitrarily close to $\mathbf{b}'$. It follows from the above argument that $H(\mathbf{b}')=H(\mathbf{a})$, and from $G(\mathbf{b}')=G(\mathbf{b})$ in Int$S$ that $H(\mathbf{b}')=H(\mathbf{b})$. Thus $H(\mathbf{a})=H(\mathbf{b})$.

The case $G(\mathbf{x})<G(\mathbf{a})$, $G(\mathbf{y})\geqslant G(\mathbf{b})$ is similar.

Since S is locally connected, the proof is similar, mutatis mutandis, if $\mathbf{a}$, $\mathbf{b}$ are both in ∂S.

The authors are indebted to a referee who eliminated an error in this lemma.

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Aequationes Math. *13* (1975), 89–96. University of Waterloo Birkhäuser Verlag, Basel

Extremal tests for scalar functions of several real variables at degenerate critical points

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1. Introduction

It is, of course, well known to students of calculus that the extremal nature of a function f at a critical point is decided if the so-called discriminant (or Hessian) given by the expression $\Delta = f_{xy}^2 - f_{xx}f_{yy}$ evaluated at the critical point is nonzero. It is also known through simple examples that, in the so-called degenerate case when $\Delta = 0$ at the point, f may have either an extremum or a saddle point. Consequently, the extremal nature of f in this case is indeterminate from a knowledge of the second derivatives at the point in question alone and higher order partial derivatives at the point must be considered. (It is interesting that during the last century a certain confusion existed concerning the degenerate case, even apparently in the minds of some renowned mathematicians. For a short account of this history of the degenerate case see [1].) Systematic, yet straightforward and simple methods by which to take into account the higher order derivatives seem, however, difficult to come by. In fact, the only method known to the author which offers an essentially complete account of this case is due to Freedman [2]. (His techniques are concerned with the solution of the equation $f(x, y) = 0$ for $x = x(y)$ but implicitly yield information about extrema as well. He also considers cases other than the degenerate case. Also in a recent paper [3] Butler and Freedman consider the case when the lowest order terms of f are cubic or higher; as stated below, we do not consider this case here.) The purpose of this note is to present a complete method for determining the extremal nature of f on the basis of its derivatives at the point in question under the two assumptions that (i) f possesses the necessary number of partial derivatives and (ii) the lowest order terms in its Taylor expansion with remainder at the point are quadratic. Under these conditions we will show how the extremal nature of f may be decided in the degenerate case through a sequence of tests each involving a discriminant and each having a degenerate case, whose occurence, however, can be followed by the next test of the sequence. Each test has the same format as the standard discriminant test using Δ, which itself may be considered as simply the first test of the sequence. Although they accomplish more or less the same ends, the details

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AMS Subject Classifications: Primary 26A57. Secondary 26A90, 26A69.

Received August 15, 1973 and, in revised form, December 11, 1973.

of our method are for the most part significantly different from those presented by Freedman in [2]. (A few particulars do overlap, however.) In addition, our method seems to be conceptually more concise in that it involves one simple algorithmic principle while Freedman's method consists of a rather long list of technical cases, some of which may loop back upon themselves. (No discussion is given in [2] concerning this looping back nor the possibility of this indefinitely happening; such a possibility is briefly discussed, but not characterized, in [3] for the case, not considered here, that the lowest order terms in f are cubic or higher.)

In Theorem 1 we describe our method as a sequence of discriminant tests. Theorem 2 contains specific information about the existence and behavior of the implicitly defined functions $f(x, y)=0$ in the case of a saddle point, as derived from our tests. Theorem 3 characterizes those analytic functions for which the sequence of tests terminates after a finite number of steps, or equivalently those for which the sequence is indefinitely inconclusive. Finally, Theorem 4 is a stronger version of Theorems 1 and 2 for the case when f is analytic.

2. Main results

Let f be a real valued function of real variables x, y which is defined and possesses at least three continuous partial derivatives in some neighborhood N of a critical point which we assume, without loss of generality, to be the origin. There is also no loss in generality in assuming $f(0, 0)=0$. The function f is said to have a *proper relative minimum* (*maximum*) at the origin if $f(x, y)>0$ (<0) in some deleted neighborhood of the origin and, in either case, is said to have a *proper relative extremum* there. If the values of $f(x, y)$ change sign in every neighborhood of the origin, then we say that f has a *saddle point* at the origin. Finally, if $f(x, y)\geqslant 0$ $(\leqslant 0)$ in some neighborhood and $f(x, y)=0$ somewhere in every neighborhood of the origin, then f has an *improper relative minimum* (*maximum*) at the origin. Under assumption (ii) above, the vanishing of both f_{xx} and f_{yy} at the origin would imply $\Delta=f_{xy}^2>0$ and the fact that f has a saddle point. Since we are only interested in the degenerate case, we may assume without loss of generality that $f_{xx}\neq 0$ at the origin. Let $i!j!a_{ij}=\partial^{i+j}f/\partial x^i y^j$ at $x=y=0$. Then, for our purposes, without any loss of generality, we may assume in everything done below that the function f has the form $f(x, y)=x^2+a_{11}xy+a_{02}y^2+o(r^2)$ in N where $r=(x^2+y^2)^{1/2}$.

To motivate briefly our method we consider Peano's well-known example (see [1]) $f(x, y)=(x-py^2)(x-qy^2)$, where p, q are constants, which is an illustration of the possibility of f having, at a degenerate critical point, either an extremum $(p=q)$ or a saddle point $(p\neq q)$. Although the nature of f at the origin is for this example quite easy to determine by inspection, one way of looking at Peano's example is to view f as a quadratic form, not in x and y, but in x and y^2. This quadratic form has discriminant $(p-q)^2$ and, hence, is indefinite if and only if $p \neq q$ and semi-definite if and only if $p=q$. For a general function f with a degenerate critical point at the origin, our method takes a hint from this example and, after changing variables so as to complete the square on its second order terms, investigates the x^2, xy^2, and y^4 terms (in the new variables) as a quadratic form. If this form is semi-definite the process in repeated, only this time to consider the x^2, xy^3, and y^6 terms; etc. To make this process precise we make the following

DEFINITION. A function f as described above is said to be *one-fold degenerate* at the origin if $\Delta_1 \equiv a_{11}^2 - 4a_{02} = 0$ and *n-fold degenerate* for $n \geqslant 2$ if the following three conditions are met:

(i) f possesses $2n$ continuous partial derivatives in N;

(ii) its Taylor expansion with remainder has the form $f(x, y) = x^2 + a_{1n}xy^n + a_{0\,2n}y^{2n} + m(x, y) + o(r^{2n})$ where $m(x, y)$ consists of all other terms of order three through $2n$ and has the form

$$m(x, y) = \sum_{\substack{i+j=3, \dots, n+1 \\ i \neq 0, 1}} a_{ij}x^i y^j + \sum_{\substack{i+j=n+2, \dots, 2n \\ i \neq 0}} a_{ij}x^i y^j ; \tag{1}$$

(iii) $\Delta_n \equiv a_{1n}^2 - 4a_{0\,2n} = 0$.

Since $\Delta_1 = \Delta$ we see that the classical degenerate case $\Delta = 0$ in the standard extremal test corresponds to f being one-fold degenerate.

It is not difficult to see that if f is n-fold degenerate for some $n \geqslant 1$ while possessing $2n+2$ continuous partial derivatives in N and if we make the change of variables

$$\bar{x} = x + (a_{1n}/2)\, y^n, \quad \bar{y} = y, \tag{2}$$

then the function f takes the form

$$f(\bar{x}, \bar{y}) = \bar{x}^2 + \bar{a}_{1\,n+1}\bar{x}\bar{y}^{n+1} + \bar{a}_{0\,2n+2}\bar{y}^{2n+2} + \bar{a}_{0\,2n+1}\bar{y}^{2n+1} + \bar{m}(\bar{x}, \bar{y}) + o(\bar{r}^{2n+2})$$
$$\bar{a}_{ij} = \sum_{k=i}^{[i+(j/n)]} (-a_{1n}/2)^{k-i} \binom{k}{i} a_{k\,j+n(i-k)} \tag{3}$$

(here $[p]$ is the largest integer less than p) where $\bar{m}$ has the form (1) with n replaced by $n+1$. This is done to complete the square on the term $x^2 + a_{1n}xy^n + a_{0\,2n}y^{2n}$ which is possible since $\Delta_n = 0$.

Although the form of the function f in the definition above looks rather formidable, it is nonetheless exactly the type which arises from an arbitrary function after n degenerate extremum tests, beginning with the familiar classical discriminant test, as described in the following theorem.

THEOREM 1. *Suppose that the function f is n-fold degenerate at the origin for some $n \geqslant 1$ and possesses $2n+2$ continuous partial derivatives in N. Suppose the change of variables* (2) *is made.*

(i) *If $\bar{a}_{0\,2n+1} \neq 0$, then f has a saddle point at the origin.*

(ii) *Suppose $\bar{a}_{0\,2n+1}=0$. Then*

(a) *f has a proper relative minimum if $\Delta_{n+1} \equiv \bar{a}_{1\,n+1}^2 - 4\bar{a}_{0\,2n+2} < 0$;*

(b) *f has a saddle point* if $\Delta_{n+1} > 0$;

(c) *the extremal nature of f is undecided if $\Delta_{n+1}=0$, but in this event f is $(n+1)$-fold degenerate.*

Notice that in the degenerate and inconclusive case (c), the fact that f is then $(n+1)$-fold degenerate allows one to reapply the theorem with n replaced by $n+1$ provided f has enough continuous partial derivatives (viz., $2n+4$). Thus, this theorem provides a systematic manner in which to continue investigating the nature of the critical point (on the basis of higher order partial derivatives at the origin) in the event of any number of degenerate cases. The possibility of indefinitely obtaining the degenerate case is discussed and characterized (for analytic functions) below.

Proof of Theorem 1. From (3), $f(0, \bar{y}) = \bar{a}_{0\,2n+1}\bar{y}^{2n+1} + o(\bar{y}^{2n+1})$ and hence, if $\bar{a}_{0\,2n+1} \neq 0$ then $f(0, \bar{y})$ changes sign with $\bar{y}$ near the origin. This proves (i). Let $\bar{x} = \bar{r}^{n+1} \cos\bar{\theta}$, $\bar{y} = \bar{r} \sin\bar{\theta}$ (where $\bar{r}$, $\bar{\theta}$ are polar coordinates in the $\bar{x}$, $\bar{y}$ plane) in (3). Because $\bar{m}$ has the form (1) with n replaced by $n+1$, this yields

$$f = \bar{r}^{2n+2} [Q(\cos\bar{\theta}, \sin^{n+1}\bar{\theta}) + o(1)]$$

where $Q(s, t) = s^2 + \bar{a}_{1\,n+1} st + \bar{a}_{0\,2n+2} t^2$. If $\Delta_{n+1} < 0$, then the quadratic form Q is positive definite and, hence, there is a constant q such that $Q(\cos\bar{\theta}, \sin^{n+1}\bar{\theta}) \geqslant q > 0$ for all $0 \leqslant \bar{\theta} \leqslant 2\pi$. Thus, for all $\bar{r} \neq 0$ small enough, $f(\bar{x}, \bar{y}) > 0$ and (a) is proved. Suppose now that $\Delta_{n+1} > 0$. We will first show that there exist values $\bar{\theta}_+$ and $\bar{\theta}_-$ for which $Q(\cos\bar{\theta}, \sin^{n+1}\bar{\theta}) > 0$ and < 0 respectively. Clearly, we may take $\bar{\theta}_+ = 0$. Since $Q(s, t) = (s - r_+ t)(s - r_- t)$ for $r_\pm = (1/2)(-\bar{a}_{1\,n+1} \pm \sqrt{\Delta_{n+1}})$ we see that $Q < 0$ for all s, t lying in an infinite sector S formed by the distinct lines $s = r_+ t$ and $s = r_- t$. Inasmuch as for any $n \geqslant 1$ the curve $s = \cos\bar{\theta}$, $t = \sin^{n+1}\bar{\theta}$, $0 \leqslant \bar{\theta} \leqslant \pi$ forms a continuous closed curve connecting $(1, 0)$ and $(-1, 0)$ in the upper half plane $t \geqslant 0$, it must intersect S for some $\bar{\theta}_-$. Now clearly, for all $\bar{r} \neq 0$ sufficiently small, $f(\bar{x}, \bar{y}) > 0$ at $\bar{\theta} = \bar{\theta}_+$ and $f(\bar{x}, \bar{y}) < 0$ at $\bar{\theta} = \bar{\theta}_-$; this proves (b). From (3) and the definition above, it is obvious that if $\bar{a}_{0\,2n+1} = \Delta_{n+1} = 0$ then f is $(n+1)$-fold degenerate. The examples $f = x^2 + xy^{n+2} + y^{2n+4}$ and $f = (x - y^{n+2})(x - 2y^{n+2})$ which have a relative proper minimum and a saddle point respectively and which both have $\Delta_{n+1} = 0$ proves (c). □

As is well known (see for example [4, §53]) the positivity of $\Delta = \Delta_1$ has a certain geometric significance relating to the two implicitly defined curves $f(x, y) = 0$;

namely, these curves have unequal tangents of slopes $(1/2)(-a_{11}\pm\sqrt{\Delta_1})$. In the degenerate case this feature is also present in an extended form which we describe in the next theorem. First we point out that if f is n-fold degenerate and the hypotheses of Theorem 1 part (ii) and (b) hold, then the relation $f(x, y)=0$ defines, for y sufficiently small, two distinct functions $x_\pm=(-a_{1n}/2)\,y^n+r_\pm y^{n+1}+y^{n+1}u_\pm(y)$ where $r_\pm=(1/2)(-\bar{a}_{1\,n+1}\pm\sqrt{\Delta_{n+1}})$ and $u_\pm$ are continuous functions in a neighborhood of $y=0$ satisfying $u_\pm(0)=0$. A proof of this fact can be constructed by setting $t=\bar{x}/\bar{y}^{n+1}$ and repeating verbatim the proof of the one-fold degenerate case as given, for example, by Goursat [4, p. 111]. It is not difficult to see what this fact says about a function f upon which the test described in Theorem 1 has been applied $m-1\geqslant 0$ times with degenerate results and an mth time with case (b) as a result. The function has, of course, a saddle point by Theorem 1, but moreover, the relation $f(x, y)=0$ (in its original variables) defines, in some neighborhood of $y=0$, two $m+1$ continuously differentiable functions $x_\pm$ intersecting at $y=0$ which have equal derivatives of all orders $1\leqslant i\leqslant m$ (given by $-\frac{1}{2}i!a_{1i}$) and distinct $(m+1)$st derivatives (given by $(m+1)!r_\pm$) at $y=0$. More specifically, if we follow the m changes of variables given by (2) which were performed in the process of performing the $m-1$ degenerate tests described by Theorem 1 then the two intersecting arcs become, in the original variables

$$x_\pm=\sum_{k=1}^{m}(-a_{1k}/2)\,y^k+r_\pm y^{m+1}+y^{m+1}u_\pm(y). \tag{4}$$

As far as saddle points are concerned, the remaining possibility is that Theorem 1 has been applied $m-1$ times with degenerate results and an mth time with the result that $\bar{a}_{0\,2m+1}\neq 0$. In this event, as pointed out by Goursat for one-fold degenerate critical points [4, p. 113] the relation $f(x, y)=0$ may define in a neighborhood of the origin either a cusp or again two intersecting curves with no other peculiarities. It can easily be shown that in this case the two branches have m equal derivatives (in the case of a cusp, one-sided derivatives) at $y=0$. Thus, we have

THEOREM 2. *Suppose f is any function with a degenerate critical point at the origin to which the test described in Theorem 1 has been applied $m-1\geqslant 0$ times with degenerate results and an* mth *time with the result that f has a saddle point at the origin*: (a) *if $\bar{a}_{0\,2m+1}\neq 0$ then $f(x, y)=0$ defines either a cusp or two intersecting curves $x_\pm=x_\pm(y)$ with m equal derivatives (one sided, in the case of a cusp) at $y=0$*; *or* (b) *if $\bar{a}_{0\,2m+1}=0$ and $\Delta_{m+1}>0$ then $f(x, y)=0$ defines two $m+1$ continuously differentiable functions $x_\pm(y)$ given by* (4) *which have m equal derivatives and different* $(m+1)$st *derivatives at $y=0$*.

(The results in this theorem are also proved in [2], if one looks hard enough, but not in the algorithmic format above.)

3. Further results

A natural question arises concerning the repeated application of Theorem 1: are there functions for which the repeated use of Theorem 1 always results in the degenerate case (c) and, hence, for which no decision about the extremal nature of the function can be made on the basis of this procedure? We will denote such functions *infinitely degenerate*. That infinitely degenerate functions exist can easily be observed by noting that, with the exception of part (c), Theorem 1 results in either a saddle point or a proper extremum for f; hence, any function possessing continuous partial derivatives of all orders which has an improper extremum at the origin is necessarily infinitely degenerate. For example, the function x^2 is infinitely degenerate. But then so is $x(x-\exp(-y^{-2}))$, which shows that an infinitely degenerate function may have a saddle point as well as an extremum. (The function $x(x-\exp(-y^{-2}))$ is infinitely degenerate because its Taylor expansion with remainder, of any order, is identical to that of x^2.) It is interesting, however, that within the class of functions analytic in N the set of infinitely degenerate functions is exactly the set of functions with improper extrema at the origin. This fact is contained in the next theorem.

THEOREM 3. *Suppose f is analytic in N (and as always is quadratic in lowest terms at the origin). Then f is infinitely degenerate if and only if it can be written as $f(x, y)=(x-\sum_1^\infty d_i y^i)^2 g(x, y)$ where g is a function analytic in N with $g(0, 0)=1$. Thus, f is infinitely degenerate if and only if it has an improper minimum at the origin. As a result, for analytic functions whose lowest order terms are quadratic, saddle points and proper extrema are always found by our procedure within some finite number of applications of Theorem* 1.

Note that if f is analytic, its lowest order terms being quadratic, and infinitely degenerate (i.e., if f has an improper extremum at the origin) then $f(x, y)=0$ defines a single analytic function $x=\sum_1^\infty d_i y^i$ near $y=0$. Here the d_i are the coefficients generated by repeated use of the change of variables (2): $d_1=-\frac{1}{2}a_{11}$, $d_2=-\frac{1}{2}\bar{a}_{12}$, etc. In proving Theorem 3 we will also obtain stronger results than those contained in Theorem 2 for analytic f; namely we can show that the cases (a) or (b) in Theorem 3 distinguish respectively the cases that $f(x, y)=0$ defines a cusp or two intersecting, analytic arcs at the origin. Thus, we will prove

THEOREM 4. *Suppose f is analytic at the origin (its lowest order terms being quadratic) and has saddle point there. Then there exists an integer $m \geqslant 1$ such that the first $m-1 \geqslant 0$ extremal tests described in Theorem* 1 *fail, but such that at the* mth *test either $\bar{a}_{0\,2m+1} \neq 0$ or $\bar{a}_{0\,2m+1}=0$ and $\Delta_{m+1}>0$. Furthermore,*

(a) *if $\bar{a}_{0\,2m+1} \neq 0$ then $f(x, y)=0$ defines a cusp at the origin consisting of two arcs both analytic only for either small $y>0$ (if $\bar{a}_{0\,2m+1}<0$) or else small $y<0$ (if $\bar{a}_{0\,2m+1}>0$)*

and both terminating at the origin with m equal one-sided derivatives c_i, $1 \leqslant i \leqslant m$, *at* $y=0$. *On the other hand,*

(b) *if* $\bar{a}_{0\,2m+1}=0$ *and* $\Delta_{m+1}>0$, *then* $f(x, y)=0$ *defines two analytic arcs for small* $|y|$ *which intersect one another at the origin and possess m equal derivatives* c_i, $1 \leqslant i \leqslant m$, *but unequal* $(m+1)$st *derivatives at* $y=0$.

The proofs of Theorems 3 and 4 depend on a theorem from the theory of functions of several complex variables (the Weierstrass preparation theorem [5, p. 68]) from which we may conclude, since f is analytic, that $f(x, y)=[x^2+a(y)\,x+b(y)]\,g(x, y)$ where $g(x, y)$ is analytic at the origin with $g(0, 0)=1$ and where a and b are analytic functions of y. The following facts are easily established (the proofs are omitted for brevity): the function f is infinitely degenerate at the origin if and only if $x^2+a(y)x+b(y)$ is; and $x^2+a(y)\,x+b(y)$ is infinitely degenerate at the origin if and only if $a^2(y)=4b(y)$ for all y in their common domain of definition.

Proof of Theorem 3. If f has an improper extremum at the origin, then as already pointed out f necessarily is infinitely degenerate; suppose conversely that f is infinitely degenerate. Then by the remarks above $f(x, y)=[y+\frac{1}{2}a(y)]^2\,g(x, y)$ where $g(0, 0)=1$, and it becomes clear that f has an improper relative minimum at the origin. □

Proof of Theorem 4. We can say immediately from Theorem 3 that if f has a saddle point at the origin then necessarily the sequence of extremal tests terminates at some finite step. Thus, either case (a) or (b) of Theorem 4 holds; the assertion of (b) follows immediately from the proof of Theorem 2. (We need only note in addition that the analyticity of f implies that the functions $u_\pm(y)$ are analytic at $y=0$, as well known implicit function theorems tell us [4, p. 399].) For part (a) we observe that the given hypotheses on f together with the remarks made above can readily be shown to yield

$$f(x, y)=\left\{[x+a(y)/2]^2+\sum_{i=2n+1}^{\infty}\left(b_i-\tfrac{1}{4}\sum_{j=1}^{i-1}a_j a_{i-j}\right)y^i\right\}g(x, y). \tag{5}$$

where $a(y)=\sum_1^\infty a_i y^i$ and $b(y)=\sum_2^\infty b_i y^i$. Since the right hand side vanishes for small $|x|$, $|y|$ if and only if the bracketed expression vanishes the implicitly defined curves are identical for these two expressions. Now the condition $\bar{a}_{0\,2n+1}\neq 0$ means that upon setting $\bar{x}=0$ at the mth test (that is, in the original variables, $x+\frac{1}{2}\sum_1^{n+1}a_i y^i=0$) the resulting power series in $\bar{y}$ has $\bar{y}^{2n+1}$ as its lowest order term. From (5) this condition is precisely $\bar{a}_{0\,2n+1}=b_{2n+1}-\frac{1}{4}\sum_{j=1}^{2n}a_j a_{2n+1-j}\neq 0$. Since $\bar{y}^{2n+1}$ is an odd power of $\bar{y}$, it is clear that $f(x, y)$ vanishes only for small $\bar{y}=y>0$ if $\bar{a}_{0\,2n+1}<0$ or only for small $y<0$ if $\bar{a}_{0\,2n+1}>0$. In fact the branches are found by setting the bracketed term in (5) equal to zero and solving for x. □

Remark. All of the above results can be extended in a straightforward manner to functions f of $n \geqslant 3$ variables $x_1, \ldots x_n$ provided the sum of the second order terms is a semi-definite quadratic form of deficiency one; i.e., in canonical variables, $f=\sum_1^{n-1} e_i x_i^2 + o(r^2)$, $e_i=\text{const.}>0$. If the deficiency d is two or more, then the method fails in that the behavior of the quadratic form in the variables $x_1, \dots, x_{n-d}$, $x_{n-d+1}^2, \dots, x_n^2$ does not determine that of f near the origin. This can be shown by the following two examples (with $n=3$): $f=(x_1+x_2x_3)^2+x_2^4+(\frac{3}{4})\,x_3^4$ and $f=x_1^2+x_2^4+x_3^4-4x_1x_2x_3$ which have a proper minimum and a saddle point at the origin respectively, while the quadratic forms (in x_1, x_2^2, and x_3^2) given by $x_1^2+x_2^4+(\frac{3}{4})\,x_3^4$ and $x_1^2+x_2^4+x_3^4$ are both positive definite.

EXAMPLE. To illustrate the repeated use of Theorem 1 consider the polynomial $f=x^2-2xy+y^2-2xy^2+2y^3+y^4-y^6+xy^6$. Since $\Delta_1=0$, the origin is a degenerate critical point. To apply Theorem 1 with $n=1$ we make the change of variables (2) with $c_1=1$ and obtain $\bar{x}^2-2\bar{x}\bar{y}^2+\bar{y}^4-\bar{y}^6+\bar{x}\bar{y}^6+\bar{y}^7$. Inasmuch as $\bar{a}_{03}=0$ and $\Delta_2=\bar{a}_{12}^2-4\bar{a}_{14}=(-2)^2-4(1)=0$, f is two-fold degenerate. To apply the theorem again with $n=2$ we make a second change of variables given by (2) (after replacing the coordinates $\bar{x}$, $\bar{y}$ with x, y for simplicity to avoid complicating the notation): $\bar{x}=x-y^2$, $\bar{y}=y$. This yields $\bar{x}^2-\bar{y}^6+\bar{x}\bar{y}^6+\bar{y}^7+\bar{y}^8$ and $\bar{a}_{05}=0$, $\Delta_2=\bar{a}_{13}^2-4\bar{a}_{06}=0-4(-1)=4>0$. From part (b) of Theorem 1 we find that this polynomial has a saddle point at the origin. Furthermore, by Theorem 4, $f(x,y)=0$ defines two analytic functions of the form $x_+=y+y^2+y^3+y^3u_+(y)$ and $x_-=y+y^2-y^3+y^3u_-(y)$ for $|y|$ sufficiently small.

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Aequationes Math. *13* (1975), 97–101. University of Waterloo Birkhäuser Verlag, Basel

On regular bipartite-preserving supergraphs

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Abstract

For a graph G with chromatic number $\chi(G) \geqslant 2$ and maximum degree $\Delta(G)$, there exists an r-regular graph H, for every $r \geqslant \Delta(G)$, such that G is an induced subgraph of H and $\chi(H) = \chi(G)$. In the case where G is bipartite, the minimum order of such a graph H is determined.

1. Introduction

Given a graph G with maximum degree $\Delta(G)$, it is not difficult to verify that there exists a supergraph H of G (i.e., G is a subgraph of H) such that H is r-regular, where $r = \Delta(G)$, and G is an induced subgraph of H. In fact, Erdös and Kelly [2, 3] have determined the minimum order of such a graph H.

If a graph G under consideration has some specified property P, then, ordinarily, an r-regular supergraph H containing G as an induced subgraph need not also possess property P. It is the object of this paper to study the above problem with the specified property being the chromatic number $\chi(G)$ of the graph H. In particular, we show that if G is a graph with $\chi(G) = n \geqslant 2$, then there exists an r-regular graph H, for every $r \geqslant \Delta(G)$, such that G is an induced subgraph of H and $\chi(H) = n$. Furthermore, in the special case where G is bipartite (i.e., $\chi(G) = 2$), we determine the minimum order of such a graph H.

2. Preliminary definitions

For basic graph theory terminology and notation, we follow [1]. The definitions of a few pertinent terms are presented here, however.

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors which may be assigned to the vertices of G so that adjacent vertices receive different colors. If $\chi(G) = n (\geqslant 2)$, then the vertex set of G can be partitioned as $V_1 \cup V_2 \cup \ldots \cup V_n$ such that every edge of G joins a vertex of some V_i to a vertex of some V_j, $i \neq j$. The sets V_i are referred to as *color classes*. In this case, we write $G = G(V_1, V_2, \ldots, V_n)$ to indicate the color classes. If $\chi(G) = 2$, then G is a *bipartite graph*.

AMS Subject Classification: Primary 05C15.
Research supported, by the first author, in part by a Faculty Research Fellowship from Western Michigan University.

Received August 22, 1973.

A subgraph G of a graph H is called an *induced subgraph* of H if every two vertices of G which are adjacent in H are also adjacent in G.

Finally, a graph G is *regular of degree* r or r*-regular* if every vertex of G has degree r.

3. The main results

First, we show that every graph G is an induced subgraph of a regular graph H having the same chromatic number as G. We denote the minimum degree of G by $\delta(G)$.

THEOREM 1. *Let G be a graph with $\chi(G)=n\geqslant 2$, and let r be a positive integer such that $r\geqslant \Delta(G)$. Then there exists an r-regular graph H containing G as an induced subgraph such that $\chi(H)=n$.*

Proof. Let $G=G(V_1, V_2,\ldots, V_n)$. If G is r-regular, we take $H=G$. Otherwise, consider another copy G' of G, where $G'=G'(V_1', V_2',\ldots, V_n')$ and V_i' is a copy of V_i for $i=1, 2,\ldots, n$. Define H_1 as the (disjoint) union of G and G' together with all edges of the type vv' (where v is in G and v' is the corresponding vertex in G') whenever the degree of v is less than r. Hence, $\delta(G)+1=\delta(H_1)\leqslant\Delta(H_1)\leqslant r$. If H_1 is r-regular, we take $H=H_1$. If H_1 is not r-regular, then we may construct H_2 from two copies of H_1, as before. We may then continue this procedure until obtaining the r-regular graph H_k, where $k=r-\delta(G)$. Since G is an induced subgraph of H_k, we take H to be the graph H_k, giving the desired result.

The preceding proof gives the order of an r-regular graph H containing G as an induced subgraph as $2^k p$, where p is the order of G and $k=r-\delta(G)$. This is, undoubtedly, far above the minimum order of such a graph H. We now consider this problem in the case of bipartite graphs.

If $G=G(V_1, V_2)$ is a bipartite graph, then we employ the symbol $\delta_i=\delta_i(G)$, $i=1, 2$, to denote the minimum degree in G among the vertices of V_i. We are now prepared to present our next result, which is reminiscent of the aforementioned theorem of Erdös and Kelly.

THEOREM 2. *Let $G=G(V_1, V_2)$ be a bipartite graph with p vertices and $q(\geqslant 1)$ edges, where $|V_1|=m$ and $|V_2|=n$. If r is a positive integer such that $r\geqslant\Delta(G)$, then the minimum order of an r-regular bipartite graph containing G as an induced subgraph is $2k$, where k is the least positive integer satisfying the following inequalities:*

(1) $k\geqslant n+r-\delta_1$;

(2) $k\geqslant m+r-\delta_2$;

(3) $k\geqslant p-[q/r]$;

(4) $k^2-(p+r)k+(mn+rn+rm-q)\geqslant 0$.

Proof. Suppose $H=H(W_1, W_2)$ is an r-regular bipartite graph containing G as induced subgraph, where $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$. Since H is regular of positive degree, it follows that $|W_1|=|W_2|$.

Let v_1 be a vertex of V_1 whose degree (in G) is δ_1. Since v_1 has degree r in H and G is an induced subgraph of H, the vertex v_1 is adjacent with $r-\delta_1$ vertices in W_2-V_2. Hence, $|W_2-V_2|=|W_2|-|V_2|\geqslant r-\delta_1$. Similarly, $|W_1-V_1|=|W_1|-|V_1|\geqslant r-\delta_2$. Thus $|W_1|\geqslant m+r-\delta_2$ and $|W_2|\geqslant n+r-\delta_1$.

Since every edge of H which is incident with a vertex of V_1 and which is not an edge of G must be incident with a vertex of W_2-V_2, it follows that $r|V_1|-q\leqslant \leqslant r(|W_2|-|V_2|)$. Hence, $rp-q\leqslant r|W_2|$ or, equivalently, $|W_2|\geqslant p-[q/r]$.

The total number of edges joining V_1 and W_2-V_2 is $rm-q$. Therefore, there is a vertex v_2 of W_2-V_2 incident with no more than $(rm-q)/(|W_2|-V_2|)$ such edges. Hence v_2 is joined to at least $r-(rm-q)/(|W_2|-n)$ vertices of W_1-V_1. Therefore, $r-(rm-q)/(|W_2|-n)\leqslant|W_1|-|V_1|$, which implies that $|W_1|^2-(p+r)|W_1|+ (mn+ +rn+rm-q)\geqslant 0$.

Hence, if we denote the order of H by $2k$ (where, then, $k=|W_1|=|W_2|$), then we have shown that k must satisfy the inequalities (1)–(4).

It remains to show that there exists an r-regular bipartite graph G^* containing G as an induced subgraph and having order $2k$, where k is the least positive integer satisfying the inequalities (1)–(4).

Let $V_1=\{x_1, x_2,\dots, x_m\}$ and $V_2=\{y_1, y_2,\dots, y_n\}$, where $\deg_G x_i\geqslant\deg_G x_{i+1}$ for $i=1, 2,\dots, m-1$ and $\deg_G y_j\geqslant\deg_G y_{j+1}$ for $j=1, 2,\dots, n-1$.

We describe the vertex set of G^*. Let $V_1^*=V_1$ if $k-m=0$, and let $V_1^*=V_1\cup \cup\{u_1, u_2,\dots, u_{k-m}\}$ if $k-m>0$, where $V_1\cup V_2$ and $\{u_1, u_2,\dots, u_{k-m}\}$ are disjoint. Similarly, we let $V_2^*=V_2$ if $k-n=0$, and let $V_2^*=V_2\cup\{v_1, v_2,\dots, v_{k-n}\}$ if $k-n>0$, where $V_1\cup V_2\cup\{u_1, u_2,\dots, u_{k-m}\}$ and $\{v_1, v_2,\dots, v_{k-n}\}$ are disjoint.

Next we describe the edge set of G^*. Since G is to be an induced subgraph of G^*, the edge set of G^* must, of course, contain the edge set of G. Now $k-m=0$ and $k-n=0$ if and only if $\delta_1=\delta_2=r$, i.e., if and only if G is r-regular. Hence, if G is r-regular, then G^* is isomorphic to G and the order of G^* is $2k$. It follows readily that $k=m=n$ is the smallest integer satisfying (1)–(4).

We assume henceforth that G is not r-regular. Thus, at least one of δ_1 and δ_2 is less than r. Without loss of generality, we assume that $\delta_1<r$, i.e., $V_2^*\neq\emptyset$. Let t be the least positive integer, $1\leqslant t\leqslant m$, such that $\deg x_t<r$. We join x_t to the vertices $v_1, v_2,\dots, v_{n_1}$, where $n_1=r-\deg x_t$. If x_{t+1} exists (i.e., if $t<m$), then we join it to $v_{n_1+1},\dots, v_{n_2}$, where $n_2=2r-\deg x_t-\deg x_{t+1}$ and where the subscripts $n_1+1,\dots, n_2$ are expressed modulo $k-n$. We continue in this manner, joining each x_i, $t\leqslant i\leqslant m$, to the appropriate $r-\deg x_i$ vertices of $V_2^*-V_2$. We note that, for each such i, there are sufficiently many vertices in $V_2^*-V_2$ since $r-\deg x_i\leqslant r-\delta_1\leqslant k-n$, the last inequality following because k satisfies (1). Denote the graph constructed thus far by H.

We claim that no vertex of H has degree exceeding r; only the vertices of $V_2^* - V_2$ need be checked. The number of edges joining V_1 and $V_2^* - V_2$ is

$$\sum_{i=1}^{m} (r - \deg x_i).$$

We verify that this sum does not exceed $r(k-n)$, which will yield the desired result because then the average of the degrees of vertices in $V_2^* - V_2$ is at most r and, by construction, $\max_{i,j} |\deg y_i - \deg y_j| \leqslant 1$ so no vertex of $V_2^* - V_2$ has degree exceeding r. Since k satisfies (3), it follows that $k \geqslant p - q/r$. This implies that $rk \geqslant r(m+n) - q$; hence, $rm - q \leqslant r(k-n)$. However, this last inequality states that

$$\sum_{i=1}^{m} (r - deg\, x_i) \leqslant r(k-n).$$

By construction, every vertex of V_1 has degree r in H. If $k = m$, then, since k satisfies (2), it follows that $m \geqslant m + r - \delta_2$ and thus $\delta_2 = r$. Hence, every vertex in V_2 has degree r. Each of the m vertices in V_1 has degree r, so H has mr edges, nr of which are incident with vertices in V_2. Thus there are $mr - nr$ edges of H incident with vertices in $V_2^* - V_2$, which has $k - n$ vertices. Now, $(mr - nr)/(k-n) = r$ so the average of the degrees in H of the vertices of V_2^* is r. Hence every vertex in V_2^* has degree exactly r.

We now assume that $k > m$. Relabel v_i, $i = 1, 2, \ldots, k-n$, as y_{n+i}. Let s be the least positive integer, $1 \leqslant s \leqslant k$, such that $\deg_H y_s < r$. We join y_s to the vertices $u_1, u_2, \ldots, u_{m_1}$, where $m_1 = r - \deg_H y_s$. If $s < k$, then we join y_{s+1} to $u_{m_1+1}, \ldots, u_{m_2}$, where $m_2 = 2r - \deg_H y_s - \deg_H y_{s+1}$, and where the subscripts $m_1 + 1, \ldots, m_2$ are interpreted modulo $k - m$. We proceed in this manner, joining each y_i for which $\deg_H y_i < r$ to the appropriate $r - \deg_H y_i$ vertices of $V_1^* - V_1$. This completes the construction of G^*.

For each i such that $1 \leqslant i \leqslant n$, there are sufficiently many vertices in $V_1^* - V_1$ which can be joined to y_i since $r - \deg_H y_i \leqslant r - \deg_G y_i \leqslant r - \delta_2 \leqslant k - m$. The number of edges of H joining V_1 and $V_2^* - V_2$ is

$$\sum_{i=1}^{m} (r - \deg x_i) = rm - q.$$

The number of edges which must be added to H to assure each vertex of V_2^* has degree r in G^* is $r(k-n) - rm + q$. By the construction, the maximum number of edges which need be added to any vertex of V_2^* is $\{(rm-q)/(k-n)\}$. Since k satisfies (4), it follows that $\{(rm-q)/(k-n)\} \leqslant k - m$; hence, there are sufficiently many vertices in $V_1^* - V_1$ to accomplish this.

The number of edges which must be added to H to assure each vertex of V_1^* has degree r is $r(k-m)-rn+q$, and this is equal to $r(k-n)-rm+q$. Thus, G^* is r-regular, and the proof is complete.

We note, in closing that examples can be given to show that each of the inequalities (1)–(4) is needed.

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Aequationes Math. *13* (1975), 103–119. University of Waterloo Birkhäuser Verlag, Basel

The dual of the cone of all convex functions on a vector space

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1. Introduction

Let Q denote the field of rational numbers. Consider a linear vector space G over Q and let X be a fixed Q-convex subset of G. A function $g:X\to R$ is said to be convex if the inequality

$$g((y+z)/2)\leqslant(g(y)+g(z))/2 \tag{1.1}$$

holds for each choice of the elements y and z in X. The class of all such convex functions will be denoted by C. One may regard C as a convex cone in the linear space Π consisting of *all* functions $g:X\to R$.

Let Σ denote the class of functions $f:X\to R$ having a finite support $\{x\in X:f(x)\neq 0\}$. The dual C^0 of C will be defined as the class of all $f\in\Sigma$ with the property that

$$\langle f,g\rangle=\sum_{x\in X} f(x)\,g(x)\geqslant 0 \quad \text{for all} \quad g\in C. \tag{1.2}$$

Clearly C^0 is a convex cone in the linear space Σ. Since all the constant functions are in C one has $\sum_{x\in X} f(x)=0$ for each $f\in C^0$.

In view of (1.1), the dual cone C^0 certainly contains all functions $f_{y,z}$ of the special form

$$\begin{aligned} f_{y,z}(x)&=1 \quad &&\text{if} \quad x=y,\\ &=1 &&\text{if} \quad x=z,\\ &=-2 &&\text{if} \quad x=(y+z)/2,\\ &=0, &&\text{otherwise,}\end{aligned} \tag{1.3}$$

(except that $f_{y,z}(x)=0$ for all $x\in X$ in case $y=z$); here, y and z are fixed elements in X. Let

$$A=\{f_{y,z}: y\in X, z\in X\}$$

be the collection of all these special elements in C^0. Also note that C itself may be regarded as the dual of A, i.e. $C=A^0$.

This research was supported in part by the National Science Foundation.
AMS Subject Classifications: Primary: 26A51, 15A39, 39A10. Secondary: 52A40, 46A20

Received August 31, 1973 and, in revised form, January 20, 1974.

The major result of the present paper (Theorem 1) says that the dual cone $C^0 = A^{00}$ of C is precisely the convex cone in Σ generated by all these special elements $f_{y,z}$. In other words, the second dual A^{00} of A coincides with the convex cone A_C in Σ which is spanned by A. Equivalently (see Section 3), A_C happens to be closed relative to the coarsest topology in Σ with the property that $\langle f, g\rangle$ is continuous in f for each fixed $g \in \Pi$. The proof of Theorem 1 is given in Section 5.

The proof implicitely contains an effective (but probably inefficient) algorithm for deciding whether or not a given $f \in \Sigma$ can be written as a finite nonnegative linear combination of elements $f_{y,z}$. In view of Theorem 1 such a representation is possible if and only if it is impossible to find a convex function $g: X \to R$ (measurable or not) such that $\sum_{x \in X} f(x)\, g(x) < 0$.

Theorem 1 says that $A^{00} = A_C$. Besides A there are many other interesting subsets D of Σ for which one would like to know exactly the second dual D^{00}. Very much simpler is the situation where instead one is interested in the iterated orthogonal complement $D^{\perp\perp}$ of D. Namely (see Section 2), this set $D^{\perp\perp}$ always coincides with the linear span D_L of D. Naturally, it may still be very hard to decide whether or not a given $f \in \Sigma$ belongs to D_L.

The proof of Theorem 1, as given in Section 5, employs, among other things, a well-known result of Blackwell [2] concerning dilatations, (compare Section 6). A generalization (Theorem 2) of Blackwell's result is presented in Section 7. It greatly simplifies the proof of Theorem 1 and, moreover, has a clear interest of its own.

2. Functional equations

Let X be a fixed set. By Π we shall denote the collection of all functions $g: X \to R$. One may regard Π as the (unrestricted) direct product

$$\Pi = \bigotimes_{x \in X} R_x$$

(with R_x as a copy of R). Let Σ denote the corresponding direct sum

$$\Sigma = \bigoplus_{x \in X} R_x .$$

An element $f \in \Sigma$ may be interpreted as a function $f: X \to R$ having a finite support $\{x \in X: f(x) \neq 0\}$. Both Σ and Π are linear vector spaces (over R). A natural basis for Σ is given by $\{\varepsilon_x : x \in X\}$ where $\varepsilon_x \in \Sigma$ is defined by

$$\begin{aligned} \varepsilon_x(\xi) &= 0 \quad \text{if} \quad \xi \in X,\ \xi \neq x; \\ &= 1 \quad \text{if} \quad \xi = x. \end{aligned}$$

Thus $f \in \Sigma$ can be written as

$$f = \sum_{x \in X} f(x)\, \varepsilon_x . \tag{2.1}$$

It is sometimes useful to identify $f\in\Sigma$ with the signed measure μ_f on X defined by

$$\mu_f(E)=\sum_{x\in E} f(x) \quad \text{for each} \quad E\subset X.$$

The linear spaces Σ and Π are in duality (see [4], p. 48, [8] p. 138) by means of the natural inner product (bilinear form)

$$\langle f, g\rangle=\sum_{x\in X} f(x)\, g(x)=\int g\, d\mu_f, \tag{2.2}$$

($f\in\Sigma$ and $g\in\Pi$), which separates points both in Σ and in Π. If $\phi:\Sigma\to R$ is a linear functional on Σ then, by (2.1),

$$\phi(f)=\sum_{x\in X} f(x)\,\phi(\varepsilon_x)=\langle f, g\rangle,$$

where $g\in\Pi$ is defined by $g(x)=\phi(\varepsilon_x)$ for all $x\in X$. Thus Π is the algebraic dual of Σ. On the other hand, if X is infinite then there exist many linear functionals ψ on Π not of the form (2.2).

If $A\subset\Sigma$ we define

$$A^{\perp}=\{g\in\Pi: \langle f, g\rangle=0 \quad \text{for all} \quad f\in A\}. \tag{2.3}$$

Thus $A^{\perp}$ is the linear manifold in Π consisting of all solutions $g:X\to R$ of the '*functional equation*'

$$\sum_{x\in X} f(x)\, g(x)=0 \quad \text{for all} \quad f\in A. \tag{2.4}$$

Similarly, if $B\subset\Pi$ we define

$$B^{\perp}=\{f\in\Sigma: \langle f, g\rangle=0 \quad \text{for all} \quad g\in B\}. \tag{2.5}$$

One has $A\subset A^{\perp\perp}$ and $A^{\perp}=A^{\perp\perp\perp}$.

LEMMA 1. *Let A be an arbitrary subset of Σ and let A_L denote the linear manifold in Σ spanned by A. Then*

$$A^{\perp\perp}=A_L. \tag{2.6}$$

Proof. Clearly $A_L\subset A^{\perp\perp}$. Let $f_0\in\Sigma$ be such that $f_0\notin A_L$. We must prove that $f_0\notin A^{\perp\perp}$.

There exists a linear functional $\phi:\Sigma\to R$ which vanishes on A_L (and thus on A) while $\phi(f_0)\neq 0$. Since Π is the algebraic dual of Σ, there exists a unique $g_0\in\Pi$ such that $\phi(f)=\langle f, g_0\rangle$ for all $f\in\Sigma$. Moreover, $g_0\in A^{\perp}$ and $\langle f_0, g_0\rangle\neq 0$ thus $f_0\notin A^{\perp\perp}$.

Remark. For subsets B of Π the set $B^{\perp\perp}$ may be much larger than B_L. For example, let $B=B_L$ consist of all functions $g:X\to R$ of finite support. Then $B^{\perp}=\{0\}$ thus $B^{\perp\perp}=\Pi\neq B$ as soon as X is infinite.

As an application of Lemma 1, let X be a subset of an additively written abelian group G and let n be a fixed positive integer. Let further Q_n denote the collection of all generalized polynomials on X of degree $\leqslant n-1$. More precisely, Q_n consists of all functions $g:X\to R$ such that

$$(\Delta_h^n g)(\xi)=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}g(\xi+jh)$$

vanishes for all ξ, h in G such that $\xi+jh\in X$ for $j=0, 1, \ldots, n$. In other words, $Q_n=(P_n)^{\perp}$ where P_n denotes the subset of Σ consisting of all functions $\Delta_{\xi,h}^n:X\to R$ of the special form

$$\Delta_{\xi,h}^n(x)=(-1)^{n-j}\binom{n}{j}\quad \text{if } x=\xi+jh \quad (j=0, 1, \ldots, n); \qquad (2.7)$$
$$=0, \quad \text{otherwise}.$$

LEMMA 2. *Let $f\in\Sigma$ be given. Then in order that*

$$\sum_{x\in X} f(x)\,g(x)=0 \quad \textit{for all} \quad g\in Q_n \qquad (2.8)$$

it is necessary and sufficient that f admits a representation of the form

$$f(x)=\sum_{\xi,h}\beta(\xi,h)\,\Delta_{\xi,h}^n(x). \qquad (2.9)$$

Proof. Apply Lemma 1 with $A=P_n$. Naturally, it is understood that the (real) coefficient $\beta(\xi,h)$ in (2.9) is equal to zero for all but finitely many pairs (ξ,h). Moreover, in (2.7) and (2.9) one only allows pairs $(\xi,h)\in G\times G$ such that $\xi+jh\in X$ for $j=0, 1, \ldots, n$.

As an application of Lemma 2 take $X=G$. It is known (see [9]) that in this case $\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}g\equiv 0$ for all $g\in Q^n$ and each choice of the elements $h_1,\ldots,h_n$ in G. Afterwards, we obtain from Lemma 2 the stronger property that the linear functional $g:\to(\Delta_{h_1}\Delta_{h_2}\cdots\Delta_{h_n}g)(\xi_0)$ on Π can be expressed as a finite linear combination of the special linear functionals $\langle\Delta_{\xi,h}^n, g\rangle=(\Delta g)_h^n(\xi)$. For instance, if $n=2$ then one such expression would be

$$2\Delta_{h_1}\Delta_{h_2}g(\xi_0)=\Delta_{h_1}^2 g(\xi_0)+\Delta_{h_2}^2 g(\xi_0)-\Delta_{h_2-h_1}^2 g(\xi_0+2h_1).$$

If $u, v \in G$ exist with $h_1 = u - v$ and $h_2 = u + v$ then another expression would be

$$\Delta_{h_1}\Delta_{h_2} g(\xi_0) = \Delta_u^2 g(\xi_0) - \Delta_v^2 g(\xi_0 + u).$$

3. Inequalities

Let X, Σ and Π be as in Section 2. If $A \subset \Sigma$ we define the dual A^0 of A by means of

$$A^0 = \{g \in \Pi : \langle f, g \rangle \geqslant 0 \quad \text{for all} \quad f \in A\}. \tag{3.1}$$

Thus A^0 is the convex cone in Π consisting of all functions $g: X \to R$ such that

$$\sum_{x \in X} f(x)\, g(x) \geqslant 0 \quad \text{for all} \quad f \in A. \tag{3.2}$$

Similarly, if $B \subset \Pi$ we define

$$B^0 = \{f \in \Sigma : \langle f, g \rangle \geqslant 0 \quad \text{for all} \quad g \in B\}. \tag{3.3}$$

One has $A \subset A^{00}$ and $A^0 = A^{000}$.

By analogy with Lemma 1, one might hope that the second dual A^{00} of A coincides with the convex cone A_C generated by A. Equivalently, one might hope that each $f_0 \in \Sigma$ with $f_0 \notin A_C$ can be separated from A_C by a linear functional $\phi: \Sigma \to R$ in the sense that $\phi(f) \geqslant 0$ for all $f \in A_C$ while $\phi(f_0) < 0$. Unfortunately, this need not always be true; compare the well-known counter-example in [3] p. 53 (problem 4).

In fact (see [3] p. 73; [8] p. 119) the required separation property holds if and only if the convex cone A_C is closed relative to the weak topology induced by the bilinear functional $\langle f, g \rangle$ on $\Sigma \times \Pi$. Thus a net $\{f_\gamma\}$ in Σ converges to $f_0 \in \Sigma$ if and only if $\lim_\gamma \langle f_\gamma, g \rangle = \langle f_0, g \rangle$ for all $g \in \Pi$.

By the way, a linear manifold in Σ is always closed. After all, it can be represented as an intersection of a suitable collection of hyperplanes $\{f \in \Sigma : \langle f, g_0 \rangle = 0\}$; compare Lemma 1 and its proof.

In general it is very difficult to see whether or not a given convex cone in Σ is closed. Consequently, if one wants to show that a given set $A \subset \Sigma$ satisfies $A^{00} = A_C$ then one is usually more or less forced to present a direct proof that each $f \in A^{00}$ can be expressed as a finite linear combination with nonnegative coefficients of elements in A.

4. The dual of the class of all convex functions

Let Q denote the field of rational numbers. From now on we shall take X as a Q-convex subset of a vector space G having Q as its field of scalars; thus if $y, z \in X$ and $0 < \alpha < 1$ is rational then $(1-\alpha)\, y + \alpha z \in X$.

Let further C denote the subset of Π which consists of all convex functions $g:X\to R$. Thus a function $g:X\to R$ belongs to C if and only if

$$g(x)\leqslant\tfrac{1}{2}(g(x+h)+g(x-h)),\tag{4.1}$$

for all $x\in X$ and $h\in G$ such that $x\pm h\in X$. Equivalently, we have $C=A^0$ where A is the subset of Σ defined by

$$A=\{\Delta^2_{\xi,h}:\xi,\ \xi+h,\ \xi+2h\in X\}.\tag{4.2}$$

Here,

$$\begin{aligned}\Delta^2_{\xi,h}(x)&=1 &&\text{if}\quad x=\xi;\\ &=-2 &&\text{if}\quad x=\xi+h;\\ &=1 &&\text{if}\quad x=\xi+2h;\\ &=0, &&\text{otherwise}.\end{aligned}\tag{4.3}$$

(provided $h\neq 0$; moreover, $\Delta^2_{\xi,0}(x)\equiv 0$). The following is the main result of the present paper.

THEOREM 1. *The dual $C^0=A^{00}$ of C coincides with the convex cone A_C generated by A. In other words, if $f\in\Sigma$ is such that*

$$\sum_{x\in X} f(x)\,g(x)\geqslant 0\quad\textit{for all}\quad g\in C\tag{4.4}$$

then f admits at least one representation as a linear combination

$$f=\sum_{\xi,h}\beta(\xi,h)\,\Delta^2_{\xi,h}\tag{4.5}$$

of elements in A with nonnegative coefficients and all but finitely many equal to zero.

It would be interesting to know how far Theorem 1 carries over to a situation where A, as in (4.2), is replaced by a collection of the form $\{D_{\xi,h}:\xi\in G,\ h\in Y\}$ where $D_{\xi,h}$ is of the form

$$\langle D_{\xi,h},g\rangle=\sum_{j=0}^{n}a_j g(\xi+T_j h).$$

Here, the coefficients $a_j\in R$ and the mappings $T_j:Y\to G$ are given.

Let us briefly consider the situation that G is a *topological* abelian group. Then one would often be more interested in the class C^* of all *continuous* and convex functions $g:X\to R$ and its dual $(C^*)^0$ which could be much larger than C^0.

Let us only consider the special case that $G\subset R$ while X is an interval. Let $f\in\Sigma$ be given. In order that $f\in(C^*)^0$ one must have $\sum f(x)\,g(x)\geqslant 0$ for all continuous convex functions $g: X \to R$. It is sufficient to restrict g to the piecewise linear functions $a+bx+\sum_j c_j(d_j-x)_+$ $(c_j \geqslant 0)$ or even to the functions $g(x)=a+bx$; $g(x)=(d-x)_+$. In this way one arrives at the known result that $f \in (C^*)^0$ if and only if

$$\sum_x f(x)=0\,; \quad \sum_x f(x)\,x=0, \tag{4.6}$$

and moreover

$$\sum_x f(x)\,(z-x)_+ \geqslant 0 \quad \text{for all} \quad z \in R \quad \text{with} \quad f(z)>0. \tag{4.7}$$

See [7] p. 405 for certain generalizations.

Suppose $f \in \Sigma$ satisfies (4.6) and (4.7), thus $f \in (C^*)^0$. In the important special case $G=Q$ each convex function $g: X \to R$ is automatically continuous on X thus $C=C^*$ and $f \in C^0$. But in general there still might exist a discontinuous convex function $g: X \to R$ such that $\sum_x f(x)\,g(x)<0$. In fact, by Theorem 1, this happens if and only if f does not admit a representation (4.5) with nonnegative coefficients.

For example, let $f_0 \in \Sigma$ be of the special form

$$\begin{aligned} f_0(x) &= -1 \quad \text{if} \quad x=0; \\ &= \alpha \qquad \text{if} \quad x=\beta; \\ &= \beta \qquad \text{if} \quad x=-\alpha; \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{4.8}$$

Here, $0<\alpha<1$ and $\beta=1-\alpha$. It is well-known that $\langle f_0, g\rangle \geqslant 0$ for all $g \in C^*$, that is, $f_0 \in (C^*)^0$.

Consider first the case that α is a rational number. It is well-known (see [6] p. 72) that then $f_0 \in C^0$, hence, f_0 must admit a representation (4.5) with nonnegative coefficients. In fact, such a representation is *implicitely contained* in every of the many known proofs that $f_0 \in C^0$ (when α is rational). One such representation would be

$$f_0=\beta \sum_{i=1}^{n} i\, \Delta^2_{-\alpha+(i-1)h,\,h}+\alpha \sum_{i=n+1}^{N-1} (N-i)\, \Delta^2_{-\alpha+(i-1)h,\,h}\,. \tag{4.9}$$

Here, n and N are positive integers such that $\alpha=n/N$ thus $1 \leqslant n<N$. Further $h=1/N$.

Next, consider the case where α is irrational. Thus α and $\beta=1-\alpha$ are rationally independent. Extending $\{\alpha, \beta\}$ to a (Hamel) basis of R over Q, one easily constructs an additive function $g: R \to R$ having prescribed values $g(\alpha)$ and $g(\beta)$; $(g(x+y)=g(x)+g(y))$. Thus one can attain that $g(\alpha)/\alpha>g(\beta)/\beta$. But then $\Delta^2_h g \equiv 0$ for all $h \in R$ (thus $g \in C$) while $\langle f_0, g\rangle<0$, showing that $f_0 \notin C^0$.

5. Proof of theorem 1.

Let $f \in \Sigma$ be fixed such that (4.4) holds. We must show that f admits a representation (4.5) with nonnegative coefficients. One may assume that $f \neq 0$. Taking $g(x) = \pm 1$ in (4.4) we see that

$$\sum_{x \in X} f(x) = 0. \tag{5.1}$$

One may as well assume that

$$\sum_{f(x)>0} f(x) = +1; \quad \sum_{f(x)<0} f(x) = -1. \tag{5.2}$$

Case (i). There is only one element $x_0 \in X$ with $f(x_0) < 0$; thus $f(x_0) = -1$. We further assume that $f(x)$ takes only rational values.

Let the positive integer N be a common denominator for the rational values $f(x)$. Then (4.4) amounts to a relation of the form

$$\frac{1}{N} \sum_{j=1}^{N} g(x_j) - g(x_0) \geqslant 0 \quad \text{for all} \quad g \in C. \tag{5.3}$$

Here $\{x_0, x_1, \ldots, x_N\}$ is a given collection of not necessarily different elements in X. One must show that there exist nonnegative numbers $\beta(\xi, h)$ [all but finitely many equal to zero; $\beta(\xi, h) > 0$ only when ξ and $\xi + 2h$ are in X] such that

$$\frac{1}{N} \sum_{j=1}^{N} g(x_j) - g(x_0) = \sum_{\xi, h} \beta(\xi, h) \, \Delta_h^2 g(\xi) \tag{5.4}$$

is an identity (valid for all $g \in \Pi$).

If $g: G \to R$ is a Q-linear function on G then (5.3) holds with the equality sign; thus $g(x_0) = g(\bar{x})$ where

$$\bar{x} = (x_1 + \cdots + x_N)/N. \tag{5.5}$$

But the Q-linear functions on G separate points, hence, $x_0 = \bar{x}$.

It is easy to prove the assertion (5.4) by induction with respect to N. One proof would use the representation (4.9) (of $f_0 \in \Sigma$ defined by (4.8) with $\alpha = n/N$) together with the identity

$$\sum_{j=1}^{N} g(x_j) - Ng(\bar{x}) = \left\{ \sum_{j=1}^{n} g(x_j) - ng(\bar{y}) \right\} + \left\{ \sum_{j=n+1}^{N} g(x_j) - mg(\bar{z}) \right\}$$
$$+ \{ ng(\bar{y}) + mg(\bar{z}) - Ng(\bar{x}) \}.$$

Here m and n denote positive integers with $m+n=N$, (such as $m=1$ and $n=N-1$). Further $\bar{y}=(x_1+\cdots+x_n)/n$ and $\bar{z}=(x_{n+1}+\cdots+x_N)/m$ thus $\bar{x}=(n/N)\,\bar{y}+(m/N)\,\bar{z}$. The use of (4.9) can even be avoided by taking $m=n=2^{k-1}$ and $N=2^k$; (this is no loss of generality as may be seen by taking some of the x_j equal to $\bar{x}$).

Another proof by induction can be read off from the identity

$$\sum_{j=1}^{N} g(x_j)-Ng(\bar{x})=\left\{\sum_{j=1}^{N-1} g(x_j)-(N-1)\,g(u)\right\}$$
$$+\{(N-2)\,g(\bar{x})+g(x_N)-(N-1)\,g(v)\}+(N-1)\,\{g(u)+g(v)-2g(\bar{x})\}.$$

Here, $u=(x_1+\cdots+x_{N-1})/(N-1)$ and $v=((N-2)\,\bar{x}+x_N)/(N-1))$; thus $(u+v)/2=\bar{x}$.

Case (ii). There is only a single element $x_0\in X$ with $f(x_0)<0$, thus $f(x_0)=-1$, and at least one of the numbers $f(x)>0$ is irrational. In this case, the assumption (4.4) takes the form

$$\sum_{j=1}^{n} \alpha_j g(x_j)-g(x_0)\geqslant 0 \quad \text{for all} \quad g\in C.$$

Here, $x_j\in X$, $\alpha_j>0$, further $\sum_{j=1}^{n}\alpha_j=1$ while at least one of the numbers α_j is irrational. In fact, all we shall need (just as in case (i)) is that for each Q-linear function $g:G\to R$ we have the equality

$$g(x_0)=\sum_{j=1}^{n} \alpha_j g(x_j).$$

Let $\{z_1, z_2,\ldots, z_N\}$ be a set of rationally independent elements in G which contains $\{x_0, x_1,\ldots, x_n\}$ in its rational span. This set in turn can be extended to a (Hamel) basis $\{z_\gamma\}$ of G over Q. Thus each element $x\in G$ admits a unique representation as a linear combination

$$x=\sum_{\gamma} g_\gamma(x)\,z_\gamma \quad \text{such that} \quad g_\gamma(x)\in Q,$$

while all but finitely many coefficients are equal to zero. It is clear that each function $g_\gamma(x)$ on G is a Q-linear function, thus,

$$g_\gamma(x_0)=\sum_{j=1}^{n} \alpha_j g_\gamma(x_j) \quad \text{for all} \quad \gamma.$$

Put $g_i(x_j)=\xi_{i,j}$ for $i=1,\ldots, N; j=0, 1,\ldots, n$. Then the $\xi_{i,j}$ are rational numbers such that

$$\xi_{i,0}=\sum_{j=1}^{n} \alpha_j\xi_{i,j} \quad \text{for} \quad i=1,\ldots, N. \tag{5.6}$$

Moreover,

$$x_j = \sum_{i=1}^{N} \xi_{i,j} z_i \quad \text{for} \quad j = 0, 1, \ldots, n. \tag{5.7}$$

Consider the compact and convex polyhedral subset S of R^n consisting of all nonnegative vectors $(\sigma_1, \ldots, \sigma_n)$ which satisfy

$$\xi_{i,0} = \sum_{j=1}^{n} \sigma_j \xi_{i,j} \quad \text{for} \quad i = 1, \ldots, N \tag{5.8}$$

and

$$1 = \sum_{j=1}^{n} \sigma_j. \tag{5.9}$$

In other words $\sigma_j \geqslant 0$ and $\sum_1^n \sigma_j X_j = X_0$ where $X_j \in R^{N+1}$ is defined by $X_j = (\xi_{1,j}, \ldots, \xi_{N,j}, 1)$, $(1 \leqslant j \leqslant n)$.

Consider for the moment an extreme point $(\sigma_1^*, \ldots, \sigma_n^*)$ of S and let I be the nonempty subset of $\{1, 2, \ldots, n\}$ such that $\sigma_j^* > 0$ for $j \in I$, $\sigma_j^* = 0$ for $j \notin I$.

We claim that the vectors $\{X_j, j \in I\}$ are linearly independent. For, otherwise, there would exist numbers t_j $(1 \leqslant j \leqslant n)$ not all zero such that $t_j = 0$ for $j \notin I$ and $\sum t_j X_j = 0$. But then both $\sigma_j' = \sigma_j^* + \varepsilon t_j$ and $\sigma_j'' = \sigma_j^* - \varepsilon t_j$ $(1 \leqslant j \leqslant n)$ would define a point of S as soon as ε is sufficiently small and the point $(\sigma_1^*, \ldots, \sigma_n^*)$ would not be an extreme point of S.

Afterwards, using that $\xi_{i,j} \in Q$ and further Cramer's rule for solving a system of linear equations, it follows that $\sigma_j^* \in Q$ for all $j = 1, \ldots, n$.

Let $(\sigma_{h,1}, \ldots, \sigma_{h,n})$ $(h = 1, \ldots, H)$ be the finitely many extreme points of S, hence, $\sigma_{h,j}$ is rational for all h and j. In view of (5.6) the point $(\alpha_1, \ldots, \alpha_n)$ belongs to S, therefore, it admits at least one representation as a convex linear combination of the extreme points of S. That is, there exist numbers $\varrho_h \geqslant 0$ such that $\sum_{h=1}^{H} \varrho_h = 1$ and

$$\alpha_j = \sum_{h=1}^{H} \varrho_h \sigma_{h,j} \quad \text{for} \quad j = 1, \ldots, n.$$

This in turn implies the identity

$$\sum_{j=1}^{n} \alpha_j g(x_j) - g(x_0) = \sum_{h=1}^{H} \varrho_h \left[\sum_{j=1}^{n} \sigma_{h,j} g(x_j) - g(x_0) \right],$$

which is valid for all $g \in \Pi$. Therefore, it suffices to prove, for each fixed h, with $1 \leqslant h \leqslant H$, that the functional

$$\sum_{j=1}^{n} \sigma_{h,j} g(x_j) - g(x_0)$$

on Π admits a representation as in (5.4) with nonnegative coefficients. And this does follow from the result already established in (i), since $\sigma_{h,j} \geqslant 0$ is rational with $\sum_{j=1}^{n} \sigma_{h,j} = 1$ (see (5.9)) and, moreover, $x_0 = \sum_j \sigma_{h,j} x_j$ (see (5.7) and (5.8)).

Case (iii). There are two or more elements $x \in X$ with $f(x) < 0$. Let $x_1, \ldots, x_m$ denote the elements $x \in X$ with $f(x) < 0$ and put $f(x_i) = -p_i$ thus $p_i > 0$ $(i = 1, \ldots, m)$. Let further $y_1, \ldots, y_n$ be the elements $x \in X$ with $f(x) > 0$ and put $f(y_j) = q_j$ thus $q_j > 0$ $(j = 1, \ldots, n)$. Hence, in view of (5.2),

$$p_i > 0, \ \sum_{i=1}^{m} p_i = 1; \quad q_j > 0, \ \sum_{j=1}^{n} q_j = 1. \tag{5.10}$$

The assumption (4.4) can now be written as

$$\sum_{i=1}^{m} p_i g(x_i) \leqslant \sum_{j=1}^{n} q_j g(y_j), \quad \text{valid for all} \quad g \in C. \tag{5.11}$$

It suffices to prove that there exist *nonnegative* real numbers $\alpha_{i,j}$ $(i = 1, \ldots, m;$ $j = 1, \ldots, n)$ such that $\sum_{j=1}^{n} \alpha_{i,j} = 1$ $(i = 1, \ldots, m)$, that further

$$g(x_i) = \sum_{j=1}^{n} \alpha_{i,j} g(y_j) \tag{5.12}$$

holds for each Q-linear function $g: G \to R$ $(i = 1, \ldots, m)$ and that finally

$$q_j = \sum_{i=1}^{m} p_i \alpha_{i,j} \tag{5.13}$$

for $j = 1, \ldots, n$. For, afterwards, one obtains the identity

$$\begin{aligned} \sum_{x \in X} f(x) g(x) &= \sum_{j=1}^{n} q_j g(y_i) - \sum_{i=1}^{m} p_i g(x_i) \\ &= \sum_{i=1}^{m} p_i \left[\sum_{j=1}^{n} \alpha_{i,j} g(y_i) - g(x_j) \right]. \end{aligned}$$

In view of the things established in case (ii), this would imply the existence of the required representation (4.5).

In fact, the existence of the numbers $\alpha_{i,j} \geqslant 0$ as above is an easy consequence of certain known results concerning dilatations; see Section 6 for further explanations. Another proof is presented in Section 7.

6. Dilatations

Let Y be a locally convex topological vector space over the reals (such as $Y = R^N$) and let $C^* = C^*(Y)$ denote the collection of all *continuous* convex functions $g: Y \to R$.

Thus each $g \in C^*$ satisfies

$$g\left(\sum_{j=1}^{n} \alpha_j x_j\right) \leqslant \sum_{j=1}^{n} \alpha_j g(x_j) \tag{6.1}$$

whenever $x_j \in Y$, $\alpha_j \geqslant 0$ and $\sum_j \alpha_j = 1$; thus the α_j need not be rational.

If μ_1 and μ_2 are probability measures on Y of finite support then μ_2 is called a *dilatation* of μ_1 if

$$\int g\, d\mu_1 \leqslant \int g\, d\mu_2 \quad \text{for all} \quad g \in C^*. \tag{6.2}$$

Let the measure μ_1 have a support $\{x_1, \ldots, x_m\}$ with corresponding masses $p_i = \mu_1(\{x_i\}) \geqslant 0$ $(i = 1, \ldots, m)$ and let μ_2 have a support $\{y_1, \ldots, y_n\}$ with masses $q_j = \mu_2(\{y_j\}) \geqslant 0$ $(j = 1, \ldots, n)$; further $\sum_i p_i = 1$ and $\sum_j q_j = 1$. Then (6.2) can be written as

$$\sum_{i=1}^{m} p_i g(x_i) \leqslant \sum_{j=1}^{n} q_j g(y_j), \quad \text{valid for each} \quad g \in C^*. \tag{6.3}$$

It follows from a result due to Blackwell [1], [2] (see [6], [10], [11] and [12] for related results) that property (6.3) is equivalent to the existence of mn nonnegative numbers α_{ij} $(i = 1, \ldots, m;\ j = 1, \ldots, n)$ such that

$$\sum_{j=1}^{n} \alpha_{ij} = 1; \quad x_i = \sum_{j=1}^{n} \alpha_{ij} y_j; \quad q_j = \sum_{i=1}^{m} p_i \alpha_{ij}, \tag{6.4}$$

(for $i = 1, \ldots, m$ and $j = 1, \ldots, n$; the sufficiency is immediate by (6.1)). We may visualize (6.4) as a Markov transformation (or 'dilatation') of the mass distribution μ_1 into the mass distribution μ_2. Here, the mass p_i at x_i is spread out over the points $y_1, \ldots, y_n$ by placing a mass $p_i \alpha_{ij}$ at the point y_j and this happens in such a way that the original centre of gravity x_i is preserved (this for each $i = 1, \ldots, m$).

End of the proof of Theorem 1. We are now in a position to prove the assertions involving (5.12) and (5.13). Thus let X be a Q-convex subset of the Q-linear space G and let C denote the collection of all convex functions $g: X \to R$. Further the $p_i > 0$, $q_j > 0$, $x_i \in X$ and $y_j \in X$ are assumed to satisfy (5.10) and (5.11).

Let $\{z_1, \ldots, z_N\}$ be a set of rationally independent elements in G containing the x_i and y_j in its rational span. Thus there exist rational numbers ξ_{ik} and η_{ik} such that

$$x_i = \sum_{k=1}^{N} \xi_{ik} z_k \quad (i = 1, \ldots, m); \qquad y_j = \sum_{k=1}^{N} \eta_{jk} z_k \quad (j = 1, \ldots, n).$$

One can extend $\{z_1, \ldots, z_N\}$ to a basis $\{z_\gamma\}$ of G over Q. Thus each $x \in G$ can be uniquely written as a finite linear combination $x = \sum_\gamma g_\gamma(x)\, x_\gamma$ with $g_\gamma: G \to Q$ as a

Q-linear function. Note that $g_k(x_i)=\xi_{ik}$ and $g_k(y_j)=\eta_{jk}$ $(i=1,\ldots,m;\ j=1,\ldots,n,\ k=1,\ldots,N)$.

Let C^* denote the class of all continuous convex functions $h:R^N\to R$. Let $h(u_1,\ldots,u_N)$ be such a function and define $g:X\to R$ by means of $g(x)=h(g_1(x),\ldots;\ g_N(x))$. Then $g\in C$ and it follows from (5.11) that

$$\sum_{i=1}^{m} p_i h(g_1(x_i),\ldots,g_N(x_i))\leqslant\sum_{j=1}^{n} q_j h(g_1(y_j),\ldots,g_N(y_j)).$$

In other words, the inequality

$$\sum_{i=1}^{m} p_i h(\xi_{i1},\ldots,\xi_{iN})\leqslant\sum_{j=1}^{n} q_j h(\eta_{j1},\ldots,\eta_{jN})$$

holds for all $h\in C^*$. It follows from the above result due to Blackwell (applied with $Y=R^N$) that there exist nonnegative numbers α_{ij} $(i=1,\ldots,m;\ j=1,\ldots,n)$ satisfying

$$\sum_{i=1}^{m} p_i\alpha_{ij}=q_j;\quad \sum_{j=1}^{n}\alpha_{ij}=1$$

and further

$$(\xi_{i1},\ldots,\xi_{iN})=\sum_{j=1}^{n}\alpha_{ij}(\eta_{j1},\ldots,\eta_{jN}),$$

that is,

$$\xi_{ik}=\sum_{j=1}^{n}\alpha_{ij}\eta_{jk},$$

$(i=1,\ldots,m;\ j=1,\ldots,n;\ k=1,\ldots,N)$.

It remains to verify (5.12). Thus let $g:G\to R$ be a Q-linear function. Since ξ_{ik} and η_{jk} are rational we have

$$\begin{aligned} g(x_i)&=g\left(\sum_{k=1}^{N}\xi_{ik}z_k\right)=\sum_{k=1}^{N}\xi_{ik}g(z_k)=\sum_{k=1}^{N}g(z_k)\sum_{j=1}^{n}\alpha_{ij}\eta_{jk}\\ &=\sum_{j=1}^{n}\alpha_{ij}\sum_{k=1}^{N}\eta_{jk}g(z_k)=\sum_{j=1}^{n}\alpha_{ij}g\left(\sum_{k=1}^{N}\eta_{jk}z_k\right)=\sum_{j=1}^{n}\alpha_{ij}g(y_j).\end{aligned}$$

This completes the proof of Theorem 1.

7. A generalization of Blackwell's theorem

Met m and n be fixed positive integers and put $I=\{1,\ldots,m\}$, $J=\{1,\ldots,n\}$. In the present section, Γ will denote a given collection of pairs (ϕ,ψ) with ϕ as a function $\phi: I \to R$ and ψ as a function $\psi: J \to R$. We further assume that the collection Γ has the following three properties.

(i) If $(\phi, \psi) \in \Gamma$ and $\lambda \geqslant 0$ is a scalar then $(\lambda\phi, \lambda\psi) \in \Gamma$.

(ii) If $(\phi_r, \psi_r) \in \Gamma$ $(r=1, 2)$ then $(\phi_1 + \phi_2, \psi_1 + \psi_2) \in \Gamma$.

(iii) If $(\phi_r, \psi_r) \in \Gamma$ $(r=1, 2)$ then $(\max(\phi_1, \phi_2), \max(\psi_1, \psi_2)) \in \Gamma$.

Thus Γ is a convex cone which is invariant under a maximum operation.

THEOREM 2. *Let $p_i > 0$ $(i=1, \ldots, m)$ and $q_j > 0$ $(j=1, \ldots, n)$ be given positive numbers. Then the following properties* (I) *and* (II) *are equivalent.*

(I) *One has*

$$\sum_{i \in I} p_i \phi(i) \leqslant \sum_{j \in J} q_j \psi(j) \quad \text{for all pairs} \quad (\phi, \psi) \in \Gamma. \tag{7.1}$$

(II) *There exist real numbers $\alpha_{i,j} \geqslant 0$ ($i \in I$ and $j \in J$) such that*

$$q_j = \sum_{i \in I} p_i \alpha_{i,j} \quad \text{for all} \quad j \in J, \tag{7.2}$$

and further

$$\phi(i) \leqslant \sum_{j \in J} \alpha_{i,j} \psi(j) \quad \text{for all} \quad (\phi, \psi) \in \Gamma, \quad \text{all} \quad i \in I. \tag{7.3}$$

Remark 1. As an immediate corollary, this theorem yields the result involving (5.12) and (5.13) which was needed at the end of the proof of Theorem 1. For, apply Theorem 2 with Γ as the collection $\Gamma = \{(\phi_g, \psi_g): g \in C\}$ where $\phi_g(i) = g(x_i)$ and $\psi_g(j) = g(y_j)$. Clearly, this collection Γ has the above properties (i), (ii) and (iii). We conclude from (5.7) and Theorem 2 that the numbers $\alpha_{i,j} \geqslant 0$ can be found so as to satisfy (7.2) and

$$g(x_i) \leqslant \sum_{j \in J} \alpha_{i,j} g(y_j) \quad \text{for all} \quad g \in C, \, i \in I. \tag{7.4}$$

Since C contains all the constant functions we have $\sum_{j \in J} \alpha_{ij} = 1$. Restricting g to the Q-linear functions one obtains (5.8).

Remark 2. Blackwell's result (6.4) follows in exactly the same way, namely, by letting $\Gamma = \{(\phi_g, \psi_g): g \in C^*\}$. Observe that the continuous linear functions $g: Y \to R$ (which belong to C^*) do separate points in Y. Consequently, property (7.4) (with C replaced by C^*) is in fact equivalent to

$$\sum_{j \in J} \alpha_{i,j} = 1; \quad \sum_{J \in J} \alpha_{i,j} y_j = x_i, \quad \text{for all} \quad i \in I.$$

Remark 3. Theorem 2 is closely related to a very general theory due to Cartier, Fell, Meyer and Strassen. See especially Theorem 53 on p. 246 in: P. A. Meyer, Probability and Potentials, Blaisdell 1966. Our proof below is much more elementary.

Proof of Theorem 2. That (II) implies (I) follows immediately by multiplying (7.3) by $p_i > 0$ and summing over i.

Conversely, suppose (I) holds. Let A denote the compact and convex subset of R^{mn} which consists of all *nonnegative* matrices $(\alpha_{i,j}; i \in I, j \in J)$ which satisfy $\sum_{i \in I} p_i \alpha_{i,j} = q_j$ for all $j \in J$, hence, $\alpha_{i,j} \leqslant q_j / p_i < \infty$.

Let us write Γ as

$$\Gamma = \{(\phi_\sigma, \psi_\sigma) : \sigma \in S_0\},$$

with S_0 as an index set. For each subset S of S_0, consider the compact and convex subset A_S of A defined by

$$A_S = \{(\alpha_{i,j}) \in A : \phi_\sigma(i) \leqslant \sum_{j \in J} \alpha_{i,j} \psi_\sigma(j) \quad \text{for all} \quad i \in I, \quad \text{all} \quad \sigma \in S\}.$$

We have to prove that A_{S_0} is non-empty. Since A_{S_0} is the intersection of all the (compact) sets A_S with S finite, and since further $A_S \cap A_{S'} = A_{S \cup S'}$ (finite intersection property), it suffices to prove that A_S is non-empty for each finite subset S of S_0.

Let S be a *fixed* finite subset of S_0. Our problem is now to show that there exist numbers $\alpha_{i,j} \geqslant 0$ $(i \in I, j \in J)$ satisfying

$$\sum_i p_i \alpha_{i,j} = q_j, \tag{7.5}$$

for $j = 1, \ldots, n$, and further

$$-\sum_j \alpha_{i,j} \psi_\sigma(j) \leqslant -\phi_\sigma(i), \tag{7.6}$$

for $i = 1, \ldots, m$ and all $\sigma \in S$. Here and in the sequel, unspecified summations will be over all $i \in I$, or all $j \in J$ or all $\sigma \in S$.

Since an equality such as (7.5) can be replaced by a pair of inequalities, this problem is precisely a linear programming type of problem of the form $By \leqslant b$, $y \geqslant 0$ where B is a matrix and y and b are vectors. As is well-known, see for instance [5], such a vector $y \geqslant 0$ exists if and only if

$$x \geqslant 0 \quad \text{and} \quad xB \geqslant 0 \quad \text{imply} \quad xb \geqslant 0, \tag{7.7}$$

(where x is a vector; the necessity of (7.7) is obvious).

In the present situation (7.5), (7.6), let $\xi_{\sigma,i}$ denote the component of the vector x which is associated with the inequality (7.6); in particular $\xi_{\sigma,i} \geqslant 0$. Let further η_j denote the component of x which is associated to the equality (7.5); it can be any real number (as can be seen by first representing (7.5) as a pair of inequalities). The condition (7.7) to be verified says in the present case that the inequality

$$\sum_j q_j \eta_j - \sum_i \sum_\sigma \phi_\sigma(i)\, \xi_{\sigma,i} \geqslant 0 \tag{7.8}$$

must be a consequence of the inequalities

$$p_i\eta_j - \sum_\sigma \psi_\sigma(j)\,\xi_{\sigma,i} \geqslant 0, \tag{7.9}$$

(all $i\in I$, $j\in J$), together with the inequality $\xi_{\sigma,i}\geqslant 0$ (all $\sigma\in S$, $i\in I$).

Consider the pair (ϕ', ψ') defined by

$$\begin{aligned}\phi'(i) &= \max_{h\in I} \frac{1}{p_h} \sum_\sigma \phi_\sigma(i)\,\xi_{\sigma,h}\,;\\ \psi'(j) &= \max_{h\in I} \frac{1}{p_h} \sum_\sigma \psi_\sigma(j)\,\xi_{\sigma,h}\,.\end{aligned} \tag{7.10}$$

In view of $\xi_{\sigma,i}\geqslant 0$ and the properties (i), (ii), (iii) of Γ we have that $(\phi', \psi')\in\Gamma$. It follows from the assumption (7.1) that

$$\sum_i p_i\phi'(i) \leqslant \sum_j q_j\psi'(j).$$

Next, by (7.9), we have for each $j\in J$ that

$$\frac{1}{p_i}\sum_\sigma \psi_\sigma(j)\,\xi_{\sigma,i} \leqslant \eta_j \quad \text{for all} \quad i\in I,$$

consequently $\psi'(j)\leqslant\eta_j$ where we used the definition (7.10) of $\psi'(j)$. Moreover, again by (7.10), $\sum_\sigma \phi_\sigma(i)\,\xi_{\sigma,h}\leqslant p_h\phi'(i)$, hence,

$$\sum_i\sum_\sigma \phi_\sigma(i)\,\xi_{\sigma,i} \leqslant \sum_i p_i\phi'(i) \leqslant \sum_j q_j\psi'(j) \leqslant \sum_j q_j\eta_j,$$

yielding the required result (7.8).

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Conservation criteria, canonical transformations, and integral invariants in the field theory of Carathéodory for multiple integral variational problems

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1. Introduction

In any attempt to construct a potentially useful Hamilton-Jacobi theory for multiple integral problems in the calculus of variations one is severely hampered by the fact that the canonical equations consist of systems of first order partial differential equations in which the derivatives appear essentially as divergences. This is in sharp contrast to the single integral theory, in which the canonical equations are represented by a system of relatively simple first order ordinary differential equations. Moreover, there exists a hierarchy of distinct field theories, each of which possesses its own peculiar canonical formalism and therefore gives rise to a distinctive Hamilton-Jacobi theory. This implies that a particular field theory must be selected even before one endeavours to come to grips with the aforementioned difficulty.

Amongst the various field theories the one proposed by Weyl ([9], [10]) is analytically the most accessible, which probably accounts for the fact that it has been used almost exclusively in applications to physical field theories. However, perhaps because of its analytical simplicity, the theory of Weyl is also somewhat inflexible, and therefore it does not seem feasible to construct satisfactory counterparts of many basic concepts which dominate the Hamilton-Jacobi theory of single integral variational problems. On the other hand, although the field theory of Carathéodory ([2]) is considerably more difficult conceptually, it is far more penetrating than that of Weyl, and, in fact, its much richer analytical apparatus offers a somewhat greater degree of flexibility, particularly as far as the canonical formalism is concerned.

It is the object of this paper to present, within the framework of the field theory of Carathéodory, a systematic development of field-theoretic concepts which are of particular relevance to physical applications. We shall begin with a brief resumé of an appropriate canonical formalism for m-fold integral problems in the calculus of variations which had been proposed previously ([6]), in terms of which the geodesic fields of Carathéodory may readily be defined. This suggests the introduction of a

AMS Subject Classifications: Primary 49C10. Secondary 70H15, 70H20.
This research was sponsored in part by NSF GP-32830.

Received October 12, 1973.

set of m 1-forms in the canonical momenta; it is shown in Section 2 that a necessary and sufficient condition in order that a family of extremals constitute a geodesic field is that these 1-forms be exact, which is equivalent to the requirement that three sets of generalized Lagrange brackets vanish. As a result of the fact that the canonical equations involve divergences, it is not possible to deduce from these equations an explicit expression which describes the evolution relative to the m independent variables of an arbitrary function of the canonical momenta (in contrast to the situation which obtains when $m=1$). Therefore, guided by the known results of Noether's theorem, special classes of functions are singled out whose divergences may indeed be specified explicitly by means of the canonical equations, which gives rise directly to criteria for conservation laws. This leads in an inevitable manner to the definition of generalized Poisson brackets. The form of the latter is somewhat surprising: the presence of a numerical factor precludes the skew-symmetry property usually associated with such brackets. This appears to be an unavoidable phenomenon which, however, does not detract from the usefulness of these brackets, as is evident, for instance, from the fact that the generalized Lagrange and Poisson brackets are related precisely as the classical theory would lead one to expect. In Section 4 the relationship between the canonical momenta at neighbouring points on an extremal is briefly examined, which in turn suggests a natural definition of generalized canonical transformations. On closer scrutiny, however, it is found that such transformations are subject to unexpected restrictions: in particular, it is shown that homogeneous canonical transformations are merely extended point transformations unless $m=1$. In the concluding section an m-form is constructed, by means of which it is shown that the divergences of certain Lagrange brackets vanish on the members of an arbitrary family of extremals, which generalizes a classical theorem due to Lagrange for the case $m=1$. Subject to the imposition of an additional condition (which is a natural one within the context of any physical field theory), this phenomenon gives rise to an integral invariant.

It is still an open question as to whether or not the field theory proposed below is susceptible to an appropriate process of quantization.

As usual, the m independent variables are denoted by t^α, while the n dependent functions are represented by x^j. [Here, and in the sequel, Latin and Greek indices range from 1 to n and from 1 to m respectively; the summation convention is operative in respect of both sets of indices.] A set of n equations of the type $x^j=x^j(t^\alpha)$ represents a subspace C_m of the configuration space R_{m+n} of the variables (t^α, x^j); it will henceforth be supposed that the functions $x^j(t^\alpha)$ are of class C^2, and we shall write $\dot{x}^j_\alpha = \partial x^j/\partial t^\alpha$. It is supposed that we are given a Lagrangian $L(t^\alpha, x^j, \dot{x}^j_\alpha)$ which is assumed to be of class C^2 in all its arguments, together with a closed, simply-connected region G in the domain R_m of the variables t^α. The fundamental integral of the underlying problem in the calculus of variations is

$$I(C_m)=\int_G L(t^\alpha, x^j, \dot{x}^j_\alpha)\, d(t), \tag{1.1}$$

where we have used the notation

$$d(t)=dt^1\wedge\cdots\wedge dt^m. \tag{1.2}$$

The Euler-Lagrange equations associated with this problem are expressed in the form

$$E_j(L)=0, \tag{1.3}$$

where the Euler-Lagrange vector is defined as

$$E_j(L)=\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^j_\alpha}\right)-\frac{\partial L}{\partial x^j}, \tag{1.4}$$

in which the operator d/dt^α denotes 'total' differentiation along C_m in the sense that, for any class C^1 function $f(t^\alpha, x^j, \dot{x}^j_\alpha)$,

$$\frac{df}{dt^\alpha}=\frac{\partial f}{\partial t^\alpha}+\frac{\partial f}{\partial x^j}\dot{x}^j_\alpha+\frac{\partial f}{\partial \dot{x}^j_\beta}\frac{\partial \dot{x}^j_\beta}{\partial t^\alpha}.$$

A vector $\mathbf{X}=(X^\alpha, X^j)$ defined at a point P of C_m is said to be transversal to C_m if it satisfies the condition

$$-H^\alpha_\beta X^\beta+\frac{\partial L}{\partial \dot{x}^j_\alpha}X^j=0, \tag{1.5}$$

where H^α_β denotes the Hamiltonian complex:

$$H^\alpha_\beta=-L\delta^\alpha_\beta+\frac{\partial L}{\partial \dot{x}^j_\alpha}\dot{x}^j_\beta. \tag{1.6}$$

It may be shown that ([7]) the set of all vectors transversal to C_m at P constitutes an n-dimensional linear space $T_n(P)$, whose orthogonal complement in R_{m+n} is denoted by $T_m(P)$. The canonical momenta at P are defined to be the m elements of a basis $\boldsymbol{\pi}^{(\alpha)}=(\pi^\alpha_\beta, \pi^\alpha_j)$ of $T_m(P)$ subject to certain restrictions, which are represented in the form

$$\pi^\alpha_j=D^{-1}p^\alpha_\varepsilon\frac{\partial L}{\partial \dot{x}^j_\varepsilon}, \tag{1.7}$$

and

$$\pi^\alpha_\beta=-D^{-1}p^\alpha_\varepsilon H^\varepsilon_\beta, \tag{1.8}$$

where

$$p^\alpha_\beta=\pi_\beta+\pi_j\dot{x}^j_\beta, \tag{1.9}$$

and

$$D = \det(p^\alpha_\beta), \tag{1.10}$$

it being supposed throughout that $D \neq 0$. The conditions (1.7) and (1.8) do not determine the vectors $\pi^{(\alpha)}$ uniquely (unless $m=1$), which entails certain definite advantages. Moreover, it is assumed that the nm relations (1.7) can be solved for the nm derivatives $\dot{x}^j_\alpha$:

$$\dot{x}^j_\alpha = \phi^j_\alpha(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h), \tag{1.11}$$

which is possible if and only if a certain determinant involving L together with its first and second order derivatives with respect to $\dot{x}^h_\varepsilon$ does not vanish. [This assumption implies that parameter-invariant problems are excluded from our considerations (cf. [5], [6] for the substantiation of these remarks).] By means of (1.11) it is then possible to construct a Hamiltonian function by writing

$$H^*(t^\alpha, x^j, \pi^\alpha_\beta, \pi^\alpha_j) = -L(t^\alpha, x^j, \phi^j_\alpha(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h)) + \det\{\pi^\alpha_\beta + \pi^\alpha_j \phi^j_\beta(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h)\}, \tag{1.12}$$

and it is easily verified that ([6]) the relations (1.8) are equivalent to the single condition

$$H^*(t^\alpha, x^j, \pi^\alpha_\beta, \pi^\alpha_j) = 0. \tag{1.13}$$

The cofactors of p^α_β in the determinant D are denoted by P^β_α:

$$p^\alpha_\varepsilon P^\varepsilon_\beta = D\delta^\alpha_\beta, \tag{1.14}$$

and it is readily shown that the derivatives of H^* satisfy the following identities:

$$\frac{\partial H^*}{\partial t^\alpha} = -\frac{\partial L}{\partial t^\alpha}, \quad \frac{\partial H^*}{\partial x^j} = -\frac{\partial L}{\partial x^j}, \quad \frac{\partial H^*}{\partial \pi^\alpha_\beta} = P^\beta_\alpha, \quad \frac{\partial H^*}{\partial \pi^\alpha_j} = P^\beta_\alpha \dot{x}^j_\beta. \tag{1.15}$$

Having disposed of these preliminaries, let us now consider an n-parameter family of subspaces $C_m(u^h)$ defined by the equations

$$x^j = x^j(t^\alpha, u^h), \tag{1.16}$$

in which the u^h denote the m parameters, it being supposed that

$$\det\left(\frac{\partial x^j}{\partial u^h}\right) = 0, \tag{1.17}$$

to that the family (1.16) covers a finite region F of R_{m+n} simply; henceforth our attention will be restricted to F. We shall assume also that the canonical momenta are given as the components of a vector field $\pi^{\alpha}(t^{\varepsilon}, u^{h})$ on F subject to the conditions (1.7) and (1.8), so that we may write

$$\pi^{\alpha}_{\beta}=\pi^{\alpha}_{\beta}(t^{\varepsilon}, u^{h}), \quad \pi^{\alpha}_{j}=\pi^{\alpha}_{j}(t^{\varepsilon}, u^{h}), \tag{1.18}$$

Because of (1.7), (1.14), and the second member of (1.15) we can express the Euler-Lagrange vector (1.4) in the form

$$E_{j}(L)=\frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{j})+\frac{\partial H^{*}}{\partial x^{j}}. \tag{1.19}$$

Also, it follows directly from the definition (1.6) that

$$\frac{dH^{\alpha}_{\beta}}{dt^{\alpha}}=-\frac{\partial L}{\partial t^{\beta}}+E_{j}(L)\,\dot{x}^{j}_{\beta}, \tag{1.20}$$

which, by virtue of (1.8), (1.14), and the first member of (1.15), may be expressed as

$$\frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{\varepsilon})+\frac{\partial H^{*}}{\partial t^{\varepsilon}}=-E_{j}(L)\,\dot{x}^{j}_{\varepsilon}. \tag{1.21}$$

This is combined with (1.19) to yield

$$\left\{\frac{\partial H^{*}}{\partial t^{\varepsilon}}+\frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{\varepsilon})\right\}+\left\{\frac{\partial H^{*}}{\partial x^{j}}+\frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{j})\right\}\dot{x}^{j}_{\varepsilon}=0, \tag{1.22}$$

this being an *identity* which is valid on any subspace $C_{m}(u^{h})$.

If, in particular, the subspaces $C_{m}(u^{h})$ are *extremals* of the integral (1.1), i.e., if they satisfy the Euler-Lagrange equations (1.3), it follows from (1.19) and (1.21) that they satisfy the *canonical equations*

$$\frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{\varepsilon})=-\frac{\partial H^{*}}{\partial t^{\varepsilon}}, \quad \frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{j})=-\frac{\partial H^{*}}{\partial x^{j}}, \tag{1.23}$$

to which the third and fourth members of (1.15) must be adjoined. It is immediately evident from the identity (1.22) that the first member of (1.23) is a direct consequence of the second.

The family (1.16) of subspaces $C_{m}(u^{h})$ is said to constitute a *geodesic field* (in the sense of Carathéodory) if there exist m functions $S^{\alpha}(t^{\beta}, x^{j})$ such that the canonical momenta are the gradients of S^{α}:

$$\pi^{\alpha}_{\beta}=\frac{\partial S^{\alpha}}{\partial t^{\beta}}, \qquad \pi^{\alpha}_{j}=\frac{\partial S^{\alpha}}{\partial x^{j}}, \tag{1.24}$$

where, in view of (1.13), it is necessary to stipulate that these so-called characteristic functions are solutions of the *generalized Hamilton-Jacobi equation*:

$$H^{*}\left(t^{\alpha}, x^{j}, \frac{\partial S^{\alpha}}{\partial t^{\beta}}, \frac{\partial S^{\alpha}}{\partial x^{j}}\right)=0. \tag{1.25}$$

We shall put

$$c^{\alpha}_{\beta}=\frac{\partial S^{\alpha}}{\partial t^{\beta}}+\frac{\partial S^{\alpha}}{\partial x^{j}}\dot{x}^{j}_{\beta}, \tag{1.26}$$

and

$$\Delta=\det(c^{\alpha}_{\beta}), \tag{1.27}$$

while the cofactor of c^{α}_{β} in Δ is denoted by C^{β}_{α}:

$$C^{\alpha}_{\varepsilon}c^{\varepsilon}_{\beta}=\Delta\delta^{\alpha}_{\beta}. \tag{1.28}$$

It is easily seen that ([5], p. 241), because of the structure of (1.26), the divergences of the cofactors vanish identically on any subspace $C_{m}(u^{h})$:

$$\frac{dC^{\alpha}_{\beta}}{dt^{\alpha}}=0, \tag{1.29}$$

by means of which it may be shown that the members $C_{m}(u^{h})$ of any geodesic field are in fact extremal subspaces. In view of (1.24), (1.9), (1.10), (1.13), (1.26), and (1.27) we have

$$c^{\alpha}_{\beta}=p^{\alpha}_{\beta}, \qquad C^{\alpha}_{\beta}=P^{\alpha}_{\beta}, \qquad D=\Delta=L \tag{1.30}$$

for a geodesic field. Thus (1.29) allows us to write the canonical equations (1.23) associated with these fields in the form

$$P^{\beta}_{\alpha}\frac{d\pi^{\alpha}_{\varepsilon}}{dt^{\beta}}=-\frac{\partial H^{*}}{\partial t^{\varepsilon}}, \qquad P^{\beta}_{\alpha}\frac{d\pi^{\alpha}_{j}}{dt^{\beta}}=-\frac{\partial H^{*}}{\partial x^{j}}. \tag{1.31}$$

It should be stressed, however, that an arbitrary field (1.16) of extremals does not necessarily constitute a geodesic field. We shall return to this point presently.

We shall conclude this section with a few remarks of a general nature, which will serve to shed some light on the geometrical significance of the canonical momenta. The tangent plane $t_m(P)$ at any point P of a class C^2 subspace $C_m: x^j = x^j(t^\alpha)$ of R_{m+n} is spanned by the m vectors

$$\mathbf{M}_{(\alpha)} = (M^\beta_\alpha, M^j_\alpha) = (\delta^\beta_\alpha, \dot{x}^j_\alpha), \tag{1.32}$$

which we shall call the *natural basis* of $t_m(P)$. Sometimes one also requires the so-called *canonical* basis of $\mathbf{P}_{(\alpha)} = P^\beta_\alpha \mathbf{M}_{(\beta)}$ of $t_m(P)$, whose components may be represented in the form

$$\mathbf{P}_{(\alpha)} = (P^\beta_\alpha, P^\beta_\alpha \dot{x}^j_\beta) = \left(\frac{\partial H^*}{\partial \pi^\alpha_\beta}, \frac{\partial H^*}{\partial \pi^\alpha_j} \right), \tag{1.33}$$

in which we have used (1.32) together with the third and fourth members of (1.15). For many purposes it is also necessary to introduce the quantities defined by

$$P^\alpha_j = -h^{-1} h^\alpha_\beta \frac{\partial L}{\partial \dot{x}^j_\beta}, \tag{1.34}$$

where h^α_β denotes the cofactor of H^β_α in the determinant $h = \det(H^\alpha_\beta)$, which is assumed to be non-vanishing. It is readily verified that the n vectors

$$\mathbf{X}_{(h)} = (X^\alpha_h, \delta^j_h) = (-P^\alpha_h, \delta^j_h), \tag{1.35}$$

defined at the point P of C_m, are transversal to C_m in the sense of (1.5); moreover, it may be shown that the $(n+m)$ vectors $\mathbf{M}_{(\alpha)}$, $\mathbf{X}_{(h)}$ constitute a linearly independent system ([7]). Hence we can regard the vectors (1.35) as a basis of the transversal plane $T_n(P)$ of C_m at P. Moreover, by construction, the canonical momenta $\boldsymbol{\pi}^{(\alpha)}$ at P represent a basis of the orthogonal complement $T_m(P)$ in R_{m+n} of $T_n(P)$, so that

$$\boldsymbol{\pi}^{(\alpha)} \cdot \mathbf{X}_{(h)} = 0, \tag{1.36}$$

which is obviously consistent with (1.7) and (1.8) by virtue of (1.34) and (1.35). We shall call $\boldsymbol{\pi}^{(\alpha)}$ the *canonical basis* of $T_m(P)$, at the same time noting that, because of (1.33), (19), and (1.14),

$$\mathbf{P}_{(\beta)} \cdot \boldsymbol{\pi}^{(\alpha)} = P^\varepsilon_\beta \pi^\alpha_\varepsilon + P^\varepsilon_\beta \dot{x}^j_\varepsilon \pi^\alpha_j = P^\varepsilon_\beta p^\alpha_\varepsilon = D\delta^\alpha_\beta, \tag{1.37}$$

which indicates that, apart from the factor D^{-1}, the canonical basis of $T_m(P)$ is *conjugate* to the canonical basis of $t_m(P)$. One would therefore be inclined to seek another basis of $T_m(P)$ which is conjugate to the natural basis $\mathbf{M}_{(\alpha)}$ of $t_m(P)$. In fact, this basis is given by

$$\mathbf{Q}^{(\beta)} = P^\beta_\varepsilon \boldsymbol{\pi}^{(\varepsilon)} = (Q^\beta_\alpha, Q^\beta_j) = (P^\beta_\varepsilon \pi^\varepsilon_\alpha, P^\beta_\varepsilon \pi^\varepsilon_j), \tag{1.38}$$

since

$$\mathbf{M}_{(\alpha)} \cdot \mathbf{Q}^{(\beta)} = \delta^\varepsilon_\alpha Q^\beta_\varepsilon + \dot{x}^j_\alpha Q^\beta_j = P^\beta_\varepsilon \pi^\varepsilon_\alpha + P^\beta_\varepsilon \pi^\varepsilon_j \dot{x}^j_\alpha = P^\beta_\varepsilon p^\varepsilon_\alpha = D\delta^\beta_\alpha, \tag{1.39}$$

as required. Because of this property we shall call the m vectors $\mathbf{Q}^{(\beta)}$ the *natural basis* of $T_m(P)$.

2. The m fundamental 1-forms

It was remarked in the previous section that an arbitrary family of extremal subspaces does not necessarily constitute a geodesic field. In order to explore this situation a little more closely, let us consider once more a family (1.16) of subspaces $C_m(u^h)$, together with the canonical momenta as represented by (1.18). We now construct the following m 1-forms on this family:

$$\omega^\alpha = \pi^\alpha_\beta dt^\beta + \pi^\alpha_j dx^j, \tag{2.1}$$

which we shall call the *fundamental* 1-*forms* since our subsequent analysis will be based almost entirely on them. Clearly

$$\left.\begin{aligned} d\omega^\alpha &= d\pi^\alpha_\beta \wedge dt^\beta + d\pi^\alpha_j \wedge dx^j \\ &= \left(\frac{\partial \pi^\alpha_\beta}{\partial t^\varepsilon} dt^\varepsilon + \frac{\partial \pi^\alpha_\beta}{\partial u^h} du^h\right) \wedge dt^\beta + \left(\frac{\partial \pi^\alpha_j}{\partial t^\varepsilon} dt^\varepsilon + \frac{\partial \pi^\alpha_j}{\partial u^h} du^h\right) \wedge \left(\frac{\partial x^j}{\partial t^\gamma} dt^\gamma + \frac{\partial x^j}{\partial u^k} du^k\right) \\ &= \left(\frac{\partial \pi^\alpha_\beta}{\partial t^\varepsilon} + \frac{\partial \pi^\alpha_j}{\partial t^\varepsilon}\frac{\partial x^j}{\partial t^\beta}\right) dt^\varepsilon \wedge dt^\beta + \left(\frac{\partial \pi^\alpha_\beta}{\partial u^h} - \frac{\partial \pi^\alpha_j}{\partial t^\beta}\frac{\partial x^j}{\partial u^h} + \frac{\partial \pi^\alpha_j}{\partial u^h}\frac{\partial x^j}{\partial t^\beta}\right) du^h \wedge dt^\beta \\ &\quad + \frac{\partial \pi^\alpha_j}{\partial u^h}\frac{\partial x^j}{\partial u^h} du^h \wedge du^k. \end{aligned}\right\} \tag{2.2}$$

This expression suggests that we introduce the following sets of *generalized Lagrange brackets* for any pair of independent parameters v and w:

$$[[v, w]]^\alpha = \frac{\partial t^\lambda}{\partial v}\frac{\partial \pi^\alpha_\lambda}{\partial w} - \frac{\partial t^\lambda}{\partial w}\frac{\partial \pi^\alpha_\lambda}{\partial v}, \quad [v, w]^\alpha = \frac{\partial x^j}{\partial v}\frac{\partial \pi^\alpha_j}{\partial w} - \frac{\partial x^j}{\partial w}\frac{\partial \pi^\alpha_j}{\partial v}, \tag{2.3}$$

which usually appear in the combination

$$\{v, w\}^\alpha = [[v, w]]^\alpha + [v, w]^\alpha. \tag{2.4}$$

Indeed, since

$$[[t^\beta, t^\varepsilon]]^\alpha = \delta^\lambda_\beta \frac{\partial \pi^\alpha_\lambda}{\partial t^\varepsilon} - \delta^\lambda_\varepsilon \frac{\partial \pi^\alpha_\lambda}{\partial t^\beta} = \frac{\partial \pi^\alpha_\beta}{\partial t^\varepsilon} - \frac{\partial \pi^\alpha_\varepsilon}{\partial t^\beta},$$

together with similar identities involving derivatives with respect to u^h, it follows that the 2-forms (2.2) may be expressed as

$$d\omega^\alpha = -\tfrac{1}{2}\{t^\beta, t^\varepsilon\}^\alpha \, dt^\beta \wedge dt^\varepsilon + \{t^\beta, u^h\}^\alpha \, dt^\beta \wedge du^h - \tfrac{1}{2}\{u^h, u^k\}^\alpha \, du^h \wedge du^k. \tag{2.5}$$

Thus, in order that $d\omega^\alpha = 0$, it is necessary and sufficient that

$$\{t^\beta, t^\varepsilon\}^\alpha = 0, \qquad \{t^\beta, u^h\}^\alpha = 0, \qquad \{u^h, u^k\}^\alpha = 0, \tag{2.6}$$

which, when written out in full, are equivalent to

$$\frac{\partial \pi^\alpha_\beta}{\partial t^\varepsilon} - \frac{\partial \pi^\alpha_\varepsilon}{\partial t^\beta} + \frac{\partial x^j}{\partial t^\beta}\frac{\partial \pi^\alpha_j}{\partial t^\varepsilon} - \frac{\partial x^j}{\partial t^\varepsilon}\frac{\partial \pi^\alpha_j}{\partial t^\beta} = 0, \tag{2.7}$$

$$\frac{\partial \pi^\alpha_\beta}{\partial u^h} + \frac{\partial x^j}{\partial t^\beta}\frac{\partial \pi^\alpha_j}{\partial u^h} - \frac{\partial x^j}{\partial u^h}\frac{\partial \pi^\alpha_j}{\partial t^\beta} \equiv \frac{\partial \pi^\alpha_\beta}{\partial u^h} + [t^\beta, u^h]^\alpha = 0, \tag{2.8}$$

$$\frac{\partial x^j}{\partial u^h}\frac{\partial \pi^\alpha_j}{\partial u^k} - \frac{\partial x^j}{\partial u^k}\frac{\partial \pi^\alpha_j}{\partial u^h} \equiv [u^h, u^k]^\alpha = 0. \tag{2.9}$$

In passing we note that

$$\frac{\partial}{\partial u^k}[t^\beta, u^h]^\alpha = \frac{\partial}{\partial t^\beta}[u^k, u^h]^\alpha + \frac{\partial}{\partial u^h}[t^\beta, u^k]^\alpha \tag{2.10}$$

identically, so that, whenever (2.9) is satisfied, we have

$$\frac{\partial}{\partial u^k}[t^\beta, u^h]^\alpha = \frac{\partial}{\partial u^h}[t^\beta, u^k]^\alpha, \tag{2.11}$$

which clearly indicates that (2.8) does not entail additional integrability conditions.

From the converse of Poincaré's Lemma it follows that when the conditions (2.6) are satisfied (i.e., $d\omega^\alpha = 0$), there exists, at least locally, a set of m 0-forms $\sigma^\alpha(t^\beta, u^h)$ such that

$$\omega^\alpha = d\sigma^\alpha = \frac{\partial \sigma^\alpha}{\partial t^\beta} dt^\beta + \frac{\partial \sigma^\alpha}{\partial u^h} du^h, \tag{2.12}$$

which, when compared with (2.1), yields

$$\pi^\alpha_\beta + \pi^\alpha_j \frac{\partial x^j}{\partial t^\beta} = \frac{\partial \sigma^\alpha}{\partial t^\beta}, \qquad \pi^\alpha_j \frac{\partial x^j}{\partial u^h} = \frac{\partial \sigma^\alpha}{\partial u^h}. \tag{2.13}$$

Moreover, since it is assumed that our family (1.16) satisfies the condition (1.17), we may solve (1.16) for $u^h = U^h(t^\beta, x^j)$, which, when substituted in (1.16), gives rise to the identities

$$x^j = x^j(t^\beta, U^h(t^\beta, x^k)) \tag{2.14}$$

in the independent variables t^β, x^k, so that

$$0 = \frac{\partial x^j}{\partial t^\beta} + \frac{\partial x^j}{\partial u^h}\frac{\partial U^h}{\partial t^\beta}, \qquad \delta^k_j = \frac{\partial x^k}{\partial u^h}\frac{\partial U^h}{\partial x^j}. \tag{2.15}$$

We may now construct the m functions $S^\alpha(t^\beta, x^j) = \sigma^\alpha(t^\beta, U^h((t^\beta, x^j))$, for which

$$\frac{\partial S^\alpha}{\partial t^\beta} = \frac{\partial \sigma^\alpha}{\partial t^\beta} + \frac{\partial \sigma^\alpha}{\partial u^h}\frac{\partial U^h}{\partial t^\beta}, \qquad \frac{\partial S^\alpha}{\partial x^j} = \frac{\partial \sigma^\alpha}{\partial u^h}\frac{\partial U^h}{\partial x^j}. \tag{2.16}$$

On eliminating the derivatives of σ^α with the aid of (2.13), we obtain

$$\frac{\partial S^\alpha}{\partial t^\beta} = \pi^\alpha_\beta + \pi^\alpha_j \left(\frac{\partial x^j}{\partial t^\beta} + \frac{\partial x^j}{\partial u^h}\frac{\partial U^h}{\partial t^\beta}\right), \qquad \frac{\partial S^\alpha}{\partial x^j} = \pi^\alpha_k \frac{\partial x^k}{\partial u^h}\frac{\partial U^h}{\partial x^j},$$

and because of (2.15) these relations reduce to (1.24).

The conditions (2.6) therefore ensure the existence of m functions $S^\alpha(t^\beta, x^j)$ such that the canonical momenta can be represented as gradients in the form (1.24). However, it does not follow that any family (1.16) of subspaces $C_m(u^h)$ satisfying (2.6) constitutes a geodesic field. For, in order that the canonical momenta as given by (1.24) be such as to ensure consistency with (1.13), it is necessary that the functions $S^\alpha(t^\beta, x^j)$, which are obtained by quadratures, satisfy the Hamilton-Jacobi equation (1.25). Since this is not necessarily implied by (2.6), we now impose an additional condition, namely that the members $C_m(u^h)$ of the family (1.16) be extremal subspaces.

Since the validity of (1.24) clearly implies that of (1.30), we infer that $dP^\beta_\alpha/dt^\alpha = 0$ under the circumstances considered here, and thus our additional requirement entails that the subspaces $C_m(u^h)$ satisfy the canonical equations in the form (1.31), or

$$P^\beta_\alpha \frac{\partial \pi^\alpha_\varepsilon}{\partial t^\beta} = -\frac{\partial H^*}{\partial t^\varepsilon}, \qquad P^\beta_\alpha \frac{\partial \pi^\alpha_j}{\partial t^\beta} = -\frac{\partial H^*}{\partial x^j}, \tag{2.17}$$

where it is to be understood that in the present context the operator $\partial/\partial t^{\beta}$, as applied to $\pi^{\varepsilon}_{\alpha}$, π^{α}_{j}, and x^{j}, refers to a given $C_m(u^h)$, i.e., to fixed values of u^h. Now let us multiply (2.7) by P^{β}_{α}, after which we apply (2.17) together with the third and fourth members of (1.15). This yields

$$\frac{\partial H^*}{\partial \pi^{\alpha}_{\beta}}\frac{\partial \pi^{\alpha}_{\beta}}{\partial t^{\varepsilon}}+\frac{\partial H^*}{\partial t^{\varepsilon}}+\frac{\partial \pi^{\alpha}_{j}}{\partial t^{\varepsilon}}\frac{\partial H^*}{\partial \pi^{\alpha}_{j}}+\frac{\partial H^*}{\partial x^{j}}\frac{\partial x^{j}}{\partial t^{\varepsilon}}=0,$$

or

$$\frac{\partial}{\partial t^{\varepsilon}}\{H^*(t^{\alpha}, x^{j}, \pi^{\alpha}_{\beta}, \pi^{\alpha}_{j})\}=0.$$

Similarly, multiplication of (2.8) by P^{β}_{α} gives

$$\frac{\partial}{\partial u^{h}}\{H^*(t^{\varepsilon}, x^{j}, \pi^{\alpha}_{\beta}, \pi^{\alpha}_{j})\}=0,$$

from which it is inferred that $H^*(t^{\varepsilon}, x^{j}, \pi^{\alpha}_{\beta}, \pi^{\alpha}_{j})$ is constant. Without loss of generality the latter may be taken to be zero, since any additive constant may be absorbed in the Lagrangian L and hence also in H^*. Thus our additional stipulation implies that the condition (1.13) and hence also the Hamilton-Jacobi Equation (1.25) is satisfied. The results of our analysis may thus be summarized in the following

THEOREM: *In order that a family* (1.16) *of extremal subspaces constitute a geodesic field, it is necessary and sufficient that the m fundamental* 1*-forms* ω^{α} *as defined by* (2.1) *be exact, or equivalently, that the conditions* (2.6) *be satisfied.*

By means of the functions $S^{\alpha}(t^{\beta}, x^{j})$ associated with a given geodesic field, an m-parameter family of n-dimensional subspaces is defined by the equations

$$S^{\alpha}(t^{\beta}, x^{j})=\Sigma^{\alpha}, \tag{2.18}$$

in which the Σ^{α} denote the parameters. This family, together with the extremals of the geodesic field, constitutes a *complete figure* in the terminology of Carathéodory. The above theorem possesses a converse in the following sense: *any set of solutions* $S^{\alpha}(t^{\beta}, x^{h})$ *of the Hamilton-Jacobi equation* (1.25) *gives rise to a complete figure* ([8]).

For any displacement (dt^{β}, dx^{j}) tangential to the manifolds (2.18) we have, by (1.24) and (2.1),

$$0=\frac{\partial S^{\alpha}}{\partial t^{\beta}}\,dt^{\beta}+\frac{\partial S^{\alpha}}{\partial x^{j}}\,dx^{j}=\pi^{\alpha}_{\beta}dt^{\beta}+\pi^{\alpha}_{j}dx^{j}=\omega^{\alpha}. \tag{2.19}$$

However, we now recall that at any point P of an extremal subspace the canonical momenta $\pi^{(\alpha)}$ constitute a basis of the orthogonal complement $T_m(P)$ of the transversal plane $T_n(P)$ at P. It therefore follows from (2.19) that the tangent plane at P of the member of the family (2.18) which passes through P must coincide with $T_n(P)$: hence the transversal planes represent envelopes of the manifolds (2.18). Indeed, it may be shown that the integrability conditions of the system of m total differential equations $\omega^\alpha=0$ are satisfied under the hypotheses of the theorem above.

3. Conservation laws and Poisson brackets

In any prospective application of the above analysis to the theory of physical fields one would be inclined to investigate specific properties of the canonical equations (1.23) which could conceivably lead to conclusions pertaining to conservation laws, i.e., to integrals of (1.23) or quantities whose divergences vanish as a result of these equations. In contrast to the single integral case, nothing can be said about derivatives of an arbitrary function $F(t^\alpha, x^j, \pi^\alpha_\beta, \pi^\alpha_j)$ of the canonical variables: this is due to the fact that the canonical equations do not yield derivatives of π^α_β and π^α_j but merely divergence-type combinations thereof. It is therefore necessary to single out certain classes of functions of π^α_β, π^α_j whose structure is such that definitive conclusions concerning their evolution as functions of t^α may be reached. A review of the known theory suggests that non-trivial functions of the required type must obviously occur in the famous theorem of Noether ([4]), which we shall now enunciate as follows:

Let the fundamental integral (1.1) be invariant under an infinitesimal r-parameter group of transformations of the type

$$\bar{t}^\alpha=t^\alpha+\zeta^\alpha_s(t^\varepsilon, x^h)\, w^s, \qquad \bar{x}^j=x^j+\zeta^j_s(t^\varepsilon, x^h)\, w^s, \tag{3.1}$$

in which w^s denotes the r parameters and ξ^α_s, ζ^j_s are given class C^1 functions of the variables t^ε, x^h. [Here $s=1, \ldots, r$, and the summation convention applies to this index also.] Then

$$\frac{d\chi^\beta_s}{dt^\beta}=0 \tag{3.2}$$

on any extremal C_m, where

$$\chi^\beta_s=-H^\beta_\varepsilon\zeta^\varepsilon_s+\frac{\partial L}{\partial \dot{x}^j_\beta}\,\zeta^j_s. \tag{3.3}$$

Because (3.2) entails the vanishing of a divergence, Noether's theorem is generally interpreted in terms of conservation laws, and accordingly the quantities (3.3) exemplify the types of functions that we are seeking for our present purposes. By means of (1.7), (1.8), (1.14), and (1.38) we can express (2.2) in the form

$$\chi_s^\beta = Q_\varepsilon^\beta \zeta_s^\varepsilon + Q_j^\beta \zeta_s^j, \tag{3.4}$$

which is a representation of the 'conserved' quantities (3.3) in terms of the elements of the natural basis (1.38) of $T_m(P)$ at each point P of the extremal C_m.

This conclusion suggests the following general approach. Let $\mathbf{F}=\mathbf{F}(t^\varepsilon, x^h)$ denote an arbitrary differentiable vector field defined on some subspace C_m of R_{m+n}. At each point P of C_m we can decompose $\mathbf{F}$ by means of the bases (1.32) and (1.35) as follows:

$$\mathbf{F} = D^{-1} F^\alpha \mathbf{M}_{(\alpha)} + F^h \mathbf{X}_{(h)}, \tag{3.5}$$

where $D^{-1}F^\alpha$ represents the components of the projections of $\mathbf{F}$ onto $t_m(P)$. But from (1.38) and (1.36) we have

$$\mathbf{X}_{(h)} \cdot \mathbf{Q}^\beta = 0, \tag{3.6}$$

so that (3.5) and (1.39) yield

$$F^\beta = \mathbf{F} \cdot \mathbf{Q}^{(\beta)}, \tag{3.7}$$

which is of the form displayed by (3.4). From (1.38) it follows that we can express F^β as

$$F^\beta = P_\alpha^\beta (\boldsymbol{\pi}^\alpha \cdot \mathbf{F}) = P_\alpha^\beta \theta^\alpha, \tag{3.8}$$

where θ^α represents expressions of the type

$$\theta^\alpha = \pi_\beta^\alpha f^\beta + \pi_j^\alpha f^j, \tag{3.9}$$

and we shall now endeavor to evaluate the divergence of F^β on C_m.

From (3.8) and (3.9) we have

$$\frac{dF^\beta}{dt^\beta} = \frac{dP_\alpha^\beta}{dt^\beta} (\pi_\varepsilon^\alpha f^\varepsilon + \pi_j^\alpha f^j) + P_\alpha^\beta \left(\frac{\partial \theta^\alpha}{\partial t^\beta} + \frac{\partial \theta^\alpha}{\partial x^j} \dot{x}_\beta^j + \frac{\partial \theta^\alpha}{\partial \pi_\lambda^\sigma} \frac{d\pi_\lambda^\sigma}{dt^\beta} + \frac{\partial \theta^\alpha}{\partial \pi_j^\varepsilon} \frac{d\pi_j^\varepsilon}{dt^\beta} \right). \tag{3.10}$$

But (3.9) gives

$$\frac{\partial \theta^\alpha}{\partial \pi_\lambda^\sigma} = \delta_\sigma^\alpha f^\lambda, \qquad \frac{\partial \theta^\alpha}{\partial \pi_j^\varepsilon} = \delta_\varepsilon^\alpha f^j, \tag{3.11}$$

so that, with the aid of the third and fourth members of (1.15), we may write (3.10) in the form

$$\frac{dF^\beta}{dt^\beta} = \frac{d}{dt^\beta} (P_\alpha^\beta \pi_\varepsilon^\alpha) f^\varepsilon + \frac{d}{dt^\beta} (P_\alpha^\beta \pi_j^\alpha) f^j + \frac{\partial H^*}{\partial \pi_\beta^\alpha} \frac{\partial \theta^\alpha}{\partial t^\beta} + \frac{\partial H^*}{\partial \pi_j^\alpha} \frac{\partial \theta^\alpha}{\partial x^j}. \tag{3.12}$$

This relation holds on *any* subspace C_m; however, on an *extremal* subspace we may apply the canonical equations (1.23), in which case (3.12) becomes

$$\frac{dF^\beta}{dt^\beta}=-\frac{\partial H^*}{\partial t^\beta}f^\beta-\frac{\partial H^*}{\partial x^j}f^j+\frac{\partial H^*}{\partial \pi^\alpha_\beta}\frac{\partial \theta^\alpha}{\partial t^\beta}+\frac{\partial H^*}{\partial \pi^\alpha_j}\frac{\partial \theta^\alpha}{\partial x^j}. \tag{3.13}$$

However, from (3.11) we obtain by contraction:

$$mf^\beta=\frac{\partial \theta^\alpha}{\partial \pi^\alpha_\beta}, \qquad mf^j=\frac{\partial \theta^\alpha}{\partial \pi^\alpha_j}, \tag{3.14}$$

so that (3.13) reduces to

$$\frac{dF^\beta}{dt^\beta}=\left(\frac{\partial \theta^\alpha}{\partial t^\beta}\frac{\partial H^*}{\partial \pi^\alpha_\beta}-\frac{1}{m}\frac{\partial \theta^\alpha}{\partial \pi^\alpha_\beta}\frac{\partial H^*}{\partial t^\beta}\right)+\left(\frac{\partial \theta^\alpha}{\partial x^j}\frac{\partial H^*}{\partial \pi^\alpha_j}-\frac{1}{m}\frac{\partial \theta^\alpha}{\partial \pi^\alpha_j}\frac{\partial H^*}{\partial x^j}\right). \tag{3.15}$$

The expressions in brackets on the right-hand side clearly suggest the following definitions of two types of *generalized Poisson brackets* of any pair of differentiable functions $F(t^\beta, x^h, \pi^\alpha_\beta, \pi^\alpha_j)$, $G(t^\beta, x^j, \pi^\alpha_\beta, \pi^j_\alpha)$:

$$((F, G))^\alpha=\frac{\partial F}{\partial t^\beta}\frac{\partial G}{\partial \pi^\alpha_\beta}-\frac{1}{m}\frac{\partial F}{\partial \pi^\alpha_\beta}\frac{\partial G}{\partial t^\beta}, \tag{3.16}$$

$$(F, G)_\alpha=\frac{\partial F}{\partial x^j}\frac{\partial G}{\partial \pi^\alpha_j}-\frac{1}{m}\frac{\partial F}{\partial \pi^\alpha_j}\frac{\partial G}{\partial x^j}, \tag{3.17}$$

together with

$$\{F, G\}_\alpha=((F, G))_\alpha+(F, G)_\alpha. \tag{3.18}$$

In terms of this notation (3.15) becomes

$$\frac{dF^\beta}{dt^\beta}=\{\theta^\alpha, H^*\}_\alpha, \tag{3.19}$$

which is the required expression for the divergence of F^β on the extremal C_m. This is the generalization of the well known classical formula of the single integral theory according to which the derivative of any differentiable function F of the canonical variables with respect to the single independent variable t along an extremal curve is given by

$$\frac{dF}{dt}=\frac{\partial F}{\partial t}+(F, H), \tag{3.20}$$

where H denotes the classical Hamiltonian.

From (3.19) we infer that a set of quantities F^{β}, which may be represented in the form (3.7), is conserved in the sense that its divergence is zero on an extremal, whenever the sum $\{\theta^{\alpha}, H^{*}\}_{\alpha}$ of the corresponding generalized Poisson bracket vanishes identically.

The above definitions of the generalized Poisson brackets appear to represent natural extensions of the classical concept of a Poisson bracket, except, perhaps, for the presence of the factor m^{-1} in the second term on the right-hand sides of (3.16) and (3.17), in consequence of which the skew-symmetry property usually associated with a Poisson bracket is destroyed. However, this phenomenon cannot be avoided: it will be seen below that it occurs inevitably whenever the generalized Poisson bracket appears.

From the definitions (3.16)–(3.18) it is immediately evident that

$$\{\pi^{\alpha}_{\varepsilon}, H^{*}\}_{\alpha} = -\frac{1}{m}\frac{\partial \pi^{\alpha}_{\varepsilon}}{\partial \pi^{\alpha}_{\beta}}\frac{\partial H^{*}}{\partial t^{\beta}} = -\delta^{\beta}_{\varepsilon}\frac{\partial H^{*}}{\partial t^{\beta}} = -\frac{\partial H^{*}}{\partial t^{\varepsilon}}, \tag{3.21}$$

together with

$$\{\pi^{\alpha}_{j}, H^{*}\}_{\alpha} = -\frac{1}{m}\frac{\partial \pi^{\alpha}_{j}}{\partial \pi^{\alpha}_{h}}\frac{\partial H^{*}}{\partial x^{h}} = -\delta^{h}_{j}\frac{\partial H^{*}}{\partial x^{h}} = -\frac{\partial H^{*}}{\partial x^{j}}. \tag{3.22}$$

Now let us identify f^{β} in (3.9) with $\pi^{\beta}_{\varepsilon}$ (for some fixed value of ε), at the same time putting $f^{j}=0$. The resulting functions F^{β} in (3.8) then become $P^{\beta}_{\alpha}\pi^{\alpha}_{\varepsilon}$, and the general formula (3.19) yields

$$\frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{\varepsilon}) = \{\pi^{\alpha}_{\varepsilon}, H^{*}\}_{\alpha}. \tag{3.23}$$

Similarly, on identifying f^{j} in (3.9) with δ^{j}_{h} (for some fixed value of h), and putting $f^{\beta}=0$, we obtain

$$\frac{d}{dt^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{j}) = \{\pi^{\alpha}_{j}, H^{*}\}_{\alpha}. \tag{3.24}$$

These are simply the canonical equations (1.23) *in Poisson bracket form*, for, in view of (3.21) and (3.22) they reduce directly to (1.23).

Let us now turn to the relationships which exist between the Poisson brackets and the Lagrange brackets as defined by (2.3) and (2.4). To this end we shall suppose that the $N=(m+1)(m+n)$ canonical variables $(t^{\alpha}, x^{j}, \pi^{\alpha}_{\beta}, \pi^{\alpha}_{j})$ are given as differentiable functions of N parameters u^{A}:

$$t^{\alpha}=t^{\alpha}(u^{A}), \quad x^{j}=x^{j}(u^{A}), \quad \pi^{\alpha}_{\beta}=\pi^{\alpha}_{\beta}(u^{A}), \quad \pi^{\alpha}_{j}=\pi^{\alpha}_{j}(u^{A}). \tag{3.25}$$

[Here upper case Latin indices $A, B, \ldots$ range from 1 to N, and the summation convention is operative in respect of these indices.] It is assumed that the functional determinant of the system (3.25) is non-vanishing, so that each u^A can be expressed in terms of the canonical variables: $u^A = u^A(t^\alpha, x^j, \pi^\alpha_\beta, \pi^\alpha_j)$. Since the canonical variables are to be regarded as entirely independent in this context, this gives rise to identities such as

$$\frac{\partial u^B}{\partial t^\alpha}\frac{\partial t^\alpha}{\partial u^C} + \frac{\partial u^B}{\partial x^j}\frac{\partial x^j}{\partial u^C} + \frac{\partial u^B}{\partial \pi^\alpha_\beta}\frac{\partial \pi^\alpha_\beta}{\partial u^C} + \frac{\partial u^B}{\partial \pi^\alpha_j}\frac{\partial \pi^\alpha_j}{\partial u^C} = \delta^B_C, \tag{3.26}$$

and

$$\frac{\partial u^A}{\partial x^j}\frac{\partial x^h}{\partial u^A} = \delta^h_j, \quad \frac{\partial u^A}{\partial x^j}\frac{\partial \pi^\beta_h}{\partial u^A} = 0, \quad \frac{\partial u^A}{\partial \pi^\alpha_j}\frac{\partial x^h}{\partial u^A} = 0, \quad \frac{\partial u^A}{\partial \pi^\alpha_j}\frac{\partial \pi^\beta_h}{\partial u^A} = \delta^\beta_\alpha \delta^j_h, \quad \text{etc.} \tag{3.27}$$

We shall now evaluate the products of various types of Lagrange and Poisson brackets. For instance, according to (3.17) and (2.3) a direct expansion gives

$$\begin{aligned}(u^A, u^B)_\alpha [u^A, u^C]^\beta &= \frac{\partial u^A}{\partial x^j}\frac{\partial x^h}{\partial u^A}\frac{\partial u^B}{\partial \pi^\alpha_j}\frac{\partial \pi^\beta_h}{\partial u^C} - \frac{\partial u^A}{\partial x^j}\frac{\partial \pi^\beta_h}{\partial u^A}\frac{\partial u^B}{\partial \pi^\alpha_j}\frac{\partial x^h}{\partial u^C} \\ &\quad - \frac{1}{m}\frac{\partial u^A}{\partial \pi^\alpha_j}\frac{\partial x^h}{\partial u^A}\frac{\partial u^B}{\partial x^j}\frac{\partial \pi^\beta_h}{\partial u^C} + \frac{1}{m}\frac{\partial u^A}{\partial \pi^\alpha_j}\frac{\partial \pi^\beta_h}{\partial u^A}\frac{\partial u^B}{\partial x^j}\frac{\partial x^h}{\partial u^C} \\ &= \frac{\partial u^B}{\partial \pi^\alpha_j}\frac{\partial \pi^\beta_j}{\partial u^C} + \frac{1}{m}\delta^\beta_\alpha \frac{\partial u^B}{\partial x^j}\frac{\partial x^j}{\partial u^C},\end{aligned} \tag{3.28}$$

where, in the last step, we have applied (3.27). Similarly it is found that

$$\left.\begin{aligned}(u^A, u^B)_\alpha [[u^A, u^C]]^\beta = 0, \quad ((u^A, u^B))_\alpha [[u^A, u^C]]^\beta = 0, \\ ((u^A, u^B))_\alpha [[u^A, u^C]]^\beta = \frac{\partial u^B}{\partial \pi^\alpha_\lambda}\frac{\partial \pi^\beta_\lambda}{\partial u^C} + \frac{1}{m}\delta^\beta_\alpha \frac{\partial u^B}{\partial t^\gamma}\frac{\partial t^\gamma}{\partial u^C}.\end{aligned}\right\} \tag{3.29}$$

The identities (3.28) and (3.29) may be combined in accordance with (2.4) and (3.18) to yield

$$\{u^A, u^B\}_\alpha \{u^A, u^C\}^\beta = \frac{\partial u^B}{\partial \pi^\alpha_j}\frac{\partial \pi^\beta_j}{\partial u^C} + \frac{\partial u^B}{\partial \pi^\alpha_\lambda}\frac{\partial \pi^\beta_\lambda}{\partial u^C} + \frac{1}{m}\delta^\beta_\alpha\left(\frac{\partial u^B}{\partial x^j}\frac{\partial x^j}{\partial u^C} + \frac{\partial u^B}{\partial t^\lambda}\frac{\partial t^\lambda}{\partial u^C}\right). \tag{3.30}$$

In particular, if we now contract over α and β, we may apply (3.26), and thus obtain

$$\{u^A, u^B\}_\alpha \{u^A, u^C\}^\alpha = \delta^B_C, \tag{3.31}$$

which is the generalization of the well known classical identity which displays the relationship between Lagrange and Poisson brackets.

The formula (3.31) contains several useful identities as special cases. For instance, the following Lagrange bracket identities result directly from (2.3) and (2.4):

$$\left.\begin{array}{llll}\{t^\varepsilon, t^\gamma\}^\alpha=0, & \{t^\varepsilon, x^h\}^\alpha=0, & \{t^\varepsilon, \pi^\beta_\gamma\}^\alpha=\delta^\alpha_\beta\delta^\gamma_\varepsilon, & \{t^\varepsilon, \pi^\beta_h\}^\alpha=0,\\ \{x^h, x^k\}^\alpha=0, & \{x^h, \pi^\beta_\gamma\}^\alpha=0, & \{x^h, \pi^\beta_k\}^\alpha=\delta^k_h\delta^\alpha_\beta, & \\ \{\pi^\beta_h, \pi^\varepsilon_k\}^\alpha=0, & \{\pi^\beta_h, \pi^\varepsilon_\gamma\}^\alpha=0, & \{\pi^\beta_\lambda, \pi^\gamma_\sigma\}^\alpha=0. & \end{array}\right\} \quad (3.32)$$

Let us now put $u^A=\pi^\varepsilon_\lambda$, $u^B=t^\sigma$, $u^C=t^\gamma$ in (3.31), after which we invoke the fourth member of (3.32), taking into account the skew-symmetry of the Lagrange brackets. This gives

$$\delta^\sigma_\gamma=\{\pi^\varepsilon_\lambda, t^\sigma\}_\alpha\,\{\pi^\varepsilon_\lambda, t^\gamma\}^\alpha=-\{\pi^\varepsilon_\lambda, t^\sigma\}_\alpha\,\delta^\alpha_\varepsilon\delta^\lambda_\gamma=-\{\pi^\alpha_\gamma, t^\sigma\}_\alpha.$$

In this manner the following Poisson bracket identities are found:

$$\left.\begin{array}{ll}\{t^\varepsilon, \pi^\beta_\gamma\}_\alpha=\delta^\beta_\alpha\delta^\varepsilon_\gamma, & \{x^h, \pi^\beta_k\}_\alpha=\delta^\beta_\alpha\delta^h_k,\\ \{\pi^\alpha_\gamma, t^\varepsilon\}_\alpha=-\delta^\varepsilon_\gamma, & \{\pi^\alpha_k, x^h\}_\alpha=-\delta^h_k,\end{array}\right\} \quad (3.33)$$

all others being zero, where it is to be noted that two sets of such identities are required as a result of the lack of skew-symmetry of the Poisson brackets.

4. Canonical transformations

One of the most profound conclusions of the classical Hamilton-Jacobi theory of single integral variational problems is embodied in the fact that the evolution of the canonical variables relative to the single independent variable t may be interpreted in terms of successive applications of infinitesimal canonical transformations. It is therefore natural to enquire as to whether this phenomenon possesses an analogue for multiple integral field theories. In order to find an answer to this question, let us consider on some given extremal subspace C_m two neighbouring points $P(t^\alpha, x^j)$, $Q(t^\alpha+\tau^\alpha, x^j+\eta^j)$, with $\eta^j=\dot{x}^j_\beta\tau^\beta$, where τ^α represents m arbitrary small constants. The canonical momenta at P and Q are denoted by $(\pi^\alpha_\beta, \pi^\alpha_j)$, $(\bar{\pi}^\alpha_\beta, \bar{\pi}^\alpha_j)$, it being recalled that these are respectively basis elements of the planes $T_m(P)$ and $T_m(Q)$. Since this defining property, together with the condition (1.13), is not sufficient to completely specify the transition from $(\pi^\alpha_\beta, \pi^\alpha_j)$ to $(\bar{\pi}^\alpha_\beta, \bar{\pi}^\alpha_j)$, we shall suppose that our extremal C_m is imbedded in a geodesic field with characteristic functions $S^\alpha(t^\beta, x^h)$, so that we may assume that the relations (1.24) are satisfied on C_m. This does not entail any loss of generality, since it has been shown that any extremal may always be imbedded globally in a geodesic field ([1]). Thus at the point P of C_m the 1-forms (2.1) may be represented as

$$\omega^\alpha=\pi^\alpha_\beta dt^\beta+\pi^\alpha_j\,dx^j=dS^\alpha, \quad (4.1)$$

while similarly at Q

$$\bar{\omega}^\alpha = \bar{\pi}^\alpha_\beta \, d\bar{t}^\beta + \bar{\pi}^\alpha_j \, d\bar{x}^j = d\bar{S}^\alpha, \tag{4.2}$$

where $\bar{S}^\alpha = S^\alpha(t^\beta + \tau^\beta, x^h + \eta^h)$. Hence, neglecting higher powers of τ^β, it is readily established that

$$\bar{\omega}^\alpha - \omega^\alpha = d\Phi^\alpha, \tag{4.3}$$

where, with the aid of (1.11), we have written

$$\Phi^\alpha(t^\varepsilon, x^h, \pi^\varepsilon_\beta, \pi^\varepsilon_j) = [\pi^\alpha_\beta + \pi^\alpha_j \phi^j_\beta(t^\varepsilon, x^h, \pi^\varepsilon_\beta, \pi^\varepsilon_h)] \, \tau^\beta. \tag{4.4}$$

Therefore, if in analogy with the classical theory we regard (4.3) as the defining equations of a canonical transformation, we may infer that *the transition from a point P to a neighbouring point Q on any extremal is governed by an infinitesimal canonical transformation.*

Although this conclusion is the obvious counterpart of the classical theorem stated above, it is not a particularly satisfactory one. This becomes evident as soon as one begins to analyze the above definition of canonical transformation from a somewhat more general point of view. In order to substantiate this remark, let us consider a system of $N = (m+1)(n+m)$ transformation equations of the type

$$\bar{t}^\alpha = \bar{t}^\alpha(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h), \qquad \bar{x}^j = \bar{x}^j(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h), \tag{4.5}$$

$$\bar{\pi}^\alpha_\beta = \bar{\pi}^\alpha_\beta(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h), \qquad \bar{\pi}^\alpha_j = \bar{\pi}^\alpha_j(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h), \tag{4.6}$$

which are assumed to be of class C^2 with non-vanishing functional determinant. In accordance with the construction above, this transformation is said to be *canonical* if there exist m functions $\Psi^\alpha(t^\varepsilon, x^h, \pi^\varepsilon_\lambda, \pi^\varepsilon_h)$ such that

$$\bar{\omega}^\alpha - \omega^\alpha = d\Psi^\alpha. \tag{4.7}$$

By virtue of Poincaré's Lemma and its converse, a necessary and sufficient condition that (4.7) be satisfied is that

$$d\bar{\omega}^\alpha - d\omega^\alpha = 0, \tag{4.8}$$

where

$$d\bar{\omega}^\alpha = d\bar{\pi}^\alpha_\beta \wedge d\bar{t}^\beta + d\bar{\pi}^\alpha_j \wedge d\bar{x}^j. \tag{4.9}$$

Each of the 1-forms which appear on the right-hand side of (4.9) are expanded in terms of the differentials dt^ε, dx^h, $d\pi^\varepsilon_\lambda$, $d\pi^\varepsilon_h$ by means of (4.5) and (4.6), which gives rise to an expression of the type

$$d\bar{\omega}^\alpha = \left(\frac{\partial \bar{\pi}^\alpha_\beta}{\partial t^\varepsilon}\frac{\partial \bar{t}^\beta}{\partial t^\gamma} + \frac{\partial \bar{\pi}^\alpha_j}{\partial t^\varepsilon}\frac{\partial \bar{x}^j}{\partial t^\gamma}\right) dt^\varepsilon \wedge dt^\gamma + \cdots$$
$$= \tfrac{1}{2}\left(\overline{[[t^\gamma, t^\varepsilon]]^\alpha} + \overline{[t^\gamma, t^\varepsilon]^\alpha}\right) dt^\varepsilon \wedge dt^\gamma + \cdots = \tfrac{1}{2}\overline{\{t^\gamma, t^\varepsilon\}^\alpha}\, dt^\varepsilon \wedge dt^\gamma + \cdots, \tag{4.10}$$

where $+\cdots$ denotes nine other 2-forms in the remaining differentials of t^ε, x^h, π^ε_λ, π^ε_h, and where the Lagrange brackets are defined as in (2.3) and (2.4) relative to the variables $\bar{t}^\alpha$, $\bar{x}^j$, $\bar{\pi}^\alpha_\beta$, $\bar{\pi}^\alpha_j$. When (4.10) is substituted in (4.8), it is seen that the validity of (4.8) is equivalent to the relations

$$\overline{\{t^\beta, \pi^\varepsilon_\gamma\}^\alpha} = \delta^\alpha_\varepsilon \delta^\gamma_\beta, \qquad \overline{\{x^h, \pi^\varepsilon_k\}^\alpha} = \delta^\alpha_\varepsilon \delta^k_h, \tag{4.11}$$

all other Lagrange brackets of this type being zero, and accordingly *these are necessary and sufficient conditions in order that the transformation* (4.5), (4.6) *be canonical.*

Now, by expansion of $d\bar{x}^j$ and $d\bar{\pi}^\alpha_j$ it is easily verified that

$$d\bar{x}^j \frac{\partial \bar{\pi}^\alpha_j}{\partial \pi^\beta_k} - d\bar{\pi}^\alpha_j \frac{\partial \bar{x}^j}{\partial \pi^\beta_k} = dt^\varepsilon\, \overline{[t^\varepsilon, \pi^\beta_k]^\alpha} + dx^h\, \overline{[x^h, \pi^\beta_k]^\alpha}$$
$$+ d\pi^\varepsilon_\lambda\, \overline{[\pi^\varepsilon_\lambda, \pi^\beta_k]^\alpha} + d\pi^\varepsilon_h\, \overline{[\pi^\varepsilon_\lambda, \pi^\beta_k]^\alpha}$$

identically, and similarly

$$d\bar{t}^\sigma \frac{\partial \bar{\pi}^\alpha_\sigma}{\partial \pi^\beta_k} - d\bar{\pi}^\alpha_\sigma \frac{\partial \bar{t}^\sigma}{\partial \pi^\beta_k} = dt^\varepsilon\, \overline{[[t^\varepsilon, \pi^\beta_k]]^\alpha} + dx^h\, \overline{[[x^h, \pi^\beta_k]]^\alpha}$$
$$+ d\pi^\varepsilon_\lambda\, \overline{[[\pi^\varepsilon_\lambda, \pi^\beta_k]]^\alpha} + d\pi^\varepsilon_h\, \overline{[[\pi^\varepsilon_h, \pi^\beta_k]]^\alpha}.$$

These identities were added, after which the conditions (4.11) are applied. It is thus found that

$$d\bar{t}^\sigma \frac{\partial \bar{\pi}^\alpha_\sigma}{\partial \pi^\beta_k} + d\bar{x}^j \frac{\partial \bar{\pi}^\alpha_j}{\partial \pi^\beta_k} - d\bar{\pi}^\alpha_\sigma \frac{\partial \bar{t}^\sigma}{\partial \pi^\beta_k} - d\bar{\pi}^\alpha_j \frac{\partial \bar{x}^j}{\partial \pi^\beta_k} = dx^h\, \overline{\{x^h, \pi^\beta_k\}^\alpha} = \delta^\alpha_\beta dx^k$$
$$= \delta^\alpha_\beta \left(\frac{\partial x^k}{\partial \bar{t}^\sigma} d\bar{t}^\sigma + \frac{\partial x^k}{\partial \bar{x}^j} d\bar{x}^j + \frac{\partial x^k}{\partial \bar{\pi}^\lambda_\sigma} d\bar{\pi}^\lambda_\sigma + \frac{\partial x^k}{\partial \bar{\pi}^\lambda_j} d\bar{\pi}^\lambda_j\right),$$

and accordingly it may be inferred that

$$\frac{\partial \bar{\pi}^\alpha_\sigma}{\partial \pi^\beta_k} = \delta^\alpha_\beta \frac{\partial x^k}{\partial \bar{t}^\sigma}, \quad \frac{\partial \bar{\pi}^\alpha_j}{\partial \pi^\beta_k} = \delta^\alpha_\beta \frac{\partial x^k}{\partial \bar{x}^j}, \quad \delta^\alpha_\lambda \frac{\partial \bar{t}^\sigma}{\partial \pi^\beta_k} = -\delta^\alpha_\beta \frac{\partial x^k}{\partial \bar{\pi}^\lambda_\sigma}, \quad \delta^\alpha_\lambda \frac{\partial \bar{x}^j}{\partial \pi^\beta_k} = -\delta^\alpha_\beta \frac{\partial x^k}{\partial \bar{\pi}^\lambda_j}. \tag{4.12}$$

The last two relations of this set entail some unexpected conclusions. Let us contract over α and β in the last member, which gives

$$\frac{\partial \bar{x}^j}{\partial \pi^\lambda_k} = -m \frac{\partial x^k}{\partial \bar{\pi}^\lambda_j}, \tag{4.13}$$

while a contraction over α and λ in the same relation yields

$$m \frac{\partial \bar{x}^j}{\partial \pi^\lambda_k} = -\frac{\partial x^k}{\partial \bar{\pi}^\lambda_j}. \tag{4.14}$$

On comparing (4.13) with (4.14), we see that

$$(m^2 - 1) \frac{\partial \bar{x}^j}{\partial \pi^\lambda_k} = 0,$$

so that either $m=1$, or

$$\frac{\partial \bar{x}^j}{\partial \pi^\lambda_k} = 0, \tag{4.15}$$

Similarly, the third member of (1.12) indicates that either $m=1$, or

$$\frac{\partial \bar{t}^\varepsilon}{\partial \pi^\lambda_k} = 0. \tag{4.16}$$

The process described in the last paragraph may be repeated three times with appropriate choice of variables. Excluding the single integral case ($m=1$), one thus obtains the following sets of relations:

$$\frac{\partial \bar{t}^\alpha}{\partial \pi^\varepsilon_\lambda} = 0, \quad \frac{\partial \bar{t}^\alpha}{\partial \pi^\varepsilon_h} = 0, \quad \frac{\partial \bar{x}^j}{\partial \pi^\varepsilon_\lambda} = 0, \quad \frac{\partial \bar{x}^j}{\partial \pi^\varepsilon_h} = 0; \tag{4.17}$$

$$\delta^\alpha_\varepsilon \frac{\partial \bar{t}^\beta}{\partial t^\gamma} = \frac{\partial \pi^\alpha_\gamma}{\partial \bar{\pi}^\varepsilon_\beta}, \quad \delta^\alpha_\varepsilon \frac{\partial \bar{t}^\beta}{\partial x^h} = \frac{\partial \pi^\alpha_h}{\partial \bar{\pi}^\varepsilon_\beta}, \quad \delta^\alpha_\varepsilon \frac{\partial \bar{x}^j}{\partial t^\beta} = \frac{\partial \pi^\alpha_\beta}{\partial \bar{\pi}^\varepsilon_j}, \quad \delta^\alpha_\varepsilon \frac{\partial \bar{x}^j}{\partial x^h} = \frac{\partial \pi^\alpha_h}{\partial \bar{\pi}^\varepsilon_j}; \tag{4.18}$$

$$\frac{\partial \bar{\pi}^\alpha_\beta}{\partial t^\varepsilon} = -\frac{\partial \pi^\alpha_\varepsilon}{\partial \bar{t}^\beta}, \quad \frac{\partial \bar{\pi}^\alpha_\beta}{\partial x^k} = -\frac{\partial \pi^\alpha_k}{\partial \bar{t}^\beta}, \quad \frac{\partial \bar{\pi}^\alpha_\beta}{\partial \pi^\varepsilon_\lambda} = \delta^\alpha_\varepsilon \frac{\partial t^\lambda}{\partial \bar{t}^\beta}, \quad \frac{\partial \bar{\pi}^\alpha_\beta}{\partial \pi^\varepsilon_k} = \delta^\alpha_\varepsilon \frac{\partial x^k}{\partial \bar{t}^\beta}; \tag{4.19}$$

$$\frac{\partial \bar{\pi}^\alpha_j}{\partial t^\beta} = -\frac{\partial \pi^\alpha_\beta}{\partial \bar{x}^j}, \quad \frac{\partial \bar{\pi}^\alpha_j}{\partial x^k} = -\frac{\partial \pi^\alpha_k}{\partial \bar{x}^j}, \quad \frac{\partial \bar{\pi}^\alpha_j}{\partial \pi^\varepsilon_\lambda} = \delta^\alpha_\varepsilon \frac{\partial t^\lambda}{\partial \bar{x}^j}, \quad \frac{\partial \bar{\pi}^\alpha_j}{\partial \pi^\varepsilon_k} = \delta^\alpha_\varepsilon \frac{\partial x^k}{\partial \bar{x}^j}. \tag{4.20}$$

The reasoning which leads from (4.11) to (4.17)–(4.20) may be reversed: hence the relations (4.17)–(4.20) *represent necessary and sufficient conditions in order that the transformations* (4.5), (4.6) *be canonical.* However, it is quite obvious that these conditions are extremely restrictive. From (4.17) it is evident that $\bar{t}^\alpha$, $\bar{x}^j$ are independent of π^ε_λ, π^ε_h, so that (4.5) reduces to the form

$$\bar{t}^\alpha = \bar{t}^\alpha(t^\varepsilon, x^h), \qquad \bar{x}^j = \bar{x}^j(t^\varepsilon, x^h). \tag{4.21}$$

Thus the derivatives $\partial t^\lambda/\partial \bar{t}^\beta$, $\partial x^k/\partial \bar{t}^\beta$, $\partial t^\lambda/\partial \bar{x}^j$, $\partial x^k/\partial \bar{x}^j$ can be expressed either as func tions $(\bar{t}^\alpha, \bar{x}^j)$ only, or of (t^ε, x^h) only. Accordingly it is possible to integrate the third and fourth members of (4.19) and (4.20) to yield respectively

$$\bar{\pi}^\alpha_\beta = \pi^\alpha_\varepsilon \frac{\partial t^\varepsilon}{\partial \bar{t}^\beta} + \pi^\alpha_h \frac{\partial x^h}{\partial \bar{t}^\beta} + \chi^\alpha_\beta(\bar{t}^\lambda, \bar{x}^l), \tag{4.22}$$

and

$$\bar{\pi}^\alpha_j = \pi^\alpha_\varepsilon \frac{\partial t^\varepsilon}{\partial \bar{x}^j} + \pi^\alpha_h \frac{\partial x^h}{\partial \bar{x}^j} + \chi^\alpha_j(\bar{t}^\lambda, \bar{x}^l), \tag{4.23}$$

where χ^α_β, χ^α_j are certain functions whose significance will become evident almost immediately. From (4.22), (4.23), and the definition (2.1) it follows that

$$\begin{aligned} \bar{\omega}^\alpha &= \bar{\pi}^\alpha_\beta d\bar{t}^\beta + \bar{\pi}^\alpha_j d\bar{x}^j \\ &= \pi^\alpha_\varepsilon \left(\frac{\partial t^\varepsilon}{\partial \bar{t}^\beta} d\bar{t}^\beta + \frac{\partial t^\varepsilon}{\partial \bar{x}^j} d\bar{x}^j \right) + \pi^\alpha_h \left(\frac{\partial x^h}{\partial \bar{t}^\beta} d\bar{t}^\beta + \frac{\partial x^h}{\partial \bar{x}^j} d\bar{x}^j \right) + \chi^\alpha_\beta d\bar{t}^\beta + \chi^\alpha_j d\bar{x}^j, \end{aligned}$$

so that

$$\bar{\omega}^\alpha - \omega^\alpha = \chi^\alpha_\beta d\bar{t}^\beta + \chi^\alpha_j d\bar{x}^j. \tag{4.24}$$

Because of (4.21) the right-hand side can now be expressed as a 1-form in dt^ε, dx^h: and hence a comparison with the defining condition (4.7) indicates that *the functions* Ψ^α *which appear in* (4.7) *can be dependent only on* t^ε, x^h. Let us denote by $\bar{\Psi}^\alpha(\bar{t}^\beta, \bar{x}^j)$ the transform of Ψ^α. Then it follows from (4.7) and (4.24) that the functions χ^α_β, χ^α_j are given by

$$\chi^\alpha_\beta = \frac{\partial \bar{\Psi}^\alpha}{\partial \bar{t}^\beta}, \qquad \chi^\alpha_j = \frac{\partial \bar{\Psi}^\alpha}{\partial \bar{x}^j}. \tag{4.25}$$

We may summarize our conclusions in the following

THEOREM: *If* $m \neq 1$, *the only transformations of the type* (4.5), (4.6) *which are canonical in the sense that the conditions*

$$\bar{\omega}^\alpha - \omega^\alpha = d\Psi^\alpha$$

are satisfied, are of the form

$$\bar{t}^{\alpha}=\bar{t}^{\alpha}(t^{\varepsilon}, x^{h}), \qquad \bar{x}^{j}=\bar{x}^{j}(t^{\varepsilon}, x^{h}),$$

$$\bar{\pi}^{\alpha}_{\beta}=\pi^{\alpha}_{\varepsilon}\frac{\partial t^{\varepsilon}}{\partial \bar{t}^{\beta}}+\pi^{\alpha}_{h}\frac{\partial x^{h}}{\partial \bar{t}^{\beta}}+\frac{\partial \bar{\Psi}^{\alpha}}{\partial \bar{t}^{\beta}}, \qquad \bar{\pi}^{\alpha}_{j}=\pi^{\alpha}_{\varepsilon}\frac{\partial t^{\varepsilon}}{\partial \bar{x}^{j}}+\pi^{\alpha}_{h}\frac{\partial x^{h}}{\partial \bar{x}^{j}}+\frac{\partial \bar{\Psi}^{\alpha}}{\partial \bar{x}^{j}},$$

where Ψ^{α} is a function of t^{ε}, x^{h} only.

In analogy with the classical theory it would seem reasonable to define a *homogeneous* canonical transformation as one for which $d\Psi^{\alpha}=0$. However, under these circumstances it follows from (4.25) that $\chi^{\alpha}_{\beta}=0$, $\chi^{\alpha}_{j}=0$, and the relations (4.22), (4.23) merely state that the canonical momenta $\boldsymbol{\pi}^{(\alpha)}$ transform as covariant vectors under the coordinate transformation (4.21) of the configuration space R_{m+n}. *A homogeneous canonical transformation is therefore nothing but an extended point transformation.*

It should be emphasized that these conclusions are valid solely for $m\neq 1$; in fact, it is quite evident from the above analysis that canonical transformations in the theory of multiple integral variational problems – at least as defined here – lack the overall significance which they enjoy in the theory of single integral problems in the calculus of variations. Moreover, alternative definitions of such transformations lead to similar disappointing conclusions. For instance, if instead of subjecting (4.5), (4.6) to the conditions (4.7), one stipulates that the transformation be such as to ensure the invariance of the canonical equations (1.23) (which represents one of the most important properties of canonical transformations when $m=1$), a long and complicated calculation again leads to the relations (4.17), whose implications are obviously unduly restrictive.

5. The integral invariant

Let us suppose once more that we are given an n-parameter family (1.16) of subspaces $C_m(u^h)$ satisfying the condition (1.17), together with a canonical momentum field (1.18) which is subject as before to the relations (1.7) and (1.8.) For the purposes of our subsequent analysis we now introduce a set of $(m-1)$-forms θ_α as defined by

$$(m-1)!\,\theta_{\alpha}=\varepsilon_{\alpha\alpha_2\ldots\alpha_m}\,dt^{\alpha_2}\wedge\cdots\wedge dt^{\alpha_m}, \tag{5.1}$$

in which $\varepsilon\ldots$ denotes the m-dimensional permutation symbol. For future reference we observe that the exterior derivatives of these forms vanish identically:

$$d\theta_{\alpha}=0, \tag{5.2}$$

while

$$dt^{\beta}\wedge\theta_{\alpha}=\delta^{\beta}_{\alpha}d(t) \tag{5.3}$$

in terms of the notation (1.2), this being an immediate consequence of the definition (5.1).

Let us now construct the following m-form:

$$\Omega=(1-m)\, Dd(t)+P^{\beta}_{\alpha}\omega^{\alpha}\wedge\theta_{\beta}, \tag{5.4}$$

where we recall that D, P^{β}_{α}, and ω^{α} are defined respectively by (1.10), (1.14), and (2.1). Now, using (1.9) and (5.3), it is seen that

$$\begin{aligned} P^{\beta}_{\alpha}\omega^{\alpha}\wedge\theta_{\beta} &= P^{\beta}_{\alpha}(\pi^{\alpha}_{\varepsilon}dt^{\varepsilon}+\pi^{\alpha}_{j}dx^{j})\wedge\theta_{\beta} \\ &= P^{\beta}_{\alpha}\left(\pi^{\alpha}_{\varepsilon}+\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial t^{\varepsilon}}\right)dt^{\varepsilon}\wedge\theta_{\beta}+P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}du^{h}\wedge\theta_{\beta} \\ &= P^{\beta}_{\alpha}P^{\alpha}_{\varepsilon}dt^{\varepsilon}\wedge\theta_{\beta}+P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}du^{h}\wedge\theta_{\beta}=mDd\,(t)+P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}du^{h}\wedge\theta_{\beta}, \end{aligned} \tag{5.5}$$

so that (5.4) can be expressed in the form

$$\Omega=Dd\,(t)+P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}du^{h}\wedge\theta_{\beta}. \tag{5.6}$$

By way of motivation we note that, for a displacement (dt^{α}, dx^{j}) tangential to a given subspace $C_{m}(u^{h})$, for which $du^{h}=0$, we have $\Omega=Dd(t)$, and thus, by virtue of (1.13),

$$\Omega=Ld(t), \tag{5.7}$$

so that Ω is simply the integrand of the fundamental integral (1.1) under these circumstances.

The general form (5.7) of Ω suggests that we introduce a set of 1-forms μ^{β} defined by

$$\mu^{\beta}=m^{-1}Ddt^{\beta}+P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}du^{h}, \tag{5.8}$$

for we may then write

$$\Omega=\mu^{\beta}\wedge\theta_{\beta} \tag{5.9}$$

in consequence of (5.3). In passing we note that the 1-forms μ^{β} and ω^{β} are related according to

$$\mu^{\beta}=P^{\beta}_{\alpha}\omega^{\alpha}-(1-m^{-1})\, Ddt^{\beta}. \tag{5.10}$$

The principal results of this section depend significantly on the explicit form of the exterior derivative of Ω, whose evaluation is simplified considerably when the expression (5.9) is used, for in view of (5.2) and (5.9) we immediately have

$$d\Omega=d\mu^{\beta}\wedge\theta_{\beta}. \tag{5.11}$$

The exterior derivative of the 1-form (5.8) is given by

$$d\mu^{\beta}=m^{-1}\frac{\partial D}{\partial t^{\varepsilon}}\,dt^{\varepsilon}\wedge dt^{\beta}+m^{-1}\frac{\partial D}{\partial u^{h}}\,du^{h}\wedge dt^{\beta}$$
$$+\frac{\partial}{\partial t^{\varepsilon}}\left(P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}\right)dt^{\varepsilon}\wedge du^{h}+\frac{\partial}{\partial u^{k}}\left(P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}\right)du^{k}\wedge du^{h},$$

and hence, with the aid of (5.1) and (5.3), the relation (5.11) becomes

$$\begin{aligned} d\Omega=&\left[\frac{\partial D}{\partial u^{h}}-\frac{\partial}{\partial t^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{j})\frac{\partial x^{j}}{\partial u^{h}}-P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial^{2}x^{j}}{\partial t^{\beta}\partial u^{h}}\right]du^{h}\wedge d(t)\\ &-\frac{\partial}{\partial u^{k}}\left(P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}\right)du^{h}\wedge du^{k}\wedge\theta_{\beta}. \end{aligned}\tag{5.12}$$

The first expression on the right-hand side may be simplified as follows. Let us substitute from (1.16) and (1.18) in (1.13), after which we differentiate with respect to u^{h}, obtaining

$$\frac{\partial H^{*}}{\partial x^{j}}\frac{\partial x^{j}}{\partial u^{h}}+\frac{\partial H^{*}}{\partial \pi^{\alpha}_{\beta}}\frac{\partial \pi^{\alpha}_{\beta}}{\partial u^{h}}+\frac{\partial H^{*}}{\partial \pi^{\alpha}_{j}}\frac{\partial \pi^{\alpha}_{j}}{\partial u^{h}}=0.$$

By means of the third and fourth members of (1.15) this can be written as

$$\frac{\partial H^{*}}{\partial x^{j}}\frac{\partial x^{j}}{\partial u^{h}}+P^{\beta}_{\alpha}\frac{\partial}{\partial u^{h}}\left(\pi^{\alpha}_{\beta}+\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial t^{\beta}}\right)-P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial^{2}x^{j}}{\partial u^{h}\partial t^{\beta}}=0,$$

or, if we use (1.9) and (1.14),

$$P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial^{2}x^{j}}{\partial u^{h}\partial t^{\beta}}=\frac{\partial D}{\partial u^{h}}+\frac{\partial H^{*}}{\partial x^{j}}\frac{\partial x^{j}}{\partial u^{h}}.\tag{5.13}$$

When this result is substituted in (5.12), the latter becomes

$$d\Omega=-\left[\frac{\partial H^{*}}{\partial x^{j}}+\frac{\partial}{\partial t^{\beta}}(P^{\beta}_{\alpha}\pi^{\alpha}_{j})\right]\frac{\partial x^{j}}{\partial u^{h}}\,du^{h}\wedge d(t)-\frac{\partial}{\partial u^{k}}\left(P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}\right)du^{h}\wedge du^{k}\wedge\theta_{\beta}.\tag{5.14}$$

Also,

$$\frac{\partial}{\partial u^{k}}\left(P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial x^{j}}{\partial u^{h}}\right)du^{h}\wedge du^{k}=\frac{\partial x^{j}}{\partial u^{h}}\frac{\partial}{\partial u^{k}}(P^{\beta}_{\alpha}\pi^{\alpha}_{j})\,du^{h}\wedge du^{k}+P^{\beta}_{\alpha}\pi^{\alpha}_{j}\frac{\partial^{2}x^{j}}{\partial u^{h}\partial u^{k}}\,du^{h}\wedge du^{k},\tag{5.15}$$

where the second term on the right-hand side vanishes identically by virtue of the symmetry of the second derivatives of x^j. This suggests that we define the *generalized Lagrange brackets of the second kind* as follows:

$$L_h{}^\beta{}_k = \frac{\partial x^j}{\partial u^h}\frac{\partial}{\partial u^k}(P^\beta_\alpha \pi^\alpha_j) - \frac{\partial x^j}{\partial u^k}\frac{\partial}{\partial u^h}(P^\beta_\alpha \pi^\alpha_j), \tag{5.16}$$

so that we may write (5.15) in the form

$$\frac{\partial}{\partial u^k}\left(P^\beta_\alpha \pi^\alpha_j \frac{\partial x^j}{\partial u^h}\right) du^h \wedge du^k = \tfrac{1}{2} L_h{}^\beta{}_k \, du^h \wedge du^k . \tag{5.17}$$

When this is substituted in (5.14), we finally obtain

$$d\Omega = -\left[\frac{\partial H^*}{\partial x^j} + \frac{\partial}{\partial t^\beta}(P^\beta_\alpha \pi^\alpha_j)\right]\frac{\partial x^j}{\partial u^h} du^h \wedge d(t) - \tfrac{1}{2} L_h{}^\beta{}_k \, du^h \wedge du^k \wedge \theta_\beta . \tag{5.18}$$

This is the explicit expression for the exterior derivative of the m-form (5.4) which we have been seeking. Let us now assume that the family of subspaces $C_m(u^h)$ consists of extremals, in which case we may apply the canonical equations (1.23), so that (5.18) reduces to

$$d\Omega = -\tfrac{1}{2} L_h{}^\beta{}_k \, du^h \wedge du^k \wedge \theta_\beta . \tag{5.19}$$

However, since $d(d\Omega)=0$, it then follows that

$$0 = \frac{\partial}{\partial t^\alpha}(L_h{}^\beta{}_k)\, dt^\alpha \wedge du^h \wedge du^k \wedge \theta_\beta + \frac{\partial}{\partial u^l}(L_h{}^\beta{}_k)\, du^l \wedge du^h \wedge du^k \wedge \theta_\beta . \tag{5.20}$$

Here it should be noted that, by virtue of the structure of (5.16), we have

$$\frac{\partial}{\partial u^l}(L_h{}^\beta{}_k)\, du^l \wedge du^h \wedge du^k = 0$$

identically, and hence (5.20) is simply equivalent to

$$\frac{\partial}{\partial t^\beta}(L_h{}^\beta{}_k)\, du^h \wedge du^k \wedge d(t) = 0,$$

where we have again made use of (5.3). We therefore conclude that *on any n-parameter family of extremals*

$$\frac{\partial}{\partial t^{\beta}}(L_{h\ k}^{\ \beta})=0. \tag{5.21}$$

This is the generalization of a classical theorem of the single integral theory (originally due to Lagrange) according to which the Lagrange brackets are constant along each curve of any congruence of extremals.

It should be remarked that this generalization involves primarily the generalized Lagrange brackets of the second kind, and not those defined by (2.3). Because of (5.16), these two types of brackets are related according to

$$L_{h\ k}^{\ \beta}=P_{\alpha}^{\beta}[u^h, u^k]^{\alpha}+\pi_j^{\alpha}\left(\frac{\partial x^j}{\partial u^h}\frac{\partial P_{\alpha}^{\beta}}{\partial u^k}-\frac{\partial x^j}{\partial u^k}\frac{\partial P_{\alpha}^{\beta}}{\partial u^h}\right), \tag{5.22}$$

so that the vanishing of one type does not in general imply the vanishing of the other. Thus, even when the family $C_m(u^h)$ of extremals constitutes a geodesic field, in which case (2.9) is satisfied, it does not follow that the Lagrange brackets of the second kind vanish. We shall return to this remark presently.

Now, in order to be able to give a geometrical interpretation to the relation (5.21), we have to single out certain variables. From amongst the m independent variables we select t^m (for instance, for the case $m=4$ we would regard t^4 as a time coordinate, and t^1, t^2, t^3 as spatial coordinates), while from amongst the n parameters u^h we select u^1, u^2, to be denoted respectively by u and v. In the configuration space S_{m+n} of the variables (t^{α}, u^h) let us now consider the subspace S_{m+2} defined by the equations $u^3=a^3,\cdots, u^n=a^n$, where $a^3,\cdots, a^n$ are arbitrary constants. (Here, and in the sequel, it is necessary to suppose that $n\geqslant 3$.) On S_{m+2} the $(m+1)$-form (5.19) reduces to

$$d\Omega=-L_{12}^{\alpha}\, du\wedge dv\wedge\theta_{\alpha}. \tag{5.23}$$

At this stage it is necessary to impose an additional assumption (which is a very natural one within the context of any physical field theory), namely that on S_{m+2}

$$L_{12}^{\alpha}\to 0 \quad \text{as} \quad t^{\alpha'}\pm\infty, \quad (\alpha'=1,\ldots,m-1), \tag{5.24}$$

for all values of u, v, and t^m. Under these circumstances it may be shown as usual ([5], p. 296) that

$$\frac{\partial}{\partial t^m}\int_{R_{m-1}} L_{12}^{m}\, d(t')=0, \tag{5.25}$$

where R_{m-1} denotes the entire domain of the variables $t^{\alpha'}$, and where

$$d(t')=dt^1\wedge\ldots\wedge dt^{m-1}. \tag{5.26}$$

The equation $t^m = T = \text{const.}$ defines an $(m+1)$-dimensional hypersurface H_{m+1} of S_{m+2}. Let g be a closed, simply-connected region with smooth boundary in the domain of the variables (u, v). The subset Z of H_{m+1} defined by the condition $(u, v) \in g$ has dimension $(m+1)$, its m-dimensional boundary being denoted by ∂Z. (The set Z may equivalently be defined as the intersection in S_{m+2} of the sets H_{m+1} and $g \times R_{m-1}$.) Moreover, since $dt^m = 0$ on H_{m+1}, it follows from the definition (5.1) that on H_{m+1}

$$\theta_1 = 0, \dots, \theta_{m-1} = 0, \quad \theta_m = (-1)^{m+1}\, d(t') \tag{5.27}$$

in the notation (5.26). Thus on Z the $(m+1)$-form (5.23) becomes

$$d\Omega = (-1)^m L^{\,m}_{12}\, du \wedge dv \wedge d(t'). \tag{5.28}$$

Let us now construct the $(m+1)$-fold integral

$$\int_Z d\Omega = (-1)^m \int_Z L^{\,m}_{12}\, du \wedge dv \wedge d(t'), \tag{5.29}$$

where it should be borne in mind that the integrand on the right-hand side is a function of (t^α, u^h), and in particular therefore also of t^m, which assumes the constant value T on Z. Thus it would appear at first sight that the value of the integral (5.29) depends on T. However, it we now invoke (5.25), we see that, to the contrary,

$$\frac{\partial}{\partial t^m} \int_Z d\Omega = 0. \tag{5.30}$$

Hence the integral (5.29) is a constant, whose value depends solely on the choice of the region g of S_2.

Also, using (5.27), (5.26) and (2.1), we observe that the fundamental m-form (5.4) assumes the following form on H_{m+1}:

$$\Omega = (-1)^{m+1}\, P^{m}_{\alpha}\omega^{\alpha}_{\alpha} \wedge d(t') = (-1)^{m+1}\, P^{m}_{\alpha}\pi^{\alpha}_{j}\, dx^j \wedge dt^1 \wedge \cdots \wedge dt^{m-1}, \tag{5.31}$$

and thus, by virtue of (5.24), an application of Stokes' theorem yields

$$\int_{\partial Z} \Omega = (-1)^{m+1} \int_{\partial Z} P^{m}_{\alpha}\pi^{\alpha}_{j}\, dx^j \wedge dt^1 \wedge \cdots \wedge dt^{m-1} = \int_Z d\Omega. \tag{5.32}$$

It therefore follows from (5.30) *that the integral on the left-hand side is an integral in-*

variant in the sense that it is a constant whose value is independent of the preferred variable t^m. An equivalent formulation of this conclusion is the statement that the value of the integral (5.32) is the same for all hypersurfaces $t^m = \text{const.}$ on which the set Z is defined once the region g in the domain of the parameters (u, v) has been fixed.

By way of comparison let us briefly consider the single integral theory, in which $m=1$. Putting the sole surviving independent variable $t^m = t$, we see that H_{m+1} reduces to the hypersurface $t = T = \text{const}$ of S_3 on which $dt = 0$. Since $P^\alpha_\beta = 1$ identically when $m=1$, it then follows from (5.31) that

$$\Omega = \omega = \pi_j \, dx^j \tag{5.33}$$

on Z, where the π_j are the classical canonical momenta $\partial L/\partial \dot{x}^j$ by virtue or (1.7), while ∂Z is simply a closed curve on H_2, of which we assume that it may be represented parametrically in the form $u = u(\tau)$, $v = v(\tau)$. Moreover, for $m=1$, the family (1.16) becomes an n-parameter congruence of extremal curves in R_{m+1}, from which a 2-parameter subset is selected in terms of (1.16):

$$x^j = x^j(t, u, v, a^3, \ldots, a^n). \tag{5.34}$$

Hence the image in R_{m+1} of Z is the closed curve c on the hyperplane $t = T$ defined by

$$x^j = x^j(T, u(\tau), v(\tau), a^3, \ldots, a^n), \tag{5.35}$$

and our conclusion concerning (5.32) is tantamount to the statement that the value of the line integral

$$\int_c \omega = \int_c \pi_j \, dx^j \tag{5.36}$$

remains constant for all values of t. This, of course, is the classical integral invariant of the single integral theory.

It has been remarked by E. Cartan ([3], p. 78) that the existence of a single relative integral invariant such as (5.36), generated by a 1-form ω, gives rise immediately to a sequence of higher order relative and absolute integral invariants, these being generated respectively by the forms

$$\omega \wedge (d\omega)^p, \quad (d\omega)^p, \qquad p = 1, \ldots, n.$$

The question therefore arises as to whether this is true also for the multiple integral theory discussed above. In this connection we observe that, in view of (5.1), (5.4), and

(5.18), products such as $\Omega \wedge \Omega$, or $d\Omega \wedge \Omega$, involve the terms $\theta_\alpha \wedge \theta_\beta$, which in turn entail exterior products of $2(m-1)$ differentials dt^α. These products, however, vanish identically unless $m \leqslant 2$. It is therefore inferred that it is not possible to construct integral invariants of higher order for arbitrary values of m by means of the fundamental m-form Ω.

In conclusion the following phenomenon should be noted. It was observed above that the generalized Lagrange brackets (5.16) of the second kind are not necessarily zero when the family $C_m(u^h)$ of subspaces constitutes a geodesic field. It would therefore appear in view of (5.32) and (5.29) that *the integral invariant* (5.32) *need not vanish when constructed on a geodesic field.* This is in sharp contrast to the corresponding state of affairs in the geodesic field theory of Weyl [9]. In the case of the latter – in which an entirely different canonical formalism must be used – it may be shown that the Lagrange brackets, whose vanishing partly characterizes a geodesic field, also occur in the integral invariant, which is therefore zero for such fields. In this sense, therefore, the geodesic field theory of Weyl is more closely analogous to the classical single integral theory than the field theory of Carathéodory.

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Zur Bestimmung von $E_n[x^{n+2m}]$

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Abstract

Remark on the estimation of $E_n[x^{n+2m}]$.
Let be

$$E_n[f] := \inf_{p\in P_n} \sup_{x\in[-1,1]} |f(x) - p(x)|$$

(P_n: set of all polynomials of degree n).
Riess-Johnson [4] proved

$$E_n[x^{n+2m}] = \frac{n^{m-1}}{2^{n+2m-1}(m-1)!}[1 + O(n^{-1})], \qquad n \text{ even}. \tag{3}$$

This degree of approximation is realized by expansion in Chebyshev polynomials and by interpolation at Chebyshev nodes.

The purpose of this paper is to give a more precise estimation by constructing the polynomial of best approximation on a finite set. This construction is easily done and one obtains the result, that the term $O(n^{-1})$ in (3) may be replaced by $\frac{1}{2}(m-1)(3m+2)n^{-1} + O(n^{-2})$.

Es sei gesetzt

$$E_n[f] := \inf_{p\in\mathscr{P}_n} \sup_{x\in[-1,1]} |f(x)-p(x)|.$$

($\mathscr{P}_n$ Menge der Polynome vom Grad n).

Das Ziel dieser Note ist der Beweis von

SATZ. *Ist m fest und durchläuft n die geraden Zahlen, so gilt*

$$E_n[x^{n+2m}] = \frac{n^{m-1}}{2^{n+2m-1}(m-1)!}\left[1 + \frac{(m-1)(3m+2)}{2}\cdot\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right]. \tag{1}$$

Die angegebene Approximationsgüte wird erreicht mit denjenigen $p\in\mathscr{P}_n$, die $f(x) = x^{n+2m}$ eingeschränkt auf

$$x_\nu := \cos\frac{\nu\pi}{n+2} \qquad \nu = 0, 1, \ldots, n+2 \tag{2}$$

am besten approximieren.

Riess-Johnson [4] haben

AMS Subject Classifications: Primary 41A25, 41A10.

Eingegangen am 25.9.1973.

$$E_n[x^{n+2m}] = \frac{n^{m-1}}{2^{n+2m-1}(m-1)!}\left[1+O\left(\frac{1}{n}\right)\right] \tag{3}$$

bewiesen und gezeigt, daß diese Approximationsgüte sowohl durch Entwicklung nach Tschebyscheff-Polynomen wie auch durch Interpolation über Tschebyscheff-Knoten erreicht werden kann. Beide Approximationsvorschriften ergeben nicht das genauere Resultat (1), wie man aus den Ergebnissen von Riess-Johnson [4] (im Fall der Interpolation auch aus (9)) ablesen kann. Die Konstruktionsvorschrift des Satzes ist also im hier behandelten Fall – wie auch vielfach sonst – den beiden obengenannten Konstruktionsvorschriften vorzuziehen, zumal auch ihre praktische Behandlung keine größeren Schwierigkeiten bietet.

Beweis. T_n bzw. U_n bezeichnen im Folgenden die Tschebyscheff-Polynome erster bzw. zweiter Art. $J[f]$ bezeichne das Interpolationspolynom von f bezüglich der Stützstellen (2), k seinen Hauptkoeffizienten.

Ist f gerade, so ist

$$p := J[f] - \frac{k}{2^{n+1}} T_{n+2}$$

ein Polynom vom Grad n, für das

$$f(x_\nu) - p(x_\nu) = (-1)^\nu \frac{k}{2^{n+1}} \qquad \nu = 0, 1, \ldots, n+2$$

gilt; somit ist p das in Rede stehende Polynom, und aus dem de la Vallée Poussinschen Satz (vgl. z.B. Meinardus [3] S. 80) folgt

$$E_n[f] \geqslant \frac{|k|}{2^{n+1}}. \tag{4}$$

Die Bestimmung von $J[x^{n+2m}]$ stützt sich auf die bekannte Entwicklung (vgl. z.B. Fox-Parker [2] S. 52)

$$x^{n+2m} = \frac{1}{2^{n+2m-1}} \sum_{\nu=0}^{n/2+m}{}' \binom{n+2m}{\nu} T_{n+2m-2\nu}(x) \tag{5}$$

wo der letzte Summand zu halbieren ist.

Wegen

$$2(1-x^2)\, U_{n+1}(x)\, U_{r-1}(x) = T_{n+2-r}(x) - T_{n+2+r}(x) \tag{6}$$

ist $J[T_{n+2+r}] = J[T_{n+2-r}] = T_{n+2-r}$ für jedes $r \in [0, n+2]$. Also gilt, falls $2m \leqslant n+4$ ist

$$J[x^{n+2m}] = \frac{1}{2^{n+2m-1}} \left\{ \sum_{\nu=0}^{m-2} \binom{n+2m}{\nu} T_{n+4-2m+2\nu} + \sum_{\nu=m-1}^{n/2+m}{}' \binom{n+2m}{\nu} T_{n+2m-2\nu} \right\}. \tag{7}$$

Hieraus bestimmt man k und hat wegen (4)

$$E_n[x^{n+2m}] \geqslant \frac{1}{2^{n+2m-1}} \binom{n+2m}{m-1}. \tag{8}$$

Um $E_n[x^{n+2m}]$ auch in der umgekehrten Richtung abschätzen zu können, stellt man $x^{n+2m} - p(x)$ mittels (5), (7), (6) in der folgenden Form dar

$$\frac{1}{2^{n+2m-1}} \left\{ -2(1-x^2)\, U_{n+1}(x) \sum_{\nu=0}^{m-2} \binom{n+2m}{\nu} U_{2m-2\nu-3}(x) + \binom{n+2m}{m-1} T_{n+2}(x) \right\}$$

$$= \frac{1}{2^{n+2m-1}} \left\{ -2 \sin(n+2)\varphi \sum_{\nu=0}^{m-2} \binom{n+2m}{\nu} \sin(2m-2\nu-2)\varphi + \binom{n+2m}{m-1} \cos(n+2)\varphi \right\}$$

wo $x = \cos\varphi$ gesetzt ist. Nun gilt bekanntlich

$$|a\cos\psi + b\sin\psi| \leqslant \sqrt{a^2+b^2}, \quad \text{also}$$

$$\sup|x^{n+2m} - p(x)| \leqslant \frac{1}{2^{n+2m-1}} \binom{n+2m}{m-1} \sqrt{1+O(n^{-2})}$$

$$= \frac{1}{2^{n+2m-1}} \binom{n+2m}{m-1} (1+O(n^{-2})).$$

Diese Beziehung ergibt zusammen mit (8) leicht die Behauptung des Satzes.

Zwei Bemerkungen seien noch hinzugefügt.

Erstens: Mit der hier benutzten Methode kann man den Fehler bei der Interpolation über Tschebyscheff-Knoten (d.h. Nullstellen von T_{n+2}) in der Form darstellen

$$\frac{T_{n+2}}{2^{n+2m-1}} \left\{ \binom{n+2m}{m-1} + 2 \sum_{\nu=0}^{m-2} \binom{n+2m}{\nu} T_{2m-2\nu-2} \right\}. \tag{9}$$

Hieraus liest man die Aussage von Riess/Johnson über die Größenordnung des Interpolationsfehlers direkt ab.

Zweitens: Während die Verschärfung von (3) zu (1) einfach zu erreichen war, dürfte eine weitere Verschärfung von (1) nur mit Hilfe ganz spezieller, auf die Struktur der Funktion $f(x)=x^{n+2m}$ zugeschnittener Verfahren möglich sein. Bernstein [1] (S. 24) hat ein solches Verfahren angegeben und auch (3) schon ausgesprochen. Mit Hilfe seiner Grundidee läßt sich – allerdings unter großem Aufwand – beweisen:

$$E_n[x^{n+2m}]=\frac{n^{m-1}}{2^{n+2m-1}(m-1)!}\left[1+\frac{(m-1)(3m+2)}{2}\cdot\frac{1}{n}+\frac{(m-1)(27m^3-19m^2-34m-48)}{24}\cdot\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)\right].$$

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Aequationes Math. *13* (1975), 155–156. University of Waterloo Birkhäuser Verlag, Basel

An identy for fixed points of permutations

Jay R. Goldman

Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$ and for $\pi \in S_n$ let $f(\pi)$ denote the number of fixed points of π. We shall prove that

$$\frac{1}{n!} \sum_{\pi \in S_n} f(\pi)^k = \sum_{i=1}^{\min(k,n)} S(k, i), \tag{1}$$

where $S(k, i)$, the Stirling number of the second kind, counts the number of partitions of a k-set into i disjoint blocks. For the case $k \leqslant n$ this identity is given by Carlitz and Scoville as a problem in [1]. In this case the right hand sides equals B_k, the kth Bell number, [2].

When $k=1$, we can interpret this identity as saying that the average number of fixed points per permutation is one. This probabilistic idea leads to a complete proof.

Every permutation with p fixed points contributes a factor of p^k to (1) and there are $\binom{n}{p} D_{n-p}$ permutations of S_n with p fixed points where $\binom{n}{p}$ counts the choice of points to be fixed and D_i denotes the number of permutations of an i-set with no fixed points. Substituting this into (1) we get

$$\sum_p p^k \binom{n}{p} D_{n-p}/n! = \sum_{i=1}^{\min(k,n)} S(k, i). \tag{2}$$

We shall prove this latter identity.

Suppose we choose a permutation of S_n at random, i.e. we put a probability measure on S_n where each permutation gets probability $1/n!$. Now $f(\pi)$ is a random variable with distribution

$$P\{f(\pi)=p\} = \binom{n}{p} D_{n-p}/n!, \quad p \leqslant n.$$

From this we see that the left hand side of (2) is just the kth moment $E(f^k)$. We now compute this moment in another way to get the right hand side of (2).

AMS Primary Subject Classification: 05A15. Secondary Subject Classifications: 05A19, 05A20.
This research was supported by the Air Force Office of Scientific Research under grant AFOSR 73–2436.

Received December 18, 1973.

Let the random variables f_i, $i=1,\dots,n$, be given by

$$f_i(\pi)=\begin{cases}1 & \text{if i is a fixed point of } \pi,\\ 0 & \text{otherwise}\end{cases}$$

Then we have $f=\sum_{i=1}^n f_i$ and we shall compute $E((f_1+f_2+\cdots+f_n)^k)$ by expanding the product. To keep track of terms in the expansion we label the parentheses; thus we compute

$$E[({}_1f_1+f_2+\cdots+f_n)\,({}_2f_1+f_2+\cdots+f_n)\dots({}_kf_1+f_2+\cdots+f_n)]. \tag{3}$$

Now a term of the product, which consists of choosing a term from each parenthesis, induces a partition on the set $P=\{(_1, (_2,\dots, (_n\}$ of parentheses, by putting two parentheses in the same block if the corresponding factors in the chosen term have the same subscript. Conversely there are $S(k, i)$ partitions of P into i blocks and to each of these partitions there are $(n)_i=n(n-1)\dots(n-i+1)$ terms of the product which induce this partition because the factors of different blocks must have distinct subscripts chosen from $\{1, 2,\dots, n\}$. If a term $f_{j_1}f_{j_2}\dots f_{j_k}$ has i distinct subscripts then

$$\begin{aligned}E(f_{j_1}\cdots f_{j_k})&=P\{f_{j_1}=1,\dots, f_{j_k}=1\}\\&=(n-i)!/n!\\&=\frac{1}{(n)_i}\end{aligned}$$

since these are $(n-i)!$ permutations satisfying the conditions. Combining the above results, those terms in the expansion of (3) which have i distinct subscripts contribute $S(k, i)\,(n)_i(1/(n)_i)=S(k, i)$ to the sum. If $k\leqslant n$ then i cannot exceed k, the number of elements to be partitioned and if $k>n$ then i cannot exceed n, since there can be at most n distinct subscripts; hence the expansion of (3) yields $\sum_{i=1}^{\min(k,n)} S(k, i)$.

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Problems and solutions

This section publishes problems and solutions believed to be new and interesting. Problems are designated by **P1**, **P2**,..., solutions by **P1S1**, **P1S2**,..., and remarks by **P1R1**, **P1R2**,.... Correspondence regarding this section should be sent to the Problems Editor, Prof. M. A. McKiernan, Faculty of Mathematics, University of Waterloo, Ont., Canada. In case several similar solutions are received, the solutions may be edited with credits given the individual contributors.

P112S1 – L. Sonneborn and J. Rätz

The answer is negative: There are countable subfields of R having property (2), *i.e., with isomorphisms* $f: K \to K^+$ *which are also homeomorphisms.* Let K_0 be the field of rational numbers. For any positive integer n, let K_n be the subfield of R generated by $K_{n-1} \cup \exp K_{n-1} \cup \ln K_{n-1}^+$. Then $K := \cup_{n=1}^{\infty} K_n$ is a countable subfield of R satisfying (2), for we have $\exp K = K^+$.

Aequationes Math. *13* (1975), 159–196. University of Waterloo Birkhäuser Verlag, Basel

Bibliographies

Works on functional equations IV

This is a continuation of a bibliography, the first three parts of which have been published in Aequationes Math. *1* (1968), 152–191, *3* (1969), 271–312, and *7* (1971), 270–319 and of the bibliography in J. Aczél's book, *Lectures on Functional Equations and Their Applications* (Academic Press, New York-London, 1966), pp. 383–496. For explanations and classification see Aequationes Math. *1* (1968), 152–155 and 190–191.

The following new sections have been added to the classification.

8.3 Equations for Operators
8.31 Classification of Equations
8.32 Homomorphisms, Derivations
8.33 Centralizers
8.34 Reynolds Operators, Averaging Operators, Smoothing Operators
8.35 Baxter's Equations
8.36 Trigonometric and other Operators

With the evolution of the theories of (algebraic) varieties and of universal algebra, works on 'unusual' algebraic identities, which were included in previous parts of this bibliography, will not be included in this fourth and further parts. – Just as in previous parts of this bibliography or in Aczél's book, functional equations with not more 'free' variables than the number of variables (places) occuring in the unknown function with most variables in the equation, are not included in this part either. For such equations, see M. Kuczma's book, *Functional Equations in a Single Variable*. Polish Scientific Publishers, Warszawa, 1968.

We would be grateful for additions and/or corrections to the present and previous bibliographies.

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Short communications

Graphs from projective planes

T. D. Parsons

Let $G(q)$ be the graph whose vertices are the points of the finite projective plane $PG(2, q)$, and where two vertices are adjacent if they are distinct and orthogonal. The structure of $G(q)$ is studied, and its automorphism group is determined. Certain subgraphs of the graphs $G(q)$ yield three infinite families of connected graphs of girth 3, and two infinite families of connected graphs of girth 5, whose automorphism groups are transitive on ordered pairs of adjacent points. Special cases are the Petersen graph, a 28-point cubic 3-transitive graph of girth 7 due to H. S. M. Coxeter, and a 36-point quintic 2-transitive graph of girth 5. The automorphism groups of these last three graphs are shown to be the projective orthogonal groups $PO_3(5)$, $PO_3(7)$, and a semidirect product $PO_3(9)\#Z_2$.

Received August 8, 1973.

Solutions of bounded variation of a linear homogeneous functional equation in the indeterminate case

Marek Cezary Zdun

In this paper solutions of bounded variation of the functional equation

$$\varphi[f(x)]=g(x)\,\varphi(x) \tag{1}$$

in the indeterminate case are investigated.

A function g is said to be of finite variation in J, or shortly $g\in B_0V[J]$, iff for every finite and closed interval $P\subset J$ we have $\operatorname{Var} g \mid P<\infty$.

A function g is said to be of bounded variation in J, or shortly $g\in BV[J]$, iff there exists a constant $M(g)<\infty$ such that for every finite and closed interval $P\subset J$ we have $\operatorname{Var} g \mid P\leqslant M(g)$.

We assume the following hypotheses in all the theorems below

(i) f is continuous and strictly increasing in an interval $J=(a, b\rangle$ and $a<f(x)<x$ for $x\in J$ (we admit $a=-\infty$);

(ii) $g\in BV[J]$ and $\lim_{x\to a^+} g(x)=1$, $\inf\{g(x)\colon x\in J\}>0$.

Let us put

$$G_n(x)=\prod_{i=0}^{n-1} g[f^i(x)], \quad x\in J, \quad n=1, 2\ldots,$$

where f^i denotes the i-th iterate of the function f.

We have the following theorems:

THEOREM 1. *If there exists* $x_0 \in J$ *such that the product* $\prod_{i=0}^{\infty} g[f^i(x_0)]$ *converges, then equation* (1) *has exactly a one-parameter family of solutions* $\varphi \in B_0 V[J]$ *and such that there exists a finite limit* $\lim_{x \to a^+} \varphi(x)$. *These solutions are given by the formula*

$$\varphi(x) = \frac{\eta}{\prod_{i=0}^{\infty} g[f^i(x)]}, \quad \text{for} \quad x \in J.$$

THEOREM 2. *If there exists an* $x_0 \in J$ *such that* $\lim_{n\to\infty} G_n(x_0) = 0$, *then equation* (1) *has a solution* $\varphi \in B_0 V[J]$ *and such that there exists a finite limit* $\lim_{x\to a^+} \varphi(x)$ *depending on an arbitrary function.*

THEOREM 3. *Suppose that there exists an* $x_0 \in J$ *such that* $\sum_{n=1}^{\infty} G_n(x_0) < \infty$ *Then equation* (1) *has a solution* $\varphi \in BV[J]$ *depending on an arbitrary function.*

THEOREM 4. *If* $\lim_{n\to\infty} G_n(x_0) = 0$ *and* $\sum_{n=1}^{\infty} G_n(x_0) = \infty$ *for an* x_0 *from* J, *then there exists at most a one-parameter family of solutions of equation* (1) *in class* $BV[J]$. *If a function* φ *satisfies equation* (1) *and* $\varphi \in BV[J]$ *then* φ *is given by the formula*

$$\varphi(x) = \eta \lim \frac{G_n(x_0)}{G_n(x)}, \quad \text{for} \quad x \in J, \quad \eta \in R.$$

Moreover an example is given showing that in above case a solution from $BV[J]$ need not exist.

Sufficient conditions for the existence of a solution of (1) in $BV[J]$ are contained in the following theorem:

THEOREM 5. *If there exists a finite limit*, $\lim_{n\to\infty} G_n(x_0)$, *for an* x_0 *from* J, *and* $\sum_{n=1}^{\infty} G_n(x_0) \operatorname{Var} g \mid (a, x_n\rangle < \infty$, *where* $x_n = f^n(x_0)$, *then equation* (1) *has a one-parameter family of solutions* $\varphi \in BV[J]$.

Received October 26, 1973.

A theorem on interpolation in Haar subspaces

T. A. Kilgore and E. W. Cheney

Let X denote the space of continuous real functions on the interval $[-1, 1]$, with norm $\|x\|=\max|x(t)|$. Let Y be an n-dimensional Haar subspace containing constants, with $n\geqslant 3$. For any n points ('nodes'), $-1\leqslant t_1<t_2<\cdots<t_n\leqslant 1$, there exist elements $y_1,\ldots, y_n$ in Y with the property $y_i(t_j)=\delta_{ij}$. The formula $Lx=\sum x(t_i)y_i$ defines a (generalized) Lagrange interpolation process. The linear operator $L:X\to Y$ has norm $\|L\|=\|\sum|y_i|\|$, and the function $\sum|y_i|$ is the 'Lebesgue Function' of L. It has been conjectured that, for some choice of the nodes, the Lebesgue function will exhibit $n+1$ local maxima of equal amplitude, and that this choice of nodes makes $\|L\|$ a minimum. We outline a proof that such a choice of nodes exists.

Let $t_0=-1$, $t_{n+1}=+1$, and let λ_i denote the maximum of the Lebesgue function on $[t_i, t_{i+1}]$ for $0\leqslant i\leqslant n$. Let K denote the simplex in R^n defined by the inequalities $-1\leqslant t_1<t_2<\cdots<t_n\leqslant 1$.

A crucial step in the proof establishes the existence of continuous maps $M_i:K\to K$ $(1\leqslant i\leqslant n)$ with the properties (a) M_i moves only t_i in the n-tuple $T=(t_1,\ldots, t_n)$; (b) $\lambda_i(M_i(T))=\lambda_{i-1}(M_i(T))$. The node vector sought is a fixed point of the map $M==M_1\circ M_2\circ\cdots\circ M_n$, and conversely.

The remainder of the proof is devoted to showing that M has a fixed point in K, despite the fact that K is not compact (and the Brouwer Theorem is not directly applicable).

A second theorem establishes the existence of symmetric nodes $(t_i=-t_{n-i+1})$ if the subspace Y has the property that $y(-t)$ belongs to Y whenever $y(t)$ does.

Received March 25, 1974.

Sharply 2- and 3-transitive groups with kernels of finite index

William Kerby

Sharply 2-transitive permutation groups are completely determined by near-domains in the sense that every such group is isomorphic to a group of transformations of the form $x\to a+mx$, $m\neq 0$, on a near-domain. The kernel of a near-domain F is the multiplicative group $\mathfrak{k}=\{k\in F^*:(a+b)k=ak+bk$, for all $a, b\in F\}$. The kernel of F leads in a natural way to the concept of kernels as subgroups of the stabilizers $G_a=\{\alpha\in G: \alpha(a)=a\}$ of the sharply 2-transitive group G associated with F. The following theorem is proved:

If G is a sharply 2-transitive group and the kernels are of finite index in the stabilizers G_a, then G is isomorphic to the group of transformations $x\to a+mx$, $m\neq 0$ on a skewfield, or G is finite. If G is sharply 3-transitive and the kernels are of finite index in the stabilizers $G_{a,b}$, then G is the group $PGL(2, K)$ for a commutative field K, or G is finite.

Received April 22, 1974.

A construction method for generalized Room squares

I. F. Blake and J. J. Stiffler

A Room square of side $(n-1)$ is an $(n-1)\times(n-1)$ array of cells which are either empty or contain an unordered pair of distinct objects from a set S of n objects such that each row and each column of the array contains each element of S exactly once and each unordered pair of elements of S appears exactly once. It is known that Room squares of all odd sides, except 3 or 5, exist. The concept of a Room square can be generalized as follows. A generalized Room square of order n and degree k is an $\binom{n-1}{k-1}\times\binom{n-1}{k-1}$ array of cells of which each cell is either empty or contains an unordered k-tuple of a set S, $|S|=n$, such that each row and each column of the array contains each element of S exactly once and the array contains each unordered k-tuple of S exactly once. A generalized Room square of order n and degree 2 is simply a Room square of side $(n-1)$.

The purpose of this paper is to present a method for constructing generalized Room squares of order n and degree 3 which has been successful for all orders n less than fifty having the necessary properties. A method is first developed to generate all unordered triples on the set $S=GF(q)\cup\{\infty\}$ where $GF(q)$ is the finite field of order q and ∞ a distinguished element satisfying the usual arithmetic properties. The method utilizes the doubly transitive affine group of transformations on $GF(q)$. For generalized Room squares of order n and degree 3 we must have $3 \mid n$ which, for our construction method implies $3 \mid (q+1)$. The method by which all unordered triples on S are generated is used to construct the generalized Room squares. The success of the method depends upon the existence of a certain set of triples, and it has not yet been proven that such a set always exists. In all cases tried, however, the set was readily found. It is conjectured that it always exists.

Received June 10, 1974.

Aequationes Math. 13 (1975), 201–228, University of Waterloo Birkhäuser Verlag, Basel

Expository papers

On multiplication and factorization of polynomials, I. Lexicographic orderings and extreme aggregates of terms

A. M. Ostrowski

Contents

Introduction

In the discussion of multiplication of polynomials the concepts of the *highest and lowest* terms of a polynomial are, in different connections, fundamental. It is therefore of interest to try, as a generalization, the definition of a general mapping, Λ, of a polynomial into an 'extreme aggregate' of its terms, by some few postulates (Sec. 33).

The classical definition of the highest terms is usually given using the so-called Lexicographic Principle which is one way of establishing an *ordering* in the set of products of powers of independent variables. We discuss, therefore, in the first part of this paper all possible orderings Ω of products of powers satisfying a simple set of postulates (Sec. 1, 2). In this connection the classical Lexicographic Principle is recognized as a special case of a very general principle.

It is then not difficult to establish a one-to-one correspondence between all orderings of the type Ω and all mappings of the type Λ (Sec. 34–38, Theorem III). Then the problem arises to discuss all realizations of postulates defining Ω. This is done in-

AMS (1970) subject classification: Primary 13-02, 13B25. Secondary 06A75.

The preparation of this paper (part I and part II) was first carried out under the contract of the Mathematical Institute, University of Basel, with the European Research Office of the US Army. Further work on this paper was done at the Faculty of Mathematics of the University of Waterloo, Canada, under the sponsorship of the National Research Council of Canada and the Swiss National Science Foundation.

Received January 21, 1975.

troducing convenient 'weight functions' and applying the Lexicographic Principle to a sequence of weight functions (sec. 8–32).

In this way, however, the same aggregates of extreme terms can be obtained for infinitely many choices of weight functions. In order to obtain a complete picture of all possibilities we introduce the *baric polyhedron of a polynomial*, which is uniquely determined by this polynomial. We show that all possible choices of aggregates correspond to different linear boundary components of the baric polyhedron (Sec. 47–52).

The second part of this paper will bring different applications of the concepts introduced here to some irreducibility problems.

A list of introduced technical terms and notations will be given at the end of part II.

It may be finally observed that the concept of the baric polyhedrons and its applications to the irreducibility problem have been the subject of a talk given at the meeting of the German Mathematical Society on September 23, 1921, in Jena. A short summary of this talk was published in the 'Jahresbericht der Deutschen Mathematikervereinigung', *30*, 2nd part, 1922, pp.98–99. Since this summary may be of interest to the reader, we reproduce it here in the original. As the reader will observe, in this summary in particular was announced a complete solution of the reducibility problem for four terms polynomials which however could not be carried through rigorously and appears to require completely new methods, in the case of a baric quadrangle.

– 1. A. Ostrowski: Über die Bedeutung der Theorie der konvexen Polyeder für die formale Algebra (Bericht).

Die vielseitigen Anwendungen, die das Prinzip der lexikographischen Anordnung in der Algebra findet, legen den Gedanken nahe, allgemein nach allen möglichen Arten zu fragen, gewisse Gliederaggregate in Polynomen so auszuzeichnen, daß diese Auszeichnung bei Multiplikation invariant bleibt, d. h. daß die ausgezeichneten Gliederaggregate sich bei Multiplikation reproduzieren, sich also ähnlich verhalten wie die höchsten Glieder von Polynomen bei einer festen Anordnung der Variabeln. Es wird also dabei jedem Polynom $P(x_1,\ldots, x_n)$ ein Gliederaggregat $\bar{P}$ aus P so zugeordnet gedacht, daß $\overline{PP_1}=\bar{P}\cdot\bar{P}_1$ ist. Alle solche Zuordnungen erhält man, wenn man ein System von Gewichtsbestimmungen G_1, $G_2,\ldots$, G_k einführt und zunächst alle Glieder von P auszeichnet, die in bezug auf G_1 die 'schwersten' sind, unter den so ausgezeichneten alle diejenigen, die in bezug auf G_2 die schwersten sind usw. Eine Übersicht über alle bei einem Polynom möglichen Fälle erhält man erst mit Hilfe einer geometrischen Konstruktion. Ordnet man jedem Potenzprodukt $x_1^{m_1}x_2^{m_2}\ldots x_n^{m_n}$ den Punkt des n-dimensionalen Raumes mit den Koordinaten $m_1,\ldots, m_k$ zu und bildet das kleinste konvexe Polyeder, das alle den einzelnen Gliedern eines Polynoms P auf diese Weise zugeordneten Punkte enthält, – das 'Gewichtspolyeder' von P – so kann man an den linearen Begrenzungsmannigfaltigkeiten des Gewichtspolyeders das Verhalten von P gegenüber einem beliebigen System von Gewichtsbestimmungen unmittelbar ablesen. Dem Produkte von zwei Polynomen entspricht die Summe ihrer Gewichtspolyeder im Sinne von Minkowski. Daraus ergeben sich Kriterien für die absolute Irreduzibilität, die nur von den Gewichtseigenschaften eines Polynoms abhängen, also nur von den Exponenten, dagegen nicht von den Koeffizienten. Die entwickelte Methode gestattet insbesondere bei allen Polynomen mit 2, 3, 4 Gliedern alle Typen der reduziblen Polynome aufzustellen.

Die ganze Art der Fragestellung hängt mit der Newton-Puiseuxschen Methode zur Reihenentwicklung algebraischer Funktionen zusammen. –

§ 1. Definition of orderings of products of powers

1. We consider in what follows $m \geqslant 1$ independent variables $x_1, \ldots, x_m$ and the products of powers (in the following denoted as *PP*) of these variables:

$$P := x_1^{\alpha_1} \ldots x_m^{\alpha_m} \tag{1}$$

with *rational integers* α_μ, which can be positive, negative or zero. The set of these *PP* will be denoted by $R[x_1, \ldots, x_m]$. Such *PP* will be called *rational*, and rational *PP* with $\alpha_\mu \geqslant 0$ will be called *integer*.

In the following, P, P_1, P_2, P_3, P_4 are arbitrary *PP* unless specified to the contrary.

2. We consider a set, Ω, of binary relations between *PP*, denoted by $\sim$, $>$ and $<$ (read respectively: is equivalent to; is higher than; is lower than) and satisfying the following postulates:

I. There exists the complete disjunction: either $P_1 \sim P_2$ or $P_1 > P_2$ or $P_1 < P_2$.

II. We have always $P_1 \sim P_1$ and from $P_1 \sim P_2$ follows $P_2 \sim P_1$, while from $P_1 > P_2$ follows $P_2 < P_1$ and vice versa, from $P_2 < P_1$ follows $P_1 > P_2$.

III. From $P_1 > P_2$ and $P_2 \gtrsim P_3$ follows $P_1 > P_3$; from $P_1 \sim P_2$ and $P_2 > P_3$ follows $P_1 > P_3$.

IV. From $P_1 > P_2$ follows $P_3 P_1 > P_3 P_2$.

Such a set Ω defines a *regular ordering* of *PP*.

3. *Observations*:

A. III remains obviously true if all $>$ signs throughout III are replaced with $<$ signs. This follows from II.

B. The transitivity holds also for equivalence relation:

$$\text{III}' \qquad (P_1 \sim P_2) \wedge (P_2 \sim P_3) \Rightarrow (P_1 \sim P_3).$$

Indeed, if we had $P_1 > P_3$ it would follow from III, since $P_3 \sim P_2$, that $P_1 > P_2$, and similarly we can disprove $P_1 < P_2$.

C. The invariance assertion IV holds not only for the relation > but also for the relations < and ~:

$$\text{IV}' \qquad (P_1 \sim P_2) \Rightarrow (P_3 P_1 \sim P_3 P_2), (P_1 < P_2) \Rightarrow (P_3 P_1 < P_3 P_2).$$

Indeed, the last assertion IV′ follows from II replacing in IV the > signs by the < signs; as to the first relation IV′, in the case that $P_1 \sim P_2$, both $P_3 P_1 > P_3 P_2$ and $P_3 P_1 < P_3 P_2$ are impossible, as these relations can be now multiplied by $1/P_3$.

D. By repeated application of IV and IV′ we obtain:

From $(P_1 > P_2, P_3 > P_4)$ or $(P_1 \sim P_2, P_3 > P_4)$ or $(P_1 > P_2, P_3 \sim P_4)$ follows $P_1 P_3 > P_2 P_4$. From $(P_1 \sim P_2, P_3 \sim P_4)$ follows $P_1 P_3 \sim P_2 P_4$.

4. It is easy to see, by repeated application of *D*, that our relations can be 'raised into any positive integer power p'. More precisely: *Let p be a positive integer. Then we have*

$$P_1^p > P_2^p \quad \text{or} \quad P_1^p \sim P_2^p \quad \text{or} \quad P_1^p < P_2^p \tag{2}$$

according as

$$P_1 > P_2 \quad \text{or} \quad P_1 \sim P_2 \quad \text{or} \quad P_1 < P_2. \tag{3}$$

We assumed here p as *positive*. If we replace p with the negative number $-p$, $p > 0$, we have to interchange > sign with < sign: *From any of the relations* (3) *follows the corresponding relation in*

$$1/P_1^p < 1/P_2^p, \; 1/P_1^p \sim 1/P_2^p, \qquad 1/P_1^p > 1/P_2^p \quad (p > 0). \tag{4}$$

5. We could define the ordering directly for the field of *integer PP*. We have then to add to the postulates I–IV the postulate

$$(P_3 P_1 > P_3 P_2) \Rightarrow (P_1 > P_2),$$

since the invariance with respect to division is then no longer contained in IV. Then, obviously, III′ and IV′ are valid again.

But then, any *rational PP*, that is (1) with partly negative α_μ, can be written as $P = P_1/P_2$ with integer *PP*, P_1 and P_2. We can then *define* for integer *PP*: $P_1, \ldots, P_4$: $P_1/P_2 \gtreqless P_3/P_4$, according as $P_1 P_4 \gtreqless P_2 P_3$.

It follows at once from IV and IV′ that this definition does not depend on the special choice of P_1, P_2, P_3, P_4. We see now easily that all postulates I–IV, III′, IV′ remain valid in the field of *rational PP*.

6. Our considerations can be further generalized, and this generalization will later allow to simplify different arguments.

We consider now the *PP* (1) admitting there *arbitrary rational exponents* α_μ. We will speak in this case of *algebraic products of powers (algebraic PP)*. The set of these *PP* will be denoted by $[x_1, \ldots, x_m]$.

No difficulty will arise due to the different values of a fractional power since we can assume in our considerations that all x_μ and their powers are *positive*.

In order to define our relations for the algebraic *PP*,

$$P_1 := x_1^{\alpha_1} \ldots x_m^{\alpha_m}, \qquad P_2 := x_1^{\beta_1} \ldots x_m^{\beta_m}, \tag{5}$$

denote by M the smallest common denominator of all α_μ and β_μ. If P_1 and P_2 are rational *PP* we take M as 1. Then we define: We have $P_1 > P_2$ or $P_1 \sim P_2$ or $P_1 < P_2$ according as $P_1^M > P_2^M$ or $P_1^M \sim P_2^M$ or $P_1^M < P_2^M$.

It follows from the above definition: *Let the positive integer N be such that both P_1^N and P_2^N are rational PP. Then we have any of the three relations $P_1 \gtreqless P_2$ according as the corresponding relation holds between P_1^N and P_2^N.*

Indeed, under our assumption we have $N = pM$ with positive integer p. But then it follows from the Sec. 4 that from any of the relations between P_1^N and P_2^N follows the corresponding relation between P_1^M and P_2^M.

7. We see now easily that all postulates I–IV, III′, IV′ are satisfied for our ordering in the field of algebraic *PP*. More generally it is easily seen that for any positive *rational p*, from any of the relations (3) between algebraic *PP*, P_1 and P_2, follow the corresponding relations in (2) and (4).

From now on we will consider generally the algebraic *PP* unless otherwise specified. Further, all exponents which will occur in the following, will be assumed to be rational numbers unless otherwise specified.

Any ordering of algebraic *PP* satisfying the postulates I–IV will be called a *regular ordering*.

The definition of Ω by the postulates I–IV does not imply directly the expressions (1) of the *PP*. It is therefore *invariant* if we apply a ‘multiplicative reversible transformation’ (*m–r-transformation*) with rational $a_{\mu\nu}$:

$$x_\mu = y_1^{a_{\mu 1}} y_2^{a_{\mu 2}} \ldots y_m^{a_{\mu m}} \qquad (\mu = 1, \ldots, m; \; |\det(a_{\mu\nu})| > 0).$$

Then each *PP* (1) assumes the shape

$$y_1^{\beta_1} \ldots y_m^{\beta_m}$$

of an algebraic *PP* from $[y_1, \ldots, y_m]$. Ω induces an ordering of the *PP* in $[y_1, \ldots, y_m]$, which shall be, of course, formally different from the ordering we obtain replacing in (1) each x_μ by the corresponding y_μ.

The discussion of all possible regular orderings in $[x_1,\ldots,x_m]$ can be interpreted in a different way if we consider, together with the *PP* (1) the set of linear forms

$$S(u):=\alpha_1 u_1+\cdots+\alpha_m u_m$$

with indeterminates $u_1,\ldots,u_m$. We will call (1) and $S(u)$ *associated*. Then to the product of two *PP* is associated the sum of their associated linear forms. Our problem is then equivalent to the problem of characterizing all 'regular' orderings in the set of all linear forms $S(u)$ with rational coefficients.

§2. Weight functions

8. If a real function $W(P)$, defined in $[x_1,\ldots,x_m]$ has the property that

$$W(P_1P_2)=W(P_1)+W(P_2), \tag{6}$$

it will be called a *weight function*. If we put then

$$w_\mu:=W(x_\mu) \quad (\mu=1,\ldots,m) \tag{7}$$

we obtain

$$W\left(x_1^{\alpha_1}\ldots x_m^{\alpha_m}\right)=w_1\alpha_1+\cdots+w_m\alpha_m. \tag{8}$$

We can obviously *define* by (8) any weight function choosing $w_1,\ldots,w_m$ as arbitrary real numbers.

The classical examples of the weight functions are given by the *dimension*:

$$w_1=\cdots=w_m=1;$$

the *degree in* x_1:

$$w_1=1,\ w_2=\cdots=w_m=0;$$

the *classical weight in the theory of symmetric functions*:

$$w_1=1,\ w_2=2,\ldots,\ w_m=m.$$

9. The coefficients w_μ in (8) can be arbitrary real numbers. If all w_μ in (8) are ra-

tional, $W(P)$ is called a *rational weight function.* It is easy to see that any weight function can be represented as a linear combination of rational weight functions. More precisely:

If r is the maximal number of the w_μ in (8) *which are linearly independent with respect to rational numbers, then $W(P)$ in* (8) *can be represented in the form*

$$W(P)=\sum_{\varrho=1}^{r} w^{(\varrho)}W^{(\varrho)}(P) \tag{9}$$

where $w^{(\varrho)}$ are linearly independent real numbers and $W^{(\varrho)}$ (P) are r rational weight functions which are linearly independent as linear forms in the α_μ.

The number r will be called the *rank* of the weight function $W(P)$.

10. It follows now that if $W(P)=0$ then for all $W^{(\varrho)}$ in (9) we have also $W^{(\varrho)}(P)=0$. We can therefore find $m-r$ PP, $P^{(1)},\ldots, P^{(m-r)}$, so that any PP with $W(P)=0$ can be written in the form $P^{(1)u_1}\ldots P^{(m-r)u_r}$ with rational exponents u_ϱ, and conversely for any such PP, W vanishes.

It follows further that if a rational weight function $W^*(P)$ has the property that it vanishes whenever $W(P)$ vanishes, then $W^*(P)$ is a linear combination of the $W^{(\varrho)}(P)$.

The rational weight functions with this property will be called *belonging to* $W(P)$. Obviously the set of all rational weight functions belonging to $W(P)$ is uniquely determined by $W(P)$.

11. We consider now an *ordered sequence of weight functions*

$$W_\kappa(x_1^{\alpha_1}\ldots x_m^{\alpha_m}):=\sum_{\mu=1}^{m} w_\mu^{(\kappa)}\alpha_\mu \quad (\kappa=1,\ldots,k). \tag{10}$$

Using the sequence (10), a regular ordering Ω of PP can be 'induced' by the 'Principle of Lexicographic Ordering', postulating that $P_1\sim P_2$ if $W_\kappa(P_1)=W_\kappa(P_2)$ $(\kappa=1,\ldots,k)$, and that $P_1>P_2$ if for the smallest κ for which $W_\kappa(P_1)\neq W_\kappa(P_2)$, we have $W_\kappa(P_1)>W_\kappa(P_2)$, that is if

$$\exists k_0: W_\kappa(P_1)=W_\kappa(P_2)\,(\kappa<k_0); \qquad W_{k_0}(P_1)>W_{k_0}(P_2). \tag{11}$$

It follows then that $P_1<P_2$ whenever

$$\exists k_0: W_\kappa(P_1)=W_\kappa(P_2)\,(\kappa<k_0); \qquad W_{k_0}(P_1)<W_{k_0}(P_2). \tag{12}$$

The properties I, II, III follow then immediately. The same holds for the property IV, if we take into account that

$$W_\kappa(PP_1)-W_\kappa(PP_2)=W_\kappa(P_1)-W_\kappa(P_2). \tag{13}$$

Observe finally that the weight functions and the ordering defined above by means of weight functions are *invariant* if we apply an $m-r$-transformation of coordinates.

12. The sequence (10) of weight functions allows certain transformations which do not change the ordering Ω.

A. Any weight function in (10) can be multiplied by any positive constant.

B. Any weight function $W_k(P)$ in (10) can be replaced with

$$\bar{W}_k(P) := W_k(P) + \sum_{\kappa<k} c_\kappa W_\kappa^*(P) \quad (k>0), \tag{14}$$

with arbitrary c_κ where each $W_\kappa^*(P)$ is a rational weight function belonging to $W_\kappa(P)$.

C. A weight function which is $\equiv 0$ can be dropped from the sequence (10), if we do not change the order of the remaining elements of (10), reducing in this way k to $k-1$.

Using these observations we can reduce the sequence (10) in such a way that each time when we choose for each $W_\kappa(P)$ a rational function $W^{(\kappa)}(P)$ belonging to it, the κ functions $W^{(\kappa)}(P)$ are linearly independent. Then the sequence (10) is called *regular*. Obviously, for a regular sequence (10) the sum of the ranks of all weight functions in (10) is $\leqslant m$. This sum is $=m$ if and only if the only *PP* which is equivalent to 1 is 1. In particular, for a regular sequence (10), always $k \leqslant m$. The number k is the '*length*' of (10). The maximum number of linearly independent rational weight functions belonging to the W_κ of (10) taken together, is called the *rank* of the sequence (10).

THEOREM I. *Any regular ordering of algebraic PP can be obtained by the lexicographic principle from a regular ordered sequence of weight functions.*

The proof of Theorem I will be given in §4.

§3. Comparability of ordered *PP*

13. We say that P_1 is *comparable with* P_2, in notation $P_1 \mathrm{c} P_2$, if there exist an $\varepsilon = \pm 1$ and two positive rational numbers λ and μ such that

$$P_1 \lesssim P_2^{\varepsilon\lambda}, P_1 \gtrsim P_2^{\varepsilon\mu}, \quad \varepsilon = \pm 1, \quad \lambda \wedge \mu > 0. \tag{15}$$

From this definition follows obviously that $P \mathrm{c} P$ (reflexivity) and that if $P_1 \mathrm{c} P_2$, then also $P_2 \mathrm{c} P_1$ (symmetry). Further, we see that if $P_1 \mathrm{c} P_2$ then also $P_1 \mathrm{c} P_2^{-1}$. Indeed, for a P_2^{-1} we obtain the relations corresponding to (15) replacing ε by $-\varepsilon$. Similarly, using the symmetry, it is seen at once that from $P_1 \mathrm{c} P_2$ follows $P_1^{-1} \mathrm{c} P_2$.

Further it follows from our definition that if $P_1 \mathrm{c} P_2$ and $P_2 \sim 1$ then also $P_1 \sim 1$.

We can now see that if $P_1 \mathrm{c} P_2$ and both P_1, P_2 are $\gtrsim 1$ then ε in (15) can be chosen as 1. This is quite obvious if P_1 or P_2 is ~ 1. We can therefore assume that both P_1 and P_2 are >1. But then the first relation (15) with $\varepsilon = -1$ would give $P_1 < 1$.

Further, it is easy to see that the comparability relation is *transitive*. Observe finally that we have always for a rational $\alpha \neq 0$, $P \mathrm{c} P^{\alpha}$.

14. From the point of view of comparability, the set of all *PP* is now decomposed into *comparability classes*, where all *PP* in a comparability class are comparable, while the *PP* of two different comparability classes are never comparable.

In particular, the comparability class containing 1 will be denoted by U. U consists of all *PP* which are equivalent to 1. If U contains only 1, it is called *trivial*, otherwise *nontrivial*.

LEMMA 1. *Assume C_1 and C_2 two different comparability classes. Denote by P_1, Q_1 two PP from C_1, which are both >1, and by P_2, Q_2 two PP from C_2 which are also >1. Then, if $Q_1 > Q_2$, we have also $P_1 > P_2$.*

– In this case we will say that the class C_1 is *higher* than the class C_2, in notation $C_1 > C_2$. This relationship is obviously transitive.

Proof. Since P_2 and Q_2 are comparable and both $\gtrsim 1$ it follows for a convenient positive rational λ

$$P_2 \lesssim Q_2^{\lambda} < Q_1^{\lambda}.$$

As $Q_1 \in C_1$ we have by (15) $Q_1^{\lambda} \lesssim P_1^{\mu}$ for a convenient positive rational μ and therefore finally $P_2 < P_1^{\mu}$. If we had then $P_1 \lesssim P_2$ it would follow $P_1 \mathrm{c} P_2$. Therefore P_1 is certainly $> P_2$. Our lemma is proved.

It follows now in particular that in the notations of our lemma we even have, for any positive rational δ:

$$P_2 < P_1^{\delta} \quad (\delta \text{ rational, } >0). \tag{16}$$

We express this, writing $P_2 \ll P_1$, $P_1 \gg P_2$.

Observe that if there exists a comparability class $C \neq U$ then always $C > U$. Indeed, in this case C must contain a P which is not equivalent to 1 and can therefore be assumed >1.

15. LEMMA 2. *Assume $k+1>1$ comparability classes*

$$C_0 < C_1 < \cdots < C_k.$$

Choose in each C_κ an arbitrary P_κ ($\kappa = 0, 1, \ldots, k$). Form, for any rational β_κ, the PP:

$$P^* := P_0^{\beta_0} P_1^{\beta_1} \ldots P_k^{\beta_k}. \tag{17}$$

Then, if β_k is $\neq 0$, we have $P^* \in C_k$ and P^* is >1 or <1 according as $P_k^{\beta_k}$ is >1 or <1.

Proof. We can assume without loss of generality that
1) all $P_\kappa > 1$;
2) $\beta_k > 0$;
3) all $\beta_\kappa \neq 0$;
4) $k \geqslant 1$.

Indeed, we can always replace a P_κ by P_κ^{-1}, P^* by P^{*-1} and leave out those C_κ for which $\beta_\kappa = 0$. Put

$$\beta := \sum_{\kappa=0}^{k} |\beta_\kappa|, \qquad \varepsilon := \frac{\beta_k}{2\beta}.$$

Then, by assumptions, we obtain $P_k^{-\varepsilon} < P_\kappa < P_k^{\varepsilon}$ $(\kappa < k)$,

$$P_k^{-\varepsilon|\beta_\kappa|} < P_\kappa^{\beta_\kappa} < P_k^{\varepsilon|\beta_\kappa|} \qquad (\kappa = 0, 1, \ldots, k-1).$$

Multiplying all these relations:

$$P_k^{\varepsilon(\beta_k - \beta)} < \frac{P^*}{P_k^{\beta_k}} < P_k^{\varepsilon(\beta - \beta_k)},$$

and it follows now at once, as $\beta_k > 0$,

$$P_k^{\beta_k/2} < P^* < P_k^{3\beta_k/2},$$

which proves our assertion.

16. Assume now a set $C_0 < C_1 < \cdots < C_k$ of $k+1$ different comparability classes. We take $C_0 = U$ if U is nontrivial. Otherwise we assume $C_0 > U$. Take from each of the C_κ in our set an element $P_\kappa \neq 1$ $(\kappa = 0, \ldots, k)$. Then obviously the relation

$$P_0^{\beta_0} \ldots P_k^{\beta_k} = 1 \tag{18}$$

is impossible unless all β_κ are $= 0$.

Indeed, to prove this, we can assume, without loss of generality that already $\beta_k \neq 0$. But then $P_k^{\beta_k} = P_0^{-\beta_0} \ldots P_{k-1}^{-\beta_{k-1}}$ is an element of $C_k > C_{k-1}$ and this contradicts Lemma 2.

Since, however, a relation of the type (18) is always possible for $k > m$ it follows that, for our set, $k \leqslant m$. We see that the total number of the comparability classes is $\leqslant m+1$, and if in particular U is nontrivial, even $\leqslant m$.

§4. Proof of the Theorem I

17. We prove first the Theorem I in the special case that U is trivial and that, besides U, there is only one comparability class C, so that all x_μ are comparable. We

can further assume that all x_μ are >1, since each x_μ can be replaced with x_μ^{-1}, as an m–r-transformation. Finally we can assume that $m>1$, as for $m=1$ obviously $W(P):=\alpha_1$ satisfies the requirements of Theorem I.

By assumption there exist for every κ, $1<\kappa\leqslant m$, two positive rational numbers λ and μ, such that

$$x_1^\lambda < x_\kappa < x_1^\mu,$$

where obviously $\lambda<\mu$, as $x_1>1$. For no rational σ, x_1^σ can be $\sim$ to x_κ, as otherwise U would contain the element $x_\kappa x_1^{-\sigma}\neq 1$. By Dedekind's axiom, the λ are separated from the μ by a number γ_κ such that for $\lambda<\gamma_\kappa<\mu$ we have always $x_1^\lambda<x_\kappa<x_1^\mu$. For $\kappa=1$ we put $\gamma_1:=1$. Then the following lemma holds.

18. LEMMA 3. *Assume the hypotheses of the preceding section, 17, and use the notation γ_κ for the constants defined in that section. Let α_κ be integers with $A:=\sum_{\kappa=1}^m |\alpha_\kappa|>0$. Put for the P given by* (1)

$$L(P):=\sum_{\kappa=1}^{m} \gamma_\kappa\alpha_\kappa.$$

Let u and v be two rational numbers satisfying $u<L(P)<v$. Then

$$x_1^u<P<x_1^v.$$

Proof. We can assume that $\sum_{\kappa=2}^m |\alpha_\kappa|>0$, as otherwise $P=x_1^{\alpha_1}$ and our assertion is evident. It suffices to prove the first inequality, as then the second one follows, replacing each α_κ with $-\alpha_\kappa$. Choose a positive rational number $\varepsilon<(L(P)-u)/A$ and then $\sigma_1:=1$ and for each $\kappa>1$ a rational σ_κ such that

$$\gamma_\kappa-\varepsilon<\sigma_\kappa<\gamma_\kappa(\alpha_\kappa>0);\quad \gamma_\kappa<\sigma_\kappa<\gamma_\kappa+\varepsilon(\alpha_\kappa<0);\ \sigma_\kappa=0(\alpha_\kappa=0). \tag{19}$$

Then, if we put $u_1:=\sum_{\kappa=1}^m \alpha_\kappa\sigma_\kappa$, obviously $u_1<L(P)$ and

$$L(P)-u_1=\sum_{\alpha_\kappa>0}\alpha_\kappa(\gamma_\kappa-\sigma_\kappa)-\sum_{\alpha_\kappa<0}\alpha_\kappa(\sigma_\kappa-\gamma_\kappa)<A\varepsilon<L(P)-u,$$

and therefore $u_1>u$. On the other hand, it follows from Sec. 18 that, for $\kappa>1$, $x_1^{\alpha_\kappa\sigma_\kappa}<x_\kappa^{\alpha_k}$ and hence $x_1^{u_1}<P$. But now, as $u<u_1$ and $x_1>1$, the first inequality of Lemma 3 follows, and this lemma is proved.

19. From the Lemma 3 it follows immediately that the γ_κ ($\kappa=1,\dots,m$) are linearly

independent with respect to rational numbers. Indeed, if we had a relation with integer β_κ:

$$\sum_{\kappa=1}^{m} \beta_\kappa \gamma_\kappa = 0, \qquad \sum_{\kappa=1}^{m} |\beta_\kappa| > 0,$$

it would follow, putting $P^* := x_1^{\beta_1} \dots x_m^{\beta_m}$, $L(P^*) = 0$ and therefore, by Lemma 3, for any positive rational p:

$$P^* < x_1^p, \qquad P^{*-1} < x_1^p.$$

But then P^* is not comparable to x_1 and must belong to U, while U is assumed trivial.

Further it follows, if $L(P) > 0$, choosing a rational p with $0 < p < L(P)$, $P > x_1^p > 1$, and similarly, if $L(P) < 0$, $P < 1$. Hence, if for two PP, P_1, P_2, we have $L(P_1) \gtrless L(P_2)$, it follows correspondingly $P_1 \gtrless P_2$. We see that the requirements of the Theorem I are satisfied if we take $W(P) := L(P)$. Theorem I is proved in the special case of Sec. 17.

20. We consider now the general case. If $m = 1$, then, for U trivial, we have just a special case of what has been already proved above. If, on the other hand, U is non-trivial, then every $P = x_1^{\alpha_1}$ belongs to U and we have no ordering at all. Here we can choose $W(P) = 0$.

We can therefore proceed by induction and assume that Theorem I has already been proved for all smaller values of m.

21. Let $C_0 < C_1 < \cdots < C_s$ be the ordered sequence of all comparability classes, where however U has to be excluded, if it is trivial. Then we can assume that $s \geqslant 1$, as otherwise this would be the case of Sec. 17, and in this case Theorem I is already proved.

We consider in particular C_0 and observe that if $P_1 \wedge P_2 \in C_0$ then $P_1 P_2$ is either 1 or lies in C_0. Indeed, for any $P > 1$ from C_1 we have for arbitrary positive rational p:

$$P^{-p} < P_1 \wedge P_2 < P^p, \qquad P^{-2p} < P_1 P_2 < P^{2p}$$

so that $P_1 P_2$ belongs to a class $< C_1$, and therefore if $P_1 P_2 \neq 1$, then $P_1 P_2 \in C_0$.

Consider the set T of all linear forms associated with the elements of C_0 as in Sec. 7. Then it follows that T is obtained from a 'module' by dropping the element 0. Let $L_1, \dots, L_t$ be the maximal number of linearly independent forms from T chosen in an arbitrary way. Then any form $L \in T$ can be written as

$$L = \sum_{\tau=1}^{t} \varrho_\tau L_\tau$$

with rational ϱ_τ.

Denote generally the PP associated with L_τ by P_τ. Then it follows that there does not exist any relation of the form

$$P_1^{\beta_1}\dots P_t^{\beta_t}=1, \qquad \sum_{\tau=1}^{t} |\beta_\tau|>0,$$

with integer β_τ, while every element of C_0 can be written as $P_1^{\varrho_1}\dots P_t^{\varrho_t}$ with rational ϱ_τ.

22. By a convenient $m-r$-transformation, as defined in Sec. 7, we can transform the $P_1,\dots, P_t$ respectively into $x_1,\dots, x_t$. We can therefore assume from the beginning that C_0 is identical with $[x_1,\dots, x_t]$, save 1 if U is trivial. But then to the general P (1) from $[x_1,\dots, x_m]$ correspond a $\bar{P}$ from $[x_{t+1},\dots, x_m]$ and a $\bar{P}'$ from $[x_1,\dots, x_t]$, so that

$$P=\bar{P}'\bar{P}, \qquad \bar{P}'=x_1^{\alpha_1}\dots x_t^{\alpha_t}, \qquad \bar{P}=x_{t+1}^{\alpha_{t+1}}\dots x_m^{\alpha_m},$$

where $\bar{P}'$ is either 1 or an element of C_0, and $\bar{P}$ is either 1 or belongs to the same comparability class $C>C_0$ as P. Obviously $\overline{(P_1/P_2)}=\bar{P}_1/\bar{P}_2$.

From Lemma 2 it follows now that $P>1$ or $P<1$ accordingly as $\bar{P}>1$ or $\bar{P}<1$, and, more generally, that $P_1>P_2$ or $P_1<P_2$ accordingly as $\bar{P}_1>\bar{P}_2$ or $\bar{P}_1<\bar{P}_2$. Denote the ordering of the $\bar{P}$, that is, that of $[x_{t+1},\dots, x_m]$ given by Ω, by the symbol $\bar{\Omega}$. Denote further by $\bar{W}(\bar{P}')$ the weight function which generates the ordering in $[x_1,\dots, x_t]$ and the existence of which has been proved in Sec. 19, unless $W(\bar{P}')\equiv 0$, that is, $C_0=U_0$. By hypothesis $\bar{\Omega}$ can be generated by a regular sequence of weight functions, $\bar{W}_1(\bar{P}),\dots, \bar{W}_k(\bar{P})$. Put now $W_\kappa(P):=\bar{W}_\kappa(\bar{P})$ $(x=1,\dots, k)$ and $W_{k+1}(P):=\bar{W}(\bar{P}')$. Then obviously the sequence

$$W_1(P),\dots, W_{k+1}(P) \tag{20}$$

generates Ω. Further (20) is *regular*. Indeed, the rational weight functions belonging to $W_{k+1}(P)$ depend only on $\alpha_1,\dots, \alpha_t$, while the rational weight functions belonging to the $W_\kappa(P)$ with $\kappa\leqslant k$ are independent of $\alpha_1,\dots, \alpha_t$.

Theorem I is proved.

§5. Structure of sequences of weight functions

23. We assume now that the ordering Ω in $[x_1,\dots, x_m]$ is generated by the sequence, of rank r,

$$W_1(P),\dots, W_k(P). \tag{21}$$

About (21) we assume that it is *irreducible*, that is to say, that none of the $W_\kappa(P)$ is a linear combination of the rational weight functions belonging to $W_1(P),\ldots, W_{\kappa-1}(P)$. Then I assert that there *exists a sequence of r rational weight functions*

$$R_1(P),\ldots, R_r(P) \tag{22}$$

and a sequence of positive integers

$$r_1<r_2<\cdots<r_k=r \tag{23}$$

so that generally

$$W_\kappa(P)=\sum_{\tau=1}^{r_\kappa} w_\kappa^{(\tau)} R_\tau(P) \quad (\kappa=1,\ldots, k). \tag{24}$$

Here the set of the linear forms (22) *is linearly independent and the following sets of constants are, each of them, linearly independent with respect to rational numbers*:

$$w_1^{(1)},\ldots, w_1^{(r_1)}; \qquad w_2^{(r_1+1)},\ldots, w_2^{(r_2)};\ldots; w_k^{(r_{k-1}+1)},\ldots, w_k^{(r_k)}. \tag{25}$$

24. In order to define the sequence (22) and the representations (24) we begin by forming a basis $R_1,\ldots, R_{r_1}$ for the rational weight functions belonging to W_1. It follows from what has been said about (9) that the corresponding $w_1^{(\tau)}$ are linearly independent.

We consider further the set of all rational weight functions belonging to W_1 or to W_2, and establish a basis for this set taking care to conserve the basis elements $R_1,\ldots, R_{r_1}$ already found. In this way we obtain additional basis elements R_τ $(r_1<\tau\leqslant r_2)$. Observe that here certainly $r_2>r_1$ as (21) is assumed irreducible. We have obtained a representation (24) with $\kappa=2$. But here it is easily seen that the coefficients of the 'new' R_τ, namely $w_2^{(r_1+1)},\ldots, w_2^{(r_2)}$, are linearly independent. For otherwise, eliminating one of these coefficients we would obtain the corresponding representation of W_2 with at the most r_2-r_1-1 additional basis elements while, by our construction, r_2 is the rank of the linear set formed from the rational weight functions belonging to W_1 or to W_2.

Proceeding in the same way further our assertion becomes evident.

25. If we assume that the sequence (21) is not only irreducible but even *regular*, as defined in Sec. 12, then obviously in (24) for each W_κ the first $R_1,\ldots, R_{r_{\kappa-1}}$ are missing, and we obtain

$$W_\kappa=\sum_\tau w_\kappa^{(\tau)} R_\tau \quad (r_{\kappa-1}<\tau\leqslant r_\kappa;\ 1\leqslant\kappa\leqslant k). \tag{26}$$

26. Write now the R_τ in (22) as linear forms in the α_μ

$$R_\tau = \sum_{\mu=1}^{m} c_{\tau\mu}\alpha_\mu \quad (1 \leqslant \tau \leqslant r).$$

Since the R_τ are linearly independent we can, if $r < m$, introduce $m-k$ further linear forms with rational coefficients,

$$R_\nu = \sum_{\mu=1}^{m} c_{\nu\mu}\alpha_\mu$$

with $r < \nu \leqslant m$ so that the determinant $|c_{\nu\mu}|$ does not vanish. If we now apply the m–r-transformation

$$x_\mu = y_1^{c_{1\mu}} \dots y_m^{c_{m\mu}} \quad (\mu = 1 \dots m) \tag{27}$$

then the PP (1) becomes

$$x_1^{\alpha_1} \dots x_m^{\alpha_m} = y_1^{\beta_1} \dots y_m^{\beta_m}$$

where

$$\beta_\nu = \sum_{\mu=1}^{m} c_{\nu\mu}\alpha_\mu \quad (\nu = 1, \dots, m).$$

We see that our R_ν become now particularly simple linear forms in the β_ν, namely

$$R_\nu = \beta_\nu \quad (\nu = 1, \dots, n).$$

We can now assume, without loss of generality, that the above transformation has been already carried out and the new variables are again denoted by $x_1, \dots, x_m$. Then we have generally $R_\tau \equiv \alpha_\tau$ $(\tau = 1, \dots, r)$, and hence

$$W_\kappa(P) = \sum_\tau w_\kappa^{(\tau)}\alpha_\tau \quad (1 \leqslant \tau \leqslant r_\kappa;\ 1 \leqslant \kappa \leqslant k), \tag{28}$$

in the case of the *irreducible* sequence (21). In the case of a *regular* sequence, we have

$$W_\kappa(P) = \sum_\tau w_\kappa^{(\tau)}\alpha_\tau \quad (r_{\kappa-1} < \tau \leqslant r_\kappa;\ 1 \leqslant \kappa \leqslant k). \tag{29}$$

27. THEOREM II. *Let the ordering Ω in $[x_1, \dots, x_m]$ be generated by an irreducible sequence of weight functions of the length k and the rank r:*

$$W_1, \dots, W_k, \tag{30}$$

and assume that

$$U \equiv C_0 < C_1 < \cdots < C_s, \quad s \geqslant 0 \tag{31}$$

is the complete sequence of comparability classes corresponding to Ω. Then

$$k = s, \tag{32}$$

$$C_\sigma = \{P: W_\kappa(P) = 0 (\kappa \leqslant k - \sigma),\ W_{k-\sigma+1}(P) \neq 0\} \quad (\sigma = 0, \ldots, s). \tag{33}$$

28. In order to prove Theorem II we introduce and discuss certain sets of PP, connected with the sequence (30). Define the sets U_κ of PP by

$$U_\kappa := \{P: W_1(P) = \cdots = W_\kappa(P) = 0\} \quad (\kappa = 1, \ldots, k),\ U_0 := [x_1, \ldots, x_m],\ U_{k+1} := \emptyset. \tag{34}$$

Put further

$$D_\kappa := U_{\kappa-1} - U_\kappa \quad (\kappa = 1, \ldots, k+1). \tag{35}$$

Obviously D_κ is characterized by

$$D_\kappa = \{P: W_1(P) = \cdots = W_{\kappa-1}(P) = 0,\ W_\kappa(P) \neq 0\} \quad (\kappa = 1, \ldots, k), \tag{36}$$

while $D_{k+1} = U_k$. Further, obviously, if $P \in D_\kappa$ then also $P^{-1} \in D_\kappa$.

29. It is easy to see that all PP lying in the same D_κ are comparable. Indeed, assume $P \wedge Q \in D_\kappa$. If $\kappa = k+1$, then $D_{k+1} = U_k$ and we have

$$W_1(P) = \cdots = W_k(P) = W_1(Q) = \cdots = W_k(Q) = 0.$$

But then $P \wedge Q$ are both comparable to 1 and we see that

$$D_{k+1} = U_k = U = C_0.$$

This proves in particular also the assertion of (33) for $\sigma = 0$.

If, on the other hand, $\kappa < k+1$, we can assume without loss of generality, that $P \wedge Q > 1$. But then we have

$$W_1(P) = \cdots = W_{\kappa-1}(P) = W_1(Q) = \cdots = W_{\kappa-1}(Q) = 0, \qquad W_\kappa(P) \wedge W_\kappa(Q) > 0.$$

Thence it is clear that we can find positive rational numbers λ and μ so that $P^\lambda < Q < P^\mu$ and therefore P and Q are comparable.

30. We are now going to show that if

$$1<P\in D_\kappa, \quad 1<Q\in D_\lambda, \quad 0<\lambda<\kappa,$$

then

$$P\ll Q. \tag{37}$$

Indeed, it follows from the assumptions about P and Q that

$$W_1(P)=\cdots=W_{\lambda-1}(P)=W_1(Q)=\cdots=W_{\lambda-1}(Q)=0, \quad W_\lambda(P)=0<W_\lambda(Q).$$

This signifies that P is lower than any rational positive power of Q.

It follows therefore that each D_κ is identical with exactly one of the comparability classes C_σ. Since $[x_1,\ldots,x_m]=\bigcup_{\kappa=1}^{k+1} D_\kappa$ we obtain in this way *all* C_σ. It follows now (32) and further

$$C_\kappa\equiv D_{k-\kappa+1} \quad (0\leqslant\kappa\leqslant k). \tag{38}$$

From (38) and (36), now (33) follows immediately and Theorem II is proved.

32. Observe that $U=C_0$ is characterized by r linear homogeneous equations between the exponents α_μ, so that the dimension of U is $m-r$. More generally, the dimension of each U_κ is $m-r_\kappa$, and further the dimension of D_κ is

$$(m-r_{\kappa-1})-(m-r_\kappa)=r_\kappa-r_{\kappa-1}.$$

Therefore by (38) the dimension of the general C_κ is $r_{k-\kappa+1}-r_{k-\kappa}$.

§6. Extreme aggregates of terms

33. We consider now the set of all polynomials in $x_1,\ldots,x_m$ with coefficients from an arbitrary field K. However, we generalize the concept of a polynomial defining an 'algebraic' polynomial as the sum of the terms $F:=\sum c_\nu P_\nu$ where c_ν are non-vanishing constants from K and P_ν *algebraic PP*. (Observe that even if the field K is of prime characteristic p, this p still could occur in the denominators of the exponents of the algebraic *PP*.) Then we will say in particular that any term $c_\nu P_\nu$ and the corresponding *PP*, P_ν, are 'contained in F' and write it $c_\nu P_\nu\in F$, $P_\nu\in F$. If we write the polynomial F in the form $\sum c_\nu P_\nu$, the *PP*, P_ν, are assumed to be *distinct PP*. If, in particular, all *PP* in F are rational (integer) we call F rational (integer). From now on by polynomial we mean an algebraic polynomial, unless otherwise specified.

We now consider a mapping, Λ, of each polynomial F upon a certain aggregate of its terms, $\bar{F}$, and assume that Λ has the following properties:

A. There is no *PP*, contained both in $\bar{F}$ and $F-\bar{F}$.

B. If $F \not\equiv 0$ then $\bar{F}$ is also $\not\equiv 0$.

C. For any couple of polynomials F_1, F_2, we have

$$\overline{F_1 F_2} = \bar{F}_1 \bar{F}_2. \tag{39}$$

We will then call $\bar{F}$ *extreme aggregate* of F.

From our postulates it follows obviously that for any *PP*: $\overline{cP} = cP$ $(c \in K)$.

For monomial polynomials, F, obviously $\bar{F} \equiv F$. We will generally denote a polynomial which is not a monomial, i.e. contains at least two different *PP*, as *proper polynomials.*

34. In the following P, P_1, P_2, P_3 will denote *general* algebraic *PP*, unless otherwise specified.

Using our mapping Λ we will now define an ordering Ω of *PP*, *induced* by Λ, and prove that Ω is a *regular ordering*.

If P_1, P_2 are two *PP* it follows from the above postulates that $\overline{P_1+P_2}$ is either P_1+P_2 or P_1 or P_2, and this is a complete disjunction. If $\overline{P_1+P_2} = P_1+P_2$ we will say that P_1 and P_2 are *equivalent*, in notation $P_1 \sim P_2$, $P_2 \sim P_1$.

If $\overline{P_1+P_2} = P_1$ we will say that P_1 is *higher than* P_2 *and* P_2 *lower than* P_1, in notation $P_1 > P_2$ and $P_2 < P_1$. And similarly if $\overline{P_1+P_2} = P_2$.

The ordering Ω defined in this way satisfies obviously the postulates I and II of Sec. 2.

Before proving that the postulates III and IV are also satisfied, we make some general observations on the ordering Ω.

35. We assume in this section that F contains the terms $c_1P_1 + c_2P_2$, $P_1 \neq P_2$. Then in the product $G := (P_1+P_2)\,F$, P_1P_2 has *exactly* the coefficient c_1+c_2.

a) If $P_1 \sim P_2$ and c_1P_1 is contained in $\bar{F}$ then c_2P_2 is also contained in $\bar{F}$.

Indeed, if c_2P_2 were missing in $\bar{F}$, under our hypotheses $\bar{G} = (P_1+P_2)\,\bar{F}$ would contain P_1P_2 with a coefficient c_1 while, if P_1P_2 occurs at all in $\bar{G}$, it must have the same coefficient c_1+c_2 as in G.

b) If $P_1 > P_2$ then c_2P_2 is not contained in $\bar{F}$.

Indeed, otherwise in $\bar{G} = P_1\bar{F}$, P_1P_2 would have the coefficient c_2 which is again $\neq c_1+c_2$.

c) If both c_1P_1 and c_2P_2 occur in $\bar{F}$ then $P_1 \sim P_2$.

Indeed, otherwise we have either $P_1 > P_2$ or $P_2 > P_1$ and one of the terms c_1P_1, c_2P_2 would be missing in $\bar{F}$, in virtue of b).

d) If c_1P_1 is contained in $\bar{F}$ but c_2P_2 is not, then $P_1>P_2$.

Indeed, $P_1\sim P_2$ would contradict a) and $P_1<P_2$ would contradict b). We see that Λ is uniquely defined by the ordering Ω induced by it.

36. We prove now that Ω satisfies the postulate IV of Sec. 2. Indeed, we have obviously

$$\overline{P_1P_3+P_2P_3}=\overline{(P_1+P_2)}\,P_3\,.$$

But if $P_1>P_2$ the right hand expression is $=P_1P_3$. Therefore by our definition the *PP*, P_1P_3, is $>P_2P_3$. This is the assertion of the postulate IV.

To prove the postulate III consider the identity

$$F:=(P_1+P_2)\,(P_2+P_3)=P_1P_2+P_1P_3+P_2P_3+P_2^2\,.$$

If $P_1>P_2$, $P_2>P_3$, we have $\bar{F}=P_1P_2$ and it follows from d) that $P_1P_2>P_2P_3$, and multiplying, in virtue of IV, by $1/P_2$, $P_1>P_3$ follows.

If $P_1>P_2$, $P_2\sim P_3$ we have $\bar{F}=P_1P_2+P_1P_3$ and it follows again from d) that $P_1P_2>P_2P_3$, $P_1>P_3$. If finally $P_1\sim P_2$, $P_2>P_3$ we have $\bar{F}=P_1P_2+P_2^2$ and it follows from d) that $P_1P_2>P_2P_3$, $P_1>P_3$.

III is proved. Further, the observations *A*, *B*, *C*, *D* made in Sec. 3 remain valid as they follow from the postulates I–IV.

We see that Ω is a *regular* ordering of *PP*.

37. If $c_1\wedge c_2\neq 0$ we will say that c_1P_1 is *higher than* ($>$) or *equivalent to* ($\sim$) or *lower than* ($<$) c_2P_2 according as $P_1>P_2$ or $P_1\sim P_2$ or $P_1<P_2$. Using this notation we can now formulate the

THEOREM III. *To a mapping Λ as defined in Sec.* 33 *corresponds a regular ordering Ω of PP such that $\bar{F}$ is always the aggregate of all highest terms of F in the sense of Ω. Any regular ordering Ω can be induced by a mapping Λ which is uniquely determined by Ω.*

38. Most assertions of this Theorem follow immediately from the Sec. 34–36.

We have now only to prove that any given Ω is induced indeed by the Λ obtained if we define as $\bar{F}$ the aggregate of the highest terms of *F*. Indeed, the properties *A* and *B* of Sec. 33 hold, obviously, and we have now to prove that the property *C* also holds.

We denote the highest terms of F_1 by $a'_\kappa Q'_\kappa$ and all other terms by $b'_\nu T'_\nu$. Similarly the highest terms of F_2 may be $a''_\lambda Q''_\lambda$ and all other terms $b''_\mu T''_\mu$. We can then write

$$F_1 = \bar{F}_1 + \sum_\nu b'_\nu T'_\nu, \qquad \bar{F}_1 := \sum_\kappa a'_\kappa Q'_\kappa,$$
$$F_2 = \bar{F}_2 + \sum_\mu b''_\mu T''_\mu, \qquad \bar{F}_2 := \sum_\lambda a''_\lambda Q''_\lambda,$$

where the sums over ν and μ could also be empty, and

$$\forall(\kappa, \kappa_1, \nu): \; Q'_\kappa \sim Q'_{\kappa_1}, \; Q'_\kappa > T'_\nu,$$
$$\forall(\lambda, \lambda_1, \mu): \; Q''_\lambda \sim Q''_{\lambda_1}, \; Q''_\lambda > T''_\mu.$$

Then we have

$$F_1 F_2 = \bar{F}_1 \bar{F}_2 + \sum_{\kappa,\mu} a'_\kappa b''_\mu Q'_\kappa T''_\mu + \sum_{\lambda,\nu} a''_\lambda b'_\nu Q''_\lambda T'_\nu + \sum_{\nu,\mu} b'_\nu b''_\mu T'_\nu T''_\mu,$$
$$\sum_{\kappa,\lambda} a'_\kappa a''_\lambda Q'_\kappa Q''_\lambda = \bar{F}_1 \bar{F}_2.$$

Observe that the product $\bar{F}_1 \bar{F}_2$ does not vanish and all its remaining terms are equivalent, by IV′ of Sec. 3.

On the other hand, if Q', Q'', $Q'Q''$ are *PP* contained respectively in $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_1\bar{F}_2$ we obviously have by virtue of IV, IV′

$$\forall(\kappa, \lambda, \mu, \nu): \; Q'Q'' > Q'_\kappa T''_\mu \wedge Q''_\lambda T'_\nu \wedge T'_\nu T''_\mu$$

and $\bar{F}_1\bar{F}_2$ is indeed the aggregate of the highest terms in F_1F_2. The property C is proved.

39. The number of the weight functions in a regular sequence – the length of this sequence – defining a regular ordering Ω depends only on Ω. If this number is 1 we will call the ordering Ω and the mapping Λ induced by Ω *monobaric.*

While there are obviously orderings which are not monobaric it is important for algebraic discussions to prove that as long as we have to do with a fixed *finite* set of *PP* it is quite sufficient to consider only monobaric orderings and mappings.

Let S be a finite set of different *PP*. A given ordering Ω induces the order relation between the elements of S, which we can call the *projection of* Ω *on* S and denote by the symbol Ω_S. If we consider then the set S^* of all polynomials formed with the *PP* from the set S (with arbitrary coefficients from K) Ω induces for each of the polynomials from S^* a mapping which will be denoted by Λ_S. We are going to prove now

THEOREM IV. *If a regular ordering* Ω *and the corresponding mapping* Λ *project upon a finite set of PP, S, and upon the corresponding set* S^* *the ordering* Ω_S *and the* *mapping Λ_S, there exists a monobaric ordering Ω' such that if the corresponding mapping is denoted by Λ' we have*

$$\Omega_S = \Omega'_S; \qquad \Lambda_S = \Lambda'_S. \tag{40}$$

40. *Proof.* Since the relations $P_1 > P_2$, $P_1 < P_2$, $P_1 \sim P_2$ can be written as

$$P_1/P_2 > 1, \quad P_2/P_1 > 1, \quad P_1/P_2 \sim 1,$$

it is sufficient to consider the effect of Ω upon those of the quotients of the *PP* from S which are $\gtrsim 1$. Denote the sequence of these quotients by Q_ν $(\nu = 1, \ldots, N)$.

To the ordering Ω may correspond a regular sequence of weight functions, of length d,

$$W_1, W_2, \ldots, W_d. \tag{41}$$

It is sufficient to show that, if $d > 1$, the sequence (41) can be replaced with a sequence containing less than d terms and corresponding to an ordering with the same effect on the Q_ν $(\nu = 1, \ldots, N)$.

41. Reordering, if necessary, the Q_ν we can assume that for non-negative N_1 and N_2:

$$W_1(Q_1) \wedge \ldots \wedge W_1(Q_{N_1}) > 0, \qquad W_1(Q_\nu) = 0 \quad (\nu > N_1);$$
$$W_2(Q_{N_1+1}) \wedge \ldots \wedge W_2(Q_{N_1+N_2}) > 0, \quad W_2(Q_\nu) = 0 \quad (\nu > N_1 + N_2).$$

If $N_1 = 0$ we can obviously drop $W_1(P)$ and reduce d. If $N_1 > 0$ we can obviously find a positive ε, so small that, putting

$$W^*(P) := W_1(P) + \varepsilon W_2(P),$$

we have

$$W^*(P) > 0, \quad (P = Q_1 \wedge \ldots \wedge Q_{N_1+N_2}).$$

But then we can replace both weight functions W_1, W_2 by $W^*(P)$ and reduce again d by 1. Theorem IV is proved.

42. We assign generally to cP, $c \neq 0$, $c \in K$, the weight of P:

$$W(cP) := W(P).$$

A polynomial in which all terms have the same weight is called *isobaric*. We assign to each isobaric polynomial F the weight of any of its terms,

$$W(F):=W(P) \quad (P\in F).$$

If a polynomial F is not isobaric it can be decomposed into isobaric aggregates of terms,

$$F=\varphi_0+\varphi_1+\cdots+\varphi_k,$$

where each φ_κ is isobaric and

$$W(\varphi_0)>W(\varphi_1)>\cdots>W(\varphi_k).$$

Then we define as $W(F)$ for a *not isobaric polynomial F* the maximum weight of all of its terms:

$$W(F):=W(\varphi_0).$$

The above decomposition will be called simply *the decomposition into isobaric aggregates*, where the single aggregates are always ordered according to decreasing weights. The isobaric aggregate φ_0 will then be called the *leading aggregate* of F. Obviously

$$W(F)=W(\varphi_0)>W(F-\varphi_0).$$

§7. Convex bodies and polyhedrons

43. In our discussion we will have to work with certain convex polyhedrons and remind first the reader of some properties of convex bodies and polyhedrons in the m-dimensional space, which will be used later.

We usually denote the general point of the m-dimensional space R^m by A and its coordinates by $\alpha_1,\ldots,\alpha_m$.

A bounded and closed set of points, C, is called a *convex body*, if it has the *convexity property*, that is if with two arbitrary points A_1, A_2 of C all points of the rectilinear segment $\langle A_1, A_2\rangle$ also belong to C. The dimension of C, $\dim C$, is the smallest integer d such that C lies in a linear d-dimensional manifold.

If $d=0$, C consists of one point only.

A *direction* η in the R^m is defined by m real numbers $w_1,\ldots, w_m$, $w_1^2+\cdots+w_m^2>0$, with the condition that η remains the same if $w_1,\ldots, w_m$ are multiplied by the same *positive* factor. If we multiply all w_μ with -1 we obtain the *opposite direction to* η, $-\eta$.

An $(m-1)$-dimensional plane normal to the direction η is the set of points A satisfying an equation

$$L_\eta(A) := \sum_{\mu=1}^{m} w_\mu \alpha_\mu = d, \tag{42}$$

where the real d can be arbitrarily chosen. If we define d by

$$d := \operatorname{Max}_{A \in C} L_\eta(A),$$

the plane (42) is called the *support plane of* C *in the direction* η. It will be denoted by E_η. It is uniquely determined by the direction η. A convex body is uniquely determined by the set of all of its supporting planes.

The set of all points of E_η common with C will be denoted by C_η,

$$C_\eta := E_\eta \cap C.$$

C_η is again a convex body and, unless $C_\eta = C$, we always have

$$\dim C_\eta < \dim C.$$

The single sets C_η will be called *linear boundary components of* C.

If $\dim C = d > 0$, the *total boundary of* C (from the m-dimensional 'point of view'), ∂C, is the union of all C_η:

$$\partial C := \bigcup_\eta C_\eta.$$

If $\dim C_\eta = 0$ the point C_η will be called a *summit* of C.

A linear boundary component of a C_η is always also a linear boundary component of C.

44. Assume C to be a d-dimensional convex body. If there exists only a finite number of different linear boundary components of C, C is called a *convex polyhedron*. To an m-dimensional convex polyhedron C there always exists a finite set of different directions $\eta_1, \ldots, \eta_N$ such that we have

$$\partial C = \sum_{\nu=1}^{N} C_{\eta_\nu},$$

that each C_{η_ν} here has the dimension $m-1$ and that different C_{η_ν} have in common at the most a linear boundary component of a dimension $<m-1$. These η_ν are uniquely determined.

If we have a finite set of points $A_1, \ldots, A_N$, then the 'smallest' convex polyhedron which contains all A_ν is the set of all points representable in the form

$$A = \sum_{\nu=1}^{N} t_\nu A_\nu \qquad (t_1 \wedge \ldots \wedge t_N \geqslant 0,\ t_1 + \cdots + t_N = 1).$$

This polyhedron will be denoted by $\langle A_1, \ldots, A_N \rangle$.

All summits of $\langle A_1, \ldots, A_N \rangle$ belong to the set of the points $\{A_1, \ldots, A_N\}$. A convex polyhedron has a finite number of summits $S_1, \ldots, S_N$ and can be always formed as $\langle S_1, \ldots, S_N \rangle$.

45. If C', C'' are two convex bodies in R^m then, if A' runs through all points of C' and A'' through all points of C'', the sum $A' + A''$ runs through all points of a convex body which is denoted by $C' + C''$.

If $A' = (\alpha_1', \ldots, \alpha_m')$, $A'' = (\alpha_1'', \ldots, \alpha_m'')$ and $\eta = (w_1, \ldots, w_m)$ is a direction in R^m then we have for the corresponding supporting planes to C' and C'',

$$L_\eta(A') - d' \leqslant 0, \qquad L_\eta(A'') - d'' \leqslant 0$$

and adding

$$L_\eta(A' + A'') - (d' + d'') \leqslant 0,$$

where the equality holds if and only if A' and A'' lie in the corresponding C_η', C_η''. It follows, for any direction η, that $L_\eta(A) = d' + d''$ is a supporting plane for $C' + C''$ and

$$(C' + C'')_\eta = C_\eta' + C_\eta''. \tag{43}$$

It is easy to see that if both C' and C'' are polyhedrons, $C' + C''$ is also a polyhedron. Indeed, in this case there are only a finite number of different ones among the terms $C_\eta' + C_\eta''$ on the right in (43).

46. Consider in particular the case $m = 2$, of a *two-dimensional plane*. Then convex polyhedrons become convex polygons. If in particular a convex polygon is a segment $\langle P_1, P_2 \rangle$, it has to be considered as consisting of two segments of equal length but opposite directions,

$$\overrightarrow{P_1 P_2} \cup \overleftarrow{P_1 P_2}.$$

We provide now our convex two-dimensional polygons with the *orientation*, going along the boundary in the *positive* sense with respect to the inside. Then it follows from the formula (43) that the oriented sides of the polygon $C'+C''$ can only have the directions occurring in the sides of C' and of C''. We obtain then $C'+C''$, decomposing C and C' into the single oriented sides and reordering these sides in the sense of increasing angle with a fixed direction.

It follows now, that if a *triangle* T is the sum of two convex polygons T_1, T_2, none of which reduces to a single point, then both T_1 and T_2 must be also triangles, *similar to* T. Generally, adding two convexe polyhedrons C', C'', all one dimensional linear boundary components of C' and C'' pass, by parallel translation, unchanged into $C'+C''$.

§8. The baric polyhedron

47. The problem of characterizing all mappings Λ in the sense of §6 finds its complete solution in the results of §6. However, in this way we do not obtain a total view of all possible extreme aggregates in, a given polynomial. Such a view can be obtained using the *baric polyhedrons* we are going to introduce now.

Assume an algebraic polynomial F given in the form $F=\sum_{\nu=1}^{n} c_\nu P_\nu$ where all c_ν are $\neq 0$ and $\in K$, while the P_ν are distinct algebraic *PP* of the form (1). To any *PP*, (1), corresponds a 'representative point', A, of R^m with coordinates $\alpha_1,\ldots,\alpha_m$. The same point is, by definition, the representative point of cP with a constant $c\neq 0$ from K. To n terms of F correspond in this way n different points $A_1,\ldots, A_n$. If we form with these points, in the sense of Sec. 44, the polyhedron,

$$C_F := \langle A_1, \ldots, A_n \rangle,$$

this polyhedron will be called the *baric polyhedron of the polynomial F*.

If for a direction η the linear boundary component $(C_F)_\eta$ contains the representative points of some terms cP of F we say that these terms '*lie on*' $(C_F)_\eta$ and write this as

$$cP \in (C_F)_\eta, \qquad P \in (C_F)_\eta.$$

48. We are now going to prove:

THEOREM V. *Let F be an algebraic polynomial. To any mapping Λ in the sense of* §6 *there exists a direction η such that $\bar{F}$ consists of all terms of F lying on $(C_F)_\eta$.*

If, for a given direction η, F^* is the sum of all terms of F lying on $(C_F)_\eta$, there exists a monobaric mapping Λ for which $F^* = \bar{F}$. More generally:

THEOREM V*. *Consider a finite number of algebraic polynomials* $F_1, \ldots, F_N$ *and the corresponding baric polyhedrons* $C_{F_\nu} =: C_\nu$ $(\nu = 1, \ldots, N)$. *Then to any mapping* Λ *corresponds a direction* η *such that, for* ν *running through* $1, \ldots, N$, $\bar{F}_\nu$ *consists of all terms of* F_ν *lying on* $(C_\nu)_\eta$.

Conversely, if we take an arbitrary direction η *and denote, for* $\nu = 1, \ldots, N$, *the sum of terms of the* F_ν *lying on* $(C_\nu)_\eta$ *by* F_ν^*, *then there exists a monobaric mapping* Λ *for which*

$$\bar{F}_\nu = F_\nu^* \qquad (\nu = 1, \ldots, N).$$

49. *Proof of Theorem V**. Assume a general mapping Λ as defined in §6. Then the mapping $F_\nu \to \bar{F}_\nu$ $(\nu = 1, \ldots, N)$ can, by virtue of Theorem IV, also be achieved by a *monobaric* mapping and we can therefore assume Λ to be *monobaric*. Let the corresponding weight function be

$$W(P) := w_1\alpha_1 + \cdots + w_m\alpha_m$$

and denote by η the direction $(w_1, \ldots, w_m)$, so that $W(P) \equiv L_\eta(A)$. Then there exist N constants g_ν such that each $\bar{F}_\nu$ is characterized by

$$W(P) = g_\nu (P \in \bar{F}_\nu),\ W(P) < g_\nu (P \in F_\nu - \bar{F}_\nu) \qquad (\nu = 1, \ldots, N). \tag{44}$$

This signifies, however, that for $\nu = 1, \ldots, N$ each plane $L_\eta(A) - g_\nu = 0$ represents a supporting plane in the direction η to C_ν, and that in particular $\bar{F}_\nu$ consists of all terms of F_ν lying on $(C_\nu)_\eta$.

Consider now an arbitrary direction $\eta := (w_1, \ldots, w_m)$. We define then for each $\nu = 1, \ldots, N$, F_ν^* as in Theorem V* and write the equation of the supporting plane in the direction η to C_ν in the form $L_\eta(A) - g_\nu = 0$. This signifies that introducing the weight function $W(P) := L_\eta(A)$ we have the relations

$$W(P) = g_\nu\ (P \in F_\nu^*),\ W(P) < g_\nu (P \in F_\nu - F_\nu^*) \qquad (\nu = 1, \ldots, N).$$

Comparing this with (44) we see that we have indeed $\bar{F} = F^*$ for the monobaric mapping Λ defined by the weight function $W(P)$. Theorem V* is proved.

50. We are going to prove now that to the *product of polynomials* corresponds the *sum of* their *baric polyhedrons*:

THEOREM VI. *If* F, G *are two algebraic polynomials in* $x_1, \ldots, x_m$ *we have*

$$C_{FG} = C_F + C_G. \tag{45}$$

The formulation of this theorem was conceived starting from Gustave Dumas' theorems on the addition of Newton's Diagrams in the case of the algebraic functions of one variable and of polynomials with p-adic coefficients (Gustave Dumas, Sur les fonctions à caractère algébraique dans le voisinage d'un point donné. Thèse, Paris, 1904; Gustave Dumas, Sur quelques cas d'irréductibilité des polynômes à coefficients rationnels, J. Math. Pures Appl. (2) *2* (1906), S. 191–258.). The reader may be reminded that for a polynomial $f(x, y)$ the Newton Diagram corresponding to $x=\infty$ is obtained from the Baric Polygon of f, taking the upper part of this polygon between the support lines parallel to the y-axis, after reflection in the x-axis.

Proof. Take a direction η and consider the weight function

$$W_\eta(P) := L_\eta(A),$$

where A is the representative point of PP, P. Let the supporting planes of C_F and C_G in the direction η be

$$L_\eta - f_\eta = 0, \qquad L_\eta - g_\eta = 0,$$

where f_η and g_η are real numbers. Then, as we have seen in Sec. 45, the support plane of $C_F + C_G$ is

$$L_\eta - (f_\eta + g_\eta) = 0. \tag{46}$$

If we use now the monobaric mapping Λ defined by $W_\eta(P)$ we have for arbitrary PP, P from F and Q from G:

$$W_\eta(P) - f_\eta \begin{cases} =0 & (P \in \bar{F}) \\ <0 & (P \in F - \bar{F}), \end{cases}$$

$$W_\eta(Q) - g_\eta \begin{cases} =0 & (Q \in \bar{G}) \\ <0 & (Q \in G - \bar{G}). \end{cases}$$

Adding the corresponding relations it follows

$$W_\eta(PQ) - (f_\eta + g_\eta) \begin{cases} =0 & (P \in \bar{F},\ Q \in \bar{G}) \\ <0 & ((P \in F - \bar{F}) \vee (Q \in G - \bar{G})). \end{cases} \tag{47}$$

51. Take now an arbitrary PP, S, from FG. Then

1) if $S \in \overline{FG}$, S can be written as PQ with P from $\bar{F}$ and Q from $\bar{G}$ and it follows from (47)

$$W_\eta(S) - (f_\eta + g_\eta) = 0 \quad (S \in \overline{FG});$$

if on the other hand

2) $S \in FG - \overline{FG}$ then S can be written as PQ, $P \in F$, $Q \in G$, where either P lies in $F - \bar{F}$ or Q lies in $G - \bar{G}$. It follows then from (47)

$$W_\eta(S) - (f_\eta + g_\eta) < 0 \qquad (S \in FG - \overline{FG}).$$

Since the same relations must hold for the supporting plane of C_{FG} in the direction η we see that C_{FG} and $C_F + C_G$ have the same supporting plane (46) in every direction and that these polyhedrons are therefore identical. Theorem V is proved.

52. Among all extreme aggregates of terms contained in a polynomial F we consider in particular those aggregates which consist of one term only. They correspond to the *summits* of C_F and will be called *S terms of F.*

Consider now a polynomial F, an S term, P^*, of F and assume that F is a product,

$$F = GH,$$

of two polynomials G, H. Consider further all weight functions $W(P)$ such that P^* is the highest term of F with respect to $W(P)$.

Then it is easy to see that there is exactly one S term of G and one S term of H which are respectively the highest terms in G and H with respect to these weight functions $W(P)$.

Indeed, taking one of the weight functions $W(P)$ and the corresponding mapping Λ we have $P^* = \bar{G}\bar{H}$ so that $\bar{G}$ and $\bar{H}$ are both monomials. Further, there exists only one couple of terms of G and H such that their product is exactly P^*. Therefore, $\bar{G}$ must be the same for all of our $W(P)$ and the same holds for $\bar{H}$.

The corresponding result holds obviously also for a product of more than two polynomials. We can speak in this connection of the S terms of the factors *corresponding to* an S term of the product.

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Aequationes Math. 13 (1975), 229–232, University of Waterloo Birkhäuser Verlag, Basel

Research papers

On the functional equation $f(x)=\sum_{j=0}^{k-1} f((x+j)/k)$ over finite rings

L. J. Dickey, H.-H. Kairies, and H. S. Shank

§1. Introduction

Functional equations of the type

$$f(x)=k^{m-1}\sum_{j=0}^{k-1} f\left(\frac{x+j}{k}\right) \tag{1}$$

have been considered by Artin, Kuwagaki, Anastassiadis, Dennler, Kairies and others. Anastassiadis [1] has shown that under certain conditions the Bernoulli polynomials B_m arise as unique solution of (1). Artin, [2], p. 34, used (1) to characterize the Gamma function by means of the Gauss multiplication theorem.

In these cases the principal solution (in Nörlund's sense) of the difference equation

$$f(x+1)-f(x)=mx^{m-1} \tag{2}$$

has been characterized by means of its corresponding multiplication theorem (1) – see [6], p. 21, 44, 53.

Dennler [3] proved that cotangent functions are the only meromorphic solutions of (1) when $m=0$. In the context considered by all of the above writers, f was assumed to have real or complex domain and codomain.

In this note it is our intention to discuss solutions of (1) where $m=0$ and $f: \mathbf{Z}_n \to \mathbf{Z}_n$, $\mathbf{Z}_n$ being the ring of integers modulo n. As far as we are aware, no one else has considered solutions of (1) over algebraic structures other than the reals or the complexes.

§2. Theorem and proof

Let $n \in \mathrm{N}$, $n \geqslant 2$ and $k, a \in \mathbf{Z}_n \backslash \{0\}$. We define U and aU by

$$U=\{x \in \mathbf{Z}_n \mid \exists p \in \mathrm{N}: x=k^p\},$$
$$aU=\{x \in \mathbf{Z}_n \mid \exists u \in U: x=au\}.$$

AMS (1970) Subject Classifications: Secondary 39A15, 39A40.

Received July 14, 1972 and, in revised form, March 12, 1974

Further define $\lambda = \lambda(n, k)$ to be the number of distinct sets of the form aU. We remark that if n is a prime, then U is a subgroup of the multiplicative group of $\mathbf{Z}_n$ and $\lambda(n, k)$ is the index of U in this group. If $(n, k) = 1$, the collection of sets $\{aU \mid a \in \mathbf{Z}_n \backslash \{0\}\}$ partitions the non-zero elements of $\mathbf{Z}_n$ into exactly λ sets, which we will call $a_1 U$, $a_2 U, \ldots, a_\lambda U$.

In the following we use the symbols i, j, k, m, n to represent simultaneously elements of $\mathbf{Z}$ and of $\mathbf{Z}_n$.

THEOREM: *Let* $(n, k) = (n, k-1) = 1$ *and* $k < n$. *Then the functional equation*

$$f(x) = \sum_{j=0}^{k-1} f\left(\frac{x+j}{k}\right), \quad f: \mathbf{Z}_n \to \mathbf{Z}_n \tag{3}$$

has exactly $\lambda(n, k)$ *independent solutions* f_i *and the general solution of* (3) *is given by*

$$f = \sum_{i=1}^{\lambda(n,k)} \alpha_i f_i, \qquad \alpha_i \in \mathbf{Z}_n.$$

A method for constructing the f_i is presented in the proof.

Proof: Since $(n, k) = 1$ there is a unique multiplicative inverse for k in $\mathbf{Z}_n$. Hence equation (3) is equivalent to the system

$$\left.\begin{aligned} f(1) &= f\left(\frac{1}{k}\right) + f\left(\frac{2}{k}\right) + \cdots + f(1) \\ f(2) &= f\left(\frac{2}{k}\right) + f\left(\frac{3}{k}\right) + \cdots + f\left(\frac{k+1}{k}\right) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f(n-1) &= f\left(\frac{n-1}{k}\right) + f(0) + \cdots + f\left(\frac{k-2}{k}\right) \\ f(0) &= f(0) + f\left(\frac{1}{k}\right) + \cdots + f\left(\frac{k-1}{k}\right). \end{aligned}\right\} \tag{4}$$

By elementary row operations we obtain an equivalent system

$$\left.\begin{aligned} f(kx+1) - f(kx) &= f(x+1) - f(x), \qquad 1 \leqslant x \leqslant n-1 \\ 0 &= \sum_{j=1}^{k-1} f\left(\frac{j}{k}\right) \end{aligned}\right\} \tag{5}$$

For each solution f of (5) there exists a $(d_1, d_2, \ldots, d_\lambda) \in \mathbf{Z}_n^\lambda$ such that

$$f(x+1) - f(x) = d_j \quad \text{for all} \quad x \in a_j U, \qquad 1 \leqslant j \leqslant \lambda.$$

Conversely choose any $(d_1, d_2, \ldots, d_\lambda) \in \mathbf{Z}_n^\lambda$; we show now that there is a unique function f satisfying

$$\left.\begin{aligned} f(x+1)-f(x)&=d_j, \qquad x\in a_jU \\ 0&=\sum_{j=1}^{k-1} f\left(\frac{j}{k}\right) \end{aligned}\right\} \tag{6}$$

The last equation of (6) can be put in the form

$$0=\sum_{j=1}^{k-1} f\left(\frac{j}{k}\right)=\sum_{j=1}^{k-1}\left[f(1)+\sum_{m=1}^{(j-k)/k}(f(m+1)-f(m))\right],$$

and, since $(n, k-1)=1$, this gives

$$f(1)=-\frac{1}{k-1}\sum_{j=1}^{k-1}\sum_{m=1}^{(j-k)/k}(f(m+1)-f(m)).$$

Thus $f(x)$, $1\leqslant x\leqslant n$, is uniquely determined by (6).

It is obvious that in this procedure different elements of $\mathbf{Z}_n^\lambda$ lead to different functions f. Clearly f is a solution of (3). Hence (3) has exactly n^λ solutions.

Now define $f_i:\mathbf{Z}_n \to \mathbf{Z}_n$, $1\leqslant i\leqslant\lambda$ to be the solution of (6) with $d_j=\delta_{ij}$, $1\leqslant j\leqslant\lambda$.

We prove that $\{f_1, f_2, \ldots, f_\lambda\}$ spans the solution set of (3): Let f be any solution of (3), defined by (6) with $d_j=\alpha_j$. Then $h:=f-\sum_{i=1}^{\lambda}\alpha_i f_i$ too is a solution of (3). Using the definition of f_i we get

$$h(x+1)-h(x)=0, \qquad 1\leqslant x\leqslant n-1,$$

hence $h(x)=h(1)$ for all $x\in\mathbf{Z}_n$. Since $(k-1)\,h(1)=0$, we have $h(x)=0$ for all $x\in\mathbf{Z}_n$, that is: $f=\sum_{i=1}^{\lambda}\alpha_i f_i$.

Finally we show that $\{f_1, f_2, \ldots, f_\lambda\}$ is independent. Suppose there exists $(\beta_1, \beta_2, \ldots, \beta_\lambda)\in\mathbf{Z}_n^\lambda$ such that $\sum_{i=1}^{\lambda}\beta_i f_i$ is the zero function. Then

$$\sum_{j=1}^{\lambda}\beta_j[f_j(x+1)-f_j(x)]=0$$

for all $x\in\mathbf{Z}_n$. Select any set a_iU and let $x_i\in a_iU$. By the definition of f_j we have

$$f_j(x_i+1)-f_j(x_i)=\delta_{ij}.$$

Thus

$$\beta_i=\sum_{j=1}^{\lambda}\beta_j\delta_{ij}=\sum_{j=1}^{\lambda}\beta_j[f_j(x_i+1)-f_j(x_i)]=0.$$

§3. Example for $n=9$, $k=5$

In this case we get $U=\{1, 5, 7, 8, 4, 2\}$, $a_1 U = 1U$ and $a_2 U = 3U = \{3, 6\}$. Hence $\lambda = \lambda(9, 5) = 2$.

System (6) becomes for f_j, $1 \leqslant j \leqslant 2$:

$$\begin{aligned}\delta_{j1} &= f_j(2)-f_j(1)=f_j(6)-f_j(5)=f_j(8)-f_j(7)\\ &= f_j(0)-f_j(8)=f_j(5)-f_j(4)=f_j(3)-f_j(2),\\ \delta_{j2} &= f_j(4)-f_j(3)=f_j(7)-f_j(6),\\ 0 &= f_j(2)+f_j(4)+f_j(6)+f_j(8).\end{aligned}$$

Representing f_j in the form $[f_j(1), f_j(2), \ldots, f_j(9)]$ we finally obtain

$$f_1 = [6, 7, 8, 8, 0, 1, 1, 2, 3] \quad \text{and} \quad f_2 = [8, 8, 8, 0, 0, 0, 1, 1, 1].$$

Thus the general solution of the functional equation

$$f(x) = \sum_{j=0}^{4} f\left(\frac{x+j}{5}\right), \quad f: \mathbf{Z}_9 \to \mathbf{Z}_9$$

is given by $f = \alpha_1 [6, 7, 8, 8, 0, 1, 1, 2, 3] + \alpha_2 [8, 8, 8, 0, 0, 0, 1, 1, 1]$.

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Aequationes Math. 13 (1975), 233–241, University of Waterloo Birkhäuser Verlag, Basel

On an exponential-cosine operator-valued functional equation

A. B. Buche

1. Introduction

Let $\mathscr{X}$ be a Banach space, and let $\mathscr{B}(\mathscr{X})$ denote the family of bounded linear operators on $\mathscr{X}$. Let $R^+=[0, \infty)$, and let I denote the identity operator on $\mathscr{X}$. A one-parameter family of operators $\{T(t), t\in R^+\}$, $T:R^+\to\mathscr{B}(\mathscr{X})$, $T(0)=I$, is said to be a *semigroup of operators* on $\mathscr{X}$, if

$$T(s+t)=T(s)\,T(t), \qquad s, t\in R^+, \tag{1}$$

(cf. Hille and Phillips [4]). Let $\{C(t), t\in R^+\}$, $C:R^+\to\mathscr{B}(\mathscr{X})$, $C(0)=I$, be a one-parameter family of operators satisfying the functional equation

$$C(t+s)+C(t-s)=2C(t)\,C(s), \qquad s, t\in R^+, s<t. \tag{2}$$

Then the family $\{C(t)\}$ is called a *regular cosine-operator* (Kurepa [6], Sova [7]). In this paper we investigate the properties of a family of operators, which is a generalization of the above two types and reduces to them as particular cases under some conditions. The generalization is called an *exponential-cosine operator* family, and is a one-parameter family of operators

$$\{S(t), t\in R^+\}, \quad S:R^+\to\mathscr{B}(\mathscr{X}), \quad S(0)=I,$$

such that

$$S(s+t)-2S(s)\,S(t)=\{S(2s)-2S^2(s)\}\,S(t-s), \quad s, t\in R^+, s<t. \tag{3}$$

It can be easily verified that the operators satisfying eq. (1) or (2) also satisfy (3). In this paper we will show that under natural conditions a solution of (3) can be obtained as a product of an operator semigroup and a regular cosine operator.

Towards this objective we introduce two infinitesimal generators. The first infinitesimal generator is defined in the same way as for operator semigroups (cf. Hille and Phillips [4]), and may be described as the operator-derivative of $S(t)$ of first order with respect to t at $t=0$. The second infinitesimal generator, which is defined, may be described as the operator-derivative of $S(t)$ of second order with respect to t at $t=0$. It is felt that these two infinitesimal generators, if they exist, completely describe the exponential-cosine operator family.

AMS (1970) Subject Classifications: Primary 39A40, 47D99.

Received March 6, 1973, and in revised, form February 5, 1974.

In Section 2, we have investigated the connections between boundedness and continuity in t of the family $\{S(t)\}$. In Section 3, we have discussed some properties of the two infinitesimal generators and derived, under certain conditions, a second order differential equation associated with the defining functional equation. In Section 4, we obtain a solution of the defining functional equation in the uniform operator topology. In Section 5, we have included some examples of the exponential-cosine operator. One of these examples is related to the scalar case. For a detailed survey of functional equations in the scalar case, see Aczél [1].

2. Boundedness and continuity properties

For operator semigroups $\{T(t)\}$ satisfying eq. (1) it is well-known that under some conditions there exist constants $M_1 > 0$ and α such that $\|T(t)\| \leqslant M_1 \cdot \exp(\alpha t)$, $t \in R^+$, (cf. Hille and Phillips [4]). Similarly for the regular cosine operator functions $\{C(t)\}$, it has been established that there exist constants $M_2 > 0$ and β such that $\|C(t)\| \leqslant M_2 \exp(\beta t)$. We now obtain a similar property for $\|S(t)\|$.

PROPOSITION 1. *Let the family* $\{S(t),\ t \in R^+\}$, $S: R^+ \to \mathscr{B}(\mathscr{X})$, $S(0) = I$, *satisfy the eq.* (3). *If* $\log \|S(t)\|$ *is bounded on the interval* $(0, t_0)$ *for each* $t_0 > 0$, *then* $\limsup_{t\to\infty} t^{-1} \log \|S(t)\| < \infty$.

Proof. By hypothesis, there exists $\omega \geqslant 0$, such that $\log \|S(t)\| \leqslant \omega t_0$ for $0 \leqslant t \leqslant t_0$. Clearly then $\sup_{0 \leqslant t \leqslant t_0/2} \log \|S(2t)\| \leqslant \omega t_0$. Putting $t = 2s$ in the eq. (3), we get $S(3s) = 2S(s)\ S(2s) + (S(2s) - 2S^2(s))\ S(s)$, and hence for $0 \leqslant s \leqslant t_0/2$, we have $\|S(3s)\| \leqslant 2 \exp(3\omega t_0) + \exp(3\omega t_0) + 2 \exp(3\omega t_0) \leqslant 5 \exp(3\omega t_0)$. By induction one can establish that, for $0 \leqslant s \leqslant t_0/2$, $\|S(ns)\| \leqslant 3^{n-1} \exp(n\omega t_0)$. (In fact assuming that the above inequality holds for positive integers $m-1$ and m, we have from eq. (3) for $0 \leqslant s \leqslant t_0/2$,

$$\begin{aligned}\|S(m+1)s\| &\leqslant \|2S(s) \cdot S(ms)\| + \|S(2s) - 2S^2(s)\|\ \ \|S((m-1)s)\| \\ &\leqslant 2 \exp(\omega t_0)\ 3^{m-1}\ \exp(m\omega t_0) + (\exp(2\omega t_0) \\ &\qquad + 2\ \exp(2\omega t_0)) \cdot 3^{m-2}\ \exp((m-1)\ \omega t_0) \\ &\leqslant 3^m \exp((m+1)\ \omega t_0)).\end{aligned}$$

Now for any real $t > 0$, select a non-negative integer n, such that $n \leqslant 2t/t_0 < n+1$. Then

$$\begin{aligned}\|S(t)\| &\leqslant 2\|S(t - nt_0/2)\|\ \ \|S(nt_0/2)\| \\ &\qquad + \|S(2t - nt_0) - 2S^2(t - nt_0/2)\|\ \ \|S((nt_0/2) - (t - nt_0/2))\| \\ &\leqslant 2\ \ \exp(\omega t_0)\ \ 3^{n-1}\ \ \exp(n\omega t_0) + 3\ \ \exp(2\omega t_0)\ \ 3^{n-1} \exp(n\omega t_0) \\ &\leqslant 3^{n+1}\ \exp((n+2)\ \omega t_0).\end{aligned}$$

Hence $\log\|S(t)\| \leqslant (n+1)\log 3+(n+2)\,\omega t_0, t>t_0$. Thus for $n\geqslant 2$,

$$t^{-1}\log\|S(t)\| \leqslant (n+2)(\log 3+\omega t_0)/t \leqslant 4\ (\log 3+\omega t_0)/t_0,$$

from which it follows that $\limsup_{t\to\infty} t^{-1}\cdot\log\|S(t)\| \leqslant 4\ (\log 3+\omega t_0)/t_0$.

COROLLARY 1. *Under the conditions of Proposition* 1, *there exist constants* $M>0$ *and* γ *such that* $\|S(t)\| \leqslant M\exp(\gamma t)$, *for all* $t\in R^+$.

In the theory of operator semigroups the continuity with respect to the real parameter t in the uniform operator topology follows from that in the neighbourhood of $t=0$. A similar property holds for the family $\{S(t)\}$ under discussion and is given by the next proposition.

PROPOSITION 2. *Let* $\{S(t), t\in R^+\}$, $S: R^+\to\mathscr{B}(\mathscr{X})$, $S(0)=I$, *be a one-parameter family of operators satisfying* (3). *If*

(i) $S(t)\,S(s)=S(s)\,S(t)$, *for all* $s, t\in R^+$,

(ii) *there exists a* $\delta>0$, *such that the family* $\{S(t)\}$ *is continuous (from the right) for* $0\leqslant t<\delta$ *in the uniform operator topology,*

then the family $\{S(t)\}$ *is continuous at all* $t\in R^+$ *in the uniform operator topology.*

Proof. First of all, we will prove that the continuity from the right at t for $0\leqslant t<\delta$ implies continuity from the right at any $t>0$.

Now the continuity from the right at $t=0$ implies the existence of a real $t'>0$ such that $\{S(t);\ 0\leqslant t\leqslant t'\}$ is bounded. It would then follow from Proposition 1 that $\{S(t),\ 0\leqslant t\leqslant t_0\}$ is uniformly bounded for any finite real $t_0\geqslant 0$. Let $M\geqslant 1$ be such that $\sup_{0\leqslant t\leqslant t_0}\|S(t)\|\leqslant M$.

Now from (3), we have from condition (i), by putting $t=2s$ and then $s=t/3$, $S(t)= =3S(t/3)\ S(2t/3)-2S^3(t/3)$, and a similar expression for $S(t+h)$. Hence for $h>0$, $S(t+h)-S(t)=3\{S((t+h)/3)-S(t/3)\}\cdot S(2t/3)+3S((t+h)/3)\ \{S((2t+2h)/3)-S(2t/3)\}-2\{S((t+h)/3)-S(t/3)\}\cdot\{S^2((t+h)/3)+S((t+h)/3)\ S(t/3)+S^2(t/3)\}$. We assume t_0 to be sufficiently larger than δ and larger than $2t$ for a fixed $t>0$. Let $t_0>3\delta/2$. Then for $0\leqslant t\leqslant t+h<3\delta/2$, and for a given $\varepsilon>0$, we can, in view of condition (ii), choose $\tau>0$, such that $\|S((t+h)/3)-S(t/3)\|\leqslant\varepsilon/12(M+M^2)$, and $\|S\times((2t+2h)/3)-S(2t/3)\|<\varepsilon/12(M+M^2)$, for $0<h<\tau<(3\delta/2)-t$, (the choice of τ may depend upon $t/3$ and $2t/3$, each of which is less than δ). Then $\|S(t+h)-S(t))\| \leqslant(6M+6M^2)\,\varepsilon/12(M+M^2)<\varepsilon$. Thus the continuity in t from the right of $S(t)$ for $0\leqslant t<\delta$ implies that for $0\leqslant t<3\delta/2$. Thus, by induction, $\{S(t)\}$ will be continuous from the right at all $t>0$. Now from eq. (3),

$$\begin{aligned}&S(t+h)-2S(t)+S(t-h)\\&\quad=2S(h)\,S(t)+(S(2h)-2S^2(h))\,S(t-h)-2S(t)+S(t-h)\\&\quad=2(S(h)-I)\,S(t)+(S(2h)-I)\,S(t-h)-2(S(h)+I)\ \ (S(h)-I)\,S(t-h)\end{aligned}$$

Hence for $0 \leqslant t-h<t<t+h<t_0$, and $h<2\tau/3$, we have

$$\|S(t+h)-2S(t)+S(t-h)\| < (2M+M+2M+2M^2)\,\varepsilon/12\,(M+M^2) < \varepsilon/2.$$

Hence $\lim_{h\to 0}\{S(t+h)-2S(t)+S(t-h)\}=0$, in the uniform operator topology for any finite t. Hence $S(t)=\lim_{h\to 0}\{S(t+h)+S(t-h)\}/2$, $t\in R^+$; or

$$\begin{aligned}\lim_{h\to 0}\{S(t)-S(t-h)\} &= -\lim_{h\to 0}\{S(t+h)-2S(t)+S(t-h)-(S(t+h)-S(t))\}\\ &=\lim_{h\to 0}\{S(t+h)-S(t)\}=0.\end{aligned}$$

Hence $\{S(t)\}$ is also continuous from the left at any finite $t>0$.

3. The infinitesimal generators and the associated differential equations

In the theory of operator semigroups, the concept of infinitesimal generator plays a central role (Hille and Phillips [4]). A single infinitesimal generator fully characterizes the operator semigroup, which is an exponential type of operator family. In case of cosine operators (Kurepa [5], [6], and Sova [7]), the (first) infinitesimal generator as defined for semigroups is identically zero, and it is the second generator which characterizes the family. In what follows we assume that $\{S(t)\}$ is a uniformly continuous in t family of operators. We now define

$$A_h=(S(h)-I)/h, \qquad h>0, \tag{4}$$

$$B_h=(S(2h)-2S(h)+I)/h^2, \qquad h>0. \tag{5}$$

The *first infinitesimal generator* A is defined by

$$A=\lim_{h\to 0} A_h, \tag{6}$$

and the *second infinitesimal generator* B is defined by

$$B=\lim_{h\to 0} B_h, \tag{7}$$

whenever the respective limits on the right hand side of (6) and (7) exist in the uniform operator topology.

Sova [7] defines the second generator by

$$B'=\lim_{h\to 0} 2(S(h)-I)/h^2. \tag{8}$$

A connection between B and B' is given by the following

PROPOSITION 3. *Let* $\{S(t),\ t\in R^+\}$, $S: R^+ \to \mathscr{B}(\mathscr{X})$, $S(0)=I$, *satisfy* (3). *If* B' *as defined by* (8) *exists in the uniform operator topology, then so does* B, *and* $B=B'$.

Proof. If B' exists, then $\lim_{h\to 0}(S(2h)-I)/h^2=\lim_{h\to 0}(S(h)-I)/(h^2/4)$. Hence $B'=\lim_{h\to 0}[\{(S(2h)-I)/h^2\}-2\{(S(h)-I)/h^2\}]=\lim_{h\to 0}(S(2h)-2S(h)+I)/h^2=B$.

As already mentioned, for a regular cosine operator $A=0$. This fact is mentioned by Sova [7], but may also be proved simply in the uniform operator topology, whether or not the family $\{S(t)\}$ satisfies (3).

PROPOSITION 4. *Let* $\{S(t),\ t\in R^+\}$, $S: R^+ \to \mathscr{B}(\mathscr{X})$, $S(0)=I$, *be a one-parameter family of operators, such that the linear operator* $B': \mathscr{X}\to\mathscr{X}$, *as defined by* (8) *exists in the uniform operator topology. Then* $A=0$.

Proof. By hypothesis of the Proposition, given $\varepsilon>0$, $\exists\delta>0$, such that for $0<h<\delta$, $\|(S(h)-I)/h^2\|<(\|B'\|+\varepsilon)/2$. Hence $\|(S(h)-I)/h\|<h(\|B'\|+\varepsilon)/2$, for $0<h<\delta$. Letting $h\to 0$, we have $\limsup_{h\to 0}\|(S(h)-I)/h\|=0$. Hence A exists, and $A=0$.

PROBLEM. If $A=0$ and B exists, then one might expect that B' also exists, and $B'=B$. We do not know any direct simple proof of this assertion, and it is deduced indirectly in the course of this paper.

In the theory of operator semigroups and cosine operator functions, the investigation is facilitated by passage to differential and integral equations, from which, under certain conditions, the solution of the original equation, in some operator topology, can be deduced. Using similar methods, we construct, in this section, a second order differential equation corresponding to (3), and use it to obtain a solution of (3) in the uniform operator topology.

THEOREM 1. *Let the family* $\{S(t), t\in R^+\}$, $S: R^+\to\mathscr{B}(\mathscr{X})$, $S(0)=I$, *be uniformly continuous in* t, *and satisfy* (3). *Assume that the derivative* $d^2S(t)/dt^2$ *exists in the uniform operator topology, for each* $t\in R^+$. *If the infinitesimal generators* A *and* B *as defined by* (6) *and* (7), *exist in the uniform operator topology, then*

$$(d^2S(t)/dt^2)=2A(dS(t)/dt)+(B-2A^2)\,S(t),\qquad t\in R^+. \tag{9}$$

Proof. Given a real $h>0$, it follows from (3), that $\{S(t+2h)-2S(t+h)+ S(t)\}/h^2 = 2(S(h)-I)\,(S(t+h)-S(t))/h^2+\{(S(2h)-2S(h)+I)/h^2\}S(t)- -2\{(S(h)-I))^2/h^2\}\,S(t)$, for all $t\in R^+$. Taking limits as $h\to 0$, on both sides, we obtain (9) in view of definitions (6) and (7) and the properties of $\mathscr{B}(\mathscr{X})$.

Note. The assumption about the existence of $d^2S(t)/dt^2$ is necessary, since that, in view of uniform continuity and hence boundedness in t, enables us to assert that the value of $d^2S(t)/dt^2$ coincides with $\lim_{h\to 0}\{S(t+2h)-2S(t+h)+S(t)\}/h^2$.

We now state a lemma.

LEMMA. *If A and B commute, so do any two of the operators A, B, $T(t)$, $C(t)$, $t\in R^+$.*

The proof of the lemma is easy and follows from the known representations of operator semigroups and cosine operator functions in uniform operator topology (Hille and Phillips [4], Kurepa [6], Sova [7]).

We now construct a solution of (9).

THEOREM 2. *Let the family $\{S(t)\}$ satisfy the hypotheses of Theorem* 1. *Let A and B commute. Then* (9) *has the solution $S(t)=T(t)\cdot C(t)$, where $\{T(t)\colon t\in R^+\}$, $T\colon R^+\to\mathscr{B}(\mathscr{X})$, $T(0)=I$, is a uniformly continuous in t semigroup of operators with A as the infinitesimal generator and $\{C(t), t\in R^+\}$, $C\colon R^+\to\mathscr{B}(\mathscr{X})$, $C(0)=I$, is a uniformly continuous in t cosine operator function with $B-A^2$ as the second generator.*

Proof. From the theory of operator semigroups and the cosine operator functions, it follows that $(dT(t)/dt)=AT(t)$, and $(d^2C(t)/dt^2)=(B-A^2)C(t), t\in R^+$. Consider $S(t)=T(t)\,C(t)$. Then

$$\begin{aligned}\{dS(t)/dt\}&=\{dT(t)/dt\}\;C(t)+T(t)\{dC(t)/dt\}\\&=AT(t)C(t)+T(t)\;\{dC(t)/dt\}.\end{aligned}$$

Then

$$\begin{aligned}\{d^2S(t)/dt^2\}&=A\{dS(t)/dt\}+AT(t)\cdot\{dC(t)/dt\}+T(t)\;\{d^2C(t)/dt^2\}\\&=A^2T(t)\;C(t)+2AT(t)\;\{dC(t)/dt\}+T(t)\;(B-A^2)\;C(t)\\&=2A^2T(t)\;C(t)+2AT(t)\;\{dC(t)/dt\}+T(t).\;(B-2A^2)\;C(t).\end{aligned}$$

Thus (9) is satisfied, as B and $T(t), t\in R^+$, will commute since A and B commute.

4. Solution of the defining functional equation

In this section we shall obtain a unique solution of (3) and (9).

THEOREM 3. *Let the family $\{S(t)\}$ satisfy the hypotheses of Theorem* 1. *Further let A and B commute. Then $S(t)=T(t)\,C(t)=C(t)\;T(t)$ is a solution of* (3), *where $\{T(t)\}$, $\{C(t)\}$ are as given by Theorem* 2.

Proof. Let $S(t)=T(t)\,C(t)$, $t\in R^+$. Then for $0<s<t$, $S(s+t)-2S(s)\,S(t) = T(s+t)\;C(s+t)-2T(s)\;C(s)\;T(t)\;C(t) = T(s+t)\;C(s+t)-2T(s)\;T(t)\;C(s)\times C(t)$, for the operators of families $\{T(t)\}$ and $\{C(t)\}$ commute in view of the fact that their generators A and $B-A^2$ commute. Thus

$$\begin{aligned}S(s+t)-2S(s)\,S(t)&=T(s+t)\{C(s+t)-2C(s)\cdot C(t)\}\\&=-T(s+t)\;C(t-s)=-T(2s)\;T(t-s)\;C(t-s)\\&=-T(2s)\,S(t-s).\end{aligned}$$

Thus $\{T(t)\, C(t)\}$ satisfies the eq. (3), since

$$S(2s)-2S^2(s)=T(2s)\cdot C(2s)-2T^2(s)\, C^2(s)=T(2s)\, \{C(2s)-2C^2(s)\}=-T(2s).$$

We have so far established that the representation $S(t)=T(t).\ C(t)$ is a solution of (3) as well as (9). Further, under certain conditions, (3) leads to (9). To complete the cycle, we proceed to show that (9) leads to the representation $S(t)=T(t)\, C(t)$.

THEOREM 4. *Let $\{S(t),\, t\in R^+\}$, satisfy the hypotheses of Theorem* 1. *If A and B commute, then $S(t)=T(t)\, C(t)$, $t\in R^+$, where the families $\{T(t)\}$, $\{C(t)\}$ are as given by Theorem* 3.

Proof. Let $E(t)=T(t)\, C(t)$, $t\in R^+$. Now $(dE(t)/dt)=AT(t)\ C(t)+T(t)\times \times(dC(t)/dt)$, $t\in R^+$. By a result of Sova [7], $dC(t)/dt$ vanishes at $t=0$. Thus $(dE(t)/dt)$ is A at $t=0$, since $T(0)=C(0)=(E(0)=I)$. Thus we have

$$\begin{aligned} S(t)-E(t)&=\int_0^t\int_0^\tau (d^2(S(\sigma)-E(\sigma))/d\sigma^2)\, d\sigma\, d\tau \\ &=\int_0^t\int_0^\tau \{2A(dS(\sigma)/d\sigma)+(B-2A^2)\, S(\sigma)- \\ &\qquad 2A(dE(\sigma)/d\sigma)-(B-2A^2)\, E(\sigma)\}\, d\sigma\, d\tau, \end{aligned}$$

since $\{E(t)\}$ satisfies (9). Thus

$$S(t)-E(t)=2A\int_0^t \{S(\tau)-E(\tau)\}\, d\tau+(B-2A^2)\int_0^t\int_0^\tau \{S(\sigma)-E(\sigma)\}\, d\sigma\, dz. \tag{10}$$

Hence repeating the process

$$\begin{aligned} S(t)-E(t)=&2A\int_0^t 2A\int_0^\tau \{S(\tau_1)-E(\tau_1)\}\, d\tau_1\, d\tau+ \\ &+2A(B-2A^2)\int_0^t\int_0^\tau\int_0^{\tau_1} \{S(\tau_2)-E(\tau_2)\}\, d\tau_2\, d\tau_1\, d\tau+ \\ &+(B-2A^2)\int_0^t\int_0^\tau \{S(\sigma)-E(\sigma)\}\, d\sigma\, d\tau \end{aligned}$$

$$= (B+2A^2) \int_0^t \int_0^\tau \{S(\tau_1) - E(\tau_1)\} \, d\tau_1 \, d\tau +$$

$$+2A(B-2A^2) \int_0^t \int_0^\tau \int_0^{\tau_1} \{S(\tau_2) - E(\tau_2)\} \, d\tau_2 \, d\tau_1 \, d\tau. \tag{11}$$

Since $\{S(t)\}$ is uniformly continuous in $t \geqslant 0$, and $S(0) = I$, there exists a $t_0 > 0$, such that $\|S(t)\|$ is bounded in $[0, t_0]$. Thus by the Proposition 1, $\|S(t)\|$ is bounded on each finite interval of t. Fixing $t > 0$, we have $\|S(\tau)\| \leqslant M_1$, for $0 \leqslant \tau \leqslant t$, for some $M_1 > 0$. Similarly $\|T(\tau)\,C(\tau)\| \leqslant M_2$, for $0 \leqslant \tau \leqslant t$, for some $M_2 > 0$. Thus $\|S(\tau) - E(\tau)\| \leqslant$ $\leqslant M_1 + M_2 = M$, say, $0 \leqslant \tau \leqslant t$. Further since A and B are bounded linear operators, $\exists$ a constant $K > 0$, such that $\|2A\| \leqslant K$, $\|B - 2A^2\| \leqslant K$. Then, from the relation (10), $\|S(t) - E(t)\| \leqslant K\ Mt + K\ Mt^2/2 = KM(t + t^2/2)$. Similarly, from the relation (11), $\|S(t) - E(t)\| \leqslant (K^2 + K)\ Mt^2/2 + K^2 Mt^3/3!$.

By induction we find that, continuing the process n times after (10),

$$\|S(t) - E(t)\| \leqslant M(K+1)^{n+1} \{t^{n+1}/(n+1)!) + (t^{n+2}/(n+2)!)\}. \tag{12}$$

(If this formula were valid, then continuing the process once more,

$$\|S(t) - E(t)\| \leqslant M\{(K+1)^{n+1}\,K + (K+1)^{n+1}\}\ t^{n+2}/(n+2)! +$$
$$+ M(K+1)^{n+1}\ Kt^{n+3}/(n+3)! \leqslant M(K+1)^{n+2}\ \{(t^{n+2}/(n+2)!) + (t^{n+3}/(n+3)!)\}$$

which can be obtained from (12) by replacing n by $n+1$). Now since $K > 0$ and $t \geqslant 0$ are fixed real numbers, the right hand side of the inequality (12) can be made arbitrarily close to zero by taking n sufficiently large. Hence $\|S(t) - E(t)\| = 0$, for all $t \in R^+$. This establishes the result.

5. Examples of the exponential-cosine operator

We now consider a few examples to illustrate the general concept.

(i) Let $\mathscr{X}$ be a Banach space, we define $\{S(t)\}$ by

$$S(t)f = (\exp(\alpha t))\,(\cosh(\beta t))f, \quad f \in \mathscr{X},\ t \in R^+,$$

where α, β are real or complex numbers. It is easy to check that $S(0) = I$, and (3) is satisfied. Further, in this case $A = \alpha$, and $B = \beta^2 + \alpha^2$.

(ii) In [2], an example of a pair of operator-valued functional equations was discussed. In the notations of that paper, the family $\{S(t)\}$ had the representation

$\{V(t)+W(t)\}/2$, where $\{V(t)\}$ and $\{W(t)\}$ are semigroups of operators of class $-(C_0)$, such that $V(t)\cdot W(s)=W(s)\,V(t)$, for all $s, t\in R^+$. We find that in this case,

$$\begin{aligned}
&S(s+t)-2S(s)\,S(t)=\\
&\quad=\{(V(s+t)+W(s+t))/2\}-(V(t)+W(t))\,(V(s)+W(s))/2=\\
&\quad=-(V(t)\,W(s)+V(s)\,W(t))/2=-W(s)\,V(s)\,(V(t-s)+W(t-s))/2=\\
&\quad=-W(s)\,V(s)\,S(t-s), \qquad s<t.
\end{aligned}$$

Further by putting $t=s$, we have $S(2s)-2S^2(s)=-W(s)\,V(s)$. Thus (3) is satisfied. If A_1 and A_2 denote the (bounded) infinitesimal generators of the families $\{V(t)\}$ and $\{W(t)\}$ respectively, then the first and second infinitesimal generators will be respectively $(A_1+A_2)/2$ and $(A_1^2+A_2^2)/2$. The second infinitesimal generator of the associated regular cosine operator function will be given by $((A_1-A_2)/2)^2$. (We note that this generator has a square root, namely, $(A_1-A_2)/2$.)

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The genus of nearly complete graphs–case 6

Jonathan L. Gross

Abstract

The genus of a complete graph equals the least integer greater than or equal to $(E-3V+6)/6$, where E and V are the numbers of edges and vertices of the graph. This paper extends the class of graphs known to have this property, concentrating on graphs whose number of vertices is congruent to 6 modulo 12.

0. Introduction

The Ringel-Youngs equation states that the genus of a complete graph equals the least integer greater than or equal to the number $(E-3V+6)/6$, where E and V are its numbers of edges and faces, a number called the 'Euler lower bound' for reasons soon explained. It follows that for every positive integer n, there is a nonvacuous (obviously finite) set of numbers p such that the removal of any set of p or fewer than p edges from the complete graph K_n yields a connected graph whose genus equals its Euler lower bound. The maximum of such numbers p is designated by $NC(n)$.

A graph on n vertices whose edge-complement in the complete graph K_n has $NC(n)$ or fewer edges is called *nearly complete*.

This paper summarizes what is known about the function $NC(n)$ and derives the inequality

$$NC(n) \geqslant 8$$

for every integer n such that $n \equiv 6$ modulo 12 and $n \geqslant 42$. A thorough reading of J. W. T. Youngs [8] is prerequisite to the understanding of this derivation. Understanding [8] depends, in turn, on reading G. Ringel and Youngs [6] and Youngs [7].

The main purpose of presenting these results is to introduce the concept of nearly complete graphs and to demonstrate the difficulty in calculating values of the function $NC(n)$.

AMS (1970) subject classifications: Primary 05C10, 55A15. Secondary 05C25, 55A10.

This research was initiated with partial support from NSF grant GJ-28403 at Columbia University and continued while the author was an IBM Postdoctoral Fellow in the Mathematical Sciences Department at the Thomas J. Watson Research Center, Yorktown Heights, N.Y. The author is also an Alfred P. Sloan Fellow.

Received May 17, 1973 and, in revised form, May 30, 1974.

1. The Euler lower bound

For any imbedding of a graph K that triangulates a surface (a 'triangular imbedding'), the number E of edges is related to the number F of faces by the equation

$$2E=3F.$$

Substituting the solution $2E/3$ for F into the Euler polyhedral formula

$$V-E+F=2-2g(K)$$

where V is the number of vertices and $g(K)$ the genus of K leads to the equation

$$g(K)=\frac{E-3V+6}{6}.$$

An arbitrary imbedding of any connected graph with no self-adjacencies or multiple adjacencies satisfies the edge-face inequality

$$2E\geqslant 3F.$$

Substituting the upper bound $2E/3$ for F into the Euler formula establishes the inequality

$$g(K)\geqslant\left\lceil\frac{E-3V+6}{6}\right\rceil.$$

The partial brackets are used to signify the ceiling function of the quantity inside, that is $\lceil x\rceil$ is the least integer greater than or equal to x. The right-hand side of the inequality is called the *Euler lower bound* for the genus of K.

For the complete graph K_n, the inequality assumes the form

$$g(K_n)\geqslant\left\lceil\frac{(n-3)(n-4)}{12}\right\rceil.$$

The right-hand side of this latter inequality is designated $I(n)$, because it is closely related to the Heawood map-coloring number $H(n)$.

The connectivity requirement mandates the special values $NC(1)=0$, $NC(2)=0$, $NC(3)=1$, and $NC(4)=2$.

For $n\geqslant 5$, an obvious lower bound on $NC(n)$ is the maximum integer k such that

$$\left\lceil\frac{(n-3)(n-4)}{12}-\frac{k}{6}\right\rceil=I(n)$$

which is denoted $m(n)$. The following table gives the value of $m(n)$ for each residue class of n modulo 12.

residue	0	1	2	3	4	5	6	7	8	9	10	11
$m(n)$	5	2	0	5	5	0	2	5	3	2	2	3

G. Ringel [5] obtained Case 5 of the Ringel-Youngs equation by constructing a triangular imbedding of $K_{12s+5}-K_2$ and attaching a handle (see also Youngs [8]). It follows that if $n\equiv 5$ modulo 12 and $n\geqslant 17$, then $NC(n)$ is at least as large as the maximum integer k such that

$$\left\lceil \frac{(n-3)(n-4)}{12} - \frac{k}{6} \right\rceil = I(n)-1$$

that is, $NC(n)\geqslant 6$. Because of the connectivity requirement, it is necessary to designate $NC(5)=3$.

Using Kuratowski's theorem, it is easy to establish that $NC(6)=2$ and $NC(7)=5$. R. Duke and G. Haggard [2] have proved that certain 8-vertex, 24-edge graphs have genus 2, from which it follows that $NC(8)=3$.

The remainder of this paper is concerned with $NC(n)$ for $n\equiv 6$ modulo 12, $n\geqslant 42$. The values of $NC(18)$ and $NC(30)$ are presently unknown.

2. The special case $n=42$

Figure 1 illustrates the five 3-edge graphs. For convenience of reference, they are labeled A, B, C, D, and E. It is shown in this section that the complement in K_{42} of each of them has genus $I(42)-1$, which implies that $NC(42)$ is at least as large as the maximum integer k such that

$$\left\lceil \frac{(n-3)(n-4)}{12} - \frac{k}{6} \right\rceil = I(n)-1$$

that is, $NC(42)\geqslant 8$.

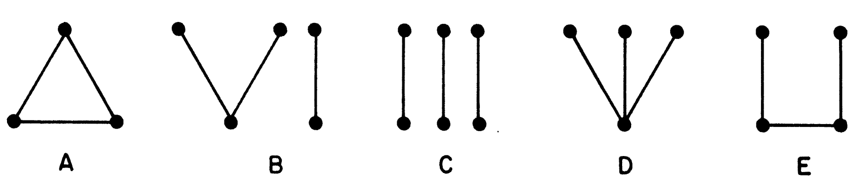

Figure 1. The five 3-edge graphs.

The graph imbedding derived, in accordance with the instructions in [8], from the current graph in Figure 19 of Youngs [8], is a triangular imbedding of $K_{42}-A$ in the surface of genus $I(42)-1$.

If the edge $(0, 1)$ is deleted from that imbedding of $K_{42}-A$ and the edge (x, z) is drawn as a diagonal of the resulting quadrilateral, then the triangular imbedding obtained is of $K_{42}-B$ in the surface of genus $I(42)-1$. It is convenient to invent the notation

$$-(0, 1)+(x, z)$$

for that modification.

An imbedding of $K_{42}-C$ is obtained by applying the modification

$$-(2, 4)+(x, y)$$

to the above imbedding of $K_{42}-B$.

An imbedding of $K_{42}-D$ is obtained from the above imbedding of $K_{42}-B$ by applying the modification sequence

$$\begin{aligned}&-(1, 38)+(x, y)\\&-(1, \ x)+(y, z).\end{aligned}$$

Obtaining an imbedding of $K_{42}-E$ is a lot trickier. It is achieved by applying the modification sequence

$$\begin{aligned}&-(27, 30)+(0, 1)\\&-(17, 18)+(27, 30)\\&-(19, \ x)+(17, 18)\end{aligned}$$

to the imbedding of $K_{42}-B$.

3. The general case $n\equiv 6$ modulo 12

Youngs [8] gives a current graph that yields a triangular imbedding of $K_{12s+6}-A$, $s\geqslant 4$. Imbeddings of the complements of graphs B, C, and D are obtained as in Section 2. An imbedding of $K_{12s+6}-E$, $s\geqslant 4$, is achieved only with the help of a new solution to the chord problem (see Youngs [8]).

To imbed $K_{12s+6}-B$ triangularly (and, therefore, minimally) in the surface of genus $I(12s+6)-1$, apply the modification

$$-(1, 9s)+(x, z)$$

to the imbedding of $K_{12s+6}-A$.

Imbed $K_{12s+6}-C$ by the modification

$$-(2, 6s+4)+(x, y)$$

on the imbedding of $K_{12s+6}-B$.

An imbedding of $K_{12s+6}-D$ is created by the modification sequence

$$-(1, 6s+2)+(x, y)$$
$$-(1, x)+(y, z)$$

on the imbedding of $K_{12s+6}-B$.

The construction of a triangular imbedding of $K_{12s+6}-E$ is begun by replacing two currents on the predetermined part of the Youngs solution [8] to Case 6 of the Heawood problem. In Figure 23, replace currents $6s+1$ and 4 by currents $6s-5$ and 1, respectively. This moves current source y from the vertex below current source x over to the immediate right of the current source z. The resulting zigzag problem is identical to the one solved by Youngs, but the current redistribution does produce a new chord problem as follows.

Points	$P=\hat{0}, 1, \ldots, \widehat{2s-2}, \ldots, \widehat{3s+1}, \ldots, 4s$
Chords	$r=1, \ldots, 2s$.

The key to the solution of this chord problem is found in the repetitive pattern shown in Figure 2.

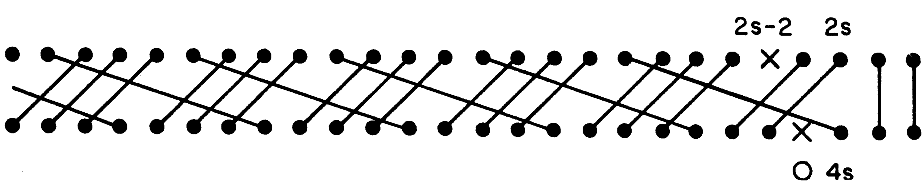

Figure 2. Main pattern in the solution to the revised chord problem.

Figure 3 shows the endplay for various residue classes of s modulo 4. Interpretation of this endplay for small values of s is not difficult. The result is a new triangular imbedding of $K_{12s+6}-A$.

A triangular imbedding of $K_{12s+6}-E$ is now obtained by applying the modification sequence

$$-(0, 3s+4)+(x, z)$$
$$-(9, 12s)+(0, 3s+4)$$
$$-(6, 9s+5)+(9, 12s)$$
$$-(x, 3s+10)+(6, 9s+5)$$

to the new triangular imbedding of $K_{12s+6}-A$.

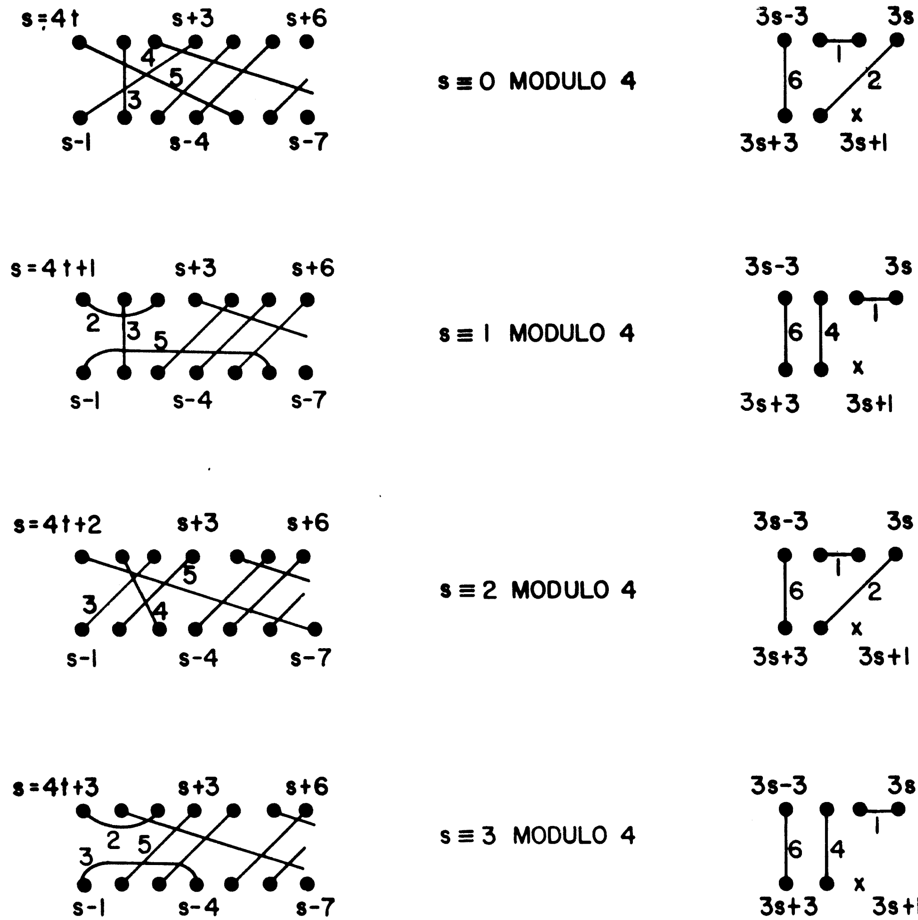

Figure 3. Endplay according to residue class in the solution to the revised chord problem.

4. Current events

The theory of current graphs given by Ringel and Youngs [6] and by Youngs [7, 8] is specialized to what is needed for the Heawood map-coloring problem.
J. L. Gross and S. R. Alpert have announced [3] and presented full details [4] of a general theory, permitting arbitrarily many rotation circuits and vortices of arbitrary order. Alpert and Gross [1] continue this theory.

Ringel has written us that his student M. Jungermann has constructed an imbedding of $K_{12s+2}-K_2$ in $I\,(12s+2)-1$, for all integers $s\geqslant 1$. (This was previously unsolved for even s, but known for odd s.) A brief computation like that for residue class 5 now implies that for $n\equiv 2$ modulo 12 and $n\geqslant 14$, $NC\,(n)\geqslant 6$.

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Aequationes Math. 13 (1975), 251–261, University of Waterloo Birkhäuser Verlag, Basel

Continuous replicative functions

Michael F. Yoder

1. Introduction

Knuth ([1], vol. 1, p. 42) proposes the problem of examining the class of functions which satisfy

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)=a_n f(nx)+b_n \tag{1}$$

for all positive integers n and all real x, a_n and b_n being independent of x.

A great many well-known functions satisfy an equation (1), such as the psi function, the Bernoulli polynomials, some trigonometric functions and others (see [1], vol. 1, p. 42, problems 39, 40).

The only functions f of interest here shall be those which are continuous, or at least piecewise continuous, partly for aesthetic reasons, but also because in such a case integral formulae involving f can be derived ([2], [3]).

The problem considered here is that of finding all continuous functions f which satisfy the functional equation (1) for some a_n, b_n; f is regarded as a complex-valued function on the real line, and $\{a_n\}$, $\{b_n\}$ are complex sequences.

In Section 2, ways of generating replicative functions from other replicative functions are outlined, and relations among the $\{a_n\}$ and $\{b_n\}$ are determined. Next it is assumed f is periodic[1]), and the Fourier series for f is determined in Section 3; in Section 4 all aperiodic functions are found which are replicative. Finally, in Section 5 a very powerful method of generating replicative functions is exhibited.

2. Preliminary definitions and results

DEFINITION.[2]) A function $f:\mathbf{R}\to\mathbf{C}$ is (a_n, b_n)-replicative if for all integer $n\geqslant 1$ and $x\in\mathbf{R}$, the equation (1) holds. If all $b_n=0$, we may abbreviate this to say f is a_n-replicative.

Remarks. (1) Either $f=c$ is constant, whence f is $(a_n, c-ca_n)$-replicative for arbitrary $\{a_n\}$, or $\{a_n\}$, $\{b_n\}$ are uniquely determined by f.

AMS (1970) Subject Classification: Primary 39A30.

[1]) 'periodic' is always taken to mean of period 1.

[2]) The normal definition of 'replicative function' is equivalent in my terminology to being '$1/n$-replicative'.

Received June 6, 1973 and, in revised form, November 12, 1973

(2) If f is (a_n, b_n)-replicative, $f+c$ is (a_n, b_n+c-ca_n)-replicative.

(3) Let f be (a_n, b_n)-replicative and g be (a_n, b'_n)-replicative. Then $c_1 f+c_2 g$ is $(a_n, c_1 b_n+c_2 b'_n)$-replicative.

(4) If f is (a_n, b_n)-replicative, df/dx is na_n-replicative; and if f is a_n-replicative, $\int_a^x f(u)\,du$ is $(a_n/n, b_n)$-replicative for some $\{b_n\}$.

(5) If f is (a_n, b_n)-replicative, then so is $g(x)=f(x-[x])$ where $[x]$ is the greatest integer less than or equal to x. To prove this note that both $a_n g(nx)+b_n$ and $1/n\sum_{k=0}^{n-1} g(x+k/n)$ are periodic with period $1/n$, hence it suffices to prove (1) with f replaced by g for $0\leqslant x<1/n$. In such a case $0\leqslant nx<1$ and $0\leqslant x+k/n<1$ for all $0\leqslant k<n$, so the equation is equivalent to

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)=a_n f(nx)+b_n$$

which is true by assumption.

In view of the first remark, f may be assumed to be nonconstant without loss of generality.

THEOREM 1. *If f is (a_n, b_n)-replicative (and nonconstant), then $a_{mn}=a_m a_n$. Furthermore at least one of the following two cases always holds:*

1. *$a_n=1$ for all n and the b_n satisfy $b_{mn}=b_m+b_n$.*
2. *There exists $c\in\mathbf{C}$ such that $b_n=c(a_n-1)$; in this case $f+c$ is a_n-replicative.*

Proof. We have

$$\begin{aligned} a_{mn} f(mnx)+b_{mn} &=\frac{1}{mn}\sum_{k=0}^{mn-1} f\left(x+\frac{k}{mn}\right)\\ &=\frac{1}{m}\sum_{k=0}^{m-1}\frac{1}{n}\sum_{j=0}^{n-1} f\left(x+\frac{j}{n}+\frac{k}{mn}\right)\\ &=\frac{1}{m}\sum_{k=0}^{m-1}\left(a_n f\left(n\left(x+\frac{k}{mn}\right)\right)+b_n\right)\\ &=b_n+\frac{a_n}{m}\sum_{k=0}^{m-1} f\left(nx+\frac{k}{m}\right)\\ &=b_n+a_n(a_m f(mnx)+b_m)\\ &=a_m a_n f(mnx)+(a_n b_m+b_n). \end{aligned}$$

Since f is not a constant, $a_{mn}=a_m a_n$ and $b_{mn}=a_n b_m+b_n$. By symmetry, $b_{mn}=a_m b_n+b_m$; hence

$$(a_n-1)\,b_m=(a_m-1)\,b_n.$$

If some $a_m \neq 1$, then

$$b_n = \frac{b_n}{a_m - 1}(a_n - 1) = c(a_n - 1)$$

for all n, as desired, and by remark (2) above $f+c$ is a_n-replicative. If all $a_n = 1$, then

$$b_{mn} = a_n b_m + b_n = b_m + b_n.$$

3. The periodic solutions

In view of the fifth remark, any solution to (1) can be made into a periodic solution; if the original f was continuous, the new g will also be continuous except perhaps for a jump at every integer. Hence g has a Fourier expansion which converges $(C, 1)$ at every point to $\frac{1}{2}(g(x^+)+g(x^-))$; this is always equal to $g(x)$ except possibly when x is an integer.

With this in mind, some definitions are in order. Let

$$\varphi_m(x) = e^{2\pi i m x}$$

$$\langle f, g \rangle = \int_0^1 f^* g \, dx$$

so that

$$\langle \varphi_m, \varphi_n \rangle = \delta_{mn}.$$

Let

$$P(a_n; x) = \sum_{n=1}^{\infty} a_n \varphi_n(x) \quad (C, 1)$$

$$Q(a_n; x) = \sum_{n=1}^{\infty} a_n \varphi_{-n}(x) \quad (C, 1)$$

provided only that the series P, Q are Cesàro-summable. Also define $\mathscr{U}(a_n)$ as the vector space (over $\mathbf{C}$) of all continuous a_n-replicative functions, and $\mathscr{P}(a_n)$ as the subspace of $\mathscr{U}(a_n)$ containing all functions of period 1.

We are now almost ready to investigate the consequences of restricting f to be continuous; before proceeding, however, one minor point must be cleared up.

THEOREM 2. *Let f be continuous and $(1, b_n)$-replicative. Then $b_n = 0$ for all n.*

Proof. We have

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) = f(0) + b_n. \tag{2}$$

Now f is Riemann integrable on $[0, 1]$, hence the left side of (2) approaches $\int_0^1 f(u)\,du$ as $n \to \infty$, therefore b_n must approach a limit also. But if some $b_m \neq 0$, then from Theorem 1

$$b_{m^2} = 2b_m, \quad b_{m^3} = 3b_m, \ldots, b_{m^k} = kb_m$$

which diverges as $k \to \infty$. Hence all $b_n = 0$.

Theorems 1 and 2 imply that without loss of generality we may assume all continuous f to be a_n-replicative if they are (a_n, b_n)-replicative.

THEOREM 3. *Let $\{a_n\}$ be a totally multiplicative sequence*[3]*). If $f \in \mathscr{P}(\alpha_n)$, then f is a linear combination of $P(a_n; x)$ and $Q(a_n; x)$ except possibly when all $a_n = 1$; in this case $\mathscr{P}(1)$ is just the space of constant functions. Thus $\mathscr{P}(a_n)$ has dimension $\leqslant 2$. Conversely, any linear combination of $P(a_n; x)$ and $Q(a_n; x)$ which is continuous is a a_n-replicative.*

Proof. We have

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) = a_n f(nx),$$

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \varphi_{mn}^*(x) = a_n f(nx)\, \varphi_{mn}^*(x).$$

So, by periodicity,

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \varphi_{mn}^*\left(x + \frac{k}{n}\right) = a_n f(nx)\, \varphi_m^*(nx)$$

$$\frac{1}{n}\int_0^1 \left(\sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right) \varphi_{mn}^*\left(x + \frac{k}{n}\right)\right) dx = \int_0^1 a_n f(nx)\, \varphi_m^*(nx)\, dx$$

$$\int_0^1 f(u)\, \varphi_{mn}^*(u)\, du = \frac{a_n}{n}\int_0^n f(u)\, \varphi_m^*(u)\, du$$

$$\langle \varphi_{mn}, f\rangle = a_n \langle \varphi_m, f\rangle.$$

In particular

$$\langle \varphi_n, f\rangle = a_n \langle \varphi_1, f\rangle$$
$$\langle \varphi_{-n}, f\rangle = a_n \langle \varphi_{-1}, f\rangle$$
$$\langle \varphi_0, f\rangle = a_n \langle \varphi_0, f\rangle$$

so if any $a_n \neq 1$, $\langle \varphi_0, f\rangle = 0$. If all $a_n = 1$, we know $\langle \varphi_0, f\rangle$ is arbitrary since 1 is 1-

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[3]) i.e. $a_{mn} = a_m a_n$ for all $m, n \geqslant 1$.

replicative. So if $a_n \neq 1$,

$$f(x) = \langle \varphi_1, f \rangle P(a_n; x) + \langle \varphi_{-1}, f \rangle Q(a_n; x).$$

For the converse, note

$$\begin{aligned} \frac{1}{m} \sum_{k=0}^{m-1} P\left(a_n; x + \frac{k}{m}\right) &= \frac{1}{m} \sum_{k=0}^{m-1} \sum_{n=1}^{\infty} a_n\, e^{2\pi i n (x + k/m)} \\ &= \sum_{n=1}^{\infty} a_n\, e^{2\pi i n x} \left(\frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i k (n/m)} \right) \\ &= \sum_{n=1}^{\infty} a_{mn}\, e^{2\pi i m n x} \\ &= \sum_{n=1}^{\infty} a_m a_n\, e^{2\pi i n (m x)} \\ &= a_m P(a_n; mx) \end{aligned}$$

where all sums are $(C, 1)$. Thus $P(a_n; x)$ is a_n-replicative, and similarly $Q(a_n; x)$ is also, whence the converse follows immediately.

Unfortunately, given a particular totally multiplicative sequence $\{a_n\}$, there seems to be no effective way of determining whether $\mathscr{P}(a_n)$ is the null space, or of dimension one, or of dimension two (obviously the latter is true if and only if $P(a_n; x)$ is continuous, since $Q(a_n; x) = P(a_n; -x)$). However a few specialized results can be proven.

COROLLARY 3.1. *If $f \in \mathscr{P}(a_n)$ is not a constant, then all of the following hold:*

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty, \tag{3}$$

$$a_n \to 0 \quad \text{as} \quad n \to \infty,$$

$$|a_n| < 1 \quad \text{for all} \quad n \neq 1.$$

Note that the second follows from the first, and the third from the second upon considering Theorem 1. To prove (3), take $f \in \mathscr{P}(a_n)$ and assume $f = c_0 + c_1 P(a_n; x) + c_2 Q(a_n; x)$ where c_1, c_2 are not both zero. (In general $c_0 = 0$, but if all $a_n = 1$ it might be nonzero). Then

$$\langle f, f \rangle = |c_0|^2 + (|c_1|^2 + |c_2|^2) \sum_{n=1}^{\infty} |a_n|^2 < \infty$$

since f is bounded on $[0, 1]$; as $|c_1|^2 + |c_2|^2 > 0$, the inequality (3) follows. This clearly shows that $\mathscr{P}(1)$ consists of just constant functions.

COROLLARY 3.2. *If*

$$\sum_{n=1}^{\infty} |a_n| < \infty, \quad \textit{then} \quad \dim \mathscr{P}(a_n) = 2.$$

This is clear, since the sums $\sum_{k=1}^{m} a_k \varphi_{\pm k}(x)$ will converge uniformly to $P(a_n; x)$ and $Q(a_n; x)$.

COROLLARY 3.3. *Assume $\dim \mathscr{P}(a_n) = 1$ where $a_n \not\equiv 1$. Then $\mathscr{P}(a_n)$ consists entirely of either even or odd functions, i.e. all elements are multiples of $P(a_n; x) \pm Q(a_n; x)$.*

Proof. If $\dim \mathscr{P}(a_n) = 1$, all functions in $\mathscr{P}(a_n)$ are multiples of some $f(x) = c_1 P(a_n; x) + c_2 Q(a_n; x)$. But $f(-x) = c_1 Q(a_n; x) + c_2 P(a_n; x)$ is also a_n-replicative and is continuous if f is, so we must have $f(-x) = cf(x)$; then obviously $c^2 = 1$, so $c = \pm 1$ and f is either even or odd, as desired.

COROLLARY 3.4. *Let $s = \sigma + it$. Then*

(a) *if* $\sigma < -1$, $\dim \mathscr{P}(n^s) = 2$

(b) *if* $\sigma \geqslant -1$ *and* $s \neq 0$, $\dim \mathscr{P}(n^s) = 0$

(c) if $s = 0$, $\mathscr{P}(n^s)$ *is the space of constant functions.*

Proof. Part (a) follows directly from 3.2 above; and (b) and (c) taken together merely state that for $\sigma \geqslant -1$, neither $R(n^s; x) = \frac{1}{2}(P(n^s; x) + Q(n^s; x))$ nor $S(n^s; x) = 1/2i(Pn^s; x) - Q(n^s; x))$ is continuous (this follows from 3.3).

If $\sigma \geqslant -\frac{1}{2}$, Corollary 3.1 tells us that $\dim \mathscr{P}(n^s) = 0$, since $\sum |n^s|^2 = \infty$. So we only need consider the case $-1 \leqslant \sigma < -\frac{1}{2}$. We have then, if R is continuous,

$$R(n^s; 0) = \sum_{n=1}^{\infty} n^s (C, \varepsilon)$$

for every $\varepsilon > 0$ (see [4], Theorem 253, p. 360). But if $\sum n^s$ is summable (C, ε), then (see [4], Theorem 76, p. 131) $\sum n^{s-\varepsilon}$ is summable $(C, 0)$, i.e. is convergent; if $\sigma \neq -1$ we may take $\varepsilon = (\sigma + 1)/2$ to reach a contradiction. If $\sigma = -1$, then

$$\sum_{n=1}^{\infty} n^{-1+it} = R(n^s; 0)\,(C, 1)$$

but the left-hand sum is not summable $(C, 1)$ (see [4], Sec. 6.11, p. 139).

So R is discontinuous; next suppose $0 < x < 1$ and consider

$$2 \sin \pi x \sum_{n=1}^{\infty} (nx)^s \sin 2\pi nx = x^s \sum_{n=1}^{\infty} n^s (\cos(2n-1)\pi x - \cos(2n+1) nx)$$

$$= x^s (\cos \pi x - 1) + x^s \sum_{n=2}^{\infty} ((n-1)^s - n^s)(1 - \cos(2n-1)\pi x).$$

The latter series is absolutely convergent, since

$$0 \leqslant 1-\cos(2n-1)\pi x = 2\sin^2(n-\tfrac{1}{2})\,\pi x < 2\pi^2 n^2 x^2$$

and for large n

$$(n-1)^s - n^s = -s(n-\tfrac{1}{2})^{s-1} + 0(n^{s-3}).$$

Hence

$$2\sin\pi x \sum_{n=1}^{\infty} (nx)^s \sin 2\pi n x = -s x^s \sum_{n=2}^{\infty} (n-\tfrac{1}{2})^{s-1}(1-\cos 2(n-\tfrac{1}{2})\,\pi x) + 0(x^{s+2})$$

and so

$$x^{s+1}\sum_{n=1}^{\infty} n^s \sin 2\pi n x = -\frac{s}{2\pi}\frac{\pi x}{\sin\pi x}\sum_{n=2}^{\infty} x((n-\tfrac{1}{2})\,x)^{s-1}(1-\cos(2n-1)\,\pi x) + 0(x^{s+2}).$$

The right-hand series and the corresponding Riemann integral

$$\int_0^{\infty} u^{s-1}(1-\cos 2\pi u)\,du$$

are both absolutely convergent, therefore

$$\lim_{x\to 0^+} x^{s+1}\sum_{n=1}^{\infty} n^s \sin 2\pi n x = -\frac{s}{2\pi}\int_0^{\infty} u^{s-1}(1-\cos 2\pi u)\,du$$

Integrating by parts yields

$$\lim_{x\to 0^+} x^{s+1}\sum_{n=1}^{\infty} n^s \sin 2\pi n x = \int_0^{\infty} u^s \sin 2\pi u\,du = (2\pi)^{-s-1}\,\Gamma(1+s)\cos\frac{s\pi}{2} \neq 0.$$

Therefore S behaves like a nonzero constant multiplied by x^{-s-1} around 0, and so is discontinuous at 0. This completes the proof of Corollary 3.4.

4. Aperiodic solutions

Let $\Delta f(x) = f(x+1) - f(x)$. Then subtracting

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right) = a_n f(nx)$$

from

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{1}{n}+\frac{k}{n}\right) = a_n f(nx+1)$$

we find

$$\Delta f(x)=na_n\,\Delta f(nx) \tag{5}$$

Thus if any $a_n=0$, then $\Delta f(x)=0$ and f is periodic. We assume here that f is aperiodic, so that Δf is not identically zero. Then $a_m\neq 0$, so that by replacing x in (5) by mx/n and then using (5) with n replaced by m we obtain

$$\Delta f\left(\frac{m}{n}x\right)=\frac{na_n}{ma_m}\,\Delta f(x). \tag{6}$$

We then derive

THEOREM 4. *Assume* $\mathscr{P}(a_n)\neq\mathscr{U}(a_n)$. *Then there is an* $s=\sigma+it$ *such that* $a_n=n^s$ *for all* n; *and also either* $\sigma<-1$ *or* $s=-1$. *Conversely, if* $\sigma<-1$, *then* $\dim\mathscr{U}(n^s)=4$, *and* $\dim\mathscr{U}(n^{-1})=1$ *so that for these cases* $\mathscr{P}(a_n)\neq\mathscr{U}(a_n)$.

Proof. By (6), if $f\in\mathscr{U}(a_n)-\mathscr{P}(a_n)$ then $a_n\neq 0$ for all n; if we could also have $\Delta f(1)=\Delta f(-1)=0$ then (6) and continuity would imply $\Delta f(x)=0$ for all x. We may therefore take $\Delta f(1)\neq 0$ without loss of generality, so $\Delta f(x)\neq 0$ for $x>0$; hence there exists a continuous function h on $(0,\infty)$ such that $\exp(h(x))=\Delta f(x)$.

Equation (5) then implies

$$\begin{aligned}\exp(h(x)-h(nx))&=na_n\\ h(x)-h(nx)&=\ln na_n+2\pi m_x i,\quad m_x\in\mathbf{Z}.\end{aligned}$$

But h is continuous, so m_x must be a constant (depending perhaps on n), and we may therefore define

$$h(x)-h(nx)=\delta_n\ln n.$$

Then in particular, $h(x)-h(mnx)=(h(x)-h(mx))+(h(mx)-h(mnx))$

$$\delta_{mn}\ln mn=\delta_m\ln m+\delta_n\ln n$$

and by induction

$$\begin{aligned}\delta_{m^k}\ln m^k&=k\,\delta_m\ln m\\ \delta_{m^k}&=\delta_m.\end{aligned}$$

Let $p>2$; if p is a power of 2, then by the above argument $\delta_p=\delta_2$. If not, then $Q=\ln p/\ln 2$ is irrational, hence there are infinitely many continued fraction convergents to Q; denote them by h_i/k_i. Then

$$\left|\frac{h_i}{k_i}-\frac{\ln p}{\ln 2}\right|<\frac{1}{k_i^2};$$

hence

$$|h_i \ln 2 - k_i \ln p| < \frac{\ln 2}{k_i} \to 0, \qquad i \to \infty,$$

so that

$$\exp(h_i \ln 2 - k_i \ln p) = 2^{h_i} p^{-k_i} \to 1, \qquad i \to \infty.$$

So

$$h(1) - h(2^{h_i} p^{-k_i}) \to 0$$

or

$$h_i \delta_2 \ln 2 - k_i \delta_p \ln p \to 0.$$

If $\delta_2 = 0$, this implies $\delta_p = 0$; otherwise

$$\frac{h_i}{k_i} - \frac{\delta_p}{\delta_2} \frac{\ln p}{\ln 2} \to 0.$$

But $h_i/k_i \to \ln p/\ln 2$, whence $\delta p/\delta_2 = 1$ and $\delta_p = \delta_2$.

So in either case $\delta_p = \delta_2$; since all δ_i's are the same we may let $\delta_2 = 1 + s$ to derive

$$\begin{aligned} na_n = \exp(h(x) - h(nx)) &= n^{1+s} \\ h(x) - h(nx) &= (1+s) \ln n \\ a_n &= n^s. \end{aligned}$$

Now, by (6)

$$\Delta f\left(\frac{m}{n}\right) = \left(\frac{m}{n}\right)^{-s-1} \Delta f(1)$$

and by continuity

$$\Delta f(x) = x^{-s-1} \Delta f(1) \quad \text{for} \quad x > 0,$$

similarly

$$\Delta f(x) = |x|^{-s-1} \Delta f(-1) \quad \text{for} \quad x < 0.$$

But Δf is continuous, so we must have either $\sigma < -1$ or $s = -1$ and $\Delta f(1) = \Delta f(-1)$; in all other cases f would be periodic.

For the converse, note that for $s = -1$, $f = x - \frac{1}{2} \in \mathscr{U}(n^s) - \mathscr{P}(n^s)$; for $\sigma < -1$, the same is true of the functions e_1, e_2 defined by

$$\begin{aligned} e_1(x) &= 0 && (x \leqslant 1) \\ \Delta e_1(x) &= x^{-s} && (x \geqslant 0) \\ e_2(x) &= 0 && (x \geqslant 0) \\ \Delta e_2(x) &= |x|^{-s} && (x \leqslant 0). \end{aligned}$$

It is easy to see that all aperiodic solutions are linear combinations of these with periodic solutions, and the statement about the dimensionality of $\mathscr{U}(n^s)$ is verified.

5. Miscellaneous results

If the condition that f is continuous is replaced with the milder one that f have only isolated discontinuities, significant deductions can still be made about the character of f. Let $\mathscr{V}(a_n)$ be the space of a_n-replicative functions with this property; then we have

THEOREM 5. *If $f \in \mathscr{V}(a_n)$, and all $a_n \neq 0$, then f has discontinuities only at integers. (This holds even when f is considered as defined over* **C**).

Proof. By (1), if f has a discontinuity at x, it has one at one of x/n, $x/n+1/n$, ..., $x/n+(n-1)/n$. If x is not a real rational, this immediately implies that there are an infinity of discontinuities within a closed disc of radius $1+|x|$ about the origin upon considering $n=p_1, p_2, \ldots$ where the p_i are distinct primes. If $x=p/q$ in lowest terms, $q \neq 1$, then one of

$$\frac{p}{q^2}, \frac{p+q}{q^2}, \ldots, \frac{p+(q-1)\,q}{q^2}$$

is a discontinuity with a larger denominator, so the same applies.

THEOREM 6. *Let $\{a_n\}$ satisfy $a_{mn}=a_m a_n$. If $\sum_{n=1} a_n\, e^{\pm 2\pi inx}$ is summable (A) to a finite limit except at isolated points, then it defines an a_n-replicative function f, provided that we allow $f(x)=\infty$ at these points.*

Proof. The proof is trivial and is left as an exercise to the reader. Note that if F is any method of summation which satisfies

$$\sum (a_n+b_n)=\sum a_n+\sum b_n \quad (F)$$
$$\sum ca_n=c\sum a_n \quad (F)$$

if $b_{mn}=a_n$ (m a fixed positive integer), $b_{mn+k}=0$ for $0<k<m$, then

$$\sum b_n=\sum a_n \quad (F)$$

then Theorem 6 is true with A replaced by F.

COROLLARY 6.1. *The functions*

$$P(n^s; x)=\sum_{n=1}^{\infty} n^s\, e^{2\pi inx} \qquad (\mathrm{A})$$
$$Q(n^s; x)=\sum_{n=1}^{\infty} n^s\, e^{-2\pi inx} \qquad (\mathrm{A})$$

which are defined (when x is not an integer) for all $s \in \mathbf{C}$, *are* n^s*-replicative.*

See [4], Sections 5.12, 6.10, and 6.11 for verification.

An interesting unresolved problem is to determine for which sequences $\{b_n\}$ there exists an f, continuous except at the integers, such that f is $(1, b_n)$-replicative. The obvious possibility $b_n = \ln n$ actually occurs; the psi function defined by

$$\psi(x) = -\lim_{n \to \infty} \left(\sum_{m=1}^{n} \frac{1}{x+m} - \ln n \right)$$

is $(1, \ln n)$-replicative (see [5], p. 330).

6. Acknowledgements and bibliography

I wish to extend my thanks and gratitude for assistance rendered during the preparation of this paper to Dr. Robert McEliece of the Jet Propulsion Laboratory; to Professor John Todd of the California Institute of Technology; to two people whom I know only as Referee #1 and Referee #2, both of whom suggested proofs of Corollary 3.4 more concise than the original (the one used is due to Referee #1); and also to Professor D. E. Knuth of Stanford University for proposing the problem in the first place.

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F-Expansions of rationals

M. S. Waterman

The ergodic theorem has been used to deduce results about the F-expansions of almost all x in $(0, 1)^n$. A simple lemma from measure theory yields some corresponding statements about the expansions of the rationals, a set of measure zero.

§1. Introduction

There have been several papers dealing with the ergodic properties of F-expansiosn of n-dimensional reals ([3, 8, 10, 11]). For $x \in (0, 1)^n$, we have

$$x = \lim_{k \to \infty} F(a_1(x) + F(a_2(x) + \cdots + F(a_k(x)) \ldots)).$$

Under certain conditions (see [11]) we have the shift T ergodic and there exists a measure $\mu \sim \lambda$, n-dimensional Lebesgue measure, such that T is ergodic and measure preserving with respect to μ. The individual ergodic theorem yields, for $f \in L_1(\mu)$,

$$\frac{1}{m} \sum_{i=0}^{m-1} f(T^i(x)) \to \int f \, d\mu \quad \text{for} \quad \text{a.a. } x.$$

In particular, if $f = I_{B(a)}$, where $B(a) = \{x : [F^{-1}(x)] = a\}$, then

$$\frac{1}{m} \sum_{i=0}^{m-1} I_{B(a)}(T^i(x)) \to \mu(B(a)) \quad \text{for} \quad \text{a.a. } x.$$

Thus ergodic theory yields a statement concerning the distribution of digits in the expansion of almost all x. Much information of a similar character can be derived.

This paper is an attempt to utilize these ergodic theory results to deduce related results concerning the rationals, a set of Lebesgue measure zero. Our results are known for one-dimensional continued fractions [5, p. 328] but do not appear to have been deduced for the general case. In this connection, see [2] for a computational treatment of one- and two-dimensional continued fractions.

AMS (1970) Subject Classifications: Primary 10K10, 10F20, 10A30. Secondary 10F25, 28A70.
This research was partially supported by National Science Foundation Grant GP-28313#1.

Received July 17, 1973 and, in revised form, April 15, 1974

§2. One-dimensional F-expansions

Our proof is based on a measure theoretic lemma and follows Knuth [5]. We define $v(E)$ to be the cardinality of the set E. The lemma is proved for n-dimensions for use in §3.

LEMMA 1. *Let* $A_m=\{0/m, 1/m,\dots, (m-1)/m\}^n$. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be countable unions of disjoint rectangles in* $[0, 1]^n$, $\mathbf{R}\cap\mathbf{S}=\phi$, *and* $\lambda(\mathbf{R}\cup\mathbf{S})=1$. *Then*

$$\lim_{m\to\infty}\frac{v(A_m\cap\mathbf{R})}{m^n}=\lambda(\mathbf{R}). \tag{1}$$

Proof. Now $\mathbf{R}=\bigcup_{i=1}^{\infty} R_i$ and $\mathbf{S}=\bigcup_{i=1}^{\infty} S_i$ where each R_i and S_i is of the form $E=\times_{l=1}^{n} I_l$, I_l an interval. Now

$$\prod_{l=1}^{n}(m\lambda(I_l)-1)\leqslant v(E\cap A_m)\leqslant\prod_{l=1}^{n}(m\lambda(I_l)+1). \tag{2}$$

Choose N such that $\lambda(\mathbf{R})<\lambda(\mathbf{R}_N)+\varepsilon$ and $\lambda(\mathbf{S})<\lambda(S_N)+\varepsilon$, where $\mathbf{R}_N=\bigcup_{i=1}^{N} R_i$ and $\mathbf{S}_N=\bigcup_{i=1}^{N} S_i$. Let $\mathbf{U}_N=(\mathbf{R}\cup\mathbf{S})'\cup\bigcup_{i>N}(R_i\cup S_i)$, where the prime $'$ denotes complement. Define $r_m=v(\mathbf{R}_N\cap A_m)$, $s_m=v(\mathbf{S}_N\cap A_m)$, $u_m=v(\mathbf{U}_N\cap A_m)$. Note that $r_m+s_m+u_m=m^n$. We easily have

$$\frac{r_m}{m^n}\leqslant\frac{r_m+u_m}{m^n}=1-\frac{s_m}{m^n}. \tag{3}$$

From (2) we can show

$$\lambda(\mathbf{R}_N)-\frac{N2^n}{m}\leqslant\frac{r_m}{m^n}$$

and obtain

$$\lambda(\mathbf{R})-\varepsilon-\frac{N2^n}{m}\leqslant\frac{r_m}{m}\leqslant\lambda(\mathbf{R})+\frac{N2^n}{m}.$$

Thus, from (3) we obtain $\lim_{m\to\infty} r_m/m^n=\lim_{m\to\infty}(r_m+u_m)/m^n=\lambda(\mathbf{R})$ and (1) therefore holds.

For our discussion of F-expansions we refer to [10] and [11]. In this section we deal one-dimensional F-expansions. In n-dimensions some measure theoretic difficulties arise. Let F be a function defining such an F-expansion. Then we define $T(x)=F^{-1}(x)-[F^{-1}(x)]$ and $a_\nu(x)=[F^{-1}(T^{\nu-1}(x))]$ where $x\in(0, 1)$. Define

$$B_\nu=B(k_1, k_2,\dots, k_\nu)=\{x\in(0, 1)\colon a_i(x)=k_i,\ i=1,\dots, \nu\}.$$

Condition (A) or (B) of Rényi [7] assumes F either decreasing or increasing and in these cases each B_v is an interval. The addition of another condition allows Rényi to show there exists a measure μ such that μ is invariant with respect to T and $C^{-1}\lambda(E)\leqslant\mu(E)\leqslant C\lambda(E)$ for all measurable sets E and some constant C depending on F only. This, of course, implies λ and μ are equivalent. Finally let $\mathbf{C}_k(a_1,\ldots,a_l)=$ $=\bigcup_{b_1,\ldots,b_k} B(b_1,\ldots,b_k,a_1,\ldots,a_l)$ which is the set of all points $x\in(0,1)$ such that $a_{k+1}(x)=a_1,\ldots,a_{k+l}(x)=a_l$.

THEOREM 1. *Suppose F satisfies condition* (A) *or* (B). *Then*

$$\lim_{m\to\infty}\frac{v(\mathbf{C}_k(a_1,\ldots,a_l)\cap A_m)}{m}=\lambda(\mathbf{C}_k(a_1,\ldots,a_l))=\lambda(T^{-k}B(a_1,\ldots,a_l)).\tag{4}$$

Proof. Let $\mathbf{R}=\mathbf{C}_k(a_1,\ldots,a_l)$ and $\mathbf{S}$ equal the union of all B_{k+l} not included in $\mathbf{R}$. Both $\mathbf{R}$ and $\mathbf{S}$ are unions of disjoint intervals, $\mathbf{R}\cap\mathbf{S}=\phi$. Also $U=\{x:a_i(x)$ is undefined for some $1\leqslant i\leqslant k+l\}$ is countable and therefore $\lambda(\mathrm{U})=0$. The requirements of Lemma 1 are satisfied and (4) follows.

COROLLARY 1. *If F satisfies condition* (A) *or* (B) *and the assumptions of Kuzmin's Theorem* [10], *then*

$$\lim_{k\to\infty}\lim_{m\to\infty}\frac{v(\mathbf{C}_k(a_1,\ldots,a_l)\cap A_m)}{m}=\mu(B(a_1,\ldots,a_l)).\tag{5}$$

where μ is the invariant measure associated with the transformation $T(x)=F^{-1}(x)-[F^{-1}(x)]$.

Proof. Theorem 2 of [10] states $\lambda(T^{-k}B(a_1,\ldots,a_l))\to\mu(B(a_1,\ldots,a_l))$.

The previous results relate the number of $x\in A_m$ with digits $a_{i+k}(x)=a_i$, $i=1,\ldots,l$, to the measure of $B(a_1,\ldots,a_l)$. The next Theorem considers the average value of a function of the digits. It is possible to prove a more general theorem here.

THEOREM 2. *Let F be as in Corollary* 1. *Suppose g is real valued and* $\sum_a|g(a)|\times|\lambda(B(a))|<\infty$ *where the summation is over all values a such that* $a=a_i(x)$ *for some* $x\in(0,1)$: *Then*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{x\in A_m} g(a_{k+1}(x))=\sum_a g(a)\,\lambda(T^{-k}B(a)),\tag{6}$$

where the first summation in (6) *is over* $x\in A_m$ *such that* $a_{k+1}(x)$ *is defined.*

Proof. Let μ be the invariant measure associated with T. From $C^{-1}\lambda(E)\leqslant\mu(E)\leqslant C\lambda(E)$ and $\mu(T^{-k}E)=\mu(E)$, we have the following equivalent inequalities:

$\sum_a |g(a)|\,\mu(B(a))<\infty$, $\sum_a |g(a)|\,\lambda(B(a))<\infty$, and $\sum_a |g(a)|\,\lambda(T^{-k}(B(a))<\infty$. Since

$$\frac{1}{m}\sum_{x\in A_m} g(a_{k+1}(x))=\sum_a g(a)\frac{\nu(\mathbf{C}_k(a)\cap A_m)}{m},$$

the result follows from Theorem 1.

COROLLARY 2. *If F satisfies the assumptions of Corollary* 1 *and Theorem* 2, *then*

$$\lim_{k\to\infty}\lim_{m\to\infty}\frac{1}{m}\sum_{x\in A_m} g(a_{k+1}(x))=\sum_a g(a)\,\mu(B(a)).$$

Proof. Since $\lambda(T^{-k}(B(a)))\leqslant C\lambda(B(a))$ by an argument of Rényi [7] and $\sum g(a)\,C\lambda(B(a))<\infty$, the Bounded Convergence Theorem gives us

$$\lim_{k\to\infty}\sum_a g(a)\,\lambda(T^{-k}(B(a)))=\sum_a g(a)\lim_{k\to\infty}\lambda(T^{-k}(B(a)))=\sum_a g(a)\,\mu(B(a)).$$

The last equality is by Theorem 2 of [10].

As an application take F such that all $a_i\geqslant 1$ and $g(m)=\log(m)$ satisfies the hypothesis of Theorem 2. Then

$$\left(\prod_{i=0}^{m-1} a_{k+1}\left(\frac{i}{m}\right)\right)^{1/m}\to\exp\left\{\sum_{l=1}^{\infty}\log(l)\,\lambda(T^{-k}B(l))\right\}$$

and

$$\lim_{k\to\infty}\lim_{m\to\infty}\left(\prod_{i=0}^{m-1} a_{k+1}\left(\frac{i}{m}\right)\right)^{1/m}=\lim_{k\to\infty}\exp\left\{\int\log(a_{k+1}(x))\,d\mu(x)\right\}$$
$$=\exp\left\{\int\log a_1(x)\,d\mu(x)\right\}.$$

Since $T(x)=1/x-[1/x]$ on $(0, 1)$ is the shift for the digits of continued fractions, our results hold for continued fraction expansions of rationals. Theorem 1 and Corollary 1 are known in the case of continued fractions [5, p. 328]. Another example is q-adic expansions where q is an integer greater than 1 and $T(x)=qx-[qx]$.

§3. The Jacobi-Perron algorithm

The Jacobi-Perron algorithm can be used to expand almost all $x\in(0, 1)^n$. The F is defined by

$$F(x)=F(x_1, x_2,\ldots, x_n)=\left(\frac{1}{x_n},\frac{x_1}{x_n},\ldots,\frac{x_{n-1}}{x_n}\right).$$

Associated with this F we have

$$T(x)=T(x_1,\dots,x_n)=\left(\frac{x_2}{x_1}-\left[\frac{x_2}{x_1}\right],\frac{x_3}{x_1}-\left[\frac{x_3}{x_1}\right],\dots,\frac{1}{x_1}-\left[\frac{1}{x_1}\right]\right).$$

$$a_1(x)=\left(\left[\frac{x_2}{x_1}\right],\left[\frac{x_3}{x_1}\right],\dots,\left[\frac{1}{x_1}\right]\right),$$

and

$$a_\nu(x)=a_1(T^{\nu-1}(x)).$$

The convergence result is

$$x=\lim_{\nu\to\infty}F(a_1(x)+F(\cdots+F(a_\nu(x))\dots)).$$

F. Schweiger has examined measure theoretic properties of the Jacobi-Perron algorithm. Among other results, he has shown there exists $\mu\sim\lambda$ such that $\mu(T^{-1}E)= =\mu(E)$ for all measurable $E\subset(0,1)^n$. In [8, 10, 11] there are references and a discussion of these and related topics.

The problem here is to establish results for the Jacobi-Perron algorithm corresponding to those of §2. Certain measure theoretic difficulties exist in the proof of those results for general F-expansions in n-dimensions but they can be overcome in certain cases. The notation of the next theorem is the n-dimensional analogue of that in §2.

THEOREM 3. *Let F and T be the transformations associated with the Jacobi-Perron algorithm. Then*

$$\lim_{m\to\infty}\frac{\nu(\mathbf{C}_k(a_1,\dots,a_l)\cap A_m)}{m^n}=\lambda(T^{-k}B(a_1,\dots,a_l)). \tag{7}$$

Proof. $\mathbf{C}_k(a_1,\dots,a_l)=\bigcup_{b_1,\dots,b_k}B(b_1,b_2,\dots,b_k,a_1,\dots,a_l)$ where the B_{k+l} are convex polytopes in $(0,1)^n$. It is this property of the Jacobi-Perron algorithm which allows us to write $\mathbf{R}$ and $\mathbf{S}$ defined below as countable unions of rectangles and apply Lemma 1. We delete $\mathfrak{U}=\{x:a_i(x)$ is undefined for some $i=1,\dots,k+l\}$. $\mathfrak{U}$ is a countable set of hyperplanes and hence $\lambda(\mathfrak{U})=0$. Thus $\mathbf{R}=\mathbf{C}_k(a_1,\dots,a_l)$ can be obtained as a countable union of rectangles as can $\mathbf{S}$ equal the union of all B_{k+l} not included in $\mathbf{R}$. Equation (2) follows from Lemma 1.

The remaining Corollaries and Theorem of §2 hold for the Jacobi-Perron algorithm. [10] contains the results necessary in these proofs. For example, we have

$$\lambda(T^{-1}B(a))=\mu(B(a))+0(\varrho^k),$$

where $\varrho=(1-1/(n+1)^n)^{1/n}$. See [4].). Thus Corollary 1 follows.

§4. Remarks

These methods can easily be extended to include β-expansions [7] and Cantor's series [9]. Most ergodic properties of expansions of reals will yield a corresponding statement about the expansion of k/m, $0 \leqslant k < m$.

The invariant measure μ for the Jacobi-Perron algorithm is unknown if $n > 1$. Corollary 1 would provide a foundation for a numerical approximation of this invariant measure. See [2] for a related treatment.

Consider any sequence $b_m \to +\infty$. Then if we define $A_m = \{0/b_m, 1/b_m, \ldots, [b_m]/b_m\}^n$, Lemma 1 holds. Therefore, Theorems 1 and 3 are valid for expansions of numbers like $k/\sqrt[3]{m}$, and, in this sense, algebraic numbers have expansions such that the digit a occurs with frequency $\mu(B(a))$. Thus, Corollary 1 holds for any such sequence. We are indebted to Professor T. S. Pitcher for this observation. See [1, p. 9] for some remarks of Professor Leon Bernstein regarding these matters.

Questions related to our work have been treated in a thesis by David B. Preston [6]. He notes that Theorem 2 holds in the case of 1-dimensional continued fractions with $g(a_1(x), \ldots, a_k(x)) = Q_{k-1}(x)/Q_k(x)$. Finally we wish to gratefully acknowledge several useful comments and suggestions by the referees.

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Aequationes Math. 13 (1975), 269–273, University of Waterloo Birkhäuser Verlag, Basel

Sur les solutions d'une équation fonctionnelle, II

Zenon Moszner

Dans la première partie de cette note [4], dans laquelle on a resous un problème de M. S. Gołąb [2], fut donnée une classe des solutions ϕ de l'équation fonctionnelle

$$\phi[\phi(\alpha, x, y), z, u] = \phi(\alpha, xz, xu + yz) \tag{1}$$

où $\phi: \Gamma \times C \times C \to \Gamma$, Γ désigne un ensemble arbitraire et C forme un corps commutatif par rapport aux opérations $+$, $\cdot$. A présent *nous donnerons la solution générale de l'équation* (1). Cette solution nous donne dans le cas général la réponse négative au problème qui fut posé à la fin de la première partie de cette note.

On voit facilement que si la fonction $\phi(\alpha, x, y)$ est une solution de l'équation (1) dans ce cas la fonction $\phi(\alpha, 1, y)$ est une solution de l'équation de translation

$$\phi[\phi(\alpha, 1, y), 1, u] = \phi(\alpha, 1, y + u) \tag{2}$$

sur l'ensemble $\Gamma \times \{1\} \times C$. Il en résulte d'après p. ex. [3] que la fonction $\phi(\alpha, 1, y)$ doit être de la forme

$$\phi(\alpha, 1, y) = g_k^{-1}[g_k(g(\alpha)) + y] \tag{3}$$

pour $g(\alpha) \in \Gamma_k$, où

a) $g(\alpha): \Gamma \to \Gamma$ est une solution de l'équation

$$g(g(\alpha)) = g(\alpha) \tag{4}$$

telle qu'il existe une décomposition de l'ensemble $g(\Gamma)$

$$g(\Gamma) = \bigcup_{k \in K} \Gamma_k$$

en ensembles Γ_k non-vides, disjoints et tels que pour chaque k de K il existe un sous-groupe C_k du groupe additif C dont l'indice est égal à la puissance de Γ_k

$$\overline{\overline{C/C_k}} = \overline{\overline{\Gamma}}_k,$$

b) g_k est une bijection arbitraire de Γ_k à l'ensemble C/C_k des classes d'équivalence (à gauche ou à droit) du groupe C par rapport au groupe C_k.

AMS (1970) Subject Classification: Primary 39A30.

Reçu le 5.9.1973, et dans une forme altérée, le 14.3.1974

On peut construire la solution générale de l'équation (1) de la manière suivante:

$$\phi(\alpha, x, y) = \begin{cases} \phi(\alpha, 1, \bar{h}_k(x) + y/x) & \text{pour} \quad x \neq 0 \quad \text{et} \quad g(\alpha) \in \Gamma_k, \\ f(\alpha) & \text{pour} \quad x = 0, \end{cases} \tag{5}$$

1) $\phi(\alpha, 1, y)$ est une solution de l'équation (2), de la forme (3), pour laquelle l'ensemble E des fibres transitives (c'est-à-dire des ensembles Γ_k dans a)), qui n'ont qu'un point, n'est pas vide,

2) Γ_k et $g(\alpha)$ ont le sens comme dans a),

3) h_k, pour chaque k de K, est un homomorphisme de $(C\backslash\{0\}, \cdot)$ dans $(C/C_k, +)$, où $+$ ici désigne l'addition das C/C_k induit par l'addition dans C, c'est-à-dire

$$h_k(x \cdot y) = h_k(x) + h_k(y) \tag{6}$$

et $\bar{h}_k(x)$ est une fonction de l'ensemble $C\backslash\{0\}$ à l'ensemble C telle que

$$\bar{h}_k(x) \in h_k(x) \qquad (\Lambda x \in C\backslash\{0\}), \tag{7}$$

4)
$$f: \Gamma \to E \tag{8}$$

est une fonction pour laquelle

$$f(\alpha) = \alpha \quad \text{sur} \quad f(\Gamma),$$

c'est-à-dire

$$f(f(\alpha)) = f(\alpha) \qquad (\Lambda \alpha \in \Gamma) \tag{9}$$

et pour laquelle

$$f(\phi(\alpha, 1, y)) = f(\alpha) \qquad (\Lambda \alpha \in \Gamma, \Lambda y \in C), \tag{10}$$

c'est-à-dire qui est stable sur l'ensemble $g^{-1}(\Gamma_k)$ pour chaque k de K.

Remarquons que la valeur de $\phi(\alpha, 1, \bar{h}_k(x) + y/x)$ ne dépend pas du choix de la fonction $\bar{h}_k(x)$ remplissant (7). En effet si nous avons w_1 et w_2 de $h_k(x)$, dans ce cas $w_1 + y/x$ et $w_2 + y/x$ appartiennent à la même classe d'équivalence de C par rapport au C_k et il résulte de (3) que

$$\phi(\alpha, 1, w_1 + y/x) = \phi(\alpha, 1, w_2 + y/x). \tag{11}$$

Vérifions que *la fonction donnée par la formule* (5) *satisfait à l'équation* (1).

Si $x \cdot z \neq 0$ nous avons d'après (5)

$$\phi[\phi(\alpha, x, y), z, u] = \phi[\phi(\alpha, 1, \bar{h}_k(x) + y/x), z, u]$$

pour $g(\alpha)\in\Gamma_k$. Nous avons d'après (3):

$$\phi(\alpha, 1, \bar{h}_k(x)+y/x)\in\Gamma_k$$

et de là d'après (4):

$$g(\phi(\alpha, 1, \bar{h}_k(x)+y/x))=\phi(\alpha, 1, \bar{h}_k(x)+y/x)\in\Gamma_k.$$

Il en résulte d'après (5), (2), (6) et (11) que

$$\begin{aligned}\phi[\phi(\alpha, 1, \bar{h}_k(x)+y/x), z, u] &= \phi[\phi(\alpha, 1, \bar{h}_k(x)+y/x), 1, \bar{h}_k(z)+u/z] \\ &= \phi[\alpha, 1, \bar{h}_k(x)+\bar{h}_k(z)+y/x+u/z] \\ &= \phi[\alpha, 1, \bar{h}_k(x\cdot z)+y/x+u/z] \\ &= \phi(\alpha, xz, xu+zy)\end{aligned}$$

puisque

$$\bar{h}_k(x)+\bar{h}_k(z)\in h_k(x)+h_k(z)=h_k(x\cdot z)$$

et

$$\bar{h}_k(x\cdot z)\in h_k(x\cdot z).$$

Si $x=0$, $z\neq 0$ nous avons d'après (5) et (8) et pour $g(f(\alpha))\in\Gamma_k$

$$\phi[\phi(\alpha, 0, y), z, u]=\phi[f(\alpha), 1, \bar{h}_k(z)+u/z]=f(\alpha)=\phi(\alpha, 0, yz).$$

Si $x\neq 0$, $z=0$ nous avons d'après (5) et (10):

$$\phi[\phi(\alpha, x, y), 0, u]=f[\phi(\alpha, 1, \bar{h}_k(x)+y/x)]=f(\alpha)=\phi(\alpha, 0, xu).$$

Si enfin $x=y=0$ nous avons d'après (5) et (9):

$$\phi[\phi(\alpha, 0, y), 0, u]=f(f(\alpha))=f(\alpha)=\phi(\alpha, 0, 0).$$

A présent nous allons démontrer que *chaque solution de l'équation* (1) *peut être donnée par la formule* (5).

Remarquons d'abord que pour la solution de l'équation (1) les formules (2) et (3) ont lieu. De là d'après (1) et (3) nous avons pour $x\neq 0$:

$$\phi(\alpha, x, y)=\phi[\phi(\alpha, 1, y/x), x, 0]=\phi[g_k^{-1}\{g_k(g(\alpha))+y/x\}, x, 0].$$

En prenant en considération la notation

$$\psi_k(W, x)\stackrel{\text{df}}{=}g_k\{\phi[g_k^{-1}(W), x, 0]\}, \tag{12}$$

définie sur $(C/C_k)\times C$, nous recevons:

$$\phi(\alpha, x, y)=g_k^{-1}\{\psi_k[g_k(g(\alpha))+y/x, x]\}. \tag{13}$$

D'après (1) et (12) on peut facilement vérifier que

$$\psi_k\{\psi_k[g_k(g(\alpha))+y/x, x]+u/z, z\}=\psi_k[g_k(g(\alpha))+y/x+u/z, xz] \tag{14}$$

pour $x\cdot z\neq 0$. En posant ici $z=1$, $h_k(x)=\psi_k(C_k, x)$ et prenant y telle que

$$g_k(g(\alpha))+y/x=C_k \tag{15}$$

nous recevons

$$\psi_k[h_k(x)+u, 1]=\psi_k[C_k+u, x]. \tag{16}$$

D'après (3) et (12) nous avons

$$\psi_k(W, 1)=W.$$

et de là (16) nous donne:

$$h_k(x)+u=\psi_k[C_k+u, x]. \tag{17}$$

Il en résulte d'après (14) et (15) que

$$h_k(x\cdot z)=h_k(x)+h_k(z),$$

donc la fonction h_k satisfait à la condition 3).

D'après (13), (17) et (3) nous avons

$$\begin{aligned}\phi(\alpha, x, y)&=g_k^{-1}[h_k(x)+g_k(g(\alpha))+y/x]\\&=g_k^{-1}[g_k(g(\alpha))+\bar{h}_k(x)+y/x]=\phi(\alpha, 1, \bar{h}_k(x)+y/x),\end{aligned} \tag{18}$$

où $\bar{h}_k(x)$ désigne une fonction arbitraire remplissant (7), donc pour $x\neq 0$ la fonction ϕ a la forme de la première formule dans (5). Le raisonnement à partir de la formule (12) jusqu'ici est une modification des concidérations dans la note [1].

En posant $x=0$ et $z=1$ dans (1) nous recevons

$$\phi[\phi(\alpha, 0, y), 1, u]=\phi(\alpha, 0, y) \quad (\Lambda y, u\in C). \tag{19}$$

Il en résulte d'après (1) et (5) que pour $z \neq 0$

$$\phi(\alpha, 0, yz) = \phi[\phi(\alpha, 0, y), z, u] = \phi[\phi(\alpha, 0, y), 1, \bar{h}_k(z) + u/z]$$
$$= \phi(\alpha, 0, y).$$

La fonction $\phi(\alpha, 0, y)$ ne dépend pas de la variable y et en posant

$$f(\alpha) \overset{\mathrm{df}}{=} \phi(\alpha, 0, 0)$$

nous recevons

$$\phi(\alpha, 0, y) = f(\alpha),$$

c'est-à-dire la deuxième partie de la formule (5).

D'après (19) nous avons que

$$f(\alpha): \Gamma \to E.$$

En posant $x = y = u = z = 0$ dans (1) nous recevons (9) et en posant $x = 1$, $u = z = 0$ dans (1) nous avons (10, c'est-à-dire la fonction $f(\alpha)$ satisfait aux conditions 4), c.q.f.d.

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A fundamental functional equation for vector lattices

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1. Introduction

The theory of vector lattices was initiated by F. Riesz, L. Kantorovich, and H. Freudenthal in the late thirties; because of its importance for functional analysis, it was rapidly developed by other authors [1]. Basically, a vector lattice is a real vector space with a lattice structure in which vector translations and scalar multiplication by positive reals are isotone in the sense that they preserve the partial order. More precisely, a real vector space V with a partial order $\leqslant$ is called an ordered vector space if $x \leqslant y$ implies $x+z \leqslant y+z$ and $\lambda x \leqslant \lambda y$ for all z in V and all scalars $\lambda > 0$. An ordered vector space V is a vector lattice (or linear lattice or lattice ordered vector space or Riesz space) if $\sup(x, y)$ and $\inf(x, y)$ exist for each pair (x, y) in $V \times V$. (for definitions see, for example, [2])

Since a semi-lattice can be defined as a set with a binary operator which is idempotent, commutative and associative, it should not be too surprising to characterize vector lattices by certain functional equations, whose solutions carry the properties of idempotence, commutativity, associativity, homogeneity etc. The purpose of this note is to introduce one such fundamental functional equation. It might be worth noting that a detailed treatment of different types of functional equations relating to generalized commutativity, associativity, homogeneity etc. can be found in [3].

2. Main Results

LEMMA. *Let V be a real vector space. A function $F: V \times V \to V$ satisfies $F(y, y) = y$, y in V, and the functional equation*

$$u + F[x, \lambda F(y, z)] = F[u + \lambda y, F(u + \lambda z, u + x)] \tag{A}$$

for all x, y, z and u in V and all scalars $\lambda > 0$ if and only if F has the following properties:

Commutativity:

$$F(y, z) = F(z, y) \quad \textit{for all} \quad y, z \tag{1}$$

Associativity:

$$F[x, F(y, z)] = F[F(x, y), z] \quad \textit{for all} \quad x, y, \quad \text{and} \quad z \tag{2}$$

AMS (1970) Subject Classifications: Primary 46A40, 39A40. Secondary 06A65, 39A30.

Received September 21, 1973 and, in revised form, February 19, 1974

Positive Homogeneity:

$$\lambda F(y, z)=F(\lambda y, \lambda z) \quad \text{for all} \quad y, z \quad \textit{and all} \quad \lambda>0 \tag{3}$$

Additive Homogeneity:

$$u+F(x, y)=F(u+x, u+y) \quad \textit{for all} \quad x, y \quad \text{and} \quad u. \tag{4}$$

Proof. Necessity: Let F be a solution to Eq. (A) with the idempotence property $F(y, y)=y$, for all y in V.

From Eq. (A), with u replaced by the zero vector, λ by 1, it follows that

$$F[x, F(y, z)]=F[y, F(z, x)] \tag{i}$$

for all x, y and z in V.

The commutativity of F can be proved through the idempotence of F and a series of applications of (i). The details follow

$$\begin{aligned} F(y, z) &= F[F(y, z), F(y, z)] \\ &= F\{y, F[z, F(y, z)]\}, \quad \text{by} \quad \text{(i)}, \\ &= F[y, F(y, z)], \ \text{by} \quad \text{(i)} \\ &= F(z, y), \quad \text{by} \quad \text{(i)}, \end{aligned}$$

where $F[z, F(y, z)]=F[y, F(z, z)]=F(y, z)$, by (i) and the idempotence; $F[y, F(y, z)]=F[z, F(y, y)]=F(z, y)$ by (i) and the idempotence.

From (i) and the commutativity it follows that

$$F[x, F(z, y)]=F[x, F(y, z)]=F[z, F(x, y)]=F[F(x, y), z].$$

Thus the associativity of F is proved.

With u replaced by the zero vector, x by λz, Eq. (A) yields

$$F[\lambda z, \lambda F(y, z)]=F[\lambda y, F(\lambda z, \lambda z)]=F(\lambda y, \lambda z). \tag{ii}$$

To establish the positive homogeneity of F, in Eq. (A) replace u by the zero vector, x by $\lambda F(y, z)$, and obtain

$$\begin{aligned} \lambda F(y, z) &= F[\lambda F(y, z), \lambda F(y, z)] \\ &= F\{\lambda y, F[\lambda z, \lambda F(y, z)]\} \\ &= F[\lambda y, F(\lambda y, \lambda z)], \quad \text{by} \quad \text{(ii)} \\ &= F(\lambda z, \lambda y), \quad \text{by} \quad \text{(i)} \quad \text{and the idempotence.} \end{aligned}$$

To prove that F is also additive homogeneous, in Eq. (A) replace λ by 1, z by y, and obtain

$$\begin{aligned} u+F(x, y) &= u+F[x, F(y, y)] \\ &= F[u+y, F(u+y, u+x)] \\ &= F(u+x, u+y), \quad \text{by (i) and the idempotence.} \end{aligned}$$

Sufficiency: A function $F: V\times V\to V$ with properties (1) through (4) clearly satisfies Eq. (A). From properties (3) and (4) it follows that $F(2y, 2y)=2F(y, y)$ and $y+F(y, y)=F(2y, 2y)$. Thus $F(y, y)=y$ for each y in V. This completes the proof of the lemma.

THEOREM. *If V is a real vector space and there exists a function $F: V\times V\to V$ satisfying $F(x, x)=x$ and $u+F[x, \lambda F(y, z)]=F[u+\lambda y, F(u+\lambda z, u+x)]$ for all x, y, z and u in V and all scalars $\lambda>0$, then V is a vector lattice relative to the partial ordering defined by setting $x\leqslant y$ whenever $F(x, y)=y$. Conversely, if V is a vector lattice, then the function $F: V\times V\to V$ defined by $F(x, y)=\sup\{x, y\}$ satisfies $F(x, x)=x$ and the above mentioned functional equation.*

Proof. Set $x\leqslant y$ whenever $F(x, y)=y$. We prove that this defines a partial ordering on V. Since $F(x, x)=x$, hence $x\leqslant x$. If $x\leqslant y$ and $y\leqslant x$, then, by property (1) in the lemma, $y=F(x, y)=F(y, x)=x$. To show that $x\leqslant y$ and $y\leqslant z$ implies $x\leqslant z$, consider $F(x, z)$. Since $F(x, y)=y$ and $F(y, z)=z$, hence, by property (2) in the lemma, $F(x, z)=F[x, F(y, z)]=F[F(x, y), z]=F(y, z)=z$, and therefore, $x\leqslant z$.

The vector space V is a (partially) ordered vector space, because $x\leqslant y$ immediately implies $x+u\leqslant y+u$ and $\lambda x\leqslant \lambda y$ for all u in V and all scalars $\lambda>0$, by properties (3) and (4) in the lemma.

We prove now that $F(x, y)$ is the supremum of x and y. From $F[F(x, y), y] =F[x, F(y, y)]=F(x, y)$ it follows that $F(x, y)\geqslant y$. Similarly $F(x, y)\geqslant x$. Hence $F(x, y)$ is an upper bound of x and y. Let z be another upper bound, so $F(x, z)=z$ and $F(y, z)=z$. Then $F[z, F(x,y)]=F[F(x, z), y]=F(z, y)=z$, therefore, $z\geqslant F(x, y)$. This shows that $F(x, y)$ is the least upper bound of x and y, i.e., $F(x, y)=\sup\{x, y\}$. It is well-known (cf. [1, p. 293]) and easy to prove that now $\inf\{x, y\}$ exists and is equal to $-\sup\{-x, -y\}$.

Conversely, suppose that V is a vector space. It is well-known [1, p. 8, p. 348] that the function $F: V\times V\to V$ defined by $F(x, y)=\sup\{x, y\}$ satisfies the properties (1) through (4) stated in the lemma, by which F in turn satisfies $F(x, x)=x$ and the equation $u+F[x, \lambda F(y, z)]=F[u+\lambda y, F(u+\lambda z, u+x)]$.

Acknowledgment

The author wishes to thank the referees for suggesting an improvement of the formulation of the main theorem.

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An asymptotic formula in the theory of compositions

Z. Star

1. Introduction

A composition of n into s parts is a partition in which the order of the parts is taken into account. For example, the compositions of 4 into two parts are (3, 1), (1, 3), (2, 2).

In this paper, compositions in which no part is to exceed r will be discussed. In the above example there is only one composition of 4 into two parts with no part exceeding 2, namely (2, 2). Let $c(n, s, r)$ be the number of compositions of n into s parts with no part exceeding r, and let $c(s, r)$ be the maximum of $c(n, s, r)$ over n for fixed s and r.

In [6] it is shown that a fairly wide class of occupancies may be enumerated by means of the $c(n, s, r)$ and Basu [2] has obtained the result that

$$\sum_{s=0}^{\infty} c(n, s, r) \sim \frac{(\beta-1)\,\beta^{n+r}}{2\beta^r - r - 1} \qquad (n \to \infty)$$

for a β satisfying

$$\frac{1}{2} < \frac{1}{\beta} < \left(\frac{2}{r+1}\right)^{1/r} < 1.$$

The estimate obtained in this paper is for the asymptotic distribution of $c(n, s, r)$ about its peak $c(s, r)$, as s approaches infinity, and is uniform in n, s, and r. The method used is similar to that of Szekeres [7].

The following two results, both trivially obtainable from the generating function $f(z) = (z + z^2 + \cdots + z^r)^s = \sum_{n=1}^{rs} c(n, s, r) z^n$, will be needed in the proof of Theorem 1. It will be assumed throughout the paper that $r \geqslant 2$. These results are:

$$c(n, s, r) = c(n-1, s, r) + c(n-1, s-1, r) - c(n-r-1, s-1, r) \tag{1}$$

and

$$c(n, s, r) = c(s(r+1) - n, s, r). \tag{2}$$

For further details of compositions the reader is referred to the work of P. A. MacMahon ([3], [4], [5]).

AMS (1970) Subject Classifications: Primary 05A20, 10A45. Secondary 41A60.

Received October 10, 1973 and, in revised form, January 11, 1974

THEOREM 1. *For given* $s \geqslant 2$ *and* r, *the solution* n_0, *of the equation* $c(n_0, s, r) = c(s, r)$, *is* $n_0 = s(r+1)/2$ *if* $s(r+1)$ *is even, and* $n_0 = (s(r+1)-1)/2$ *if* $s(r+1)$ *is odd.*

Proof. For fixed r, let $k(n, s) = c(n, s, r)$. The case when $r+1$ is even will be proven. For odd $r+1$ the proof is similar.

We begin by proving that if $2 \leqslant s \leqslant n \leqslant s(r+1)/2$ then

$$k(n, s) - k(n-1, s) > 0. \tag{3}$$

The proof is by induction on s. When $s = 2$ the left hand side of (3) is

$$k(n, 2) - k(n-1, 2) = k(n-1, 1) - k(n-r-1, 1)$$

by (1). But $k(n-1, 1)$ is greater than zero since $n-1 \geqslant 1$, and $k(n-r-1, 1)$ equals zero since $n \leqslant r+1$. Hence (3) is true when $s = 2$. Now suppose that (3) is true for any given s greater than 1.

By (1) once more,

$$k(n, s+1) - k(n-1, s+1) = k(n-1, s) - k(n-r-1, s)$$

and the right hand side is greater than zero provided $s \leqslant n-1 \leqslant s(r+1)/2$, that is, provided $s+1 \leqslant n \leqslant s(r+1)/2+1$.

Consider now, values of n such that $s(r+1)/2+2 \leqslant n \leqslant (s+1)(r+1)/2$. Put $n = s(r+1)/2 + n_1$, where $2 \leqslant n_1 \leqslant (r+1)/2$.

Then

$$\begin{aligned} k(n, s+1) - k(n-1, s+1) &= k\left(\frac{s(r+1)}{2} + n_1 - 1, s\right) - k\left(\frac{s(r+1)}{2} + n_1 - r - 1, s\right) \\ &= k\left(\frac{s(r+1)}{2} - n_1 + 1, s\right) - k\left(\frac{s(r+1)}{2} + n_1 - r - 1, s\right) \end{aligned}$$

by (2).

That is,

$$k(n, s+1) - k(n-1, s+1) = k(N, s) - k(N-q, s)$$

where

$$N = \frac{s(r+1)}{2} - n_1 + 1 \quad \text{and} \quad q = r + 2 - 2n_1.$$

But $s < N < s(r+1)/2$ and $q > 0$ so that, by the induction hypothesis, the right hand side is greater than zero. This proves (3). The theorem now follows immediately from (3) by using (2).

2. The asymptotic formula

In this section, the c_t will denote positive constants independent of n, r or s, but possibly dependent upon some preassigned constant such as the number of terms in a given expansion. The same will be true for the positive constant L, where $g(x)=0(h(x))$ is used throughout in the sense that $|g(x)|<L|h(x)|$. In deriving the asymptotic formula, in view of (2) it suffices to consider only those values of n which do not exceed $s(r+1)/2$. By applying Cauchy's theorem to the generating function $f(z)$ and choosing a path which passes near to a suitable saddle point in the direction of steepest descent, we obtain

THEOREM 2. *For given* K *and* β *let* $n=\frac{1}{2}(s-k)(r+1)$ where $0\leqslant k\leqslant Ks^{\beta}$, and $0\leqslant\beta<\frac{1}{2}$. *Then uniformly in* n, r *and* s *as* s *approaches infinity*

$$c(n,s,r)=\frac{1}{\sqrt{\pi}}\left(\frac{6}{r^2-1}\right)^{1/2}\cdot\frac{r^s}{s^{1/2}}\left[1+\frac{h_{1,0}(r)+h_{1,1}(r)k^2}{s}+\cdots+\frac{\sum\limits_{j=0}^{(m-1)}h_{m-1,j}(r)k^{2j}}{s^{m-1}}+0\left(\frac{1+k^{2m}}{s^m}\right)\right].$$

where $h_{t,j}(r)$ *is a rational function of* r *of* $0(1)$. *In particular,*

$$h_{1,0}=\frac{-3(r^2+1)}{20(r^2-1)},\qquad h_{1,1}=\frac{-3(r+1)}{2(r-1)},$$

$$h_{2,0}=\frac{-(13r^4-134r^2+13)}{1120(r^2-1)^2},\qquad h_{2,1}=\frac{9(r^2+1)}{8(r-1)^2},$$

$$h_{2,2}=\frac{9(r+1)^2}{8(r-1)^2}.$$

Proof. By Cauchy's theorem $c(n,s,r)=(2\pi i)^{-1}\int_U f(z)z^{-n-1}\,dz$ where U is the unit circle with centre 0, and $f(z)$ is the generating function. Hence

$$c(n,s,r)=\frac{r^s}{2\pi}\int\limits_{-\pi}^{\pi}\exp\left\{s\log\left(\frac{\sin\frac{r\theta}{2}}{\frac{r\theta}{2}}\right)-s\log\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)+\tfrac{1}{2}ik(r+1)\theta\right\}d\theta.$$

Dissect the integral into

$$I_1+I_2-I_3=\int_{-r^{-1}s^{-\mu}}^{r^{-1}s^{-\mu}}+\int_{r^{-1}s^{-\mu}}^{\pi}-\int_{-r^{-1}s^{-\mu}}^{-\pi}, \quad \text{where} \quad \max(\beta, \tfrac{1}{4})<\mu<\tfrac{1}{2},$$

then using the expansion ([1], formula 4.3.71, p. 75)

$$\log\left(\frac{\sin z}{z}\right)=\sum_{t=1}^{\infty}\frac{(-1)^t\, 2^{2t}B_{2t}\cdot z^{2t}}{2t\cdot(2t)\,!}, \; |z|<\pi, \tag{4}$$

where B_{2t} is the $2t$-th Bernoulli number, we have

$$I_1=\int_{-r^{-1}s^{-\mu}}^{r^{-1}s^{-\mu}} e^{-s(r^2-1)\,\theta^2/24}\left\{1+\sum_{t=1}^{m-1}\sum_{q=0}^{t} k^{2q}p_{q,t} + \psi(r, s, \theta)+0\left(s^m r^{4m}\theta^{4m}+k^{2m}r^{2m}\theta^{2m}\right)\right\} d\theta \tag{5}$$

where

$$p_{q,t}=\sum_{l=1}^{t-q} a_{q,l,t}s^l\theta^{2(t+l)}, \qquad q=1, 0, \ldots, t-1,$$

$$p_{t,t}=a_t\theta^{2t}, \quad a_{q,l,t}=0\left(r^{2t+2l}\right), \quad a_t=0\left(r^{2t}\right)$$

and ψ is an odd function of θ.
But

$$\int_{s^{1/2-\mu}}^{\infty} x^{2(t+l)}\, e^{-x^2}\, dx=0\left(s^{(1/2-\mu)\,(2t+2l-1)}\exp\left(-s^{1-2\mu}\right)\right),$$

and $t+l=0, 1, 2, \ldots, (t\geqslant l)$, so that

$$\int_{r^{-1}s^{-\mu}}^{\infty} a_{q,l,t}\, s^l\theta^{2(t+l)}\, e^{-s(r^2-1)\,\theta^2/24}\, d\theta=0\left(s^{-[2\mu(t-l)+(4\mu-1)\,l+(1-\mu)]}\exp\left(-s^{1-2\mu}\right)\right).$$

We may therefore replace the limits of integration in (5) by $-\infty$ and ∞ respectively, provided $s\geqslant c_1$. This, together with the result

$$\int_0^{\infty} x^{2n}\, e^{-ax^2}\, dx=\frac{1.3.\ldots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}},$$

gives the general form of the theorem, the contribution of $1/s^t$ coming from the term $\sum_{q=0}^{t} k^{2q} p_{q,t}$ in (5).

In particular, for $m=3$,

$$I_1=\int_{-\infty}^{\infty} e^{-s(r^2-1)\theta^2/24}\left\{1-\left[\frac{(r^4-1)s\theta^4}{2880}+\tfrac{1}{8}k^2(r+1)^2\theta^2\right]\right.$$
$$+\left[\frac{-(r^6-1)s\theta^6}{181440}+\frac{(r^4-1)^2s^2\theta^8}{16588800}+\frac{k^2(r+1)^2(r^4-1)s\theta^6}{23040}\right.$$
$$\left.\left.+\frac{k^4(r+1)^4\theta^4}{384}\right]+0(s^3r^{12}\theta^{12}+k^6r^6\theta^6)\right\}d\theta,$$

provided $s\geqslant c_2$, giving the result explicitly to the second remainder term.

It remains to prove that the contributions of I_2 and I_3 are negligible.

$$|I_2|\leqslant\int_{r^{-1}s^{-\mu}}^{\pi}\exp\left[s\left\{\log\left(\frac{\sin\frac{r\theta}{2}}{\frac{r\theta}{2}}\right)-\log\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)\right\}\right]d\theta=\left(\int_{r^{-1}s^{-\mu}}^{\pi/r}+\int_{\pi/r}^{\pi}\right)=I_4+I_5.$$

$$I_4=\int_{r^{-1}s^{-\mu}}^{\pi/r}\exp\left(s\left\{\sum_{t=1}^{\infty}\frac{(-1)^t B_{2t}\cdot(r^{2t}-1)\theta^{2t}}{2t\cdot(2t)!}\right\}\right)d\theta \tag{6}$$

by (4),

$$<\int_{r^{-1}s^{-\mu}}^{\pi/r} e^{-s(r^2-1)\theta^2/24}\,d\theta,$$

since all the coefficients in (6) are negative.

Hence

$$I_4<\int_{r^{-1}s^{-\mu}}^{\pi/r} e^{-(r^2-1)s^{1-2\mu}/24r^2}\,d\theta=0(\exp(-c_3s^{1-2\mu}).$$

Also,

$$I_5=\int_{\pi/r}^{\pi}\exp\left(s\left\{\tfrac{1}{2}\log\left(\frac{1-\cos r\theta}{1-\cos\theta}\right)-\log r\right\}\right)d\theta,$$

and

$$\max\left(\frac{1-\cos r\theta}{1-\cos\theta}\right), \quad \pi/r\leqslant\theta\leqslant\pi$$

occurs at $\theta=\pi/r$, so that using the same expansion as in (6),

$$I_5<\int_{\pi/r}^{\pi} e^{-s(r^2-1)\pi^2/24r^2}\,d\theta=0(\exp(-c_4 s)).$$

Hence, both $|I_2|$ and $|I_3|$ are $0(\exp(-c_5 s^{1-2\mu}))$.

3. Acknowledgements

I wish to thank the referees for their detailed and helpful comments. This work was supported by a grant from the Research and Study Travel Division of the Academic Development Fund of the University of New Brunswick.

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Aequationes Math. 13 (1975), 285–288, University of Waterloo Birkhäuser Verlag, Basel

On extending solutions of a functional equation

Karol Baron

Adopting the usual conventions, by $\mathscr{X}^S$, where $\mathscr{X}$ and S are arbitrary sets, we shall denote the set of all functions from S into $\mathscr{X}$ with the Tychonoff topology in the case where $\mathscr{X}$ is a topological space and by $\triangle_{s\in S}\, g_s$ the diagonal of a family of transformations $\{g_s: s\in S\}$ (cf. [1], Ch. 2, §3).

The aim of this paper is to prove the following theorems (cf. [2], th. 3).

THEOREM 1. *Let X, Y, S and $U\subset X$ be arbitrary sets, $h: X\times Y^S\to Y$ and $f_s: X\to X$, $s\in S$, arbitrary functions. If $f_s(U)\subset U$ for every $s\in S$ and*

(i) *for every $x\in X$ there exists a positive integer k such that for every $s_1,\ldots,s_k\in S$*

$$f_{s_1}\circ\cdots\circ f_{s_k}(x)\in U,$$

then for every solution $\varphi_0: U\to Y$ of the functional equation

$$\varphi(x)=h(x, \mathop{\triangle}_{s\in S} \varphi\circ f_s(x)) \tag{1}$$

there exists exactly one solution $\varphi: X\to Y$ of this equation such that

$$\varphi(x)=\varphi_0(x), \qquad x\in U. \tag{2}$$

Moreover, if X and Y are topological spaces, U is open, h, f_s, $s\in S$, and φ_0 are continuous functions and

(ii) *for every open set V, $U\subset V\subset X$, $\bigcap\{f_s^{-1}(V): s\in S\}$ is open,*
then φ is also continuous.

THEOREM 2. *Let X be a closed and convex subset of a finite dimensional Banach space normed by $\|\cdot\|$, $U\subset X$ an open set (in X) and Y, S and $h: X\times Y^S\to Y$, $f_s: X\to X$, $s\in S$, arbitrary sets and functions, respectively. If $f_s(U)\subset U$ for every $s\in S$ and $\{f_s: s\in S\}$ is a locally equicontinuous family such that*

$$\sup\{\|f_s(x)-\xi\|: s\in S\}<\|x-\xi\|, \qquad x\in X, x\neq\xi \tag{3}$$

holds for a certain $\xi\in U$ then for every solution $\varphi_0: U\to Y$ of equation (1) there exists a solution $\varphi: X\to Y$ of this equation such that condition (2) is fulfilled.

AMS (1970) Subject Classification: Primary 39A15.

Received October 26, 1973 and, in revised form, February 15, 1974

Moreover, if Y is a topological space and h is a continuous function, then for every continuous solution $\varphi_0 : U \to Y$ of the equation (1) *there exists a continuous solution $\varphi : X \to Y$ of this equation such that* (2) *holds.*

Proof of Theorem 1. We define the sequence of sets

$$U_0 = U, \quad U_{k+1} = \bigcap \{f_s^{-1}(U_k) : s \in S\}, \qquad k = 0, 1, 2, \ldots. \tag{4}$$

As a direct consequence of the above definition we get

$$f_s(U_{k+1}) \subset U_k, \qquad s \in S, \quad k = 0, 1, 2, \ldots. \tag{5}$$

Then, because $f_s(U_0) \subset U_0$ for every $s \in S$, it follows that $U_0 \subset f_s^{-1}(U_0)$ for every $s \in S$ and in view of (4), $U_0 \subset U_1$. By induction

$$U_k \subset U_{k+1}, \qquad k = 0, 1, 2, \ldots. \tag{6}$$

Moreover,

$$X = \bigcup \{U_k : k = 0, 1, 2, \ldots\} \tag{7}$$

in a point of hypothesis (i).

Now, we define the sequence of functions $\{\varphi_k\}$, $\varphi_k : U_k \to Y$, $k = 0, 1, 2, \ldots$, by the formula

$$\varphi_{k+1}(x) = h(x, \underset{s \in S}{\triangle} \varphi_k \circ f_s(x)), \qquad x \in U_{k+1}, k = 0, 1, 2, \ldots, \tag{8}$$

where $\varphi_0 : U_0 \to Y$ is the given solution of (1). It follows from property (5) that this definition is correct. We shall show that

$$\varphi_{k+1}(x) = \varphi_k(x), \qquad x \in U_k, k = 0, 1, 2, \ldots. \tag{9}$$

This relation is obvious for $k = 0$, since φ_0 is a solution of equation (1) in U_0. Supposing (9) for a $k \geqslant 0$ and applying (6), (8) and (5), we have for every $x \in U_{k+1}$

$$\varphi_{k+2}(x) = h(x, \underset{s \in S}{\triangle} \varphi_{k+1} \circ f_s(x)) = h(x, \underset{s \in S}{\triangle} \varphi_k \circ f_s(x)) = \varphi_{k+1}(x),$$

thus (9) is valid for every $k = 0, 1, 2, \ldots$. Recalling (6), (7) and (9) we may define the function $\varphi : X \to Y$ by

$$\varphi(x) = \varphi_k(x), \qquad x \in U_k, k = 0, 1, 2, \ldots. \tag{10}$$

It is clear that φ fulfils equation (1) in X and that it yields the unique extension of φ_0 to a solution of (1) in all X.

Now, if X and Y are topological spaces, U is open, h, f_s, $s \in S$, and φ_0 are continuous functions and hypothesis (ii) is satisfied, then the sets U_k, $k=0, 1, 2, \ldots$, defined by (4) are open sets in view of (6) and $\varphi_k: U_k \to Y$, $k=0, 1, 2, \ldots$, given by the formula (8) are continuous functions. Therefore, the function $\varphi: X \to Y$ defined by (10) is continuous. Thus Theorem 1 is proved.

Remarks. (1) The hypothesis (i) in Theorem 1 cannot be replaced by

(iii) *for every $x \in X$ there exists a positive integer k such that for every $s \in S$, $f_s^k(x) \in U$.*[1])

Indeed, let $\varphi_0: (-1, 1) \to \mathbf{R}$ be a solution of the equation

$$\varphi(x) = \tfrac{1}{2}\varphi[f_1(x)] + 2\varphi[f_2(x)] + x,$$

where

$$f_1(x) = \begin{cases} 1 & \text{for} \quad x=-1 \\ sx & \text{for} \quad -1<x<1, \\ 0 & \text{for} \quad x=1 \end{cases} \qquad f_2(x) = \begin{cases} 0 & \text{for} \quad x=-1 \\ sx & \text{for} \quad -1<x<1, \\ -1 & \text{for} \quad x=1 \end{cases} \qquad s \in (0, 1),$$

(the existence of such a solution follows for example from Theorem 2.10 of monograph [3]) and suppose that this solution may be extended to a solution $\varphi: [-1, 1] \to \mathbf{R}$ of this equation. Then $\varphi(0)=0$ and

$$\varphi(-1) = \tfrac{1}{2}\varphi(1) - 1$$
$$\varphi(1) = 2\varphi(-1) + 1.$$

Thus, although hypothesis (iii) is satisfied the solution φ_0 cannot be extended to a solution $\varphi: [-1, 1] \to \mathbf{R}$.

(2) The hypothesis

(iv) *X is a metric space with a metric ϱ, $U \subset X$ is an open set and $f_s: X \to X$, $s \in S$, fulfils*

$$\varrho(f_s(x), \xi) \leqslant \gamma(\varrho(x, \xi)), \qquad x \in X, s \in S, \tag{11}$$

with a $\xi \in U$ and a continuous and increasing function $\gamma: [0, \infty) \to [0, \infty)$ such that

$$\gamma(t) < t, \qquad t \in (0, \infty), \tag{12}$$

guarantees that (i) *is satisfied.*

In fact, for every $x \in X$, $k=0, 1, 2, \ldots$, and $s_1, \ldots, s_k$ from S

$$\varrho(f_{s_1} \circ \cdots \circ f_{s_k}(x), \xi) \leqslant \gamma^k(\varrho(x, \xi)),$$

as well as

$$\lim_{k \to \infty} \gamma^k(t) = 0, \qquad t \in [0, \infty)$$

(cf. [3], Th. 0.4).

---

[1]) Here, and in remark 2 below, the upper indexes denote functional iterates.

(3) The hypothesis (ii) is fulfilled whenever

(v) *f_s, $s\in S$, are continuous self-mappings of a topological space X, $U\subset X$ and for every open set V, $U\subset V\subset X$, the family $\{X\setminus f_s^{-1}(V): s\in S\}$ is locally finite* (cf. [1], p. 193, Th. 1).

(4) The condition

$$\|f_s(x)-\xi\|\leqslant\gamma(\|x-\xi\|),\qquad x\in X,\ s\in S,$$

where $\gamma:[0,\infty)\to[0,\infty)$ fulfils (12), implies (3).

Proof of Theorem 2. Let V be a maximal (in the inclusion sense) open set such that $U\subset V\subset X$, $f_s(V)\subset V$ for every $s\in S$ and there exists a solution $\varphi: V\to Y$ (being continuous whenever Y is a topological space and φ_0 is a continuous function) of equation (1) which satisfies (2) (we use here the Kuratowski-Zorn Lemma). It is enough to show that $V=X$. Suppose for the indirect proof that $V\neq X$ and let us choose an $\bar{x}\in\partial V\cap\partial B$, where B is the greatest open ball centered at ξ and contained in V. In view of the condition (3) and the local equicontinuity of $\{f_s: s\in S\}$ there exists a neighbourhood W of $\bar{x}$ such that $f_s(W)\subset B$ for every $s\in S$. Thus we may define the function $\bar{\varphi}: V\cup W\to Y$ by

$$\bar{\varphi}(x)=h(x,\ \mathop{\triangle}_{s\in S}\varphi\circ f_s(x)),\qquad x\in V\cup W.$$

This function fulfils equation (1) and the condition

$$\bar{\varphi}(x)=\varphi_0(x),\qquad x\in U$$

(and, of course, it is a continuous function whenever h, f_s for $s\in S$, and φ are continuous functions). This ends the proof.

I would like to thank Professor M. Kuczma for several helpful discussions concerning the results contained in this paper.

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Modular Hadamard matrices and related designs, III

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Abstract

This paper continues the investigations presented in two previous papers on the same subject by the author and A. T. Butson. Modular Hadamard matrices having n odd and $h \equiv -1 \pmod{n}$ are studied for a few values of the parameters n and h. Also, some results are obtained for the two related combinatorial designs. These results include: a discussion on the known techniques for constructing pseudo (v, k, λ)-designs; the fact that the existence of one of the two related designs always implies the existence of the other; and some information about the column sums of the incidence matrix of each of the two 'maximal' cases of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs.

1. Introduction

Modular Hadamard matrices (or $H(n,h)$ matrices) and their related combinatorial designs were introduced in [1]. (The reader is referred to [1] and [2] for all the pertinent definitions and notation not explained in the present paper.) A part of the work presented in [1] and [2] was motivated by the study of some classes of modular Hadamard matrices having the parameter n odd and $h \equiv 1 \pmod{n}$. If h is odd and h and n are relatively prime, then a necessary condition for the existence of an $H(n, h)$ matrix is that n be odd and h be a quadratic residue of n [1, Theorem 2.2]. This work was initiated with the study of some classes of $H(n, h)$ matrices having n odd and $h \equiv -1 \pmod{n}$. When $h > 1$ is odd, then -1 is not a quadratic residue of n if $n \equiv 3 \pmod{4}$. These observations yield a nonexistence result, part of Theorem 2.3. The modular Hadamard matrices studied here lead to the same types of combinatorial designs as did the classes of matrices studied in [1] and [2].

Section 3 is concerned with pseudo (v, k, λ)-designs. There are exactly four known techniques for constructing a pseudo (v, k, λ)-design from a given (v', k', λ')-design. For each one of these four techniques, there is a known necessary and sufficient condition which ensures that a given pseudo (v, k, λ)-design can be obtained from some (v', k', λ')-design by one of the aforementioned techniques; this information is collected for summary in Theorem 3.1. Also, it is shown that the transpose of the incidence matrix of a primary pseudo (v, k, λ)-design leads to (after an appropriate permutation of its rows) the incidence matrix of an $(m', v', k_1', \lambda_1', k_2', \lambda_2', f', \lambda_3')$-design. Finally, it is observed that neither of the two necessary conditions in Theorem 3.1 in [1] is sufficient for the existence of a pseudo (v, k, λ)-design.

AMS (1970) Subject Classifications: Primary 05B20, 05B30. Secondary 05B05, 62K10.

Received November 27, 1973

Section 4 is concerned with the column sums of the incidence matrices of the two possible 'maximal' cases of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs, $m=v$ and $m=v+1$ [cf. 1, Theorem 4.1]. The theorems presented in this section do not definitively determine the structure of these incidence matrices; however, it is hoped that these results will be helpful in the future study of such matrices.

The paper is concluded with a few simple examples of classes of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs.

2. $H(n, h)$ matrices having $h \equiv -1 \pmod{n}$ and n odd

An h by h matrix H with entries ± 1 is called a *modular Hadamard matrix* if the inner product of each two distinct row vectors is a multiple of a fixed (positive) integer n; that is, each two distinct row vectors are 'orthogonal modulo n.' Such a matrix will be referred to as an $H(n, h)$ matrix, with parameters n and h. With this notation, an ordinary Hadamard matrix of order h will be an $H(0, h)$ matrix.

An $H(n, h)$ matrix is said to be in 'standard form' provided that its first row consists entirely of $+1$'s. Let H_h be a $(1, -1)$-matrix having the first row consisting of all $+1$'s. For each $i, j=2, \ldots, h$ let k_i denote the number of $+1$'s in the ith row, and let λ_{ij} denote the number of times the ith and jth rows have a $+1$ in the same column, $i \neq j$. Then necessary and sufficient conditions that H be an $H(n, h)$ matrix are

$$\begin{aligned} 2k_i &\equiv h \pmod{n} \\ 4\lambda_{ij} &\equiv h \pmod{n} \end{aligned} \tag{2.1}$$

[1, Theorem 2.1].

In this section $H(4q+1, h)$ matrices are studied for $h=2n-1$ and $3n-1$. When $n=4q+3$, $H(4q+3, h)$ matrices are shown not to exist for $h=an-1$ and a even; also, $H(4q+3, h)$ matrices are studied for $h=3n-1$.

THEOREM 2.1. *If an* $H(4q+1, 8q+1)$ *matrix exists, then* $32q^2+16q+1$ *must be a square. Moreover, if* $32q^2+16q+1=D^2$, *then the existence of any one of the following one is equivalent to the existence of any of the other two: an* $H(4q+1, 8q+1)$ *matrix, a pseudo* $(8q+1, 2q, q)$*-design, and an* $(8q+1, (8q+1-D)/2, (8q+1-D)/2-q)$*-design.*

Let H be an $H(n, h)$ matrix in standard form and having $n=4q+1$ and $h=2n-1 =8q+1$. In this case, the only solution (having $2k_i \leqslant h$) of the Congruences (2.1) is $k_i=2q$ and $\lambda_{ij}=q$. Hence, if A is the matrix obtained from H by removing the first row of all $+1$'s and changing each -1 to 0, then

$$\begin{aligned} AA^T &= qI+qJ, \quad \text{and} \\ AJ_{8q+1,1} &= 2qJ_{8q,1}. \end{aligned}$$

Thus, A is the incidence matrix of a pseudo $(8q+1, 2q, q)$-design. The result is now a consequence of Theorems 3.3 and 3.4 in [1].

THEOREM 2.2. *The existence of an* $H(4q+1, 12q+2)$ *matrix is equivalent to the existence of an* $(m', v', k_1', \lambda_1', k_2', \lambda_2', f', \lambda_3')$*-design, where* $m'=12q+1$, $v'=12q+2$, $k_1'=2q$, $\lambda_1'=q$, $k_2'=6q+1$, $\lambda_2'=5q+1$, $f'=f$ *(where* f *is some fixed integer,* $0\leqslant f \leqslant 12q+1$*), and* $\lambda_3'=q$.

Let H be an $H(n,h)$ matrix in standard form and having $n=4q+1$ and $h=3n-1=12q+2$. Following the same procedure as in the proof of Theorem 2.1, one is led to a (0, 1)-matrix B of the form

$$B=\begin{bmatrix} M_{f,12q+2} \\ N_{12q+1-f,12q+2} \end{bmatrix},$$

where

$$BB^T=\begin{bmatrix} MM^T & qJ \\ qJ & NN^T \end{bmatrix},$$

$$MM^T=qI+qJ,$$

$$NN^T=\begin{bmatrix} 6q+1 & \mu_{12} & \cdots & \mu_{1,12q+1-f} \\ \mu_{21} & 6q+1 & \cdots & \mu_{2,12q+1-f} \\ \vdots & \vdots & & \vdots \\ \mu_{12q+1-f,1} & \mu_{12q+1-f,2} & \cdots & 6q+1 \end{bmatrix},$$

each μ_{rs} is either q or $5q+1$, and f is a fixed integer, $0\leqslant f\leqslant 12q+1$. Then, using an argument similar to that which precedes Theorem 3.4 in [2], one may show that each μ_{rs} may be assumed to be $5q+1$ (without loss of generality).

THEOREM 2.3. *If* $n=4q+3$, $h=an-1$, *and* a *is even, then* $H(n,h)$ *matrices cannot exist (because* h *is not a quadratic residue of* n*). The only* $H(4q+3, 12q+8)$ *matrices are the* $H(0, 12q+8)$ *matrices (because the only pertinent solution of the Congruences* (2.1) *is* $k_i=6q+4$ *and* $\lambda_{ij}=3q+2$*).*

It is well known that a necessary condition for the existence of an $H(0, h)$ matrix (an ordinary Hadamard matrix of order h) having $h\geqslant 3$ is that $h=4t$ for some positive integer t; moreover, it is a well known conjecture, unresolved to this writing, that this condition is also sufficient. If $h\geqslant 3$, then a necessary condition for the existence of an $H(n, h)$ matrix is that $h\equiv 4t \pmod{n}$ for some integer t [1, Theorem 2.2]; but this condition is not sufficient (because, for example, $H(3, 11)$ matrices do not exist [2, Theorem 2.10]):

COROLLARY 2.1. *The condition that* $h\equiv 4t \pmod{n}$ *for some integer* t *is not sufficient to ensure the existence of an* $H(n, h)$ *matrix having* $h\geqslant 3$.

3. Pseudo (v, k, λ)-designs

Let $X=\{x_1,\ldots,x_v\}$, and let $X_1,\ldots,X_{v-1}$ be subsets of X. The subsets $X_1,\ldots,X_{v-1}$ are said to form a *pseudo* (v, k, λ)*-design* if each X_j $(1\leqslant j\leqslant v-1)$ has k elements; each two distinct X_i, X_j $(1\leqslant i, j\leqslant v-1)$ intersect in λ elements; and $0<\lambda<k<v-1$.

A pseudo (v, k, λ)-design is called *primary* or *nonprimary* according to whether its parameters satisfy $v\lambda\neq k^2$ or $v\lambda=k^2$, respectively. A nonprimary pseudo (v, k, λ)-design is equivalent to a (v', k', λ')-design [1, Theorem 3.5].

The incidence matrix of a pseudo (v, k, λ)-design can be obtained from the incidence matrix A of a given (v', k', λ')-design by any one of the following four techniques:

1. a column of $+1$'s is adjoined to A;
2. a column of 0's is adjoined to A;
3. a row is discarded from A; or
4. a row is discarded from A and then the k' columns of A which had a $+1$ in the discarded row are complemented (0's and $+1$'s are interchanged in these columns).

These four are the only known techniques for the construction of pseudo (v, k, λ)-designs. For each one of the above techniques there is an arithmetical condition on the parameters v, k, λ which is necessary and sufficient for a given primary pseudo (v, k, λ)-design to be obtained from a (v', k', λ')-design by one of the aforementioned techniques. This is the content of:

THEOREM 3.1. *The incidence matrix of a given primary pseudo* (v, k, λ)*-design can be obtained from the incidence matrix of some* (v', k', λ')*-design by the* ith $(1\leqslant i\leqslant 4)$ *technique above if and only if the parameters* v, k, λ *satisfy the respective* ith *condition below:*

1. $(k-1)(k-2)=(\lambda-1)(v-2)$;
2. $k(k-1)=\lambda(v-2)$;
3. $k(k-1)=\lambda(v-1)$; *or*
4. $k=2\lambda$.

This theorem is essentially proven in [1]; for $i=1$, it is Theorem 3.8 in [1]; for $i=3$, it is the result in [1, Theorem 3.9]; for $i=4$, it is a consequence of [1, Theorem 3.4]; and for $i=2$, it follows from the fact that the parameters v, k, λ of a pseudo (v, k, λ)-design satisfy $k(k-1)=\lambda(v-2)$ if and only if the parameters $\bar{v}$, $\bar{k}$, $\bar{\lambda}$ of the complementary design (its incidence matrix is obtained from the incidence matrix of the given design by replacing each 0 by $+1$ and each $+1$ by 0) satisfy $(\bar{k}-1)(\bar{k}-2)=(\bar{\lambda}-1)(\bar{v}-2)$.

A primary pseudo (v, k, λ)-design is said to be of *type* i $(1\leqslant i\leqslant 4)$ if its parameters satisfy the ith equation in the statement of Theorem 3.1. It is not difficult to construct examples of pseudo (v, k, λ)-designs of each of these four types that are not of any of the other three types. For example, a pseudo (8, 5, 3)-design is of type 1 only; a pseudo (6, 4, 3)-design is of type 2 only; a pseudo (11, 5, 2)-design is of type 3 only; and a pseudo (13, 6, 3)-design is of type 4 only. It is possible for a pseudo (v, k, λ)-design to be of more than one type; for example, a pseudo (5, 4, 3)-design is of both type 1 and type 3.

The condition that the parameters v, k, λ satisfy the ith $(1 \leqslant i \leqslant 4)$ equation in the statement of Theorem 3.1 is *not* sufficient to ensure the existence of a pseudo (v, k, λ)-design, since none of these conditions is sufficient to ensure the existence of a (v', k', λ')-design with the appropriate parameters v', k', λ'. However, it is known [1, Theorem 3.7] that each primary pseudo (v, k, λ)-design with $\lambda = 1$ can be obtained from some (v', k', λ')-design (including 'degenerate' cases such as $\lambda' = 0$) by one of the four techniques mentioned in the second paragraph of this section.

The transpose of the incidence matrix of a primary pseudo (v, k, λ)-design leads to (after an appropriate permutation of its rows) the incidence matrix of an $(m', v', k_1', \lambda_1', k_2', \lambda_2', f', \lambda_3')$-design, which is the other combinatorial design introduced in [1]. This result is stated in the next theorem, which has also been obtained independently by Woodall [3, pp. 682–684] (the terminology used by Woodall is not the same as that used here). The argument given by this author is essentially the same as that given by Woodall, and therefore no details of the proof of the next theorem will be presented in this paper.

THEOREM 3.2. *The existence of a primary pseudo* (v, k, λ)*-design implies the existence of an* $(m', v', k_1', \lambda_1', k_2', \lambda_2', f', \lambda_3')$*-design, where* $m' = v$, $v' = v - 1$, $k_1' = k_1$, $\lambda_1' = k_1 - k + \lambda$, $k_2' = k_2$, $\lambda_2' = k_2 - k + \lambda$, $\lambda_3' = k_1 r_2/(v - f) = k_2 r_1/f$, *where* k_1 *and* k_2 *are the two distinct column sums of the incidence matrix* A *of the primary pseudo* (v, k, λ)*-design,* $r_1 + r_2 = k$, *and* $f' = f$ *is the number of columns of* A *having sum* k_1.

As a consequence of Theorem 4.1 in [1] one now has:

COROLLARY 3.1. *A necessary condition for the existence of a primary pseudo* (v, k, λ)*-design is that* $(k_1 r_2/(v - f))^2 = (k_2 r_1/f)^2 > (k_1 - k + \lambda)(k_2 - k + \lambda)$, *where these parameters are as in the statement of Theorem* 3.2.

The following two conditions are necessary for the existence of pseudo (v, k, λ)-designs [1, Theorem 3.1]:

$$\begin{aligned} &\text{if } v \text{ is even, then } vt - (v-1)k^2 \text{ must be a square; and} \\ &\text{if } v \text{ is odd, then } (k-\lambda)\,[vt - (v-1)k^2] \text{ must be a square,} \end{aligned} \tag{3.1}$$

where $t = k + \lambda(v - 2)$. However, it can be shown that neither pseudo (54, 14, 4)-designs (their existence is equivalent to the existence of (53, 13, 3)-designs) nor pseudo (73, 16, 4)-designs (in this case there is no integral solution for p in Equation (3.8) in [1]) can exist. This yields:

COROLLARY 3.2. *Neither of the two necessary conditions in* (3.1) *is sufficient for the existence of a pseudo* (v, k, λ)*-design.*

4. $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs

An $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design must have $m \leqslant v$ if $\lambda_3^2 \leqslant \lambda_1\lambda_2$, or $m \leqslant v+1$ if $\lambda_3^2 > \lambda_1\lambda_2$ [1, Theorem 4.1]. This section is concerned with the column sums of the incidence matrices of each of the two possible 'maximal' cases, $m=v$ and $m=v+1$. In both cases an equation is obtained involving the sum s_j of the jth column of the incidence matrix A of the design and the sum s_j' of the jth column in the 'top part' of A; also, some information is obtained on the parameter λ_3.

Throughout this section, the incidence matrix A of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design will be assumed to have the form

$$A = \begin{bmatrix} M_{f,v} \\ N_{m-f,v} \end{bmatrix},$$

where

$$AA^T = \begin{bmatrix} MM^T & \lambda_3 J \\ \lambda_3 J & NN^T \end{bmatrix},$$

and

$$MM^T = (k_1 - \lambda_1) I + \lambda_1 J,$$
$$NN^T = (k_2 - \lambda_2) I + \lambda_2 J.$$

Also, the jth column sums of A and M will be denoted by s_j and s_j', respectively.

THEOREM 4.1. *For each* j $(1 \leqslant j \leqslant v)$ *the parameters* s_j *and* s_j' *associated with an* $(m=v, v, \mathrm{k}_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$*-design are related by the following equation, provided that* $s_j' \neq 1$:

$$\begin{aligned} s_j = s_j' - &\{[k_2 + (v-f-1)\lambda_2]\lambda_1 - (v-f)\lambda_3^2\} \\ &\times s_j' / \{[k_1 + (f-1)\lambda_1]\lambda_3 - f\lambda_1\lambda_3\} \\ &- \{[k_1 + (f-1)\lambda_1][k_2 + (v-f-1)\lambda_2] - f\lambda_3^2(v-f)\} \\ &\times (k_1 - \lambda_1) / \{[k_1 + (f-1)\lambda_1]\lambda_3 - f\lambda_1\lambda_3\}(f-1)(s_j' - 1). \end{aligned}$$

If $s_j' = 1$ *and* $v = f$, *then the design must be a* $(v' = v, k' = 1, \lambda' = 0)$*-design. Finally, if* $s_j' = 1$ *and* $v \neq f$, *then* $\lambda_3 > 0$ *and*

$$\lambda_3 = \{[k_1 + (f-1)\lambda_1][k_2 + (v-f-1)\lambda_2] / f(v-f)\}^{1/2}.$$

Let A be the incidence matrix of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design having $m=v$, and let B be the matrix obtained from A by adjoining a last row of all 0's except for a $+1$ in the jth position, $1 \leqslant j \leqslant v$. It follows that BB^T is singular, and therefore $\det BB^T=0$. But also, it is possible to compute $\det BB^T$ directly and obtain

$$\det BB^T=(k_1-\lambda_1)^{f-1}(k_2-\lambda_2)^{v-f-1}D,$$

where

$$\begin{aligned} D=&\{[k_2+(v-f-1)\lambda_2]\lambda_1-(v-f)\lambda_3^2\}(f-1)s_j'(s_j'-1)/(k_1-\lambda_1)\\ &+\{[k_1+(f-1)\lambda_1]\lambda_3-f\lambda_1\lambda_3\}(f-1)(s_j-s_j')(s_j'-1)/(k_1-\lambda_1)\\ &+[k_1+(f-1)\lambda_1][k_2+(v-f-1)\lambda_2]-f\lambda_3^2(v-f). \end{aligned} \tag{4.1}$$

Because $k_1>\lambda_1$ and $k_2>\lambda_2$, it follows that $D=0$. This yields an equation relating s_j and s_j', whenever $s_j'\neq 1$. If $s_j'=1$, then it follows from (4.1) that

$$\lambda_3^2 f(v-f)=[k_1+(f-1)\lambda_1][k_2+(v-f-1)\lambda_2].$$

Now, if $v=f$, then the design under consideration is a (v', k', λ')-design, where $v'=v$, $k'=k_1=s_j'=1$, and $\lambda'=\lambda_1=0$. And, if $v\neq f$, then λ_3 must be positive (it will be seen in Section 5 that it is possible to have a class of $(m', v', k_1', \lambda_1', k_2', \lambda_2', f', \lambda_3')$-designs having $\lambda_3'=0$), and

$$\lambda_3=\{[k_1+(f-1)\lambda_1][k_2+(v-f-1)\lambda_2]/f(v-f)\}^{1/2}.$$

The above result easily yields the following corollary, where the number of distinct column sums in M is assumed to be exactly one (so that M is the incidence matrix of a $(b', v', r', k', \lambda')$-design), and this one column sum is different from 1.

COROLLARY 4.1. *If M is the incidence matrix of a $(b'=v, v'=f, r'=k_1, k'=s\neq 1, \lambda'=\lambda_1)$-design, then the number of distinct column sums of A is exactly one.*

When $m=v+1$, the other 'maximal' case, a similar relationship between s_j and s_j' can be obtained; in fact, this relationship is simpler in this case.

THEOREM 4.2. *An $(m=v+1, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design must have either*

$$s_j=[k_2+(v-f)\lambda_2+f\lambda_3]s_j'/f\lambda_3, \quad \textit{or}$$
$$s_j'=1, \quad \textit{for} \quad 1\leqslant j\leqslant v.$$

Moreover, λ_3 is always positive and

$$\lambda_3=\{[k_1+\lambda_1(f-1)][k_2+\lambda_2(v-f)]/(v+1-f)f\}^{1/2}.$$

Let A be the incidence matrix of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design having $m=v+1$ (and, as a consequence of Theorem 4.1 in [1], $\lambda_3^2>\lambda_1\lambda_2$). It follows that

AA^T is singular, so that $\det AA^T=0$. But one may compute $\det AA^T$ directly to obtain

$$\det AA^T=(k_1-\lambda_1)^{f-1}(k_2-\lambda_2)^{v-f}\{k_1(k_2-\lambda_2)+(v+1-f)\\ \times(k_1-\lambda_1)\lambda_2+(f-1)(k_2-\lambda_2)\lambda_1+(v+1-f)f(\lambda_1\lambda_2-\lambda_3^2)\}.$$

Now, because $k_1>\lambda_1$ and $k_2>\lambda_2$, it follows that

$$\lambda_3=\{[k_1+\lambda_1(f-1)]\,[k_2+\lambda_2(v-f)]/(v+1-f)f\}^{1/2}; \tag{4.2}$$

and, hence, such a design must have $\lambda_3>0$.

Let B be the matrix obtained from A by adjoining a last row of all 0's except for a $+1$ in the jth position, $1\leqslant j\leqslant v$. Then, as before, $\det BB^T=0$, and one can compute $\det BB^T$ directly to obtain

$$\det BB^T=(k_1-\lambda_1)^{f-1}\,(k_2-\lambda_2)^{v-f}D,$$

where

$$D=\{[k_2+(v-f)\lambda_2]\lambda_1-(v+1-f)\lambda_3^2\}(f-1)s_j'(s_j'-1)/(k_1-\lambda_1)\\ +\{[k_1+(f-1)\lambda_1]\lambda_3-f\lambda_1\lambda_3\}(f-1)(s_j-s_j')(s_j'-1)/(k_1-\lambda_1)\\ +\{[k_1+(f-1)\lambda_1]\,[k_2+(v-f)\lambda_2]-f\lambda_3^2(v+1-f)\}. \tag{4.3}$$

It is possible to simplify D, and this will be done presently. First, it follows from Equation (4.2) that the third term on the right hand side of (4.3) is 0. Next, (4.2) is used to substitute for λ_3^2 in the first term on the right hand side of (4.3). Now, using the facts that D must be 0, $k_1>\lambda_1$, and $f>1$ (one *always* has $f\geqslant 1$; however, for the case under present consideration, it is possible to show that $f>1$), one obtains that either

$$s_j=[k_2+(v-f)\lambda_2+f\lambda_3]\,s_j'/f\lambda_3, \quad \text{or}\\ s_j'=1.$$

5. Some examples of classes of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs

In [1] and [2] there is no specific mention of any class of examples of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs. Some simple examples will be presented here mainly to exhibit some infinite classes of these designs, and not so much to present novel construction techniques for combinatorial designs. Only (v, k, λ)-designs will be used in the following examples; however, other well known combinatorial designs could also be used (e.g., the group divisible designs having two groups).

EXAMPLE 5.1. If A is the incidence matrix of a (v', k', λ')-design, then A is also the incidence matrix of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design, where $m=v=f=v'$, $k_1=k_2=k'$, and $\lambda_1=\lambda_2=\lambda_3=\lambda'$.

EXAMPLE 5.2. Let A be the incidence matrix of a (v', k', λ')-design. For a fixed integer f' $(1 \leqslant f' \leqslant v')$ let B be the matrix obtained from A by complementing (interchange 0's and $+1$'s) the first f' rows of A. Then B is the incidence matrix of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design, where $m = v = v'$, $k_1 = v' - k'$, $\lambda_1 = v' - 2k' + \lambda'$, $k_2 = k'$, $\lambda_2 = \lambda'$, $f = f'$, and $\lambda_3 = k' - \lambda'$.

EXAMPLE 5.3. Let A be the incidence matrix of a (v', k', λ')-design. Permute the rows of A, if necessary, so that the first column has all $+1$'s in the first k' positions. Let A' denote the matrix thus obtained. Next, complement the k' by $v' - 1$ submatrix of A' consisting of the first k' rows, except the first column. If B denotes the matrix thus obtained from A, then B is the incidence matrix of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design, where $m = v = v'$, $k_1 = v' - k' + 1$, $\lambda_1 = v' - 2k' + \lambda' + 1$, $k_2 = k'$, $\lambda_2 = \lambda'$, $f = k'$, and $\lambda_3 = k' - \lambda'$.

EXAMPLE 5.4. Let A be the incidence matrix of a (v', k', λ')-design, and let θ denote the matrix having all entries equal to 0. If

$$B = \begin{bmatrix} A & \theta \\ \theta & A \end{bmatrix},$$

then B is the incidence matrix of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design, where $m = v = 2v'$, $k_1 = k_2 = k'$, $\lambda_1 = \lambda_2 = \lambda'$, $f = v'$, and $\lambda_3 = 0$.

EXAMPLE 5.5. Let A be the incidence matrix of a (v', k', λ')-design, and let $\bar{A}$ denote the complement (interchange 0's and $+1$'s) of A. If θ is as in the above example and

$$B = \begin{bmatrix} A & \theta \\ \theta & \bar{A} \end{bmatrix},$$

then B is the incidence matrix of an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design, where $m = v = 2v'$, $k_1 = k'$, $\lambda_1 = \lambda'$, $k_2 = v' - k'$, $\lambda_2 = v' - 2k' + \lambda'$, $f = v'$, and $\lambda_3 = 0$.

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Orthogonal designs II

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Abstract

Orthogonal designs are a natural generalization of the Baumert-Hall arrays which have been used to construct Hadamard matrices. We continue our investigation of these designs and show that orthogonal designs of type $(1, k)$ and order n exist for every $k<n$ when $n=2^{t+2}\cdot 3$ and $n=2^{t+2}\cdot 5$ (where t is a positive integer). We also find orthogonal designs that exist in every order $2n$ and others that exist in every order $4n$.

Coupled with some results of earlier work, this means that the *weighing matrix conjecture* 'For every order $n\equiv 0 \pmod 4$ there is, for each $k\leqslant n$, a square $\{0, 1, -1\}$ matrix $W=W(n, k)$ satisfying $WW^t=kI_n$' is resolved in the affirmative for all orders $n=2^{t+1}\cdot 3$, $n=2^{t+1}\cdot 5$ (t a positive integer).

The fact that the matrices we find are skew-symmetric for all $k<n$ when $n\equiv 0 \pmod 8$ and because of other considerations we pose three other conjectures about weighing matrices having additional structure and resolve these conjectures affirmatively in a few cases.

In an appendix we give a table of the known results for orders $\leqslant 64$.

§0. Introduction

An orthogonal design of order n and type $(s_1, s_2, \ldots, s_l)$ $(s_i>0)$ on the commuting variables $x_1, x_2, \ldots, x_l$ is an $n\times n$ matrix A with entries from $\{0, \pm x_1, \ldots, \pm x_l\}$ such that

$$AA^t=\sum_{i=1}^{l} (s_i x_i^2)\, I_n.$$

Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$.

In [2], where this was first defined and many examples and properties of such designs were investigated, we mentioned that

$$A^tA=\sum_{i=1}^{l} (s_i x_i^2)\, I_n$$

and so our alternative description of A applies equally well to the columns of A. We also showed in [2] that $l\leqslant \varrho(n)$, where $\varrho(n)$ (Radon's function) is defined by

$$\varrho(n)=8c+2^d$$

AMS (1970) Subject Classification: Primary 05B20, 05B15.
This work was supported in part by NRC grant 8488.

Received May 6, 1974.

when

$$n=2^a\cdot b, \quad b \text{ odd}, \quad a=4c+d \quad 0\leqslant d<4.$$

In [2] we also showed that if there is an orthogonal design or order n and type (a, b) then

(i) $n\equiv 2 \pmod 4 \Rightarrow b/a=c^2$ for some rational number c

(ii) $n=4t$, t odd $\Rightarrow b/a$ is a sum of $\leqslant$three rational squares.

DEFINITION. A *weighing matrix of weight k and order n*, is a square $\{0, 1, -1\}$ matrix, A, of order n satisfying

$$AA^t=kI_n.$$

In [2] we showed that the existence of an orthogonal design of order n and type $(s_1,\ldots, s_l)$ is equivalent to the existence of weighing matrices $A_1,\ldots, A_l$, of order n, where A_i has weight s_i and the matrices, $\{A_i\}_{i=1}^l$, satisfy the matrix equation

$$XY^t+YX^t=0$$

in pairs. In particular, the existence of an orthogonal design of order n and type $(1, k)$ is equivalent to the existence of a skew-symmetric weighing matrix of weight k and order n.

It is conjectured that:

(I) for $n\equiv 0 \pmod 4$ there is a weighing matrix of weight k and order n for every $k\leqslant n$.

(II) for $n\equiv 0 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k<n$ (equivalently there is an orthogonal design of type $(1, k)$ in order n for every $k<n$).

(III) for $n\equiv 4 \pmod 8$ there is a skew-symmetric weighing matrix of order n for every $k<n$, where k is the sum of $\leqslant$three squares of integers (equivalently, there is an orthogonal design of type $(1, k)$ in order n for every $k<n$ which is the sum of $\leqslant$three squares of integers. In other words, the necessary condition, given above in (ii), for the existence of an orthogonal design of type $(1, k)$ in order n, $n\equiv 4 \pmod 8$, is also sufficient).

(IV) for $n\equiv 2 \pmod 4$ there is a skew-symmetric weighing matrix for every weight $k<n-1$ when k is a square (equivalently, the necessary condition given above for the existence of an orthogonal design of type $(1, k)$ in order n (see (i) above), is also sufficient.)

Conjecture (I) is an extension of the Hadamard conjecture (i.e. for every $n\equiv 0 \pmod 4$ there is a $\{1, -1\}$ matrix, H, of order n satisfying $HH^t=nI_n$), while (II) and (III) generalize the conjecture that for every $n\equiv 0 \pmod 4$ there is a Hadamard matrix, H, of order n with the property that $H=I_n+S$ where $S=-S^t$.

Conjecture (I) was established in [4] for $n\in\{4, 8, 12,\ldots, 32, 40\}$ and in [1] for $n=2^t$. Conjecture (II) was established in [2, Theorem 17] for $n=2^t$ ($t\geqslant 3$), while Conjecture III was established in [2] for $n=4$, 12. Conjecture IV was established in [2] for $n=6$, 10, 14.

In this paper we establish conjectures (II) and (III) (and as a consequence (I)) for $n=2^{t+1}\cdot 3$, $n=2^{t+1}\cdot 5$, t a positive integer. We also establish conjecture (III) for $n=28$ separately. In addition, for every order $n\equiv 0 \pmod 4$ we give 'segments' on which the conjecture is true. These segments grow with n.

Let R be the back diagonal matrix. Then an orthogonal design or weighing matrix is said to be *constructed from two circulant matrices A and B* if it is of the form

$$\begin{bmatrix} A & B \\ B^t & -A^t \end{bmatrix}$$

and to be of Goethals-Seidel type if it is of the form

$$\begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^tR & -C^tR \\ -CR & -D^tR & A & B^tR \\ -DR & C^tR & -B^tR & A \end{bmatrix}$$

where A, B, C, D are circulant matrices.

§1. Known results and new applications

In this section we would like to list some of the results from [2] that we shall use and also give some new applications of them.

PROPOSITION 1. [2, Corollary to Construction 22]. *If there is an orthogonal design of type* $(1, l)$ *in order n then there is an orthogonal design of type* $(1, 1, l, l)$ *in order* $2n$ *and of type* $(1, 1, 2, l, l, 2l)$ *in order* $4n$.

The following is an easy corollary and was not mentioned in [2]. We shall use it extensively in this paper.

COROLLARY. *If there are orthogonal designs of type* $(1, k)$, $1\leqslant k\leqslant l$, *in order n then there are orthogonal designs of type* $(1, m)$ *in order* $2n$ *for* $1\leqslant m\leqslant 2l+1$. *In particular, if there are orthogonal designs of type* $(1, k)$, $1\leqslant k\leqslant n-1$, *in order n then there are orthogonal designs of type* $(1, m)$, $1\leqslant m\leqslant 2^t n-1$, *in order* $2^t n$, *t a positive integer.*

PROPOSITION 2. [2, Corollary to Proposition 6]. *If there is an orthogonal design of order n and type* $(s_1, \ldots, s_l)$ *then there are orthogonal designs of type* $(\varepsilon_1 s_1, \ldots, \varepsilon_l s_l)$ *in order* $2n$, *where* $\varepsilon_i = 1$ *or* 2.

We can now establish

THEOREM 3. *Let n be any number of the form* $2^t \cdot 3$, *t a positive integer. Then*

(a) *If* $t=1$, *then conjecture IV is true.*

(b) *If* $t=2$, *then conjectures I and III are true.*

(c) *If* $t \geqslant 3$, *then conjecture II (and consequently conjecture I) is true.*

Proof. As we mentioned in the introduction, (a) was verified in [2] while (b) was verified in [1] and [2]. Thus we need only consider (c).

From (b) we have orthogonal designs of type $(1, k)$ in order 12 for $1 \leqslant k \leqslant 11$, $k \neq 7$. Proposition 1 then gives designs of type $(1, m)$ in order 24 for $1 \leqslant m \leqslant 23$, $m \neq 14, 15$. From (a) we have a design of type (1, 4) in order 6 and so by Proposition 1 a design of type (1, 1, 2, 4, 4, 8) in order 24. By setting the variables in this design equal to each other, or to zero, we obtain designs of type (1, 14), (1, 15) in order 24. Thus, conjecture II (and I) are true for $n=24$. Now the corollary to Proposition 1 gives the full result.

THEOREM 4. (a) *There are orthogonal designs of type* (1, 1) *and* (1, 4) *in order* $2n$ *for every integer* $n \geqslant 3$.

(b) *There is an orthogonal design of type* (1, 9) *in order* $2n$ *for every integer* $n \geqslant 6$ *(except possibly for* $n=9, 11$*).*

(c) *There is an orthogonal design of type* (1, 1, 1, 4) *in every order* $4n$, $n \geqslant 2$.

(d) *There are orthogonal designs of type* (1, 1, 2, 8) *and* (1, 1, 1, 9) *in every order* $4n$, $n \geqslant 3$.

Proof. The proofs of these statements all follow the same pattern. They will follow from the observation that if A and B are orthogonal designs of type $(s_1, \ldots, s_l)$ having orders n and m respectively, then $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is an orthogonal design of the same type having order $n+m$.

(a) In [2] we showed that orthogonal designs of type (1, 1) and (1, 4) exist in orders 6, 8 and 10. Since every even integer $\geqslant 6$ is a non-negative linear combination of these three integers the result will follow from our remark above.

(b) In [2] we showed that an orthogonal design of type (1, 9) exists for orders 12, 14, 16. Since there is an orthogonal design of type (1, 4) in order 10, we have, by Proposition 1, an orthogonal design of type (1, 1, 4, 4) in order 20. Thus, there is an orthogonal design of type (1, 9) in order 20. Now observe that every even integer $2n$, $n \geqslant 12$, is a non-negative linear combination of 12, 14, 16 and 20 to finish off the proof here.

(c) We have, in [2], exhibited a design of type (1, 1, 1, 4) in orders 8 and 12. Since every integer $4n$, $n \geqslant 2$, is a non-negative linear combination of these two integers we are finished with this part.

(d) In [2] we exhibited designs of type (1, 1, 2, 8) and (1, 1, 1, 9) in orders 12 and 16. If we can exhibit designs of this type in order 20 we will be done. We construct four circulant matrices A, B, C, D such that $AA^t + BB^t + CC^t + DD^t = kI_5$ where k is, in the first instance $x_1^2 + x_2^2 + x_3^2 + 9x_4^2$, and in the second instance $k = x_1^2 + x_2^2 + 2x_3^2 + 8x_4^2$. We then use these matrices in the Goethals-Seidel array:

k	(1, 1, 1, 9)	(1, 1, 2, 8)
1*st Row of A*	$x_1\ 0\ x_4\ -x_4\ 0$	$x_1\ 0\ x_4\ -x_4\ 0$
1*st Row of B*	$x_2\ 0\ x_4\ -x_4\ 0$	$x_2\ 0\ x_4\ -x_4\ 0$
1*st Row of C*	$x_3\ -x_4\ 0\ 0\ x_4$	$x_3\ 0\ x_4\ x_4\ 0$
1*st Row of D*	$x_4\ x_4\ x_4\ 0\ 0$	$-x_3\ 0\ x_4\ x_4\ 0$

Remark. (1) We strongly suspect that there are designs of type (1, 9) in orders 18 and 22.

(2) Our method of proof shows that any orthogonal design that appears in orders 12 and 16 also appears in order $4n$, $n \geqslant 6$. Thus, in addition to the designs mentioned above there are designs of type (1, 1, 5, 5), (1, 2, 2, 4), (1, 2, 3, 6) and (2, 2, 2, 2) in every order $4n$, $n = 3, 4$, $n \geqslant 6$. [See [2] for the description of these designs in orders 12 and 16].

COROLLARY 1. *There are orthogonal designs of type* $(1, k)$ *in order* $4n$, $n \geqslant 3$, *for* $1 \leqslant k \leqslant 11$, $k \neq 7$.

Proof. From (c) and (d) above we have designs of type (1, 1, 2, 8) and (1, 1, 1, 4) in all the orders $4n$, $n \geqslant 3$. Now set the variables in these designs equal to each other or to zero to obtain the statement in this Corollary.

COROLLARY 2. *There is an orthogonal design of type* $(1, k)$ *in every order* $8n$, $n \geqslant 3$, for $k \in \{1, \ldots, 23\}$.

Proof. By using Proposition 1 and the designs of type (1, 1) and (1, 4) in order $2n$, $n \geqslant 3$, we obtain designs of type (1, 1, 2, 1, 1, 2) and (1, 1, 2, 4, 4, 8) in every order $8n$, $n \geqslant 3$. These two designs give designs of type $(1, k)$ for $1 \leqslant k \leqslant 19$. By Corollary 1 we have designs of type (1, 10) and (1, 11) in order $4n$, $n \geqslant 3$ and an application of Proposition 1 then gives designs of type (1, 1, 10, 10) and (1, 1, 11, 11) in order $8n$, $n \geqslant 3$ and so the proof is complete.

COROLLARY 3. *There is an orthogonal design of type* $(1, k)$ *in order* $8n, n=6, 7, 8, 10$ *or* $n \geq 12$ for $k \in \{1, \ldots, 23, 27, \ldots, 30, 36, \ldots, 39\}$.

Proof. For $1 \leq k \leq 23$ we appeal to Corollary 2. By Theorem 4, part (b), we have a design of type $(1, 9)$ in order $2n$, $n=6, 7, 8, 10$, $n \geq 12$ and so by Proposition 1 we have a design of type $(1, 1, 2, 9, 9, 18)$ in these orders. Setting the variables in this design equal to each other, or to zero will then prove the corollary.

§2. Golay sequences and orthogonal designs

Let $X=\{[a_{11}, \ldots, a_{1n}], [a_{21}, \ldots, a_{2n}], \ldots, [a_{m1}, \ldots, a_{mn}]\}$ be m sequences of integers of length n.

DEFINITION. (1) The *non-periodic auto-correlation function of the family of sequences* X (denoted N_X) is a function from the set of integers $\{1, 2, \ldots, n-1\}$ to Z (the integers) where

$$N_X(j)=\sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j}+a_{2,i}a_{2,i+j}+\cdots+a_{m,i}a_{m,i+j}).$$

Note that if the following collection of m matrices of order n is formed

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{11} & & a_{1,n-1} \\ & & \ddots & \\ 0 & & & \\ & & & a_{11} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ & a_{21} & & a_{2,n-1} \\ & & \ddots & \\ 0 & & & \\ & & & a_{21} \end{bmatrix}, \ldots, \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \\ & \ddots & & a_{m,n-1} \\ & & \ddots & \\ 0 & & & \\ & & & a_{m1} \end{bmatrix},$$

then $N_X(j)$ is simply the sum of the inner products of rows 1 and $j+1$ of these matrices.

(2) The *periodic auto-correlation function of the family of sequences* X (denoted P_X) is a function from the set of integers $\{1, 2, \ldots, n-1\}$ to Z where

$$P_X(j)=\sum_{i=1}^{n} (a_{1,i}a_{1,i+j}+a_{2,i}a_{2,i+j}+\cdots+a_{m,i}a_{m,i+j})$$

where we assume the second subscript is actually chosen from the complete set of residues $\bmod(n)$, $\{1, 2, \ldots, n\}$.

We can interpret the function P_X in the following way: Form the m circulent matrices which have first rows respectively, $[a_{11}\, a_{12} \ldots a_{1n}]$, $[a_{21}\, a_{22} \ldots a_{2n}], \ldots,$

$[a_{m1}\, a_{m2} \dots a_{mn}]$, then $P_X(j)$ is the sum of the inner products of rows 1 and $j+1$ of these matrices.

LEMMA 5. *Let X be a family of sequences as above, then*

$$P_X(j) = N_X(j) + N_X(n-j), \quad j = 1, \dots, n-1.$$

COROLLARY. *If* $N_X(j)=0$ *for all* $j=1,\dots,n-1$ *then* $P_X(j)=0$ *for all* $j=1,\dots,n-1$.

Note. $P_X(j)$ may equal 0 for all $j=1,\dots,n-1$ even though the $N_X(j)$ are not.

DEFINITION. If $X=\{[a_1,\dots,a_n], [b_1,\dots,b_n]\}$ are two sequences where $a_i, b_j \in \{1, -1\}$ and $N_X(j)=0$ for $j=1,\dots,n-1$ then the sequences in X are called *Golay complementary sequences of length n.*

We note that if X is as above and A is the circulant formed by $[a_1,\dots,a_n]$ and B the circulant formed by $[b_1,\dots,b_n]$ then

$$AA^t + BB^t = \sum (a_i^2 + b_i^2)\, I_n.$$

Consequently, such matrices may be used in the Goethals-Seidel array to obtain Hadamard matrices.

EXAMPLE. $X=\{[1, -1], [1, 1]$ are Golay complementary sequences of length 2.

By results of R. J. Turyn [3], Golay complementary sequences exist having length r for

$$r = 2^a \cdot 10^b \cdot 26^c, \quad a, b, c$$

non-negative integers.

Since our interest is in orthogonal designs we shall not be restricted to sequences with entries only ± 1, but shall allow 0's also. One very simple remark is in order. If we have a collection of sequences, X, (each having length n) such that $N_X(j)=0$, $j=1,\dots,n-1$, then we may augment each sequence at the beginning with k zeroes and at the end with l zeroes so that the resulting collection, (say $\bar{X}$), of sequences having length $k+n+l$ still has $N_{\bar{X}}(j)=0$, $j=1,\dots,k+n+l-1$. More interesting is the following result of Turyn.

PROPOSITION 6. *Let* $X=\{[a_1,\dots,a_n], [b_1,\dots,b_n]\}$ *be Golay complementary sequences. Then the sequences in*

$$X' = \left\{\left[\frac{a_1+b_1}{2}, \dots, \frac{a_n+b_n}{2}\right], \left[\frac{a_1-b_1}{2}, \dots, \frac{a_n-b_n}{2}\right]\right\}$$

satisfy

(i) $N_{X'}(j)=0, 1 \leqslant j \leqslant n-1$.

(ii) *exactly half of the* $(a_i+b_i/2$ *are equal to* 0 *and exactly half of the* $(a_i-b_i)/2$ *equal* 0, *all others* $= \pm 1$.

Thus, if we let $X=\{g_r, h_r\}$ represent Golay complementary sequences of length r we obtain a new pair of sequences of length r, which we denote g_r', h_r' each having exactly $r/2$ non-zero members which can be chosen from $\{1, -1\}$ and such that if $X'=\{g_r', h_r'\}$ then $N_{X'}(j)=0, 1 \leqslant j \leqslant r-1$.

Some more notation will be necessary. If g_r denotes a sequence of integers of length r then by xg_r we mean the sequence of length r obtained from g_r by multiplying each member of g_r by x. We let $\bar{0}_r$ denote a sequence consisting of r zeroes and $|g_r|$ denote the sum of the absolute values of the elements of g_r.

THEOREM 7. *Let r be any number of the form* $2^a \cdot 10^b \cdot 26^c$, *a, b, c non-negative integers, and let n be any integer* $>r$. *Then*

(i) *There are orthogonal designs of order* $4n$ and *types* $(1, 1, 2r)$ *and* $(1, 1, r)$.

If, in addition n is odd, *then*

(ii) *there are orthogonal designs of order* $4n$ *and types* $(1, 4, r)$ *and* $(1, 4, 2r)$.

Proof. (i) Let r be as above and let $X=\{g_r, h_r\}$ be Golay complementary sequences of length r. Consider the four circulant matrices A_i, $i=1, 2, 3, 4$ of order n having first rows respectively

$$[x_1\, \bar{0}_{n-1}],\quad [x_2\, \bar{0}_{n-1}],\quad [\bar{0}_{n-r}\, x_3 g_r],\quad [\bar{0}_{n-r}\, x_3 h_r].$$

If

$$Y=\{[x_1, \bar{0}_{n-1}],\ [x_2, \bar{0}_{n-1}],\ [\bar{0}_{n-r}, x_3 g_r],\ [\bar{0}_{n-r}, x_3 h_r]\}$$

then $N_j(Y)=0, 1 \leqslant j \leqslant n-1$ and so we have

$$\sum_{i=1}^{4} A_i A_i^t = (x_1^2 + x_2^2 + 2r\ x_3^2)\, I_n.$$

Thus, the A_i may be used in the Goethals-Seidel array to give an orthogonal design of order $4n$ and type $(1, 1, 2r)$.

If we now replace g_r and h_r by g_r' and h_r' (as in Proposition 6) we obtain an orthogonal design of order $4n$ and type $(1, 1, r)$.

(ii) Let n be odd, $n>r$ and consider the sequences of length n in

$$Y=\{[x_1, \bar{0}_s, x_2, -x_2, \bar{0}_s],\ [0, \bar{0}_s, x_2, x_2, \bar{0}_s]\}$$

(where $s=(n-3)/2$). We claim that $P_Y(j)=0, 1 \leqslant j \leqslant n-1$. This is easy to see since the circulant matrix formed by the first sequence in Y has the form $x_1 I_n + U$, where $U = -U^t$ (note that this is the only place we use the fact that n is odd).

Now let g_r, h_r be Golay complementary sequences of length r and let $X=[\bar{0}_{n-r}, x_3g_r]$, $[\bar{0}_{n-r}, x_3h_r]$. We thus use the sequences in X and Y, as in (i), to obtain orthogonal designs of order $4n$ and types $(1, 4, 2r)$ and $(1, 4, r)$.

THEOREM 8. *Let $X=\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a collection of four $\{0, 1, -1\}$-sequences of length n for which $P_X(j)=0$, $1\leqslant j\leqslant n-1$. Suppose further that the circulant matrices generated by $\alpha_1, \alpha_2, \alpha_3$ are skew-symmetric. Then,* (i) *there is an orthogonal design of order $4n$ and type $(1, 1, 1, |\alpha_1|+|\alpha_2|+|\alpha_3|+|\alpha_4|)$.*

If, in addition, $N_X(j)=0$, $1\leqslant j\leqslant n-1$, then (ii) *this design exists in every order $4(n+2k)$, $k\geqslant 0$.*

Proof. In order that the hypothesis on $\alpha_1, \alpha_2, \alpha_3$ be satisfied we must have

$$\begin{aligned}\alpha_1&=[0, a_1,\dots, a_{n-1}]=[0, g_{n-1}],\\ \alpha_2&=[0, b_1,\dots, b_{n-1}]=[0, h_{n-1}]\\ \alpha_3&=[0, c_1,\dots, c_{n-1}]=[0, l_{n-1}]\end{aligned}$$

where $a_{1+j}=-a_{(n-1)-j}$, $b_{1+j}=-b_{(n-1)-j}$, $c_{1+j}=-c_{(n-1)-j}$, $0\leqslant j\leqslant n-1$. Form four circulant matrices, A_i, $i=1, 2, 3, 4$, whose first rows are, respectively,

$$[x_1\ x_4g_{n-1}],\quad [x_2\ x_4h_{n-1}],\quad [x_3\ x_4l_{n-1}],\quad [x_4\alpha_4].$$

Now $A_i=x_iI_n+B_i$, $i=1, 2, 3$ and $B_i=-B_t^i$; thus,

$$\sum_{i=1}^{4} A_iA_i^t=(x_1^2+x_1^2+x_3^2)\, I_n+\sum_{i=1}^{3} B_iB_i^t+A_4A_4\,.$$

Since $P_X(j)=0$, we thus obtain

$$\sum_{i=1}^{3} B_iB_i^t+A_4A_4^t=(|\alpha_1|+|\alpha_2|+|\alpha_3|+|\alpha_4|)\, x_4^2\cdot I_n\,.$$

We may thus use these matrices, A_i in the Goethals-Seidel array to obtain the first part of the theorem.

The final assertion of the theorem follows from the observation we made earlier, that a collection of sequences whose non-periodic auto-correlation function was identically zero could be augmented, front or back (or both) by zero sequences to obtain longer sequences with the same property. If, in our case, we add sequences of *equal* length to the front *and* back of the given sequences then we preserve their skew-symmetric character. These remarks, coupled with the proof of (i) will then constitute a proof of (ii).

Table 1

	α_1	α_2	α_3	α_4	Remarks
i)	[0, 1, 1, −, 1, −, −]	$\alpha_2=\alpha_1$	[1, 1, 1, 0, 1, 0, 0]	[−, 1, 1, 0, 1, 0, 0]	$P(j)=0,\ 1\leqslant j\leqslant 6$ $N(2)\neq 0$
ii)	[0, 1, 1, −, 1, −, −]	$\alpha_2=\alpha_1$	$\alpha_3=\alpha_1$	[−, 1, 1, 1, 1, 1, 1]	$P(j)=0,\ 1\leqslant j\leqslant 6$ $N(1)\neq 0$
iii)	[0, 1, 1, −, 1, −, −]	[0, 1, 1, −, −, 1, 1]	[0, 1, 1, 1, −, 1, −]	[0, 1, −, 1, 1, 1, 1]	$N(j)=0,\ 1\leqslant j\leqslant 6$
iv)	[0, 1, −, 1, −]	[0, 1, 1, −, −]	[0, 1, −, −, 1]	[0, 1, 1, 1, 1]	$N(j)=0,\ 1\leqslant j\leqslant 4$
v)	[0, 0, 1, −, 0]	[0, 0, 1, 1, 0]	[−, 1, 1, 1, 1]	[0, 1, −, −, 1]	$P(j)=0,\ 1\leqslant j\leqslant 4$ $N(1)\neq 0$
vi)	[0, 1, −, 1, −]	[−, 0, 1, 1, 0]	[−, 1, 1, 1, 1]	[0, 0, 1, 1, 0]	$P(j)=0,\ 1\leqslant j\leqslant 4$ $N(1)\neq 0$
vii)	[0, 1, −, 1, −]	[0, 1, 1, −, −]	[−, 1, 1, 1, 1]	$\alpha_4=\alpha_3$	$P(j)=0,\ 1\leqslant j\leqslant 4$ $N(1)\neq 0$
viii)	[0, 1, 1, −, 1, −, −]	[0, 0, 1, 0, 0, −, 0]	$\alpha_3=\alpha_2$	[0, 1, 1, −, 1, 1, 1]	$N(j)=0,\ 1\leqslant j\leqslant 6$
ix)	[0, −, 1, −, −, 1, 1, −, 1]	[0, 1, −, −, −, 1, 1, 1, −]	[0, −, 1, 1, 1, 1, 1, 1, −]	[0, 1, −, 1, 1, 1, 1, −, 1]	$N(j)=0,\ 1\leqslant j\leqslant 8$
x)	[0, 1, −, 1, 1, 1, −, −, −, 1, −]	[−, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0]			$P(j)=0,\ 1\leqslant j\leqslant 10$ $N(2)\neq 0$

Remarks. (a) We would have an analogous theorem if only α_1 (or only α_1 and α_2) generated a skew-symmetric circulant.

(b) There is a completely analogous theorem if there are only two sequences of length n and the first has the skew-symmetric character described above. We just use the array $\begin{bmatrix} A & B \\ -B^t & A^t \end{bmatrix}$, mentioned in the introduction, in place of the Goethals-Seidel array. To facilitate the references we shall explicitly state:

COROLLARY. *Let* $X=\{\alpha_1, \alpha_2\}$ *be two* $\{0, 1, -1\}$ *sequences of length n for which* $P_X(j)=0$, $1\leqslant j\leqslant n-1$. *Suppose that the circulant matrix generated by* α_1 *is skew-symmetric. Then, there is an orthogonal design of order* $2n$ *and type* $(1, |\alpha_1|+|\alpha_2|)$.

If, in addition, $N_X(j)=0$, $1\leqslant j\leqslant n-1$, *then this design exists in every order* $2(n+2k)$, $k\geqslant 0$.

We now give some examples illustrating the use of Theorem 8 and its corollary.

These examples then give the following:

PROPOSITION 9. *The following orthogonal designs exist in the orders stated*:

i) $(1, 1, 20)$ *in order* 28
ii) $(1, 1, 1, 25)$ *in order* 28
iii) $(1, 24)$ *in orders* $4(7+2k)$, $k\geqslant 0$
iv) $(1, 1, 16)$ *in orders* $4(5+2k)$, $k\geqslant 0$
v) $(1, 13)$ *in order* 20
vi) $(1, 14)$ *in order* 20
vii) $(1, 1, 18)$ *in order* 20
viii) $(1, 1, 1, 16)$ *in orders* $4(7+2k)$, $k\geqslant 0$
ix) $(1, 1, 32)$ *in orders* $4(9+2k)$, $k\geqslant 0$
x) $(1, 16)$ *in order* 22.

Proof. Use the sequences in Table 1 as indicated in Theorem 8 or its corollary.

§3. Some applications

THEOREM 10. *Let* $n=2^t\cdot 5$, $t>0$.

(a) *If* $t=1$ *then conjecture IV is true.*

(b) *If* $t=2$ *then conjectures III and I are true.*

(c) *If* $t\geqslant 3$ *then conjectures II and I are true.*

Proof. (a) As mentioned in the introduction, this was established in [2].

(b) Corollary 1 to Theorem 4 gives designs of type $(1, k)$ in order 20 for $1\leqslant k\leqslant 11$, $k\neq 7$. By Theorem 7 (ii), there is an orthogonal design of type $(1, 4, 2\cdot 4)$ in order 20 which gives $(1, k)$ for $k=12$. Proposition 9 then gives $(1, k)$ in order 20 for $k=13$, 14, 16, 17, 18, 19. Since 7 and 15 are not the sum of $\leqslant$three squares this proves conjecture III for 20 and also conjecture I.

(c) From Propositions 1 and 2 and from (b) of this Theorem we obtain designs of type $(1, k)$ in order 40 for $1 \leqslant k \leqslant 39$, $k \neq 15, 30, 31$. By Corollary 2 to Theorem 4 we have the result for $k=15$. If we can find orthogonal designs of type (1, 30) and (1, 31) in order 40 then the proof of (c) will be completed by repeated application of the corollary to Proposition 1.

To do this we need the following well-known fact:

LEMMA 11. *Let X, Y be two back-circulant matrices of order n, n* odd. *Suppose that the circulant matrices generated by the first rows of X and Y respectively are symmetric, then $XY^t = YX^t$.*

We recall that there is an orthogonal design of order 8 and type (1, 1, 1, 1, 1, 1, 1, 1), (see [5] for this classical design derived from the Cayley Numbers) on the variables $y_1, \ldots, y_8$. We shall substitute a circulant matrix for y_1 and back-circulants for $y_2, \ldots, y_8$; where if A_i is the matrix being substituted for y_i we have:

A_1	with first row	x_1	0	0	0	0
A_2	with first row	x_2	0	0	0	0
A_3	with first row	0	x_2	x_2	x_2	x_2
A_4	with first row	$-x_2$	x_2	x_2	x_2	x_2
$A_5 = A_6$	with first row	x_2	$-x_2$	x_2	x_2	$-x_2$
$A_7 = A_8$	with first row	x_2	x_2	$-x_2$	$-x_2$	x_2.

If $X, Y \in \{A_1, \ldots, A_8\}$ then $XY^t = YX^t$ by Lemma 11 or by the fact that if X is circulant and Y is back-circulant then $XY^t = YX^t$.

This then gives an orthogonal design of order 40 and type (1, 30) since

$$\sum_{i=1}^{8} A_i A_i^t = (x_1^2 + 30 x_2^2)\, I_n .$$

We use the same procedure to obtain a design of order 40 and type (1, 31). This time, let

A_1	have first row	x_1	0	x_2	$-x_2$	0
A_2	have first row	0	0	x_2	x_2	0
$A_3 = A_4 = A_5$	have first row	$-x_2$	x_2	x_2	x_2	x_2
$A_6 = A_7 = A_8$	have first row	0	x_2	$-x_2$	$-x_2$	x_2.

PROPOSITION 12. (a) *Conjecture IV is true for $n = 2 \cdot 7$.*

(b) *For $n = 4 \cdot 7$ conjectures III and I are true.*

(c) *For* $n=8\cdot 7$, *there is an orthogonal design of type* $(1, k)$ *for* $1 \leqslant k \leqslant 55$, *except possibly for* $k=46, 47$.

Proof. (a) was established in [2].

(b) We first establish conjecture III. This will then yield a proof of conjecture I for 28 (although this was already done in [4]). As in the proof of Theorem 10 (b) we have orthogonal designs of type $(1, k)$, $1 \leqslant k \leqslant 12$, $k \neq 7$. From Proposition 9 we obtain orthogonal designs in order 28 of type $(1, k)$ for $k=16, 17, 18, 20, 21, 24, 25, 26, 27$.

Now in [2] we found circulant matrices

i) A_1, A_2 of order 7 such that

$$\sum_{i=1}^{2} A_i A_i^t = (x_1^2 + 4x_2^2)\, I_7$$

ii) B_1, B_2 of order 7 such that

$$\sum_{i=1}^{2} B_i B_i^t = (y_1^2 + 9y_2^2)\, I_7$$

iii) C_1, C_2 of order 7 such that

$$\sum_{i=1}^{2} C_i C_i^t = 13 I_7 .$$

Hence, if we use A_1, A_2, B_1, B_2 in the Goethals-Seidel array we obtain a design of type $(1, 1, 4, 9)$ in order 28, Using B_1, B_2, B_1, B_2 in the Goethals-Seidel array gives a design of type $(1, 1, 9, 9)$ in order 28, while if we use B_1, B_2, $y_3 C_1$, $y_3 C_2$ we obtain an orthogonal design of type $(1, 9, 13)$. These last three yield designs of type $(1, k)$ (not already listed) for $k=13, 14, 19, 22$. Since 7, 15 and 23 are not the sum of $\leqslant$ three squares this proves conjecture III for $n=28$.

(c) Corollary 3 to Theorem 4 gives designs of type $(1, k)$ in order 56 $k \in \{1, \ldots, 23, 27, \ldots, 30, 36, \ldots, 39\}$. Using Proposition 1 and the designs we found in (b) above we can fill in $k=24, 25, 32, \ldots, 35, 40, \ldots, 45, 48, \ldots, 55$. Using Proposition 2 and the design of type $(1, 26)$ in order 28 we have the result for $k=26$ in order 56. The only gaps left are for $k=31, 46, 47$. The design of type $(1, 9, 13)$ in order 28 yields (by Proposition 2) a design of type $(1, 18, 13)$ in order 56. This gives us $(1, 31)$ in order 56. We have been unable to find designs of type $(1, k)$ in order 56 for $k=46, 47$.

With the results of these last two sections we can now sharpen the corollaries to Theorem 4.

PROPOSITION 13. *There are orthogonal designs of order* $4n$, *n odd and type* $(1, k)$ *when*

(a) $n \geqslant 3$, $k \in \{1, \ldots, 6, 8, \ldots, 11\}$.

(b) $n \geqslant 5$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, 17\}$.

(c) $n \geq 7$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, 17, 18, 20, 24\}$
(d) $n \geq 9$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, 17, 18, 20, 21, 24, 32, 33\}$.
(e) $n \geq 11$, $k \in \{1, \ldots, 6, 8, \ldots, 14, 16, \ldots, 21, 24, 32, 33\}$.

The other corollaries admit of generalization in the same fashion. For the reader's benefit and for future reference we give, in an appendix, the status of our conjectures for small n.

Appendix

In the table given below we shall use the following ideas. If the conjecture is not applicable to that order we write N.A.; if the conjecture is verified for that order we shall put T in the table. If there are still values to test we shall list them. We shall not deal with conjecture IV in this table.

The first few numbers which are not the sum of $\leq$ three squares are:

7, 15, 23, 28, 31, 39, 47, 55, 60, 92, 112.

Order	I	II	III
4	T	N.A.	T
8	T	T	N.A.
12	T	N.A.	T
16	T	T	N.A.
20	T	N.A.	T
24	T	T	N.A.
28	T	N.A.	T
32	T	T	N.A.
36	23, 31	N.A.	19, 22, 25, 26, 27, 29, 30, 34
40	T	T	N.A.
44	23, 29, 31, 35, 41	N.A.	22, 25, 27, 29, 30, 34, 35, 37, 38, 40, 41, 42
48	T	T	N.A.
52	26–31, 35–38, 40–47, 49, 50	N.A.	25, 26, 27, 30, 34–38, 40–46, 48–50
56	47	46, 47	N.A.
60	29–31, 35–58	N.A.	29, 30, 34–38, 40–46, 48–54, 56–58
64	T	T	N.A.

Order	IV	Order	IV
2	T	26	16
6	T	30	16,25
10	T	34	16,25
14	T	38	25,36
18	9,16	42	25,36
22	9		

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Short communications

Autoaddition formulae and algebraic groups

Christopher L. Morgan

The notion of an autoaddition formula is defined for continuous functions from open subsets of a topological group G to algebraic varieties as follows:

DEFINITION (Autoaddition Formula). Let U, V, and W be non-empty open subsets of G, and let X be an algebraic variety of $\mathscr{k}$. Let f be a continuous function from W to X, and let F be a $\mathscr{k}$-rational function from $X \times X$ to X which is defined at each point of $f(U) \times f(V)$.

If W contains U, V, and UV, and if

$$f(uv) = F(f(u), f(v)),$$

for all (u, v) in $U \times V$, we say that:

$$[f; F, U \times V, X]$$

is an autoaddition formula for f on $U \times V$.

This definition is essentially the same as the one given in a previous paper by the author, entitled *Addition Formulae for Field-Valued Continuous Functions on Topological Groups* (to appear in *Aequationes Mathematicae*).

The major result obtained is a classification of all functions which have such an autoaddition formula:

THEOREM 5 (Structure of Autoaddition Formulae). *Let U and V be neighborhoods of e in G such that $U = V^{-1}$ and UV is connected. Let $[f; F, U \times V, X]$ be an autoaddition formula. Then there is an abelian variety A over $\mathscr{k}$; a commutative linear group K over $\mathscr{k}$; a linear group M over $\mathscr{k}$; a $\mathscr{k}$-rational function ϱ from $K \times A \times M$ to X; $\mathscr{k}$-rational functions α in $Z^2(A, K)$ and μ in $Z^2(M, K)$; local homomorphisms λ_A in $Z^1(G, A)$ and λ_M in $Z^1(G, M)$; and a continuous function λ_K in $C^1(G, K)$ defined on a neighborhood of e in G such that*

$$f = \varrho(\lambda_K, \lambda_A, \lambda_M)$$
$$\delta_1(\lambda_K) = \alpha(\lambda_A, \lambda_A)\,\mu(\lambda_M, \lambda_M).$$

Furthermore, λ_A *and* λ_M *have dense images.*

If G is commutative, then M is trivial.

Note. The terminology used in the statement of this theorem involves a local group cohomology theory which is described in Section 6 of the paper.

The method is to relate an algebraic group to each autoaddition formula and then to use Chevalley's theorem on the classification of algebraic groups to classify the autoaddition formulae.

Received September 23, 1973.

The intersection numbers of a complex

A. K. Dewdney

With certain families of sets, one associates the intersection graph which displays at a glance, so to speak, all the intersections among sets in the family. This notion is generalized to n-dimensional complexes as follows: Let S be a finite set and let $F = \{S_i : i \in I\}$ be a family of distinct non-empty subsets of S. The nth *intersection complex* of F, denoted by $\Omega_n(F)$ is the complex whose vertices are the sets S_i and whose simplexes are the collections $\{S_{i_0}, S_{i_1}, \ldots, S_{i_m}\}$ of distinct sets S_{i_j} for which $\bigcap_{j=0}^{m} S_{i_j} \neq \emptyset$, $m \leqslant n$. The *intersection graph* of F is just $\Omega_1(F)$. The nth *intersection number* of K, denoted by $\omega_n(K)$, is the minimum cardinality of a set S for which there exists a family F such that $K = \Omega_n(F)$. An $(n+2)$*-hedron* is an n-complex having $n+2$ vertices and $n+2$ n-simplexes.

An elementary result on intersection numbers says that if G is a connected graph with at least four vertices, then $\omega_1(G) = q(G)$ if and only if G contains no triangles, where $q(G)$ is the number of edges in G. A complex K is *taut* if every two vertices have distinct stars. A subcomplex L of K is isolated if no vertex of L lies in a simplex of K-L. Denoting the number of principal simplexes of K by $\alpha(K)$, the above graph-theoretic result is generalized as follows:

THEOREM. *If K is a taut n-complex, then $\omega_n(K) = \alpha(K)$ if and only if K contains no $(n+2)$-hedra when $n > 1$ and K contains no non-isolated $(n+2)$-hedra when $n = 1$.*

Received November 21, 1973.

The generalized Cauchy equation for operator–valued functions

A. B. Buche and H. L. Vasudeva

Let $\mathfrak{X}$ denote a Banach space and let $\mathscr{B}(\mathfrak{X})$ denote the family of bounded linear operators on $\mathfrak{X}$. Let $\{S(t); t \in R^+\}$, $S: R^+ \to \mathscr{B}(\mathfrak{X})$ be a one-parameter family of operators and let $H: \mathscr{B}(\mathfrak{X}) \times \mathscr{B}(\mathfrak{X}) \to \mathscr{B}(\mathfrak{X})$ be a function. The family $\{S(t)\}$ is said to be a *generalized Cauchy system* if it satisfies the generalized Cauchy equation $S(s+t) = H(S(s), S(t))$, $s, t \in R^+$. Let φ and ψ be real-valued functions defined on R^+ such that (i) $\lim_{t \to 0^+} \varphi(t) = 0$, (ii) φ is a non-negative strictly monotone increasing continuous function on $R^+ - \{0\}$, (iii) ψ is a non-negative function, bounded on each compact set of R^+, (iv) $\|H(S(s_1), S(t)) - H(S(s_2), S(t))\| \leqslant \varphi(\|S(s_1) - S(s_2)\|)$. $\psi(\|S(t)\|)$, $s_1, s_2, t \in R^+$. Then, in the uniform operator topology, the Lebesgue measurability of S on $R^+ - \{0\}$ implies continuity of S on $R^+ - \{0\}$.

Received February 11, 1974.

Pairs of edge-disjoint Hamiltonian circuits

Branko Grünbaum and Joseph Malkevitch

Several authors have conjectured that there exists a pair of edge-disjoint Hamiltonian circuits in every 4-valent 4-connected graph, or at least in each planar graph of this kind. A negative answer to the general conjecture follows from a recent result of G. H. J. Meredith; the planar variant was recently refuted by Pierre Martin. We provide simpler counterexamples to both versions of the conjecture. One of our methods may also be used to establish that every 3-connected, 3-valent, cyclically-4-connected planar graph has a simple circuit that contains at least $\frac{3}{4}$ of the vertices, while for some such graphs (with arbitrarily many vertices) no simple circuit contains more than $\frac{76}{77}$ of the vertices.

Received March 25, 1974.

Matrix summability and a generalized Gibbs phenomenon

J. A. Fridy

Consider a real-valued function sequence $f = \{f_k\}$ that converges to φ on a deleted neighborhood of α. If there is a subsequence $\{f_{k(j)}\}$ and a number sequence x such that $\lim_j x_j = \alpha$ and either

$$\lim_j f_{k(j)}(x_j) > \limsup_{x \to \alpha} \varphi(x) \quad \text{or} \quad \lim_j f_{k(j)}(x_j) < \liminf_{x \to \alpha} \varphi(x),$$

then f is said to display the *Gibbs phenomenon* at α. If A is a (real) summability matrix, then Af denotes the function sequence given by $(Af)_n(x)=\sum_{k=0}^{\infty} a_{nk} f_k(x)$. If Af displays the Gibbs phenomenon whenever f does, then A is said to be *GP-preserving*. By replacing $f_k(x)$ with $f_k(x_j)\equiv F_{k,j}$, the Gibbs phenomenon is viewed as a property of the matrix F, and GP-preserving properties are determined by properties of the matrix product AF.

THEOREM. *If A is GP-preserving and $\{k(j)\}_{j=0}^{\infty}$ is any infinite set of column indices, then*

$$\lim_n (\sup_j |a_{n,k(j)}|)>0,$$

and

$$\limsup_n \left| \sum_{j=0}^{\infty} a_{n,k(j)} \right| >0.$$

THEOREM. *If A is a nonnegative regular matrix, then A is GP-preserving for nonnegative function sequences if and only if, for any set $\{k(j)\}_{j=0}^{\infty}$ of column indices,*

$$\limsup_{n,j} |a_{n,k(j)}|>0.$$

THEOREM. *Let T be a lower triangular matrix with a nonvanishing principal diagonal sequence; if T is stronger than ordinary convergence, then T is not GP-preserving.*

THEOREM. *Suppose that A is a matrix (other than the identity matrix) belonging to one of the following classes: Nörlund means, Nörlund-type means, Euler-Knopp means, or Taylor means; than A is not GP-preserving.*

THEOREM. *The Nörlund matrix corresponding to the nonnegative sequence p is GP-preserving for nonnegative function sequences if and only if $\sum_k p_k$ is convergent; the corresponding Nörlund-type matrix is GP-preserving for nonnegative function sequences if and only if* $\lim_n (p_n/\sum_{k=0}^{n} p_k)\neq 0$.

THEOREM. *If ψ is a function of bounded variation on* $[0, 1]$ *such that* $\int_0^1 [1-t)]^{-1/2} |d\psi(t)|<\infty$, *then the corresponding Hausdorff matrix H_ψ satisfies* $\lim_n(\max_k\{H_\psi[n, k]\})=0$, *so H_ψ is not GP-preserving.*

Received April 19, 1974.

On a theorem by Kronecker

A. M. Ostrowski

(Dedicated to Olga Taussky-Todd)

Let

$$F_i := \sum_{\mu=0}^{m_i} a_\mu^{(i)} x^{m_i-\mu} \, (i=1,2), \qquad F_1F_2 := \sum_{\lambda=0}^{m_1+m_2} c_\lambda x^{m_1+m_2-\lambda},$$
$$S := a_{\mu_1}^{(1)} a_{\mu_2}^{(2)} \qquad (0<\mu_1+\mu_2<m_1+m_2).$$

Then by Kronecker's theorem, for a convenient $r=r(\mu_1, \mu_2)$,

$$S^r + \sum_{\varrho=1}^{r} K_\varrho S^{r-\varrho} = 0,$$

where each K_ϱ is homogeneous of dimension ϱ in the c_λ with integer coefficients.

The article contains a simple proof of this theorem and beyond that the precise determination of r in function of μ_1, μ_2, further the proof of a bound for r given without proof by Macaulay and depending only on m_1 and m_2, and finally the discussion of cases, where Macaulay's bound is exact.

Received June 6, 1974.

The construction of orthogonal k-skeins and latin k-cubes

Trevor Evans

A k-skein is the k-dimensional analogue of a quasigroup. In the finite case it consists of a set $N=\{1, 2, \ldots, n\}$ and a k-ary operation $a: N^k \to N$ such that, in an equation $a(x_1, x_2, \ldots, x_k)=x_{k+1}$, if any k of the x_i are given then the remaining element is uniquely determined as an element of N. The operation table of a finite k-skein is a latin k-cube, i.e. a k-dimensional cubical array $\{1, 2, \ldots, n\}$ with n^k cells, each occupied by one of $\{1, 2, \ldots, n\}$, such that in each line of cells parallel to an edge of the cube each element in $\{1, 2, \ldots, n\}$ occurs exactly once. A k-set of finite k-skeins (or latin k-cubes) $a_1, a_2, \ldots, a_k$ of order n is said to be orthogonal if the k equations

$a_i(x_1, x_2, \ldots, x_k)=a_i(x_1', x_2', \ldots, x_k')$, $i=1, 2, \ldots, k$ imply that $x_i=x_i'$, $i=1, 2, \ldots,$

k. In other words, if k orthogonal latin k-cubes are superimposed, then in the resulting n^k cells, each k-tuple of elements from $\{1, 2, \ldots, n\}$ occurs exactly once.

THEOREM 1. *Let* $\{a_1, a_2, \ldots, a_l\}$, $\{b_1, b_2, \ldots, b_m\}$ *be sets of orthogonal l-skeins and m-skeins on* $\{1, 2, \ldots, n\}$, *and let* $k = l + m - 1$. *The set of k-skeins,* c_{ij}, $i = 1, \ldots, l$; $j = 1, \ldots, m$, *where* $c_{ij}(x_1, x_2, \ldots, x_k) = a_i(x_1, x_2, \ldots, x_{l-1}, b_j(x_l, x_{l+1}, \ldots, x_k))$ *contains* ml^{m-1} *k-sets of orthogonal k-skeins.*

COROLLARY. *For any order* $n \neq 2, 6$, *and any* $k > 2$, *there exists a k-set of orthogonal k-skeins.*

Other methods of constructing orthogonal sets of k-skeins are also described.

We note that Arkin [1] has recently constructed three orthogonal latin cubes of order 10 (or rather four such, every three of which are orthogonal) and in [2] Arkin and Straus extend this to the construction for $k > 2$ of a k-set of orthogonal latin k-cubes for every order $n \neq 2, 6$.

A q-set of orthogonal latin k-cubes is a set of q latin cubes, every k of which are orthogonal. For each q and k, a variety (equational class) is described (in terms of its operations and identities) such that a certain specified q of the operations in each algebra form a q-set of orthogonal k-skeins. Conversely, each orthogonal q-set of k-skeins arises in this way.

THEOREM 2. *For each q, k, there is a variety such that its finite algebras are all q-sets of orthogonal latin k-cubes.*

COROLLARY. *If q-sets of latin k-cubes of orders* n_1, n_2 *exist, then so does a q-set of latin k-cubes of order* $n_1 n_2$.

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Received October 17, 1974.