Polylogarithmic functional equations: A new category of results developed with the help of computer algebra (MACSYMA).

Terms and Conditions

The Goettingen State and University Library provides access to digitized documents strictly for noncommercial educational, research and private purposes and makes no warranty with regard to their use for other purposes. Some of our collections are protected by copyright. Publication and/or broadcast in any form (including electronic) requires prior written permission from the Goettingen State- and University Library.

Each copy of any part of this document must contain there Terms and Conditions. With the usage of the library's online system to access or download a digitized document you accept the Terms and Conditions. Reproductions of material on the web site may not be made for or donated to other repositories, nor may be further reproduced without written permission from the Goettingen State- and University Library.

For reproduction requests and permissions, please contact us. If citing materials, please give proper attribution of the source.

Contact

Niedersächsische Staats- und Universitätsbibliothek Göttingen
Georg-August-Universität Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen
Germany
Email: gz@sub.uni-goettingen.de
Polylogarithmic functional equations: A new category of results developed with the help of computer algebra (MACSYMA)

L. Lewin and E. Rost

Abstract. The interrelation of polylogarithmic functional equations and certain numerical results, known as "ladders", is discussed, and leads to a consideration of three new, single-variable functional equations at the second order. Two of these families each contain six leading terms whose interrelationship constitutes a constraint on the integration process, but the third has only a single leading term with no such constraints. It is shown how this functional equation can be integrated to the third order, and the process reduced to an algorithm — actually a sequence of instructions — for incorporation into a computer program for symbolic manipulation. The procedure utilizes results from Kummer's equations to cancel out, in sequence, terms which do not vanish, or do vanish, with the variable z. Arguments are all of the form \( \pm z^p(1-z)^q(1+z)^r \), and the process is "algebraicized" by using a \((p,q,r,s)\) notation (with \( s = \pm 1 \)) to represent such terms. Application of the procedure leads to an integration to the fourth and fifth orders, the latter exhibiting 55 transcendental terms. The first step for the transition to the sixth order can also be achieved but the subsequent steps are frustrated by the restricted forms that the Kummer equations take at the fifth order — it is not possible to create the needed equations in a form which vanishes with \( z \), this corresponding to the elimination of the \( \zeta(5) \) constant in the extension of the numerically determined ladders to the sixth and higher orders. The existence of the higher-order ladders strongly suggests functional equations at these orders, but the present process has not yet been successful in finding them. The new equations have, however, produced ladders that were inaccessible from Kummer's equations, and had heretofore been only obtainable numerically, up to the fifth order. The method which was developed should be capable of generalization to other systems of equations characterized by the appearance of arguments with recurrent factors. Some new feature, however, will need to be determined before the barrier to the sixth order can be breached.

Introduction

The last few years have seen the publication of a number of papers [1–4] exploring the interrelation of polylogarithmic functional properties and certain numerical results.


Manuscript received March 5, 1985.
Some of these numerical results, which have important order-independent characteristics, can be derived from Kummer's functional equations. Others apparently can not, and have been designated "inaccessible". Yet others, obtained from the order-independent property by extrapolation, represent the first non-trivial equations for orders 6 through 9 to be recorded. In view of Wechsung's proof [5] that Kummer's equations can not be extended beyond the fifth order, the existence of these results poses something of a challenge. In particular, since the form of these numerical equations appears to be quite independent of whether or not they are derived from Kummer's formulas, it raises the question of whether both the "inaccessible" results, and those for order greater than 5, can be derived from some other functional equations, and if so, what form they might take.

The beginning of an answer to these questions is given in reference [4], to which the reader is referred for many important details. Essentially, three distinct families of single-variable functional equations were developed for the dilogarithm and shown, at the level of the second order, to be able to produce four of the heretofore inaccessible results. It is not known whether these three families of equations stem from a common source — perhaps a two-variable functional equation of some form. However, their further integration to at least the fifth order is a formidable problem, in part because of the very large number of terms to be handled. This integration is achieved here for the simplest of the three categories, utilizing the computer algebra system [6] MACSYMA for computation with symbolic forms. It is the opinion of the authors that this task could not have been completed "by hand", and that the synergistic interaction with the computer aided substantially in understanding the structure of the integration process.

1. Definitions

1.1 Kummer's Function. Kummer defined the function $\Lambda_n(z)$ by

$$
\Lambda_n(z) = \int_0^z \frac{\log^{n-1}|u|}{1 + u} \, du
$$

(1)

As explained in reference [4], the use of $|u|$ to keep the logarithm real when $u$ is real and negative is a device used also for Rogers' function. We shall here simply omit the modulus symbol with the understanding that $\log(u)$ is to be interpreted as $1/2 \log(u^2)$ in all of what follows.

Clearly the differential of $\Lambda_n$ is an elementary function, and Kummer used the functional equations of $\log^{n-1}$ to help generate equations for $\Lambda_n$. In particular, for the fifth order we have
\[
\log^4(a^2b) + \log^4(b^2a) + \log^4(a/b) = 9[\log^4(ab) + \log^4(a) + \log^4(b)],
\]

(2)

a result which will be referred to later. Functional equations for \( \Lambda_n \) involve that function, homogeneous powers of logarithms, and certain constants related to \( \zeta(m) \), \( 1 < m \leq n \).

1.2 The Polylogarithm. Inside the unit circle, the polylogarithm is defined by the series

\[
\text{Li}_n(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^n}.
\]

(3)

It satisfies the recurrence relation

\[
\text{Li}_n(z) = \int_0^z \frac{\text{Li}_{n-1}(u)\,du}{u}
\]

(4)

and this, together with \( \text{Li}_1(z) = -\log(1 - z) \) extends the range outside of the unit circle. Functional equations for \( \text{Li}_n \) involve that function, homogeneous powers of logarithms, and constants related to \( \zeta(m) \), \( 1 < m \leq n \).

\( \text{Li}_n \) is related to \( \Lambda_n \) via the formula

\[
\text{Li}_n(z) = \frac{(-1)^n}{(n-1)!} \Lambda_n(-z) + \sum_{r=1}^{n-1} \frac{(-1)^{r-1} \log^r(z) \text{Li}_{n-r}(z)}{r!}.
\]

(5)

Thus, any equation involving \( \Lambda_n(z) \) translates into a corresponding equation for \( \text{Li}_n(-z) \). It is significant that such equations are of "pure" form, i.e., do not contain \( \text{Li}_m \) with \( 1 < m < n \). (This is related to the order-independent property referred to earlier).

1.3 Rogers' Function. Rogers [7] introduced a function

\[
L_2(z) = \text{Li}_2(z) + 1/2 \log(z) \log(1 - z).
\]

(6)

It has the property of removing all logarithmic terms when functional equations for \( \text{Li}_2 \) are expressed in terms of \( L_2 \).

A generalization to order \( n \) was given in reference [4], and can be written as either

\[
\text{Li}_n(x) = L_n(x) + \sum_{r=1}^{n-2} \frac{\log^r(x) L_{n-r}(x)}{r!} - \frac{1}{n!} \log^{n-1}(x) \log(1 - x)
\]

(7)

or
\[ Li_n(x) = L_n(x) + \sum_{r=1}^{n-2} \frac{(-1)^{r-1} \log^r(x) Li_{n-r}(x)}{r!} + \frac{(-1)^{n-1}(n-1)}{n!} \log^{n-1}(x) \log(1-x). \]  

(8)

Any equation for \( Li_n(z) \) translates into a similar equation for \( L_n(z) \), but free of logarithmic terms, and containing only \( \zeta(n) \) for the constant involved.

1.4 Polylogarithmic Ladder. This involves a sum of \( Li_n \) of certain powers of an algebraic number \( u \), defined as the root of a given polynomial equation. The ladder is defined by [4]

\[ L_n(N, u) \equiv \frac{Li_n(u^N)}{N^{n-1}} - \sum_{r \neq N} A_r \frac{Li_n(u^r)}{r^{n-1}} + B_0 \frac{\log^n(u)}{n!} + \sum_{m=2}^n B_m \zeta(m) \frac{\log^{n-m}(u)}{(n-m)!}, \]  

(9)

where the constants are determined implicitly by the equation (known as the base equation) defining \( u \). The quantity \( N \), called the index, is the highest power of \( u \) occurring in the ladder. Several different permissible values of \( N \) may be determined by the base equation.

For a certain significant subset of algebraic quantities \( u \), the ladder has the property that it is zero with a simple rational value of \( B_n \). If derived from a functional equation for \( Li_n \), this property becomes apparent from that equation. In other cases (the so-called inaccessible ones) the property emerges from a numerical calculation, carried out to sufficient accuracy.

The ladder result remains valid, if the order is decreased, with the same coefficients \( A_m \) and \( B_m \). With some refinements, [3] the order can also be increased, leading to the new results for \( n > 5 \); 18 such formulas have been deduced to date for three different algebraic bases. In the present studies only real values of \( u \) between zero and unity are considered. Probably, comparable results exist for the other roots of the base equations.

2. Base Equations

2.1 Permissible Forms. Since Kummer’s equations [8] involve two variables, \( x \) and \( y \), in various product combinations of powers of \( x, 1 - x, y, \) and \( 1 - y \), the only form
of the base equation which will permit every transcendental term to have its argument expressed as a power of an algebraic number \( u \) is

\[ u^m + u^n = 1, \]  

(10)

where \( m \) and \( n \) are integers. By taking, for example, \( x = u^m \), \( 1 - x = u^n \), we obtain the required power form. Kummer's equations are based on the harmonic group

\[ z, \quad 1/z, \quad 1 - z, \quad 1 - 1/z, \quad 1/(1 - z), \quad 1 - 1/(1 - z). \]  

(11)

This gives a limited range of selections for \( x \) and \( y \), and a corresponding set of ladders.

A slightly different state of affairs prevails for the second and third orders, since the quantities \( 1 - x \) and \( 1 - y \) occur only in the ratio \((1 - x)/(1 - y)\). This gives a somewhat broader range of base equations, as discussed in reference 1. However, this is not available for the higher orders, leading to an apparent loss when the order is increased from 3 to 4. Since this non-availability is related to the utilization of Kummer's formulas, it does not necessarily follow that the loss is absolute — some important exceptions are known, but are deduced numerically, and therefore fall into the inaccessible category.

2.2 The Quantities \( \rho \) and \( \omega \). These two algebraic numbers have been discussed extensively in the references. They are the relevant solutions, respectively, of

\[ u^2 + u = 1, \quad u = \rho = (\sqrt{5} - 1)/2 \]  

(12)

\[ u^3 + u^2 = 1, \quad u = \omega. \]  

(13)

A re-arrangement of (12) and (13) gives

\[ \rho^2 = 1 - \rho, \quad \rho^{-1} = 1 + \rho \]  

(14)

\[ \omega^4 = 1 - \omega, \quad \omega^{-2} = 1 + \omega. \]  

(15)

There is some indirect evidence that ladder results for \( \rho \) and \( \omega \) have a lot in common. If it is supposed that new functional equations exist from which results for both of these sets of ladders can be deduced, then (14) and (15) suggest arguments \( v \) of the form

\[ v = \pm z^p(1 - z)^q(1 + z)^r; \quad p, q, r \text{ are positive or negative integers (including zero).} \]  

(16)

Despite many attempts, no useful form, other than (16), of a simple rational algebraic character, has been found that applies to both \( \rho \) and \( \omega \). Although it is far from clear
that the form (16) is mandatory, all of the successful new results to date for $\rho$ and $
abla$ have been of this character.

2.3 The Quantity $\theta$. This algebraic number satisfies

$$u^3 + u = 1; \quad u \equiv \theta.$$  \hfill (17)

A rearrangement gives

$$\theta^3 = 1 - \theta, \quad \theta^{-1} = 1 + \theta^2$$  \hfill (18)

and suggests arguments $v$ of the form

$$v = \pm z^r(1 - z)^q(1 + z^2)^r.$$  \hfill (19)

The structure of (19) is quite distinct from that of (16), and gives rise to a distinct category of new equations. Although forms different from (19) can be constructed that give powers of $\theta$ when $z = \theta$, none of these, so far, has given any usable result.

2.4 The Indices $N$. The ladder index, together with its factors $r$ in (9), are determined, for each base, implicitly from the base equation, though not in any obvious way. The ladders for $\rho$, $\omega$, and $\theta$ have been extensively explored, and it is unlikely that indices other than the currently known ones exist. They are:

$$\rho: \ 1, 2, 6, 12, 20, 24$$

$$\theta: \ 3, 4, 6, 10, 14, 18, 24$$  \hfill (20)

$$\omega: \ 2, 3, 5, 8, 12, 14, 18, 20, 28, 30, 42$$

Of these, 20 and 24 for $\rho$, 24 for $\theta$, and 42 for $\omega$, do not appear to be accessible from Kummer's equations. For $Li_2$, results for $\rho$ with indices 20 and 24 were found by Coxeter [9] and Phillips [10] by methods not involving the functional equations for $Li_2$. No analytical results for these indices for order greater than 2, nor any at all for 24 ($\theta$), or 42 ($\omega$), are known in the literature. Moreover, the only ladder indices surviving for $n > 4$ for $\rho$ are 12, 20, and 24; whilst, for $\theta$ and $\omega$, some results also apparently get "lost" at the fourth and fifth orders — this is probably related to the occurrence of $1 - x$ and $1 - y$ in the formulas in combinations other than solely their ratio, as discussed earlier.
3. Functional Equations

3.1 The Three New Categories. In reference [4] it was shown how clues from this rather meager index information were sufficient to lead to three new families of functional equations. Although the first two are common to both $\rho$ and $\omega$, one was developed primarily to be able to construct ladders of index 20 for $\rho$: it will here be referred to as the $\rho$-family. The second generates ladders of index 42 for $\omega$, as well as index 24 for $\rho$, and is referred to as the $\omega$-family. The third, which generates ladders of index 24 for $\theta$, is referred to as the $\theta$-family.

The main difference between the $\rho$-family and the other two is that the latter two involve six new and different arguments, related by (2). These will be referred to as "head" terms. There seems to be a sort of hierarchy of terms with these head terms (currently) at the top. Kummer's equation at the 5th order can be put in the form [11]

$$\sum \sum \Lambda_5 \left[ \frac{-X(1-X)}{Y(1-Y)} \right] - \sum \sum 18 \Lambda_5 \left[ \frac{-X}{Y} \right] + 36 \sum \{ \Lambda_5(-X) + \Lambda_5(-Y) \} = L,$$

(21)

where the summation is taken over the complete harmonic group for $X$ and $Y$, and $L$ is a logarithmic term. The terms stemming from $[-X(1-X)/Y(1-Y)]$ constitute the lead terms in this equation.

The development of the $\theta$- and $\omega$-families will be presented later in an extensive survey [12; 13] related to these expressions. Here, it will suffice to note that, at a minimum, the character of the head terms must be such that they can be suitably combined and mutually integrated to produce equations of higher order. This is a necessary, though by no means a sufficient condition for the existence of functional equations of higher order involving these arguments. It is not, in fact, too difficult to verify that the necessary relationship exists between them up to the fifth order, but no further. On the other hand, the head term in the $\rho$-family consists of but a single argument; all the other arguments come from Kummer's equations. There is therefore no comparable head term restriction and, as it turns out, everything can be integrated, as far as the fifth order. Beyond that something additional is needed — perhaps a superstructure with a new type of head term. Thus the present study has the promise of solving all the inaccessible $\rho$, $\omega$, and $\theta$ ladders up to the fifth order, but falls short of breaching the barrier between the fifth and sixth orders. The remainder of this paper will be devoted to the development of the $\rho$-family of equations.

3.2 Index-20 Ladder. As already indicated, Coxeter has obtained results for the dilogarithm (though not utilizing any dilogarithmic functional equation) which can be presented as an index-20 ladder. We designate rearrangements coming from using
the inversion formula (connecting arguments $z$ and $1/z$) and the duplication formula (connecting arguments $z^2$ with $z$ and $-z$) as "trivial". An argument $z^{20}$ can be trivially obtained from an argument $-z^{10}$. However, when the dilogarithm is differentiated it leads to $\log(1+z^{10})$ and the factorization of this form does not lead to any useful formulas. We need something of the form (16) which can give both an $N = 20$ $\rho$-ladder and also a permissible $\omega$-ladder, with index taken from (20). The only form so far discovered, apart from the unusable $-z^{10}$, is that discussed in reference [4], and leads to a consideration of the structure

$$y = Li_2\left(-z^7 \frac{1-z}{1+z}\right).$$  

(22)

On differentiation one gets

$$\frac{dy}{dz} = \left[\frac{7}{z} - \frac{1}{1-z} - \frac{1}{1+z}\right][\log(1+z) - \log(1+z + z^7 - z^8)].$$  

(23)

The entire success depends now on the possibility of usefully factorizing $1+z + z^7 - z^8$. "Useful" here implies the existence of factors which, when later integrated, give ladder-type arguments, i.e., of the form (16). Now it is easily verified that

$$1 + z + z^7 - z^8 = (1 + z^2)(1 + z - z^3)(1 - z^2 + z^3)$$  

(24)

and this result is the key to all the subsequent equations. We can write $1 + z - z^3$ as either $(1+z)[1-z^3/(1+z)]$ or as $1 + z(1-z)(1+z)$ whilst $1 - z^2 + z^3$ can be similarly expressed as $(1-z)(1+z)[1 + z^3/(1-z)(1+z)]$ or as $1 - z^2(1-z)$. If dilogarithms of corresponding arguments are differentiated and inserted into (23) modified by (24), then it is found that all the various terms can be collected together so as to be integrable. There are, in fact, more terms to be treated than there are free coefficients to handle them — this feature occurs throughout — and the process works only because the terms occur in just the right combinations. Why this is so is unknown to us, but this fortunate feature enables the integrations to proceed, leading to

$$Li_2\left[-z^7 \frac{1-z}{1+z}\right] = 2Li_2[z^2(1-z)] + Li_2\left[\frac{-z^3}{1-z^2}\right] + 2Li_2\left[\frac{z^3}{1+z}\right] + Li_2[-z(1-z^2)]$$

$$+ \frac{7}{4}Li_2(z^4) - \frac{9}{4}Li_2(z^2) + \frac{1}{2}Li_2\left[\frac{1-z}{1+z}\right] - \frac{1}{2}Li_2\left[-z \frac{1+z}{1-z}\right]$$

$$+ \frac{5}{4}\log^2(1+z) + \frac{1}{4}\log^2(1-z) + \frac{3}{2}\log(1+z)\log(1-z).$$  

(25)
For $z = \rho$ this leads to the needed $N = 20$ ladder, first discovered by Coxeter. [9] For $z = \omega$ we get an already known $\omega$-ladder of index 28, whilst $z = -\rho$ or $z = -\omega$ also give nothing new. At the second order this seems to be all that the new formula reveals.

3.3 Integration to the Third Order. If $Li_3\left[\frac{-z^7(1-z)}{1+z}\right]$ is differentiated, it gives

$$\left[\frac{7}{z} - \frac{1}{1-z} - \frac{1}{1+z}\right]Li_2\left[\frac{-z^7(1-z)}{1+z}\right]$$

The $Li_2$ term is eliminated via (25) and the groups of terms headed by $1/z$, $1/(1-z)$, and $1/(1+z)$ are segregated; they clearly are not individually integrable. The technique, which subsequently became identifiable as an algorithm for use with the computer program MACSYMA, is to concentrate on the group headed by $1/z$, to find or create $Li_3$ functional equations carrying exactly the same arguments, to differentiate them, and to use those new terms, headed by $1/z$, to attempt to cancel out all arguments other than those that are a simple power of $z$ (these are directly integrable.) If and when this has been done the remnant of terms headed by $1/(1-z)$ and $1/(1+z)$ are collected, equations in $Li_2$ are utilized to attempt to remove all terms with arguments involving powers of $z$ other than zero, and if this can be done, the remaining terms are integrated. As before, there are more terms to be cancelled than there are free coefficients, but they occur in just the right combinations to permit the process to work.

The $Li_2$ and $Li_3$ equations utilized in the above process all come from Kummer's formulas by taking $x = \pm z$ and $y = z$ or $z^2$, or such other combinations of these as correspond to the terms in the harmonic group (11). There are very many such formulas, but only a specific few of them carry the requisite arguments, so the transition from $n = 2$ to $n = 3$ was not too difficult to achieve, once the correct modus operandi had been found. The resulting equation was found to be

$$Li_3\left[\frac{-z^7(1-z)}{1+z}\right] - 7\left\{Li_3[z^2(1-z)] + \frac{1}{3}Li_3\left[\frac{-z^3}{1-z^2}\right] + \frac{2}{3}Li_3\left[\frac{z^3}{1+z}\right] + Li_3[-z(1-z^2)]\right\}$$

$$+ \frac{7}{16}Li_3(z^4) - \frac{9}{8}Li_3(z^2) + \frac{1}{2}Li_3\left[\frac{z-1-z}{1+z}\right] + \frac{1}{2}Li_3\left[\frac{-1+z}{1-z}\right]$$

$$+ 2Li_3\left[\frac{-z^2}{(1+z)(1-z)}\right] + 2Li_3\left[\frac{-z}{(1-z)(1+z)}\right] + 6Li_3(z) - 6Li_3\left(\frac{1}{1+z}\right)$$

$$+ 4Li_3[-z(1-z^2)] - \frac{2}{3}Li_3\left[\frac{-z^3}{1-z^2}\right] - \frac{4}{3}Li_3\left[\frac{z^3}{1+z}\right] + 6Li_3(1-z)$$

$$+ Li_3[z^2(1-z)] - \frac{1}{16}Li_3\left[\frac{(1-z)^4}{(1+z)^4}\right] + \frac{3}{8}Li_3\left[\frac{(1-z)^2}{(1+z)^2}\right] =$$

(cont.)
\[ \begin{align*}
&= \frac{5}{16} \xi(3) + \frac{1}{2} \xi(2)[11 \log(1 + z) + 13 \log(1 - z)] + \frac{1}{4} \log(z)[11 \log^2(1 + z) \\
&+ 2 \log(1 + z) \log(1 - z) - 13 \log^2(1 - z)] + \frac{1}{12} [37 \log^3(1 - z) \\
&+ 33 \log^2(1 - z) \log(1 + z) + 75 \log(1 - z) \log^2(1 + z) - \log^3(1 + z)]
\end{align*} \] 
\tag{26}

Some simplification is possible, but this form gives some information on the derivation. The main modifications needed later are given in section 6.1. Equation (26) yields the needed third-order $\rho$-ladder of index 20, but nothing else of current interest.

This is about as far as the method can be readily taken without a more systematic approach. The difficulty now becomes the very large number of terms to be handled, and it was at this stage that resort was made to the MACSYMA program [6] for symbolic computation. This required, among other things, for the ad hoc integration process to be made into a precise algorithm appropriate for a computer.

4. Preparation for MACSYMA

4.1 Differentiation. By straightforward differentiation, it is found that

\[ \frac{d}{dz} Li_n[\pm z^p(1 - z)^q(1 + z)^r] = \left[ \frac{p}{z} - \frac{q}{1 - z} + \frac{r}{1 + z} \right] Li_{n-1}[\pm z^p(1 - z)^q(1 + z)^r]. \]
\tag{27}

Although it is possible to program a direct translation of (27), it is not very convenient since MACSYMA would perform an unwanted "simplification" of the first factor on the right. Moreover, once the differentiation has been achieved, the role of $z$ as a variable is reduced to that of being a mere carrier of the powers $p, q,$ and $r$. Since these powers are the dominant feature of the calculation, a change of notation is warranted which largely "algebraizes" the problem.

4.2 The $(pqrs)$ Notation. In (27), the order $n$ plays no particular role and it was soon discovered that in any given procedure it was not necessary to specify it — it could always be recovered from the context when needed.

Because of the duplication formulas which connect variables $v, -v, \text{ and } v^2$, the minus signs in the arguments of (27) can be converted into plus signs. However, it is not convenient to do this in every case, and some arguments with negative signs are retained. In order to specify the argument unambiguously, we introduce a further parameter $s$ which takes on the values $+1$ or $-1$ according to the sign of the argument. The ensuing simplified notation is then given by the definition
\[ M(p, q, r, s) \equiv Li_n[\pm z^p(1-z)^q(1+z)^r]; \quad s = \pm 1. \] (28)

The duplication formulas can then be used to eliminate undesired linear relations between the different \( M \)'s. For the \( n \)th order this formula can be put into the form

\[ M(p, q, r, -1) = 2^{1-n} M(2p, 2q, 2r, 1) - M(p, q, r, 1). \] (29)

A differentiation operator \( D \) is defined so that (27) can be written as

\[ DM(p, q, r, s) = [p, -q, r] M(p, q, r, s). \] (30)

In this context \([p, -q, r]\) implies a set of three segregated terms in which the coefficients of a term are multiplied by \( p \), \(-q\), and \( r \) respectively. (Factors \( 1/z \), \( 1/(1-z) \), and \( 1/(1+z) \) are implied but their presence is not important in the ensuing calculations.)

4.3 Rogers' Function \( L_n \). One of the very substantial chores in the procedure outlined in section 3 is keeping track of and integrating the subsidiary logarithmic terms. The use of Rogers' function would eliminate this, but unfortunately it does not satisfy a recurrence formula like (4) on which (27) and (30) are based. The bookkeeping chore increases rapidly in magnitude as the order increases, and since the main concern is to establish the nature of the transcendental terms in any presumptive equation, the procedure adopted was to work with \( Li_n \), to ignore all subsidiary logarithms and constants, and then to interpret a resulting functional equation as applying to \( L_n \). Only an integration constant, easily found by putting \( z = 0 \), is then needed. The use of (8) will then, in principle, produce the needed equation of \( Li_n \), including all the subsidiary logarithms. The subsequent steps to be discussed are thereby significantly simplified.

4.4 Integration. There is no corresponding formula to (30) for integration. An integration by parts, operating on the \( z^p \) part of the argument, is

\[ \int_0^z Li_{n-1}[\pm z^p(1-z)^q(1+z)^r] \frac{dz}{z} = \frac{1}{p} Li_n[\pm z^p(1-z)^q(1+z)^r] \]

\[ -\frac{1}{p} \int_0^z \left[ \frac{-q}{1-z} + \frac{r}{1+z} \right] Li_{n-1}[\pm z^p(1-z)^q(1+z)^r] \, dz. \] (31)

This can be expressed in the form

\[ IM(p, q, r, s) = \frac{1}{p} M(p, q, r, s) - I[0, -q/p, r/p] M(p, q, r, s), \] (32)
where $I$ is an integration operator. In order for the integration in (31) to be possible, $p$ must be positive, so any arguments with negative $p$ are first replaced by ones with positive $p$ by using the inversion formula.

The procedure outlined above also requires that $p$ should not be zero. Such terms must first be cancelled out by utilizing any suitable pre-existing functional equations in appropriate linear combinations. To date this has always been found possible to achieve, even at the transition to the sixth order, by utilizing various forms of Kummer's equations. As elsewhere, there have been more terms to handle than free coefficients, so that the necessary equations are overdetermined. But relations among the coefficients of these terms have been such that the equations in fact do have a valid solution.

The effect of the process is to produce a starting sequence of terms all of whose arguments vanish with $z = 0$; there is therefore no constant of integration to be determined since the expression vanishes with $z$. Now, one of the requirements encountered [3] when extrapolating a ladder to higher order is that ladders at odd order $(2m + 1)$ should first be combined to eliminate any $\zeta(2m + 1)$ term. This is paralleled by the preceding procedure which does so, however, both for odd and for even orders. (The ladder extrapolation process makes no such requirements when the order is even.) The prescribed process clearly obviates any need to take explicit account of integration constants.

The production of the term $(1/p)M(p, q, r, s)$, by integration by parts, as on the right side of (32), is an operation of which we make frequent use — in the remainder of the paper we shall refer to this procedure as "partial-integration".

5. An Algorithm for the Transition from Order $n$ to $(n + 1)$

5.1 List all possible single-variable equations of order $n$ obtainable from Kummer's equations using the variables of section 3.3. These seem to be the only possible ones giving the $(pqrs)$ form except for $x = y = z^2$, but this latter form introduces several arguments not heretofore encountered and thus does not seem to be a useful variant.

5.2 Remove all redundant equations from the list of section 5.1, expressing the final result in terms of linearly independent equations. In practice there are many redundant equations and some can be excluded easily. At the fourth order, for example, Kummer's equation is symmetrical in $x$ and $y$, so interchange can give no new results. However, there are also linear combinations of several equations which are difficult to detect "by hand". In general we must consider possible linear relationships

$$\sum_{i} C_i \sum_{j} a_{ij} M(p_i q_j r_i s_j) = 0 \quad (33)$$
with constants $C_i$, where $i$ denotes one of the Kummer equations and the $j$-summation gives the generation of terms in that equation. A program using MACSYMA was written to analyze (33) and set up the appropriate linear system in (many) equations with (somewhat fewer) unknowns. This task required picking out and taking apart the terms in the several equations and used the built-in LINSOLVE facility to provide the possible non-trivial solutions $C_i$. The residual set of independent equations is called the basic-set at order $n$.

5.3 From the above-deduced basic-set, determine all independent linear combinations which are free of $p = 0$ arguments. These will be known as the residual-set. This procedure is computerized by a natural extension of the one used in section 5.2.

5.4 Starting with the equation with the head term, known as the head-equation, e.g., (26) for $n = 3$, cancel out all terms with arguments having $p = 0$, using members of the basic-set. If this cannot be done the process stops. If it can be done, and if the residual-set of section 5.3 is non-empty, then the cancellation is non-unique and some parameters are available and may be chosen to achieve some simplicity in the form of what we now call a modified head-equation.

5.5 Partial-integrate the modified head-equation and all members of the residual-set, and combine them with as-yet undetermined coefficients, called the primary-set. This combination of terms, when the coefficients in the primary-set have been determined, will constitute the bulk of the new equation at order $(n + 1)$, and will be called the bulk-expression. It is possible to create it in this way because all terms with $p = 0$ have been removed, and terms with negative $p$ have been expressed with arguments with positive $p$ by the inversion relation.

5.6 Differentiate the bulk-expression and segregate the groups of terms headed by $1/z$, $-1(1 - z)$, and $1/(1 + z)$, to be known as the $P$-, $Q$- and $R$-terms, respectively. By construction, the $P$-term is identically zero.

5.7 To the $Q$-term add linear combinations of the basic-set and the head-equation with as yet undetermined coefficients, called the secondary-set. Do the same for the $R$-term, with a different secondary-set of coefficients. These now constitute the modified $Q$- and $R$-terms.

5.8 Extract from the modified $Q$- and $R$-terms the complete set of net coefficients of all terms for which the arguments have $p \neq 0$ and equate all such coefficients to zero so as to generate a set of equations for the determination of the primary and two secondary sets of coefficients. Up to the fifth order it was found that this set of
equations, though very highly overdetermined, had a valid solution including some available undetermined parameters.

5.9 Solve the equations of section 5.8, put the values for the primary set into the bulk-expression, and put the values for all sets into the modified \(Q\)- and \(R\)-terms. These are now reduced to a set of terms all of whose arguments have \(p = 0\), and are called the remnant \(Q\)- and \(R\)-terms.

5.10 Multiply the remnant \(Q\)- and \(R\)-terms by \(-1/(1 - z)\) and \(1/(1 + z)\) respectively and integrate, determining any outstanding coefficients to make this possible. For example, \([a/(1 - z) + b(1 + z)]Li_n(1 - z^2)\) requires \(a = -b\), whereupon it integrates to \(b\ Li_{n+1}(1 - z^2)\); whilst \([c/(1 - z)]Li_n(1 + z)\) is not integrable in the present context, and requires \(c = 0\).

5.11 Subtract the integrated terms of section 5.10 from the bulk-expression, and determine any needed integration constants by taking \(z = 0\). The resulting expression, by construction, differentiates to zero, and therefore becomes the sought head-equation at the \((n + 1)\)th order. The method of construction makes it possible for it to contain some arbitrary multiples of Kummer's equations at order \((n + 1)\), and these can be dealt with in any convenient manner.

5.12 Although the preceding algorithm was developed for the \(\rho\)-family, it applies with very little change to the \(\omega\)- and \(\theta\)-families. The main difference is due to the existence of, not one head-term, but six coupled head-terms in each family. However, the details of this coupling have already been determined, and if this information is used, the algorithm should proceed as scheduled.

A minor change for the \(\theta\)-family is the presence of terms in \(1 + z^2\) rather than \(1 + z\). The \((pqrs)\) notation is retained unaltered; however, the selection for Kummer's equation is more limited, and comprises \(x = z\), \(y = z\), or \(y = -z^2\), plus the harmonic group variations. The only other change is the relatively trivial one for the logarithmic differential, which is now \(2z/(1 + z^2)\), and with a corresponding impact on the integration of the remnant \(Q\) and \(R\) terms.

6. Transition from Third to Fourth Order

6.1 Modification of Equation (26). Some simplification is possible by combining a few terms, but the main alteration needed is to remove terms not vanishing with \(z\), i.e., the terms with arguments in \([(1 - z)/(1 + z)]^4\), \([(1 - z)/(1 + z)]^2\), \((1 - z)\), and \(1/(1 + z)\). The first two are handled via equation (6.110) of reference 11, the last two
by taking \( x = z \) in (6.10) and \( x = -z \) in (6.11) of the same reference. The resulting equation is somewhat simpler, with 14 transcendental terms:

\[
\begin{align*}
Li_3 \left[ -z \frac{1 - z}{1 + z} \right] &- 6Li_3 [z^2(1 - z)] - 3Li_3 \left[ -z^2 \frac{1 - z^2}{1 - z^2} \right] - 6Li_3 \left[ \frac{z^3}{1 + z} \right] \\
- 3Li_3 [z(1 - z^2)] - 3Li_3 (z^4) + 9Li_3 (z^2) - 4Li_3 \left[ z \frac{1 - z}{1 + z} \right] + 3Li_3 \left[ -z \frac{1 + z}{1 - z} \right] \\
+ 6Li_3 \left[ \frac{z}{1 + z} \right] - 6Li_3 \left[ \frac{-z}{1 - z} \right] - 6Li_3 (z) + 2Li_3 \left[ -z^2 \frac{1}{(1 + z)^2(1 - z)} \right] \\
+ 2Li_3 \left[ \frac{-z}{(1 - z)^2(1 + z)} \right]
\end{align*}
\]

\[= 2[\log^3 (1 + z) + \log^3 (1 - z)] + 3 \log (1 - z) \log (1 + z)[\log (1 + z) + 2 \log (1 - z)]\]

(34)

The \((pqrs)\) form of Equation (34) is

\[
M(7, 1, -1, -1) - 6M(2, 1, 0, 1) - 3M(3, -1, -1, -1) - 6M(3, 0, -1, 1)
- 3M(1, 1, 1, -1) - 3M(4, 0, 0, 1) + 9M(2, 0, 0, 1) - 4M(1, 1, -1, 1)
+ 3M(1, -1, 1, -1) + 6M(1, 0, -1, 1) - 6M(1, -1, 0, -1) - 6M(1, 0, 0, 1)
+ 2M(2, -1, -2, -1) + 2M(1, -2, -1, -1) = 0,
\]

(35)

where the \(M(\ )\) may be taken to stand for Rogers' function \(L_3\).

Equation (35) is the modified head-equation discussed in section 5.4.

6.2 The Basic Set of Equations at \( n = 3 \). These are obtainable from (6.107) of reference 11. Three forms of this equation come by taking \( y, 1 - y, \) and \(-(1 - y)/y\) for \( y \). In these, \( y \) takes on the set \( \{z, -z, z^2\} \) and \( x \) takes on the set \( \{z, 1 - z, z/(1 - z), -z, 1 + z, z/(1 + z)\} \) giving a total of 54 functional equations, each with nine transcendental terms. The substitution and subsequent simplification were done entirely by the computer thereby avoiding transcription and algebraic errors. Of the 54 equations, 30 were found to form the basic set, the remainder being redundant. From these 30, 18 combinations were found that were free of \( p = 0 \) terms and which constitute the residual set.

6.3 Fourth-order Equation. The head-equation (35) together with the residual set (in practice only a few of them are essential) are partial-integrated and the procedures
of sections 5.5 to 5.11 are followed. Apart from handling the large number of terms (for which the computer is very convenient), no special problems occur, and the resulting expression can be written in terms of Rogers’ function $L_4$ with 28 transcendental terms:

\[
L_4 \left[ -z^7 \frac{1-z}{1+z} \right] = L_4 \left[ \frac{-z^5}{1+z} \right] + 3L_4 \left[ -z^4(1-z) \right] - \frac{9}{8} L_4(z^6) + \frac{21}{4} L_4(z^4) \\
-3L_4 \left[ \frac{-z^3}{1-z^2} \right] + 6L_4 \left[ \frac{z^3}{1+z} \right] + 6L_4[z^2(1-z)] - 3L_4 \left[ -z(1-z^2) \right] \\
+ 3L_4 \left[ \frac{-z^2}{1-z^2} \right] - L_4 \left[ \frac{-z^2}{(1+z)^2(1-z)} \right] + L_4 \left[ \frac{-z}{(1-z)^2(1+z)} \right] - \frac{99}{8} L_4(z^2) \\
+ 9L_4 \left[ \frac{-z^2}{(1-z)} \right] + 28L_4 \left[ \frac{z(1-z)}{(1+z)} \right] - 21L_4 \left[ \frac{-z(1+z)}{(1-z)} \right] - 3L_4 \left[ \frac{-z}{(1-z)^2} \right] \\
+ 9L_4 \left[ -z(1+z) \right] - 18L_4 \left[ \frac{z}{(1+z)} \right] + 12L_4 \left[ \frac{-z}{(1-z)} \right] + 3L_4 \left[ \frac{z}{(1+z)^2} \right] \\
- 18L_4 \left[ \frac{-z^2}{1+z} \right] - 18L_4[z(1-z)] - \frac{3}{4} L_4 \left[ \frac{(1-z)^4}{(1+z)^4} \right] + 9L_4 \left[ \frac{(1-z)^2}{(1+z)^2} \right] \\
+ 3L_4(1-z^2) - 18L_4(1-z) - 12L_4 \left[ \frac{1}{(1+z)} \right] + \frac{75}{4} L_4(1). \tag{36}
\]

With $z = \pm \rho$, equation (36) gives combinations of ladders of index 12, already known from Kummer’s equations, and the sought-for ladder of index 20. It may be noted that the last six terms in (36) have $p = 0$.

7. Transition from the Fourth to the Fifth Order

7.1 The Basic Set at $n = 4$. Kummer’s equation is given by (7.90) of reference 11, and an extremely large number of equations, many of then redundant, or exhibiting trivial variations, can be created by the variable changes of section 6.2. As mentioned there, it was necessary to use the computer to search for a linearly independent set; this set was found to involve 15 equations, of which 7 combinations free of $p = 0$ terms could be constructed. Combinations to isolate each of the $p = 0$ terms of (36) [except for $L_4(1)$] were also constructed and used to clear (36) of all $p = 0$ terms. The resulting relation is the modified head-equation, ready for partial-integration.

7.2 The Fifth-Order Equation. Unlike (34) for $n = 3$, for which the modified
equation is marginally simpler than (26), the modified head-equation at \( n = 4 \) contains considerably more transcendental terms than (36). However, the process of sections 5.5 to 5.11 can be carried out without difficulty using the equations of section 7.1, leading to an equation at \( n = 5 \). This equation contains some \( p = 0 \) terms, and, in the particular form generated, there are five in the following combination:

\[
-72L_5[(1 - z)^2(1 + z)] + 153L_5[(1 - z)(1 + z)^2] - 162L_5(1 - z^2) \\
+ 297L_5(1 - z) - 378L_5(1 + z).
\]

(37)

7.3 Modified Fifth-Order Equation. Kummer's equation at \( n = 5 \) is given by equation (17) of reference 3. Because, as exhibited in (21), the fifth-order equation involves all combinations of the harmonic group, only five independent equations can be constructed. These come from \( x = y = z; x = y = -z; x = -y = z; x = z^2, y = \pm z \). (The variant \( x = y = z^2 \) is ineffective.) Each of these five equations involves a different \( p = 0 \) term, so no combinations involving only terms with arguments with \( p \neq 0 \) can be constructed. The two equations from \( x = z^2, y = \pm z \) are the only ones to give the arguments \( (1 - z)^2(1 + z) \) and \( (1 + z)^2(1 - z) \) of (37). As encountered throughout this study, the equations to eliminate terms, like the above five, are very much over-determined. It was therefore a pleasant surprise to discover that the pair of coefficients needed to cancel the two specified arguments also eliminated the remaining three. The resulting modified head-equation, with 55 transcendental terms, is

\[
9L_5\left[ -z^7(1 - z) \right] \left( \frac{(1 + z)}{(1 + z)} \right) = -\frac{27}{2}L_5(z^6) + 64L_5\left[ -\frac{z^5}{1 + z} \right] - 16L_5\left[ \frac{z^5}{1 - z} \right] \\
+ 11L_5\left[ -\frac{z^5}{(1 - z)(1 + z)^3} \right] - 8L_5\left[ \frac{z^5}{(1 - z)^3(1 + z)} \right] + 56L_5\left[ -z^4(1 - z) \right] \\
- 8L_5\left[ -z^4(1 + z) \right] + \frac{135}{2}L_5(z^4) - 16L_5\left[ \frac{z^4}{(1 - z)^2(1 + z)^3} \right] \\
+ 28L_5\left[ \frac{z^4}{(1 - z)^3(1 + z)^2} \right] + 90L_5\left[ -\frac{z^3(1 - z)}{(1 + z)} \right] + 36L_5\left[ \frac{z^3}{1 + z} \right] \\
+ 72L_5\left[ -\frac{z^3}{(1 + z)^3} \right] - 36L_5\left[ \frac{z^3(1 + z)}{(1 - z)} \right] + 72L_5\left[ -\frac{z^3}{1 - z} \right] + 144L_5\left[ \frac{z^3}{1 - z^2} \right] \\
- 306L_5\left[ -\frac{z^3}{1 - z^2} \right] + 72L_5\left[ -\frac{z^3}{(1 - z)(1 + z)^2} \right] - 72L_5\left[ \frac{z^3}{(1 - z)^2(1 + z)} \right] \\
- 72L_5\left[ \frac{z^3}{(1 - z)^3} \right] - 36L_5[z^2(1 - z)] + 144L_5[z^2(1 + z)] + \frac{9}{2}L_5(z^2)
\]

(cont.)
\[-944L_5 \left[ \frac{z^2}{1 + z} \right] - 1656L_5 \left[ \frac{-z^2}{1 + z} \right] - 324L_5 \left[ \frac{z^2}{(1 + z)^2} \right] + 452L_5 \left[ \frac{z^2}{1 - z} \right] \]
\[+ 1224L_5 \left[ \frac{-z^2}{1 - z} \right] - 144L_5 \left[ \frac{-z^2}{(1 - z)(1 + z)^2} \right] + 324L_5 \left[ \frac{z^2}{(1 - z)^2} \right] \]
\[+ 72L_5 \left[ \frac{-z^2}{(1 - z)^2(1 + z)} \right] + 19L_5 \left[ -z(1 - z)^3(1 + z) \right] - 16L_5 \left[ z(1 - z)(1 + z)^3 \right] \]
\[+ 72L_5 \left[ z(1 - z^2) \right] - 234L_5 \left[ -z(1 - z^2) \right] - 936L_5 \left[ z(1 - z) \right] \]
\[-1024L_5 \left[ -z(1 - z) \right] - 180L_5 \left[ \frac{z(1 - z)}{(1 + z)} \right] + 532L_5 \left[ z(1 + z) \right] \]
\[+ 504L_5 \left[ -z(1 + z) \right] - 1872L_5 \left[ z \right] + 1080L_5 \left[ \frac{z}{1 + z} \right] - 648L_5 \left[ \frac{-z}{1 + z} \right] \]
\[+ 296L_5 \left[ \frac{z}{(1 + z)^2} \right] + 135L_5 \left[ \frac{-z(1 + z)}{(1 - z)} \right] + 648L_5 \left[ \frac{z}{1 - z} \right] - 1080L_5 \left[ \frac{-z}{1 - z} \right] \]
\[-1188L_5 \left[ \frac{z}{1 - z^2} \right] + 216L_5 \left[ \frac{-z}{1 - z^2} \right] + 144L_5 \left[ \frac{z}{(1 - z)(1 + z)^2} \right] \]
\[+ 344L_5 \left[ \frac{-z}{(1 - z)^2(1 + z)} \right] - 216L_5 \left[ \frac{-z}{(1 - z)^2(1 + z)} \right] + 20L_5 \left[ \frac{z}{(1 - z)^2(1 + z)^3} \right] \]
\[-8L_5 \left[ \frac{-z}{(1 - z)^3(1 + z)^2} \right]. \quad (38)\]

The ordering of the terms in equation (38) is in a \((pqrs)\) hierarchy which occurs automatically in the computer algebra performed. It has the disadvantage, however, of separating structurally-related terms, and of inter-mixing the lead and subsidiary terms arising from the use of Kummer’s results.

### 7.4 Non-extension to the Sixth Order

Although the first step, elimination of \(p = 0\) terms, was readily accomplished, the next step was halted by the impossibility of creating any residual equation (no \(p = 0\) terms) at the fifth order from Kummer’s formulas. The only other available equation, that obtainable from (38) by replacing \(z\) by \(-z\), and called the *adjoint equation*, is insufficient. Any further progress requires additional equations involving the *same* arguments as those in (38), as generated by Kummer’s equation. Whether such equations exist without introducing yet further arguments is not currently known. One of the difficulties is the large number of variants that can be constructed, making it very time consuming to determine whether
a supposedly new result is really novel. At this time no such formulas are known, and the generation of a sixth degree equation may be a more formidable operation than had been expected.

Equation (38) has been checked by differentiation. At $z = \pm \rho$ it gives the needed ladder of index 20 and the already known ladder of index 12; the latter, as a consequence of a confluence of the various coefficients of terms in (38), is generated (with a factor 39) in a manner that indirectly provides an additional check on the correctness of the equation. The $\zeta(5)$ contribution comes from terms in $L_5(\pm p^0)$, the net zero power coming via many different combinations, of which $(p, q, r) = (1, 0, 1)$ is the simplest. With $z = \pm w$, (38) yields two of the missing $w$-ladders discussed earlier.

8. Conclusions

The algorithm discussed in this paper is potentially a very powerful one, and can be used, with modifications, for a variety of equations in which the terms involve recurrent factors. This need not be limited to single-variable arguments. The key ingredients are the discovery of a suitable starting formula, and the generation, via Kummer’s equations or otherwise, of a pool of results for incorporation into the functional equations as the order is increased. But it appears that something additional may be needed to go from the fifth to the sixth order.

REFERENCES

[6] MACSYMA. A large symbolic manipulation program developed at the MIT Laboratory for Computer Science. MACSYMA is a trademark of Symbolics, Inc.
[12] Lewin, L. and Abouzahra, M., Polylogarithmic ladders to the sixth order for the base $\theta$, $(\theta^3 + \theta = 1)$. To be published.

[13] Lewin, L. and Abouzahra, M., Polylogarithmic ladders to the eighth order for the base $\omega$, $(\omega^3 + \omega^2 = 1)$. To be published.

Dept. of Electrical & Computer Eng.  
and Dept. of Physics,  
University of Colorado,  
Boulder, CO 80309  
USA.