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On the Functional Equation of Certain Dirichlet Series

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Introduction

In a previous paper [1], we have indicated a possibility to find some relations between the zeta functions associated to distinct quaternion algebras over distinct basic fields, which are analogous to the "decomposition theorem" in the case of L-functions of algebraic number fields. The purpose of the present note is to give an analytical support, which was suggested by G. Shimura, of the problem in the simplest case (a real quadratic extension over the rational number field). To state it, let us repeat here the problem more specifically[1].

Let $\varphi(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series of Hecke type associated to a cusp form of weight k with respect to a unit group Γ of an order in an indefinite quaternion algebra A (which may or may not be a division algebra) over $\mathbb{Q}$. We assume that $\varphi(s)$ has a "usual" Euler product and satisfies a functional equation. Let $F=\mathbb{Q}(\sqrt{D})$ be a real quadratic number field and let χ be a class character with respect to $F/\mathbb{Q}$. Put $\varphi(s,\chi)=\sum_{n=1}^{\infty} \chi(n)\, a_n n^{-s}$. Now the problem is to find a quaternion algebra B over F such that $\varphi(s)\cdot\varphi(s,\chi)$ is one of the Dirichlet series associated to the cusp forms of weight k with respect to a unit group of an order in B.

Although the present investigation deals with only the case of a quadratic extension over $\mathbb{Q}$, it is of course possible to generalize our problem to the case of a higher field extension with a more general basic field[2].

Now, by Shimizu [8, Theorem 4.3], any zeta function (with the identity representation) associated to an order (of square-free level)

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[1] In [1], the problem has been raised from the point of view of algebraic geometry and we restricted ourselves to the case of cusp form of weight 2. However, it is naturally extended to the case of an *arbitrary* weight.

[2] According to a communication from R. P. Langlands, he has formulated our theorem in the case of quadratic extension of an arbitrary A-field (in Weil's sense) in the adele language.

in B comes from an order in a quaternion algebra of discriminant (1) over F. Then, in our problem, it will be natural to take $B=M_2(F)$ and the Hilbert modular group over F as its unit group. On the other hand, by means of the properties of Euler product of $\varphi(s)$ and $\varphi(s,\chi)$, $\varphi(s)\cdot\varphi(s,\chi)$ can be expressed in the following form with suitable coefficients $C_{\mathfrak{a}}$ which are defined for every integral ideal $\mathfrak{a}$ in F (see text 1.1):

$$\varphi(s)\cdot\varphi(s,\chi)=\sum_{\mathfrak{a}} C_{\mathfrak{a}}\cdot N\mathfrak{a}^{-s},$$

where $\mathfrak{a}$ ranges over all integral ideals in F. Moreover, for a Grössen-character ξ of F, we can define a Dirichlet series $D(s,\varphi,\chi,\xi)$ by

$$D(s,\varphi,\chi,\xi)=\sum_{\mathfrak{a}} \xi(\mathfrak{a})\cdot C_{\mathfrak{a}}\cdot N\mathfrak{a}^{-s}.$$

Now one can certainly generalize the result of Weil [11] to the case of the Hilbert modular group over F. Therefore, if one proves a functional equation for $D(s,\varphi,\chi,\xi)$ for an *arbitrary* Grössen-character ξ of F, one can possibly show that our $\varphi(s)\cdot\varphi(s,\chi)(=D(s,\varphi,\chi,1))$ is a Dirichlet series associated to certain Hilbert modular form. In the present paper, we restrict ourselves to the case where *the conductor of* ξ *is one.* Under this restriction[3], we can prove a functional equation for $D(s,\varphi,\chi,\xi)$, taking $A=M_2(\mathbb{Q})$, $\Gamma=SL_2(\mathbb{Z})$ (see text §1 Theorem). This can be done by expressing $D(s,\varphi,\chi,\xi)$ as a convolution of $\varphi(s)$ and L-function of F and making use of a method of Rankin (see text §2). Further, by means of the "Mellin transform", our theorem shows that the corresponding function to $\varphi(s)\cdot\varphi(s,\chi)$ admits a transformation formula of Hilbert modular type (see text §3).

We would like to add here one remark. In view of the theory of quaternion algebras in Eichler [2] and [3], it may be possible to formulate our problem in an algebraic way. The problem is to find the relations between the zeta functions associated to distinct *definite quaternion algebras* over distinct basic fields, or equivalently, the relations between the "Anzahlmatrizen" (= Brandt matrices) over distinct basic fields.

Notation. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the ring of rational integers, the rational number field, the real number field, and the complex number field. For a subring $X(\ni 1)$ of $\mathbb{R}$, we denote by $M_2(X)$ the ring of 2×2 matrices with entries in X and $SL_2(X)$ the group of elements of $M_2(X)$ whose determinant is equal to 1. For every element $\alpha\in\mathbb{C}$, we denote by $\bar{\alpha}$ the complex conjugation of α.

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[3] It may be possible to apply our proof to the case of an *arbitrary* Grössen-character; however, we could not make clear this point in the present paper.

1. Statement of the Result

1.1. We let $\mathfrak{H}$ denote the upper half complex plane:

$$\mathfrak{H}=\{\tau\in\mathbb{C}\,|\,\mathrm{Im}(\tau)>0\}.$$

Let $f(\tau)$ be a cusp form on $\mathfrak{H}$ with respect to $\Gamma=SL_2(\mathbb{Z})$ of weight k: $f(\tau)=\sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$. Let $\varphi(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ be the corresponding Dirichlet series. Then $\varphi(s)$ converges absolutely for $\mathrm{Re}(s)>\sigma$ with a suitable constant $\sigma>0$ and $\varphi(s)$ can be continued holomorphically to the whole s-plane. Now we assume that $\varphi(s)$ can be expressed in the form of an Euler product

$$\varphi(s)=\prod_p (1-a_p p^{-s}+p^{k-1-2s})^{-1}$$

and the following functional equation holds:

$$\varphi^*(s)=(-1)^{k/2}\varphi^*(k-s) \qquad \text{with } \varphi^*(s)=\Gamma(s)(2\pi)^{-s}\varphi(s).$$

Let $F=\mathbb{Q}(\sqrt{D})$ be a real quadratic number field with the discriminant D. We assume that *the class number of F is one*. We denote by "$'$" the conjugation of $F/\mathbb{Q}$. Let χ be a class character with respect to F. Put $\varphi(s,\chi)=\sum_{n=1}^{\infty}\chi(n)a_n n^{-s}$. Define the sequence of numbers $\{C_{\mathfrak{a}}\}$, for integral ideals $\mathfrak{a}$ in F, in the following manner: For prime ideals $\mathfrak{p}$ in F, we put

$$C_{\mathfrak{p}}=C_{\mathfrak{p}'}=a_p \qquad \text{if } \mathfrak{p}\cdot\mathfrak{p}'=(p) \qquad (\text{not necessarily } \mathfrak{p}\neq\mathfrak{p}'),$$
$$C_{\mathfrak{p}}=a_p^2-2p^{k-1} \qquad \text{if } \mathfrak{p}=(p),$$

and define

$$C_{\mathfrak{a}}=C_{\mathfrak{p}^e}=C_{\mathfrak{p}}\cdot C_{\mathfrak{p}^{e-1}}-N\mathfrak{p}^{k-1}C_{\mathfrak{p}^{e-2}} \qquad \text{if } \mathfrak{a}=\mathfrak{p}^e,$$
$$C_{\mathfrak{a}}=\prod_i C_{\mathfrak{p}_i^{e_i}} \qquad \text{if } \mathfrak{a}=\prod_i \mathfrak{p}_i^{e_i}.$$

Then we have, for a constant $\sigma>0$,

$$\varphi(s)\varphi(s,\chi)=\sum_{\mathfrak{a}} C_{\mathfrak{a}} N\mathfrak{a}^{-s}, \qquad \mathrm{Re}\, s>\sigma,$$

by means of the relations of Hecke operators. Moreover, let ξ_m $(m\in\mathbb{Z})$ be a Grössen-character of F which is defined by $\xi_m(\mathfrak{a})=|\alpha/\alpha'|^{\frac{m\pi i}{\log\varepsilon_0}}$, for every ideal $\mathfrak{a}=(\alpha)$ in F, where ε_0 is the fundamental unit in F, $\varepsilon_0>1$. Put

$$D(s,\varphi,\chi,\xi_m)=\sum_{\mathfrak{a}}\xi_m(\mathfrak{a})\cdot C_{\mathfrak{a}}\cdot N\mathfrak{a}^{-s},$$

and

$$D^*(s, \varphi, \chi, \xi_m) = (2\pi)^{-2s}\,\Gamma(s+im\kappa)\,\Gamma(s-im\kappa)\,D(s, \varphi, \chi, \xi_m),$$

where $\kappa = \pi/\log \varepsilon_0$.

1.2. **Theorem.** *In the above situation, $D(s, \varphi, \chi, \xi_m)$ converges absolutely for $\mathrm{Re}(s) > \sigma_1$ with a suitable constant $\sigma_1 > 0$; $D(s, \varphi, \chi, \xi_m)$ can be continued holomorphically to the whole s-plane; $D(s, \varphi, \chi, \xi_m)$ can be expressed in the form of an Euler product*

$$(1.2.1)\qquad D(s, \varphi, \chi, \xi_m) = \prod_{\mathfrak{p}} \left(1 - \xi_m(\mathfrak{p})\, C_{\mathfrak{p}}\, N\mathfrak{p}^{-s} + \xi_m(\mathfrak{p})^2\, N\mathfrak{p}^{k-1-2s}\right)^{-1}$$

and the following functional equation holds:

$$(1.2.2)\qquad D^*(s, \varphi, \chi, \xi_m) = D^{k-2s} \cdot D^*(k-s, \varphi, \chi, \xi_m).$$

1.3. Here we remark that the Γ-factor and the constant factor (except for a minor difference) of the above functional equation coincide with those of zeta function associated to $B = M_2(F)$ and its unit group of a maximal order, which is given by Shimura [9, Theorem 1].

2. Proof of Theorem

2.1. The notation being as in §1. We note that, from the definition of $C_{\mathfrak{a}}$, $|C_{\mathfrak{a}}| < c_1 N\mathfrak{a}^{c_2}$ for every $\mathfrak{a}$ with suitable positive constants c_1, c_2. Therefore $D(s, \varphi, \chi, \xi_m)$ converges absolutely for $\mathrm{Re}(s) > \sigma_1$ with a suitable constant $\sigma_1 > 0$. Then, again by the definition of $C_{\mathfrak{a}}$, one can easily check (1.2.1).

2.2. Now we shall give the lemma which is essential to prove (1.2.2). To state it, let us introduce the following notation. For a Grössen-character ξ of F, let us denote by $\xi_{\mathbb{Q}}$ the restriction of ξ to $\mathbb{Q}$. For the L-function $L_F(s, \xi)$ of F, we put

$$L_F(s, \xi) = \sum_{\mathfrak{a}} \xi(\mathfrak{a})\, N\mathfrak{a}^{-s} = \sum_{n=1}^{\infty} t_n n^{-s}$$

with $t_n = \sum_{N\mathfrak{a}=n} \xi(\mathfrak{a})$. Also put

$$\zeta(s, \chi\xi_{\mathbb{Q}}) = \sum_{n=1}^{\infty} \chi(n)\, \xi_{\mathbb{Q}}(n)\, n^{-s}.$$

2.3. **Lemma.** *Let $\{C_{\mathfrak{a}}\}$ be as in* 1.1 *and ξ be an arbitrary Grössen-character of F. Put $D(s, \varphi, \chi, \xi) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) \cdot C_{\mathfrak{a}} \cdot N\mathfrak{a}^{-s}$. Then we have*

$$(2.3.1)\qquad D(s, \varphi, \chi, \xi) = \zeta(2s+1-k, \chi\xi_{\mathbb{Q}}) \cdot \sum_{n=1}^{\infty} a_n t_n n^{-s}$$

for $\mathrm{Re}\, s > \sigma_2$ *with a suitable constant* $\sigma_2 > 0$.

Proof. First, it can be easily seen that there exists a positive constant σ_2 such that the both sides of (2.3.1) are absolutely convergent for $\mathrm{Re}(s)>\sigma_2$. Thus, by the same reason as mentioned in 2.1, $D(s, \varphi, \chi, \xi)$ has an Euler product

$$D(s, \varphi, \chi, \xi)=\prod_{\mathfrak{p}}(1-\xi(\mathfrak{p})\, C_{\mathfrak{p}}\, N\mathfrak{p}^{-s}+\xi(\mathfrak{p})^2\, N\mathfrak{p}^{k-1-2s})^{-1}$$

for $\mathrm{Re}(s)>\sigma_2$. Moreover, we note that, by the assumption of $\varphi(s)$ and the definition of t_n, $\sum_{n=1}^{\infty} a_n t_n n^{-s}$ is equal to $\prod_p \left(\sum_{\nu=0}^{\infty} a_{p^\nu} t_{p^\nu} p^{-\nu s}\right)$, where p ranges over all prime numbers. Let p be a prime number. Then we have to prove

$$\text{(2.3.2)}\qquad \left(\sum_{\nu=0}^{\infty} a_{p^\nu} t_{p^\nu} p^{-\nu s}\right) \prod_{\mathfrak{p}|p}(1-\xi(\mathfrak{p})\, C_{\mathfrak{p}}\, N\mathfrak{p}^{-s}+\xi(\mathfrak{p})^2\, N\mathfrak{p}^{k-1-2s}) = 1-\chi(p)\,\xi_{\mathbf{Q}}(p)\, p^{k-1-2s}.$$

In fact, we can check it separately, the cases where $(p)=\mathfrak{p}\,\mathfrak{p}'$ (not necessarily $\mathfrak{p}\neq\mathfrak{p}'$) and $(p)=\mathfrak{p}$, making essential use of a formula

$$(*)\qquad a_p \cdot a_{p^\nu}=a_{p^{\nu+1}}+p^{k-1}\, a_{p^{\nu-1}}$$

which is well-known in the theory of Hecke operators. The case where $(p)=\mathfrak{p}$ is much simpler. Put $U=\xi_{\mathbf{Q}}(p)\, p^{-2s}$. Since $t_{p^\nu}=0$ (if ν is odd) or $=\xi_{\mathbf{Q}}(p^{\nu/2})$ (if ν is even), the left-hand side of (2.3.2) becomes

$$\text{(2.3.3)}\qquad \begin{aligned} &\left(\sum_{\nu=0}^{\infty} a_{p^{2\nu}} U^\nu\right) [1-(a_p^2-2p^{k-1})\, U+p^{2(k-1)}\, U^2] \\ &\quad =1+[a_{p^2}-(a_p^2-2p^{k-1})]\, U \\ &\qquad +\sum_{\nu=2}^{\infty} [a_{p^{2\nu}}-(a_p^2-2p^{k-1})\, a_{p^{2(\nu-1)}}+p^2\, a_{p^{2(\nu-2)}}]\, U^\nu. \end{aligned}$$

Here, by $(*)$, the coefficient of U on the right-hand side of (2.3.3) is equal to p^{k-1}, and the other coefficients for higher term of U^ν ($\nu\geqq 2$) are all zero. This settles (2.3.2) in this case. By the same way, we can verify the remaining case with more lengthy computation. Therefore it may be omitted.

2.4. From 2.3, our task is reduced to prove a functional equation for $\zeta(2s+1-k, \chi)\left(\sum_{n=1}^{\infty} a_n t_n n^{-s}\right)$, since the restriction of ξ_m to $\mathbf{Q}$ is the identity. To do it, we make use of a method of Rankin [7]. (Roughly speaking, one can express $\sum_{n=1}^{\infty} a_n t_n n^{-s}$ as a convolution of two automorphic forms corresponding to $\varphi(s)$ and $L_F(s, \xi_m)$; furthermore, a functional equation for the convolution product can be derived from that of certain Eisenstein series.) For this purpose, here we consider a real analytic automorphic function $g(\tau, \xi_m)$ attached to the L-functions of F, referring to Maass [6] for details. Hereafter, we let $\tau = x + iy$ denote a variable of the upper half complex plane $\mathfrak{H}$. Let $\mathfrak{o}$ be the ring of integers in $F = \mathbb{Q}(\sqrt{D})$ and let E (resp. E_+) be the group of all (resp. all totally positive) units in $\mathfrak{o}$. Put

(2.4.1) $$g(\tau, \xi_m) = \sum_{\substack{\mu \in \mathfrak{o}/E_+ \\ \mu \neq 0}} \xi_m(\mu)\, y^{\frac{1}{2}} K_{im\kappa}(2\pi |N\mu|\, y)\, e^{2\pi i N\mu x}$$

with $\xi_m = |\mu/\mu'|^{im\kappa}$ and a Bessel function $K_{im\kappa}(\ )$ which is defined in [6]. In the following we write simply $g(\tau)$ for $g(\tau, \xi_m)$. Then, in view of Maass [6, Satz 1 and its application p. 145, p. 146], we know that $g(\tau)$ has the following properties:

(2.4.2) *For $\tau \in \mathfrak{H}$, $g(\tau)$ is real analytic with respect to x and y and $g(\tau) = O(y^{\iota_1})$ (resp. $= O(y^{-\iota_2})$) uniformly in x, as $y \to \infty$ (resp. as $y \to 0$) with suitable positive constants ι_1, ι_2.*

(2.4.3) $$g\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)\, g(\tau),$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{D} \right\}$.

(2.4.4) $$g\left(\frac{-1}{D\tau}\right) = g(\tau).$$

In fact, (2.4.3) can be obtained by applying the same argument of the proof of Hecke [5, Satz 7, p. 447] to [6, (101), p. 160] and also (2.4.4) is a direct consequence of [6, (28), p. 146]. (Remark. By considering the functional equations for L-functions of F, one may also derive these properties at once from extending the method of Weil [11] to the real analytic case. It will be an exercise.)

2.5. **Lemma.** *The notation being as above. Let $f(\tau)$ be a cusp form of weight k with respect to $SL_2(\mathbb{Z})$, which is associated to $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$;*

$f(\tau)=\sum_{n=1}^{\infty} a_n e^{2\pi i n \tau}$. Then we have

$$\int_0^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\tau)\overline{g(\tau)}\, y^{s-\frac{3}{2}}\, dx\, dy \tag{2.5.1}$$
$$=\sqrt{\pi}(4\pi)^{-s}\frac{\Gamma(s+im\kappa)\Gamma(s-im\kappa)}{\Gamma(s+\frac{1}{2})}\sum_{n=1}^{\infty} a_n t_n n^{-s}$$

for $\mathrm{Re}(s)>\sigma_3$ *with a suitable positive constant* σ_3.

Proof. We have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\tau)\overline{g(\tau)}\, dx=\sum_{n=1}^{\infty} a_n t_n e^{-2\pi n y} y^{\frac{1}{2}} K_{im\kappa}(2\pi n y),$$

where $\tau=x+iy$, $y>0$. Here we know, by [4, 7.7.3 (26), p. 50],

$$\int_0^{\infty} e^{-ax} x^{\mu-1} K_\nu(ax)\, dx=\sqrt{\pi}(2a)^{-\mu}\frac{\Gamma(\mu+\nu)\Gamma(\mu-\nu)}{\Gamma(\mu+\frac{1}{2})}$$

for $\mathrm{Re}(\mu+\nu)>0$, $a>0$. Therefore

$$\sqrt{\pi}(4\pi)^{-s}\frac{\Gamma(s+im\kappa)\Gamma(s-im\kappa)}{\Gamma(s+\frac{1}{2})}\sum_{n=1}^{\infty} a_n t_n n^{-s}$$
$$=\sum_{n=1}^{\infty} a_n t_n \int_0^{\infty} e^{-2\pi n y} y^{s-1} K_{im\kappa}(2\pi n y)\, dy$$
$$=\int_0^{\infty} y^{s-\frac{3}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\tau)\overline{g(\tau)}\, dx\, dy.$$

By (2.4.2), the inversion of the order of integration is justified for $\mathrm{Re}(s)>\sigma_3$ with a suitable positive constant σ_3 (cf. Rankin [7, p. 360 (4.9.3)]).

2.6. Let $\mathscr{D}_{\Gamma_0(D)}$ be a fundamental domain for $\Gamma_0(D)$ and σ_3 being as in 2.5. Then by the same argument of Rankin [7] and by (2.4.3), we have

$$\int_0^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\tau)\overline{g(\tau)}\, y^{s-\frac{3}{2}}\, dx\, dy \tag{2.6.1}$$
$$=\iint_{\mathscr{D}_{\Gamma_0}(D)} y^{s-\frac{3}{2}} f(\tau)\overline{g(\tau)}\sum_{\substack{c=0\\ c\equiv 0(D)\\ (cD,d)=1}}^{\infty}\ \sum_{d=-\infty}^{\infty}\chi(d)\frac{(c\tau+d)^k}{|c\tau+d|^{2s+1}}\, dx\, dy$$
$$(\mathrm{Re}\, s>\sigma_3),$$

where c (resp. d) ranges over non-negative (resp. all) integers. We note that the right-hand side of (2.6.1) does not depend on the choice of

$\mathscr{D}_{\Gamma_0(D)}$ for $\Gamma_0(D)$. Now combining (2.3.1), (2.5.1) and (2.6.1), our next task is to prove the functional equation for the right-hand side of (2.6.1) multiplying a suitable Γ-factor and $\zeta(2s+1-k,\chi)$. Here we note

$$2\zeta(2s+1-k,\chi)\sum_{\substack{c=0\\ c\equiv 0(D)\\ (cD,d)=1}}^{\infty}\sum_{d=-\infty}^{\infty}\chi(d)\frac{(c\tau+d)^k}{|c\tau+d|^{2s+1}}=\sum_{\substack{c,d\\ c\equiv 0(D)}}{}'\chi(d)\frac{(c\tau+d)^k}{|c\tau+d|^{2s+1}},$$

where the prime on the summation symbol means to omit the term (0, 0). Now we put for a positive integer l

$$G_1(s,\tau,\chi,l)=\sum_{\substack{c,d\\ c\equiv 0(D)}}{}'\chi(d)\frac{(c\tau+d)^l}{|c\tau+d|^{2s+l}},$$

$$G_2(s,\tau,\chi,l)=\sum_{c,d}{}'\chi(c)\frac{(c\tau+d)^l}{|c\tau+d|^{2s+l}},$$

moreover

$$\Lambda_i(s,\tau,\chi,l)=\left(\frac{y}{\pi}\right)^s\Gamma\left(s+\frac{l}{2}\right)G_i(s,\tau,\chi,l)\qquad(i=1,2).$$

Then these functions are holomorphically continued to the whole s-plane and the following functional equation holds.

$$\text{(2.6.2)}\qquad \Lambda_1(s,\tau,\chi,l)=D^{\frac{1}{2}-2s}\Lambda_2(1-s,\tau,\chi,l).$$

In fact, following the notation of Siegel [10, p. 71], our Λ_1,Λ_2 can be expressed in the form

$$\Lambda_1(s,\tau,\chi,l)=D^{-2s}\sum_{v=0}^{D-1}\chi(v)\,\varphi\left(s,\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\frac{v}{D}\\0\end{pmatrix},\tau,l\right)$$

$$\Lambda_2(s,\tau,\chi,l)=D^{-\frac{1}{2}}\sum_{v=0}^{D-1}\chi(v)\,\varphi\left(s,\begin{pmatrix}\frac{v}{D}\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\tau,l\right)$$

and it is known

$$\varphi\left(s,\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\frac{v}{D}\\0\end{pmatrix},\tau,l\right)=\varphi\left(1-s,\begin{pmatrix}\frac{v}{D}\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},\tau,l\right).$$

From this, (2.6.2) follows immediately.

2.7. From (2.6.2), the transformation $\tau \to \omega(D)\,\tau = \frac{-1}{D\tau}$ on τ induces naturally the transformation $s \to 1-s$ on s. For this reason, now let us start over with $f(\tau)$ replaced by $f_1(\tau)$:

$$f_1(\tau) = f(\tau) + D^{k/2} f(D\tau).$$

Then, $f_1(\tau)$ has the following property: For every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$,

$$\text{(2.7.1)} \qquad \begin{aligned} f_1\left(\frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^k f_1(\tau), \\ f_1\left(\frac{-1}{D\tau}\right) &= D^{k/2}\, \tau^k f_1(\tau). \end{aligned}$$

Replacing $f(\tau)$ by $f_1(\tau)$ in (2.5.1) and (2.6.1), we get

$$\text{(2.7.2)} \qquad \begin{aligned} &(1+D^{k/2-s})\, D^*(s, \varphi, \chi, \xi_m) \\ &= (1+D^{k/2-s})(2\pi)^{-2s}\, \Gamma(s+im\kappa)\, \Gamma(s-im\kappa)\, D(s, \varphi, \chi, \xi_m) \\ &= 2^{-1}[E:E_+]^{-1}\, \pi^{-k/2} \iint_{\mathscr{D}_{\Gamma_0(D)}} y^{k/2-2} f_1(\tau)\, \overline{g(\tau)} \\ &\cdot \Lambda_1(s-k/2+\tfrac{1}{2}, \tau, \chi, k)\, dx\, dy, \end{aligned}$$

since $\varphi(s)$ is replaced by $(1+D^{k/2-s})\,\varphi(s)$.

Let $\Gamma^*(D)$ be the group generated by $\Gamma_0(D)$ and $\begin{pmatrix} 0 & -1 \\ D & 0 \end{pmatrix}$ and $\mathscr{D}_{\Gamma^*(D)}$ be a fundamental domain of $\Gamma^*(D)$ on $\mathfrak{H}$. Then $\mathscr{D}_{\Gamma_0(D)}$ is decomposed into disjoint union

$$\text{(2.7.3)} \qquad \mathscr{D}_{\Gamma_0(D)} = \mathscr{D}_{\Gamma^*(D)} \cup \omega(D)^{-1}\, \mathscr{D}_{\Gamma^*(D)}.$$

Decompose the integral of (2.7.2) by (2.7.3). Then by (2.7.1) and (2.4.4), we have

$$\text{(2.7.4)} \qquad \begin{aligned} &(1+D^{k/2-s})\, D^*(s, \varphi, \chi, \xi_m) \\ &= 2^{-1}[E:E_+]^{-1}\, \pi^{-k/2} \iint_{\mathscr{D}_{\Gamma^*(D)}} y^{k/2-2} f_1(\tau)\, \overline{g(\tau)} \\ &\cdot [\Lambda_1(s-k/2+\tfrac{1}{2}, \tau, \chi, k) \\ &+ D^{k/2-s-\frac{1}{2}} \cdot \Lambda_2(s-k/2+\tfrac{1}{2}, \tau, \chi, k)]\, dx\, dy. \end{aligned}$$

Here, put $\Lambda(s,\tau,\chi)=\Lambda_1(s-k/2+\frac{1}{2},\tau,\chi,k)+D^{k/2-s-\frac{1}{2}}\Lambda_2(s-k/2+\frac{1}{2},\tau,\chi,k)$. Then by (2.6.2), we get

$$\Lambda(s,\tau,\chi)=D^{3k/2-3s}\Lambda(k-s,\tau,\chi). \tag{2.7.5}$$

Therefore we have

$$\begin{aligned}
&(1+D^{k/2-s})\,D^*(s,\varphi,\chi,\xi_m)\\
&=2^{-1}[E:E_+]^{-1}\pi^{-k/2}\iint_{\mathscr{D}_{\Gamma^*(D)}} y^{k/2-2} f_1(\tau)\overline{g(\tau)}\,\Lambda(s,\tau,\chi)\,dx\,dy\\
&=2^{-1}[E:E_+]^{-1}\pi^{-k/2}D^{3k/2-3s}\iint_{\mathscr{D}_{\Gamma^*(D)}} y^{k/2-2} f_1(\tau)\overline{g(\tau)}\,\Lambda(k-s,\tau,\chi)\,dx\,dy\\
&=D^{3k/2-3s}(1+D^{k/2-(k-s)})\,D^*(k-s,\varphi,\chi,\xi_m).
\end{aligned}$$

Thus, we have $D^*(s,\varphi,\chi,\xi_m)=D^{k-2s}D^*(k-s,\varphi,\chi,\xi_m)$. This completes the proof of our theorem.

3. Mellin Transform

3.1. Throughout this section, we deal with, in general, a totally real algebraic number field F of degree n. For every element $\mu\in F$, we denote by $\mu^{(\nu)}$ the conjugate element of μ for $\nu=1,\dots,n$. Let $\mathfrak{o}$ be the ring of integers in F and E be the group of all units of $\mathfrak{o}$. For every subset S of F, we denote by S_+ the set of all totally positive elements in S. Denote by $\mathbb{R}_+$ the group of all positive elements in $\mathbb{R}$. Then E_+ can be considered as a subset of $\mathbb{R}_+^n$ (the product of n copies of $\mathbb{R}_+$) by the identification of $F\otimes_{\mathbb{Q}}\mathbb{R}$ with $\mathbb{R}^n$. Now, by the unit theorem, we fix an isomorphism ε of $\mathbb{Z}^{n-1}$ (the product of $n-1$ copies of $\mathbb{Z}$) onto E_+. Then ε can be extended to an isomorphism of $\mathbb{R}^{n-1}$ into $\mathbb{R}_+^n$, and we put for $r=(r_1,\dots,r_{n-1})\in\mathbb{R}^{n-1}$,

$$(\varepsilon(r))_\nu=e^{\sum_{\lambda=1}^{n-1}c_{\nu\lambda}r_\lambda},\qquad(\nu=1,\dots,n), \tag{3.1.1}$$

where $(\varepsilon(r))_\nu$ denotes the ν-th component of $\varepsilon(r)$. Here we note that

$$\sum_{\nu=1}^{n}c_{\nu\lambda}=0,\qquad(\lambda=1,\dots,n-1). \tag{3.1.2}$$

For $x\in\mathbb{R}_+^n$, $m=(m_1,\dots,m_{n-1})\in\mathbb{Z}^{n-1}$, we define a Grössen-character $\xi_m(x)$ by

$$\xi_m(x)=\prod_{\nu=1}^{n}x_\nu^{\alpha_\nu(m)} \tag{3.1.3}$$

with $\sum_{\nu=1}^{n}c_{\nu\lambda}\,\alpha_\nu(m)=2\pi i\,m_\lambda$ $(\lambda=1,\dots,n-1)$ and $\sum_{\nu=1}^{n}\alpha_\nu(m)=0$.

3.2. Here we consider the Mellin transform of

$$\prod_{\nu=1}^{n} \Gamma(s+\alpha_\nu(m))\, t_\nu^{-(s+\alpha_\nu(m))} = \int_0^\infty \cdots \int_0^\infty e^{-\sum_{\nu=1}^{n} t_\nu x_\nu} \prod_{\nu=1}^{n} x_\nu^{s+\alpha_\nu(m)-1}\, dx_1 \ldots dx_n, \tag{3.2.1}$$

which we need in the next 3.3. Instead of the variables x_ν $(\nu=1, \ldots, n)$ we introduce the variables u, y_λ $(\lambda=1, \ldots, n-1)$ by the relation

$$x_\nu = u\, e^{\sum_{\lambda=1}^{n-1} c_{\nu\lambda} y_\lambda}.$$

Then, by (3.1.2) and (3.1.3), the right-hand side of (3.2.1) is equal to $\Delta \int_0^\infty \Phi(u)\, u^{ns-1}\, du$, where

$$\Delta = \begin{vmatrix} 1\, c_{11} \ldots c_{1\,n-1} \\ \ldots \\ 1\, c_{n1} \ldots c_{n\,n-1} \end{vmatrix}$$

and

$$\Phi(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-u\langle t, \varepsilon(y)\rangle]\, e^{2\pi i \langle m, y\rangle}\, dy_1 \ldots dy_{n-1}.$$

Putting $v=u^n$, we have

$$\prod_{\nu=1}^{n} \Gamma(s+\alpha_\nu(m))\, t_\nu^{-(s+\alpha_\nu(m))} = \int_0^\infty \Psi(v)\, v^{s-1}\, dv \tag{3.2.2}$$

with

$$\Psi(v) = n^{-1} \Delta \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[-v^{1/n} \langle t, \varepsilon(y)\rangle]\, e^{2\pi i \langle m, y\rangle}\, dy_1 \ldots dy_{n-1}. \tag{3.2.3}$$

On the other hand, by the theory of Mellin transform

$$\Psi(v) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left\{ \prod_{\nu=1}^{n} \Gamma(s+\alpha_\nu(m))\, t_\nu^{-(s+\alpha_\nu(m))} \right\} v^{-s}\, ds \tag{3.2.4}$$

for $\mathrm{Re}(s) = \sigma_0 > 0$. Comparing (3.2.3) and (3.2.4), we have

$$\begin{aligned} n\Delta^{-1}\Psi(1) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\langle t, \varepsilon(y)\rangle}\, e^{2\pi i \langle m, y\rangle}\, dy_1 \ldots dy_{n-1} \\ &= \frac{n\Delta^{-1}}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left\{ \prod_{\nu=1}^{n} \Gamma(s+\alpha_\nu(m))\, t_\nu^{-(s+\alpha_\nu(m))} \right\} ds. \end{aligned} \tag{3.2.5}$$

Now we put

$$\varphi(t)=\sum_{\gamma\in E_+} e^{-\langle t,\gamma\rangle}, \qquad t=(t_\nu).$$

Here, $\langle t,\gamma\rangle=\sum_{\nu=1}^{n} t_\nu\gamma^{(\nu)}$. Then, by the Poisson summation formula and (3.2.5), we have, for $t_\nu\geqq\sigma_0>0$ $(\nu=1,\dots,n)$

$$\varphi(t)=\sum_{a\in\mathbb{Z}^{n-1}} e^{-\langle t,\varepsilon(a)\rangle}$$

(3.2.6)
$$=\sum_{m\in\mathbb{Z}^{n-1}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\langle t,\varepsilon(y)\rangle}e^{2\pi i\langle m,y\rangle}\,dy_1\dots dy_{n-1}$$

$$=\sum_{m\in\mathbb{Z}^{n-1}}\frac{n\Delta^{-1}}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left\{\prod_{\nu=1}^{n}\Gamma(s+\alpha_\nu(m))\,t_\nu^{-(s+\alpha_\nu(m))}\right\}ds.$$

3.3. The consideration in 3.2 gives us the Mellin transform of a Dirichlet series associated to F, which is analogous to that of the elliptic modular case [5, p. 591]. In fact, let $\{c(\mu)\}$ be a sequence of complex numbers, which is defined for a full representative system of $\mathfrak{o}_+/E_+$. We assume that each $c(\mu)$ does not depend on the choice of a representative $\mu \bmod E_+$ and satisfies, for every μ, $|c(\mu)|\leqq MN(\mu)^{\iota}$ for some positive constants M, ι. Now following the notation in 3.1, we define a Dirichlet series $\varphi_F(s;\xi_m)$ by

$$\varphi_F(s;\xi_m)=\sum_{\mu\in\mathfrak{o}_+/E_+}\xi_m(\mu)\,c(\mu)\,N(\mu)^{-s}, \qquad \mathrm{Re}(s)>\iota.$$

Put

$$\varphi_F^*(s;\xi_m)=(2\pi)^{-ns}\prod_{\nu=1}^{n}\Gamma(s+\alpha_\nu(m))\,\varphi(s;\xi_m).$$

For this $\varphi_F(s;\xi_m)$ we define the function $h(\tau)=h(\tau_1,\dots,\tau_n)$ on $\mathfrak{H}^n$ (the product of n copies of $\mathfrak{H}$):

$$h(\tau)=\sum_{\mu\in\mathfrak{o}_+/E_+}c(\mu)\sum_{\gamma\in E_+}e^{2\pi i\langle\mu,\gamma\tau\rangle},$$

where $\langle\mu,\gamma\tau\rangle=\sum_{\nu=1}^{n}\mu^{(\nu)}\gamma^{(\nu)}\tau_\nu$. Then, as in [11], the following (3.3.1) and (3.3.2) are equivalent:

(3.3.1) *For every* $m\in\mathbb{Z}^{n-1}$, (i) $\varphi_F^*(s;\xi_m)$ *is continued holomorphically to the whole s-plane and* $\varphi_F^*(s;\xi_m)$ *is bounded for every vertical strip* $\sigma'\leqq\mathrm{Re}(s)\leqq\sigma''$; (ii) $\varphi_F^*(s;\xi_m)$ *satisfies a functional equation of the* *form*

$$\varphi_F^*(s;\xi_m)=C\cdot N(\delta)^{k/2-s}\,\xi_m(\delta)\,\varphi_F^*(k-s;\xi_{-m}),$$

where $\delta\in\mathfrak{o}_+$, $k>0$, *and* $C(\neq 0)$ *are constants which do not depend on* $m\in\mathbb{Z}^{n-1}$.

(3.3.2) $$h(\tau)=C\cdot(-1)^{nk/2}\,N(\delta)^{-k/2}(\tau_1\ldots\tau_n)^{-k}\,h\left(\frac{-1}{\delta\tau}\right),$$

for $\tau=(\tau_1,\ldots,\tau_n)\in\mathfrak{H}^n$, *where* $h\left(\frac{-1}{\delta\tau}\right)=h\left(\frac{-1}{\delta^{(1)}\tau_1},\ldots,\frac{-1}{\delta^{(n)}\tau_n}\right)$.

In fact, we can prove it easily from the following (3.3.3) and (3.3.4). Take $w=(w_1,\ldots,w_n)\in\mathbb{R}_+^n$ and put $(t)=(\mu^{(1)}w_1,\ldots,\mu^{(n)}w_n)$ in (3.2.6) for $\mu\in\mathfrak{o}_+$. Then we have

$$\begin{aligned} h(iw)&=h(iw_1,\ldots,iw_n)\\ &=\sum_{\mu\in\mathfrak{o}_+/E_+} c(\mu)\,\varphi\big(2\pi(\mu^{(1)}w_1,\ldots,\mu^{(n)}w_n)\big)\\ &=\frac{n\Delta^{-1}}{2\pi i}\sum_{\mu\in\mathfrak{o}_+/E_+}\sum_{m\in\mathbb{Z}^{n-1}}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left[\prod_{\nu=1}^{n}\Gamma(s+\alpha_\nu(m))(2\pi\mu^{(\nu)}w_\nu)^{-(s+\alpha_\nu(m))}\right]ds \qquad (3.3.3)\\ &=\frac{n\Delta^{-1}}{2\pi i}\sum_{m\in\mathbb{Z}^{n-1}}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\varphi_F^*(s;\xi_m)\prod_{\nu=1}^{n}w_\nu^{-s-\alpha_\nu(m)}\,ds.\end{aligned}$$

Conversely, we have

(3.3.4) $$\varphi_F^*(s;\xi_m)=\int_{E_+\backslash\mathbb{R}_+^n} h(iw)\,\xi_m(w)\prod_{\nu=1}^{n}w_\nu^{s-1}\,dw_1\ldots dw_n.$$

3.4. Now returning to our Theorem 1.2, we can apply 3.3 to $D(s,\varphi,\chi,\xi_m)$. *By* 1.2, *our Dirichlet series* $D(s,\varphi,\chi,\xi_m)$ *satisfies* (3.3.1) *with* $C=1$, $\delta=D$. *Thus, the corresponding function* $h(\tau)$ *has the same property as* (3.3.2).

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Extensions of Abelian Sheaves and Eilenberg-MacLane Algebras*

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To my parents

Introduction

The first unified treatment of extensions of algebraic groups is to be found in Serre's book [19], following work by Weil, Barsotti and Rosenlicht. Serre defines, for G_1, G_2 (reduced) algebraic groups, the group $\mathrm{Ext}^1(G_1, G_2)$ as Yoneda did for abstract groups: it is the group of equivalence classes of short exact sequences

$$0 \to G_2 \to H \xrightarrow{f} G_1 \to 0 \tag{0.1}$$

(i.e. G_2 is the kernel of the separable homomorphism of algebraic groups f), two such sequences lying in the same equivalence class if there exists a group homomorphism g such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \to & G_2 & \to & H & \xrightarrow{f} & G_1 & \to & 0 \\ & & \| & & \downarrow{\scriptstyle g} & & \| & & \\ 0 & \to & G_2 & \to & H' & \xrightarrow{f'} & G_1 & \to & 0 \end{array} \tag{0.2}$$

Serre then computed these groups when G_1 and G_2 were the various simple algebraic groups. The main tool for this computation was the exact sequence of groups:

$$0 \to H^2_{\mathrm{reg}}(G_1, G_2)_s \xrightarrow{i} \mathrm{Ext}^1(G_1, G_2) \xrightarrow{j} H^1_{\mathrm{et}}(G_1, G_2) \tag{0.3}$$

where the right hand group is the first étale cohomology group of the scheme G_1 with values in G_2, i.e. the group of G_2 – principal homogeneous space over the scheme G_1; the map j is given by the fact that via the sequence (0.1), $H \xrightarrow{f} G_1$ is a G_2 – principal homogeneous space over G_1.

The kernel of j is composed of the classes of those sequences (0.1) which are trivial as homogeneous spaces, i.e. those for which the homomorphism f admits a section morphism s (so that $f \circ s = 1_{G_1}$), with s not necessarily a homomorphism.

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This kernel is shown to be equal to the group $Z^2(G_1, G_2)$ of morphisms $h: G_1 \times G_1 \to G_2$ satisfying a certain symmetric cocycle condition modulo coboundary morphisms $h = \partial g$ where g is a morphism: $G_1 \xrightarrow{g} G_2$. These cocycle and coboundary formulas are in complete analogy with those one obtains when one considers G_1 and G_2 as abstract groups, forgetting their extra structure as varieties, and computes the group of commutative extensions of G_1 by G_2 (see for example [14] Chapter 4 § 4).

The groups $\mathrm{Ext}^1(G_1, G_2)$ are computed by noticing that (0.3) became simpler by specific choices of G_1 and G_2 ensuring the vanishing of the first or the last term of (0.3), followed by explicit computation of the other term.

The main aim of Oort's book [18] is to extend the results of [19] from the category of commutative group varieties to the category of commutative group schemes; the groups $\mathrm{Ext}^1(G_1, G_2)$ are computed, where G_1 and G_2 are simple commutative group schemes over an algebraically closed field, and where Ext^1 is defined as above, but with objects and morphisms lying in this new category. Also the groups $\mathrm{Ext}^i(G_1, G_2)$ are defined, for i any positive integer in an analogous manner via long exact sequences of commutative group schemes

$$0 \to G_2 \to H_i \to \cdots \to H_1 \to G_1 \to 0$$

with a certain equivalence relation. These groups are computed and in particular it is shown that, for any two commutative algebraic group schemes G_1, G_2 over an algebraically closed field k,

$$\mathrm{Ext}^i(G_1, G_2) = 0 \qquad \text{for } i \geqq 3.$$

However, this point of view is not entirely satisfactory for the following reasons. First of all, it would be of interest to extend the results above to the case of commutative group schemes defined over an arbitrary base scheme, and far less is known about such group schemes than about schemes defined over a field. In particular, there is no general structure theorem known which would allow us to restrict ourselves to the consideration of specific "simple" group schemes. Secondly, and this is the main consideration, the definition of Ext^i by Yoneda extensions is not very natural, and one would prefer a definition of Ext^i as the ith derived functor of Hom. This cannot be achieved in the category of commutative group schemes since this category is in general not abelian (unless the base is a field) and never has enough injectives.

There is however a standard way out of this sort of difficulty, which is due to Grothendieck: the category of commutative group schemes over a base S is a full subcategory of the category $\mathcal{S}$ of abelian sheaves in the faithfully flat finite presentation ($fppf$) topology on S (see [7]

Exposé IV 6.3 for a definition of this topology). The embedding

$$((\text{Comm. Grp. Sch.}))\hookrightarrow\mathcal{S}$$

is given by sending a group G to the contravariant functor h_G on S represented by G. Thus

$$h_G(T)=\text{Hom}_S(T,G) \tag{0.4}$$

for any scheme T/S. h_G is an $fppf$ sheaf by a standard theorem on descent theory ([10] exposé VIII Theorem 5.2).

The category $\mathcal{S}$ is an $AB5$, $AB3^*$ category with enough injectives ([1] Chapter II Theorem 1.6, Miscellany 1.8), so one can use in it the general machinery of homological algebra. In particular, for any F_1, F_2 in $\mathcal{S}$, define

$$F_2 \to \text{Ext}^i(F_1, F_2)$$

to be the ith derived functor of

$$F_2 \to \text{Hom}(F_1, F_2).$$

It is natural to ask what these groups $\text{Ext}^i(F_1, F_2)$ are, when F_1 and F_2 are representable (i.e. lie in the subcategory of commutative group schemes). In particular, if $S=\text{Spec}(k)$, does one get groups computed by Oort? Clearly this is the case when one considers $\text{Ext}^0(F_1, F_2)= \text{Hom}(F_1, F_2)$, since the subcategory of commutative group schemes is full. Similarly, the answer is positive whenever F_2 is affine over S, or $S=\text{Spec}(k)$ in the case $i=1$ (see [18] Proposition 17.4 for the case F_2 affine/S) by the theory of descent. However, for $i\geqq 2$, it is not at all clear that these groups are the same.

The aim of the present work is to give a method for computing the groups $\text{Ext}^i(F_1, F_2)$ whenever F_1 is representable. This method may be considered a natural generalization of [19] Chapter 7 § 1.4 and 3.14 – 15. In particular our exact sequence (0.3) is replaced by a spectral sequence, and (0.3) becomes the first three terms of the five term exact sequence associated in low degrees to the spectral sequence. A similar interpretation of (0.3) via a different spectral sequence was given in [22] Exposé 9. The methods employed in that exposé were of influence on the present work.

The main observation which enables one to compute the groups $\text{Ext}^i(F_1, F_2)$ is the following: the groups $\text{Ext}^i(\mathbf{Z}[F], F_2)$ can easily be computed when F is representable ($\mathbf{Z}[F]$ is the "free abelian sheaf" on F). In fact

$$\text{Ext}^i(\mathbf{Z}[F], F_2)=H^i_{fppf}(G, F_2) \tag{0.5}$$

where G is the scheme representing F.

One then defines for every such F a complex $X=(X_i)$ with values in $\mathcal{S}$, with each X_i of the form

$$X_i=\mathbf{Z}[G_{i,1}]\times\mathbf{Z}[G_{i,2}]\times\cdots\times\mathbf{Z}[G_{i,r_i}]$$

with all the $G_{i,j}$ representable whenever F is. The complex X has the property that its lowest non-zero homology sheaf is isomorphic to F, and all the higher homology sheaves are torsion sheaves, which can be computed in terms of F. The complex X is modelled after the complex $A(\Pi, n)$ defined by iterated bar construction for any abelian group Π by Eilenberg and MacLane in [8].

Using a spectral sequence derived from this complex and (0.5) one gets information about the group $\mathrm{Ext}^i(F, F_2)$ in terms of cohomological properties of the scheme representing F.

One can use the technique just outlined to verify that certain groups are torsion; one proves that, under certain hypotheses on the base scheme S, the groups $\mathrm{Ext}^i(A, G_{mS})$, $\mathrm{Ext}^i(G_{mS}, G_{mS})$, $\mathrm{Ext}^i(G_{aS}, G_{mS})$, $\mathrm{Ext}^i(A, B)$, $\mathrm{Ext}^i(G_{mS}, B)$ and $\mathrm{Ext}^i(G_{aS}, B)$ are torsion, where A, B are abelian schemes over S and G_{mS} (resp. G_{aS}) is the multiplicative (resp. the additive) group on S, and i is any integer greater than 1. In fact, they are at most p-torsion, where p is any of the residual characteristics of S.

It is natural to ask wheter the p-torsion components of these groups also vanish. It has been shown in [4] that this is not always the case and that the p-torsion phenomena are more subtle than one might have been led to expect from [18]. It would be of particular interest to compute $\mathcal{E}xt^i(A, G_{mS})$: as pointed out by Tate in [21], this is connected with his duality theorems. The question was also raised by Artin and Mazur in [3] with related applications to the autoduality of the Jacobian in mind. It is hoped that the methods presented in this work will in a not too distant future yield a complete answer to this question.

Our work is divided into two chapters. In the first one the general theory is built up: after a reminder of the complex $A(\Pi, n)$ of Eilenberg-MacLane (§ 1), we define a similar complex with values in the category of abelian presheaves (§ 2). We then pass to an analogous complex in the category of abelian sheaves (§ 3) defined for every abelian sheaf F. In § 4 the spectral sequence alluded to above is defined, and using (0.5) its simpler form for F representable is exhibited (§ 5). In the second chapter this machinery is applied to some specific computations, and the titles of the paragraphs are self-explanatory. An effort has been made, throughout Chapter I, to make statements in more generality than was necessary for the computations of Chapter II, in the hope of future use.

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Chapter I

§1. Eilenberg-MacLane Algebras

We review parts of [6, 8]; for any abelian group G, a chain complex $A(G,1)$ is defined: in dimension q, it is the free abelian group on the q-fold product of G with itself:

$$A(G,1)_q = \mathbf{Z}[G \times \cdots \times G], \qquad A(G,1)_0 = \mathbf{Z}.$$

The boundary $\partial_q\colon A(G,1)_q \to A(G,1)_{q-1}$ is defined on the generators of the free abelian group, and extended by linearity:

$$\begin{aligned}\partial_q([x_1,\dots,x_q]) = [x_2,\dots,x_q] + \sum_{i=1}^{q-1}(-1)^i[x_1,\dots,x_i+x_{i+1},\dots,x_q]\\ +(-1)^q[x_1,\dots,x_{q-1}]\end{aligned} \tag{1.1}$$

with $q>1$, $\partial_1([x_1])=1$ where $[x_1,\dots,x_q]$ is a typical generator of $A(G,1)_q$ ($x_i\in G$ for all i), and an augmentation $\alpha\colon A(G,1)\to\mathbf{Z}$ is defined to be the identity on $A(G,1)_0$ and trivial elsewhere. The map defined by

$$\begin{aligned} S\colon G &\to A(G,1)_1\\ S(x) &= [x]\end{aligned} \tag{1.2}$$

is called the suspension.

A multiplication can be defined on $A(G,1)$ and with this extra structure $A(G,1)$ is a differential graded augmented algebra on $\mathbf{Z}$ (DGA-algebra for short; see [6] Exposé 2 §1 for the precise definition of DGA-algebra).

Also, the free abelian group on G, $\mathbf{Z}[G]$, can be trivially considered as a DGA-algebra by assigning degree zero to every element. The process by which the DGA-algebra $A(G,1)$ was obtained from the DGA-algebra $\mathbf{Z}[G]$ is called the bar construction and can be applied to any DGA-algebra. We write $A(G,1)=B(\mathbf{Z}[G])$. In particular, it can be applied to the DGA-algebra $A(G,1)$ and yields a DGA-algebra $A'(G,2)=B(A(G,1))$. Iterating this process one defines for any integer n, a DGA-algebra $A'(G,n)$ (see [8] I Chapter II, §7 for the precise definition of the bar construction and [6] Exposé 3 for a more general notion of construction).

It is preferable to modify this slightly: for any DGA-algebra G, define the normalized bar construction $B_N(G)$:

$$B_N(G)=B(G)/DB(G)$$

where $DB(G)$ is the degenerate subcomplex (see [8] I, § 8 for the definition of the degeneracy operators). Following [8] define for any abelian group G

$$A(G,n)=B_N(A(G,n-1)) \tag{1.3}$$

where $A(G,1)$ was defined previously.

For each integer $i>0$, there is a DGA-monomorphism of degree $+1$ called the suspension

$$S: A(G,i)\to A(G,i+1). \tag{1.4}$$

This suspension commutes with the boundary (up to a $(-1)^q$ term).

One also checks that $A(G,n)_d=0$ for $0<d<n$. In fact $A(G,n)$, which we here define to be a chain complex, actually posesses the additional structure of a CSS-complex. Furthermore any group homomorphism $f\colon G\to G'$ induces a morphism

$$A(f,n)\colon A(G,n)\to A(G',n) \tag{1.5}$$

of CSS-complexes. We define a new chain complex $A^n(G)$:

$$\begin{aligned} A^n(G)_q&=A(G,n)_{q+n-1} && \text{for } q\neq 1-n\\ A^n(G)_q&=0 && \text{for } q=1-n. \end{aligned} \tag{1.6}$$

The new boundary operator $\partial'\colon A^n(G)_q\to A^n(G)_{q-1}$ is

$$\partial'=(-1)^q\,\partial \tag{1.7}$$

where ∂ is the corresponding boundary for $A(G,n)$. In this new collection of complexes S preserves degree and commutes with the boundary; it may therefore be regarded as an inclusion:

$$A^1(G)\subset A^2(G)\dots A^n(G)\subset\cdots. \tag{1.8}$$

The union of these complexes is by definition the complex $A(G)$. We describe this complex more precisely: for any integer q, let $I=(k_1,\dots,k_{r-1})$ be any ordered set of positive integers subject to the condition

$$\sum_{i=1}^{r-1} k_i=q-1.$$

For such an I, write $A_I(G)=\mathbf{Z}[G\times\cdots\times G]$ for the free abelian group on the r-fold product of G with itself (in the future we will write $\#I$ for the cardinality $r-1$ of I). Let $\mathcal{I}_q$ be the set of all I satisfying the condition above. We then have:

$$A(G)_q = \prod_{I \in \mathscr{I}_q} A_I(G). \tag{1.9}$$

To keep track of it, a generator of $A_I(G)$ is denoted $[g_1 \underset{k_1}{|} \dots \underset{k_{r-1}}{|} g_r]$ rather than the more usual $[g_1, \dots, g_r]$. The boundary map of $A(G)$ is complicated and at each $A_I(G)$ it depends actually on I and not just on $\# I$. For more details, see [8] I. $A^n(G)$ can now also be described simply: $A^n(G)_q$ is the product of those $A_I(G)$ where I satisfies the additional condition $k_i \leqq n$ for all k_i in I.

There is an analogous complex $A_N(G)$ which is a normalized version of $A(G)$:

$$A_N(G) = \bigcup_n A_N^n(G) \tag{1.10}$$

where $A_N^n(G)$ is obtained from $A_N(G, n)$ exactly as $A^n(G)$ was from $A(G, n)$ in (1.6) and the definition of $A_N(G, n)$ is by iteration:

$$A_N(G, n) = B_N(A_N(G, n-1))$$
$$A_N(G, 1) = B_N(\mathbf{Z}[G]).$$

More simply, $A_N(G)$ is obtained from $A(G)$ by factoring out the subcomplex generated by elements

$$[g_1 \underset{k_1}{|} \dots \underset{k_{r-1}}{|} g_r] \quad \text{with} \quad g_i = 1 \quad \text{for some } i,\ 1 \leqq i \leqq r. \tag{1.11}$$

There are morphisms of complexes f and g

$$A(G) \underset{g}{\overset{f}{\rightleftarrows}} A_N(G) \quad \text{with} \quad f \circ g = 1_{A_N(G)}, \qquad g \circ f \simeq 1_{A(G)}. \tag{1.12}$$

Moreover both f and g are natural, and so is the homotopy $\Phi: g \circ f \simeq 1_{A(G)}$, in other words for any homomorphism of abelian groups

$$h: G \to G'$$

the following diagram (and analogous ones for g and Φ) commutes:

$$\begin{array}{ccc} A(G) & \xrightarrow{f} & A_N(G) \\ {\scriptstyle A(h)} \downarrow & & \downarrow {\scriptstyle A_N(h)} \\ A(G') & \xrightarrow{f} & A_N(G') \end{array} \tag{1.13}$$

where the vertical maps are defined by

$$A(h): [x_1 \underset{k_1}{|} \dots \underset{k_{r-1}}{|} x_r] = [h(x_1) \underset{k_1}{|} \dots \underset{k_{r-1}}{|} h(x_r)]$$

and similarly for $A_N(h)$ (see [9] p. 52, [8] I Corollary 12.2).

From the description of $A^n(G)$ and $A(G)$ given above in terms of $A_I(G)$, it is immediately clear that $A^n(G)_q = A(G)_q$ for $q<n+2$. In particular

$$H_q(A(G)) = H_q(A^n(G)) \quad \text{for } q<n+1. \tag{1.14}$$

By the definition (1.6) of $A^n(G)$,

$$H_q(A^n(G)) = H_{n+q-1}(A(G,n)). \tag{1.15}$$

The chain equivalence f of (1.12) induces a functorial isomorphism

$$H_q(f)\colon H_q(A(G)) \xrightarrow{\approx} H_q(A_N(G)). \tag{1.16}$$

The isomorphisms (1.14), (1.15) and (1.16) reduce the problem of computing $H_q(A_N(G))$ to computing $H_{n+q-1}(A(G,n))$ for any $n>q-1$. This however has been done in [6] where more generally $H_i(A(G,j))$ is calculated for any two integers i and j. For our purposes it will only be necessary to consider the case $j \leqq i < 2j$. Here one gets the so-called stable homology groups. Their name derives from the fact that for any j, the suspension map induces an isomorphism

$$H_q(S)\colon H_q(A(G,j)) \xrightarrow{\approx} H_{q+1}(A(G,j+1))$$

for $j \leqq q < 2j$. This is in fact an immediate consequence of (1.14) since the left hand side of the equation is independent of n.

In order to state the main theorem on the computation of the stable homology groups in the form most convenient for our purposes, it will be necessary to introduce the theory of Steenrod operations, as presented in [6].

We refer to [6] Exposé 14 §1 for the definition of a complete semi-simplicial complex (*CSS*-complex for short) and a *CSS*-morphism, and state the following

Definition. *An additive homological operation (modulo p) consists in a collection of homomorphisms $T(X)$, one for every CSS-complex X*

$$T(X)\colon H_{n+q}(X;\mathbf{Z}/p) \to H_n(X;\mathbf{Z}/p)$$

n and q given such that for any CSS-morphism $f\colon X \to X'$, the following diagram commutes:

$$\begin{array}{ccc} H_{n+q}(X;\mathbf{Z}/p) & \xrightarrow{T(X)} & H_n(X;\mathbf{Z}/p) \\ \downarrow{\scriptstyle H_{n+q}(f)} & & \downarrow{\scriptstyle H_n(f)} \\ H_{n+q}(X';\mathbf{Z}/p) & \xrightarrow{T(X')} & H_n(X';\mathbf{Z}/p). \end{array}$$

In [6] Theorem 3, for every $k \geqq 1$ and every prime p, a certain specific collection of homological operations $(T_n)_{n\in\mathbf{Z}}$ is defined

$$T_n(X)\colon H_{n+2k(p-1)}(X;\mathbf{Z}/p) \to H_n(X;\mathbf{Z}/p).$$

The transposes of these homological operations, which are automatically what is known as cohomological operations, are well known: they are nothing else than the Steenrod operations modulo p (see [20] for a different, axiomatic, approach to these).

Another well known homological operation is the Bockstein operation; this is the connecting homomorphism in the homology exact sequence associated to the exact coefficient sequence

$$0 \to \mathbf{Z}/p \to \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0.$$

Thus the Bockstein homomorphism β_p has the form

$$\beta_p(X)\colon H_{q+1}(X;\mathbf{Z}/p) \to H_q(X;\mathbf{Z}/p).$$

We now relabel these specific operations and various compositions of them in the following way: fix a prime p; for every integer a with $a \equiv 0$ or $1 \bmod (2p-2)$, write

$$a=(2p-2)k+u \quad \text{with} \quad u=0 \quad \text{(resp. 1);}$$

for any such a, define the collection of homological operations St_a^p by

$$St_a^p(X)=T_n(X)\colon H_{n+a}(X;\mathbf{Z}/p) \to H_n(X;\mathbf{Z}/p) \quad \text{when} \quad a=(2p-2)k \tag{1.17}$$

(resp. $St_a^p(X)=St_{a-1}^p(X)\circ \beta_p(X)\colon H_{n+a}(X;\mathbf{Z}/p) \to H_n(X;\mathbf{Z}/p)$). To compose these operations even further, we need the following

Definition. *Let p be any prime; a p-admissible sequence of (stable) degree q is a sequence of non-negative integers $(a_1, \dots, a_i, \dots)$ satisfying the following conditions;*

$$\begin{array}{lll} 1. & a_i \equiv 0 \text{ or } 1 \bmod (2p-2) & \text{for all } i \\ 2. & a_i \geqq p\, a_{i+1} & \\ 3. & \sum_i a_i = q. & \end{array} \tag{1.18}$$

Such a sequence is said to be of second type if the last non-zero integer a_i (towards the right) is equal to 1. A sequence is called of first type whenever it is not of second type.

Let $I=(a_1, \dots, a_i, \dots)$ be an admissible sequence of degree q; the homological operation St_I^p is defined as follows:

$$\begin{aligned} &St_I^p(X)\colon H_{n+q}(X;\mathbf{Z}/p) \to H_n(X;\mathbf{Z}/p) \\ &St_I^p(X)=St_{a_i}^p \circ St_{a_{i-1}}^p \circ St_{a_{i-2}}^p \circ \cdots \circ St_{a_1}^p \end{aligned} \tag{1.19}$$

with a_i the last non zero integer of I.

For any $n>q$ and any admissible sequence I of degree q of first type (resp. of second type), and for any abelian group G, define θ_I^p as the following composite homomorphism:

$$H_{n+q}(G,n;\mathbf{Z}) \to H_{n+q}(G,n;\mathbf{Z}/p) \xrightarrow{St_I^p(A(G,n))} H_n(G,n;\mathbf{Z}/p) \approx G_p \tag{1.20}$$

(resp.)

$$H_{n+q}(G,n;\mathbf{Z}) \to H_{n+q}(G,n;\mathbf{Z}/p) \xrightarrow{St_J^p(A(G,n))} H_{n+1}(G,n;\mathbf{Z}/p) \approx {}_pG \tag{1.21}$$

(where, for any abelian group H, $H_i(G,j;H)$ is shorthand for $H_i(A(G,j);H)$). We explain the various terms of (1.20) and (1.21): J is the p-admissible sequence of first type obtained from the p-admissible sequence of second type I by replacing the last non-zero term $a_i=1$ of I by $a_i=0$; G_p (resp. ${}_pG$) is the cokernel (resp. the kernel) of the p-th power endomorphism on G; the left hand side homomorphism of (1.20) and (1.21) is induced by the coefficient homomorphism $\mathbf{Z}\to\mathbf{Z}/p$; the middle homomorphisms St^p were defined above; as for the right hand homomorphisms, which are in fact isomorphisms, it is more convenient to define their inverses. In the case (1.20) this is the composite:

$$G_p \xrightarrow{\approx} G\otimes\mathbf{Z}/p \xrightarrow{\approx} H_1(G,1;\mathbf{Z}/p) \xrightarrow{\approx} H_2(G,2;\mathbf{Z}/p) \xrightarrow{\approx} \cdots \to H_n(G,n;\mathbf{Z}/p) \tag{1.22}$$

where the first map is the obvious one and all the others are induced on the homology by the suspension maps described in (1.4), and are all isomorphisms in the dimensions considered, as remarked earlier.

In the case (1.21) it is the following composite isomorphism:

$${}_pG \xrightarrow{\approx} H_2(G,1;\mathbf{Z}/p) \xrightarrow{\approx} H_3(G,2;\mathbf{Z}/p) \xrightarrow{\approx} \cdots \xrightarrow{\approx} H_{n+1}(G,n;\mathbf{Z}/p). \tag{1.23}$$

Here again all the maps are the suspension isomorphisms, except for the first one, which is known as the transpotence; it is defined in [6] Exposé 6 §4, and shown to be an isomorphism in Exposé 6 Theorem 4.

It is now possible to describe entirely the goups $H_{n+q}(G,n;\mathbf{Z})$ for $q<n$.

For $q=0$, this is easy since we know the suspensions induce the isomorphisms

$$G \xrightarrow{\approx} H_1(G,1;\mathbf{Z}) \xrightarrow{\approx} H_2(G,2;\mathbf{Z}) \xrightarrow{\approx} \cdots \xrightarrow{\approx} H_n(G,n;\mathbf{Z}). \tag{1.24}$$

For $q>0$ the situation is entirely described in [5] II Theorem 7:

Theorem 1 (Cartan): *For all n, $H_n(G,n;\mathbf{Z})\approx G$. For any two integers $q\geqq 1$, $n>q$, $H_{n+q}(G,n;\mathbf{Z})$ is a torsion group. Let $L_q(G,p)$ denote its p-primary component. For any p-admissible sequence I of first type (resp.*

of second type) of degree q satisfying the additional condition $a_1 \equiv 0 \bmod (2p-2)$, θ_I^p sends $L_q(G,p)$ onto G_p (resp. ${}_pG$). Let N_I^p denote the intersection of the kernels of the θ_J^p for all p-admissible sequences J of either type of degree q with $a_1 \equiv 0 \bmod (2p-2)$, with $J \neq I$. Then $L_q(G,p)$ is the direct sum of the N_I^p and θ_I^p is an isomorphism of N_I^p on G_p (resp. ${}_pG$).

The groups $H_{n+q}(G,n;\mathbf{Z})$ were first computed in [6] Exposé 11 Theorem 2. The more elaborate description given in Theorem 1 above involving Steenrod operations is preferable for our purposes for the following reason: we need to know that the computation of $H_{n+q}(G,n;\mathbf{Z})$ is functorial in G, and this is easy to check from Theorem 1. Specifically we check that the decomposition

$$L_q(G,p) \approx \coprod_I N_I^p. \tag{1.25}$$

and the isomorphisms

$$\theta_I^p: N_I^p \to G_p \qquad (\text{resp. } \theta_I^p: N_I^p \to {}_pG) \tag{1.26}$$

are functorial in G; it is clearly sufficient to check that the homomorphisms θ_I^p are functorial, i.e. that for any homomorphism of abelian groups $h: G \to G'$, the following large diagram (and a similar one for admissible words of second type) commutes:

$$\begin{array}{ccccccc} H_{n+q}(G,n;\mathbf{Z}) & \to & H_{n+q}(G,n;\mathbf{Z}/p) & \xrightarrow{St_I^p} & H_n(G,n;\mathbf{Z}/p) & \rightleftarrows & G_p \\ \downarrow{\scriptstyle H_{n+q}(f)} & & \downarrow{\scriptstyle H_{n+q}(f)} & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle h'} \\ H_{n+q}(G',n;\mathbf{Z}) & \to & H_{n+q}(G',n;\mathbf{Z}/p) & \longrightarrow & H_n(G',n;\mathbf{Z}/p) & \rightleftarrows & G'_p \end{array} \tag{1.27}$$

where $f = A(h,n)$ is the morphism of CSS-complexes induced by h as in (1.5) and h' is the homomorphism induced by h on G_p. We check that each of the three squares of (1.27) commutes. This is obvious for the first square. The second square commutes because St_I^p is a homological operation and so commutes with the CSS-morphism f. As for the third square, it is sufficient to check that the square obtained by replacing the horizontal isomorphisms by their inverses commutes. These inverse maps were defined as composites of other isomorphisms in (1.22) (resp. (1.23)). We are reduced to showing that the following diagrams commute:

$$\begin{array}{ccc} G & \xrightarrow{S} & H_1(G,1;\mathbf{Z}) \\ {\scriptstyle h}\downarrow & & \downarrow \\ G' & \xrightarrow{S} & H_1(G',1;\mathbf{Z}) \end{array} \qquad \begin{array}{ccc} {}_pG & \xrightarrow{\psi} & H_2(G,1;\mathbf{Z}) \\ {\scriptstyle h}\downarrow & & \downarrow \\ {}_pG' & \xrightarrow{\psi} & H_2(G',1;\mathbf{Z}). \end{array} \tag{1.28}$$

For the left hand diagram, this is immediately clear from the definition of the suspension S. For the right hand one, one has to go back to the definition of the transpotence ([6] p. 6–05). Commutativity then follows immediately from Exposé 6, prop. 2 of [6].

As an illustration of theorem 1, we compute $H_{n+q}(G, n; \mathbf{Z})$ for small values of q, and any $n>q$:

$H_n(G, n; \mathbf{Z}) \approx G$

$H_{n+1}(G, n; \mathbf{Z}) = 0$

$H_{n+2}(G, n; \mathbf{Z}) \approx G_2$ corresponding to the admissible word (2,0)

$H_{n+3}(G, n; \mathbf{Z}) \approx {}_2G$ corresponding to the admissible word (2,1)

$H_{n+4}(G, n; \mathbf{Z}) \approx G_2 \oplus G_3$

corresponding to the admissible word (4,0), (4,0)

$H_{n+5}(G, n; \mathbf{Z}) \approx {}_2G \oplus {}_3G$

corresponding to the admissible word (4,1), (4,1)

etc. ....

§ 2. The Complex in $\mathcal{P}_C$

All the results of § 1, and in particular Theorem 1, were functorial in the abelian group G. This allows us to repeat all the constructions of § 1 with G replaced by a presheaf of abelian groups.

Let C be a category, $\mathcal{P}_C$ the category of presheaves of abelian groups (also called abelian presheaves) on C. $\mathcal{P}_C$ is an abelian category with enough injectives satisfying the axioms $AB\,5$, $AB\,3^*$ on the existence of sums and products.

For any object P of the category $\mathcal{Q}_C$ of presheaves of sets on C, define the abelian presheaf $\mathbf{Z}[P]$ as follows: set, for any $T \in C$,

$$\mathbf{Z}[P](T) = \mathbf{Z}[P(T)]$$

where the right hand term is the free abelian group on $P(T)$ and define, for any morphism $f: T' \to T$, $\mathbf{Z}[P](f)$ to be the obvious homomorphism induced by linearity from the map of sets $P(f): P(T) \to P(T')$.

The functor $G \to \mathbf{Z}[G]$ from the category of sets to the category of abelian groups is the adjoint of the forgetful functor F from the category of abelian groups to the category of sets:

$$\mathrm{Hom}_{ab}(\mathbf{Z}[G], H) \approx \mathrm{Hom}_{\mathrm{sets}}(G, FH) \tag{2.1}$$

where G is any set and H any abelian group. From (2.1) one deduces immediately the corresponding fact for presheaves:

$$\mathrm{Hom}_{\mathcal{P}_C}(\mathbf{Z}[P], P') \approx \mathrm{Hom}_{\mathcal{Q}_C}(P, FP')$$

for any $P \in \mathcal{Q}_C$, $P' \in \mathcal{P}_C$, F being now the forgetful functor from $\mathcal{P}_C$ to $\mathcal{Q}_C$.

For any $P \in \mathscr{P}_C$ and any integer $i > 0$, we define a chain complex with values in $\mathscr{P}_C$ (in fact a CSS-complex with values in $\mathscr{P}_C$), denoted $A(P, i)$; for any $T \in C$, define

$$A(P, i)(T) = A\big(P(T), i\big) \tag{2.2}$$

where $A(P(T), i)$ is the chain complex defined in (1.3) for the abelian group $P(T)$.

For any morphism $f: T' \to T$ in C, define a morphism of chain complexes $A(P, i)(f)$: $A(P, i)(T) \to A(P, i)(T')$ by

$$A(P, i)(f) = A(P(f), i) \tag{2.3}$$

where $A(P(f), i)$ was defined in (1.5).

Define analogously, for any $P \in \mathscr{P}_C$, the chain complexes with values in $\mathscr{P}_C$, $A^n(P)$, $A(P)$, $A_N(P)$:

$$A^n(P)(T) = A^n(P(T)),$$

$$A(P)(T) = A(P(T)), \tag{2.4}$$

$$A_N(P)(T) = A_N(P(T)) \tag{2.5}$$

where in each case the complex on the right hand side of the equation was defined in § 1, and with the morphisms defined as in (2.3).

For future reference, here is a more explicit definition of $A(P)$ and $A_N(P)$. We mimic the definition given in § 1. For every positive integer q, we defined in § 1 the set $\mathscr{I}_q$ of I satisfying

$$\sum_{i=1}^{r} k_i = q - 1 \tag{2.6}$$

where $I = \{k_1, \ldots, k_r\}$. For any I in $\mathscr{I}_q$, define $A_I(P)$ by

$$A_I(P) = \mathbf{Z}[P \times \cdots \times P] \qquad (\# I + 1)\text{-fold product}. \tag{2.7}$$

It is then clear from (1.9) that

$$A(P)_q = \prod_{I \in \mathscr{I}_q} A_I(P). \tag{2.8}$$

Another more familiar way of writing $A_I(P)$ might be

$$A_I(P) = \mathbf{Z}[P \underset{k_1}{|} \ldots \underset{k_r}{|} P] \quad \text{where} \quad I = \{k_1, \ldots, k_r\}. \tag{2.9}$$

The boundary maps in $A(P)$ are complicated. They are precisely the ones one would get if one used notation (2.9) and P were an abelian group, instead of an abelian presheaf.

For any $I \in \mathscr{I}_q$, let $\{e_i\}_{i=1}^{\# I+1}$ be the collection of morphisms of presheaves

$$e_i\colon \mathbf{Z}[P \times \cdots \times P] \to A_I(P) \qquad (\# I\text{-fold product of } P\text{'s}) \tag{2.10}$$

induced by the morphism of presheaves of sets

$$e_i\colon P \times P \times \cdots \times P \to A_I(P)$$

with

$$e_i(T)\colon P(T) \times \cdots \times P(T) \to A_I(P)(T)$$

$$e_i(T)(x_1, \ldots, x_{\# I}) = (x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{\# I}).$$

Define the presheaf $A_{N,I}(P)$ as the quotient of $A_I(P)$ by the union of the images of the e_i. Then

$$A_N(P)_q = \prod_{I \in \mathscr{I}_q} A_{N,I}. \tag{2.11}$$

Definition (2.2) implies that the homology presheaf $H_q(A(P, i))$, (which we will also write $H_q(P, i)$ or $H_q(P, i;\ \mathbf{Z})$) is given by

$$H_q(A(P, i))(T) = H_q(A(P(T), i))$$

$$H_q(A(P, i))(f) = H_q(A(P(f), i))$$

for any object T and any morphism f in C.

Define the presheaf $\mathbf{Z}/p[P]$ in the obvious manner; it is clear what one means by $H_q(A(P, i); \mathbf{Z}/p)$, for any prime p. In fact, for any T in C

$$H_q(A(P, i); \mathbf{Z}/p)(T) = H_q(A(P(T), i); \mathbf{Z}/p).$$

We now set about computing the presheaves $H_{n+q}(A(P, n))$ for $q < n$. For any prime p and any P in $\mathscr{P}_C$, define the morphism of presheaves θ_I^p

$$\theta_I^p\colon H_{n+q}(P, n; \mathbf{Z}) \to P_p \tag{2.12}$$

(resp.)

$$\theta_I^p\colon H_{n+q}(P, n; \mathbf{Z}) \to {}_pP \tag{2.13}$$

where I is any p-admissible sequence of first type (resp. of second type), and P_p is the cokernel (resp. the kernel) of the pth power map on P by

$$\theta_I^p(T) = \theta_I^p(P(T))\colon\ H_{n+q}(P(T), n; \mathbf{Z}) \to P(T)_p$$

$$(\text{resp. } \theta_I^p(T) = \theta_I^p(P(T))\colon\ H_{n+q}(P(T), n; \mathbf{Z}) \to {}_pP(T)).$$

This actually defines a morphism of presheaves because of the commutativity of diagram (1.27). We now state

Theorem 2. *Let $q \geqq 1, n > q$; P a presheaf in $\mathscr{P}_C$. The presheaf $H_n(P, n; \mathbf{Z})$ is isomorphic to P. $H_{n+q}(P, n; \mathbf{Z})$ is a torsion presheaf. Let $L_q(P, p)$ be its p-primary component. For any p-admissible sequence I of first type (resp.*

of second type) of degree q with the additional condition $a_1 \equiv 0 \bmod 2p-2$, θ_I^p *sends* $L_q(P, p)$ *onto* P_p *(resp.* ${}_pP$*). Let* N_I^p *be the intersection of the kernels of the* θ_J^p *for all such p-admissible sequences J of degree q of either type, with* $J \neq I$*. Then* $L_q(P, p)$ *is the direct sum of the* N_I^p *and* θ_I^p *is an isomorphism of* N_I^p *on* P_p *(resp.* ${}_pP$*).*

Proof. We check that $H_{n+q}(P, n; \mathbf{Z})(T)$ is a torsion group for all T: by (2.2) this is a consequence of the corresponding statement in Theorem 1. Similarly for the other assertions of the theorem: we have well-defined morphisms

$$P \to H_n(P, n; \mathbf{Z}),$$

$$\oplus N_I^p \to L_q(P, p), \tag{2.14}$$

$$\theta_I^p\colon N_I^p \to P_p \quad (\text{resp. } N_I^p \to {}_pP). \tag{2.15}$$

To check these are isomorphisms in the category of presheaves, it is sufficient to check that for all T, they induce isomorphisms of abelian groups

$$P(T) \xrightarrow{\approx} H_n(P(T), n; \mathbf{Z})$$

$$\oplus N_I^p(T) \xrightarrow{\approx} L_q(P, p)(T)$$

$$\theta_I^p(T)\colon N_I^p(T) \xrightarrow{\approx} P_p(T) \quad (\text{resp. } {}_pP(T)).$$

These are isomorphisms by Theorem 1, and this finishes the proof of Theorem 2.

Our aim is to compute the homology of $A_N(P)$. In fact we claim that

$$H_q(A_N(P)) \approx H_{n+q-1}(A(P, n)) \quad \text{for all } n > q-1. \tag{2.16}$$

Clearly $H_{n+q-1}(A(P, n)) \approx H_q(A(P))$ since the relevant degrees of the chain complexes are isomorphic. It remains to prove that

$$H_q(A(P)) \approx H_q(A_N(P)).$$

This is actually clear by the usual method of reducing to the case of abelian groups where it is just (1.16). We will however need later the additional fact that $A(P)$ and $A_N(P)$ are chain equivalent: there exist morphisms f and g of abelian presheaves, and a homotopy Φ:

$$A(P) \underset{g}{\overset{f}{\rightleftarrows}} A_N(P) \tag{2.17}$$

with $f \circ g = 1$ and $\Phi\colon g \circ f \simeq 1$. For any $T \in C$, f and g are defined by

$$A(P(T)) \underset{g(T)}{\overset{f(T)}{\rightleftarrows}} A_N(P(T))$$

where these maps are the f and g of (1.12) in the case $G = P(T)$. This actually defines morphisms of presheaves by commutativity of diagram

(1.13) and a similar one for g. The homotopy Φ is defined in the same way, since the definition in the category of abelian groups is natural.

§ 3. *The Complex in* $\mathcal{S}$

Theorem 2 was an analogue of Theorem 1 in the category $\mathcal{P}_C$. In this paragraph, we deduce from Theorem 2 that an analogue also exists in the category of abelian sheaves on some topology.

C now stands for a topology, $\mathcal{P}=\mathcal{P}_C$ the category of presheaves on C, $\mathcal{S}$ the category of abelian sheaves on C. The natural inclusion

$$i\colon \mathcal{S}\to\mathcal{P} \tag{3.1}$$

has an adjoint

$$\#\colon \mathcal{P}\to\mathcal{S}.$$

$\#(P)$ is called the associated sheaf of the presheaf P. $\mathcal{S}$ is an $AB\,5$, $AB\,3^*$ category with enough injectives, and $\#$ is an exact functor ([1] Chapter II, Theorem 1.6).

Let $\mathcal{T}$ be the category of sheaves of sets on C. The forgetful functor $j\colon \mathcal{S}\to\mathcal{T}$ has an adjoint $\mathcal{T}\to\mathcal{S}$ sending F in $\mathcal{T}$ to $\mathbf{Z}[F]$ defined below in (3.3). The adjointness property means that for F in $\mathcal{T}$, G in $\mathcal{S}$

$$\mathrm{Hom}_{\mathcal{S}}(\mathbf{Z}[F],G)\approx \mathrm{Hom}_{\mathcal{T}}(F,jG). \tag{3.2}$$

For F in $\mathcal{T}$, define

$$\mathbf{Z}[F]=\#\,\mathbf{Z}[iF] \tag{3.3}$$

where $\mathbf{Z}[iF]$ was defined in § 2. It is easy to check, using the adjointness of i and $\#$, that $\mathbf{Z}[F]$ satisfies (3.2). We now define the complexes of sheaves $A(F,r)$, $A^n(F)$, $A(F)$ and $A_N(F)$, for any positive integers n, r, and any abelian sheaf F:

$$A(F,r)=\#\,A(iF,r), \tag{3.4}$$

$$A^n(F)=\#\,A^n(iF), \tag{3.5}$$

$$A(F)=\#\,A(iF), \tag{3.6}$$

$$A_N(F)=\#\,A_N(iF) \tag{3.7}$$

where the right hand sides of the equations were defined in (2.2)–(2.5).

More explicitly,

$$\begin{aligned} A(F)_q &= \#\,A(iF)_q=\#\prod_{I\in\mathcal{I}_q} A_I(iF) \qquad \text{(by (2.7))}\\ &= \prod_{I\in\mathcal{I}_q}\#\,\mathbf{Z}[iF\times\cdots\times iF]=\prod_{I\in\mathcal{I}_q}\#\,\mathbf{Z}[i(F\times\cdots\times F)]\\ &= \prod_{I\in\mathcal{I}_q}\mathbf{Z}[F\times\cdots\times F].\end{aligned}$$

If we define, in analogy with (2.8)

$$A_I(F) = \mathbf{Z}[F \times \cdots \times F] \qquad (\#I+1\text{-fold product}) \tag{3.8}$$

one gets

$$A(F)_q = \prod_{I \in \mathscr{I}_q} A_I(F); \tag{3.9}$$

similarly

$$A_N(F)_q = \prod_{I \in \mathscr{I}_q} A_{N,I}(F)$$

where $A_{N,I}(F)$ is the quotient in $\mathscr{S}$ of $A_I(F)$ by the images of morphisms e_i defined exactly as in (2.10).

One can also write $A_I(F) = \mathbf{Z}[F \underset{k_1}{|} \ldots \underset{k_r}{|} F]$, where $I = \{k_1, \ldots, k_r\}$. The boundary map is defined as for abelian groups.

Since $\#$ is an exact functor, it commutes with homology. Thus

$$H_q(A(F,r)) = H_q(\#(A(iF,r)) = \#H_q A(iF,r)). \tag{3.10}$$

The computation of $H_q(A(F,r))$ is thus reduced to "taking the associated sheaf of Theorem 2", applied to the presheaf iF. Specifically, for any F in $\mathscr{S}$, and any p-admissible sequence I of first type (resp. of second type), there is a morphism of sheaves θ_I^p:

$$\theta_I^p\colon H_{n+q}(A(F,r);\mathbf{Z}) \to F_p \qquad (\text{resp.} \to {}_pF) \tag{3.11}$$

where F_p (resp. ${}_pF$) is the cokernel (resp. the kernel) in $\mathscr{S}$ of the pth power map on F: one just takes $\theta_I^p = \#\theta_I^p$, where θ_I^p was the morphism defined in (2.12)–(2.13) for the presheaf $P = iF$. By definition, θ_I^p lands in $\#_p iF$ (resp. $\#\mathrm{i}F_p$). In order to be able to write (3.11), it is necessary to note that

$$\#(iF)_p = F_p \quad \text{and} \quad \#_p(iF) = {}_pF.$$

We note state

Theorem 3. *Let $q \geqq 1, n > q$; for any F in $\mathscr{S}$ the following is true: the sheaf $H_n(F,n;\mathbf{Z})$ is isomorphic to F. $H_{n+q}(F,n;\mathbf{Z})$ is a torsion sheaf. Let $L_q(F,p)$ be its p-primary component. For any p-admissible sequence I of first type (resp. of second type) of degree q with the additional condition $a_1 \equiv 0 \bmod (2p-2)$, θ_I^p sends $L_q(F,p)$ onto F_p (resp. ${}_pF$). Let N_I^p be the intersection of the kernels of the θ_J^p, for all such sequences J of either type, with $J \neq I$. Then $L_q(F,p)$ is the direct sum of the N_I^p and θ_I^p is an isomorphism of N_I^p on F_p (resp. ${}_pF$).*

The proof of the theorem is clear, using (3.10) and Theorem 2. Also we have

$$H_{n+q-1}(F,n) \approx H_q(A(F)) \approx H_q(A_N(F));$$

indeed, there is a chain equivalence

$$A(F) \underset{g}{\overset{f}{\rightleftarrows}} A_N(F) \tag{3.12}$$

induced by (2.17).

§ 4. The Associated Spectral Sequence

We construct the spectral sequence alluded to in the introduction. Let $X=(X_q, d_q)$ be a complex in $\mathscr{S}$ and G an object of $\mathscr{S}$. Since $\mathscr{S}$ has enough injectives, we can choose an injective resolution $I=(I^q, \partial^q)$ of G:

$$G \xrightarrow{\varepsilon} I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} I^2 \rightarrow \cdots.$$

Consider the double complex $K=\sum_{p,q\geqq 0} K^{p,q}$, where

$$K^{p,q}=\mathrm{Hom}(X_p, I^q) \tag{4.1}$$

with the differential

$$d'\colon\ K^{p,q} \rightarrow K^{p+1,q} \tag{4.2}$$

$$(\text{resp. } d''\colon\ K^{p,q} \rightarrow K^{p,q+1})$$

induced by d_{p+1} (resp. $(-1)^p\, \partial^q$).

View K as a simple complex (K_n, d_n) by writing

$$K_n=\sum_{p+q=n} K^{p,q}; \qquad d=d'+d''. \tag{4.3}$$

There is a spectral sequence associated to this double complex (see for example [14] Chapter 11 § 6). There are two filtrations

$$'E_2^{p,q}={}'H^p({}''H^q(K)), \tag{4.4}$$

$$''E_2^{p,q}={}''H^p({}'H^q(K)) \tag{4.5}$$

and their common abutment is the cohomology $H^{p+q}(K_*)$ of the total complex. In our case

$$''H^q(K^{i*})=H^q(\mathrm{Hom}(X_i, I^*))=\mathrm{Ext}^q(X_i, G) \tag{4.6}$$

since Ext^q is by definition the qth derived functor of Hom and I^* is an injective resolution of G.

Thus, by (4.4)

$$'E_2^{p,q}=H^p(n\mapsto \mathrm{Ext}^q(X_n, G)). \tag{4.7}$$

Similarly

$$'H^q(K^{*j})=H^q(\mathrm{Hom}(X_*, I^j)).$$

Since I^j is injective, the functor $G \to \operatorname{Hom}(G, I^j)$ is by definition exact. It therefore commutes with homology, so

$$'H^q(K^{*j}) = \operatorname{Hom}(H_q(X), I^*).$$

Hence

$$''E_2^{p,q} = H^p(\operatorname{Hom}(H_q(X)), I^*) = \operatorname{Ext}^p(H_q(X), G). \tag{4.8}$$

To sum up, we have described a spectral sequence with $'E_2^{p,q}$ and $''E_2^{p,q}$ as in (4.7) and (4.8), and with abutment

$$H(K_*) = H\Big(\sum_{p+q=n} \operatorname{Hom}(X_p, I^q)\Big).$$

In general, this spectral sequence does not degenerate for either of the two filtrations. It can be thought of as the natural generalization to our context of the universal coefficient theorem of the topologists.

Note. One obtains a similar spectral sequence from the double complex

$$K^{p,q} = \mathscr{H}om(X_p, I^q) \tag{4.9}$$

(where $\mathscr{H}om$ is the associated sheaf of Hom for some topology). The whole argument given above carries through without change. In order to get the analogue of (4.8), one needs to know that the functor from $\mathscr{S}$ to $\mathscr{S}$:

$$G \to \mathscr{H}om(G, I)$$

is exact, for any injective sheaf I. This is proved in [2] I Exposé V, Proposition 3.10.

In this case

$$'E_2^{p,q} = H^p(n \mapsto \mathscr{E}xt^q(X_n, G)), \tag{4.10}$$

$$''E_2^{p,q} = \mathscr{E}xt^p(H_q(X), G). \tag{4.11}$$

§ 5. *A Special Case*

We look at the spectral sequence of § 4, when $X = A_N(F)$, for some F in $\mathscr{S}$. By (4.7)–(4.8)

$$'E_2^{p,q} = H^p(n \mapsto \operatorname{Ext}^q(A_N(F)_n, G)), \tag{5.1}$$

$$''E_2^{p,q} = \operatorname{Ext}^p(H_q(A_N(F)), G). \tag{5.2}$$

In order to compute $'E_2^{p,q}$, it is necessary to make the following additional assumptions, which will hold for all of this paragraph:

1. We choose for the topology T the *fppf* topology on an arbitrary base scheme S: $\operatorname{Cat}(T) = \{\text{finitely presented schemes}/S\}$; a fundamental system of covering families of some X in $\operatorname{Cat}(T)$ is given by surjective families of flat morphisms, locally of finite presentation $\{X_i \to X\}$.

2. The sheaf F is representable in Cat(T), i.e. there exists a finitely presented commutative group scheme $h: M \to S$, with

$$F \approx h_M.$$

These are the most convenient hypotheses for the applications we have in mind, but the computations of the present paragraph will work just as well for any other choice of a topology T, so long as hypothesis 2. is correspondingly modified.

Lemma 1. *Let T, F, M be as above; then for any $q \geqq 0$, and any G in $\mathscr{S}$,*

$$\operatorname{Ext}^q(\mathbf{Z}[F], G) \approx H^q_{fppf}(M, G). \tag{5.3}$$

Proof. By definition, $G \to \operatorname{Ext}^q(\mathbf{Z}[F], G)$ is the qth derived functor of $G \to \operatorname{Hom}(\mathbf{Z}[F], G)$. Similarly $G \to H^q_{fppf}(M, G)$ is the qth derived functor of $G \to H^0(M, G)$. We are thus reduced to checking (5.3) in the case $q=0$. Now, by (3.2),

$$\operatorname{Hom}_{\mathscr{S}}(\mathbf{Z}[F], G) \approx \operatorname{Hom}_{\mathscr{T}}(F, jG). \tag{5.4}$$

The well known (and trivial) Yoneda lemma states that for any F in $\mathscr{T}$ representable ($F \approx h_M$), and any H in $\mathscr{T}$,

$$\operatorname{Hom}_{\mathscr{T}}(h_M, H) \approx H(M) = H^0(M, H). \tag{5.5}$$

Putting (5.4) and (5.5) together, we get (5.3) for $q=0$, and hence for all q.

By (3.8), (3.9), (5.3), and the fact that a product of representable sheaves is represented by the product of the corresponding schemes, one has

$$\operatorname{Ext}^i(A(F)_q, G) = \coprod_{I \in \mathscr{I}_q} \operatorname{Ext}^i(A_I(F), G) = \coprod_{I \in \mathscr{I}_q} H^i(M^{\# I+1}, G) \tag{5.6}$$

where for any positive integer s, we write

$$M^s = M \underset{S}{\times} M \underset{S}{\times} \cdots \underset{S}{\times} M \qquad (s\text{-fold product}).$$

For each q, we have, by definition of $A_N(F)$, the split exact sequence

$$0 \to D(A(F))_q \to A(F)_q \underset{g}{\overset{f}{\rightleftarrows}} A_N(F)_q \to 0 \tag{5.7}$$

where f and g are as in (3.12), and $D(A(F))_q$ is the "sheaf of degenerate q-cells". Since (5.7) splits, it induces for each i a short exact sequence

$$0 \to \operatorname{Ext}^i(A_N(F)_q, G) \to \operatorname{Ext}^i(A(F)_q, G) \to \operatorname{Ext}^i(D(A(F))_q, G) \to 0. \tag{5.8}$$

Define $\# I+1$ morphisms $e_j: M^{\# I} \to M^{\# I+1}$:

$$e_j\colon\ M^{\# I} \approx M \underset{S}{\times} \cdots M \underset{S}{\times} S \underset{S}{\times} M \underset{S}{\times} \cdots \underset{S}{\times} M \xrightarrow{1 \underset{S}{\times} \cdots \underset{S}{\times} \varepsilon \underset{S}{\times} 1 \cdots \underset{S}{\times} 1} M^{\# I+1}$$

where ε is the zero section morphism of the group scheme M. These morphisms induce homomorphisms, also denoted e_j, on the cohomology:

$$e_j\colon\ H^i(M^{\#I+1}, G) \to H^i(M^{\#I}, G),$$

and hence a homomorphism

$$e_I = (e_1, \ldots, e_{\#I+1})\colon\ H^i(M^{\#I+1}, G) \to H^i(M^{\#I}, G)^{\#I+1}.$$

We write $H^i_N(M^{\#I+1}, G) = \ker e_I = \bigcap_j \ker e_j$. (5.8) now becomes, with the help of (5.3)

$$\begin{aligned} &0 \to \operatorname{Ext}^i\big(A_N(F)_q, G\big) \\ &\to \coprod_{I\in\mathscr{I}_q} H^i(M^{\#I+1}, G) \xrightarrow{\coprod_I e_I} \coprod_{I\in\mathscr{I}_q} H^i(M^{\#I}, G)^{\#I+1} \to 0 \end{aligned} \tag{5.9}$$

in other words

$$\operatorname{Ext}^i\big(A_n(F)_q, G\big) = \coprod_{I\in\mathscr{I}_q} H^i_N(M^{\#I+1}, G).$$

The boundary morphism $d_{q+1}\colon A(F)_{q+1} \to A(F)_q$ induces a coboundary

$$d^{q,i}\colon\ \operatorname{Ext}^i\big(A_N(F)_q, G\big) \to \operatorname{Ext}^i\big(A_N(F)_{q+1}, G\big).$$

This we do not write down precisely; it is precisely obtained from the boundary operator of the complex $A(G)$ (G an abelian group) via the following translation: a boundary operator in $A(G)$ is the sum of maps such as

$$r_j\colon\ [x_1 \underset{k_1}{|} \ldots x_j \underset{1}{|} x_{j+1} \ldots \underset{k_{r-1}}{|}\ x_r] \to [x_1 \underset{k_1}{|} \ldots x_j + x_{j+1} \ldots \underset{k_{r-1}}{|}\ x_r], \tag{5.10}$$

projections, and shuffles:

$$s_\pi\colon\ [x_1 \underset{k_1}{|} \ldots \underset{k_{r'-1}}{|}\ x_{r'}] \to [x_{\pi(1)} \underset{k_1}{|} \ldots \underset{k_{r'-1}}{|}\ x_{\pi(r')}] \tag{5.11}$$

where π is a certain permutation of $(1, \ldots, r')$. To the operator r_j corresponds the homomorphism induced on the ith cohomology group by the morphism of schemes

$$r_j\colon\ M^{\#I+1} \to M^{\#I}$$

$$r_j = 1 \underset{S}{\times} \cdots \underset{S}{\times} m \underset{S}{\times} \cdots \underset{S}{\times} 1$$

where the jth factor m of r_j is the multiplication morphism of the group scheme M. s_π corresponds to the homomorphism induced on the ith cohomology group by the morphism $s_\pi\colon M^{\#I+1} \to M^{\#I+1}$ permuting the factors as in (5.11). Taking the alternating sum of these various homomorphisms in the manner prescribed by the bar construction, one obtains the coboundary

$$d^{q,i}\colon \coprod_{I\in\mathscr{I}_q} H^i(M^{\#I+1},G)\to \coprod_{I'\in\mathscr{I}_{q+1}} H^i(M^{\#I'+1},G).$$

Restricting the operator $d^{q,i}$ to $\coprod_{I\in\mathscr{I}_q} H^i_N(M^{\#I+1},G)$, we finally obtain a complex which we call $X(M,G;i)$:

$$\cdots\xrightarrow{d^{q-1,i}}\coprod_{I\in\mathscr{I}_q} H^i_N(M^{\#I+1},G)\xrightarrow{d^{q,i}}\coprod_{I'\in\mathscr{I}_{q+1}} H^i_N(M^{\#I'+1},G)\xrightarrow{d^{q+1,i}}\cdots. \tag{5.12}$$

The pth cohomology group of the complex $X(M,G;q)$ is $'E_2^{p,q}$.

One computes in the same way the $'E_2^{p,q}$ term of the spectral sequence (4.9) when F is representable, using (4.11). It is sufficient, as above, to compute $\mathscr{E}xt^p(\mathbf{Z}[F],G)$ when F is represented by $h\colon M\to S$.

$$T\to\mathscr{E}xt^p(\mathbf{Z}[F],G)(T)$$

is the associated sheaf of the presheaf $T\to \mathrm{Ext}^p_T(\mathbf{Z}[F]|T,G|T)$ where $G|T$ denotes the restriction of the sheaf G to (Sch/T) (see [2] Exposé V, Proposition 4.1). Now, by Lemma 1 of § 5,

$$\mathrm{Ext}^p_T(\mathbf{Z}[F]|T,G|T)=\mathrm{Ext}^p_T(\mathbf{Z}[F|T],G|T)=H^p(M\times_S T,G).$$

$T\to\mathscr{E}xt^p(\mathbf{Z}[F],G)(T)$ is thus the associated sheaf of the presheaf $T\to H^p(M\times_S T,G)$. In other words

$$\mathscr{E}xt^p(\mathbf{Z}[F],G)\approx R^p h_* G.$$

Using this, the computation of (4.10) is easy: let $h^s\colon M^s\to S$ be the structure morphism. There are $\#I+1$ obvious morphisms

$$e_i\colon R^q h_*^{\#I+1}\to R^q h_*^{\#I}.$$

Define $R^q_N h_*^{\#I+1}=\bigcap_i \ker(e_i)$. One gets as above a complex (this time in $\mathscr{S}$), called $\mathscr{X}(M,G;q)$:

$$\cdots\to\coprod_{I\in\mathscr{I}_q} R^q_N h_*^{\#I+1}G\to\coprod_{I'\in\mathscr{I}_{q+1}} R^q_N h_*^{\#I'+1}G\to\cdots. \tag{5.13}$$

This complex is in fact nothing else than the associated sheaf of the complex $X(M,G;q)$ of (5.12). The pth cohomology sheaf of the complex of sheaves $\mathscr{X}(M,G;q)$ is the sheaf $'E_2^{p,q}$ of (4.10).

Chapter II

§ 6. The General Method

In the following applications of the spectral sequences of Chapter I we will always use the following technique: since $H_1(A_N(F))$ is isomorphic to F, we have

$$''E_2^{p,1}=\mathrm{Ext}^p(F,G)$$

in the spectral sequence of Chapter I § 4 corresponding to the abelian sheaves F and G. Also, by Theorem 3 $H_s(A_N(F))$ is a torsion group for $s \neq 1$, and hence by (5.2) the groups $''E_2^{r,s}$ are torsion for $s \neq 1$ and any r. Since the spectral sequence considered is first quadrant and hence converges, it suffices, in order to show that $''E_2^{p,1}$ is a torsion group, to check that the abutment $H^{p+1}(K_*)$ is a torsion group. But this last group can be calculated by the use of the $'E$ filtration: it is clearly sufficient to verify that $'E_2^{r,s}$ is a torsion group for all r, s satisfying the condition $r+s=p+1$. $'E_2^{r,s}$ is calculated (for F representable) by (5.12). These computations, which follow in various special cases, are non-formal and rely on the cohomological properties of the scheme representing F.

It is clear that the method just outlined will also work for $\mathscr{E}xt^p(F, G)$ if one uses the second spectral sequence of § 4 instead of the first one; one need just replace the word group by the word sheaf everywhere in the above discussion, and the reference to (5.12) by (5.13).

§ 7. $\mathrm{Ext}^i(A, G_m)$

Let A be an abelian scheme over a regular noetherian base scheme S, $h: A \to S$ being the structure morphism. G_{mS} is the usual multiplicative group scheme over S: $G_{mS} = \mathrm{Spec}\, S[t, t^{-1}]$.

It is trivial that $\mathrm{Hom}_S(A, G_{mS}) = 0$, since $h_* \mathcal{O}_A = \mathcal{O}_S$ and G_{mS} is affine over S. A well known theorem is that the following $fppf$ sheaves are isomorphic:

$$\mathscr{E}xt_S^1(A, G_{mS}) \approx \mathscr{P}ic_{A/S}^{\tau}; \tag{7.1}$$

for a proof of this, see [19] VII.16 Theorem 6 for the case $S = \mathrm{Spec}(k)$ and [18] Theorem 18.1 for the general case. These two results are of course quite independent of the regularity assumption above.

We now show that under our hypotheses, $\mathrm{Ext}_S^i(A, G_{mS})$ is a torsion group for all $i \geqq 2$. As pointed out in § 6, it suffices to check that the corresponding $'E_2^{p,q}$ is a torsion group for $p+q \geqq 3$. The cases $q=0$, $q=1$ and $q \geqq 2$ are treated separately.

1. $q=0$.

In the notation of (5.12), $'E_2^{p,0}$ is the pth cohomology group of the complex $X(A, G_m; 0)$. In this case the much stronger statement that every cochain of the complex is trivial holds: for any integer s, $h_*^s \mathcal{O}_{A^s} = \mathcal{O}_S$, and thus, since G_{mS} is affine over S,

$$H^0(A^s, G_{mS}) \approx H^0(S, G_{mS}).$$

With the notation of § 5, all the homomorphisms e_j are now the identity map, and hence

$$H_N^0(A^s, G_{mS}) = \bigcap_j \ker e_j = 0;$$

since the cochains are sums of such elements, this shows they are all trivial.

2. $q=1$.

The complex $X(A, G_{mS}; 1)$ has the form

$$\cdots \to \coprod_{I\in\mathcal{I}_n} H^1_N(A^{\#I+1}, G_{mS}) \xrightarrow{d^{n,1}} \coprod_{I'\in\mathcal{I}_{n+1}} H^1_N(A^{\#I'+1}, G_{mS}) \to \cdots$$

with

$$H^1_N(A^s, G_{mS}) = \bigcap_i \ker e_i$$

where

$$e_i\colon \operatorname{Pic}(A^s) \to \operatorname{Pic}(A^{s-1})$$

is the map pulling the s-hypercube A^s to an $s-1$ hyperface via the ith zero section. The theorem of the cube is precisely the statement that $H^1_N(A^3, G_{mS})=0$ (see for example [16]). An easy consequence of this theorem is that $H^1_N(A^s, G_{mS})=0$ for any $s \geqq 3$. The complex $X(A, G_{mS}; 1)$ thus simplifies to a complex $X'(A, G_{mS}; 1)$ whose nth term is the sum of the groups $H^1_N(A^{\#I+1}, G_{mS})$, the summation running over all $I\in\mathcal{I}_n$ subject to the additional condition $\#I+1 \leqq 2$. For any given $n>1$, there is only one such I in $\mathcal{I}_n$: it is $I=(n-1)$. $X'(A, G_{mS}; 1)$ has the form

$$\begin{aligned} 0 \to H^1_N(A, G_m) &\xrightarrow{d^{1,1}} H^1_N(A\times A, G_m) \to \cdots \\ \cdots \to H^1_N(A\times A, G_m) &\xrightarrow{d^{n,1}} H^1_N(A\times A, G_m) \ldots \end{aligned} \tag{7.2}$$

the boundaries $d^{n,1}$ are induced from the boundary on the relevant cells of $A(F)$. The corresponding boundary maps on $A(G)$ (G an abelian group) are

$$\mathbf{Z}[G \underset{n}{|} G] \to \mathbf{Z}[G \underset{n-1}{|} G] \to \cdots \to \mathbf{Z}[G \underset{1}{|} G] \to \mathbf{Z}[G]$$

$$\partial[x \underset{i}{|} y] = [x \underset{i-1}{|} y] + (-1)^{i-1}[y \underset{i-1}{|} x], \qquad i>1,\ x, y\in G, \tag{7.3}$$

$$\partial[x \underset{1}{|} y] = [x]+[y]-[x+y]. \tag{7.4}$$

Note that

$$H^1_N(A, G_{mS}) = \operatorname{Pic}(A)/\operatorname{Pic}(S).$$

Also

$$H^1_N(A \underset{S}{\times} A, G_{mS}) = \ker\big(\operatorname{Pic}(A \underset{S}{\times} A) \xrightarrow{(e_1, e_2)} \operatorname{Pic}(A)\times \operatorname{Pic}(A)\big)$$

so it is precisely the group $\operatorname{Div}\operatorname{Corr}_S(A, A)$ of divisorial correspondences on A. It is now clear from (7.4) and the translation indicated in § 5, that the coboundary $d^{1,1}$ of (7.2) is:

$$\begin{aligned} d^{1,1} = \rho\colon\ & \operatorname{Pic}(A)/\operatorname{Pic}(S) \to \operatorname{Div}\operatorname{Corr}_S(A, A) \\ d^{1,1}(x) = & -m^*(x)+p_1^*(x)+p_2^*(x). \end{aligned}$$

Similarly, one sees that $d^{i,1}$: Div $\mathrm{Corr}_S(A, A) \to$ Div $\mathrm{Corr}_S(A, A)$ is:

$$d^{i,1}(x) = x + (-1)^{i+1} s^*(x) \tag{7.5}$$

where s^* is the homomorphisms induced on Pic by the morphisms $s\colon A \underset{S}{\times} A \to A \underset{S}{\times} A$ permuting the factors $(i > 1)$.

We examine the cohomology of $X'(A, G_{mS}; 1)$. The first cohomology group, which is of no interest for our purposes, is

$$\ker(d^{1,1}) = \mathrm{Pic}^{\tau}(A)/\mathrm{Pic}^{\tau}(S)$$

(this is in fact the proof of (7.1)).

We claim $'E_2^{p,1}$ is a torsion group for $p+1 \geqq 3$. This is equivalent to showing that p-th cohomology group is torsion for $p \geqq 2$ in (7.2). In fact, we show it is 2-torsion: for $p \geqq 3$, this is clear from the description (7.5) of the boundary and the formula

$$2x = \big(x + s^*(x)\big) + \big(x - s^*(x)\big)$$

for all x in Div $\mathrm{Corr}_S(A, A)$.

The second cohomology group is the group of symmetric divisorial correspondences, modulo $\rho\big(\mathrm{Pic}(A)\big)$. This is well known to be at most a 2-torsion group: let Δ^*: $\mathrm{Pic}\big(A \underset{S}{\times} A\big) \to \mathrm{Pic}(A)$ be the homomorphism induced by the diagonal morphism. Then, for any symmetric x in Div $\mathrm{Corr}_S(A, A)$,

$$\rho\,\Delta^* x = 2x$$

(see for example [16]).

3. $q \geqq 2$.

In this case $X(A, G_{mS}; q)$ has the usual form (5.12). It clearly suffices to show that, for any integer $s > 0$, the group $H^q_N(A^s, G_{mS})$ is a torsion group. Even better, we show that $H^q(A^s, G_{mS})$ is a torsion group: since G_{mS} is a smooth group scheme, we have by [12] appendix, Theorem 11.7

$$H^q_{fppf}(A^s, G_{mS}) = H^q_{\mathrm{et}}(A^s, G_{mS}).$$

Since S is by hypothesis noetherian and regular, and A^s is smooth over S, A^s is a noetherian regular scheme. But [11] Proposition 1.4 states that, for any Noetherian regular scheme X, $H^q_{\mathrm{et}}(X, G_m)$ is a torsion group, whenever $q \geqq 2$. This finishes the proof of case 3) and with it the proof that $\mathrm{Ext}^i(A, G_{mS})$ is a torsion group for $i \geqq 2$.

We now sketch the proof of a related result, thus partly answering a question of Artin and Mazur ([3], p. 17). Let $X = \mathrm{Spec}(R)$, where R is a discrete valuation ring with function field K and residue class field k. Let A be an abelian variety defined over $\mathrm{Spec}(K)$. There exists a smooth commutative group scheme over X, defined in a canonical manner, which is written $N(A)$ (the Neron minimal model of A). Its generic fibre is A, and its special fibre can be quite complicated. Also, there is a unique subgroup scheme of $N(A)$, also defined over X, which has same generic fibre as $N(A)$, but whose special fibre is the connected component of the special fibre of $N(A)$. We call this subgroup scheme $N^c(A)$, and have the following exact sequence of $fppf$ sheaves over X:

$$0 \to N^c(A) \to N(A) \to F \to 0. \tag{7.6}$$

The sheaf F is not representable. It is a torsion sheaf, and this is all we will need to know about it.

We claim that $\mathrm{Ext}^i_R(N(A), G_{mR})$ is a torsion group for any $i \geqq 2$. The long exact sequence of Ext's derived from (7.6), together with the fact that F is torsion, implies immediately that this is equivalent to showing that $\mathrm{Ext}^i_R(N^c(A), G_{mR})$ is a torsion group.

To show this we use the spectral sequence of § 5: $'E_2^{p,q}$ is a torsion group for $q \geqq 2$, just as in the abelian scheme case, since $N^c(A)$ is smooth over X and hence regular. $'E_2^{p,0} = 0$, since the global sections of $N^c(A)$ are constant (as they are contained in the global sections of the generic fibre). Finally, the injection of the generic fibre A into $N^c(A)$ induces an isomorphism

$$i^*: \mathrm{Pic}(N^c(A)) \xrightarrow{\approx} \mathrm{Pic}(A)$$

and this reduces the computation of the groups $'E_2^{p,1}$ to the corresponding computation with $N^c(A)$ replaced by the abelian variety A/K which we made earlier in this paragraph.

§ 8. $\mathrm{Ext}^i(G_a, G_m)$ *and* $\mathrm{Ext}^i(G_m, G_m)$

The proofs here proceed in exactly the same manner as in § 7, and are in fact easier. For simplicity we assume $S = \mathrm{Spec}(k)$, but this is clearly not indispensable; one could take S to be any regular scheme and consider the group schemes G_{aS} and G_{mS}.

We prove that the groups $\mathrm{Ext}^i(G_a, G_m)$ are torsion groups for all integers i (this is of course non trivial only in characteristic 0, since G_a is p-torsion in characteristic p), and that the groups $\mathrm{Ext}^i(G_m, G_m)$ are torsion groups for $i \geqq 1$. The cases $i=0$ and $i=1$ are of course well known.

It suffices to check that the corresponding groups $'E_2^{p,q}$ are torsion. For $q \geqq 2$, this is (as in § 7) because the smoothness of G_a and G_m insures that, for all integers s

$$H^q(G_m^s, G_m) \quad \text{and} \quad H^q(G_a^s, G_m)$$

are torsion groups ([11], Proposition 1.4). $'E_2^{p,1} = 0$ since

$$\mathrm{Pic}(G_m^s) = \mathrm{Pic}(G_a^s) = 0$$

as G_m^s and G_a^s are affine schemes whose rings are *UFD*'s. We look at $'E_2^{p,0}: H^0(G_m^s, G_m)$ is equal to the group of invertible elements in the ring $k[x_1, \ldots, x_s, x_1^{-1}, \ldots, x_s^{-1}]$. The only such elements are the monomials $a x_1^{n_1} \ldots x_s^{n_s}$ where the n_i are not necessarily positive integers. The normalization condition is that this monomial must be identically equal to 1 whenever one of the indeterminates x_i is replaced by 1. Thus

$$H_N^0(G_m^s, G_m) = 0 \qquad \text{for } s > 1.$$

Also $H^0(G_a^s, G_m) = k^*$, and so $H_N^0(G_a^s, G_m) = 0$.

§ 9. $\mathrm{Ext}^i(A, B)$, $\mathrm{Ext}^i(G_a, B)$, $\mathrm{Ext}^i(G_m, B)$

Let A and B be abelian varieties over a field k. We show that $\mathrm{Ext}^i(A, B)$ for $i \geqq 1$, and $\mathrm{Ext}^i(G_m, B)$ and $\mathrm{Ext}^i(G_a, B)$ for all i, are torsion groups. This is of course trivial for the last group in characteristic $p \neq 0$.

We show that $'E_2^{p,0} = 0$ in all three cases. In fact

$$H_N^0(A^s, B) = 0 \qquad \text{for } s > 1.$$

$$H_N^0(G_m^s, B) = 0 \qquad \text{for all } s.$$

$$H_N^0(G_a^s, B) = 0 \qquad \text{for all } s.$$

For the first group, this follows directly from the rigidity lemma ([15] Chapter 6 Corollary 6.3) which states that a morphism from a product of abelian schemes to a group scheme is the sum of morphisms defined on each factor. For the two other groups, this is even simpler: every morphism from a rational variety to an abelian variety is constant ([13] Chapter II Theorem 4 corollary).

Also $'E_2^{p,q}$ is a torsion group for $q \geqq 1$ in all three cases: more generally, let X be a noetherian regular connected scheme over a field k, then $H^i(X, B)$ is a torsion group for $i \geqq 1$: as in § 7 and § 8, B smooth implies that

$$H^i(X, B) = H_{\mathrm{et}}^i(X, B).$$

Let i: $\mathrm{Spec}(K) \to X$ be the generic point of X.

The canonical morphism of étale sheaves over X:

$$B \to i_*(B_K)$$

is an isomorphism of sheaves: it is injective since every étale extension of X is regular, so a fortiori integral; it is surjective since every morphism from an open set of a regular variety to an abelian variety can be extended to a morphism from the whole variety to the abelian variety ([13] Chapter II, Theorem 2). Thus

$$H^q(X, B) \approx H^q(X, i_* B_K).$$

But this last group is torsion by [11] Lemma 1.1.

§ 10. Remarks

The results of the previous paragraphs can be somewhat improved; associated to the short exact sequences

$$0 \to {}_nA \to A \xrightarrow{n} A \to 0$$

$$0 \to \mu_n \to G_m \xrightarrow{n} G_m \to 0$$

where A is an abelian variety, one has for every abelian sheaf G long exact sequences

$$\cdots \to \operatorname{Ext}^{i-1}({}_nA, G) \to \operatorname{Ext}^i(A, G) \xrightarrow{n} \operatorname{Ext}^i(A, G) \to \cdots \qquad (10.1)$$

$$\cdots \to \operatorname{Ext}^{i-1}(\mu_n, G) \to \operatorname{Ext}^i(G_m, G) \xrightarrow{n} \operatorname{Ext}^i(G_m, G) \to \cdots. \qquad (10.2)$$

Whenever G is G_m or an abelian variety B, we have seen that for $i \geqq 2$, $\operatorname{Ext}^i(A, G)$ (resp. $\operatorname{Ext}^i(G_m, G)$) is a torsion group. By (10.1) (resp. (10.2)), we see that its n-torsion part, for any integer n, lies in the image of $\operatorname{Ext}^{i-1}({}_nA, G)$ (resp. $\operatorname{Ext}^{i-1}(\mu_n, G)$). Whenever we can show that this last group is trivial, it is an immediate consequence that the corresponding n-torsion component of $\operatorname{Ext}^i(A, G)$ (resp. $\operatorname{Ext}^i(G_m, G)$) vanishes.

This can be seen in certain special cases: if $S = \operatorname{Spec}(k)$ where k is an algebraically closed field, then for n a prime different from the characteristic of k (or n any prime when $\operatorname{char}(k) = 0$), ${}_nA$ and μ_n take on simple forms

$${}_nA = (\mathbf{Z}/n)^{2 \dim A} \qquad \mu_n = \mathbf{Z}/n$$

where by $\mathbf{Z}/n$ we mean the finite group scheme representing the constant sheaf $\mathbf{Z}/n$. We are reduced to showing that, for $i \geqq 1$,

$$\operatorname{Ext}^i(\mathbf{Z}/n, G_m) = 0 \quad \text{and} \quad \operatorname{Ext}^i(\mathbf{Z}/n, B) = 0.$$

But the sheaf $\mathbf{Z}/n$ has the two term resolution

$$0 \to \mathbf{Z} \xrightarrow{n} \mathbf{Z} \to \mathbf{Z}/n \to 0.$$

This resolution is acyclic since, by Chapter I, § 5, Lemma 1, for $i \geqq 1$, $\operatorname{Ext}^i(\mathbf{Z}, G_m) = H^i(k, G_m) = 0$, and similarly with G_m replaced by B. This proves our assertion for $i > 1$. The case $i = 1$ can be checked directly.

Another special case is the case $i = 1$. It is then proved in [18] that, for H any finite group scheme over k,

$$\operatorname{Ext}^1_k(H, G_m) = 0$$

and, though this is computed in the category of group schemes, it implies the same result in the category of *fppf* sheaves, as was pointed out in the introduction, since G_m is affine.

It is not true that $\mathrm{Ext}_k^1(H, B)=0$ for any H. However, the following sequence (which is (10.1) in low dimensions) is exact:

$$\mathrm{Ext}^1(A, B) \xrightarrow{n} \mathrm{Ext}^1(A, B) \to \mathrm{Ext}^1({}_nA, B) \to 0$$

for all primes n (see [17]). Thus by the exactness of (10.1) $\mathrm{Ext}^2(A, B)$ has no n-torsion part for any n and is therefore trivial.

To sum up, the above arguments show that, for k an algebraically closed field, the groups $\mathrm{Ext}_k^i(A, G_m)$, $\mathrm{Ext}_k^i(G_m, G_m)$, $\mathrm{Ext}^i(G_m, B)$ and $\mathrm{Ext}^i(A, B)$ are at most p-torsion groups, where $p=\mathrm{char}(k)$, for $i \geqq 3$. They are trivial for $i=2$ (and for $i \geqq 2$ when k is of characteristic zero).

The vanishing of the group $\mathrm{Ext}^2(A, G_m)$ has an interesting geometric consequence; let us consider the low terms of the spectral sequence which we studied in detail in § 7. Since $'E_2^{0,2} = {}'E_2^{3,0} = 0$, we have the exact sequence

$$0 \to {}'E_2^{2,1} \to H^3(K_*). \tag{10.3}$$

We claim that $H^3(K)=0$. It is clearly sufficient to check that $''E_2^{3,0} = {}''E_2^{2,1} = {}''E_2^{1,2} = {}''E_2^{0,3} = 0$. Now

$''E_2^{3,0}=0$ trivially,

$''E_2^{2,1}=\mathrm{Ext}^2(A, G_m)=0$ as we have just seen,

$''E_2^{1,2}=0$ since $H_2(A_N(F))=0$ for any abelian sheaf F, and in particular when F is our abelian variety A.

$''E_2^{0,3}=0$ since, for any F, $H_3(A_N(F))=F_2$ and when F is an abelian variety A the 2th power morphism (in fact the nth power morphism, for any n) is an isogeny and hence has trivial cokernel.

Thus, by (10.3), we have shown that $'E_2^{2,1}=0$. We computed this group in § 7 and saw that it was the cokernel of the homomorphism

$$\rho: \ \mathrm{Pic}(A) \to \mathrm{Symm\ Div\ Corr}(A, A)$$

$$\rho(x) = p_1^*(x) + p_2^*(x) - m^*(x)$$

where the right hand term of course stands for the group of symmetric divisorial correspondences on A. The kernel of ρ is $\mathrm{Pic}^\tau(A)$ by the theorem of the square (see [16]). We have thus shown that ρ induces ρ: $NS(A) \approx \mathrm{Symm\ Div\ Corr}(A, A)$. For another proof, see [16] § 20 Theorem 2 and § 23 Theorem 3.

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Über Adams-Operationen I

Werner End (Heidelberg)

§ 1. Einleitung

Ist E ein komplexes Vektorraumbündel über X, so zerlegt der p-Zykel $g=(1\dots p)$ die Tensorpotenz $\bigotimes^p E$ in Eigenräume $\operatorname{Ker}(g-\zeta^j)$, $\zeta=\exp(2\pi i/p)$. Atiyah hat in [3] gezeigt, daß für eine Primzahl p die Adams-Operationen $\psi^p: K(X)\to K(X)$ durch $\psi^p[E]=[\operatorname{Ker}(g-1)]-[\operatorname{Ker}(g-\zeta)]$ gegeben sind. Mit dieser Beschreibung geben Atiyah in [3] und Dold (unveröffentlicht) neue elementare Beweise für die wichtigsten Eigenschaften der ψ^p. Weder Atiyah noch Dold verwenden dabei das tiefliegende Spaltungsprinzip.

Hat man für allgemeinere K-Theorien ähnliche Formeln?

Wir geben in dieser Arbeit an, wie eine solche Formel für die $KR_\Gamma(X)$-Theorie aussieht, verfeinern Dolds Methode und leiten die fundamentalen Eigenschaften der ψ^p her.

Die Theorie $KR_\Gamma(X)$ [Γ = kompakte topologische Gruppe, X = parakompakter $\Gamma\times\mathbf{Z}_2$-Raum] wird in §2 diskutiert. Sie umfaßt $K_\Gamma=KU_\Gamma$, KO_Γ und Atiyahs KR aus [4], ist aber ein Spezialfall von Karoubis $KR_{\hat\Gamma}$ aus [7], 1.2.5–1.2.6, die mit einer $\mathbf{Z}_2$-graduierten Gruppe $\hat\Gamma$ definiert wird [$\hat\Gamma$ nicht notwendig von der Form $\Gamma\times\mathbf{Z}_2$].

Wir beschreiben Operationen ψ^p in der $KR_\Gamma(X)$-Theorie: Zunächst sei p eine ungerade Primzahl. Wir gehen aus von einem „reellen" Vektorraumbündel E [= Objekt in der Kategorie $VR_\Gamma(X)$ vgl. § 2]. Die Permutationen

$$g=(1\ \dots\ p)\quad\text{und}\quad t=\begin{pmatrix}1 & \dots & p\\ -1 & \dots & -p\end{pmatrix}$$

[sie erzeugen die Diedergruppen D_{2p}] opertieren auf $\bigotimes^p E$. Wir konstruieren die „reellen" Unterbündel

$$A_j := \{\operatorname{Ker}(g-\zeta^j)\oplus\operatorname{Ker}(g-\zeta^{-j})\}\cap\operatorname{Ker}(t-1),\qquad j\not\equiv 0 \bmod p,$$
$$A_0 := \operatorname{Ker}(g-1).$$

Dann definieren wir $\psi^p[E]$ durch

$$\psi^p[E]=[A_0]-[A_1].$$

Diese Formel definiert auch ψ^2. In diesem Fall verstehen wir unter A_1 den Eigenraum $\mathrm{Ker}(g+1)$; $g=(12)$. Wir bemerken, daß A_j komplex isomorph zu $\mathrm{Ker}(g-\zeta^j)$ und zu $\mathrm{Ker}(g-\zeta^{-j})$ ist. Es hat aber keinen Sinn, nach einem „reellen" Isomorphismus zu fragen, weil $\mathrm{Ker}(g-\zeta^{\mp j})$ i.A. keine „reellen" Unterbündel von $\bigotimes^p E$ sind. In § 4 sehen wir den Grund, warum man zur Konstruktion von $\psi^p (p>2)$ in der $KR_\Gamma(X)$-Theorie nicht allein mit dem p-Zykel $(1 \ldots p)$ auskommt.

In § 3 stellen wir eine Liste von Eigenschaften dieser Operationen zusammen, ähnlich wie in [1], S. 611, Theorem 4.1. Wir werden unter anderem zeigen $\psi^p \psi^q = \psi^q \psi^p$ für zwei Primzahlen p und q und können dann nach Dold ψ^k durch die Formel $\psi^k = \psi^{p_1} \ldots \psi^{p_i}$ definieren, wenn $k = p_1 \ldots p_i$ eine Zerlegung von k in Primzahlen ist.

Wir schließen mit folgenden Bemerkungen, deren Beweise in [5] durchgeführt sind. Mit Atiyahs Methoden aus [3] kann man zeigen, daß das vorhin definierte ψ^k mit dem Newtonpolynom $Q_k(\lambda_1, \ldots, \lambda_k)$ übereinstimmt. Man hat noch eine bemerkenswerte Formel für $\psi^p[E]$, wobei p keine Primzahl mehr zu sein braucht.

$$\psi^p[E] = \sum_{j|p} \mu(p/j)\,[A_j]$$

$j|p$ bedeutet: j ist Teiler von p; μ ist die Möbiusfunktion und die A_j sind definiert wie im Primzahlfall, außer $A_{p/2}$ falls p gerade ist, womit dann der Eigenraum $\mathrm{Ker}(g+1)$ bezeichnet wird.

§ 2. Die KR_Γ-Theorie

Die KR_Γ-Theorie ist eine leichte Verallgemeinerung der KR-Theorie von Atiyah; vgl. [4].

Ein „reelles" Γ-Vektorraumbündel $E=(F, J)$ über dem $\Gamma \times \mathbf{Z}_2$-Raum X $\big(=$Objekt der Kategorie $VR_\Gamma(X)\big)$ besteht aus einem $\mathbf{R}$-$\Gamma \times \mathbf{Z}_2$-Vektorraumbündel F und einer komplexen Struktur $J\colon F \to F$, so daß Elemente von Γ komplex linear (d.h. verträglich mit J) operieren und das von Null verschiedene Element α von $\mathbf{Z}_2$ antilinear operiert (d.h. $\alpha J = -J\alpha$). $KR_\Gamma(X)$ bezeichnet die Grothendieckgruppe von $VR_\Gamma(X)$. Wir schreiben $X=(Y, \alpha)$, wobei Y ein Γ-Raum und $\alpha\colon Y \to Y$ eine Γ-äquivariante Involution ist. Wir nennen Operationen von α „reelle" Strukturen. Die Theorie KR_Γ umfaßt $KU_\Gamma = K_\Gamma$ and KO_Γ. Man hat; vgl. [4], S. 371 und 376; natürliche Isomorphismen

$$KO_\Gamma(Y) \cong KR_\Gamma(Y, \alpha = \mathrm{id}_Y) \quad \text{und} \quad KU_\Gamma(Y) \cong KR_\Gamma(Y \times S^{1,0}),$$

wobei $S^{1,0}$ die 0-Sphäre $S^0 = \{1, -1\}$ mit der Vertauschung beider Punkte als „reeller" Struktur bezeichnet.

Vergißt man die $\mathbf{Z}_2$-Struktur α von $E=(F,J)\in VR_\Gamma(X)$, so erhält man die Komplexifizierung $c\colon KR_\Gamma(Y,\alpha)\to KU_\Gamma(Y)$. c wird induziert von der $\Gamma\times\mathbf{Z}_2$ äquivarianten Abbildung

$$C\colon Y\times S^{1,0}\to X=(Y,\alpha),\qquad C(y,1)=y,\qquad C(y,-1)=\alpha y.$$

Es gibt 2 Möglichkeiten die Reellifizierung $r_O\colon KU_\Gamma(Y)\to KO_\Gamma(Y)$ zu verallgemeinern. Man erhält:

$$r\colon KU_\Gamma(Y)\to KR_\Gamma(X) \quad \text{und} \quad \rho\colon KR_\Gamma(X)\to KO_{\Gamma\times\mathbf{Z}_2}(X)$$

r ist in [4], S. 371 wie folgt definiert. Ist E ein komplexes Vektorraumbündel über Y [d.h. E Objekt von $VU_\Gamma(Y)$], so kann man auf $E\oplus\alpha^*\bar{E}$ eine kanonische antilineare Operation von α definieren und erhält ein Objekt $rE\in VR_\Gamma(X)$.

ρ wird induziert, indem man dem Objekt $E=(F,J)\in VR_\Gamma(X)$ das Objekt $F\in VO_{\Gamma\times\mathbf{Z}_2}(X)$ zuordnet.

Setzt man $X=(Y,\alpha=\mathrm{id}_Y)$ [resp. $X=Y\times S^{1,0}$], dann liefert r [resp. ρ] das alte r_O.

§ 3. Eigenschaften der ψ^p

In § 1 haben wir für Primzahlen p Operationen ψ^p beschrieben. Wir müssen vorsichtig sein, denn wir haben nur Operationen

$$\bar{\psi}^p\colon \overline{VR}_\Gamma(X)\to KR_\Gamma(X)$$

durch $\bar{\psi}^p(E)=[A_0]-[A_1]$ definiert. Dabei bezeichnet $\overline{VR}_\Gamma(X)$ den Halbring der Isomorphieklassen von Objekten aus $VR_\Gamma(X)$. Wir bezeichnen ein Objekt von $VR_\Gamma(X)$ und seine Isomorphieklasse in $\overline{VR}_\Gamma(X)$ mit dem gleichen Buchstaben.

Satz.

(3.1) *$\bar{\psi}^p$ induziert $\psi^p\colon KR_\Gamma(X)\to KR_\Gamma(X)$.*

(3.2) *ψ^p hängt in natürlicher Weise von Γ und X ab.*

(3.3) *ψ^p ist ein Ringhomomorphismus.*

(3.4) *$\psi^p\psi^q=\psi^q\psi^p$; p, q Primzahlen.*

(3.5) *$\psi^p(x)=x^p$, wenn x die Klasse eines Linienbündels ist.*

(3.6) *$\psi^p(x)\equiv x^p \bmod p$ für alle $x\in KR_\Gamma(X)$.*

(3.7) *ψ^p ist verträglich mit der Komplexifizierung c und den beiden Reellifizierungen r und ρ; d.h. die folgenden 3 Diagramme sind kommutativ:*

(3.7.1)
$$\begin{array}{ccc} KR_\Gamma(Y,\alpha) & \xrightarrow{c} & KU_\Gamma(Y) \\ \big\downarrow{\scriptstyle \psi^p} & & \big\downarrow{\scriptstyle \psi_U^p} \\ KR_\Gamma(Y,\alpha) & \xrightarrow{c} & KU_\Gamma(Y) \end{array}$$

(3.7.2)
$$\begin{array}{ccc} KU_\Gamma(Y) & \xrightarrow{r} & KR_\Gamma(Y,\alpha) \\ \big\downarrow{\scriptstyle \psi_U^p} & & \big\downarrow{\scriptstyle \psi^p} \\ KU_\Gamma(Y) & \xrightarrow{r} & KR_\Gamma(Y,\alpha) \end{array}$$

(3.7.3)
$$\begin{array}{ccc} KR_\Gamma(X) & \xrightarrow{\rho} & KO_{\Gamma\times \mathbf{Z}_2}(X) \\ \big\downarrow{\scriptstyle \psi^p} & & \big\downarrow{\scriptstyle \psi_O^p} \\ KR_\Gamma(X) & \xrightarrow{\rho} & KO_{\Gamma\times \mathbf{Z}_2}(X). \end{array}$$

Dabei ergeben sich ψ_U^p[1], ψ_O^p, *wenn man beachtet, daß* KU_Γ, $KO_{\Gamma\times \mathbf{Z}_2}$ *Spezialfälle von* KR_Γ *sind.*

(3.8) *Die Operation* ψ^p *im Charakterring* $R(\Gamma)$ *der kompakten topologischen Gruppe* Γ *ist durch* $(\psi^p\chi)(a)=\chi(a^p)$ *gegeben; dabei ist* χ *ein Charakter von* Γ *und* $a\in\Gamma$. ψ^p *bildet den Unterring* $RO(\Gamma)$ *in sich ab.*

(3.9) *Für jeden* $\Gamma\times\mathbf{Z}_2$*-Raum* X, *den wir als* $G\times\Gamma\times\mathbf{Z}_2$*-Raum auffassen, auf dem* G *trivial operiert, ist das folgende Diagramm kommutativ:*

$$\begin{array}{ccc} RO(G)\otimes KR_\Gamma(X) & \xrightarrow{m} & KR_{G\times\Gamma}(X) \\ \big\downarrow{\scriptstyle \psi^p\otimes\psi^p} & & \big\downarrow{\scriptstyle \psi^p} \\ RO(G)\otimes KR_\Gamma(X) & \xrightarrow{m} & KR_{G\times\Gamma}(X). \end{array}$$

Die Definition von m *geben wir in* §4.

§ 4. Gruppentheoretische Hintergründe

Dieser Abschnitt dient zur Erläuterung. Zum Verständnis der folgenden Paragraphen ist nur erforderlich, daß für die Diedergruppen D_{2p} die Bedingungen des Satzes (4.1) erfüllt sind und daß somit für $G=D_{2p}$ die Abbildung m aus Satz (4.2) bijektiv ist.

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[1] Man überlegt sich leicht, daß dieses ψ_U^p dann mit Atiyahs ψ^p; vgl. § 1 übereinstimmt.

(4.1) Sei G eine endliche Gruppe. Wir bezeichnen mit $\mathbf{R}(G)$ [$\mathbf{C}(G)$] die reelle [komplexe Gruppenalgebra] mit $RO(G)$ [$RU(G)$] den reellen [komplexen] Darstellungsring von G. Die Komplexifizierung $c: RU(G) \to RO(G)$ ist injektiv, weil $rc = 2\,\mathrm{id}$ und $RO(G)$ torsionsfrei ist; dabei ist $r: RU(G) \to RO(G)$ die Reellifizierung. Wir identifizieren $RU(G)$ mit dem Charakterring $R(G)$ und $RO(G)$ vermöge c mit einem Unterring von $R(G)$. Für jeden irreduziblen $\mathbf{R}(G)$-Modul W ist $F_W := \mathrm{Hom}_G(W, W)$ ein Schiefkörper endlichen Grades über $\mathbf{R}$ und somit nach einem Satz von Frobenius zu $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ isomorph. Für einen $\mathbf{R}(G)$-Modul E haben wir einen natürlichen Isomorphismus

$$\sum_{W \in \mathfrak{W}} \mathrm{Hom}_G(W, E) \underset{F_W}{\otimes} W \xrightarrow{\cong} E,$$

wobei W ein vollständiges Repräsentantensystem $\mathfrak{W}$ irreduzibler $\mathbf{R}(G)$-Moduln durchläuft. Diese Formel gilt auch für $\mathbf{R}$-G-Vektorraumbündel E, wobei G trivial auf der Basis operiert.

Ist V ein irreduzibler $\mathbf{C}(G)$-Modul, dann ist eine der folgenden Bedingungen erfüllt:

(1) $V \not\cong \overline{V}$.

(2) $V \cong \mathbf{C} \underset{\mathbf{R}}{\otimes} W$, wobei W ein irreduzibler $\mathbf{R}(G)$-Modul ist.

(3) V kommt von einem quaternionalen G-Modul her.

Ist χ der Charakter von V, dann ist der Schur-Index

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = 0, 1, -1,$$

wenn V (1), (2), (3) erfüllt. Einen Beweis hierfür findet man in [6], S. 190ff; man beachte, daß der

$$\text{Schur-Index} = \frac{1}{|G|} \sum_{g \in G} (\psi^2 \chi)(g)$$

ist.

Sind n_1, n_2, n_3 die Anzahlen der Isomorphieklassen, die (1), (2), (3) erfüllen, so gilt:

$$\mathrm{Rang}\, RO(G) = n_1 + n_2 + n_3; \qquad \mathrm{Rang}\, RU(G) = 2n_1 + n_2 + n_3$$

$$RU(G)/c\,RO(G) = \underbrace{\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}}_{n_1\text{-mal}} \oplus \underbrace{\mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2}_{n_2\text{-mal}}.$$

(4.1) **Satz.** *Folgende Aussagen über G sind äquivalent:*

(a) *$c: RO(G) \to RU(G)$ is bijektiv.*

(b) *Jeder irreduzible $\mathbf{C}(G)$-Modul ist zu $\mathbf{C} \underset{\mathbf{R}}{\otimes} W$ isomorph, wobei W ein irreduzibler $\mathbf{R}(G)$-Modul ist.*

(c) *Für jeden irreduziblen* $\mathbf{R}(G)$*-Modul ist* $F_W := \mathrm{Hom}_G(W, W) \cong \mathbf{R}$.

(d) *Der Schur-Index von jedem primitiven Charakter ist* 1.

Die Bedingungen dieses Satzes sind erfüllt für $\mathbf{Z}_2$, S_k, D_{2k} und nicht erfüllt für $\mathbf{Z}_k (k \geqq 3)$, für die Quaternionengruppe der Ordnung 8, für jede Gruppe ungerader Ordnung $\geqq 3$, für jede Gruppe, in der es ein Element gibt, das nicht zu seinem Inversen konjugiert ist.

Die Beweise hierfür sind sehr einfach und werden deshalb ausgelassen.

(4.2) Nach [3], S. 167 hat man einen natürlichen Isomorphismus $KU_G(Y) \cong RU(G) \otimes KU(Y)$; Y = trivialer G-Raum. Wir leiten ein analoges Resultat für die KR_Γ-Theorie her. Sei X ein $\Gamma \times \mathbf{Z}_2$ Raum, den wir als $G \times \Gamma \times \mathbf{Z}_2$-Raum auffassen. Wir interpretieren einen $\mathbf{R}(G)$-Modul als $\mathbf{C}(G)$-Modul V mit einer Konjugation α der Periode 2. Ist $E \in VR_\Gamma(X)$, so kann man $V \otimes E$ als Objekt von $VR_{G \times \Gamma}(X)$ auffassen. Man erhält eine natürliche Abbildung

$$m: RO(G) \otimes KR_\Gamma(X) \to KR_{G \times \Gamma}(X).$$

(4.2) **Satz.** *m ist bijektiv für jedes* Γ *und jeden* $\Gamma \times \mathbf{Z}_2$ *Raum X genau dann, wenn* $c: RO(G) \to RU(G)$ *bijektiv ist. Die zu m inverse Abbildung m' wird induziert von*

$$VR_{G \times \Gamma}(X) \ni A \mapsto \sum_{L \in \mathfrak{L}} L \otimes \mathrm{Hom}_G(L, A) \in RO(G) \otimes KR_\Gamma(X),$$

wobei $\mathfrak{L}$ *ein vollständiges Repräsentantensystem irreduzibler* $\mathbf{C}(G)$*-Moduln ist.*

Beweis. Ist m bijektiv für alle Γ und X, so speziell für Γ = triviale Gruppe, $X = S^{1,0}$. In diesem Fall ist m die Komplexifizierung $c: RO(G) \to RU(G)$.

Umgekehrt: Die Abbildung

$$a_A: \sum_{L \in \mathfrak{L}} L \otimes \mathrm{Hom}_G(L, A) \to A \quad a_A(v \otimes f) = f(v)$$

ist bijektiv für jedes $A \in VU_{G \times \Gamma}(X)$.

Ist nun $A \in VR_{G \times \Gamma}(X)$, so erhebt sich die Frage, ob man auf den Definitionsbereich von a_A eine „reelle" Struktur bringen kann, so daß a_A eine „reelle" Abbildung wird.

c bijektiv bedeutet, daß wir auf jedem $L \in \mathfrak{L}$ eine Konjugation α auswählen können. $\alpha \in \mathbf{Z}_2$ operiert auf $L \otimes \mathrm{Hom}_G(L, A)$ durch

$$\alpha(v \otimes f) = \alpha v \otimes \alpha f.$$

Man erinnere sich: Wenn $f \in \mathrm{Hom}_G(L, A_x)$ ist, so ist $\alpha f \in \mathrm{Hom}_G(L, A_{\alpha x})$ durch $(\alpha f)(v) = \alpha f(\alpha v)$ definiert.

Die Betrachtung von a_A lehrt uns, daß m' wohldefiniert und daß $m m' = \mathrm{id}$ ist.

Sind $L, L' \in \mathfrak{L}$, $E \in VR_\Gamma(X)$, dann kann man zeigen $\mathrm{Hom}_G(L, L' \otimes E) \cong 0$ oder $\cong E$, wenn $L \neq L'$ oder $L = L'$. Das zeigt $m' m = \mathrm{id}$.

§ 5. Diedergruppen

Die Diedergruppe D_{2n} hat die Ordnung $2n$ und wird von 2 Elementen g und t erzeugt, die den Relationen $g^n = g\,t\,g\,t = t^2 = 1$ genügen. Die zyklische Gruppe $\mathbf{Z}_n$, die von g erzeugt wird, ist Normalteiler in D_{2n}. Wir haben $D_4 = \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Ist $n > 2$, so fassen wir D_{2n} als Untergruppe von S_n auf, die von den Permutationen

$$g = (1 \dots n) \quad \text{und} \quad t = \begin{pmatrix} 1 & \dots & n \\ -1 & \dots & -n \end{pmatrix}$$

erzeugt wird. Wir definieren Charaktere ε, τ, τ_j [σ nur falls n gerade] durch die Formeln:

$$\varepsilon(g^k) = \varepsilon(g^k t) = 1; \qquad \tau(g^k) = -\tau(g^k t) = 1;$$

$$\tau_j(g^k) = \zeta^{jk} + \zeta^{-jk}, \qquad \tau_j(g^k t) = 0, \qquad \zeta = \exp(2\pi i/n);$$

$$[\sigma(g^k) = \sigma(g^k t) = (-1)^k, \text{ nur wenn } n \text{ gerade}].$$

Diese Charaktere gehören zu den $\mathbf{C}(D_{2n})$-Moduln

$$\mathbf{1} = (\mathbf{C} \mid g = t = \mathrm{id}); \qquad T = (\mathbf{C} \mid g = \mathrm{id}, t = -\mathrm{id});$$

$$T_j = (\mathbf{C}^2 \mid g\,v = M^j v,\ t\,v = N\,v,\ v \in \mathbf{C}^2),$$

wobei

$$M = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}, \qquad \gamma = 2\pi/n$$

$$N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2, \mathbf{C});$$

$$S = (\mathbf{C} \mid g = -\mathrm{id}, t = \mathrm{id})$$

Jeder dieser Moduln trägt eine kanonische Konjugation.

Die primitiven Charaktere von D_{2n} sind:

(a) $\varepsilon, \tau, \tau_1, \dots, \tau_{(n-1)/2}$, wenn n ungerade ist.

(b) $\varepsilon, \tau, \sigma, \sigma\tau, \tau_1, \dots, \tau_{n/2-1}$, wenn n gerade ist.

Wir sehen, daß $c\colon RO(D_{2n}) \to RU(D_{2n})$ bijektiv ist und somit hat man nach Satz (4.2) einen natürlichen Isomorphismus

$$m'_n\colon KR_{D_{2n} \times \Gamma}(X) \to R(D_{2n}) \otimes KR_\Gamma(X).$$

§ 6. Eine neue Beschreibung der $\bar{\psi}^p$

(6.1) Das Ziel dieses Abschnittes ist es, $\bar{\psi}^p\colon \overline{VR}_\Gamma(X) \to KR_\Gamma(X)$ in 3 Abbildungen zu zerlegen: $\bar{\psi}^p = (u_p \otimes 1)\, a_p \bigotimes^p$. Für $p > 2$ schreiben wir uns das in einem Diagramm hin.

$$\begin{array}{ccccc} & & \overbrace{\qquad\qquad\qquad\qquad}^{a_p} & & \\ \overline{VR}_{S_p\times\Gamma}(X) & \xrightarrow{i_p} & KR_{D_{2p}\times\Gamma}(X) & \xrightarrow{m'_p} & R(D_{2p})\otimes KR_\Gamma(X) \\ \uparrow{\scriptstyle \otimes^p} & & & & \downarrow{\scriptstyle u_p\otimes 1} \\ \overline{VR}_\Gamma(X) & & \overset{\overline{\psi}^p}{\dashrightarrow} & & \mathbf{Z}\otimes KR_\Gamma(X) \end{array}$$

a_p faktorisiert wieder in 2 Abbildungen $a_p = m'_p i_p$.

i_p ist die kanonische Abbildung „Übergang zur Klasse in $KR_{D_{2p}\times\Gamma}(X)$“
u_p: $R(D_{2p}) \to \mathbf{Z}$ ist der Homomorphismus von Abelschen Gruppen, der durch $u_p(\varepsilon) = u_p(\tau) = -u_p(\tau_1) = 1$, $u_p(\tau_2) = \cdots = u_p(\tau_{(p-1)/2}) = 0$ gegeben ist. u_p bildet $\tau - \varepsilon$, $\delta = \varepsilon + \tau_1 + \cdots + \tau_{(p-1)/2}$ und $\tau\delta$ nach Null ab.

Wenn $p=2$ ist, ersetzen wir in dem obigen Diagramm D_{2p} durch $\mathbf{Z}_2$. In diesem Fall ist $u_2: R(\mathbf{Z}_2) \to \mathbf{Z}$ durch $u_2(\varepsilon) = -u_2(\tau) = 1$ definiert. Dabei ist ε der triviale und τ der nicht triviale Charakter von $\mathbf{Z}_2$.

Setzen wir $G_p := D_{2p}$ für Primzahlen $p>2$ und $G_2 := \mathbf{Z}_2$, und ersetzen wir D_{2p} in dem obigen Diagramm durch G_p, so erhalten wir ein Diagramm, das für alle Primzahlen gilt.

Wir bemerken, daß $\otimes^p$ multiplikativ, a_p additiv und multiplikativ, $u_p \otimes 1$ additiv ist. Trotzdem werden wir zeigen, daß $\overline{\psi}^p$ sowohl additiv wie auch multiplikativ ist.

(6.2) Die Abbildung m'_p kann man nach (4.2) und §5 durch

$$m'_p([A]) = \varepsilon \otimes [A_0^+] + \tau \otimes [A_0^-] + \sum_{j=1}^{(p-1)/2} \tau_j \otimes [\tilde{A}_j]$$

beschreiben, wobei

$$A_0^+ = \mathrm{Hom}_{D_{2p}}(\mathbf{1}, A), \qquad A_0^- = \mathrm{Hom}_{D_{2p}}(T, A), \qquad \tilde{A}_j = \mathrm{Hom}_{D_{2p}}(T, A).$$

Lemma. *Man hat folgende „reelle“ Isomorphismen:*

(a) $A_0^+ \oplus A_0^- = \mathrm{Hom}_{D_{2p}}(\mathbf{1} \oplus T, A) = \mathrm{Hom}_{D_{2p}}(T_0, A) \cong \mathrm{Ker}(g-1)$

(b) $A_j \cong \mathrm{Hom}_{D_{2p}}(T_j, A) = \tilde{A}_j$, wobei die A_j in §1 definiert sind.

Beweis. Wir zeigen nur (b). T_j ist als Vektorraum $\mathbf{C}^2$. Wir bezeichnen mit e_1, e_2 die Einheitsvektoren. Ist $f \in \tilde{A}_j = \mathrm{Hom}_{D_{2p}}(T_j, A)$, dann ist $f(e_1) \in A_j \subset A$, da

$$\begin{aligned} 2f(e_1) &= f(e_1) - if(e_2) \oplus f(e_1) + if(e_2) \in \mathrm{Ker}(g-\zeta^j) \oplus \mathrm{Ker}(g-\zeta^{-j}) \\ &= (1+t)\big(f(e_1) - if(e_2)\big) \in \mathrm{Ker}(t-1). \end{aligned}$$

Man sieht leicht, daß $f \mapsto f(e_1)$ „reell“ ist und $\tilde{A}_j$ isomorph auf A_j abbildet.

(6.3) Wir wollen nun a_p explizit beschreiben.

Lemma. *Ist* $A \in VR_{S_p \times \Gamma}(X)$ *und* j *eine Zahl, die nicht durch* p *teilbar ist, so induziert*

$$t_j := \begin{pmatrix} 1 & \dots & p \\ 1j & & pj \end{pmatrix}$$

[*untere Zeile* mod p] *einen „reellen" Isomorphismus von* $\mathrm{Hom}_{D_{2p}}(T_j, A)$ *auf* $\mathrm{Hom}_{D_{2p}}(T_1, A)$.

Man benutze zum Beweis $g^j t_j = t_j g$ und die Beschreibung von T_j aus § 5.

Mit Hilfe der in diesem Paragraphen gewonnenen Ergebnisse können wir a_p folgendermaßen aufschreiben:

$$a_p(A) = \varepsilon \otimes ([A_0] - [A_1]) + (\tau - \varepsilon) \otimes [A_0^-] + \delta \otimes [A_1],$$

wobei

$$A_0 = A_0^+ \oplus A_0^- \cong \mathrm{Ker}(g-1), \qquad A_1 \cong \tilde{A}_1 = \mathrm{Hom}_{D_{2p}}(T_1, A),$$

δ wie in (6.1).

Ist $A = \bigotimes^p E$, so können wir schreiben:

$$a_p \bigotimes^p E = a_p(A) = \varepsilon \otimes \bar{\psi}^p(E) + (\tau - \varepsilon) \otimes [A_0^-] + \delta \otimes [A_1].$$

$u_p \otimes 1$ auf beide Seiten angewandt ergibt:

$$(u_p \otimes 1)\, a_p \bigotimes^p E = \bar{\psi}^p(E).$$

§ 7. Beweis der einfachen Teile des Satzes aus § 3

In diesem Abschnitt setzen wir voraus, daß $\bar{\psi}^p$: $\overline{VR}_\Gamma(X) \to KR_\Gamma(X)$ additiv ist. Wir werden das in § 10 beweisen.

(7.1) *Beweise von* (3.1), (3.2), (3.5), (3.7.1):

Weil $\bar{\psi}^p$ additiv ist, wird ψ^p induziert. Aus der Definition von $\bar{\psi}^p$ geht hervor, daß es in natürlicher Weise von Γ und X abhängt. Wir haben $\psi^p([L]) = [L]^p$ für ein Linienbündel L, weil S_p trivial auf $\bigotimes^p L$ operiert und somit haben wir $\bigotimes^p L = \mathrm{Ker}(g-1) = A_0$ und $A_1 = 0$. ψ^p ist mit der Komplexifizierung c verträglich, weil c von der „reellen" Abbildung C induziert wird; vgl. § 2.

(7.2) *Beweis von* (3.6):

Weil sich jedes Element $x \in KR_\Gamma(X)$ als $[E] - [F]$ schreiben läßt, genügt es für Elemente der Form $[E]$ zu zeigen:

$$\psi^p[E] = [E]^p \bmod p.$$

Sei $p > 2$; $A = \bigotimes^p E$ liefert in $R(D_{2p}) \otimes KR_\Gamma(X)$ das Element

$$\varepsilon \otimes \bar{\psi}^p E + (\tau - \varepsilon) \otimes [A_0^-] + \delta \otimes [A_1]$$

und in $KR_\Gamma(X)$ das Element

$$[\bigotimes^p E] = [E]^p = \varepsilon(1)\bar{\psi}^p(E) + (\tau-\varepsilon)(1)[A_0^-] + \delta(1)[A_1]$$

$$[\chi(1) = \text{Grad des Charakters } \chi,\ 1 \in D_{2p}]$$

Weil $\varepsilon(1)=\tau(1)=1$, $\delta(1)=p$, ergibt sich:

$$[E]^p = \bar{\psi}^p(E) + p[A_1]$$

Den Fall $p=2$ beweist man analog.

(7.3) *$\bar{\psi}^p$ ist multiplikativ:*

Wir behandeln den Fall $p>2$. Seien $E, F \in VR_\Gamma(X)$, $A = \bigotimes^p E$, $B = \bigotimes^p F \in VR_{S_p \times \Gamma}(X)$. Die Abbildung a_p aus §6 ist multiplikativ; somit ergibt sich

$$a_p(A \otimes B) = a_p(A)\, a_p(B) = \{\varepsilon \otimes \bar{\psi}^p(E) + (\tau-\varepsilon) \otimes [A_0^-] + \delta \otimes [A_1]\}$$
$$\cdot \{\varepsilon \otimes \bar{\psi}^p(F) + (\tau-\varepsilon) \otimes [B_0^-] + \delta \otimes [B_1]\}$$

man benutze:

$$(\tau-\varepsilon)\delta = \tau-\varepsilon, \qquad \delta^2 = p + \frac{p-1}{2}(\tau-\varepsilon), \qquad (\tau-\varepsilon)^2 = 2(\varepsilon-\tau)$$

und erhält:

$$a_p(A \otimes B) = \varepsilon \otimes \bar{\psi}^p(E)\bar{\psi}^p(F) + (\tau-\varepsilon) \otimes \cdots + \delta \otimes \cdots.$$

Wendet man $u_p \otimes 1$ auf beide Seiten an, so ergibt sich:

$$\bar{\psi}^p(E \otimes F) = (u_p \otimes 1)\, a_p \bigotimes^p (E \otimes F) = (u_p \otimes 1)\, a_p(A \otimes B) = \bar{\psi}^p(E)\,\bar{\psi}^p(F).$$

(7.4) *Beweis von* (3.9)*:*

Wir faktorisieren m in 3 Abbildungen

$$m\colon KR_G(\cdot) \otimes KR_\Gamma(X) \xrightarrow{p_G^* \otimes p_\Gamma^*} KR_{G\times\Gamma}(\cdot) \otimes KR_{G\times\Gamma}(X)$$
$$\xrightarrow{j^* \otimes 1} KR_{G\times\Gamma}(X) \otimes KR_{G\times\Gamma}(X) \xrightarrow{\mu} KR_{G\times\Gamma}(X).$$

Dabei sind p_Γ, p_G die Projektionen von $G\times\Gamma$ auf die Faktoren Γ, G; $j\colon X \to \cdot$ die Abbildung von X auf einen Punkt und μ die Multiplikation in $KR_{G\times\Gamma}(X)$. Man beachte, daß $RO(G) \cong KR_G(\cdot)$. Nun benutze man, daß ψ^p natürlich (also verträglich mit $p_G^* \otimes p_\Gamma^*$ und $j^* \otimes 1$) und daß ψ^p multiplikativ (also verträglich mit μ) ist.

§ 8. Beweis von (3.8)

Sei χ ein Charakter von Γ; $d\colon \Gamma \to GL(n, \mathbf{C})$ ein Homomorphismus, dessen Charakter χ ist. Der p-Zykel $g=(1\ldots p)$ operiert auf $\bigotimes^p \mathbf{C}^n$. $\operatorname{Ker}(g-1)$ und $\operatorname{Ker}(g-\zeta)$ sind $\mathbf{C}-\Gamma$-Moduln. Ihre Charaktere bezeichnen wir mit χ_0 resp. χ_1. Wir haben nun:

$$\psi^p \chi = \chi_0 - \chi_1.$$

Ist $e_1, \ldots, e_n$ eine Basis von $\mathbf{C}^n$, dann ist

$$B=\{e_I=e_{i_1}\otimes\cdots\otimes e_{i_p}\,|\,I=(i_1,\ldots,i_p)\in\{1,\ldots,n\}^p\}$$

eine Basis für $\mathbf{C}^n$. Der p-Zykel operiert auf B und zerlegt B in Orbits, die aus p oder aus einem Element bestehen. Sei

$$\{x_1=e_1\otimes\cdots\otimes e_1,\ldots,x_n=e_n\otimes\cdots\otimes e_n,\ e_{I_1},\ldots,e_{I_j}\}$$

ein Repräsentantensystem für die Orbits. Dann bilden

$$x_1,\ldots,x_n,\ x_{I_1}=\frac{1}{p}\sum_{k=1}^{p} g^k e_{I_1},\ldots,x_{I_j}=\frac{1}{p}\sum_{k=1}^{p} g^k e_{I_j}$$

eine Basis für $\operatorname{Ker}(g-1)$ und

$$y_{I_1}=\frac{1}{p}\sum_{k=1}^{p}\zeta^{-k} g^k e_{I_1},\ldots,y_{I_j}=\frac{1}{p}\sum_{k=1}^{p}\zeta^{-k} g^k e_{I_j}$$

eine Basis für $\operatorname{Ker}(g-\zeta)$.

Sei $a\in\Gamma$ ein festes Element. Dann können wir annehmen, daß $d(a)$ Diagonalgestalt hat; $d(a)=\operatorname{Diag}(\alpha_1,\ldots,\alpha_n)$, $\alpha_i\in\mathbf{C}$. Mit α_I bezeichnen wir das Produkt $\alpha_{i_1}\cdot\cdot\alpha_{i_p}$, wenn $I=(i_1,\ldots,i_p)$. Dann operiert $a\in\Gamma$ auf $\operatorname{Ker}(g-1)$ durch

$$a\,x_1=\alpha_1^p x_1,\ldots,a\,x_n=\alpha_n^p x_n,\ldots,a\,x_{I_1}=\alpha_{I_1}x_{I_1},\ldots,a\,x_{I_j}=\alpha_{I_j}x_{I_j}$$

und auf $\operatorname{Ker}(g-\zeta)$ durch

$$a\,y_{I_1}=\alpha_{I_1}y_{I_1},\ldots,a\,y_{I_j}=\alpha_{I_j}y_{I_j}$$

gegeben. Es ist also:

$$\chi_0(a)=\alpha_1^p+\cdots+\alpha_n^p+\alpha_{I_1}+\cdots+\alpha_{I_j};\qquad \chi_1(a)=\alpha_{I_1}+\cdots+\alpha_{I_j}.$$

Somit erhält man:

$$(\psi^p\chi)(a)=\chi_0(a)-\chi_1(a)=\alpha_1^p+\cdots+\alpha_n^p=\chi(a^p).$$

(8.1) **Lemma.** *Seien p, q Primzahlen $p\neq q$, $p>2$. Für ψ^q: $R(D_{2p})\to R(D_{2p})$ hat man:*

$$\psi^q(\tau_j)=\tau_{jq},\qquad \psi^q(\tau)=\tau,\qquad \psi^q(\delta)=\delta,\qquad \textit{wenn } q>2,$$

$$\psi^2(\tau_j)=\tau_{2j}+(\varepsilon-\tau),\qquad \psi^2(\tau)=\varepsilon,\qquad \psi^2\delta=\delta+\frac{p-1}{2}(\varepsilon-\tau).$$

Das rechnet man nach, indem man beide Seiten auf die Elemente von D_{2p} anwendet.

§ 9. Spezielle D_{2p}-Objekte

Sei p eine ungerade Primzahl. Wir geben ein nützliches Kriterium an, wann die Klasse eines Objektes aus $VR_{D_{2p}\times\Gamma}(X)$ in $R(D_{2p})\otimes KR_\Gamma(X)$ ein Element repräsentiert, das unter $u_p\otimes 1$ auf Null geht.

Sei I eine endliche Menge, auf der D_{2p} operiert und $(E_i | i \in I)$ eine Familie von Objekten aus $VR_\Gamma(X)$. Wir setzen voraus, daß es zu jedem $i \in I$ und $h \in D_{2p}$ einen $VR_\Gamma(X)$-Morphismus $h_i\colon E_i \to E_{h(i)}$ gibt, so daß $1_i = \mathrm{id}_{E_i}$ und $h'_{h(i)} h_i = (h' h)_i$. $\sum_{i \in I} E_i$ wird zu einem Objekt aus $VR_{D_{2p} \times \Gamma}(X)$, wenn man die D_{2p}-Struktur definiert durch $h(x_i | i \in I) = (y_i | i \in I)$, wobei $y_i = h\, x_{h(i)}$, $x_i \in E_i$. (Wir lassen ab jetzt die Indizes an den Elementen von D_{2p} weg.)

Definition. Bestehen die Orbits von I unter der Operation von D_{2p} aus p oder $2p$ Elementen, so nennen wir $\sum_{i \in I} E_i$ ein spezielles D_{2p}-Objekt.

Lemma. *Ist $\sum_{i \in I} E_i$ ein spezielles D_{2p}-Objekt, so gilt*

$$\sum_{i \in I} E_i \cong \mathbf{C}(\mathbf{Z}_p)(1+t) \otimes E \oplus \mathbf{C}(\mathbf{Z}_p)(1-t) \otimes F,$$

wobei E, F zwei geeignete „reelle" Bündel sind.

Die Charaktere der D_{2p}-Modulin $\mathbf{C}(\mathbf{Z}_p)(1+t)$, $\mathbf{C}(\mathbf{Z}_p)(1-t)$ sind δ, $\tau\,\delta$. Diese werden von u_p nach Null abgebildet.

Folgerung. Ein spezielles D_{2p}-Objekt von $VR_{D_{2p} \times \Gamma}(X)$ liefert in $R(D_{2p}) \otimes KR_\Gamma(X)$ ein Element, das unter $u_p \otimes 1$ auf 0 geht.

Beweis. Wir setzen voraus, daß D_{2p} transitiv auf I operiert und unterscheiden 2 Fälle:

1. *Fall:* I bestehe aus $2p$ Elementen. Dann kann man annehmen $I = D_{2p}$ und D_{2p} operiert durch Linkstranslation auf sich. Ein $VR_{D_{2p} \times \Gamma}(X)$-Isomorphismus

$$f\colon\ \mathbf{C}(D_{2p}) \otimes E_1 \to \sum_{i \in I} E_i,$$

wird durch $f(h \otimes e) = (e_i | i \in D_{2p})$, $e_h = h\, e$, $e_i = 0$, wenn $i \neq h$ definiert. Dabei ist $h \in D_{2p}$, $e \in E_1$. $\mathbf{C}(D_{2p})$ ist als D_{2p}-Modul zu $\mathbf{C}(\mathbf{Z}_p)(1+t) \oplus \mathbf{C}(\mathbf{Z}_p)(1-t)$ isomorph.

2. *Fall:* I bestehe aus p Elementen. Dann können wir annehmen, daß $I = \{0, 1, \ldots, p-1\}$, $g(i) = i+1 \bmod p$, $t(i) = p - i$. Die Abbildung

$$f\colon\ \mathbf{C}(D_{2p}) \otimes E_0 \to \sum_{i \in I} E_i,$$

die durch $f(h \otimes e) = (e_i | i \in I)$, $e_{h(0)} = h\, e$, $e_i = 0$, wenn $i \neq h(0)$ definiert ist, ist „reell" und induziert einen Isomorphismus

$$f'\colon\ \mathbf{C}(\mathbf{Z}_p)(1+t) \otimes E_0^+ \oplus \mathbf{C}(\mathbf{Z}_p)(1-t) \otimes E_0^- \to \sum_{i \in I} E_i.$$

Dabei sind $E_0^\pm$ die Eigenräume von $t\colon E_0 \to E_0$ zu den Eigenwerten ± 1. Das sieht man so ein: Ist $e_k \in E_k$, so gilt

$$f[g^k(1+t)\otimes(1+t)\,g^{-k}e_k + g^k(1-t)\otimes(1-t)\,g^{-k}e_k] = 4(0,\ldots,e_k,\ldots,0).$$

Das zeigt uns, daß f' surjektiv ist. Aus Dimensionsgründen folgt dann, daß f' bijektiv ist.

§ 10. $\bar{\psi}^p$ ist additiv

Der Fall $p=2$ wird übergangen. Wir behandeln den Fall $p>2$. Sind $E_0, E_1 \in VR_\Gamma(X)$, so gilt in $VR_{S_p\times\Gamma}(X)$

$$\otimes^p(E_0 \oplus E_1) = \otimes^p E_0 \oplus \otimes^p E_1 \oplus F,$$

wobei

$$F = \sum_{i\in I} E_{i_1}\otimes\cdots\otimes E_{i_p} \in VR_{S_p\times\Gamma}(X),$$

$$i=(i_1,\ldots,i_p)\in I := \{0,1\}^p - \{(0,\ldots,0),(1,\ldots,1)\}.$$

F liefert ein spezielles D_{2p}-Objekt in $VR_{D_{2p}\times\Gamma}(X)$ und wird unter $u_p\otimes 1$ nach Null abgebildet. Somit haben wir:

$$\begin{aligned}\bar{\psi}^p(E_0\oplus E_1) &= (u_p\otimes 1)\,a_p \otimes^p(E_0\oplus E_1)\\ &= (u_p\otimes 1)\,a_p[\otimes^p E_0 \oplus \otimes^p E_1 \oplus F]\\ &= (u_p\otimes 1)\,a_p \otimes^p E_0 + (u_p\otimes 1)\,a_p\otimes^p E_1\\ &= \bar{\psi}^p(E_0)+\bar{\psi}^p(E_1).\end{aligned}$$

§ 11. $\psi^p\psi^q = \psi^q\psi^p$; p, q Primzahlen (Beweis von (3.4))

Wir fassen S_{pq} als Permutationsgruppe der pq Gitterpunkte $G = \{(i,j) \mid 1\leqq i\leqq q,\ 1\leqq j\leqq p\}$ auf. $S_q\times S_p$ betrachten wir als Untergruppe von S_{pq}; $(a,b)\in S_q\times S_p$ wirkt auf (i,j) durch $(a,b)(i,j) = (a(i), b(j))$. Mit $\iota\colon S_q\times S_p \to S_{pq}$ bezeichnen wir diese kanonische Inklusion.
Wie in (6.1) bezeichnen wir mit G_p die Diedergruppe D_{2p}, wenn $p>2$ und mit G_2 die zyklische Gruppe Z_2. Um die folgenden Diagramme aufschreiben zu können, führen wir folgende Abkürzungen ein:

$$\bar{V} := VR_\Gamma(X),\qquad K := KR_\Gamma(X),\qquad V_G := VR_{G\times\Gamma}(X),\qquad K_G := KR_{G\times\Gamma}(X)$$

Wir definieren 2 Abbildungen $\bar{\psi}^{pq}, \bar{\psi}^{q,p}\colon \overline{VR}_\Gamma(X) \to KR_\Gamma(X)$:

$$\begin{aligned}\bar{\psi}^{pq}\colon\ &\bar{V} \xrightarrow{\iota^*\otimes^{pq}} \bar{V}_{S_q\times S_p} \to K_{G_q\times G_p} \xleftarrow{\cong}\\ &R(G_q)\otimes R(G_p)\otimes K \xrightarrow{u_q\otimes u_p\otimes 1} \mathbf{Z}\otimes\mathbf{Z}\otimes K \cong K,\\ \bar{\psi}^{q,p}\colon\ &\bar{V} \xrightarrow{\otimes^p} \bar{V}_{S_p} \xrightarrow{a_p} R(G_p)\otimes K\\ &\xrightarrow{\psi^q\otimes\psi^q} R(G_p)\otimes K \xrightarrow{u_p\otimes 1} \mathbf{Z}\otimes K \cong K.\end{aligned}$$

Wir zeigen:

(11.1) $$\bar{\psi}^{q,p} = \psi^q \bar{\psi}^p,$$

(11.2) $$\bar{\psi}^{q,p} = \bar{\psi}^{pq}.$$

Da $\bar{\psi}^{pq}$ symmetrisch ist in p und q folgt: $\psi^q \bar{\psi}^p = \psi^p \bar{\psi}^q$

Beweis von (11.1). Sei $E \in VR_\Gamma(X)$ und $A = \bigotimes^p E \in VR_{S_p \times \Gamma}(X)$. Wir haben zu berechnen:

$$\bar{\psi}^{q,p}(E) = (u_p \otimes 1)(\psi^q \otimes \psi^q)\, a_p(A),$$

wobei

$$a_p(A) = \varepsilon \otimes \bar{\psi}^p(E) + (\tau - \varepsilon) \otimes [A_0^-] + \delta \otimes [A_1], \qquad \text{falls } p > 2$$

und

$$a_2(A) = \varepsilon \otimes [A_0] + \tau \otimes [A_1] = \varepsilon \otimes \bar{\psi}^2(E) + (\tau - \varepsilon) \otimes [A_1].$$

Man benutze Lemma (8.1) und unterscheide 3 Fälle:

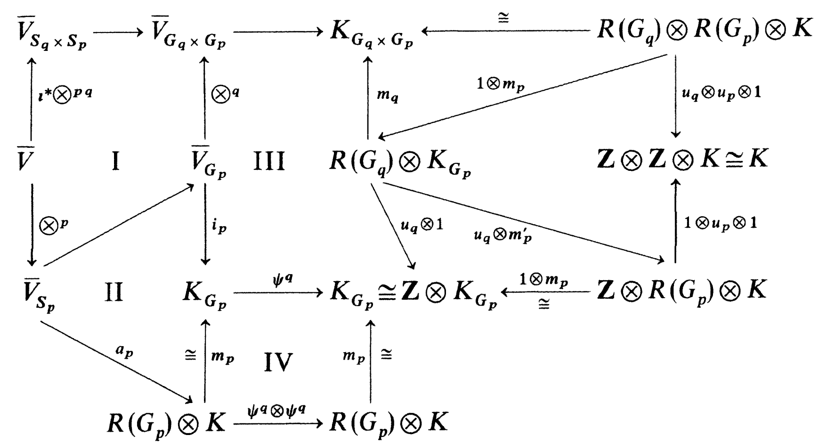
$$p > 2,\ q > 2; \qquad p = 2,\ q > 2; \qquad p > 2,\ q = 2.$$

Es folgt sofort $\bar{\psi}^{q,p}(E) = \psi^q \bar{\psi}^p(E)$.

Beweis von (11.2). Dieser Beweis ist eine große Diagrammjagd: Wir fügen noch folgende Liste von Begründungen für die Kommutativität des folgenden Diagrammes bei:

I Rechenregeln über das Tensorprodukt.
II Definition von a_p; vgl. (6.1).
III Definition von $\psi^q \colon K_{G_p} \to K_{G_p}$.
IV nach (3.9).

$\bar{\psi}^{pq}$ erscheint nun auf dem oberen und $\bar{\psi}^{q,p}$ auf dem unteren Weg.



§ 12. Verträglichkeit mit der Reellifizierung r (Beweis von (3.7.2))

Sei $X=(Y,\alpha)$ ein $\Gamma\times\mathbf{Z}_2$-Raum. ψ_U^p wird von $\bar{\psi}_U^p$ induziert, das durch die nachfolgende Zeile definiert ist:

$$\bar{\psi}_U^p\colon\ \overline{VU_\Gamma}(Y)\xrightarrow{\otimes^p} VU_{S_p\times\Gamma}(Y)\to KU_{D_{2p}\times\Gamma}(Y)$$
$$\xrightarrow{m'_p} R(D_{2p})\otimes KU_\Gamma(Y)\xrightarrow{u_p\otimes 1}\mathbf{Z}\otimes KU_\Gamma(Y)\cong KU_\Gamma(Y).$$

Man betrachte nun die Diagramme:

$$\begin{array}{ccccc}
\bar{\psi}^p\colon\ \overline{VR_\Gamma}(X) & \xrightarrow{\otimes^p} & \overline{VR}_{S_p\times\Gamma}(X) & \xrightarrow{\beta} & KR_\Gamma(X)\\
\uparrow r & \mathrm{I} & \uparrow r & \mathrm{II} & \uparrow r\\
\bar{\psi}_U^p\colon\ \overline{VU_\Gamma}(Y) & \xrightarrow{\otimes^p} & \overline{VU}_{S_p\times\Gamma}(Y) & \xrightarrow{\beta_U} & KU_\Gamma(Y).
\end{array}$$

Dabei ist $\beta=(u_p\otimes 1)\,a_p$; vgl. § 6; β_U ist analog definiert wie β. Diagramm II ist kommutativ; Diagramm I ist nicht kommutativ.

Sei $E\in VU_\Gamma(Y)$. Wir zeigen, daß in $VR_{S_p\times\Gamma}(X)$ gilt

$$\otimes^p r\,E\cong r\otimes^p E\oplus F,$$

wobei F ein spezielles D_{2p}-Objekt liefert. Somit wird F nach (9.1) unter $\beta=(u_p\otimes 1)\,a_p$ nach Null abgebildet. An Hand des obigen Diagrammes überlegt man sich nun, daß

$$\bar{\psi}^p\,r\,E=r\,\bar{\psi}_U^p\,E.$$

In $VU_{S_p\times\Gamma}(Y)$ gilt:

$$\otimes^p(E\oplus\alpha^*\bar{E})=\otimes^p E\oplus\alpha^*\,\overline{\otimes^p E}\oplus\sum_{(j_1,\dots,j_p)\in J}E_{j_1}\otimes\cdots\otimes E_{j_p},$$

wobei

$$J=\{0,1\}^p-\{(0,\dots,0),(1,\dots,1)\},\qquad E_0:=E,\ E_1:=\alpha^*\bar{E}.$$

Auf $\{0,1\}$ hat man die Involution, die durch $\bar{0}=1$, $\bar{1}=0$ gegeben ist. Sie induziert auf J eine Involution, die J in Orbits zerlegt, die alle aus 2 Elementen bestehen. Wir bezeichnen mit I die Menge dieser Orbits. Nun sieht man, daß in $VR_{S_p\times\Gamma}(X)$ gilt:

$$\otimes^p r\,E=r\otimes^p E\oplus\sum_{i\in I}r(F_i).$$

Dabei ist $F_i=E_0\otimes E_{j_2}\otimes\cdots\otimes E_{j_p}$, wenn

$$i=\{(0,j_2,\dots,j_p),(1,\bar{j}_2,\dots,\bar{j}_p)\}\in I.$$

Man beachte, daß I eine S_p-Menge ist, weil die S_p-Struktur auf J mit der oben beschriebenen Involution auf J verträglich ist.

Die Orbits unter der D_{2p}-Operation bestehen aus p oder $2p$ Elementen. Somit ist

$$F := \sum_{i \in I} r(F_i)$$

ein spezielles D_{2p}-Objekt.

§ 13. Verträglichkeit mit ρ (Beweis von (3.7.3))

Die Verträglichkeit mit ρ beweist man analog wie die Verträglichkeit mit r. Man benutzt

$$VO_{\Gamma \times \mathbf{Z}_2} = VR_{\Gamma \times \mathbf{Z}_2}(X, \mathrm{id}_X);$$

das bedeutet: Man interpretiert ein $\mathbf{R} - \Gamma \times \mathbf{Z}_2$-Vektorraumbündel als $\mathbf{C} - \Gamma \times \mathbf{Z}_2$-Vektorraumbündel mit einer Konjugation, die über id_X liegt.

Ist $E = (F, J) \in VR_\Gamma(X)$, so ist jetzt $\rho(E) \in VR_{\Gamma \times \mathbf{Z}_2}(X, \mathrm{id}_X)$. $\rho(E)$ ist als $\mathbf{C} - \Gamma \times \mathbf{Z}_2$-Vektorraumbündel isomorph zu

$$\mathbf{C} \otimes_{\mathbf{R}} E = E \oplus \bar{E} = (F \oplus F, J \oplus -J).$$

Die „reelle" Struktur auf $\rho(E)$ ist durch die Konjugation in $\mathbf{C}$ oder gleichbedeutend durch Vertauschen der beiden Summanden

$$(f_1, f_2) \mapsto (f_2, f_1), \qquad f_1, f_2 \in F$$

gegeben. Jetzt zeigt man analog wie für r, daß in $VR_{S_p \times \Gamma \times \mathbf{Z}_2}(X, \mathrm{id}_X)$ gilt:

$$\bigotimes^p \rho E = \rho \bigotimes^p E \oplus H,$$

wobei H ein spezielles D_{2p}-Objekt liefert.

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Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen

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Bekanntlich läßt sich eine quadratintegrierbare Funktion auf dem Einheitskreis genau dann zu einer holomorphen Funktion auf der Einheitsscheibe fortsetzen, wenn alle ihre negativen Fourier-Koeffizienten verschwinden. Ebenfalls weiß man, daß eine differenzierbare Funktion f auf der Sphäre

$$S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid \sum |z_i|^2 = 1\}, \qquad n \geqq 2,$$

die Randwerte einer holomorphen Funktion auf der Vollkugel darstellt, falls f die auf S^{2n-1} beschränkten Cauchy-Riemann-Gleichungen erfüllt. Diese beiden Aussagen werde ich in der vorliegenden Arbeit zu Behauptungen über nichtkompakte, hermitesch symmetrische Räume (Cartan-Gebiete) verallgemeinern.

Nun sei D ein solcher hermitesch symmetrischer Raum des nichtkompakten Typs. Mit G soll die Einheitskomponente der Automorphismengruppe von D bezeichnet werden, und mit K die Untergruppe all der Abbildungen, die einen ausgewählten Punkt $0 \in D$ festhalten. Wie Cartan und Harish-Chandra gezeigt haben [3–5], kann man D so als beschränktes Gebiet im $\mathbf{C}^n$ einbetten, daß 0 dem Nullpunkt entspricht und K linear operiert. Der Schilow-Rand S von D ist dann eine Bahn der Gruppe K auf dem topologischen Rand ∂D [8]. Da K kompakt ist, existiert auf S ein invariantes Maß ds; L^2 sei der Hilbert-Raum der quadratintegrierbaren, komplex-wertigen Funktionen auf S. Die abgeschlossene Hülle derjenigen $f \in L^2$, die sich zu holomorphen Funktionen auf einer Umgebung von $D \cup \partial D$ fortsetzen lassen, bildet einen K-invarianten Unterraum $H^2 \subset L^2$. Ich werde H^2 explizit beschreiben und in K-irreduzible Komponenten aufspalten. Insbesondere können die Funktionen in H^2 durch das Verschwinden gewisser Fourier-Integrale gekennzeichnet werden; und wenn keiner der einfachen Faktoren von D ein Tubengebiet im Sinne von Koecher [7] ist, läßt sich H^2 auch durch die auf S beschränkten Cauchy-Riemann-Gleichungen charakterisieren. Das verallgemeinert die Aussagen über den Einheitskreis und die Einheitssphäre.

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Der Einfachheit halber nehme ich an, daß D nicht zerlegbar ist. Es bereitet keine großen Schwierigkeiten, die Ergebnisse auf den reduziblen Fall zu übertragen. Man wähle einen maximalen Torus $H \subset K$; H ist dann auch eine Cartan-Untergruppe für G. Ich bezeichne die Lie-Algebren von G, K, H mit $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$, und deren Komplexifizierungen mit $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$. Der holomorphe und der antiholomorphe Tangentenraum von $D \simeq G/K$ am Punkt eK entsprechen jeweils einer Ad K-invarianten, abelschen Unteralgebra $\mathfrak{p}_+$ und $\mathfrak{p}_-$ in $\mathfrak{g}$; $\mathfrak{g}$ ist die direkte Summe von $\mathfrak{k}$, $\mathfrak{p}_+$, $\mathfrak{p}_-$. Unter der adjungierten Darstellung von H zerfällt $\mathfrak{g}$ als

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha. \tag{1}$$

Hierbei ist $\Delta \subset \mathfrak{h}^*$ die Menge der von Null verschiedenen Wurzeln, und

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [y, x] = \langle \alpha, y \rangle x \text{ für jedes } y \in \mathfrak{h}\}.$$

Eine Wurzel α heißt kompakt oder nichtkompakt, je nachdem ob $\mathfrak{g}^\alpha \subset \mathfrak{k}$ oder $\mathfrak{g}^\alpha \subset \mathfrak{p}_+ \oplus \mathfrak{p}_-$. Man kann ein System von positiven Wurzeln $P \subset \Delta$ auswählen, so daß $\mathfrak{g}^\alpha \subset \mathfrak{p}_+$ genau wenn α positiv und nichtkompakt ist.

Nun sei γ_1 die niedrigste positive, nichtkompakte Wurzel, und dann – induktiv – γ_k die niedrigste unter denjenigen positiven, nichtkompakten Wurzeln, die zu $\gamma_1, \ldots, \gamma_{k-1}$ senkrecht stehen. So erhält man schließlich eine maximale Folge $\gamma_1, \ldots, \gamma_r$ von paarweise orthogonalen, positiven, nichtkompakten Wurzeln; diese Konstruktion stammt von Harish-Chandra [4]. Mit $\hat{K}$ bezeichne ich die Menge der Äquivalenzklassen irreduzibler Darstellungen von K, und mit I die Untermenge der Klassen, deren höchste Gewichte als

$$-n_1\gamma_1 - n_2\gamma_2 - \cdots - n_r\gamma_r,$$

mit

$$n_i \in \mathbf{Z}, \quad n_1 \geqq n_2 \geqq \cdots \geqq n_r \geqq 0, \tag{2}$$

ausgedrückt werden können. Jedes Element $i \in \hat{K}$ bestimmt einen Charakter χ_i.

Um zu erklären, wie die auf S beschränkten Cauchy-Riemann-Gleichungen auch für Funktionen $f \in L^2$ sinnvoll sind, ordne ich einem jeden solchen f die Distribution

$$T_f\colon g \to \int_S f g \, ds, \qquad g \in C^\infty(S)$$

zu. Alle C^∞-Differentialoperatoren, insbesondere die zu S tangentialen antiholomorphen Vektorfelder, kann man auf T_f einwirken lassen.

Das Ergebnis dieser Arbeit besteht aus dem folgenden

Satz. *Eine Funktion $f \in L^2$ liegt dann und nur dann in H^2, wenn*

$$\int_K \chi_i(k) f(k s) \, dk = 0 \tag{3}$$

für jedes $i \in \hat{K} - I$ und $s \in S$. Falls D nicht als Tubengebiet realisiert werden kann, ist dazu ebenfalls notwendig und hinreichend, daß f die auf S beschränkten Cauchy-Riemann-Gleichungen erfüllt. Alle irreduziblen Darstellungen von K, die in H^2 vorkommen, gehören zu I, und jede der Darstellungen in I ist — bis auf Äquivalenz — genau einmal in H^2 enthalten.

Einige Bemerkungen sind angebracht. Der Schilow-Rand eines Tubengebietes ist total reell; in diesem Fall werden die beschränkten Cauchy-Riemann-Gleichungen trivial und können also sicher nicht H^2 charakterisieren. Statt das Verschwinden der Integrale (3) für alle $i \in \hat{K} - I$ zu fordern, genügt es, nur die irreduziblen Darstellungen zu betrachten, die in L^2 vorkommen. Wenn D ein Tubengebiet ist, sind das diejenigen Darstellungen, deren höchste Gewichte in dem von $\gamma_1, \gamma_2, \ldots, \gamma_r$ erzeugten Gitter liegen. Auch sonst kann man die Menge der Darstellungen, die in L^2 enthalten sind, näher eingrenzen. Doch darauf möchte ich hier nicht eingehen. Aus dem Satz folgen natürlich ähnliche Behauptungen auch für reduzible Gebiete; die Herleitung sei dem Leser überlassen. Analog zu H^2 kann man die Räume H^p, $1 \leqq p < \infty$, definieren, sowie die Algebra A der stetigen Funktionen auf S, die sich zu stetigen Funktionen auf $D \cup \partial D$, holomorph auf D, fortsetzen lassen. Für diese Räume gilt der Satz, mutatis mutandis, ebenfalls; und zwar ergibt sich das aus der Tatsache, daß in einer stetigen Darstellung einer kompakten Lie-Gruppe auf einem Banach-Raum die endlichdimensionalen, irreduziblen Teilräume einen dichten Unterraum aufspannen.

Moore und Wolf hatten mich auf die Möglichkeit aufmerksam gemacht, H^2 für Tubengebiete durch das Verschwinden geeigneter Fourier-Integrale zu beschreiben, sobald man die Zerlegung von H^2 unter K kennt. Das hat mein Interesse an der Frage der Randwerte geweckt. Für die vier Cartanschen Reihen ist die Zerlegung von H^2 in Hua [6] — allerdings ein wenig versteckt — von Fall zu Fall beschrieben.

Den Beweis des Satzes führe ich zunächst auf die folgenden drei Behauptungen zurück:

a) Die Darstellungen aus der Menge I kommen — bis auf Äquivalenz — jeweils höchstens einmal in L^2 vor.

b) Nun sei D nicht als Tubengebiet realisierbar, und $B \subset L^2$ sei der Unterraum der Funktionen, die den auf S beschränkten Cauchy-Riemann-Gleichungen genügen. In der Zerlegung von B unter K sind nur die Darstellungen aus der Menge I vertreten.

c) Im k-ten symmetrischen Tensorprodukt von $\mathfrak{p}_-$ sind alle die Darstellungen genau einmal enthalten, deren höchste Gewichte die Form (2) haben, mit $\sum n_i = k$, und keine andere Darstellungen.

Durch Harish-Chandra's Einbettung entspricht D einem beschränkten, konvexen Gebiet in $\mathfrak{p}_+$. Eine holomorphe Funktion, definiert auf einer Umgebung einer kompakten, konvexen Menge $C \subset \mathbf{C}^n$, kann gleichmäßig auf C durch Polynome approximiert werden. Daher sind die Beschränkungen der Polynome dicht in H^2. Andererseits verschwindet ein Polynom nur dann auf S, wenn es selber Null ist. Folglich zerfällt H^2 unter K in die gleichen irreduziblen Komponenten wie die Algebra der Polynome. Die Killing-Form identifiziert $\mathfrak{p}_-$ mit dem Dualraum von $\mathfrak{p}_+$; also ist die symmetrische Tensoralgebra von $\mathfrak{p}_-$ der Algebra der Polynomfunktionen auf $\mathfrak{p}_+$ als K-Modul isomorph. Die Behauptung c) führt jetzt zum letzten Teil des Satzes.

Wegen a) und der soeben bewiesenen Aussage beschreiben die Gln. (3) genau die abgeschlossene Hülle der Polynomalgebra in L^2, nämlich H^2; dies folgt aus den bekannten Orthogonalitätsrelationen für kompakte Gruppen. Der in b) definierte Hilbert-Raum B enthält offensichtlich H^2 als abgeschlossenen Unterraum. Wären H^2 und B ungleich, müßte deswegen mindestens eine irreduzible Darstellung von K öfter in B vorkommen als in H^2; das aber widerspräche a), b) und dem letzten Teil des Satzes. Es genügt also, a), b) und c) zu beweisen.

Dazu werden einige Ergebnisse von Korányi und Wolf [8] benötigt. Für jedes $\alpha \in \Delta$ bildet $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha} \oplus [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ eine dreidimensionale Unteralgebra von $\mathfrak{g}$. Man wähle eine Basis $e_\alpha \in \mathfrak{g}^\alpha$, $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$, $h_\alpha \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$, so daß

$$\begin{aligned} \bar{e}_\alpha &= \pm e_{-\alpha}, \quad & [e_\alpha, e_{-\alpha}] &= h_\alpha, \\ [h_\alpha, e_\alpha] &= 2 e_\alpha, \quad & [h_\alpha, e_{-\alpha}] &= -2 e_{-\alpha} \end{aligned} \tag{4}$$

(das Überstreichen bedeutet komplexe Konjugation bezüglich $\mathfrak{g}_0$). Durch Harish-Chandra's Einbettung von D in $\mathfrak{p}_+$ entspricht

$$e_+ = e_{\gamma_1} + e_{\gamma_2} + \cdots + e_{\gamma_r}$$

einem Punkt auf dem Schilow-Rand S. Also ist S die Bahn von K durch e_+; M sei die Isotropiegruppe, $\mathfrak{m}_0$ deren Lie-Algebra, und $\mathfrak{m}$ die Komplexifizierung von $\mathfrak{m}_0$. Der Automorphismus

$$\tau = \operatorname{Ad} \exp\left(\tfrac{1}{2} \pi \sqrt{-1}\,(e_+ + \bar{e}_+)\right)$$

der Lie-Algebra $\mathfrak{g}$ hat die folgenden Eigenschaften:

$$\tau^2(\mathfrak{k}) = \mathfrak{k}, \qquad \tau^4 = 1,$$

$$\mathfrak{m} = \{x \in \mathfrak{k} \mid \tau(x) = x\}.$$

Wenn man

$$\mathfrak{u} = \{x \in \mathfrak{k} \mid \tau^2(x) = x\},$$

$$\mathfrak{q}_+ = \mathfrak{k} \cap \tau(\mathfrak{p}_+), \qquad \mathfrak{q}_- = \mathfrak{k} \cap \tau(\mathfrak{p}_-) = \bar{\mathfrak{q}}_+$$

setzt, dann gilt $\mathfrak{k} = \mathfrak{u} \oplus \mathfrak{q}_+ \oplus \mathfrak{q}_-$, $(\mathfrak{k}, \mathfrak{u})$ ist ein hermitesch symmetrisches Paar, und unter der natürlichen Identifizierung $S \simeq K/M$ entspricht $\mathfrak{q}_-$

den antiholomorphen, zu S tangentialen Vektoren am Punkt $e_+ \simeq eM \in K/M$. Man kann D genau dann als Tubengebiet realisieren, wenn $\mathfrak{k} = \mathfrak{u}$. Da τ einen involutiven Automorphismus von $\mathfrak{u}$ definiert, ist

$$\mathfrak{u} = \mathfrak{m} \oplus \mathfrak{r},$$

mit

$$\mathfrak{r} = \{x \in \mathfrak{k} \mid \tau(x) = -x\},$$

eine Cartan-Zerlegung. Die Cartan-Unteralgebra $\mathfrak{h} \subset \mathfrak{k}$ liegt ganz in $\mathfrak{u}$; sie zerfällt als

$$\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-,$$

wobei

$$\mathfrak{h}_+ = \mathfrak{h} \cap \mathfrak{m}, \qquad \mathfrak{h}_- = \mathfrak{h} \cap \mathfrak{r}. \tag{5}$$

Eine Wurzel $\alpha \in \Delta$, die auf $\mathfrak{h}_-$ verschwindet, ist zu $\gamma_1, \ldots, \gamma_r$ streng orthogonal [4], d.h. weder $\alpha + \gamma_i$ noch $\alpha - \gamma_i$ sind Wurzeln. Deswegen ist τ auf $\mathfrak{g}^\alpha$ die Identität, und $\mathfrak{m} \oplus \mathfrak{h}_-$ enthält den Zentralisator von $\mathfrak{h}_-$ in $\mathfrak{k}$. Also:

$$\mathfrak{h}_- \text{ ist maximal abelsch in } \mathfrak{r}. \tag{6}$$

Es sollte auch noch erwähnt werden, daß

$$\mathfrak{h}_+ = \{x \in \mathfrak{h} \mid \langle \gamma_i, x \rangle = 0,\ 1 \leqq i \leqq r\}. \tag{7}$$

Das folgende Lemma stammt sinngemäß von Cartan [2]; es ist vielleicht angebracht, es auch hier zu beweisen:

Lemma 1. *Man betrachte eine Cartan-Zerlegung* $\mathfrak{u} = \mathfrak{m} \oplus \mathfrak{r}$ *der komplexen, reduktiven Lie-Algebra* $\mathfrak{u}$, *und eine Cartan-Unteralgebra* $\mathfrak{h} \subset \mathfrak{u}$ *mit den Eigenschaften* (5) *und* (6). *Ein System von positiven Wurzeln für* $(\mathfrak{u}, \mathfrak{h})$ *soll so ausgewählt sein, daß die Beschränkung einer positiven Wurzel auf* $\mathfrak{h}_-$ *entweder verschwindet oder positiv ist. Ferner sei* π *eine irreduzible Darstellung von* $\mathfrak{u}$ *auf dem Vektorraum* V, $V' \subset V$ *bezeichne den eindimensionalen Unterraum, der dem höchsten Gewicht entspricht, und*

$$V_{\mathfrak{m}} = \{v \in V \mid \pi(\mathfrak{m})\, v = 0\}.$$

Dann hat jeder Vektor $v \in V_{\mathfrak{m}}$, $v \neq 0$, *in der Zerlegung unter* $\mathfrak{h}$ *eine von Null verschiedene Komponente in* V'.

Beweis. Die Wurzelvektoren, deren zugehörige Wurzeln eine negative Beschränkung auf $\mathfrak{h}_-$ haben, spannen eine nilpotente Unteralgebra $\mathfrak{n} \subset \mathfrak{u}$ auf, und

$$\mathfrak{u} = \mathfrak{n} \oplus \mathfrak{h}_- \oplus \mathfrak{m}$$

ist eine Iwasawa-Zerlegung. Deswegen gilt

$$\mathscr{U}(\mathfrak{u}) = \mathscr{U}(\mathfrak{n})\, \mathscr{U}(\mathfrak{h}_-)\, \mathscr{U}(\mathfrak{m});$$

dabei bezeichnet $\mathscr{U}(\mathfrak{u})$ die universelle einhüllende Algebra von $\mathfrak{u}$, $\mathscr{U}(\mathfrak{n})$ diejenige von $\mathfrak{n}$, usw. Aus der Irreduzibilität von π folgt

$$V' \subset V = \pi(\mathscr{U}(\mathfrak{u}))\, v = \pi(\mathscr{U}(\mathfrak{n})\, \mathscr{U}(\mathfrak{h}_-))\, v,$$

falls $v \in V_{\mathfrak{m}}$, $v \neq 0$. Nun schreibe man v als die Summe $\mathfrak{h}_-$-invarianter Komponenten,

$$v = v_0 + v_1 + \cdots + v_m,$$

und zwar sei $v_0 \in V'$ die Komponente, die dem höchsten Gewicht entspricht. Weil $\mathfrak{n}$ von negativen Wurzelvektoren erzeugt wird, haben die Vektoren in $\pi(\mathscr{U}(\mathfrak{n})\, \mathscr{U}(\mathfrak{h}_-))\, v_i$, für $i \neq 0$, sicher keine Komponente in V'. Daher $v_0 \neq 0$.

Korollar (Cartan [2]). *Der Unterraum $V_{\mathfrak{m}}$ ist höchstens eindimensional. Falls $V_{\mathfrak{m}} \neq 0$, verschwindet das höchste Gewicht auf $\mathfrak{h}_+$.*

Gäbe es nämlich zwei linear unabhängige Vektoren in $V_{\mathfrak{m}}$, so hätte eine geeignete lineare Kombination keine Komponente in dem eindimensionalen Raum V'. Und wenn $\mathfrak{m}$ trivial auf einen Vektor v operiert, dann muß $\mathfrak{h}_+$ trivial auf jede Komponente in der Zerlegung unter $\mathfrak{h}$ operieren.

Eine positive Wurzel α von $(\mathfrak{u}, \mathfrak{h})$ verschwindet laut Moore [9] entweder auf $\mathfrak{h}_-$, oder ihre Beschränkung hat die Form $\frac{1}{2}(\gamma_i - \gamma_j)$, $i > j$, und ist deshalb positiv. Die Voraussetzungen von Lemma 1 sind also erfüllt. Die Lie-Algebra $\mathfrak{u}$ ist die Komplexifizierung der Algebra $\mathfrak{u}_0 = \mathfrak{u} \cap \mathfrak{k}_0$. Ich bezeichne die zusammenhängende Untergruppe von K, die $\mathfrak{u}_0$ entspricht, mit U, und die Isotropieuntergruppe von K am Punkt $e_+ \in S$ mit M, wie zuvor. Die Einheitskomponente der Gruppe M ist in U enthalten, weil $\mathfrak{m} \subset \mathfrak{u}$; M braucht aber nicht zusammenhängend zu sein.

Lemma 2. *Es sei π eine irreduzible Darstellung von U auf dem Vektorraum V. Der Unterraum*

$$V_{U \cap M} = \{v \in V \mid \pi(U \cap M)\, v = v\}$$

ist höchstens eindimensional, und wenn $V_{U \cap M} \neq 0$, dann hat das höchste Gewicht von π die Form

$$-\sum n_i \gamma_i, \qquad n_i \in \mathbf{Z}, \qquad n_1 \geqq n_2 \geqq \cdots \geqq n_r. \tag{8}$$

Beweis. Die erste Behauptung ist eine Folge des vorhergehenden Korollars; ebenso die Tatsache, daß das höchste Gewicht auf $\mathfrak{h}_+$ verschwindet, falls $V_{U \cap M} \neq 0$. Wegen (7) ist es dann eine lineare Kombination

$$-\sum a_i \gamma_i, \qquad a_i \in \mathbf{R}.$$

Für $1 \leqq i < j \leqq r$ gibt es eine positive Wurzel α von $(\mathfrak{u}, \mathfrak{h})$, deren Beschränkung auf $\mathfrak{h}_-$ mit $\frac{1}{2}(\gamma_j - \gamma_i)$ übereinstimmt. Deswegen gilt

$$a_1 \geqq a_2 \geqq \cdots \geqq a_r.$$

Man rechnet leicht nach, daß

$$m_i = \exp(\sqrt{-1}\,\pi\, h_{\gamma_i}) \in M$$

(vgl. (4)); m_i ist aber offensichtlich auch ein Element der Cartan-Untergruppe $H \subset U$. Lemma 1 zufolge muß das höchste Gewicht von π am Punkt m_i den Wert 1 annehmen:

$$1 = \exp\langle \sum_j a_j \gamma_j,\ \sqrt{-1}\,\pi\, h_{\gamma_i}\rangle = \exp(2\pi\sqrt{-1}\, a_i).$$

Alle a_i sind also ganze Zahlen, wie behauptet.

Jedem $i \in \hat{K}$ entsprechend wähle ich eine Darstellung auf dem Vektorraum W_i aus; W_i^* ist der Darstellungsraum der dualen Darstellung. Für $j \in \hat{U}$ (= Menge der Äquivalenzklassen irreduzibler Darstellungen von U) haben V_j und V_j^* analoge Bedeutung. Das Symbol einer Darstellung, gefolgt von dem Subskript U oder M, bezeichnet den Unterraum der U- oder M-invarianten Vektoren. Der Einfachheit halber setze ich das Gleichheitszeichen zwischen isomorphe Darstellungen von K, und das Zeichen $\subset$ zwischen Darstellungen, von denen die erste in die zweite eingebettet werden kann. Schließlich sei J die Untermenge derjenigen $j \in \hat{U}$, deren höchste Gewichte die Form (8) haben. Unter Benutzung des Satzes von Peter-Weyl, des Frobeniusschen Reziprozitätssätzes und von Lemma 2 ergibt sich dann

$$\begin{aligned} L^2 &= L^2(K)_M \subset L^2(K)_{U\cap M} \\ &= \bigoplus_{i\in\hat{K}} W_i \otimes (W_i^*)_{U\cap M} \\ &= \bigoplus_{i\in\hat{K}} \bigoplus_{j\in\hat{U}} W_i \otimes \mathrm{Hom}(V_j^*, W_i^*)_U \otimes (V_j^*)_{U\cap M} \qquad (9) \\ &\subset \bigoplus_{i\in\hat{K}} \bigoplus_{j\in J} W_i \otimes \mathrm{Hom}(V_j^*, W_i^*)_U . \end{aligned}$$

Die Behauptung a) ist jetzt eine Folge von

Lemma 3. *Für $i \in I$ und $j \in J$ ist* $\mathrm{Hom}(V_j, W_i)_U$ *höchstens eindimensional und nur dann von Null verschieden, wenn die beiden Darstellungen dasselbe höchste Gewicht haben.*

Beweis. Wenn D ein Tubengebiet ist, gibt es nichts zu beweisen, weil $U = K$. Ich nehme deshalb an, daß D kein Tubengebiet ist, und daß $\mathrm{Hom}(V_j, W_i)_U \neq 0$. Dann kann V_j U-invariant in W_i eingebettet werden, also $V_j \subset W_i$. Ich betrachte W_i als Darstellungsraum für die Lie-Algebra $\mathfrak{k}$ und wähle Vektoren $v \in V_j$, $w \in W_i$, die den jeweiligen höchsten Gewichten entsprechen. Durch wiederholtes Einwirken positiver Wurzelvektoren, etwa $e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_r}$, gelangt man von v zu einem Vielfachen von w. Weil jeder positive Wurzelvektor $e_\alpha \in \mathfrak{u}$ v zum Verschwinden bringt, sind entweder v und w einander proportional — und genau das muß ich zeigen, um das Lemma zu beweisen —, oder aber mindestens eine der positiven, kompakten Wurzeln $\alpha_1, \ldots, \alpha_m$ ist nicht eine Wurzel von $(\mathfrak{u}, \mathfrak{h})$. Daß dies nicht zutreffen kann, sieht man folgendermaßen. Als die Differenz der beiden höchsten Gewichte ist $\sum \alpha_i$ eine Linearkombination von $\gamma_1, \ldots, \gamma_r$. Eine positive, kompakte Wurzel ist genau dann nicht eine Wurzel von $(\mathfrak{u}, \mathfrak{h})$, wenn ihre Beschränkung auf $\mathfrak{h}_-$ die Form $-\frac{1}{2}\gamma_j$ hat [9]. Es genügt, das nächste Lemma anzuführen, das auch später von Nutzen sein wird.

Es sei $\alpha_1, \ldots, \alpha_k$ eine Aufzählung der positiven, kompakten Wurzeln α mit

$$\alpha|_{\mathfrak{h}_-} = -\tfrac{1}{2}\gamma_j, \qquad 1 \leqq j \leqq r,$$

und $\beta_1, \ldots, \beta_k$ eine Aufzählung der positiven, nichtkompakten Wurzeln β mit

$$\beta|_{\mathfrak{h}_-} = \tfrac{1}{2}\gamma_j, \qquad 1 \leqq j \leqq r.$$

Solche Wurzeln existieren genau dann, wenn D kein Tubengebiet ist. In diesem Fall gibt es eine einfache Wurzel in der Menge $\{\alpha_1, \ldots, \alpha_k\}$, etwa α_1; die einzigen positiven Wurzeln, für die α_1 mit positivem Koeffizienten vorkommt, wenn sie als lineare Kombination der einfachen Wurzeln geschrieben werden, sind $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ (Moore [9]). Daraus folgt:

Lemma 4. *Keine lineare Kombination von $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ mit nichtnegativen Koeffizienten, von denen nicht alle verschwinden, kann linear durch die übrigen Wurzeln ausgedrückt werden.*

Für den Beweis von b) sei daran erinnert, daß der Raum der antiholomorphen, zu S tangentialen Vektoren am Punkt $eM \in K/M \simeq S$ der Lie-Algebra $\mathfrak{q}_-$ entspricht. Deswegen kann B mit dem Hilbert-Raum all der $f \in L^2(K)$ identifiziert werden, die von rechts M-invariant sind, und auf die jedes $x \in \mathfrak{q}_-$ trivial operiert, wenn x als links-invariantes, komplex-wertiges Vektorfeld auf K betrachtet wird. Fädelt man diese Beschreibung durch die Isomorphismen in (9), so findet man, daß B in die direkte Summe

$$\bigoplus_{i \in \hat{K}} \bigoplus_{j \in J} W_i \otimes \mathrm{Hom}(V_j^*, W_i^*)_{\mathfrak{u} \oplus \mathfrak{q}_-}$$

eingebettet werden kann. Hier ist $\mathrm{Hom}(V_j^*, W_i^*)_{\mathfrak{u} \oplus \mathfrak{q}_-}$ folgendermaßen zu verstehen: W_i^* und V_j^* sind auf natürliche Weise Darstellungsräume für die Lie-Algebren $\mathfrak{k}$ und $\mathfrak{u}$; da $\mathfrak{u}$ $\mathfrak{q}_-$ normalisiert, darf man V_j^* auch als Darstellungsraum für $\mathfrak{u} \oplus \mathfrak{q}_-$ auffassen, wobei $\mathfrak{q}_-$ trivial operiert. Nun sei $V_j^* \subset W_i^*$ eine $\mathfrak{u} \oplus \mathfrak{q}_-$-invariante Einbettung, und $v \in V_j^*$ sei ein Vektor, der dem niedrigsten Gewicht von V_j^* entspricht. Weil alle Wurzelvektoren $e_{-\alpha}$, α kompakt und positiv, v zum Verschwinden bringen, muß v auch dem niedrigsten Gewicht von W_i^* entsprechen. Der Raum der

$\mathfrak{u} \oplus \mathfrak{q}_-$-invarianten Homomorphismen $V_j^* \to W_i^*$ ist also höchstens eindimensional und nur von Null verschieden, wenn V_j^* und W_i^* dasselbe niedrigste Gewicht haben, d.h. wenn V_j und W_i dasselbe höchste Gewicht haben. Da D kein Tubengebiet ist, gibt es eine kompakte, positive Wurzel α, so daß

$$\alpha = \nu - \tfrac{1}{2}\gamma_r, \qquad \nu \perp \gamma_1, \ldots, \gamma_r$$

(Moore [9]). Deswegen ist (8) nur dann das höchste Gewicht einer irreduziblen Darstellung von K, falls $n_r \geqq 0$. Damit ist b) bewiesen. Das vorhergehende Argument ist übrigens ein sehr spezieller Fall des verallgemeinerten Satzes von Borel-Weil [1].

Nun zu c)! Wenn D als Tubengebiet realisiert werden kann, fallen die Gruppen U und K zusammen, und gemäß Lemma 2 kommen nur solche irreduziblen Darstellungen von K in L^2 vor, deren höchste Gewichte die Form (8) haben. Wenn andererseits D nicht ein Tubengebiet ist, sind nur Darstellungen aus der Menge I in B enthalten, wie gerade gezeigt worden ist. Dadurch, daß man die Elemente der symmetrischen Tensoralgebra von $\mathfrak{p}_-$ als Polynome auffaßt und auf S beschränkt, bettet man diese Algebra in L^2, bzw. B, ein. In beiden Fällen gilt deshalb:

Lemma 5. *Das höchste Gewicht einer jeden irreduziblen Darstellung von K, die in der symmetrischen Tensoralgebra von $\mathfrak{p}_-$ vorkommt, verschwindet auf $\mathfrak{h}_+$.*

Es sei Λ_+ die Menge der dominanten Gewichte, χ_λ der Charakter des irreduziblen K-Moduls mit dem höchsten Gewicht $\lambda \in \Lambda_+$, und $m_{k,\lambda}$ die Multiplizität dieses K-Moduls im k-ten symmetrischen Tensorprodukt von $\mathfrak{p}_-$. Ich zähle die positiven, nichtkompakten Wurzeln als $\beta_1, \ldots, \beta_n$ auf. Dann besteht auf H die Identität

$$\sum_\lambda m_{k,\lambda}\chi_\lambda = \sum_{l_i} \prod_i \exp(-l_i \beta_i);$$

$(l_1, \ldots, l_n)$ läuft dabei über alle n-tupel natürlicher Zahlen mit $\sum l_i = k$. Durch mehrfaches Anwenden der Summenformel für die geometrische Reihe erhält man

$$\sum_{k,\lambda} t^k m_{k,\lambda}\chi_\lambda = \prod_i (1 - t\, e^{-\beta_i})^{-1}, \tag{10}$$

für $|t| < 1$. Ich bezeichne den Raum der konvergenten Fourier-Reihen auf H, die zu allen e^λ, $\lambda \in \Lambda_+$, orthogonal sind, mit N. Aus Weyls Charakterformel [10] folgt

$$\prod_j (1 - e^{-\alpha_j})\chi_\lambda \equiv e^\lambda \bmod N;$$

$\alpha_1, \ldots, \alpha_l$ sind die positiven, kompakten Wurzeln. Folglich:

$$\prod_j (1 - e^{-\alpha_j}) \prod_i (1 - t\, e^{-\beta_i})^{-1} \equiv \sum_{k,\lambda} t^k m_{k,\lambda} e^\lambda \bmod N. \tag{11}$$

Lemma 6. *Es genügt, die Behauptung* c) *für Tubengebiete zu beweisen.*

Beweis. Falls nötig, sollen die Wurzeln $\alpha_1, \dots, \alpha_l$, $\beta_1, \dots, \beta_n$ so umnumeriert werden, daß $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_k$ die gleiche Bedeutung haben wie in Lemma 4. Lemma 5 behauptet, daß $m_{k,\lambda}=0$, wenn λ nicht eine lineare Kombination von $\gamma_1, \dots, \gamma_r$ ist; und wegen Lemma 4 kann der Faktor

$$\prod_{j\leqq k}(1-e^{-\alpha_j})\prod_{i\leqq k}(1-t\,e^{-\beta_i})^{-1}$$

auf der linken Seite von (11) nur durch das konstante Glied 1 Summanden der Art e^{μ}, $\mu=\sum n_i\gamma_i$, zur Fourier-Reihe beisteuern. Modulo N darf dieser Faktor deshalb ausgelassen werden. Weil

$$\{\pm\alpha_{k+1}, \dots, \pm\alpha_l, \pm\beta_{k+1}, \dots, \pm\beta_n\}$$

das Wurzelsystem eines Tubengebietes ist [9], habe ich, was die Behauptung c) anbelangt, D durch ein Tubengebiet ersetzt.

Lemma 7. *Die Behauptung* c) *stimmt für* $r=1$ *und* $r=2$.

Beweis. Ich darf annehmen, daß D ein Tubengebiet ist. Für $r=1$ muß D dann die Einheitsscheibe in $\mathbf{C}$ sein; das folgt leicht aus Moores Beschreibung der Wurzeln [9]. In diesem Fall trifft c) offensichtlich zu. Wenn $r=2$, können die positiven, nichtkompakten Wurzeln außer γ_1, γ_2 als

$$\tfrac{1}{2}(\gamma_1+\gamma_2)+\mu_i, \qquad 1\leqq i\leqq p, \quad \text{mit} \quad \mu_i(\mathfrak{h}_-)=0$$

dargestellt werden [9]. Nun seien $\xi_1, \dots, \xi_q$ die positiven, kompakten Wurzeln, die auf $\mathfrak{h}_-$ verschwinden. Dann sind

$$\tfrac{1}{2}(\gamma_2-\gamma_1)+\mu_i, \qquad 1\leqq i\leqq p$$

die übrigen positiven, kompakten Wurzeln. Ich definiere formal

$$z_i=(t\,e^{-\gamma_i})^{\frac{1}{2}}, \qquad i=1,2,$$
$$a_i=e^{-\mu_i}, \qquad 1\leqq i\leqq p,$$

so daß

$$\begin{aligned}\prod_{1\leqq j\leqq l}(1-e^{-\alpha_j})&\prod_{1\leqq i\leqq n}(1-t\,e^{-\beta_i})^{-1}\\ &=\prod_{1\leqq i\leqq q}(1-e^{-\xi_i})(1-z_1^2)^{-1}(1-z_2^2)^{-1}\\ &\quad\cdot\prod_{1\leqq i\leqq p}\{(1-a_i z_1^{-1} z_2)(1-a_i z_1 z_2)^{-1}\}.\end{aligned} \tag{12}$$

Um die Multiplizitäten $m_{k,\lambda}$ zu berechnen, müssen die Nenner als Potenzreihen in z_1, z_2 ausgedrückt werden; wegen (11) und Lemma 5 sind, modulo N, nur die Summanden von Interesse, in denen z_1 mindestens so oft vorkommt wie z_2, und ohne einen Koeffizienten e^μ, $\mu(\mathfrak{h}_-)=0$, $\mu \neq 0$. In allen Summanden der Laurant-Reihe von

$$\prod_{1 \leqq i \leqq q} (1-e^{-\xi_i})(1-z_2^2)^{-1} \prod_{2 \leqq i \leqq p} \{(1-a_i z_1^{-1} z_2)(1-a_i z_1 z_2)^{-1}\}$$

ist z_2 mit einer mindestens ebenso hohen Potenz vertreten wie z_1, während in

$$(1-z_1^2)^{-1}(1-a_1 z_1^{-1} z_2)(1-a_1 z_1 z_2)^{-1} = (1-z_1^2)^{-1} - a_1 z_1^{-1} z_2 (1-a_1 z_1 z_2)^{-1}$$

nur $(1-z_1^2)^{-1}$ höhere Potenzen von z_1 als von z_2 liefert. Modulo N kann deshalb im zweiten Produkt auf der rechten Seite von (12) der Faktor für $i=1$ durch $(1-z_1^2)^{-1}$ ersetzt werden. Wiederholt man dieses Argument für $i=2, \ldots, p$, so erhält man schließlich

$$\begin{aligned}
&\prod_{1 \leqq j \leqq l} (1-e^{-\alpha_j}) \prod_{1 \leqq i \leqq n} (1-t\, e^{-\beta_i})^{-1} \\
&\quad \equiv \prod_{1 \leqq i \leqq q} (1-e^{-\xi_i})(1-z_1^2)^{-1}(1-z_2^2)^{-1} \\
&\quad \equiv (1-z_1^2)^{-1}(1-z_2^2)^{-1} \equiv \sum_{0 \leqq i \leqq j} z_1^{2j} z_2^{2i} \bmod N.
\end{aligned}$$

Der zweite Schritt ist dadurch gerechtfertigt, daß in der Fourier-Reihe von

$$\prod_{1 \leqq i \leqq q} (1-e^{-\xi_i})$$

alle Summanden außer der Konstante 1 die Form e^μ, $\mu(\mathfrak{h}_-)=0$, $\mu \neq 0$, haben. Also gilt

$$\sum_{k,\lambda} t^k m_{k,\lambda} e^\lambda = \sum_{0 \leqq i \leqq j} t^{i+j} \exp(-j\gamma_1 - i\gamma_2),$$

und damit ist c) für $r=2$ bewiesen.

Lemma 8. *Die Behauptung* c) *stimmt, falls* $\mathfrak{h}_+ = 0$.

Beweis. Aus Moores Beschreibung der Wurzeln [9] folgt dann nämlich, daß D die Siegelsche obere Halbebene sein muß. Die positiven, kompakten Wurzeln sind

$$\tfrac{1}{2}(\gamma_i - \gamma_j), \qquad 1 \leqq j < i \leqq r,$$

und die positiven, nichtkompakten Wurzeln

$$\tfrac{1}{2}(\gamma_i + \gamma_j), \qquad 1 \leqq j \leqq i \leqq r.$$

Ich definiere formal

$$z_i = (t\, e^{-\gamma_i})^{\frac{1}{2}},$$

und erhalte so

$$\begin{aligned}
&\prod_j (1-e^{-\alpha_j}) \prod_i (1-t\, e^{-\beta_i})^{-1} \\
&= (z_1^{r-1} z_2^{r-2} \dots z_{r-1})^{-1} \prod_{1 \leqq i < j \leqq r} (z_i - z_j) \prod_{1 \leqq i \leqq j \leqq r} (1 - z_i z_j)^{-1} \\
&= (z_1^{r-1} z_2^{r-2} \dots z_{r-1})^{-1} \sum \delta_{1\,2\dots r}^{i_1 i_2 \dots i_r} z_{i_1}^{r-1} z_{i_2}^{r-2} \dots z_{i_{r-1}} \\
&\quad \cdot (1 - z_{i_1}^2)^{-1} (1 - z_{i_1}^2 z_{i_2}^2)^{-1} \dots (1 - z_{i_1}^2 z_{i_2}^2 \dots z_{i_r}^2)^{-1} \\
&= (z_1^{r-1} z_2^{r-2} \dots)^{-1} \sum \delta_{1\,2\dots r}^{i_1 i_2 \dots i_r} z_{i_1}^{r-1} z_{i_2}^{r-2} \dots z_{i_{r-1}} \\
&\quad \cdot \sum_{0 \leqq j_r \leqq j_{r-1} \leqq \dots \leqq j_1} z_{i_1}^{2 j_1} z_{i_2}^{2 j_2} \dots z_{i_r}^{2 j_r}
\end{aligned}$$

(vgl. (11)); für den zweiten Schritt habe ich Theorem 1.1.1 von Hua [6] benutzt. Modulo N kommt es in der Potenzreihe nur auf die Summanden der Form

$$z_1^{j_1} z_2^{j_2} \dots z_r^{j_r}, \qquad \text{mit } j_1 \geqq j_2 \geqq \dots \geqq j_r$$

an. Also gilt

$$\begin{aligned}
&\prod_j (1-e^{-\alpha_j}) \prod_i (1-t\, e^{-\beta_i})^{-1} \\
&\equiv \sum_{0 \leqq j_r \leqq j_{r-1} \leqq \dots \leqq j_1} t^{j_1 + \dots + j_r} \exp(-j_1 \gamma_1 - \dots - j_r \gamma_r) \bmod N,
\end{aligned}$$

was zu zeigen war.

Lemma 9. *Wenn D ein Tubengebiet ist, mit $\mathfrak{h}_+ \neq 0$ und $r > 2$, dann sind $\frac{1}{2}(\gamma_i + \gamma_j)$ und $\frac{1}{2}(\gamma_i - \gamma_j)$, für $i \neq j$, keine Wurzeln.*

Beweis. Für jedes Paar $i \neq j$, $1 \leqq i, j \leqq r$, kommt entweder $\frac{1}{2}(\gamma_i - \gamma_j)$ oder ein Ausdruck der Art

$$\tfrac{1}{2}(\gamma_i - \gamma_j) + \mu, \; \mu(\mathfrak{h}_-) = 0, \qquad \mu \neq 0, \tag{13}$$

möglicherweise auch beide, als kompakte Wurzel vor [9]. Es muß mindestens eine Wurzel (13) geben, sonst wäre $\mathfrak{h}_+$ Null. Angenommen, es existierte eine Wurzel $\frac{1}{2}(\gamma_p - \gamma_q)$, $p \neq q$, und r sei größer als 2; dann gäbe es drei verschiedene ganze Zahlen i, j, k zwischen 1 und r, und Wurzeln $\alpha = \frac{1}{2}(\gamma_i - \gamma_j) + \mu$, $\alpha' = \frac{1}{2}(\gamma_j - \gamma_k)$. Weil $(\alpha, \alpha) > (\alpha', \alpha')$ und $(\alpha, \alpha') < 0$, müßte z. B. auch

$$\alpha + 2\alpha' = \tfrac{1}{2}(\gamma_i + \gamma_j) - \gamma_k + \mu$$

als Wurzeln auftreten, und das ist unmöglich [9]. Die Wurzeln $\frac{1}{2}(\gamma_i - \gamma_j)$, $\frac{1}{2}(\gamma_i + \gamma_j)$, mit $i \neq j$, kommen immer gemeinsam vor, und somit ist der Beweis erbracht.

Wegen der Lemmata 6–9 darf ich mich auf die folgende Situation beschränken:

$$\begin{gathered} D \text{ ist als Tubengebiet realisierbar, und} \\ \tfrac{1}{2}(\gamma_i \pm \gamma_j) \notin \Delta \qquad \text{für } 1 \leqq i < j \leqq r. \end{gathered} \tag{14}$$

Das sei von jetzt ab immer vorausgesetzt. Ich bezeichne die Weyl-Gruppe von $(\mathfrak{k}, \mathfrak{h})$ mit W, und das Vorzeichen eines jeden $w \in W$ mit ε_w. Weiter definiere ich

$$\Delta_0 = \{\alpha \in \Delta \mid \alpha(\mathfrak{h}_-) = 0\},$$

$$\rho_0 = \text{halbe Summe aller positiven } \alpha \in \Delta_0.$$

Lemma 10. *Für jede Permutation σ von $\{1, 2, \ldots, r\}$ gibt es ein $w \in W$, so daß $\varepsilon_w = +1$, $w(\rho_0) = \rho_0$, und*

$$w(\gamma_i) = \gamma_{\sigma i}, \qquad 1 \leqq i \leqq r.$$

Beweis. Es genügt, eine Transposition σ,

$$\sigma i = j, \qquad \sigma j = i, \qquad \sigma k = k \qquad \text{für } k \neq i, j,$$

zu betrachten. Wegen (14) gibt es eine kompakte Wurzel α der Art (13); $\alpha' = \frac{1}{2}(\gamma_i - \gamma_j) - \mu$ ist dann ebenfalls eine kompakte Wurzel. Weil $\alpha + 2\gamma_j$ nicht eine Wurzel sein kann, müssen α und die γ_k die gleiche Länge haben, also

$$(\mu, \mu) = \left(\tfrac{1}{2}(\gamma_i - \gamma_j), \tfrac{1}{2}(\gamma_i - \gamma_j)\right). \tag{15}$$

Nun sei $w' \in W$ das Produkt der beiden Spiegelungen an den Ebenen $\alpha = 0$, $\alpha' = 0$; w' hat das Vorzeichen $+1$, läßt $\mathfrak{h}_+$ und $\mathfrak{h}_-$ invariant, und permutiert die γ_k auf die gewünschte Weise. Weil die Wurzeln $\frac{1}{2}(\gamma_p - \gamma_q) \pm \nu$, mit $\nu(\mathfrak{h}_-) = 0$, $\nu \neq 0$, $p \neq q$, immer paarweise auftreten, vertauscht w' das Vorzeichen einer geraden Zahl von kompakten, positiven Wurzeln in $\Delta - \Delta_0$, und deshalb auch einer geraden Zahl von positiven Wurzeln in Δ_0. In der Weyl-Gruppe des Wurzelsystems Δ_0 gibt es daher ein Element w'', mit Vorzeichen $+1$, das genau dieselben positiven Wurzeln in Δ_0 negativ macht wie w'. Dann hat $w = w' w''^{-1}$ alle verlangten Eigenschaften.

Wegen (14) können die positiven, kompakten Wurzeln, die nicht zu Δ_0 gehören, als

$$\tfrac{1}{2}(\gamma_i - \gamma_j) \pm \mu_{ijk}, \qquad 1 \leqq j < i \leqq r,\ 1 \leqq k \leqq s \tag{16}$$

aufgezählt werden [9]; dabei gilt $\mu_{ijk}(\mathfrak{h}_-) = 0$, $\mu_{ijk} \neq 0$. Daß s von i und j unabhängig ist, folgt aus Lemma 10, und wegen der Beobachtung (15) im Beweis dieses Lemmas haben die μ_{ijk} alle dieselbe Länge, nämlich

$$(\mu_{ijk}, \mu_{ijk}) = \tfrac{1}{2}(\gamma_i, \gamma_i). \tag{17}$$

Die Menge der positiven, nichtkompakten Wurzeln besteht aus

$$\gamma_1, \ldots, \gamma_r; \quad \tfrac{1}{2}(\gamma_i+\gamma_j)\pm\mu_{ijk}, \qquad 1\leqq j<i\leqq r,\ 1\leqq k\leqq s \tag{18}$$

[9]. Die Reflektion an einer Wurzel (16) vertauscht γ_j und $\tfrac{1}{2}(\gamma_i+\gamma_j)\pm\mu_{ijk}$. Andererseits kann W jede Permutation von $\{\gamma_1, \ldots, \gamma_r\}$ erwirken. Das beweist:

Lemma 11. *Unter der Voraussetzung* (14) *operiert* W *transitiv auf die Menge der positiven, nichtkompakten Wurzeln.*

Lemma 12. *Für* $i>j$ *stimmen die beiden Zahlenmengen*

$$\{\pm(\mu_{ijk}, \rho_0)\,|\,1\leqq k\leqq s\} \quad und \quad \left\{\pm\frac{k}{2}(\gamma_1,\gamma_1)\,|\,k=0,1,\ldots,s-1\right\}$$

überein.

Beweis. In Anbetracht von Lemma 10 genügt es, diese Behauptung für $i=2, j=1$ zu begründen. Aus (17) folgt

$$\left(\tfrac{1}{2}(\gamma_2-\gamma_1)+\mu_{21k}, \tfrac{1}{2}(\gamma_2-\gamma_1)\pm\mu_{21m}\right)>0$$

für $1\leqq k, m\leqq s$, $k\neq m$. Also sind Summe und Differenz von μ_{21k}, μ_{21m} Wurzeln, die auf $\mathfrak{h}_-$ verschwinden müssen, d.h.

$$\mu_{21k}\pm\mu_{21m}\in\Delta_0 \qquad \text{für } k\neq m. \tag{19}$$

Unter den Wurzeln (16), mit $i=2, j=1$, gibt es eine einfache, kompakte Wurzel [9], etwa $\tfrac{1}{2}(\gamma_2-\gamma_1)+\mu_{211}$. Dann müssen $\pm\mu_{21k}-\mu_{211}$, $k\geqq 2$, positive Wurzeln in Δ_0 sein, die notwendigerweise ein positives inneres Produkt mit ρ_0 haben. Daher:

$$|(\mu_{21k}, \rho_0)|\leqq -(\mu_{211}, \rho_0), \qquad 1\leqq k\leqq s.$$

Um (μ_{211}, ρ_0) zu berechnen, definiere ich

$$\rho = \text{halbe Summe aller positiven, kompakten Wurzeln}. \tag{20}$$

Wegen der Beschreibung (16) der positiven, kompakten Wurzeln außerhalb von Δ_0 findet man

$$\rho=\rho_0+s\sum_{i>j}\tfrac{1}{2}(\gamma_i-\gamma_j)=\rho_0+s\sum_i\left(i-\frac{r+1}{2}\right)\gamma_i. \tag{21}$$

Das innere Produkt einer einfachen Wurzel mit ρ ist bekanntlich das halbe Quadrat der Länge der Wurzel, so daß

$$\begin{aligned}(\mu_{211}, \rho_0)&=\left(\tfrac{1}{2}(\gamma_2-\gamma_1)+\mu_{211}, \rho_0\right)\\ &=\left(\tfrac{1}{2}(\gamma_2-\gamma_1)+\mu_{211}, \rho-s\sum_i\left(i-\frac{r+1}{2}\right)\gamma_i\right)\\ &=\tfrac{1}{2}(\gamma_1,\gamma_1)-\tfrac{1}{2}s(\gamma_1,\gamma_1)=-\frac{s-1}{2}(\gamma_1,\gamma_1).\end{aligned}$$

Weil ρ_0 ein nichtsinguläres Gewicht des Wurzelsystems Δ_0 ist, folgt aus (17) und (19)

$$2(\mu_{21k} \pm \mu_{21m}, \rho_0)/(\gamma_1, \gamma_1) \in \mathbf{Z} - \{0\} \qquad \text{für } k \neq m.$$

Die Menge

$$\{\pm 2(\mu_{21k}, \rho_0)/(\gamma_1, \gamma_1) \mid k = 1, \dots, s\}$$

besteht demnach aus mindestens $2s-1$ verschiedenen ganzen Zahlen zwischen $\pm(s-1)$, also genau aus $0, \pm 1, \pm 2, \dots, \pm(s-1)$.

Korollar. *Ein Gewicht der Form*

$$\rho_0 + a_1 \gamma_1 + a_2 \gamma_2 + \dots + a_r \gamma_r, \qquad a_i \in \mathbf{R},\ a_i - a_j \in \mathbf{Z}$$

ist genau dann nichtsingulär (bezüglich $\mathfrak{k}$*), wenn* $|a_i - a_j| \geqq s$ *für alle* $i \neq j$.

Die Wurzeln in Δ_0 stehen nämlich auf keinen Fall zu einem solchen Gewicht senkrecht, während die übrigen positiven, kompakten Wurzeln zur Menge (16) gehören.

Lemma 13. *In der Situation* (14) *ist die Behauptung* c) *mit der Identität*

$$\begin{aligned} &\prod_\alpha (e^{\alpha/2} - e^{-\alpha/2}) \prod_\beta (1 - t e^{-\beta})^{-1} \\ &= \frac{1}{r!} \sum_W \varepsilon_w\, w \{ e^{\rho_0} \prod_i (t e^{-\gamma_i})^{-s(r-1)/2} \prod_i (1 - t e^{-\gamma_i})^{-1} \} \end{aligned} \tag{22}$$

gleichbedeutend; dabei laufen α *und* β *über alle kompakte, bzw. nichtkompakte, positive Wurzeln.*

Beweis. Die Behauptung c) besagt, daß die Multiplizitäten $m_{k,\lambda}$ in (10) 1 oder 0 sind, je nachdem ob λ die Form (2) hat, mit $\sum_i n_i = k$, oder nicht. Wenn man χ_λ durch Weyl's Charakterformel [10] ausdrückt und dabei (21) benützt, so werden c) und die Gleichung

$$\begin{aligned} &\prod_\alpha (e^{\alpha/2} - e^{-\alpha/2}) \prod_\beta (1 - t e^{-\beta})^{-1} \\ &= \sum_{0 \leqq n_r \leqq n_{r-1} \leqq \dots \leqq n_1} t^{n_1 + \dots + n_r} \sum_W \varepsilon_w\, w \left\{ \exp\left(\rho_0 - \sum_i \left(n_i - s i + s \frac{r+1}{2} \right) \gamma_i \right) \right\} \end{aligned} \tag{23}$$

äquivalent. Die rechte Seite kann in

$$\sum_W \varepsilon_w\, w \{ e^{\rho_0} \prod_i (t e^{-\gamma_i})^{-s(r-1)/2} \sum_{n_i} \prod_i (t e^{-\gamma_i})^{n_i} \} \tag{24}$$

umgeformt werden; die innere Summe erstreckt sich über alle $n_i \in \mathbf{Z}$ mit der Einschränkung

$$n_1 \geqq n_2 + s,\ n_2 \geqq n_3 + s, \dots, n_{r-1} \geqq n_r + s, \qquad n_r \geqq 0.$$

Wegen Lemma 10 darf man in (24) die γ_i auf beliebige Weise permutieren und deshalb auch den Durchschnitt über alle Permutationen bilden. Dieser Durchschnitt unterscheidet sich von

$$\frac{1}{r!}\sum_W \varepsilon_w\, w\left\{e^{\rho_0}\prod_i (t\,e^{-\gamma_i})^{-s(r-1)/2}\sum_{n_i\geqq 0}\prod_i (t\,e^{-\gamma_i})^{n_i}\right\}$$
$$=\frac{1}{r!}\sum_W \varepsilon_w\, w\left\{e^{\rho_0}\prod_i (t\,e^{-\gamma_i})^{-s(r-1)/2}\prod_i (1-t\,e^{-\gamma_i})^{-1}\right\}$$

durch eine Reihe mit Gliedern der Form

$$t^k\sum_W \varepsilon_w\, w(e^{\mu}),\qquad \mu \text{ singulär};$$

das folgt aus dem Korollar zu Lemma 12. Ein solcher Ausdruck aber verschwindet, so daß die rechten Seiten von (22) und (23) übereinstimmen.

Ich werde (22) durch Induktion über r beweisen. Dazu ist es allerdings nötig, die Voraussetzungen ein wenig allgemeiner zu fassen. Und zwar soll $\mathfrak{g}$ halbeinfach, und nicht mehr unbedingt einfach sein; wohl aber muß $\mathfrak{g}_0$ genau einen einfachen, nichtkompakten Faktor enthalten, und $(\mathfrak{g}, \mathfrak{k})$ soll natürlich noch ein hermitesch symmetrisches Paar sein, so daß (14) zutrifft. Alle Lemmata bleiben auch so weiterhin gültig. Lemma 7 liefert die Induktionsverankerung. Der Induktionsschritt beruht auf:

Lemma 14. *Für Unbestimmte $x_1, \dots, x_n$ über einem Körper gilt*

$$\prod_{1\leqq i\leqq n}(1-x_i)^{-1}=\sum_{1\leqq k\leqq n} x_k^{n-1}(1-x_k)^{-1}\prod_{j\neq k}(x_k-x_j)^{-1}.$$

Übrigens ist diese Identität, in verkleideter Form, genau die Behauptung c) für die Einheitskugel im $\mathbf{C}^n$.

Beweis. Ich ersetze x_i durch $t x_i$, wobei t eine zusätzliche Unbestimmte bezeichnet, und multipliziere die Nenner aus. Dadurch wird die Behauptung zur Gleichung

$$t^{n-1}\prod_{i<j}(x_i-x_j)=\sum_k(-1)^{k-1}x_k^{n-1}\prod_{l\neq k}(t-x_l)\prod_{i,j\neq k;\,i<j}(x_i-x_j).$$

Die beiden Seiten sind Polynome in t vom Grad $n-1$, die an den n Stellen $t=x_1, \dots, x_n$ übereinstimmen, und die deshalb zusammenfallen müssen.

Von jetzt ab laufen α und β als freie Indices immer über die Mengen der kompakten, bzw. nichtkompakten, positiven Wurzeln, mit den jeweils angezeigten Einschränkungen. Es sei daran erinnert, daß $n=\dim\mathfrak{p}_+=$ Zahl der positiven, nichtkompakten Wurzeln. Aus dem Lemma folgt

$$\prod_\beta (1-t\,e^{-\beta})^{-1}=\sum_\beta e^{-(n-1)\beta}(1-t\,e^{-\beta})^{-1}\prod_{\beta'\neq\beta}(e^{-\beta}-e^{-\beta'})^{-1}.$$

Wegen Lemma 11 kann man, anstatt über β zu summieren, $\beta=\gamma_r$ festhalten und über W summieren, muß dann aber durch $|W'|$, die Ordnung der Gruppe

$$W'=\{w\in W\,|\,w(\gamma_r)=\gamma_r\},$$

dividieren:

$$\prod_{\beta}(1-t\,e^{-\beta})^{-1}=|W'|^{-1}\sum_{W} w\{e^{-(n-1)\gamma_r}(1-t\,e^{-\gamma_r})^{-1}\prod_{\beta\neq\gamma_r}(e^{-\gamma_r}-e^{-\beta})^{-1}\}.$$

Weil die Fourier-Reihe

$$\prod_{\alpha}(e^{\alpha/2}-e^{-\alpha/2})=e^{\rho}\prod_{\alpha}(1-e^{-\alpha})$$

(vgl. (20)) bezüglich W alternierend ist, führt die letzte Identität zu

$$\begin{aligned}&\prod_{\alpha}(e^{\alpha/2}-e^{-\alpha/2})\prod_{\beta}(1-t\,e^{-\beta})^{-1}\\&=|W'|^{-1}\sum_{W}\varepsilon_w\,w\{e^{\rho}(1-t\,e^{-\gamma_r})^{-1}\prod_{\beta\neq\gamma_r}(e^{\beta-\gamma_r}(e^{\beta-\gamma_r}-1)^{-1})\prod_{\alpha}(1-e^{-\alpha})\}.\end{aligned}\tag{25}$$

Wenn β über die positiven, nichtkompakten Wurzeln außer γ_r läuft, die nicht auf γ_r senkrecht stehen, läuft $\gamma_r-\beta$ über die positiven, kompakten und nicht zu γ_r orthogonalen Wurzeln. Die Zahl dieser Wurzeln ist gerade (16). Die rechte Seite von (25) kann deshalb in

$$\begin{aligned}&(-1)^{n-1}|W'|^{-1}\\&\cdot\sum_{W}\varepsilon_w\,w\{e^{\rho}(1-t\,e^{-\gamma_r})^{-1}\prod_{\beta}e^{\beta-\gamma_r}\prod_{\beta\perp\gamma_r}(1-e^{\beta-\gamma_r})^{-1}\prod_{\alpha\perp\gamma_r}(1-e^{-\alpha})\}\\&=|W'|^{-1}\sum_{W}\varepsilon_w\,w\{e^{\rho}(1-t\,e^{-\gamma_r})^{-1}\prod_{\beta\not\perp\gamma_r}e^{\beta-\gamma_r}\prod_{\beta\perp\gamma_r}(1-e^{\gamma_r-\beta})^{-1}\prod_{\alpha\perp\gamma_r}(1-e^{-\alpha})\}\end{aligned}\tag{26}$$

umgeformt werden.

In der Bezeichnungsweise von (1) definiere ich

$$\mathfrak{g}'=\mathfrak{h}\oplus\sum_{\alpha\perp\gamma_r}(\mathfrak{g}^{\alpha}\oplus\mathfrak{g}^{-\alpha})\oplus\sum_{\beta\perp\gamma_r}(\mathfrak{g}^{\beta}\oplus\mathfrak{g}^{-\beta}),$$

$$\mathfrak{g}_0'=\mathfrak{g}\cap\mathfrak{g}_0,\qquad \mathfrak{k}'=\mathfrak{g}'\cap\mathfrak{k}.$$

Dann ist $(\mathfrak{g}',\mathfrak{k}')$ ein hermitesch symmetrisches Paar vom Rang $(r-1)$, und $\mathfrak{g}_0'$ hat genau einen einfachen, nichtkompakten Faktor. Die apostrophierten Symbole ρ', W' usw. sollen dieselbe Bedeutung in bezug auf $\mathfrak{g}'$ haben wie die entsprechenden Symbole ρ, W, usw. ohne Apostroph in bezug auf $\mathfrak{g}$. Die Induktionsvoraussetzung liefert die Gleichung

$$\begin{aligned}&e^{\rho'}\prod_{\alpha\perp\gamma_r}(1-e^{-\alpha})\prod_{\beta\perp\gamma_r}(1-t\,e^{-\beta})^{-1}\\&\quad=\frac{1}{(r-1)!}\sum_{W'}\varepsilon_w\,w\{e^{\rho_0}\prod_{i<r}(t\,e^{-\gamma_i})^{-s(r-2)/2}\prod_{i<r}(1-t\,e^{-\gamma_i})^{-1}\},\end{aligned}$$

die natürlich gültig bleibt, wenn t durch e^{γ_r} ersetzt wird. Benutzt man diese Identität in (26) und bemerkt dabei, daß

$$e^{\rho-\rho'}(1-t\,e^{-\gamma_r})\prod_{\beta\perp\gamma_r} e^{\beta-\gamma_r}$$

W'-invariant ist, so erhält man

$$\begin{aligned}&\prod_{\alpha}(e^{\alpha/2}-e^{-\alpha/2})\prod_{\beta}(1-t\,e^{-\beta})^{-1}\\ &=\frac{1}{(r-1)!}\sum_{W}\varepsilon_w\,w\Big\{e^{\rho-\rho'}(1-t\,e^{-\gamma_r})^{-1}\prod_{\beta\perp\gamma_r}e^{\beta-\gamma_r}\\ &\quad\cdot e^{\rho_0}\prod_i (e^{\gamma_r-\gamma_i})^{-s(r-2)/2}\prod_{i<r}(1-e^{\gamma_r-\gamma_i})^{-1}\Big\}.\end{aligned} \tag{27}$$

Die Beschreibung (16) und (18) der positiven, kompakten und nichtkompakten Wurzeln zeigt, daß

$$\begin{aligned}&e^{\rho-\rho'}\prod_{\beta\perp\gamma_r}e^{\beta-\gamma_r}\prod_i(e^{\gamma_r-\gamma_i})^{-s(r-2)/2}\\ &=\prod_i(e^{\gamma_r-\gamma_i})^{-s(r-1)/2}=(t\,e^{-\gamma_r})^{s\,r(r-1)/2}\prod_i(t\,e^{-\gamma_i})^{-s(r-1)/2}.\end{aligned}$$

Dadurch wird die rechte Seite von (27) zu

$$\begin{aligned}&\frac{1}{(r-1)!}\sum_W\varepsilon_w\,w\Big\{e^{\rho_0}(1-t\,e^{-\gamma_r})^{-1}(t\,e^{-\gamma_r})^{s\,r(r-1)/2+r-1}\\ &\quad\cdot\prod_i(t\,e^{-\gamma_i})^{-s(r-1)/2}\prod_{i<r}(t\,e^{-\gamma_r}-t\,e^{-\gamma_i})^{-1}\Big\}.\end{aligned}$$

Wegen Lemma 10 dürfen $\gamma_1,\dots,\gamma_r$ beliebig permutiert werden, so daß

$$\begin{aligned}&\prod_\alpha(e^{\alpha/2}-e^{-\alpha/2})\prod_\beta(1-t\,e^{-\beta})^{-1}\\ &=\frac{1}{r!}\sum_W\varepsilon_w\,w\Big\{e^{\rho_0}\prod_i(t\,e^{-\gamma_i})^{-s(r-1)/2}\sum_k\big[(1-t\,e^{-\gamma_k})^{-1}\\ &\quad\cdot(t\,e^{-\gamma_k})^{s\,r(r-1)/2+r-1}\prod_{i\neq k}(t\,e^{-\gamma_k}-t\,e^{-\gamma_i})^{-1}\big]\Big\}\\ &=\frac{1}{r!}\sum_W\varepsilon_w\,w\Big\{e^{\rho_0}\prod_i(t\,e^{-\gamma_i})^{-s(r-1)/2}\sum_k\big[(1-t\,e^{-\gamma_k})^{-1}\\ &\quad\cdot(t\,e^{-\gamma_k})^{r-1}\prod_{i\neq k}(t\,e^{-\gamma_k}-t\,e^{-\gamma_i})^{-1}\big]\Big\}\\ &+\frac{1}{r!}\sum_W\varepsilon_w\,w\Big\{e^{\rho_0}\prod_i(t\,e^{-\gamma_i})^{-s(r-1)/2}\sum_k\big[(1-t\,e^{-\gamma_k})^{-1}\\ &\quad\cdot(t\,e^{-\gamma_k})^{r-1}\big((t\,e^{-\gamma_k})^{s\,r(r-1)/2}-1\big)\prod_{i\neq k}(t\,e^{-\gamma_k}-t\,e^{-\gamma_i})^{-1}\big]\Big\}.\end{aligned} \tag{28}$$

Der erste Summand im letzten Ausdruck stimmt aufgrund von Lemma 14 mit der rechten Seite von (22) überein. Um den Induktionsschritt zu vollenden, muß ich also zeigen, daß der zweite Summand identisch verschwindet.

Lemma 15. *Für jede positive ganze Zahl l ist die rationale Funktion*

$$\sum_{1\leqq k\leqq n} x_k^{n-1}(1-x_k^l)(1-x_k)^{-1}\prod_{i\neq k}(x_k-x_i)^{-1}$$

ein Polynom in $x_1, \ldots, x_n$, dessen totaler Grad $l-1$ nicht übersteigt.

Beweis. Die rationale Funktion kann als Quotient f/g ausgedrückt werden, wobei das Polynom

$$f=f(x_1,\ldots,x_n)=\sum_{1\leqq k\leqq n}(-1)^{k-1}(x_k^{n-1}+\cdots+x_k^{n+l-2})\prod_{i<j;\, i,j\neq k}(x_i-x_j)$$

den Grad

$$n+l-2+(n-1)(n-2)/2=l-1+n(n-1)/2$$

hat, und

$$g=g(x_1,\ldots,x_n)=\prod_{i<j}(x_i-x_j)$$

ein Polynom $n(n-1)/2$-ten Grades ist. Offensichtlich verschwindet f auf den Hyperebenen $x_i=x_j$, $i<j$; deshalb kann f durch g geteilt werden.

Wie aus dem Lemma folgt, kann der zweite Summand auf der rechten Seite in (28) als Linearkombination von Gliedern

$$\sum_W \varepsilon_w\, w\{e^{\rho_0}\prod_i (t\, e^{-\gamma_i})^{-s(r-1)/2+n_i}\} \tag{29}$$

geschrieben werden, mit

$$n_i\in\mathbf{Z},\qquad n_i\geqq 0,\qquad n_1+n_2+\cdots+n_r<s\,r(r-1)/2. \tag{30}$$

Wegen des Korollars zu Lemma 12 ist ein Gewicht

$$\sum_i (s(r-1)/2-n_i)\,\gamma_i$$

singulär, falls (30) zutrifft. Also verschwinden alle Glieder (29). Damit ist c) bewiesen.

Zusatz zur Korrektur: B. Kostant hat mir ein unveröffentlichtes Resultat mitgeteilt, das die Behauptung c) als Spezialfall enthält. Und zwar sei $\mathfrak{b}=\mathfrak{m}\oplus\mathfrak{n}$ eine parabolische Unteralgebra einer halbeinfachen, komplexen Lie-Algebra; dabei bezeichnet $\mathfrak{n}$ das Nilradikal von $\mathfrak{b}$ und $\mathfrak{m}$ ein reduktives Komplement für $\mathfrak{n}$ in $\mathfrak{b}$. Durch die adjungierte Darstellung operiert $\mathfrak{m}$ auf $\mathscr{Z}$, das Zentrum der universellen einhüllenden Algebra von $\mathfrak{n}$. Kostants Resultat besagt nun, daß jede irreduzible Darstellung von $\mathfrak{m}$ höchstens einmal in $\mathscr{Z}$ enthalten ist; weiterhin lassen sich diejenigen Darstellungen, die tatsächlich vorkommen, genau kennzeichnen. Im vorliegenden Fall setze man $\mathfrak{m}=\mathfrak{k}$, $\mathfrak{n}=\mathfrak{p}_-$. Weil $\mathfrak{p}_-$ abelsch ist, fällt $\mathscr{Z}$ mit der symmetrischen Tensoralgebra von $\mathfrak{p}_-$ zusammen. Kostants Beschreibung der Algebra $\mathscr{Z}$ führt dann sofort zur Behauptung c).

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Smoothings of Sphere Bundles over Spheres in the Stable Range

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The general problem of classifying up to orientation-preserving diffeomorphism those smooth manifolds homeomorphic to a given manifold is probably too complicated to be treated in any effective uniform manner. The first case to be treated was of course the sphere, where everything has been established modulo computation of the Adams spectral sequence. The next case considered was a product of two spheres, where the author and DeSapio independently reduced the classification to homotopy theory with the exception of determining the action of ∂P_{4k+2} on some $(4k+1)$-dimensional products. It is completely straightforward to extend this classification to the k-sphere bundles associated to $(k+1)$-plane bundles over S^n which have nowhere zero cross sections and satisfy $k<n$; this was done in the author's thesis and in [1]. The latter paper also announced results in the case $n \leqq k \leqq n+2$. We shall more generally give results in the case $k \geqq n$ which apparently diverge from [1] in some respects.

We make two remarks for completeness. A classification of smoothings of bundles with $k<n$ (not necessarily having cross sections) may be derived from [3, § 5]. Finally, throughout this paper the phrases "combinatorially equivalent" and "homeomorphic" are interchangeable by the Hauptvermutung for simply connected closed manifolds with torsion free homology and dimension at least 5.

1. The Case $k=n$

We shall be interested in the following situation. Let $n \geqq 3$ and $k \geqq 2$, and let $\xi \in \pi_{n-1}(SO_{k+1})$ classify a vector bundle over S^n. The associated disk and sphere bundles will be denoted by $D(\xi)$ and $S(\xi)$ respectively. Let $\xi_0 \in \pi_{n-1}(SO_k)$ be a class which maps to ξ under the canonical map; if $k \geqq n$ such a class always exists.

Let M be an oriented closed manifold. The *inertia group* of M, denoted by $I(M)$, consists of those homotopy spheres Σ such that $M \# \Sigma$ is orientation-preservingly diffeomorphic to M. The manifolds M and N are said to be *almost diffeomorphic* if and only if for some Σ, there is a diffeomorphism from $M \# \Sigma$ to N. In Section 2 we shall need the subgroup

$I_0(M) \subset I(M)$ of homotopy spheres Σ such that some diffeomorphism from $M \# \Sigma$ to M induces the same mapping in homology as the obvious homeomorphism from $M \# \Sigma$ to M.

We shall first consider the case $k=n$; clearly $S(\xi)$ is then an $(n-1)$-connected $2n$-manifold and hence the results of Wall [9] are applicable.

Theorem 1. a) *Any smooth manifold M homeomorphic to $S(\xi)$ is almost diffeomorphic to $S(\xi)$.*

b) *Suppose $n \not\equiv 1 \bmod 8$. Then $I(S(\xi))$ consists of those homotopy spheres which bound plumbings of $D(\xi)$ with $D(\tau)$ for some $\tau \in \pi_n(SO_n)$.*

c) *Suppose $n \equiv 1 \bmod 8$ and $\xi \neq 0$. Then $I(S(\xi))$ consists of those homotopy spheres which bound $(n-1)$-connected $(2n+1)$-manifolds. If $\xi = 0$, then $I = 0$.*

Proof. Let $e_1, e_2 \in H_n(S(\xi)) = \mathbf{Z} \oplus \mathbf{Z}$ be generators such that e_1 comes from the fiber and e_2 comes from a cross section with normal bundle ξ_0. In the notation of [9], if α is the normal bundle mapping and λ is the intersection pairing, then $\alpha(e_1) = 0$, $\alpha(e_2) = \xi_0$, and $\lambda(e_1, e_2) = 1$.

Statement a) follows trivially from [9]. The idea behind statement b) is to show that if h is an orientation-preserving almost diffeomorphism of $S(\xi)$, then $h_* e_1 = e_1$. (We exclude the case $\xi = 0$, since the result in that case is known [8, 2.1].) If this is true then the argument in the case $k<n$ goes through to show that Σ bounds a plumbing (see the proof of [8, 2.5] in the case $\xi = 0$, which generalizes to $k<n$ and $S(\xi)$ has a cross section).

Write $h_* e_1 = p e_1 + r e_2$. Since h_* is an isomorphism, p and r are relatively prime. However, these integers must also satisfy $\alpha(p e_1 + r e_2) = 0$. If n is even, we claim that $r = 0$. Suppose $n \equiv 0 \bmod 4$; then $\alpha(p e_1 + r e_2) = r \xi_0 + x \tau$ is not stably trivial if $r \neq 0$. Suppose $n \equiv 2 \bmod 4$; then it is easy to compute that $\alpha(p e_1 + r e_2)$ has a nontrivial Euler class if $pr \neq 0$. In the latter case $p = 0$ and $\det h_* = +1$ imply $r = \pm 1$, which again means that $\alpha(p e_1 + r e_2) = \alpha(\pm e_2)$ is not stably trivial. Therefore, it is always true that $r = 0$; since $\det h_* = +1$, it also follows that $p = \pm 1$. If $p = -1$, then $\xi = -\xi$ and upon composing with a diffeomorphism g of $S(\xi)$ such that $g_* = -1$, we obtain a new almost diffeomorphism for which $p = +1$; hence without loss of generality we may now suppose that $p = 1$. Since $\xi = 0$ unless n is even or $n \equiv 1 \bmod 8$, this concludes the proof of b).

We now prove c); as before let $h_* e_1 = p e_1 + r e_2$. Again, p and r are relatively prime. Furthermore, r must be even by normal bundle considerations. It follows that if $\Sigma \in I(S(\xi))$, then Σ bounds a manifold Q formed by attaching a handle to $D(\xi)$ along the class $p e_1 + r e_2 \in H_n(S(\xi))$. Notice that the only nontrivial homology of the highly connected manifold Q is $H_n(Q) = \mathbf{Z}_r$. This proof of the "only if" portion of 2 c) is of course a special case of [11, p. 289].

Conversely, if p and r are integers as above, then we can form Q. If the canonical copy of $D(\xi)$ is removed from Int Q, we obtain a cobordism P from $S(\xi)$ to a homotopy sphere Σ; as in [8, 1.1], P may be altered to a corbordism P' between $S(\xi)\# -\Sigma$ and S^{2n}. Let V be obtained from P' by adding a disk along S^{2n}. It follows at once that V has the homotopy type of S^n; specifically, if $a, b \in \mathbf{Z}$ are such that $ap+br=1$, then the image of $-be_1+ae_2$ in $H_n(V)$ is a generator. Since the integer a must be odd, this class has a nontrivial normal bundle which must be ξ since $\pi_{n-1}(SO_{n+1})=Z_2$. Hence V is orientation-preservingly diffeomorphic to $D(\xi)$ and consequently $\partial V=S(\xi)\# -\Sigma$ is orientation-preservingly diffeomorphic to $S(\xi)$.

Thus it remains to show that any boundary of an $(n-1)$-connected $(2n+1)$-manifold may be obtained by attaching a handle to $D(\xi)$ to form a manifold Q as in the preceding paragraph. In this paragraph we shall assume that $r>0$. Let $q\colon H_n(Q)\to \mathbf{Q}/2\mathbf{Z}$ be the self-intersection mapping defined in [11, p. 274]; if v generates $H_n(Q)=\mathbf{Z}_r$ it follows that $q(v)=p/r$. Let $\mathfrak{g}(Q)\in \mathbf{Z}/8\mathbf{Z}=\mathbf{Z}_8$ be the "Grothendieck element" defined in [11, Theorem 9, p. 288]; recall that if $0 \neq \chi \in H_n(Q)$ satisfies $2\chi=0$, then $\mathfrak{g}(Q)=2q(\chi)$ as elements of $\mathbf{Z}/4\mathbf{Z}$. If we choose $r=2$ so that χ generates $H_n(Q)$, then

$$\mathfrak{g}(Q)=2(p/2)=p \mod 4,$$

where p is odd. Thus if $\theta\colon \mathbf{Z}_8\to\Gamma_{2n}$ is the obstruction homomorphism [11, p. 287], we have shown that $\Sigma=\theta\,\mathfrak{g}(Q)=\theta(\text{odd})$ generates the subgroup of homotopy spheres which bound highly connected $(2n+1)$-manifolds. Thus c) follows, since we know that $\Sigma\in I(S(\xi))$ from the preceding paragraph.

Remark. The statement of b) is true for $n\equiv 1 \mod 8$ *if and only if θ is the zero homomorphism.* For if Q is a plumbing, then $\mathfrak{g}(Q)\in 4\mathbf{Z}_8$ by [11, p. 296]. (Compare [1, Proposition 5].)

2. The Case $k>n$

In order to state our results more conviently, we introduce some notation. Suppose the manifold T is formed by closing up the plumbing of two disk bundles over n- and k-dimensional homotopy spheres. Then $H_*(T)\cong H_*(S^n\times S^k)$, and the images of the orientation classes of the spheres provide a canonical orientation for T. If T' is another such manifold, a homotopy equivalence $h\colon T'\to T$ will be called *strongly orientation preserving* if it maps the spherical orientation classes of T' to those of T.

Theorem 2. *Let $\xi\in\pi_{n-1}(SO_{k+1})$ with $k>n$.*

a) *Any smooth manifold M combinationally equivalent to $S(\xi)$ is formed by closing up a manifold obtained by plumbing $S^k_\beta \times D^n$ with $D(\xi_0)$, where $\beta \in \Gamma_k$.*

b) *For β and ξ as above denote a chosen plumbing by $T(\beta, \xi)$. Then $T(\beta, \xi)$ and $T(\beta', \xi)$ are strongly orientation-preservingly almost diffeomorphic if and only if there is an embedding h of $S^k_{\beta'}$ into $T(\beta, \xi)$ with trivial normal bundle such that $h_*[S^k_{\beta'}] = i_*[S^k_\beta]$.*

c) *The group $I_0(T(\beta, \xi))$ has index at most 2 in the inertia group, and we have the formula*

$$I_0(T(\beta, \xi)) = I(S^k_\beta \times S^n) + I(S(\xi)).$$

Part c) of Theorem 2 was originally proved for the special case $n < k \leqq 2n-3$ by Sato [7, 6.11, p. 31].

Remarks 1. The choice of $T(\beta, \xi)$ is unique up to almost diffeomorphism.

2. Let $F^n_k \subseteq \Gamma_k$ be the subgroup of homotopy k-spheres which embed in $\mathbf{R}^{n+k}$ with trivial normal bundle. Then by 2 b) it is immediate that the almost diffeomorphism class of $T(\beta, \xi)$ only depends on the congruence class of β mod F^n_k. The author knows of no examples where $T(\beta, \xi)$ and $T(\beta', \xi)$ are almost diffeomorphic but $\beta \not\equiv \beta'$ mod F^{n+1}_k. (See Theorem 3.)

2'. To expand upon Remark 2, if ξ is fiber homotopically trivial, then by Sullivan theory the condition $\beta \equiv \beta'$ mod F^{n+1}_k is necessary for $T(\beta, \xi)$ and $T(\beta', \xi)$ to be strongly orientation-preservingly almost diffeomorphic. If in addition the Pontrjagin class of ξ is divisible by a suitable number depending on n and k, then this condition is also sufficient.

3. If there is an orientation-preserving almost diffeomorphism of $T(\beta, \xi)$ to itself which is not strongly so, then clearly $\xi = -\xi$ so that $\xi = 0$ or $n \equiv 1, 2 \bmod 8$.

The inertia group of $S(\xi)$ is given by the next result.

Proposition. *Let $k > n$. Then the homotopy sphere Σ is in $I(S(\xi))$ if and only if Σ bounds a manifold Q formed from $D(\xi)$ by adding a handle along a k-sphere in $S(\xi)$ which intersects a cross section once.*

The proof is identical to that of the case $k < n$. Obviously, just as in the case $k < n$, Q may be any plumbing of $D(\xi)$ with an n-sphere bundle over a homotopy $(k+1)$-sphere. However, more spheres may be obtained since the handle may wrap around the cross section by an element of $\pi_k(S^n)$. (Again compare [1, Proposition 5].)

The next result, first proved by Munkres [5, p. 189], now follows easily.

Corollary. *Let M be homeomorphic to $S(\xi)$. Then for any $\gamma \in \Gamma_{k+1}$, we have*

$$\gamma J(\xi) \in I(M) \subseteq \Gamma_{n+k}.$$

Proof. In the above proposition, take Q to be the plumbing of $S^{k+1}_\gamma \times D^n$ with $D(\xi)$. The result then follows from 2c).

Part b) of Theorem 2 takes a somewhat more pleasant form in the metastable range $n < k \leqq 2n-3$. From the fact that homotopy classes are representable by embeddings which are unique up to regular homotopy, we have mappings

$$F\colon \pi_k(S^n) \times \pi_{n-1}(SO_k) \to \pi_{k-1}(SO_n),$$

$$\varphi^n_k\colon \Gamma_k \to \pi_{k-1}(SO_n)$$

defined in [10] and [2] respectively. Let $\partial\colon \pi_k(S^n) \to \pi_{k-1}(SO_n)$ be the boundary map of the fibration $SO_n \to SO_{n+1} \to S^n$.

Theorem 3. *The manifolds $T(\beta, \xi)$ and $T(\beta', \xi)$ are strongly orientation-preservingly almost diffeomorphic if and only if for some $x \in \pi_k(S^n)$,*

$$\varphi^n_k(\beta) - \varphi^n_k(\beta') = F(x, \xi_0) + \partial x.$$

Proof of Theorem 2 a). This follows by methods like those of [8]. The piecewise differentiable embedding of S^n with normal bundle ξ_0 may be approximated by a smooth embedding f with the same normal bundle since $k > n$. Furthermore, the piecewise differentiable fiber inclusion $g_0 = S^k \times R^n \to M$ induces a smoothing of $S^k \times R^n$ which is of the form $S^k_\beta \times R^n$ by the Cairns-Hirsch theorem; consequently, there is a smooth embedding $g\colon S^k_\beta \to M$ with trivial normal bundle. Since the images of f and g have intersection number 1, we may assume that they intersect transversely in a single point since $k > n \geqq 3$. By a standard construction [12, 3.13, p. 65], there is an embedding of the plumbing of $S^k_\beta \times D^n$ with $D(\xi_0)$ into M. If M_0 is M with an open disk removed, by the h-cobordism theorem we may assume that the image of this embedding is M_0. This proves 2 a).

Proof of 2 b). This is similar to a). Suppose $f\colon T(\beta', \xi) \# \Sigma \to T(\beta, \xi)$ is an appropriate diffeomorphism. Let $g\colon S^k_{\beta'} \to T(B', \xi) \# \Sigma$ be a standard embedding. Then $h = fg$ has the right normal bundle and homological property. Conversely, given the embedding h, let $k\colon S^n \to T(\beta, \xi)$ be a standard embedding. As before, the images of h and k intersect transversely in a single point, and an embedding of the plumbing of $S^k_{\beta'} \times D^n$ with $D(\xi_0)$ into $T(\beta, \xi)$ is obtained. Thus the plumbing, which is $T(\beta', \xi) - \operatorname{Int} D^{n+k}$, is strongly orientation-preservingly diffeomorphic to $T(\beta, \xi) - \operatorname{Int} D^{n+k}$ by the same argument used in a).

The main geometric argument behind the proof of 2 c) appears in the next result.

Lemma. *In the above notation, there is an orientation-preserving diffeomorphism from* $T(\beta, \xi) \# -T(\beta, \xi)$ *to* $S^k_\beta \times S^n \# S(\xi)$.

Proof. It is immediate that there is an almost diffeomorphism by a standard change of basis trick which we shall exhibit for subsequent reference. Let x, y be the preferred generators of the n-dimensional homology $T(\beta, \xi)$ and $T(-\beta, \xi)$ respectively, and let a, b generate the k-dimensional homology. The intersection pairings on $T(\beta, \xi)$ and $-T(\beta, \xi)$ satisfy the relations $xa = -yb = 1$ and $xb = ya = 0$. Thus there is an embedding of the manifold

$$S^k_\beta \times S^n \# S(\xi) - \operatorname{Int} D^{n+k}$$

in $T \# -T$ which sends the respective n-dimensional homology generators to $x - y$ and y, and sends the k-dimensional generators to a and $a - b$. This establishes the existence of an orientation-preserving diffeomorphism from $T \# -T$ to $S^k_\beta \times S^n \# S(\xi) \# \Sigma$.

The proof will be completed by noticing that $\Sigma \in I_0(T(\beta, \xi))$. Let U be the connected sum cobordism from the disjoint union $T \cup -T$ to $T \# -T$. If handles are added to $T \# -T$ along the classes $x - y$ and $a - b$ in the obvious manner, one obtains a cobordism V from $T \cup -T$ to Σ. By a trivial generalization of [8, 1.1] this may be altered to a cobordism V' from $T \cup -(T \# \Sigma)$ to S^{n+k}, and the latter cobordism may be closed along the sphere to give a manifold W with boundary $T \cup -(T \# \Sigma)$. It is immediate that W is an h-cobordism and that the diffeomorphism between the ends is strongly orientation-preserving.

Proof of 2 c). The group I/I_0 is obviously a subquotient of

$$\operatorname{Aut} H^*(T(\beta, \xi)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

and the fact that orientations are preserved implies that it is in fact a subquotient of $\mathbf{Z}_2$.

Suppose that $\Sigma \in I(S^k_\beta \times S^n) + I(S(\xi))$; notice that the latter group is contained in $I_0(S^k_\beta \times S^n \# S(\xi))$. By the standard type of argument (e.g., [8, 2.5]) we can add two handles to the connected sum cobordism U from $T \cup -T$ to $T \# -T = S^k_\beta \times S^n \# S(\xi)$ and obtain a new cobordism V from $T \cup -T$ to Σ. The proof that $\Sigma \in I_0(T)$ now proceeds by an argument identical to that of the lemma.

Conversely, suppose that $\Sigma \in I_0(T)$. Then upon taking connected sums with $-T$ we obtain an orientation preserving diffeomorphism.

$$h\colon S^k_\beta \times S^n \# S(\xi) \# \Sigma \to S^k_\beta \times S^n \# S(\xi)$$

such that, in the notation of the lemma, $h_*(x-y)=x-y$ and $h_*(a-b)=a-b$. Thus Σ is obtained from $S^k_\beta \times S^n \# S(\xi)$ by adding handles along $x-y$ and $a-b$. Since the spheres upon which the handles are attached may be assumed disjoint, we may first attach along $x-y$ and then along $a-b$.

By general position we may assume that h maps the n-sphere represented by $x-y$ onto the standard $S^n \subset S^k_\beta \times S^n$. Thus the effect of adding a handle is to form a cobordism from $S^k_\beta \times S^n \# S(\xi)$ to $\Sigma_1 \# S(\xi)$, where $\Sigma_1 \in I(S^k_\beta \times S^n)$ by [8, 2.5]; the cobordism is in fact a cobordism connected sum of the trivial cobordism on $S(\xi)$ with the obvious one from $S^k_\beta \times S^n$ to Σ_1. This implies that Σ is formed from $S(\xi) \# \Sigma$ by adding a $(k+1)$-dimensional handle along a sphere which intersects the cross section once. According to the Proposition proved previously, $\Sigma \# -\Sigma_1 \in I(S(\xi))$. This completes the proof.

Proof of Theorem 3. This is merely a translation of 2 b) into the metastable range. Suppose N^{n+k} is an $(n-1)$-connected manifold and $n \leqq k \leqq 2n-3$. Then there is a well-defined normal bundle mapping

$$\alpha\colon \pi_k(N) \to \pi_{k-1}(SO_n)$$

defined in [10, p. 254]; actually it is just a generalization of the map constructed in [9]. Suppose $N = T(\beta, \xi)$; then $\pi_k(N) = \pi_k(S^k) \oplus \pi_k(S^n)$ and using the formal properties proved in [10] we have the formula

$$\alpha(m, x) = m\,\varphi(\beta) + F(x, \xi_0) + m\,\partial x$$

where φ, F, and ∂ are defined in the paragraph preceding the statement of Theorem 3.

The condition of 2b) on embeddings of S^k_β with trivial normal bundle is clearly equivalent to the existence of comparable embeddings of S^n with normal bundle $-\varphi(\beta')$. Since we are in the metastable range, such embeddings exist if and only if for some $x \in \pi_k(S^n)$ we have that $\alpha(1, x) = -\varphi(\beta')$. The formula of Theorem 3 now follows from the more general formula presented above.

Example. Let $\xi \in \pi_{11}(SO_{17})$ be such that $J(\xi)$ has odd order. Then there are two almost diffeomorphism classes of smooth manifolds combinatorially equivalent to $S(\xi)$. Since $\Gamma_{16} = \mathbf{Z}_2$, we have that $\beta J(\xi) = 0$ and hence all plumbings of $S^{16}_\beta \times D^{12}$ and $D(\xi_0)$ may be closed. If η^* generates Γ_{16}, then it is well-known that $\varphi(\eta^*) \neq 0$ [2]; since $\pi_{16}(S^{12}) = 0$, it follows that $T(\eta^*, \xi)$ and $S(\xi)$ are not almost diffeomorphic. The inertia groups of both classes are identical by 2c) and [8, 2.7]. Other examples of this type may be constructed using $\pi_{19}(SO_{33})$, $\pi_{19}(SO_{34})$, and $\pi_{27}(SO_{33})$. In fact there are always such examples with at least two almost diffeomorphism classes whenever $h_1 h_k$ and $h_1^2 h_k$ survive in the Adams spectral sequence [4].

Problem. Is $T(\beta, \xi)$ the total space of an S^k_β bundle over S^n? This is connected to some results of Novikov [6]. The only general result the author knows is that if $\beta \in \partial P_{k+1}$ and $k \geqq 31$, then $S(2\xi)$ is the total space of an S^k_β bundle over S^n provided $k \geqq n+4$. A proof of this fact will appear in a forthcoming paper by the author.

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A Generalization to the Non-Separable Case of Takesaki's Duality Theorem for C^*-Algebras

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Abstract. Takesaki [5] poses the question of how much information about a C^*-algebra A is contained in its representation theory. He gives it a precise meaning in the following setting: One can furnish the set $\mathrm{Rep}(A:H)$ of all representations of A in a suitable Hilbert space H with a topology, with an action of the unitary group G of $B(H)$ on it, and with an addition. The set A^F of operator fields $\mathrm{Rep}(A:\mathrm{H})\to B(H)$ commuting with the action of G and addition, called the admissible operator fields, turn out to form a W^*-algebra isomorphic to the bidual of A with Arens multiplication or with the universal enveloping von Neumann algebra of A. Takesaki shows in the separable case that A can be identified in A^F as the set of continuous admissible operator fields, and leaves the same question open for arbitrary C^*-algebras. Changing the structures on $\mathrm{Rep}(A:H)$ slightly, it is shown here that this result obtains in the general case as well. The proof proceeds along the lines set up in [5] but makes no use of the representation theory of NGCR algebras.

I. Preparation and Notation

Let A be a C^*-algebra, A' its dual and A'' its bidual. We consider A as contained in A'' and denote the duality by $x, \alpha \to \langle x|\alpha\rangle$ ($x\in A'$, $\alpha\in A''$). The involution $*$ of A gives rise to involutions in A' and A'', again denoted by $*$. Finally, let $B(A)$ denote the space of positive linear functionals on A of norm not exceeding one, with the weak*-topology inherited from A'. The following lemma is a collection of results spread over the literature.

Lemma. (i) *Every element x of A' of norm not greater than one can be written uniquely in the form $x=x_1-x_2+i(x_3-x_4)$ with $x_i\in B(A)$, $|x_1-x_2|=|x_1|+|x_2|$, $|x_3-x_4|=|x_3|+|x_4|$.*

(ii) *For every element x of A' there are: a representation π, sum of at most four cyclic representations π_i, and vectors ξ,η in the space of π such that $\langle x|a\rangle=(\pi(a)\xi|\eta)$ for all $a\in A$.*

(iii) *The restriction $\hat\alpha$ of an element α of A'' to $B(A)$ is an affine-linear, norm-continuous function on $B(A)$ to $\underline{C}$ vanishing in 0 and $\alpha\to\hat\alpha$ is an isomorphism of A'' onto the vector space $AN_0(B(A):\underline{C})$ of all these functions satisfying $|\alpha|\geqq|\hat\alpha|\geqq\frac{1}{4}|\alpha|$, where $|\hat\alpha|$ is the sup-norm of $\hat\alpha$.*

(iv) *The restriction of $\hat{}$ to $A\subset A''$ is an isomorphism of the vector space A onto the space $AC_0(B(A):\underline{C})$ of all affine-linear, continuous functions on $B(A)$ vanishing in* 0.

Proof. (i): x splits uniquely into its hermitian and anti-hermitian parts, and to those one applies Theorem 12.3.4 of Dixmier [2]. It is sufficient to prove (ii) for $|x| \leqq 1$ (cf. Takeda [4]). Split x into x_i as in (i), select cyclic representations π_i of A with totalizing vectors ξ_i such that $x_i(a) = (\pi_i(a)\,\xi_i|\xi_i)$. Then $\pi := \bigoplus \pi_i$; $\xi := \xi_1 - \xi_2 + i(\xi_3 - \xi_4)$; $\eta = \sum \xi_i$ produce x as stated. Everything in statement (iii) is obvious from (i), except possibly that $\hat{}$ is onto. For this, let $f \in AN_0(B(A):\underline{C})$ and extend f by linearity to a function x on A', using (i). It is easy to check that x is in A'' and has restriction f to $B(A)$. (iv): A can be identified in A'' as the set of weakly continuous elements of A''. Hence $\hat{A} \subset AC_0(B(A):\underline{C})$. As A is complete, the inequality in (iii) yields that $\hat{A}$ is complete with sup-norm and hence closed. Let μ be a linear functional of norm one on $AC_0(B(A):\underline{C})$ vanishing on $\hat{A}$. μ can be extended to all continuous functions on $B(A)$, decomposed into a linear combination of positive measures of norm not greater than one on $B(A)$ and is hence (cf. Bauer [1], Satz 2.5.1) of the form $\mu = \varepsilon_{x_1} - \varepsilon_{x_2} + i(\varepsilon_{x_3} - \varepsilon_{x_4})$ on $AN_0(B(A):\underline{C})$. From $\mu(\hat{A}) = \{0\}$ one deduces $x_1 = x_2$ and $x_3 = x_4$ and μ vanishes on $AN_0(B(A):\underline{C})$, also. This shows that $\hat{A} = AC_0(B(A), \underline{C})$.

It is obvious that the isomorphism $\hat{}$ carries hermitian elements of A'' into real-valued functions.

II. The Structure of Rep $(A:H)$

1) Given the C^*-algebra A we choose once and for all a Hilbert space H of dimension sufficiently high to ensure that every cyclic representation of A is unitarily equivalent to some representation in a subspace of H. The inner product in H is $(\,|\,)$. The set $B(H)$ of all bounded linear operators on H, equipped with the strong, ultrastrong, weak, or ultraweak topology will be denoted by $B_s(H)$, $B_{us}(H)$, $B_w(H)$, or $B_{uw}(H)$ respectively.

Let Rep $(A:H)$ denote the set of all repesentations of A in H. It is a space of functions $A \to B(H)$, and we furnish it with the topology of pointwise convergence:

Lemma. *The topologies of pointwise convergence for functions* $A \to B_s(H)$, $A \to B_{us}(H)$, $A \to B_w(H)$, $A \to B_{uw}(H)$ *coincide on* Rep $(A:H)$.

Proof. It is obviously sufficient to show that pointwise weak convergence on Rep $(A:H)$ entrains pointwise ultrastrong convergence. Let the net π^μ in Rep $(A:H)$ convergence pointwise-$B_w(H)$ to $\pi \in$ Rep $(A:H)$, and let $a \in A$, $\xi \in H$. The equation

$$|\pi^\mu(a)\,\xi - \pi(a)\,\xi|^2$$

$$= (\pi^\mu(a^* a)\,\xi|\xi) + (\pi(a^* a)\,\xi|\xi) - (\pi^\mu(a)\,\xi|\pi(a)\,\xi) - (\pi(a)\,\xi|\pi^\mu(a)\,\xi) \to 0$$

shows that π^μ converges also pointwise-$B_s(H)$ to π. For every $a \in A$, the net $\pi^u(a)$ is bounded and hence converges strongly if and only if it converges ultrastrongly. That is, π^μ converges pointwise-$B_{us}(H)$.

Henceforth, $\mathrm{Rep}(A:H)$ denotes the topological space consisting of the set of representations as above together with the topology of pointwise convergence in any of the topologies on $B(H)$ considered above. $\mathrm{Rep}(A:H)$ is, for instance, a Hausdorff space.

2) For $\pi \in \mathrm{Rep}(A:H)$, denote by H_π its essential subspace, i.e., the closure in H of $\pi(A)H$, and by p_π the projection of H onto H_π. We have

$$\pi(a) = p_\pi \pi(a) = \pi(a) p_\pi \qquad \text{for all } a \in A.$$

π is said to be cyclic with totalizing vector ξ if $\xi \in H_\pi$ and $\pi(A)\xi$ is dense in H_π, and the couple (π, ξ) is called a representative couple for the positive linear form $a \to (\pi(a)\xi \mid \xi)$ on A. The assumption on H then says that every positive linear form has a representative couple in $\mathrm{Rep}(A:H)$.

For two representations π, π' in $\mathrm{Rep}(A:H)$ with H_π orthogonal to $H_{\pi'}$, we define their sum $\pi + \pi'$ by

$$(\pi + \pi')(a) = \pi(a) + \pi'(a) \qquad \text{for all } a \in A.$$

It is easy to check that $\pi + \pi'$ is, again, in $\mathrm{Rep}(A:H)$, with essential subspace $H_\pi \oplus H_{\pi'}$. (Let D be the subset of $\mathrm{Rep}(A:H) \times \mathrm{Rep}(A:H)$ on which addition is defined. Then addition is a continuous mapping from D to $\mathrm{Rep}(A:H)$.)

3) Let $\pi \in \mathrm{Rep}(A:H)$ and $u \in B(H)$ a partial isometry with initial projector $u^* u$ greater than p_π. Then $\pi^u: a \to u\,\pi(a)\,u^*$ is, again, a representation of A in H, with $p_\pi = u\,p_\pi$. We have $(\pi^u)^{u^*} = \pi$, and we say that two representations π, π' in $\mathrm{Rep}(A:H)$ are equivalent (under u) if $\pi' = \pi^u$ for some partial isometry u. It is easy to see that this defines, indeed, an equivalence relation on $\mathrm{Rep}(A:H)$.

The structures on $\mathrm{Rep}(A:H)$ here defined, topology, addition and equivalence, resemble closely those imposed on $\mathrm{Rep}(A:H)$ by Takesaki [4], and Ernest [3] in his definition of the "big group algebra" of a locally compact group. There are, however, small differences, though significant enough to enable us to prove Takesaki's conjecture in the new setting. While the topology is the same, while our definition of addition differs only insignificantly from Takesaki's, the notion of equivalence of representations here is stronger, such that more representations are equivalent in our setting. This is due to the fact that Takesaki allows only unitary mappings in the definition of equivalence. However, the notion used here seems natural enough, as it corresponds to the usual one, when no Hilbert space is fixed aforehand. Its usefulness lies in that it permits to establish the following crucial result:

4) **Proposition.** *Let $x \in B(A)$ and (π, ξ) a representative couple for x in* $\mathrm{Rep}(A:H)$.

(i) *For every neighborhood V of π in* $\mathrm{Rep}(A:H)$ *there is a neighborhood U of x in $B(A)$ such that every y in U has a representative couple (ρ, η) with ρ in V.*

(ii) *If A has an identity e and W is a neighborhood of ξ in H, then U can be chosen such that every $y \in U$ has a representative couple (ρ, η) with $\rho \in V$ and $\eta \in W$.*

For the proof, we need the following lemma, which is proved in Dixmier ([2], 3.5.6).

Lemma. *Let $\xi_1, \ldots, \xi_n$ be a finite number of vectors in a Hilbert space H. For every $\varepsilon > 0$ there is a $\delta > 0$ such that for every collection $\eta_1, \ldots, \eta_n$ of vectors in H with "δ-nearly the same orthogonality relations", i.e., with*

$$|(\xi_i | \xi_j) - (\eta_i | \eta_j)| < \delta \qquad 1 \leqq i, j \leqq n,$$

there is a unitary operator $u: H \to H$ such that

$$|u\eta_i - \xi_i| < \varepsilon, \qquad 1 \leqq i \leqq n,$$

i.e., "u shifts η_i ε-close to ξ_i".

Proof of the Proposition. A basis for the neighborhood filter of π is given by the sets

$$\{\rho \,\|\, \rho(a_i)\,\zeta_j - \pi(a_i)\,\zeta_j| < 12\,\varepsilon\}$$

where $a_1, \ldots, a_I$ are elements of A of norm not exceeding one and $\zeta_1, \ldots, \zeta_P$ are vectors in H of norm not exceeding one. It is no restriction to assume $\zeta_1 = \zeta (=0$, if $\pi = 0)$ [and, if A has an identity e, that $a_1 = e$].

Splitting ζ_j into its part ξ'_j in H_π and its part ξ''_j orthogonal to H_π, we see that the sets of the form

$$\{\rho \,\|\, \rho(a_i)\,\xi'_j - \pi(a_i)\,\xi'_j| < 6\varepsilon\} \cap \{\rho \,\|\, \rho(a_i)\,\xi''_j| < 6\varepsilon\}$$

(a_i as before; $\xi'_j \in H_\pi$, $\xi'_1 = \xi$, $\xi''_j \in H_\pi^\perp$, the ξ', ξ'' of norm smaller or equal to one) are, again, a basis for the neighborhood filter at π.

Now choose $b_1, b_2, \ldots, b_J$ in A such that $|\xi'_j - \pi(b_j)\,\xi| < 2\varepsilon$. [If $e \in A$, choose $b_1 = e$.] Putting $\pi(b_j)\,\xi = \xi_j$, a little calculation shows that the sets

$$V' := \{\rho \,|\, |\rho(a_i)\,\xi_j - \pi(a_i)\,\xi_j| < 2\varepsilon\} \cap \{\rho \,|\, |\rho(a_i)\,\xi_j''| < 6\varepsilon\}$$

($a_i \in A$ as before; $\xi_j = \pi(b_j)\,\xi$ for some $b_j \in A$; $\xi_j'' \in H_\pi^\perp$) form. again, a basis at π.

Now let u be a partial isometry: $H \to H$ whose initial projector $u^* u$ is the identity operator, whose final projector $u u^*$ is the projector onto the orthogonal complement of the ξ_j'' $(j = 1, \ldots, J)$, and which leaves the vectors $\xi_j, \xi_{ij} := \pi(a_i)\,\xi_j$ fixed $(j = 1, \ldots, J; i = 1, \ldots, I)$. Such a partial isometry can be found by splitting H into the span H^1 of the ξ_j, ξ_{ij} and its orthogonal complement H^2 and taking the sum of the identity operator of H^1 and a partial isometry of H^2 meeting the first two requirements for u. Such exists, as H and hence H^2 has infinite dimension.

With such a $u: H \to H$ fixed, if ρ is in

$$V := \{\rho \,|\, |\rho(a_i)\,\xi_j - \pi(a_i)\,\xi_j| < 2\varepsilon\}$$

then ρ^u is in V'. Indeed, $\rho^u: a \to u\,\rho(a)\,u^*$ annihilates the ξ_j'', $(j = 1, \ldots, J)$, and

$$\begin{aligned} |\rho^u(a_i)\,\xi_j - \pi(a_i)\,\xi_j| &= |u\,\rho(a_i)\,u^*\,\xi_j - \pi(a_i)\,\xi_j| \\ &= |u^*\,u\,\rho(a_i)\,\xi_j - u^*\,\pi(a_i)\,\xi_j| \\ &= |\rho(a_i)\,\xi_j - \pi(a_i)\,\xi_j| < 2\varepsilon. \end{aligned}$$

The considerations up to here amount to showing that it is no restriction to assume that the neighborhood V of π in the statement is of the form

$$V = \{\rho \,|\, |\rho(a_i)\,\xi_j - \xi_{ij}| < 2\varepsilon\}$$

(a_i as above, $\xi_j = \pi(b_j)\,\xi$, $\xi_{ij} = \pi(a_i)\,\xi_j$, $a_1 = b_1 = e$, if $e \in A$).

To construct the neighborhood U of x with the required properties, consider the $J + IJ$ vectors ξ_j, ξ_{ij} of H $(i = 1, \ldots, I, j = 1, \ldots, J)$. They have the following inner products:

$$\begin{aligned} (\xi_j \,|\, \xi_{j'}) &= (\pi(b_{j'}^*\,b_j)\,\xi \,|\, \xi) = \langle x \,|\, b_{j'}^*\,b_j \rangle \\ (\xi_j \,|\, \xi_{ij'}) &= (\pi(b_{j'}^*\,a_i^*\,b_j)\,\xi \,|\, \xi) = \langle x \,|\, b_{j'}^*\,a_i^*\,b_j \rangle \\ (\xi_{ij} \,|\, \xi_{i'j'}) &= (\pi(b_{j'}^*\,a_{i'}^*\,a_i\,b_j)\,\xi \,|\, \xi) = \langle x \,|\, b_{j'}^*\,a_{i'}^*\,a_i\,b_j \rangle. \end{aligned}$$

Find now, in the sense of the lemma above, a $\delta > 0$ such that any set η_j, η_{ij} $(i = 1, \ldots, I,\ j = 1, \ldots, J)$ of vectors in H with δ-nearly the same orthogonality relations as the ξ_i, ξ_{ij} can be shifted ε-close to the ξ_i, ξ_{ij} by a unitary map u. With this δ, define the neighborhood U of x by

$$U = \{y \in B(A) \,|\, |x(a) - y(a)| < \delta \quad \text{for } a = b_{j'}^* b_j, \; b_{j'}^* a_i^* b_j, \; b_{j'}^* a_{i'}^* a_i b_j \quad (j, j' = 1, \ldots, J; \; i, i' = 1, \ldots, I)\}.$$

U has the properties required in the statement of the theorem: Let $y \in U$, and let (ρ, η) be a representative couple for y in Rep$(A:H)$. The vectors $\eta_j := \rho(b_j)\eta$, $\eta_{ij} := \rho(a_i b_j)\eta$ have δ-nearly the same orthogonality relations as the ξ_j, ξ_{ij}. We find a unitary operator $u: H \to H$ such that $|u\eta_j - \xi_j| < \varepsilon$ and $|u\eta_{ij} - \xi_{ij}| < \varepsilon$ $(j = 1, \ldots, J; \; i = 1, \ldots, I)$. The couple $(\rho^u, u\eta)$ is again representative for y, and it is in V:

$$|\rho^u(a_i)\xi_j - \pi(a_i)\xi_j| = |u\rho(a_i)u^*\xi_j - \pi(a_i)\xi_j|$$
$$\leqq |u\rho(a_i)(u^*\xi_j - \eta_j)| + |u\eta_{ij} - \xi_{ij}| < 2\varepsilon.$$

[If $e \in A$ then $\eta_1 = \rho(b_1)\eta = \rho(e)\eta = p_\rho \eta = \eta$ and $\xi_1 = \xi$ and hence $|u\eta - \xi| = |u\eta_1 - \xi_1| < \varepsilon$ and $u\eta$ lies in W for sufficiently small ε.]

III. Admissible Operator Fields

1) **Definition** (Takesaki [5], Ernest [3]). *A function T: Rep$(A:H) \to B(H)$ is an admissible operator field if*

(i) $\|T\| := \sup\{|T(\pi)|; \pi \in \text{Rep}(A:H)\} < \infty$.

(ii) $T(\pi) = p_\pi T(\pi) = T(\pi) p_\pi$.

(iii) $T(\pi + \pi') = T(\pi) + T(\pi')$ *for* $(\pi, \pi') \in D$.

(iv) $T(\pi^u) = u\,T(\pi)\,u^*$.

It is obvious that the set A^F of all admissible operator fields is a C^*-algebra under pointwise addition, multiplication, involution, and with the norm $\| \;\; \|$.

An element a of A defines an element $\tilde{a}: \pi \to \pi(a)$ of A^F and the map $a \to \tilde{a}$ is an isomorphism of the C^*-algebra A into A^F. The fields $\tilde{a}\,(a \in A)$ are obviously continuous with respect to any of the four topologies on $B(H)$ under consideration in Section II. Takesaki's problem, to be solved in the next section, is to show that, on the other hand, $\tilde{A}$ can be identified in A^F as the subset of precisely all the continuous fields in A^F.

2) One can define analogues to the weak, ultraweak, strong, and ultrastrong topologies on a von Neumann algebra on A^F by the pseudo-

norms $T\to|(T(\pi)\,\xi\,|\,\eta)|$, ($\pi\in\operatorname{Rep}(A:H)$, $\xi,\eta\in H$); $T\to|T(\pi)\,\xi|$ (π,ξ as before); etc. (Cf. [3, 5].) A^F is the closure of A in each of these topologies. We establish as much of this statement as will be needed later. If e is a projector in $B(H_\pi)$ ($\pi\in\operatorname{Rep}(A:H)$) commuting with the restriction of $\pi(A)$ to H then e causes a decomposition of H into two orthogonal closed invariant subspaces, and (iii) shows that $T(\pi)$ commutes with e. That is, $T(\pi)$ is in the von Neumann algebra spanned by $\pi(A)$ in H and is hence a limit in $B_w(H)\dots B_{us}(H)$ of elements $\pi(a)$ (a in A^e, the algebra obtained from A by adjoining a unit element). According to Kaplanski's theorem ([2], Ch. I, § 3, No. 5), $T(\pi)$ can even be approximated by elements $\pi(a)$ of norm not exceeding the norm of $T(\pi)$.

3) Let T be in A^F, x in $B(A)$ and select a representative couple (π_x,ξ_x) for x. The number $(T(\pi_x)\,\xi_x\,|\,\xi_x)$ does not depend on the choice of (π_x,ξ_x), as a simple calculation shows: another representative couple for x is of the form $(\pi_x^u, u\,\xi_x)$ for some partial isometry u, and the claim results from (iv). Using (iii) and (ii), it is equally easy to show that the function $\lambda(T)\colon x\to(T(\pi_x)\,\xi_x\,|\,\xi_x)$ is affine-linear on $B(A)$ and vanishes in 0. Indeed, let $0<\alpha<1$ and $x=\alpha\,y+(1-\alpha)\,z$ (x,y,z in $B(A)$). If (π_y,ξ_y) and (π_z,ξ_z) are representative couples for y and z respectively, chosen such that $p_{\pi_y}\cdot p_{\pi_z}=0$, then a representative couple (π_x,ξ_x) for x is obtained by setting $\xi_x=\alpha^{\frac12}\,\xi_y+(1-\alpha)^{\frac12}\,\xi_z$ and π_x the subrepresentation of $\pi_y+\pi_z$ which has cyclic vector ξ_x. One gets

$$\begin{aligned}\langle x,\lambda(T)\rangle&=(T(\pi_x)\,\xi_x\,|\,\xi_x)=(T(\pi_y+\pi_z)\,\xi_x\,|\,\xi_x)\\&=\big((T(\pi_y)+T(\pi_z))\,(\alpha^{\frac12}\,\xi_y+(1-\alpha)^{\frac12}\,\xi_z)\,|\,\alpha^{\frac12}\,\xi_y+(1-\alpha)^{\frac12}\,\xi_z\big)\\&\overset{*}{=}(T(\pi_y)\,\alpha^{\frac12}\,\xi_y\,|\,\alpha^{\frac12}\,\xi_y)+(T(\pi_z)\,(1-\alpha)^{\frac12}\,\xi_z\,|\,(1-\alpha)^{\frac12}\,\xi_z)\\&=\langle(\alpha\,y+(1-\alpha)\,z,\lambda(T))\rangle.\end{aligned}$$

The equality $\overset{*}{=}$ holds because of (ii).

Let $\lambda''(T)$ denote the unique linear extension of $\lambda(T)$ to the whole of A'. Then $\langle x,\lambda''(T)\rangle$ can be calculated as follows. If (π,ξ,η) is a triple representing x in the manner of Lemma 1, (ii), then

$$\langle x,\lambda''(T)\rangle=(T(\pi)\,\xi\,|\,\eta). \tag{$*$}$$

Indeed, the proof of Lemma 1 shows that $\lambda''(T)$ defined by ($*$) is a linear extension of $\lambda(T)$ to A'.

Proposition. (i) *The map λ: $T\to\lambda(T)$ is a linear isomorphism of A^F onto $AN_0(B(A)\colon\underline{C})$.*

(ii) *The linear isomorphism λ'' of A^F onto A'' such that $\lambda''(T)$ is the unique linear extension of $\lambda(T)$ to A' is an isometry of Banach spaces and preserves involution.*

(iii) $\lambda(\tilde{a})=\hat{a}$ *and* $\lambda''(\tilde{a})=a$ *for all* $a\in A$. *Hence* $\lambda(\tilde{A})=\hat{A}$ *and* $\lambda''(\tilde{A})=A\subset A''$.

Proof. If x is in $B(A)$ and represented in the form $a\to(\pi(a)\xi\,|\,\xi)$ then $|x|=|\xi|^2$. One sees this letting run a through an approximate identity. Taking into account Lemma 1, (ii) and (i), one sees that if x in A' converges in the norm zero and is represented in the form $a\to(\pi(a)\xi\,|\,\eta)$ then $\xi\to 0$ and $\eta\to 0$ and hence $\langle x,\lambda''(T)\rangle\to 0$. That is, $\lambda''(T)$ is, indeed in A'' and $\lambda(T)$ is an $AN_0(B(A):\underline{C})$.

Linearity of λ and λ'' is obvious. If $\lambda(T)=0$ then $(T(\pi)\xi\,|\,\xi)=0$ for all cyclic representations (π,ξ), hence $T=0$. Thus λ and λ'' are one-to-one.

To see that λ'' and with it λ is onto, let $\pi\in\mathrm{Rep}(A:H)$ and $\xi,\eta\in H$ and put $x^\pi_{\xi\eta}=(\pi(\cdot)\xi\,|\,\eta)$. The form $\xi,\eta\to\langle x^\pi_{\xi\eta},\alpha\rangle$ is sesquilinear and bounded for α in A'' and defines an operator $T(\pi)$ such that $\langle x^\pi_{\xi\eta},\alpha\rangle=(T(\pi)\xi\,|\,\eta)$. It is clear that $T:\pi\to T(\pi)$ is an admissible operator field with $\lambda''(T)=\alpha$.

It remains to be shown that λ'' is isometric. First, as $x^\pi_{\xi\eta}$ has norm not exceeding $|\xi||\eta|$, we get $|\lambda''(T)|=\sup\{|\langle x,\lambda''(T)\rangle|;\ x$ in A', $|x|<1\}\geqq\sup\{|(T(\pi)\xi\,|\,\eta)|;\ \pi$ in $\mathrm{Rep}(A:H)$, ξ,η in H, $|\xi|<1$, $|\eta|<1\}=\sup\{|T(\pi)|;\ \pi$ in $\mathrm{Rep}(A:H)\}=\|T\|$. On the other hand, suppose $x=x^\pi_{\xi\eta}$ and T have norm not exceeding one. As $T(\pi)$ can be approximated weakly in $B(H_\pi)$ by operators $\pi(a)$ (a in A^e, cf. Section 2) above), we have $|\langle x,\lambda''(T)\rangle|=|(T(\pi)\xi\,|\,\eta)|=\lim|(\pi(a)\xi\,|\,\eta)|\leqq|x^\pi_{\xi\eta}|=1$, and $\lambda''(T)$ has norm not exceeding one. That is, $|\lambda''(T)|=\|T\|$.

4) The proposition above shows that A^F as defined here is isomorphic to the algebra of admissible operator fields, as defined by Takesaki [5], Ernest [3], despite the difference in the structures imposed here and there on $\mathrm{Rep}(A:H)$. We list without proof several properties of A^F which have been established by these authors:

(i) λ'' is an isomorphism of the W^*-algebra A^F onto A'' with Arens multiplication.

(ii) A^F has a faithful representation onto the universal enveloping von Neumann algebra $\bar{A}$ of A which extends the canonical representation of A into $\bar{A}$.

(iii) Universal property of A^F: If π is a *-homomorphism of A into a W^*-algebra M then there is a unique normal extension $\tilde{\pi}$ of π to A^F into M.

(iv) A is of Type I if and only if A^F is.

IV. Identification of $\tilde{A}$ in A^F

Theorem. *Let $T \in A^F$. Equivalent are:*

(i) $T \in \tilde{A}$.

(ii) *T is continuous from* $\mathrm{Rep}(A:H)$ *to* $B_{us}(H)$.

(iii) *T is continuous from* $\mathrm{Rep}(A:H)$ *to* $B_s(H)$.

(iv) *T is continuous from* $\mathrm{Rep}(A:H)$ *to* $B_{uw}(H)$.

(v) *T is continuous from* $\mathrm{Rep}(A:H)$ *to* $B_w(H)$.

That is, A is isomorphic under $\tilde{}$ to the C^*-algebra of all continuous admissible operator fields.

Proof. It is obvious that (i) implies (ii) through (v). It is sufficient to show that (v) implies (i). To do that, we first assume A has an identity e. Let $T \in A^F$ be weakly continuous. According to the proposition in the last section it is sufficient to show that $\lambda(T)$ is continuous.

Let (x^μ) be a net in $B(A)$ converging to some $x \in B(A)$. Let (π^μ, ξ^μ), (π, η) be representative couples for x^μ and x respectively such that $\xi^\mu \to \xi$ and $\pi^\mu \to \pi$. They exist due to Proposition II, 4). We find

$$\begin{aligned}
|\langle x^\mu | \lambda(T)\rangle - \langle x | \lambda(T)\rangle| &= |(T(\pi^\mu)\,\xi^\mu | \xi^\mu) - (T(\pi)\,\xi | \xi)| \\
&\leqq |(T(\pi^\mu)\,\xi^\mu | \xi^\mu) - (T(\pi^\mu)\,\xi^\mu | \xi)| + |(T(\pi^\mu)\,\xi^\mu | \xi) - (T(\pi^\mu)\,\xi | \xi)| \\
&\quad + |(T(\pi^\mu)\,\xi | \xi) - (T(\pi)\,\xi | \xi)| \\
&\leqq \|T\|\,|\xi^\mu|\,|\xi^\mu - \xi| + \|T\|\,|\xi^\mu - \xi|\,|\xi| + |((T(\pi^\mu) - T(\pi))\,\xi | \xi)| \to 0.
\end{aligned}$$

If A does not have an identity, we consider A as a subalgebra of the C^*-algebra A^e obtained from A by adjoining an identity e. To every representation π in $\mathrm{Rep}(A^e:H)$ there corresponds naturally the representation $\pi_{|A}$ of A in H, the restriction of π to A. The mapping $\pi \to \pi_{|A}$ is clearly continuous from $\mathrm{Rep}(A^e:H)$ to $\mathrm{Rep}(A:H)$. If T is continuous in A^F then $T': \pi \to T(\pi_{|A})$ is a continuous admissible field in A^{eF} and hence $\lambda(T')$ is a continuous function on $B(A^e)$. We have, for $x \in B(A^e)$, $\langle x | \lambda(T')\rangle = \langle T'(\pi_x)\,\xi_x | \xi_x\rangle = \langle T(\pi_{x|A})\,\xi_x | \xi_x\rangle = \langle x_{|A} | \lambda(T)\rangle$. In the latter equation, $x_{|A}$ denotes the restriction of x to A, and it holds as $\langle x_{|A} | a\rangle = \langle x | a\rangle = \langle \pi_x(a)\,\xi_x | \xi_x\rangle = \langle \pi_{x|A}(a)\,\xi_x | \xi_x\rangle$ for all a in A. This shows that the function $x \to \langle x_{|A} | \lambda(T)\rangle$ is continuous on $B(A^e)$. But as $B(A^e)$ and $B(A)$ are both compact Hausdorff and $x \to x_{|A}$ is continuous and onto, $B(A)$ has the quotient topology under this map and we deduce from the fact that the composite map $x \to x_A \to \langle x_A | \lambda(T)\rangle$ is continuous on $B(A^e)$ that the map $y \to \langle y | \lambda(T)\rangle$ is continuous on $B(A)$. That is, $\lambda(T)$ is in $\hat{A}$ and hence (cf. Proposition III, 2), (iii)) T is in $\tilde{A}$.

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Kohomologiegruppen und Konstantenreduktion in Funktionenkörpern

K. KIYEK* (Saarbrücken)

Einleitung

Es sei K ein Körper, $\mathfrak{o}$ ein noetherscher Teilring von K, welcher japanisch ist (bzw. $K/\mathfrak{o}$ separabel erzeugbar). Die Riemannsche Fläche X von $K/\mathfrak{o}$ kann in natürlicher Weise mit einer Garbe von Ringen $\mathscr{A}$ versehen werden. Es wird gezeigt, daß die Kohomologiegruppen $H^p(X, \mathscr{A})$ als induktiver Limes der Kohomologiegruppen $H^p(Y, \mathcal{O}_Y)$, Y projektives Modell von $K/\mathfrak{o}$, erhalten werden können, und daß das gleiche auch für die Čechschen Kohomologiegruppen gilt. Hieraus folgt, daß der kanonische Homomorphismus $\check{H}^p(X, \mathscr{A}) \to H^p(X, \mathscr{A})$ bijektiv ist. Ist speziell $\mathfrak{o}$ von endlicher Krullscher Dimension, $\dim(\mathfrak{o}) = n$, und universeller Kettenring, r der Transzendenzgrad von K über dem Quotientenkörper k von $\mathfrak{o}$, so ist $H^{r+n}(X, \mathscr{A})$ ein endlich erzeugter $\mathfrak{o}$-Modul. Im Falle $\mathfrak{o} = k$ heißt die k-Dimension dieser Kohomologiegruppe das geometrische Geschlecht $g(K/k)$ von K/k. Ist $\mathfrak{o}$ ein einrangig-diskreter Bewertungsring, v die zugehörige Bewertung, V eine Funktionalfortsetzung von v auf K, so wird gezeigt, daß $g(K/k) \geqq g(\bar{K}/\bar{k})$, falls V prim über v. Hierbei bedeuten Querstriche Restbildung mod V. Dabei wird ein Ergebnis aus [6] benutzt. Diese Ungleichung ist eine Verallgemeinerung des von Deuring und allgemeiner von Lamprecht [7] bewiesenen Satzes, daß das Geschlecht eines algebraischen Funktionenkörpers einer Veränderlichen (also $r = 1$) unter geeigneten Voraussetzungen bei Konstantenreduktion höchstens abnehmen kann.

Die Arbeit lehnt sich teilweise an Snapper [11] an, der jedoch nur die algebraischen Bewertungsringe von K über einem Körper $\mathfrak{o}$ und nur die Čechschen Kohomologiegruppen betrachtet. So mußte die von Snapper entwickelte Theorie in diesem allgemeineren Zusammenhang zum Teil neu dargestellt werden.

Im einzelnen gliedert sich die Arbeit wie folgt. In den Abschnitten 1 – 2 werden induktive Systeme von Kohomologiegruppen behandelt, die aus einem projektiven System geringter Räume gewonnen werden. Als Nebenergebnis wird dabei u.a. gezeigt, daß in einer Klasse topologischer

* Die Ergebnisse dieser Arbeit erweitern Resultate aus der Habilitationsschrift. Der Deutschen Forschungsgemeinschaft bin ich für ihre finanzielle Unterstützung dankbar.

Räume, welche die Klasse noetherscher Räume umfaßt, der kanonische Homomorphismus

$$\varinjlim H^p(X, \mathscr{A}_i) \to H^p(X, \mathscr{A}), \qquad \mathscr{A} = \varinjlim \mathscr{A}_i,$$

bijektiv ist. Abschnitt 4 bringt die oben erwähnte Darstellung der Čechschen Kohomologiegruppen und Kohomologiegruppen der Riemannschen Fläche als induktive Limites. In Abschnitt 5 wird der Endlichkeitssatz bewiesen. Abschnitt 6 bringt einiges über die Reduktion von Modellen und die Reduktion der Riemannschen Fläche. Hier sei auf das kommutative Diagramm mit exakten Zeilen und Spalten (6.9.5) hingewiesen. Zum Schluß wird die Formel $g(K/k) \geq g(\bar{K}/\bar{k})$ bewiesen.

Dem Referenten danke ich für einige Bemerkungen zu Abschnitt 2.

1. Kohomologiegruppen

(1.1) Es sei X ein topologischer Raum, $\mathfrak{U}$ die Familie der offenen Mengen von X. Es sei weiter I eine teilweise geordnete, aufsteigend gefilterte Indexmenge. Ist $(\mathscr{F}_i, \theta_{ij})_{i \in I}$ ein induktives System von Garben von Mengen auf X, so heißt die zu der Prägarbe $(\varinjlim(\mathscr{F}_i(U)))_{U \in \mathfrak{U}}$ gehörige Garbe $\mathscr{F}$ der induktive Limes des induktiven Systems $(\mathscr{F}_i)_{i \in I}$: $\mathscr{F} = \varinjlim \mathscr{F}_i$.

Es sei $((X_i, \mathscr{A}_i), f_{ij})_{i \in I}$, $f_{ij} = (\varphi_{ij}, \theta_{ij})$, ein projektives System geringter Räume. Sei (X, φ_i) der projektive Limes von $(X_i, \varphi_{ij})_{i \in I}$; dann ist $(\varphi_i^*(\mathscr{A}_i), \varphi_i^*(\theta_{ij}^\#))_{i \in I}$ ein induktives System von Garben von Ringen auf X. Ist $(\mathscr{A}, \theta_i)$ der induktive Limes dieses Systems, so ist $((X, \mathscr{A}), f_i)$, $f_i = (\varphi_i, \theta_i)$, der projektive Limes in der Kategorie der geringten Räume des gegebenen projektiven Systems geringter Räume. Für jedes $x \in X$ ist $\mathscr{A}_x = \varinjlim (\mathscr{A}_i)_{\varphi_i(x)}$.

(1.2) Ist $f = (\varphi, \theta): (X, \mathscr{A}) \to (Y, \mathscr{B})$ ein Morphismus geringter Räume, so bestimmt f einen Bihomomorphismus

(1.2.1) $$\theta_p: H^p(Y, \mathscr{B}) \to H^p(X, \mathscr{A})$$

der Kohomologiegruppen und ebenso einen Bihomomorphismus

(1.2.2) $$\check{\theta}_p: \check{H}^p(Y, \mathscr{B}) \to \check{H}^p(X, \mathscr{A})$$

der Čechschen Kohomologiegruppen und es kommutiert ([4], $\mathbf{0}_{\text{III}}$, § 12.1)

(1.2.3) $$\begin{array}{ccc} \check{H}^p(Y, \mathscr{B}) & \longrightarrow & H^p(Y, \mathscr{B}) \\ \downarrow{\scriptstyle \check{\theta}_p} & & \downarrow{\scriptstyle \theta_p} \\ \check{H}^p(X, \mathscr{A}) & \longrightarrow & H^p(X, \mathscr{A}); \end{array}$$

hier sind die horizontalen Pfeile die kanonischen Homomorphismen der Čechschen Kohomologie in die Kohomologie ([3], Ch. III, (5.9)).

Es sei nun $((X_i, \mathscr{A}_i), f_{ij})_{i\in I}$, $f_{ij}=(\varphi_{ij}, \theta_{ij})$, ein projektives System geringter Räume, $(X, \mathscr{A})$ der projektive Limes. Dann liefern die Abbildungen (1.2.1), (1.2.2) und das kommutative Diagramm (1.2.3) induktive Systeme $(H^p(X_i, \mathscr{A}_i))_{i\in I}$, $(\check{H}^p(X_i, \mathscr{A}_i))_{i\in I}$ von Moduln über dem induktiven System von Ringen $(\Gamma(X_i, \mathscr{A}_i))_{i\in I}$ sowie ein induktives System von Homomorphismen $(\check{H}^p(X_i, \mathscr{A}_i))_{i\in I} \to (H^p(X_i, \mathscr{A}_i))_{i\in I}$, also einen Homomorphismus

(1.2.4) $$\varinjlim \check{H}^p(X_i, \mathscr{A}_i) \to \varinjlim H^p(X_i, \mathscr{A}_i).$$

Weiterhin bestimmen die Abbildungen (1.2.1), (1.2.2) Bihomomorphismen

(1.2.5) $$\varinjlim H^p(X_i, \mathscr{A}_i) \to H^p(X, \mathscr{A}),$$

(1.2.6) $$\varinjlim \check{H}^p(X_i, \mathscr{A}_i) \to \check{H}^p(X, \mathscr{A})$$

und (1.2.3) führt zu einem kommutativen Diagramm

(1.2.7) $$\begin{array}{ccc} \varinjlim \check{H}^p(X_i, \mathscr{A}_i) & \to & \varinjlim H^p(X_i, \mathscr{A}_i) \\ \downarrow & & \downarrow \\ \check{H}^p(X, \mathscr{A}) & \to & H^p(X, \mathscr{A}); \end{array}$$

hier ist der untere horizontale Pfeil der kanonische Homomorphismus.

(1.3) Es werden hinreichende Bedingungen dafür angegeben, daß (1.2.6) ein Isomorphismus ist. Für jedes $i\in I$ sei $\mathfrak{S}(X_i)=\mathfrak{S}_i$ eine Menge von beliebig feinen, aufsteigend gefilterten Überdeckungen von X_i, so daß $\check{H}^p(X_i, \mathscr{A}_i)=\varinjlim_{\mathfrak{U}\in\mathfrak{S}_i} H^p(\mathfrak{U}, \mathscr{A}_i)$. Die Summenmenge $\mathfrak{S}=\sum_i \mathfrak{S}_i$ wird teilweise geordnet: Ist $\mathfrak{U}_i\in\mathfrak{S}_i$, $\mathfrak{V}_j\in\mathfrak{S}_j$, so $\mathfrak{U}_i\leq\mathfrak{V}_j$, falls $i\leq j$ und $\mathfrak{V}_j$ feiner als $f_{ij}^{-1}(\mathfrak{U}_i)$ ist. $\mathfrak{S}$ ist dann aufsteigend gefiltert, und für $\mathfrak{U}_i\leq\mathfrak{V}_j$ gibt es einen kanonischen Bihomomorphismus

(1.3.1) $$\theta_{ij,p}(\mathfrak{U}_i, \mathfrak{V}_j)\colon\ H^p(\mathfrak{U}_i, \mathscr{A}_i) \to H^p(\mathfrak{V}_j, \mathscr{A}_j).$$

Wird $(\Gamma(X_i, \mathscr{A}_i))_{i\in I}$ als induktives System von Ringen über $\mathfrak{S}$ aufgefaßt, so bestimmt (1.3.1) ein induktives System von Moduln

$$(H^p(\mathfrak{U}_i, \mathscr{A}_i), \theta_{ij,p}(\mathfrak{U}_i, \mathfrak{V}_j))$$

über dem induktiven System von Ringen $(\Gamma(X_i, \mathscr{A}_i))$. Sei $A=\varinjlim \Gamma(X_i, \mathscr{A}_i)$.

Es ist leicht zu sehen, daß der durch die Homomorphismen $H^p(\mathfrak{U}_i, \mathscr{A}_i) \to \check{H}^p(X_i, \mathscr{A}_i)$ induzierte A-Homomorphismus

(1.3.2) $$\varinjlim_{\mathfrak{S}} H^p(\mathfrak{U}_i, \mathscr{A}_i) \longrightarrow \varinjlim_{I} \check{H}^p(X_i, \mathscr{A}_i)$$

bijektiv ist und daß

(1.3.3)
$$\begin{array}{ccc} \varinjlim\limits_{\mathfrak{S}} H^p(\mathfrak{U}_i, \mathscr{A}_i) & \xrightarrow{\sim} & \varinjlim\limits_{I} \check{H}^p(X_i, \mathscr{A}_i) \\ & \searrow \quad \swarrow & \\ & \check{H}^p(X, \mathscr{A}) & \end{array}$$

kommutiert; hier ist der linke schräge Pfeil der durch die Homomorphismen $H^p(\mathfrak{U}_i, \mathscr{A}_i) \to \check{H}^p(X, \mathscr{A})$ induzierte Homomorphismus von A-Moduln

(1.3.4)
$$\varinjlim_{\mathfrak{S}} H^p(\mathfrak{U}_i, \mathscr{A}_i) \to \check{H}^p(X, \mathscr{A}).$$

(1.4) **Proposition.** *Es sei* $(X_i, \mathscr{A}_i)_{i\in I}$ *ein projektives System geringter Räume,* $((X, \mathscr{A}), f_i)$, $f_i = (\varphi_i, \theta_i)$, *der projektive Limes, p eine feste natürliche Zahl. Dann ist der kanonische Homomorphismus* (1.3.4)

$$\varinjlim_{\mathfrak{S}} H^p(\mathfrak{U}_i, \mathscr{A}_i) \to \check{H}^p(X, \mathscr{A})$$

bijektiv, falls die beiden folgenden Bedingungen erfüllt sind:

i) *Die Räume X und* X_i *sind quasikompakt, die Abbildungen* $\varphi_i\colon X \to X_i$ *sind abgeschlossen oder surjektiv.*

ii) *Der kanonische Homomorphismus*

$$H^p(\mathfrak{U}_i, \mathscr{A}_i) \to H^p(f_i^{-1}(\mathfrak{U}_i), \mathscr{A})$$

ist für jedes $\mathfrak{U}_i \in \mathfrak{S}$ *bijektiv.*

Zum Beweis vgl. ([2], Ch. X, § 3, Lemma 3.7; [11], St. 4.3). Man wählt $\mathfrak{S}_i$ als Menge der endlichen, offenen Überdeckungen von X_i und zeigt, daß $\mathfrak{S}' = \{\varphi_i^{-1}(\mathfrak{U}_i) \mid \mathfrak{U}_i \in \mathfrak{S}_i, i \in I\}$ eine kofinale Teilmenge der Menge aller endlichen offenen Überdeckungen von X ist.

(1.5) **Korollar.** *Sind die Bedingungen* i) *und* ii) *aus* (1.4) *erfüllt, so ist der kanonische Homomorphismus* (1.2.6)

$$\varinjlim \check{H}^p(X_i, \mathscr{A}_i) \to \check{H}^p(X, \mathscr{A})$$

bijektiv.

Das folgt sofort aus (1.4) wegen (1.3.3).

(1.6) Es sei wieder $((X_i, \mathscr{A}_i), f_{ij})_{i\in I}$ ein projektives System geringter Räume. Für jedes $i \in I$ sei ein $\mathscr{A}_i$-Modul $\mathscr{F}_i$ und für $i \leqq j$ ein f_{ij}-Morphismus $\mathscr{F}_j \to \mathscr{F}_i$ gegeben. Dann ist klar, was unter einem induktiven System $(\mathscr{F}_i)_{i\in I}$ von Moduln über dem projektiven System $((X_i, \mathscr{A}_i))_{i\in I}$ zu verstehen ist. Ist $((X, \mathscr{A}), f_i)$, $f_i = (\varphi_i, \theta_i)$, der projektive Limes dieses Systems, $\mathscr{F} = \varinjlim \varphi_i^*(\mathscr{F}_i)$, so hat $\mathscr{F}$ die Struktur eines $\mathscr{A}$-Moduls, und es ist leicht zu sehen, daß $\mathscr{F}$ mit dem induktiven Limes $\varinjlim f_i^*(\mathscr{F}_i)$ von $\mathscr{A}$-Moduln identifiziert werden kann. Die in den vorhergehenden Abschnitten durchgeführten Überlegungen bleiben sinngemäß auch für die Moduln $\mathscr{F}_i$ richtig.

2. Induktive Limites von Kohomologiegruppen

(2.1) In diesem Abschnitt werden hinreichende Bedingungen dafür angegeben, daß der kanonische Homomorphismus (1.2.5) bijektiv ist. Alle Literaturangaben beziehen sich auf [3], Ch. II. Alle Garben sind Garben abelscher Gruppen. Für einen topologischen Raum X bezeichnet $\mathfrak{Q}(X)$ die Menge der offenen, quasikompakten Teilmengen von X.

(2.2) Ein topologischer Raum X heiße *fastnoethersch*, falls er die beiden folgenden Bedingungen erfüllt:

(F 1) Die Familie $\mathfrak{Q}(X)$ ist eine Basis von X.

(F 2) Der Durchschnitt zweier Mengen aus $\mathfrak{Q}(X)$ liegt wieder in $\mathfrak{Q}(X)$.

Jeder offene, quasikompakte Unterraum eines fastnoetherschen Raumes ist fastnoethersch; jede Vereinigung offener, fastnoetherscher Unterräume eines topologischen Raumes ist fastnoethersch (damit auch jedes Präschema, da ein affines Schema stets fastnoethersch ist). Noethersche Räume sind fastnoethersch (in einem noetherschen Raum ist jede offene Menge quasikompakt), Produkte fastnoetherscher, quasikompakter Räume sind fastnoethersch.

(2.3) **Proposition.** *Es sei $(X_i, \varphi_{ij})_{i\in I}$ ein projektives System fastnoetherscher Räume, (X, φ_i) der projektive Limes. Sind die Abbildungen φ_i und φ_{ij} quasikompakt*[1]*, so ist der Raum X fastnoethersch, und jedes $U\in\mathfrak{Q}(X)$ ist von der Form $\varphi_i^{-1}(U_i)$ für ein geeignetes $i\in I$ und $U_i\in\mathfrak{Q}(X_i)$.*

Die Mengen der Form $\varphi_i^{-1}(U_i)$, $U_i\in\mathfrak{Q}(X_i)$, $i\in I$, sind eine Basis von X, also ist (F 1) erfüllt. Jedes $U\in\mathfrak{Q}(X)$ ist folglich von der Form $U=\varphi_{i_1}^{-1}(U_{i_1})\cup\cdots\cup\varphi_{i_n}^{-1}(U_{i_n})$, $U_{i_l}\in\mathfrak{Q}(X_{i_l})$, $i_l\in I$, $l=1,\dots,n$. Ist $j\in I$ mit $j\geqq i_l$, $l=1,\dots,n$, so ist $V_l=\varphi_{i_l j}^{-1}(U_{i_l})\in\mathfrak{Q}(X_j)$ und $U=\varphi_j^{-1}(V_1)\cup\cdots\cup\varphi_j^{-1}(V_n)$. Hieraus folgt sofort (F 2).

(2.4) **Proposition.** *Es sei $(\mathscr{A}_i)_{i\in I}$ ein induktives System von Garben auf dem fastnoetherschen Raum X. Für jedes $U\in\mathfrak{Q}(X)$ ist der kanonische Homomorphismus $\varinjlim \Gamma(U,\mathscr{A}_i)\to\Gamma(U,\mathscr{A})$ bijektiv.*

Die Injektivität ist im Beweis von Th. 3.10.1 enthalten: Von dem dortigen X (dessen Stelle jetzt U einnimmt) wird nur die Quasikompaktheit benutzt. Die Surjektivität ergibt sich aus dem Beweis von Th. 3.10.1

[1] Eine stetige Abbildung $f\colon X\to Y$ zweier topologischer Räume heißt quasikompakt, wenn $f^{-1}(V)\in\mathfrak{Q}(X)$ für jedes $V\in\mathfrak{Q}(Y)$.

für den Fall eines noetherschen Raumes: Für das dortige X wird unser U gewählt, $U=\bigcup_{i=1}^{n} U_i$ mit $U_i \in \mathfrak{Q}(X)$ gesetzt und beachtet, daß die loc. cit. konstruierte Menge $V_p \cap U_{p+1}$ als endliche Vereinigung der nach Voraussetzung quasikompakten Mengen $U_j \cap U_{p+1}$, $j=1,\ldots,p$, wieder quasikompakt ist.

(2.5) Eine Garbe $\mathscr{A}$ auf einem fastnoetherschen Raum X heißt *fastwelk*, falls für alle $U \subset V$ aus $\mathfrak{Q}(X)$ die Restriktion $\Gamma(V,\mathscr{A}) \to \Gamma(U,\mathscr{A})$ surjektiv ist. Eine welke Garbe auf X ist fastwelk; eine fastwelke Garbe auf einem noetherschen Raum ist welk.

(2.6) **Proposition.** *Sei*

$$0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$$

eine exakte Sequenz von Garben auf dem fastnoetherschen Raum X. Ist $\mathscr{A}'$ fastwelk, so ist für jedes $U \in \mathfrak{Q}(X)$ die Sequenz

$$0 \to \Gamma(U,\mathscr{A}') \to \Gamma(U,\mathscr{A}) \to \Gamma(U,\mathscr{A}'') \to 0$$

exakt.

Beim Beweis von Th. 3.1.2 werden das dortige V quasikompakt und statt $U \cap V$ eine darin enthaltene Menge aus $\mathfrak{Q}(X)$ gewählt.

Hiermit folgen im Fall eines fastnoetherschen Raumes Cor. zu Th. 3.1.2 und Th. 3.1.3, wenn welk durch fastwelk, $\Gamma_\Phi(\ )$ durch $\Gamma(U,\ )$ mit $U \in \mathfrak{Q}(X)$ ersetzt werden.

Die kanonische Auflösung $\mathscr{C}^\bullet(X,\mathscr{A})$ in [3], Abschnitt 4.3 ist auf einem fastnoetherschen Raum fastwelk; damit ergibt sich Th. 4.4.3 in folgender Form:

(2.7) **Proposition.** *Für jede fastwelke Garbe $\mathscr{A}$ auf einem fastnoetherschen Raum X ist*

$$H^p(U,\mathscr{A})=0 \qquad \text{für } p \geqq 1,\ U \in \mathfrak{Q}(X).$$

Hieraus schließt man wie in Th. 4.7.1:

(2.8) **Satz.** *Es sei X ein quasikompakter, fastnoetherscher Raum, $\mathscr{A}$ eine Garbe auf X. Ist $\mathscr{L}^\bullet$ eine fastwelke Auflösung von $\mathscr{A}$, so ist der kanonische Homomorphismus*

$$H^\bullet(\Gamma(X,\mathscr{L}^\bullet)) \to H^\bullet(X,\mathscr{A})$$

bijektiv.

Aus (2.4) folgt wie in [3], S. 163, daß auf einem fastnoetherschen Raum jeder induktive Limes fastwelker Garben wieder fastwelk ist und hiermit schließlich wie beim Beweis nach Th. 4.12.1, in dem der Fall eines noetherschen Raumes behandelt wird:

(2.9) **Satz.** *Ist X ein quasikompakter, fastnoetherscher Raum,* $(\mathscr{A}_i)_{i\in I}$ *ein induktives System von Garben auf* X, $\mathscr{A}=\varinjlim \mathscr{A}_i$, *so ist der kanonische Homomorphismus*

$$\varinjlim H^p(X,\mathscr{A}_i)\to H^p(X,\mathscr{A})$$

bijektiv.

Für den Rest dieses Abschnittes sei $\big((X_i,\mathscr{A}_i),(\varphi_{ij},\theta_{ij})\big)_{i\in I}$ ein projektives System geringter Räume, $\big((X,\mathscr{A}),(\varphi_i,\theta_i)\big)$ der projektive Limes.

(2.10) **Proposition.** *Es seien die folgenden Bedingungen erfüllt:*

i) *Die Räume* X_i *sind fastnoethersch, die Abbildungen* φ_i *und* φ_{ij} *sind quasikompakt, die Abbildungen* φ_i *sind surjektiv;*

ii) *die Garben* $\mathscr{A}_i$ *sind fastwelk;*

dann ist der Raum X fastnoethersch und die Garbe $\mathscr{A}$ *fastwelk.*

Die erste Behauptung ist (2.3). Seien nun $U\subset V$ aus $\mathfrak{Q}(X)$. Es wird das kommutative Diagramm

$$\begin{array}{ccc} \Gamma(V,\mathscr{A}) & \longrightarrow & \Gamma(U,\mathscr{A}) \\ \uparrow\wr & & \uparrow\wr \\ \varinjlim \Gamma\big(V,\varphi_i^*(\mathscr{A}_i)\big) & \longrightarrow & \varinjlim \Gamma\big(U,\varphi_i^*(\mathscr{A}_i)\big) \end{array} \tag{2.10.1}$$

betrachtet, in dem die vertikalen Pfeile nach (2.4) Isomorphismen sind. Sei s ein Schnitt von $\mathscr{A}$ über U; er ist durch einen Schnitt s_i von $\varphi_i^*(\mathscr{A}_i)$ über U bestimmt. Nach (2.3) gibt es ein $j\in I$ und $U_j\subset V_j$ aus $\mathfrak{Q}(X_j)$ mit $U=\varphi_j^{-1}(U_j)$, $V=\varphi_j^{-1}(V_j)$. Es werde $k\in I$ mit $k\geqq i$, $k\geqq j$ gewählt. Das Bild s_k von s_i in $\Gamma\big(U,\varphi_k^*(\mathscr{A}_k)\big)$ bestimmt ebenfalls den Schnitt s von $\mathscr{A}$ über U. In dem kommutativen Diagramm

$$\begin{array}{ccc} \Gamma(V_k,\mathscr{A}_k) & \longrightarrow & \Gamma(U_k,\mathscr{A}_k) \\ \downarrow\wr & & \downarrow\wr \\ \Gamma\big(V,\varphi_k^*(\mathscr{A}_k)\big) & \longrightarrow & \Gamma\big(U,\varphi_k^*(\mathscr{A}_k)\big) \end{array} \tag{2.10.2}$$

mit $U_k=\varphi_{jk}^{-1}(U_j)$, $V_k=\varphi_{jk}^{-1}(V_j)$ sind wegen der Surjektivität von φ_k die vertikalen Pfeile Isomorphismen. Der Schnitt s_k bestimmt einen Schnitt s'_k von $\mathscr{A}_k$ über U_k, der sich – da $\mathscr{A}_k$ fastwelk ist – zu einem Schnitt t'_k von $\mathscr{A}_k$ über V_k fortsetzen läßt. Dieser Schnitt liefert nach (2.10.2) einen Schnitt t_k von $\varphi_k^*(\mathscr{A}_k)$ über V; der dadurch gemäß (2.10.1) bestimmte Schnitt von $\mathscr{A}$ über V ergibt bei Restriktion auf U gerade s.

(2.11) **Satz.** *Es sei die Bedingung* i) *von* (2.10) *erfüllt und X quasikompakt. Dann ist der kanonische Homomorphismus* (1.2.5)

$$\varinjlim H^p(X_i,\mathscr{A}_i)\to H^p(X,\mathscr{A})$$

bijektiv.

Es sei $\mathscr{L}_i^\bullet = \mathscr{C}^\bullet(X_i, \mathscr{A}_i)$ die kanonische Auflösung von $\mathscr{A}_i$. Die φ_{ij}-Homomorphismen θ_{ij}: $\mathscr{A}_i \to (\varphi_{ij})_* (\mathscr{A}_j)$ $(i \leq j)$ liefern ein induktives System von Komplexen $(\Gamma(X_i, \mathscr{L}_i^\bullet))_{i \in I}$, für dessen Kohomologie

(2.11.1) $$\varinjlim H^\bullet(X_i, \mathscr{A}_i) = H^\bullet(\varinjlim \Gamma(X_i, \mathscr{L}_i^\bullet))$$

gilt. Es ist nach (2.10) $\mathscr{L}^\bullet = \varinjlim \varphi_i^* (\mathscr{L}_i^\bullet)$ eine fastwelke Auflösung von $\mathscr{A}$, also nach (2.9)

(2.11.2) $$H^\bullet(\Gamma(X, \mathscr{L}^\bullet)) \xrightarrow{\sim} H^\bullet(X, \mathscr{A})$$

bijektiv. In dem kommutativen Diagramm

$$\begin{array}{ccccccccc} 0 \longrightarrow & \varinjlim \Gamma(X_i, \mathscr{L}_i^0) & \longrightarrow & \varinjlim \Gamma(X_i, \mathscr{L}_i^1) & \longrightarrow & \varinjlim \Gamma(X_i, \mathscr{L}_i^2) & \longrightarrow \cdots \\ & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\ 0 \longrightarrow & \Gamma(X, \mathscr{L}^0) & \longrightarrow & \Gamma(X, \mathscr{L}^1) & \longrightarrow & \Gamma(X, \mathscr{L}^2) & \longrightarrow \cdots \end{array}$$

sind die vertikalen Pfeile wegen der Bijektivität von

$$\Gamma(X_i, \mathscr{L}_i^\bullet) \to \Gamma(X, \varphi_i^*(\mathscr{L}^\bullet)) \quad \text{und} \quad \varinjlim \Gamma(X, \varphi_i^*(\mathscr{L}^\bullet)) \to \Gamma(X, \mathscr{L}^\bullet)$$

Isomorphismen. Aus (2.11.1) und (2.11.2) folgt dann die Behauptung.

3. Die Riemannsche Fläche

In diesem Abschnitt wird vorausgesetzt: Es ist K ein Körper, $\mathfrak{o}$ ein Teilring von K. Für die im folgenden ohne Beweis angeführten Aussagen siehe [13], Ch. VI, §§ 17, 18.

(3.1) Es sei $X = X(K/\mathfrak{o})$ die Menge der Bewertungsringe von K, welche $\mathfrak{o}$ umfassen; hier wird K selbst ebenfalls als Bewertungsring aufgefaßt. Für $x \in X$ wird der Bewertungsring auch mit B_x bezeichnet. Es sei $\mathsf{A}(K/\mathfrak{o})$ die Menge der endlich erzeugten $\mathfrak{o}$-Algebren in K und für $A \in \mathsf{A}(K/\mathfrak{o})$ sei $S(A)$ die Menge der $x \in X$ mit $B_x \supset A$. Es ist $S(A_1) \cap S(A_2) = S(A)$, $A_1, A_2 \in \mathsf{A}(K/\mathfrak{o})$, A der von A_1 und A_2 erzeugte Ring. Die $S(A)$ werden als Basis der offenen Mengen einer Topologie auf X, der Zariski-Topologie, gewählt. Der topologische Raum X heißt die *Riemannsche Fläche von* $K/\mathfrak{o}$. Er hat folgende Eigenschaften:

i) Für $x \in X$ ist $\overline{\{x\}}$ die Menge der $y \in X$, für die B_y eine Spezialisierung von B_x ist.

ii) Jede offene Menge enthält den Punkt ζ mit $B_\zeta = K$; X ist irreduzibel und ζ ein allgemeiner Punkt.

iii) Nach i) ist X ein T_0-Raum, folglich ist ζ der einzige allgemeine Punkt von X.

iv) X ist quasikompakt.

Für $A \in \mathsf{A}(K/\mathfrak{o})$ ist die auf $S(A)$ induzierte Topologie gerade die Topologie der Riemannschen Fläche K/A. Aus iv) entnimmt man deshalb:

(3.2) **Proposition.** *Die Riemannsche Fläche $X = X(K/\mathfrak{o})$ ist ein quasikompakter, fastnoetherscher Raum.*

(3.3) Für jedes offene U in X sei $\mathscr{A}(U) = \bigcap_{x \in U} B_x$; für $U \subset V$ sei $\mathscr{A}(V) \to \mathscr{A}(U)$ die Inklusion. Es ist leicht zu sehen, daß das System $(\mathscr{A}(U))$ eine Garbe von Ringen über X ist, welche mit $\mathscr{A}$ bezeichnet wird. Für jedes $x \in X$ ist der Halm $\mathscr{A}_x$ zu B_x isomorph; es sind also die Halme der Garbe $\mathscr{A}$ quasilokale Ringe.

(3.4) Es sei $\mathrm{Loc}(K/\mathfrak{o})$ die Menge der quasilokalen Ringe von K, welche $\mathfrak{o}$ umfassen. Die Mengen $L(A) = \{$Menge der quasilokalen Ringe von K, welche A umfassen, $A \in \mathsf{A}(K/\mathfrak{o})\}$, lassen sich als Basis der offenen Mengen einer Topologie auf $\mathrm{Loc}(K/\mathfrak{o})$ wählen. Dann stimmt die auf X als Untermenge von $\mathrm{Loc}(K/\mathfrak{o})$ induzierte Topologie gerade mit der Topologie von X (als Riemannsche Fläche) überein. Für $y \in \mathrm{Loc}(K/\mathfrak{o})$ wird der zugehörige Ring auch mit A_y bezeichnet.

Für jeden Ring A, $\mathfrak{o} \subset A \subset K$, sei $T(A) = \{A_{\mathfrak{p}}, \mathfrak{p} \in \mathrm{Spec}(A)\}$. $T(A)$ ist eine Teilmenge von $\mathrm{Loc}(K/\mathfrak{o})$. Wird $T(A)$ mit der induzierten Topologie versehen, so ist $\mathfrak{p} \to A_{\mathfrak{p}}$ ein Homöomorphismus von $\mathrm{Spec}(A)$ auf $T(A)$.

Sind M, M' zwei Teilmengen von $\mathrm{Loc}(K/\mathfrak{o})$, so dominiert M' die Menge M, wenn jeder quasilokale Ring aus M' mindestens einen quasilokalen Ring aus M dominiert. Eine Teilmenge M von $\mathrm{Loc}(K/\mathfrak{o})$ heißt vollständig (separiert), wenn für jedes $x \in X$ der Ring B_x mindestens (höchstens) einen Ring A_y, $y \in M$, dominiert. Wir sagen kurz: x dominiert y.

(3.5) Eine Teilmenge Y von $\mathrm{Loc}(K/\mathfrak{o})$ heißt projektives Modell von $K/\mathfrak{o}$, wenn gilt: es gibt ein endliches System von Null verschiedener Elemente $\{a_0 = 1, a_1, \dots, a_n\}$ aus K so, daß

$$Y = \bigcup_{i=1}^{n} T(A_i), \qquad A_i = \mathfrak{o}[a_0/a_i, \dots, a_n/a_i].$$

Für $y \in Y$ wird der dem Punkt y entsprechende lokale Ring auch mit $\mathcal{O}_y$ bezeichnet. Es wird Y mit der induzierten Topologie versehen. Die Menge Y ist vollständig und separiert, also die Dominationsabbildung $\varphi_Y = \varphi\colon X \to Y$ erklärt. Sie ist stetig, abgeschlossen und surjektiv.

Sei t_0 ein über K transzendentes Element, $t_i = t_0 a_i$, $i = 1, \dots, n$, und R der graduierte Ring $\mathfrak{o}[t_0, \dots, t_n]$. Dann ist $\mathfrak{p} \to R_{(\mathfrak{p})}$ ein Homöomorphismus von $\mathrm{Proj}(R)$ auf Y.

(3.6) Für jedes offene U in dem projektiven Modell Y wird $\mathcal{O}_Y(U) = \bigcap_{y \in U} \mathcal{O}_y$ gesetzt. Dann ist $(\mathcal{O}_Y(U))$ eine Garbe von Ringen über Y, die mit $\mathcal{O}_Y$ bezeichnet wird; für jedes $y \in Y$ ist der Halm von $\mathcal{O}_Y$ im Punkt y zum

Ring $\mathcal{O}_y$ isomorph. Die Bijektion $\mathfrak{p} \to R_{(\mathfrak{p})}$ liefert gleichzeitig einen Isomorphismus der Garben $\tilde{R}$ und $\mathcal{O}_Y$; es kann also das projektive Schema $(\operatorname{Proj}(R), \tilde{R})$ mit $(Y, \mathcal{O}_Y)$ identifiziert werden. Unter einem projektiven Modell von $K/\mathfrak{o}$ soll künftig ein projektives Schema $(Y, \mathcal{O}_Y)$ der eben behandelten Art verstanden werden.

(3.7) Wird $y \in Y$ von $x \in X$ dominiert, so ist jedenfalls $\mathcal{O}_y \subset \mathscr{A}_x$, also für jedes offene U in Y

$$\Gamma(U, \mathcal{O}_Y) = \bigcap_{y \in U} \mathcal{O}_y \subset \bigcap_{x \in \varphi^{-1}(U)} \mathscr{A}_x = \Gamma(\varphi^{-1}(U), \mathscr{A}) = \Gamma(U, \varphi_*(\mathscr{A})),$$

und man hat einen Garbenhomomorphismus

$$\theta_Y = \theta\colon\ \mathcal{O}_Y \to \varphi_*(\mathscr{A}).$$

Für $x \in X$ ist $\theta_x^\# : \mathcal{O}_{\varphi(x)} \to \mathscr{A}_x$ gerade die der Domination von $\varphi(x)$ durch x entsprechende Inklusion, d.h. $\theta_x^\#$ ist ein lokaler Homomorphismus. Damit ist ein Morphismus geringter Räume $f_Y = (\varphi_Y, \theta_Y)$: $(X, \mathscr{A}) \to (Y, \mathcal{O}_Y)$ in der Kategorie der geringten Räume, deren Halme quasilokale Ringe sind, konstruiert.

(3.8) Es seien $(Y, \mathcal{O}_Y)$, $(Z, \mathcal{O}_Z)$ zwei projektive Modelle, und die Menge Z werde von der Menge Y dominiert. Die Dominationsabbildung $\varphi_{ZY} = \varphi\colon Y \to Z$ ist stetig, abgeschlossen und surjektiv. Wie eben konstruiert man einen Garbenhomomorphismus

$$\theta_{ZY} = \theta\colon \mathcal{O}_Z \to \varphi_*(\mathcal{O}_Y),$$

und für jedes $y \in Y$ ist $\theta_y^\# : \mathcal{O}_{\varphi(y)} \to \mathcal{O}_y$ gerade die der Domination von $\varphi(y)$ durch y entsprechende Inklusion, d.h. $\theta_y^\#$ ist ein lokaler Homomorphismus. Damit ist $f_{ZY} := (\varphi_{ZY}, \theta_{ZY})$: $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ ein Morphismus von Schemata.

Es ist nun $((Y, \mathcal{O}_Y), f_{YZ})$ ein projektives System geringter Räume über der teilweise geordneten, aufsteigend gefilterten Indexmenge $\mathfrak{Y}'$, deren Elemente gerade die Y sind.

(3.9) **Proposition.** *Man hat einen kanonischen Isomorphismus geringter Räume*

$$(X, \mathscr{A}) \to \varprojlim (Y, \mathcal{O}_Y).$$

Es ist bekannt, daß der topologische Raum X mit dem projektiven Limes $\varprojlim Y$ identifiziert werden kann und die kanonischen Abbildungen $\varphi_Y\colon X \to Y$ gerade die Dominationsabbildungen sind. Für jedes offene U in X ist $((\varphi_Y)^*(\mathcal{O}_Y))(U) = \bigcap_{y \in \varphi_Y^{-1}(U)} \mathcal{O}_y$, wie aus der Konstruktion der Garbe $(\varphi_Y)^*(\mathcal{O}_Y)$ folgt. Man hat Inklusionen $((\varphi_Y)^*(\mathcal{O}_Y))(U) \xrightarrow{\subset} \mathscr{A}(U)$, $Y \in \mathfrak{Y}'$, und es ist

$$\varinjlim ((\varphi_Y)^*(\mathcal{O}_Y))(U) \to \mathscr{A}(U)$$

bijektiv. Das folgt auch in dem hier – gegenüber [11] allgemeineren – betrachteten Fall wie in [11], St. 12.1 aus

$$B_x = \bigcup_{Y \in \mathfrak{Y}'} \mathcal{O}_{\varphi_Y(x)}, \qquad x \in X$$

(vgl. [11], S. 45).

4. Die Kohomologiegruppen der Riemannschen Fläche

(4.1) Es werden die in Abschnitt 3 gemachten Voraussetzungen spezialisiert. Mit k wird der Quotientenkörper von $\mathfrak{o}$ bezeichnet. K sei eine endlich erzeugte Erweiterung von k. Von dem Ring $\mathfrak{o}$ wird verlangt:

(R1) $\mathfrak{o}$ ist noethersch,

(R2) $\mathfrak{o}$ ist ein japanischer Ring (d.h. der ganze Abschluß von $\mathfrak{o}$ in einem beliebigen, endlich algebraischen Erweiterungskörper von k ist ein endlich erzeugter $\mathfrak{o}$-Modul. Ist K/k separabel erzeugbar, so ist diese Bedingung unnötig (vgl. Nagata [8], Part II, S. 419, Prop. 4), da sie nur dazu dient, die Existenz eines abgeleiteten normalen Modells sicherzustellen).

Die Menge der projektiven Modelle von $K/\mathfrak{o}$ wird wieder mit $\mathfrak{Y}'$ bezeichnet; die Menge der projektiven Modelle von $K/\mathfrak{o}$, deren Funktionenkörper mit K übereinstimmt, sei $\mathfrak{Y}$. $\mathfrak{Y}$ ist wieder aufsteigend gefiltert.

(4.2) **Proposition.** *$\mathfrak{Y}$ ist ein kofinales Untersystem von $\mathfrak{Y}'$.*

In der Tat: Ist $Y' \in \mathfrak{Y}'$, $\{a_1, \ldots, a_m\}$ ein Erzeugendensystem von K/k und keines der a_i gleich Null, Y das durch $\{1, a_1, \ldots, a_m\}$ bestimmte projektive Modell, so hat das Verbindungsmodell $J(Y, Y')$ ([13], S. 120) den Funktionenkörper K und dominiert Y'.

(4.3) Zu jedem projektiven Modell $Y \in \mathfrak{Y}$ existiert das abgeleitete normale Modell $N(Y)$ ([13], S. 128); aus der Konstruktion ist ersichtlich, daß $N(Y)$ mit der Normalisation des Schemas Y ([4], **II**, (6.3.8)) identifiziert werden kann. Es sei $\mathfrak{Y}_n$ die Menge der normalen Modelle aus $\mathfrak{Y}$.

(4.4) **Proposition.** *$\mathfrak{Y}_n$ ist ein kofinales Untersystem von $\mathfrak{Y}$.*

Das folgt aus der Tatsache, daß Y von $N(Y)$ dominiert wird.

(4.5) **Proposition.** *Der kanonische Homomorphismus* (1.2.6)

$$\varinjlim_{\mathfrak{Y}} \check{H}^p(Y, \mathcal{O}_Y) \to \check{H}^p(X, \mathscr{A}), \qquad p \geqq 0,$$

ist ein Isomorphismus von $\mathfrak{o}$-Moduln.

Es wird gezeigt, daß für die Menge $\mathfrak{Y}_n$ die Voraussetzungen von (1.5) erfüllt sind. Die dortige Bedingung i) ist erfüllt ((3.1), iv), (3.5)). Sei $Y \in \mathfrak{Y}$, $\mathfrak{U} \in \mathfrak{S}(Y)$ ($\mathfrak{S}(Y)$ ist die Menge der endlichen, offenen Überdeckungen von Y), $\mathfrak{U}=(U_i)_{1\leqq i\leqq n}$,

$$\alpha\colon C^\bullet(\mathfrak{U},\mathcal{O}_Y)\to C^\bullet\big(f_Y^{-1}(\mathfrak{U}),\mathscr{A}\big)$$

der kanonische Homomorphismus. Sei s eine p-Kokette aus $C^\bullet(\mathfrak{U},\mathcal{O}_Y)$; für alle $i_0,\ldots,i_p$ aus $\{1,\ldots,n\}$ ist $s(i_0,\ldots,i_p)$ also ein Schnitt von $\mathcal{O}_Y$ über $U_{i_0}\cap\cdots\cap U_{i_p}=U$; $s(i_0,\ldots,i_p)$ ist folglich ein Element von $\Gamma(U,\mathcal{O}_Y)\xrightarrow{\subset}\Gamma\big(f_Y^{-1}(U),\mathscr{A}\big)$, und es ist $\alpha(s)$ offenbar genau dann Null, wenn $s=0$ ist, also α injektiv (das gilt übrigens auch unter den allgemeineren Voraussetzungen von Abschnitt 3). Ist Y normal, so ist $\Gamma(U,\mathcal{O}_Y)$ ganz abgeschlossen, also $\Gamma(U,\mathcal{O}_Y)=\Gamma\big(f_Y^{-1}(U),\mathscr{A}\big)$, und damit α surjektiv, d.h.: Für die Menge $\mathfrak{Y}_n$ ist ii) von (1.5) richtig.

Der zweite Teil dieses Beweises zeigt:

(4.6) **Proposition.** *Es werde das projektive Modell Y von dem projektiven Modell Z dominiert, $f\colon Z\to Y$, und es sei $\mathfrak{U}$ eine offene Überdeckung von Y. Dann gilt:*

a) *Der kanonische Homomorphismus*

$$C^p(\mathfrak{U},\mathcal{O}_Y)\to C^p\big(f^{-1}(\mathfrak{U}),\mathcal{O}_Z\big),\qquad p\geqq 0,$$

ist injektiv.

b) *Ist Y normal, so ist dieser Homomorphismus bijektiv und folglich*

$$H^p(\mathfrak{U},\mathcal{O}_Y)\to H^p\big(f^{-1}(\mathfrak{U}),\mathcal{O}_Z\big),\qquad p\geqq 0,$$

ein Isomorphismus.

c) *Wird in* b) *Z durch X, f durch die Dominationsabbildung ersetzt, so bleibt die Aussage von* b) *richtig.*

(4.7) **Proposition.** *Der kanonische Homomorphismus* (1.2.5)

$$\varinjlim_{\mathfrak{Y}} H^p(Y,\mathcal{O}_Y)\to H^p(X,\mathscr{A}),\qquad p\geqq 0,$$

ist ein Isomorphismus von $\mathfrak{o}$-Moduln.

Ist $Y\in\mathfrak{Y}_n$, U eine offene, affine Menge in Y, zum Ring A gehörig, so ist $\varphi_Y^{-1}(U)=S(A)$ quasikompakt. Die Behauptung folgt aus (2.10), (3.5) und (4.4).

(4.8) **Satz.** *Der kanonische Homomorphismus*

$$\check{H}^p(X,\mathscr{A})\to H^p(X,\mathscr{A}),\qquad p\geqq 0,$$

ist ein Isomorphismus von $\mathfrak{o}$-Moduln.

Das folgt aus (1.2.7), (4.5), (4.6) und der Tatsache, daß für jedes $Y\in\mathfrak{Y}$ der kanonische Homomorphismus

$$\check{H}^p(Y,\mathcal{O}_Y)\to H^p(Y,\mathcal{O}_Y),\qquad p\geqq 0,$$

bijektiv ist, da es sich um Schemata handelt.

5. Endliche Erzeugbarkeit gewisser Kohomologiegruppen

Es seien weiterhin die Voraussetzungen (R 1), (R 2) aus Abschnitt 4 erfüllt.

(5.1) **Proposition.** *Es ist $H^0(X, \mathscr{A})$ ein endlich erzeugter $\mathfrak{o}$-Modul.*

Das folgt aus $H^0(X, \mathscr{A}) = \Gamma(X, \mathscr{A})$ und der Tatsache, daß der ganze Abschluß von $\mathfrak{o}$ in K ein endlich erzeugter $\mathfrak{o}$-Modul ist.

(5.2) Es sei $Y \in \mathfrak{Y}$, a ein von Null verschiedenes Element aus K, $Z(a)$ das zu $\{1, a\}$ gehörige projektive Modell. Es wird

(5.2.1) $$\Pi(a)(Y) = J(Y, Z(a)), \qquad \Omega(a)(Y) = N(\Pi(a)(Y))$$

gesetzt. Sind $a_1, \ldots, a_n$ endlich viele von Null verschiedene Elemente aus K, so wird zur Abkürzung

(5.2.2) $$\Pi(a_1, \ldots, a_n) = \Pi(a_1) \circ \Pi(a_2) \circ \cdots \circ \Pi(a_n),$$

(5.2.3) $$\Omega(a_1, \ldots, a_n) = \Omega(a_1) \circ \Omega(a_2) \circ \cdots \circ \Omega(a_n)$$

geschrieben. Wie in [11], St. 18.1 zeigt man dann:

(5.3) **Proposition.** *Es sei $Y \in \mathfrak{Y}$. Dann sind*

$$\Pi(Y) := \{\Pi(a_1, \ldots, a_n)(Y)\}, \qquad \Omega(Y) := \{\Omega(a_1, \ldots, a_n)(Y)\},$$

wo $a_1, \ldots, a_n$ jeweils alle endlichen Systeme von Null verschiedener Elemente aus K durchläuft, kofinale Teilmengen von $\mathfrak{Y}$.

(5.4) **Proposition.** *Es sei $Y \in \mathfrak{Y}$, a ein von Null verschiedenes Element aus K, $Z = \Pi(a)(Y)$. Für jede offene, affine Überdeckung $\mathfrak{U}$ von Y ist der kanonische Homomorphismus*

$$H^p(f^{-1}(\mathfrak{U}), \mathcal{O}_Z) \to H^p(Z, \mathcal{O}_Z), \qquad p \geqq 0,$$

bijektiv ($f\colon Z \to Y$ ist die Dominationsabbildung).

Es sei $\mathfrak{U} = (U_i)_{i \in I}$, $V_i = f^{-1}(U_i)$. Es ist

$$H^p(V_{i_0} \cap \cdots \cap V_{i_p}, \mathcal{O}_Z) = 0 \qquad \text{für } p \geqq 1$$

zu zeigen ([3], Ch. II, Cor. zu Th. 5.4.1). Da der Durchschnitt zweier offener, affiner Mengen von Y wieder affin ist, folgt das aus (zum Beweis vgl. auch [11], Th. 16.1)

(5.4.1) **Lemma.** *Für jedes offene, affine U in Y ist $H^p(f^{-1}(U), \mathcal{O}_Z) = 0$ für $p \geqq 1$.*

Es sei $V=f^{-1}(U)$. Wird $A=\Gamma(U,\mathcal{O}_Y)$ gesetzt, so ist $V=V_1\cup V_2$, $V_i=T(A_i)$, $A_1=A[a]$, $A_2=A[1/a]$; es kann V mit dem zu $\{1,a\}$ gehörigen projektiven Modell von K/A identifiziert werden, und $\mathfrak{V}=(V_1, V_2)$ ist eine offene, affine Überdeckung von V. Sei α ein 1-Kozykel aus dem Komplex $C^1(\mathfrak{V},\mathcal{O}_Z|V)$. Das Element $\alpha(1,2)=-\alpha(2,1)$ aus $\Gamma(V_1\cap V_2,\mathcal{O}_Z)$ wird in der Form $\alpha(1,2)=b+b'$, $b\in A[a]$, $b'\in A[1/a]$, geschrieben (es ist $\Gamma(V_1\cap V_2,\mathcal{O}_Z)=A[a,1/a]$). Die Nullkokette β, $\beta(1)=-b$, $\beta(2)=b'$, hat den 1-Kozykel α als Bild. Es folgt $H^p(\mathfrak{V},\mathcal{O}_Z|V)=0$ für $p\geqq 1$, so daß sich aus der Bijektivität von $H^p(\mathfrak{V},\mathcal{O}_Z|V)\to H^p(V,\mathcal{O}_Z)$ die Behauptung von (5.4.1) ergibt.

(5.5) Von nun an wird für den Rest dieses Abschnittes von dem Ring $\mathfrak{o}$ neben den Bedingungen (R 1) und (R 2) zusätzlich verlangt:

(R 3) $\mathfrak{o}$ hat endliche (Krullsche) Dimension: $\dim(\mathfrak{o})=n$;

(R 4) $\mathfrak{o}$ ist ein universeller Kettenring ([4], **IV**, (5.6.2)).

Die Bedingung (R 4) ist für jeden noetherschen regulären Ring erfüllt.

Der Transzendenzgrad von K/k sei $r\geqq 1$.

(5.6) **Proposition.** *Für jedes $Y\in\mathfrak{Y}$ ist*

(5.6.1) $$\dim(Y)=n+r.$$

Sei $S=\operatorname{Spec}(\mathfrak{o})$, $f\colon Y\to S$ der kanonische Morphismus. Da es zu jedem Primideal $\mathfrak{p}$ von $\mathfrak{o}$ einen Bewertungsring von K gibt, welcher $\mathfrak{o}$ umfaßt und in $\mathfrak{p}$ zentriert, ist $f(Y)=S$. Dann folgt (5.6.1) aus ([4], **IV**, (5.6.6)).

(5.7) **Proposition.** *Es sei $Y\in\mathfrak{Y}$, $N(Y)$ die Normalisierung von Y. Dann ist der kanonische Homomorphismus* (1.2.1)

$$\theta_{n+r}\colon H^{n+r}(Y,\mathcal{O}_Y)\to H^{n+r}(N(Y),\mathcal{O}_{N(Y)})$$

surjektiv.

Zum Beweis wird benötigt:

(5.7.1) **Lemma** (vgl. [10], Lemma 1, S. 244). *Es sei X ein projektives Schema über dem Ring $\mathfrak{o}$, zum graduierten Ring $S=\mathfrak{o}[t_0,\dots,t_m]$ gehörig ($t_i\in S_1$), Y ein abgeschlossenes Unterschema von der Dimension s. Dann gibt es $s+1$ homogene Elemente $f_0,\dots,f_s$ aus S_+ mit*

$$Y\subset\bigcup_{i=0}^{s} D_+(f_i)\ ^2.$$

Ist $s=-1$, so ist nichts zu zeigen. Es sei (5.7.1) für alle abgeschlossenen Unterschemata von X der Dimension $<s$ bewiesen. Sei $\mathfrak{a}$ das größte graduierte Ideal in S, so daß $Y=\operatorname{Proj}(S')$, $S'=S/\mathfrak{a}$, und $\varphi\colon S\to S'$ der kanonische Homomorphismus. Die irreduziblen Komponenten

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[2] Zur Bezeichnung vgl. ([4], **II**, § 2).

von Y seien $Y_1, \ldots, Y_k$. Zu jedem i, $1 \leqq i \leqq k$, wähle man einen Punkt $y_i \in Y_i$; dann gibt es ein homogenes $f \in S_+$ mit $y_i \in D_+(f)$ ([1], Ch. III, § 1, Prop. 8). Sei f vom Grad d. Es kann $f' = \varphi(f)$ mit einem Schnitt über Y des invertierbaren $\mathcal{O}_Y$-Moduls $\widetilde{S'(d)}$ identifiziert werden. Da $V_+(f')$ nach Wahl von f keine irreduzible Komponente Y_i enthält, ist $V_+(f')$ in Y von der Kodimension 1 ([4], **IV**, (5.1.8)), also hat $V_+(f) \cap Y$ eine Dimension $< s$ ([4], $\mathbf{0}_{\mathrm{IV}}$, (14.2.2)). Nach Induktionsannahme gibt es homogene Elemente $f_0, \ldots, f_{s-1}$ aus S_+ mit $V_+(f) \cap Y \subset \bigcup_{i=0}^{s-1} D_+(f_i)$; folglich ist

$$Y \subset \bigcup_{i=0}^{s} D_+(f_i) \qquad \text{mit } f_s = f.$$

Nun zum Beweis von (5.7). Die Menge Y' der normalen Punkte von Y ist offen ([4], **IV**, (6.13.2)); ist f: $N(Y) \to Y$ der kanonische Morphismus, so ist für jedes offene affine $U \subset Y'$ $V = f^{-1}(U)$ offen und affin in $N(Y)$ und $\Gamma(U, \mathcal{O}_Y) = \Gamma(V, \mathcal{O}_{N(Y)})$. Das Komplement F von Y' in Y is abgeschlossen und $\dim(F) < n + r$. In der Tat: Ist $(U_i)_{1 \leqq i \leqq m}$ eine endliche Überdeckung von Y durch offene affine Mengen U_i, zu Ringen A_i gehörig, so ist $(F \cap U_i)_{1 \leqq i \leqq m}$ eine offene Überdeckung von F und $\dim(F) = \max(\dim(F \cap U_i))$. Es ist $F \cap U_i$ abgeschlossen in U_i, also von der Form $\operatorname{Spec}(A_i/\mathfrak{a}_i)$, $\mathfrak{a}_i$ ein Ideal in A_i. Nun ist $\mathfrak{a}_i \neq 0$, da der allgemeine Punkt von Y nicht in F liegt.

Der weitere Beweis wird wie bei Snapper ([11], St. 17.4) geführt. Es wird eine offene affine Überdeckung $\mathfrak{U}$ von Y so konstruiert, daß der kanonische Homomorphismus

$$\theta_{n+r}\colon\ H^{n+r}(\mathfrak{U}, \mathcal{O}_Y) \to H^{n+r}\big(f^{-1}(\mathfrak{U}), \mathcal{O}_{N(Y)}\big)$$

surjektiv ist. Aus der Bijektivität von

$$H^{n+r}(\mathfrak{U}, \mathcal{O}_Y) \xrightarrow{\ \sim\ } H^{n+r}(Y, \mathcal{O}_Y),$$

$$H^{n+r}\big(f^{-1}(\mathfrak{U}), \mathcal{O}_{N(Y)}\big) \xrightarrow{\ \sim\ } H^{n+r}\big(N(Y), \mathcal{O}_{N(Y)}\big)$$

ergibt sich die Behauptung von (5.7).

Nach (5.7.1) werden offene affine Mengen $U_1, \ldots, U_{n+r}$ in Y so gewählt, daß $F \subset \bigcup_{i=1}^{n+r} U_i$. Seien weiter $U_{n+r+1}, \ldots, U_s$ offene affine Mengen in Y so, daß $Y' = \bigcup_{i=n+r+1}^{s} U_i$. Es sei $\mathfrak{U} = (U_i)_{1 \leqq i \leqq s}$ die Überdeckung von Y durch offene affine Mengen. Es wird eine $(n+r)$-Kokette β von

$$C^{n+r}\big(f^{-1}(\mathfrak{U}), \mathcal{O}_{N(Y)}\big)$$

betrachtet. Es ist $\beta(i_0, \ldots, i_{n+r})$ ein Schnitt von $\mathcal{O}_{N(Y)}$ über

$$V = V_{i_0} \cap \cdots \cap V_{i_{n+r}}, \qquad V_i = f^{-1}(U_i).$$

Sind die Indizes $i_0, \ldots, i_{n+r}$ alle verschieden, so ist $U = U_{i_0} \cap \cdots \cap U_{i_{n+r}} \subset Y'$, so daß nach der oben durchgeführten Überlegung $\beta(i_0, \ldots, i_{n+r})$ mit einem Schnitt $\alpha(i_0, \ldots, i_{n+r})$ von $\mathcal{O}_Y$ über U identifiziert werden kann. Sind die Indizes $i_0, \ldots, i_{n+r}$ nicht alle verschieden, so ist $\beta(i_0, \ldots, i_{n+r}) = 0$; es wird $\alpha(i_0, \ldots, i_{n+r}) = 0$ gesetzt, und α ist eine $(n+r)$-Kokette von $C^{n+r}(\mathfrak{U}, \mathcal{O}_Y)$ mit $\theta_{n+r}(\alpha) = \beta$. Ist β ein Kozykel, d.h. $d^{n+r}(\beta) = 0$, so ist $\theta_{n+r}(d^{n+r}(\alpha)) = 0$, und aus (4.6) a) folgt $d^{n+r}(\alpha) = 0$.

(5.8) **Satz.** *Erfüllt der Ring* $\mathfrak{o}$ *die Bedingungen* (R 1)–(R 4), *so ist* $H^{n+r}(X, \mathscr{A})$ *ein endlich erzeugter* $\mathfrak{o}$-*Modul und* $H^p(X, A) = 0$ *für* $p > n+r$.

Die zweite Behauptung ergibt sich unmittelbar aus (5.7.1) und (5.6) in Verbindung mit (4.6) und (4.8). Zum Beweis der ersten Aussage benötigen wir (zum Beweis dieses Lemmas im Falle eines Körpers R s. [11], St. 18.2)

(5.8.1) **Lemma.** *Es sei* R *ein Ring,* (M_i, f_{ij}) *ein induktives System von* R-*Moduln,* $M = \varinjlim M_i$, f_i: $M_i \to M$ *die kanonischen Homomorphismen. Es gebe ein* $i_0 \in I$, *so daß* $f_{i_0 i}$: $M_{i_0} \to M_i$ *surjektiv ist für alle* $i \geqq i_0$. *Dann gilt:*

a) *Alle* f_{ij} *und alle* f_i, $i_0 \leqq i \leqq j$, *sind surjektiv.*

b) *Sind alle* M_i *endlich erzeugte* R-*Moduln, so ist* M *ein endlich erzeugter* R-*Modul.*

c) *Haben alle* M_i *endliche Länge* $l(M_i)$, *so hat auch* M *endliche Länge; es ist* $l(M_i) \geqq l(M)$ *für* $i \geqq i_0$, *und es gibt ein* $j \in I$ *mit* $l(M_j) = l(M)$.

Nun zum Beweis von (5.8). Es sei $Y \in \mathfrak{Y}_n$. Für jedes $t \geqq 0$ sei Ω_t die Teilmenge von $\Omega(Y)$, deren Elemente durch höchstens t Elemente $a_1, \ldots, a_t$ aus K bestimmt sind. Es werde induktiv angenommen, daß für jedes $Z \in \Omega_t$ $H^{n+r}(Y, \mathcal{O}_Y) \to H^{n+r}(Z, \mathcal{O}_Z)$ surjektiv ist. Sei $Z'' \in \Omega_{t+1}$, also $Z'' = N(Z')$, $Z' = \Pi(a)(Z)$ für geeignetes $a \in K$ und $Z \in \Omega_t$. Es ist

$$H^p(Z, \mathcal{O}_Z) \to H^p(Z', \mathcal{O}_{Z'}), \qquad p \geqq 0,$$

bijektiv nach (4.7), (6.4),

$$H^{n+r}(Z', \mathcal{O}_{Z'}) \to H^{n+r}(Z'', \mathcal{O}_{Z''})$$

surjektiv nach (5.7). Aus (5.3) und (5.8.1) b) folgt die Behauptung, da die $H^p(Z, \mathcal{O}_Z)$ für $Z \in \mathfrak{Y}$ endlich erzeugte $\mathfrak{o}$-Moduln sind ([4], **III**, (2.2.2)).

(5.9) Es werde die Riemannsche Fläche von K/k, k der Quotientenkörper von $\mathfrak{o}$, mit X^* bezeichnet, die Strukturgarbe von X^* mit $\mathscr{A}^*$ und das System der projektiven Modelle von K/k, die K als Funktionenkörper besitzen, mit $\mathfrak{Y}^*$. Dann folgt aus dem Beweis von (5.8) und (5.8.1)c)

(5.10) **Korollar.** *Es ist* $H^r(X^*, \mathscr{A}^*)$ *ein endlichdimensionaler Vektorraum über* k. *Für jedes* $Y \in \mathfrak{Y}^*$ *ist*

(5.10.1) $$\dim_k(H^r(Y, \mathcal{O}_Y)) \geqq \dim_k(H^r(X^*, \mathscr{A}^*)),$$

und es gibt ein $Z \in Y^*$ *mit* $\dim_k(H^r(Z, \mathcal{O}_Z)) = \dim_k(H^r(X^*, \mathscr{A}^*))$.

Dieses Resultat ist im wesentlichen das Ergebnis von Snapper ([11], Theorem 18.1), dort allerdings nur für die Čechschen Kohomologiegruppen bewiesen.

(5.11) **Definition.** Die endliche k-Dimension des Vektorraumes $H^r(X^*, \mathscr{A}^*)$ heißt das *geometrische Geschlecht* von K/k und wird mit $g(K/k)$ bezeichnet.

(5.12) Ist Y ein projektives Modell von $K/\mathfrak{o}$, so kann $Y \otimes_{\mathfrak{o}} k$ mit einem projektiven Modell von K/k identifiziert werden, und es ist klar, daß $\mathfrak{Y}^* = \{Y \otimes_{\mathfrak{o}} k \mid Y \in \mathfrak{Y}\}$, analog für $\mathfrak{Y}_n^*$. Da für jedes $Y \in \mathfrak{Y}$ $H^p(Y, \mathcal{O}_Y) \otimes_{\mathfrak{o}} k$ mit $H^p(Y', \mathcal{O}_{Y'})$, $Y' = Y \otimes_{\mathfrak{o}} k$, identifiziert werden kann ([4], **III**, (1.4.15)), folgt aus $\varinjlim H^p(Y, \mathcal{O}_Y) \xrightarrow{\sim} H^p(X, \mathscr{A})$ und der Verträglichkeit des induktiven Limes mit Tensorproduktbildung, daß

$$H^p(X, \mathscr{A}) \otimes_{\mathfrak{o}} k \xrightarrow{\sim} H^p(X^*, \mathscr{A}^*)$$

bijektiv ist.

(5.13) Ist k in K algebraisch abgeschlossenen und K/k vom Transzendenzgrad 1, so stimmt das geometrische Geschlecht $g(K/k)$ mit dem im Riemann-Rochschen Satz auftretenden Geschlecht überein. Das ergibt sich etwa aus Rosenlicht [9], wenn man beachtet, daß für diesen Fall stets singularitätenfreie, projektive Modelle von K/k existieren.

6. Reduktion des Funktionenkörpers

(6.1) Ist P ein quasilokaler Ring mit dem maximalen Ideal $\mathfrak{m}(P)$, so heißt der natürliche Homomorphismus $P \to P/\mathfrak{m}(P)$ der durch P definierte Homomorphismus; er wird mit φ_P bezeichnet. Ist M eine Menge von in P enthaltenen Ringen, so wird die Menge der homomorphen Bilder der Ringe aus M bei φ_P mit $\varphi_P(M)$ bezeichnet und das Bild von M bei φ_P genannt.

(6.2) Es sei $(Y, \mathcal{O}_Y)$ ein projektives Modell von $K/\mathfrak{o}$, $\mathfrak{o}$ wie in Abschnitt 3 ein beliebiger Teilring von K, Y' eine abgeschlossene irreduzible Teilmenge des topologischen Raumes Y mit dem allgemeinen Punkt y', der dem lokalen Ring $\mathcal{O}_{y'} = P$ entspricht. Wie in [11], § 19 zeigt man, daß die Menge $\varphi_P(Y')$ von Ringen über $\varphi_P(\mathfrak{o})$ in dem Körper $\varphi_P(P)$ die Menge der zu einem projektiven Modell von $\varphi_P(P)/\varphi_P(\mathfrak{o})$ gehörigen Ringe ist. Der so erhaltene topologische Raum $\bar{Y} = \varphi_P(Y')$ ist zu Y' isomorph; sei $\mathcal{O}_{\bar{Y}}$ die Strukturgarbe von $\bar{Y}$. Ist $\mathscr{I}_Y$ die quasikohärente Idealgarbe, welche das abgeschlossene Unterschema Y' von Y definiert, so hat man eine exakte Sequenz

(6.2.1) $$0 \to \mathscr{I}_Y \to \mathcal{O}_Y \to \mathscr{B}_Y \to 0,$$

und es ist $\mathscr{B}_Y | Y' = \mathcal{O}_{Y'}$. Für $y \in Y \setminus Y'$ ist $\mathscr{I}_y = \mathcal{O}_{Y,y}$, für $y \in Y'$ ist $\mathscr{I}_y = \mathcal{O}_{Y,y} \cap \mathfrak{m}(P)$. Folglich sind die geringten Räume $(Y', \mathcal{O}_{Y'})$ und $(\bar{Y}, \mathcal{O}_{\bar{Y}})$ isomorph.

Man hat eine exakte Kohomologiesequenz

(6.2.2) $$\cdots \to H^p(Y, \mathscr{I}_Y) \to H^p(Y, \mathcal{O}_Y) \to H^p(Y, \mathscr{B}_Y) \to H^{p+1}(Y, \mathscr{I}_Y) \to \cdots,$$

und es können

(6.2.3) $$H^p(Y, \mathscr{B}_Y), \quad H^p(Y', \mathcal{O}_{Y'}), \quad H^p(\bar{Y}, \mathcal{O}_{\bar{Y}})$$

identifiziert werden.

(6.3) Von nun an sei $\mathfrak{o}$ ein noetherscher Bewertungsring, der die Bedingung (R 2) aus (3.1) erfüllt (oder es sei K/k separabel erzeugbar). Es sei v die zum Bewertungsring $\mathfrak{o}$ gehörige (einrangig diskrete) Bewertung, V eine Funktionalfortsetzung von v auf K ([10], S. 309). Der zu V gehörige Bewertungsring von $K/\mathfrak{o}$ werde B genannt. Der Körper $\bar{K} = B/\mathfrak{m}(B)$ ist ein algebraischer Funktionenkörper vom Transzendenzgrad r über $\bar{k} = \mathfrak{o}/\mathfrak{m}(\mathfrak{o})$; der ihm entsprechende geringte Raum sei $(\bar{X}, \bar{\mathscr{A}})$. Es sei X' der Abschluß in X des zu B gehörigen Punktes $x' \in X$. Dann vermittelt der Homomorphismus $B \to \bar{K}$ eine Bijektion $f'\colon X' \to \bar{X}$ von der Menge der in B enthaltenen Bewertungsringe von $K/\mathfrak{o}$, welche Spezialisierungen von B sind, auf die Menge der Bewertungsringe von $\bar{K}/\bar{k}$, und es ist klar, daß f' ein Homöomorphismus ist. Für jedes $y \in X'$ sei $\mathscr{I}_y = B_y \cap \mathfrak{m}(B)$, für $y \notin X'$ sei $\mathscr{I}_y = \mathscr{A}_y$. Dann definieren die $\mathscr{I}_y$ eine Idealgarbe $\mathscr{I}_X$ und die durch die exakte Sequenz

(6.3.1) $$0 \to \mathscr{I}_X \to \mathscr{A} \to \mathscr{B}_X \to 0$$

definierte Garbe $\mathscr{B}_X$ kann als Garbe über X' aufgefaßt werden. Wird $\mathscr{A}' = \mathscr{B}_X | X'$ gesetzt, so kann $(X', \mathscr{A}')$ mit $(\bar{X}, \bar{\mathscr{A}})$ identifiziert werden. Insbesondere sind die Kohomologiegruppen

(6.3.2) $$H^p(X, \mathscr{B}_X), \quad H^p(X', \mathscr{A}'), \quad H^p(\bar{X}, \bar{\mathscr{A}})$$

alle zueinander isomorph.

(6.4) Sei $(Y, \mathcal{O}_Y)$ ein projektives Modell von $K/\mathfrak{o}$, y' das Bild von x' bei der Dominationsabbildung $X \to Y$, P_Y der zugehörige lokale Ring, Y' der

Abschluß von y' in Y. Dann kann $\varphi_{P_Y}(Y')$ mit einem projektiven Modell $\overline{Y}$ von $\overline{K}/\overline{k}$ identifiziert werden, und man hat ein kommutatives Diagramm von Morphismen geringter Räume

$$(6.4.1)\qquad \begin{array}{ccc} (X', \mathscr{A}') & \xrightarrow{\sim} & (\overline{X}, \overline{\mathscr{A}}) \\ \downarrow & & \downarrow \\ (Y', \mathcal{O}_{Y'}) & \xrightarrow{\sim} & (\overline{Y}, \mathcal{O}_{\overline{Y}}). \end{array}$$

Es sei weiter $\mathfrak{Y}^0$ die Menge der projektiven Modelle Y von $K/\mathfrak{o}$, so daß $\overline{Y}$ den Körper $\overline{K}$ als Körper der rationalen Funktionen besitzt.

Ähnlich wie in [11], St. 20.1, zeigt man:

(6.5) **Proposition.** *$\mathfrak{Y}^0$ ist ein kofinales Untersystem von $\mathfrak{Y}$.*

Sind Y, Z aus $\mathfrak{Y}$ und wird Y von Z dominiert, so hat mit $\overline{Y}$ auch $\overline{Z}$ den Körper $\overline{K}$ als Körper der rationalen Funktionen. Es genügt also, die Existenz eines $Y \in \mathfrak{Y}$ mit $Y \in \mathfrak{Y}^0$ nachzuweisen. Seien $b_1, \dots, b_n$ von Null verschiedene Elemente aus V, so daß ihre Restklassen $\bar{b}_1, \dots, \bar{b}_n \pmod{\mathfrak{m}(B)}$ den Körper $\overline{K}$ über $\bar{k}$ erzeugen und Y das zu $\{1, b_1, \dots, b_n\}$ gehörige projektive Modell von $K/\mathfrak{o}$. Dann hat $\overline{Y}$ den Funktionenkörper $\overline{K}$. Ist $Y \notin \mathfrak{Y}$, so gibt es $Z \in \mathfrak{Y}$ so, daß Y von Z dominiert wird (4.2), und es ist $Z \in \mathfrak{Y}^0$.

(6.6) Die teilweise geordnete, aufsteigend gefilterte Menge der projektiven Modelle von $\overline{K}/\bar{k}$, welche $\overline{K}$ als Funktionenkörper haben, wird mit $\overline{\mathfrak{Y}}$ bezeichnet. Nach (6.5) hat man eine Abbildung

$$(6.6.1)\qquad \pi\colon \mathfrak{Y}^0 \to \overline{\mathfrak{Y}}, \quad \text{definiert durch } Y \mapsto \overline{Y}.$$

(6.7) **Proposition.** *Die Abbildung π ist surjektiv. Sind Y, Z aus $\mathfrak{Y}$, so ist*

$$(6.7.1)\qquad \overline{J(Y, Z)} = J(\overline{Y}, \overline{Z}).$$

Wird Y von Z dominiert, so wird $\overline{Y}$ von $\overline{Z}$ dominiert.

Die Surjektivität und (6.7.1) folgen unmittelbar aus der Konstruktion von π. Werden Z', z' bezüglich Z genau so bestimmt wie Y', y' bezüglich Y, so wird y' von z' dominiert, und jeder Punkt von Z' dominiert einen Punkt von Y'. Hieraus folgt der letzte Teil der Behauptung.

(6.8) Für jedes $Y \in \mathfrak{Y}$ hat man die exakte Sequenz (6.3.1)

$$(6.8.1)\qquad 0 \to \mathscr{I}_Y \to \mathcal{O}_Y \to \mathscr{B}_Y \to 0$$

von $\mathcal{O}_Y$-Moduln, in der also $\mathscr{B}_Y | Y' = \mathcal{O}_{Y'}$. Wird Y von Z dominiert: $f\colon Z \to Y$, so hat man ein kommutatives Diagramm

$$(6.8.2)\qquad \begin{array}{ccccccccc} 0 \to & \mathscr{I}_Y & \to & \mathcal{O}_Y & \to & \mathscr{B}_Y & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & f_*(\mathscr{I}_Z) & \to & f_*(\mathcal{O}_Z) & \to & f_*(\mathscr{B}_Z) & \to 0, \end{array}$$

in dem der mittlere vertikale Pfeil der durch den Morphismus $(Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ bestimmte Homomorphismus ist. In der Tat: Sei $y \in Y$, $z \in Z$, und $f(z) = y$. Ist $y \notin Y'$, so $z \notin Z'$, und $\mathcal{I}_y = \mathcal{O}_y$, $\mathcal{I}_z = \mathcal{O}_z$. Ist $y \in Y'$, so ist $z \in Z'$, $\mathcal{O}_z \supset \mathcal{O}_y$ und $\mathcal{I}_y = \mathfrak{m}(B) \cap \mathcal{O}_y \subset \mathfrak{m}(B) \cap \mathcal{O}_z = \mathcal{I}_z$.

Der Homomorphismus $\mathcal{B}_Y \to f_*(\mathcal{B}_Z)$ induziert durch Restriktion einen Morphismus $(Z', \mathcal{O}_{Z'}) \to (Y', \mathcal{O}_{Y'})$, und es kommutiert

(6.8.3)
$$\begin{array}{ccc} (Z', \mathcal{O}_{Z'}) & \xrightarrow{\sim} & (\bar{Z}, \mathcal{O}_{\bar{Z}}) \\ \downarrow & & \downarrow \\ (Y', \mathcal{O}_{Y'}) & \xrightarrow{\sim} & (\bar{Y}, \mathcal{O}_{\bar{Y}}). \end{array}$$

(6.9) Es sei $\mathcal{I}$ die Idealgarbe von $\mathcal{A}$, welche als induktiver Limes der $(\mathcal{I}_Y)$ erhalten wird, analog $\mathcal{B}$ die Garbe von $\mathcal{A}$-Moduln, welche als induktiver Limes der $(\mathcal{B}_Y)$ erhalten wird. Dann hat man eine exakte Sequenz

(6.9.1)
$$0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{B} \to 0$$

von $\mathcal{A}$-Moduln. Es ist leicht zu sehen, daß $\mathcal{B}$ mit $\mathcal{B}_X$ und $\mathcal{I}$ mit $\mathcal{I}_X$ identifiziert werden kann (beachte (6.8.3)).

Aus (4.8) und (6.3.2) folgt, daß die kanonischen Homomorphismen

(6.9.2)
$$\check{H}^p(X, \mathcal{A}) \xrightarrow{\sim} H^p(X, \mathcal{A}), \qquad \check{H}^p(X, \mathcal{B}_X) \xrightarrow{\sim} H^p(X, \mathcal{B}_X), \quad p \geqq 0,$$

bijektiv sind. Es wird gezeigt, daß auch

(6.9.3)
$$\check{H}^p(X, \mathcal{I}_X) \xrightarrow{\sim} H^p(X, \mathcal{I}_X), \qquad p \geqq 0,$$

bijektiv ist, was sich aus (4.7) und (1.2.7) ergibt, falls

(6.9.4)
$$\varinjlim \check{H}^p(Y, \mathcal{I}_Y) \to \check{H}^p(X, \mathcal{I}_X), \qquad p \geqq 0,$$

bijektiv ist. Dazu beschränkt man sich auf das kofinale System $\mathfrak{Y}_n$ der normalen Modelle. Ist $\mathfrak{U} = (U_i)_{i \in I}$ eine offene Überdeckung von $Y \in \mathfrak{Y}_n$, $f: X \to Y$ die Dominationsabbildung, so induziert die Bijektion $C^\bullet(\mathfrak{U}, \mathcal{O}_Y) \to C^\bullet(f^{-1}(\mathfrak{U}), \mathcal{A})$ (vgl. (4.6) c)) einen injektiven Bihomomorphismus $C^\bullet(\mathfrak{U}, \mathcal{I}_Y) \to C^\bullet(f^{-1}(\mathfrak{U}), \mathcal{I}_X)$. Sei β eine p-Kokette aus $C^p(f^{-1}(\mathfrak{U}), \mathcal{I}_X)$, α die p-Kokette aus $C^p(\mathfrak{U}, \mathcal{O}_Y)$, welche das Urbild von β ist. Sei $U = U_{i_0} \cap \cdots \cap U_{i_p}$; dann ist $\alpha(i_0, \ldots, i_p)$ ein Schnitt s von $\mathcal{O}_Y$ über U. Es wird $s_y \in \mathcal{I}_y$ für jedes $y \in Y$ gezeigt. Für $y \notin Y'$ ist nichts zu zeigen. Sei also $y \in U \cap Y'$, $x \in X'$ und $f(x) = y$. Da $(\beta(i_0, \ldots, i_p))_x$ in $\mathfrak{m}(B) \cap B_x$ liegt, liegt s_y in $\mathcal{O}_y \cap \mathfrak{m}(B) = \mathcal{I}_y$. Damit ist (6.9.4) ein Isomorphismus.

Aus (6.3.1), (6.8.1) erhält man das kommutative Diagramm

$$\begin{array}{ccccccccc} \cdots\to \varinjlim H^p(Y,\mathscr{I}_Y) & \to & \varinjlim H^p(Y,\mathcal{O}_Y) & \to & \varinjlim H^p(Y,\mathscr{B}_Y) & \to & \varinjlim H^{p+1}(Y,\mathscr{I}_Y) & \to\cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots\to H^p(X,\mathscr{I}_X) & \to & H^p(X,\mathscr{A}) & \to & H^p(X,\mathscr{B}_X) & \to & H^{p+1}(X,\mathscr{I}_X) & \to\cdots \end{array} \tag{6.9.5}$$

von $\mathfrak{o}$-Moduln mit exakten Zeilen, in dem die vertikalen Pfeile Isomorphismen sind.

(6.10) Nun wird vorausgesetzt: V ist prim über v. Das soll genauer heißen: Es gibt eine Transzendenzbasis $(\xi_i)_{1\leqq i\leqq r}$ von K/k so, daß V Funktionalfortsetzung von v bezüglich $\xi_1,\dots,\xi_r$ ist und $[K:k(\xi)]=[\bar{K}:\bar{k}(\bar{\xi})]$[3]. Es ist dann V unverzweigt über v und V die einzige Fortsetzung der Funktionalbewertung v' von $k(\xi)$ bezüglich $\xi_1,\dots,\xi_r$ auf K.

(6.11) **Satz.** *Ist V prim über v, so ist $g(K/k)\geqq g(\bar{K}/\bar{k})$.*

Sei Y das durch $\{\xi_0=1,\xi_1,\dots,\xi_r\}$ definierte projektive Modell von $K/\mathfrak{o}$; $\mathfrak{o}_i=\mathfrak{o}[\xi_0/\xi_i,\dots,\xi_r/\xi_i]$, so daß $Y=\bigcup T(\mathfrak{o}_i)$. Sei weiter Z ein Modell aus $\Omega(Y)$ (5.3). Dann ist die Menge Z' der $z\in Z$, so daß $\mathcal{O}_z$ den Ring $\mathfrak{o}$ dominiert, eine abgeschlossene, irreduzible Teilmenge von Z (Z also ein $\mathfrak{m}(\mathfrak{o})$-einfaches Modell im Sinne von Yanagihara ([12], S. 140)). Ist nämlich $Z=\bigcup T(A_j)$, A_j ganz abgeschlossener noetherscher Ring, so genügt es zu zeigen, daß das erzeugende Element π von $\mathfrak{m}(\mathfrak{o})$ in A_j entweder ein Primideal erzeugt oder Einheit von A_j ist. Ist π keine Einheit in A_j, so sei

$$(\pi)=\mathfrak{p}_1^{(n_1)}\cap\cdots\cap\mathfrak{p}_s^{(n_s)},$$

$\mathfrak{p}_i$ minimale Primideale von A_j, die Primärzerlegung von π ([13], Ch. V, § 6, Th. 14, Cor. 1). Da die Ringe $(A_j)_{\mathfrak{p}_1},\dots,(A_j)_{\mathfrak{p}_s}$ diskrete Bewertungsringe sind und jeder Ring A_j mindestens einen Ring $\mathfrak{o}_i$ enthält, liefert jeder der Ringe $(A_j)_{\mathfrak{p}_1},\dots,(A_j)_{\mathfrak{p}_s}$ eine Bewertung, welche Fortsetzung von v' ist. Also ist $s=1$, $n_1=1$ und (π) ein Primideal. Es wird dann weiter durch π ein graduiertes Primideal in einem geeigneten, zu Z gehörigen graduierten Ring erzeugt, und ([6], (4.9)) zeigt, daß

$$\dim_k(H^r(Z^*,\mathcal{O}_{Z^*}))=\dim_{\bar{k}}(H^r(\bar{Z},\mathcal{O}_{\bar{Z}})),\qquad Z^*=Z\otimes_{\mathfrak{o}}k.$$

Aus (5.10), (5.12) und (6.7) folgt die Behauptung des Satzes.

(6.12) Sind insbesondere unter der Voraussetzung von (6.10) k in K und $\bar{k}$ in $\bar{K}$ algebraisch abgeschlossen, so ist im Falle $r=1$ das Resultat von Deuring und Lamprecht ([7], S. 258, Kor. 1) von neuem bewiesen, vgl. (5.13).

[3] Querstriche bedeuten Restbildung mod V.

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Neighborhoods of Hyperbolic Sets

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§ 1. Introduction

In this paper we study the asymptotic behavior of points near a compact hyperbolic set of a C^r diffeomorphism ($r \geqq 1$) f: $M \to M$, M being a compact manifold. The purpose of our study is to complete the proof of Smale's Ω-stability Theorem by demonstrating (2.1), (2.4) of [6].

Ω denotes the set of non-wandering points for f. Smale's Axiom A requires [5]:

(a) Ω has a hyperbolic structure,

(b) the periodic points are dense in Ω.

Hyperbolic structure, the stable manifold of Ω, and fundamental neighborhoods are discussed in §§ 2 and 5.

The result of [6] proved here is:

If f obeys Axiom A *then there exists a proper fundamental neighborhood V for the stable manifold of Ω such that the union of the unstable manifold of Ω and the forward orbit of V contains a neighborhood of Ω in M.*

As a consequence we have:

If f obeys Axiom A *then any point whose orbit stays near Ω is asymptotic with a point of Ω.*

Section 8 of the mimeographed version of [1] contains a generalization of the above results with an incorrect proof. A correct generalization is:

(1.1) Theorem. *If Λ is a compact hyperbolic set then $W^u(\Lambda) \cup 0_+ V$ contains a neighborhood U of Λ, where V is any fundamental neighborhood for $W^s(\Lambda)$ and $0_+ V = \bigcup_{n \geqq 0} f^n(V)$. If Λ has local product structure then a proper fundamental neighborhood may be found and any point whose forward orbit lies in U is asymptotic with some point of Λ.*

Theorem (1.1) is proved in § 5, local product structure is discussed in [5] and in § 2. In § 7 we prove the analogous theorems for flows.

Here is an example, due to Bowen, of a compact hyperbolic set Λ which does not have local product structure, has no proper fundamental neighborhood and for which there are points asymptotic to Λ without being asymptotic with any point of Λ.

Consider the Cantor set C as the space of sequences of zeros and ones with a decimal point:

$$x=(\dots x_{-2}\,x_{-1}\,.\,x_0\,x_1\dots) \qquad x_j=0 \text{ or } 1.$$

The elements of C can be thought of as maps $\mathbf{Z}\to\{0,1\}$. The compact open topology makes C a metrizable compact space. A distance is given by

$$d(x,y)=\sum_{-\infty}^{\infty}\frac{|x_i-y_i|}{2^{|i|}}.$$

The shift map $\sigma\colon C\to C$ moves the decimal point one space to the right:

$$\sigma(\dots x_{-1}\,.\,x_0\,x_1\dots)=(\dots x_{-1}\,x_0\,.\,x_1\dots).$$

σ is a homeomorphism. By [4] (C,σ) is conjugate to the restriction of a diffeomorphism $f\colon S^2\to S^2$ to a certain compact invariant hyperbolic set K. Hence the topological properties of C reflect themselves exactly in K.

Consider the subset of C, and correspondingly of K,

$$\Lambda=\{x\in C\colon \text{any finite maximal string of 0's is of even length}\}.$$

Clearly Λ is a compact σ-invariant subset of C and the periodic points are dense in Λ. However, there are points $c\in C$ such that

$$d(\sigma^n c,\Lambda)\to 0 \qquad \text{as } n\to\pm\infty$$

$$d(\sigma^n c,\sigma^n x)\not\to 0 \qquad \text{as } n\to\pm\infty \text{ for any } x\in\Lambda.$$

That is, there are points of C tending to Λ but not asymptotic with any point of Λ. Even worse, there are periodic points of $C-\Lambda$ whose entire orbits lie arbitrarily close to Λ. This behavior is opposite to that of (1.1).

Define such c as follows. Put odd maximal strings of zeros of increasing length on both sides of the decimal point

$$\dots 1000001000101\,.\,1010001000001\dots.$$

Given any $m>0$, $\sigma^n c$ has at most one entry of 1 in $[-m,m]$ for large enough $|n|$. The sequence x with zeros except at this entry belongs to Λ and

$$d(\sigma^n c,x)\leqq\sum_{|i|\geqq m}1/2^{|i|}=1/2^{m-1}$$

which is arbitrarily small when m is large. Hence $\sigma^n c\to\Lambda$. On the other hand, for any $x\in C$ if $d(\sigma^n c,\sigma^n x)\to 0$ as $|n|\to\infty$ then

$$x=c \text{ to the right of some entry if } n\to\infty$$

$$x=c \text{ to the left of some entry if } n\to-\infty.$$

Either way, x can not belong to Λ.

A periodic point $p \in C$ is a sequence which endlessly repeats a finite block. If this block is of the form $10\ldots0$ with $2m+1$ zeros then the orbit of p lies within a distance $1/2^{m-1}$ of Λ by the same reasoning as before. Thus, the existence of periodic points as claimed above is clear.

§2. Local Product Structure

Several notions and results from [1, 5] should be recalled.

A compact set $\Lambda \subset M$ is hyperbolic for the diffeomorphism $f: M \to M$ if $f\Lambda = \Lambda$ and Tf leaves invariant a continuous splitting $T_\Lambda M = E^s \oplus E^u$, expanding E^u and contracting E^s. That is

$$|Tf(v)| \leqq \tau\, |v| \qquad \text{if } v \in E^s$$
$$|Tf(v)| \geqq \tau^{-1}\, |v| \qquad \text{if } v \in E^u$$

for some constant $\tau, 0 < \tau < 1$, and some Riemannian metric on M. The constant τ is called the skewness.

Through such a Λ pass families of smooth unstable and stable manifolds tangent to E^u, E^s at Λ [1]. The unstable manifold of size ε through $p \in \Lambda$ is called $W_\varepsilon^u(p)$, the stable one $W_\varepsilon^s(p)$. This "size ε" refers to the radius measured in the tangent space at p:

$$W_\varepsilon^u(p) = \exp_p(\operatorname{graph} g_p)$$

where $g_p: E_p^u(\varepsilon) \to E_p^s(\varepsilon)$ is a smooth map whose graph has slope $\leqq 1$ with $g_p(0) = 0$, $Tg_p(0) = 0$. $E_p^u(\varepsilon)$, $E_p^s(\varepsilon)$ are the ε-discs in E_p^u, E_p^s. Similarly for $W_\varepsilon^s(p)$.

The families $W_\varepsilon^u = \{W_\varepsilon^u(p) \mid p \in \Lambda\}$, $W_\varepsilon^s = \{W_\varepsilon^s(p) \mid p \in \Lambda\}$ are overflowing invariant in the following sense: $f^{-1} W_\varepsilon^u(p) \subset W_\varepsilon^u(f^{-1}p)$ and $f W_\varepsilon^s(p) \subset W_\varepsilon^s(fp)$. They are expanding from Λ and contracting to Λ in the sense that

$$\bigcap_{n \geqq 0} f^{-n} W_\varepsilon^u = \Lambda = \bigcap_{n \geqq 0} f^n W_\varepsilon^s$$

for $W_\varepsilon^u = \bigcup_{p \in \Lambda} W_\varepsilon^u(p)$ and $W_\varepsilon^s = \bigcup_{p \in \Lambda} W_\varepsilon^u(p)$. We call $W_\varepsilon^u = W_\varepsilon^u \Lambda$ the local unstable manifold of Λ. Similarly, $W_\varepsilon^s = W_\varepsilon^s \Lambda$ is called the local stable manifold of Λ. Notice that this terminology differs from that of Smale in [5].

If p is a hyperbolic fixed point (that is $\Lambda = p$ in the preceding discussion) and V is a local submanifold transverse to its stable manifold then locally f^n presses V toward the unstable manifold of p, $W^u(p)$. If V has the same dimension as $W^u p$ then as $n \to \infty$, $f^n V \to W^u(p)$ in the C^1

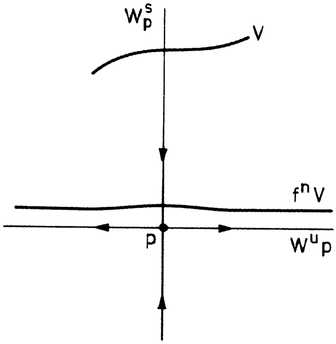

Fig. 1. f^n flattens V toward $W^u p$

sense, locally. This is the content of the λ-lemma [3] and is illustrated in Fig. 1.

A direct consequence of the λ-lemma is the

(2.1) *Cloud Lemma. If f: $M \to M$ is a diffeomorphism and p, q are hyperbolic periodic points with $W^u(p) \pitchfork W^s(q) \neq \emptyset$, $W^s(p) \pitchfork W^u(q) \neq \emptyset$ then these points of intersection are non-wandering.*

Proof [5]. See Fig. 2 for n so large that p, q are fixed points of f^n. The set U is a neighborhood of $x \in W^u(q) \cap W^s(p)$. We have drawn some iterates $f^{nk} U$ and shown how they must re-intersect U eventually, by the λ-lemma.

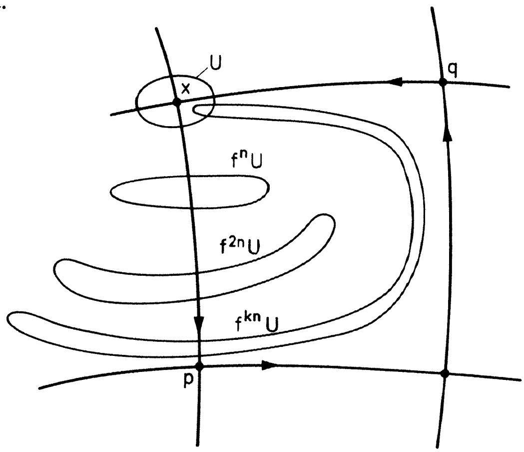

Fig. 2. f^{kn} U re-intersects U

Definition. A hyperbolic set Λ has local product structure if, for some $\varepsilon > 0$,

$$W^u_\varepsilon(p) \cap W^s_\varepsilon(p') \subset \Lambda$$

for all $p, p' \in \Lambda$.

(2.2) *Local Product Structure Theorem. If f obeys Axiom* A *then f has local product structure on Ω.*

Proof. Let $\varepsilon > 0$ be small enough that the 3ε-local stable and unstable manifolds through points of Ω are given by the stable manifold theory of [1]. Let $x, x' \in \Omega$ have $y \in W^s_\varepsilon(x) \cap W^s_\varepsilon(x')$. By [1] the intersection is transverse, consists of a single point, and the same is true of the intersection $W^s_{2\varepsilon}(x) \cap W^u_{2\varepsilon}(x')$. (See Fig. 3.)

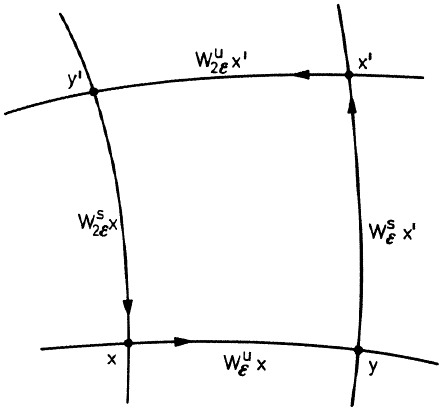

Fig. 3. Local product structure

Approximate x, x' by periodic points p, p'. By continuity of the stable and unstable manifolds, the intersections

$$W^u_{3\varepsilon}(p) \cap W^s_{3\varepsilon}(p'), \qquad W^s_{3\varepsilon}(p) \cap W^u_{3\varepsilon}(p')$$

continue to be transverse and to consist of single points, q, q' (see Fig. 4). By the Cloud Lemma, $q, q' \in \Omega$. As q is arbitrarily near y and Ω is a closed set, y belongs to Ω.

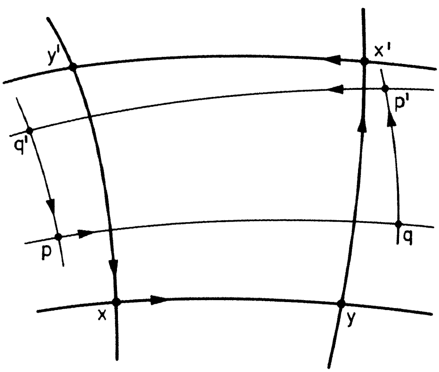

Fig. 4. Stable and unstable manifolds for periodic p, p' approximating x, x'

Corollary of the Proof. If Λ is any hyperbolic set isolated from $\Omega-\Lambda$ and the periodic points are dense in Λ then f has local product structure at Λ.

The following key lemma shows how local product structure simplifies the topology of $W_\varepsilon^s = \bigcup_{p\in\Lambda} W_\varepsilon^s(p)$.

(2.3) Lemma. *W_δ^s is a neighborhood of Λ in W_ε^s if $0<\delta\leqq\varepsilon$, ε is small, and Λ has local product structure.*

Proof. Suppose, on the contrary, that there existed a sequence of points $x_n \in W_\varepsilon^s(p_n) - W_\delta^s$ converging to some $p\in\Lambda$. As in (2.2), when ε is small this implies that $W_{2\varepsilon}^s(p_n) \overline{\pitchfork} W_{2\varepsilon}^u(p)$ in a single point, say q_n. By local product structure $q_n\in\Lambda$ and $d(x_n, q_n)\to 0$. Thus $x_n \in W_\delta^s(q_n)$ when n is large, contradicting our assumption.

§3. Fundamental Domains

A fundamental domain for W_ε^s is a compact set $D\subset W_\varepsilon^s$ such that

$$W_\varepsilon^s - \Lambda \subset 0_+ D$$

where $0_+ D = \bigcup_{n\geqq 0} f^n D$, i.e. the forward orbit of D. A fundamental domain for W_ε^u is a compact set $D\subset W_\varepsilon^u$ such that

$$W_\varepsilon^u - \Lambda \subset 0_- D$$

where $0_- D = \bigcup_{n\geqq 0} f^{-n} D$. If in addition D is disjoint from Λ, we call D a *proper* fundamental domain.

(3.1) Lemma. *The set $D^s = Cl(W_\varepsilon^s - f W_\varepsilon^s)$ is a fundamental domain for W_ε^s and $D^u = Cl(W_\varepsilon^u - f^{-1} W_\varepsilon^u)$ is one for W_ε^u.*

Proof. $0_+ D^s \supset W_\varepsilon^s - \bigcap_{n\geqq 0} f^n W_\varepsilon^s = W_\varepsilon^s - \Lambda$. The proof for D^u is similar.

(3.2) Lemma. *If Λ has local product structure then D^s, D^u are proper fundamental domains.*

Proof. There is a δ, $0<\delta<\varepsilon$, such that $f W_\varepsilon^s \supset W_\delta^s$. By (2.3), W_δ^s is a neighborhood of Λ in W_ε^s. Therefore $D^s = Cl(W_\varepsilon^s - f W_\varepsilon^s)$ is disjoint from Λ. Similarly for D^u.

Question. Do the following conditions imply the existence of proper fundamental domain for $W^s \Lambda$:

(a) the periodic points are dense in Λ,

(b) there exists a neighborhood U of Λ such that if $x\in U$ and $0_+(x)\subset U$ then $\omega(x)\subset\Lambda$, where $\omega(x)$ is the ω-limit set of x.

§4. Semi Invariant Disc Families

In this section we state and prove the basic technical theorem required for (1.1).

The space $\mathrm{Emb}(D^u, M)$ of all embeddings of the closed u-disc into M may be thought of as a fiber bundle over M. The fiber at $x \in M$ is the set of all embeddings $e: D^u \to M$ such that $e(0) = x$. We put the uniform topology on $\mathrm{Emb}(D^u, M)$.

Definition. If $a: X \to \mathrm{Emb}(D^u, M)$ is a continuous section over $X \subset M$ then $\mathscr{A} = \{\text{image } a(x) \mid x \in X\}$ is a (continuous) u-disc family through X.

Clearly $W_\delta^u = \{W_\delta^u(p) \mid p \in \Lambda\}$ is a u-disc family for any small $\delta > 0$.

The following lemma may be thought of as a type of Inverse Function Theorem for certain u-disc families. By $R^m(\varepsilon)$ we mean the disc of radius ε in R^m, centered at the origin.

(4.1) Lemma. *If $u+s=m$ and $\mathscr{A} = \{A(y) \mid y \in R^s(\delta)\}$ is a u-disc family through $0 \times R^s(\delta) \subset R^m$ with $A(y) = \text{graph } g_y$ for $g_y: R^u(\varepsilon) \to R^s$, $g_y(0) = y$ then $\bigcup_{y \in R^s} A(y)$ is a neighborhood of* 0 *in R^m.*

Proof. Continuity of $\mathscr{A}$ implies that $g: (x, y) \mapsto (x, g_y x)$ is a continuous map from $B = R^u(\varepsilon) \times R^s(\delta)$ into R^m. It suffices to prove that $g \mid \partial B$ is homotopic to the inclusion map $\partial B \hookrightarrow R^m - 0$. Such a homotopy is given by

$$G_t(x, y) = \big(x, (1-t)\, g_y + t\, y\big).$$

The curves $G_t(x, y)$ never pass through 0 when $(x, y) \in \partial B$ since $x = 0$ implies $g_y(0) = y$ and so $G_t(x, y) = (0, y) \neq 0$. Thus, (4.1) is proved.

Now we may proceed to the main theorem of this section.

(4.2) Theorem. *For any small $\delta > 0$ there is a u-disc family $\tilde{W}_\delta^u$ through N, a neighborhood of Λ in M, which reduces to W_δ^u at Λ and is semi-invariant in the sense that*

$$\tilde{W}_\delta^u(f x) \subset f\, \tilde{W}_\delta^u(x) \qquad \textit{for any } x \in N \cap f^{-1} N.$$

Moreover for any $p \in \Lambda$, $\bigcup_{y \in W_\varepsilon^s p} \tilde{W}_\delta^u(y)$ is a neighborhood of p in M.

Remark. If f is C^r we can make $\tilde{W}_\delta^u$ a continuous family of C^r u-discs.

Remark. We cannot expect any sort of uniqueness for $\tilde{W}_\delta^u$, as simple examples show with Λ taken to be one point. It is unknown whether $\tilde{W}_\delta^u$ can be chosen to foliate a neighborhood of Λ.

The proof of (4.2) is similar to the existence of $\{W_\varepsilon^u(p) \mid p \in \Lambda\}$ in [1]. All estimates needed here were proved in [1]. We must deal with graph transforms induced by maps of one space to another instead of to itself. We outline what must be re-interpreted.

Let $E=E_1\times E_2$, $F=F_1\times F_2$ be Banach spaces, equiped with the product norms

$$|(x_1, x_2)|=\max(|x_1|, |x_2|).$$

Define $\mathscr{G}_\varepsilon(E)$ to be the space of maps g: $E_1(\varepsilon)\to E_2$ such that

$$g(0)=0, \qquad L(g)\leqq 1$$

where L denotes the Lipschitz constant and $E(\varepsilon)$ denotes the closed ball of radius ε. The metric in $\mathscr{G}_\varepsilon(E)$ is $|g-g'|=\sup_x |g\,x-g'\,x|$ and is complete. Define $\mathscr{G}_\varepsilon(F)$ similarly.

Let f: $E(\varepsilon)\to F$, $f(0)=0$ be a map. It might happen that for every $g\in\mathscr{G}_\varepsilon(E)$ there exists $h\in\mathscr{G}_\varepsilon(F)$ such that

$$\text{graph } h\subset f(\text{graph } g).$$

If so, h is uniquely determined by g and we call h the graph transform of g by f,

$$h=\Gamma_f\, g.$$

The next theorem was proved in [1, §4].

(4.3) Theorem. *Let T: $E\to F$ be an isomorphism sending E_1 onto F_1, E_2 onto F_2 when $E_1\times E_2=E$, $F_1\times F_2=F$ as above. Suppose $\|T_2\|\leqq\tau$, $\|T_1^{-1}\|\leqq\tau$, $0<\tau<1$ for $T_i=T|E_i$, $i=1,2$. If f: $E(\varepsilon)\to F$ is sufficiently near $T|E(\varepsilon)$ then Γ_f: $\mathscr{G}_\varepsilon(E)\to\mathscr{G}_\varepsilon(F)$ is defined and has $L(\Gamma_f)<1$. The precise condition on f is*

$$f(0)=0 \quad \textit{and} \quad L(f-T)<\frac{1-\tau}{1+\tau}.$$

Before proving (4.2) we need a simple extension lemma.

(4.4) Lemma. *Let E, M be manifolds and π: $E\to M$ a fiber bundle. If s: $X\to E$ is a section over a closed $X\subset M$ then s extends to $\bar{s}$: $N\to E$, a section over a neighborhood of X.*

Proof. Since E is an ANR (absolute neighborhood rectract) and M is a normal space, s extends to a map s_0: $N_0\to E$ when N_0 is a neighborhood of X in M. Let d be a metric on M. Choosing small neighborhoods $N\subset N_0$ of X makes the function $d(\pi\, s_0\, x, x)$ small if $x\in N_0$. Because M is an ANR, this makes $\pi\, s_0|N$: $N\to M$ homotopic, rel X, to the inclusion i_N: $N\hookrightarrow M$. The Covering Homotopy Theorem [7] shows that $s_0|N$ is homotopic, rel X, to a map $\bar{s}$: $N\to E$ such that $\pi\,\bar{s}=i_N$. This proves the lemma.

Proof of (4.2). Let the skewness of $Tf|T_\Lambda M$ be $<\tau$. By (4.4) there exists a neighborhood N_0 of Λ and extensions $\bar{E}^u$, $\bar{E}^s$ of the bundles E^u, E^s to N_0 for they are sections of the Grassmann bundles $G^u(M)$, $G^s(M)$ restricted to Λ. Taking a smaller N_0 if necessary we have $\bar{E}^u\oplus\bar{E}^s=T_{N_0}M$ since $E^u\oplus E^s=T_\Lambda M$.

Applying (4.4) again we can find a "connector" θ, that is a continuous family of isomorphisms $\theta(x, x')\colon M_x \to M_{x'}$ defined for all $(x, x') \in M \times M$ sufficiently near the diagonal Δ of $\Lambda \times \Lambda$ such that

$$\theta(x, x) = I_x, \qquad \theta(x, x')\bar{E}^u_x = \bar{E}^u_{x'}, \qquad \theta(x, x')\bar{E}^s_x = \bar{E}^s_{x'}.$$

In fact $GL(\bar{E}^u)$ is a bundle over $N_0 \times N_0$ whose fiber over (x, x') is the space of all isomorphisms $\bar{E}^u_x \to \bar{E}^u_{x'}$ and the map $(x, x) \to I_x$ is a section of $GL(\bar{E}^u)$ over Δ. By (4.4) this section extends giving a section θ^u. Similarly we get a section θ^s. Combining the two as $\theta^u \oplus \theta^s$ we get the connector θ.

Of course $\bar{E}^u \oplus \bar{E}^s$ is not likely to be Tf-invariant. Let $(Tf)_x\colon M_x \to M_{fx}$ be defined, with respect to this splitting, by the matrix

$$(Tf)_x = \begin{pmatrix} A_x & B_x \\ C_x & K_x \end{pmatrix}$$

for $x \in f^{-1} N_0 \cap N_0$. Consider the linear map $T_x\colon M_x \to M_{fx}$ defined on $f^{-1} N_0 \cap N_0$ by

$$T_x = \begin{pmatrix} A_x & 0 \\ 0 & K_x \end{pmatrix}.$$

At Λ the entries B and C vanish and $T = Tf$.

For any $N \subset f^{-1} N_0 \cap N_0$ the map $T\colon T_N M \to T_{fN} M$ defined by $T|M_x = T_x$ is a bundle isomorphism leaving $\bar{E}^u \oplus \bar{E}^s$ invariant and equal Tf at Λ.

Let $\bar{f} = \exp^{-1} \circ f \circ \exp\colon M(\delta) \to TM$, where $M(\delta) = \bigcup_{x \in M} M_x(\delta)$. For small enough δ and a small enough neighborhood $N \subset \bar{N}_0$ of Λ, $\bar{f}$ is well defined and

$$\sup_{x \in N} L((\bar{f} - T) | M_x(\delta)) = \xi$$

satisfies the inequality of (4.3), $\xi < (1 - \tau)/(1 + \tau)$. For when $s \to 0$ and $N \to \Lambda$ we are merely seeing how well Tf approximates $\bar{f}$ at Λ. By (4.3) $\Gamma_{\bar{f}}\colon \mathcal{G}_\delta(T_N M) \to \mathcal{G}_\delta(T_{fN} M)$ is defined and has Lipschitz constant < 1.

We may assume N chosen to be a smooth manifold with boundary. If V is a neighborhood of fN with $V \subset N_0$ and $\rho\colon V \to fN$ a retraction then $\theta(\rho x, x)$ will be defined for V small enough and ρ near enough the inclusion $i_V\colon V \hookrightarrow M$.

Define $h\colon V \to N$ by $h = f^{-1} \circ \rho$. For each $x \in V$ define $(\theta, \bar{f})_x\colon M_{hx}(\delta) \to M_x$ by

$$(\theta, \bar{f})_x = \theta(\rho x, x) \circ \bar{f} | M_{hx}(\delta)$$

and define $(\theta, T)_x\colon M_{hx} \to M_x$ by

$$(\theta, T)_x = \theta(\rho x, x) \circ T_{hx}.$$

Since θ is continuous, and equals I_x at Λ, $(\theta, T)_x$ may be made arbitrarily near $Tf | T_\Lambda M$ by taking N small, V near N, and ρ near i_V. That is

$$\sup_{x \in V} L((\theta, \bar{f})_x - (\theta, T)_x) = \bar{\xi}$$

can be made near ξ and in particular can be made $<(1-\tau)(1+\tau)$. But (θ, T) preserves $\bar{E}^u$ and $\bar{E}^s$ because both θ and T do. Taking N small enough assures (θ, T) has skewness $\leqq \tau$ because $Tf | T_\Lambda M$ has skewness $<\tau$. Hence we may apply (4.3) to conclude that $(\theta, \bar{f})$ defines a graph-transform $\Gamma_{(\theta, \bar{f})}: \mathcal{G}_\delta(T_N M) \to \mathcal{G}_\delta(T_V M)$ with Lipschitz constant <1.

Let $\varphi: M \to [0, 1]$ be continuous with $\varphi = 0$ off V and $\varphi = 1$ on fN. Define $F: \mathcal{G}_\delta(T_N M) \to \mathcal{G}_\delta(T_N M)$ by sending $g \in \mathcal{G}_\delta(T_N M)$ onto the element of $\mathcal{G}_\delta(T_N M)$ which, when restricted to $\bar{E}_x^u(\delta)$ equals

$$\begin{array}{ll} 0 & \text{if } x \in N - V, \\ \varphi(x)\, \Gamma_{(\theta, \bar{f})_x}(g) & \text{if } x \in V \cap N. \end{array}$$

If $x \in \partial V \cap N$, we know that $\varphi(x) = 0$ and so the two definitions agree. Note that $\Gamma_{(\theta, \bar{f})_x}(g)$ is a map $\bar{E}_x^u(\delta) \to \bar{E}_x^s(\delta)$ and so our definition of F is well stated. Also note that F is *not* globally a graph transform: it does not cover a map $N \to N$.

Clearly F carries $\mathcal{G}_\delta(T_N M)$ into itself because $\Gamma_{(\theta, f)}$ carries $\mathcal{G}_\delta(T_N M)$ into $\mathcal{G}_\delta(T_V M)$ and multiplication by φ, $0 \leqq \varphi \leqq 1$, can only help. In the computation of the Lipschitz size of the map $F(g)_x: \bar{E}_x^u(\delta) \to \bar{E}_x^s(\delta)$, φ is constant. For the same reason, $L(F) \leqq L(\Gamma_{(\theta, f)}) < 1$. Hence F has a unique fixed point, say g^*.

Put $\tilde{W}_\delta^u(x) = \exp_x(\text{graph}(g^* | M_x(\delta)))$, where $x \in f^{-1} N \cap N$. For such x, it is clear that $f \tilde{W}_\delta^u(x) \supset \tilde{W}_\delta^u(fx)$ by the definition of F. On Λ we have just gone through the construction of W_δ^u in [1] again and so $\tilde{W}_\delta^u(p) = W_\delta^u(p)$ by uniqueness of $\{W_\delta^u(p) | p \in \Lambda\}$. Thus $\tilde{W}_\delta^u$ is a semi-invariant family of u-discs extending W_δ^u. It remains to show that $\bigcup_{y \in W_\varepsilon^s(p)} \tilde{W}_\delta^u(y)$ is a neighborhood of p for $p \in \Lambda$.

There is a diffeomorphism $j: M_p(\varepsilon) \to M_p(\varepsilon)$ such that $(Tj)_p = I_p$ and $j(\exp_p^{-1}(W_\varepsilon^s p)) = E_p^s(\varepsilon)$, $j(\exp_p^{-1}(W_\varepsilon^u p)) = E_p^u(\varepsilon)$. So

$$\{j \circ \exp^{-1}(\tilde{W}_\delta^u y) | y \in W_\mu^s p\}$$

is a u-disc family through $E_p^s(\mu)$. Its radius may be slightly less than δ, but it is greater than $\delta/2$, say, if μ is small enough. Also, for small δ these u-discs will be graphs of maps $E_p^u(\delta/2) \to E_p^s(\varepsilon)$ having slope $\leqq 2$. By (4.3) this u-disc family includes a neighborhood of p in $M_p(\varepsilon)$ and hence its $\exp_p \circ j^{-1}$ image $\{\tilde{W}_\delta^u(y) | y \in W_\mu^s p\}$ contains a neighborhood of p in M, since $\exp_p \circ j^{-1}$ is a local diffeomorphism at p. This proves (4.2).

Remark. In a similar manner we could obtain an invariant family of linear graphs $G_x\colon \bar{E}^u_x \to \bar{E}^s_x$. These determine a subbundle of $T_N M$ semi-invariant under Tf, and thus a semi-invariant extension of $E^s \oplus E^u$ to $\tilde{E}^u \oplus \tilde{E}^s = T_N M$. Proceeding with the construction of $\tilde{W}^u_\delta$ we find that $\tilde{W}^u_\delta(y)$ is tangent to $\tilde{E}^u_y$ at y for y near Λ. In fact, it can be shown that $\tilde{W}^u_\delta(y)$ is C^r when f is C^r by imitating the proof of smoothness of unstable manifolds in either [1] or [2]. In this way one can prove.

(4.5) Theorem. *The semi-invariant family $\tilde{W}^u_\delta$ of (4.2) can be chosen to be a continuous family of C^r u-discs when f is C^r.*

§ 5. Fundamental Neighborhoods and the Proof of the Main Theorem

As before, let Λ be a hyperbolic set for $f \in \mathrm{Diff}(M)$. A *fundamental neighborhood* for W^s_ε is a compact neighborhood of a fundamental domain of W^s_ε. Similarly for W^u_ε.

The following theorem was stated in § 8 of the mimeographed version of [1] but was proved incorrectly.

(5.1) Theorem. *If V is a fundamental neighborhood for W^s_ε then $W^u_\varepsilon \cup 0_+ V$ is a neighborhood of Λ in M.*

From (5.1) we can finish the proof of (1.1). By (3.1), (3.2) it suffices to prove the following

(5.2) Theorem. *If $W^u_\varepsilon \Lambda$ has a proper fundamental domain D then Λ has a neighborhood U such that if $0_+ x \subset U$ then $x \in W^s_\varepsilon(p)$ for some $p \in \Lambda$.*

Proof. Let V be a compact neighborhood of D disjoint from Λ. Set

$$U = (W^s_\varepsilon \cup 0_- V) - V$$

where $0_- V = \bigcup_{n \geqq 0} f^{-n} V$. By (5.1) applied to f^{-1}, U is a neighborhood of Λ. If $f^n(x) \in U$ for all $n \geqq 0$ then $x \notin 0_- V$ for otherwise $f^n(x) \in V$ for some $n \geqq 0$, but $f^n(x) \in U \subset M - V$ which is a contradiction. Therefore $x \in W^s_\varepsilon$, proving the theorem.

Proof of (5.1). Let D be a fundamental domain for W^s_ε and let V be a neighborhood of it. We must show that $W^u_\varepsilon \cup 0_+ V$ is a neighborhood of Λ in M.

Consider the semi-invariant u-disc family $\tilde{W}^u_\delta$ constructed in (4.2) where δ has been chosen so small that $\tilde{W}^u_\delta(y) \subset V$ for any $y \in D$ and $0 < \delta \leqq \varepsilon$.

Clearly

$$W^u_\varepsilon \cup 0_+ V \supset W^u_\varepsilon \cup \bigcup_{y \in D} \tilde{W}^u_\delta(0_+ y)$$

by the semi-invariance of $\tilde{W}^u_\delta$. But $W^s_\varepsilon = \Lambda \cup \bigcup_{y\in D} (0_+ y)$ and so

$$W^u_\varepsilon \cup \bigcup_{y\in D} \tilde{W}^u_\delta(0_+ y) \supset \bigcup_{y\in W^s_\varepsilon} \tilde{W}^u_\delta(y).$$

But for each $p\in\Lambda$, $\bigcup_{y\in W^s_\varepsilon} \tilde{W}^u_\delta(y) \supset \bigcup_{y\in W^s_\varepsilon p} \tilde{W}^u_\delta(y)$ which is a neighborhood of p in M by (4.2). Hence $W^u_\varepsilon \cup 0_+ V$ is a neighborhood of Λ and (5.1) is proved.

§ 6. The Pseudo-Hyperbolic Case

The goal of this section is to generalize (4.2) to pseudo hyperbolic invariant sets of diffeomorphisms. We will apply this theorem to extend to flows the main results of the earlier sections about diffeomorphisms.

Recall that a compact subset Λ of M is pseudo hyperbolic for $f\in \mathrm{Diff}(M)$ if $f\Lambda=\Lambda$ and Tf leaves invariant a continuous spitting $T_\Lambda M = E_1 \oplus E_2$ such that $\|(Tf)^{-1}|E_1\| < k$, $\|Tf|E_2\| < \ell$, and $k\ell < 1$. For $f\in \mathrm{Diff}^r(M)$, $r\geqq 1$, Theorem 3B.5 of [2] says

(6.1) Theorem. *If $k<1$ then there exists $\varepsilon>0$ and a unique f-invariant C^r regular family $W_{1,\varepsilon}=\{W_{1,\varepsilon}(p): p\in\Lambda\}$ of submanifolds of size ε such that $W_{1,\varepsilon}(p)$ is tangent to $E_1(p)$ for each $p\in\Lambda$. The manifold $W_{1,\varepsilon}(p)$ is characterized by $x\in W_{1,\varepsilon}(p)$ if and only if*

$$d(f^{-n}x, f^{-n}p) \leqq \varepsilon \qquad \text{for all } n\geqq 0$$

$$\lim_{n\to\infty} \frac{d(f^{-n}x, f^{-n}p)}{k^n} = 0.$$

We extend W_1 to a disc family through a neighborhood of Λ as in § 4.

(6.2) Theorem. *If Λ is as in (6.1) then for any small $\delta>0$ there is a disc family $\tilde{W}_{1,\delta}$ through N, a neighborhood of Λ, which reduces to $W_{1,\delta}$ at Λ and is semi-invariant in the sense that*

$$\tilde{W}_{1,\delta}(fx) \subset f\tilde{W}_{1,\delta}(x) \qquad \text{for } x\in N\cap f^{-1}N.$$

Proof. The proof of (6.2) is essentially the same as that of (4.2), except that a new complete metric on the function spaces $\mathscr{G}_\varepsilon(E)$, $\mathscr{G}_\varepsilon(F)$, etc. must be used. It was introduced in [2, § 3B] and is

$$|g-g'|_* = \sup_{x\neq 0} |gx - g'x|/|x|.$$

The estimates demonstrating the analogue of (4.3) in the pseudo hyperbolic case with $k<1$ are contained in [2, § 3B] and are similar to those in [1]. Once this analogue of (4.3) is proved, the proof of (6.2) is exactly the same as that of (4.2).

§ 7. Flows

Let $\{\varphi_t\}$ be a C^r flow on M, $r \geqq 1$. A compact invariant subset $\Lambda \subset M$ is said to be hyperbolic for $\{\varphi_t\}$ if, for every $t > 0$, $T\varphi_t$ leaves invariant a continuous splitting

$$T_\Lambda M = E^u \oplus E^\varphi \oplus E^s$$

expanding E^u and contracting E^s, where E^φ is the tangent bundle to the orbits of the flow. If X is the vector field generating the flow then E^φ_x is the 0-dimensional or 1-dimensional space spanned by X_x. The equality $(T\varphi_t)_x(X_x) = X_{\varphi_t x}$ is valid for any smooth flow $\{\varphi_t\}$, so invariance of E^φ is automatic. In [5] and [2] details and equivalent definitions are given.

Note that Λ is a pseudo hyperbolic set for φ_t, $t \neq 0$. If $t > 0$ then $E_1 = E^u$ and $E_2 = E^\varphi \oplus E^s$. If $t < 0$ then $E_1 = E^s$ and $E_2 = E^u \oplus E^\varphi$. In either case we can choose $k < 1$ where $\|(T\varphi_t | E_1)^{-1}\| < k$.

Applying (6.1) to φ_1, we get a unique φ_1-invariant family through Λ, $W^u_\varepsilon = \{W^u_\varepsilon p \mid p \in \Lambda\}$ tangent to E^u. By uniqueness, as in [2], W^u_ε is locally φ_t invariant for all t. Applying (6.1) to φ_{-1}, we get a unique φ_{-1} invariant family through Λ, $W^s_\varepsilon = \{W^\varepsilon_\varepsilon p : p \in \Lambda\}$ tangent to E^s and W^s_ε is φ_t-invariant for all t. For any orbit $\mathcal{O} \subset \Lambda$ we put

$$W^u_\varepsilon \mathcal{O} = \bigcup_{p \in \mathcal{O}} W^u_\varepsilon p, \qquad W^s_\varepsilon \mathcal{O} = \bigcup_{p \in \mathcal{O}} W^s_\varepsilon p.$$

(7.1) Theorem. *The* $\{W^u_\varepsilon \mathcal{O}\}$, $\{W^s_\varepsilon \mathcal{O}\}$ *form invariant families of* C^r *manifolds. The families* $\{W^u_\varepsilon p \mid p \in \mathcal{O}\}$, $\{W^s_\varepsilon p : p \in \mathcal{O}\}$ *form* C^r *fibrations of* $W^u_\varepsilon \mathcal{O}$, $W^s_\varepsilon \mathcal{O}$.

Proof. Since $\varphi_t W^u_\varepsilon p \supset W^u_\varepsilon(\varphi_t p)$ for $t > 0$ and $\varphi_t W^s_\varepsilon p \supset W^s_\varepsilon(\varphi_t p)$ for $t < 0$, (7.1) is a direct consequence of (6.1).

We say that Λ has local product structure if

$$W^s_\varepsilon \cap W^u_\varepsilon = \Lambda$$

for some $\varepsilon > 0$. (2.3) and (3.1) generalize at once to

(7.2) Proposition. *Let* $\{\varphi_t\}$ *be a flow satisfying Axiom* A′ [5], *i.e. the nonwandering set* Ω *is hyperbolic and the periodic orbits are dense in* Ω. *Then* Ω *has local product structure.*

(7.3) Lemma. *If* $t > 0$ *and* Λ *is hyperbolic for* $\{\varphi_t\}$ *then the set* $D^s = Cl(W^s_\varepsilon - \varphi_t W^s_\varepsilon)$ *is a fundamental domain for* $W^s_\varepsilon \Lambda$. *If* Λ *has local product structure,* D^s *is proper.*

We also have

(7.4) Theorem. *If* $D \subset W^s_\varepsilon \Lambda$ *is a fundamental domain for some* φ_τ, $\tau > 0$, *and* V *is a neighborhood of* D. *Then the set*

$$W^u_\varepsilon \Lambda \cup 0_+ V$$

is a neighborhood of Λ *in* M.

Proof. Extend the disc family W_ε^u to a disc family $\tilde{W}_\varepsilon^u$, semi-invariant by φ_τ, through a neighborhood of Λ. This can be done by (6.2). Then apply (4.1) as in (4.2). Since $0_+ V = \bigcup_{t \geqq 0} \varphi_t V \supset \bigcup_{n \geqq 0} \varphi_{n\tau} V$, the theorem is proved.

(7.2, 7.3, 7.4) provide the analogue of (1.1) for flows.

Remark. (7.4) holds for normally hyperbolic foliations as defined in [2]. The proof is the same.

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Über die analytische Cohomologiegruppe $H^{n-1}(\mathbb{P}_n \setminus A, \mathcal{F})$

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0.

Ziel dieser Arbeit ist ein Beweis folgender Aussage:

Es sei $A \subset \mathbb{P}_n$ eine nirgends diskrete singularitätenfreie algebraische Teilmenge mit $k \geqq 1$ Zusammenhangskomponenten und $\mathcal{F}$ eine kohärente analytische Garbe über $\mathbb{P}_n \setminus A$. Dann gilt für die analytische Cohomologie:

$$\dim_{\mathbb{C}} H^{n-1}(\mathbb{P}_n \setminus A, \mathcal{F}) = (k-1) \cdot \dim_{\mathbb{C}} H^0(\mathbb{P}_n \setminus A, \mathcal{H}om(\mathcal{F}, \mathcal{K})),$$

($\mathcal{K}$ kanonische Garbe über $\mathbb{P}_n$). Insbesondere verschwindet $H^{n-1}(\mathbb{P}_n \setminus A, \mathcal{F})$, wenn A zusammenhängt.

Eine entsprechende Aussage für algebraische Cohomologiegruppen mit Koeffizienten in algebraischen Garben stammt von Hartshorne [6, Theorem 7.5]. Sie ist insofern allgemeiner, als sie die Singularitätenfreiheit von A nicht benötigt. Wesentliche Hilfsmittel bei ihrem Beweis sind Serre-Dualität und ein Fortsetzungssatz für Schnitte in invertierbaren Garben. Der Beweis unserer Aussage verläuft ähnlich. Wir benutzen Serre-Dualität, etwa in der Form [9], und müssen dazu vorher wissen, daß $H^{n-1}(\mathbb{P}_n \setminus A, \mathcal{F})$ endlich-dimensional ist. Um dies sicherzustellen, benötigen wir die Singularitätenfreiheit von A und die Endlichkeitsaussage [2, Korollar zu Satz 3]. Ein weiteres wesentliches Hilfsmittel beim Beweis ist ein Fortsetzungssatz für Schnitte in der Garbe $\mathcal{H}om(\mathcal{F}, \mathcal{K})$, der sich aus Ergebnissen von Rossi [7] und Rothstein [8] sowie der Lösung [3] des Levi-Problems nach Docquier-Grauert ergibt.

Den Anstoß zur vorliegenden Note gab eine Frage von Hartshorne[1]. Wäre nämlich $H^2(\mathbb{P}_3 \setminus A, \mathcal{F}) \neq 0$ für eine kohärente analytische Garbe $\mathcal{F}$ über dem Komplement einer zusammenhängenden Kurve $A \subset \mathbb{P}_3$, so könnte A nicht Durchschnitt zweier algebraischer Flächen sein. Und das Problem, eine solche Kurve zu finden, ist bis jetzt noch ungelöst.

Noch einiges zur Notation: Eine komplexe Mannigfaltigkeit X ist stets eine Mannigfaltigkeit mit abzählbarer Basis. Die Strukturgarbe über X sei $\mathcal{O}_X$, das kanonische Bündel K_X. Die Garbe der holomorphen

[1] Ihm habe ich auch zu danken für seine wertvollen Verbesserungsvorschläge zu einer ersten Version dieser Note.

Schnitte in einem holomorphen Bündel F bezeichnen wir mit $\mathscr{F}$. Den Träger einer Garbe $\mathscr{F}$ bezeichnen wir mit $|\mathscr{F}|$, den Träger eines Schnittes $g \in H^0(X, \mathscr{F})$ mit $|g|$.

1.

Fortsetzung holomorpher Funktionen über affinen Teilmengen des $\mathbb{P}_n$. — Ziel dieses Abschnitts ist der Beweis des folgenden Fortsetzungssatzes.

Satz 1. *Es sei* $V \subset \mathbb{P}_n$ *eine zusammenhängende Umgebung der abgeschlossenen algebraischen Menge* $A \subset \mathbb{P}_n$ *mit* $\dim A \geqq 1$. $H \subset \mathbb{P}_n$ *sei eine Hyperfläche. Dann ist die Einschränkungsabbildung*

$$H^0(\mathbb{P}_n \setminus H, \mathcal{O}) \to H^0(V \setminus H, \mathcal{O})$$

surjektiv.

Zum Beweis benutzen wir die beiden folgenden Fortsetzungseigenschaften holomorpher Funktionen:

I. (Rothstein [8].) *Es seien* $G_0 \subset G \subset \mathbb{C}^n$ *zwei Gebiete derart, daß die Einschränkungsabbildung*

$$H^0(G, \mathcal{O}) \to H^0(G_0, \mathcal{O})$$

surjektiv ist. Dann ist auch für jede Hyperfläche $\hat{H} \subset G$ *die Einschränkungsabbildung*

$$H^0(G \setminus \hat{H}, \mathcal{O}) \to H^0(G_0 \setminus \hat{H}, \mathcal{O})$$

surjektiv.

II. (Rossi [7, Beweis von Theorem 3.1].) *Es sei* $\hat{A} \subset \mathbb{C}^{n+1}$ *analytisch,* $0 \in \hat{A}$ *und* $\dim_0 \hat{A} \geqq 2$. *Ist* G_0 *eine zusammenhängende Umgebung von* $\hat{A} \setminus \{0\}$, *dann existiert eine Umgebung* $G \subset \mathbb{C}^{n+1}$ *von* $\hat{A}$, $G_0 \subset G$, *so, daß*

$$H^0(G, \mathcal{O}) \to H^0(G_0, \mathcal{O})$$

surjektiv ist.

Beweis von Satz 1. Es sei $\pi: \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}_n$ die natürliche Projektion. $\hat{A} := \overline{\pi^{-1} A}$ ist analytisch in $\mathbb{C}^{n+1}$ und nach Voraussetzung $\dim_0 A \geqq 2$. $\hat{H} := \overline{\pi^{-1} H}$ ist eine Hyperfläche in $\mathbb{C}^{n+1}$. $G_0 := \pi^{-1} V$ ist eine zusammenhängende Umgebung von $\hat{A} \setminus \{0\}$. Wegen I und II gibt es also eine Umgebung G von $\hat{A}$ so, daß jede in $G_0 \setminus \hat{H}$ holomorphe Funktion nach $G \setminus \hat{H}$ fortgesetzt werden kann. Da 0 innerer Punkt von G ist, gilt $\pi(G \setminus \hat{H}) = \mathbb{P}_n \setminus H$.

Sei nun $f \in H^0(V \setminus H, \mathcal{O})$ beliebig vorgegeben. Dann ist $\hat{f} := f \circ \pi$ in $G_0 \setminus \hat{H}$ holomorph und invariant unter allen Homothetien

$$G_0 \setminus \hat{H} \ni z \mapsto c \cdot z \in G_0 \setminus \hat{H}, \qquad c \in \mathbb{C}^*.$$

$\hat{f}$ besitzt eine holomorphe Fortsetzung auf $G \setminus \hat{H}$, die wegen des Identitätssatzes für holomorphe Funktionen ebenfalls unter allen Homothetien invariant ist, und somit eine holomorphe Fortsetzung von f nach $\mathbb{P}_n \setminus H$ definiert.

2.

Existenzgebiete von holomorphen Schnitten in torsionsfreien Garben. – Dieser Abschnitt dient der präzisen Definition solcher Existenzgebiete.

Definition 1. Es sei X eine komplexe Mannigfaltigkeit. Eine stetige Abbildung $\varphi\colon B \to X$ eines hausdorffschen topologischen Raumes B in X heißt *Bereich über* X, wenn sie lokal topologisch ist. Ist φ injektiv, so heißt der Bereich *schlicht*. Ist B zusammenhängend, so heißt $\varphi\colon B \to X$ ein *Gebiet* über X.

Als komplexe Mannigfaltigkeit ist X metrisierbar. Mit Hilfe einer Metrik auf X kann man Kugeln in B wie in [5, S. 44] definieren. Wie in [5, IG, Proposition 2] erhält man dann

Lemma 1. *Es sei $\varphi\colon B \to X$ ein Bereich über X und B zusammenhängend. Dann besitzt B eine abzählbare Basis.*

Korollar. *Es sei $\varphi\colon B \to X$ ein Gebiet über X. Dann gibt es auf B eine eindeutig bestimmte komplexe Struktur, so, daß $\varphi\colon B \to X$ lokal biholomorph ist.*

Nun sei $\mathscr{F}$ eine $\mathcal{O}_X$-Garbe über X ohne Torsion. $E(\mathscr{F})$ sei der topologische Raum der Keime lokaler Schnitte in $\mathscr{F}$. Ist $\varphi\colon E(\mathscr{F}) \to X$ die kanonische Projektion, so ist φ ein Bereich über X. Jeder Keim $f \in E(\mathscr{F})$ bestimmt eindeutig eine Zusammenhangskomponente $E(f) \subset E(\mathscr{F})$. $E(f)$ heißt das *Existenzgebiet* von f. Da $E(\mathscr{F})$ nach Voraussetzung hausdorffsch ist, ist $E(f)$ ein Gebiet über X.

3.

Garben, welche dem Kontinuitätssatz genügen. Wir setzen

$$D := \{z \in \mathbb{C}^n\colon |z_1| \leqq 1 \text{ und } |z_\nu| < 1 \text{ für } \nu = 2, \ldots, n\}$$

$$\delta D := \{z \in D\colon |z_1| = 1\},$$

$$\bar{\delta} D := \{z \in \bar{D}\colon |z_1| = 1\}.$$

Definition 2. Es seien X eine komplexe Mannigfaltigkeit und $\mathscr{H}$ eine torsionsfreie $\mathcal{O}_X$-Garbe. Wir sagen, *$\mathscr{H}$ genügt dem Kontinuitätssatz*, wenn folgendes gilt: Ist ψ eine biholomorphe Abbildung (einer offenen Umgebung) von $\bar{D}$ in X, so ist die Einschränkung

$$H^0(\psi \bar{D}, \mathscr{H}) \to H^0(\psi(\bar{\delta} D \cup D), \mathscr{H})$$

surjektiv.

Mit dieser Definition formuliert sich die Lösung [3] des Levischen Problems nach Docquier-Grauert so:

Lemma 2. *Es seien X eine Steinsche Mannigfaltigkeit und $\mathscr{H}$ eine kohärente $\mathcal{O}_X$-Garbe, die dem Kontinuitätssatz genügt. Dann ist jedes Existenzgebiet $E(f)$, $f \in E(\mathscr{H})$, Steinsch.*

Beweis. Sei $\psi: \bar{D} \to X$ eine biholomorphe Abbildung in X, sei $\chi: \bar{\delta} D \cup D \to E(f)$ stetig mit $\varphi \circ \chi = \psi | \bar{\delta} D \cup D$ ($\varphi: E(f) \to X$ Projektion). Dann ist $\chi(\bar{\delta} D \cup D)$ eine Schnittfläche in $\mathscr{H}$ über $\psi(\bar{\delta} D \cup D)$. Wegen der Voraussetzung an $\mathscr{H}$ kann man sie über $\psi \bar{D}$ fortsetzen, d.h., es gibt ein stetiges $\bar{\chi}: \bar{D} \to E(f)$ mit $\varphi \circ \bar{\chi} = \psi$. Also ist $E(f)$ pseudokonvex im Sinne von Docquier-Grauert [3, Definition 15] und nach [3, Satz 10] Steinsch.

Wir wollen dies auf die Garbe $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{K}_X)$ anwenden und zeigen daher zunächst:

Lemma 3. *Es seien $\mathscr{G}$ eine kohärente und $\mathscr{F}$ eine lokal freie $\mathcal{O}_X$-Garbe. Für die $\mathcal{O}_X$-Garbe $\mathscr{H} := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F})$ über X gilt dann*

i) *$\mathscr{H}$ ist torsionsfrei;*

ii) *$\mathscr{H}$ genügt dem Kontinuitätssatz.*

Beweis. i): Da $\mathscr{G}$ kohärent ist, gibt es lokale Auflösungen

$$p\mathcal{O} \to \mathscr{G} \to 0,$$

die eine Inklusion

$$\mathscr{H} = \mathscr{H}om(\mathscr{G}, \mathscr{F}) \hookrightarrow \mathscr{H}om(p\mathcal{O}, \mathscr{F}) = p\mathscr{F}$$

induzieren.

ii) Es sei ψ eine biholomorphe Abbildung von $\bar{D}$ in X. Weil $\psi \bar{D} \subset X$ kompakt ist, gibt es nach Theorem A eine freie Auflösung

$$p\mathcal{O} | \psi \bar{D} \to q\mathcal{O} | \psi \bar{D} \to \mathscr{G} | \psi \bar{D} \to 0.$$

Da $\mathscr{F}$ über $\psi \bar{D}$ zu $s\mathcal{O}_X$ isomorph ist, $s = \operatorname{rang} \mathscr{F}$, erhält man hieraus eine exakte Sequenz

$$0 \to \mathscr{H} \to s \cdot q\mathcal{O} \to \mathscr{R} \to 0,$$

wo $\mathscr{R}$ als Untergarbe von $s \cdot p\mathcal{O}$ torsionsfrei ist.

Die Einschränkungshomomorphismen bilden ein kommutatives Diagramm

$$\begin{array}{ccccccc}
0 \to & H^0(\chi \bar{D}, \mathscr{H}) & \to & H^0(\chi \bar{D}, q\mathcal{O}) & \to & H^0(\chi \bar{D}, \mathscr{R}) \\
& \downarrow s & & \downarrow t & & \downarrow u \\
0 \to & H^0(\chi(\bar{\delta} D \cup D), \mathscr{H}) & \to & H^0(\chi(\bar{\delta} D \cup D), q\mathcal{O}) & \to & H^0(\chi(\bar{\delta} D \cup D), \mathscr{R})
\end{array}$$

mit exakten Zeilen. t ist bijektiv (Kontinuitätssatz für holomorphe Funktionen) und u ist injektiv, da $\mathscr{R}$ torsionsfrei ist. Daraus folgt die Surjektivität von s [2].

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[2] Diese Schlußweise wurde mir freundlicherweise von Herrn H. Kerner mitgeteilt.

4.

Anwendung: Zariski-offene Mengen in $\mathbb{P}_n$. — Es sei $A \subset P_n$ algebraisch und abgeschlossen, und $C := \mathbb{P}_n \setminus A$. Die Zusammenhangskomponenten von A seien A_i, $1 \leqq i \leqq k$. Wir definieren die folgenden Trägerfamilien von abgeschlossenen Teilmengen C' aus C:

$$\Phi_i := \{C' \subset C\colon C' \text{ abgeschlossen in } C \cup A_i\}.$$

$\Phi_1 \cap \cdots \cap \Phi_k$ ist dann die Familie Φ der kompakten Teilmengen von C. Ist $\mathscr{G}$ eine Garbe über C, so setzen wir

$$H^0(\partial_i C, \mathscr{G}) := \varinjlim \{H^0(C \setminus C', \mathscr{G})\colon C' \in \Phi_i\},$$
$$H^0(\partial C, \mathscr{G}) := \varinjlim \{H^0(C \setminus C', \mathscr{G})\colon C' \in \Phi\}.$$

Offenbar ist $H^0(\partial C, \mathscr{G}) = \bigoplus_1^k H^0(\partial_i C, \mathscr{G})$. Man hat natürliche Einschränkungsabbildungen

$$r_i\colon H^0(C, \mathscr{G}) \to H^0(\partial_i C, \mathscr{G}), \qquad r\colon H^0(C, \mathscr{G}) \to H^0(\partial C, \mathscr{G}).$$

Lemma 4. *Es sei* $\dim A_i > 0$ *und* $\mathscr{G}$ *eine kohärente analytische Garbe über* C, *die dem Kontinuitätssatz genügt. Dann ist die Abbildung* r_i *surjektiv.*

Beweis. Wir stellen A dar als genauen Durchschnitt von endlich vielen (höchstens $n+1$) Hyperflächen $H_j \subset \mathbb{P}_n$. Dann halten wir eine zusammenhängende offene Umgebung $V \subset \mathbb{P}_n$ fest. Da für Schnitte in torsionsfreien Garben der Identitätssatz gilt, genügt es, folgende Aussage zu beweisen:

Lemma 5. *Es sei* $H \subset \mathbb{P}_n$ *eine Hyperfläche, und* V *eine zusammenhängende offene Umgebung von* A_i. *Dann ist die Einschränkungsabbildung*

$$H^0(\mathbb{P}_n \setminus H, \mathscr{G}) \to H^0(V \setminus H, \mathscr{G})$$

surjektiv.

Beweis. Es sei also $f \in H^0(V \setminus H, \mathscr{G})$ und $E(f)$ das Existenzgebiet von f über $\mathbb{P}_n \setminus H$. Nach Lemma 2 ist $E(f)$ steinsch.

Wir zeigen: $E(f)$ ist ein schlichtes Gebiet. Dazu mögen etwa zwei Punkte $x_1 \neq x_2 \in E(f)$ gegeben sein, die über dem gleichen Punkt $x \in \mathbb{P}_n \setminus H$ liegen. Da $E(f)$ holomorph-separabel ist, gibt es ein $g \in H^0(E(f), \mathcal{O})$ mit $g(x_1) \neq g(x_2)$. Die Einschränkung $g | V \setminus H$ besitzt nach Satz 1 eine holomorphe Fortsetzung g_0 nach ganz $\mathbb{P}_n \setminus H$. Die auf $E(f)$ geliftete Funktion g' stimmt nun in $V \setminus H \subset E(f)$ mit g überein, nimmt aber in x_1 und x_2 gleiche Werte an. Dies ist ein Widerspruch zum Identitätssatz für holomorphe Funktionen!

Somit ist $E(f)$ ein Steinsches Teilgebiet von $\mathbb{P}_n \setminus H$, das $V \setminus H$ umfaßt. Wäre $E(f) \neq \mathbb{P}_n \setminus H$, so gäbe es auf $E(f)$ (und damit auf $V \setminus H$) eine holomorphe Funktion, die man nicht nach ganz $\mathbb{P}_n \setminus H$ fortsetzen kann. Da dies wegen Satz 1 unmöglich ist, muß $E(f) = \mathbb{P}_n \setminus H$ sein, und Lemma 5 ist bewiesen.

5.

Cohomologie mit Trägerfamilien Φ_i. – Wir berechnen hier die Gruppe $H^1_{\Phi_i}(\mathbb{P}_n \setminus A, \mathscr{F})$. Dazu sei $A \subset \mathbb{P}_n$ algebraisch und nirgends diskret, $C := \mathbb{P}_n \setminus A$ und $\mathscr{F}$ eine kohärente $\mathcal{O}_C$-Garbe, die dem Kontinuitätssatz genügt. Zunächst zeigen wir

Lemma 6. *Der natürliche Homomorphismus γ_i: $H^1_{\Phi_i}(C, \mathscr{F}) \to H^1(C, \mathscr{F})$ verschwindet, falls $A \neq \emptyset$.*

Beweis. Wir wählen $n+1$ Hyperflächen $H(j)$, $1 \leqq j \leqq n+1$, die genau A als gemeinsamen Durchschnitt enthalten. Die Mengen $C(j) := \mathbb{P}_n \setminus H(j)$, $1 \leqq j \leqq n+1$, bilden dann eine Steinsche Überdeckung von C. Jede Cohomologieklasse $[c] \in H^1(C, \mathscr{F})$ wird repräsentiert durch einen alternierenden Čech-Cozyklus

$$c = \big(c(j_1, j_2)\big)_{1 \leqq j_1, j_2 \leqq n+1}, \qquad c(j_1, j_2) \in H^0\big(C(j_1) \cap C(j_2), \mathscr{F}\big).$$

Falls $[c] \in \operatorname{im} \gamma_i$, so gibt es eine zusammenhängende Umgebung $V \subset \mathbb{P}_n$ von A_i mit $[c] \,|\, V \setminus A = 0$.

Bekanntlich folgt hieraus

$$c(j_1, j_2) \,|\, V \cap C(j_1) \cap C(j_2) = b(j_1) - b(j_2), \qquad b(j) \in H^0\big(V \cap C(j), \mathscr{F}\big). \quad (1)$$

Nach Lemma 5 kann man die $b(j)$ zu holomorphen Schnitten $b(j) \in H^0\big(C(j), \mathscr{F}\big)$ fortsetzen. Wegen des Identitätssatzes gilt dann (1) in ganz $C(i) \cap C(j)$. Also ist c Corand der Cokette $\big(b(j)\big)$ und $[c] = 0$.

Korollar. *Für alle κ, $1 \leqq \kappa \leqq k$, ist*

$$\dim H^1_{\Phi_1 \cap \dots \cap \Phi_\kappa}(C, \mathscr{F}) = (\kappa - 1) \cdot \dim H^0(C, \mathscr{F}).$$

Beweis. Ist $V \subset \mathbb{P}_n$ eine abgeschlossene Umgebung von $A_1 \cup \dots \cup A_\kappa$, so setzen wir

$$\Phi_i(V) := \{C' \in \Phi_i \colon C' \cap V = \emptyset\}.$$

Man hat eine exakte Sequenz [4, Théorème 4.10.1, S. 190]

$$0 \to H^0(C, \mathscr{F}) \to H^0(V \setminus A, \mathscr{F}) \to H^1_{\Phi_1(V) \cap \dots \cap \Phi_\kappa(V)}(C, \mathscr{F}) \to H^1(C, \mathscr{F}) \to \cdots.$$

Beachtet man

$$\bigoplus_1^\kappa H^0(\partial_i C, \mathscr{F}) = \varinjlim_V H^0(V \setminus A, \mathscr{F})$$

$$H^1_{\Phi_1 \cap \dots \cap \Phi_\kappa}(C, \mathscr{F}) = \varinjlim_V H^1_{\Phi_1(V) \cap \dots \cap \Phi_\kappa(V)}(C, \mathscr{F}),$$

so ergibt der Grenzübergang wegen der Exaktheit des induktiven Limes eine exakte Sequenz

$$0 \to H^0(C, \mathscr{F}) \to \bigoplus_{1}^{\kappa} H^0(\partial_i C, \mathscr{F}) \to H^1_{\Phi_1 \cap \ldots \cap \Phi_\kappa}(C, \mathscr{F}) \xrightarrow{\gamma'} H^1(C, \mathscr{F}) \to \cdots.$$

Mit γ_i verschwindet auch γ'. Aus Lemma 4 folgt die Behauptung.

6.

Ext^1 *mit Trägern.* – Ist Ψ eine Trägerfamilie auf einer komplexen Mannigfaltigkeit, so bezeichnen wir mit $\mathrm{Ext}^q_\Psi(X; \mathscr{G}, \mathscr{K})$ die abgeleiteten Funktoren des Funktors $\mathrm{Hom}_\Psi(X; \mathscr{G}, \mathscr{K}) = H^0_\Psi(X, \mathscr{H}om_{\mathcal{O}}(\mathscr{G}, \mathscr{K}))$.

Es sei wieder $A \subset \mathbb{P}_n$ algebraisch, nirgends diskret, und $C = \mathbb{P}_n \setminus A$. Mit $H(v)$, $v \in \mathbb{Z}$, bezeichnen wir das $\mathbb{C}^*$-Bündel über $\mathbb{P}_n$, das zu einer v-fach gezählten Hyperebene gehört. Ist $\mathscr{G}$ eine analytische Garbe, so sei $\mathscr{G}(v) := \mathscr{G} \otimes_{\mathcal{O}} \mathscr{H}(v)$. Jedes $s \in H^0(\mathbb{P}_n, \mathscr{H}(v))$, $v \geqq 0$, definiert einen Homomorphismus φ_s und eine exakte Sequenz

$$0 \to \mathcal{O} \xrightarrow{\varphi_s} \mathcal{O}(v) \to \mathscr{B} \to 0.$$

Dabei ist der Träger $|\mathscr{B}|$ von $\mathscr{B}$ gerade die Nullstellenmenge $N(s)$ von s. Der durch Tensorieren mit $\mathscr{G}$ induzierte Homomorphismus $\varphi_s(\mathscr{G})$: $\mathscr{G} \to \mathscr{G}(v)$ ist i.a. nicht injektiv.

Lemma 7. *Für jede kohärente $\mathcal{O}$-Garbe $\mathscr{G}$ über C ist*

$$\mathrm{Ext}^1_{\Phi_i}(C; \varphi_s(\mathscr{G}), \mathscr{K}):\ \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}(v), \mathscr{K}) \to \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}, \mathscr{K}), \qquad v \geqq 0,$$

injektiv.

Beweis. Man hat ein kommutatives Diagramm

$$\begin{array}{ccc} \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}(v), \mathscr{K}) & = & \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}, \mathscr{K}(-v)) \\ {\scriptstyle \mathrm{Ext}^1_{\Phi_i}(C;\, \varphi_s(\mathscr{G}), \mathscr{K})} \downarrow & & \downarrow {\scriptstyle \mathrm{Ext}^1_{\Phi_i}(C;\, \mathscr{G}, \varphi_s(\mathscr{K}(-v)))} \\ \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}, \mathscr{K}) & = & \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}, \mathscr{K}) \end{array}$$

und von der exakten Sequenz

$$0 \to \mathscr{K}(-v) \xrightarrow{\varphi_s(\mathscr{K}(-v))} \mathscr{K} \to \mathscr{K} \otimes_{\mathcal{O}} \mathscr{B} \to 0$$

her eine exakte Sequenz

$$\mathrm{Hom}_{\Phi_i}(C; \mathscr{G}, \mathscr{K} \otimes \mathscr{B}) \to \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}, \mathscr{K}(-v)) \to \mathrm{Ext}^1_{\Phi_i}(C; \mathscr{G}, \mathscr{K}).$$

Wäre $\mathrm{Hom}_{\Phi_i}(C; \mathscr{G}, \mathscr{K} \otimes \mathscr{B}) \neq 0$, so enthielte die Garbe $\mathscr{H}om(\mathscr{G}, \mathscr{K} \otimes \mathscr{B})$ wegen $|\mathscr{B}| \cap A_i \neq \emptyset$ Schnitte h mit Trägern $|h|$ der Codimension $\geqq 2$. Dann enthielte auch die Garbe $\mathscr{B}$ lokale Schnitte c mit Trägern $|c|$ der

Codimension $\geqq 2$. Ausgehend von der exakten $\mathcal{O}$-Sequenz

$$0 \to \mathcal{O} \to \mathcal{O}(\nu) \to \mathcal{B} \to 0$$

erhielte man dann einen Widerspruch zum zweiten Riemannschen Hebbarkeitssatz für Schnitte in $\mathcal{O}$, w.z.b.w.

Um $\mathrm{Ext}^1_{\Phi_i}$ selbst zu berechnen, bedienen wir uns der Spektralsequenz [4, Théorème 7.3.3, S. 264]

$$E_2^{p,q} = H^p_{\Phi_i}(C, \mathcal{E}xt^q(\mathcal{G}, \mathcal{K})) \Rightarrow \mathrm{Ext}^r_{\Phi_i}(C; \mathcal{G}, \mathcal{K})$$

und der daraus resultierenden exakten Sequenzen [4, S. 265]

$$\begin{aligned} 0 &\to H^1_{\Phi_i}(C, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{K})) \to \mathrm{Ext}^1_{\Phi_i}(C; \mathcal{G}, \mathcal{K}) \to \\ &\to H^0_{\Phi_i}(C, \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{G}, \mathcal{K})) \to \cdots, \end{aligned} \tag{2}$$

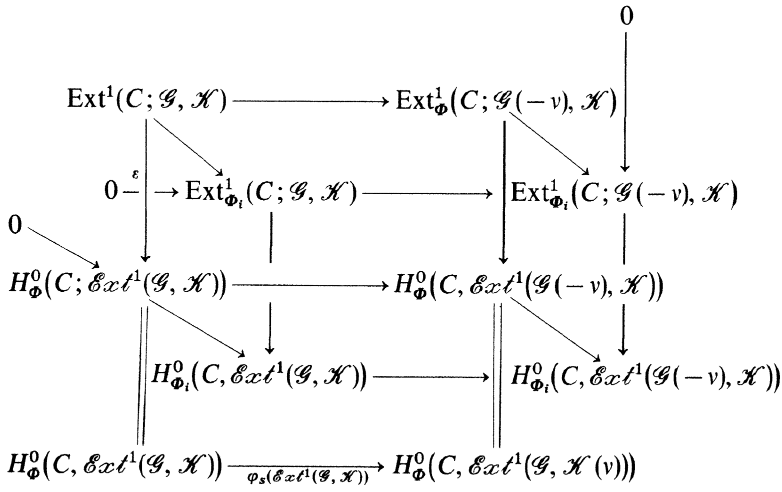
$$\begin{aligned} 0 &\to H^1_{\Phi}(C, \mathcal{H}om_{\mathcal{O}}(\mathcal{G}, \mathcal{K})) \to \mathrm{Ext}^1_{\Phi}(C; \mathcal{G}, \mathcal{K}) \to \\ &\xrightarrow{\varepsilon} H^0_{\Phi}(C, \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{G}, \mathcal{K})) \to \cdots. \end{aligned} \tag{3}$$

Lemma 8. *Der Homomorphismus ε verschwindet.*

Beweis. Es sei $h \in \operatorname{im} \varepsilon$. Dann ist $|h|$ eine kompakte analytische Menge in C. Es gibt nach dem Hilbertschen Nullstellensatz ein $\nu \geqq 0$ und ein $s \in H^0(\mathbb{P}_n, \mathcal{H}(\nu))$ mit

$$\varphi_s(\mathcal{E}xt^1(\mathcal{G}, \mathcal{K}))(h) = 0.$$

Wir betrachten das kommutative Diagramm



Wegen Lemma 7 ist die zweite Zeile exakt. Die Spalte rechts ist wegen (2) und Korollar zu Lemma 6 (im Falle $\kappa = 1$) exakt. Daraus ergibt sich durch Diagramm-Chasing $h = 0$, w.z.b.w.

Zusammen erhalten wir

Satz 2. *Es ist* $\dim \operatorname{Ext}^1_\Phi(C; \mathscr{G}, \mathscr{K}) = \dim H^1_\Phi(C, \mathscr{H}om(\mathscr{G}, \mathscr{K})) = (k-1) \cdot \dim H^0(C, \mathscr{H}om(\mathscr{G}, \mathscr{K})) < \infty$.

Beweis. Die erste Gleichung folgt aus (3) mit Lemma 8, die zweite aus Lemma 3 ii) und Korollar zu Lemma 6, wenn man dort $\kappa = k$ setzt.

Die Endlichkeit ist trivial, wenn A zusammenhängt, da dann $k-1$ verschwindet. Hängt A nicht zusammen, so muß die Menge A mindestens die Codimension 2 besitzen. C enthält also eine kompakte algebraische Kurve und ist damit pseudokonkav im Sinne von Andreotti [1]. (Hierzu siehe etwa [2, Satz 1].) Da $\mathscr{H}om(\mathscr{G}, \mathscr{K})$ torsionsfrei ist (Lemma 4 i)), ergibt sich die Endlichkeit aus [1].

7.

Serre-Dualität. — Es sei X eine rein n-dimensionale komplexe Mannigfaltigkeit, $\mathscr{G}$ eine kohärente $\mathcal{O}_X$-Garbe und Φ die Familie der kompakten Teilmengen von X. Dann gibt es eine natürliche Paarung [9, S. 15]

$$H^q(X, \mathscr{G}) \otimes \operatorname{Ext}^{n-q}_\Phi(X; \mathscr{G}, \mathscr{K}_X) \to \mathbb{C}.$$

Diese Paarung ist nicht immer perfekt. Aber es gilt beispielsweise

Lemma 9. *Die $\mathbb{C}$-Vektorräume $H^q(X, \mathscr{G})$ und $H^{q+1}(X, \mathscr{G})$ seien endlichdimensional. Dann ist obige Paarung perfekt.*

Beweis (Suominen [9]). Wie im Beweis von [9, Theorem 4.6] werde eine abzählbare Überdeckung $\mathfrak{U}$ von X durch offene Steinsche Mengen gewählt, über denen $\mathscr{G}$ eine endliche freie Auflösung besitzt. Der Beweis verläuft dann wie der von [9, Theorem 4.5], da die $(q-1)$-te und die q-te Ableitung im Čech-Komplex $\check{C}^\cdot(\mathfrak{U}, \mathscr{G})$ abgeschlossenes Bild besitzen.

Die Endlichdimensionalität von $H^n(\mathbb{P}_n \setminus A, \mathscr{G})$ und $H^{n-1}(\mathbb{P}_n \setminus A, \mathscr{G})$ folgt aber aus [2, Satz 3], falls A nirgends diskret und singularitätenfrei ist. Somit erhalten wir das gewünschte Ergebnis.

Satz 3. *Es sei $A \subset \mathbb{P}_n$ eine nirgends diskrete algebraische Menge ohne Singularitäten. Die Anzahl der Zusammenhangskomponenten von A sei k. Für jede kohärente analytische Garbe $\mathscr{G}$ über $C = \mathbb{P}_n \setminus A$ gilt dann*

$$\dim_{\mathbb{C}} H^{n-1}(C, \mathscr{G}) = (k-1) \dim_{\mathbb{C}} H^0(C, \mathscr{H}om_{\mathcal{O}}(\mathscr{G}, \mathscr{K})).$$

Bemerkung. Es ist dem Verfasser nicht bekannt, ob C streng $(n-1)$-pseudokonvex ist, wenn A Singularitäten besitzt. Wäre dies der Fall, so erhielte man die Endlichkeit von $H^{n-1}(C, \mathscr{F})$ auch hier aus dem

Théorème de Finitude von Andreotti und Grauert, und Satz 3 wäre auch in diesem Falle richtig. So zeigt unser Beweis jedoch nur, daß das topologische Dual des $\mathbb{C}$-Vektorraumes $H^{n-1}(\mathbb{P}_n \setminus A, \mathscr{G})$, letzterer versehen mit der kanonischen Quotiententopologie, endlich-dimensional ist (bzw. verschwindet, falls $k=1$).

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Affine Duality and Cofiniteness

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§ 0. Introduction

The purpose of this paper is to answer some questions raised by Grothendieck in his algebraic geometry seminar of 1962 [SGA 2, exposé XIII, conjectures 1.1 and 1.2]. They have to do with finiteness properties of the local cohomology modules of a noetherian ring with respect to an ideal J. For a local ring, with J the maximal ideal, such finiteness theorems were proved earlier in the same seminar. The problem is to extend these results to the non-local case.

Our results show that in general the local cohomology modules do not satisfy the expected finiteness conditions. However, using the derived category of Verdier [RD, Ch. I], we can prove finiteness theorems for the complex which represents all of the local cohomology modules at once.

We first review the local situation. Then we formulate questions about the general situation, and give an example to show that the finiteness conditions are not always satisfied. In Sections 4 and 5 we give positive results, phrased in terms of the derived category. In Sections 6 and 7, we give two special cases in which the answers to the original questions are affirmative.

Throughout most of the paper, we will restrict our attention to a regular noetherian ring A (i.e., a ring, all of whose localizations are regular local rings) of finite Krull dimension, which is complete with respect to the J-adic topology, where J is an ideal. The regular case is technically simpler to handle, and exhibits most of the interesting

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features of the theory. The interested reader can deduce results for a non-regular ring by expressing it as a quotient of a regular ring and using our results for the regular ring.

§ 1. The Local Situation

Let A be a complete local noetherian ring, with maximal ideal $\mathfrak{m}$, and residue field $k=A/\mathfrak{m}$. Let E be an injective hull of k over A. We recall various results due to Matlis [8] and Grothendieck [LC].

Proposition 1.1. *For an A-module N, the following conditions are equivalent.*

(i) *N satisfies the descending chain condition* (dcc).

(ii) *N is a submodule of E^n, the direct sum of n copies of E, for some n.*

(iii) *There is an A-module M if finite type such that $N \cong \mathrm{Hom}(M, E)$.*

(iv) $\mathrm{Supp}\, N \subseteq V(\mathfrak{m})$, *and* $\mathrm{Hom}(k, N)$ *is finite type.*

(iv bis) $\mathrm{Supp}\, N \subseteq V(\mathfrak{m})$, *and* $\mathrm{Ext}^i(k, N)$ *is finite type, for all i.*

(v) $\mathrm{Supp}\, N \subseteq V(\mathfrak{m})$, *and* $\mathrm{Hom}(N, E)$ *is of finite type.*

Proof. The equivalence of (i), (ii), and (iii) is proved by Matlis [8, § 4]. For the other implications, note (iv)⇒(ii), because the injective hull of N must be a sum of finitely many copies of E; if N satisfies (dcc), then it has a resolution by finite direct sums of copies of E, whence all the $\mathrm{Ext}^i(k, N)$ are of finite type. Note (v)⇒(iii), taking $M = \mathrm{Hom}(N, E)$. One has only to show if $\mathrm{Supp}\, N \subseteq V(\mathfrak{m})$, and $\mathrm{Hom}(N, E)=0$, then $N=0$, which is clear.

Definition. If N satisfies the equivalent conditions of the proposition, we say it is *cofinite* (or $\mathfrak{m}$-*cofinite*, to achieve a distinction with other notions to appear below).

Note that the cofinite modules form an abelian subcategory of the category of all A-modules, which is closed under formation of submodules, quotient modules, and extensions.

Proposition 1.2. *The functor $D = \mathrm{Hom}_A(\cdot, E)$ (or $D_{\mathfrak{m}}$, to be more precise) gives an antiequivalence of the category* (acc) *of modules of finite type over A, with the category* (dcc) *of cofinite modules. That is, D maps an element of either category into the other; it is exact and contravariant, and the natural map* $\mathrm{id} \to D \circ D$ *is an isomorphism on each of the categories.*

Proof. Matlis [8, § 4]. See also [LC, p. 61].

Proposition 1.3. (Local Duality.) *Let A be a complete regular local ring of dimension n, and let M be a module of finite type. Then there is a* *natural isomorphism, for each* $i>0$

$$H^i_{\mathfrak{m}}(M) \xrightarrow{\simeq} D_{\mathfrak{m}}(\operatorname{Ext}^{n-i}_A(M,A)).$$

Here $H^i_{\mathfrak{m}}(M)$ *denote the* i^{th} *local cohomology module of* M *with supports in* $\mathfrak{m}$ [LC, § 1].

Proof. Grothendieck [LC, Thm. 6.3].

Corollary 1.4. *Let* A *be an arbitrary complete local ring as before, and let* M *be a module of finite type. Then the local cohomology modules* $H^i_{\mathfrak{m}}(M)$ *are cofinite, for all* i.

Proof. Write A as a quotient of a regular local ring B. Then M is also a B-module. The local cohomology module can be calculated over B [LC, 5.7], so we can apply the duality theorem, and observe that $\operatorname{Ext}^{n-i}(M,A)$ is of finite type.

Corollary 1.5. *Let* A, M *be as in Corollary* 1.4. *Then*

$$\operatorname{Hom}_A(k, H^i_{\mathfrak{m}}(M))$$

and

$$\operatorname{Ext}^j_A(k, H^i_{\mathfrak{m}}(M))$$

are of finite type over A, *for all* i, j.

§ 2. Four Questions and One Theorem

Our problem is, to what extent can we generalize the situation just described to the case of a noetherian ring A, complete with respect to a J-adic topology, where J is an ideal of A? Can we find a category of "J-cofinite" modules, having support in $V(J)$, the variety of J in Spec A, analogous to the $\mathfrak{m}$-cofinite modules above? Are these modules dual in some way to the finite-type modules? Are local cohomology modules of finite-type modules J-cofinite? Is there a suitable generalization of the local duality theorem?

We must make these questions more precise. In fact, part of the problem is to know how to formulate the generalizations. We will find that the most obvious generalizations fail, and that we must deal with complexes of modules, rather than individual modules, to get valid results.

We begin with the last result of the previous section, which admits an obvious generalization. In the statements below, we deal always with a noetherian ring A, and an ideal J of A, such that A is complete in the J-adic topology.

First Question. Let M be an A-module of finite type. Are the modules

$$\operatorname{Hom}_A(A/J, H_J^i(M))$$

and

$$\operatorname{Ext}_A^j(A/J, H_J^i(M))$$

also of finite type?

To define a category of J-cofinite modules, we attempt to use condition (iv bis) of Proposition 1.1 as a definition. This leads to the

Second Question. Do the A-modules N satisfying the condition

(∗) $\operatorname{Supp} N \subseteq V(J)$, and $\operatorname{Ext}_A^j(A/J, N)$ finite type for all j

form an abelian subcategory of the category of all A-modules?

Another approach to J-cofiniteness is to generalize condition (iii) of Proposition 1.1. First let us generalize the duality theorem, so as to find an appropriate analogue for the injective module E. Let us assume that A is a regular ring (i.e., all its localizations are regular local rings). Then consideration of the hypothetical duality theorem, applied to the module $M = A$, shows that we should replace E by the complex $E^\bullet = \Gamma_J(I^\bullet)$, where $I^\bullet$ is an injective resolution of A. Here Γ_J means the submodule of elements with supports in $V(J)$. The duality theorem, phrased in the language of the derived category [RD, Ch. I] is then as follows (cf. also [RD, V. 6.2], which is a reformulation of the local duality theorem in the language of the derived category).

Theorem 2.1. *Let A be a regular ring of finite Krull dimension, and let J be an ideal. Let $E^\bullet = \Gamma_J(I^\bullet)$, where $I^\bullet$ is an injective resolution of A. For any A-module M, there is a natural map*

$$\Gamma_J(M) \to \operatorname{Hom}_A(\operatorname{Hom}_A(M, I^\bullet), E^\bullet)$$

of complexes of modules. This gives rise to a morphism of functors, for $M^\bullet \in \mathcal{D}(A)$, the derived category of the category of A-modules,

$$\theta\colon \mathbf{R}\Gamma_J(M^\bullet) \to \mathbf{R}\operatorname{Hom}(\mathbf{R}\operatorname{Hom}(M^\bullet, I^\bullet), E^\bullet).$$

This morphism is an isomorphism for all $M^\bullet \in \mathcal{D}_{\mathrm{ft}}(A)$, where the subscript "ft" *indicates that the cohomology modules $H^i(M^\bullet)$ of the complex $M^\bullet$ must be A-modules of finite type.*

Proof. Since A has finite Krull dimension, both sides are way-out functors on both sides [RD, I § 7]. Thus by the lemma on way-out functors [RD, I.7.1] we reduce to proving that θ is an isomorphism for $M^\bullet = A$, the ring itself. But this is true by definition of $E^\bullet$. Note that in this theorem, we do not assume that A is complete for the I-adic topology.

We now consider the function $D_J = \mathbf{R}\,\mathrm{Hom}(\cdot, E^\bullet)$ on $\mathscr{D}(A)$, the derived category of the category of A-modules. In attempting to generalize the notion of cofiniteness, we are now led to the

Third Question. Let A be a regular ring, complete with respect to a J-adic topology, and let $E^\bullet, D_J$ be as above. Is the functor D_J fully faithful on the category $\mathscr{D}_{\mathrm{ft}}(A)$ of complexes $M^\bullet$ with finite-type cohomology modules? Is the natural map

$$\mathrm{id} \to D_J \circ D_J$$

an isomorphism for complexes $M^\bullet \in \mathscr{D}_{\mathrm{ft}}(A)$?

In anticipation of the affirmative answer to this question (Theorem 4.1 below), we make the

Definition. A complex $N^\bullet \in \mathscr{D}(A)$ is *J-cofinite* if there is an $M^\bullet \in \mathscr{D}_{\mathrm{ft}}(A)$ such that $N^\bullet \cong D_J(M^\bullet)$. We denote the subcategory of cofinite complexes by $\mathscr{D}(A, J)_{\mathrm{cof}}$. It follows from Theorem 4.1 below that it is a full triangulated subcategory of $\mathscr{D}(A)$.

Example. If M is an A-module of finite type, then $\mathbf{R}\Gamma_J(M)$ is a cofinite complex, since it is isomorphic to $D_J\, D(M)$, and $D(M) = \mathrm{Hom}(M, I^\bullet)$ is in $\mathscr{D}_{\mathrm{ft}}(A)$.

Fourth Question. With A, J as above, does there exist an abelian category $\mathscr{M}_{\mathrm{cof}}$ of A-modules, such that elements $N^\bullet \in \mathscr{D}(A, J)_{\mathrm{cof}}$ are characterized by the property "$H^i(N^\bullet) \in \mathscr{M}_{\mathrm{cof}}$ for all i"?

Answers. We will see below that the answer to the first, second, and fourth questions is no in general. The answer to the third question is yes. If the variety of J, $V(J)$ in Spec A has either codimension one or dimension one, then the other questions also have affirmative answers (Sections 6 and 7).

§ 3. An Example

Let k be a field. Let $A = k[x, y]\,[\![u, v]\!]$ be a formal power series ring in two variables over a polynomial ring in two variables. Let $J = (u, v)$. Then A is noetherian, and complete in the J-adic topology. Let $M = A/(xu + yv)$. Then M is a module of finite type. We will show that

$$\mathrm{Hom}_A(A/J, H_J^2(M))$$

is not of finite type. This answers the first question. Later we will see that this same example also gives counter-examples to the second and fourth questions.

First we calculate the local cohomology modules of the ring A itself. We have

$$H_J^i(A)=\begin{cases}0 & \text{for } i\neq 2\\ N & \text{for } i=2,\end{cases}$$

where N is the free $k[x,y]$-module generated by the symbols $\{u^i v^j | i,j<0\}$, and where the A-module structure is given by the formulae

$$u\cdot u^i v^j=\begin{cases}u^{i+1} v^j & \text{if } i+1<0\\ 0 & \text{if } i+1=0,\end{cases}$$

$$v\cdot u^i v^j=\begin{cases}u^i v^{j+1} & \text{if } j+1<0\\ 0 & \text{if } j+1=0.\end{cases}$$

We calculate these local cohomology modules as follows. First we express them as a direct limit of Ext groups [LC, 2.8]:

$$H_J^i(A)=\varinjlim \operatorname{Ext}_A^i(A/J^{(n)}, A),$$

where $J^{(n)}=(u^n, v^n)$. This sequence of ideals is cofinal with the sequence of powers of J, so we may use it instead of the powers. We take a projective resolution of $A/J^{(n)}$ by the Koszul complex

$$0\to A\xrightarrow{v^n,\,-u^n} A\oplus A\xrightarrow{u^n,\,v^n} A\to A/J^{(n)}\to 0.$$

This allows us to calculate

$$\operatorname{Ext}_A^i(A/J^{(n)}, A)\cong\begin{cases}0 & \text{if } i\neq 2\\ A/J^{(n)} & \text{if } i=2.\end{cases}$$

Furthermore, under this isomorphism, the maps of the direct limit are $A/J^{(n)}\to A/J^{(n+1)}$ by multiplication by uv. We achieve an identification of the module

$$H_J^2(A)=\varinjlim A/J^{(n)}$$

with the module N described above, by identifying the element $u^i v^j\in N$ with the class of $u^{i+n} v^{j+n}$ in $A/J^{(n)}$, for n sufficiently large.

Now we use the exact sequence

$$0\to A\xrightarrow{xu+yv} A\to M\to 0$$

and the resulting sequence

$$H_J^2(A)\xrightarrow{xu+yv} H_J^2(A)\to H_J^2(M)\to 0$$

of local cohomology to calculate $H_J^2(M)$. Let $\varphi: N \to N$ be multiplication by $xu + yv$. Let $a_n \in N$ be the element

$$a_n = y^{n-1} u^{-n} v^{-1},$$

for $n = 1, 2, \ldots$. Then one checks easily, using the explicit description of N above, that $a_n \notin \operatorname{im} \varphi$, but $(x, y, u, v)\, a_n \subseteq \operatorname{im} \varphi$. Thus the images in coker φ of the elements a_n generate a submodule of $H_J^2(M)$ which is isomorphic to k^∞, a direct sum of infinitely many copies of k. It follows that the module

$$\operatorname{Hom}_A(A/J, H_J^2(M))$$

also contains a submodule isomorphic to k^∞, and so cannot be finitely generated. Thus the answer to the first question is negative.

This same example also provides a negative answer to the second question. For an easy calculation shows that

$$\operatorname{Hom}(A/J, N) \cong A/J,$$
$$\operatorname{Ext}^j(A/J, N) = 0 \qquad \text{for } j > 0$$

so that our module N satisfies the condition $(*)$ of the second question. However, the cokernel of the map $\varphi: N \to N$ above does not, so the modules satisfying $(*)$ do not form an abelian subcategory of the category of all A-modules.

Finally, this example shows also that the answer to the fourth question is negative. Indeed, by Theorem 2.1 we find that the complex $\mathbf{R}\Gamma_J(M)$ is a cofinite complex, since it is isomorphic to D_J of the complex $\mathbf{R}\operatorname{Hom}(M, I^\bullet)$, whose cohomology groups are $\operatorname{Ext}^i(M, A)$, which are of finite type. Hence, if such a set of A-modules $\mathcal{M}_{\mathrm{cof}}$ existed, then the module $H_J^2(M)$ discussed above would have to be in the set $\mathcal{M}_{\mathrm{cof}}$. This in turn would imply that the complex consisting of the single module $H_J^2(M)$, is also a cofinite complex. But as we will see below (Theorem 5.1), for any cofinite complex $N^\bullet$, the modules $\operatorname{Ext}_A^j(A/J, N^\bullet)$ are of finite type. In particular, we would have

$$\operatorname{Hom}(A/J, H_J^2(M))$$

of finite type, which is a contradiction.

§ 4. Affine Duality

Let A be a regular ring of finite Krull dimension, complete with respect to a J-adic topology. Let $E^\bullet = \Gamma_J(I^\bullet)$, where $I^\bullet$ is an injective resolution of A, and let D_J be the functor $\mathbf{R}\operatorname{Hom}(\cdot, E^\bullet)$ on the derived category $\mathscr{D}(A)$ of the category of A-modules. The following theorem gives an affirmative answer to the third question raised in section two.

Theorem 4.1 (Affine duality). *The functor D_J is fully faithful on the subcategory $\mathscr{D}_{\mathrm{ft}}(A)$ of complexes with finite type cohomology modules. Denoting its essential image by $\mathscr{D}(A, J)_{\mathrm{cof}}$, the category of "$J$-cofinite complexes", we have furthermore that the natural morphism of functors $\mathrm{id} \to D_J \circ D_J$ is an isomorphism, for complexes in either of the categories $\mathscr{D}_{\mathrm{ft}}(A)$ or $\mathscr{D}(A, J)_{\mathrm{cof}}$.*

Proof. We need prove only that for $M^\bullet \in \mathscr{D}_{\mathrm{ft}}(A)$, the natural map

$$M^\bullet \to D_J\, D_J(M^\bullet)$$

is an isomorphism, since the other two statements follow formally from this result.

By the lemma on way-out functors [RD, I.7.1] we reduce to the case $M^\bullet = A$. Thus we must show that the natural map of complexes

$$A \to \mathrm{Hom}(E^\bullet, E^\bullet)$$

induces an isomorphism of cohomology modules. Our technique is to work over A/J^n, and then pass to the limit.

Let $E_n^\bullet = \mathrm{Hom}(A/J^n, E^\bullet)$. Then $E^\bullet = \varinjlim E_n^\bullet$, since the components of the complex $E^\bullet$ all have support in $V(J)$. Now

$$\begin{aligned} \mathrm{Hom}(E^\bullet, E^\bullet) &= \mathrm{Hom}(\varinjlim E_n^\bullet, E^\bullet) \\ &= \varprojlim \mathrm{Hom}(E_n^\bullet, E^\bullet) \\ &= \varprojlim \mathrm{Hom}(E_n^\bullet, E_n^\bullet). \end{aligned}$$

The last equality follows from the fact that anything in $E_n^\bullet$ is annihilated by J^n, and so its image by any homomorphism of $E_n^\bullet$ into $E^\bullet$ must land in $E_n^\bullet$.

Now $E_n^\bullet$ is a dualizing complex for A/J^n [RD, V §2]. Indeed, since A is a regular ring, $I^\bullet$ is a dualizing complex for A [RD, V. 2.2], and so $E_n^\bullet = \mathrm{Hom}(A/J^n, I^\bullet)$ is a dualizing complex for A/J^n [RD, V. 2.3]. Therefore, by definition of a dualizing complex, the natural map

$$A/J^n \to \mathrm{Hom}(E_n^\bullet, E_n^\bullet)$$

is an isomorphism in the derived category, i.e., it induces an isomorphism of cohomology modules. Furthermore, these isomorphisms are compatible with the maps of the inverse system above.

Thus we have

$$H^i(\mathrm{Hom}(E_n^\bullet, E_n^\bullet)) = \begin{cases} A/J^n & \text{for } i=0 \\ 0 & \text{otherwise.} \end{cases}$$

Since A is complete, we have $A \cong \varprojlim A/J^n$. Thus to finish the proof, it will be sufficient to show that the inverse limit commutes with taking cohomology in the inverse system of complexes $(\mathrm{Hom}(E_n^\bullet, E_n^\bullet))$. For this, by [EGA, 0_{III} 13.2.3] it is sufficient to verify that

a) for each k, the inverse system $\mathrm{Hom}^k(E_n^\bullet, E_n^\bullet)$ satisfies (ML), which is true, because the maps $E_n^\bullet \to E_{n+1}^\bullet$ are injective, and the complex $E^\bullet$ is composed of injective modules, so the maps in the inverse system are surjective, and

b) for each i, the inverse system $H^i(\mathrm{Hom}(E_n^\bullet, E_n^\bullet))$ satisfies (ML), which is true because for $i \neq 0$ it is 0, and for $i=0$ it is the system (A/J^n) which has surjective maps.

We include for future reference a method of calculating the functors $H^i(D_J(M))$.

Proposition 4.2. *There is a natural functorial isomorphism, for all A-modules M of finite type*

$$H^i(D_J(M)) \xrightarrow{\approx} \varinjlim_k \mathrm{Ext}_A^i(M/J^k M, A).$$

Proof. We express $E^\bullet = \Gamma_J(I^\bullet)$ as $\varinjlim_k \mathrm{Hom}(A/J^k, I^\bullet)$. Then

$$D_J(M) = \mathrm{Hom}(M, \varinjlim \mathrm{Hom}(A/J^k, I^\bullet)),$$

and since M is of finite type, this is

$$= \varinjlim \mathrm{Hom}(M, \mathrm{Hom}(A/J^k, I^\bullet))$$
$$= \varinjlim \mathrm{Hom}(M/J^k M, I^\bullet).$$

Now taking H^i commutes with $\varinjlim$, and $I^\bullet$ is an injective resolution of A, so we get the desired result.

§ 5. A Characterization of Cofinite Complexes

In this section we give a characterization of cofinite complexes, which is the analogue, for complexes, of the property of modules suggested in the Second Question. The hypotheses on A are the same as in Section 4.

Theorem 5.1. *Let $N^\bullet \in \mathcal{D}^+(A)$ be a complex of A-modules. Then $N^\bullet$ is cofinite if and only if*

a) Supp $H^i(N^\bullet) \subseteq V(J)$ *for each* i, *and*

b) $\mathrm{Ext}^j(A/J, N^\bullet)$ *is of finite type over* A, *for each* j.

Proof. First suppose $N^\bullet$ is cofinite. Then by definition, there is an $M^\bullet \in \mathcal{D}_{\mathrm{ft}}(A)$ such that $N^\bullet \cong D_J(M^\bullet)$. By the method of way-out functors [RD, I.7] we reduce to proving a) and b) for the case $M^\bullet = A$, or $N^\bullet = E^\bullet$. But $E^\bullet$ has support in $V(J)$, so a) is clear, and $\mathrm{Ext}^i(A/J, E^\bullet) = \mathrm{Ext}^i(A/J, I^\bullet) = \mathrm{Ext}^i(A/J, A)$ which is indeed of finite type.

Now let $N^\bullet \in \mathcal{D}^+(A)$ be a complex satisfying conditions a) and b). Using condition a), we may assume that $N^\bullet$ is a bounded-below complex of injective A-modules with support in $V(J)$. Let $M^\bullet = \mathrm{Hom}(N^\bullet, E^\bullet)$. We will show that the cohomology modules $H^i(M^\bullet)$ are of finite type over A.

For each $p \geqq 0$, let $N_p^\bullet = \mathrm{Hom}(A/J^p, N^\bullet)$. Then $N^\bullet = \varinjlim N_p^\bullet$, and the maps of the direct system are injective. (Throughout this proof, we deal with actual complexes, since direct and inverse limits do not exist in the derived category.) Then $M^\bullet = \varprojlim \mathrm{Hom}(N_p^\bullet, E^\bullet)$, and the maps in the inverse system are surjective, since $E^\bullet$ is a complex of injective A-modules.

We define a filtration on the complex $M^\bullet$ as follows: for $p \geqq 0$,

$$M_p^\bullet = \ker\left(M^\bullet \to \mathrm{Hom}(N_p^\bullet, E^\bullet)\right).$$

Then

$$M^\bullet = M_0^\bullet \supseteq M_1^\bullet \supseteq \cdots.$$

One sees easily from the construction that this filtration is compatible with the J-adic filtration on A, namely, that $J^p M_q^\bullet \subseteq M_{p+q}^\bullet$ for all p, q. Furthermore, I claim there are isomorphisms $M_p^\bullet \cong M^\bullet \otimes J^p$ in the derived category. (Here $\otimes$ denotes the left derived functor of the tensor product [RD, II §4].) Indeed, for each p, since $N_p^\bullet$ has support in $V(J)$, $\mathrm{Hom}(N_p^\bullet, E^\bullet) = \mathrm{Hom}(N_p^\bullet, I^\bullet) = D(N_p^\bullet)$. Similarly $M^\bullet = D(N^\bullet)$. Now by [RD, V. 2.6, p. 263], since $N_p^\bullet = \mathrm{Hom}(A/J^p, N^\bullet)$, we have

$$D(N_p^\bullet) \cong D(N^\bullet) \otimes A/J^p = M^\bullet \otimes A/J^p.$$

Now it follows from the exact sequence

$$0 \to J^p \to A \to A/J^p \to 0$$

that there is an isomorphism $M_p^\bullet \cong M^\bullet \otimes J^p$ in the derived category.

In particular, $M^\bullet \otimes A/J$ is isomorphic in the derived category to $D(\mathrm{Hom}(A/J, N^\bullet))$, and hence by hypothesis b) of the theorem, and the fact that D sends $\mathscr{D}_{\mathrm{ft}}(A)$ into itself, the modules $H^i(M^\bullet \otimes A/J)$ are of finite type over A.

Thus $M^\bullet$ and its filtration satisfy the hypotheses of the lemma below. Admitting the lemma for the moment, we conclude that the modules $H^i(M^\bullet)$ are of finite type over A, and so $M^\bullet \in \mathscr{D}_{\mathrm{ft}}(A)$.

To show that $N^\bullet$ is cofinite, we will show that the natural map $N^\bullet \to D_J(M^\bullet)$ is an isomorphism. Let $N'^\bullet$ be the third side of a triangle built on this map. Now $D_J(M^\bullet)$ is cofinite by definition, and so satisfies conditions a) and b) of the theorem, by the easy implication. So by the long exact sequence of cohomology of this triangle, we see that $N'^\bullet$ also satisfies the conditions a) and b). Applying the functor D_J to this triangle, and using Theorem 4.1, we find that $D_J(N'^\bullet)=0$. Hence $D_J(N'^\bullet)\otimes A/J=0$. As above, this is isomorphic to $D(\mathrm{Hom}(A/J, N'^\bullet))$, and since $\mathrm{Hom}(A/J, N'^\bullet)\in \mathscr{D}_{\mathrm{ft}}(A)$, we deduce that $\mathrm{Hom}(A/J, N'^\bullet)=0$, which implies that $N'^\bullet=0$ since it has support in $V(J)$. Thus $N^\bullet \cong D_J(M^\bullet)$ is cofinite.

It remains only to prove the following

Lemma 5.2. *Let A be a commutative noetherian ring, complete with respect to a J-adic topology. Let $M^\bullet$ be a complex of A-modules, bounded above, with a filtration*

$$M^\bullet = M_0^\bullet \supseteq M_1^\bullet \supseteq M_2^\bullet \supseteq \cdots$$

by subcomplexes, such that $J^p M_q^\bullet \subseteq M_{p+q}^\bullet$ for all $p, q \geqq 0$. Assume furthermore

(i) *for each p there is an isomorphism $M_p^\bullet \cong M^\bullet \otimes J^p$ in the derived category,*

(ii) *the natural map $M^\bullet \to \varprojlim (M^\bullet/M_p^\bullet)$ is an isomorphism, and*

(iii) *the modules $H^i(M^\bullet \otimes A/J)$ are of finite type over A.*

Then the modules $H^i(M^\bullet)$ are also of finite type over A.

Proof[1]. We will consider the spectral sequence (E_r^{pq}, E^n) of the filtered complex $M^\bullet$ [EGA, 0_{III} 11.2]. We have

$$E_1^{pq} = H^{p+q}(\mathrm{gr}_p(M^\bullet))$$

and

$$E^n = H^n(M^\bullet),$$

with the filtration on E^n given by

$$F_p(E^n) = \mathrm{im}\big(H^n(M_p^\bullet) \to H^n(M^\bullet)\big).$$

[1] This proof is inspired by the work of Shih, Grothendieck, and Jouanolou on projective systems of spectral sequences. See [EGA 0_{III} 13.7] and [7].

For each $\alpha > 0$ we will also consider the spectral sequence $(E_{r\alpha}^{pq}, E_\alpha^n)$ of the filtered complex $M^\bullet/M_\alpha^\bullet$. These form a projective system of spectral sequences, and there is a natural map of the first spectral sequence into this projective system.

We will now proceed in several steps to study the convergence of these spectral sequences (cf. [EGA, 0_{III} 13.5]). Because the filtration of $M^\bullet$ is compatible with the J-adic filtration of A, there is a natural structure of graded $\operatorname{gr}(A)$-module on

$$E_r^{(n)} = \sum_{p+q=n} E_r^{pq} \qquad \text{for each } r,\ 1 \leqq r \leqq \infty$$

and similarly on

$$E_{r\alpha}^{(n)} = \sum_{p+q=n} E_{r\alpha}^{pq} \qquad \text{for each } \alpha.$$

1. The $\operatorname{gr}(A)$-modules $E_r^{(n)}$ and $E_{r\alpha}^{(n)}$ are of finite type for all n, α, and all $r < \infty$.

Indeed, since $\operatorname{gr}(A)$ is noetherian, and $E_r^{(n)}$ is the homology of the complex $E_{r-1}^{(n)}$ (and similarly for α), it will be sufficient to treat the case $r = 1$. Now

$$E_1^{(n)} = \sum_{0 \leqq p} H^n(\operatorname{gr}_p(M^\bullet)),$$

and

$$E_{1\alpha}^{(n)} = \sum_{0 \leqq p < \alpha} H^n(\operatorname{gr}_p(M^\bullet)).$$

Thus it is sufficient to consider the first one.

By hypothesis (i) above there is an isomorphism in the derived category $\operatorname{gr}_p(M^\bullet) \cong M^\bullet \otimes_A J^p/J^{p+1}$. Since J^p/J^{p+1} is an A/J-module, this is also isomorphic to $M'^\bullet \otimes_{A/J} J^p/J^{p+1}$, where $M'^\bullet = M^\bullet \otimes_A A/J$. Now by hypothesis (iii), $M'^\bullet$ is a complex with finite-type cohomology modules. So we may assume that $M'^\bullet$ is a complex of free finite-type A/J-modules. Now

$$E_1^{(n)} = \sum_p H^n(M'^\bullet \otimes J^p/J^{p+1}) = H^n(M'^\bullet \otimes \operatorname{gr}(A))$$

which is clearly of finite type over $\operatorname{gr}(A)$.

2. For r sufficiently large, depending on n (resp. n and α), we have

$$E_r^{(n)} = E_{r+1}^{(n)} = \cdots$$

$$(\text{resp. } E_{r\alpha}^{(n)} = E_{r+1,\alpha}^{(n)} = \cdots).$$

Indeed, the increasing family of submodules $B_k(E_1^{(n)})_{k \geqq 2}$ is stationary (resp. ...) and so is the family of submodules $B_k(E_1^{(n+1)})$ of $E_1^{(n+1)}$. Thus for r sufficiently large, depending on n, the maps d_r in and out of $E_r^{(n)}$ are zero, and the conclusion follows.

3. For each α, the spectral sequence $(E_{r\alpha}^{pq}, E_\alpha^n)$ is biregular [EGA, 0_{III} 11.1.3]. Indeed, since the filtration on the complex $M^\bullet/M_\alpha^\bullet$ is finite, this spectral sequence is regular. Combining with step 2 above shows that it is biregular. In particular since $E_{1\alpha}^{pq}=0$ for $p<0$ and $p \geqq \alpha$, we have

$$E_{r\alpha}^{pq} = E_{r+1,\alpha}^{pq} = \cdots = E_{\infty,\alpha}^{pq}$$

if $r>p$ and $r \geqq \alpha - p$.

4. For fixed p, q, r, the projective system $E_{r\alpha}^{pq}$ is essentially constant. Indeed, the module $E_{r\alpha}^{pq}$ depends only on the modules $E_{1\alpha}^{p'q'}$ for a finite number of pairs (p', q'), and these are independent of α provided $\alpha > p'$. To be precise, we have

$$E_{r\alpha}^{pq} = E_{r,\alpha+1}^{pq} = \cdots = E_r^{pq} \qquad \text{for } \alpha \geqq p+r.$$

5. For fixed p, q, the projective system $E_{\infty,\alpha}^{pq}$ is essentially constant. Indeed, let $\alpha > 2p$, and also so large that for $r=\alpha-p$, $E_r^{(n)} = E_{r+1}^{(n)} = \cdots$, which is possible by step 2 above. Then $r>p$, so we have $E_{\infty\alpha}^{pq} = E_{r\alpha}^{pq}$ by step 3, and $E_{r\alpha}^{pq} = E_r^{pq}$ by step 4. Now this is constant as r increases, so $E_{\infty\alpha}^{pq}$ is constant as α increases.

6. The inverse system E_α^n satisfies (ML). Indeed, it has a finite filtration whose associated graded is $\operatorname{gr}(E_\alpha^n) = \sum_{p+q=n} E_{\infty\alpha}^{pq}$. The conclusion follows easily from step 5 and [EGA, 0_{III} 13.4.3].

7. The natural map $E^n \to \varprojlim E_\alpha^n$ is an isomorphism. This follows from hypothesis (ii) of the lemma, the fact that the inverse system of complexes $M^\bullet/M_\alpha^\bullet$ has surjective maps, step 6, and [EGA, 0_{III} 13.2.3].

8. The filtration on E^n is separated. Indeed,

$$F_\alpha(E^n) = \operatorname{im}\bigl(H^n(M_\alpha^\bullet) \to H^n(M^\bullet)\bigr).$$

The exact sequence of complexes

$$0 \to M_\alpha^\bullet \to M^\bullet \to M^\bullet/M_\alpha^\bullet \to 0$$

gives an exact sequence of cohomology

$$H^n(M_\alpha^\bullet) \to H^n(M^\bullet) \to H^n(M^\bullet/M_\alpha^\bullet),$$

so we have also

$$F_\alpha(E^n) = \ker(E^n \to E^n_\alpha).$$

But by step 7, intersection of these kernels is zero.

9. For each p, q, the natural map

$$E^{pq}_\infty \to \varprojlim E^{pq}_{\infty\alpha}$$

is an isomorphism. Since $E^{(n)}_\infty$ is the associated graded of E^n, and $E^{(n)}_{\infty\alpha}$ is the associated graded of E^n_α, the result will follow from step 7, provided we know that the filtration on E^n is the same as the one induced from the inverse system E^n_α. This follows from the fact that for any β, the filtered complex $M^\bullet_\beta$ satisfies the same hypotheses as $M^\bullet$, so

$$H^n(M^\bullet_\beta) = \varprojlim H^n(M^\bullet_\beta / M^\bullet_\alpha).$$

Thus the filtrations are the same.

10. For r as large as in step 2, we have $E^{(n)}_r = E^{(n)}_\infty$. It is sufficient to show for each $p+q=n$, that $E^{pq}_r = E^{pq}_\infty$. Take $s > p, r$. Then $E^{pq}_r = E^{pq}_s = E^{pq}_{\infty\alpha}$ where $\alpha = p+s$ by steps 3 and 4. As s increases, so does α, so in the limit we have E^{pq}_∞ by step 9.

Now we can finish our proof. $H^n(M^\bullet)$ is a filtered module over the J-adic complete ring A. The filtration is separated (step 8), and the associated graded $E^{(n)}_\infty$ is of finite type over $\mathrm{gr}(A)$ (steps 1 and 10). It follows that $H^n(M^\bullet)$ is a module of finite type [1, Ch. III, Cor 1, p. 41].

Q.E.D.

§ 6. The Case of a Hypersurface

Throughout this section, we let A be a regular ring of finite Krull dimension, complete with respect to a J-adic topology, and we assume furthermore that J is generated by a single non-zero-divisor x of A. In this case we will see that all four questions have affirmative answers.

Proposition 6.1. *The set of A-modules N satisfying the condition*

(∗) $\operatorname{Supp} N \subseteq V(J)$, *and* $\operatorname{Ext}^j(A/J, N)$ *of finite type for all* j

forms an abelian subcategory of the category of all A-modules, which is stable under extensions. (This answers the second question.)

Proof. Clearly the condition is preserved under extensions, i.e., if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence, and if N' and N'' satisfy (∗), so does N. We must show that if $f\colon N_1 \to N_2$ is a homomorphism of modules N_1 and N_2, each satisfying (∗), then the kernel and cokernel of f also satisfy (∗). Clearly they have support in $V(J)$.

Since J is generated by a single non-zero-divisor x of A, there is a projective resolution of A/J of length one:

$$0 \to A \xrightarrow{x} A \to A/J \to 0.$$

Hence for any A-module M, $\operatorname{Ext}^j(A/J, M)=0$ for $j \neq 0, 1$. Now by considering the long exact sequence of Ext, with A/J as the first argument, associated to the short exact sequences

$$0 \to \ker f \to N_1 \to \operatorname{im} f \to 0$$
$$0 \to \operatorname{im} f \to N_2 \to \operatorname{coker} f \to 0,$$

we find that $\operatorname{Ext}^j(A/J, \ker f)$ and $\operatorname{Ext}^j(A/J, \operatorname{coker} f)$ are of finite type, for $j=0, 1$, so that $\ker f$ and $\operatorname{coker} f$ also satisfy $(*)$.

Definition. We denote the abelian category described in the proposition by $\mathscr{M}(A, J)_{\text{cof}}$, or $\mathscr{M}_{\text{cof}}$ for short, and we call its objects *J-cofinite modules.*

Proposition 6.2. *Let $N^\bullet \in \mathscr{D}^+(A)$. Then $N^\bullet$ is a cofinite complex if and only if $H^i(N^\bullet) \in \mathscr{M}_{\text{cof}}$ for all i. (This answers the fourth question.)*

Proof. By Theorem 5.1, $N^\bullet$ is cofinite if and only if $\operatorname{Supp} H^i(N^\bullet) \subseteq V(J)$ for all i, and $\operatorname{Ext}^j(A/J, N^\bullet)$ is of finite type, for all j. The spectral sequence

$$E_2^{pq} = \operatorname{Ext}^p(A/J, H^q(N^\bullet)) \Rightarrow E^n = \operatorname{Ext}^n(A/J, N^\bullet)$$

degenerates, since $E_2^{pq}=0$ for $p \neq 0, 1$. Thus $E_2^{pq} = E_\infty^{pq}$ for all p, q, and we have short exact sequences

$$0 \to E_2^{1, n-1} \to E^n \to E_2^{0, n} \to 0$$

for all n. Thus the initial terms E_2^{pq} of the spectral sequence are all of finite type if and only if the abutment terms E^n are all of finite type. We conclude that $N^\bullet$ is cofinite if and only if $H^i(N^\bullet) \in \mathscr{M}_{\text{cof}}$ for all i.

Corollary 6.3. *If M is an A-module of finite type, then the modules*

$$\operatorname{Ext}^j(A/J, H_J^i(M))$$

are of finite type for all i, j. (This answers the first question.)

Proof. By Theorem 2.1, the complex $\mathbf{R}\Gamma_J(M)$ is cofinite, so its cohomology modules $H_J^i(M)$ are in $\mathscr{M}_{\text{cof}}$, from which the conclusion follows.

Remarks. One can show that the results of this section remain valid under the weaker hypothesis that $\operatorname{Spec} A - V(J)$ is affine. For in that case $H^i_J(M)=0$ for $i \neq 0, 1$ and all A-modules M. Now by Theorem 2.1, a complex $N^\bullet \in \mathscr{D}^+(A)$ is cofinite if and only if it is of the form $\mathbf{R}\Gamma_J(M^\bullet)$ for some $M^\bullet \in \mathscr{D}^+_{\mathrm{ft}}(A)$. Using the techniques of Section 7 below applied to the functor $\mathbf{R}\Gamma_J$ instead of D_J, one gets the results.

§ 7. The Case of a Curve

In this section we will consider a complete regular local ring A, with maximal ideal $\mathfrak{m}$, and we will take for J a prime ideal $\mathfrak{p}$ such that $\dim A/\mathfrak{p}=1$. In this case again we find that the answer to all four questions is yes.

We fix our notation for this section: let $I^\bullet$ be an injective resolution of A. Let $E^\bullet = \Gamma_{\mathfrak{p}}(I^\bullet)$. Then $E^\bullet$ exists only in degerees $d-1$ and d, where $d=\dim A$; in fact, $E^\bullet$ is a complex $I_{\mathfrak{p}} \to I_{\mathfrak{m}}$, where $I_{\mathfrak{p}}$ is an injective hull of $A/\mathfrak{p}$ over A, $I_{\mathfrak{m}}$ is an injective hull of $k=A/\mathfrak{m}$ over A, $I_{\mathfrak{p}}$ is in degree $d-1$, and $I_{\mathfrak{m}}$ is in degree d. We denote by $D_{\mathfrak{p}}$ the functor $\operatorname{Hom}(\cdot, E^\bullet)$ on the derived category. If M is an A-module, then the complex $D_{\mathfrak{p}}(M)$ has at most two non-zero cohomology modules, in degrees $d-1$ and d. We will denote these as follows: $D^{d-1}_{\mathfrak{p}}(M)=H^{d-1}(D_{\mathfrak{p}}(M))$; $D^d_{\mathfrak{p}}(M)=H^d(D_{\mathfrak{p}}(M))$.

For our proof we will first characterize those A-modules of finite type for which $D^{d-1}_{\mathfrak{p}}=0$ or $D^d_{\mathfrak{p}}=0$. We begin with some preliminaries about topologies on a module. Let A be a ring, $\mathfrak{p}$ a prime ideal, and M an A-module. Then the $\mathfrak{p}$-*adic topology* on M is the linear topology on M which has the submodules $\mathfrak{p}^n M$, for $n=1, 2, 3, \ldots$ as a base for the neighborhoods of 0. We define the $\mathfrak{p}$-*symbolic topology* on M as follows: let $\varphi: M \to M_{\mathfrak{p}}$ be the natural map of M into its localization at $\mathfrak{p}$. Then we take as a base for the neighborhoods of 0, the submodules $\varphi^{-1}(\mathfrak{p}^n M_{\mathfrak{p}})$ of M. Clearly $\varphi^{-1}(\mathfrak{p}^n M) \supseteq \mathfrak{p}^n M$, so that the $\mathfrak{p}$-adic topology is stronger than the $\mathfrak{p}$-symbolic topology. A general question, whose solution is quite complicated, is to determine when the $\mathfrak{p}$-adic topology is equivalent to the $\mathfrak{p}$-symbolic topology. We give a solution to this problem in the case which interests us.

Proposition 7.1. *Let A, $\mathfrak{m}$ be a complete local noetherian ring, let $\mathfrak{p}$ be a prime ideal of A such that* $\dim A/\mathfrak{p}=1$, *and let M be an A-module of finite type. Then the $\mathfrak{p}$-adic topology on M is equivalent to the $\mathfrak{p}$-symbolic topology on M if and only if every associated prime ideal of M is contained in $\mathfrak{p}$.*

Lemma 7.2. (Chevalley's theorem for a module.) *Let A, $\mathfrak{m}$ be a complete local noetherian ring, and let M be an A-module of finite type. Then the $\mathfrak{m}$-adic topology on M is minimal among all separated linear topologies* *on M: in other words, if $M_1 \supseteq M_2 \supseteq \cdots$ is a sequence of submodules of M, with $\bigcap M_n = 0$, then each submodule $\mathfrak{m}^s M$, for $s = 1, 2, \ldots$, contains one of the submodules M_n, for $n = n(s)$.*

Proof. The proof of this result is identical to the proof of Chevalley's theorem for the ring A itself, given in [9, vol. II, Thm. 13, pp. 270–271], if one replaces the ring by the module M.

Proof of the Proposition. Consider the kernel M' of the natural map φ of M into its localization $M_\mathfrak{p}$ at $\mathfrak{p}$. Its consists of all elements of M whose support does not contain $V(\mathfrak{p})$. Thus it is zero if and only if every associated prime ideal of M is contained in $\mathfrak{p}$. If $M' = 0$, then $M \subseteq M_\mathfrak{p}$, and so $\bigcap \varphi^{-1}(\mathfrak{p}^n M_\mathfrak{p}) = 0$, because $\bigcap \mathfrak{p}^n M_\mathfrak{p} = 0$ in $M_\mathfrak{p}$. Hence by the lemma, the $\mathfrak{p}$-symbolic topology of M is stronger than the $\mathfrak{m}$-adic topology. Now consider the noetherian decomposition of the submodule $\mathfrak{p}^n M$ of M:

$$\mathfrak{p}^n M = \varphi^{-1}(\mathfrak{p}^n M_\mathfrak{p}) \bigcap M_n'',$$

where M_n'' is primary for $\mathfrak{m}$, since $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{p}$ properly. Thus $M_n'' \supseteq \mathfrak{m}^s M$ for some s, and $\mathfrak{m}^s M \supseteq \varphi^{-1}(\mathfrak{p}^t M_\mathfrak{p})$ for some $t = t(s)$. Thus

$$\mathfrak{p}^n M \supseteq \varphi^{-1}(\mathfrak{p}^{\max(n,t)} M_\mathfrak{p}),$$

so the two topologies are equal.

Conversely, if the two topologies are equal, then the $\mathfrak{p}$-symbolic topology must be separated, since the $\mathfrak{m}$-adic one is, so $M' = 0$, and the condition of the proposition is satisfied.

Now we will characterize those modules of finite type for which $D_\mathfrak{p}^{d-1} = 0$ or $D_\mathfrak{p}^d = 0$. The notations are those of the beginning of this section.

Lemma 7.3. *Let M be an A-module of finite type. Then $D_\mathfrak{p}^{d-1}(M) = 0$ if and only if no associated prime of M is contained in $\mathfrak{p}$.*

Proof. First suppose that no associated prime of M is contained in $\mathfrak{p}$. Then $M_\mathfrak{p} = 0$. Since $I_\mathfrak{p}$ is an $A_\mathfrak{p}$-module, every map of M to $I_\mathfrak{p}$ factors through $M_\mathfrak{p}$, and so is 0. Hence the complex $D_\mathfrak{p}(M)$ consists of the single module $\operatorname{Hom}(M, I_\mathfrak{m})$ in degree d, and so its H^{d-1} is 0.

Conversely, suppose $D_\mathfrak{p}^{d-1}(M) = 0$. Then $D_\mathfrak{p}(M)$ is isomorphic in the derived category to the single module $D^d(M)$ in degree d. Furthermore, this module has support in $V(\mathfrak{m})$, so that $D_\mathfrak{p}^{d-1} D_\mathfrak{p}^d(M) = 0$, and $D_\mathfrak{p}^d D_\mathfrak{p}^d(M) = \operatorname{Hom}(D_\mathfrak{p}^d(M), I_\mathfrak{m})$. Now by Theorem 4.1, the natural map $M \to D_\mathfrak{p} D_\mathfrak{p}(M)$ is an isomorphism. On the other hand, by affine duality with respect to $\mathfrak{m}$, we see that $D_\mathfrak{p}^d(M) = \operatorname{Hom}(M, I_\mathfrak{m})$, by Proposition 1.1(v). Therefore $\operatorname{Hom}(M, I_\mathfrak{p})=0$, and so no associated prime of M is contained in $\mathfrak{p}$ (if $\mathfrak{q}\in\operatorname{Ass} M$, $\mathfrak{q}\subseteq\mathfrak{p}$, then there is a submodule M' of M, $M'\cong A/\mathfrak{q}$. Map $A/\mathfrak{q}\to A/\mathfrak{p}\subseteq I_\mathfrak{p}$. This extends by the injectivity of $I_\mathfrak{p}$ to a non-zero map $M\to I_\mathfrak{p}$.)

Lemma 7.4. *Let M be an A-module of finite type. Then $D_\mathfrak{p}^d(M)=0$ if and only if every associated prime of M is contained in $\mathfrak{p}$.*

Proof. First suppose that every associated prime of M is contained in $\mathfrak{p}$. Using Proposition 4.2 we can write

$$D_\mathfrak{p}^d(M)=\varinjlim_k \operatorname{Ext}_A^d(M/\mathfrak{p}^k M, A).$$

Now by Proposition 7.1, the $\mathfrak{p}$-adic topology in M is equivalent to the $\mathfrak{p}$-symbolic topology. Thus we can replace $\mathfrak{p}^k M$ by $M_k=\varphi^{-1}(\mathfrak{p}^k M_\mathfrak{p})$ in the above expression, where $\varphi\colon M\to M_\mathfrak{p}$ is the natural map, and we still obtain the same limit. Note that $\mathfrak{m}\notin\operatorname{Ass}(M/M_k)$. Hence depth $M/M_k\geqq 1$, and so the homological dimension of $M/M_k\leqq n-1$, since A is regular. Thus $\operatorname{Ext}^d(M/M_k, A)=0$, and so $D_\mathfrak{p}^d(M)=0$. (This generalizes an argument in [6, pp. 417–418].)

Conversely, suppose that $D_\mathfrak{p}^d(M)=0$, and suppose by way of contradiction that there is a prime ideal $\mathfrak{q}\in\operatorname{Ass} M$, $\mathfrak{q}\nsubseteq\mathfrak{p}$. Choose a submodule M' of M, with $M'\cong A/\mathfrak{q}$. Since $D_\mathfrak{p}^d$ is contravariant and right exact, we have $D_\mathfrak{p}^d(M')=0$ also. But by the previous lemma, $D_\mathfrak{p}^{d-1}(M')=0$, so we find that $D_\mathfrak{p}(M')=0$ in the derived category. But this is impossible in view of the affine duality theorem.

Definition. We denote by $\mathscr{M}(A,\mathfrak{p})_{\text{cof}}$ or $\mathscr{M}_{\text{cof}}$ for short, the set of A-modules N satisfying the condition

(*) $\operatorname{Supp} N\subseteq V(\mathfrak{p})$ and $\operatorname{Ext}^j(A/\mathfrak{p}, N)$ is of finite type, for all j.

It is clear that if $0\to N'\to N\to N''\to 0$ is an exact sequence of A-modules, and if any two are in $\mathscr{M}_{\text{cof}}$, then so is the third.

Theorem 7.5. *Let $N^\bullet\in\mathscr{D}^+(A)$. Then $N^\bullet$ is cofinite if and only if $H^i(N^\bullet)\in\mathscr{M}_{\text{cof}}$ for all i. (This, together with the following result, answers the fourth question.)*

Proof. Suppose first that $N^\bullet$ is cofinite. Then there is a complex $M^\bullet\in\mathscr{D}_{\text{ft}}^-(A)$ with $N^\bullet\cong\mathscr{D}_\mathfrak{p}(M^\bullet)$. There is a spectral sequence

$$E_2^{pq}=H^p(D_\mathfrak{p}(H^{-q}(M^\bullet))) \Rightarrow E^n=H^n(D_\mathfrak{p}(M^\bullet)),$$

which can be obtained as the first spectral sequence of the double complex $\operatorname{Hom}(M^\bullet, E^\bullet)$. It degenerates, since $E_2^{pq}=0$ for $p\neq d-1, d$, where

$d = \dim A$, and so gives rise to short exact sequences

$$0 \to E_2^{d, n-d} \to E^n \to E_2^{d-1, n-d+1} \to 0$$

for each n. Thus to show $E^n \in \mathcal{M}_{\mathrm{cof}}$, it will be sufficient to show that $E_2^{pq} \in \mathcal{M}_{\mathrm{cof}}$ for all p and q. In other words, we reduce to the case where the complex $M^\bullet$ consists of a single A-module M of finite type.

Given M of finite type, let M' be the submodule consisting of all $m \in M$ whose support does not contain $V(\mathfrak{p})$. Let $M'' = M/M'$. Then we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

and furthermore, no associated prime of M' is contained in $\mathfrak{p}$, and every associated prime of M'' is contained in $\mathfrak{p}$. Applying the functor $D_{\mathfrak{p}}$, and taking the corresponding long exact sequence of cohomology, and using Lemmas 7.3 and 7.4, we obtain isomorphisms

$$D_{\mathfrak{p}}^{d-1}(M'') \xrightarrow{\approx} D_{\mathfrak{p}}^{d-1}(M)$$
$$D_{\mathfrak{p}}^{d}(M) \xrightarrow{\approx} D_{\mathfrak{p}}^{d}(M').$$

Furthermore, all other $D_{\mathfrak{p}}^i$ applied to M', M, and M'', are zero. Thus the complexes consisting of the single modules $D_{\mathfrak{p}}^{d-1}(M)$ and $D_{\mathfrak{p}}^{d}(M)$ are cofinite complexes, and so by Theorem 5.1 they are in $\mathcal{M}_{\mathrm{cof}}$.

Conversely, let $N^\bullet \in \mathcal{D}^+(A)$ be given, such that $H^i(N^\bullet) \in \mathcal{M}_{\mathrm{cof}}$ for all i. Then $\mathrm{Ext}^j(A/\mathfrak{p}, H^i(N^\bullet))$ is of finite type for all j. From the spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p(A/\mathfrak{p}, H^q(N^\bullet)) \Rightarrow E^n = \mathrm{Ext}^n(A/\mathfrak{p}, N^\bullet)$$

we deduce that the abutment terms E^n are also of finite type. Therefore $N^\bullet$ is a cofinite complex, by Theorem 5.1.

Proposition 7.6. *The set $\mathcal{M}_{\mathrm{cof}}$ is closed under kernels and cokernels, and so forms an abelian subcategory of the category of all A-modules. (This answers the second question.)*

Proof. Let $f: N_1 \to N_2$ be a homomorphism of modules in $\mathcal{M}_{\mathrm{cof}}$. Then by the theorem, each of them is a cofinite complex. Hence the third side of a triangle built on f is also a cofinite complex, so again by the theorem, its cohomology modules, which are just the kernel and cokernel of f, are in $\mathcal{M}_{\mathrm{cof}}$.

Corollary 7.7. *If M is an A-module of finite type, then the modules $\mathrm{Ext}^j(A/\mathfrak{p}, H_{\mathfrak{p}}^i(M))$ are also of finite type. (This answers the first question.)*

Proof. Copy the proof of Corollary 6.3.

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Points singuliers d'un opérateur différentiel analytique

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Soient X une variété analytique réelle, E, F des fibrés vectoriels analytiques de base X et D un opérateur différentiel analytique d'ordre k de E dans F. On associe à D des morphismes de fibrés de jets

$$p_l(\varphi): J_{k+l}(E) \to J_l(F)$$

où $p_0(\varphi)=\varphi$; posons $R_{k+l}=\ker p_l(\varphi)$ (cf. [2], [3] et [4]). Si pour tout $l \geqq 0$, $p_l(\varphi)$ est de rang localement constant, et si pour tout $l, m \geqq 0$, l'application $\pi_{k+l}: R_{k+l+m} \to R_{k+l}$ est de rang localement constant, nous dirons que D est complètement régulier. Dans [3] et [4], nous avons montré l'existence de solutions analytiques d'un opérateur différentiel complètement régulier.

On dira que $x \in X$ est un point singulier d'un morphisme analytique $\psi: E \to F$ de fibrés vectoriels s'il existe une suite $\{y_\alpha\}_{\alpha \in A}$ qui converge vers x telle que

$$\operatorname{rang} \psi_x < \operatorname{rang} \psi_{y_\alpha}$$

pour tout $\alpha \in A$. L'ensemble des points singuliers de ψ est un sous-ensemble analytique strict de X. On dira que $x \in X$ est un point régulier de ψ s'il n'est pas un point singulier.

Définition. On dit que $x \in X$ est un point singulier de D s'il existe un entier $m \geqq 0$ tel que x soit un point singulier de $p_m(\varphi)$, ou s'il existe des entiers m, $l \geqq 0$ tels que x soit un point régulier de $p_{m+l}(\varphi)$ et de $p_m(\varphi)$ et singulier de $\pi_{k+m}: R_{k+m+l} \to R_{k+m}$ (ce qui a un sens puisque R_{k+m+l} et R_{k+m} sont des fibrés vectoriels au-dessus d'un voisinage de x).

Le but de cet article est de démontrer que *l'ensemble Y des points singuliers de D est un sous-ensemble analytique strict de X*. L'ensemble Y est donc le plus petit sous-ensemble fermé de X tel que la restriction de D à $X-Y$ soit complètement régulière. Plus précisément,

Théorème. *Soit D un opérateur différentiel analytique d'ordre k.*

(i) *L'ensemble Y des points singuliers de D est un sous-ensemble analytique strict de X.*

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(ii) *Pour toute partie compacte K de X, il existe des entiers $\mu_0 \geqq k$, $\lambda_0 \geqq 0$ tels que Y soit l'ensemble $Y_1 \cup Y_2$, où*

$$Y_1 = \bigcup_{0 \leqq m \leqq \mu_0 + \lambda_0 - k} \{x \in K \mid x \text{ est un point singulier de } p_m(\varphi)\}$$

$$Y_2 = \bigcup_{\substack{0 \leqq l \leqq \lambda_0 \\ k \leqq m \leqq \mu_0}} \left\{ x \in K \,\middle|\, \begin{array}{l} x \text{ est un point régulier de } p_{m+l-k}(\varphi) \text{ et } p_{m-k}(\varphi) \\ x \text{ est un point singulier de } \pi_m\colon R_{m+l} \to R_m \end{array} \right\}.$$

En particulier, le fait que D soit complètement régulier sur un ouvert précompact ne dépend que d'un nombre fini de prolongements $p_l(\varphi)$ de φ.

On adoptera le point de vue de Malgrange [6] et l'on travaillera avec les équations elles-mêmes ce qui permet une simplification de l'exposition. Les méthodes de [3] seront reprises de ce point de vue et interviendront d'une manière essentielle.

Nous employons la terminologie et les notations de [2] et [3]. L'auteur tient à remercier A. Douady de lui avoir signalé les résultats de Frisch [1].

I. Opérateurs différentiels

Posons

$$D_k(E) = \operatorname{Hom}(J_k(E), 1)$$

où 1 est le fibré trivial sur X de dimension 1; c'est le fibré des opérateurs différentiels d'ordre k de E dans 1. Si $l \geqq 0$, on identifiera $D_k(E)$ à un sous-fibré de $D_{k+l}(E)$ en considérant un opérateur différentiel d'ordre k comme un opérateur différentiel d'ordre $k+l$. Soit T le fibré tangent de X et $S^k T$ la k-ième puissance symétrique de T; on écrit

$$S\,T = \bigoplus_{k \geqq 0} S^k\,T.$$

Le quotient $D_{k+1}(E)/D_k(E)$ s'identifie à $S^{k+1}\,T \otimes E^*$. Soit $\mathcal{O}_X$ le faisceau des germes de fonctions analytiques réelles sur X. On notera $\mathscr{E}$ le faisceau des germes de sections analytiques d'un fibré vectoriel E.

Soit $D_1 = D_1(1)$. L'application

$$\begin{gathered} \mathscr{D}_1 \otimes_{\mathcal{O}_X} \mathscr{D}_k(\mathscr{E}) \to \mathscr{D}_{k+1}(\mathscr{E}) \\ d \otimes u \mapsto d \circ u \end{gathered} \tag{1}$$

est induite par la composition d'opérateurs différentiels. En considérant $\mathcal{O}_X$ comme le faisceau des opérateurs différentiels scalaires d'ordre zéro, $\mathscr{D}_1$ devient un $\mathcal{O}_X$-bimodule et $\mathcal{O}_X$ un sous-module de $\mathscr{D}_1$. Nous avons une suite exacte

$$0 \to \mathcal{O}_X \to \mathcal{D}_1 \to \mathcal{T} \to 0;$$

les deux structures de $\mathcal{O}_X$-modules sur $\mathcal{T}$ provenant de celles de $\mathcal{D}_1$ sont les mêmes. De plus, la composition d'opérateurs différentiels définit une structure de faisceau de $\mathbb{R}$-algèbres de Lie sur $\mathcal{D}_1$; en effet, le commutateur de deux opérateurs différentiels scalaires d'ordre 1 est aussi d'ordre 1.

Le morphisme $p_l(\varphi)$ induit une application duale

$$\varphi: D_l(F) \to D_{k+l}(E)$$

qui est tout simplement l'application $u \mapsto u \circ \varphi$, si φ est considéré comme un opérateur différentiel d'ordre k de E dans F. Ces applications sont compatibles avec l'action de $\mathcal{D}_1$ donnée par (1), c'est-à-dire, le diagramme suivant

$$\begin{array}{ccc} \mathcal{D}_1 \otimes_{\mathcal{O}_X} \mathcal{D}_l(\mathcal{F}) & \to & \mathcal{D}_{l+1}(\mathcal{F}) \\ \downarrow {\scriptstyle 1\otimes\varphi} & & \downarrow {\scriptstyle \varphi} \\ \mathcal{D}_1 \otimes_{\mathcal{O}_X} \mathcal{D}_{k+l}(\mathcal{E}) & \to & \mathcal{D}_{k+l+1}(\mathcal{E}) \end{array}$$

est commutatif (cf. Malgrange [6]).

La suite exacte

$$0 \to \mathcal{D}_m(\mathcal{E}) \cap (\mathcal{D}_{m-k+l}(\mathcal{F}) \cdot \varphi) \to \mathcal{D}_m(\mathcal{E}) \to \mathcal{S}_m^{(l)} \to 0$$

définit $\mathcal{S}_m^{(l)}$; on écrit $\mathcal{S}_m = \mathcal{S}_m^{(0)}$. L'injection $\mathcal{D}_m(\mathcal{E}) \to \mathcal{D}_{m+l}(\mathcal{E})$ induit une application $\mathcal{S}_m^{(r)} \to \mathcal{S}_{m+l}^{(r)}$ dont l'image est isomorphe à $\mathcal{S}_m^{(r+l)}$. En particulier, $\mathcal{S}_m^{(l)}$ est isomorphe à l'image de $\mathcal{S}_m \to \mathcal{S}_{m+l}$. Le faisceau $\mathcal{S}_m^{(l+r)}$ est un quotient de $\mathcal{S}_m^{(l)}$, pour $r \geqq 0$. La suite exacte de $\mathcal{O}_X$-modules

$$0 \to \mathcal{N}_m^{(l)} \to \mathcal{S}_m^{(l)} \to \mathcal{S}_{m+1}^{(l)} \to \mathcal{M}_{m+1}^{(l)} \to 0 \tag{2}$$

définit $\mathcal{N}_m^{(l)}$ et $\mathcal{M}_{m+1}^{(l)}$. Tous les faisceaux considérés sont des $\mathcal{O}_X$-modules cohérents. On a

$$\mathcal{N}_m^{(l)} = \frac{\mathcal{D}_m(\mathcal{E}) \cap (\mathcal{D}_{m-k+l+1}(\mathcal{F}) \cdot \varphi)}{\mathcal{D}_m(\mathcal{E}) \cap (\mathcal{D}_{m-k+l}(\mathcal{F}) \cdot \varphi)}$$

et les suites exactes

$$0 \to \mathcal{N}_m^{(l)} \to \mathcal{S}_m^{(l)} \to \mathcal{S}_m^{(l+1)} \to 0, \tag{3}$$

$$\mathcal{S}_m \to \mathcal{S}_{m+1}^{(l)} \to \mathcal{M}_{m+1}^{(l)} \to 0. \tag{4}$$

On écrit $\mathcal{N}_m = \mathcal{N}_m^{(0)}$ et $\mathcal{M}_m = \mathcal{M}_m^{(0)}$.

Posons

$$\mathscr{L}_m^{(l)} = \frac{\mathscr{D}_m(\mathscr{E}) \cap \left(\mathscr{D}_{m-k+l}(\mathscr{F}) \cdot \varphi\right)}{\mathscr{D}_{m-1}(\mathscr{E}) \cap \left(\mathscr{D}_{m-k+l-1}(\mathscr{F}) \cdot \varphi\right)}$$

et $\mathscr{L}_m = \mathscr{L}_m^{(0)}$. Le diagramme commutatif et exact

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & 0 & & 0 & & \mathscr{N}_m^{(l)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \mathscr{D}_m(\mathscr{E}) \cap \left(\mathscr{D}_{m-k+l}(\mathscr{F}) \cdot \varphi\right) & \to & \mathscr{D}_m(\mathscr{E}) & \to & \mathscr{S}_m^{(l)} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & \mathscr{D}_{m+1}(\mathscr{E}) \cap \left(\mathscr{D}_{m-k+l+1}(\mathscr{F}) \cdot \varphi\right) & \to & \mathscr{D}_{m+1}(\mathscr{E}) & \to & \mathscr{S}_{m+1}^{(l)} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathscr{L}_{m+1}^{(l)} & \to & \mathscr{S}^{m+1} \mathscr{T} \otimes \mathscr{E}^* & \to & \mathscr{M}_{m+1}^{(l)} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

induit une suite exacte de $\mathscr{O}_X$-modules

$$0 \to \mathscr{N}_m^{(l)} \to \mathscr{L}_{m+1}^{(l)} \to \mathscr{S}^{m+1} \mathscr{T} \otimes \mathscr{E}^* \to \mathscr{M}_{m+1}^{(l)} \to 0. \tag{5}$$

Si $l=0$, on a la suite exacte

$$0 \to \mathscr{N}_m \to \mathscr{L}_{m+1} \to \mathscr{S}^{m+1} \mathscr{T} \otimes \mathscr{E}^* \to \mathscr{M}_{m+1} \to 0. \tag{6}$$

De plus, la suite

$$0 \to \mathscr{L}_m^{(l)} \to (\mathscr{S}^m \mathscr{T} \otimes \mathscr{E}^*) \oplus \mathscr{L}_{m+l} \tag{7}$$

est exacte. Le diagramme

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & 0 & & 0 & & \mathscr{N}_m \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathscr{D}_{m-k}(\mathscr{F}) & \xrightarrow{\varphi} & \mathscr{D}_m(\mathscr{E}) & \longrightarrow & \mathscr{S}_m \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathscr{D}_{m-k+1}(\mathscr{F}) & \xrightarrow{\varphi} & \mathscr{D}_{m+1}(\mathscr{E}) & \longrightarrow & \mathscr{S}_{m+1} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to \mathscr{P}_{m-k+1} & \to & \mathscr{S}^{m-k+1} \mathscr{T} \otimes \mathscr{F}^* & \xrightarrow{\sigma(\varphi)} & \mathscr{S}^{m+1} \mathscr{T} \otimes \mathscr{E}^* & \longrightarrow & \mathscr{M}_{m+1} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

est exact et commutatif, si $\mathscr{P}_{m-k+1}$ est le noyau de

$$\sigma(\varphi) \colon \mathscr{S}^{m-k+1} \mathscr{T} \otimes \mathscr{F}^* \to \mathscr{S}^{m+1} \mathscr{T} \otimes \mathscr{E}^*.$$

Ce diagramme induit une application surjective

$$\mathscr{P}_{m-k+1} \to \mathscr{N}_m \to 0. \tag{8}$$

Remarquons que $\mathscr{L}_m^{(l)}$ et $\mathscr{P}_m$ sont des $\mathscr{O}_X$-modules cohérents. On écrit

$$\mathscr{M}^{(l)} = \bigoplus_{m \geqq 0} \mathscr{M}_m^{(l)}, \quad \mathscr{N}^{(l)} = \bigoplus_{m \geqq 0} \mathscr{N}_m^{(l)}, \quad \mathscr{L}^{(l)} = \bigoplus_{m \geqq 0} \mathscr{L}_m^{(l)}, \quad \mathscr{P} = \bigoplus_{m \geqq 0} \mathscr{P}_m.$$

Les applications (1) induisent des applications

$$\mathscr{D}_1 \otimes_{\mathscr{O}_X} \mathscr{M}_m^{(l)} \to \mathscr{M}_{m+1}^{(l)}, \quad \mathscr{D}_1 \otimes_{\mathscr{O}_X} \mathscr{N}_m^{(l)} \to \mathscr{N}_{m+1}^{(l)},$$

$$\mathscr{D}_1 \otimes_{\mathscr{O}_X} \mathscr{L}_m^{(l)} \to \mathscr{L}_{m+1}^{(l)}, \quad \mathscr{D}_1 \otimes_{\mathscr{O}_X} \mathscr{P}_m \to \mathscr{P}_{m+1}.$$

Par passage au quotient $\mathscr{T}$ de $\mathscr{D}_1$, ces applications induisent des applications

$$\mathscr{T} \otimes_{\mathscr{O}_X} \mathscr{M}_m^{(l)} \to \mathscr{M}_{m+1}^{(l)}, \quad \mathscr{T} \otimes_{\mathscr{O}_X} \mathscr{N}_m^{(l)} \to \mathscr{N}_{m+1}^{(l)},$$

$$\mathscr{T} \otimes_{\mathscr{O}_X} \mathscr{L}_m^{(l)} \to \mathscr{L}_{m+1}^{(l)}, \quad \mathscr{T} \otimes_{\mathscr{O}_X} \mathscr{P}_m \to \mathscr{P}_{m+1}.$$

Comme $\mathscr{D}_1$ est un faisceau d'algèbres de Lie, on vérifie que ces applications proviennent de structures de $\mathscr{S}\mathscr{T}$-modules gradués sur $\mathscr{M}^{(l)}$, $\mathscr{N}^{(l)}$, $\mathscr{L}^{(l)}$, $\mathscr{P}$ respectivement. Les suites exactes de $\mathscr{O}_X$-modules (5), (6), (7) et (8) induisent les suites exactes

$$0 \to \mathscr{N}^{(l)} \to \mathscr{L}^{(l)} \to \mathscr{S}\mathscr{T} \otimes \mathscr{E}^* \to \mathscr{M}^{(l)} \to 0 \tag{9}$$

$$0 \to \mathscr{N} \to \mathscr{L} \to \mathscr{S}\mathscr{T} \otimes \mathscr{E}^* \to \mathscr{M} \to 0 \tag{10}$$

$$0 \to \mathscr{L}^{(l)} \to (\mathscr{S}\mathscr{T} \otimes \mathscr{E}^*) \oplus \mathscr{L} \tag{11}$$

$$0 \to \mathscr{P} \to \mathscr{S}\mathscr{T} \otimes \mathscr{F}^* \tag{12}$$

$$\mathscr{P} \to \mathscr{N} \to 0 \tag{13}$$

de $\mathscr{S}\mathscr{T}$-modules gradués.

II. Points réguliers d'un opérateur différentiel analytique

Soit $\mathscr{Q}_{m+l}^m$ le conoyau de $\mathscr{S}_m \to \mathscr{S}_{m+l}$; c'est un $\mathscr{O}_X$-module cohérent.

Lemme 1.

$$X - Y = \left\{ x \in X \,\middle|\, \begin{array}{ll} \forall\, m \geqq k, & \mathscr{S}_{m,x} \textit{ est un } \mathscr{O}_{X,x}\textit{-module libre} \\ \forall\, m \geqq k,\ l \geqq 0, & \mathscr{Q}_{m+l}^m \textit{ est un } \mathscr{O}_{X,x}\textit{-module libre} \end{array} \right\}.$$

Démonstration. C'est immédiat à partir du théorème des syzygies [5].

Lemme 2.

$$X - Y = \left\{ x \in X \,\middle|\, \forall\, m \geqq k,\ l \geqq 0, \begin{array}{l} \mathscr{S}_{m,x}^{(l)} \textit{ est un } \mathscr{O}_{X,x}\textit{-module libre} \\ \mathscr{M}_{m+1,x}^{(l)} \textit{ est un } \mathscr{O}_{X,x}\textit{-module libre} \end{array} \right\}.$$

Démonstration. Le diagramme

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{S}_m & \rightarrow & \mathcal{S}^{(l)}_{m+1} & \rightarrow & \mathcal{M}^{(l)}_{m+1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{S}_m & \rightarrow & \mathcal{S}_{m+l+1} & \rightarrow & \mathcal{Q}^m_{m+l+1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{Q}^{m+1}_{m+l+1} & \rightarrow & \mathcal{Q}^{m+1}_{m+l+1} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

est exact et commutatif. Le lemme découle du lemme 1, du fait que $\mathcal{Q}^m_{m+1} = \mathcal{M}_{m+1}$ et de l'exactitude de ce diagramme.

Lemme 3. *Pour toute partie K de X compacte et semi-analytique, l'anneau $\Gamma(K, \mathcal{S}\mathcal{T})$ est noethérien.*

Démonstration. Il existe un fibré trivial V sur X tel que T soit un fibré analytique quotient de V. D'après Frisch [1], l'anneau $A = \Gamma(K, \mathcal{O}_X)$ est noethérien, puisque K est de Stein; donc si $S\,V$ est l'algèbre symétrique de V, l'anneau $\Gamma(K, \mathcal{S}\mathcal{V})$ est une algèbre de polynômes sur A et donc est noethérien. L'anneau quotient $\Gamma(K, \mathcal{S}\mathcal{T})$ de $\Gamma(K, \mathcal{S}\mathcal{V})$ est donc aussi noethérien.

Lemme 4. *Soit K une partie de X compacte et semi-analytique. Les $\Gamma(K, \mathcal{S}\mathcal{T})$-modules $\Gamma(K, \mathcal{M}^{(l)})$ et $\Gamma(K, \mathcal{N}^{(l)})$ sont de type fini.*

Démonstration. D'après le théorème B et l'exactitude des suites (9), (10), (11), (12) et (13), nous avons les suites exactes de $\Gamma(K, \mathcal{S}\mathcal{T})$-modules

$$0 \rightarrow \Gamma(K, \mathcal{N}^{(l)}) \rightarrow \Gamma(K, \mathcal{L}^{(l)}) \rightarrow \Gamma(K, \mathcal{S}\mathcal{T} \otimes \mathcal{E}^*) \rightarrow \Gamma(K, \mathcal{M}^{(l)}) \rightarrow 0,$$

$$0 \rightarrow \Gamma(K, \mathcal{N}) \rightarrow \Gamma(K, \mathcal{L}) \rightarrow \Gamma(K, \mathcal{S}\mathcal{T} \otimes \mathcal{E}^*) \rightarrow \Gamma(K, \mathcal{M}) \rightarrow 0,$$

$$0 \rightarrow \Gamma(K, \mathcal{L}^{(l)}) \rightarrow \Gamma(K, \mathcal{S}\mathcal{T} \otimes \mathcal{E}^*) \oplus \Gamma(K, \mathcal{L}),$$

$$0 \rightarrow \Gamma(K, \mathcal{P}) \rightarrow \Gamma(K, \mathcal{S}\mathcal{T} \otimes \mathcal{E}^*),$$

$$\Gamma(K, \mathcal{P}) \rightarrow \Gamma(K, \mathcal{N}) \rightarrow 0.$$

Le lemme 3 et ces suites montrent, puisque $\Gamma(K, \mathcal{S}\mathcal{T} \otimes \mathcal{E}^*)$ et $\Gamma(K, \mathcal{S}\mathcal{T} \otimes \mathcal{F}^*)$ sont des $\Gamma(K, \mathcal{S}\mathcal{T})$-modules de type fini, que tous les modules qui figurent dans ces suites sont aussi de type fini.

Lemme 5. *Soient $\mathcal{G} = \bigoplus_{l \geqq 0} \mathcal{G}_l$ un $\mathcal{S}\mathcal{T}$-module gradué, K une partie de X compacte et semi-analytique. Si chaque $\mathcal{G}_l$ est un $\mathcal{O}_X$-module cohérent et si $\Gamma(K, \mathcal{G})$ est un $\Gamma(K, \mathcal{S}\mathcal{T})$-module de type fini, alors il existe un* *entier β tel que, quelque soit x dans l'intérieur de K, $\{\mathcal{G}_x$ est un $\mathcal{O}_{X,x}$-module libre$\}$ $\Leftrightarrow$ $\{\forall\, 0 \leqq l \leqq \beta$, $\mathcal{G}_{l,x}$ est un $\mathcal{O}_{X,x}$-module libre$\}$.*

Démonstration. Pour tout module gradué

$$H = \bigoplus_{l \geqq 0} H_l,$$

posons

$$H(q)_l = H_{l+q}, \qquad H(q) = \bigoplus_{l \geqq 0} H(q)_l.$$

Ecrivons $B = \Gamma(K, \mathcal{S}\mathcal{T})$. D'après le lemme 3, il existe des entiers p_0, p_1, q_0, q_1 et une suite exacte de B-modules gradués

$$B^{p_1} \xrightarrow{\gamma} B(q_1)^{p_0} \to \Gamma(K, \mathcal{G}(q_0)) \to 0,$$

le morphisme γ étant de degré zéro. Ce morphisme provient de morphismes de fibrés vectoriels définis sur un voisinage de K

$$\gamma_l \colon (S^l\, T)^{p_1} \to (S^{q_1+l}\, T)^{p_0}, \qquad l \geqq 0,$$

dont on notera les conoyaux Q_l. De plus

$$\gamma = \bigoplus_{l \geqq 0} \gamma_l \colon (S\,T)^{p_1} \to \big(S\,T(q_1)\big)^{p_0} \tag{14}$$

est un morphisme de $S\,T$-modules gradués de dégré zéro dont le conoyau est

$$Q = \bigoplus_{l \geqq 0} Q_l.$$

Si

$$\mathcal{H} = \bigoplus_{l \geqq 0} \mathcal{H}_l$$

est le conoyau de l'application induite $\gamma \colon (\mathcal{S}\,\mathcal{T})^{p_1} \to \big(\mathcal{S}\,\mathcal{T}(q_1)\big)^{p_0}$, d'après le théorème B, on a

$$\Gamma(K, \mathcal{H}) \cong \Gamma\big(K, \mathcal{G}(q_0)\big)$$

comme B-modules, et d'après le théorème A

$$\mathcal{H}_x \cong \mathcal{G}(q_0)_x, \qquad x \in K$$

comme $\mathcal{O}_{X,x}$-modules. Examinons le morphisme (14). D'après le lemme 3.1 de [2] (cf. [3]), il existe un entier α qui ne dépend que de la dimension n de X, p_1, p_0 et q_1 tel que sur K le complexe de Koszul de Q soit acyclique en degrés supérieurs à α, c'est-à-dire, que les suites

$$0 \to \Lambda^n\, T \otimes Q_l \to \cdots \xrightarrow{\delta^*} \Lambda^2\, T \otimes Q_{l+n-2} \xrightarrow{\delta^*} T \otimes Q_{l+n-1} \xrightarrow{\delta^*} Q_{l+n} \to 0$$

soient exactes sur K, pour tout $l \geqq \alpha$. Il en résulte que $x \in \mathring{K}$ est un point régulier de γ_l pour tout $l \geqq 0$, si et seulement si c'est un point régulier de γ_l pour $0 \leqq l \leqq \alpha + n$. Mais $x \in \mathring{K}$ est un point régulier de γ_l si et seulement si $\mathcal{H}_{l,x}$ est un $\mathcal{O}_{X,x}$-module libre. Donc, l'entier $\beta = q_0 + \alpha + n$ a la propriété requise.

Lemme 6. *Pour toute partie compacte K de X, il existe un entier $l_0 \geqq 0$ tel que pour tout $x \in K$, $m \geqq k$, $l \geqq l_0$,*

$$\begin{aligned} \mathscr{M}^{(l)}_{m+1,x} &= \mathscr{M}^{(l_0)}_{m+1,x}, \\ \mathscr{S}^{(l)}_{m,x} &= \mathscr{S}^{(l_0)}_{m,x}, \\ \mathscr{N}^{(l)}_{m,x} &= 0. \end{aligned} \tag{15}$$

et

Démonstration. On peut supposer que K est semi-analytique; écrivons $B = \Gamma(K, \mathscr{S}\mathscr{T})$. Posons $\mathscr{M}^{(-1)} = \mathscr{S}\mathscr{T} \otimes \mathscr{E}^*$. Puisque, pour $l \geqq 0$, $\mathscr{M}^{(l)}$ est un quotient de $\mathscr{M}^{(l-1)}$ comme $\mathscr{S}\mathscr{T}$-modules, d'après le théorème B, le B-module $\Gamma(K, \mathscr{M}^{(l)})$ est un quotient du B-module $\Gamma(K, \mathscr{M}^{(l-1)})$. Comme $\Gamma(K, \mathscr{S}\mathscr{T} \otimes \mathscr{E}^*)$ est un B-module de type fini, il existe un entier $l_1 \geqq 0$, tel que pour $l \geqq l_1$,

$$\Gamma(K, \mathscr{M}^{(l)}) = \Gamma(K, \mathscr{M}^{(l_1)}).$$

D'après le théorème A, pour tout $x \in K$, $l \geqq l_1$,

$$\mathscr{M}^{(l)}_x = \mathscr{M}^{(l_1)}_x.$$

Comme toute famille filtrante croissante de sous-faisceaux analytiques cohérents de $\mathscr{S}_k$ est stationnaire sur le compact K (cf. Frisch [1]), il existe un entier $l_0 \geqq l_1$ tel que, pour tout $x \in K$, $l \geqq l_0$,

$$\mathscr{S}^{(l)}_{k,x} = \mathscr{S}^{(l_0)}_{k,x}.$$

Montrons que l'égalité (15) est vraie par récurrence sur m. C'est vrai pour $m = k$. Supposons que (15) est vrai pour $m \geqq k$. Comme l'égalité $\mathscr{S}^{(l+1)}_{m,x} = \mathscr{S}^{(l)}_{m,x}$ est équivalente à $\mathscr{N}^{(l)}_{m,x} = 0$, le diagramme

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{S}^{(l_0)}_{m,x} & \to & \mathscr{S}^{(l)}_{m,x} & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathscr{S}^{(l_0)}_{m+1,x} & \to & \mathscr{S}^{(l)}_{m+1,x} & \to & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \to & \mathscr{M}^{(l_0)}_{m+1,x} & \to & \mathscr{M}^{(l)}_{m+1,x} & \to & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

est exact et commutatif, pour $x \in K$, $l \geqq l_0$. Il en résulte que $\mathscr{S}^{(l)}_{m+1,x} = \mathscr{S}^{(l_0)}_{m+1,x}$.

Lemme 7. *Pour toute partie compacte K de X, il existe des entiers $m_0 \geqq k$, $l_0 \geqq 0$ tels que*

$$K \cap (X - Y) = \left\{ x \in K \,\middle|\, \begin{array}{l} \forall\, k \leqq m \leqq m_0,\ 0 \leqq l \leqq l_0,\ \mathscr{S}^{(l_0)}_{k,x},\ \mathscr{M}^{(l)}_{m+1,x},\ \mathscr{N}^{(l)}_{m,x} \\ \text{sont des } \mathscr{O}_{X,x}\text{-modules libres} \end{array} \right\}.$$

Démonstration. D'après les lemmes 2 et 6,

$$K \cap (X-Y) = \left\{ x \in K \,\middle|\, \begin{array}{l} \forall\, m \geqq k, 0 \leqq l \leqq l_0, \mathscr{S}^{(l)}_{m,x}, \mathscr{M}^{(l)}_{m+1,x} \\ \text{sont des } \mathscr{O}_{X,x}\text{-modules libres} \end{array} \right\}$$

où l_0 est l'entier donné par le lemme 6. En fait, l'on a

$$K \cap (X-Y) = \left\{ x \in K \,\middle|\, \begin{array}{l} \forall\, m \geqq k, 0 \leqq l \leqq l_0, \mathscr{S}^{(l_0)}_{k,x}, \mathscr{M}^{(l)}_{m+1,x}, \mathscr{N}^{(l)}_{m,x} \\ \text{sont des } \mathscr{O}_{X,x}\text{-modules libres} \end{array} \right\}.$$

L'inclusion $\supset$ suit de l'exactitude des suites (2). Soit $x \in K$; si $\mathscr{S}^{(l_0)}_{k,x}$, $\mathscr{M}^{(l_0)}_{m+1,x}$, pour $m \geqq k$, sont des $\mathscr{O}_{X,x}$-modules libres, l'exactitude de la suite

$$0 \to \mathscr{S}^{(l_0)}_{m,x} \to \mathscr{S}^{(l_0)}_{m+1,x} \to \mathscr{M}^{(l_0)}_{m+1,x} \to 0$$

implique que $\mathscr{S}^{(l_0)}_{m,x}$ est un $\mathscr{O}_{X,x}$-module libre pour $m \geqq k$. Les suites exactes (3) montrent que si, pour tout $m \geqq k$, $\mathscr{S}^{(l+1)}_{m,x}$, $\mathscr{N}^{(l)}_{m,x}$ sont des $\mathscr{O}_{X,x}$-modules libres, alors $\mathscr{S}^{(l)}_{m,x}$ est un $\mathscr{O}_{X,x}$-module libre pour tout $m \geqq k$.

On applique maintenant le lemme 5 aux $\mathscr{S}\mathscr{T}$-modules $\mathscr{M}^{(l)}$ et $\mathscr{N}^{(l)}$, ce que l'on peut faire d'après le lemme 4. Puisque tout point $x \in K$ possède des voisinages compacts et semi-analytiques, il existe donc un entier m_0 tel que, pour $0 \leqq l \leqq l_0$, $\mathscr{M}^{(l)}_{m+1,x}$ est un $\mathscr{O}_{X,x}$-module libre pour tout $m \geqq k$ si et seulement si $\mathscr{M}^{(l)}_{m+1,x}$ est un $\mathscr{O}_{X,x}$-module libre pour $k \leqq m \leqq m_0$, et tel que $\mathscr{N}^{(l)}_{m,x}$ est un $\mathscr{O}_{X,x}$-module libre pour tout $m \geqq k$ si et seulement si $\mathscr{N}^{(l)}_{m,x}$ est un $\mathscr{O}_{X,x}$-module libre pour tout $k \leqq m \leqq m_0$.

Démonstration du théorème. Il découle du lemme 7 que

$$K \cap (X-Y) = \left\{ x \in K \,\middle|\, \begin{array}{l} \forall\, k \leqq m \leqq m_0+1, 0 \leqq l \leqq l_0+1, \mathscr{S}_{m+l,x}, \\ \mathscr{Q}^m_{m+l,x} \text{ sont des } \mathscr{O}_{X,x}\text{-modules libres} \end{array} \right\}$$

à l'aide des suites exactes (2) et des suites exactes

$$0 \to \mathscr{M}^{(l)}_{m+1} \to \mathscr{Q}^m_{m+l+1} \to \mathscr{Q}^{m+1}_{m+l+1} \to 0.$$

Donc il suit que pour tout $x \in X$, il existe un voisinage compact K_0 de x tel que $Y \cap K_0$ soit un sous-ensemble analytique strict de $\mathring{K}_0$. De plus, si l'on pose $\mu_0 = m_0 + 1$, $\lambda_0 = l_0 + 1$, l'on a

$$K \cap (X-Y) = \left\{ x \in K \,\middle|\, \begin{array}{l} \forall\, 0 \leqq m \leqq \mu_0 + \lambda_0 - k, \ x \text{ est un point} \\ \text{régulier de } p_m(\varphi) \\ \forall\, k \leqq m \leqq \mu_0, 0 \leqq l \leqq \lambda_0, \ x \text{ est un point} \\ \text{régulier de } \pi_m\colon R_{m+l} \to R_m \end{array} \right\}.$$

Il en résulte immédiatement (ii) et que Y est fermé, et donc que Y est un sous-ensemble analytique strict de X.

Remarque. Si X est une variété complexe et D un opérateur différentiel $\mathbb{C}$-analytique, nos résultats sont encore vrais.

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Ein Satz über ganze Funktionen und Irrationalitätsaussagen

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§1. Einleitung

Gelfond [2] bewies einen Satz, der besagt, daß eine ganze transzendente Funktion, die an allen Stellen β^n ($\beta > 1$ eine feste natürliche Zahl; $n=1, 2, \ldots$) ganze rationale Zahlen als Werte annimmt, nicht zu langsam anwachsen kann.

Hier soll nun ein ähnlicher Satz über ganze transzendente Funktionen gezeigt werden, der es erlaubt, Irrationalitätsaussagen für eine gewisse Klasse derartiger Funktionen zu machen. *Dabei bezeichnet K in der ganzen Arbeit einen fest vorgegebenen imaginär-quadratischen Zahlkörper und q eine feste Zahl aus K mit Norm* $(q) > 1$.

Satz 1. *q sei ganz und $f(z) = \sum_{n=0}^{\infty} c_n z^n$ sei eine ganze transzendente Funktion. Es seien alle $c_n \in K$ und es gebe ganze $b_n \in K$ ($b_n \neq 0$) derart, daß $b_n c_\nu$ für $\nu = 0, \ldots, n$ ganz ist und daß gilt: $|b_n| \leqq |q|^{\lambda n^2 + o(n^2)}$ mit festem $\lambda \geqq 0$. Ferner gebe es ein $a \in K (a \neq 0)$ derart, daß $f(a q^{-m}) \in K$ ist für alle $m \geqq 0$, und ganze Zahlen $T_m \in K$ ($T_m \neq 0$) derart, daß $T_m f(a q^{-\kappa})$ ganz ist für $\kappa = 0, \ldots, m$ und daß gilt: $|T_m| \leqq |q|^{\mu m^2 + o(m^2)}$ mit festem $\mu \geqq 0$. Dann ist*

$$\rho^* := \overline{\lim_{r \to \infty}} \frac{\log M(r)}{\log^2 r} \geqq \left(\frac{1}{\lambda} + \frac{1}{\mu}\right) \frac{1}{4 \log |q|} \qquad \left(M(r) = \operatorname*{Max}_{|z| \leqq r} |f(z)|\right). \quad \text{(i)}$$

Bemerkung. Den in (i) auftretenden Ausdruck ρ^* haben wir in [1, S. 278] *Quasiwachstumsordnung* von $f(z)$ genannt. Obige Ungleichung (i) bedeutet: Unter den Voraussetzungen des Satzes ist die Quasiwachstumsordnung von $f(z)$ gleich ∞, falls $\lambda \mu = 0$, bzw. entweder ∞ oder eine Zahl, die nicht kleiner als der Ausdruck rechts in (i) ist, falls $\lambda \mu \neq 0$.

Wir geben nun einige Korollare zu Satz 1 an, die Irrationalitätsaussagen machen; es ließen sich noch eine ganze Reihe ähnlicher Resultate aus Satz 1 ableiten [1].

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[1] Satz 1 und Korollar 5 wurden mit Beweisskizze auf der Jahrestagung der DMV am 24.9.1969 in Darmstadt vorgetragen.

Korollar 1. *Sei q ganz und die ganze Funktion $f_1(z)$ genüge der Funktionalgleichung*

$$f_1(z) = (1 - z/q)\, f_1(z/q) \quad mit \quad f_1(0) = 1; \tag{ii}$$

die komplexe Zahl a sei $\neq 0$ und $\neq q^n$ für alle natürlichen n. Dann ist nicht gleichzeitig $a \in K$ und $f_1(a) \in K$.

Dieses Korollar ist nichts anderes als Satz 1 von [1]; hieraus gewinnt man ganz leicht das

Korollar 2. *$p(n)$ bezeichne die Anzahl der Partitionen von n für jedes natürliche n. Dann ist für jedes ganze q*

$$\sum_{n=1}^{\infty} p(n)\, q^{-n} \notin K.$$

Korollar 3. *Sei q ganz und die ganze Funktion $f_2(z)$ genüge der Funktionalgleichung*

$$f_2(z) = f_2(z/q) + z/q\, f_2(z/q^2) \quad mit \quad f_2(0) = 1; \tag{iii}$$

die komplexe Zahl a sei $\neq 0$. Dann sind nicht gleichzeitig $a \in K$, $f_2(a) \in K$ und $f_2(a/q) \in K$.

Eine Bezeichnung. Sei $m \geqq 2$ eine natürliche Zahl und seien μ_1, μ_2 zwei verschiedene Zahlen der Folge $0, 1, \ldots, m-1$; mit $p(n; m; \mu_1, \mu_2)$ bezeichnen wir dann die Anzahl der Partitionen von n ($n \geqq 1$) in natürliche Zahlen, die sämtliche $\equiv \mu_1$ oder $\equiv \mu_2 \pmod m$ sind. Nach Vorwegnahme dieser Bezeichnung können wir das nächste Korollar kurz formulieren.

Korollar 4. *Für jedes ganze q ist*

$$\sum_{n=1}^{\infty} p(n; 5; 1, 4)\, q^{-n} \notin K \quad oder \quad \sum_{n=1}^{\infty} p(n; 5; 2, 3)\, q^{-n} \notin K.$$

Korollar 5. *Sei q ganz, $f_3(z)$ die Funktion $\sum_{n=0}^{\infty} z^n q^{-\frac{1}{2}n(n-1)}$ und a eine komplexe Zahl $\neq 0$. Dann ist nicht gleichzeitig $a \in K$ und $f_3(a) \in K$.*

Besitzt K eindeutige Primfaktorzerlegung, so ist Korollar 5 in einem alten Satz von Tschakaloff [5] enthalten; besitzt K jedoch keine eindeutige Primfaktorzerlegung, so macht der Tschakaloffsche Satz keinerlei Aussage.

Unter geeigneten Voraussetzungen kann man die Korollare 1 bis 5 auch dann beweisen, wenn q nicht ganz ist. Hierauf werden wir in §4 eingehen. Dort wird ferner die Frage untersucht, wie gut sich Zahlen, die aufgrund eines unserer Korollare nicht aus K sind, durch Zahlen aus K approximieren lassen. Man kann z.B. folgendes Resultat erhalten:

Sei q ganzrational und $|q| \geqq 2$; ferner sei a eine rationale Zahl $\neq 0$. Dann ist die Irrationalzahl $\sum_{n=0}^{\infty} a^n q^{-\frac{1}{2}n(n-1)}$ keine Liouville-Zahl.

§2. Beweis von Satz 1

Dem Beweis von Satz 1 schicken wir ein einfaches Lemma voraus, dessen Beweis analog demjenigen für die Ganzheit der Binomialkoeffizienten verläuft.

Hilfssatz. *α und β seien beliebige ganze Zahlen eines algebraischen Zahlkörpers K^*; jedoch sei $|\alpha| \neq |\beta|$. Für jedes natürliche k ist*

$$\prod_{\lambda=1}^{\kappa} (\alpha^{k-\lambda+1} - \beta^{k-\lambda+1})(\alpha^{\lambda} - \beta^{\lambda})^{-1} \qquad (\kappa = 1, \ldots, k)$$

ganz in K^*.

Zum Beweis von Satz 1 machen wir die Annahme, die Quasiwachstumsordnung ρ^* von $f(z)$ sei endlich und es sei $\rho^* < \left(\frac{1}{\lambda} + \frac{1}{\mu}\right) \frac{1}{4 \log |q|}$. Wir geben uns ein $\varepsilon > 0$ beliebig vor; o.B.d.A. können wir es so voraussetzen, daß gilt:

$$\rho^* + \varepsilon < \left(\frac{1}{\lambda+\varepsilon} + \frac{1}{\mu+\varepsilon}\right) \frac{1}{4 \log |q|}. \tag{1}$$

Wir entwickeln $f(z)$ in eine Interpolationsreihe nach den Stellen 0 und $a q^{-\kappa} (\kappa \geqq 0)$, indem wir die Folge $\{z_n\}_{n=1,2,\ldots}$ der Interpolationsstellen geeignet wählen.

Für jedes natürliche k werde $j(k)$ definiert durch

$$j(k) = \left[\frac{\mu+\varepsilon}{\lambda+\varepsilon} k\right] \qquad (k = 1, 2, \ldots), \tag{2}$$

wobei $[x]$ die größte ganze Zahl $\leqq x$ bezeichnet. Dann gibt es zu jedem vorgegebenen natürlichen n entweder kein oder genau ein natürliches ν derart, daß $n = \nu + j(\nu)$ gilt. Nun definieren wir

$$z_n = \begin{cases} 0, & \text{falls kein} \\ a q^{1-\nu}, & \text{falls ein} \end{cases} \Bigg\} \ \nu \text{ existiert derart, daß } n = \nu + j(\nu). \tag{3}$$

Wegen i) von Hilfssatz 1 aus [1] konvergiert die Interpolationsreihe für $f(z)$ nach diesen Stellen z_n in der ganzen z-Ebene. Es ist also

$$f(z) = A_0 + \sum_{n=1}^{\infty} A_n \prod_{\nu=1}^{n} (z - z_\nu) \tag{4}$$

mit

$$A_{n-1} = \frac{1}{2\pi i} \int_{C_n} f(\zeta) \prod_{\nu=1}^{n} (\zeta - z_\nu)^{-1} \, d\zeta \qquad (n = 1, 2, \ldots), \tag{5}$$

wobei C_n einen Kreis um $\zeta = 0$ bezeichnet, der $z_1, \ldots, z_n$ in seinem Innern enthält. Wählen wir $C_n = \left\{\zeta \,\middle|\, |\zeta| = \exp\left(\frac{n}{2(\rho^*+\varepsilon)}\right)\right\}$, so ist jedenfalls für

$n \geqq n_0(\rho^*, a, \varepsilon)$ die soeben genannte Bedingung für C_n erfüllt und wir erhalten durch Abschätzen aus (5) mit Rücksicht auf $|z_\nu| \leqq |a|$ (vgl. (3)):

$$|A_{n-1}| \leqq \exp\left(-\frac{n^2}{4(\rho^* + \varepsilon)} + c_0\, n\right) \quad \text{für} \quad n \geqq n_0 \tag{6}$$

mit einem $c_0 = c_0(\rho^*, \varepsilon) > 0$.

Wenden wir andererseits den Residuensatz auf das Integral in (5) an, so haben wir zu unterscheiden, ob n die Form $k + j(k)$ hat oder nicht. Im ersten Fall ergibt sich aus

$$A_{k+j(k)-1} = \frac{1}{2\pi i} \int_{C_{k+j(k)}} f(\zeta)\, \zeta^{-j(k)} \prod_{\kappa=1}^{k} (\zeta - a\, q^{1-\kappa})^{-1}\, d\zeta$$

der folgende Ausdruck

$$\begin{aligned} A_{k+j(k)-1} = q^{\frac{1}{2}k(k-1)} \Bigg\{ &(-1)^k \sum_{\sigma=0}^{j(k)-1} c_{j(k)-1-\sigma}\, a^{-k-\sigma} \sum_{\substack{\sigma_1, \ldots, \sigma_k \geqq 0 \\ \sigma_1 + \cdots + \sigma_k = \sigma}} q^{\sum_{\kappa=1}^{k} (\kappa-1)\sigma_\kappa} \\ &+ a^{-k-j(k)+1} \sum_{\kappa=1}^{k} \frac{(-1)^{\kappa-1} f(a\, q^{1-\kappa})\, q^{(\kappa-1)(j(k)-1+\kappa/2)}}{\prod_{\rho=1}^{\kappa-1} (q^\rho - 1) \prod_{\rho=1}^{k-\kappa} (q^\rho - 1)} \Bigg\}, \end{aligned} \tag{7}$$

falls $k \geqq k_0(\lambda, \mu, \varepsilon)$ ist; k_0 so, daß $j(k_0) \geqq 1$. Definieren wir nun

$$A^*_{k+j(k)-1} = q^{-\frac{1}{2}k(k-1)} A_{k+j(k)-1}, \tag{8}$$

so sind die $A^*_{k+j(k)-1}$ Zahlen aus K, die wir betragsmäßig nach unten abschätzen können, wenn sie nicht Null sind. Dazu haben wir ganze, von 0 verschiedene $B_{k+j(k)-1} \in K$ zu bestimmen derart, daß die $A^*_{k+j(k)-1} B_{k+j(k)-1}$ ganz sind. Setzt man $a = s/S$ mit ganzen, von 0 verschiedenen $s, S \in K$ in (7) ein, beachtet Hilfssatz 1 sowie die Voraussetzungen unseres Satzes 1, so erkennt man leicht, daß man

$$B_{k+j(k)-1} = s^{k+j(k)-1}\, b_{j(k)-1}\, T_{k-1} \prod_{\rho=1}^{k-1} (q^\rho - 1) \tag{9}$$

nehmen kann. Hieraus folgt für $k \geqq k_0$:

$$|B_{k+j(k)-1}| \leqq \exp\left(\left(\frac{\mu+\varepsilon}{\lambda+\varepsilon}(\lambda + \mu + 2\varepsilon) + \frac{1}{2}\right) k^2 \log|q| + c_1\, k\right) \tag{10}$$

mit $c_1 = c_1(a, \lambda, \mu, \varepsilon) > 0$.

Machen wir nun die Annahme, für unendlich viele $k\,(k \geqq k_0)$ sei $A_{k+j(k)-1} \neq 0$. Dann folgt aus (6), (8) und (10) für $k \geqq k_0$:

$$1 \leqq |A^*_{k+j(k)-1} B_{k+j(k)-1}| \\ \leqq \exp\left(\left((\mu+\varepsilon)\log|q| - \frac{\lambda+\mu+2\varepsilon}{4(\rho^*+\varepsilon)(\lambda+\varepsilon)}\right)\frac{\lambda+\mu+2\varepsilon}{\lambda+\varepsilon}k^2 + c_2\,k\right) \tag{11}$$

mit $c_2 = c_2(\rho^*, q, a, \lambda, \mu, \varepsilon) > 0$. Wegen (1) ist hier der Koeffizient von k^2 in $\exp(\ldots)$ negativ, so daß die Ungleichung (11) höchstens endlich viele Lösungen k haben kann. Daher sind alle A_{n-1} gleich Null, für die n genügend groß ist und die Form $k+j(k)$ hat. Hat n nicht diese Form, so verläuft der Beweis völlig analog. Unsere zu Anfang des Beweises für Satz 1 gemachte Annahme über ρ^* führt wegen (4) zur Folgerung, daß sich $f(z)$ auf ein Polynom reduzieren muß. Da aber $f(z)$ als ganz transzendent vorausgesetzt war, ist Satz 1 vollständig bewiesen.

§ 3. Beweis der Korollare 1 bis 5

Wir werden in diesem Paragraphen sehen, wie leicht sich unser Satz 1 zur Gewinnung von Irrationalitätsaussagen anwenden läßt.

Zu Korollar 1. Aus der Funktionalgleichung (ii) gewinnt man sofort

$$c_0 = 1 \quad \text{und} \quad c_n = (-1)^n \prod_{\nu=1}^{n} (q^\nu - 1)^{-1} \quad (n \geqq 1), \tag{12}$$

woraus man mit Hilfe von Teil ii) des Hilfssatzes 1 in [1] auf die Quasiwachstumsordnung ρ_1^* von $f_1(z)$ schließen kann:

$$\rho_1^* = \frac{1}{2\log|q|}. \tag{13}$$

Alle c_n sind aus K und für die b_n aus Satz 1 können wir wegen (12) wählen

$$b_0 = 1 \quad \text{und} \quad b_n = \prod_{\nu=1}^{n} (q^\nu - 1) \quad (n \geqq 1). \tag{14}$$

Wir nehmen an, es gäbe ein $a \in K$, $a \neq 0$ und $\neq q^n$ derart, daß auch $f_1(a) \in K$ ist. Dann sind wegen

$$f_1(a\,q^{-k}) = f_1(a)\,q^{\frac{1}{2}k(k+1)} \prod_{\kappa=1}^{k} (q^\kappa - a)^{-1} \quad (k \geqq 1) \tag{15}$$

alle $f_1(a\,q^{-k})\in K$ $(k\geqq 0)$. Ist $a=s/S$, $f_1(a)=t/T$ mit ganzen $s, S, t, T\in K$, so sieht man, daß man die T_m aus Satz 1 folgendermaßen wählen kann:

$$T_0=T \quad \text{und} \quad T_m=T\prod_{\kappa=1}^{m}(q^{\kappa}-1) \qquad (m\geqq 1). \tag{16}$$

Aus (14) und (16) schließt man auf $\lambda\leqq\frac{1}{2}$, $\mu\leqq\frac{1}{2}$ und daher wegen (i) von Satz 1 auf $\rho_1^*\geqq 1/\log|q|$ entgegen (13).

Zu Korollar 2. Für jedes komplexe x mit $|x|<1$ ist nach [3, S. 272]

$$\left(\prod_{\nu=1}^{\infty}(1-x^{\nu})\right)^{-1}=1+\sum_{n=1}^{\infty}p(n)\,x^{n}. \tag{17}$$

Setzt man hierin $x=q^{-1}$ und beachtet $f_1(z)=\prod_{\nu=1}^{\infty}(1-z\,q^{-\nu})$, so folgt

$$\sum_{n=1}^{\infty}p(n)\,q^{-n}=-1+\left(f_1(1)\right)^{-1}$$

und nach Korollar 1 ist $f_1(1)\notin K$.

Zu Korollar 3. Wieder sieht man aus der Funktionalgleichung (iii) leicht, wie die Taylorkoeffizienten von $f_2(z)$ aussehen:

$$c_0=1 \quad \text{und} \quad c_n=q^{-\frac{1}{2}n(n-1)}\prod_{\nu=1}^{n}(q^{\nu}-1)^{-1} \qquad (n\geqq 1), \tag{18}$$

woraus sich ergibt

$$\rho_2^*=\frac{1}{4\log|q|}; \tag{19}$$

$$b_0=1 \quad \text{und} \quad b_n=q^{\frac{1}{2}n(n-1)}\prod_{\nu=1}^{n}(q^{\nu}-1) \qquad (n\geqq 1). \tag{20}$$

Ist $a\neq 0$ und setzt man $P_0(1/a,q)=1$; $Q_0(1/a,q)=0$; $P_1(1/a,q)=0$; $Q_1(1/a,q)=1$, so kann man $f_2(a\,q^{-\kappa})$ ausdrücken als

$$f_2(a\,q^{-\kappa})=P_\kappa(1/a,q)\,f_2(a)+Q_\kappa(1/a,q)\,f_2(a/q) \qquad (\kappa\geqq 0), \tag{21}$$

wobei die P_κ und Q_κ für $\kappa\geqq 2$ Polynome in $1/a$ bzw. q von den genauen Graden $\kappa-1$ bzw. $\frac{1}{2}\kappa(\kappa-1)$ und mit ganzrationalen Koeffizienten sind. Dies folgt durch Induktion über κ aus der Funktionalgleichung (iii). Nimmt man an, es gäbe ein $a\in K$, $a\neq 0$ derart, daß auch $f_2(a)\in K$ und $f_2(a/q)\in K$ ist, so sind wegen (21) sogar alle $f_2(a\,q^{-\kappa})\in K$ $(\kappa\geqq 0)$. Setzt man $a=s/S$, $f_2(a)=t/T$, $f_2(a/q)=u/U$ in (21) ein, so erkennt man, daß man die T_m wählen kann zu

$$T_0=T \quad \text{und} \quad T_m=s^{m-1}\,T\,U \quad \text{für} \quad m\geqq 1. \tag{22}$$

Aus (20) und (22) schließt man auf $\lambda\leqq 1$, $\mu=0$ und also nach Satz 1 auf $\rho_2^*=\infty$ entgegen (19).

Zu Korollar 4. Nach (18) ist

$$f_2(1)=1+\sum_{n=1}^{\infty} q^{-n^2}\prod_{\nu=1}^{n}(1-q^{-\nu})^{-1}$$

und

$$f_2(1/q)=1+\sum_{n=1}^{\infty} q^{-n^2-n}\prod_{\nu=1}^{n}(1-q^{-\nu})^{-1}.$$

Hier kann man die rechten Seiten den Identitäten von Rogers und Ramanujan [3, S. 288] zufolge umformen zu

$$f_2(1)=\left(\prod_{n=0}^{\infty}(1-q^{-5n-1})(1-q^{-5n-4})\right)^{-1}$$

bzw.

$$f_2(1/q)=\left(\prod_{n=0}^{\infty}(1-q^{-5n-2})(1-q^{-5n-3})\right)^{-1}.$$

Aus diesen Produktdarstellungen findet man leicht

$$f_2(1)=1+\sum_{n=1}^{\infty} p(n;5;1,4)\,q^{-n}$$

bzw.

$$f_2(1/q)=1+\sum_{n=1}^{\infty} p(n;5;2,3)\,q^{-n}$$

und hieraus folgt Korollar 4, wenn man Korollar 3 mit $a=1$ anwendet.

Zu Korollar 5. Aus $f_3(z)=\sum_{n=0}^{\infty} z^n q^{-\frac{1}{2}n(n-1)}$ liest man ab:

$$b_n=q^{\frac{1}{2}n(n-1)}, \tag{23}$$

$$\rho_3^*=\frac{1}{2\log|q|}. \tag{24}$$

Ferner folgt, daß $f_3(z)$ der Funktionalgleichung

$$f_3(z)=1+z\,f_3(z/q) \tag{25}$$

genügt. Hieraus bekommt man für $a\neq 0$ und alle $\kappa\geqq 1$

$$f_3(a\,q^{-\kappa})=P_\kappa(1/a,q)+a^{-\kappa}q^{\frac{1}{2}\kappa(\kappa-1)}f_3(a), \tag{26}$$

wo P_κ ein Polynom in $1/a$ bzw. q vom genauen Grad κ bzw. $\frac{1}{2}\kappa(\kappa-1)$ mit ganzrationalen Koeffizienten ist. Nimmt man an, es gäbe ein $a\in K$, $a\neq 0$, für das auch $f_3(a)\in K$ ist, so sind nach (26) sogar alle $f_3(a\,q^{-\kappa})\in K$ $(\kappa\geqq 0)$. Setzt man $a=s/S$, $f_3(a)=t/T$, so kann man T_m wegen (26) wählen zu

$$T_m=T s^m \qquad (m\geqq 0). \tag{27}$$

Aus (23) und (27) folgt $\lambda\leqq\frac{1}{2}$, $\mu=0$ und daraus mit (i) in Satz 1: $\rho_3^*=\infty$ entgegen (24).

§ 4. Weitere Ergebnisse

Wir setzen jetzt $q = q_1/q_2$ mit ganzen $q_1, q_2 \in K$ $(q_2 \neq 0)$, wobei q_1 und q_2 nur Einheiten von K als Teiler gemeinsam haben mögen. Dann definieren wir γ durch

$$\log |q_2| = \gamma \log |q_1|,$$

so daß $0 \leqq \gamma < 1$ gilt, und zwar $\gamma = 0$ genau dann, wenn q ganz ist.

Beschränkt man nun γ nach oben durch eine gewisse positive Zahl $1/\Gamma$, die kleiner als 1 ist und nur von den Parametern λ und μ aus Satz 1 abhängt, so kann man einen zu Satz 1 analogen Satz auch für solche (nicht notwendig ganze) $q \in K$ beweisen, für die $|q_1| > |q_2|^\Gamma$ ist. Eine genauere Formulierung dieses Satzes geben wir nicht, da sich bei direkter Behandlung der Funktionen $f_1(z)$, $f_2(z)$ bzw. $f_3(z)$ aus den Korollaren 1, 3 bzw. 5 bessere Resultate bezüglich Γ ergeben als sie aus der Verallgemeinerung von Satz 1 zu erhalten sind. Das liegt daran, daß bei einer Verschärfung der Ergebnisse in dieser Richtung die spezielle Form der c_n und der $f(a\, q^{-m})$ stärker zum Tragen kommt.

Zunächst läßt sich Korollar 1 verbessern zu

Satz 2. *q habe eine Darstellung $q = q_1/q_2$ mit $|q_1| > |q_2|^{\frac{7}{3}}$; die ganze Funktion $f_1(z)$ genüge der Funktionalgleichung*

$$f_1(z) = (1 - z/q)\, f_1(z/q) \quad \textit{mit} \quad f_1(0) = 1$$

und die komplexe Zahl a sei $\neq 0$ und $\neq q^n$ für alle natürlichen n. Dann ist $a \notin K$ oder $f_1(a) \notin K$.

Es ist durchaus zu vermuten, daß Satz 2 auch dann gilt, wenn man die Konstante $\frac{7}{3}$ im Exponenten von $|q_2|$ durch die bestmögliche Konstante 1 ersetzt. Doch reicht dazu unsere Beweismethode nicht hin. Dieselbe Bemerkung gilt auch für die Konstante $\frac{12}{7}$ im folgenden Satz 3, der das Korollar 3 verschärft.

Satz 3. *q habe eine Darstellung $q = q_1/q_2$ mit $|q_1| > |q_2|^{\frac{12}{7}}$; die ganze Funktion $f_2(z)$ genüge der Funktionalgleichung*

$$f_2(z) = f_2(z/q) + z/q\, f_2(z/q^2) \quad \textit{mit} \quad f_2(0) = 1$$

und die komplexe Zahl a sei $\neq 0$. Dann ist $a \notin K$ oder $f_2(a) \notin K$ oder $f_2(a/q) \notin K$.

Aus den Sätzen 2 bzw. 3 folgen selbstverständlich auch Verbesserungen der Korollare 2 bzw. 4. Natürlich ließe sich auch Korollar 5 auf gewisse nicht ganze $q \in K$ ausdehnen.

Zum Beweis der Sätze 2 bzw. 3 hat man lediglich in (6) die aus (13) bzw. (19) bekannten ρ^* einzutragen; in (7) wählt man am günstigsten in beiden Fällen $j(k) = k$ und setzt die aus (12) bzw. (18) bekannten

Taylorkoeffizienten und die aus (15) bzw. (21) bekannten $f(a\,q^{1-\kappa})$ ($\kappa=1,\dots,k$) ein. Danach trägt man in (7) $q=q_1/q_2$, $a=s/S$, $f(a)=t/T$ und gegebenenfalls $f(a/q)=u/U$ ein und ermittelt erneut Nenner B für die Interpolationskoeffizienten. Der Rest des Beweises verläuft dann wie in §2.

Die negativen Aussagen, die die Sätze 2 bzw. 3 machen, werden in den folgenden Sätzen 4 bzw. 5 positiv gewandt.

Satz 4. *q habe eine Darstellung $q=q_1/q_2$; γ werde definiert durch $\log|q_2|=\gamma\cdot\log|q_1|$ und es sei $0\leqq\gamma<\frac{3}{7}$. Ferner werde für diese γ gesetzt: $g_1(\gamma)=1+4/(3-7\gamma)$. $f_1(z)$ und a seien wie in Satz 2 vorausgesetzt und überdies sei $a\in K$ und $\varepsilon>0$ beliebig vorgegeben. Dann hat die Ungleichung*

$$|f_1(a)-P/Q|\leqq|Q|^{-(g_1(\gamma)+\varepsilon)}$$

nur endlich viele Lösungen P/Q mit ganzen $P, Q\in K$ $(Q\neq 0)$.

Satz 5. *q habe eine Darstellung $q=q_1/q_2$; γ werde definiert durch $\log|q_2|=\gamma\log|q_1|$ und es sei $0\leqq\gamma<\frac{7}{12}$. Ferner werde für diese γ gesetzt: $g_2(\gamma)=1+5/(7-12\gamma)$. $f_2(z)$ und a seien wie in Satz 3 vorausgesetzt und überdies sei $a\in K$ und $\varepsilon>0$ beliebig vorgegeben. Dann haben die Ungleichungen*

$$|f_2(a)-P/Q|\leqq|Q|^{-(g_2(\gamma)+\varepsilon)},\qquad |f_2(a/q)-P^*/Q|\leqq|Q|^{-(g_2(\gamma)+\varepsilon)}$$

nur endlich viele Lösungen P/Q, P^*/Q mit ganzen $P, P^*, Q\in K$ $(Q\neq 0)$.

Aus diesen Sätzen kann man wieder Folgerungen ziehen. Ist q in Satz 4 ganz, so hat man $\gamma=0$ und $g_1(0)=\frac{7}{3}$ und also gilt

Korollar 6. *Ist q ganz, $f_1(z)$ und a wie in Satz 2 vorausgesetzt und außerdem $a\in K$ und $\varepsilon>0$ beliebig vorgegeben. Dann hat die Ungleichung*

$$|f_1(a)-P/Q|\leqq|Q|^{-(\frac{7}{3}+\varepsilon)}$$

nur endlich viele Lösungen P/Q.

Dies Ergebnis verschärft Satz 2 von [1] erheblich bezüglich des Exponenten von $|Q|$ in Richtung auf die bestmögliche Schranke $|Q|^{-(2+\varepsilon)}$. Beachtet man $f_1(a)=\prod_{k=1}^{\infty}(1-a\,q^{-k})$ und die Definition der Liouville-Zahlen (vgl. etwa [4, S. 176]), so gewinnt man aus Satz 4

Korollar 7. *q sei rational und habe eine Darstellung $q=q_1/q_2$ mit ganzrationalen q_1, q_2 $(q_2>0)$ und es sei $|q_1|>q_2^{\frac{7}{3}}$. a sei rational, aber $\neq 0$ und $\neq q^n$ für alle natürlichen n. Dann ist die Irrationalzahl $\prod_{k=1}^{\infty}(1-a\,q^{-k})$ keine Liouville-Zahl.*

Ein Beweis der Sätze 4 und 5 soll hier nicht gegeben werden; die Methode ist die gleiche, die der Autor in §8 von [1] angewandt hat. Eine enge Beziehung zwischen Quasiwachstumsordnung ρ^* und den Interpolationskoeffizienten A_n spielt dabei die entscheidende Rolle. Dieser Zusammenhang wurde in ii) von Hilfssatz 1 aus [1] formuliert.

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Cogroups and Suspensions*

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Introduction

It has been known for a long time that any $(n-1)$-connected CW-complex of dimension $\leqq 2n-1$ has the homotopy type of a suspension. More recently, it has been proved [4] that an $(n-1)$-connected CW-complex X of dimension $\leqq 3n-3$ has the homotopy type of a suspension if, and only if, it carries a comultiplication; moreover, the homotopy equivalence relating X to the suspension is primitive, i.e., homomorphic with respect to the given comultiplication on X and suspension comultiplication. The main purpose of this paper is to achieve the next step: We prove that an $(n-1)$-connected CW-complex of dimension $\leqq 4n-5$ has the primitive homotopy type of a suspension if, and only if, it carries a comultiplication which is homotopy associative. Thus, at each stage an extra assumption is introduced and the dimension restriction is relaxed by $n-2$. Our proof splits into two parts: First, without any connectivity or dimension assumption, we show that a space carrying a homotopy associative comultiplication with inversion is a coalgebra over the cotriple $\Sigma\Omega$ (in the sense of [5]), thus answering a question [7; p. 213] which had been open for some time; then, a coalgebra satisfying appropriate connectivity and dimension assumptions is shown to have the primitive homotopy type of a suspension.

Dual results, elucidating the relationship between homotopy associative H-spaces and loop spaces, have been obtained, essentially, by means of two methods: that of the infinite reduced product of James, and that provided by the Sugawara-Dold-Lashof quasi-fibration (see, e.g., [7; p. 202] and [10, 11], respectively). None of these methods dualizes to the present situation, and this may account for the great delay in its solution; similarly, the arguments used here do not seem to dualize and, hence, do not lead to alternative proofs of the known results on H-spaces. One main difference between the two situations is the following: there is a 1:1 correspondence between homotopy classes of comultiplications on X and homotopy classes of coretractions $X \to \Sigma\Omega X$, whereas the set of homotopy classes of multiplications on an H-space Y is usually smaller than the set of homotopy classes of retractions $\Omega\Sigma Y \to Y$ (if $Y = S^3$, the former contains 12 elements whereas the latter is infinite).

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1. Comultiplications and Coretractions

We consider spaces of the based homotopy type of a CW-complex; all maps and homotopies are supposed to preserve base-points; in any space, the base-point is denoted by $*$. It follows from [8] that the constructions we will perform do not lead outside the class of spaces considered. A *comultiplication* on X is a map σ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \vee X \\ & {}_{\Delta}\searrow \quad \swarrow_{j} & \\ & X \times X & \end{array}$$

homotopy commutes; Δ is the diagonal map, and j the inclusion of the axes into the Cartesian product. With any space X we may associate $\Sigma\Omega X$, the (reduced) suspension of the loop space of X; the natural projection $p\colon \Sigma\Omega X \to X$ is given by $p\langle s, \omega\rangle = \omega(s)$. A *coretraction* is a map $\gamma\colon X \to \Sigma\Omega X$ such that $p \circ \gamma \simeq 1$, the identity map of X. For any A, the suspension comultiplication and coretraction

$$S\colon \Sigma A \to \Sigma A \vee \Sigma A \quad \text{and} \quad \Sigma e\colon \Sigma A \to \Sigma\Omega\Sigma A$$

are given by

$$\begin{aligned} S\langle s, a\rangle &= (\langle 2s, a\rangle, *) && \text{if } 0 \leqq 2s \leqq 1, \quad \text{and} \quad e(a)(t) = \langle t, a\rangle. \\ &= (*, \langle 2s-1, a\rangle) && \text{if } 1 \leqq 2s \leqq 2, \end{aligned}$$

The generalized homotopy groups of any space Y consist of homotopy classes of maps of arbitrary suspensions into Y with (possibly noncommutative) multiplication induced in the classical way by S. The following result is partially known (cf. [7; p. 209–212]).

Theorem 1.1. *The map $(p \vee p) \circ S\colon \Sigma\Omega X \to X \vee X$ induces monomorphisms of generalized homotopy groups and a bijection between homotopy classes of coretractions and homotopy classes of comultiplications on X.*

Proof. Let W be the fibred product of $\Delta\colon X \to X \times X$ and $j\colon X \vee X \to X \times X$; thus,

$$W = \{(x, \lambda, y) \in X \times (X \times X)^I \times (X \vee X) \mid \Delta(x) = \lambda(0), \lambda(1) = j(y)\}$$

and the projections $f\colon W \to X$, $g\colon W \to X \vee X$ are fibre maps. The map

$$\varphi\colon\ V = \{\xi \in X^I \mid \xi(0) = *\} \cup \{\xi \in X^I \mid \xi(1) = *\} \to W$$

given by

$$\varphi(\xi) = (\xi(\tfrac{1}{2}), \lambda, (\xi(1), \xi(0))) \quad \text{with} \quad \lambda(t) = \left(\xi\left(\frac{1+t}{2}\right), \xi\left(\frac{1-t}{2}\right)\right) \cdot$$

is a homeomorphism of the subspace V of the function space X^I onto W and, hence, $f\circ\varphi$ and $g\circ\varphi$ are fibre maps. The fibre $(g\circ\varphi)^{-1}(*)$ coincides with the subset ΩX of V, and the inclusion map $\Omega X\to V$ is obviously nullhomotopic. Therefore, $g\circ\varphi$ induces monomorphisms of generalized homotopy groups. It follows easily from [8] that the map

$$\begin{aligned} \varepsilon\colon \Sigma\Omega X\to V \text{ given by } \varepsilon\langle s,\omega\rangle &= \omega_{0,2s} \quad \text{if} \quad 0\leqq 2s\leqq 1,\\ &= \omega_{2s-1,1} \quad \text{if} \quad 1\leqq 2s\leqq 2,\end{aligned}$$

where $\omega_{a,b}(t)=\omega\big(a(1-t)+b\,t\big)$, is a homotopy equivalence; also, $g\circ\varphi\circ\varepsilon=(p\vee p)\circ S$ and $f\circ\varphi\circ\varepsilon=p$. This proves the first assertion. For any coretraction γ, $(p\vee p)\circ S\circ\gamma$ is obviously a comultiplication. The properties of fibred products imply that any comultiplication σ pulls back to a cross-section of $f\circ\varphi$, hence to a coretraction γ satisfying $(p\vee p)\circ S\circ\gamma\simeq\sigma$. Finally, if γ_1,γ_2 are coretractions with $(p\vee p)\circ S\circ\gamma_1\simeq(p\vee p)\circ S\circ\gamma_2$, then $\gamma_1\circ p\simeq\gamma_2\circ p$ by the first assertion, so that

$$\gamma_1\simeq\gamma_1\circ p\circ\gamma_1\simeq\gamma_2\circ p\circ\gamma_1\simeq\gamma_2.$$

A *co-H-space* is a space with a comultiplication; a *coretract* (of $\Sigma\Omega$) is a space with a coretraction. Let f be any map and consider the diagrams

$$\begin{array}{ccc} \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\psi} \\ X & \xrightarrow{f} & Y \end{array} \qquad \begin{array}{ccc} X\vee X & \xrightarrow{f\vee f} & Y\vee Y \\ {\scriptstyle\sigma}\uparrow & & \uparrow{\scriptstyle\tau} \\ X & \xrightarrow{f} & Y \end{array} \tag{1}$$

in which γ, ψ are coretractions and σ, τ the corresponding comultiplications; from the relations $f\circ p=p\circ\Sigma\Omega f$ and $(\Sigma\Omega f\vee\Sigma\Omega f)\circ S=S\circ\Sigma\Omega f$ we then obtain

$$(f\vee f)\circ\sigma\simeq(p\vee p)\circ S\circ\Sigma\Omega f\circ\gamma \quad\text{and}\quad \tau\circ f\simeq(p\vee p)\circ S\circ\psi\circ f.$$

Thus, the second square homotopy commutes if so does the first; since X is dominated by $\Sigma\Omega X$, the first assertion in 1.1 reveals that, conversely, the first square homotopy commutes if so does the second; we then say that f is *primitive* with respect to γ and ψ, or σ and τ.

2. Cogroups and Coalgebras

A comultiplication σ on X is homotopy *associative* if $(\sigma\vee 1)\circ\sigma\simeq(1\vee\sigma)\circ\sigma\colon X\to X\vee X\vee X$. An *inversion* for σ is a map η such that the composites

$$X\xrightarrow{\sigma}X\vee X\xrightarrow{1\vee\eta}X\vee X\xrightarrow{\nabla}X \quad\text{and}\quad X\xrightarrow{\sigma}X\vee X\xrightarrow{\eta\vee 1}X\vee X\xrightarrow{\nabla}X$$

are both nullhomotopic; ∇ is the folding map. A (homotopy) *cogroup* is a space carrying a homotopy associative comultiplication with inversion. A (homotopy) *coalgebra* (over the cotriple $(\Sigma\Omega, p, \Sigma e)$ in the sense of [5]) is a space X with a coretraction γ such that the diagram

$$\begin{array}{ccc} \Sigma\Omega X & \xrightarrow{\Sigma\Omega\gamma} & \Sigma\Omega\Sigma\Omega X \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\Sigma e} \\ X & \xrightarrow{\gamma} & \Sigma\Omega X \end{array} \tag{2}$$

homotopy commutes. The main purpose of this section is to prove that these two concepts are equivalent. For this we need some preparations.

Let $k \geqq 2$ be an integer, and let X_i be copies of a given space X, $1 \leqq i \leqq k$. Let $\pi\colon W \to X_1 \vee \cdots \vee X_k$ be the fibre map induced by the folding map from the contractible path-space fibration over X. Define

$$\Phi_k\colon \Sigma\Omega X_1 \vee \cdots \vee \Sigma\Omega X_{k-1} \to X_1 \vee \cdots \vee X_k$$

by

$$\begin{aligned} \Phi_k(*, \ldots, \langle s, \omega_i\rangle, \ldots, *) &= (*, \ldots, \omega_i(2s), \ldots, *) \quad \text{if} \quad 0 \leqq 2s \leqq 1, \\ &= (*, \ldots, *, \omega_k(2-2s)) \quad \text{if} \quad 1 \leqq 2s \leqq 2, \end{aligned}$$

where ω_k is the replica of ω_i in ΩX_k. (Thus, Φ_k is dual to the map $X_1 \times \cdots \times X_k \to \Omega\Sigma X_1 \times \cdots \times \Omega\Sigma X_{k-1}$ which sends $(x_1, \ldots, x_k)$ to $(\xi_1 - \xi_k, \ldots, \xi_{k-1} - \xi_k)$, where $\xi_i = e(x_i)$ and "$-$" denotes loop subtraction).

Lemma 2.1. *There is a homotopy equivalence θ yielding commutativity in the diagram*

$$\begin{array}{ccc} \Sigma\Omega X_1 \vee \cdots \vee \Sigma\Omega X_{k-1} & \xrightarrow{\Phi_k} & X_1 \vee \cdots \vee X_k \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\pi} & \\ & W & \end{array}$$

Moreover, ϕ_k induces monomorphisms of generalized homotopy groups.

Proof. Let PX denote the space of all paths in X emanating from the base-point. Then $W = \bigcup W_i$, where $1 \leqq i \leqq k$ and

$$W_i = \{((*, \ldots, x_i, \ldots, *), \xi) \mid x_i \in X_i, \xi \in PX, x_i = \xi(1)\}$$

is a subspace of $(X_1 \vee \cdots \vee X_k) \times PX$. Obviously, every W_i is homeomorphic to PX, hence contractible, and the intersection of any two of them is $(*, \ldots, *) \times \Omega X$. Let CA denote the (reduced) cone over any space A; a point in CA is denoted by sa, the vertex corresponds to $s=0$, and the base $1A$ is homeomorphic to A. Let

$$V = C\Omega X_1 \cup \cdots \cup C\Omega X_k$$

result from the disjoint union of the k cones by point-wise identification of their bases with ΩX. It follows easily from [8] that the map

$$\varepsilon\colon\ V \to W \text{ given by } \varepsilon(s\omega_i) = ((*, \dots, \omega_i(s), \dots, *), \omega_{is}),$$

where $\lambda_s(t) = \lambda(st)$ for any path λ, is a homotopy equivalence. The inverse of the homotopy equivalence $V \to \Sigma\Omega X_1 \vee \cdots \vee \Sigma\Omega X_{k-1}$ obtained by collapsing $C\Omega X_k$ to a point is the map

$$\psi\colon\ \Sigma\Omega X_1 \vee \cdots \vee \Sigma\Omega X_{k-1} \to V$$

given by

$$\begin{aligned} \psi(*, \dots, \langle s, \omega_i \rangle, \dots, *) &= 2s\,\omega_i \in C\Omega X_i && \text{if } 0 \leqq 2s \leqq 1, \\ &= (2-2s)\,\omega_k \in C\Omega X_k && \text{if } 1 \leqq 2s \leqq 2, \end{aligned}$$

where ω_k is the replica of ω_i in ΩX_k. Since

$$\pi((*, \dots, x_i, \dots, *), \xi) = (*, \dots, x_i, \dots, *),$$

the homotopy equivalence $\theta = \varepsilon \circ \psi$ behaves as asserted. Since the folding map has a cross-section, π induces monomorphisms of generalized homotopy groups and, therefore, so does Φ_k.

The preceding description of the homotopy type of W agrees with that in [9] where, however, no explicit description of the maps involved is given. Define $M\colon\ \Omega X \times \Omega X \to \Omega(X \vee X)$ by $M(\alpha, \beta) = (\alpha, *) + (*, \beta)$, where "+" denotes loop addition.

Theorem 2.2. *Let $\gamma\colon X \to \Sigma\Omega X$ be a coretraction, and let $\sigma\colon X \to X \vee X$ be the corresponding comultiplication. The following assertions are equivalent:*

(i) *σ is homotopy associative and admits an inversion η;*

(ii) *γ is primitive with respect to σ and S;*

(iii) *(X, γ) is a coalgebra;*

(iv) $\Sigma\Omega\sigma \circ \gamma \simeq \Sigma M \circ \Sigma\Delta \circ \gamma\colon X \to \Sigma\Omega(X \vee X)$.

Proof. Consider the diagram

$$\begin{array}{ccccc} & & X & & \\ & & \uparrow{\scriptstyle \iota} & & \\ \Sigma\Omega X & \xrightarrow{\Phi_2} & X \vee X & \xrightarrow{\nabla} & X \\ {\scriptstyle \psi}\uparrow & & \uparrow{\scriptstyle 1 \vee \eta} & & \\ X & \xrightarrow{\sigma} & X \vee X & & \end{array} \tag{3}$$

where l projects on the left term. By the definition of inversion, $\nabla\circ(1\vee\eta)\circ\sigma\simeq 0$ and, by the first assertion in 2.1, there results a map ψ yielding homotopy commutativity in the square. Hence, the relations $l\circ\Phi_2\simeq p$ and $l\circ(1\vee\eta)\circ\sigma\simeq 1$ imply that ψ is a coretraction. Introduce the diagram

$$\begin{array}{ccccccc} \Sigma\Omega X & \xrightarrow{S} & \Sigma\Omega X\vee\Sigma\Omega X & \xrightarrow{1\vee J} & \Sigma\Omega X\vee\Sigma\Omega X & \xrightarrow{\Phi_3} & X\vee X\vee X \\ \uparrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\psi\vee\psi} & & & & \\ X & \xrightarrow{\sigma} & X\vee X & & & & \end{array} \tag{4}$$

where J is suspension inversion given by $J\langle s,\omega\rangle=\langle 1-s,\omega\rangle$, and let

$$E\colon X\vee X\to X\vee X\vee X \quad\text{and}\quad T\colon X\vee X\to X\vee X$$

be inclusion into the end terms and the switching map, respectively. The definition of Φ_k yields

$$\Phi_3=(1\vee 1\vee\nabla)\circ(1\vee T\vee 1)\circ(\Phi_2\vee\Phi_2) \quad\text{and}\quad \Phi_2\circ J=T\circ\Phi_2,$$

and it follows easily that

$$\Phi_3\circ(1\vee J)=(1\vee T)\circ(1\vee\nabla\vee 1)\circ(\Phi_2\vee\Phi_2).$$

Let $\varphi=(1\vee\eta)\circ\sigma$. Since S and σ are both homotopy associative,

$$(1\vee\nabla\vee 1)\circ(\Phi_2\vee\Phi_2)\circ S\simeq E\circ\Phi_2 \quad\text{and}\quad (1\vee\nabla\vee 1)\circ(\varphi\vee\varphi)\circ\sigma\simeq E\circ\varphi.$$

Since (3) homotopy commutes, $\Phi_2\circ\psi\simeq\varphi$ so that

$$\Phi_3\circ(1\vee J)\circ S\circ\psi\simeq(1\vee T)\circ E\circ\varphi\simeq\Phi_3\circ(1\vee J)\circ(\psi\vee\psi)\circ\sigma. \tag{5}$$

(The preceding argument is suggested by the following: since (3) homotopy commutes, the dual of ψ sends $e(a)-e(b)$ into $a-b$ and, using the description of the dual of Φ_k, the maps $\Phi_3\circ(1\vee J)\circ S\circ\psi$ and $\Phi_3\circ(1\vee J)\circ(\psi\vee\psi)\circ\sigma$ appear as dual to the composites

$$x,y,z\to\xi-\zeta,\ \eta-\zeta\to\xi-\zeta,\ \zeta-\eta\to(\xi-\zeta)+(\zeta-\eta)\simeq\xi-\eta\to x-y,$$
$$x,y,z\to\xi-\zeta,\ \eta-\zeta\to\xi-\zeta,\ \zeta-\eta\to$$
$$x-z,\ z-y\to(x-z)+(z-y)\simeq x-y.)$$

Since $1\vee J$ is a homotopy equivalence and since X is dominated by a suspension, the second assertion in 2.1 enables us to cancel out $\Phi_3\circ(1\vee J)$ in (5); this proves that the square in (4) homotopy commutes. Since ψ is a coretraction, and since σ corresponds to γ, we now obtain

$$(p\vee p)\circ S\circ\psi\simeq(p\vee p)\circ(\psi\vee\psi)\circ\sigma\simeq\sigma\simeq(p\vee p)\circ S\circ\gamma,$$

and 1.1 implies $\psi\simeq\gamma$. Thus, the square in (4) homotopy commutes with ψ replaced by γ, and (i) $\Rightarrow$ (ii) is proved. Conversely, the primitivity of γ readily implies that σ is homotopy associative and, also, that $p\circ J\circ\gamma$ is an inversion for σ. The next equivalence follows from the discussion at the end of the previous section upon setting $Y=\Sigma\Omega X$, $f=\gamma$, $\psi=\Sigma e$, and $\tau=S$ in (1), and noting that S is the comultiplication corresponding to the coretraction Σe. To obtain the last equivalence, consider (2), note that

$$\Omega(p\vee p)\circ\Omega S\circ e=M\circ\Delta \quad\text{and}\quad \Omega(p\vee p)\circ\Omega S\circ\Omega\gamma\simeq\Omega\sigma,$$

and observe that the map $\Omega(p\vee p)\circ\Omega S$, hence also its suspension, has a left homotopy inverse as implied by the fact that $\Omega X\to\Sigma\Omega X\to X\vee X$, in the proof of 1.1, may be regarded as a fibration with fibre contractible in the total space.

Corollary 2.3. *In the category of spaces of the based homotopy type of a CW-complex and based homotopy classes of maps, $\Sigma\Omega Y$ is the "free" cogroup "cogenerated" by the object Y under p.*

Proof. Let X be a cogroup and $f\colon X\to Y$ any map. By 2.2, X has a primitive coretraction γ and so $\Sigma\Omega f\circ\gamma$ is a homomorphism satisfying $p\circ\Sigma\Omega f\circ\gamma\simeq f$. Moreover, if $h\colon X\to\Sigma\Omega Y$ is any homomorphism satisfying $p\circ h\simeq f$, then, as in (1), $\Sigma e\circ h\simeq\Sigma\Omega h\circ\gamma$ so that $h=\Sigma\Omega p\circ\Sigma e\circ h\simeq\Sigma\Omega f\circ\gamma$.

3. Coalgebras and Suspensions

Consider the diagrams

$$\begin{array}{ccc} D & \xrightarrow{h} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & L \end{array} \qquad\qquad \begin{array}{ccccc} \Sigma D & \xrightarrow{\Phi} & W & \xrightarrow{k} & \Sigma B \\ & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Sigma\beta} \\ & & \Sigma A & \xrightarrow{\Sigma\alpha} & \Sigma L \end{array} \tag{6}$$

where α and β are any maps, D and W are the fibred products of α,β and $\Sigma\alpha,\Sigma\beta$ respectively, and

$$\Phi\langle s,(a,\lambda,b)\rangle=(\langle s,a\rangle,\langle s,\lambda\rangle,\langle s,b\rangle) \quad\text{if } \alpha(a)=\lambda(0),\ \lambda(1)=\beta(b);$$

g,h,j,k are the projections, and $\langle s,\lambda\rangle$ stands for the path in ΣL given by $\langle s,\lambda\rangle(t)=\langle s,\lambda(t)\rangle$.

Lemma 3.1. *Suppose A and B are 0-connected. If α is p-connected and β is q-connected, then Φ is $(p+q+1)$-connected $(p\geqq 1, q\geqq 1)$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} \Sigma D & \xrightarrow{\Sigma h} & \Sigma B & \longrightarrow & \Sigma B\cup C\Sigma D & \xleftarrow{\eta} & \Sigma(B\cup CD) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \varphi} & & \\ W & \xrightarrow{k} & \Sigma B & \longrightarrow & \Sigma B\cup CW & & \downarrow{\scriptstyle \Sigma\theta} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \Sigma\beta} & & \downarrow{\scriptstyle \psi} & & \\ \Sigma A & \xrightarrow{\Sigma\alpha} & \Sigma L & \longrightarrow & \Sigma L\cup C\Sigma A & \xleftarrow{\eta} & \Sigma(L\cup CA) \end{array}$$

where the cones are attached in the obvious way, and φ, ψ, θ are the induced maps; η reverses the order of suspension and cone parameters. The relative Serre theorem applied to the squares in (6) reveals that θ and ψ are, respectively, homology $(p+q+1)$- and $(p+q+3)$-connected. Since each η is a homeomorphism and their square commutes, it follows that φ is homology $(p+q+2)$-connected, and the five lemma now implies that Φ is homology, hence also homotopy, $(p+q+1)$-connected.

For the next result we refer to the diagram

$$\begin{array}{ccccccc} E & \xrightarrow{\theta} & \Sigma\Omega X & \xrightarrow{p_X} & X & \xrightarrow{j} & \mathscr{X} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\Sigma\Omega f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\Gamma} & \Sigma\Omega Y & \xrightarrow{p_Y} & Y & = & Y \end{array}$$

where f is any map, Γ is a coretraction, and E the fibred product of Γ and $\Sigma\Omega f$ with projections ψ and θ; $\mathscr{X}, \pi$, and j result by converting f into a homotopically equivalent fibre map so that

$$\mathscr{X} = \{(x, \eta) \in X \times Y^I \mid f(x) = \eta(1)\}, \quad \pi(x, \eta) = \eta(0), \quad j(x) = (x, \eta_x),$$

where $\eta_x(t) = f(x)$.

Lemma 3.2. *If X is $(n-1)$-connected and f is m-connected, $m \geqq n-1 \geqq 1$, then $p_X \circ \theta$ is $(m+n-1)$-connected.*

Proof. Since j is a homotopy equivalence, the connectivity of $p_X \circ \theta$ coincides with that of $j \circ p_X \circ \theta$. The latter is homotopic to the map Φ: $E \to \mathscr{X}$ given by

$$\Phi(y, \lambda, z) = \big(p_X(z), p_Y \circ \lambda\big) \quad \text{where} \quad \Gamma(y) = \lambda(0), \quad \lambda(1) = \Sigma\Omega f(z),$$

and $\pi \circ \Phi = p_Y \circ \Gamma \circ \psi$. Since ψ and π are both fibre maps, and since $p_Y \circ \Gamma \simeq 1$, the five lemma reveals that the connectivity of Φ coincides with that of its restriction φ: $\psi^{-1}(*) \to \pi^{-1}(*)$. For any map g: $A \to B$, let $F(g)$ denote the fibre space over A induced by g from the contractible path space fibration over B. Since φ clearly coincides with the map $F(\Sigma\Omega f) \to F(f)$ induced by p_X and p_Y, its connectivity is easily seen to coincide with that of the map $F(p_X) \to F(p_Y)$ induced by f. It is well known [1] that there is a natural homotopy equivalence of the join $\Omega A * \Omega A$ into $F(p_A)$, and so the connectivity of φ coincides with that of $\Omega f * \Omega f$ which is $m+n-1$.

We write $\dim X \leqq r$ to indicate that the homotopy type of X contains an r-dimensional CW-complex. Also, γ is a fixed coretraction on X with respect to which primitivity may be defined.

Theorem 3.3. *Suppose X is $(n-1)$-connected and* $\dim X \leqq 4n-5$, $n \geqq 2$. *If (X, γ) is a coalgebra, then X has the primitive homotopy type of a suspension.*

Proof. Set $D_1 = \Omega X$ and $\gamma_1 = \gamma$. Suppose D_i and γ_i: $X \to \Sigma D_i$ are defined for some $i \geqq 1$, and introduce the diagrams

$$\begin{array}{ccc} D_{i+1} & \xrightarrow{h_i} & \Omega X \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle \Omega\gamma_i} \\ D_i & \xrightarrow{e_i} & \Omega\Sigma D_i \end{array} \qquad \begin{array}{ccccccc} \Sigma D_{i+1} & \xrightarrow{\Phi_i} & W_i & \xrightarrow{\theta_i} & \Sigma\Omega X & \xrightarrow{p} & X \\ & & {\scriptstyle \psi_i}\downarrow & & \downarrow{\scriptstyle \Sigma\Omega\gamma_i} & & \downarrow{\scriptstyle \gamma_i} \\ & & \Sigma D_i & \xrightarrow{\Sigma e_i} & \Sigma\Omega\Sigma D_i & \xrightarrow{p_i} & \Sigma D_i \end{array}$$

where $e_i = e$, $p_i = p$, D_{i+1} and W_i are the fibred products of e_i, $\Omega\gamma_i$ and Σe_i, $\Sigma\Omega\gamma_i$, respectively, g_i, h_i, ψ_i, θ_i are the projections, and Φ_i is as in 3.1. Since (2) homotopy commutes, the universality of fibred products yields a map ε: $X \to W_1$ such that $\psi_1 \circ \varepsilon = \gamma_1 = \theta_1 \circ \varepsilon$. Since p is $(2n-1)$-connected, $\Omega\gamma_1$ is $(2n-3)$-connected and, since so is also e_1, 3.1 implies that Φ_1 is $(4n-5)$-connected. Therefore, and since $\dim X \leqq 4n-5$, ε lifts to a map γ_2 with $\Phi_1 \circ \gamma_2 \simeq \varepsilon$. By 3.2, $p \circ \theta_1$ is $(3n-3)$-connected and, since $p \circ \theta_1 \circ \Phi_1 \circ \gamma_2 \simeq 1$, γ_2 is $(3n-4)$-connected. Therefore, by 3.2, $p \circ \theta_2$ is $(4n-5)$-connected and, since e_2 is $(2n-3)$-connected, Φ_2 is $(5n-7)$-connected by 3.1; thus, $p \circ \theta_2 \circ \Phi_2$ is $(4n-5)$-connected. Since the homology group $H_{4n-5}(X)$ is free, the homology decomposition result in [4] yields a connected CW-complex Y and a map f: $Y \to D_3$ such that

$$\begin{array}{rl} f_*\colon H_q(Y) \to H_q(D_3) & \text{is isomorphic for } q < 4n-6, \\ (p \circ \theta_2 \circ \Phi_2 \circ \Sigma f)_*\colon H_{4n-5}(\Sigma Y) \to H_{4n-5}(X) & \text{is isomorphic,} \\ H_q(Y) = 0 & \text{for } q \geqq 4n-5. \end{array}$$

Therefore, and since $H_q(X) = 0$ for $q > 4n-5$, $F = p \circ \theta_2 \circ \Phi_2 \circ \Sigma f$: $\Sigma Y \to X$ is a homotopy equivalence. Since $\theta_1 \circ \Phi_1 = \Sigma h_1$ and $\psi_2 \circ \Phi_2 = \Sigma g_2$, $\gamma \circ F \simeq \Sigma(h_1 \circ g_2 \circ f)$ and the relation $\gamma \circ F \simeq \Sigma\Omega F \circ \Sigma e$, the primitivity of F, follows now from 2.3 since both sides are homomorphisms $\Sigma Y \to \Sigma\Omega X$ whose compositions with p are homotopic.

Remark 3.4. The preceding proof reveals that the restriction on $\dim X$ could be relaxed to $\dim X \leqq k(n-2)+3$ provided all maps in the sequence $\gamma_1, \ldots, \gamma_{k-2}$ were available. We have no neat formulation of the assumptions yielding this sequence for $k > 4$. In case X is a suspension, and γ is suspension coretraction, the entire sequence is easily constructed, as expected. Also, the same type of argument reveals that, under appropriate connectivity and dimension assumptions, coalgebras over the (obviously defined) cotriple $\Sigma^k \Omega^k$ have the primitive homotopy type of k-fold suspensions.

Corollary 3.5. *Suppose X is an $(n-1)$-connected co-H-space of dimension $\leqq 4n-5$, $n \geqq 2$. Then, X has the primitive homotopy type of a suspension if and only if it is homotopy associative.*

This is an immediate consequence of 2.2, 3.3, and of the following

Proposition 3.6. *A 1-connected co-H-space X has an inversion on each side; the two are homotopic if X is homotopy associative.*

Proof. Dualize the classical argument [7; Prop. 17.3] yielding inversions on 0-connected H-spaces, noting that the maps

$$X \vee X \xrightarrow{1 \vee \sigma} X \vee X \vee X \xrightarrow{\nabla \vee 1} X \vee X$$

and

$$X \vee X \xrightarrow{\sigma \vee 1} X \vee X \vee X \xrightarrow{1 \vee \nabla} X \vee X$$

induce isomorphisms of homology groups and, therefore, have homotopy inverses provided $\pi_1(X \vee X)=0$.

4. Examples

For any odd prime p, let $S^3 \cup e^{2p+1}$ result by attaching a $(2p+1)$-cell to the 3-sphere via an element of order p in $\pi_{2p}(S^3)$; it is known [3] that these are co-H-spaces which fail to have the homotopy type of a suspension. Since $7=4.3-5$, Corollary 3.5 implies

Proposition 4.1. *No comultiplication on $S^3 \cup e^7$ is homotopy associative.*

The similar result for any p was first proved in [2] by means of entirely different arguments. Any element of order 9 in $\pi_{15}(S^5)$ has vanishing Hilton-Hopf invariants [6] but fails to be a suspension. Therefore, by [3], attaching a 16-cell to S^5 by means of such an element produces a co-H-space which, again, fails to have the homotopy type of a suspension.

Conjecture 4.2. *$S^5 \cup e^{16}$ admits homotopy associative comultiplications.*

If it were true, this would reveal that the dimension restriction in 3.5 is best possible. That some dimension restriction is, anyhow, necessary in 3.5 is shown by the next result.

Proposition 4.3. *$(S^3 \vee S^{15}, \sigma)$ fails to have the primitive homotopy type of a suspension for at least 16 homotopy classes of homotopy associative comultiplications σ.*

Proof. Let $\alpha \in \pi_{15}(S^5)$ be fixed, and let $\beta = H_0(\alpha) \in \pi_{15}(S^9)$ be its first Hilton-Hopf invariant. For any space Y and any elements $f, g \in \pi_3(Y)$, let $p = p(f, g) = [f, g] \in \pi_5(Y)$ and

$$\Phi(f, g) = p \circ \alpha + [f, [f, p]] \circ \beta + [g, [f, p]] \circ \beta + [g, [g, p]] \circ \beta \in \pi_{15}(Y),$$

where $[\,,]$ denotes the Whitehead product. Let $X = S^3 \vee S^{15}$, and let

$$i_1, i_2 \in \pi_3(X \vee X), \qquad h_1, h_2, h_3 \in \pi_3(X \vee X \vee X),$$
$$j_1, j_2 \in \pi_{15}(X \vee X), \qquad k_1, k_2, k_3 \in \pi_{15}(X \vee X \vee X)$$

be the obvious inclusions. Define a map $\sigma\colon X \to X \vee X$ by

$$\sigma | S^3 = i_1 + i_2, \qquad \sigma | S^{15} = j_1 + j_2 + \Phi(i_1, i_2).$$

Since $p(i_1, i_2)$ is killed by inclusion into $X \times X$, σ is a comultiplication. Naturality of the Whitehead product reveals that

$$(\sigma \vee 1) \circ \sigma | S^{15} = k_1 + k_2 + k_3 + \Phi(h_1, h_2) + \Phi(h_1 + h_2, h_3),$$
$$(1 \vee \sigma) \circ \sigma | S^{15} = k_1 + k_2 + k_3 + \Phi(h_2, h_3) + \Phi(h_1, h_2 + h_3).$$

Hence, σ is homotopy associative if and only if $D = 0$, where

$$D = \Phi(h_1, h_2) + \Phi(h_1 + h_2, h_3) - \Phi(h_2, h_3) - \Phi(h_1, h_2 + h_3) \in \pi_{15}(X \vee X \vee X).$$

Suppose $H_1(\alpha) = H_2(\alpha) = 0 \in \pi_{15}(S^{13})$, where H_1 and H_2 are the next Hilton-Hopf homomorphisms, so that [6; p. 166]

$$([h_m, h_n] + [h_q, h_r]) \circ \alpha$$
$$= [h_m, h_n] \circ \alpha + [h_q, h_r] \circ \alpha + [[h_m, h_n], [h_q, h_r]] \circ \beta.$$

For dimension reasons, β is a suspension and, hence, fulfils the left-distributive law; also, $\beta = -\beta$ since $\pi_{15}(S^9) = Z_2$ [12]. Therefore, using the bilinearity of Whitehead products, we may now express D as a sum of 4-fold Whitehead products each composed with β and, signs being irrelevant after composition, the relation $D = 0$ follows readily by repeated application of the Jacobi identity. Since $H_1 = -H_2$ [6; p. 170] and $\pi_{15}(S^5) = Z_{72} \times Z_2$, $\pi_{15}(S^{13}) = Z_2$ [12], there are at least 72 elements $\alpha \in \pi_{15}(S^5)$ with $H_1(\alpha) = H_2(\alpha) = 0$. Also, since

$$[i_1, i_2],\ [i_1, [i_1, [i_1, i_2]]],\ [i_2, [i_1, [i_1, i_2]]],\ [i_2, [i_2, [i_1, i_2]]] \qquad (7)$$

are all "basic" products, the main result in [6] implies that distinct α's yield homotopically distinct comultiplications σ. We have thus produced a set C of at least 72 homotopy classes of homotopy associative comultiplications on $S^3 \vee S^{15}$. Moreover, since every product in (7) contains an even number of factors, each $\sigma \in C$ satisfies

$$\sigma \circ \theta \simeq (\theta \vee \theta) \circ \sigma \quad \text{where} \quad \theta = (-1) \vee 1 \colon S^3 \vee S^{15} \to S^3 \vee S^{15}. \qquad (8)$$

The result will follow if we prove that $(S^3 \vee S^{15}, \sigma)$ has the primitive homotopy type of a suspension for at most 56 homotopy classes of comultiplications σ satisfying (8). Let σ be a fixed comultiplication, and suppose that $(S^3 \vee S^{15}, \sigma)$ has the primitive homotopy type of $(\Sigma Y, S)$ for some arbitrary space Y. Then, it follows easily that the singular polytope $K(Y)$ of Y is connected, and that $(S^3 \vee S^{15}, \sigma)$ also has the primitive homotopy type of $(\Sigma K(Y), S)$. The homology decomposition argument in [3; p. 443] reveals now that $(S^3 \vee S^{15}, \sigma)$ also has the primitive homotopy type of $(\Sigma A, S)$ for some 1-connected CW-complex A. Again, an easy homology decomposition argument reveals that such an A has the homotopy type of $S^2 \cup e^{14}$, where the cell is attached by means of some element in the kernel of the suspension $\pi_{13}(S^2) \to \pi_{14}(S^3)$. According to [12; Ch. VII and XIII], this kernel is isomorphic to $Z_4 \times Z_3$; also, $S^2 \cup e^{14}$ has the same homotopy type whether the cell is attached by means of α or of $-\alpha$. Therefore, the 12 elements in the kernel yield at most 7 possible homotopy types A_m for A, and the diagram

$$\begin{array}{ccc} S^3 \vee S^{15} & \xrightarrow{\sigma} & S^3 \vee S^{15} \vee S^3 \vee S^{15} \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle h \vee h} \\ \Sigma A_m & \xrightarrow{S} & \Sigma A_m \vee \Sigma A_m \end{array} \tag{9}$$

homotopy commutes for some $m \in \{1, \ldots, 7\}$ and some homotopy equivalence h. Since $\pi_{15}(S^3) = Z_2 \times Z_2$ [12], the formulae

$$\varphi | S^3 = i \quad \text{and} \quad \varphi | S^{15} = \pm j + i \circ f,$$

where i, j are the obvious inclusions and $f \in \pi_{15}(S^3)$, define 8 self-equivalences φ_n of $S^3 \vee S^{15}$. For each m, select a homotopy equivalence $h_m: \Sigma A_m \to S^3 \vee S^{15}$. Then, $h \simeq \xi \circ h_m$ for some self-equivalence ξ of $S^3 \vee S^{15}$ and, clearly

$$\xi \simeq \varphi_n \quad \text{or} \quad \xi \simeq \theta \circ \varphi_n \tag{10}$$

for some $n \in \{1, \ldots, 8\}$; here, θ is as in (8). Let k_m, η, and ψ_n be homotopy inverses of h_m, ξ, and φ_n, respectively. Since (9) homotopy commutes

$$\sigma \simeq (\xi \vee \xi) \circ (h_m \vee h_m) \circ S \circ k_m \circ \eta$$

and, since $\theta \circ \theta = 1 \vee 1$, (8) and (10) readily imply

$$\sigma \simeq (\varphi_n \vee \varphi_n) \circ (h_m \vee h_m) \circ S \circ k_m \circ \psi_n$$

with $1 \leqq m \leqq 7$, $1 \leqq n \leqq 8$. Thus, any σ satisfying (8) and for which $(S^3 \vee S^{15}, \sigma)$ has the primitive homotopy type of a suspension is homotopic to one of 56 comultiplications.

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A Remark on the Sato-Tate Conjecture

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Let k be a number field, and let A be an abelian curve defined over k which has no complex multiplications (over any extension of k). At a prime $\mathfrak{p}$ of k where A has good reduction, the non-trivial part of the zeta-function of A may be written

$$(1-a_{\mathfrak{p}} N\mathfrak{p}^{-s}+N\mathfrak{p}^{1-2s})^{-1}=(1-\varepsilon_{\mathfrak{p}} N\mathfrak{p}^{\frac{1}{2}-s})^{-1}(1-\bar{\varepsilon}_{\mathfrak{p}} N\mathfrak{p}^{\frac{1}{2}-s})^{-1},$$

where $1+N\mathfrak{p}-a_{\mathfrak{p}}$ is the number of points on the reduction of A modulo $\mathfrak{p}$ which are rational over the residue field of $\mathfrak{p}$ and $\varepsilon_{\mathfrak{p}}$ has absolute value 1. Following Tate [3], we define

$$L_0(s)=\prod_{\mathfrak{p}}(1-N\mathfrak{p}^{-s})^{-1},$$

$$L_\nu(s)=\prod_{\mathfrak{p}}(1-\varepsilon_{\mathfrak{p}}^{\nu} N\mathfrak{p}^{-s})^{-1}(1-\bar{\varepsilon}_{\mathfrak{p}}^{\nu} N\mathfrak{p}^{-s})^{-1}$$

for $\nu\geqq 1$ and $\sigma=\mathrm{Re}(s)>1$, and where (as in the rest of this paper) $\mathfrak{p}$ runs over all primes of k where A has good reduction. Note that $L_0(s)$ is the zeta-function of k, except for a finite number of factors, and so is entire except for a simple pole at $s=1$. In general, nothing is known about analytic continuation of the functions $L_\nu(s)$. Let us also define functions $M_\nu(s)$ for $\sigma>1$, $\nu\geqq 1$, by

$$M_\nu=L_\nu L_{\nu-2} L_{\nu-4}\cdots =\begin{cases} L_0 L_2 L_4\cdots L_\nu & (\nu \text{ even})\\ L_1 L_3\cdots L_\nu & (\nu \text{ odd}).\end{cases}$$

According to Tate's conjectures [3], $M_\nu(s)$ should be holomorphic and non-zero on the line $\sigma=1$, for $\nu\geqq 1$. In this paper we show that this would follow if we knew (as unfortunately we do not) the analytic continuation of the functions $M_\nu(s)$:

Theorem. Suppose that $M_\nu(s)$ is holomorphic in $\sigma>\frac{1}{2}-\delta$, with $\delta>0$, for all $\nu\geqq 1$. Then $M_\nu(s)$ has no zeroes on the line $\sigma=1$.

It would then follow, by Serre [2, p. 1–26], that the angles $\theta_{\mathfrak{p}}$, where $\varepsilon_{\mathfrak{p}}=\exp(\pm i\theta_{\mathfrak{p}})$, $0\leqq\theta_{\mathfrak{p}}\leqq\pi$, are uniformly distributed in $[0,\pi]$ relative to the measure $\frac{2}{\pi}\sin^2\theta\, d\theta$, which is the Sato-Tate conjecture for A.

Lemma. Let $f(s)$ be holomorphic in $\sigma>1-\delta$, except for a pole of order $e\geqq 0$ at $s=1$; assume $f(s)$ has at least one zero in $\sigma>1-\delta$. Suppose that $f(s)=\exp(g(s))$, with

$$g(s)=\sum_{n=1}^{\infty} c_n n^{-s},$$

for $\sigma>1$, with $c_n\geqq 0$ for all n. Then $f(s)$ has no zero of order $\geqq e$ on the line $\sigma=1$. In particular, $e\geqq 1$.

Proof. Suppose that $f(s)$ has a zero of order $\geqq e$ at $s=1+it_0$. Then it also has one at $s=1-it_0$, since $g(s)$, and hence $f(s)$, is real on the real axis. Then

$$h(s)=f(s)^2 f(s+it_0) f(s-it_0)$$

is regular at $s=1$. Now for $\sigma>1$, $h(s)=\exp(k(s))$, with

$$\begin{aligned} k(s) &= \sum_{n=1}^{\infty} c_n n^{-s}(2+n^{-it_0}+n^{it_0}) \\ &= \sum_{n=1}^{\infty} c_n n^{-s}(n^{it_0/2}+\overline{n^{it_0/2}})^2 \\ &= \sum_{n=1}^{\infty} c'_n n^{-s}, \end{aligned}$$

where $c'_n\geqq 0$. Let σ_0 be the first real zero of $h(s)$, $1-\delta<\sigma_0\leqq 1$, if it exists; otherwise take $\sigma_0=1-\delta$. Then $k(s)$ can be continued analytically to the segment (σ_0,∞) of the real axis, and so

$$k(s)=\sum_{n=1}^{\infty} c'_n n^{-s}$$

for $\sigma>\sigma_0$, by a standard lemma on Dirichlet series with real non-negative coefficients (cf., e.g., [1, p. 127]). But then, for real $s=\sigma>\sigma_0$,

$$\log|h(\sigma)|=\log h(\sigma)=\sum c'_n n^{-\sigma}\geqq 0,$$

and so $|h(\sigma)|\geqq 1$. Hence $h(\sigma_0)\neq 0$, by continuity, and so $\log h(s)=k(s)$ is analytic in $\sigma>1-\delta$, contrary to the hypothesis that $f(s)$ has a zero in $\sigma>1-\delta$.

Proof of Theorem. If ε has absolute value 1, put $\alpha_\nu=\varepsilon^\nu+\bar{\varepsilon}^\nu$ for $\nu\geqq 1$; then we have the inequality

$$(2\nu+1)+2\nu\alpha_1+(2\nu-1)\alpha_2+\cdots+\alpha_{2\nu}\geqq 0, \qquad (*)$$

which holds since the left side is $(1+\alpha_1+\alpha_2+\cdots+\alpha_\nu)^2$, as one checks by induction.

Now consider the function

$$H_\nu = (M_0 M_1 M_2 \cdots M_{2\nu-1})^2 M_{2\nu}$$
$$= L_0^{2\nu+1} L_1^{2\nu} L_2^{2\nu-1} \cdots L_{2\nu},$$

entire except for a pole of order $\leqq 2$ at $s=1$, and with at least one zero (coming from one zero of M_0 on $\sigma=\frac{1}{2}$). Then, for $\sigma>1$,

$$\log H_\nu(s) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{(2\nu+1)+2\nu\,\alpha(\mathfrak{p},m)+\cdots+\alpha(\mathfrak{p},2\nu m)}{m\,N\mathfrak{p}^{ms}}$$

where $\alpha(\mathfrak{p},m)=\varepsilon_{\mathfrak{p}}^m+\bar{\varepsilon}_{\mathfrak{p}}^m$. Thus, by (*), $\log H_\nu(s)$ is a real Dirichlet series with non-negative coefficients. Hence, by the lemma, $M_{2\nu-1}(s)$ has no zeroes on the line $\sigma=1$, since a non-real zero would give a double zero of $H_\nu(s)$, while a zero at $s=1$ would make $H_\nu(s)$ regular at $s=1$, contrary to the last statement in the lemma.

This proves the theorem for odd subscripts. For the even case, the proof is much the same, taking this time

$$G_\nu = M_0 M_2 M_4 \cdots M_{4\nu}$$
$$= L_0^{2\nu+1} L_2^{2\nu} L_4^{2\nu-1} \cdots L_{4\nu-2}^2 L_{4\nu},$$

entire except for a pole of order $\leqq 1$ at $s=1$, with at least one zero, and with $\log G_\nu(s)$ a real Dirichlet series with non-negative coefficients, again using (*). Hence $M_2, M_4, \ldots, M_{4\nu}$ have no zeroes on $\sigma=1$, which proves the theorem.

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Local Euler Characteristics

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One purpose of this paper is to give formulae counting the number of isomorphically distinct principal homogeneous spaces [6, IX § 4] of certain group schemes over certain rings. To this end we develop a rather precise "deformation theory" for fairly general group schemes (Prop. (3.5)). As a consequence, if R is a local ring of finite cardinality, and F is a quasi-projective, faithfully flat, commutative group scheme over R, we obtain formulae (4.4), (4.5) which give us this number, in terms of the number of rational points of F over R.

It follows from our theory that if D is the ring of integers in a finite extension of Q_p, and F is a group scheme over D, as above, then there are at most a finite number of distinct principal homogeneous spaces for F over D (4.7). The main theorem (8.1) computes this number in the case where F is a finite flat commutative group scheme over D. We obtain

$$\# H^1(D, F) = \# F(D) \cdot \|\mathrm{disc}_{G/R}\|^{1/g}$$

where g is the rank of G over R, $\mathrm{disc}_{G/R}$ is the relative discriminant, and $\| \ \|$ denotes normalized absolute value in D. That is, $\|a\| = (D : aD)$, for $a \in D$. We have denoted the set of isomorphism classes of principal homogeneous spaces F over D by $H^1(D, F)$. (The notation is reasonable. For given any noetherian D, this set may be identified in a natural manner with the one-dimensional cohomology of F over $\operatorname{Spec} D$, computed either for the (fppf)- or the (fpqf)-site [8, IV 6.3] using the theorem of faithfully flat descent for the fibre category of affine schemes [6, **1**, VIII 2.1]. If D is hensel local, it may be computed using the site T_{fin} (see § 3).)

We give two proofs of this formula. The first uses the fact (§ 5) that F can be imbedded in a smooth group scheme, and then reduces this formula to a problem in the theory of analytic groups. The second, which proves a slightly more general result, procedes by means of the deformation theory (§§ 1–4), and a formula which counts the number of rational points of F over artinian rings (§ 7).

In § 5 we exploit the above-mentioned imbedding result to obtain some general results concerning the flat cohomology of F.

We are thankful to A. Grothendieck for suggesting a proof of (5.1)(i), to Rasala for providing us with a proof of (6.4), and to the referee for his valuable suggestions.

§1

The purpose of §§1–4 is to set forth results about derivations and their "higher dimensional analogue", which pertain especially to group schemes. The five main sources for the theory of the cotangent complex is [1, 4, 9; O_{IV*}, §§18, 20; 12, 15]. We follow Quillen's treatment, mainly, extending it to topological algebras.

Throughout, all rings are commutative with identity; all topological rings are linear, that is, they possess a fundamental system of neighborhoods composed of ideals. We recall some basic definitions (for a more detailed approach see [9; O_{IV} §§18, 20]).

Let R and W be topological rings such that W is a topological R-algebra. If M is a topological W-module define $\mathrm{Dercont}_R(W, M)$ as the R-module of continuous R-derivations from W to M.

If R, W, and M are all discrete, call an *extension* of W by M an exact sequence

$$0 \to M \xrightarrow{i} E \xrightarrow{u} W \to 0$$

where u is a homomorphism of R-algebras such that $(\ker u)^2 = 0$, and where i induces an isomorphism of W-modules $M \xrightarrow{\sim} \ker u$. $\mathrm{Ker}\, u$ has the W-module structure induced by u.

Designate by $\mathrm{Exalcom}_R(W, M)$ the set of isomorphism classes of extensions of W by M. Thus we have defined a tri-functor, contravariant in R and W and covariant in M [see 9, O_{IV} 18.3.5]. As this functor preserves products in the variable M, it is a functor in abelian groups.

Let R and W be topological rings, and assume that $p: R \to W$ makes W into a topological R-algebra (i.e. p is continuous). Let M be a discrete W-module which is annihilated by an open ideal K_0 of W.

Let

$$\varepsilon_R(W, M) = \varinjlim \mathrm{Exalcom}_{R/J}(W/K, M)$$

where the limit is taken over pairs (J, K) of open ideals in R and W respectively such that $\rho(J) \subseteq K \subseteq K_0$. $\varepsilon_R(W, M)$ is denoted $\mathrm{Exalcotop}_R(W, R)$ in [9]. It has an R-module structure. (Note that in general the group does not have an interpretation as a set of extensions.)

We remark that under the conditions of the last paragraph there is a canonical isomorphism [9, O_{IV} 20.3.5].

$$\varinjlim \mathrm{Der}_{R/J}(W/K, M) \xrightarrow{\sim} \mathrm{Dercont}_R(W, M).$$

Here, of course, $\mathrm{Der}_A(B, M)$ is the A module of ordinary A-derivations from B to M.

§2. Cotangent Complexes

Quillen has defined a sequence of functors $D_R^q(W, M)$ for all $q \geqq 0$ which specialize for $q=0$, 1 to $\mathrm{Der}_R(W, M)$ and $\mathrm{Exalcom}_R(W, R)$ respectively. In checking certain facts that we need for the latter two functors, we have found it neater to prove things for $D_R^q(W, M)$, arbitrary q. We recall, very briefly, Quillen's construction [15, I. 2.8].

Let R be a ring, W an R-algebra (both discrete).

Quillen shows that there is a free simplicial R-algebra P. and homomorphisms such that

$$R \xrightarrow{i} \mathrm{P.} \xrightarrow{p} W \tag{2.1}$$

is a cofibrant factorization. This merely means that the homomorphism p of simplicial R-algebras (W being viewed as the "constant" simplicial algebra) induces isomorphisms on the homology groups of the underlying chain complexes [15, I. 2.1].

Such a diagram is seen to be unique up to simplicial homotopy. We say that P. is a projective simplicial R-algebra resolution of W. One then sets

$$\mathbf{L}D_{W/R} = D_{\mathrm{P.}/A} \underset{\mathrm{P.}}{\otimes} W.$$

This is the simplicial W-module obtained by applying the functor $Y \mapsto D_{Y/A} = \Omega^1_{Y/A}$ (= Y-module of A-differential of Y) dimension-wise to P. The underlying chain complex of $\mathbf{L}D_{W/R}$ is called the contangent complex of W and is unique up to chain homotopy equivalence. Define now, for M a W-module

$$D_R^q(W, M) = H^q(\mathrm{Hom}_W(\mathbf{L}D_{W/R}, M))$$

One checks directly [15, 3.12] that there are canonical isomorphisms

$$\mathrm{Der}_R(W, M) \xleftarrow{\sim} D_R^0(W, M),$$

$$\mathrm{Exalcom}_R(W, M) \xleftarrow{\sim} D_R^1(W, M).$$

In the next section we will need the following result.

(2.2) **Proposition.** *Let R be noetherian, and assume that W is a finite type R-algebra. Then there exists a free simplicial R-algebra resolution of W which is finitely generated in each dimension.*

The proof follows from an examination of the proof for the existence of a resolution [14, Ch. I § 4 Prop. 3]. (See also loc. cit. Remark 4 p. 4.11 and [15, Ch. II Remark 9.3].)

Before generalizing the preceding to topological rings, we should emphasize again that in the sequel we work only with D^0 and D^1, (which may be defined independently). In fact, the algebras W that we will consider will be relative complete intersections, and in this case it is known [15, Ch. II 10.7] that $D^q_R(W, M)=0$ for $q\geqq 2$. However we will never use this.

The Topological Case. Let R and W be topological rings, $p: R\to W$ a continuous homomorphism, and M a discrete W-module which is killed by an open ideal in W. As before define

$$D^q_R(W, M)=\varinjlim D^q_{R/J}(W/K, M)$$

where the limit is taken as in the definition of

$$\varepsilon_R(W, M)=\operatorname{Exalcotop}_R(W, M).$$

By the preceeding remarks one has canonical homomorphisms

$$D^0_R(W, M)\xrightarrow{\sim}\operatorname{Dercont}_R(W, M),$$

$$D^1_R(W, M)\xrightarrow{\sim}\varepsilon_R(W, M).$$

(2.3) **Proposition.** *Let $R\to W$ and $W\to V$ be continuous homomorphisms of topological rings, and let M be a discrete V-module annihilated by an open ideal of V. Then there is a long exact sequence*

$$0\to D^0_W(V, M)\to D^0_R(V, M)\to D^0_R(W, M)\to D^1_W(V, M)\to\cdots.$$

Proof. For R, W, V discrete the statement is known [15, § 4 Cor. 4.17]. The topological case is immediate from the exactness of $\varinjlim$.

A link between the discrete and topological cases is given by the following observation: Let W be an R-algebra, $I\subset W$ an ideal, and let $\hat{W}$ denote the topological completion of W with respect to the ideal I. Let M be a W-module annihilated by I. Then,

(2.4) $$D^q_R(W, M)\xrightarrow[\approx]{}D^q_R(\hat{W}, M)\qquad\text{for } q=0, 1.$$

This observation is immediate for $q=0$, and follows from [9] $\mathrm{O_{IV}}$ 18.4.2.1 for $q=1$.

Although we shall not use this for q other than 0, 1, one might note that (2.4) is valid for all q if W is noetherian. For then, using [15] Ch. II Th. 9.7 one finds that the direct limit, as n goes to infinity, of the cohomology of the complexes,

$$\operatorname{Hom}_W(\mathbf{L}D_{(W/I^n)/W}, M)$$

vanishes.

A Nominal Extension to the Global Case. Given a locally noetherian scheme X over R, and a quasi-coherent sheaf $\mathscr{F}$ which has the property that its support lies in an open subscheme of X which is affine, we can define $D^q_R(X, \mathscr{F})$ to be $D^q_R(W, M)$ where Spec(W) is an affine containing the support of $\mathscr{F}$, and M is the module of global sections of $\mathscr{F}$ over Spec(W). This definition is independent of the choice of affine, Spec(W), in the light of (2.4). Thus we have a tri-functor, $D^q_-(-,-)$ on the appropriate category of triples.

We conclude this section with

(2.5) **Lemma.** *Let X be a locally noetherian scheme flat over R, a (discrete) ring. Let R_0/R be an extension of (discrete) rings. Let X_0/R_0 denote the base change of X/R to R_0, and suppose given a quasi-coherent sheaf $\mathscr{F}_0$ over X_0. Finally, suppose that the triples $(R_0, X_0, \mathscr{F}_0)$ and $(R, X, \mathscr{F})$ satisfy the above condition, where $\mathscr{F}$ is the quasi-coherent sheaf on X which is the direct image of $\mathscr{F}_0$.*

Then, the natural map,

$$D^q_R(X, \mathscr{F}) \to D^q_{R_0}(X_0, \mathscr{F}_0)$$

is an isomorphism, for all q.

Proof. This is just [15] Ch. II 4.9.

§3

We now restrict our setting somewhat. Let R be a discrete noetherian local ring with I an ideal in R. Assume that M is an R-module annihilated by I. Let W be a topological R-algebra, and $f\colon W \to R/I$ a continuous R-algebra homomorphism. M may be considered as a W-module via f. If $R \to R'$ is a finite morphism (R' discrete), set $W' = W \underset{R}{\otimes} R'$ (with the tensor product topology, [9] 0_{I} 7.7.2) and $M' = M \underset{R}{\otimes} R'$. Define

$$d^q_W(R') = D^q_{R'}(W', M').$$

Thus, d^q_W is a functor on the category of finite R-algebras. Note that the notation omits M. There is a natural homomorphism of R'-modules,

$$d^q_W(R) \otimes R' \to d^q_W(R').$$

If a topological R-algebra possesses a system of ideal neighborhoods K of zero, such that W/K is of finite type over R, we shall say that W is topologically of finite type over R.

(3.1) **Proposition.** *Let R'/R be a flat extension of discrete rings. The homomorphism*

$$d^q_W(R) \otimes R' \to d^q_W(R')$$

is an isomorphism under either of the following assumptions:

(i) *R'/R is finite flat.*

(ii) *W is topologically of finite type over R.*

Proof. Recall [15, Ch. I (4.7)] that for B a discrete R-algebra, and R'/R flat there is an isomorphism

$$\mathbf{L}D_{B'/R'} \xleftarrow{\sim} \mathbf{L}D_{B/R} \underset{R}{\otimes} R'.$$

Under the assumption that either R'/R is finite, or B is of finite type over R, we have:

$$\begin{aligned} D^q_{R'}(B', M') &= H^q(\mathrm{Hom}_{B'}(\mathbf{L}D_{B'/R'}, M')) \\ &= H^q(\mathrm{Hom}_{B'}(\mathbf{L}D_{B/R} \otimes R', M')) \\ &= H^q(\mathrm{Hom}_B(\mathbf{L}D_{B/R}, M) \otimes R') \\ &= D^q_R(B, M) \otimes R'. \end{aligned}$$

The last equality uses flatness of R'/R. The equality preceding is immediate if R'/R is finite flat. If B is of finite type over R, then by Proposition (2.2) $\mathbf{L}D_{B/R}$ can be taken to be of finite type over B in each dimension and the equality again follows.

Now note that since R' is discrete, a fundamental system of ideals in W' is given by those of the form $\mathrm{Im}(K \otimes R')$ where K runs over a fundamental system in W (definition of tensor product topology). Moreover, since R' is flat

$$\frac{W'}{\mathrm{Im}(K \otimes R')} \xleftarrow{\sim} W/K \otimes R'.$$

Thus,

$$d^q_W(R') = \underset{R}{\varinjlim}\, D^q_{R'}(W/K \otimes R', M').$$

The hypothesis that W is topologically finitely generated implies that each quotient W/K is of finite type. Hence the above computation applies, and we have

$$\begin{aligned} d^q_W(R') &= \varinjlim D^q_{R'}(W/K \otimes R', M') \\ &= \varinjlim (D^q_R(W/K, M) \otimes R') \\ &= (\varinjlim D^q_R(W/K, M)) \otimes R' \\ &= d^q_W(R) \otimes R' \end{aligned}$$

where, by the preceeding remarks, all limits are taken over the set of open ideals K in W which annihilate M.

Let us introduce the "finite site" over R, which we will denote T_{fin}. The underlying category of T_{fin} consists in affine schemes, finite and flat over R. Covering families are taken to be arbitrary surjective families of flat morphisms.

It follows immediately from (3.1) and the theorem of faithfully flat descent for modules [6; **1**, VIII Cor. 1.5] that d^q_W is a sheaf of abelian groups for the topology T_{fin}. Moreover,

(3.2) **Corollary.** *d^q_W is an acyclic sheaf for the site T_{fin} in the sense that the* r-th *cohomology group of d^q_W computed for the site T_{fin} vanishes, for all $r>0$.*

Proof. It suffices to show the following: Given S/R any finite fla extension, and S'/S any finite, faithfully flat extension (i.e. a covering morphism for the site T_{fin}) then the associated Cech cochain complex $C^\bullet(S'/S, d^q_W)$ is an acyclic complex.

For one can then show that $H^r(T_{\text{fin}}/R', d^q_W)=0$ for $r>0$, and any R'/R which is finite flat, using the Leray Spectral Sequence for the Cech covering S'/S and induction on r (this argument is the well known "Lemma of Cartan").

To see that $C^\bullet(S'/S, d^q_W)$ is acyclic, note that this complex is isomorphic to $C^\bullet(S'/S, \mathbb{G}_a)\otimes_R d^q_W(R)$ by (3.1), and this latter complex is acyclic by [5, I, B Lemma 1.1].

The sheaves d^q_W $(q=0,1)$ will play an important role in the sequel.

Now assume, in addition to the above, that $I^2=0$. Let F be a covariant functor (set valued) defined on the category of finite R-algebras. Define [as in 8; III, §0] F^+ as the functor

$$F^+(R')=F(R'\underset{R}{\otimes} R/I).$$

The canonical morphism $R'\to R'\otimes R/I$ yields a morphism of functors i: $F\to F^+$.

Consider two cases where F is representable:

(a) *The Topological Case.* There is a separated and complete topological R-algebra W such that $F(R')\cong \operatorname{Hom}_{\text{cont-}R}(W, R')$.

Thus we may say: "F is representable by the affine formal scheme $\operatorname{Spf}(W)$" [9, I 10.1].

(b) *The Global Case.* There is a scheme X over R such that $F(-)\cong \operatorname{Hom}_R(-, X)$.

Let R'/R be an extension, and let the superscript $'$ denote base change to R'. In the topological case, if λ: $W\to R/I$ is a homomorphism, then [8, III Prop. 0.2] the set of continuous homomorphisms $W'\to R'$ which reduce to λ' when projected to R'/I' is (either empty, or) principal homogeneous over the abelian group $D^0_{R'}(W', I')$, where I' is regarded as a W'-module via λ'.

In the global case if we are given $\lambda \in F^+(R)$, then $i^{-1}(\lambda')$ is a principal homogeneous space over the abelian group $D^0_{R'}(X', \mathscr{F}')$ where $\mathscr{F}'$ is the quasi-coherent sheaf obtained by taking the direct image of the quasi-coherent sheaf associated to the module I' over $R/I \otimes R'$, via the morphism λ': $\mathrm{Spec}(R/I \otimes R') \to X'$.

If F is a group-valued functor, and λ the zero section, let $d^q_F(R')$ denote $D^q_{R'}(W', I')$ in the topological case, and $D^q_{R'}(X', \mathscr{F}')$ in the global case.

In either case, we obtain that the sequence,

$$0 \to d^0_F \to F \to F^+ \tag{3.3}$$

is an exact sequence of sheaves of groups for the site T_{fin}. (See [8, III Cor. 0.9].) That the morphism $d^0_F \to F$ is a homomorphism of group functors may be seen either by (loc. cit. 0.8) or from the next lemma, which will be needed in its generality, later on.

To set up some terminology, let $\Lambda \subset W$ denote the kernel of the zero section λ. If $K \subset \Lambda$ is any ideal neighborhood of zero in W, we may find an ideal neighborhood $J \subset K$ such that the group law of the functor F determines a (comultiplication) homomorphism

$$W/J \xrightarrow[\mu]{} W/K \otimes_R W/K.$$

Let i_1, i_2: $W/J \to W/K \otimes_R W/K$ denote the maps induced by

$$i_1\colon\ x \to x \otimes 1,$$
$$i_2\colon\ x \to 1 \otimes x.$$

If φ: $W_1 \to W_2$ is any homomorphism, let φ^q denote the induced homomorphism, $d^q_{W_2}(R') \to d^q_{W_1}(R')$ where M is any W_2-module.

(3.4) **Lemma.** *Let M be any R-module, regarded as a W-module via the zero section λ. Fixing this M, consider the three maps,*

$$i^q_1, i^q_2, \mu^q\colon\ d^q_{W/K \otimes_R W/K}(R') \to d^q_{W/J}(R').$$

Then

$$\mu^q = i^q_1 + i^q_2 \qquad \textit{for } q = 0, 1.$$

Proof. For $q=0$ this is immediate using that

$$\mu(x) = i_1(x) + i_2(x) + \Sigma\, a \otimes b$$

where the a's and b's occurring in the right-hand expression are all in the image of the augmentation ideal $\Lambda \subset W$.

Consider, now, $q=1$, and note that it suffices to prove it for $R=R'$. Let $P \xrightarrow{s} W/J$ be a homomorphism over R, of a polynomial ring P over R, which induces isomorphisms,

$$P/\mathscr{J} \xrightarrow{\approx} W/J, \qquad P/\mathscr{K} \xrightarrow{\approx} W/K$$

for ideals $\mathscr{J} \subset \mathscr{K}$ of P, and such that the composition $\varepsilon = \lambda\, s$ is the homomorphism sending every polynomial in P to its constant term. Let $\Gamma = \ker(\varepsilon)$. Find a lifting m: $P \to \bar{P} = P \otimes_R P$ of the homomorphism μ: $P/\mathscr{J} \to P/\mathscr{K} \otimes P/\mathscr{K}$ such that if we denote

$$\delta(x) = m(x) - 1 \otimes x - x \otimes 1$$

then $\delta(P) \subset \Gamma \otimes \Gamma$. This m is attainable merely by choosing, for each of the generating variables x_i of P, a lifting $m(x_i)$ of $\mu(x_i)$ such that $\delta(x_i) \in \Gamma \otimes \Gamma$. Let f denote, in turn, each of the three induced homomorphisms

$$i_1, i_2, m\colon \mathscr{J} \to P \otimes \mathscr{K} + \mathscr{K} \otimes P = \bar{\mathscr{K}} \subset \bar{P}$$

and let φ denote, correspondingly, the homomorphisms i_1, i_2, μ.

Then we have commutative diagrams,

$$\begin{array}{ccc} \mathrm{Hom}_P(\mathscr{J}, M) & \xleftarrow{f'} & \mathrm{Hom}_{\bar{P}}(\bar{\mathscr{K}}, M) \\ \downarrow & & \downarrow \\ D^1_R(W/J, M) & \xleftarrow{\varphi^1} & D^1_R(W/K \otimes W/K, M) \end{array}$$

where f' is the map induced by f, and the vertical surjective maps are the natural ones [e.g. 12, 3.1.2].

Consequently we will be done if we show $m' = i'_1 + i'_2$. But this equality follows from the observation that

$$\delta(\mathscr{J}) \subset \bar{\mathscr{K}} \cap (\Gamma \otimes \Gamma) = \Gamma \otimes \mathscr{K} + \mathscr{K} \otimes \Gamma \subset \bar{P}$$

and that any $\bar{P}$-homomorphism $g\colon \bar{\mathscr{K}} \to M$ annihilates $\Gamma \otimes \mathscr{K} + \mathscr{K} \otimes \Gamma$.

If F is a group-valued functor, representable in the global case by a flat scheme X/R, or, in the topological case, by a topological R-algebra W possessing a fundamental system of ideal neighborhoods $K \subset W$ such that W/K is flat over R, we shall describe a homomorphism of group-valued functors,

$$F^+ \xrightarrow{\eta} d^1_F.$$

We do this in the global case, the definition being quite analogous in the topological case, and we shall only give $\eta(u)$ for $u \in F^+(R)$, the description of $\eta(u)$ for $u \in F^+(R')$ being identical, after base change to R', for any flat extension R'/R. Let $R_0 = R/I$, and the subscript 0 will denote base change to R_0. The letter I will also denote the quasi-coherent sheaf over $\mathrm{Spec}(R_0)$ associated to the R_0-module I. The element u determines morphisms of schemes,

$$\begin{array}{ccc} \mathrm{Spec}(R_0) & \xrightarrow{u} & X \\ & {\scriptstyle u_0}\searrow \quad \nearrow & \\ & X_0 & \end{array}$$

By means of the group law on the scheme X_0, we may find an R_0-isomorphism $\varphi: X_0 \to X_0$ such that $\varphi u_0 = \lambda_0$, the zero-section of X_0.

Let $\tau \in D_R^1(R_0, I)$ denote the class of the extension

$$0 \to I \to R \to R_0 \to 0.$$

Consider the image $u_* \tau \in D_R^1(X, u_* I)$. By definition, $\eta(u)$ is the element of $d_F^1(R)$ which corresponds to $u_* \tau$ via the sequence of isomorphisms,

$$D_R^1(X, u_* I) \xrightarrow{\approx} D_{R_0}^1(X_0, u_{0*} I) \xrightarrow[\approx]{\varphi} D_{R_0}^1(X_0, \lambda_{0*} I) \xrightarrow{\approx} D_R^1(X, \lambda_* I) = d_F^1(R).$$

The unlabelled isomorphisms in the above line come from (2.5). That η is a homomorphism follows from an application of (3.4) for $q=1$. It is also straightforward to check that $\eta(u)=0$ if and only if u extends to an R-morphism, $\mathrm{Spec}(R) \to X$. Thus we have the exact sequence of presheaves of groups,

$$0 \to d_F^0 \to F \to F^+ \xrightarrow{\eta} d_F^1.$$

(3.5) **Proposition.** *(The Deformation Sequence.)*

Suppose F is represented either

(a) *(Global case:) by a group scheme X/R which is locally of finite type and flat, or*

(b) *(Topological case:) by an affine formal group scheme* $\mathrm{Spf}(W)$ *where W is a complete local topological R-algebra which is topologically of finite type, and topologically flat over R* [8, $\mathrm{VII_B}$ 1.3.1]. *Then*

$$0 \to d_F^0 \to F \to F^+ \xrightarrow{\eta} d_F^1 \to 0$$

is an exact sequence of sheaves for the site T_{fin}.

Proof. The map η is defined in both cases. To see that it is defined in case (b) one must assure oneself that W possesses a fundamental system of ideal neighborhoods K of zero, such that W/K is flat over R. This follows from the Dieudonne-Cartier classification theorem (e.g. loc. cit. (5.4), and explicitly (6.2) below). Thus we must show η to be surjective. Without loss of generality we may concentrate on the topological case. For, in the global case, we may replace X by $\hat{X}$, its formal completion along the zero-section, and surjectivity of the morphism η associated to $\hat{X}=\mathrm{Spf}(W)$ implies surjectivity of the morphism η associated to X.

To show surjectivity of the sheaf morphism η, for a topological R-algebra W, as in case (b), we shall show that for any artinian base ring (which we will again call R), and any element $x \in d^1_W(R)$, there is a finite flat extension R'/R such that x' is in the image of η (where $'$ denotes the base change to R').

Let x be represented by an extension $0 \to I \to E \to W/K \to 0$ where $K \subset W$ is an open ideal. Find a power series ring P in m variables over R and a surjective continuous R-homomorphism $P \to E \times_{W/K} W$. By composing this homomorphism with projection onto the second factor, we will obtain an isomorphism α: $P/J \to W$, where J is some open ideal in P. Note that J is transverse-regular, relative to R ([9, IV 19.2] provides the theory of transverse-regularity, relative to a base).

To see this, it suffices to see that W can be expressed in some way as the quotient of a power series ring over R by an ideal which is transverse-regular over R [9, IV 19.3.2], and this latter fact is well known (Explicitly (6.2) below).

Thus we may suppose J generated by a sequence $(f_1, \dots, f_r)$ $r \leqq m$ which is transverse-regular, relative to R. Consider the R-homomorphism obtained by projecting $P \to E \times_{W/K} W$ onto the first factor, β: $P \to E$. Then $\beta(f_i) = \gamma_i \in I$, and hence the γ_i may be regarded as elements of the ring R. Form the sequence $(g_1, \dots, g_r)$ of elements in P, $g_i = f_i - \gamma_i$, $i = 1, \dots, r$.

It follows immediately from [9, 0_{IV} 15.1.16] that this is a transverse-regular sequence, relative to R, in P. Thus, the R-algebra $V = P/(g_1, \dots, g_r)$ is a flat complete intersection [9, IV 19.3] over R.

We claim that if R'/R is a finite flat extension, then x' is in the image of η if and only if there is a (continuous) R-homomorphism $V \to R'$.

This will then conclude the proof of (3.5), for the transverse-regular sequence $(g_1, \dots, g_r)$ may be extended to a transverse regular sequence $(g_1, \dots, g_m)$ relative to R, and then $R' = V/(g_{r+1}, \dots, g_m)$ is finite and flat over R, and there is, quite evidently, an R-homomorphism $V \to R'$.

We shall establish the above claim only in the direction that we need it; the opposite direction may be seen by a mirror image of our argument. We shall also only consider the case $R' = R$; the general case reduces to this by a base change.

Suppose, then, φ: $V \to R$ is an R-homomorphism. Then composition with the natural projection $P \to V$ yields an R-homomorphism $\varphi: P \to R$, which we denote by the same letter. Since φ annihilates the ideal

$(g_1, \ldots, g_r)$ it follows that $\varphi | J = \beta$. We thus obtain a diagram,

$$\begin{array}{ccc} J & \xrightarrow{\beta} & I \\ \downarrow & & \downarrow \\ P & \xrightarrow{\varphi} & R \\ \downarrow & & \downarrow \\ W & \xrightarrow{u} & R/I \end{array}$$

where u is the induced R-homomorphism. We show that $x = \eta(u)$.

Let $K' \subset K$ be an open ideal in the kernel of u. Then x pulls back to an element represented by an extension given by the right-hand column in

$$\begin{array}{ccc} J & \xrightarrow{\beta} & I \\ \downarrow & & \downarrow \\ P & \longrightarrow & E' \\ \downarrow & & \downarrow \\ W & \longrightarrow & W/K' \end{array}$$

where we have surjectivity of the middle horizontal arrow, by construction. A comparison of the above two diagrams show that they induce a diagram,

$$\begin{array}{ccc} I & \xrightarrow{=} & I \\ \downarrow & & \downarrow \\ E' & \longrightarrow & R \\ \downarrow & & \downarrow \\ W/K' & \xrightarrow{u} & R/I \end{array}$$

which, after chasing through the definition of η, tells us that $x = \eta(u)$. q.e.d.

Remarks. We are thankful to the referee for providing us with the above definition of η which is an improvement on our original definition, and has allowed us to establish (3.5) in its present generality.

As the referee noted, it is plausible that (3.5) could be refined, in the case where F is a commutative group-valued functor, to yield an isomorphism in the appropriate derived category between the complex of sheaves $0 \to F \to F^+ \to 0$, and a complex of sheaves described entirely in terms of the cotangent complex of F.

(3.6) **Corollary.** *Let F be a commutative group-valued functor represented either*

(a) (Global case:) by a quasi-projective, faithfully flat group scheme X/R, or

(b) (Topological case:) by an affine formal group Spf(W) *where W is a complete local topological R-algebra which is topologically of finite type and topologically faithfully flat over R.*

Then we have the exact sequence:

$$0 \to d_F^0(R) \to F(R) \to F(R/I) \to d_F^1(R) \to H^1(R, F) \to H^1(R/I, F) \to 0$$

where cohomology is computed for the site T_{fin}.

Proof. In the light of (3.2), and (3.5) we must show that the natural map $k\colon H^1(R, F^+) \to H^1(R/I, F)$ is an isomorphism. We do this without assuming F commutative.

The quasi-projectivity assumption in case (a), which undoubtedly can be weakened, will be used to insure that any element of $H^1(R/I, F)$ may be represented as a scheme Y_0 over R/I which is a principal homogeneous space for X_0 over R/I, where the subscript 0 refers to base change to R/I [8, VIII Cor. 7.7].

We concentrate on case (a), the proof for case (b) being quite analogous.

Injectivity of k. An element $x \in H^1(R, F^+)$ may be expressed as a Cech cohomology class for a covering U/R [6, **4**, V (App.) 4.1].

Since

$$\check{H}^1(U, F^+) = \check{H}^1(U_0, F) \to H^1(R/I, F)$$

where the second map is an injection (loc. cit.), the map k is injective.

Surjectivity of k.

Let $x_0 \in H^1(R/I, F)$ be represented by the principal homogeneous space Y_0, as mentioned above. By [9, IV 19.3.9 (ii)], Y_0 is a flat complete intersection over R/I. We shall sketch a proof of the following extension property: There is a finite Zariski-open cover of Y_0, such that if we let Z_0 denote the union of its members, there is a flat complete intersection Z over R such that Z_0 is the base change of Z to R/I.

To see this, let $y \in Y_0$, and find an open neighborhood U_0 of y, which admits a closed immersion over R_0, $U_0 \to S_0$, where S_0 is smooth over R_0 and U_0 is cut out by a sequence of sections (f_i^0) which are transverse-regular, relative to R_0, at y. Invoking [9] IV 18.1.1, and possibly cutting down to a smaller open subscheme, we may suppose that $S_0 = S \times_R R_0$, where S is smooth over R, and the sequence (f_i^0) is the restriction of a sequence (f_i) of $\mathcal{O}_S$-sections. It follows from [9, 0_{IV} 15.1.16] that (f_i) is again transverse-regular, relative to R, at y. They therefore cut out a scheme U/R which is a flat, complete intersection over R, in a neighborhood of y.

Cut down to such a neighborhood, which we again call U. Let Z denote the disjoint union of a finite number of such U's so that the family of U_0's cover Y_0.

Since Z is a faithfully flat local complete intersection over R, we may find an R'/R which is faithfully flat, such that there is an R-morphism, $\mathrm{Spec}(R') \to Z$. For this R'/R we may express x_0 as a Cech cohomology class, $x_0 \in \check{H}^1(R'_0/R_0, F)$, since there is an R_0-morphism, $\mathrm{Spec}(R'_0) \to Y_0$. But $\check{H}^1(R'/R, F^+) = \check{H}^1(R'_0/R_0, F)$, and therefore x_0 is in the image of k. q.e.d.

§ 4. Euler Characteristics over Artinian Rings

(4.1) **Proposition.** *Let $P = R[x_1 \dots x_n]$, let J be an ideal in P and set $B = P/J$. Suppose that $J = (f_1 \dots f_k)$ and that $f_1 \dots f_k$ is a regular sequence for P. If M is any B-module there is an exact sequence*

$$0 \to D_R^0(B, M) \to M^n \xrightarrow{\Phi} M^k \to D_R^1(B, M) \to 0,$$

where Φ has the matrix $\Phi = (\partial f_i / \partial x_j)_{ij}$ (with entries in $\mathrm{Hom}_B(M, M)$).

Proof. By [15] Chapter I Proposition 5.4 there is an exact sequence

$$0 \to D_R^0(B, M) \to \mathrm{Der}_R(P, M) \to \mathrm{Hom}_B\left(\frac{J}{J^2}, M\right) \to D_R^1(B, M) \to 0.$$

It is well-known that the module of Kähler differentials $D_{P/R}$ is a free P module on the elements dx_i. Hence $\mathrm{Der}_R(R, M)$ identifies with M^n under the obvious map. On the other hand the assumption that $f_1 \dots f_k$ is a regular sequence implies that the map

$$\oplus^k B \to \frac{J}{J^2}$$

given by $(b_1 \dots b_k) \mapsto \Sigma f_i b_i$ is an isomorphism of B modules (immediate from one of the definitions of regularity [9, 0_{IV} 15.1]). Hence $\mathrm{Hom}_B(J/J^2, M)$ identifies in the obvious way with M^k. Finally as the map $\mathrm{Der}_R(P, M) \to \mathrm{Hom}_B(J/J^2, M)$ is restriction, one just checks through the isomorphism to obtain the Φ is as stated.

(4.2) **Corollary.** *If M is a B-module of finite cardinality C, $\#(D_R^0(B, M)) = \#(D_R^1(B, M)) \cdot C^{n-k}$.*

In the computations below we will stick to F as in case (a) above. We would get analogous computations for F as in case (b). Let, then, F denote a quasi-projective, flat group scheme over R, a local ring of finite cardinality. Let $m \subset R$ denote the maximal ideal, and $k = R/m$ the residue field. Let F_k denote the reduction of the group scheme F to k. Let F^0 denote the connected component subgroup scheme of F_k, and $F^{\text{ét}} = F_k/F^0$ the etale quotient group scheme.

Let $\chi(R', F)$ denote the Euler characteristic, $\# H^0(R', F)/\# H^1(R', F)$ which is defined if both numerator and denominator are finite.

(4.3) **Proposition.** *$\chi(R, F)$ is defined, and*

$$\chi(R, F) = c^t \chi(k, F) \quad \textit{where} \quad c = \operatorname{card}(m), \quad \textit{and} \quad t = \dim_R F.$$

Remarks. To compute $\chi(k, F)$, we first note that $H^1(k, F^0)$ is trivial. The reason for this is the following: Let $G = F^0$. Since k is perfect, G_{red} is again a group scheme. It is smooth and connected. Thus by [7, III § 11], $H^1(k, G_{\text{red}})$ may be computed using galois cohomology, and therefore it vanishes, by Lang's theorem [11]. If N is the kernel of the surjective morphism $G \to G_{\text{red}}$, then N is a finite connected group scheme. We shall show that $H^1(k, N)$ vanishes for any such group scheme over any perfect field k. For, a principal homogeneous space for N over k is represented by the spectrum of a finite dimensional algebra W over k, which has the property that the base change of W to the algebraic closure of k is a local ring. By the elementary structure theory for such algebras it follows that there is a k-homomorphism $W \to k'$, where k'/k is a finite purely inseparable field extension. Since k is perfect, $k' = k$, and therefore the principal homogeneous space has a section over k. Thus, by half-exactness of the functor H^1, we obtain that $H^1(k, G)$ is trivial, as was to be shown. Thus the natural map

$$H^1(k, F) \to H^1(k, F^{\text{ét}})$$

is an injection. Since the range, which is given by the one-dimensional galois cohomology of the finite galois module $F^{\text{ét}}$, is clearly a finite group, so is the domain, and $\chi(k, F)$ is therefore always defined. Now recall that for any commutative group scheme G over k, $H^q(k, G) = 0$ for $q \geqq 2$. This is so by the following argument. It is true for smooth commutative group schemes G, for then [7, III, § 11] the cohomology groups above may be computed as galois cohomology, and the above fact is well known for galois cohomology. One checks it for α_p and μ_p using the exact sequences,

$$0 \to \alpha_p \to \mathbb{G}_a \to \mathbb{G}_a \to 0,$$
$$0 \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

The assertion then follows, using the structure theory of commutative group schemes over k, and half-exactness of the functors $H^q(k, -)$.

Consequently, since $H^2(k, F^0)$ vanishes, one learns that the morphism $H^1(k, F) \to H^1(k, F^{\text{ét}})$ is an isomorphism, and therefore $\chi(k, F) = F^0(k)$.

As an application, we have the following useful formulae:

(4.4) $\# H^1(R, F) = c^{-t} \# F(R) / \# F^0(k)$.

(4.5) If F is a commutative finite flat group scheme over R, $\# H^1(R, F) = \# F(R)$.

Proof of (4.3). Since, by the preceding remarks, $\chi(k, F)$ is defined, we obtain the formula of (4.3) for R/m^n by induction on n, using (3.6) and (4.2).

An Application. Let D be the ring of integers in a field which is finite over Q_p.

By a generalization of the technique of § 7 below, Shuck has proven a theorem [23] which implies the following asymptotic formula: Let F be a flat scheme of finite type over D such that every closed point has dimension t in its fibre over D, and such that F is smooth over the generic point of Spec(D). Then there is a constant h such that for large enough n,

$$\# F(D/m^n) = h \operatorname{card}(m/m^n)^t$$

where $m \subset D$ is the maximal ideal.

Since the generic point of Spec(D) is of characteristic zero, any quasi-projective flat commutative group scheme F over D satisfies the hypotheses of the theorem of Shuck (using the theorem of Cartier [8, $\mathrm{VII_B}$ 3.3]). If one combines the above formula with (4.3), one learns that the order of $H^1(D/m^n, F)$ is finite, and independent of n, for large n. Therefore the maps,

$$H^1(D/m^{n+1}, F) \to H^1(D/m^n, F) \tag{4.6}$$

which are surjective by (3.6), are actually isomorphisms for sufficiently large n.

(4.7) **Corollary.** *For n large enough, the natural morphism,*

$$H^1(D, F) \to H^1(D/m^n, F)$$

is injective. Consequently, $H^1(D, F)$ *is finite.*

Proof. It suffices to show that $H^1(D, F) \to \varprojlim_n H^1(D/m^n, F)$ is injective. But this reduces to showing that a principal homogeneous space for F over D, which has a section over D/m^n for every n, actually has a section over D. This is an easy application of a theorem of Greenberg [3].

Remarks. We expect that the above corollary can be sharpened in many ways. For example, the case where $D = k[[t]]$, where k is a finite field, and F is smooth over the field of fractions of D, should be included. We also expect that the above morphism is actually an isomorphism, for n large enough. Finally, one should give a formula which describes the order of $H^1(D, F)$ in terms of elementary scheme-theoretic invariants of F.

When F is a finite flat commutative group scheme, we shall sharpen the above corollary in the indicated ways.

(4.8) **Corollary.** *Let D be a complete discrete valuation ring with finite residue field. Let F be a finite flat commutative group scheme over D, which is etale over the field of fractions of D. Then*

$$H^1(D, F) \to H^1(D/m^n, F)$$

is an isomorphism for large enough n.

Proof. Injectivity may be seen as in the proof of (4.7), with one change: One needn't make use of the theorem of Shuck. One need only notice directly, by an application of Newton's lemma [3, § 3] that $F(D) \to F(D/m^n)$ is an isomorphism for large n.

To show surjectivity, we must show that $H^1(D, F) \to \varprojlim_n H^1(D/m^n, F)$ is surjective. Suppose we are given an element in the inverse image. This amounts to being given a sequence of principal homogeneous spaces Y_n for F over D/m^n, for each n, such that Y_{n+1} reduces to Y_n over D/m^n. Let $Y_n = \mathrm{Spec}(W_n)$. Then the W_n's are free (say of rank r) over D/m^n, and we have (D/m^n)-algebra homomorphisms $W_{n+1} \otimes_D D/m^n \xrightarrow[\varepsilon_n]{\approx} W_n$ which preserve the "principal homogeneous space structure for F". Let $W = \varprojlim_n W_n$, where the limit is taken via the ε_n's. The D-algebra W is a free module of rank r over D, and clearly $Y = \mathrm{Spec}(W)$ represents a principal homogeneous space for F over D, which, as an element of $H^1(D, F)$, maps to the given element in the inverse image.

§ 5. Computation of Flat Cohomology of Certain Group Schemes

Let S be a locally noetherian scheme, and F any finite flat commutative group scheme over S.

(5.1) **Proposition.** (i) *If S is affine, there is an exact sequence of group schemes over S,*

$$0 \to F \to G_0 \to G_1 \to 0$$

where G_0 and G_1 are smooth commutative group schemes of finite type over S.

(ii) *If, as in* [7, III Appendix], *$p: S_{\mathrm{fppf}} \to S_{\text{ét}}$ denotes the morphism of the* fppf *site to the etale site over S, then $R^i p_*(F)$ vanishes for $i > 1$.*

(iii) *Let $S_0 \subset S$ be a closed subscheme such that F is etale over $S - S_0$. If Z is any scheme, let $d(Z)$ denote the smallest integer such that $H^r(Z_{\text{ét}}, E) = 0$ for all abelian torsion sheaves E, and $r > d(Z)$.*

Then $H^r(S_{\mathrm{fppf}}, F) = 0$ if $r > \max(d(S), d(S_0) + 1)$.

Proof. We first show that our entire proposition follows from the following assertion: If S is affine, there is a smooth commutative affine group scheme G of finite type over S, and an imbedding of F into G, over S.

For we may obtain the exact sequence of (i) by taking $G_0 = G$, and $G_1 = G/F$. Since G is affine of finite type, and F is proper, the quotient group scheme G_1 exists. [See 16, Th. (iii), or 8, $\mathrm{VI_B}$]. The group scheme G_1 is then again of finite type. One sees easily that it is formally smooth, since G is. Consequently G_1 is smooth [9, IV, 17.5.1].

To obtain (ii) one then uses the fact that $R^i p_* G_0 = R^i p_* G_1 = 0$ for $i > 0$ [7, III App.]. Since one has an exact sequence of the type (5.1)(i) over every affine open subscheme of S, one gets (ii).

One obtains (iii) from the Spectral Sequence,

$$H^r(S_{\text{ét}}, R^s p_* F) \Rightarrow H^{r+s}(S_{\text{fppf}}, F)$$

using (ii), plus the observation that $R^1 p_* F$ has support on S_0, for we may apply [7, III App.] to the smooth group scheme F over $S - S_0$.

To establish our initial assertion, we begin by recalling.

a) Let S be a noetherian scheme, X projective and flat over S, Y quasi-projective over S. Then $\underline{\mathrm{Hom}}_S(X, Y)$ is representable, and is locally of finite type over S. $\underline{\mathrm{Hom}}_S(X, Y)$ can be realized as an open subscheme of $\underline{\mathrm{Hilb}}_{X \times_S Y/S}$.

The proof of a) is [5, IV] Th. 3.1, plus variants 4a and 4c. That $\underline{\mathrm{Hom}}_S(X, Y)$ is locally of finite type follows from the fact that $\underline{\mathrm{Hilb}}_{X \times_S Y/S}$ is the direct sum of quasi-projective S-schemes (variant 4a), hence is locally of finite type.

If X is finite and flat over S, and Y is affine over S, then $\underline{\mathrm{Hom}}_S(X, Y)$ can be constructed in a purely elementary fashion, without recourse to Hilbert schemes (cf. loc.cit., Remark, pp. 24–25). In fact, one can reduce to the case where $S = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(B)$, and $X = \mathrm{Spec}(A)$, with A a free R-module.

Choose a basis of A, $e_1, \ldots, e_n$. Define the constants c_{ijk}, and a_i by the equations

$$e_i e_j = \sum_{k=1}^{n} c_{ijk} e_k \qquad c_{ijk} \in R,$$

$$1_A = \sum_{i=1}^{n} a_i e_i \qquad a_i \in R.$$

Let $A^\vee$ denote the R-module dual to A, with dual basis $T_1, \ldots, T_n$.

Let $\Sigma = \mathrm{Symm}_R(B \otimes_R A^\vee)$ be the symmetric R-algebra on the module $B \otimes_R A^\vee$. Let $I \subset \Sigma$ denote the ideal generated by the elements,

$$a_i - 1 \otimes T_i \qquad i = 1, \ldots, n,$$

$$b_1 b_2 \otimes T_k - \sum_{i,j=1}^{n} c_{ijk} b_1 \otimes T_i \otimes b_2 \otimes T_j$$

for all choices of $b_1, b_2 \in B$, $k = 1, \ldots, n$.

It is an exercise to check that the R-algebra $D=\Sigma/I$ represents the functor, $C \mapsto \mathrm{Hom}_{C\text{-alg}}(B \otimes C, A \otimes C)$. Thus, in this case, $\underline{\mathrm{Hom}}_S(X, Y)$ is represented by an affine scheme over S. Let us show the slightly modified assertion:

b) Let S be a noetherian affine scheme. Let X be finite and flat over S. Let Y be affine and of finite type over S. Then $\underline{\mathrm{Hom}}_S(X, Y)$ is affine and of finite type over S. Moreover, if Y is smooth, then so is $\underline{\mathrm{Hom}}_S(X, Y)$.

To see this, note that Y is quasi-projective [9, II 5.3.4i)] and we may apply a) to obtain that $\underline{\mathrm{Hom}}_S(X, Y)$ is locally of finite type. Since it is also affine over S, it is of finite type. Therefore, for the last assertion we must check that $\underline{\mathrm{Hom}}_S(X, Y)$ is formally smooth [9, IV 17.1.1]. Since $\underline{\mathrm{Hom}}_S(X, Y)$ is affine, it suffices to check formal smoothness using, as test objects, $\mathrm{Spec}(C/J) \subset \mathrm{Spec}(C)$, where C is an R-algebra, and J is an ideal of square zero. Letting $X=\mathrm{Spec}(A)$, $Y=\mathrm{Spec}(B)$, as before, we must check that

$$\mathrm{Hom}_{R\text{-alg}}(B, A \otimes C) \to \mathrm{Hom}_{R\text{-alg}}(B, A \otimes C/J)$$

is surjective, for all such C, and J. But the kernel of the surjective homomorphism, $A \otimes C \to A \otimes C/J$, is of square zero, and B is formally smooth, so the above is surjective, and we have shown b).

Note that if Y is a group scheme, so is $\underline{\mathrm{Hom}}_S(X, Y)$ via the group structure on $\underline{\mathrm{Hom}}_S(X, Y)(S') = \mathrm{Hom}_S(X \times_S S', Y)$. Let $\hat{}$ denote Cartier duality, and consider the morphisms of functors,

$$F \xrightarrow[\approx]{} \underline{\mathrm{Hom}}_{S\text{-groups}}(\hat{F}, G_{m,S}) \hookrightarrow \underline{\mathrm{Hom}}_S(\hat{F}, G_{m,S}) = G.$$

The first isomorphism comes from the Cartier duality theorem, and the second is the natural monomorphism. By b), G is a smooth affine commutative group scheme of finite type. This concludes the proof of (5.1).

If R is a complete local noetherian ring, and F is a formal commutative group scheme over R, topologically flat [8, VII_B 1.3.1] and connected, we may regard F as a sheaf (loc. cit. 1.5) for the site T_{fin}, $F(R') = \mathrm{Hom}_{\mathrm{cont}\text{-}R}(W, R')$, where $F=\mathrm{Spf}(W)$. We say that F is *(smoothly) resolvable* if it admits a resolution,

$$(5.2) \qquad 0 \to F \to G_0 \to G_1 \to 0$$

where the G_i are formal commutative lie groups [13, § 2]. This is an exact sequence of sheaves for the site T_{fin} [13, 2.5], and the G_i are acyclic as sheaves of abelian groups for T_{fin} [13, Th. 2.6].

§ 6. The Affine Algebras of Formal Group Schemes

Let R be a complete discrete valuation ring with finite residue field, k, of characteristic p. Let K denote its field of fractions.

(6.1) **Lemma.** *Let G be a connected finite flat group scheme over R. Suppose $G \otimes_R K$ is etale over K. Let W denote the affine R-algebra of G. Then we may write*

$$W \cong R[x_1, \dots, x_N]/(\Phi_1, \dots, \Phi_N)$$

where Φ_j are polynomials such that $\Phi_j \equiv x_j^{p^{\lambda_j}} \pmod{m}$, $j=1, \dots, N$ (for specific positive integers λ_j) and $(\Phi_1, \dots, \Phi_N)$ is $R[x_1, \dots, x_N]$-regular.

Proof. The fibre $W \underset{R}{\otimes} k$ being a finite connected group scheme over k is covered by the classification theorem of Dieudonné-Cartier [2] and is isomorphic to an algebra of the sort,

$$W/\pi W \cong W \underset{R}{\otimes} k \xrightarrow{\approx} k[x_1, \dots, x_N]/(x_1^{p^{\lambda_1}}, \dots, x_N^{p^{\lambda_N}})$$

for exponents $\lambda_j > 0$.

Consider a diagram,

$$\begin{array}{ccc} I \subset R[x_1, \dots, x_N] & \xrightarrow{\gamma} & W \\ h\downarrow \quad\quad \downarrow & & \downarrow g \\ \bar{I} \cdot \subset k[x_1, \dots, x_N] & \xrightarrow{\bar{\gamma}} & k[x_1, \dots, x_N]/(x_1^{p^{\lambda_1}}, \dots, x_N^{p^{\lambda_N}}) \end{array}$$

where γ is any lifting, guaranteed to exist since g is surjective. I, $\bar{I}$ are the kernels of γ, $\bar{\gamma}$ respectively. We have that γ is surjective by Nakayama's lemma since W is finite over R. The homomorphism $h: I \to \bar{I}$ is surjective. Also one has $I \underset{R}{\otimes} k \xrightarrow{\approx} \bar{I}$ since W is flat over R, Choose liftings $\Phi_j \in I$ of the elements $x_j^{p^{\lambda_j}} \in \bar{I}$. Thus,

$$h(\Phi_j) = x_j^{p^{\lambda_j}}.$$

If $\Phi = (\Phi_1, \dots, \Phi_N)$ is the ideal generated by the Φ_j, we have that $\Phi \subset I$ and Φ generates I mod π. Since I is a module of finite type over $R[x_1, \dots, x_N]$ Nakayama's lemma gives $\Phi = I$. q.e.d.

In the same realm of application of the Dieudonné-Cartier classification theorem [8, $\mathrm{VII_B}$; 2], we have

(6.2) **Lemma.** *Let G be a complete local, topologically flat, group scheme, topologically of finite type over R, and dimension t over R. Then the affine algebra W of G may be written:*

$$W \approx R[\![x_1, \dots, x_n]\!]/(\Phi_1, \dots, \Phi_k)$$

where $n-k=t$, and $\bar{\Phi}_i = \bar{x}_i^{p^{\lambda_i}}$, $\lambda_i > 0$, $i=1, \dots, k$.

Proof. The proof is a slight modification of the proof of (6.1). One constructs the diagram,

$$\begin{array}{ccc} I\subset R[\![x_1,\dots,x_n]\!] & \xrightarrow{\gamma} & W \\ {\scriptstyle h}\downarrow \qquad \downarrow & & \downarrow \\ \bar I\subset k[\![x_1,\dots,x_n]\!] & \xrightarrow{\bar\gamma} & k[\![x_1,\dots,x_n]\!]/(x_1^{p^{\lambda_1}},\dots,x_k^{p^{\lambda_k}}) \end{array}$$

and obtains that λ is surjective using completeness of $R[\![x_1,\dots,x_u]\!]$, then the remainder of the proof of (6.1) may be repeated.

Let G be a connected finite flat group scheme over R whose fibre over K is étale. By virtue of (6.1), the theory of the different, as in the Appendix below, applies to the affine R-algebra W, of G. Suppose, as in (A.15) below, that $W=R[\![x_1,\dots,x_N]\!]/(\Phi_1,\dots,\Phi_N)$ where Φ is a regular $R[\![x_1,\dots,x_N]\!]$-sequence. If $\alpha\colon W\to R$ is any R-homomorphism, denote by $(d\Phi)_\alpha$ the matrix $(\partial\Phi_i/\partial x_j)$ evaluated at the point $(\alpha(x_1),\dots,\alpha(x_N))$. We therefore have

$$\|\det(d\Phi)_\alpha\|=\|\alpha(\delta_{W/R})\|, \tag{6.3}$$

by (A.15). If $e\colon W\to R$ denotes the zero-section of the group scheme G, there is an automorphism of W over R, sending e to any other section α, and consequently from (6.3) we get

$$\|\det(d\Phi)_e\|=\|\alpha(\delta_{W/R})\|. \tag{6.4}$$

As an application of (6.4) we shall deduce that

$$\|\mathrm{disc}_{G/R}\|=\|\det(d\Phi)_e\|^g, \tag{6.5}$$

where g is the rank of G over R.

To see this, it suffices to pass to a discrete valuation ring which is a finite flat extension of R, over which the g sections of $G\otimes_R\bar K/\bar K$ are rational. Then use (A.2) and (A.14).

§ 7. Counting Solutions in Certain Artin Rings

Let R be a complete local noetherian domain, with finite residue field. If R^N denotes the product of N copies of R, $I\subset R$ an ideal, we denote by $I\cdot R^N\subset R^N$ the product of N copies of I. Normalized haar measure in R will be denoted μ.

Consider $\Phi\colon R^N\to R^N$, $(\Phi=(\Phi_1,\dots,\Phi_N))$ a polynomial function with coefficients in R which is transverse-regular at $0\in R^N$. This means, in effect, that $\Phi^{-1}(0)$ consists in a finite set, and if $\Theta\in\Phi^{-1}(0)$, $(d\Phi)_\Theta=(\partial\Phi/\partial x)|_{\Theta(x)}$ has nonvanishing determinant in R.

For S any R-algebra, set

$$F(S) = \mathrm{Hom}_{R\text{-alg}}(R[x_1, \ldots, x_N]/(\Phi_1, \ldots, \Phi_N), S).$$

Thus $F(S) = \{(s_1, \ldots, s_N) \in S^N;\ \Phi(s_1, \ldots, s_N) = 0\}$.

If $S = R/I$ for $I \subset R$ an open ideal, we therefore have:

$$\begin{aligned} \# F(R/I) &= \mu\{\Phi^{-1}(IR^N)\}/\mu(IR^N) \\ &= \mu\{\Phi^{-1}(IR^N)\}/\mu(I)^N. \end{aligned} \tag{7.1}$$

To continue the computation, we suppose R a discrete valuation ring with uniformizer π.

(7.2) **Lemma.** *Let R be a discrete valuation ring with finite residue field, and suppose Φ satisfies the transverse-regularity condition cited above. Then there is an integer s_0 such that if $s \geqq s_0$,*

$$\# F(R/\pi^s) = \sum_{\Theta} \|\det d\Phi_\Theta\|^{-1}$$

where π is a uniformizer, $\| \ \|$ is normalized absolute value and Θ runs through the roots of Φ in R^N.

Proof. Let K be the field of fractions of R, and endow the vector space K^N with the norm $\|x\| = \max_{i=1,\ldots,N} \|x_i\|$ for $x = (x_1, \ldots, x_N)$.

Let us introduce the compact open discs

$$B_s(z) = \{x \in R^N \mid \|x - z\| \leqq \|\pi^s\|\}.$$

(7.2) will follow from the following results.

(7.3) **Lemma.** *Let Θ be a root of Φ in R^N; then there is an integer $r \geqq 0$ such that for $s \geqq r$, $\Phi | B_s(\Theta)$ is a homeomorphism with image*

$$\Phi(B_s(\Theta)) = d\Phi_\Theta(B_s(0)).$$

Proof. Since $d\Phi_\Theta$ is invertible at Θ it is invertible in some neighborhood $B_{r'}(\Theta)$ of Θ. By the Taylor expansion of Φ we may write for $x + \Theta \in B_{r'}(\Theta)$.

$$\Phi(x + \Theta) = d\Phi_\Theta(x) + H(x)$$

where H is an analytic function (a polynomial in our present case) composed of terms of degree greater than or equal to two.

First assume that $d\Phi_\Theta = I$. Then the fact that $H(x)$ has no linear terms implies that there is an $r \geqq r'$ such that for $x \in B_r(0)$, $\|H(x)\| < \|x\|$. Since the norm is ultrametric we have $\|\Phi(x + \Theta)\| = \|x\|$. Hence $\Phi(B_x(\Theta)) \subseteq B_s(0)$ if $s \geqq r$. However $\Phi | B_r(\Theta)$ is invertible, and the inverse must be of the form (on $B_r(0)$)

$$\Phi^{-1}(x) = \Theta + x + H'(x).$$

As this function (by the argument above) takes $B_s(0)$ to $B_s(\Theta)$ we have that $\Phi|B_s(\Theta)$ is a homeomorphism onto $B_s(0)$ for all $s \geqq r$.

In the general case we may write

$$\Phi|B_{r'}(\Theta) = d\Phi_\Theta \circ \Psi$$

where Ψ is as above. If r is then chosen as above we have that for $s \geqq r$ $\Phi|B_s(\Theta)$ is a homeomorphism onto

$$\begin{aligned}\Phi\big(B_s(\Theta)\big) &= d\Phi_\Theta\big(\Psi(B_s(\Theta)\big)\\ &= d\Phi_\Theta\big(B_s(0)\big).\end{aligned}$$

(7.4) **Corollary.** *If* $s \geqq r$

$$\mu\big(\Phi(B_s(\Theta))\big) = \|\det d\Phi_\Theta\| \; \mu\big(B_s(0)\big).$$

Proof. Set $B_s = B_s(0)$, and $L = (d\Phi)_\Theta$. Then L is an R-endomorphism of the free R-module B_s. Granted our conditions above, we have that L is an isomorphism of K^N, so the R-modules LB_0 and LB_s are free of rank N. Thus $(B_0 : LB_0)$ and $(B_s : LB_s)$ are finite, and the snake lemma applied to the obvious diagram tells us that they are, in fact, equal. Since s is as in (7.3), B_s is the disjoint union of $(B_s : LB_s)$ cosets of the open subgroup LB_s. Therefore by (7.3) and our above remark, $\mu(B_s) = (B_0 : LB_0) \cdot \mu(\Phi(B_s(\Theta)))$. We are reduced to showing that $(B_0 : LB_0) = \|\det L\|^{-1}$. But, to compute $\|\det L\|$ we may use any two R-bases of B_0. By the structure of finitely generated modules over a principal ideal domain, we know that there are bases (e_i) and (f_i) of B_0 such that $Le_i = a_i f_i$. Hence $\|\det L\| = \|\prod_i a_i\|$; but the latter term is, by definition of normalized absolute value, just $(B_0 : LB_0)$.

We now return to the proof of (7.2). Chose r so that

i) $B_r(\Theta)$ are disjoint,

ii) the conclusion of the lemma is valid for all roots Θ.

For $s \geqq r$ set $C_\Theta = \Phi^{-1}(B_s(0)) \cap B_r(\Theta)$. By (7.3) C_Θ is precisely the set $d\Phi_\Theta^{-1}(B_s(0))$. Hence

$$\begin{aligned}\mu(C_\Theta) &= \mu\big(d\Phi_\Theta^{-1}(B_s(0))\big)\\ &= \|\det(d\Phi)_\Theta\|^{-1} \cdot \mu\big(B_s(0)\big).\end{aligned}$$

Since $\Phi^{-1}(B_s(0)) = \bigcup C_\Theta$ (disjoint), we have

$$\frac{\mu(\Phi^{-1}(B_s(0)))}{\mu(B_s(0))} = \sum_\Theta \|\det(d\Phi)_\Theta\|^{-1}.$$

Lemma (7.2) immediately follows once we remember that $B_s(0) = \pi^s R^N$, and invoke (7.1).

§ 8. The Main Theorem

Let R be a complete discrete valuation ring with finite residue field k, and field of fractions K. Set $S = \operatorname{Spec} R$.

(8.1) **Proposition.** *Let G be a commutative finite flat group scheme over R of rank g. Suppose that $G \otimes_R K$ is etale over K. Then the flat cohomology groups $H^q(S, G)$ vanish for $g \geq 2$, all cohomology groups are finite, and*

$$\chi(R, G) = \|\operatorname{disc}_{G/R}\|^{1/g} = \|e(\delta_{W/R})\|$$

where W is the affine algebra of G, e: $W \to R$, is the zero section, and $\delta_{W/R}$ the different.

Remarks. (a) The cohomology groups vanish beyond dimension 1 by (5.1)(iii). $H^1(S, G)$ is a finite group by (4.8).

(b) We may reduce to connected group schemes G by considering the exact sequence,

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0.$$

Since $G^{\text{ét}}$ is smooth, its flat cohomology is isomorphic to its cohomology computed for the étale site over $\operatorname{Spec} R$ [7, III App. § 11], which is isomorphic to the galois cohomology over k, of the galois module associated to $G^{\text{ét}}$ [6; **4** VIII 8,6]. Since k is finite and $G^{\text{ét}}$ is a torsion-module it is easy to see [20, Ch. XIII Prop. 1,2] that the cohomology groups of $G^{\text{ét}}$ vanish beyond dimension 1, and since $G^{\text{ét}}$ is finite, $\chi(G^{\text{ét}}) = 1$.

(c) The second equality of (8.1) is nothing but (6.5).

(d) It remains to prove the first equality in the case where G is connected. For this we give two proofs. Our first proof is a fairly direct computation in the theory of analytic groups, but is applicable only in the case where K is a finite extension of Q_p, for p a prime. The second proves the general theorem, and makes no use of a smooth resolution of G.

The first Proof. (K of characteristic zero.)

Using (5.2) we may find a resolution

(8.2)
$$0 \to G \to G_0 \to G_1 \to 0.$$

Write

$$G_0 = \operatorname{Specf}(R[\![y_1, \ldots, y_n]\!]),$$
$$G_1 = \operatorname{Specf}(R[\![x_1, \ldots, x_n]\!])$$

where augmentations ε are given by $y_j \mapsto 0$; $x_i \mapsto 0$, and the finite flat homomorphism $G_0 \to G_1$ is given by a homomorphism,

$$\Psi\colon R[\![x_1, \ldots, x_n]\!] \to R[\![y_1, \ldots, y_n]\!].$$

The affine algebra of G, being formed by tensor product from the diagram,

$$\begin{array}{c} R \\ \uparrow{\scriptstyle\varepsilon} \\ R[\![x_1, \ldots, x_n]\!] \xrightarrow{\Psi} R[\![y_1, \ldots, y_n]\!], \end{array}$$

is:

(8.3) $W = R[\![y_1, \ldots, y_n]\!]/(\Psi_1, \ldots, \Psi_n)$, where $\Psi_i = \Psi(x_i)$. Since the sequence $(x_1, \ldots, x_n)$ is regular for $R[\![x_1, \ldots, x_n]\!]$, and Ψ makes $R[\![y_1, \ldots, y_n]\!]$ a finite free module over $R[\![x_1, \ldots, x_n]\!]$, we have that $(\Psi_1, \ldots, \Psi_n)$ is regular for $R[\![y_1, \ldots, y_n]\!]$.

Let $\bar{K}$ be an algebraic closure of the field of fractions K of R, and consider R-homomorphisms, $W \to \bar{K}$. We may find a finite field extension K'/K such that all the above homomorphisms take their values in K'. There are g distinct such homomorphisms and they may be identified with points $\Theta = (\Theta_1, \ldots, \Theta_n) \in R'^{(n)}$ which are zeros of Ψ_i, $i = 1, \ldots, n$. We refer to Θ as a root of Ψ in R' (R' is the integral closure of R in K'. Since W is local, and $0 = (0, \ldots, 0)$ is a root of Ψ, it follows that all roots of Ψ actually lie in $(m')^{(n)}$, where m' is the maximal ideal of R'.)

Let $|G_i|$ denote the formal group determined by G_i, $i = 0, 1$ [cf. 19, § 6]. We have an identification of points,

$$G_i(R) \cong m^{(n)} = |G_i|$$

and the group scheme structure on G_i determines, in an obvious way, a formal group law on $|G_i|$. The map Ψ induces a map of analytic groups, $\Phi \colon |G_0| \to |G_1|$.

Let the symbol $(d\Phi)_0$ denote the endomorphism of the tangent space of the analytic manifold $m^{(n)}$ at the origin, induced by Φ and the identifications. Upon choice of the standard bases, the matrices $(d\Phi)_0$ and $(d\Psi)_\varepsilon$ coincide. Hence our first proof will be concluded when we establish

(8.4) **Lemma.**

$$\chi = \frac{\# \ker \Phi}{\# \operatorname{cok} \Phi} = \|\det(d\Phi)_0\|.$$

Proof. The first equality follows since (6.2) is an acyclic resolution of sheaves for the site T_{fin} (5.2).

The second is a modification of an excercise in [19, 4.41] for whose proof we are indebted to R. Rasala. We sketch this proof below.

We may identify the tangent space to $|G_i|$ at the origin caninically with K^n.

Since both $|G_0|$ and $|G_1|$ are commutative, the Lie algebra structures $\mathscr{G}_i$ induced on K^n are just K^n regarded as an abelian Lie algebra. Hence $CH(\mathscr{G}_i) = K^n$ under addition, where $CH(\mathscr{G}_i)$ is the Campbell-Hausdorff group associated to $|G_i|$ [cf. 19, § 4]. By the general theory [19, § 7 Th. 1 Cor. 2] there exist local isomorphisms of analytic groups (both designated by exp) exp: $K^n \to |G_i|$ inducing the identity on the Lie algebras.

Set $U_\delta = \{x \in K^n\colon \operatorname{Max} \|x_i\| < \delta\}$.

We let U_δ^+ denote U_δ as a subgroup under addition, and U_δ^* (resp. V_δ^*) denote U_δ regarded as a subgroup of $|G_0|$ (resp. $|G_1|$). Since and $(d\Phi)_0$ have coefficients in R, they map U_δ into itself.

The following two assertions are quite often used in the theory of ultrametric formal groups, and may easily be established:

(a) The identity map identifies cosets of $U_1^+ \bmod U_\delta^+$ with cosets of U_1^* with U_δ^*, if δ is sufficiently small. Indeed, this is so because the formula giving multiplication by x in $U_1^* = m^{(n)}$ may be expressed as $x + Z + g(Z)$, where $g(Z)$ is a power series containing only terms of degree at least 2 and has coefficients in R.

(b) If δ is sufficiently small we have the commutative diagram

$$\begin{array}{ccc} U_\delta^+ & \xrightarrow{(d\Phi)_0} & U_\delta^+ \\ \exp \big\downarrow \approx & & \approx \big\downarrow \exp \\ U_\delta^* & \xrightarrow{\Phi} & U_\delta^* \end{array}$$

where the vertical maps are isomorphisms.

Now find a δ sufficiently small so that (a), (b) hold for δ, and $U_\delta^* \cap \ker \Phi$ is trivial. This is possible since $\ker \Phi$ is finite. Then

$$(U_1^* : U_\delta^*) = (\Phi U_1^* : \Phi U_\delta^*) \cdot \# \ker \Phi$$

and therefore

$$\frac{\# \ker \Phi}{\# \operatorname{cok} \Phi} = \frac{(U_1^* : U_\delta^*)}{(V_1^* : \Phi U_1^*)(\Phi U_1^* : \Phi U_\delta^*)} = \frac{(U_1^* : U_\delta^*)}{(V_1^* : \Phi U_\delta^*)} = \frac{(U_1^+ : U_\delta^+)}{(U_1^+ : (d\Phi)_0\, U_\delta^+)}$$

$$= \frac{1}{(U_\delta^+ : (d\Phi)_0\, U_\delta^+)} = \|\det (d\Phi)_0\|.$$

In the above line of equalities, all but the last follow immediately from (a), (b). The last is wellknown, and may be seen by the argument of (7.4) for example.

The Second Proof. (K general.)

From (7.2) and (6.3), (6.4), we obtain

$$\# G(R/\pi^s) = \# G(R) \cdot \|\operatorname{disc}_{G/R}\|^{-1/g}$$

for s sufficiently large, and π a uniformizer of R.

From (4.5), (4.8) one has

$$\# G(R/\pi^s) = \# H^1(S, G), \qquad \text{for } s \text{ sufficiently large.} \quad \text{q.e.d.}$$

§ 9. Applications of the Euler Characteristic Formula

We retain the situation of the previous section. Thus G is a commutative finite flat group scheme over R, étale at the general point of Spec R. Let $\hat{G}$ denote the Cartier dual of G, which we suppose is also etale at the generic point of Spec R. Our first application is to provide an alternate proof of the local flat duality theorem [13]. Reverting to the terminology of § 1 of [13] we denote by $H^1(G)$ the *flat* cohomology group; by $h^1(G)$ the galois cohomology of the galois module $G \underset{R}{\otimes} K$; by *, Pontrjagin duality. We have maps

$$0 \to H^1(G) \xrightarrow[a]{} h^1(G) \xrightarrow[b]{} H^1(\hat{G})^* \to 0 \tag{9.1}$$

given as follows: The map a is the composition,

$$H^1(R, G) \to H^1(K, G) \xrightarrow[\approx]{} h^1(G)$$

where the first arrow is the map induced by the inclusion of Spec$(K) \to S$, and the second is the canonical identification of galois cohomology with the etale cohomology over Spec(K). We have used that $G \otimes_R K$ is etale over K to identify $H^1(K, G)$ with the corresponding etale cohomology group [7, III § 11].

The map b is the composition,

$$h^1(G) \xrightarrow[\approx]{\tau} h^1(\hat{G})^* \xrightarrow{(\hat{a})^*} H^1(\hat{G})^*$$

where τ is the Tate local duality isomorphism [21, Ch. II § 5 Th. 2] induced by cup-product.

a is injective, and b is surjective as is easily seen (using only that R is integrally closed). One also checks that $b \circ a = 0$ [cf. 13].

(9.2) **Corollary.** (Local Flat Duality.)

(9.1) *is exact.*

As was shown in [13], the problem boils down to the inequality:

$$\# H^1(G) \cdot \# H^1(\hat{G}) \geqq \# h^1(G)$$

or equivalently,

$$\frac{\# H^1(G)}{\# H^0(G)} \cdot \frac{\# H^1(\hat{G})}{\# H^0(\hat{G})} \geqq \frac{\# h^1(G)}{\# h^0(G) \cdot \# h^2(G)}$$

using Tate duality. Tate's Euler characteristic theorem [25, Th. 2.2] evaluates the r.h.s. as being $(R : gR)$. But (8.1) gives the l.h.s. as being $\|\text{disc}\, G \cdot \text{disc}\, \hat{G}\|^{-1/g}$, where we have denoted the discriminant $\text{disc}_{G/R}$ by disc G.

An elementary computation gives $\text{disc}\, G \cdot \text{disc}\, \hat{G} = g^g$, [e.g., 17], from which (9.2) follows.

A second application of (8.1) is the following [see 17, Ch. V Th. 1).

(9.3) **Proposition.** *Let R be the ring of integers in a local number field, and assume that R contains the p-th roots of* 1, *where p is the characteristic of the residue field of R. Let G be a finite flat commutative group scheme of rank p such that $\# H^0(G)=p$. Then there is a non-trivial homomorphism $t\colon G\to\mu_p$, and any such t induces a diagram with α an isomorphism*

$$\begin{array}{ccc} H^1(G) & \xrightarrow{\ t\ } & H^1(\mu_p)=\dfrac{U}{U^p} \\ & \searrow^{\alpha}_{\sim} & \uparrow \\ & & \dfrac{U^{(n)}\cdot U^p}{U^p} \end{array}$$

where U is the group of units in R, $U^{(n)}$ the n-th group of the standard filtration on U, [20, Ch. IV §2] *and where*

$$n=\frac{p\,e(K/Q_p)-\operatorname{ord}_R(\operatorname{disc} G)}{p-1}.$$

In the above formula, $e(K/Q_p)$ is the ramification index.

Proof. It is shown in [17] (Ch. III Cor. of Prop. 6) that t factors through $\dfrac{U^{(n)}\cdot U^p}{U^p}$. By (8.1) we know that

$$\begin{aligned}\# H^1(G)&=\# H^0(G)\,\|\operatorname{disc} G\|^{-1/p}\\ &=p\,q^{\frac{\operatorname{ord}(\operatorname{disc} G)}{p}}\end{aligned}$$

where $q=\#\dfrac{R}{m}$ (it is a fact that $(p-1)p\,|\operatorname{ord}\operatorname{disc} G)$. Hence to show that α is an isomorphism one only needs to prove that the number of elements in $\dfrac{U^{(n)}\cdot U^p}{U^p}$ is also $p\,q^{\frac{\operatorname{ord}(\operatorname{disc} G)}{p}}$. However, in general, (loc. cit. Ch. I Prop. 6 Cor. 1) for $0\leq i\leq \dfrac{e}{p-1}=m$

$$\#\frac{U^{(pi)}\cdot U^p}{U^p}=p\,q^{(p-1)(m-i)}.$$

In the above $i=m-\dfrac{\operatorname{ord}\operatorname{disc} G}{p}$, and we are done.

Appendix. The Different and the Discriminant

Although we might have used the general theory of residues, [e.g. 10], to obtain the desired properties of the different of a relative local complete intersection, the following presentation, due to Tate, is elementary, self-contained, and swift enough.

Assume that R is any ring and that A is an R-algebra which is free and of finite rank over R.

(A.1) **Definition.** *If* $\mathrm{Hom}_R(A, R)$ *is a free* A *module of rank* 1 *with* λ *as a basis we may write*

$$Tr_{A/R} = \delta_{A/R}\,\lambda \qquad \delta_{A/R} \in A$$

(where $Tr_{A/R}\colon A \to R$ *is given by* $Tr_{A/R}(b) =$ *trace of "multiplication by* b*"). Then* $\delta_{A/R}$ *is called a different of* A/R.

(A.2) **Proposition.** *If* $\delta_{A/R}$ *is a different of* A/R *then*

$$(\mathrm{Norm}_{A/R}(\delta_{A/R})) = \mathrm{disc}_R\, A.$$

Proof. Let $a_1, \ldots, a_n$ be an R basis of A. Then $\lambda\, a_i$ is an R basis of $\mathrm{Hom}_R(A, R) = A^*$. Since A^* and A are isomorphic R-modules under the map which takes Φ to $\sum \Phi(a_i)\, a_i$, it is immediate that $\det(\lambda(a_i\, a_j))$ is a unit in R. If one then writes

$$a_i = \sum \gamma_{ik}\, a_k, \qquad \gamma_{ij} \in R$$

we have $\det(\gamma_{ij}) = N(\delta)$, and that

$$\begin{aligned}
\mathrm{disc}_R\, A &= \det\big(Tr_{A/R}(a_i\, a_j)\big) = \det\big(\lambda(a_i\, a_j)\big) \\
&= \det\big(\textstyle\sum \gamma_{ik}\, \lambda(a_k\, a_j)\big) \\
&= \det(\gamma_{ik}) \cdot \det \lambda(a_k\, a_j) \\
&= N(\delta) \cdot (\text{unit}).
\end{aligned}$$

(A.3) **Theorem** (Tate). *Hypotheses:* R *is a ring,* A *an* R*-algebra* $\underline{f} = (f_1, \ldots, f_N) \subset A$ *a regular* A*-sequence,* $A \xrightarrow{\alpha} A/\underline{f} = C$ *the canonical projection. Suppose* C *is free and of finite rank over* R. *Suppose further that the kernel of the homomorphism,*

$$\beta\colon A \underset{R}{\otimes} C \xrightarrow{(\alpha,\, 1_C)} C$$

is generated by $(g_1, \ldots, g_N)$ *a regular* $A \underset{R}{\otimes} C$*-sequence.*

Conclusions: 1. $\mathrm{Hom}_R(C, R)$ *is a free* C*-module of rank* 1. (*The module-structure on* $\mathrm{Hom}_R(C, R)$ *being given by* $(c\,\Phi)(x) = \Phi(c \cdot x)$.)

2. *Write*

(A.4) $$f_i = \sum b_{ij} g_j, \quad b_{ij} \in B = A \underset{R}{\otimes} C.$$

Then there is a C-module generator $\lambda \in \mathrm{Hom}_R(C, R)$ *such that,*

(A.5) $$\alpha(\lambda'(d)) = 1$$

where $\lambda' = \lambda \underset{R}{\otimes} 1_A \in \mathrm{Hom}_A(B, A)$ *is our notation for base change, and* $d = \det(b_{i,j})$.

3. *(Characterization.) For* $\gamma = c \cdot \lambda$, *any* $c \in C$, *we have*

(A.6) $$\alpha(\gamma'(d)) = c.$$

4. $\delta_{C/R} = \beta(d)$.

Proof. 1. Consider the exterior algebra complexes (Koszul complexes) $K^A(\underline{f})$ and $K^B(\underline{g})$ associated to the regular sequences $\underline{f}$ of A and $\underline{g}$ of B [18, Ch. IV § 1, 2]. Thus, if we choose generators $u_j \in K_1^A(f_j)$ and $v_j \in K_1^B(g_j)$ such that $d_1(u_j) = f_j$; $d_1(v_j) = g_j$ we have $K^A(\underline{f})$ is the free graded differential exterior A-algebra generated in dimension 1 by $(u_1, \ldots, u_N)$ and $K^B(\underline{g})$ is the free differential graded exterior B-algebra generated by $(v_1, \ldots, v_N)$.

There is a unique differential graded algebra homomorphism $\Phi\colon K^A(\underline{f}) \to K^B(\underline{g})$ lifting the canonical map. $\Phi_0\colon A \to B$, and given in dimension one by:

(A.7) $$\Phi_1(u_i) = \sum b_{ij} v_j.$$

We then have:

(A.8) $$\Phi_N(u_1 \otimes \cdots \otimes u_N) = d \cdot v_1 \otimes \cdots \otimes v_N.$$

The hypotheses of regularity insures that $K^A(\underline{f})$ is a free A-resolution of $A/(\underline{f}) = C$, and $K^B(\underline{g})$ is a free B-resolution of $B/(\underline{g}) = C$ [18, IV Prop. 2]. Since C is free over R, B is free over A. Thus both complexes $K^A(\underline{f})$ and $K^B(\underline{g})$ provide free A-resolutions of C.

(A.9) **Lemma.** *We have an isomorphism*

$$h\colon \frac{\mathrm{Hom}_A(B, A)}{(\underline{g}) \cdot \mathrm{Hom}_A(B, A)} \xrightarrow{\sim} C$$

given by $h(\Phi) = \alpha(\Phi(d))$.

Proof. Φ induces a chain map on cochain complexes,

$$\mathrm{Hom}_A(K^B(\underline{g}), A) \xrightarrow{\Phi^*} \mathrm{Hom}_A(K^A(\underline{f}), A)$$

which must be an isomorphism for cohomology, and in particular fo N-dimensional cohomology. One computes

$$\begin{array}{ccc} H^n_A(K^B(\underline{g}), A) & \xrightarrow{\Phi^*_n} & H^n_A(K^A(\underline{f}), A) \\ \| & & \| \\ \dfrac{\mathrm{Hom}_A(B, A)}{(\underline{g})\,\mathrm{Hom}_A(B, A)} & \xrightarrow{h} & A/\underline{f}. \end{array}$$

If we take $h = \Phi^*_n$, and use (A.8), the lemma follows.

Consider, now, the composite

$$\begin{array}{ccccc} & & \mathrm{Hom}_A(B, A) & & \\ & \nearrow & & \searrow{\scriptstyle h} & \\ \mathrm{Hom}_R(C, R) & & \xrightarrow[t]{\qquad\qquad} & & C \end{array}$$

where the first map, $\Phi \mapsto \Phi'$ is base change.

(A.10) **Lemma.** *t is an isomorphism of C-modules;* $t(\Phi) = \alpha(\Phi'(d))$.

Proof. First, that it is a morphism of C-modules: Let $c \in C$, $c = \alpha(a)$ for some $a \in A$.

$$t(\lambda c) = \alpha(\lambda'(c \otimes 1)\, d); \qquad c\, t(\lambda) = \alpha(a \cdot \lambda'(d)).$$

We are reduced, then, to checking:

$$\alpha(\lambda'((c \otimes 1 - 1 \otimes a) \cdot d)) = 0$$

which is true, since:

a) $c \otimes 1 - 1 \otimes a \in (g_1, \ldots, g_N)$,

b) $\det(b_{ij})\, g_i = \sum b'_{ji} f_i$ for $b'_{ji} \in B$ yielding,

c) $(c \otimes 1 - 1 \otimes a) \cdot d = \sum b''_{ji} f_i$ for $b''_{ji} \in B$.

One checks easily that t is surjective. It follows that t is an isomorphism since $\mathrm{Hom}_R(C, R)$ and C are free R-modules of equal (finite) rank.

Lemma (A.10) concludes the proof of 1.

2. Take $\lambda = t^{-1}(1)$.

3. Immediate from (2).

4. By Definition 1 and the characterization formula (3), we must check:

$$\alpha(Tr_{C'/R}(d)) = \beta(d). \qquad \text{(A.11)}$$

Using the formula for t in (A.10), it suffices to check:

$$t(Tr_{C/R}) = \beta(d). \tag{A.12}$$

Choosing dual R-bases $\{\gamma_u\}$ of C and $\{\lambda_v\}$ of $\mathrm{Hom}_R(C, R)$, we have that $\{\gamma'_u\}$ is an A-basis of B, and we write:

$$d = \sum_{v=1}^{N} a_v \gamma'_v,$$

where $a_v = \lambda'_v(d)$. Also, $Tr_{C/R} = \sum \gamma_v \cdot \lambda_v$,
where the $\cdot$ denotes C-module multiplication in $\mathrm{Hom}_R(C, R)$. We now show (A.12):

$$\begin{aligned} t(Tr_{C/R}) &= \sum \gamma_v \cdot t(\lambda_v) \\ &= \sum \alpha(\lambda'_v(d)) \cdot \gamma_v \\ &= \sum \alpha(a_v) \cdot \gamma_v \\ &= \beta(d). \end{aligned}$$

The first equality in the above string uses that t preserves C-multiplication, and the other equalities are immediate. q.e.d.

We now may prove two corollaries which are used in the preceding.

(A.13) **Corollary.** *Let R be a complete discrete valuation ring and let $W = \dfrac{R[x_1 \dots x_N]}{(\Phi_1 \dots \Phi_N)}$ be an R-algebra which is finite and free over R. Assume that $W \underset{R}{\otimes} K$ is étale over K, and that $\Phi_1 \dots \Phi_N$ is regular for $R[x_1 \dots x_N]$. Then* $\det(d\Phi) \in W$ *is a different for W/R and*

$$\mathrm{disc}_R\, W = \Big(\prod_{\Theta} \det d\Phi_\Theta\Big)$$

where Θ runs over all the roots of Φ in $\overline{K}^N$, ($\overline{K}$ an algebraic closure of K).

Proof. We claim first that $\det(d\Phi) \in W$ is a different for W. This is immediate from the theorem: for set $A = R[x_1, \dots, x_N]$, $C = W$; then $B = C[x_1, \dots, x_N]$, and it is immediate that if $\bar{x}_i$ is the residue class of x_i in W we have $\ker(B) = (x_1 - \bar{x}_1, \dots, x_N - \bar{x}_N)$ which is B-regular.

Thus if we write

$$\Phi_i = \sum b_{ij}(x_j - \bar{x}_j)$$

$$\frac{\partial \Phi_i}{\partial x_j} = b_{ij} + k_{ij} \qquad \text{with} \quad k_{ij} \in \ker \beta.$$

Therefore $\delta_{C/R} = \det(d\Phi) \in C$.

Note now that

$$\prod_{\Theta} \det d\Phi_{\Theta} = \prod_{\sigma \in \operatorname{Hom}_{\mathrm{alg}}(W, \bar{K})} \sigma(\det d\Phi).$$

Thus we are reduced to the following lemma

(A.14) **Lemma.** *For W/R finite, free $/R$ with $W \otimes_R K$ étale$/K$. the norm is given by*

$$N_{W/R}(a) = \prod_{\sigma \in \operatorname{Hom}_{\mathrm{alg}}(W, \bar{K})} \sigma(a).$$

Proof. Since the norm from W to R of a is the same as the norm from $W \otimes K$ to K, and since $\operatorname{Hom}_{\mathrm{alg}}(W \otimes K, \bar{K})$ is naturally isomorphic to $\operatorname{Hom}_{\mathrm{alg}}(W, \bar{K})$ we are reduced to the case when R is a field. W is then the direct product of separable field extensions, and one checks that the formula is valid if it is valid for each factor. Hence we may assume that W/R is a separable field extension; then the result is Bourbaki Algebre Chapter VIII § 12 no. 2 Proposition 4.

Finally we prove the "formal analogue" of (A.13).

(A.15) **Corollary.** *Let R be a complete discrete valuation ring and let $W = \dfrac{R[[x_1, \ldots, x_N]]}{(\Phi_1, \ldots, \Phi_N)}$ be a finite free R algebra, étale at the generic point, with $(\Phi_1, \ldots, \Phi_N)$ a regular $R[[x_1, \ldots, x_N]]$-sequence. Then* $\det(d\Phi) \in W$ *is a different for W/R and*

$$\operatorname{disc}_R W = \prod_{\Theta} \det(d\Phi_{\Theta}).$$

Proof. Is essentially word for word the same as (A.13), once one notes that the right hand side makes sense because all Θ are in the maximal ideal of the integral closure of R in some finite extension of K.

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A General Strong Law*

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§ 1. Introduction

Recently Komlós [4] proved the interesting fact that given any norm-bounded sequence F of functions in L^1 there is a subsequence F_0 of F and a function $\bar{f} \in L^1$ such that for any further sub-sequence f_n of F_0

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} (f_k - \bar{f}) = 0 \qquad \text{a.e.}$$

The main purpose of this paper is to establish the following generalization:

Theorem 1. *Let (S, Σ, P) be a measure space and F a sequence from $L^p(S, \Sigma, P)$, $0<p<2$, and let $\sup_{f\in F} \int |f|^p < \infty$. Then there is a sub-sequence F_0 of F and a function $\bar{f} \in L^p$ such that for any sub-sequence f_n of F_0,*

$$\lim_{n\to\infty} n^{-1/p} \sum_{k=1}^{n} (f_k - \bar{f}) = 0 \qquad \text{a.e.} \tag{1}$$

Further $\bar{f} \equiv 0$ is always a possible choice if $0<p<1$. In general F_0 cannot be chosen in such a way as to ensure L^p-convergence in (1). *If however there is a sub-sequence G such that $\{|f|^p : f \in G\} \subset L^1$ is weakly sequentially compact* (this means in the case of a bounded measure P that G is an uniformly integrable family i.e. $\lim_{N\to\infty} \int_{\{|f|>N\}} |f|^p \, dP = 0$ uniformly for $f \in G$) *then L^p-convergence in* (1) *can be ensured as well.*

Actually the proof of Theorem 1 in the case of an arbitrary measure follows immediately from the special case of probability measures and so in the body of our proof we shall suppose $P(S)=1$. Indeed, since denumerable integrable families have σ-finite supports one could first reduce the theorem to the σ-finite case and then replace the σ-finite measure by an equivalent probability measure. We leave out the details of this latter argument.

The following simple example shows that the condition of uniform integrability is indeed necessary for the L^p-convergence part of Theorem 1.

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Example. Let $f_n = n^{2/p}$ on $[1/n+1,\ 1/n)$ and $=0$ outside on $[0, 1]$, $n=1, 2, \ldots$. Then $\frac{1}{2} \leqq \int |f_n|^p = n^2/n(n+1) \leqq 1$ and $\Sigma f_n < \infty$ everywhere since the f_n's have disjoint supports. Hence for any subsequence

$$n_1 < n_2 < \cdots \lim_{k\to\infty} k^{-1/p} \sum_{j=1}^{k} f_{n_j} = 0.$$

But

$$k^{-1} \int \left(\sum_{j=1}^{k} f_{n_j}\right)^p = k^{-1} \sum_{j=1}^{k} n_j^2/n_j(n_j+1) \geqq \tfrac{1}{2} \nrightarrow 0.$$

It is also clear that the subtraction of any function $\bar{f} \in L^p$ from the f_{n_j}'s will not induce L^p-convergence.

We note the following corollary to Theorem 1:

Corollary. *If* $\{f_n\}$ *is a sequence of symmetrically dependent random variables* (r.v.) *over a probability space* (S, Σ, P) *and if* $f_n \in L^p$, $0<p<2$, *then*

$$\lim_{n\to\infty} n^{-1/p} \sum_{k=1}^{n} (f_k - \Theta) = 0 \qquad \textit{a.e. and in } L^p$$

where $\Theta = 0$ *if* $0<p<1$ *and* $\Theta = E(f_1 | I)$, $I =$ *tail* σ-*field of* $\{f_n\}$ *if* $1 \leqq p < 2$.

By symmetric dependence is meant that

$$P\{(f_1, \ldots, f_k) \in B\} = P\{(f_{n_1}, \ldots, f_{n_k}) \in B\}$$

for any $k \geqq 1$, $n_1 \neq \cdots \neq n_k$ and any k-dimensional Borel set B. The existence of the limit above is immediate from Theorem 1. The identification of Θ as the indicated conditional expectation is known and routine. If the f_n's are independent, identically distributed r.v.'s then the a.e. convergence part of the corollary corresponds to the classical theorems of Kolmogorov (for $p=1$) and Marcinkiewicz (see Loève [5]). L^p-convergence in this case has recently been proved by Pyke and Root [6]. A general L^p-convergence theorem for martingales is proved in [2] from which also the above corollary can be easily deduced. Note that for $0<p<1$, the convergence in the above corollary follows solely from the identical distribution of the f_n's. This has been noted by Sawyer [8] pp. 165 and it follows from our Lemmas 1 and 2 as well.

We remark that known facts of elementary probability theory prove that in the range $0<p<2$, Theorem 1 is the best possible and that the analogue of Theorem 1 for $p=2$ is false. In Section 5 we indicate the best possible substitute for Theorem 1 in the case $p=2$, a theorem due to Révész [7].

As a final comment we point out a mistake in a wellknown paper of Banach and Saks [1]. There the authors after having proved that a weakly convergent sequence f_n in L^p, $1<p<\infty$ possesses a sub-sequence whose arithmetic averages converge strongly, claim that this is false for $p=1$.

Indeed on pp. 55 of [1] they write down a presumed counter-example. It turns out that their counterexample is invalid since the sequence which they claim to be weakly convergent to 0 is not so. In fact, Theorem 1 for $p=1$ (i.e. Komlós' theorem) proves that the Banach-Saks theorem is valid for L^1. On the other hand in the Banach space $C[0, 1]$ the Banach-Saks theorem is known not to be valid, Schreier [9]. Is there any natural characterization of these Banach spaces in which the Banach-Saks theorem is valid? It is interesting to add here that a famous theorem of Mazur says that if in the theorem of Banach-Saks one permits arbitrary convex combinations then one can get strong convergence in any Banach-space.

We should like to indicate here our indebtedness to the methods implicit in Komlós' work [4]. In particular the idea of proving Lemma 1 and using conditional expectations is taken from [4]. Originally we had given a proof for $p \neq 1$, $0<p<2$, based on methods of the theory of orthogonal functions e.g. as in the proof of the Rademacher-Menchov theorem (Zygmund [10]). Since the details of this proof were too clumsy and further did not yield the general theorem for $p=1$ we preferred to resort to the methods of [4].

§ 2. Preliminaries

As pointed out in the introduction we shall be working exclusively in a probability space (S, Σ, P). Hence we can freely use the standard probabilistic devices like conditional expectations etc. In particular we shall need the following version of the Doob-martingale convergence theorem: see Loève [5].

Theorem 0. *If* $f_n \in L^2$, $n \geqq 1$ *and* $\Theta_n = E(f_n | f_1, \ldots, f_{n-1})$ *then* $\sum_n (f_n - \Theta_n)$ *converges a.e. and in* L^2 *if* $\sum_n \int f_n^2 < \infty$.

For some of the work on L^p-convergence, $1<p<2$, we shall need the following lemma proved recently by Esseen and Von Bahr [3]. A simple proof is given in Chatterji [2].

Lemma 0. *If* $f_j \in L^p$, $1 \leqq j \leqq n$, $1 \leqq p \leqq 2$ *and if* $E(f_j | f_1, \ldots, f_{j-1}) = 0$ *for* $j = 2, \ldots, n$, *then*

$$\int |f_1 + \cdots + f_n|^p \leqq 2 \sum_{j=1}^{n} \int |f_j|^p.$$

Actually a slightly better constant than 2 is possible (see [2] or [3]) but this is of no consequence to us here.

We use in the sequel $\int f$ instead of $\int f(s)\, P(ds)$, the letter A for finite constants, positive, possibly different each time, immaterial to the argument at hand and the symbol $C\{\ldots\}$ for the function which is one on the set determined by the parentheses and zero outside.

In Section 3 we present four lemmas which may be of some methodological interest in themselves. Lemma 1 for the case $p=1$ is implicit in Komlós [4]. This crucial lemma is actually a reflection of the weak-compactness of distribution functions on the line with a bounded p-th moment. In Section 4 we prove our main Theorem 1 by means of the lemmas in Sections 3. In the final Section 5 we consider the case $p=2$.

§ 3. Some Lemmas

Lemma 1. *If F is a sequence in L^p, $0<p<2$ with $\sup_{f\in F} \int |f|^p < \infty$ then there is a sub-sequence F_0 of F such that for any sub-sequence $\{f_n\}$ of F_0 one has*

(a) $\sum_n P(|f_n| > n^{1/p}) < \infty$;

(b) $\sum_n n^{-2/p} \int |f_n|^2 \cdot C\{|f_n| \leqq n^{1/p}\} < \infty$

and

(c) $\sum_n n^{-1/p} \int |f_n| \cdot C\{|f_n| \leqq n^{1/p}\} < \infty$ *if* $0<p<1$

or

$\sum_n n^{-1/p} \int |f_n| \cdot C\{|f_n| > n^{1/p}\} < \infty$ *if* $1<p<2$.

Proof. Let $\sup_{f\in F} \int |f|^p = B < \infty$. Repeated use of the Bolzano-Weierstraß theorem and the familiar diagonal-choice procedure allows us to choose a sequence $f_n \in F$ such that

$$\lim_{n\to\infty} P(k < |f_n| \leqq k+1) = a_k \qquad \text{for } k=0, 1, 2, \ldots$$

and

$$P(k < |f_n| \leqq k+1) \leqq a_k + 2^{-k} \qquad \text{for } 0 \leqq k \leqq n^{2/p}. \tag{2}$$

Since

$$B \geqq \int |f_n|^p \geqq \sum_{k=1}^{\infty} k^p P\{k < |f_n| \leqq k+1\}$$

we have by Fatou's lemma (for series)

$$\sum_1^{\infty} k^p a_k \leqq \liminf_{n\to\infty} \sum_{k=1}^{\infty} k^p P\{k < |f_n| \leqq k+1\} \leqq B$$

hence

$$\sum_1^{\infty} k^p a_k < \infty. \tag{3}$$

We shall prove that the sub-sequence $F_0 = \{f_n : n \geqq 1\}$ satisfies the assertions of Lemma 1. Actually we shall show, using only the inequalities (2) and (3), that the norm-bounded sequence F_0 in L^p itself satisfies (a)–(c). Since any subsequence of F_0 will satisfy inequalities (2) and (3) as well, it too will fulfill (a)–(c) of Lemma 1, thus proving the latter.

Now

$$P(|f_n|>n^{1/p})\leqq \sum_{n^{1/p}-1<k<n^{2/p}} P(k<|f_n|\leqq k+1)+P(|f_n|\geqq n^{2/p})$$
$$\leqq \sum_{n^{1/p}-1<k<n^{2/p}} (a_k+2^{-k})+n^{-2}B$$

by (2) and Markov's inequality.

So

$$\sum_{n=1}^{\infty} P(|f_n|>n^{1/p})\leqq \sum_n \sum_{n^{1/p}-1<k<n^{2/p}} (a_k+2^{-k})+A$$
$$\leqq \sum_{k=1}^{\infty}(a_k+2^{-k})\cdot \sum_{k^{p/2}<n<(k+1)^p} 1+A$$
$$\leqq \sum_{k=1}^{\infty}(k+1)^p(a_k+2^{-k})+A$$

hence $\sum_{n=1}^{\infty} P(|f_n|>n^{1/p})<\infty$ by (3); this proves (a).

Again as above

$$\int |f_n|^2\cdot C\{|f_n|\leqq n^{1/p}\}\leqq \sum_{0\leqq k\leqq n^{1/p}} (k+1)^2\cdot P\{k<|f_n|\leqq k+1\}$$
$$\leqq \sum_{0\leqq k\leqq n^{1/p}} (k+1)^2(a_k+2^{-k}) \qquad \text{by (2)}$$

so that (here we use the fact $0<p<2$)

$$\sum_n n^{-2/p}\int |f_n|^2\cdot C\{|f_n|\leqq n^{1/p}\}\leqq \sum_n n^{-2/p}\sum_{0\leqq k\leqq n^{1/p}}(k+1)^2(a_k+2^{-k})$$
$$\leqq \sum_{k=0}^{\infty}(k+1)^2(a_k+2^{-k})\sum_{n\geqq k^p} n^{-2/p}$$
$$\leqq A\cdot \sum_{k=0}^{\infty}(k+1)^2(a_k+2^{-k})(k^p)^{-\frac{2}{p}+1}$$
$$\leqq A\cdot \sum_{k=0}^{\infty}(k+1)^p(a_k+2^{-k})<\infty \qquad \text{by (3).}$$

This proves (b). Similarly if $0<p<1$

$$\sum_{n=1}^{\infty} n^{-1/p}\cdot\int |f_n|\cdot C\{|f_n|\leqq n^{1/p}\}\leqq \sum_n n^{-1/p}\sum_{0\leqq k\leqq n^{1/p}}(k+1)(a_k+2^{-k})$$
$$=\sum_{k=0}^{\infty}(k+1)(a_k+2^{-k})\sum_{n\geqq k^p} n^{-1/p}$$
$$\leqq A\cdot\sum_{k=0}^{\infty}(k+1)^p(a_k+2^{-k})<\infty \qquad \text{by (3);}$$

this proves (c) for $0<p<1$.

To prove (c) if $1<p<2$ we note first that (with $p^{-1}+q^{-1}=1$)

$$\int |f_n| \cdot C\{|f_n| \geqq n^{2/p}\} \leqq \|f_n\|_p \cdot [P(|f_n| \geqq n^{2/p})]^{1/q} \leqq A \cdot n^{-2/q}$$

by applying Hölder's and Markov's inequalities successively. Hence

$$\sum_n n^{-1/p} \int |f_n| \cdot C\{|f_n| \geqq n^{2/p}\} \leqq A \cdot \sum_n n^{-\frac{1}{p}-\frac{2}{q}} = A \cdot \sum n^{-1-1/q} < \infty.$$

Also as before

$$\begin{aligned} &\sum_n n^{-1/p} \int |f_n| \cdot C\{n^{1/p} < |f_n| < n^{2/p}\} \\ &\leqq \sum_{n=1}^{\infty} n^{-1/p} \sum_{n^{1/p}-1<k<n^{2/p}} (k+1)(a_k+2^{-k}) \\ &\leqq \sum_{k=0}^{\infty} (k+1)(a_k+2^{-k}) \sum_{k^{p/2}<n<(k+1)^p} n^{-1/p} \\ &\leqq A \cdot \sum_{k=0}^{\infty} (k+1)(a_k+2^{-k}) \{(k+1)^p\}^{-\frac{1}{p}+1} \\ &= A \cdot \sum_{k=0}^{\infty} (k+1)^p (a_k+2^{-k}) < \infty \qquad \text{by (3).} \end{aligned}$$

The foregoing two inequalities clearly establish (c) for $1<p<2$ and this proves Lemma 1.

Lemma 2. *If* $0<p<1$ *and* $f_n \in L^p$ *is a sequence which satisfies* (a)–(c) *of Lemma* 1 *then*

$$\lim_{n\to\infty} n^{-1/p} \sum_{k=1}^{n} f_k = 0 \quad \text{a.e.}$$

There is L^p*-convergence if* $\lim_{n\to\infty} \int |f_n|^p \cdot C\{|f_n| > n^{1/p}\} = 0$, *in particular if the* $|f_n|^p$*'s are uniformly integrable.*

Proof. Define $\tilde{f}_n = f_n \cdot C\{|f_n| \leqq n^{1/p}\}$ and $\bar{f}_n = f_n - \tilde{f}_n$.

By (a) and Borel-Cantelli lemma, for a.e. x in S, there exists an integer $N(x) \geqq s$ such that $f_n(x) = \tilde{f}_n(x)$ for $n \geqq N(x)$. So to prove a.e. convergence above it suffices to prove that $\lim_{n\to\infty} n^{-1/p} \sum_1^n \tilde{f}_k = 0$ a.e. For this, note that in the identity

$$\sum_n n^{-1/p} \tilde{f}_n = \sum_n n^{-1/p} (\tilde{f}_n - \beta_n) + \sum_n n^{-1/p} \cdot \beta_n$$

where $\beta_n = E(\tilde{f}_n | f_1, \ldots, f_{n-1})$, the first series on the right converges a.e. and in L^2 by the martingale convergence theorem (0) since by (b) we have

$\sum_n n^{-2/p} \int |\tilde{f}_n|^2 < \infty$. The second series converges a.e. and in L^1 since

$$\sum_n n^{-1/p} \int |\beta_n| \leqq \sum_n n^{-1/p} \int |\bar{f}_n| < \infty \qquad \text{by (c).}$$

Thus $\sum_n n^{-1/p} \tilde{f}_n$ converges a.e. and in L^1 which proves (by the so-called Kronecker's lemma) that $\lim_{n\to\infty} n^{-1/p} \sum_1^n \tilde{f}_k = 0$ a.e. and in L^1. This proves the a.e.-convergence part of Lemma 2. To prove the other part note that $n^{-1/p} \sum_1^n f_k = n^{-1/p} \sum_1^n \tilde{f}_k + n^{-1/p} \sum_1^n \bar{f}_k$ and that the first expression on the right converges to 0 in L^1 and hence in L^p, $0<p<1$. It remains only to show that the second term goes to 0 in L^p i.e. $\lim_{n\to\infty} n^{-1} \int \left|\sum_1^n \bar{f}_k\right|^p = 0$. But since $0<p<1$, we have

$$\begin{aligned} 0 \leqq \limsup_{n\to\infty} n^{-1} \int \left|\sum_1^n \bar{f}_k\right|^p &\leqq \lim_{n\to\infty} n^{-1} \sum_1^n \int |\bar{f}_k|^p \\ &= \lim_{n\to\infty} \int |\bar{f}_n|^p = \lim_{n\to\infty} \int |f_n|^p \cdot C\{|f_n| > n^{1/p}\} \\ &= 0 \end{aligned}$$

if the further assumption of Lemma 2 is valid. This completes the proof of Lemma 2.

Lemma 3. *If* $f_n \in L^p$, $1<p<2$, *satisfy* (a)–(c) *of Lemma* 1 *then*

$$\lim_{n\to\infty} n^{-1/p} \sum_1^n (f_k - \Theta_k) = 0 \quad a.e.$$

where $\Theta_k = E(f_k | f_1, \dots, f_{k-1})$. *There is* L^p*-convergence as well if the* $|f_n|^p$*'s are uniformly integrable* (or a little more generally if $\lim_{n\to\infty} \int |f_n|^p \cdot C(|f_n| > n^{1/p}) = 0$).

Proof. As before, define $\tilde{f}_n = f_n \cdot C\{|f_n| \leqq n^{1/p}\}$, $\bar{f}_n = f_n - \tilde{f}_n$ and $\beta_n = E(\tilde{f}_n | f_1, \dots, f_{n-1})$. Then in the identity

$$\sum_n n^{-1/p}(f_n - \Theta_n) = \sum_n n^{-1/p}(\tilde{f}_n - \beta_n) + \sum_n n^{-1/p}(\bar{f}_n + \beta_n - \Theta_n)$$

the first series on the right converges a.e. and in L^2 (hence in L^p, $p<2$) by the martingale convergence theorem (0) and condition (b) of Lemma 1 and the second converges (absolutely) a.e. and in L^1 since

$$\sum_n n^{-1/p} \int |\bar{f}_n + \beta_n - \Theta_n| \leqq 2 \sum_n n^{-1/p} \int |\bar{f}_n| < \infty$$

by condition (c) of Lemma 1. This proves the a.e. convergence part of Lemma 3. To prove L^p-convergence under the indicated condition, it suffices to prove that

$$n^{-1/p}\sum_{1}^{n}(\bar{f}_k+\beta_k-\Theta_k)\to 0 \quad \text{in } L^p.$$

Since $E(\bar{f}_k|f_1,\dots,f_{k-1})=\Theta_k-\beta_k$, the sequence $\{\bar{f}_k+\beta_k-\Theta_k\}$ is a martingale-difference sequence and so by the Esseen-von Bahr inequality (Lemma 0)

$$n^{-1}\int\left|\sum_{1}^{n}\bar{f}_k+\beta_k-\Theta_k\right|^p \leqq 2n^{-1}\sum_{1}^{n}\int|\bar{f}_k+\beta_k-\Theta_k|^p$$

$$\leqq 8n^{-1}\sum_{1}^{n}\int|\bar{f}_k|^p\to 0$$

as $n\to\infty$ since $\int|\bar{f}_k|^p\to 0$ by the supplementary assumption of Lemma 3. This completes the proof of Lemma 3.

Although we shall not need it, we state for the sake of completeness the analogue of Lemmas 2 and 3 for $p=1$. We keep the notation as above. Then

Lemma 4. *If* $f_n\in L^1$ *satisfies* (a) *and* (b) *of Lemma* 1 *for* $p=1$ *and if further* $E(\bar{f}_n|f_1,\dots,f_{n-1})\to 0$ *a.e. then*

$$\lim_{n\to\infty} n^{-1}\sum_{1}^{n}(f_k-\Theta_k)=0 \quad a.e.$$

There is L^1*-convergence as well if* $\lim_{n\to\infty}\int|\bar{f}_n|=0$, *in particular, if the* f_n*'s are uniformly integrable.*

The proof of Lemma 4 can be read off instantly from that of Lemma 3 and hence we omit it.

Note that the proofs above show that for any sequence c_n with $n^{1/p}/c_n\to 0$, $\lim_{n\to\infty} c_n^{-1}\sum_{k=1}^{n}(f_k-\Theta_k)=0$ in L^p (and naturally a.e. as well) without any uniform integrability conditions. This comment makes the Banach-Saks theorem referred to in the introduction a consequence of Theorem 1 for $1\leqq p<2$.

§ 4. Proof of Theorem 1

We come now to the proof of Theorem 1. We divide the proof into two parts, the first for $0<p<1$, the second for $1<p<2$. The case $p=1$ is just Komlós' theorem [4] and we omit its proof.

The case $0<p<1$ is the easiest. Here we choose a suitable subsequence F_0 as guaranteed by Lemma 1 and then Lemma 2 completes the proof in this case.

In the case $1<p<2$, we first choose a sub-sequence $\{f_n\}$ of F and $\bar{f}\in L^p$ such that $f_n\to\bar{f}$ weakly in L^p. This can be done because of the norm-boundedness of F. Now we choose a sequence g_n of measurable simple functions (i.e. each g_n takes only a finite number of values) such that $\|(f_n-\bar{f})-g_n\|_p<2^{-n}$. Then we know by standard arguments that $\sum_n(f_n-\bar{f}-g_n)$ is absolutely convergent a.e. and in L^p. Also $g_n\to 0$ weakly in L^p and the $|g_n|^p$'s will be uniformly integrable if the $|f_n|^p$'s are. It will therefore be sufficient to prove the theorem for the sequence $\{g_n\}$. By passing to a further sub-sequence we can ensure that the sequence $G_0=\{g_n\}$ satisfies (a), (b) and (c) of Lemma 1 and further satisfies

$$\text{(d)}\qquad |E(g_n|g_{i_1},\dots,g_{i_k})|\leqq 2^{-n}$$

for any choice of $i_1,\dots,i_k\leqq n-1$.

Condition (d) can be assured by the following argument. Since the original sequence $g_n\to 0$ weakly in L^p, we have $\int_A g_n\to 0$ for any $A\in\Sigma$. The family of all sets in all the Borel fields generated by $g_{i_1},\dots,g_{i_k}$ for all choices of $i_1,\dots,i_k\leqq n-1$ is finite and so the conditional expectations in (d) are of the form $[P(A)]^{-1}\int_A g_n$ on certain (finite number of) subsets. Hence a suitable choice of a sub-sequence will ensure (d). It is also clear that if G_0 satisfies (a)–(d) then any sub-sequence of G_0 will do the same. For the sake of simplicity of presentation, we complete the proof for the sequence g_n itself. This is now very easy. By Lemma 3 we have

$$n^{-1/p}\sum_1^n(g_k-\Theta_k)\to 0\quad\text{a.e.},$$

where $\Theta_k=E(g_k|g_1,\dots,g_{k-1})$ and by condition (d), $\sum_1^\infty|\Theta_k|\leqq 1$ a.e. So $n^{-1/p}\sum_1^n g_k\to 0$ a.e. That L^p-convergence is ensured under the supplementary condition follows from Lemma 3 similarly. Thus the proof for $1<p<2$ and of the theorem is thus completed.

§ 5. Case $p=2$

Here we can prove the following theorem, essentially that of Révész [7]. Our proof following the methods of this paper is much simpler than that of Révész.

Theorem 2. *If $f_n\in L^2$ converges weakly to $f\in L^2$ then there is a sub-sequence $n_1<n_2<\cdots$ such that $\sum_k c_k(f_{n_k}-f)$ converges a.e. and in L^2 for every sequence c_k with $\sum_k|c_k|^2<\infty$.*

Corollary 1. *Given a bounded sequence F in L^2 we can choose a sub-sequence $\{f_n\}$ of F and a function $f \in L^2$ such that whenever $\sum_k |c_k|^2 < \infty$, the series $\sum_n c_n(f_n - f)$ converges a.e. and in L^2. Hence if $c_n \downarrow 0$ then*

$$\lim_{n\to\infty} c_n \sum_{j=1}^{n} (f_{k_j} - f) = 0$$

a.e. and in L^2 for every sub-sequence f_{k_j} of f_n.

Corollary 2. *From any orthogonal sequence $f_n \in L^2$ one can always choose a sub-sequence $\{f_{n_k}\}$ such that $\sum_k c_k f_{n_k}$ converges a.e. and in L^2 whenever $\sum_k |c_k|^2 < \infty$.*

This follows immediately from Theorem 2 because an orthonormal sequence $f_n \to 0$ weakly in L^2.

In the case of special orthonormal series Corollary 2 has been known (see Zygmund [10]) for a long time. E.g. for the trigonometric system any lacunary sub-sequence $n_{k+1}/n_k \geqq q > 1$ satisfies Corollary 2. For the Walsh-Paley sequence, the Rademacher functions provide an appropriate sub-sequence. Of course for the trigonometric case the famous recent a.e. convergence theorem for Fourier Series of L. Carleson shows that the original sequence itself satisfies Corollary 2.

For the proof of Theorem 2 we first choose a sequence g_n of simple functions such that $\|(f_n - f) - g_n\|_2 \leqq 2^{-n}$. It is clear that if Theorem 2 has been proved for a sub-sequence g_{n_k} of $\{g_n\}$ then the corresponding sub-sequence f_{n_k} will satisfy the assertion of Theorem 2. Since g_n's are simple functions converging to zero weakly in L^2, we can, by an argument used before, choose a sub-sequence g_{n_k} such that

$$|E(g_{n_k} | g_{n_1}, \ldots, g_{n_{k-1}})| \leqq 2^{-k}, \quad k \geqq 1.$$

Now suppose $\sum_k |c_k|^2 < \infty$ and consider the series $\sum_k c_k(g_{n_k} - \Theta_k)$ where $\Theta_k = E(g_{n_k} | g_{n_1}, \ldots, g_{n_{k-1}})$. This series converges a.e. and in L^2 by the martingale convergence Theorem 0 because the weakly convergent sequence (g_n) is norm-bounded and therefore

$$\sum_k |c_k|^2 \int |g_{n_k}|^2 \leqq A \cdot \sum_k |c_k|^2 < \infty.$$

Since $\sum_k |\Theta_k| \leqq 1$, it follows that $\sum_k c_k g_{n_k}$ is convergent a.e. and in L^2. This completes the proof of Theorem 2.

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On Some Results Concerning Uniform Approximation

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In [2], Glicksberg proved a very interesting result directly related to uniform approximation, which was in turn extended by Garnett and Glicksberg in [1] (see Theorem 1 below). Here we show that these results can be derived in a very brief manner from the general theory developed in [3]. Moreover our proof does not require at all the use of the (extended) von Neumann minimax theorem which plays a fundamental role in the proofs given in [2, 1]. We also indicate an extension of the above mentioned Garnett-Glicksberg result.

A shall denote a function algebra on a compact Hausdorff space X. Let $\mathcal{M}(A)$ denote the spectrum (Gelfand space) of A, and for $\gamma \in \mathcal{M}(A)$, write $M_\gamma = \{$all representing measures for γ on X, relative to $A\}$. We shall use freely the notions and notations of [3]. Let us however recall a few basic facts. One defines, for $0<p<+\infty$, $|||f|||_p^p$ as $\sup_{\mu \in M_\gamma} \int |f|^p \, d\mu$ provided the latter makes sense and is finite, and then defines $L^p(\gamma)$ with the usual type of identificationes. To each $\gamma \in \mathcal{M}(A)$ there are thus associated Hardy spaces $H^p(\gamma)$ defined as being the closure of A in $L^p(\gamma)$. For details concerning these spaces and the conjugation operator "$*$" – defined modulo γ-null functions for $u \in \operatorname{Re} A$, by $u + i\, {}^*u \in A$, $\gamma(u + i\, {}^*u)$ real, and then extended to a larger domain $D(*) \subset \operatorname{Re} L^1(\gamma)$ – we refer the reader to [3]. The "unique extension space $\mathscr{E}_\gamma = \{u \in C_{\boldsymbol{R}}(X) : \int u \, d\mu =$ constant as μ varies in $M_\gamma\}$ plays a crucial role, and we recall explicitely the following result of [3] (Theorem I.6.2 and proof; see also I.6.4):

$$\text{If } u \in \mathscr{E}_\gamma, \text{ then } u \in D(*) \text{ and } u + i\, {}^*u \in H^2(\gamma).$$

1. Theorem (Garnett-Glicksberg)[1]. *Suppose A and B are function algebras, $A \subset B \subset C(X)$, and there is no non-trivial completely singular element in $A^\perp$. Then $A = B$ iff: 1) $\mathcal{M}(A) = \mathcal{M}(B)$; 2) $M_\gamma(A) = M_\gamma(B) \; \forall \gamma \in \mathcal{M}(A)$, or equivalently (under $\mathcal{M}(A) = \mathcal{M}(B)$) $\operatorname{Re} B \subset \mathscr{E}_\gamma(A) \; \forall \gamma \in \mathcal{M}(A)$.*

Proof. Let $f \in B$, $f = u + i v$ with u and v real. By assumption $u \in \mathscr{E}_\gamma(A)$. Hence $u + i\, {}^*u \in H^2(\gamma, A) \subset H^2(\gamma, B)$, and so ${}^*u - v \in H^2(\gamma, B)$ and is real,

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[1] A slightly more general version is given in [1] and can be treated similarly to what we do in Theorem 1; see Remarks 5 below. Also notice that we use here the terminology "completely singular" (used in [2]), for what is called "totalement singulier" in [3].

therefore constant γ. a.e. by I.4.3 of [3]. Thus, $f=u+iv\in H^2(\gamma, A), \forall f\in B$, and hence $B=A$ by I.7.4 of [3]. Q.E.D.

2. Corollary (Glicksberg). *Suppose A and B are function algebras, $A\subset B\subset C(X)$, and there is no non-trivial completely singular element in $A^\perp$. Then $A=B$ iff:* 1) $\mathcal{M}(A)=\mathcal{M}(B)$; 2) $(\operatorname{Re} A)^\perp=(\operatorname{Re} B)^\perp$, i.e. *the real-orthogonal measures for* Re A *and* Re B *are the same.*

This corollary obtained earlier by Glicksberg in [2], follows from Theorem 1 since it is easily seen that if $\mathcal{M}(A)=\mathcal{M}(B)$, $(\operatorname{Re} A)^\perp=(\operatorname{Re} B)^\perp$ implies $M_\gamma(A)=M_\gamma(B)\ \forall\gamma\in\mathcal{M}(A)$.

The following two results are obtained in [2] via the minimax theorem, and used in deriving the theorem (and corollary) above. It seems of interest to show how they too can be obtained without appeal to the minimax, which we do below.

3. Lemma ([2]). *If $f\in C(X)$ and $f\in H^2(\mu)\ \forall\mu\in M_\gamma$, then $f\in H^2(\gamma)$.*

Proof. For arbitrary μ_1, μ_2 in M_γ, we have $f\in H^2(\mu)$ where $\mu=\frac{1}{2}(\mu_1+\mu_2)$. So $\exists f_n\in A, f_n\to f$ in $H^2(\mu)$. This implies $f_n\to f$ in $H^2(\mu_1)$ and $H^2(\mu_2)$, and therefore

$$\int f\,d\mu_1=\lim_n \int f_n\,d\mu_1=\lim_n \gamma(f_n)=\lim_n \int f_n\,d\mu_2=\int f\,d\mu_2.$$

Hence $\int f\,d\mu$ is constant for $\mu\in M_\gamma$, and therefore $u=\operatorname{Re} f$ and $v=\operatorname{Im} f$ are in $\mathcal{E}_\gamma$. We may suppose $\gamma(u)=\gamma(v)=0$. Now $u+i\,{}^*u\in H^2(\gamma)$ since $u\in\mathcal{E}_\gamma$. In particular ${}^*u-v\in H^2(\mu)\forall\mu\in M_\gamma$, from which we have ${}^*u=v\,\mu$. a. e. (see lines 5 and 6 of the Proof I.7.5 of [3]). Thus $v={}^*u$ γ.a.e., and $f=u+iv\in H^2(\gamma)$. Q.E.D.

The above and I.3.10 of [3] give the following

4. Corollary ([2] Corollary 2.3). *If $f\in C(X)$ and belongs to $H^2(\mu)$ for each $\mu\in M_\gamma$, then $\exists$ a sequence $\{f_n\}$ in the $\|f\|_\infty-$ ball of A, such that $f_n\to f$ μ.a.e. for all $\mu\in M_\gamma$.*

5. *Remarks.* It is clear from [3] (see I.7.4), that in Theorem 1 one can replace "$\mathcal{M}(A)=\mathcal{M}(B)$" and "$\forall\gamma\in\mathcal{M}(A)$" by "One γ in each part of $\mathcal{M}(A)$ is in $\mathcal{M}(B)$, and for these γ...", as is done in [2], without any change in our proof. Also, as in Theorem 1.9 of [1] for instance, one can replace in Theorem 1 above "there is no non-trivial completely singular element in $A^\perp$" by "there exists no completely singular extreme point in ball $A^\perp$", and our proof is adapted to that situation without difficulty via the Krein-Milman theorem. Again, one can derive without appeal to the minimax, via Lemma 3 above, the characterization given in [2] for A provided $\nexists$ completely singular extreme points in ball $A^\perp$:

$$A=\{f\in C(X):\ f\in H^2(\mu, A),\ \text{all } \mu\in M_\gamma \text{ for one } \gamma \text{ in each part of } \mathcal{M}(A)\}.$$

Also other related results of [2], [1], which instead of making assumptions on completely singular elements of $A^{\perp}$, suppose the existence of an appropriate $F \subset A$, of elements invertible in $C(X)$, such that $F^{-1} \cup A$ generates $C(X)$, can be given proofs shortened along the above lines and not requiring the minimax theorem.

One also sees through the arguments used above that weak-star compactness (as holds for M_γ) is not essential for the validity of a result of the kind of Theorem 1; explicitely, we give the following extension.

6. Theorem. *Suppose A and B are function algebres, $A \subset B \subset C(X)$, and $\nexists$ any completely singular extreme point in ball $A^{\perp}$. Then $A = B$ iff:* 1) *One γ in each part of $\mathscr{M}(A)$ is in $\mathscr{M}(B)$.* 2) *For each such γ there is M convex and weak-star dense in $M_\gamma(A)$ such that* Re B *is in the* $L^1(\mu)$ *closure of* Re A *for every $\mu \in M$.*

Proof. If $u \in \operatorname{Re} B$, the convexity argument used in the proof of Lemma 3 above, shows that for each M as described in the hypothesis $\int u \, d\mu$ is constant as μ ranges over M, and since M is weak-star dense, therefore constant on M_γ. Hence $u \in \mathscr{E}_\gamma$. Thus $\operatorname{Re} B \subset \mathscr{E}_\gamma$ for one γ in each part, and in this case it follows that $B = A$ (we are back to the situation covered in Theorem 1, Remarks 5, i.e. the Garnett-Glicksberg result). Q.E.D.

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Sur la formule des caractères de H. Weyl

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Dans cet article, on se propose de montrer que la formule des caractères se déduit simplement du théorème de Bott énoncé dans [2].

Dans toute la suite, on désigne par k un corps de caractéristique 0 (cette dernière hypothèse n'est utilisée qu'à partir du n° 2), par G un k-groupe semi-simple déployé simplement connexe, par T un tore maximal déployé de G, par B un groupe de Borel contenant T.

1. Images directes et réciproques de représentations

Pour tout k-schéma algébrique X où G opère, on note $\mathscr{K}_G(X)$ la catégorie des G-modules cohérents sur X, c'est-à-dire des $\mathscr{O}_X$-modules cohérents $\mathscr{E}$ munis d'un relèvement à $\mathscr{E}$ de l'opération de G (cf. [3], § 3, déf. 1.6 *mutatis mutandis*); le groupe de Grothendieck de cette catégorie abélienne est noté $K_G(X)$, l'image de l'objet $\mathscr{E}$ de $\mathscr{K}_G(X)$ dans $K_G(X)$ est notée $\mathrm{cl}(\mathscr{E})$.

Soit P un sous-groupe parabolique de G; notons $\mathscr{R}(P)$ la catégorie abélienne des représentations linéaires de dimension finie de P, $R(P)$ le groupe de Grothendieck de $\mathscr{R}(P)$ et $\mathrm{cl}(V)$ l'image dans $R(P)$ de l'objet V de $\mathscr{R}(P)$. Considérons les deux foncteurs additifs

$$\Phi\colon\ \mathscr{R}(P)\to\mathscr{K}_G(G/P)$$
$$\Psi\colon\ \mathscr{K}_G(G/P)\to\mathscr{R}(P)$$

définis comme suit:

à la représentation $\rho\colon P\to GL(V)$, Φ fait correspondre le fibré associé $G\times^P V$ qui est un $\mathscr{O}_{G/P}$-module localement libre (car la fibration $G\to G/P$ est localement triviale); au G-module $\mathscr{E}$ sur G/P, Ψ fait correspondre la représentation de P dans la fibre de $\mathscr{E}$ au point marqué de G/P, déduite de l'opération de G sur $\mathscr{E}$. On vérifie sans difficultés que ces deux foncteurs sont quasi-inverses l'un de l'autre. Il en résulte en particulier que tout objet de $\mathscr{K}_G(G/P)$ est un $\mathscr{O}_{G/P}$-module localement libre et que Φ induit un *isomorphisme de groupes*

$$\varphi_P\colon\ R(P)\to K_G(G/P).$$

En outre, si l'on munit $R(P)$ et $K_G(G/P)$ de leurs structures d'anneaux déduites du produit tensoriel, φ_P est un *isomorphisme d'anneaux.*

Si G opère sur les k-schémas algébriques X et Y et si $f\colon X \to Y$ est un morphisme (*resp.* un morphisme *propre*) compatible avec les opérations de G, on définit une application $f^*\colon K_G(Y) \to K_G(X)$ (*resp.* une application $f_!\colon K_G(X) \to K_G(Y)$) par

$$f^*(\mathrm{cl}(\mathscr{F})) = \mathrm{cl}(f^*(\mathscr{F}))$$

(*resp.* $f_!(\mathrm{cl}(\mathscr{E})) = \sum (-1)^n \mathrm{cl}(\mathscr{R}^n f_*(\mathscr{E}))$), où $f^*(\mathscr{F})$ est l'image réciproque du $\mathscr{O}_Y$-module $\mathscr{F}$ (*resp.* où $\mathscr{R}^n f_*(\mathscr{E})$ est la n-ième image directe supérieure du $\mathscr{O}_X$-module $\mathscr{E}$), que l'on munit de sa structure naturelle de G-module, déduite de la structure de G-module de $\mathscr{F}$ (*resp.* $\mathscr{E}$). Si f est propre, et si $y \in K_G(Y)$ est la classe d'un $\mathscr{O}_Y$-module localement libre, on a aussitôt

$$f_!(f^*(y)\, x) = y\, f_!(x), \qquad x \in K_G(X).$$

En particulier, considérons deux sous-groupes paraboliques P et Q de G tels que $P \subset Q$. Désignons par $i\colon P \to Q$ l'inclusion, par $f\colon G/P \to G/Q$ la projection canonique et par $i^*\colon R(Q) \to R(P)$ l'homomorphisme de restriction. On a un diagramme commutatif:

$$\begin{array}{ccc} R(Q) & \xrightarrow[\varphi_Q]{\sim} & K_G(G/Q) \\ {\scriptstyle i^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ R(P) & \xrightarrow[\varphi_P]{\sim} & K_G(G/P). \end{array}$$

On définit un homomorphisme de groupes $i_!\colon R(P) \to R(Q)$ par le diagramme commutatif:

$$\begin{array}{ccc} R(P) & \xrightarrow[\varphi_P]{\sim} & K_G(G/P) \\ {\scriptstyle i_!}\big\downarrow & & \big\downarrow{\scriptstyle f_!} \\ R(Q) & \xrightarrow[\varphi_Q]{\sim} & K_G(G/Q); \end{array}$$

on a donc

$$i_!(i^*(\beta)\,\alpha) = \beta\, i_!(\alpha), \qquad \alpha \in R(P), \qquad \beta \in R(Q). \tag{1}$$

Par exemple, si $P = B$ et $Q = G$, on a $R(P) = R(B) = R(T)$, $G/Q = \operatorname{Spec} k = \mathrm{pt}$, d'où un diagramme commutatif:

$$\begin{array}{ccc} R(G) & \xrightarrow[\varphi_G]{\sim} & K_G(\mathrm{pt}) \\ {\scriptstyle i^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ R(T) & \xrightarrow[\varphi_B]{\sim} & K_G(G/B) \\ {\scriptstyle i_!}\big\downarrow & & \big\downarrow{\scriptstyle f_!} \\ R(G) & \xrightarrow[\varphi_G]{\sim} & K_G(\mathrm{pt}), \end{array}$$

où i^* est l'homomorphisme de restriction et où, si $\rho\colon T \to GL(V)$ est une représentation de T, on a

$$i_!(\mathrm{cl}(\rho)) = \sum (-1)^n \mathrm{cl}(H^n(G/B, G \times^B V)). \tag{2}$$

2. La formule des caractères

Notons M le groupe des caractères de T; pour $\chi \in M$, soit e^χ l'élément de $R(T)$ obtenu en faisant opérer T sur k grâce à χ. Il est clair ([4], n° 3.4) que $R(T)$ s'identifie ainsi à l'algèbre $\mathbf{Z}[M]$ du groupe M. D'autre part, si W désigne le groupe de Weyl de G relativement à T, on sait ([4], n° 3.6) que $i_*: R(G) \to R(T)$ est *injectif* et que son image est $R(T)^W = \{x \in R(T) | wx = x$ pour tout $w \in W\}$. Pour calculer $i_!$: $R(T) \to R(G)$, il suffit donc de calculer $i^* i_!: R(T) \to R(T)$. Notons J l'endomorphisme de $R(T)$ défini par

$$J(x) = \sum_{w \in W} \varepsilon_w\, w(x),$$

où, pour $w \in W$, on désigne par ε_w la signature de w. Soit $\rho \in M$ la demi-somme des racines de B relativement à T. D'après [1], § 3, n° 3, prop. 2, $J(x e^\rho)$ est divisible par $J(e^\rho)$ pour tout $x \in R(T)$ et on a $J(x e^\rho)/J(e^\rho) \in R(T)^W$.

Théorème. *Pour* $x \in R(T)$, *on a* $i^* i_!(x) = J(x e^\rho)/J(e^\rho)$.

En particulier, prenons $x = e^\chi$, où χ est un *poids dominant*. Si $\varphi_B(e^\chi) = \mathscr{L}(\chi)$ est l'élément correspondant de $R_G(G/B)$, on a $H^n(G/B, \mathscr{L}(\chi)) = 0$ pour $n > 0$ ([2], n° 8) et la représentation de G dans $H^0(G/B, \mathscr{L}(\chi))$ est simple de plus haut poids χ. D'où, d'après le théorème et la formule (2):

Corollaire (formule des caractères). *Si* χ *est un poids dominant et si* E_χ *est un G-module simple de plus haut poids* χ, *on a dans* $R(T)$

$$i^*(\mathrm{cl}(E_\chi)) = J(e^{\chi+\rho})/J(e^\rho).$$

3. Démonstration

D'après la formule (1) et le fait que $i_*(R(G)) = R(T)^W$, on a

$$i^* i_!(x y) = x\, i^* i_!(y), \qquad x \in R(T)^W, \qquad y \in R(T). \tag{3}$$

D'autre part, d'après [2], n° 8,

$$i_!(1) = 1. \tag{4}$$

Lemme. *Pour* $x \in R(T)$ *et* $w \in W$, *on a*

$$i_!(w(x e^\rho) e^{-\rho}) = \varepsilon_w\, i_!(x) \tag{5}$$

où ε_w *est la signature de* w.

Il suffit de démontrer (5) lorsque $x = e^\chi$ où χ est un caractère de T. Alors $i_!(e^\chi) = \sum (-1)^n \mathrm{cl}(H^n(G/B, \mathscr{L}(\chi)))$. D'après [2], n° 7, on a, pour tout $n \in \mathbf{Z}$,

$$\mathrm{cl}(H^n(G/B, \mathscr{L}(\chi))) = \mathrm{cl}(H^{n+\ell(w)}(G/B, \mathscr{L}(w(\chi+\rho)-\rho))),$$

où $\ell(w)$ est la longueur de w (relativement aux symétries fondamentales associées à B), donc

$$\varepsilon_w\, i_!(e^\chi)=(-1)^{\ell(w)}\, i_!(e^\chi)=\sum_n (-1)^{n+\ell(w)}\,\mathrm{cl}\big(H^{n+\ell(w)}(G/B, \mathscr{L}(w(\chi+\rho)-\rho))\big)$$
$$=i_!(e^{w(\chi+\rho)-\rho})=i_!\big(w(e^\chi e^\rho)\, e^{-\rho}\big).$$

Démontrons maintenant le théorème. D'après (5), on a

$$\mathrm{Card}(W)\cdot i_!(x)=\sum_{w\in W}\varepsilon_w\, i_!\big(w(x\, e^\rho)\, e^{-\rho}\big)=i_!\big(J(x\, e^\rho)\, e^{-\rho}\big).$$

Comme $J(x\, e^\rho)/J(e^\rho)\in R(T)^W$, on a d'après (3)

$$\mathrm{Card}(W)\, i^*\, i_!(x)=\big(J(x\, e^\rho)/J(e^\rho)\big)\cdot i^*\, i_!\big(J(e^\rho)\, e^{-\rho}\big),$$

donc

$$i^*\, i_!(x)=\alpha\, J(x\, e^\rho)/J(e^\rho),$$

où $\alpha\in R(T)\otimes\mathbf{Q}$ est indépendant de x. Faisant $x=1$ et utilisant (4), on voit que $\alpha=1$, d'où le théorème.

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Derivations of Matroid C^*-Algebras

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Introduction. The main object of this paper is to give an elementary proof that every derivation of a matroid C^*-algebra with unit is inner (Theorem 14). (A C^*-algebra A is called matroid ([2], Definition 1.1) if for every $x_1, \ldots, x_n \in A$ and $\varepsilon > 0$ there exists a sub-C^*-algebra B of A isomorphic to a matrix algebra (that is, to the C^*-algebra of all $r \times r$ complex matrices for some integer $r > 0$) such that the distance from $x_1, \ldots, x_n$ to B is less than ε (that is, there exist $y_1, \ldots, y_n \in B$ with $\|x_i - y_i\| \leqq \varepsilon$, $i = 1, \ldots, n$).) Sakai, in [8], proves a more general result, and Sakai in [7], Lance in [5] consider special cases; all three of these papers use von Neumann algebra techniques. The present proof is based on the following observation, purely algebraic in nature. Let A be a C^*-algebra, and suppose that B is a sub-C^*-algebra of A isomorphic to a matrix algebra. Then every derivation from B into A is inner (that is, is implemented by an element of A). Example 15 shows that this need not hold if B is just assumed to be matroid with unit.

For certain matroid C^*-algebras without unit, examples of outer derivations are given in 16.

1. **Lemma.** *Let A be a C^*-algebra with unit* 1, *and suppose that B is a sub-C^*-algebra of A containing* 1 *and isomorphic to a matrix algebra. Let D be a derivation from B into A. Then there exists $y \in A$ such that $D = \operatorname{ad} y | B$. If $DB \subset B$ then y may be chosen $\in B$.*

Proof. Let $(e_{ij})_{i,j=1,\ldots,n}$ be a system of matrix units for B. Set $y = \sum_{i=1}^n (D e_{i1}) e_{1i}$. Then $y \in A$, and if $DB \subset B$ then $y \in B$.

We have $y e_{ij} = (D e_{i1}) e_{1j}$ $(i, j = 1, \ldots, n)$.

Since $D1 = D(1^2) = (D1)1 + 1(D1) = 2D1$, $D1 = 0$.

Since $1 \in B$, we must have $1 = \sum_{i=1}^n e_{ii}$; therefore

$$\begin{aligned} 0 = D\left(\textstyle\sum_{i=1}^n e_{ii}\right) &= D\left(\textstyle\sum_{i=1}^n e_{i1} e_{1i}\right) = \textstyle\sum_{i=1}^n D(e_{i1} e_{1i}) \\ &= \textstyle\sum_{i=1}^n \left((D e_{i1}) e_{1i} + e_{i1} D e_{1i}\right) = y + \textstyle\sum_{i=1}^n e_{i1} D e_{1i}. \end{aligned}$$

Hence $e_{ij} y = -e_{i1} D e_{1j}$ $(i, j = 1, \ldots, n)$.

Now we have

$$y e_{ij} - e_{ij} y = (D e_{i1}) e_{1j} + e_{i1} D e_{1j} = D(e_{i1} e_{1j}) = D e_{ij} \qquad (i, j = 1, \ldots, n).$$

By linearity, $D = \operatorname{ad} y | B$.

Remark. The above proof works equally well if A is an arbitrary algebra with unit over a commutative ring k with unit and B is a subalgebra of A containing the unit of A and isomorphic to the algebra of all $r \times r$ matrices over k for some integer $r > 0$. The assumption that A have a unit is in fact unnecessary, but will be satisfied whenever Lemma 1 is applied in this paper.

2. The following lemma can easily be deduced from a theorem of Tomiyama [11]; a somewhat more direct proof is given below. The proof of (iii) is due to Stinespring [10].

Lemma. *Let A be a C^*-algebra with unit* 1 *and suppose that B is a sub-C^*-algebra of A containing* 1 *and isomorphic to a matrix algebra. Let f be a state on A (that is, a positive linear functional on A such that $f(1)=1$). Let B' denote the commutant of B in A. Then there exists a unique linear mapping $P\colon A \to B$ such that $P(x\,y) = x\,f(y)$ whenever $x \in B$ and $y \in B'$. P has the following properties:*

(i) $P(x^*) = (P\,x)^* \quad (x \in A)$;

(ii) $P(x\,y) = x\,P\,y,\ P(y\,x) = (P\,y)\,x \quad (x \in B,\ y \in A)$;

(iii) $P(A^+) = B^+$;

(iv) $(P\,x)^*(P\,x) \leqq P(x^*\,x) \quad (x \in A)$;

(v) $\|P\| = 1$.

Proof. Let $(e_{ij})_{i,j=1,\dots,n}$ be a system of matrix units for B. Then if $x \in A$ we have $x = \sum_{i,j} e_{ij}\,x_{ij}$ with $x_{ij} = \sum_k e_{ki}\,x\,e_{jk} \in B'$ $(i,j=1,\dots,n)$. Hence P, if it exists, is unique.

Let us show that $P\colon A \to B$ defined by $P\,x = \sum_{i,j} e_{ij}\,f\big(\sum_k e_{ki}\,x\,e_{jk}\big)$ $(x \in A)$ fulfills the requirements.

If $x \in A$ and $x = \sum_{i,j} e_{ij}\,x_{ij}$ with all $x_{ij} \in B'$ then we must have $x_{ij} = \sum_k e_{ki}\,x\,e_{jk}$ $(i,j=1,\dots,n)$. Let $y \in B'$. Then $P(e_{ij}\,y) = e_{ij}\,f(y)$ $(i,j=1,\dots,n)$, and by linearity, $P(x\,y) = x\,f(y)$ for all $x \in B$.

If $x \in A$, then

$$P(x^*) = \textstyle\sum_{i,j} e_{ij}\,f\big(\sum_k e_{ki}\,x^*\,e_{jk}\big) = \sum_{i,j} e_{ij}\,\overline{f\big(\sum_k e_{jk}^*\,x\,e_{ki}^*\big)}$$
$$= \big(\textstyle\sum_{i,j} e_{ji}\,f\big(\sum_k e_{kj}\,x\,e_{ik}\big)\big)^* = (P\,x)^*;$$

thus (i) is proved.

Let us prove (ii). By linearity in y, we need consider only the case $y = y_1\,y_2$ with $y_1 \in B$, $y_2 \in B'$. Then $P(x\,y) = P(x\,y_1\,y_2) = x\,y_1\,P\,y_2 = x\,P(y_1\,y_2) = x\,P\,y$, and $P(y\,x) = P(y_1\,y_2\,x) = P(y_2\,y_1\,x) = (P\,y_2)\,y_1\,x = P(y_2\,y_1)\,x = (P\,y)\,x$.

Next, (iii). Suppose that $y \in A$; we must show that $P(y^* y) \in B^+$. We have $y = \sum_{i,j} e_{ij} y_{ij}$ with all $y_{ij} \in B'$; hence

$$y^* y = \sum_{i,j,k,l} (e_{ij} y_{ij})^* e_{kl} y_{kl} = \sum_{i,j,k,l} e_{ji} e_{kl} y_{ij}^* y_{kl} = \sum_{j,k,l} e_{jl} y_{kj}^* y_{kl};$$

so $P(y^* y) = \sum_{j,k,l} e_{jl} f(y_{kj}^* y_{kl})$. To show that $P(y^* y) \geqq 0$ we must show for arbitrary $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ that $\sum_{j,k,l} \bar{\lambda}_j \lambda_l f(y_{kj}^* y_{kl}) \geqq 0$; that is, $f(\sum_k (\sum_j \lambda_j y_{kj})^* (\sum_l \lambda_l y_{kl})) \geqq 0$; this follows from the positivity of f.

Now, (iv). From $(x - Px)^* (x - Px) \geqq 0$ follows

$$\begin{aligned} 0 &\leqq P((x - Px)^* (x - Px)) = P(x^* x - (Px)^* x - x^* Px + (Px)^* (Px)) \\ &= P(x^* x) - (Px)^* (Px) - P(x^*)(Px) + (Px)^* (Px) \\ &= P(x^* x) - (Px)^* (Px); \end{aligned}$$

that is, $(Px)^* (Px) \leqq P(x^* x)$.

Finally, (v). If $x \in A$ then $0 \leqq (Px)^* (Px) \leqq P(x^* x)$ (by (iv)). Since $x^* x \leqq \|x^* x\|$, we have (by (iii)) $P(x^* x) \leqq P(\|x^* x\|) = f(\|x^* x\|) = \|x^* x\|$. Thus $0 \leqq (Px)^* (Px) \leqq \|x^* x\|$; hence $\|(Px)^* (Px)\| \leqq \|x^* x\|$, and $\|Px\| = \|(Px)^* (Px)\|^{\frac{1}{2}} \leqq \|x^* x\|^{\frac{1}{2}} = \|x\|$.

3. **Lemma.** *Let A be a matroid C^*-algebra with unit 1. Then there exists a unique normalized trace t on A (that is, positive linear functional t on A such that $t(1) = 1$ and $t(xy) = t(yx)$ $(x, y \in A)$).*

Proof. If A is separable then the assertion of the lemma is proved in Theorem 2.5 of [2] (and, as the author points out, would be easy to prove more directly).

For the nonseparable case, we shall use the fact established in the proof of Theorem 1.6 of [2] that for any sequence $x_1, x_2, \ldots$ in A there exists a separable matroid sub-C^*-algebra B of A such that $x_n \in B$ $(n = 1, 2, \ldots)$. Suppose that $x \in A$ and that B_1, B_2 are separable matroid sub-C^*-algebras of A such that $1, x \in B_1, B_2$. Then there exists a third separable matroid sub-C^*-algebra B of A which contains B_1 and B_2. Hence we may unambiguously define $t(x)$ for $x \in A$ to be the value at x of the unique normalized trace on any separable matroid sub-C^*-algebra B of A such that $1, x \in B$. Since for any $x, y \in A$ there exists a separable matroid sub-C^*-algebra B of A such that $1, x, y \in B$, it follows that t is a normalized trace on A. If t' is another normalized trace on A, then t' must agree with t on any separable matroid sub-C^*-algebra of A containing 1; hence $t' = t$.

Remark. A similar argument shows that if A is any matroid C^*-algebra there exists on A^+ a faithful lower semicontinuous trace with dense ideal of definition, and that such a trace is unique up to a scalar multiple.

4. **Lemma.** *Let A be a C^*-algebra with unit 1 which is generated by a sequence $(A_i)_{i\in\mathbf{N}}$ of mutually commuting sub-C^*-algebras, each containing 1 and isomorphic to a matrix algebra. For each $I\subset\mathbf{N}$, let A_I denote the sub-C^*-algebra with unit generated by $\bigcup_{i\in I}A_i$ (so that $A_\emptyset=\mathbf{C}$, and $A_\mathbf{N}=A$). If $J\subset\mathbf{N}$ is finite then A_J is isomorphic to a matrix algebra. The C^*-algebra A is matroid (with unit). If $I_1\subset I_2\subset\mathbf{N}$, then $A'_{I_1}\cap A_{I_2}=A_{I_2-I_1}$ (the case that I_1 is finite is treated in* [6], *Lemma* 3.2*). Let t denote the unique normalized trace on A. If $I\subset\mathbf{N}$ then there exists a unique bounded linear mapping $P=P_I$: $A\to A_I$ such that $P(xy)=x\,t(y)$ whenever $x\in A_I$ and $y\in A'_I$; P satisfies* 2(i) *to* 2(v) *with $B=A_I$ (again, this can be deduced from* [11], *but we shall prove it using only* 2*). If $I\subset\mathbf{N}$ and $x\in A$ then $P_I x=\lim_{J\subset I, J\,\mathrm{finite}}P_J x$. If $I_1, I_2\subset\mathbf{N}$ then $P_{I_1}P_{I_2}=P_{I_1\cap I_2}$. If $J\subset\mathbf{N}$ is finite then there exist unitaries $u_1,\dots,u_k\in A_J$ such that*

$$P_{\mathbf{N}-J}\,x=\frac{1}{k}\sum_{i=1}^{k}u_i x u_i^* \quad (x\in A).$$

Proof. Let J be a finite subset of $\mathbf{N}$. If S_j is a system of matrix units for A_j ($j\in J$), then the set S of all products $\prod_{j\in J, x_j\in S_j}x_j$ is a system of matrix units for A_J; therefore A_J is isomorphic to a matrix algebra. Since $\bigcup_{J\subset\mathbf{N}, J\,\mathrm{finite}}A_J$ is dense in A, the C^*-algebra generated by $(A_i)_{i\in\mathbf{N}}$, it follows that A is matroid.

Let J_1 and J_2 be finite subsets of $\mathbf{N}$ with $J_1\subset J_2$. Then $A_{J_2-J_1}\subset A'_{J_1}\cap A_{J_2}$; let us show that $A_{J_2-J_1}=A'_{J_1}\cap A_{J_2}$. Let $(e_{ij})_{i,j=1,\dots,n}$ be a system of matrix units for A_{J_1}. Suppose that $x\in A'_{J_1}\cap A_{J_2}$. Since A_{J_2} is generated by A_{J_1} and $A_{J_2-J_1}$, we have $x=\sum_{i,j}e_{ij}x_{ij}$ with all $x_{ij}\in A_{J_2-J_1}$. Since $x\in A'_{J_1}$, we deduce that $x_{ij}=0$ if $i\neq j$ and that x_{ii} is independent of i; hence $x\in A_{J_2-J_1}$.

Let J be a finite subset of $\mathbf{N}$. Then by 2 there exists a unique linear mapping $P=P_J$: $A\to A_J$ such that $P(xy)=x\,t(y)$ whenever $x\in A_J$ and $y\in A'_J$; P satisfies 2(i) to 2(v) with $B=A_J$.

Let J_1 and J_2 be finite subsets of $\mathbf{N}$; let us show that $P_{J_1}P_{J_2}=P_{J_1\cap J_2}$. By uniqueness of $P_{J_1\cap J_2}$ it suffices to show that for $x\in A_{J_1\cap J_2}$ and $y\in A'_{J_1\cap J_2}$ we have $P_{J_1}P_{J_2}(xy)=x\,t(y)$. We have $P_{J_1}P_{J_2}(xy)=P_{J_1}(xP_{J_2}y)=xP_{J_1}P_{J_2}y$, so we have only to show that $P_{J_1}P_{J_2}y=t(y)$ ($y\in A'_{J_1\cap J_2}$). Let $(e_{ij})_{i,j=1,\dots,n}$ be a system of matrix units for $A_{J_1\cap J_2}$, and let $(f_{kl})_{k,l=1,\dots,m}$ be a system of matrix units for $A_{J_2-J_1\cap J_2}$. Then $(e_{ij}f_{kl})_{i,j=1,\dots,n;\,k,l=1,\dots,m}$ is a system of matrix units for A_{J_2}. Let $y\in A'_{J_1\cap J_2}$. Then (cf. proof of 2) $y=\sum_{i,j,k,l}e_{ij}f_{kl}y_{ijkl}$ with all $y_{ijkl}\in A'_{J_2}$. From $ye_{i'j'}=e_{i'j'}y$ ($i',j'=1,\dots,n$), we deduce that $y_{ijkl}=0$ if $i\neq j$ and that y_{iikl} is independent of i. Setting $y_{iikl}=y_{kl}$ ($i=1,\dots,n$) we have $y=\sum_{k,l}f_{kl}y_{kl}$, with all $y_{kl}\in A'_{J_2}$. Therefore $P_{J_2}y=\sum_{k,l}f_{kl}t(y_{kl})$. Since each $f_{kl}\in A'_{J_1}$, we have

$$P_{J_1}P_{J_2}y=\sum_{k,l}t(f_{kl})\,t(y_{kl})=\frac{1}{m}\sum_k t(y_{kk}).$$

We deduce that $P_{J_1} P_{J_2} | A'_{J_1 \cap J_2}$ is a normalized trace on $A'_{J_1 \cap J_2}$; denote it by f. Then the mapping

$$A \ni \sum e_{ij} x_{ij} \mapsto \frac{1}{n} \sum f(x_{ii})$$

(where x_{ij} denotes an element of $A'_{J_1 \cap J_2}$) is a normalized trace on A, so by uniqueness is equal to t. Hence $f = t | A'_{J_1 \cap J_2}$; that is, $P_{J_1} P_{J_2} y = t(y)$ $(y \in A'_{J_1 \cap J_2})$.

Now let I be an arbitrary subset of $\mathbf{N}$. Let us show that for every $x \in A$, the net $(P_J x)_{J \subset I, J \text{ finite}}$ is convergent. Since $\bigcup_{J \subset \mathbf{N}, J \text{ finite}} A_J$ is dense in A, and since all P_J have bound 1, it suffices to suppose that $x \in A_{J_x}$ for some finite $J_x \subset \mathbf{N}$. We shall now show more than is necessary; namely, that if $J \subset I$, J is finite, and $J \supset I \cap J_x$ then $P_J x = P_{I \cap J_x} x$. We have $I \cap J_x = J \cap J_x$, whence by the result of the preceding paragraph, $P_J x = P_J P_{J_x} x = P_{J \cap J_x} x = P_{I \cap J_x} x$.

For $x \in A$ and $I \subset \mathbf{N}$, set $P_I x = \lim_{J \subset I, J \text{ finite}} P_J x$. Then P_I is a linear mapping from A into A_I. Since $\|P_J\| \leqq 1$ for all finite $J \subset I$, we have $\|P_I\| \leqq 1$. Suppose that $x \in A_I$ and $y \in A'_I$; let us show that $P_I(x y) = x t(y)$. Let $\varepsilon > 0$. Then there exist a finite $J_\varepsilon \subset I$ and an $x_\varepsilon \in A_{J_\varepsilon}$ such that $\|x - x_\varepsilon\| < \varepsilon$. Hence $\|P_I(x y) - P_I(x_\varepsilon y)\| \leqq \|x y - x_\varepsilon y\| < \varepsilon \|y\|$. Next, $P_I(x_\varepsilon y) = \lim_{J \subset I, J \text{ finite}} P_J(x_\varepsilon y) = x_\varepsilon t(y)$ (since if $J_\varepsilon \subset J \subset I$ and J is finite then $y \in A'_I \subset A'_J$ and $x_\varepsilon \in A_{J_\varepsilon} \subset A_J$). Therefore

$$\|P_I(x y) - x_\varepsilon t(y)\| < \varepsilon \|y\|;$$

hence

$$\begin{aligned} \|P_I(x y) - x t(y)\| &\leqq \|P_I(x y) - x_\varepsilon t(y)\| + \|x_\varepsilon t(y) - x t(y)\| \\ &< \varepsilon \|y\| + \varepsilon |t(y)|. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $P_I(x y) = x t(y)$.

To show uniqueness, suppose that $I \subset \mathbf{N}$ and that $P: A \to A_I$ is a bounded linear mapping such that $P(x y) = x t(y)$ whenever $x \in A_I$ and $y \in A'_I$. Then $P = P_I$ because the algebra generated by A_I and A'_I contains $\bigcup_{i \in \mathbf{N}} A_i$ and is therefore dense in A.

Let $I \subset \mathbf{N}$, and let us verify 2(i) to 2(v) for $P = P_I$ and $B = A_I$. 2(v) has already been established. If I is finite, then 2(i) to 2(iv) hold by 2. By passage to the limit we get 2(i), 2(iii) and 2(iv) for arbitrary I. To prove 2(ii), it suffices since $\bigcup_{J \subset I, J \text{ finite}} A_J$ is dense in A_I to suppose that $x \in A_{J_x}$ for some finite $J_x \subset I$. Then $P_J(x y) = x P_J(y)$ whenever $J_x \subset J \subset I$ and J is finite; passing to the limit we get $P_I(x y) = x P_I y$.

Suppose that $I_1 \subset I_2 \subset \mathbf{N}$; let us show that $A'_{I_1} \cap A_{I_2} = A_{I_2 - I_1}$. We have $A_{I_2 - I_1} \subset A_{I_2}$ and $A_{I_2 - I_1} \subset A'_{I_1}$. Suppose conversely that $y \in A_{I_2}$ and $y \in A'_{I_1}$; we must show that $P_{I_2 - I_1} y = y$. Since $P_{I_2} y = y$, we have $y =$

$\lim_{J\subset I_2, J \text{ finite}} P_J y$. By continuity, it suffices to show for each finite $J\subset I_2$ that $P_{I_2-I_1} P_J y = P_J y$. Let $J\subset I_2$, J finite be fixed. Then $y\in A'_{I_1}\subset A'_{I_1\cap J}$; hence by 2(ii), $P_J y\in A'_{I_1\cap J}$ (if $x\in A_{I_1\cap J}$ then $x\in A_J$, and $x P_J y = P_J(x y) = P_J(y x) = (P_J y)x$). Since J and $I_1\cap J$ are finite, we have proven already that $A'_{I_1\cap J}\cap A_J = A_{J-I_1\cap J}$. Therefore $P_J y\in A_{J-I_1\cap J}$. Since $J\subset I_2$ we have $J-I_1\cap J\subset I_2-I_1$, so that $P_{I_2-I_1} x = x$ for $x\in A_{J-I_1\cap J}$; in particular, $P_{I_2-I_1} P_J y = P_J y$.

Suppose that $I_1, I_2\subset \mathbf{N}$; let us show that $P_{I_1} P_{I_2} = P_{I_1\cap I_2}$. Let $x\in A$. We have $P_{I_2} x = \lim_{J_2\subset I_2, J_2 \text{ finite}} P_J x$, and $P_{I_1\cap I_2} x = \lim_{J\subset I_1\cap I_2, J \text{ finite}} P_J x = \lim_{J_2\subset I_2, J_2 \text{ finite}} P_{I_1\cap J_2} x$. For fixed finite $J_2\subset I_2$ we have

$$P_{I_1\cap J_2} x = \lim_{J\subset I_1\cap J_2, J \text{ finite}} P_J x = \lim_{J_1\subset I_1, J_1 \text{ finite}} P_{J_1\cap J_2} x$$
$$= \lim_{J_1\subset I_1, J_1 \text{ finite}} P_{J_1} P_{J_2} x = P_{I_1} P_{J_2} x.$$

Hence $P_{I_1\cap I_2} x = \lim_{J_2\subset I_2, J_2 \text{ finite}} P_{I_1} P_{J_2} x = P_{I_1} P_{I_2} x$.

Finally, let $J\subset \mathbf{N}$ be finite, and let us construct unitaries $u_1, \ldots, u_k\in A_J$ such that

$$P_{\mathbf{N}-J} x = \frac{1}{k} \sum_{i=1}^k u_i x u_i^* \qquad (x\in A).$$

Let $(e_{ij})_{i,j=1,\ldots,n}$ be a system of matrix units for A_J. Let $k'=2^n$ and let $u'_1, \ldots, u'_{k'}$ denote the linear combinations of the e_{ii} $(i=1, \ldots, n)$ with coefficients ± 1 (the diagonal symmetries of A_J). Then

$$\frac{1}{k'} \sum_{i=1}^{k'} u'_i e_{i'j'} u'^*_i = \delta_{i'j'} e_{i'j'} \qquad (i', j' = 1, \ldots, n).$$

(A slightly more general result than this was proved by Davis in Theorem 3.1 of [0].) Next, for $i=0, \ldots, n-1$, set $v_i = \sum_{j=1}^n e_{jj+i}$, where $e_{jj+i} = e_{jj+i-n}$ if $j+i>n$. Then each v_n is unitary $\in A_J$, and we have

$$\frac{1}{n} \sum_{i=1}^n v_i e_{i'i'} v_i^* = \frac{1}{n} \qquad (i' = 1, \ldots, n).$$

Now set $k=nk'$ and let $u_1, \ldots, u_k$ denote the unitaries $v_i u'_j$ $(0\leqq i\leqq n-1, 1\leqq j\leqq k')$. Then we have

$$\frac{1}{k} \sum_{i=1}^k u_i e_{i'j'} u_i^* = \frac{1}{n} \delta_{i'j'} = t(e_{i'j'}) \qquad (i', j' = 1, \ldots, n),$$

and by linearity

$$\frac{1}{k} \sum_{i=1}^k u_i x u_i^* = t(x) \qquad \text{for all } x\in A_J.$$

To show that

$$P_{\mathbf{N}-J}\, x = \frac{1}{k} \sum_{i=1}^{k} u_i\, x\, u_i^* \qquad (x \in A)$$

we may suppose by linearity that $x = x_1 x_2$ with $x_1 \in A_J$, $x_2 \in A'_J = A_{\mathbf{N}-J}$. Then

$$P_{\mathbf{N}-J}\, x = P_{\mathbf{N}-J}(x_1 x_2) = t(x_1)\, x_2 = \frac{1}{k} \sum_{i=1}^{k} u_i\, x_1\, u_i^*\, x_2$$

$$= \frac{1}{k} \sum_{i=1}^{k} u_i\, x_1\, x_2\, u_i^* = \frac{1}{k} \sum_{i=1}^{k} u_i\, x\, u_i^*.$$

5. **Lemma.** *Let A and $(A_i)_{i \in \mathbf{N}}$ be as in 4, and let D be a derivation of A. Let $\varepsilon > 0$; then there exists a derivation D_ε of A such that $\|D - D_\varepsilon\| \leqq \varepsilon$ and $D_\varepsilon \bigcup_{J \subset \mathbf{N},\, J \text{ finite}} A_J \subset \bigcup_{J \subset \mathbf{N},\, J \text{ finite}} A_J$.*

Proof. Let us denote D by D_0. We shall construct a sequence D_1, $D_2, \ldots$ of derivations of A and a sequence $J_1 \subset J_2 \subset \cdots$ of finite subsets of $\mathbf{N}$ with union $\mathbf{N}$ such that for each $n = 1, 2, \ldots$ we have:

(i) $D_n | A_{J_n} = P_{J_{n+1}} D_{n-1} | A_{J_n}$;

(ii) $\|D_n - D_{n-1}\| < \dfrac{\varepsilon}{2^n}$.

Set $J_1 = \{0\}$. Fix $k = 1, 2, \ldots$ and suppose that we are given, for $1 \leqq n < k$, a derivation D_n of A and a finite set $J_{n+1} \subset \mathbf{N}$ such that $J_1 \subset \cdots \subset J_k$, (i) and (ii) hold for $1 \leqq n < k$, and in addition $n \in J_{n+1}$, $1 \leqq n < k$. By 1 there exists $y \in A$ such that $D_{k-1} | A_{J_k} = \operatorname{ad} y | A_{J_k}$. By 4, there exists $J_{k+1} \supset J_k \cup \{k\}$, finite, such that

$$\|y - P_{J_{k+1}} y\| < \frac{\varepsilon}{2 \cdot 2^k}.$$

Set $D_k = D_{k-1} + \operatorname{ad}(P_{J_{k+1}} y - y)$. Then D_k is a derivation of A, and $D_k | A_{J_k} = \operatorname{ad} P_{J_{k+1}} y | A_{J_k}$. By 2(ii), $\operatorname{ad} P_{J_{k+1}} y | A_{J_k} = P_{J_{k+1}} \operatorname{ad} y | A_{J_k}$; hence

$$D_k | A_{J_k} = P_{J_{k+1}} D_{k-1} | A_{J_k}.$$

Also,

$$\|D_k - D_{k-1}\| \leqq 2 \|P_{J_{k+1}} y - y\| < \frac{\varepsilon}{2^k}.$$

We now have, for $1 \leqq n < k+1$, a derivation D_n of A and a finite set $J_{n+1} \subset \mathbf{N}$ such that $J_1 \subset \cdots \subset J_{k+1}$, (i) and (ii) hold for $1 \leqq n < k+1$, and in addition $n \in J_{n+1}$, $1 \leqq n < k+1$. Therefore, by induction, there exist a sequence $D_1, D_2, \ldots$ of derivations of A and a sequence $J_1 \subset J_2 \subset \cdots$ of finite subsets of $\mathbf{N}$ such that for each $n = 1, 2, \ldots$ we have (i) and (ii), and in addition $n - 1 \in J_n$, so that $\bigcup_{n=1,2,\ldots} J_n = \mathbf{N}$.

From (ii), we see that the sequence $D_1, D_2, \ldots$ is Cauchy; set $D_\varepsilon = \lim D_n$. Then D_ε is a derivation of A. For each $n=1,2,\ldots$ we have $D - D_n = \sum_{k=1}^n (D_{k-1} - D_k)$; hence

$$\|D - D_n\| \leqq \sum_{k=1}^n \|D_{k-1} - D_k\| < \sum_{k=1}^n \frac{\varepsilon}{2^k} < \varepsilon;$$

passing to the limit we get $\|D - D_\varepsilon\| \leqq \varepsilon$.

It remains to prove that $D_\varepsilon \bigcup_{J \subset \mathbf{N}, J \text{ finite}} A_J \subset \bigcup_{J \subset \mathbf{N}, J \text{ finite}} A_J$. Fix $n=1,2,\ldots$. For each $p=1,2,\ldots$ we have, by (i),

$$\begin{aligned} D_{n+p}|A_{J_n} &= P_{J_{n+p+1}} D_{n+p-1}|A_{J_n} = \cdots = P_{J_{n+p+1}} \cdots P_{J_{n+1}} D_{n-1}|A_{J_n} \\ &= P_{J_{n+1}} D_{n-1}|A_{J_n}. \end{aligned}$$

Letting $p \to \infty$, we have $D_\varepsilon|A_{J_n} = P_{J_{n+1}} D_{n-1}|A_{J_n}$, whence $D_\varepsilon A_{J_n} \subset A_{J_{n+1}}$. Since every finite $J \subset \mathbf{N}$ is contained in J_n for some $n=1,2,\ldots$, the assertion is proved.

6. **Lemma.** *Let A, $(A_i)_{i \in \mathbf{N}}$ be as in 4. Let D be a derivation of A. Suppose that $D A_i \subset A_i$ for each $i \in \mathbf{N}$. Then D is inner.*

Proof. By 1, for each $i \in \mathbf{N}$ there exists $y_i \in A_i$ such that $D|A_i = \operatorname{ad} y_i|A_i$.

Define $D^*: A \to A$ by $D^* x = (D(x^*))^*$ $(x \in A)$; D^* is a derivation of A. D is called skew adjoint if $D = -D^*$. The derivations $\frac{D - D^*}{2}$ and $\frac{D + D^*}{2i}$ are both skew adjoint, and $D = \frac{D - D^*}{2} + i\frac{D + D}{2i}$, so to prove that D is inner we may suppose that D is skew adjoint.

Since $D = -D^*$, we have $D|A_i = -(D|A_i)^*$ $(i \in \mathbf{N})$. Since $D|A_i = \operatorname{ad} y_i|A_i$, we have $(D|A_i)^* = \operatorname{ad}(-y_i)^*|A_i$. Hence

$$D|A_i = \tfrac{1}{2}(D|A_i - (D|A_i)^*) = \operatorname{ad} \frac{y_i + y_i^*}{2}\bigg| A_i,$$

so we may suppose that $y_i = y_i^*$ $(i \in \mathbf{N})$.

Replacing y_i by $y_i - t(y_i)$ $(i \in \mathbf{N})$, we have, for each $i \in \mathbf{N}$, $y_i = y_i^*$, $t(y_i) = 0$, and $\operatorname{ad} y_i|A_i = D|A_i$.

Let $i \in \mathbf{N}$ be fixed. Let $(e_{pq})_{p,q=1,\ldots,n}$ be a system of matrix units for A_i such that $y_i = \sum_{k=1}^n \lambda_k e_{kk}$ with all $\lambda_k \in \mathbf{R}$. Since $t(y_i) = 0$, there exist j_1, j_2, $1 \leqq j_1, j_2 \leqq n$, such that $\lambda_{j_2} - \lambda_{j_1} = \beta_i \geqq \|y_i\|$. Set $x_i = e_{j_2 j_1}$. Then $D x_i = (\operatorname{ad} y_i) x_i = \beta_i x_i$.

Let J be a finite subset of $\mathbf{N}$. Then $\prod_{i \in J} x_i$ is a nonzero partial isometry, so $\|\prod_{i \in J} x_i\| = 1$.

By the product rule for D, we have $D\prod_{i\in J}x_i=(\sum_{i\in J}\beta_i)\prod_{j\in J}x_j$. By [9] (see [1], Chapitre III, §9, Lemme 3 for a slightly simpler proof), D is bounded. Hence $\sum_{i\in J}\|y_i\|\leqq\sum_{i\in J}\beta_i=\|D\prod_{i\in J}x_i\|\leqq\|D\|$. This proves that $\sum_{i\in\mathbf{N}}\|y_i\|\leqq\|D\|$.

Set $y=\sum_{i\in\mathbf{N}}y_i\in A$. Let $i\in\mathbf{N}$; then $y_j\in A_i'$ for $j\neq i$; hence $\operatorname{ad}y|A_i=\operatorname{ad}y_i|A_i=D|A_i$. By the product rule, linearity and continuity, $\operatorname{ad}y=D$.

7. **Lemma.** *Let A be an algebra, let $B_1\subset B_2$ be subalgebras of A, and let D be a derivation of A such that $DB_1\subset B_2$. Then $D(B_2')\subset B_1'$.*

Proof. If x commutes with y and Dy then Dx commutes with y. (This is proved easily by applying D to both sides of the equation $xy=yx$.) The lemma follows immediately.

8. **Lemma.** *Let A, $(A_i)_{i\in\mathbf{N}}$ be as in 4. Let D be a derivation of A. Suppose that $D\bigcup_{J\subset\mathbf{N},J\text{ finite}}A_J\subset\bigcup_{J\subset\mathbf{N},J\text{ finite}}A_J$ (we are using the notation of 4). Then D is inner.*

Proof. From the hypothesis we may construct a sequence $J_0\subset J_1\subset\cdots$ of finite subsets of $\mathbf{N}$ with union $\mathbf{N}$ such that $DA_{J_n}\subset A_{J_{n+1}}$ $(n\in\mathbf{N})$. Let us apply 7 with $B_1=A_{J_n}$, $B_2=A_{J_{n+1}}$ to deduce that $D(A'_{J_{n+1}})\subset A'_{J_n}$ $(n\in\mathbf{N})$.

For $n\in\mathbf{N}$, set $I_n=J_n-J_{n-1}$ (where $J_{-1}=\emptyset$). Then $DA_{I_n}\subset A_{J_{n+1}}$ and also $DA_{I_n}\subset A'_{J_{n-2}}$ (where $J_{-2}=J_{-1}=\emptyset$). By 4 we have $A'_{J_{n-2}}\cap A_{J_{n+1}}=A_{J_{n+1}-J_{n-2}}=A_{I_{n-1}\cup I_n\cup I_{n+1}}$, so we have shown:

(*) $DA_{I_n}\subset A_{I_{n-1}\cup I_n\cup I_{n+1}}$ $(n\in\mathbf{N})$ (where $I_{-1}=\emptyset$).

Let $I\subset\mathbf{N}$. Denote $P_I D|A_I$ by D_I. Then by 2(ii), D_I is a derivation of A_I.

We shall use the following remark five times. Suppose that $n_0<n_1<\cdots$ is a strictly increasing sequence in $\mathbf{N}\cup\{-1\}$, with $n_0=-1$, and set $I=\mathbf{N}-\bigcup_{i\in\mathbf{N}}I_{n_i}$. Then the hypotheses of 6 are fulfilled with $A=A_I$, $A_i=A_{\bigcup_{n_i<n<n_{i+1}}I_n}$ $(i\in\mathbf{N})$ and $D=D_I$, for by (*) we have for $i\in\mathbf{N}$,

$$\begin{aligned}D_I A_{\bigcup_{n_i<n<n_{i+1}}I_n}&=P_I D A_{\bigcup_{n_i<n<n_{i+1}}I_n}\\&=P_I P_{\bigcup_{n_i\leqq n\leqq n_{i+1}}I_n} D A_{\bigcup_{n_i<n<n_{i+1}}I_n}\\&=P_{I\cap\bigcup_{n_i\leqq n\leqq n_{i+1}}I_n} D A_{\bigcup_{n_i<n<n_{i+1}}I_n}\\&=P_{\bigcup_{n_i<n<n_{i+1}}I_n} D A_{\bigcup_{n_i<n<n_{i+1}}I_n}\subset A_{\bigcup_{n_i<n<n_{i+1}}I_n}.\end{aligned}$$

Therefore D_I is inner.

Let $\mathbf{Z}_2$ denote the group $\{0,1\}$ (with addition modulo 2) and for $j\in\mathbf{Z}_2$ set $I^j=\bigcup_{n\in\mathbf{N},n\equiv j(\mathrm{mod}\,2)}I_n$. Then the previous paragraph is applicable with $I=I^j$ $(j\in\mathbf{Z}_2)$, so there exists $y_j\in A_{I^j}$ such that $D_{I^j}=\operatorname{ad}y_j|A_{I^j}$ $(j\in\mathbf{Z}_2)$.

Let us show that $(*)$ holds with D replaced by $D-\mathrm{ad}(y_0+y_1)$. It suffices to show for $n\in\mathbf{N}$ that $\mathrm{ad}(y_0+y_1)A_{I_n}\subset A_{I_n}$. Let $n\in\mathbf{N}$; then $I_n\subset I^{\bar n}$ for some $\bar n\in\mathbf{Z}_2$. We have

$$\begin{aligned}\mathrm{ad}\, y_{\bar n} A_{I_n} &= P_{I^{\bar n}} D A_{I_n} = P_{I^{\bar n}} P_{I_{n-1}\cup I_n\cup I_{n+1}} D A_{I_n}\\ &= P_{I^{\bar n}\cap(I_{n-1}\cup I_n\cup I_{n+1})} D A_{I_n} = P_{I_n} D A_{I_n}\subset A_{I_n},\end{aligned}$$

and $y_{\bar n+1}\in A_{I^{\bar n+1}}\subset A'_{I_n}$, so $\mathrm{ad}\, y_{\bar n+1} A_{I_n}=0$; hence $\mathrm{ad}(y_0+y_1)A_{I_n}\subset A_{I_n}$.

Let us show that $(D-\mathrm{ad}(y_0+y_1))_{I_n}=0$ $(n\in\mathbf{N})$. Let $n\in\mathbf{N}$; then $I_n\subset I^{\bar n}$ for some $\bar n\in\mathbf{Z}_2$. We have $P_{I_n} D|A_{I_n}=P_{I_n\cap I^{\bar n}} D|A_{I_n}=P_{I_n} P_{I^{\bar n}} D|A_{I_n}=P_{I_n}\mathrm{ad}\, y_{\bar n}|A_{I_n}$, and $y_{\bar n+1}\in A_{I^{\bar n+1}}\subset A'_{I_n}$, so $\mathrm{ad}\, y_{\bar n+1}A_{I_n}=0$; hence $P_{I_n}(D-\mathrm{ad}(y_0+y_1))|A_{I_n}=0$.

Since to prove that D is inner it suffices to prove that $D-\mathrm{ad}(y_0+y_1)$ is inner, we may suppose that D satisfies in addition to $(*)$ the following condition:

$(**)$ $D_{I_n}=0$ $(n\in\mathbf{N})$.

Now let $\mathbf{Z}_3$ denote the group $\{0,1,-1\}$ (with addition modulo 3), and for $j\in\mathbf{Z}_3$ set $I^j=\bigcup_{n\in\mathbf{N},\, n\not\equiv j (\mathrm{mod}\ 3)} I_n$. Then for $j\in\mathbf{Z}_3$ there exists $y_j\in A_{I^j}$ such that $D_{I^j}=\mathrm{ad}\, y_j|A_{I^j}$.

Let us show that $(*)$ holds with D replaced by $D-\mathrm{ad}(y_0+y_1+y_{-1})$. It suffices to show that $\mathrm{ad}(y_0+y_1+y_{-1})A_{I_n}\subset A_{I_{n-1}\cup I_n\cup I_{n+1}}$ $(n\in\mathbf{N})$. Let $n\in\mathbf{N}$; then $I_n\cap I^{\bar n}=\emptyset$ for some $\bar n\in\mathbf{Z}_3$. We have $A_{I^{\bar n}}\subset A'_{I_n}$, so $\mathrm{ad}\, y_{\bar n}A_{I_n}=0$. If $j=\pm 1$ then $I_n\subset I^{\bar n+j}$, whence

$$\begin{aligned}\mathrm{ad}\, y_{\bar n+j}A_{I_n} &= D_{I^{\bar n+j}}A_{I_n}=P_{I^{\bar n+j}} D A_{I_n}=P_{I^{\bar n+j}} P_{I_{n-1}\cup I_n\cup I_{n+1}} D A_{I_n}\\ &= P_{I^{\bar n+j}\cap(I_{n-1}\cup I_n\cup I_{n+1})} D A_{I_n}=P_{I_n\cup I_{n-j}} D A_{I_n}\subset A_{I_n\cup I_{n-j}}.\end{aligned}$$

Therefore $\mathrm{ad}(y_0+y_1+y_{-1})A_{I_n}\subset A_{I_{n-1}\cup I_n\cup I_{n+1}}$.

Let us show that $(D-\mathrm{ad}(y_0+y_1+y_{-1}))_{I_n\cup I_{n+1}}=0$ $(n\in\mathbf{N})$. Let $n\in\mathbf{N}$; then $I_n\cap I^{\bar n}=\emptyset$ for some $\bar n\in\mathbf{Z}_3$. We have $I_n\cup I_{n+1}\subset I^{\bar n-1}$, so

$$\begin{aligned}P_{I_n\cup I_{n+1}} D|A_{I_n\cup I_{n+1}} &= P_{(I_n\cup I_{n+1})\cap I^{\bar n-1}} D|A_{I_n\cup I_{n+1}}=P_{I_n\cup I_{n+1}} P_{I^{\bar n-1}} D|A_{I_n\cup I_{n+1}}\\ &= P_{I_n\cup I_{n+1}} D_{I^{\bar n-1}}|A_{I_n\cup I_{n+1}}=P_{I_n\cup I_{n+1}}\mathrm{ad}\, y_{\bar n-1}|A_{I_n\cup I_{n+1}}.\end{aligned}$$

If $j=0$ or 1 then

$$\begin{aligned}P_{I_n\cup I_{n+1}}\mathrm{ad}\, y_{\bar n+j}|A_{I_n\cup I_{n+1}} &= \mathrm{ad}\, P_{I_n\cup I_{n+1}} y_{\bar n+j}|A_{I_n\cup I_{n+1}}\\ &= \mathrm{ad}\, P_{I_n\cup I_{n+1}} P_{I^{\bar n+j}} y_{\bar n+j}|A_{I_n\cup I_{n+1}}\\ &= \mathrm{ad}\, P_{(I_n\cup I_{n+1})\cap I^{\bar n+j}} y_{\bar n+j}|A_{I_n\cup I_{n+1}}\\ &= \mathrm{ad}\, P_{I_{n+1-j}}\, y_{\bar n+j}|A_{I_n\cup I_{n+1}}.\end{aligned}$$

Since $n+1-j \not\equiv n+j \pmod 3$ we have $I_{n+1-j} \subset I^{\bar{n}+j}$; hence

$$\begin{aligned}\operatorname{ad} P_{I_{n+1-j}} y_{\bar{n}+j}|A_{I_{n+1-j}} &= P_{I_{n+1-j}} \operatorname{ad} y_{\bar{n}+j}|A_{I_{n+1-j}} = P_{I_{n+1-j}} D_{I^{\bar{n}+j}}|A_{I_{n+1-j}} \\ &= P_{I_{n+1-j}} P_{I^{\bar{n}+j}} D|A_{I_{n+1-j}} = P_{I_{n+1-j} \cap I^{\bar{n}+j}} D|A_{I_{n+1-j}} \\ &= P_{I_{n+1-j}} D|A_{I_{n+1-j}} = D_{I_{n+1-j}} = 0 \ \text{(by (**))}.\end{aligned}$$

Also, since $n+1-j \neq n+j$, we have $\operatorname{ad} P_{I_{n+1-j}} y_{\bar{n}+j}|A_{I_{n+j}} = 0$. Thus

$$\operatorname{ad} P_{I_{n+1-j}} y_{\bar{n}+j}|(A_{I_{n+j}} \cup A_{I_{n+1-j}}) = 0;$$

that is,

$$\operatorname{ad} P_{I_{n+1-j}} y_{\bar{n}+j}|(A_{I_n} \cup A_{I_{n+1}}) = 0.$$

By the product law, $\operatorname{ad} P_{I_{n+1-j}} y_{\bar{n}+j}|A_{I_n \cup I_{n+1}} = 0$. We have shown that $P_{I_n \cup I_{n+1}}(D - \operatorname{ad}(y_0 + y_1 + y_{-1}))|A_{I_n \cup I_{n+1}} = 0$.

Since to prove that D is inner it suffices to prove that $D - \operatorname{ad}(y_0 + y_1 + y_{-1})$ is inner, we may suppose that D satisfies in addition to (*) the following condition (which is stronger than (**)):

(***) $D_{I_n \cup I_{n+1}} = 0 \ (n \in \mathbf{N})$.

We shall show that (*) and (***) in fact imply that $D = 0$. We have $D A_{I_0} = P_{I_0 \cup I_1} D A_{I_0} = D_{I_0 \cup I_1} A_{I_0} = 0$. Hence by 7 with both B_1 and B_2 taken to be A_{I_0}, $D A_{\mathbf{N}-I_0} \subset A_{\mathbf{N}-I_0}$ (by 4, $A_{\mathbf{N}-I_0} = A'_{I_0}$). Then

$$\begin{aligned}D A_{I_1} &= P_{\mathbf{N}-I_0} D A_{I_1} = P_{\mathbf{N}-I_0} P_{I_0 \cup I_1 \cup I_2} D A_{I_1} \\ &= P_{(\mathbf{N}-I_0) \cap (I_0 \cup I_1 \cup I_2)} D A_{I_1} = P_{I_1 \cup I_2} D A_{I_1} = D_{I_1 \cup I_2} A_{I_1} = 0.\end{aligned}$$

Proceeding in this way we get $D A_{I_n} = 0$ for all $n \in \mathbf{N}$. By the product law, linearity, and continuity (cf. proof of 6), $D = 0$.

9. **Lemma.** *Let A, $(A_i)_{i \in \mathbf{N}}$ be as in 4. Let $y \in A$ be such that $t(y) = 0$. Then $\|y\| \leqq \|\operatorname{ad} y\|$.*

Proof. Since $y = \lim_{J \subset \mathbf{N}, J \text{ finite}} P_J y$, to show that $\|y\| \leqq \|\operatorname{ad} y\|$ it suffices to show for each finite $J \subset \mathbf{N}$ that $\|P_J y\| \leqq \|\operatorname{ad} y\|$. We shall show more; namely, $\|P_J y\| \leqq \|P_J \operatorname{ad} y|A_J\|$ ($J \subset \mathbf{N}$, J finite).

First of all, by 2(ii) we have, for a fixed finite set $J \subset \mathbf{N}$, $P_J \operatorname{ad} y|A_J = \operatorname{ad} P_J y|A_J$. Set $\|\operatorname{ad} P_J y|A_J\| = M$; we must show that $\|P_J y\| \leqq M$. If $u \in A_J$ is unitary, then $\|(P_J y) u - u P_J y\| \leqq M$; equivalently, $\|P_J y - u(P_J y) u^*\| \leqq M$. Hence if $u_1, \ldots, u_k \in A_J$ are unitary then

$$\|P_J y - \frac{1}{k} \textstyle\sum_{i=1}^k u_i (P_J y) u_i^*\| \leqq M.$$

By 4 we may chose $u_1, \ldots, u_k$ so that

$$\frac{1}{k} \textstyle\sum_{i=1}^k u_i (P_J y) u_i^* = P_{\mathbf{N}-J} P_J y = P_\emptyset y = t(y) = 0;$$

hence $\|P_J y\| \leqq M$.

10. **Lemma.** *Let A, $(A_i)_{i\in\mathbf{N}}$ be as in* 4. *Let D be a derivation of A. Then D is inner.*

Proof. By 5 and 8 there exist a sequence $D_1, D_2, \ldots$ of derivations of A and a sequence $y_1, y_2, \ldots$ in A such that $\|D-D_n\| \to 0$ and, for each $n=1,2,\ldots$, $D_n = \operatorname{ad} y_n$. Replacing each y_n by $y_n - t(y_n)$ we may suppose that $t(y_n)=0$, $n=1,2,\ldots$.

Let $n, m = 1, 2, \ldots$. Then $D_n - D_m = \operatorname{ad}(y_n - y_m)$ and $t(y_n - y_m) = t(y_n) - t(y_m) = 0$; hence by 9, $\|y_n - y_m\| \leqq \|D_n - D_m\|$.

Since $\|D_n - D_m\| \to 0$ we have $\|y_n - y_m\| \to 0$; set $y = \lim y_n \in A$. Then $\|\operatorname{ad} y - \operatorname{ad} y_n\| \to 0$ and $\|D - \operatorname{ad} y_n\| = \|D - D_n\| \to 0$, so $D = \operatorname{ad} y$.

11. **Lemma.** *Let A be a C^*-algebra with unit 1. Suppose that $(e_{ij})_{i,j=1,\ldots,p}$ and $(e'_{ij})_{i,j=1,\ldots,p}$ are systems of matrix units in A such that $\sum e_{ii} = \sum e'_{ii} = 1$. Then for every $\varepsilon > 0$ there exists a $\delta = \delta(p, \varepsilon)$ such that if $\|e_{ij} - e'_{ij}\| < \delta$ $(i,j=1,\ldots,p)$ then there exists a unitary $u \in A$ with $u e_{ij} u^* = e'_{ij}$ $(i,j=1,\ldots,p)$ and $\|u-1\| < \varepsilon$.*

Proof. Set

$$\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{8p}\right).$$

As in the proof of Lemma 1.8 of [3], we may construct a partial isometry $w \in A$ such that $w^* w = e_{11}$, $w w^* = e'_{11}$, and

$$\|w - e_{11}\| < \frac{\varepsilon}{2p}.$$

Set $u = \sum_{k=1}^p e'_{k1} w e_{1k}$. Then u is unitary, and

$$\begin{aligned} u e_{ij} u^* &= \textstyle\sum_{k=1}^p e'_{k1} w e_{1k} e_{ij} \sum_{l=1}^p e_{l1} w^* e'_{1l} \\ &= e'_{i1} w e_{1i} e_{ij} e_{j1} w^* e'_{1j} = e'_{ij} \qquad (i,j=1,\ldots,p). \end{aligned}$$

Also,

$$\begin{aligned} \|u-1\| &= \|\textstyle\sum_{k=1}^p (e'_{k1} w e_{k1} - e_{kk})\| \leqq \sum_{k=1}^p \|e'_{k1} w e_{1k} - e_{kk}\| \\ &\leqq \textstyle\sum_{k=1}^p (\|e_{k1} e_{11} e_{1k} - e_{kk}\| + \|e'_{k1} e_{11} e_{1k} - e_{k1} e_{11} e_{1k}\| \\ &\quad + \|e'_{k1} w e_{1k} - e'_{k1} e_{11} e_{1k}\|) \leqq \textstyle\sum_{k=1}^p \left(0 + \frac{\varepsilon}{8p} + \frac{\varepsilon}{2p}\right) < \varepsilon. \end{aligned}$$

Remark. The construction in Lemma 1.8, [3] of w satisfying the above conditions can be considerably simplified; we may just set $w = e'_{11} e_{11} a^{\frac{1}{2}}$, where a is the inverse of $e_{11} e'_{11} e_{11}$ in $e_{11} A e_{11}$ (then $\|w - e_{11}\| < \varepsilon/4p$).

12. **Lemma.** *Let A be a C^*-algebra with unit* 1, *and let B be a sub-C^*-algebra of A containing* 1 *and isomorphic to a matrix algebra. Let D be a derivation from B into A. Then there exists $y \in A$ such that $D = \operatorname{ad} y|B$ and $\|y\| \leqq \|D\|$.*

Proof. By 1 there exists $y_0 \in A$ such that $D = \operatorname{ad} y_0|B$.

By 4 there exist unitaries $u_1, \ldots, u_k \in B$ such that

$$\frac{1}{k} \sum_{i=1}^k u_i x u_i^* = t(x) \qquad (x \in B),$$

where t is the unique normalized trace on B. Set

$$y = y_0 - \frac{1}{k} \sum_{i=1}^k u_i y_0 u_i^*.$$

Let us show that

$$\frac{1}{k} \sum_{i=1}^k u_i y_0 u_i^* \in B'.$$

Let $(e_{ij})_{i,j=1,\ldots,n}$ be a system of matrix units for B. Then $y_0 = \sum_{i,j} e_{ij} y_{ij}$ with all $y_{ij} \in B'$ (cf. proof of 2). We have

$$\begin{aligned}\frac{1}{k} \sum_{l=1}^k u_l y_0 u_l^* &= \frac{1}{k} \sum_{l=1}^k u_l \left(\sum_{i,j} e_{ij} y_{ij}\right) u_l^* \\ &= \sum_{i,j} \left(\frac{1}{k} \sum_l u_l e_{ij} u_l^*\right) y_{ij} = \sum_{i,j} t(e_{ij}) y_{ij} \in B'.\end{aligned}$$

Since $\frac{1}{k} \sum_{i=1}^k u_i y_0 u_i^* \in B'$, we have $\operatorname{ad} y|B = \operatorname{ad} y_0|B = D$.

For $i = 1, \ldots, k$ we have $\|y_0 - u_i y_0 u_i^*\| = \|y_0 u_i - u_i y_0\| = \|D u_i\| \leqq \|D\|$; hence

$$\|y\| = \|y_0 - \frac{1}{k} \sum_{i=1}^k u_i y_0 u_i^*\| \leqq \|D\|.$$

13. **Lemma.** *Let A be a matroid C^*-algebra with unit* 1, *and let D be a derivation of A. Suppose that B is a sub-C^*-algebra of A containing* 1 *and isomorphic to a matrix algebra such that $DB = 0$. Then $\|D\| \leqq 2\|D|B'\|$.*

Proof. First, by 7 with $B_1 = B_2 = B$, we have $D(B') \subset B'$.

Next, if e is a minimal projection of B then B' is isomorphic to eAe; hence by Theorem 1.5 of [2], B' is matroid.

Let $\varepsilon > 0$. Let $x \in A$ be such that $\|Dx\| > (1-\varepsilon)\|D\| \|x\|$. Let $(e_{ij})_{i,j=1,\ldots,p}$ be a system of matrix units for B. Let B_1 be a sub-C^*-algebra of A isomorphic to a matrix algebra such that x and e_{ij} $(i, j = 1, \ldots, p)$ are arbitrarily close to B_1; by 1.2 of [2] we may suppose that $1 \in B_1$. By Lemma 1.10 of [3], if e_{ij} is close enough to B_1 $(i, j = 1, \dots, p)$ there exist matrix units $(e'_{ij})_{i,j=1,\dots,p}$ in B_1 such that $\|e_{ij} - e'_{ij}\|$ is arbitrarily small $(i, j = 1, \dots, p)$. Hence by 11 there exists a unitary $u \in A$ such that $u e_{ij} u^* = e'_{ij}$ $(i, j = 1, \dots, p)$ and $\|1 - u\|$ is arbitrarily small. Set $B_2 = u^* B_1 u$. Then $B_2 \supset B$ and x is arbitrarily close to B_2. By continuity we may suppose that $\|D x_2\| > (1 - \varepsilon) \|D\| \, \|x_2\|$ for some $x_2 \in B_2$; that is, $\|D|B_2\| > (1 - \varepsilon) \|D\|$.

By 12 (with the A, B, and D of 12 replaced by the present B', $B' \cap B_2$, and $D|B' \cap B_2$, respectively), there exists $y \in B'$ such that $D|B' \cap B_2 = \operatorname{ad} y|B' \cap B_2$ and $\|y\| \leqq \|D|B' \cap B_2\|$.

Next, B_2 is the algebra generated by B and $B' \cap B_2$ (cf. proof of 2). Since D and $\operatorname{ad} y$ are both 0 on B and they agree on $B' \cap B_2$, it follows by the product rule and linearity that they agree on B_2.

We now have $(1 - \varepsilon)\|D\| < \|D|B_2\| = \|\operatorname{ad} y|B_2\| \leqq 2\|y\| \leqq 2\|D|B' \cap B_2\| \leqq 2\|D|B'\|$; since $\varepsilon > 0$ is arbitrary, we have $\|D\| \leqq 2\|D|B'\|$.

14. **Theorem.** *Let A be a matroid C^*-algebra with unit* 1, *and let D be a derivation of A. Then D is inner.*

Proof. We shall construct a sequence $B_0 \subset B_1 \subset \cdots$ of sub-C^*-algebras of A containing 1 and isomorphic to matrix algebras such that for all $n \in \mathbf{N}$:

(i) $\|D|B'_n \cap B_{n+1}\| \geqq \frac{1}{2}\|D|B'_n\|$;

(ii) $\|(D - P_{B_{n+1}} D)|B_n\| < \dfrac{1}{n+1}$

(where $P_{B_{n+1}}$ is the P given by 2 with $B = B_{n+1}$ and f the unique normalized trace of A).

Set $B_0 = \mathbf{C}$. Suppose that $k = 1, 2, \dots$ and that there exist $B_0 \subset B_1 \subset \cdots \subset B_{k-1}$ such that (i) and (ii) hold for $0 \leqq n < k - 1$. Let $x \in B'_{k-1}$ be such that $\|D x\| > \frac{1}{2}\|D|B'_{k-1}\| \, \|x\|$. Since B'_{k-1} is isomorphic to eAe if e is a minimal projection of B_{k-1}, B'_{k-1} is matroid ([2], Theorem 1.5). Therefore, there exists a sub-C^*-algebra A_k of B'_{k-1} isomorphic to a matrix algebra such that x is arbitrarily close to A_k; by 1.2 of [2] we may suppose that $1 \in A_k$. By continuity we may suppose that there exists $y \in A_k$ such that $\|D y\| > \frac{1}{2}\|D|B'_{k-1}\| \, \|y\|$; that is, $\|D|A_k\| > \frac{1}{2}\|D|B'_{k-1}\|$.

Let R_k be the algebra generated by B_{k-1} and A_k; since B_{k-1} and A_k commute and are each isomorphic to a matrix algebra, it follows that R_k is isomorphic to a matrix algebra. Let $(e_{ij})_{i,j=1,\dots,p}$ be a system of matrix units for R_k. By 1, there exists $z \in A$ such that $D|B_{k-1} = \operatorname{ad} z|B_{k-1}$. Let C_k be a sub-C^*-algebra of A isomorphic to a matrix algebra such that z and e_{ij} $(i, j = 1, \dots, p)$ are arbitrarily close to C_k; by 1.2 of [2] we may suppose that $1 \in C_k$. By Lemma 1.10 of [3] we may suppose that there exist matrix units $(e'_{ij})_{i,j=1,\dots,p}$ in C_k such that $\|e_{ij}-e'_{ij}\|$ is arbitrarily small $(i,j=1,\dots,p)$. By 11, there exists a unitary $u\in A$ such that $\|u-1\|$ is arbitrarily small and $u\,e_{ij}\,u^*=e'_{ij}\,(i,j=1,\dots,p)$. Let $B_k=u^*\,C_k\,u$. Then $R_k\subset B_k$; hence $B_{k-1}\subset B_k$ and $A_k\subset B_k\cap B'_{k-1}$. Since $\|D|A_k\|>\frac{1}{2}\|D|B'_{k-1}\|$ we have $\|D|B_k\cap B'_{k-1}\|>\frac{1}{2}\|D|B'_{k-1}\|$. Moreover, z is arbitrarily close to B_k, so we may suppose that $\|(D-P_{B_k}D)|B_{k-1}\|<k^{-1}$ (if $w\in B_k$ satisfies $\|w-z\|<(4k)^{-1}$ then $\|P_{B_k}z-w\|=\|P_{B_k}z-P_{B_k}w\|\leqq\|w-z\|<(4k)^{-1}$, so $\|z-P_{B_k}z\|<(4k)^{-1}+(4k)^{-1}=(2k)^{-1}$; then $\|(D-P_{B_k}D)|B_{k-1}\|=\|(\operatorname{ad}z-P_{B_k}\operatorname{ad}z)|B_{k-1}\|=\|(\operatorname{ad}z-\operatorname{ad}P_{B_k}z)|B_{k-1}\|\leqq 2\|z-P_{B_k}z\|<k^{-1}$). We now have $B_0\subset B_1\subset\cdots\subset B_k$ such that (i) and (ii) hold for $0\leqq n\leqq k$. Therefore, by induction, there exists a sequence $B_0\subset B_1\subset\cdots$ with the required properties.

Let B denote the closure of $\bigcup_{n\in\mathbf{N}}B_n$. Then B satisfies the hypotheses for A in 10 (with $A_i=B_i\cap B'_{i-1}$ $(i\in\mathbf{N})$, where $B_{-1}=\mathbf{C}$). From (ii) it follows that $\|(D-P_{B_m}D)|B_n\|<m^{-1}$ if $0\leqq n<m$; hence $DB_n\subset B$ $(n\in\mathbf{N})$; hence $DB\subset B$. Therefore by 10 there exists $y\in B$ such that $D|B=\operatorname{ad}y|B$.

Let us show that $D=\operatorname{ad}y$. Let $\varepsilon>0$. Then since $y\in B$, there exists by 4, $k\in\mathbf{N}$ such that $\|y-P_{B_k}y\|<\varepsilon$. Using (i), we have

$$\begin{aligned}\|(D-\operatorname{ad}y)|B'_k\|&\leqq\|D|B'_k\|+\|\operatorname{ad}y|B'_k\|\leqq 2\|D|B'_k\cap B_{k+1}\|+\|\operatorname{ad}y|B'_k\|\\&=2\|\operatorname{ad}y|B'_k\cap B_{k+1}\|+\|\operatorname{ad}y|B'_k\|\leqq 3\|\operatorname{ad}y|B'_k\|\\&=3\|\operatorname{ad}(y-P_{B_k}y)|B'_k\|\leqq 6\|y-P_{B_k}y\|<6\varepsilon.\end{aligned}$$

Also, $(D-\operatorname{ad}y)|B_k=0$. Therefore by 13 (with $B=B_k$ and $D=D-\operatorname{ad}y$), $\|D-\operatorname{ad}y\|<12\varepsilon$. Since $\varepsilon>0$ is arbitrary, we have proved that $D=\operatorname{ad}y$.

15. *Example.* Let A be a C^*-algebra and let B be a sub-C^*-algebra of A. Suppose that B is separable matroid with unit, not isomorphic to a matrix algebra, suppose that A is isomorphic to the tensor product of B and B', and suppose that there exists an infinite set of mutually orthogonal projections in B'. Then there exists an outer derivation from B into A.

To construct such a derivation, let $(B_i)_{i\in\mathbf{N}}$ be a sequence of mutually commuting sub-C^*-algebras generating B, such that each B_i is isomorphic to a matrix algebra, contains the unit of B and also contains an element x_i of norm one and trace zero (such a sequence exists by Theorem 1.6 of [2]). Let $(e_i)_{i\in\mathbf{N}}$ be a sequence of mutually orthogonal nonzero projections in B'. Then there exists a unique derivation D from B into A such that $D|B_i=\operatorname{ad}x_i\,e_i|B_i$ $(i\in\mathbf{N})$. D is an outer derivation.

16. *Example.* Let A be a C^*-algebra and let B be a sub-C^*-algebra of A such that the algebra generated by B and B' is dense in A. Then (because of [1], Chapitre III, § 9, Corollaire to Théorème 1) every deri-

vation of B can be extended to a unique derivation of A which is 0 on B'. Suppose that B is simple (e.g. matroid) and that there exists a family $(e_i)_{i\in I}$ of mutually orthogonal nonzero projections in B such that $\bigcup_{J\subset I, J \text{ finite}}(\sum_{i\in J} e_i)B(\sum_{i\in J} e_i)$ is dense in B. Then A has outer derivations. To construct one, let I_1 be a subset of I; then there exists a unique derivation D of A such that

$$D|(\textstyle\sum_{i\in J} e_i)B(\sum_{i\in J} e_i) = \mathrm{ad}\sum_{i\in J\cap I_1} e_i|(\sum_{i\in J} e_i)B(\sum_{i\in J} e_i)$$

for every finite $J\subset I$, and such that $D(B')=0$. If both I_1 and $I-I_1$ are infinite, D is an outer derivation of A.

17. *Problems. 17.1.* Let B be an algebra over $\mathbf{C}$ satisfying the condition: whenever A is an algebra over $\mathbf{C}$ and $A\supset B$ then every derivation from B into A is inner. If B is simple, must B be isomorphic to a matrix algebra? In general, must B be isomorphic to a finite product of matrix algebras?

17.2. Let A be a C^*-algebra, let B be a sub-C^*-algebra of A, and let D be a derivation from B into A. Must D be bounded? (This is known if $A=B$ [9], and we have used it. If $B\subset$ centre (A) then D must be 0 ([4], p. 21). If both B and B' are separable matroid with unit and if the algebra generated by B and B' is dense in A then the techniques of 4 may be used to show that D is closed, hence bounded.) If A is a von Neumann algebra, must D be inner?

17.3. Let A be a matroid C^*-algebra without unit. Must A have outer derivations? (Example 16 shows that the answer is yes for a large number of cases, including (in view of [2]) the case that A is separable and the case that A is finite. Therefore (cf. 3) the only case remaining is that A is nonseparable and there exists on A^+ a faithful lower semicontinuous trace with dense ideal of definition distinct from A.)

17.4. Let A be a simple C^*-algebra without unit. Must A have outer derivations?

I would like to thank Professor Israel Halperin for pointing out to me that, given a derivation of a matroid C^*-algebra with unit, it should be possible to construct an element which implements it by fitting together elements which implement it locally (that is, on sub-C^*-algebras isomorphic to matrix algebras).

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Concerning My Paper on the Boundary Behavior of Minimal Surfaces

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My proof of Kellogg's theorem for solutions of Plateau's problem (these Inventiones math. **8**, 313–333 (1969)) can still be shortened:

The relation $\mathfrak{x}_\rho(e^{i\vartheta})\,\mathfrak{y}'(x(e^{i\vartheta}))=0$ from p. 328 is available in almost all boundary points as soon as it is known that the functions $f_j(w)$ belong to class $C^{0,1}(\bar{P})$. The proof of this fact is accomplished on the bottom of p. 324. Let now $w=1$ be the point described in the middle of p. 325. From

$$\mathfrak{x}_\rho(w)\,\mathfrak{y}'(x(w))=x_\rho(w)+\psi'(x(w))\,y_\rho(w)+\chi'(x(w))\,z_\rho(w)=0, \qquad w=e^{i\vartheta}$$

and $x_\rho(1)=0$ the inequality

$$|x_\rho(e^{i\vartheta})-x_\rho(1)|\leqq\mathscr{C}_4\,|\psi'(x(e^{i\vartheta}))|+\mathscr{C}_4\,|\chi'(x(e^{i\vartheta}))|\leqq\mathscr{C}_1\,\mathscr{C}_4^{1+\alpha}\,|\vartheta|^\alpha$$

follows for a.a. ϑ in $|\vartheta|\leqq\vartheta_0$. Since $\rho\, x_\rho(w)=\mathrm{Re}\,[w f_1'(w)]$ (see p. 329), Lemma 5 tells us that $f_1(w)$ satisfies the same inequality as the functions $f_2(w)$ and $f_3(w)$ on top of p. 325:

$$|f_1''(\rho)|\leqq\mathscr{C}_5(1-\rho)^{\alpha-1} \qquad \text{for } \tfrac{1}{2}\leqq\rho<1.$$

The application of argument $\mathscr{A}$ and Lemma 3 now imply that the functions $f_j(w)$, as well as the position vector $\mathfrak{x}(w)$, belong to class $C^{1,\alpha}(\bar{P})$.

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Whiteheadgruppe topologischer Räume*

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$\pi_1(X)$ ist die Fundamentalgruppe des topologischen Raumes X, Wh ist der Whitehead-Funktor von Gruppen in abelsche Gruppen [3]. Diese Arbeit enthält eine von $\pi_1(X)$ unabhängige Interpretation der Gruppe $Wh(\pi_1(X))$, die auf den geometrischen Konstruktionen in [4] basiert (vgl. auch [2]). In Abschnitt 1 wiederholen wir die Definition der Relation $P = DQ$ rel K von [4] und ordnen jedem CW-Komplex K eine Whiteheadgruppe $Wh(K)$ zu. In Abschnitt 2 beweisen wir einen allgemeinen Summensatz für die Whitehead-Torsion, der die "product- and sum-theorems" von [1] als Spezialfälle enthält: Sind $f_n\colon K_n \to L_n$ ($n=0,1,2$) Homotopieäquivalenzen zwischen endlichen zusammenhängenden CW-Komplexen mit $K_0 = K_1 \cap K_2$, $L_0 = L_1 \cap L_2$ und $f_1|K_0 = f_2|K_0 = f_0$, so ist die zusammengesetzte Abbildung $f\colon K = K_1 \cup K_2 \to L = L_1 \cup L_2$ eine Homotopieäquivalenz, und ihre Whitehead-Torsion ist

$$\tau(f) = i_{1*}\tau(f_1) + i_{2*}\tau(f_2) - i_{0*}\tau(f_0)$$

($i_n =$ Inklusion $L_n \subset L$); das folgt aus Satz 2, aus der Definition von $\tau(f)$ in [3] und aus Abschnitt 3, in dem ein Isomorphismus $Wh(K) \to Wh(\pi_1(K))$ konstruiert wird.

Indem man relative CW-Komplexe über X betrachtet, kann man jedem topologischen Raum X eine Whiteheadgruppe $Wh(X)$ zuordnen und eine entsprechende Theorie entwickeln; das ist der Inhalt von Abschnitt 4.

1. Die Gruppe *Wh*(*K*)

Sei K ein CW-Komplex, nicht notwendig endlich. Ein K-Komplex ist ein CW-Komplex P, der K als co-endlichen Teilkomplex enthält (d.h. es ist eine Abbildung $i\colon K \to P$ gegeben, die K isomorph auf einen Teilkomplex von P abbildet, und $P - i(K)$ besteht aus nur endlich vielen Zellen). Seien P' und P'' K-Komplexe; P' ist ein einfacher Teilkomplex von P'', wenn gilt:

i) P' ist Teilkomplex von P'' und $P'' - P'$ besteht aus genau zwei Zellen e^q und e^{q-1},

* Prof. Dr. Wolfgang Franz zum 65. Geburtstag.

ii) es gibt eine charakteristische Abbildung $f\colon I^q \to P''$ der Zelle e^q mit $f(J^{q-1}) \subset P'$, so daß $f|I^{q-1}$ charakteristische Abbildung von e^{q-1} ist; dabei ist $J^{q-1} = \overline{\dot{I}^q - I^{q-1}}$.

Zwei K-Komplexe P und Q sind vom gleichen *einfachen Homotopietyp relativ* K, $P \sim Q$ rel K, wenn es K-Komplexe $P_0 = P, P_1, \ldots, P_{n-1}, P_n = Q$ gibt, so daß P_{i-1} einfacher Teilkomplex von P_i oder P_i einfacher Teilkomplex von P_{i-1} ist $(i = 1, \ldots, n)$; welcher der beiden Fälle eintritt, kann von Index zu Index verschieden sein. Mit den Bezeichnungen von [4]: $P \sim Q$ rel K genau dann, wenn es eine formale Deformation $D\colon P \to Q$ rel K gibt. Das ist eine Äquivalenzrelation zwischen K-Komplexen. $Sh(K)$ ist die Menge der Äquivalenzklassen, $s(P, K) \in Sh(K)$ ist die Äquivalenzklasse des K-Komplexes P. Definiere eine Addition in $Sh(K)$ durch $s(P, K) + s(Q, K) = s(P \cup_K Q, K)$, wobei $P \cup_K Q$ aus der disjunkten Vereinigung von P und Q entsteht, indem man die Teilkomplexe $K \subset P$ und $K \subset Q$ identifiziert. $Sh(K)$ ist eine kommutative Halbgruppe mit neutralem Element $0 = s(K, K)$. Eine zelluläre Abbildung $f\colon K \to L$, L ein CW-Komplex, induziert einen Homomorphismus $f_*\colon Sh(K) \to Sh(L)$ durch $f_* s(P, K) = s(L \cup_f P, L)$, wobei $L \cup_f P$ aus der disjunkten Vereinigung von L und P entsteht, indem man jedem Punkt $x \in K$ mit $f(x) \in L$ identifiziert. Ist $g\colon L \to M$ ebenfalls zellulär, M ein CW-Komplex, so $(gf)_* = g_* f_*$. Ferner gilt das Homotopieaxiom: $f_1 \simeq f_2\colon K \to L$, f_1 und f_2 zellulär, impliziert $f_{1*} = f_{2*}$.

Die einzig nichttriviale Aussage ist das Homotopieaxiom (vgl. Lemma 13 in [4]). Wir brauchen es nur für die zellulären Abbildungen $i_n\colon K \to K \times I$, $i_n(x) = (x, n)$, zu beweisen $(n = 0, 1)$. Sei P ein K-Komplex, seien $e_1, \ldots, e_m$ die Zellen von $P - K$, so geordnet, daß $\dim(e_{i-1}) \geqq \dim(e_i)$, und sei

$$Q = P \times I \quad \text{und} \quad P_i = P \times I - \bigcup_{j=1}^{i} e_j \times [0, 1).$$

Dann ist in der Folge

$$Q \supset P_1 \supset \cdots \supset P_{m-1} \supset P_m = K \times I \cup P \times 1$$

von $(K \times I)$-Komplexen jeder Komplex einfacher Teilkomplex des vorangehenden, also $s(Q, K \times I) = s(P_m, K \times I) = i_{1*} s(P, K)$. Ebenso $s(Q, K \times I) = i_{0*} s(P, K)$, somit $i_{0*} = i_{1*}$. – Man kann jetzt $f_*\colon Sh(K) \to Sh(L)$ auch für nicht-zelluläre Abbildungen $f\colon K \to L$ definieren, nämlich durch $f_* = f_{1*}$, wobei f_1 zelluläre Approximation von f ist.

Wir definieren die *Whiteheadgruppe von* K, $Wh(K)$, als die größte in $Sh(K)$ enthaltene Gruppe, d. h. als die Menge der Elemente von $Sh(K)$, die ein Inverses bei der Addition besitzen; der folgende Satz beschreibt diese Elemente:

Satz 1. *$s(P,K)\in Wh(K)$ genau dann wenn K Deformationsretrakt von P ist.*

Beweis. Ist $s(P,K)\in Wh(K)$, so gibt es einen K-Komplex Q mit $P\cup_K Q\sim K$ rel K. Dann ist K Deformationsretrakt von $P\cup_K Q$ und folglich auch Deformationsretrakt von P. Zum Beweis der Umkehrung benötigen wir ein Lemma. Seien P und Q K-Komplexe, P Teilkomplex von Q, K Deformationsretrakt von P. Sei $r\colon P\to K$ eine Retraktion. Dann gilt (vgl. Lemma 7.4 in [3]):

Lemma 1. $s(Q,K)=s(P,K)+r_* s(Q,P)$.

Beweis. Wir dürfen r zellulär annehmen. Sei $i\colon K\to P$ die Inklusion. $i\,r\colon P\to P$ ist homotop zur Identität, also ist $(i\,r)_*$ die Identität von $Sh(P)$. Es folgt $P\cup_{ir}Q\sim Q$ rel P. Dann ist insbesondere $P\cup_{ir}Q\sim Q$ rel K, und das ist Lemma 1, da der K-Komplex $P\cup_{ir}Q$ die rechte Seite der zu beweisenden Formel repräsentiert.

Beweis von Satz 1. Sei wieder $r\colon P\to K$ eine zelluläre Retraktion, und sei M_r der Abbildungszylinder von r relativ K mit der induzierten Zellzerlegung. M_r ist der Quotientenraum von $P\times I$ bezüglich der Identifikationen $(x,1)=r(x)$ und $(y,t)=y$ für $x\in P$, $y\in K$ und $t\in I$. P ist als $P\times 0$ in M_r eingebettet. Seien $e_1,\dots,e_m$ mit $\dim(e_{i-1})\geqq\dim(e_i)$ die Zellen von $P-K$. M_r besteht aus den Zellen von P und den Zellen $e_i\times(0,1)$, $i=1,\dots,m$. Setze

$$Q_i=M_r-\bigcup_{j=1}^{i} e_j\cup e_j\times(0,1).$$

Dann ist in der Folge

$$M_r\supset Q_1\supset\cdots\supset Q_{m-1}\supset Q_m=K$$

von K-Komplexen jeder Komplex einfacher Teilkomplex des vorangehenden, also $s(M_r,K)=s(K,K)=0$ (vgl. Lemma 11 in [4]). Aus Lemma 1, auf $K\subset P\subset M_r$ angewandt, folgt: $r_* s(M_r,P)$ ist das Inverse von $s(P,K)$.

2. Der allgemeine Summensatz

Wir beweisen eine Formel zur Berechnung von $s(P,K)\in Sh(K)$ in dem Fall, daß P und K wie folgt in Teilkomplexe zerfallen:

$$P=P_1\cup P_2 \quad\text{mit}\quad P_1\cap P_2=P_0,$$
$$K=K_1\cup K_2 \quad\text{mit}\quad K_1\cap K_2=K_0,$$
$$K_1\subset P_1,\quad K_2\subset P_2,\quad\text{also}\quad K_0\subset P_0.$$

Sei $i_n\colon K_n\subset K$ die Inklusion $(n=0,1,2)$.

Satz 2. *Ist K_0 Deformationsretrakt von P_0, so ist*

$$s(P, K) = i_{1*}\, s(P_1, K_1) + i_{2*}\, s(P_2, K_2) - i_{0*}\, s(P_0, K_0).$$

Beweis. Sei $Q_n = K \cup P_n$; dann ist $P = Q_1 \cup Q_2$ und $Q_1 \cap Q_2 = Q_0$, somit

$$s(P, Q_0) = s(Q_1, Q_0) + s(Q_2, Q_0).$$

K ist Deformationsretrakt von Q_0, sei $r\colon Q_0 \to K$ eine Retraktion. Aus Lemma 1, auf $K \subset Q_0 \subset P$ bzw. $K \subset Q_0 \subset Q_n$ angewandt, folgt:

$$s(P, K) = s(Q_0, K) + r_*\, s(P, Q_0),$$
$$s(Q_n, K) = s(Q_0, K) + r_*\, s(Q_n, Q_0).$$

Aus allen vier Formeln ($n = 1, 2$) zusammen folgt

$$s(P, K) = s(Q_1, K) + s(Q_2, K) - s(Q_0, K),$$

und wegen $s(Q_n, K) = i_{n*}\, s(P_n, K_n)$ ist das die Behauptung (das Element $-s(Q_0, K)$ existiert nach Satz 1).

Es folgt aus den Ergebnissen des nächsten Abschnitts, daß Satz 2 den in [1] bewiesenen Summensatz als Spezialfall enthält. Wir zeigen, daß auch der Produktsatz von [1] aus Satz 2 folgt. Sei $P = P_1 \cup \cdots \cup P_m$ und $K = K_1 \cup \cdots \cup K_m$ mit $K_i \subset P_i$, so daß jeder nichtleere Durchschnitt $K_{i_1} \cap \cdots \cap K_{i_r}$ Deformationsretrakt von $P_{i_1} \cap \cdots \cap P_{i_r}$ ist. Aus Satz 2 folgt durch Induktion nach m:

$$s(P, K) = \sum_{r \geqq 1} (-1)^{r+1} \sum f_{i_1 \dots i_r}\, s(P_{i_1} \cap \cdots \cap P_{i_r}, K_{i_1} \cap \cdots \cap K_{i_r}),$$

wobei über alle Indexfolgen $i_1, \dots, i_r$ mit $1 \leqq i_1 < \cdots < i_r \leqq m$ und $K_{i_1} \cap \cdots \cap K_{i_r} \neq \emptyset$ zu summieren ist; $f_{i_1 \dots i_r}$ ist von der Inklusion $K_{i_1} \cap \cdots \cap K_{i_r} \subset K$ induziert. Sei L ein zusammenhängender Simplizialkomplex mit den abgeschlossenen Simplexen $L_1, \dots, L_m$. Für $s(P, K) \in Wh(K)$ berechne $s(P \times L, K \times L) \in Wh(K \times L)$ mit $P \times L = (P \times L_1) \cup \cdots \cup (P \times L_m)$ und $K \times L = (K \times L_1) \cup \cdots \cup (K \times L_m)$ nach der obigen Formel. Da jeder nichtleere Durchschnitt $L_{i_1} \cap \cdots \cap L_{i_r}$ ein L_n ist für ein gewisses n, ist

$$f_{i_1 \dots i_r}\, s\big(P \times (L_{i_1} \cap \cdots \cap L_{i_r}), K \times (L_{i_1} \cap \cdots \cap L_{i_r})\big) = i_{n*}\, s(P \times L_n, K \times L_n)$$

mit $i_n\colon K \times L_n \subset K \times L$. Sei $y_n \in L_n$ Basisecke und $f_n\colon K \to K \times L_n$, $f_n(x) = (x, y_n)$. Aus Lemma 1, auf $K \times L_n \subset P \times y_n \cup K \times L_n \subset P \times L_n$ angewandt, folgt

$$s(P \times L_n, K \times L_n) = s(P \times y_n \cup K \times L_n, K \times L_n) = f_{n*}\, s(P, K);$$

denn offenbar ist $s(P \times L_n, P \times y_n \cup K \times L_n) = 0$. Daher:

$$i_{n*} s(P \times L_n, K \times L_n) = i_{n*} f_{n*} s(P, K) = i_* s(P, K)$$

mit $i: K \to K \times L$, $i(x) = (x, y_1)$; $i \simeq i_n f_n$ für alle n, weil L zusammenhängend ist. Insgesamt ergibt sich

$$s(P \times L, K \times L) = \sum_{r \geqq 1} (-1)^{r+1} a_r(L)\, i_* s(P, K),$$

wo $a_r(L)$ die Anzahl der Indexfolgen $1 \leqq i_1 < \cdots < i_r \leqq m$ ist mit $L_{i_1} \cap \cdots \cap L_{i_r} \neq \emptyset$. Der Zahlenfaktor ist die Eulersche Charakteristik von L:

$$s(P \times L, K \times L) = \chi(L)\, i_* s(P, K).$$

Das ist der Produktsatz, zunächst allerdings nur bewiesen, wenn L ein Simplizialkomplex ist. Er bleibt jedoch richtig, wenn man L durch einen endlichen zusammenhängenden CW-Komplex ersetzt: Das folgt leicht aus [4], wonach jeder endliche CW-Komplex vom einfachen Homotopietyp eines endlichen Simplizialkomplexes ist.

3. Die Isomorphie $Wh(K) \approx Wh(\pi_1(K))$

K sei zusammenhängend (andernfalls ist $Wh(K)$ die direkte Summe der Whiteheadgruppen der Zusammenhangskomponenten von K), $x_0 \in K$ sei Basisecke, $\pi = \pi_1(K, x_0)$ und $Z\pi$ der Gruppenring von π über den ganzen Zahlen. Für $q \geqq 3$ und $m \geqq 1$ sei

$$K_{q,m} = K \cup e_1^{q-1} \cup \cdots \cup e_m^{q-1} \qquad (\dot{e}_1^{q-1} = \cdots = \dot{e}_m^{q-1} = x_0)$$

die Ein-Punkt-Vereinigung von K und m $(q-1)$-Sphären. Die Homotopiegruppe $\pi_{q-1} = \pi_{q-1}(K_{q,m}, x_0)$ ist bezüglich der Operation von $\pi_1(K_{q,m}, x_0) \approx \pi$ ein $Z\pi$-Modul. Sei F_i: $(I^{q-1}, \dot{I}^{q-1}) \to (K_{q,m}, x_0)$ charakteristische Abbildung von e_i^{q-1} und $a_i \in \pi_{q-1}$ das von ihr repräsentierte Element. Sei $A = (x_{ij})$ eine (m, m)-Matrix über $Z\pi$ und $b_i = \sum_j x_{ij} a_j \in \pi_{q-1}$.

$$f_i: (\dot{I}^q, J^{q-1}) \to (K_{q,m}, x_0)$$

sei eine zelluläre Abbildung, so daß $f_i | I^{q-1}$ das Element b_i repräsentiert. Dann ist

$$P_q(f_1, \ldots, f_m) = K_{q,m} \cup_{f_1} e_1^q \cup \cdots \cup_{f_m} e_m^q$$

ein K-Komplex mit Teilkomplex $K_{q,m}$.

Lemma 2. *Die Zuordnung* $A \to s(P_q(f_1, \ldots, f_m), K)$ *induziert einen Homomorphismus* S_q: $Wh(\pi) \to Wh(K)$.

Beweis. Sei $P_i = P_q(f_1, \ldots, f_m) - e_i^q$ und h_i: $K_{q,m} \subset P_i$ die Inklusion. Sei $x \in \pi$, $y \in Z\pi$ und g_i: $(\dot{I}^q, J^{q-1}) \to (K_{q,m}, x_0)$ eine zelluläre Abbildung, so daß $g_i | I^{q-1}$ das Element $\pm x b_i + y b_j \in \pi_{q-1}$ repräsentiert $(j \neq i)$. Sei

$s_0 = s(I^q, \dot{I}^q) \in Sh(\dot{I}^q)$. Es ist

$$s\big(P_q(f_1, \ldots, f_i, \ldots, f_m), P_i\big) = (h_i f_i)_*(s_0)$$
$$s\big(P_q(f_1, \ldots, g_i, \ldots, f_m), P_i\big) = (h_i g_i)_*(s_0).$$

Nach Definition von g_i ist $h_i g_i \simeq h_i f_i \colon \dot{I}^q \to P_i$ (freie Homotopie). Aus dem Homotopieaxiom folgt, daß die Komplexe $P_q(f_1, \ldots, f_i, \ldots, f_m)$ und $P_q(f_1, \ldots, g_i, \ldots, f_m)$ die gleiche Klasse in $Sh(P_i)$, also auch in $Sh(K)$, repräsentieren. Somit ist $s\big(P_q(f_1, \ldots, f_m), K\big) = s_A$ unabhängig von der Wahl der Repräsentanten f_i der b_i, und s_A ändert sich nicht bei elementaren Zeilenumformungen der Matrix A. Insbesondere kann man im Fall

$$A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

annehmen, daß $f_m | I^{q-1} = F_m$ ist. Dann ist $P_q(f_1, \ldots, f_{m-1})$ einfacher Teilkomplex von $P_q(f_1, \ldots, f_m)$ und $s_A = s_B$. Damit ist gezeigt: Durch $\{A\} \to s_A$, wo $\{A\} \in Wh(\pi)$ die Restklasse der regulären Matrix A über $Z\pi$ ist, ist eine Zuordnung $S_q \colon Wh(\pi) \to Sh(K)$ definiert. Gegeben reguläre Matrizen A und B über $Z\pi$, sei

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Dann ist $\{C\} = \{A\} + \{B\}$ und offenbar auch $s_C = s_A + s_B$. Folglich ist S_q homomorph, daher Bild $(S_q) \subset Wh(K)$.

Lemma 3. *Zu $s \in Wh(K)$ und jeder hinreichend großen Zahl q gibt es eine reguläre Matrix A über $Z\pi$ mit $S_q(\{A\}) = s$.*

Beweis. Nach Lemma 15 von [4] gibt es einen Repräsentanten P von s, so daß $P - K$ nur aus Zellen der Dimension q und $q-1$ besteht (wegen $s \in Wh(K)$ ist K Deformationsretrakt von jedem Repräsentanten von s nach Satz 1). Nach homotoper Deformation der Klebeabbildungen der Zellen von $P - K$ kann man annehmen, daß $K \cup P^{q-1} = K_{q,m}$ ist für ein $m \geqq 1$ (P^{q-1} ist das $(q-1)$-Gerüst von P). Der Randoperator

$$d \colon \pi_q(P, K_{q,m}, x_0) \to \pi_{q-1}(K_{q,m}, K, x_0)$$

in der Homotopiesequenz des Tripels $K \subset K_{q,m} \subset P$ ist isomorph. Beide Homotopiegruppen sind freie $Z\pi$-Moduln, die Zellen von $P - K$ definieren ausgezeichnete Basen. Ist A die Matrix von d bezüglich dieser Basen, so $s = s_A$ und A ist regulär.

Satz 3. *$S_q \colon Wh(\pi) \to Wh(K)$ ist isomorph und $S_3 = -S_4 = S_5 = -S_6 = \cdots$.*

Beweis. Für jeden K-Komplex P mit K Deformationsretrakt von P ist die Whitehead-Torsion $\tau(P, K) \in Wh(\pi)$ definiert (vgl. [3]), und die Zuordnung $P \to \tau(P, K)$ induziert einen Homomorphismus $T \colon Wh(K) \to$

$Wh(\pi)$ (vgl. Theorem 3.1 von [3]). Ist A eine reguläre Matrix über $Z\pi$, so ist K nach Lemma 2 und Satz 1 Deformationsretrakt von $P_q(f_1, \ldots, f_m)$, und die Torsion ist leicht auszurechnen: $\tau(P_q(f_1, \ldots, f_m), K) = (-1)^q\{A\}$. Das bedeutet $TS_q = (-1)^q$, und hieraus und aus Lemma 3 folgt, daß T isomorph ist und $T^{-1} = (-1)^q S_q$. Daraus Satz 3.

Bemerkung. T ist ein natürlicher Isomorphismus. Eine Abbildung $f: K \to L$ induziert Homomorphismen $f_*: Wh(K) \to Wh(L)$ und $f_1: \pi \to \pi' = \pi_1(L, f(x_0))$. f_1 induziert einen Homomorphismus $f_\#: Wh(\pi) \to Wh(\pi')$. Dann ist $Tf_* = f_\# T$.

4. Whiteheadgruppe topologischer Räume

Die Definition der Gruppe $Wh(K)$ in Abschnitt 1 benutzt die gegebene Zellzerlegung von K. Nach Abschnitt 3 ist $Wh(K)$ unabhängig von dieser Zellzerlegung. Wir geben eine von der Zellzerlegung unabhängige Definition dieser Gruppe.

Sei X ein topologischer Raum, und seien P' und P'' endliche relative CW-Komplexe über X. P' ist einfacher Teilkomplex von P'', wenn die Bedingungen i) und ii) von Abschnitt 1 erfüllt sind. Zwei endliche relative CW-Komplexe P und Q über X sind vom gleichen *einfachen Homotopietyp relativ* X, $P \sim Q$ rel X, wenn es relative CW-Komplexe $P_0 = P, P_1, \ldots, P_{m-1}, P_m = Q$ über X gibt, so daß $P_{i-1} \subset P_i$ oder $P_i \subset P_{i-1}$ einfacher Teilkomplex ist $(i = 1, \ldots, m)$. Das ist eine Äquivalenzrelation. $Sh(X)$ ist die Menge der Äquivalenzklassen und $s(P, X) \in Sh(X)$ ist die Klasse des endlichen relativen CW-Komplexes P über X. Mit der Addition $s(P, X) + s(Q, X) = s(P \cup_X Q, X)$ ist $Sh(X)$ eine kommutative Halbgruppe mit neutralem Element $0 = s(X, X)$. Eine Abbildung $f: X \to Y$ induziert einen Homomorphismus $f_*: Sh(X) \to Sh(Y)$ durch $f_* s(P, X) = s(Y \cup_f P, Y)$, und für $g: Y \to Z$ gilt $(gf)_* = g_* f_*$. Es gilt das Homotopieaxiom. Wir definieren die *Whiteheadgruppe von* X, $Wh(X)$, als die größte in $Sh(X)$ enthaltene Gruppe. Es gilt sinngemäß Satz 1. Lemma 1 lautet jetzt wie folgt: Sei X Deformationsretrakt von P und $r: P \to X$ eine Retraktion. Ist dann $P \subset Q$ Teilkomplex, so ist Q relativer CW-Komplex über $|P|$ und

$$s(Q, X) = s(P, X) + r_* s(Q, |P|). \tag{1'}$$

(Dabei ist $|P|$ der dem Komplex P zugrundeliegende Raum; (1′) folgt am einfachsten durch Induktion nach der Zahl der Zellen in $Q - P$; der Beweis von Abschnitt 1 überträgt sich nicht.)

Die Konstruktion von $S_q: Wh(\pi) \to Wh(X)$ überträgt sich, Lemma 2 und 3 bleiben richtig. Wenn auf X die Theorie des universellen Überlagerungsraumes anwendbar ist, läßt sich für jeden endlichen relativen CW-Komplex P über X mit X Deformationsretrakt von P analog zu [3] die Whitehead-Torsion $\tau(P, X) \in Wh(\pi)$ definieren. Dann bleibt Satz 3 richtig.

Sei K ein CW-Komplex und $|K|$ der zugrundeliegende Raum. Jeder K-Komplex ist ein endlicher relativer CW-Komplex über $|K|$, und aus $P \sim Q$ rel K, P und Q K-Komplexe, folgt $P \sim Q$ rel $|K|$. Durch $s(P, K) \to s(P, |K|)$ wird folglich ein Homomorphismus $V: Sh(K) \to Sh(|K|)$ definiert; V vergißt die Zellstruktur von K. V bildet $Wh(K)$ isomorph auf $Wh(|K|)$ ab.

Satz 4. *$V: Sh(K) \to Sh(|K|)$ ist isomorph.*

Beweis. Zu einem endlichen relativen CW-Komplex P über $|K|$ erhält man durch zelluläre Approximation der Klebeabbildungen der Zellen von $P - |K|$ einen K-Komplex Q mit $P \sim Q$ rel $|K|$; man benutzt das Homotopieaxiom und Lemma 14 von [4]. Daher ist V epimorph. Seien P und Q K-Komplexe mit $P \sim Q$ rel $|K|$. Dann gibt es relative CW-Komplexe $P_1, \ldots, P_{m-1}$ über $|K|$, so daß $P_{i-1} \subset P_i$ oder $P_i \subset P_{i-1}$ einfacher Teilkomplex ist ($i = 1, \ldots, m$ und $P_0 = P$, $P_m = Q$). $f_i: P_{i-1} \to P_i$ sei die Inklusion im ersten Fall und eine Retraktion im zweiten. Dann ist $f = f_m \ldots f_1: P \to Q$ eine zelluläre Homotopieäquivalenz zwischen relativen CW-Komplexen über $|K|$ mit $f(x) = x$ für $x \in |K|$ (f ist i. allg. keine zelluläre Abbildung des CW-Komplexes P in den CW-Komplex Q). $g: P \to Q$ mit $g(x) = x$ für $x \in K$ sei zelluläre Approximation von f, und M_f, M_g seien die Abbildungszylinder von f bzw. g relativ $|K|$. M_f ist relativer CW-Komplex über $|P|$, M_g ist CW-Komplex mit Teilkomplex P. Durch Induktion nach der Anzahl m der Faktoren in $f = f_m \ldots f_1$ folgt leicht, daß $s(M_f, |P|) = 0$. Wegen $g \simeq f$ ist dann auch $s(M_g, |P|) = 0$ (das folgt aus (1'); vgl. den Beweis von Lemma 7.7 in [3]). Aber $s(M_g, P) \in Wh(P)$, da g Homotopieäquivalenz ist, und aus $0 = s(M_g, |P|) = Vs(M_g, P)$ darf man daher auf $s(M_g, P) = 0$ schließen, d.h. $M_g \sim P$ rel P. Nach Lemma 11 in [4] ist $M_g \sim Q$ rel Q, vgl. den Beweis von Satz 1. Beides zusammen impliziert $P \sim Q$ rel $P \cap Q$, also $P \sim Q$ rel K wegen $K \subset P \cap Q$. Somit ist V monomorph.

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Une remarque sur les idéaux de fonctions différentiables

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1. Introduction

Soient X un ouvert de $\mathbf{R}^n$ et $\mathscr{E}$ (resp. $\mathscr{O}$) le faisceau des germes de fonctions de classe $\mathscr{C}^\infty$ (resp. analytiques) sur X, à valeurs complexes. Pour tout point $a \in X$, $a=(a_1, \ldots, a_n)$, désignons par F_a l'anneau des séries formelles $\mathbf{C}[[x_1-a_1, \ldots, x_n-a_n]]$ et par $f_a \mapsto \hat{f}_a$ l'application $\mathscr{E}_a \to F_a$ (ou $\mathscr{O}_a \to F_a$): «série de Taylor en a».

Soit $M \in \mathrm{Hom}(X;\, \mathscr{O}^q, \mathscr{O}^p)$ une matrice à coefficients analytiques sur X; le résultat suivant est connu (Malgrange [2]):

Soit $f \in \mathscr{E}(X)^p$; pour qu'il existe $g \in \mathscr{E}(X)^q$ vérifiant $M g = f$, il faut et il suffit qu'en tout point $a \in X$, la condition suivante soit satisfaite:

$$C(a):\ \textit{Il existe } \gamma_a \in F_a^q \textit{ tel qu'on ait: } \hat{f}_a = \hat{M}_a \gamma_a.$$

Il est facile de voir sur des exemples que la conclusion reste vraie si l'on fait seulement l'hypothèse que $C(a)$ est vérifié en «suffisamment de points a». Par exemple, si $X=\mathbf{R}^2$, un $f \in \mathscr{E}(\mathbf{R}^2)$ est de la forme $x_1 g$ si et seulement si l'on a $f(0, x_2)=0$; par conséquent, il suffit ici d'imposer $C(a)$ pour un ensemble dense sur la droite $x_1=0$. D'une façon générale, posons la définition suivante:

Définition 1.1. *Un sous-ensemble Y de X sera dit «M-dense» si tout $f \in \mathscr{E}(X)^p$, vérifiant $C(a)$ en tout point $a \in Y$, est de la forme $M g$, avec $g \in \mathscr{E}(X)^q$* [Autrement dit, si la condition «$\forall\, a \in Y$, $C(a)$» entraîne «$\forall\, a \in X$, $C(a)$»].

Le but de cette note est de donner une caractérisation des ensembles M-denses; l'outil essentiel est un argument de «semi-continuité de la dimension», déjà utilisé dans des questions analogues par Tougeron [4] (voir l'Appendice de cet article). Cette question m'avait été posée par F. Pham, ce dont je le remercie avec plaisir.

2. Caractérisation des ensembles M-denses

Désignons par L le conoyau du morphisme de faisceaux $M: \mathscr{O}^q \to \mathscr{O}^p$; en tensorisant par $\mathscr{E}$, on obtient une suite exacte:

$$\mathscr{E}^q \to \mathscr{E}^p \to L \underset{\mathscr{O}}{\otimes} \mathscr{E} \to 0$$

et, par partition de l'unité, une suite exacte pour les sections:

$$\mathscr{E}^q(X) \to \mathscr{E}^p(X) \to \Gamma(X, L \underset{\mathcal{O}}{\otimes} \mathscr{E}) \to 0.$$

On déduit de là que la propriété «Y est M-dense» ne dépend en fait que de L, et peut s'exprimer ainsi:

«*Soit* $\varphi \in \Gamma(X, L \underset{\mathcal{O}}{\otimes} \mathscr{E})$; *pour qu'on ait* $\varphi = 0$, *il faut et il suffit que*, $\forall a \in Y$, *l'image* $\hat{\varphi}_a$ *de* φ_a *dans* $L_a \underset{\mathcal{O}_a}{\otimes} F_a = \hat{L}_a$ *par l'application* (identité $\underset{\mathcal{O}_a}{\otimes}$ série de Taylor en a) *soit nulle*».

Posons $V = \operatorname{supp}(L) = \{a \in X \mid L_a \neq 0\}$; en tout point $a \in X - V$, la dernière condition est trivialement vérifiée; par conséquent, on peut supposer (quitte à remplacer Y par $Y \cap V$) que l'on a $Y \subset V$, ce que nous ferons désormais. Pour tout point $a \in V$, soient $\mathfrak{P}_{a,i}$ $(1 \leqq i \leqq l(a))$ les idéaux premiers de $\mathcal{O}_a$ associés à L_a (Bourbaki [1]), et soit $V_{a,i}$ le germe en a de sous-ensemble analytique-réel de X défini par $\mathfrak{P}_{a,i}$ (i.e. le germe des zéros de $\mathfrak{P}_{a,i}$). Désignons encore par V_a le germe de V en a; la proposition suivante est bien connue:

Proposition 2.1. *On a* $V_{a,i} \subset V_a$, *et* $V_a = \bigcup_i V_{a,i}$ $(1 \leqq i \leqq l(a))$.

Rappelons sa démonstration; soit $0 = \bigcap L_{a,i}$ une décomposition primaire réduite de 0 dans L_a, avec $L_{a,i}$ primaire pour $\mathfrak{P}_{a,i}$; soit L_i un sous-faisceau analytique cohérent de L, défini au voisinage de a, tel qu'on ait $(L_i)_a = L_{a,i}$; on a encore $0 = \bigcap L_i$, donc l'application diagonale $L \to \oplus (L/L_i)$ est injective, d'où $V_a = \bigcup \operatorname{supp}(L/L_i)_a$. Tout revient donc à montrer qu'on a $(L/L_i)_a = V_{a,i}$, c'est-à-dire qu'on est ramené au cas où L_a est coprimaire pour un idéal $\mathfrak{P}_a$.

Dans ce cas, on sait qu'il existe une suite de composition $L_a = \bar{L}_{a,1} \supset \bar{L}_{a,2} \supset \cdots \supset \bar{L}_{a,p+1} = 0$ telle que $\bar{L}_{a,i}/\bar{L}_{a,i+1}$ soit monogène et isomorphe à $\mathcal{O}_a/\mathfrak{Q}_{a,i}$, $\mathfrak{Q}_{a,i}$ idéal premier de $\mathcal{O}_a$ contenant $\mathfrak{P}_a$; on peut aussi supposer $\mathfrak{Q}_{a,p} = \mathfrak{P}_a$, puisque L_a contient un sous-module monogène isomorphe à $\mathcal{O}_a/\mathfrak{P}_a$; soient $\bar{L}_i$ des sous-faisceaux cohérents de L, définis au voisinage de a, vérifiant $(\bar{L}_i)_a = \bar{L}_{a,i}$; on a encore, au voisinage de a, $\bar{L}_i/\bar{L}_{i+1} \simeq \mathcal{O}/\mathfrak{Q}_i$, $\mathfrak{Q}_i$ un faisceau cohérent d'idéaux vérifiant $(\mathfrak{Q}_i)_a = \mathfrak{Q}_{a,i}$; par suite, on a, avec $\mathfrak{P} = \mathfrak{Q}_p$:

$$\operatorname{supp}(L)_a = \bigcup_i \operatorname{supp}(L_i/L_{i+1})_a = \bigcup_i \operatorname{supp}(\mathcal{O}/\mathfrak{Q}_i)_a = \operatorname{supp}(\mathcal{O}/\mathfrak{P})_a,$$

ce qui démontre la proposition.

Remarque 2.2. La décomposition précédente de V_a ne doit pas être confondue avec sa «décomposition en germes irréductibles»; les deux décompositions n'ont presque aucun rapport pour les raisons suivantes:

D'une part, les $V_{a,i}$ ne sont pas forcément irréductibles; ensuite il se peut que l'on ait $V_{a,i} \subset V_{a,j}$, pour $i \neq j$, soit que l'on ait $\mathfrak{P}_{a,i} \supset \mathfrak{P}_{a,j}$ («composantes immergées» de la décomposition primaire), soit encore simplement parce que, en analytique-réel, le «Nullstellensatz» est faux. Naturellement, dans le cas analytique-complexe, de ces trois accidents, seul le second peut se produire.

Théorème 2.3. *Les propriétés suivantes sont équivalentes:*

a) *Y est M-dense;*

b) *Pour tout point $a \in V$ et tout $i \leqq l(a)$,*

on a $V_{a,i} \cap Y_a \neq \emptyset$;

(autrement dit: soit $\tilde{V}_{a,i}$ un représentant de $V_{a,i}$ au voisinage de a; alors $\tilde{V}_{a,i} \cap Y$ est adhérent à a).

Démontrons d'abord a) $\Rightarrow$ b). Raisonnons par l'absurde, et supposons qu'il existe un point $a \in V$ et un i tels qu'on ait $V_{a,i} \cap Y_a = \emptyset$; il existe φ, section de L au voisinage de a tel que le noyau de l'application $f \mapsto f\varphi : \mathcal{O}_a \to L_a$ soit égal à $\mathfrak{P}_{a,i}$; au voisinage de a, soit $\mathfrak{P}_i$ le faisceau cohérent d'idéaux de $\mathcal{O}$ tel que $(\mathfrak{P}_i)_a = \mathfrak{P}_a$; l'application $f \mapsto f\varphi$ donne encore une suite exacte $0 \to \mathfrak{P}_i \to \mathcal{O} \to L$; l'application naturelle $L \to L \underset{\mathcal{O}}{\otimes} \mathscr{E}$ est injective (parce que, pour tout b, $\mathscr{E}_b$ est fidèlement plat sur $\mathcal{O}_b$, cf. Malgrange [2]; ou, plus simplement, parce que l'application composée $L_b \to L_b \underset{\mathcal{O}_b}{\otimes} \mathscr{E}_b \to L_b \underset{\mathcal{O}_b}{\otimes} F_b$ est injective, F_b étant fidèlement plat sur $\mathcal{O}_b$, cf. Bourbaki [1]); on peut donc considérer φ comme une section au voisinage de a de $L \underset{\mathcal{O}}{\otimes} \mathscr{E}$, dont le germe en a est $\neq 0$; soit $\psi \in \mathscr{E}(X)$, égal à 1 au voisinage de a, et à support compact: si le support de ψ est assez voisin de a, $\psi\varphi$ peut être considéré comme une section de $L \underset{\mathcal{O}}{\otimes} \mathscr{E}$ sur X entier; si de plus $\operatorname{supp}(\psi) \cap \tilde{V}_{a,i} \cap Y = 0$, $\tilde{V}_{a,i}$ un représentant convenable de $V_{a,i}$ au voisinage de 0, on aura $(\psi\varphi)_b = 0$ pour tout $b \in Y$; a fortiori $(\widehat{\psi\varphi})_b = 0$; mais $(\psi\varphi)_a \neq 0$, ce qui montre que Y n'est pas M-dense.

Démontrons maintenant b) $\Rightarrow$ a). Le théorème étant local, il suffit de le démontrer au voisinage d'un point $a \in X$ fixé. Nous allons procéder pour cela en deux étapes:

Etape 1. Supposons qu'on ait $L = \mathcal{O}/\mathfrak{J}$, $\mathfrak{J}$ faisceau cohérent d'idéaux, avec $\mathcal{O}_a/\mathfrak{J}_a$ réduit, i.e. $\mathfrak{J}_a$ intersection d'idéaux premiers; soit f une section de $\mathscr{E}$ au voisinage de a, vérifiant, pour tout point $b \in Y$ voisin de a: $\hat{f}_b \in \mathfrak{J}_b \underset{\mathcal{O}_b}{\otimes} F_b \, (= \hat{\mathfrak{J}}_b)$. Il suffit de montrer qu'on a $\hat{f}_a \in \hat{\mathfrak{J}}_a$: en effet, d'après un théorème classique de Cartan-Oka, en tout point $c \in V$, assez voisin de a, $\mathcal{O}_c/\mathfrak{J}_c$ est réduit, et par conséquent, le même raisonnement, appliqué à c au lieu de a, montrera qu'on a $\hat{f}_c \in \hat{\mathfrak{J}}_c$, ce qui est le résultat cherché.

On a $\mathfrak{J}_a = \bigcap \mathfrak{P}_{a,i}$ $(1 \leq i \leq l(a))$, donc, par platitude de F_a sur $\mathcal{O}_a$: $\hat{\mathfrak{J}}_a = \bigcap \hat{\mathfrak{P}}_{a,i}$; par hypothèse, on a, pour chaque i: $Y \cap V_{a,i} \neq 0$; ceci montre qu'il suffit de traiter le cas où $\mathfrak{J}_a$ est premier, et, plus précisément, de démontrer le lemme suivant:

Lemme 2.4. *Soient $\mathfrak{J}$ un faisceau cohérent d'idéaux, avec $\mathfrak{J}_a$ premier, Y un sous-ensemble de $V = \operatorname{supp}(\mathcal{O}/\mathfrak{J})$, avec $Y_a \neq \emptyset$, et f une section de $\mathscr{E}$ au voisinage de a, vérifiant, $\forall b \in Y$: $\hat{f}_b \in \hat{\mathfrak{J}}_b$; alors, on a $\hat{f}_a \in \hat{\mathfrak{J}}_a$.*

Posons $k = \dim \mathcal{O}_a/\mathfrak{J}_a$; on sait que, en tout point $b \in V$ voisin de a, on a encore $k = \dim \mathcal{O}_b/\mathfrak{J}_b$, donc $k = \dim F_b/\hat{\mathfrak{J}}_b$. Pour tout point $b \in X$, voisin de a, soit $\mathfrak{R}_b$ l'idéal de F_b engendré par $\hat{\mathfrak{J}}_b$ et $\hat{f}_b$; on sait (Tougeron [4]) que la fonction $b \mapsto \dim F_b/\mathfrak{R}_b$ est semicontinue supérieurement au voisinage de a (en un point où $\mathfrak{R}_b = F_b$, on pose par exemple $\dim F_b/\mathfrak{R}_b = -1$); par hypothèse, pour $b \in Y$, on a $\mathfrak{R}_b = \hat{\mathfrak{J}}_b$, donc, puisque Y est adhérent à a, on aura $\dim F_a/\mathfrak{R}_a \geq k$.

D'après un théorème de Nagata-Zariski, $\hat{\mathfrak{J}}_a$ est premier (voir par exemple une démonstration due à Serre, dans Malgrange [3]); d'autre part, on a $\dim F_a/\hat{\mathfrak{J}}_a = k$, et $\mathfrak{R}_a \supset \hat{\mathfrak{J}}_a$; il en résulte qu'on a nécessairement $\mathfrak{R}_a = \hat{\mathfrak{J}}_a$, d'où $\hat{f}_a \in \hat{\mathfrak{J}}_a$, ce qui démontre le lemme et achève la première étape.

Etape 2; le cas général. Soit $a \in V$, et soit φ une section de $L \underset{\mathcal{O}}{\otimes} \mathscr{E}$, au voisinage de a, vérifiant, pour tout $b \in Y$ voisin de a: $\hat{\varphi}_b = 0$. Pour établir le théorème il suffit de montrer qu'on a $\hat{\varphi}_a = 0$. En raisonnant comme dans l'étape précédente, il suffit de traiter le cas où L_a est co-primaire pour un certain idéal $\mathfrak{P}_a$. Reprenons alors les notations de la proposition 2.1, fin de la démonstration. Nous allons démontrer, par récurrence descendante sur i $(1 \leq i \leq p+1)$, le résultat suivant:

Lemme 2.5. *Il existe g_i, analytique au voisinage de a, avec $g_{i,a} \notin \mathfrak{P}_a$, possédant la propriété suivante; si φ est une section de $\bar{L}_i \otimes \mathscr{E}$ au voisinage de a, vérifiant, $\forall b \in Y$: $\hat{\varphi}_b = 0$, on a $g_i \varphi = 0$.*

Pour $i = p+1$, le résultat est trivial, puisque $\bar{L}_{p+1} = 0$. Supposons alors le résultat acquis pour $i+1$, et démontrons le pour i; soit ψ l'image de φ dans l'espace des sections de $(\bar{L}_i/\bar{L}_{i+1}) \underset{\mathcal{O}}{\otimes} \mathscr{E} \simeq \mathscr{E}/\mathfrak{Q}_i \mathscr{E}$; si $\mathfrak{Q}_i = \mathfrak{P}$, par la première étape, on a $\psi = 0$, donc en fait φ est une section de $L_{i+1} \underset{\mathcal{O}}{\otimes} \mathscr{E}$ (noter que, par platitude de $\mathscr{E}$ sur $\mathcal{O}$, l'application $\bar{L}_{i+1} \underset{\mathcal{O}}{\otimes} \mathscr{E} \to \bar{L}_i \underset{\mathcal{O}}{\otimes} \mathscr{E}$ est injective; on peut donc considérer le premier comme un sous-faisceau du second); dans ce cas, le lemme est vrai avec $g_i = g_{i+1}$. Si au contraire $\mathfrak{Q}_{i,a} \neq \mathfrak{P}_a$, il existe h analytique au voisinage de a, avec $h_a \in \mathfrak{Q}_{i,a}$ et $h_a \notin \mathfrak{P}_a$; on aura $h\psi = 0$ au voisinage de a, donc $h\varphi$ est une section de $\bar{L}_{i+1} \underset{\mathcal{O}}{\otimes} \mathscr{E}$ et le lemme sera vrai avec $g_i = h g_{i+1}$.

Achevons maintenant la démonstration: soit φ une section de $L \underset{\mathcal{O}}{\otimes} \mathcal{E}$ au voisinage de a, vérifiant, $\forall\, b \in Y, \hat{\varphi}_b = 0$; on a $g_1\ \varphi = 0$, donc en particulier $\hat{g}_{1,a}\, \hat{\varphi}_a = 0$. Comme $g_{1,a} \notin \mathfrak{P}_a$, l'homothétie $g_{1,a}$: $L_a \rightarrow L_a$ est injective; par suite (platitude de F_a sur $\mathcal{O}_a$) l'homothétie $\hat{g}_{1,a}$: $\hat{L}_a \rightarrow \hat{L}_a$ est aussi injective; par suite on a $\hat{\varphi}_a = 0$, ce qui démontre le théorème.

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Zur Theorie der Deformationen kompakter komplexer Räume

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Das Ziel der vorliegenden Arbeit ist es, die folgende Verallgemeinerung eines Satzes von Grauert und Fischer [4] herzuleiten (Satz 4.9): Ist $p: X \to S$ eine eigentliche platte holomorphe Abbildung und sind alle Fasern von p paarweise isomorph, so ist p in jedem Punkt von S trivial, falls S reduziert ist.

Da die Fasern beliebige kompakte komplexe Räume sein dürfen, sind die potential-theoretischen Methoden von Kodaira-Spencer nicht anwendbar. Verfasser benützt vor allem das Ergebnis von Douady [2], mehr noch dessen relativierte Form von [14]. Hieraus wird folgendes Resultat hergeleitet, welches ein Haupthilfsmittel zum Beweis von (4.9) darstellt:

Ist $p: X \to S$ platt und eigentlich, $\mathbf{Coh}(S)$ (resp. $\mathbf{Coh}(X)$) die Kategorie der kohärenten Modulgarben auf S (resp. X), so besitzt der Funktor „analytisches Urbild" $p^*: \mathbf{Coh}(S) \to \mathbf{Coh}(X)$ einen linksadjungierten Funktor ${}_*p: \mathbf{Coh}(X) \to \mathbf{Coh}(S)$.

Außerdem wird gezeigt:

Ist $p: X \to S$ eine eigentliche holomorphe Abbildung, so ist p genau dann platt, wenn der Funktor „Basiserweiterung" $p^+: \mathbf{An}_{/S} \to \mathbf{An}_{/X}$ einen rechtsadjungierten Funktor besitzt. (Hierbei ist $\mathbf{An}_{/S}$ die Kategorie der „komplexen Räume über S", analog ist $\mathbf{An}_{/X}$ definiert.) Vgl. zu diesem Satz auch [8].

§ 0. Bezeichnungen

(0.1) Ist $\mathbf{A}$ eine Kategorie, so wird mit $\hat{\mathbf{A}}$ die Kategorie der kontravarianten Funktoren von $\mathbf{A}$ in $\mathbf{Ens}$ bezeichnet. Man hat einen Funktor $h: A \mapsto \hat{A}$, $f \mapsto (g \mapsto Hom(g, f))$. h ist voll-treu, daher läßt sich, und das geschieht im folgenden stets, $\mathbf{A}$ als volle Unterkategorie von $\hat{\mathbf{A}}$ auffassen. Ein Objekt aus $\hat{\mathbf{A}}$ heißt darstellbar, wenn es zu einem Objekt von $\mathbf{A}$ isomorph ist.

Ist S ein Objekt von $\mathbf{A}$, so wird mit $\mathbf{A}_{/S}$ die Kategorie der „Objekte über S" bezeichnet. Die Objekte von $\mathbf{A}_{/S}$ sind die Morphismen $p: X \to S$; ist $p: X \to S$ ein Objekt von $\mathbf{A}_{/S}$, so wird kurz X als Objekt über S bezeichnet. Sind $p: X \to S$, $p': X' \to S$ Objekte von $\mathbf{A}_{/S}$, so wird die Menge der $\mathbf{A}_{/S}$-Morphismen durch $Hom_S(X, X') := \{f \in Hom(X, X') : p' f = p\}$ gegeben.

(0.2) Im folgenden sei **A** eine Kategorie mit endlichen Faserprodukten; konkret wird es sich stets um die Kategorie **An** der separierten komplexen Räume handeln.

Ist $p\colon X \to S$ ein Morphismus, so hat man den Funktor „Basiserweiterung": $p^+\colon \mathbf{A}_{/S} \to \mathbf{A}_{/X}$, $f \mapsto f \times_S id_X$. Für p^+T, p^+f wird oft auch: $T_{(X)}, f_{(X)}$ geschrieben.

Im folgenden wichtig ist der Funktor

$$p_+\colon \mathbf{A}_{/X} \to \widehat{\mathbf{A}_{/S}}, \quad f \mapsto (g \mapsto Hom_X(p^+ g, f)).$$

Für Objekte Y über X, T über S ist also $(p_+ Y)(T) = Hom_X(T_{(X)}, Y)$.

(0.2.1) Falls für alle Objekte Y über X der Funktor $p_+ Y$ darstellbar ist, besitzt p^+ einen rechtsadjungierten Funktor, welcher ohne Gefahr der Konfusion wieder mit p_+ bezeichnet wird; dies folgt aus dem Auswahlaxiom.

(0.2.2) Ist $p\colon X \to S$ ein Objekt über S, Y ein weiteres Objekt über S, so definiert man:

$$\mathbf{Hom}_S(X, Y) := p_+(Y_{(X)}).$$

Ist T ein Objekt über S, so hat man eine funktorielle Bijektion $\mathbf{Hom}_S(X, Y)(T) = Hom_S(T \times_S X, Y)$. $\mathbf{Isom}_S(X, Y)$ wird als Objekt von $\widehat{\mathbf{A}_{/S}}$ definiert durch: $\mathbf{Isom}_S(X, Y)(T) := Isom_T(X_{(T)}, Y_{(T)})$. (Sind U, U' Objekte über T, so ist $Isom_T(U, U')$ die Menge der invertierbaren Elemente von $Hom_T(U, U')$.)

(0.2.3) Sind $\mathbf{Hom}_S(X, X)$, $\mathbf{Hom}_S(X, Y)$, $\mathbf{Hom}_S(Y, X)$ und $\mathbf{Hom}_S(Y, Y)$ darstellbar, so ist nach [1] auch $\mathbf{Isom}_S(X, Y)$ darstellbar.

(0.2.4) Ist Z ein Objekt über X, Y ein Objekt über S, und sind $p\colon X \to S$, $q\colon S' \to S$ Objekte über S, so gilt nach [1] Exp. I, 1.7, wenn man $p' := p_{(S')}$ setzt:

$$q^+(p_+ Z) \cong p'_+(Z \times_X p^+ S'),$$

$$q^+ \mathbf{Hom}_S(X, Y) \cong \mathbf{Hom}_{S'}(X_{(S')}, Y_{(S')}),$$

$$q^+ \mathbf{Isom}_S(X, Y) \cong \mathbf{Isom}_{S'}(X_{(S')}, Y_{(S')}).$$

(Hierbei ist natürlich q^+ zu einem Funktor $\widehat{\mathbf{A}_{/S}} \to \widehat{\mathbf{A}_{/S'}}$ fortgesetzt zu denken.)

(0.2.5) Ist e das Finalobjekt von **A**, so ist ja $\mathbf{A} = \mathbf{A}_{/e}$; sind X, Y Objekte von **A**, so setzt man $\mathbf{Hom}(X, Y) := \mathbf{Hom}_e(X, Y)$.

§ 1. Die Relativierung des Satzes von Douady

(1.1) Es sei $X \to S$ eine holomorphe Abbildung. Es soll nun ein Objekt $\mathbf{H}_{X/S}$ aus $\widehat{\mathbf{An}_{/S}}$ definiert werden: Für ein Objekt T aus $\mathbf{An}_{/S}$ sei $\mathbf{H}_{X/S}(T)$ die Menge der abgeschlossenen analytischen Unterräume Y von $X_{(T)}$, für welche die Komposition $Y \to X_{(T)} \to T$ platt und eigentlich ist. Ist $f: T \to T'$ ein $\mathbf{An}_{/S}$-Morphismus, Y ein Element von $\mathbf{H}_{X/S}(T')$, so ist $Y \times_{T'} T \in \mathbf{H}_{X/S}(T)$; man hat also eine Abbildung

$$\mathbf{H}_{X/S}(f)\colon\ \mathbf{H}_{X/S}(T') \to \mathbf{H}_{X/S}(T).$$

Damit ist $\mathbf{H}_{X/S}$ ein Objekt von $\widehat{\mathbf{An}_{/S}}$.

Ist $\mathbf{H}_{X/S}$ darstellbar, so bezeichnen wir das Objekt aus $\mathbf{An}_{/S}$, welches zu $\mathbf{H}_{X/S}$ isomorph ist, auch mit $\mathbf{H}_{X/S}$. Man hat dann einen abgeschlossenen analytischen Unterraum $\mathbf{Z}_{X/S}$ von $\mathbf{H}_{X/S} \times_S X$, die Komposition $\mathbf{Z}_{X/S} \to \mathbf{H}_{X/S} \times_S X \to \mathbf{H}_{X/S}$ ist platt und eigentlich und es gilt: Ist T ein Raum über S, $Y \in \mathbf{H}_{X/S}(T)$, so gibt es genau einen S-Morphismus $f: T \to \mathbf{H}_{X/S}$, derart daß $Y = f^+ \mathbf{Z}_{X/S}$ ist.

Ist e das Finalobjekt von $\mathbf{An}$, so sei $\mathbf{H}_X := \mathbf{H}_{X/e}$, $\mathbf{Z}_X := \mathbf{Z}_{X/e}$.

(1.2) Nach Douady [2] ist $\mathbf{H}_X$ darstellbar für jeden komplexen Raum X. Sind T, X Objekte von $\mathbf{An}_{/S}$, $Y \in \mathbf{H}_{X/S}(T)$, so läßt sich wegen der universellen Eigenschaft von $\mathbf{H}_X$ der Unterraum $Y \to T \times_S X \hookrightarrow T \times X$ als Element von $Hom_S(T, \mathbf{H}_X \times S)$ auffassen. Man hat also eine Abbildung $\mathbf{H}_{X/S}(T) \to Hom_S(T, \mathbf{H}_X \times S)$, die offenbar funktoriell in T ist, also hat man einen $\widehat{\mathbf{An}_{/S}}$-Morphismus $s: \mathbf{H}_{X/S} \to \mathbf{H}_X \times S$. Nach Pourcin [14] ist $\mathbf{H}_{X/S}$ darstellbar und s eine abgeschlossene Einbettung. Verfasser möchte hierfür den Beweis nun skizzieren; durch Verwendung eines Resultats von Frisch [6] läßt sich der Beweis an einer Stelle kürzer als in [14] führen.

(1.3) Im folgenden wird öfters die folgende Tatsache ([2], Prop. 1 von § 10) verwendet: Sind $p: X \to S$, $q: Y \to S$ platt und eigentlich, $h \in Hom_S(X, Y)$, so ist die Menge $S' := \{s \in S : h_s \in Isom(X_s, Y_s)\}$ offen und $h_{(S')}$ ist ein Isomorphismus.

(Hierbei ist $X_s := X \times_S \{s\}$, $h_s := h \times_S id_{\{s\}}$ und diese Bezeichnungsweise wird auch im folgenden stets verwendet.)

(1.4) **Satz** (vgl. [14]). *Ist $p: X \to S$ eine platte eigentliche holomorphe Abbildung und Y ein abgeschlossener analytischer Unterraum von X, so ist $p_+ Y$ darstellbar.*

Beweis. $i: Y \to X$ sei die Injektion. $p\,i$ ist eigentlich, also gibt es nach [6] einen Raum S' über S mit folgenden Eigenschaften: i) $Y_{(S')} \to S'$ ist platt, ii) ist T ein Raum über S und ist $Y_{(T)} \to T$ platt, so gibt es genau einen S-Morphismus von T in S'. $j := i_{(S')} \in Hom_{S'}(Y_{(S')}, X_{(S')})$; $S'' :=$

$\{s \in S' : j_s$ ist Isomorphismus$\}$. Nach (1.3) ist S'' offen in S'. Es läßt sich schnell verifizieren, daß S'' als Raum über S isomorph zu $p_+ Y$ ist.

(1.5) **Satz** (Pourcin [14]). *Ist $f: X \to S$ eine holomorphe Abbildung, so ist $\mathbf{H}_{X/S}$ darstellbar.*

Beweisskizze. Sei $p: \mathbf{Z}_X \to \mathbf{H}_X$ der kanonische Morphismus, $q := p \times id_S$, $j: \mathbf{Z}_X \times S \to \mathbf{H}_X \times X \times S$ sei die Injektion. Z' sei der Differenzkern von

$$\mathbf{Z}_X \times S \underset{f\,pr_2 j}{\overset{pr_3 j}{\rightrightarrows}} S,$$

d.h. die Sequenz

$$Z' \to \mathbf{Z}_X \times S \underset{f\,pr_2 j}{\overset{pr_3 j}{\rightrightarrows}} S$$

sei exakt. Es ist jetzt prinzipiell einfach zu zeigen, daß es einen $\widehat{\mathbf{An}_{/S}}$-Isomorphismus $h: \mathbf{H}_{X/S} \to q_+ Z'$ gibt, derart daß $s = l\,h$ ist (s war in (1.2) definiert, l sei der kanonische Morphismus). Aus (1.4) folgt die Behauptung.

§ 2. Eine Charakterisierung der platten eigentlichen holomorphen Abbildungen

(2.1) **Satz.** *Es sei $p: X \to S$ eine eigentliche holomorphe Abbildung. Dann sind die folgenden Aussagen äquivalent:*

a) *p ist platt.*

b) *Für jedes Object Y aus $\mathbf{An}_{/X}$ ist $p_+ Y$ darstellbar.*

c) *Es gibt einen zu $p^+: \mathbf{An}_{/S} \to \mathbf{An}_{/X}$ rechtsadjungierten Funktor.*

d) *Für jeden Epimorphismus f in $\mathbf{An}_{/S}$ ist $f \times_S id_X$ ein Epimorphismus in $\mathbf{An}_{/X}$.*

In [8, 9] hat Grothendieck gezeigt, wie mit Verwendung von (1.5) b) aus a) folgt.

Beweis. 1. Aus a) folgt b): Ist Y ein Raum über X, T ein Raum über S, so ist $p_+ Y(T) = Hom_X(T_{(X)}, Y) \cong Hom_{T_{(X)}}(T_{(X)}, Y \times_X T_{(X)})$. Nun ist $Y \times_X T_{(X)} = Y \times_S T$; man hat also eine in T funktorielle Bijektion von $(p_+ Y)(T)$ auf die Menge aller abgeschlossenen analytischen Unterräume $W \hookrightarrow T \times_S Y$, für welche die Komposition $W \to T \times_S Y \to T \times_S X$ ein Isomorphismus ist.

Der Morphismus $Y \to X$ liefert einen Morphismus $a: \mathbf{H}_{Y/S} \times_S Y \to \mathbf{H}_{Y/S} \times_S X$; es sei $\mathbf{Z}_{Y/S} \xrightarrow{j} \mathbf{H}_{Y/S} \times_S Y \to \mathbf{H}_{Y/S}$ der kanonische Morphismus (er wird mit $p_{Y/S}$ bezeichnet). Dann ist $p_{(\mathbf{H}_{Y/S})}\, a\, j = p_{Y/S}$ und damit ist $a\,j$ ein Morphismus über $\mathbf{H}_{Y/S}$. Aus (1.3) und den obigen Ausführungen folgt nun, daß $H' := \{h \in \mathbf{H}_{Y/S} : (a\,j)_h$ ist Isomorphismus$\}$ offen ist und daß $p_+ Y \cong H'$ ist.

2. Aus (0.2.1) folgt, daß c) durch b) impliziert wird.

3. Daß d) durch c) impliziert wird, ist trivial.

4. d) impliziert a): Für einen komplexen Raum Z sei $\mathbf{Coh}(Z)$ die Kategorie der kohärenten Modulgarben auf Z. Ist $\mathscr{F}$ ein Objekt von $\mathbf{Coh}(Z)$, so sei $I_Z(\mathscr{F}) := (Z, \mathcal{O}_Z + \mathscr{F})$ die triviale Erweiterung von Z durch $\mathscr{F}$ (vgl. [1] Exp. 2, 2). Der Morphismus $\mathcal{O}_Z \to \mathcal{O}_Z + \mathscr{F}$, $a \mapsto (a, 0)$, liefert einen Morphismus $I_Z(\mathscr{F}) \to Z$, außerdem hängt $I_Z(\mathscr{F})$ funktoriell von $\mathscr{F}$ ab, man hat also einen Funktor I_Z: $\mathbf{Coh}(Z)^0 \to \mathbf{An}_{/Z}$. Man überlegt sich leicht, daß ein Morphismus f in $\mathbf{Coh}(Z)$ genau dann injektiv ist, wenn $I_Z(f)$ ein Epimorphismus ist.

Sei nun f ein Monomorphismus in $\mathbf{Coh}(S)$. Dann ist $p^+ I_S(f)$ ein Epimorphismus; da das Diagramm

$$\begin{array}{ccc} \mathbf{Coh}(S)^0 & \xrightarrow{I_S} & \mathbf{An}_{/S} \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p^+} \\ \mathbf{Coh}(X)^0 & \xrightarrow{I_X} & \mathbf{An}_{/X} \end{array}$$

kommutiert, ist also $I_X(p^* f)$ ein Epimorphismus, daher ist $p^* f$ injektiv. Damit ist p^* ein exakter Funktor, also ist p platt.

(2.2) **Folgerung.** *Sei p: $X \to S$ eine platte eigentliche holomorphe Abbildung, Y ein Raum über S. Dann gilt:*

i) $\mathbf{Hom}_S(X, Y)$ *ist darstellbar.*

ii) *Wenn $Y \to S$ platt und eigentlich ist, so ist* $\mathbf{Isom}_S(X, Y)$ *darstellbar und gestattet eine offene Einbettung in* $\mathbf{Hom}_S(X, Y)$.

Beweis. i) ergibt sich aus (2.1) und (0.2.2). Zu ii): Daß $\mathbf{Isom}_S(X, Y)$ darstellbar ist, folgt aus (0.2.3), daß der kanonische Morphismus $\mathbf{Isom}_S(X, Y) \to \mathbf{Hom}_S(X, Y)$ eine offene Einbettung ist, folgt aus (1.3).

(2.3) **Corollar.** *Ist $X \to S$ eine holomorphe Abbildung, so ist s: $\mathbf{H}_{X/S} \to \mathbf{H}_X \times S$ eine abgeschlossene Einbettung.*

In der Tat: Im Beweis von (1.5) ist l eine abgeschlossene Einbettung, da p_+ ein rechtsadjungierter Funktor ist, also ist auch s eine abgeschlossene Einbettung.

§ 3. Eine Modul-theoretische Eigenschaft platter eigentlicher Morphismen

Für einen komplexen Raum Z sei $\mathbf{Lin}(Z)$ die Kategorie der linearen Faserräume über Z im Sinne von [3]. V_Z: $\mathbf{Coh}(Z)^0 \to \mathbf{Lin}(Z)$ sei der kanonische Funktor (vgl. [3]). T_Z: $\mathbf{Lin}(Z) \to \mathbf{An}_{/Z}$ sei der Funktor, der die algebraische Struktur vergißt. Ist p: $X \to S$ ein Morphismus in $\mathbf{An}$, so liefert das Faserprodukt einen additiven Funktor $\mathbf{Lin}(S) \to \mathbf{Lin}(X)$, der ebenfalls mit p^+ bezeichnet wird. Ist p platt und eigentlich, L ein Objekt von $\mathbf{Lin}(X)$, so wird $p_+(T_X L)$ ein Objekt von $\mathbf{Lin}(S)$, wenn man für einen Raum Z über S der Menge $Hom_S(Z, p_+ T_X L)$, die ja isomorph zu $Hom_X(p^+ Z, T_X L)$ ist, die $\Gamma(Z, \mathcal{O}_Z)$-Modul-Struktur von $Hom_X(p^+ Z, T_X L)$ einpflanzt.

(3.1) Damit hat man einen additiven Funktor $\mathbf{Lin}(X) \to \mathbf{Lin}(S)$ definiert (falls $X \to S$ platt und eigentlich ist), der wieder mit p_+ bezeichnet wird und der zu $p^+: \mathbf{Lin}(S) \to \mathbf{Lin}(X)$ rechtsadjungiert ist.

(3.2) **Satz.** *Sei $p: X \to S$ ein eigentlicher platter Morphismus in* **An**. *Dann gibt es einen additiven Funktor ${}_*p$:* $\mathbf{Coh}(X) \to \mathbf{Coh}(S)$, *der linksadjungiert zu p^* ist und für den gilt: $V_S\, {}_*p \cong p_+ V_X$.*

Beweis. Dies folgt aus (3.1) und der Tatsache ([3, 15]), daß für jeden komplexen Raum Z der Funktor V_Z eine Äquivalenz von Kategorien ist.

(3.3) **Satz.** *Sei $p: X \to S$ ein platter eigentlicher Morphismus in* **An**, $\mathcal{F} \xrightarrow[f]{} \mathcal{G} \xrightarrow[g]{} \mathcal{H} \longrightarrow 0$ *eine exakte Sequenz in* $\mathbf{Coh}(X)$. *Dann ist die Sequenz ${}_*p\mathcal{F} \to {}_*p\mathcal{G} \to {}_*p\mathcal{H} \to 0$ exakt.*

Beweis. Dies folgt daraus, daß ${}_*p$ einen rechtsadjungierten Funktor besitzt.

(3.4) Ist Z ein komplexer Raum, $\mathcal{F}$ eine Modulgarbe auf Z, so sei $\mathcal{F}' := \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{O}_Z)$.

(3.5) **Satz.** *Ist $p: X \to S$ eine platte eigentliche holomorphe Abbildung, $\mathcal{F}$ ein Objekt aus* $\mathbf{Coh}(X)$, *so hat man einen natürlichen Isomorphismus $p_*(\mathcal{F}') = ({}_*p\mathcal{F})'$.*

Beweis. Ist U offen in S, so ist $(p_*(\mathcal{F}'))(U) = \mathcal{F}'(U_{(X)}) \cong Hom_X(U_{(X)}, V_X\mathcal{F}) = Hom_S(U, p_+ V_X\mathcal{F}) \cong Hom_S(U, V_S({}_*p\mathcal{F})) = ({}_*p\mathcal{F})'(U)$.

(3.6) Sei $p: X \to S$ eine platte eigentliche holomorphe Abbildung, $s \in S$, $q_s: X_s \to \{s\}$ die triviale Abbildung, $i_s: X_s \to X$ die Injektion. Dann gilt für jede kohärente Modulgarbe $\mathcal{F}$ auf X:

$$(p_+ V_X\mathcal{F}) \times_S \{s\} \cong (q_s)_+ V_{X_s}(i_s^* \mathcal{F}).$$

Dies ergibt sich sofort aus (0.2.4).

(3.7) **Satz.** *Sei $p: X \to S$ eine platte eigentliche holomorphe Abbildung, S sei reduziert; $\mathcal{F}$ sei ein Objekt von* $\mathbf{Coh}(X)$. *Dann ist ${}_*p\mathcal{F}$ genau dann lokalfrei, wenn die Abbildung $S \to \mathbf{Z}$, $s \mapsto [H^0(X_s, (\mathcal{F}|X_s)') : \mathbf{C}]$ lokalkonstant ist.*

Beweis. ${}_*p\mathcal{F}$ ist wegen (3.2) genau dann lokalfrei, wenn $p_+ V_X\mathcal{F}$ ein Vektorraumbündel ist. Dies ist genau dann der Fall, wenn die Abbildung $s \mapsto dim(p_+ V_X \mathcal{F} \times_S \{s\})$ lokal-konstant ist. Die Behauptung folgt nun aus (3.5) und (3.6).

§ 4. Platte eigentliche Abbildungen mit paarweise isomorphen Fasern

(4.1) **Definition.** *Sei* $f: Y \to S$ *eine holomorphe Abbildung,* $y \in Y$.

a) f *heißt offen in* y, *wenn* $f(U)$ *eine Umgebung von* $f(y)$ *ist, falls* U *eine Umgebung von* y *ist.*

b) f *besitzt einen Schnitt durch* y, *wenn folgendes gilt: Es gibt eine offene Umgebung* U *von* $f(y)$ *und ein* $g \in Hom_S(U, Y)$, *derart daß* $g(fy) = y$ *ist.*

(4.2) **Hilfssatz.** *Sei* $f: Y \to S$ *eine holomorphe Abbildung,* S *sei reduziert, für alle* $s \in S$ *sei* Y_s *eine reindimensionale Mannigfaltigkeit,* $\dim Y_s = n$ *sei von* s *unabhängig. Dann gilt:*

i) *Ist* f *in* y *offen, so besitzt* f *einen Schnitt durch* y.

ii) *Ist die Topologie von* Y *abzählbar, so gibt es ein* $y \in Y$, *derart daß* f *in* y *offen ist.*

Beweis. Nach ([7], Satz 2.2) gilt Folgendes: Zu jedem $y \in Y$ gibt es eine offene Umgebung V^y von y, eine offene Umgebung U^y von $f(y)$, einen Polyzylinder P^y im $\mathbf{C}^n$ und eine abgeschlossene Einbettung $h^y: V^y \to U^y \times P^y$, derart, daß für alle $z \in V^y$ gilt: $pr_1 h^y z = fz$. Für jedes $s \in f(V^y)$ ist also $h_s^y: V^y \cap Y_s \to \{s\} \times P^y$ ein Isomorphismus. Da $pr_1: U^y \times P^y \to U^y$ bezüglich der Zariski-Topologien offen ist, ist also $f(V^y)$ ein analytischer Unterraum von U^y.

Beweis von i). Ist f in y offen, so kann man V^y, P^y, U^y und h^y so wählen, daß sogar $f(V^y) = U^y$ ist. Also ist h^y bijektiv. Da $U^y \times P^y$ reduziert ist, ist h^y ein Isomorphismus; hieraus folgt sofort die Behauptung.

Beweis von ii). Man kann o.E.d.A. annehmen, daß S ein Polyzylinder ist. Da Y abzählbare Topologie hat, gibt es eine abzählbare Teilmenge $Y' \subset Y$, derart daß $\bigcup_{y \in Y'} V^y = Y$ ist. Also ist $S = \bigcup_{y \in Y'} f(V^y)$. Nach dem Satz von Baire gibt es nun ein $y \in Y'$, derart daß $f(V^y) = U^y$ ist. Wie oben folgt nun die Behauptung.

(4.3) Sind X, Y komplexe Räume, so kann man $Hom(X, Y)$ mit der Topologie von Kaup [11] versehen; dieser topologische Raum wird mit $Hol(X, Y)$ bezeichnet.

Ist X kompakt, so ist die identische Abbildung $Hol(X, Y) \to$ $\mathbf{Hom}(X, Y)$ stetig. Dies ist in [2] bewiesen, falls X reduziert ist. Ist X nicht reduziert, so lassen sich der Beweis von § 10, Lemma 1 und Teil i) des Beweises von § 10, Theorem 2 in [2] übertragen, indem man folgendes benützt:

Sei X ein analytischer Unterraum von $U = \mathring{U} \subset \mathbf{C}^n$, K, K' seien $\mathcal{O}_X$-privilegierte Polyzylinder in U, $K' \subset \mathring{K}$. Dann ist die kanonische Abbildung $H^0(\mathring{K}, \mathcal{O}_X) \to B(K', \mathcal{O}_X)$ stetig; also ist auch $H^0(X, \mathcal{O}_X) \to$

$B(K', \mathcal{O}_X)$ stetig. (Dabei haben $H^0(\mathring{K}, \mathcal{O}_X)$, $H^0(X, \mathcal{O}_X)$ die kanonischen Topologien, vgl. [5].)

Ist X kompakt und hat Y abzählbare Topologie, so hat $Hol(X, Y)$ nach [11] abzählbare Topologie, also hat auch $\mathbf{Hom}(X, Y)$ abzählbare Topologie.

(4.4) **Lemma.** *Sei $X \to S$ eine platte eigentliche holomorphe Abbildung, X_0 sei ein kompakter komplexer Raum, S habe abzählbare Topologie. Dann hat $\mathbf{Isom}_S(S \times X_0, X)$ abzählbare Topologie.*

Beweis. Da $X \to S$ eigentlich ist, hat X abzählbare Topologie. Nach [17], Beweis von Satz 1.1, läßt sich $\mathbf{Hom}_S(S \times X_0, X)$ als endliches Faserprodukt von S, X und Räumen der Gestalt $\mathbf{Hom}(X_0, Z)$ darstellen, wobei Z abzählbare Topologie besitzt. Die Behauptung folgt nun aus (4.3) und (2.2).

(4.5) Sei $p\colon X \to S$ eine platte eigentliche holomorphe Abbildung, $s \in S$. p heißt trivial in s, wenn es eine Umgebung U von s und einen spurpunkttreuen Isomorphismus $U \times X_s \to X \times {}_S U$ gibt. p heißt trivial, wenn p in jedem Punkt von S trivial ist.

Seien nun $s, s' \in S$, und sei X_s isomorph zu $X_{s'}$. $f\colon \mathbf{Isom}_S(S \times X_s, X) \to S$ sei der kanonische Morphismus. Dann gilt: p ist genau dann in s' trivial, wenn es ein $y \in \overset{-1}{f}(s')$ gibt, derart daß f einen Schnitt durch y besitzt. Dies folgt sofort aus (0.2.4).

Es wird noch eine Tatsache benötigt, deren einfacher Beweis dem Leser überlassen wird:

(4.6) Ist $g\colon S' \to S$ eine surjektive stetige und abgeschlossene Abbildung, $s \in S$, V eine Umgebung von $\overset{-1}{g}(\{s\})$, so ist $g(V)$ eine Umgebung von s.

(4.7) **Lemma.** *Seien $p\colon X \to S$, $g\colon S' \to S$ holomorphe Abbildungen, p sei eigentlich und platt, g sei surjektiv, $p_{(S')}\colon X_{(S')} \to S'$ sei trivial. Dann ist auch p trivial, falls eine der folgenden Bedingungen erfüllt ist:*

a) *g ist platt.*

b) *g ist abgeschlossen und S ist reduziert.*

Beweis. Sei $s \in S$; es ist zu zeigen, daß p in s trivial ist.

$$Y := \mathbf{Isom}_S(S \times X_s, X), \qquad f\colon Y \to S$$

sei der kanonische Morphismus; $Y' := g^+ Y$, $f' := g^+ f$, $g' := f^+ g$. Man hat also ein kommutatives Diagramm

$$\begin{array}{ccc} Y & \xleftarrow{\;g'\;} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \xleftarrow{\;g\;} & S' \end{array}$$

Wegen (0.2.4) ist $Y' = \mathbf{Isom}_{S'}(S' \times X_s, X_{(S')})$. Ist $s' \in S'$, $g(s') = s$, so gibt es eine offene Umgebung U von s' und ein $h \in Isom_U(U \times X_s, X_{(U)})$, also gibt es wegen (0.2.4) einen $\mathbf{An}_{/U}$-Isomorphismus $Y'_{(U)} \to U \times \mathbf{Isom}(X_s, X_s)$.

a) Sei nun g platt. In jedem $y \in Y'$ mit $f'(y) \in U$ ist f' platt, also ist f in jedem $y \in Y_s$ platt. Nun ist $Y_s = \mathbf{Isom}(X_s, X_s)$, also ist Y_s eine Mannigfaltigkeit. Wegen (1.8) in [16] besitzt also f durch jedes $y \in Y_s$ einen Schnitt. Aus (4.5) folgt nun die Behauptung.

b) Sei g abgeschlossen und S reduziert. Es sei $y \in Y$, $f(y) = s$; wegen (4.2) reicht es zu zeigen, daß f offen in y ist. Sei also V eine Umgebung von y. Dann ist $f'(\overset{-1}{g}{}'(V))$ eine Umgebung von $\overset{-1}{g}(\{s\})$. Nach (4.6) ist also $f(V) = (g f')(\overset{-1}{g}{}'(V))$ eine Umgebung von s.

Ist X ein komplexer Raum über S, $s \in S$, so wird mit $\mathscr{T}_s$ die Garbe der Keime von holomorphen kontravarianten Vektorfeldern auf X_s bezeichnet. Nun können endlich die eigentlichen Hauptsätze dieser Arbeit bewiesen werden.

(4.8) **Satz.** *Sei $p: X \to S$ eine eigentliche platte holomorphe Abbildung, S sei reduziert, lokal-irreduzibel und zusammenhängend. Es gebe ein $s_0 \in S$, in welchem p trivial ist. Dann ist p trivial, falls die Abbildungen $S \ni s \mapsto [H^0(X_s, \mathcal{O}_{X_s}):\mathbf{C}]$, $S \ni s \mapsto [H^0(X_s, \mathscr{T}_s):\mathbf{C}]$ konstant sind.*

Beweis. 1. S sei eine Mannigfaltigkeit. Man hat die kanonische exakte Sequenz $E: p^*\Omega_S \xrightarrow{\delta} \Omega_X \to \Omega_{X/S} \to 0$ (vgl. [8]). Nach (3.3) ist die Sequenz ${}_*pE$: ${}_*pp^*\Omega_S \to {}_*p\Omega_X \to {}_*p\Omega_{X/S} \to 0$ exakt. Da p in s_0 trivial ist, gibt es eine Umgebung U von s_0, derart daß $\delta | X \times_S U$ linksinversibel ist. Also ist $ker({}_*p\delta)_s = 0$ für $s \in U$. Da $[H^0(X_s, \mathcal{O}_{X_s}):\mathbf{C}]$ nicht von s abhängt, ist ${}_*pp^*\Omega_S$ lokalfrei (nach (3.7)). Der Identitätssatz liefert also: $ker\, {}_*p\delta = 0$. Also ist die Sequenz

$$(4.8.1) \qquad 0 \to {}_*pp^*\Omega_S \to {}_*p\Omega_X \to {}_*p\Omega_{X/S} \to 0 \quad \text{exakt.}$$

Da die analytische Beschränkung von $\Omega_{X/S}$ auf X_s isomorph zu Ω_{X_s} ist (vgl. etwa [8]) und weil $[H^0(X_s, \mathscr{T}_s):\mathbf{C}]$ nicht von s abhängt, ist wegen (3.7) die Garbe ${}_*p\Omega_{X/S}$ lokalfrei. Die Sequenz (4.8.1) spaltet also lokal auf, aus (3.5) folgt also, daß $p_*(\delta')$: $p_*\Omega'_X \to p_*(p^*\Omega_S)'$ surjektiv ist. Ist also $s \in S$, D der Keim eines Vektorfeldes in s, so gibt es eine offene Umgebung U von s, derart daß sich D zu einem Vektorfeld auf $X_{(U)}$ liften läßt. Nach [12] ist also p in s trivial.

2. S besitzt Singularitäten: $S' := \{s \in S: p \text{ ist trivial in } s\}$; S' ist offen und nicht leer. Sei $s \in \overline{S'}$. Dann kann man eine offene Umgebung V von s so wählen, daß gilt:

i) Die Menge der regulären Punkte von V ist zusammenhängend.

ii) Es gibt eine Auflösung $g\colon M \to V$ der Singularitäten von V im Sinne von [10].

M ist dann zusammenhängend und aus Teil 1 dieses Beweises folgt zusammen mit (4.7), daß $s \in S'$ ist. Also ist S' abgeschlossen und daher ist $S' = S$.

(4.9) **Satz.** *Sei $p\colon X \to S$ eine platte eigentliche holomorphe Abbildung, S sei reduziert; für alle $s, t \in S$ sei X_s isomorph zu X_t. Dann ist p trivial.*

Beweis. 1. S sei eine Mannigfaltigkeit. Nach (4.5), (4.4) und (4.2) gibt es auf jeder Zusammenhangskomponente von S einen Punkt, in dem p trivial ist. Wegen (4.8) ist also p trivial.

2. S sei ein reduzierter Raum. Zu jedem $s \in S$ gibt es eine offene Umgebung U und eine Auflösung $g\colon M \to U$ der Singularitäten von U im Sinne von [10]. Die Behauptung folgt nun aus (4.7).

(4.10.1) (4.9) wurde in [4] für den Spezialfall bewiesen, daß S und alle Fasern von p Mannigfaltigkeiten sind.

(4.10.2) In [13] wird ein Beweis von (4.9) angegeben für den Spezialfall, daß alle Fasern von p reduziert sind und daß p in jedem Punkt von X lokaltrivial ist. Es scheint mir aber in [13] an der Stelle, wo der Fall, daß S eine Mannigfaltigkeit ist, betrachtet wird, der Beweis nicht ganz einleuchtend zu sein.

(4.10.3) Läßt man in (4.9) die Voraussetzung, daß S reduziert sei, fallen, so wird die Aussage natürlich falsch. Man betrachte einfach eine nichtverschwindende infinitesimale Deformation eines kompakten Raumes (vgl. [17]).

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Epimorphisms and Surjectivity

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1. Introduction

Suppose that $\mathscr{C}$ is a category whose objects are sets (possibly with further additional structure) and whose morphisms (as well as satisfying perhaps other requirements) are mappings in the usual sense of the underlying sets. Many familiar categories are of this sort. For a morphism $\theta: A \to B$ in $\mathscr{C}$ we can on the one hand say that θ is *surjective* if the image of θ (as a set) is the whole of B, or alternatively we can define θ to be an *epimorphism* (a purely categorical notion) if for every pair of morphisms φ, φ' in $\mathscr{C}$ from B to any other object C, $\varphi\,\theta = \varphi'\,\theta$ implies $\varphi = \varphi'$. It is clear that a surjective map is always an epimorphism, and the question we shall be concerned with throughout this paper is whether, for various concrete examples of categories, the converse is true. As we shall see, for the category of commutative C^*-algebras this converse is equivalent to the classical Stone-Weierstrass theorem, and thus the statement that in the category of C^*-algebras epimorphisms are necessarily surjective would be a natural generalisation of the Stone-Weierstrass result. This was our initial motivation for this investigation; the question as to the behaviour of other categories arose later as being of interest in itself.

We shall from the outset make two further simple assumptions about the categories we shall be working with, which will in practise normally be realised. Firstly suppose that for any pair of morphisms φ, $\varphi': B \to C$, the set $\{b \in B: \varphi(b) = \varphi'(b)\}$ is an object of $\mathscr{C}$. And secondly, suppose that each subset of an object B in $\mathscr{C}$ is contained in a unique smallest subobject of B in $\mathscr{C}$. In these circumstances it is easily seen that a morphism $\theta: A \to B$ is an epimorphism if and only if the morphism embedding the subobject of B generated by the image of θ in B is an epimorphism. Thus the question of whether epimorphisms are necessarily surjective becomes equivalent to the question of whether any epimorphically embedded subobject of an object is necessarily the whole object. The question therefore becomes the following: given A, B objects of $\mathscr{C}$ with A a proper subset of B, does there exist an object C of $\mathscr{C}$ and morphisms φ, φ' equal on A but unequal on B?

Needless to say, one can ask of a category of sets the dual question of whether monomorphisms are necessarily injective, but this appears to be generally much easier to answer.

2. The Stone-Weierstrass Theorem

The Stone-Weierstrass theorem classically stated says that a self-adjoint subalgebra A of the algebra $C_0(X)$ of all complex-valued continuous functions vanishing at infinity on the locally compact Hausdorff space X is dense in $C_0(X)$ in the topology of uniform convergence on X if and only if (i) given distinct points $x, y \in X$ there is a function f in A with $f(x) \neq f(y)$, and (ii) given any point $x \in X$, there is a function f in A such that $f(x) \neq 0$.

Now algebras of the form $C_0(X)$ with their norm topology of uniform convergence on X are exactly commutative C^*-algebras, and the maps $f \to f(x)$, or $f \to 0$ are exactly the multiplicative linear functionals on $C_0(X)$, that is to say the morphisms (in the category of commutative C^*-algebras) of $C_0(X)$ into the complex numbers $\mathbf{C}$. Thus the classical formulation of the Stone-Weierstrass theorem above may be rephrased as follows: If A is a proper C^*-subalgebra of the C^*-algebra B there exist morphisms $\theta, \varphi: B \to \mathbf{C}$ equal on A but not equal on the whole of B. In other words, epimorphisms in the category of commutative C^*-algebras are necessarily surjective.

To reinforce this connection between the Stone-Weierstrass theorem and the surjectivity of epimorphisms it is worthwhile to give another example, where in fact, for essentially the same reason, both results fail.

Let X be a finite-dimensional real compact $C^{(\infty)}$-manifold, and consider the algebra $B = C^{(\infty)}(X)$ of all complex-valued $C^{(\infty)}$-functions on X, taken in its topology of uniform convergence of derivatives on X. Then B is a complete locally convex (though not Banach) *-algebra with jointly continuous multiplication.

In B the Stone-Weierstrass theorem fails. A further condition is required, and in fact we have (see [8]) that a self-adjoint subalgebra A is dense in B if and only if (i) given distinct points $x, y \in X$, there is a function f in A with $f(x) \neq f(y)$, (ii) given any point $x \in X$, there is a function f in A with $f(x) \neq 0$, and (iii) given any point $x \in X$ and any direction α at x, there is a function f in A such that $D_\alpha f(x) \neq 0$, where D_α denotes the derivative in the direction α.

Take now any point x in X and any direction α at x. The set $A = \{f: D_\alpha f(x) = 0\}$ is a closed self-adjoint subalgebra of $C^{(\infty)}(X) = B$, containing 1 and separating the points of X but not the whole of $C^{(\infty)}(X)$. Now let φ be any (algebraic) morphism $C^{(\infty)}(X) \to C^{(\infty)}(Y) = C$, or more generally into any algebra C of functions on Y say. Let g be any $C^{(\infty)}$-function on X such that $D_\alpha g(x) \neq 0$ and $g(x) = 0$; then A and g together generate (algebraically) B, so φ is determined by its values on A and its value at g. However g^2 and g^3 are elements of A, so that $\varphi(g)^2 = \varphi(g^2)$ and $\varphi(g)^3 = \varphi(g^3)$ are determined by the values of φ on A. At each point of Y, either $\varphi(g^2) = 0$, in which case $\varphi(g) = 0$, or $\varphi(g^2) \neq 0$, in which case

$\varphi(g)=\varphi(g^3)/\varphi(g^2)$. Hence $\varphi(g)$ is determined by the restriction of φ to A, and hence φ is determined on the whole of B by its values on A. Thus A is epimorphically embedded (at least with respect to algebras of functions).

3. Von Neumann Algebras and C^*-Algebras

In order to solve the problem for C^*-algebras it is necessary first to deal with the category of von Neumann algebras. Here a certain caution is necessary in deciding what are the relevant morphisms to choose for our category, and the natural choice is to insist that not only the *-algebra structure is preserved but further that the morphisms are normal. Recall that $\Phi: \mathfrak{A} \to \mathfrak{B}$ is *normal* if for each increasing net (T_λ) of positive elements of $\mathfrak{A}$, $\Phi(\sup T_\lambda) = \sup \Phi(T_\lambda)$. Equivalently, the normal morphisms are those which are ultraweakly continuous. (For a general reference for the theory of von Neumann algebras see [3].)

The category of von Neumann algebras and normal morphisms has been studied by Guichardet [5], and he mentions as outstanding the question of whether epimorphisms are necessarily surjective ([5], p. 44).

Since we are dealing with normal morphisms the subset of a von Neumann algebra $\mathfrak{B}$ on which two morphisms from $\mathfrak{B}$ are equal is a weakly closed subalgebra of $\mathfrak{B}$. Therefore our category satisfies the assumptions made in § 1, and it suffices to prove that an epimorphically embedded subalgebra of $\mathfrak{B}$ must be the whole of $\mathfrak{B}$. The proof of this that we give below is due to Lorenz.

Proposition 1. *In the category of von Neumann algebras and normal morphisms, epimorphisms are necessarily surjective.*

Proof. Suppose that $\mathfrak{A}$ is a proper von Neumann subalgebra of the von Neumann algebra $\mathfrak{B}$ on H, and suppose first that the identity 1 of $\mathfrak{B}$ is in $\mathfrak{A}$. Then by von Neumann's double commutant theorem $\mathfrak{A}''=\mathfrak{A}$, where $\mathfrak{A}''$ denotes the double commutant of $\mathfrak{A}$, hence $\mathfrak{A}'' \neq \mathfrak{B}''$, and so $\mathfrak{A}' \neq \mathfrak{B}'$. Thus $\mathfrak{B}'$ is a proper von Neumann subalgebra of the von Neumann algebra $\mathfrak{A}'$. But any C^*-algebra with identity, and in particular any von Neumann algebra, is generated algebraically by its unitary elements (see [3], Ch. 1, § 1.3), so we can find a unitary operator U on H which is in $\mathfrak{A}'$ but not in $\mathfrak{B}'$. Then the maps $T \to T$, $T \to U^* T U$ are normal morphisms $\mathfrak{B} \to \mathscr{L}(H)$, equal on $\mathfrak{A}$, but unequal on $\mathfrak{B}$. Therefore $\mathfrak{A}$ is not epimorphically embedded.

Suppose now that $1 \notin \mathfrak{A}$. $\mathfrak{A}$ contains its principal identity E, which is an identity element for $\mathfrak{A}$, and so $E \neq 1$. It will clearly suffice to prove that the subalgebra $E\mathfrak{B}E$ is not epimorphically embedded in $\mathfrak{B}$. Since E is a projection, $U=2E-1$ is unitary. Therefore $T \to T$ and $T \to U^* T U$ give two normal morphisms $\mathfrak{B} \to \mathscr{L}(H)$ equal on $E\mathfrak{B}E$. These will be unequal on $\mathfrak{B}$ unless $E \in \mathfrak{B}'$, in which case E will be a central projection in $\mathfrak{B}$. But then $T \to T$ and $T \to ET$ are two normal morphisms $\mathfrak{B} \to \mathscr{L}(H)$ equal on $E\mathfrak{B}E$ but unequal on $\mathfrak{B}$.

The proposition is thus proved and it is now easy to deduce the C^*-algebra result.

Proposition 2. *In the category of C^*-algebras epimorphisms are necessarily surjective.*

Proof. The technique is to work *via* the enveloping von Neumann algebra of a C^*-algebra. If A is any Banach algebra, then the second dual A'' of A may be given (in two ways) a product under which it becomes a Banach algebra (see [1]). If A is a C^*-algebra then these two products coincide, an involution can be defined on A'', and the resulting Banach $*$-algebra becomes in fact a C^*-algebra. Moreover A'' may be represented (by the universal representation) as a von Neumann algebra, and the ultraweak topology is given just by the weak topology $\sigma(A'', A')$. A'' is called the *enveloping von Neumann algebra* of A (for details see [2], § 12).

Suppose now that A is a proper C^*-subalgebra of the C^*-algebra B. A and B are both embedded in the usual way in their second duals, and the inclusion map $A \to B$ induces an injective map $A'' \to B''$ embedding A'' as a C^*-subalgebra of B''. This map is $(\sigma(A'', A'), \sigma(B'', B'))$-continuous, so that A'' becomes a von Neumann subalgebra of B''. Since the map $A \to B$ is not surjective, neither is $A'' \to B''$, and so A'' is a proper von Neumann subalgebra of B''. Therefore by Proposition 1 there are two ultraweakly continuous maps $B'' \to \mathscr{L}(H)$ (where H is in fact the representation space for the universal representation of B) which coincide on A'' but differ on B''. Since B is $\sigma(B'', B')$-dense in B'', these maps differ on B. Thus, restricting to the subspace B of B'', we obtain two distinct morphisms $B \to \mathscr{L}(H)$ which coincide on A. Hence the result is proved.

It is interesting to compare this generalisation of the Stone-Weierstrass theorem with the other possible generalisations that have been proposed (see [2], § 11). The deepest and most difficult of these is the result of Glimm [4] that if a C^*-subalgebra A of B separates the weak closure of the pure state space of B together with 0 then $A = B$. A generalisation, or proposed generalisation, closer to ours, however, is that suggested by Fell. He defined a C^*-subalgebra A of B to be *rich* if each irreducible representation of B remains irreducible when considered as a representation of A, and if inequivalent irreducible representations of B remain inequivalent when considered as representations of A (see [2], p. 223). It is easy to see that if A is epimorphically embedded in B, then A is rich. However whether A rich implies $A = B$ is unknown (except in the special case of A postliminal), and if this were generally true then Glimm's result could be improved to require only separation of the pure state space of B (together with 0) by A.

4. Groups

Proposition 3. *In the category of groups epimorphisms are necessarily surjective. Likewise in the category of finite groups.*

Proof. It is possible to give a proof for the category of groups using free products with amalgamation, however the technique used here, of split extensions, is more immediately adaptable to the category of finite groups.

Suppose that G is a group and H a proper subgroup of G. We must find a group K and two morphisms $G \to K$ equal on H but unequal on G. Let us look for K which is the semidirect product $A \cdot G$ of a normal subgroup A with G. Writing ${}_g a = g a g^{-1}$ for $g \in G$, $a \in A$, we can regard A as a (not necessarily commutative) G-module, and K as the split extension of A by G. If $\theta: G \to K$ is of the form $\theta(g) = \varphi(g)\, g$, where φ is a map $G \to A$, then θ is a morphism if and only if φ satisfies the cocycle condition

$$\varphi(g\,g') = \varphi(g)\; {}_g\varphi(g') \qquad (g, g' \in G).$$

This is fulfilled in particular if φ is a coboundary, in other words if φ is of the form $\varphi(g) = \psi^{-1}\; {}_g\psi$ for some $\psi \in A$. To the identity $1 \in A$ corresponds the embedding of G in $A \cdot G$, and the map θ will coincide with this on H but be distinct on G if and only if ${}_h\psi = \psi$ for $h \in H$, but ${}_g\psi \neq \psi$ for some g in G.

Thus we are reduced to finding a G-module A having an element fixed by H but not fixed by the whole of G. But this is easily found, and we shall give two possible answers.

Firstly take $A = G^G$, the group of all maps of G into itself, the group operation being defined pointwise. The G-module action we take to be right translation on the domain, so that ${}_x\psi$ for $x \in G$ is defined by $({}_x\psi)(y) = \psi(yx)$. Now take ψ such that $\psi(h) = 1$ for $h \in H$, and $\psi(g) = $ some $g_0 \neq 1$ for $g \in G \backslash H$. It is clear that ψ is fixed by H but not by any element outside H.

A second, and perhaps simpler, example is given by $A = (\mathbf{Z}/2\mathbf{Z})^G$, where as before the group operation is pointwise and the G-module action is by right translation on the domain. Writing $\mathbf{Z}/2\mathbf{Z}$ as the multiplicative group $\{\pm 1\}$, we take $\psi(h) = 1$ for $h \in H$, and $\psi(g) = -1$ for $g \in G \backslash H$. In this example A is in fact commutative.

Note that if G is finite then either of the examples we have taken for A is finite, so that $K = A \cdot G$ is finite, and we remain in the category of finite groups throughout.

5. Lie Algebras

We now consider the category of Lie algebras, where, surprisingly, the situation is more complicated.

Suppose that we are given a Lie algebra $\mathfrak{g}$ and a proper subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We try the same method as in the proof of Proposition 3, now phrased in terms of Lie algebras. So let V be a $\mathfrak{g}$-module and consider the split extension $\mathfrak{k}=V+\mathfrak{g}$ of V by $\mathfrak{g}$. A map $\theta\colon \mathfrak{g}\to\mathfrak{k}$ of the form $\theta(x)=\varphi(x)+x$, where φ is a map $\mathfrak{g}\to V$, will be a morphism if and only if φ satisfies the cocycle condition

$$\varphi([x,x'])={}_x\varphi(x')-{}_{x'}\varphi(x) \qquad (x,x'\in\mathfrak{g}).$$

We again take φ a coboundary, so that $\varphi(x)={}_x\psi$ for some ψ in V, and to get a map θ that coincides with the embedding $\mathfrak{g}\to\mathfrak{k}$ on $\mathfrak{h}$ but not on $\mathfrak{g}$, we require ψ such that ${}_y\psi=0$ for $y\in\mathfrak{h}$, but ${}_x\psi\neq0$ for some x in $\mathfrak{g}$. Thus the problem is reduced to finding a $\mathfrak{g}$-module V containing an element annihilated by $\mathfrak{h}$ but not annihilated by the entire algebra $\mathfrak{g}$. And to find such a module is considerably less trivial than in the case of groups.

Choose a basis $x_1,\dots,x_n$ of $\mathfrak{g}$ such that $x_{p+1},\dots,x_n$ is a basis of $\mathfrak{h}$. (We give the argument for finite-dimensional $\mathfrak{g}$. The general case is then clear.) By the Poincaré-Birkhoff-Witt theorem monomials

$$x^k=x_1^{k_1}x_2^{k_2}\dots x_n^{k_n},$$

with k running over all n-tuples of non-negative integers, form a basis of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. Let V be the subspace spanned by the x^k with $k_{p+1}=\dots=k_n=0$, and denote by π the projection $U(\mathfrak{g})\to V$ defined by the basis x^k of $U(\mathfrak{g})$. We define a $\mathfrak{g}$-operation on V as follows: For $\psi\in V$, $x\in\mathfrak{g}$ define ${}_x\psi=\pi(x\psi)$, where $x\psi$ is just the ordinary product in $U(\mathfrak{g})$.

To prove that this is a module action we have to show

$$\pi([x,x']\,\psi)=\pi\big(x\,\pi(x'\psi)\big)-\pi\big(x'\,\pi(x\psi)\big).$$

And since $\pi([x,x']\psi)=\pi(x\,x'\psi)-\pi(x'x\psi)$, it will suffice to prove that $\pi\big(x\,\pi(x'\psi)\big)=\pi(x\,x'\psi)$.

Now $x'\psi=\pi(x'\psi)+\psi'$, where ψ' is a linear combination of elements x^k with $k_i>0$ for some $i>p$; so it suffices to prove that $\pi(x\,x^k)=0$ for any such k. Now such x^k may be written $x^h x^l$ with $h_i=0$ for $i>p$, $l_i=0$ for $i\leqq p$, and $l\neq0$. We may then write $x\,x^h$ as a linear combination of elements $x^{h'}x^{l'}$ with $h'_i=0$ for $i>p$, and $l'_i=0$ for $i\leqq p$; so $x\,x^k$ is a linear combination of elements $x^{h'}x^{l'}x^l$.

Consider the product $x^{l'}x^l$. Since $\mathfrak{h}$ is a subalgebra this may be written as a linear combination of elements $x^{l''}$ with $l''_i=0$ for $i\leqq p$, and since the terms of degree greater than zero form an ideal in $U(\mathfrak{h})$, $l\neq0$ will imply $l''\neq0$. Thus $x\,x^k$ is finally a linear combination of elements $x^{h'}x^{l''}$ with $l''\neq0$, and since all these map to zero under π we will have $\pi(x\,x^k)=0$, proving our result.

Now let ψ denote the identity element 1 in V. We have that for y in $\mathfrak{h}$, y is a linear combination of $x_{p+1}, \ldots, x_n$, and ${}_y\psi = \pi(y) = 0$. While for $1 \leqq i \leqq p$, we have ${}_{x_i}\psi = \pi(x_i) = x_i$. Thus we have proved

Proposition 4. *In the category of Lie algebras epimorphisms are necessarily surjective.*

6. Finite-Dimensional Lie Algebras

By analogy with the case of finite groups we may ask whether in the category of finite-dimensional Lie algebras epimorphisms are necessarily surjective. However, as we shall shortly see, this is not the case.

Note first that for the proof of the preceeding section to go through, in order that $\mathfrak{k}$ should be finite-dimensional, V would have to be a finite-dimensional $\mathfrak{g}$-module. And in the above construction, even for finite-dimensional $\mathfrak{g}$, this is not the case.

We shall begin, however, by seeing how far we can proceed in the finite-dimensional case and still retain a positive result. We shall consider in turn the cases of Abelian, nilpotent, soluble and semisimple algebras. One simplifying remark that applies throughout the finite-dimensional case is that every proper subalgebra is contained in a maximal proper subalgebra, and so the problem reduces to proving that no maximal proper subalgebra is epimorphically embedded.

Abelian Lie algebras are just vector spaces, and it is trivial to see that epimorphisms are surjective. The nilpotent case is also easily dealt with.

Proposition 5. *Epimorphisms are surjective in the category of finite-dimensional nilpotent Lie algebras.*

Proof. Let $\mathfrak{h}$ be a maximal proper subalgebra of the finite-dimensional nilpotent Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ being nilpotent, the normaliser of $\mathfrak{h}$ in $\mathfrak{g}$ is properly larger than $\mathfrak{h}$, and hence equals $\mathfrak{g}$. Thus $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, and the canonical projection π and the zero map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ give two morphisms equal on $\mathfrak{h}$ but distinct on $\mathfrak{g}$.

In fact since $\mathfrak{g}/\mathfrak{h}$ is nilpotent and nonzero, $(\mathfrak{g}/\mathfrak{h})'$ is a proper subspace of $\mathfrak{g}/\mathfrak{h}$, and we can find a nonzero linear functional f on $\mathfrak{g}/\mathfrak{h}$ vanishing on $(\mathfrak{g}/\mathfrak{h})'$. Then the composition $f\pi$ and the zero map give two morphisms from $\mathfrak{g}$ to the ground field equal on $\mathfrak{h}$ but unequal on $\mathfrak{g}$.

Proposition 6. *In the category of finite-dimensional soluble Lie algebras epimorphisms are necessarily surjective.*

Proof. Suppose that $\mathfrak{h}$ is a maximal proper subalgebra of the finite-dimensional soluble Lie algebra $\mathfrak{g}$. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ then the proof above applies. So suppose that $\mathfrak{h}$ is not an ideal in $\mathfrak{g}$, in which case $\mathfrak{h}$ does not contain $\mathfrak{g}'$, and $\mathfrak{h} + \mathfrak{g}'$, being a subalgebra properly larger than $\mathfrak{h}$, must be the whole of $\mathfrak{g}$.

Suppose — and we shall obtain a contradiction from this — that $\mathfrak{h}$ does not contain $\mathfrak{g}''$. Then $\mathfrak{h}+\mathfrak{g}''$ is a subalgebra properly larger than $\mathfrak{h}$, and hence $\mathfrak{h}+\mathfrak{g}''=\mathfrak{g}$. Write $\mathfrak{g}'^2=[\mathfrak{g}',\mathfrak{g}']=\mathfrak{g}''$, and inductively $\mathfrak{g}'^{n+1}=[\mathfrak{g}',\mathfrak{g}'^n]$. We have $\mathfrak{g}'\subset\mathfrak{g}=\mathfrak{h}+\mathfrak{g}'^2$, and an inductive proof gives $\mathfrak{g}'^n\subset\mathfrak{h}+\mathfrak{g}'^{n+1}$ for all n.

By our assumption that h is not an ideal we have $\mathfrak{g}'\not\subset\mathfrak{h}$. But since $\mathfrak{g}$ is soluble, $\mathfrak{g}'$ is nilpotent (see [6], p. 51; we are assuming henceforth that the ground field is of characteristic zero) and for some n we will have $\mathfrak{g}'^n=0\subset\mathfrak{h}$. Hence there exists p such that $\mathfrak{g}'^p\not\subset\mathfrak{h}$, but $\mathfrak{g}'^{p+1}\subset\mathfrak{h}$. However we then have $\mathfrak{g}'^p\subset\mathfrak{h}+\mathfrak{g}'^{p+1}\subset\mathfrak{h}$, a contradiction.

It follows that our hypothesis that $\mathfrak{h}$ does not contain $\mathfrak{g}''$ is untenable, so in fact $\mathfrak{h}\supset\mathfrak{g}''$ and we have

$$[\mathfrak{h}\cap\mathfrak{g}',\mathfrak{g}]=[\mathfrak{h}\cap\mathfrak{g}',\mathfrak{h}+\mathfrak{g}']\subset[\mathfrak{h},\mathfrak{h}]+\mathfrak{g}''\subset\mathfrak{h}\cap\mathfrak{g}'.$$

Therefore $\mathfrak{h}\cap\mathfrak{g}'$ is an ideal in $\mathfrak{g}$.

Now let $\bar{\mathfrak{g}}=\mathfrak{g}/(\mathfrak{h}\cap\mathfrak{g}')$, $\bar{\mathfrak{h}}=\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{g}')$. Then $\bar{\mathfrak{h}}$ is a maximal proper subalgebra of $\bar{\mathfrak{g}}$, which is soluble, and $\bar{\mathfrak{h}}$ is not an ideal in $\bar{\mathfrak{g}}$. Furthermore $\bar{\mathfrak{h}}\cap\bar{\mathfrak{g}}'=0$; therefore $\bar{\mathfrak{g}}=\bar{\mathfrak{h}}+\bar{\mathfrak{g}}'$ is a direct sum decomposition of $\bar{\mathfrak{g}}$ as a vector space, and $\bar{\mathfrak{h}}'\subset\bar{\mathfrak{h}}\cap\bar{\mathfrak{g}}'=0$, so that $\bar{\mathfrak{h}}$ is Abelian.

Each $\bar{x}\in\bar{\mathfrak{g}}$ is uniquely expressible as $\bar{y}+\bar{x}'$ with $\bar{y}\in\bar{\mathfrak{h}}$, $\bar{x}'\in\bar{\mathfrak{g}}'$. Define maps θ, φ: $\mathfrak{g}\to\bar{\mathfrak{g}}$ by $\theta(x)=x+h\cap\mathfrak{g}'=\bar{x}$, and $\varphi(x)=\bar{y}$. θ is clearly a Lie algebra morphism, and it easily seen that φ also is. Since $\mathfrak{h}\not\supset\mathfrak{g}'$, there exists $x'\in\mathfrak{g}'$ with $x'\notin\mathfrak{h}$, and $\theta(x')=\bar{x}'\neq 0$, while $\varphi(x')\in\varphi(\mathfrak{g}')=0$, so that θ and φ are distinct on $\mathfrak{g}$. However $\theta(y)=\bar{y}=\varphi(y)$ for $y\in\mathfrak{h}$, so $\theta=\varphi$ on $\mathfrak{h}$. Thus $\mathfrak{h}$ is not epimorphically embedded in $\mathfrak{g}$, and the proposition is proved.

Note that if $\mathfrak{h}$ is a maximal proper subalgebra which is not an ideal, then $\mathfrak{g}=\mathfrak{h}+\mathfrak{g}'$ so that morphisms into the ground field alone do not suffice to distinguish $\mathfrak{h}$ from $\mathfrak{g}$.

The next obvious class to consider is the semisimple algebras, but here the result fails and we shall give an example of a proper subalgebra of a semisimple Lie algebra which is embedded epimorphically (in the category of finite-dimensional algebras).

Let $\mathfrak{g}$ be the three-dimensional simple Lie algebra with basis x, y, z such that $[x,y]=y$, $[x,z]=-z$, $[y,z]=x$; and let $\mathfrak{h}$ be the subalgebra spanned by x and y. We assert that $\mathfrak{h}$ is epimorphically embedded in $\mathfrak{g}$ in the category of finite-dimensional Lie algebras.

For suppose not, Then there exists a finite-dimensional Lie algebra $\mathfrak{k}$ containing elements x, y, z, z' with $z\neq z'$ and such that $[x,y]=y$, $[x,z]=-z$, $[y,z]=x$, $[x,z']=-z'$, $[y,z']=x$. Since $\mathfrak{g}$ is simple and $z\neq z'$ implies $x\neq 0$, we have that x, y, z are linearly independent.

Put $u_0=z-z'$, so we have $[x,u_0]=-u_0$, $[y,u_0]=0$; and then inductively define $u_n=[z,u_{n-1}]$. It is straightforward to prove by induction

that
$$[y, u_n] = -\tfrac{1}{2} n(n+1) u_{n-1}, \qquad [x, u_n] = -(n+1) u_n.$$

It follows that the elements x, y, z, u_n $(n \geqq 0)$ are linearly independent. For otherwise some u_n will be a linear combination of x, y, z, u_i $(i<n)$. So applying ad y, we see that if $n>0$, u_{n-1} is a linear combination of x, y, u_i $(i<n-1)$. Hence u_0 will be a linear combination of x, y, z; say $u_0 = ax+by+cz$. Then $[x, u_0] = -u_0$ implies $a=b=0$, and then $[y, u_0]=0$ implies $c=0$. Thus $u_0=0$ contrary to assumption.

Hence $\mathfrak{k}$ must be infinite-dimensional, which is contrary to hypothesis, and therefore $\mathfrak{h}$ is epimorphically embedded in $\mathfrak{g}$.

Unfortunately we have no complete description of the epimorphically embedded subalgebras of a given finite-dimensional semisimple Lie algebra. However we do have

Proposition 7. *Any finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero contains a proper epimorphically embedded subalgebra.*

Proof. Let $\mathfrak{g}$ be such an algebra. Take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and take a system of positive roots with respect to $\mathfrak{h}$ with root vectors $x_\alpha, x_{-\alpha} (\alpha>0)$, so that

$$[y, x_\alpha] = \alpha(y)\, x, \qquad [y, x_{-\alpha}] = -\alpha(y)\, x \quad (y \in \mathfrak{h}).$$

Then we assert that the maximal soluble subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ generated by $\mathfrak{h}$ and the vectors x_α $(\alpha>0)$ is an epimorphically embedded subalgebra of $\mathfrak{g}$.

For let θ be any morphism of $\mathfrak{g}$ into a finite-dimensional Lie algebra $\mathfrak{k}$. For each $\alpha>0$, the vectors x_α, $x_{-\alpha}$ and α, considered as an element of $\mathfrak{h}$, form a three-dimensional simple Lie algebra isomorphic to that considered above. Thus the values of θ on x_α and α suffice to determine its value on $x_{-\alpha}$. Hence the values of θ on $\mathfrak{s}$ determine its values on all $x_{-\alpha}$, and hence on the whole of $\mathfrak{g}$.

It is not, by the way, true that any epimorphically embedded subalgebra necessarily contains a maximal soluble subalgebra $\mathfrak{s}$. For example consider the simple algebra of type G_2. This has positive roots of the form $\alpha, \beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha, 2\beta+3\alpha$, where α, β are simple roots. Then $\mathfrak{h}$, x_α and $x_{\beta+3\alpha}$ span a subalgebra $\mathfrak{m}$ which is in fact metabelian (that is $\mathfrak{m}''=0$) and which, we assert, is epimorphically embedded in $\mathfrak{g}$.

For let θ be any morphism of $\mathfrak{g}$ into a finite-dimensional algebra. The values of θ on x_α, $x_{\beta+3\alpha}$ and $\mathfrak{h}$ determine its values on $x_{-\alpha}$ and $x_{-\beta-3\alpha}$. But $[x_{-\alpha}, x_{\beta+3\alpha}]$ is some nonzero multiple of $x_{\beta+2\alpha}$, so the value of θ on $x_{\beta+2\alpha}$ is determined. Likewise so are the values of $x_{\beta+\alpha}$ and x_β. Then $[x_\beta, x_{\beta+3\alpha}]$ is a nonzero multiple of $x_{2\beta+3\alpha}$, so the value of θ is determined on $x_{2\beta+3\alpha}$, hence on $\mathfrak{s}$, and hence on $\mathfrak{g}$.

A like situation apertains generally. For a semisimple Lie algebra of rank r over an algebraically closed field of characteristic zero, we may always find a metabelian subalgebra $\mathfrak{m}$ of dimension $2r$, spanned by $\mathfrak{h}$ and r of the positive root vectors, such that $\mathfrak{m}$ is epimorphically embedded in $\mathfrak{g}$.

We shall see later by contrast that for compact real Lie algebras proper epimorphically embedded subalgebras do not exist.

7. Locally Compact Groups

We are indebted to Hofmann for observing that the negative result for finite-dimensional Lie algebras of the previous section implies that the category of locally compact groups with continuous morphisms also has epimorphisms which are not surjective.

Proposition 8. *Let $G = SL(2, \mathbf{R})$ and let H be the soluble subgroup of all 2×2 real matrices*

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \qquad (a \neq 0).$$

Then H is epimorphically embedded in G.

Proof. Suppose that we have two continuous morphisms θ, φ from G to some locally compact group K equal on H but distinct on G. By [7], Chapter 4, K has an open subgroup K' such that every neighbourhood of the identity in K' contains a normal subgroup N with K'/N a Lie group. Since G is connected it is mapped into K' by both θ and φ, and so we can suppose that $K' = K$. There exists some g in G with $\theta(g) \neq \varphi(g)$, so there is some symmetric neighbourhood of the identity in K not containing $\theta(g)\,\varphi(g)^{-1}$ and we can take a normal subgroup N contained in this neighbourhood with K/N a Lie group. Then composing θ and φ with the canonical projection $K \to K/N$, we have two maps of G into a Lie group which are equal on H but distinct on G. Thus we can suppose K above taken to be a Lie group. The maps θ, φ induce maps $\bar{\theta}, \bar{\varphi}$ from $\mathfrak{g}$ to the Lie algebra $\mathfrak{k}$ of K in the usual way. These differ on $\mathfrak{g}$ since the image of the exponential map generates G, and are equal on $\mathfrak{h}$. But since $\mathfrak{k}$ is finite-dimensional and $\mathfrak{g}$ is the three-dimensional simple algebra with subalgebra $\mathfrak{h}$ that we considered above, this is a contradiction. Thus H is epimorphically embedded.

8. Compact Groups and Lie Algebras

The question that now naturally arises is how far must we specialise the category of locally compact groups to obtain positive results. We have already seen in Proposition 3 that epimorphisms are surjective in the category of discrete groups, and it is trivial to see that the same is true of the category of locally compact Abelian groups. We now ask what the situation is in the category of compact groups.

Proposition 9. *Epimorphisms are necessarily surjective in the category of compact groups.*

Proof. We use the same split extension technique that was applied in the proof of Proposition 3. For this we require (given a compact group G and a proper closed subgroup H) to find a (not necessarily commutative) G-module A having an element fixed by H but not fixed by the whole of G. Since we are now in the category of compact groups we must require in addition that the module action $G \times A \to A$ be jointly continuous, and that A (ignoring the G-structure) be a compact topological group. Our aim, then, is to construct such an A.

Let us denote by $R(G)$ the subspace of the space $C(G)$ of continuous functions on G spanned (algebraically) by the coefficients of the continuous irreducible representations of G. These representations are all finite-dimensional and equivalent to unitary representations; and since any finite-dimensional unitary representation is completely reducible into a sum of irreducible components, we see, by means of tensor products, that $R(G)$ is in fact a subalgebra of $C(G)$. Furthermore, if π is a finite-dimensional representation so is its complex conjugate $\bar{\pi}$, and thus $R(G)$ is a self-adjoint subalgebra of $C(G)$. $R(G)$ contains the identity function 1, and by the Peter-Weyl theorem it separates the points of G, therefore by the Stone-Weierstrass theorem it is dense in $C(G)$.

$C(G)$ is a G-module by right translation: $({}_x f)(y) = f(yx)$ for $f \in C(G)$ and $x, y \in G$. And $R(G)$ is a submodule in this action since $\pi(yx) = \pi(y)\,\pi(x)$, and so

$$_x(\pi_{ij}) = \sum_{k=1}^{n} \pi_{kj}(x)\,\pi_{ik},$$

where $n = \dim \pi$.

Note that the elements of $C(G)$ fixed by H are precisely those functions which are constant on the left cosets xH of H. For $f \in C(G)$ define $\tilde{f} \in C(G)$ constant on the left cosets of H by

$$\tilde{f}(x) = \int_H f(xy)\, d\mu_H(y),$$

where μ_H is Haar measure on H normalised to have total mass 1. $R(G)$ is invariant under this operation since

$$\tilde{\pi}_{ij}(x) = \int_H \pi_{ij}(xy)\, d\mu_H(y) = \sum_{k=1}^{n} \Big(\int_H \pi_{kj}(y)\, d\mu_H(y)\Big)\, \pi_{ik}(x).$$

We assert that there is a function f in $R(G)$ fixed under H but not fixed under the whole of G. For suppose not. Any $g \in C(G)$ can be uniformly approximated by $f \in R(G)$, and then $\tilde{g}$ is approximated by $\tilde{f}$. But $\tilde{f}$ is fixed under H and so, by assumption, under G; therefore $\tilde{g}$ is fixed under G. But since it is possible to find a function in $C(G)$ fixed under H but not fixed under the whole of G, this gives a contradiction.

Since the space spanned by π_{ik} for $k=1$ to n is G-invariant, since the representation induced on this space is exactly π, since this space is also invariant under $f \to \tilde{f}$, and since these spaces for varying i and π are independent, it follows that we can find f in such a subspace fixed under H but not under G. In other words, there is a continuous irreducible unitary representation π of G on some finite dimensional Hilbert space, V say, and a vector $v \in V$ fixed under H but not under G.

If V is one-dimensional, then H is contained in the proper closed normal subgroup $\ker \pi$ of G, and in the usual way maps $G \to G/\ker \pi$ will separate H and G.

Suppose that $\dim V > 1$. π induces a representation of G on $\mathrm{Hom}(V, V)$ by $T \to \pi(x)\, T \pi(x)^{-1}$ for $x \in G$, and the unitary group A of V is invariant under this action. A is compact and the induced module action is jointly continuous; finally the unitary operator $U = 2(v \otimes v) - 1$ is an element of A fixed under H but not fixed under G. (For otherwise the one-dimensional subspace generated by v would be invariant under G, contradicting π irreducible and $\dim V > 1$.)

As a corollary to this result we obtain.

Proposition 10. *Epimorphisms are necessarily surjective in the category of real compact Lie algebras.*

Proof. Let $\mathfrak{g}$ be a real compact Lie algebra, and $\mathfrak{h}$ be a proper subalgebra of $\mathfrak{g}$. We may take a compact connected Lie group G having $\mathfrak{g}$ as Lie algebra, and then corresponding to $\mathfrak{h}$ there will be a Lie subgroup H of G. The difficulty of course is that H may not be closed in G; let $\bar{H}$ denote its closure.

Suppose first that $\bar{H} = G$, so that H is dense in G. Now a Lie subgroup of G is normal if and only if its Lie algebra is invariant under the adjoint representation of G. But $\mathfrak{h}$ is invariant under the adjoint representation of H, and H is dense in G. Therefore $\mathfrak{h}$ is invariant under the adjoint representation of G, and so H is normal in G. Thus $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, and two morphisms $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ will separate $\mathfrak{h}$ and $\mathfrak{g}$.

If now $\bar{H}$ is a proper subgroup of G, we can find a compact group K and two morphisms $G \to K$ equal on $\bar{H}$ but distinct on G. Then the same argument as in the proof of Proposition 8 shows that we can take K to be a Lie group and obtain two morphisms from $\mathfrak{g}$ to the Lie algebra $\mathfrak{k}$ of K which are equal on $\mathfrak{h}$ but distinct on $\mathfrak{g}$. $\mathfrak{k}$ will be compact since K is.

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Existence de traces pour les éléments d'espaces de distributions définis comme domaines d'opérateurs différentiels maximaux

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Introduction

Soient Ω un ouvert de $\mathbf{R}^n$, A un opérateur différentiel d'ordre m dans Ω, et Γ un ouvert de la frontière $\partial\Omega$ de Ω. On note $E(\Omega)=\{u\in L^2(\Omega);\ Au\in L^2(\Omega)\}$, le domaine de l'opérateur maximal associé à A dans $L^2(\Omega)$.[1]

On montre que, sous certaines hypothèses, les éléments de E ont des traces successives γ_i sur Γ, qui appartiennent aux espaces de Sobolev $H^{-\frac{1}{2}-i}(\Gamma)$, pour $0\leqq i\leqq m-1$.

Des résultats de ce type sont connus (cf. [3]), pour des opérateurs A elliptiques par exemple, mais obtenus généralement par des méthodes globales, alors que la méthode utilisée ici est «locale».

On donne d'abord les hypothèses utilisées (que l'on prend assez fortes afin de dégager le plus clairement possible la méthode) et le résultat correspondant. Puis on décompose la démonstration en plusieurs étapes: Localisation du problème – Passage à $\mathbf{R}^n$ – Méthode par dualité – Obtention des traces successives. Enfin on signale quelques extensions possibles, dont une rédaction détaillée sera faite ultérieurement. Certains des résultats démontrés ici ont été annoncés dans une note aux C. R. Acad. Sci. (cf. [2]).

Nous remercions M.S. Baouendi pour l'aide qu'il nous a apportée dans ce travail.

1. Hypothèses et résultat

On fait les hypothèses suivantes:

H$_1$) Les coefficients de l'opérateur A sont de classe C^m sur $\bar{\Omega}$ et $\bar{\Omega}$ est une variété à bord de classe C^m.

H$_2$) $\bar{\Gamma}$ n'est caractéristique pour l'opérateur A en aucun point.

H$_3$) On peut trouver un ouvert V qui soit l'intersection de Ω avec un voisinage de $\bar{\Gamma}$, tel que $\mathfrak{D}(\bar{V})$ soit dense dans $E(V)$ pour la norme du graphe.

[1] Les notations utilisées sont celles de [3], par exemple.

On obtient alors le résultat:

Théorème 1. *Sous les hypothèses* H_1), H_2), H_3) *l'application* γ_i *de* $\mathfrak{D}(\bar{V})$ *dans* $\mathfrak{D}(\bar{\Gamma})$ *qui à* $u \in \mathfrak{D}(\bar{V})$ *fait correspondre la trace de* $\frac{\partial^i u}{\partial v i}$ *sur* Γ, *se prolonge en une application linéaire continue de* $E(\Omega)$ *dans* $H^{-\frac{1}{2}-i}(\Gamma)$, *pour* $0 \leqq i \leqq m-1$.

Remarque 1. Les hypothèses de régularité H_1) ne sont utiles qu'au voisinage de $\bar{\Gamma}$.

On signalera au paragraphe 5 l'extension aux cas où $\bar{\Gamma}$ peut être caractéristique.

L'hypothèse de densité H_3) peut être encore affaiblie et on peut même s'en passer pour définir les traces; cette hypothèse est cependant commode ici et est vérifiée dans des cas très généraux (cf. [1]) (par exemple pour A elliptique ou parabolique ou à coefficients constants, pour certains opérateurs hyperboliques à coefficients variables ... etc.).

2. Localisation du problème

Il existe un difféomorphisme θ de classe C^m d'un ouvert de $\mathbf{R}^n$ sur un voisinage de Γ tel que: $V = \theta(R)$ avec

$$R = \{(x, y);\ x = (x_1, \ldots, x_{n-1}) \in I \text{ ouvert de } \mathbf{R}^{n-1} \text{ et } y \in]0, 1[\}$$

et $\theta(I) = \Gamma$; et toutes les dérivées partielles de θ et de θ^{-1} d'ordre $\leqq m$ soient bornées sur R et sur V.[2]

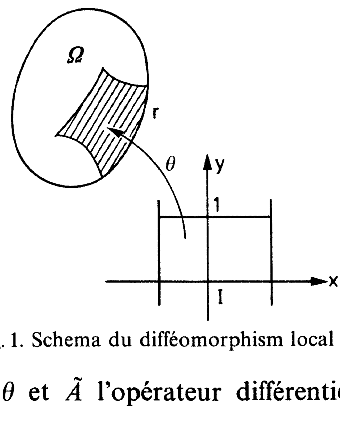

Fig. 1. Schema du difféomorphism local θ

On pose $\tilde{u} = u|_V \circ \theta$ et $\tilde{A}$ l'opérateur différentiel sur R défini par $\tilde{A}(u|_V \circ \theta) = A u|_V \circ \theta$.

On est ramené à chercher les traces pour $y = 0$ des éléments de l'espace: $\tilde{E}(R) = \{u \in L^2(R);\ \tilde{A} u \in L^2(R)\}$.

[2] On peut toujours se ramener à cette situation en considérant au lieu de Γ les parties d'un recouvrement ouvert de Γ.

Comme $\overline{\Gamma}$ est non caractéristique pour l'opérateur A, on peut supposer:

$$\tilde{A} = D_y^m + \sum_{\alpha < m} D_y^\alpha A_\alpha,$$

où A_α désigne un opérateur différentiel par rapport aux variables $(x_i)_{1 \leqq i \leqq n-1}$, d'ordre $\leqq m - \alpha$.

En écrivant

$$D_y^m u = f - \sum_{\alpha < m} D_y^\alpha A_\alpha u$$

pour $u \in \tilde{E}(R)$ avec $f \in L^2$ on obtient que les éléments u de $\tilde{E}(R)$ vérifient:

$$\text{(a)} \qquad \begin{cases} u(x, y) \in L^2(0, 1; L^2(I)) \\ D_y^m u \in \sum_{i=0}^{m-1} H^{-i}(0, 1; H^{-m+i}(I)). \end{cases}$$

(Remarquons que les éléments qui vérifient (a) sont bien plus généraux que ceux de $\tilde{E}(R)$, en particulier la fonction f pouvant être prise dans un espace plus grand que $L^2(R)$.)

On va montrer que si u vérifie (a), alors u a une trace pour $y = 0$ en un sens que l'on précisera.

On sait que les éléments de l'espace

$$\{u \in L^2(0, 1; L^2(I)); D_y^m u \in L^2(0, 1; H^{-m}(I))\}$$

ont une trace pour $y = 0$ dans l'espace $H^{-\frac{1}{2}}(I)$. On peut, dans certains cas, se ramener à cette situation par changement de variables pour supprimer les dérivées mixtes.

On va utiliser une méthode différente, qui nécessite d'abord quelques lemmes préliminaires de densité et de prolongement.

3. Résultats de densité et prolongement

Ces résultats sont démontrés dans le cas général dont nous aurons besoin plus tard (pour les traces d'ordre supérieur).

On note, pour $0 \leqq p$ entier $\leqq m - 1$,

$$E_{m,p}(R) = \{u \in H^{-p}(R); D_y^{m-p} u \in F_{m,p}\}$$

avec

$$F_{m,p} = \sum_{i=0}^{m-p-1} H^{-i}(0, 1; H^{-m+i}(I)).$$

Il est immédiat que $\mathfrak{D}(R)$ est dense dans $F_{m,p}$ et que:

$$F'_{m,p} = \bigcap_{i=0}^{m-p-1} H_0^i(0, 1; H_0^{m-i}(I)) \text{ et } F_{m,p} \text{ est réflexif.}$$

On démontre alors le résultat:

Lemme 1. *L'espace $\mathfrak{D}(\bar{R})$ est dense dans l'espace $E_{m,p}(R)$ (muni de la norme naturelle).*

Démonstration. On va démontrer que toute forme linéaire continue sur $E_{m,p}(R)$, nulle sur $\mathfrak{D}(\bar{R})$, est identiquement nulle. Toute forme linéaire continue T sur $E_{m,p}(R)$ peut s'écrire, avec $f \in H_0^p(R)$ et $g \in F'_{m,p}$:

$$T(u) = \langle f, u \rangle_{H_0^p(R), H^{-p}(R)} + \langle g, D_y^{m-p} u \rangle_{F'_{m,p}, F_{m,p}} \quad \text{pour tout } u \in E_{m,p}.$$

On suppose que, pour tout $u \in \mathfrak{D}(\bar{R})$, on a $T(u) = 0$. En considérant les seules fonctions u de $\mathfrak{D}(R)$, on a:

$$(-1)^{m-p} D^{m-p} g + f = 0 \quad \text{dans } \mathfrak{D}'(R),$$

donc:

$$D^{m-p} g \in H_0^p(R)$$

et par ailleurs:

$$g \in \bigcap_{i=0}^{m-p-1} H_0^i\big(0, 1; H_0^{m-i}(I)\big).$$

Il en résulte que $g \in H^m(R)$ et pour $y = 0$ ou 1, les traces de $D_y^k g$ sont nulles pour $0 \leqq k \leqq m-1$, *sauf éventuellement pour* $k = m-p-1$, et il suffit d'écrire $T(u) = 0$ avec maintenant u arbitraire dans $\mathfrak{D}(\bar{R})$ pour obtenir finalement que toutes les traces de g sur ∂R jusqu'à l'ordre $m-1$ sont nulles, donc $g \in H_0^m(R)$.

Il existe donc une suite $(\varphi_n)_{n \in \mathbf{N}}$ de fonctions de $\mathfrak{D}(R)$ qui converge vers g dans $H^m(R)$ et en particulier

$$f = (-1)^{m-p} D^{m-p} g = \lim_{n \to \infty} (-1)^{m-p} D^{m-p} \varphi_n \quad \text{dans } H_0^p(R).$$

Donc, pour tout $u \in E_{m,p}$, on a:

$$T(u) = \lim_{n \to \infty} [\langle (-1)^{m-p} D^{m-p} \varphi_n, u \rangle + \langle \varphi_n, D^{m-p} u \rangle]$$

$$T(u) = 0.$$

Ce qui démontre le lemme.

On utilise aussi le résultat:

Lemme 2. *On peut construire un opérateur linéaire continu Π de l'espace $E_{m,p}(R)$ dans l'espace*

$$E_{m,p}(\mathbf{R}^n) = \left\{ u \in H^{-p}(\mathbf{R}^n);\, D_y^{m-p} u \in \sum_{i=0}^{m-p-1} H^{-i}\big(\mathbf{R}; H^{-m+i}(\mathbf{R}^{n-1})\big) \right\},$$

tel que: $\Pi u|_R = u$ pour tout $u \in E_{m,p}(R)$.

Démonstration. a) On passe de $E_{m,p}(R)$ à $E_{m,p}(]0,1[\times \mathbf{R}^{n-1})$ sans difficulté; il suffit d'utiliser un opérateur P_1 de prolongement de $H^{-r}(I)$ à $H^{-r}(\mathbf{R}^{n-1})$ valable pour $0 \leqq r \leqq m$ (ce qui existe si on suppose I, donc Γ, assez régulier).

b) On considère l'opérateur P défini sur les fonctions régulières sur $\mathbf{R}^n_+ = \mathbf{R}^{n-1} \times \mathbf{R}_+$ par:

$$P u(x,y) = \begin{cases} u(x,y) & \text{si } y>0 \\ \sum_{i=1}^{m-1} \lambda_i u(x, -\alpha_i y) & \text{si } y<0 \end{cases}$$

où les α_i sont des réels distincts strictement positifs et les λ_i seront déterminés par la condition:

$$\|P u\|_{E_{m,p}(\mathbf{R}^n)} \leqq C \|u\|_{E_{m,p}(\mathbf{R}^n_+)}. \tag{1}$$

On vérifie d'abord que P est continu pour les normes induites par $H^{-r}(\mathbf{R}^n_+)$ et $H^{-r}(\mathbf{R}^n)$ pour $0 \leqq r \leqq p$, pourvu que l'on ait:

$$\begin{aligned} &\sum_{i=1}^{m-1} \lambda_i \alpha_i^{-1} = -1 \\ &\dots\dots\dots\dots\dots\dots \\ &\sum_{i=1}^{m-1} \lambda_i \alpha_i^{-1} (-\alpha_i)^{-p+1} = -1. \end{aligned} \tag{2}$$

Ceci se voit par transposition[3].

On a alors:

$$D_y^{m-p}(P u)(x,y) = \begin{cases} (D_y^{m-p} u)(x,y) & \text{si } y>0 \\ \sum_{i=1}^{m-1} \lambda_i (-\alpha_i)^{m-p} (D_y^{m-p} u)(x, -\alpha_i y) & \text{si } y<0. \end{cases}$$

Notons Q l'opérateur de prolongement défini par

$$Q v(x,y) = \begin{cases} v(x,y) & \text{si } y>0 \\ \sum_{i=1}^{m-1} \lambda_i (-\alpha_i)^{m-p} v(x, -\alpha_i y) & \text{si } y<0, \end{cases}$$

et exprimons que Q est continu pour les normes induites par

$$H^{-i}(\mathbf{R}_+, H^{-m+i}(\mathbf{R}^{n-1})) \quad \text{et} \quad H^{-i}(\mathbf{R}, H^{-m+i}(\mathbf{R}^{n-1}))$$

[3] On montre que l'application transposée ${}^tP$ est continue de $H^r(\mathbf{R}^n)$ dans $H_0^r(\mathbf{R}^n_+)$ et vérifie ${}^tP\,\tilde{\varphi} = \varphi$ pour tout $\varphi \in H_0^r(\mathbf{R}^n_+)$, $\tilde{\varphi}$ désignant l'élément de $H^r(\mathbf{R}^n)$ obtenu en prolongeant φ par 0 pour $y<0$.

pour $i=0, 1, \dots, m-p-1$; cela se traduit par les conditions:

$$\begin{aligned} &\sum_{i=1}^{m-1} \lambda_i(-\alpha_i)^{m-p-1}=1 \\ &\dots\dots\dots\dots\dots \\ &\sum_{i=1}^{m-1} \lambda_i(-\alpha_i)=1. \end{aligned} \tag{3}$$

Le système (2) (3) est un système de Cramer par rapport aux variables $\lambda_1, \dots, \lambda_{m-1}$, donc le prolongement P est construit pour les fonctions régulières; comme ces fonctions sont denses dans $E_{m,p}(\mathbf{R}^n_+)$ on a construit un prolongement P de $E_{m,p}(\mathbf{R}^n_+)$ dans l'espace $E_{m,p}(\mathbf{R}^n)$.

c) Soient φ_1 et φ_2 deux fonctions de $\mathfrak{D}([0,1[)$ et $\mathfrak{D}(]0,1])$ respectivement; on montre que l'application $u \mapsto \varphi_i u$ est continue de $E_{m,p}(]0,1[\times \mathbf{R}^{n-1})$ dans lui-même, pour $i=1,2$. On choisira φ_1 et φ_2 telles que $\varphi_1+\varphi_2=1$ sur $[0,1]$.

d) Soit $u \in E_{m,p}(R)$; on obtient $P_1 u \in E_{m,p}(]0,1[\times \mathbf{R}^{n-1})$; puis

$$\begin{aligned} &\varphi_1 P_1 u \in E_{m,p}(\mathbf{R}^n_+) \\ &\varphi_2 P_1 u \in E_{m,p}(\mathbf{R}^n_-) \quad \text{avec } \mathbf{R}^n_- =]-\infty, 1[\times \mathbf{R}^{n-1} \\ &v_1 = P(\varphi_1 P_1 u) \in E_{m,p}(\mathbf{R}^n) \\ &v_2 = P_-(\varphi_2 P_1 u) \in E_{m,p}(\mathbf{R}^n), \end{aligned}$$

où P_- désigne le prolongement analogue à P, $\mathbf{R}^n_+$ étant remplacé par $\mathbf{R}^n_-$.

Enfin, on pose $\Pi u = v_1 + v_2$.

Remarque 2. Il aurait été possible d'éviter ce prolongement, mais il est plus agréable de travailler dans $\mathbf{R}^n$, pour les techniques de dualité et relèvement des traces, que l'on va utiliser.

4. Méthode par dualité

On démontre d'abord un résultat de «relèvement».

Proposition 1. *L'application* T: $\varphi \to \delta_{(y=0)} \otimes \varphi$ *est linéaire continue de* $H^{p+\frac{1}{2}}(\mathbf{R}^{n-1})$ *dans*

$$E'_{m,p}(\mathbf{R}^n) = \left\{ f + D^{m-p} g;\, f \in H^p(\mathbf{R}^n) \text{ et } g \in \bigcap_{i=0}^{m-p-1} H^i(\mathbf{R}, H^{m-i}(\mathbf{R}^{n-1})) \right\}.$$

Démonstration. On sait cf. [3] qu'il existe un relèvement continu R de $H^{p+\frac{1}{2}}(\mathbf{R}^{n-1})$ dans $H^m(\mathbf{R}^n_+)$ qui à φ associe $R\varphi = W$ vérifiant:

$$\begin{aligned} &\gamma_0 W = \dots = \gamma_{m-p-2} W = 0 \\ &\gamma_{m-p-1} W = \varphi \\ &\gamma_{m-p} W = \dots = \gamma_{m-1} W = 0 \end{aligned}$$

$\gamma_i W$ désigne $(D^i_y W)(x,0)$.

On note $\tilde{W}$ la distribution obtenue en prolongeant W par 0 pour $y<0$. On a:

$$D_y^{m-p}\tilde{W}=\widetilde{D_y^{m-p}W}+\delta_{(y=0)}\otimes\varphi.$$

Donc

$$\delta_{(y=0)}\otimes\varphi=f+D_y^{m-p}g \quad \text{avec}$$
$$f=-\widetilde{D_y^{m-p}W}$$
$$g=\tilde{W}.$$

Il reste à vérifier:

$$\widetilde{D_y^{m-p}W}\in H^p(\mathbf{R}^n)$$

$$\tilde{W}\in\bigcap_{k=0}^{m-p-1}H^k(\mathbf{R},H^{m-k}(\mathbf{R}^{n-1})).$$

On a:

1) La fonction $D_y^{m-p}W$ a toutes ses traces d'ordre $<p$ nulles pour $y=0$, donc $\widetilde{D_y^{m-p}W}\in H^p(\mathbf{R}^n)$ et on a:

$$\|\widetilde{D_y^{m-p}W}\|_{H^p(\mathbf{R}^n)}\leqq C_1\|W\|_{H^m(\mathbf{R}^n_+)}\leqq C_2\|\varphi\|_{H^{p+\frac{1}{2}}(\mathbf{R}^{n-1})},$$

2) La fonction W a toutes ses traces sur $y=0$ nulles jusqu'à l'ordre $m-p-2$, donc

$$\tilde{W}\in\bigcap_{k=0}^{m-p-1}H^k(\mathbf{R},H^{m-k}(\mathbf{R}^{n-1}))$$

et on a:

$$\|\tilde{W}\|\leqq C_3\|\varphi\|_{H^{p+\frac{1}{2}}(\mathbf{R}^{n-1})}.$$

La proposition est complètement démontrée.

Par transposition, on déduit de la proposition 1 un résultat de «traces».

Théorème 2. *L'application γ_0 de $\mathfrak{D}(\mathbf{R}^n)$ dans $\mathfrak{D}(\mathbf{R}^{n-1})$, qui à $u\in\mathfrak{D}(\mathbf{R}^n)$ fait correspondre sa trace pour $y=0$, se prolonge par continuité en une application linéaire continue de $E_{m,p}(\mathbf{R}^n)$ dans $H^{-\frac{1}{2}-p}(\mathbf{R}^{n-1})$.*

Démonstration. On démontre d'abord (même méthode que pour le lemme 1 ou régularisation par convolution) que $\mathfrak{D}(\mathbf{R}^n)$ est dense dans $E_{m,p}(\mathbf{R}^n)$.

On vient de montrer que l'application T est linéaire continue de $H^{p+\frac{1}{2}}(\mathbf{R}^{n-1})$ dans $E'_{m,p}$; donc l'application transposée ${}^tT$ est linéaire continue de $E_{m,p}(\mathbf{R}^n)$ dans $H^{-\frac{1}{2}-p}(\mathbf{R}^{n-1})$. Il suffit de montrer que la restriction à $\mathfrak{D}(\mathbf{R}^n)$ coïncide avec la trace. Or, pour $f\in\mathfrak{D}(\mathbf{R}^n)$, on a:

$$\forall u\in H^{\frac{1}{2}+p}(\mathbf{R}^{n-1}),\ \langle{}^tTf,u\rangle=\langle f,Tu\rangle_{E_{m,p},E'_{m,p}}$$
$$=\langle f,\delta_{(y=0)}\otimes u\rangle.$$

Comme f est régulière, ceci peut s'écrire aussi

$$\langle {}^tTf, u\rangle = \int_{\mathbf{R}^{n-1}} f(x,0)\, u(x)\, dx = \langle \gamma_0 f, u\rangle .$$

Donc la restriction de ${}^tT$ à $\mathfrak{D}(\mathbf{R}^n)$ est γ_0.

Donc le théorème 2 est démontré. En utilisant les résultats de prolongement, on en déduit immédiatement:

Théorème 2′. *L'application γ_0 de $\mathfrak{D}(\bar{R})$ dans $\mathfrak{D}(I)$ se prolonge par continuité en une application linéaire continue de $E_{m,p}(R)$ dans $H^{-\frac{1}{2}-p}(I)$.*

5. Application aux traces des éléments de $E(R)$

a) On a immédiatement, d'après le théorème 2′ pour $p=0$, que les éléments de $\tilde{E}(R)$ ont une trace γ_0 pour $y=0$ et que cette trace est dans $H^{-\frac{1}{2}}(I)$.

b) *Recherche des traces d'ordre supérieur.*

Il est facile de voir que les éléments de $E_{m,0}(R)$ (ou $E_{m,0}(\mathbf{R}^n)$) n'ont pas nécessairement de traces d'ordre supérieur à 0, au sens précédent. Par exemple, on peut vérifier que $u \in L^2(\mathbf{R}^2)$ et $D_y^2 u \in H^{-1}(\mathbf{R}, H^{-1}(\mathbf{R}))$ n'impliquent pas que $\gamma_1 u$ existe pour $y=0$, en prenant u quelconque dans $H^1(\mathbf{R}^2)$.

Il faut donc reprendre le problème plus tôt.

Pour $u \in \tilde{E}(R)$ on peut écrire, avec $f \in L^2(R)$:

$$D_y^{m-1}(D_y u + A_{m-1} u) = f - \sum_{\alpha \leqq m-2} D_y^\alpha A_\alpha u .$$

On pose: $v = D_y u + A_{m-1} u$.

On obtient que v vérifie:

$$(\mathrm{a}_1) \qquad \begin{cases} v(x,y) \in H^{-1}(R) \\ D_y^{m-1} v \in \sum_{i=0}^{m-1} H^{-i}(0,1; H^{-m+i}(I)). \end{cases}$$

Donc la distribution v appartient à $E_{m,1}(R)$ et sa trace pour $y=0$ est définie dans $H^{-\frac{3}{2}}(I)$, d'après le théorème 2′. On a donc obtenu la trace d'une dérivée oblique de u. On en déduit la trace de toute dérivée oblique et en particulier de $D_y u$; en effet:

On raisonne sur les fonctions de $\mathfrak{D}(\bar{R})$.

L'application $u \mapsto \gamma_0(A_{m-1} u)$ est continue pour les normes induites par $\tilde{E}(R)$ et $H^{-\frac{3}{2}}(I)$, car l'opérateur A_{m-1} étant un opérateur différentiel par rapport aux seules variables $x_1, \dots, x_{n-1}$, on a $\gamma_0 A_{m-1} u = A_{m-1}(\gamma_0 u)$ pour $y=0$.

On en déduit que $u \mapsto \gamma_0 D_y u$ est continue pour les normes induites par $\tilde{E}(R)$ et $H^{-\frac{3}{2}}(I)$, et de façon générale, pour obtenir les traces $\gamma_0 D_y^\alpha u$ pour $u \in \tilde{E}(R)$ et $1 \leqq \alpha \leqq m-1$, on procède par récurrence, en utilisant le théorème 2′ pour $1 \leqq p \leqq m-1$. On en déduit les traces successives sur Γ des éléments de $E(R)$ (pour les traces d'ordre $\geqq 1$, il faut limiter la classe des difféomorphismes permettant de localiser le problème; cf. paragraphe 2).

6. Quelques généralisations possibles et remarques

1. On remarque d'abord que la théorie faite ici à partir de $L^2(\Omega)$ peut se faire de la même manière à partir de $L^p(\Omega)$ avec $1<p<\infty$ et même à partir de certains espaces avec poids.

2. *Bord caractéristique.*

Pour l'étude des traces des éléments de $E(\Omega)$, il peut se présenter diverses situations, où les difficultés rencontrées ne sont pas de même nature. Signalons, par exemple dans $\mathbf{R}^n_+$, le cas où l'opérateur A est de la forme:

(1) $\quad Au = y^\gamma D_y^m u + \sum_{\alpha \leqq m-1} D_y^\alpha A_\alpha u, \qquad$ ou

(2) $\quad Au = x^\gamma D_y^m u + \sum_{\alpha \leqq m-1} D_y^\alpha A_\alpha u, \qquad$ avec $\gamma > 0$,

et A_α désignant un opérateur différentiel par rapport aux variables $x_1, \dots, x_{n-1}$ d'ordre au plus $m-\alpha$. Dans le cas (2) on peut obtenir les traces successives de u dans des espaces d'interpolation entre des espaces de Sobolev avec poids, en suivant la méthode ci-dessus; tandis que dans le cas (1) on peut obtenir les traces successives de u (ou de $y^\beta u$ avec β convenable) dans des espaces $H^s(\mathbf{R}^{n-1})$, mais il faut adapter, la méthode utilisée ici (en particulier pour le résultat de relèvement, cf. proposition 1).

3. Si on ne peut démontrer que les fonctions régulières sont denses dans $E(\Omega)$ (plus précisément, l'hypothèse H_3), on peut cependant définir les traces des éléments de $E(\Omega)$ sur Γ. On montre que si $u \in \tilde{E}(R)$, alors u est une fonction de y presque partout égale à une fonction continue à valeurs dans $H^{-\frac{1}{2}}(I)$; en composant avec θ^{-1}, on montre que $u \in E(\Omega)$ est presque partout égale à une fonction continue (en variable normale) à valeurs dans $H^{-\frac{1}{2}}$ (en variables tangentielles). Ceci suppose le choix de difféomorphismes θ privilégiés.

4. *Surjectivité des traces.* Les espaces de traces obtenus sont les meilleurs possibles lorsqu'on ne fait aucune hypothèse supplémentaire sur l'opérateur A, et dans certains cas (par exemple: A elliptique) l'ap-

plication $\gamma=(\gamma_0, \ldots, \gamma_{m-1})$ est surjective de $E(\Omega)$ sur

$$\prod_{i=0}^{m-1} H^{-\frac{1}{2}-i}(\Gamma).$$

Cependant, pour certains opérateurs A, les espaces de traces correspondants peuvent être plus petits, et il n'y a donc pas surjectivité de γ en général (exemple: $\Omega=\mathbf{R}^2_+$, $\Gamma=\{(x,y);\ y=0\}$ et $A=D_y$); dans ces cas, on peut toujours adapter la méthode utilisée ici pour obtenir les meilleurs espaces de traces.

5. On peut aussi chercher par la même méthode les traces sur Γ des éléments de l'espace $E^s(\Omega)=\{u\in H^s(\Omega);\ A u\in L^2(\Omega)\}$ avec s réel >0. Pour $u\in E^s(\Omega)$, on obtient les traces $\gamma_i u$ dans $H^{s-i-\frac{1}{2}}$ pour $0\leqq i<s-\frac{1}{2}$, sans avoir à utiliser l'hypothèse «$A u\in L^2$».

Pour $s-\frac{1}{2}\leqq i$ (ce qui implique $s\leqq m-\frac{1}{2}$), on écrit encore:

$$D_y^{m-i}(D_y^i u+D_y^{i-1} A_1 u+\cdots+A_i u)=f-D_y^{m-i-1} A_{i+1} u-\cdots-A_m u$$
$$u\in H^s(R).$$

Ce qui donne, en posant $v=D_y^i u+\cdots+A_i u$,

$$v\in H^{s-i}(R)$$
$$D_y^{m-i} v\in H^{-m+i+1}(0,1;H^{-s-i-1}(I))+\cdots+L^2(0,1;H^{s-m}(I))$$

où on trouve encore, en généralisant les résultats des paragraphes 3 et 4, que $\gamma_0 v\in H^{s-i-\frac{1}{2}}$, donc que $\gamma_i u\in H^{s-i-\frac{1}{2}}(I)$.

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Algebraic K-Theory and Quadratic Forms

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The first section of this paper defines and studies a graded ring $K_* F$ associated to any field F. By definition, $K_n F$ is the target group of the universal n-linear function from $F^\bullet \times \cdots \times F^\bullet$ to an additive group, satisfying the condition that $a_1 \times \cdots \times a_n$ should map to zero whenever $a_i + a_{i+1} = 1$. Here $F^\bullet$ denotes the multiplicative group $F - 0$.

Section 2 constructs a homomorphism $\partial \colon K_n F \to K_{n-1} \bar{F}$ associated with a discrete valuation on F with residue class field $\bar{F}$. These homomorphisms ∂ are used to compute the ring $K_* F(t)$ of a rational function field, using a technique due to John Tate.

Section 3 relates $K_* F$ to the theory of quadratic modules by defining certain "Stiefel-Whitney invariants" of a quadratic module over a field F of characteristic $\neq 2$. The definition is closely related to Delzant [5].

Let W be the Witt ring of anisotropic quadratic modules over F, and let $I \subset W$ be the maximal ideal, consisting of modules of even rank. Section 4 studies the conjecture that the associated graded ring

$$(W/I, I/I^2, I^2/I^3, \ldots)$$

is canonically isomorphic to $K_* F / 2 K_* F$. Section 5 computes the Witt ring of a field $F(t)$ of rational functions.

Section 6 describes the conjecture that $K_* F / 2 K_* F$ is canonically isomorphic to the cohomology ring $H^*(G_F; Z/2Z)$; where G_F denotes the Galois group of the separable closure of F. An appendix, due to Tate, computes $K_* F / 2 K_* F$ for a global field.

Throughout the exposition I have made free use of unpublished theorems and ideas due to Bass and Tate. I want particularly to thank Tate for his generous help.

§1. The Ring $K_* F$

To any field F we associate a graded ring

$$K_* F = (K_0 F, K_1 F, K_2 F, \ldots)$$

as follows. By definition, $K_1 F$ is just the multiplicative group $F^\bullet$ written additively. To keep notation straight, we introduce the canonical isomorphism

$$l \colon F^\bullet \to K_1 F,$$

where $l(a\,b)=l(a)+l(b)$. Then $K_* F$ is defined to be the quotient of the tensor algebra

$$(Z,\ K_1 F,\ K_1 F \otimes K_1 F,\ K_1 F \otimes K_1 F \otimes K_1 F, \ldots)$$

by the ideal generated by all $l(a) \otimes l(1-a)$, with $a \neq 0, 1$. In other words each $K_n F, n \geqq 2$, is the quotient of the n-fold tensor product $K_1 F \otimes \cdots \otimes K_1 F$ by the subgroup generated by all $l(a_1) \otimes \cdots \otimes l(a_n)$ such that $a_i + a_{i+1} = 1$ for some i.

In terms of generators and relations, $K_* F$ can be described as the associative ring with unit which is generated by symbols $l(a)$, $a \in F^{\bullet}$, subject only to the defining relations $l(a\,b)=l(a)+l(b)$ and $l(a)\, l(1-a)=0$.

Explanation. This definition of the group $K_2 F$ is motivated by work of R. Steinberg, C. Moore, and H. Matsumoto on algebraic groups; and has already been the object of much study. (Compare references [2–4, 7–9, 17].) For $n \geqq 3$, the definition is purely ad hoc. Quite different definition of K_n for $n \geqq 3$ have been proposed by Swan [18] and by Nobile and Villamayor [11]; but no relationship between the various definitions is known.

First let us describe some fundamental properties of the ring $K_* F$. (Examples will be given in §§ 1.5 – 1.8.)

Lemma 1.1. *For every $\xi \in K_m F$ and every $\eta \in K_n F$, the identity*

$$\eta\, \xi = (-1)^{mn}\, \xi\, \eta$$

is valid in $K_{m+n} F$.

Proof (following Steinberg). Clearly it suffices to consider the case $m=n=1$. Since $-a=(1-a)/(1-a^{-1})$ for $a \neq 1$, we have

$$\begin{aligned} l(a)\, l(-a) &= l(a)\, l(1-a) - l(a)\, l(1-a^{-1}) \\ &= l(a)\, l(1-a) + l(a^{-1})\, l(1-a^{-1}) = 0. \end{aligned}$$

Hence the sum $l(a)\, l(b) + l(b)\, l(a)$ is equal to

$$\begin{aligned} & l(a)\, l(-a) + l(a)\, l(b) + l(b)\, l(a) + l(b)\, l(-b) \\ &= l(a)\, l(-a\,b) + l(b)\, l(-a\,b) \\ &= l(a\,b)\, l(-a\,b) = 0; \end{aligned}$$

which completes the proof.

Here are two further consequences of this argument:

Lemma 1.2. *The identity $l(a)^2 = l(a)\, l(-1)$ is valid for every $l(a) \in K_1 F$.*

For the equation $l(a)\, l(-a)=0$ implies that $l(a)^2 = l(a)\big(l(-1)+l(-a)\big)$ must be equal to $l(a)\, l(-1)$.

Lemma 1.3. *If the sum $a_1+\dots+a_n$ of non-zero field elements is equal to either* 0 *or* 1, *then $l(a_1)\dots l(a_n)=0$.*

Proof by Induction on n. The statement is certainly true for $n=1,2$; so we may assume that $n\geqq 3$. If $a_1+a_2=0$, then the product $l(a_1)\,l(a_2)$ is already zero. But if $a_1+a_2\neq 0$, then the equation

$$a_1/(a_1+a_2)+a_2/(a_1+a_2)=1$$

implies that

$$\big(l(a_1)-l(a_1+a_2)\big)\big(l(a_2)-l(a_1+a_2)\big)=0.$$

Multiplying by $l(a_3)\dots l(a_n)$, and using 1.1 and the inductive hypothesis that

$$l(a_1+a_2)\,l(a_3)\dots l(a_n)=0,$$

the conclusion follows.

Here is an application.

Theorem 1.4. *The element* -1 *is a sum of squares in F if and only if every positive dimensional element of K_*F is nilpotent.*

Proof. If -1 is not a sum of squares, then F can be embedded in a real closed field, and hence can be ordered. Choosing some fixed ordering, define an n-linear mapping from $K_1F\times\dots\times K_1F$ to the integers modulo 2 by the correspondence

$$l(a_1)\times\dots\times l(a_n)\mapsto \frac{1-\operatorname{sgn}(a_1)}{2}\cdots\frac{1-\operatorname{sgn}(a_n)}{2}.$$

Evidently the right hand side is zero whenever $a_i+a_{i+1}=1$. Hence this correspondence induces a homomorphism

$$K_nF\to Z/2Z;$$

which carries $l(-1)^n$ to 1. This proves that the element $l(-1)$ is not nilpotent.

Conversely, if say $-1=a_1^2+\dots+a_r^2$, then it follows from 1.3 that

$$l(-a_1^2)\dots l(-a_r^2)=0;$$

hence

$$l(-1)^r\equiv 0 \bmod 2K_rF.$$

Since $2l(-1)=0$, it follows immediately that $l(-1)^{r+1}=0$.

For any generator $\gamma=l(a_1)\dots l(a_n)$ of the group K_nF, it follows from 1.2 that γ^s is equal to a multiple of $l(-1)^{n(s-1)}$. Hence $\gamma^s=0$ whenever $n(s-1)>r$. Similarly, for any sum $\gamma_1+\dots+\gamma_k$ of generators, the power $(\gamma_1+\dots+\gamma_k)^s$ can be expressed as a linear combination of monomials $\gamma_1^{i_1}\dots\gamma_k^{i_k}$ with $i_1+\dots+i_k=s$. Choosing $s>k$, note that each such monomial is a multiple of $l(-1)^{n(s-k)}$. If $s>k+r/n$, it follows that $(\gamma_1+\dots+\gamma_k)^s=0$; which completes the proof.

To conclude this section, the ring K_*F will be described in four interesting special cases.

Example 1.5 (Steinberg). If the field is finite, then $K_2F=0$. In fact K_1F is cyclic, say of order $q-1$; so § 1.1 implies that K_2F is either trivial or of order $\leqq 2$, according as q is even or odd. But, if q is odd, then an easy counting argument shows that 1 is the sum of two quadratic non-residues in F; from which it follows that $K_2F=0$. This implies, of course, that $K_nF=0$ for $n>2$ also.

Example 1.6. Let R be the field of real numbers. Then every K_nR, $n\geqq 1$, splits as the direct sum of a cyclic group of order 2 generated by $l(-1)^n$, and a divisible group generated by all products $l(a_1)\ldots l(a_n)$ with $a_1, \ldots, a_n>0$. This is easily proved by induction on n, using the argument of § 1.4 to show that $l(-1)^n$ is not divisible.

Example 1.7. Let F be a *local field* (i.e. complete under a discrete valuation with finite residue class field), and let m be the number of roots of unity in F. Calvin Moore [10] proves that K_2F is the direct sum of a cyclic group of order m and a divisible group.

We will show that K_nF is divisible for $n\geqq 3$. Consider the algebra K_*F/pK_*F over Z/pZ; where p is a fixed prime. If p does not divide m, then Moore's theorem clearly implies that $K_2F/pK_2F=0$. Suppose that p does divide m. We claim then that:

(1) the vector space K_1F/pK_1F has dimension $\geqq 2$ over Z/pZ;

(2) the vector space K_2F/pK_2F has dimension 1; and

(3) for each $\alpha\neq 0$ in K_1F/pK_1F there exists β in K_1F/pK_1F so that $\alpha\beta\neq 0$.

In fact (1) is clear; (2) follows from Moore's theorem; and (3) is an immediate consequence of the classical theorem which asserts that, for each $a\in F^\bullet$ which is not a p-th power, there exists b so that the p-th power norm residue symbol $(a, b)_F$ is non-trivial. (See for example [20, p. 260].) The correspondence

$$l(a)\, l(b)\mapsto (a, b)_F$$

clearly extends to a homomorphism from K_2F to the group of p-th roots of unity. So, taking $\alpha\equiv l(a)$, $\beta\equiv l(b)$, the conclusion (3) follows.

Proof that every generator $\alpha\beta\gamma$ of K_3F/pK_3F is zero. Given α, β, γ one can first choose $\beta'\neq 0$ so that $\alpha\beta'=0$ (using (1) and (2)), and then choose γ' so that $\beta'\gamma'=\beta\gamma$ (using (2) and (3)). The required equation

$$\alpha\beta\gamma=\alpha\beta'\gamma'=0$$

follows.

Thus $K_3F/pK_3F=0$ for every prime p; which proves that K_3F is divisible.

Example 1.8. Let F be a *global field* (that is a finite extension of the field Q of rational numbers, or of the field of rational functions in one indeterminate over a finite field). Let F_v range over all local or real completions of F. The complex completions (if any) can be ignored for our purposes. The inclusions $F \to F_v$ induce a homomorphism

$$K_2 F \to \oplus_v K_2 F_v/(\text{max. divis. subgr.}),$$

where each summand on the right is finite cyclic by 1.6 and 1.7. Bass and Tate [3] have shown that the kernel of this homomorphism is finitely generated, but the precise structure of the kernel is not known. Moore has shown that the cokernel is isomorphic to the group of roots of unity in F.

The structure of $K_n F$ is not known for $n \geqq 3$, but Tate has proved the following partial result: *The quotient* $K_n F/2K_n F$ *maps isomorphically to the direct sum, over all real completions* F_v, *of*

$$K_n F_v/2K_n F_v \cong Z/2Z.$$

Thus the dimension of $K_n F/2K_n F$ as a mod 2 vector space is equal to the number of real completions. Tate's proof of this result is presented in the Appendix.

It may be conjectured that the subgroup $2K_n F$ is actually zero for $n \geqq 3$, so that $K_n F$ itself is a vector space over $Z/2Z$. As an example, for the field Q of rational numbers the isomorphism

$$K_n Q \cong Z/2Z$$

for $n \geqq 3$ can be established by methods similar to those of §2.3.

§ 2. Discrete Valuations and the Computation of $K_* F(t)$

Suppose that a field F has a discrete valuation v with residue class field $\bar{F}$ $(=\bar{F}_v)$. The group of *units* (elements u with $\mathrm{ord}_v u = 0$) will be denoted by U, and the natural homomorphism $U \to \bar{F}^\bullet$ by $u \mapsto \bar{u}$. An element π of $F^\bullet$ is *prime* if $\mathrm{ord}_v \pi = 1$.

Lemma 2.1. *There exists one and only one homomorphism* $\partial = \partial_v$ *from* $K_n F$ *to* $K_{n-1} \bar{F}$ *which carries the product* $l(\pi)\, l(u_2) \ldots l(u_n)$ *to* $l(\bar{u}_2) \ldots l(\bar{u}_n)$ *for every prime element* π *and for all units* $u_2, \ldots, u_n$. *This homomorphism* ∂ *annihilates every product of the form* $l(u_1) \ldots l(u_n)$.

(For $n = 1$ the defining property is to be that $\partial\, l(\pi) = 1$.)

Remarks. Evidently ∂ is always surjective. For $n = 1$ the homomorphism ∂ can essentially be identified with the homomorphism

$$\mathrm{ord}_v \colon F^\bullet \to Z;$$

and for $n=2$ it is closely related to the classical "tame symbol"

$$\pi^i u_1, \pi^j u_2 \mapsto \overline{(-1)}^{ij} \bar{u}_2^i / \bar{u}_1^j$$

which is utilized for example in [3].

To begin the proof, note that any unit u_1 can be expressed as the quotient $\pi u_1/\pi$ of two prime elements. So the property

$$\partial(l(u_1)\dots l(u_n))=0$$

follows immediately from the defining equation.

Proof of Uniqueness. Choose a prime element π. Since $F^\bullet$ is generated by π and U, it follows that $K_n F$ is generated by products of the form $l(\pi)^r l(u_{r+1})\dots l(u_n)$. If $r=1$, then the image of any such product under ∂ has been specified; and if $r>1$ then using the identity $l(\pi)^r = l(\pi)\, l(-1)^{r-1}$ it is also specified. But if $r=0$, then any such product maps to zero. This proves that ∂ is unique, if it exists.

Proof of Existence[1]. It will be convenient to introduce an indeterminate symbol x which is to anticommute with all elements of $K_1 \bar{F}$. Given any n-tuple of elements

$$l(\pi^{i_1} u_1), \dots, l(\pi^{i_n} u_n) \in K_1 F,$$

construct a sequence of elements $\varphi_j \in K_j \bar{F}$ by the formula

$$(x\, i_1 + l(\bar{u}_1))\dots(x\, i_n + l(\bar{u}_n)) = x^n \varphi_0 + x^{n-1}\varphi_1 + \dots + \varphi_n.$$

Evidently each φ_j is n-linear as a function of $l(\pi^{i_1} u_1), \dots, l(\pi^{i_n} u_n)$. Now consider the linear combination

$$\varphi = l(\overline{-1})^{n-1}\varphi_0 + l(\overline{-1})^{n-2}\varphi_1 + \dots + \varphi_{n-1}.$$

Thus $\varphi \in K_{n-1}\bar{F}$, and evidently φ is also linear as a function of each $l(\pi^{i_j} u_j)$.

If two successive $\pi^{i_j} u_j$ add up to 1, we will prove that $\varphi=0$. This will show that the correspondence

$$l(\pi^{i_1} u_1)\dots l(\pi^{i_n} u_n) \mapsto \varphi$$

1 *Added in Proof.* A much better construction of the homomorphism ∂ has been suggested by Serre. Adjoin to the ring $K_* \bar{F}$ a new symbol ξ of degree 1 which is to anticommute with the elements of $K_1 \bar{F}$, and to satisfy the identity $\xi^2 = \xi\, l(\overline{-1})$, but is to satisfy no other relations. Thus the enlarged ring $(K_* \bar{F})\,[\xi]$ is free over $K_* \bar{F}$ with basis $\{1, \xi\}$. It is not difficult to show that the correspondence

$$l(\pi^i u) \mapsto i\,\xi + l(\bar{u})$$

extends uniquely to a ring homomorphism θ_π from $K_* F$ to this enlarged ring. Now, setting

$$\theta_\pi(\alpha) = \psi(\alpha) + \xi\,\partial(\alpha)$$

with $\psi(\alpha)$ and $\partial(\alpha)$ in $K_* \bar{F}$, we obtain the required homomorphism ∂.

is well defined and extends to a homomorphism

$$K_n F \to K_{n-1} \bar{F}.$$

Since it is clear that $l(\pi u)\, l(u_2) \dots l(u_n)$ maps to $l(\bar{u}_2) \dots l(\bar{u}_n)$, this will complete the proof.

To avoid complicated notation, we will carry out details only for the case

$$\pi^{i_1} u_1 + \pi^{i_2} u_2 = 1.$$

There are four possibilities to consider.

If $i_1 > 0$, then it follows easily that

$$i_2 = 0, \quad \bar{u}_2 = 1.$$

Hence the factor $x\, i_2 + l(\bar{u}_2)$ is zero and it certainly follows that $\varphi = 0$.

The case $i_1 = 0$, $i_2 > 0$ is disposed of similarly.

If $i_1 = i_2 = 0$, then $\bar{u}_1 + \bar{u}_2 = \bar{1}$, hence

$$(x\, i_1 + l(\bar{u}_1))\,(x\, i_2 + l(\bar{u}_2)) = 0,$$

so again $\varphi = 0$.

Finally suppose that $i_1 < 0$. Then clearly $i_1 = i_2$ and $\bar{u}_2 = -\bar{u}_1$. In this case the product $(x\, i_1 + l(\bar{u}_1))\,(x\, i_2 + l(\bar{u}_2))$ evidently simplifies to

$$x^2\, i_1^2 + x\, i_1\, l(\overline{-1}) + 0.$$

Hence the expression $\sum x^{n-j} \varphi_j$ can be written as

$$x\,(x\, i_1^2 + i_1\, l(\overline{-1}))\,(x\, i_3 + l(\bar{u}_3)) \dots (x\, i_n + l(\bar{u}_n)).$$

Cancelling the initial x, and then substituting $l(\overline{-1})$ for the remaining x's, we evidently obtain an expression for φ. But this substitution carries $x\, i_1^2 + i_1\, l(\overline{-1})$ to $l(\overline{-1})\, i_1^2 + i_1\, l(\overline{-1}) = 0$. So $\varphi = 0$ in this case also; which completes the proof of 2.1.

A similar argument proves the following.

Lemma 2.2. *Choosing some fixed prime element π, there is one and only one ring homomorphism*

$$\psi\colon\ K_* F \to K_* \bar{F}$$

which carries $l(\pi^i u)$ to $l(\bar{u})$ for every unit u.

In fact ψ is defined by the rule

$$l(\pi^{i_1} u_1) \dots l(\pi^{i_n} u_n) \mapsto l(\bar{u}_1) \dots l(\bar{u}_n).$$

Details will be left to the reader. Evidently this homomorphism ψ is less natural than ∂, since it depends on a particular choice of π.

Now let F be an arbitrary field. We will use 2.1 and 2.2 to study the field $F(t)$ of rational functions in one indeterminate over F.

Each monic irreducible polynomial $\pi \in F[t]$ gives rise to a (π)-adic valuation on $F(t)$ with residue class field $F[t]/(\pi)$. Here (π) denotes the prime ideal spanned by π. Hence there is an associated surjection

$$\partial_\pi\colon\ K_n F(t) \to K_{n-1} F[t]/(\pi).$$

Theorem 2.3. *These homomorphisms ∂_π give rise to a split exact sequence*

$$0 \to K_n F \to K_n F(t) \to \oplus K_{n-1} F[t]/(\pi) \to 0,$$

where the direct sum extends over all non-zero prime ideals (π).

This theorem is essentially due to Tate. In fact the proof below is an immediate generalization of Tate's proof for the special case $n=2$.

Proof. Keeping n fixed, let $L_d \subset K_n F(t)$ be the subgroup generated by those products $l(f_1)\dots l(f_n)$ such that $f_1, \dots, f_n \in F[t]$ are polynomials of degree $\leqq d$. Thus

$$L_0 \subset L_1 \subset L_2 \subset \cdots$$

with union $K_n F(t)$. Using the homomorphism

$$\psi_\pi\colon\ K_n F(t) \to K_n F$$

of 2.2, where π is any monic (irreducible) polynomial of degree 1, we see easily that L_0 is a direct summand of $K_n F(t)$, naturally isomorphic to $K_n F$.

Let π be a monic irreducible polynomial of degree d. Then each element $\bar{g}$ of the quotient $F[t]/(\pi)$ is represented by a unique polynomial $g \in F[t]$ of degree $<d$.

Lemma 2.4. *There exists one and only one homomorphism*

$$h_\pi\colon\ K_{n-1} F[t]/(\pi) \to L_d/L_{d-1}$$

which carries each product $l(\bar{g}_2)\dots l(\bar{g}_n)$ to the residue class of $l(\pi)\, l(g_2)\dots l(g_n)$ modulo L_{d-1}.

Proof. First consider the correspondence

$$l(\bar{g}_2) \times \cdots \times l(\bar{g}_n) \mapsto l(\pi)\, l(g_2)\dots l(g_n) \bmod L_{d-1}$$

from $K_1 F[t]/(\pi) \times \cdots \times K_1 F[t]/(\pi)$ to L_d/L_{d-1}. We will show that this correspondence is linear, for example as a function of $\bar{g}_2$. Suppose that

$$g_2 \equiv g_2' g_2'' \bmod (\pi),$$

where g_2, g_2', g_2'' are polynomials of degree $<d$. Then

$$g_2 = \pi f + g_2' g_2''$$

where f is also a polynomial of degree $<d$. Hence, if $f\neq 0$,

$$1=\pi f/g_2+g'_2 g''_2/g_2$$

and therefore

$$\big(l(\pi)+l(f)-l(g_2)\big)\big(l(g'_2)+l(g''_2)-l(g_2)\big)=0.$$

Multiplying on the right by $l(g_3)\dots l(g_n)$, and then reducing modulo L_{d-1}, we obtain

$$l(\pi)\big(l(g'_2)+l(g''_2)-l(g_2)\big)\, l(g_3)\dots l(g_n)\equiv 0.$$

Since the case $f=0$ is straight forward, this proves that our correspondence is $(n-1)$-linear.

To prove that this correspondence gives rise to a homomorphism

$$l(\bar g_2)\dots l(\bar g_n)\mapsto l(\pi)\, l(g_2)\dots l(g_n)$$

from $K_{n-1}\bar F$ to L_d/L_{d-1}, it is now only necessary to note that the image is zero whenever $\bar g_j+\bar g_{j+1}=\bar 1$ and hence $g_j+g_{j+1}=1$. This proves 2.4.

Lemma 2.5. *The homomorphisms ∂_π give rise to an isomorphism between L_d/L_{d-1} and the direct sum of $K_{n-1}F[t]/(\pi)$ as π ranges over monic irreducible polynomials of degree d.*

Proof. Inspection shows that each ∂_π induces a homomorphism

$$L_d/L_{d-1}\to K_{n-1}F[t]/(\pi).$$

Furthermore it is clear that the composition

$$K_{n-1}F[t]/(\pi)\xrightarrow{h_\pi} L_d/L_{d-1}\xrightarrow{\partial_{\pi'}} K_{n-1}F[t]/(\pi')$$

is either the identity or zero, according as $\pi=\pi'$ or $\pi\neq\pi'$. So to complete the argument we need only to show that L_d/L_{d-1} is generated by the images of the h_π.

Consider any generator of L_d, expressed as a product $l(f_1)\dots l(f_s)\, l(g_{s+1})\dots l(g_n)$ where $f_1,\dots,f_s$ have degree d and $g_{s+1},\dots,g_n$ have degree $<d$. If $s\geqq 2$ then we can set

$$f_2=-af_1+g$$

with $a\in F^\bullet$ and degree $g<d$. If $g\neq 0$ it follows that

$$af_1/g+f_2/g=1$$

hence

$$\big(l(a)+l(f_1)-l(g)\big)\big(l(f_2)-l(g)\big)=0.$$

Thus the product $l(f_1)\, l(f_2)$ can be expressed as a sum of terms

$$l(f_1)\, l(g)+l(g)\, l(f_2)-l(a)\, l(f_2)+l(a)\, l(g)-l(g)^2,$$

each of which involves at most one polynomial of degree d. A similar situation obtains when $g=0$. It follows, by induction on s, that every element of L_d can be expressed, modulo L_{d-1}, in terms of products $l(f_1)\,l(g_2)\dots l(g_n)$ where only f_1 has degree d. If f_1 is irreducible, then setting $f=a\pi$ this product evidently belongs to the image of h_π. But if f_1 is reducible then the product is congruent to zero modulo L_{d-1}. Thus L_d/L_{d-1} is generated by the images of the homomorphisms h_π, which completes the proof of 2.5.

An easy induction on d now shows that the homomorphisms ∂_π induce an isomorphism from L_d/L_0 to the direct sum of $K_{n-1}F[t]/(\pi)$, taken over all monic irreducible π of degree $\leqq d$. Passing to the direct limit as $d\to\infty$, this completes the proof of Theorem 2.3.

To conclude this section, let us record a similar, but easier statement.

Lemma 2.6. *Suppose that a field E is complete under a discrete valuation with residue class field $\bar{E}=F$. Then for any prime p distinct from the characteristic of $\bar{E}$ there is a natural split exact sequence*

$$0\to K_nF/pK_nF\to K_nE/pK_nE\xrightarrow{\partial} K_{n-1}F/pK_{n-1}F\to 0.$$

Proof. If a unit of E maps to 1 in F, then it has a p-th root. Hence the correspondence $l(\bar{u})\mapsto l(u) \bmod pK_1E$ is well defined. This correspondence extends to a ring homomorphism

$$K_*F\to K_*E/pK_*E.$$

Further details will be left to the reader.

§ 3. The Stiefel-Whitney Invariants of a Quadratic Module

For the rest of this paper we will only be interested in the quotient of the ring K_*F by the ideal $2K_*F$. To simplify the notation, let us set

$$k_nF=K_nF/2K_nF.$$

Thus k_*F is a graded algebra over $Z/2Z$, with $k_1F\cong F^\bullet/F^{\bullet 2}$. *We will always assume that F has characteristic* $\neq 2$.

The symbol $k_\Pi F$ will stand for the algebra consisting of all formal series $\xi_0+\xi_1+\xi_2+\cdots$ with $\xi_i\in k_iF$. Thus $k_\Pi F$ is additively isomorphic to the cartesian product $k_0F\times k_1F\times k_2F\times\cdots$.

Let M be a *quadratic module* over F. That is M is a finite dimensional vector space with a non-degenerate symmetric bilinear inner product. Then M is isomorphic to an orthogonal direct sum $\langle a_1\rangle\oplus\cdots\oplus\langle a_r\rangle$ of one dimensional modules. Here $\langle a\rangle$ denotes the one dimensional quadratic module such that the inner product of a suitable basis vector with itself is a.

Define the *Stiefel-Whitney invariant*

$$w(M) \in k_{\Pi} F$$

of a quadratic module $M \cong \langle a_1 \rangle \oplus \cdots \oplus \langle a_r \rangle$ by the formula

$$w(M) = (1 + l(a_1))(1 + l(a_2)) \dots (1 + l(a_r)).$$

Thus $w(M)$ can be written as

$$1 + w_1(M) + \cdots + w_r(M)$$

where $w_i(M)$, the i-th *Stiefel-Whitney invariant*, is equal to the i-th elementary symmetric function of $l(a_1), \dots, l(a_r)$ considered as an element of $k_i F$.

Evidently w_1 is just the classical "discriminant" of M, and w_2 is closely related to the classical Hasse-Witt invariant.

Remark. This definition is very similar to the definition proposed by Delzant [5]. However Delzant's Stiefel-Whitney classes belong to the cohomology $H^*(G_F; Z/2Z)$ of the maximal Galois extension of F. They are precisely the images of our w_i under a canonical homomorphism

$$k_* F \to H^*(G_F; Z/2Z)$$

which is described in § 6.

Lemma 3.1. *The invariant* $w(M)$ *is a well defined unit in the ring* $k_{\Pi} F$ *and satisfies the Whitney sum formula*

$$w(M \oplus N) = w(M)\, w(N).$$

Proof. Just as in the classical proof that the Hasse-Witt invariant is well defined, it suffices to consider the rank 2 case. (Compare O'Meara [12, p. 150].) Suppose then that

$$\langle a \rangle \oplus \langle b \rangle \cong \langle \alpha \rangle \oplus \langle \beta \rangle.$$

Then the discriminant ab must be equal to $\alpha\beta$ multiplied by a square; or in other words

$$l(a) + l(b) \equiv l(\alpha) + l(\beta) \mod 2K_1 F. \tag{4}$$

Furthermore, the equation $\alpha = a x^2 + b y^2$ must have a solution $x, y \in F$. Since the case $x = 0$ or $y = 0$ is easily disposed of, we may assume that $x \neq 0$, $y \neq 0$. Then the equation

$$1 = a x^2/\alpha + b y^2/\alpha$$

implies that

$$0 = (l(a) + 2l(x) - l(\alpha))(l(b) + 2l(y) - l(\alpha))$$
$$\equiv (l(a) - l(\alpha))(l(b) - l(\alpha)) \mod 2K_2 F.$$

Rearranging terms, and then substituting (4) this implies that

$$\begin{aligned} l(a)\,l(b) &\equiv l(\alpha)\big(l(a)+l(b)-l(\alpha)\big) \\ &\equiv l(\alpha)\,l(\beta) \quad \bmod 2K_2F; \end{aligned}$$

which completes the proof.

Remark. Delzant shows that a quadratic module over a number field is determined up to isomorphism by its rank and Stiefel-Whitney cohomology classes. But Scharlau points out that the corresponding statement for an arbitrary field is false. The same statements, proofs, and examples apply to our Stiefel-Whitney invariants.

Now let us introduce the *Witt-Grothendieck ring* $\hat{W}F$, consisting of all formal differences $M-N$ of quadratic modules over F; where $M-N$ equals $M'-N'$ if and only if the orthogonal direct sum $M\oplus N'$ is isomorphic to $M'\oplus N$. (Compare [5, 14].) The product operation in $\hat{W}F$ is characterized by the identity

$$\langle a\rangle\langle b\rangle=\langle a\,b\rangle.$$

The *augmentation ideal*, consisting of all $M-N$ in $\hat{W}F$ with $\operatorname{rank} M=\operatorname{rank} N$, will be denoted by $\hat{I}F$, and its n-th power by $\hat{I}^n F$.

Evidently the function w extends uniquely to a homomorphism from the additive group of $\hat{W}F$ to the multiplicative group of units in $k_\Pi F$; where

$$w(M-N)=w(M)/w(N)$$

by definition.

Next consider a generator

$$\xi=(\langle a_1\rangle-\langle 1\rangle)(\langle a_2\rangle-\langle 1\rangle)\dots(\langle a_n\rangle-\langle 1\rangle) \tag{5}$$

of the ideal $\hat{I}^n F$. Let $t=2^{n-1}$.

Lemma 3.2. *The Stiefel-Whitney invariant w of such a product ξ is equal to either*

$$1+l(a_1)\dots l(a_n)\,l(-1)^{t-n}$$

or

$$\big(1+l(a_1)\dots l(a_n)\,l(-1)^{t-n}\big)^{-1}$$

according as n is odd or even.

Proof. Multiplying out the formula (5), we obtain

$$\xi=\sum\pm\langle a_1^{\varepsilon_1}\dots a_n^{\varepsilon_n}\rangle,$$

to be summed as $\varepsilon_1,\dots,\varepsilon_n$ range over 0 and 1. Here and subsequently, $\pm$ stands for the sign $(-1)^{\varepsilon_1+\dots+\varepsilon_n+n}$. Therefore

$$w(\xi)=\prod\big(1+\varepsilon_1\,l(a_1)+\dots+\varepsilon_n\,l(a_n)\big)^{\pm1}.$$

Consider the corresponding product

$$(6) \qquad \prod (1+\varepsilon_1 x_1+\cdots+\varepsilon_n x_n)^{\pm 1}$$

in the ring of formal power series with mod 2 coefficients in n indeterminates. If we substitute 0 for some x_i, then evidently this product becomes 1. Hence the product (6) must be equal to

$$1+x_1 \dots x_n f(x_1, \dots, x_n)$$

for some formal power series f. Therefore

$$\begin{aligned} w(\xi) &= 1+l(a_1)\dots l(a_n) f(l(a_1), \dots, l(a_n)) \\ &= 1+l(a_1)\dots l(a_n) f(l(-1), \dots, l(-1)); \end{aligned}$$

using § 1.2.

To compute the power series $f(l(-1), \dots, l(-1))$ it suffices to substitute $x_1=\cdots=x_n=x$ in (6), so as to compute $f(x, \dots, x)$. Evidently the product reduces to either $(1+x)^t$ or $(1+x)^{-t}$ according as n is odd or even; where $t=2^{n-1}$. For n odd it follows that

$$1+x^n f(x, \dots, x)=(1+x)^t=1+x^t,$$

so that

$$f(x, \dots, x)=x^{t-n};$$

and a similar computation can be carried out for n even. This completes the proof.

Corollary 3.3 *If $t=2^{n-1}$, then the invariants $w_1, \dots w_{t-1}$ annihilate the ideal $\hat{I}^n F$, while w_t induces a homomorphism*

$$w_t\colon \hat{I}^n F/\hat{I}^{n+1} F \to k_t F$$

which carries the product

$$(\langle a_1\rangle-\langle 1\rangle)\dots(\langle a_n\rangle-\langle 1\rangle)$$

to $l(a_1)\dots l(a_n)\, l(-1)^{t-n}$.

Proof. Since the elements $\langle a\rangle-\langle 1\rangle$ form an additive set of generators for $\hat{I}F$, it is clear that the n-fold products of such elements generate $\hat{I}^n F$. The conclusion now follows immediately.

Remark 3.4. These formulas suggest that the Stiefel-Whitney invariants are not independent of each other. In fact the following is true: *If n is not a power of 2, then $w_n(M)$ can be expressed as a product $w_r(M)\, w_{n-r}(M)$ where r is the highest power of 2 dividing n.*

(Compare also [14, § 2.2.2].) The proof can be outlined as follows. Interpreting w_s as an elementary symmetric function, and using § 1.2, it is not difficult to show that

$$w_r w_s = \sum (i, r-i, s-i)\, w_{r+s-i}\, l(-1)^i,$$

to be summed over $0 \leqq i \leqq \mathrm{Min}\{r, s\}$. Here (i, j, k) stands for the trinomial coefficient $(i+j+k)!/i!\,j!\,k!$. But if r is a power of 2, and if

$$s \equiv 0 \bmod 2r,$$

then this identity takes the simple form

$$w_r w_s = w_{r+s}$$

which completes the outlined proof.

§4. The Surjection $K_n/2K_n \to I^n/I^{n+1}$

Let F be a field of characteristic $\neq 2$. The Witt ring $W = WF$ can be defined as the quotient $\hat{W}/H$, where $\hat{W}$ is the Witt-Grothendieck ring of §3, and H is the free cyclic additive group spanned by $\langle 1 \rangle \oplus \langle -1 \rangle$. Clearly H is an ideal, so that W is a ring. Note that the augmentation ideal $\hat{I}$ in $\hat{W}$ maps bijectively to a maximal ideal in W. This image ideal will be denoted by $I = IF$.

(Remark. The utility of working with W, rather than $\hat{W}$, will become apparent only in §5.)

As in §3, we set $k_n F = K_n F/2K_n F$. This will sometimes be abbreviated as $k_n = K_n/2K_n$.

Theorem 4.1. *There is one and only one homomorphism*

$$s_n\colon\ k_n F \to I^n F/I^{n+1} F$$

which carries each product $l(a_1) \ldots l(a_n)$ in $k_n F$ to the product

$$(\langle a_1 \rangle - \langle 1 \rangle) \ldots (\langle a_n \rangle - \langle 1 \rangle)$$

modulo $I^{n+1} F$. The homomorphisms s_1 and s_2 are bijective (*compare* [13]); *and every s_n is surjective.*

Proof. The correspondence

$$l(a_1) \times \cdots \times l(a_n) \mapsto \prod (\langle a_i \rangle - \langle 1 \rangle) \quad \bmod I^{n+1}$$

from $K_1 \times \cdots \times K_1$ to I^n/I^{n+1} is n-linear since

$$\langle a \rangle - \langle 1 \rangle + \langle b \rangle - \langle 1 \rangle \equiv \langle a\,b \rangle - \langle 1 \rangle \quad \bmod I^2.$$

Furthermore, if $a_i + a_{i+1} = 1$ then an easy computation shows that

$$(\langle a_i \rangle - \langle 1 \rangle)(\langle a_{i+1} \rangle - \langle 1 \rangle) = 0,$$

so the image is zero. Thus this correspondence gives rise to a homomorphism $K_n \to I^n/I^{n+1}$. This homomorphism annihilates $2K_n$ since

$$2\,l(a_1)\dots l(a_n) = l(a_1^2)\,l(a_2)\dots l(a_n)$$

with

$$\langle a_1^2\rangle - \langle 1\rangle = 0.$$

Thus we have shown that the homomorphism

$$s_n\colon\ K_n/2K_n \to I^n/I^{n+1}$$

exists and is well defined. This homomorphism is clearly surjective, since the elements $\langle a\rangle - \langle 1\rangle$ form an additive set of generators for the ideal I.

Now let $t = 2^{n-1}$, and consider the homomorphism

$$w_t\colon\ I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1} \to k_t$$

of § 3.3. Evidently the composition $w_t \circ s_n$ is just multiplication by $l(-1)^{t-n}$.

But if n equal 1 or 2, then $t = n$, and the appropriate statement is that $w_n \circ s_n$ is the identity. This shows that s_1 and s_2 are bijective; which completes the proof of 4.1.

Remark 4.2. For $n > 2$, this argument proves the following: *If multiplication by $l(-1)^{t-n}$ carries $k_n F$ injectively into $k_t F$, then the homomorphism*

$$s_n\colon\ k_n F \to I^n/I^{n+1}$$

is necessarily bijective.

Evidently there are two key questions in relating k_* to the Witt ring W. Let F be any field of characteristic $\neq 2$.

Question 4.3. *Is the homomorphism $s_n\colon k_n F \to I^n/I^{n+1}$ bijective for all values of n?*

Question 4.4. *Is the intersection of the ideals I^n equal to zero?* (Compare [13, 14].)

This section will conclude by proving two preliminary results. (See also §§ 5.2 and 5.8.)

Lemma 4.5. *If F is a global field, or a direct limit of global fields, then both questions have affirmative answers.*

Proof. Using Tate's explicit computation of $k_* F$ for a global field (§ 1.8 or the Appendix), we see that multiplication by $l(-1)$ induces isomorphisms

$$k_3 F \to k_4 F \to k_5 F \to \cdots.$$

Together with § 4.2, this proves that s_n is bijective in the case of a global field. The corresponding statement for a direct limit follows immediately.

As to the intersection of the ideals I^n, first note that each embedding of F in the real field gives rise to a ring homomorphism

$$WF \to WR \cong Z$$

called the *signature*. Note that an element of $I^3 F$ is zero if and only if its signature at every embedding $F \to R$ is zero. In the case of a global field, this statement follows immediately from the Hasse-Minkowski theorem; and for the direct limit of a sequence

$$F_1 \subset F_2 \subset F_3 \subset \cdots$$

of global fields it follows easily using the isomorphisms

$$W \varinjlim F_\alpha = \varinjlim WF_\alpha$$

and

$$\mathrm{Emb}(\varinjlim F_\alpha, R) = \varprojlim \mathrm{Emb}(F_\alpha, R).$$

But each such signature carries the ideal IF to $2Z$, and hence carries the intersection of the ideals $I^n F$ to $\bigcap 2^n Z = 0$. This completes the proof.

Lemma 4.6. *Now suppose that F is a field such that $k_2 F$ has at most two distinct elements. Then again the s_n are bijective and $\bigcap I^n = 0$.*

Notice that this includes the case of a finite, or local, or real closed, or quadratically closed field; as well as any direct limit of such fields.

Proof. If k_1, modulo the null-space of the pairing $k_1 \otimes k_1 \to k_2$, has dimension $\neq 1$, then Kaplansky and Shaker [6] show that a quadratic module is completely determined by its rank, discriminant, and Hasse-Witt invariant. It follows that $I^3 = 0$. But just as in § 1.7 one sees that $k_3 = 0$. Since s_1 and s_2 are already known to be bijective, it certainly follows that every s_n is bijective.

On the other hand if k_1 modulo this null-space has dimension 1, then it is easy to define the "signature" of a quadratic module, and to show that the rank, discriminant, and signature form a complete invariant. (Compare [6, Lemma 1].) Since the signature of an element in I^n is divisible by 2^n, it follows that $\bigcap I^n = 0$. Futhermore, techniques similar to those of § 1.4 show that k_n is cyclic of order 2, generated by $l(-1)^n$, for every $n \geqq 2$; hence § 4.2 implies that every s_n is bijective. This completes the proof.

§ 5. The Witt Ring of a Rational Function Field

This section will study the Witt ring, using constructions very similar to those of § 2.

First consider a field E which is complete under a discrete valuation v, with residue class field $\bar{E}$ of characteristic $\neq 2$. Let π be a prime element.

Theorem of Springer. *The Witt ring WE contains a subring W_0 canonically isomorphic to $W\bar{E}$. Furthermore WE, splits additively as the direct sum of W_0 and $\langle\pi\rangle W_0$.*

In fact W_0 can be defined as the subring generated by $\langle u\rangle$ as u ranges over units of E, and the isomorphism $W_0 \to W\bar{E}$ is defined by the correspondence $\langle u\rangle \mapsto \langle\bar{u}\rangle$.

For the proof, see T.A. Springer [16]. Since $\langle\pi\rangle^2 = \langle 1\rangle$, it follows that the ring WE is completely determined by $W\bar{E}$.

Corollary 5.1. *There is a split exact sequence*

$$0 \to W\bar{E} \to WE \xrightarrow{\partial} W\bar{E} \to 0,$$

where the first homomorphism carries $\langle\bar{u}\rangle$ to $\langle u\rangle$, and where ∂ is defined by the conditions

$$\partial\langle u\rangle = 0, \qquad \partial\langle\pi u\rangle = \langle\bar{u}\rangle.$$

Note however that ∂ depends on the particular choice of the prime element π.

The proof is straightforward.

Corollary 5.2. *If the questions* 4.3 *and* 4.4 *have affirmative answers for the residue class field $\bar{E}$, then they also have affirmative answers for E.*

Proof. It will be convenient to identify $W\bar{E}$ with the sub-ring $W_0 \subset WE$. Note that the ideal IE then splits as a direct sum

$$IE = I\bar{E} \oplus (\langle\pi\rangle - \langle 1\rangle)\, W\bar{E}.$$

It follows inductively that

$$I^n E = I^n \bar{E} \oplus (\langle\pi\rangle - \langle 1\rangle)\, I^{n-1}\bar{E}.$$

Hence the sequence 5.1 gives rise to a split exact sequence

$$(7_n) \qquad 0 \to I^n\bar{E} \to I^n E \to I^{n-1}\bar{E} \to 0.$$

Consider the diagram

$$\begin{array}{ccccc} k_n\bar{E} & \longrightarrow & k_n E & \longrightarrow & k_{n-1}\bar{E} \\ \downarrow & & \downarrow & & \downarrow \\ I^n\bar{E}/I^{n+1}\bar{E} & \to & I^n E/I^{n+1}E & \to & I^{n-1}\bar{E}/I^n\bar{E}, \end{array}$$

where the top sequence comes from §2.6, the vertical arrows from §4.1, and the bottom sequence is the quotient of (7_n) by (7_{n+1}). Checking that this diagram is commutative, and then applying the Five Lemma, the conclusion follows.

Now consider a field $E=F(t)$ of rational functions. For each monic irreducible $\pi\in F[t]$ we can form the π-adic completion E_π, with residue class field

$$\bar{E}_\pi \cong F[t]/(\pi).$$

Let

$$\partial_\pi\colon\ WE\to W\bar{E}_\pi$$

denote the composition of the natural map $WE\to WE_\pi$ with the homomorphism ∂ of 5.1. Evidently $\partial_\pi\langle u\rangle=0$ and $\partial_\pi\langle\pi u\rangle=\langle\bar{u}\rangle$.

Theorem 5.3. *These homomorphisms ∂_π give rise to a split exact sequence*

$$0\to WF\to WE\to \bigoplus W\bar{E}_\pi\to 0,$$

where $E=F(t)$, and where the summation extends over all monic irreducible polynomials π in $F[t]$.

The proof will be based on the Tate technique already utilized in §2.3. Let $L_d\subset WE$ denote the subring generated by all $\langle f\rangle$ such that $f\in F[t]$ is a polynomial of degree $\leqq d$. Thus

$$L_0\subset L_1\subset L_2\subset\cdots$$

with union WE. Additively, L_d is generated by all products $\langle f_1\ldots f_s\rangle$ where the f_i are polynomials of degree $\leqq d$.

Note that L_0 is just the image of the natural homomorphism $WF\to WE$.

Lemma 5.4. *In fact WF maps bijectively to L_0. Furthermore L_0 is a retract of WE under a ring homomorphism*

$$\rho\colon\ WE\to WF\cong L_0.$$

Proof. Choose some monic polynomial π of degree 1, and define ρ by the conditions

$$\rho\langle u\rangle=\langle\bar{u}\rangle,\qquad \rho\langle\pi u\rangle=\langle\bar{u}\rangle.$$

Here u denotes any unit with respect to the (π)-adic valuation. It follows from Springer's theorem, applied to the (π)-adic completion, that ρ is a well defined ring homomorphism. Since the composition

$$WF\to WE\xrightarrow{\ \rho\ } WF$$

is the identity, this proves 5.4.

Now suppose that $d\geqq 1$.

Lemma 5.5. *The additive group L_d is generated, modulo L_{d-1}, by expressions $\langle\pi g_1\ldots g_s\rangle$ where π is an irreducible polynomial of degree d, and $g_1,\ldots,g_s$ are polynomials of degree $<d$. Furthermore if f is the poly-*

nomial of degree $<d$ *defined by*

$$f \equiv g_1 \dots g_s \qquad \bmod (\pi),$$

then

$$\langle \pi f \rangle \equiv \langle \pi\, g_1 \dots g_s \rangle \qquad \bmod L_{d-1}.$$

Proof. First note that the identity

$$\langle a+b \rangle = \langle a \rangle + \langle b \rangle - \langle a\, b(a+b) \rangle \tag{8}$$

holds in the Witt ring of any field.

Consider a generator $\langle f_1 \dots f_r g_1 \dots g_s \rangle$ of L_d, where the polynomials $f_1, \dots, f_r$ are distinct, monic of degree d, and where $g_1, \dots, g_s$ have degree $<d$. If $r \geqq 2$, then defining a polynomial h of degree $<d$ by

$$f_1 = f_2 + h,$$

the identity (8) becomes

$$\langle f_1 \rangle = \langle f_2 \rangle + \langle h \rangle - \langle f_1 f_2 h \rangle.$$

Multiplying by $\langle f_2 \dots f_r g_1 \dots g_s \rangle$ and cancelling all squared factors, it follows that $\langle f_1 \dots f_r g_1 \dots g_s \rangle$ is equal to

$$\langle f_3 \dots g_s \rangle + \langle h f_2 \dots g_s \rangle - \langle f_1\, h f_3 \dots g_s \rangle.$$

Since each of these terms has at most $r-1$ factors of degree d, it follows by induction on r that L_d is generated, modulo L_{d-1}, by expressions $\langle f g_1 \dots g_s \rangle$ where f is monic of degree d and the g_i have degree $<d$. We may clearly assume that f is irreducible.

Consider then such a generator $\langle \pi\, g_1 \dots g_s \rangle$ with π monic and irreducible. Setting

$$g_1 g_2 \equiv h \qquad \bmod (\pi)$$

with degree $h<d$, we have

$$g_1 g_2 = h + \pi\, k$$

for some k of degree $<d$, hence

$$\langle g_1 g_2 \rangle = \langle h \rangle + \langle \pi\, k \rangle - \langle \pi\, k\, h\, g_1 g_2 \rangle.$$

Multiplying by $\langle \pi\, g_3 \dots g_s \rangle$, this shows that $\langle \pi\, g_1 \dots g_s \rangle$ is equal to

$$\langle \pi\, h\, g_3 \dots g_s \rangle + \langle k\, g_3 \dots g_s \rangle - \langle k\, h\, g_1 \dots g_s \rangle \equiv \langle \pi\, h\, g_3 \dots g_s \rangle \qquad \bmod L_{d-1}.$$

An easy induction now completes the proof of 5.5.

Now consider the field $\bar{E}_\pi = F[t]/(\pi)$, where π is monic irreducible of degree d. For each residue class $\bar{f}$ modulo (π), let f denote the unique polynomial of degree $<d$ representing $\bar{f}$.

Lemma 5.6. *The correspondence*

$$\langle \bar{f} \rangle \mapsto \langle \pi f \rangle \qquad \mathrm{mod}\, L_{d-1}$$

gives rise to a homomorphism from $W\bar{E}_\pi$ *to* L_d/L_{d-1}.

Proof. For any field F of characteristic $\neq 2$ it is not difficult to show that the additive group of WF has a presentation in terms of generators $\langle a \rangle$, where a ranges over $F^\bullet$, subject only to the relations

$$\langle a\, b^2 \rangle = \langle a \rangle,$$
$$\langle a+b \rangle = \langle a \rangle + \langle b \rangle - \langle a\, b(a+b) \rangle,$$
$$\langle 1 \rangle + \langle -1 \rangle = 0,$$

and their consequences. But, substituting $\bar{E}_\pi$ for F, each such relation in $W\bar{E}_\pi$ maps to a valid relation in L_d/L_{d-1}. Thus if

$$f g(f+g) \equiv h \qquad \mathrm{mod}\,(\pi),$$

where f, g, h are non-zero polynomials of degree $<d$, then the relation

$$\langle \bar{f} + \bar{g} \rangle = \langle \bar{f} \rangle + \langle \bar{g} \rangle - \langle \bar{h} \rangle$$

in $W\bar{E}_\pi$ corresponds to the relation

$$\begin{aligned} \langle \pi(f+g) \rangle &= \langle \pi f \rangle + \langle \pi\, g \rangle - \langle \pi f g(f+g) \rangle \\ &\equiv \langle \pi f \rangle + \langle \pi\, g \rangle - \langle \pi\, h \rangle \qquad \mathrm{mod}\, L_{d-1}; \end{aligned}$$

making use of Lemma 5.5. Similarly, if $f g^2 \equiv k$ mod (π), then the relation $\langle \bar{f} \rangle = \langle \bar{k} \rangle$ corresponds to $\langle \pi f \rangle = \langle \pi f g^2 \rangle \equiv \langle \pi\, k \rangle$. Finally, the relation $\langle 1 \rangle + \langle -1 \rangle = 0$ corresponds to $\langle \pi \rangle + \langle -\pi \rangle = 0$. So it follows that the correspondence $\langle \bar{f} \rangle \mapsto \langle \pi f \rangle$ mod L_{d-1} does indeed define a homomorphism from $W\bar{E}_\pi$ to L_d/L_{d-1}. This proves 5.6.

Proof of Theorem 5.3. The argument is very similar to that in §2.5. First one checks that the composition

$$W\bar{E}_\pi \to L_d/L_{d-1} \xrightarrow{\partial_{\pi'}} W\bar{E}_{\pi'}$$

is either the identity or zero according as $\pi = \pi'$ or $\pi \neq \pi'$. Using 5.5, it follows that L_d/L_{d-1} splits canonically as the direct sum of those $W\bar{E}_\pi$ for which degree $\pi = d$.

Now induction on d shows that the homomorphisms ∂_π give rise to an isomorphism

$$L_d/L_0 \to \bigoplus_{\text{degree}\, \pi \leq d} W\bar{E}_\pi .$$

Passing to the direct limit as $d \to \infty$, this completes the proof of 5.3.

Remark. More generally, suppose that E is a finite extension field of $F(t)$. Every valuation v of E which is trivial on F gives rise to a homomorphism

$$\partial_v\colon WE \to W\bar{E}_v,$$

well defined up to multiplication by a unit of the form $\langle \bar{e} \rangle$. It would be very interesting to know something about the kernel and cokernel of the associated homomorphism

$$WE \to \bigoplus W\bar{E}_v.$$

For the special case $E = F(t)$, both kernel and cokernel turn out to be isomorphic to WF.

Perhaps one may find some clue by applying the analogous construction to a global field. As an example, for the field Q of rationals, there is an additive isomorphism

$$WQ \to Z \oplus (Z/2Z) \oplus \bigoplus_{p\,\mathrm{odd}} W(Z/pZ),$$

using the signature and the correspondence

$$\langle q \rangle \mapsto \mathrm{ord}_2\, q \quad \bmod 2$$

to map to the first two summands, and using the homomorphisms ∂_p for the third.

Now let us bring the multiplicative structure of W into Theorem 5.3. Again let $E = F(t)$.

Lemma 5.7. *The sequence* 5.3 *gives rise to an exact sequence*

$$0 \to I^n F \to I^n E \to \bigoplus I^{n-1} \bar{E}_\pi \to 0$$

for any $n \geqq 1$.

Proof. The proof of 5.2 shows that each ∂_π maps $I^n E$ to $I^{n-1}\bar{E}_\pi$. Consider any generator

$$\eta = (\langle \bar{f}_2 \rangle - \langle \bar{1} \rangle) \dots (\langle \bar{f}_n \rangle - \langle \bar{1} \rangle)$$

of $I^{n-1}\bar{E}_\pi$. Let degree $\pi = d$. Then the product

$$\xi = (\langle \pi \rangle - \langle 1 \rangle)(\langle f_2 \rangle - \langle 1 \rangle) \dots (\langle f_n \rangle - \langle 1 \rangle)$$

in $I^n E$, where each representative f_i has degree $< d$, satisfies $\partial_\pi \xi = \eta$, and satisfies $\partial_{\pi'} \xi = 0$ for every $\pi' \neq \pi$ with degree $\pi' \geqq d$.

Now, given any element (η_π) of $\bigoplus I^{n-1}\bar{E}_\pi$, let

$$d_0 = \mathrm{Max}\,\{\text{degree}\ \pi \mid \eta_\pi \neq 0\}.$$

Then it follows by induction on d_0 that (η_π) is the image of some element in $I^n E$.

To prove exactness in the middle of the sequence 5.7, consider any $\xi \in I^n E$ which maps to zero in $\oplus I^{n-1} \bar{E}_\pi$. According to 5.3, ξ comes from some element ζ of WF. Now apply the homomorphism ρ of §5.4. Evidently ρ maps $I^n E$ into $I^n F$, and evidently $\rho(\xi)=\zeta$. This proves that $\zeta \in I^n F$; which completes the proof of 5.7.

Corollary 5.8. *If the questions* 4.3 *and* 4.4 *have affirmative answers for every finite extension* $\bar{E}_\pi$ *of a field* F, *then they have affirmative answers for the field* $E=F(t)$ *of rational functions.*

The proof is completely analogous to that of 5.2.

§6. Relations with Galois Cohomology

The following construction is due to Bass and Tate. For any field F of characteristic $\neq 2$, let F_s be a separable closure, and let $G=G_F$ be the Galois group of F_s over F. Then the exact sequence

$$1 \to \{\pm 1\} \to F_s^\bullet \xrightarrow{2} F_s^\bullet \to 1,$$

upon which G operates, leads to an exact sequence

$$H^0(G; F_s^\bullet) \xrightarrow{2} H^0(G; F_s^\bullet) \to H^1(G; \{\pm 1\}) \to H^1(G; F_s^\bullet)$$

of cohomology groups; where the right hand group is zero. Identifying the first two groups with $F^\bullet$, and substituting $Z/2Z$ for $\{\pm 1\}$, this yields

$$F^\bullet \xrightarrow{2} F^\bullet \xrightarrow{\delta} H^1(G; Z/2Z) \to 0.$$

The quotient $F^\bullet/F^{\bullet 2}$ can of course be identified with $k_1 F$.

Lemma 6.1 (Bass, Tate). *The isomorphism* $l(a) \mapsto \delta(a)$ *from* $k_1 F$ *to* $H^1(G; Z/2Z)$ *extends uniquely to a ring homomorphism*

$$h_F\colon\ k_* F \to H^*(G; Z/2Z).$$

Proof. It is only necessary to verify that each of the defining relations $l(a)\, l(1-a)=0$ for the ring $k_* F$ maps to a valid relation

$$\delta(a)\, \delta(1-a)=0$$

in $H^2(G; Z/2Z)$. But in fact, if we identify $H^2(G; Z/2Z)$ with the set of elements of order 2 in the Brauer group $H^2(G; F_s^\bullet)$, then $\delta(a)\, \delta(b)$ corresponds to the quaternion algebra associated with a, b. (Compare Delzant [5].) Since the quaternion algebra associated with a, $1-a$ splits, the relation $\delta(a)\, \delta(1-a)=0$ follows.

Remark. Bass and Tate [3] also consider the more general homomorphism associated with the sequence

$$1 \to \{m\text{-th roots of } 1\} \to F_s^\bullet \xrightarrow{m} F_s^\bullet \to 1,$$

but we will only be interested in the case $m=2$.

I do not know of any examples for which the homomorphism $h=h_F$ fails to be bijective. Here is a list of special cases.

Lemma 6.2. *If the field F is finite, or local, or global, or real closed, then the homomorphism*

$$h_F\colon k_* F \to H^*(G; Z/2Z)$$

is bijective. Furthermore if F is the direct limit of subfields F_α, and if each h_{F_α} is bijective, then h_F is bijective.

Proof. The finite, local, and real closed cases are straightforward. (Compare §1, together with Serre [15, II, pp. 10–20].) Suppose then that F is a global field. Bass and Tate [3] prove that the homomorphism

$$h_2\colon k_2 F \to H^2(G; Z/2Z)$$

is bijective. But for $n \geqq 3$ the group $H^n(G; Z/2Z)$ has been completely described by Tate [19, §3.1]. Comparing with Tate's computation of $k_n F$, as described in §1.8 or the Appendix, it follows that h_n is bijective also.

Finally, the statement for direct limits follows easily from [15, I, p. 9]. This completes the proof.

Here is one final partial result. Let $F((t))$ be the field of formal power series in one variable over F.

Theorem 6.3. *If h_F is bijective, then $h_{F((t))}$ is bijective.*

Proof. We will concentrate on the characteristic p case, leaving characteristic zero to the reader. Recall that $p \neq 2$.

Let V be the maximal tamely ramified extension of $F((t))$. (Compare Artin [1, pp. 70, 81].) Then V can be obtained from $F_s((t))$ by adjoining $t^{1/r}$ for every integer r prime to p. The Galois group G_V is a pro-p-group; and the quotient $G_{F_s((t))}/G_V$, which we denote briefly by $G_{V/F_s((t))}$, is isomorphic to $\varprojlim (Z/rZ)$, taking the inverse limit over integers r prime to p. Hence the mod 2 cohomology group

$$H^n G_{F_s((t))} \cong H^n G_{V/F_s((t))}$$

is cyclic of order 2 for $n=0, 1$, and is zero otherwise.

Clearly there is an exact sequence

$$1 \to G_{F_s((t))} \to G_{F((t))} \to G_F \to 1.$$

Dividing the first two groups by G_V, we obtain a sequence

$$1 \to G_{V/F_s((t))} \to G_{V/F((t))} \to G_F \to 1$$

which is actually split exact, since each automorphism of F_s over F lifts uniquely to an automorphism of V which keeps each $t^{1/r}$ fixed.

The associated cohomology spectral sequence now gives rise to a split exact sequence

$$0 \to H^n G_F \to H^n G_{V/F((t))} \to H^{n-1} G_F \to 0.$$

Note that the middle group is canonically isomorphic to $H^n G_{F((t))}$.

With a little work one can check that the homomorphism $H^n G_{F((t))} \to H^{n-1} G_F$ carries each product $\delta(t)\,\delta(u_2)\ldots\delta(u_n)$ to $\delta(\bar{u}_2)\ldots\delta(\bar{u}_n)$. Hence the following diagram is commutative:

$$\begin{array}{ccccc} k_n F & \longrightarrow & k_n F((t)) & \longrightarrow & k_{n-1} F \\ \downarrow & & \downarrow & & \downarrow \\ H^n G_F & \longrightarrow & H^n G_{F((t))} & \longrightarrow & H^{n-1} G_F. \end{array}$$

(Compare § 2.6.) Applying the Five Lemma, the conclusion 6.3 follows.

Appendix: $K_*/2K_*$ for a Global Field

The arguments in this appendix are due to Tate.

Let F be a global field of characteristic $\neq 2$. We will again use the abbreviation $k_* F$ for the algebra $K_* F/2 K_* F$.

The group $k_2 F$ has been computed by Bass and Tate as follows.

Lemma A.1. *There is an exact sequence*

$$0 \to k_2 F \to \oplus\, k_2 F_v \to Z/2Z \to 0,$$

where the summation extends over all completions F_v of F. Here the homomorphism $k_2 F \to k_2 F_v$ is induced by inclusion, and the homomorphism $k_2 F_v \to Z/2Z$ is injective.

In fact we recall from § 1 that the group $k_2 F_v$ is cyclic of order 2, unless F_v is the complex field in which case $k_2 F_v$ is clearly zero. The composition

$$k_2 F \to k_2 F_v \subset Z/2Z$$

evidently carries each generator $l(a)\,l(b)$ of $k_2 F$ to either 0 or 1 according as the quadratic Hilbert symbol $(a, b)_v$ is trivial or not.

For the proof, we refer to Bass and Tate [3]. (Alternatively, this lemma can be proved by comparing the isomorphism $k_2 F \cong I^2/I^3$ of § 4.1 with standard descriptions of the Witt ring of a global field.)

Theorem A.2 (Tate). *For $n \geqq 3$ the natural homomorphism*

$$k_n F \to \oplus k_n F_v$$

is an isomorphism.

Here the group $k_n F_v$ is cyclic of order 2 if F_v is the real field, and is zero otherwise. So it follows that the groups

$$k_3 F \cong k_4 F \cong k_5 F \cong \cdots$$

are finite, and in fact are zero unless F has a real completion.

To prove A.2, first consider any homomorphism Φ from $k_n F$ to the multiplicative group $\{\pm 1\}$. The image $\Phi(l(a_1) \dots l(a_n))$ will be denoted briefly by $\varphi(a_1, \dots, a_n)$. Thus φ is a symmetric function of n variables, multiplicative in each variable, and $\varphi(a_1, \dots, a_n) = 1$ whenever $a_1 + a_2 = 1$.

If $n = 2$, then it follows from A.1 that any such function $\varphi(a, b)$ can be expressed in terms of the Hilbert symbols $(a, b)_v$ as a product

$$\varphi(a, b) = \prod_v (a, b)_v^{\varepsilon_v},$$

where each exponent ε_v is 0 or 1. These exponents are well defined except that we may simultaneously replace each ε_v by $1 - \varepsilon_v$.

Now suppose that $n = 3$. For fixed c the correspondence

$$a, b \mapsto \varphi(a, b, c)$$

can be described as above. Thus there exist exponents $\varepsilon_v(c)$ so that

$$\varphi(a, b, c) = \prod_v (a, b)_v^{\varepsilon_v(c)}.$$

Fixing b and c, consider the idele (d_v) whose v-th component is

$$d_v = b^{\varepsilon_v(c)} c^{\varepsilon_v(b)}.$$

Using the symmetry relation

$$\varphi(a, b, c) = \varphi(a, c, b)$$

it follows that

$$\prod_v (a, d_v)_v = 1. \tag{9}$$

We will need the following classical result.

Lemma A.3. *If an idele (d_v) satisfies the product formula* (9) *for every non-zero field element a, then (d_v) can be expressed as the product of a field element d and the square of an idele.*

This is proved for example in Weil [20, p. 262].

Thus, given field elements b and c, we can construct the idele (d_v), and hence the field element d, so that

(10) $$d \in b^{\varepsilon_v(c)} c^{\varepsilon_v(b)} F_v^{\bullet 2}$$

for every v.

Consider the extension field $F(\sqrt{b}, \sqrt{c})$. Since d is a square in every completion of this field, it follows that d is a square in the field $F(\sqrt{b}, \sqrt{c})$ itself. By Kummer theory, this implies that d can be expressed as $b^i c^j$ times the square of an element of F. Here the exponents i and j are equal to 0 or 1. The assertion (10) now implies that

(11) $$b^{\varepsilon_v(c)-i} c^{\varepsilon_v(b)-j} \in F_v^{\bullet 2}$$

for every v.

Lemma A.4. *If v and w are discrete valuations (i.e. corresponding to finite primes), then $\varepsilon_v(c) = \varepsilon_w(c)$ for all c.*

Proof. Note that the groups $F_v^\bullet/F_v^{\bullet 2}$ and $F_w^\bullet/F_w^{\bullet 2}$ both have order at least 4. So given c it is possible to choose b so that the image of b in $F_v^\bullet/F_v^{\bullet 2}$ is independent of c, and simultaneously so that the image of b in $F_w^\bullet/F_w^{\bullet 2}$ is independent of c. Thus (11) implies that

$$\varepsilon_v(c) - i = 0, \qquad \varepsilon_w(c) - i = 0;$$

which proves A.4.

Proof of Theorem A.2. Replacing every $\varepsilon_v(c)$ by $1 - \varepsilon_v(c)$ if necessary, we may assume that $\varepsilon_v(c) = 0$ for every discrete valuation v. Hence in the formula

$$\varphi(a, b, c) = \prod_v (a, b)_v^{\varepsilon_v(c)},$$

we need only take the product over real completions of F. It follows that $\varphi(a, b, c) = 1$ unless there exists a real completion at which both a and b are negative.

But this is true for every φ. So it follows that:

Lemma A.5. *The product $l(a)\, l(b)\, l(c) \in k_3 F$ is zero unless there exists a real completion at which both a and b are negative.*

The rest of the proof is easy. Let $v_1, \ldots, v_r$ be the real valuations, and let $e_1, \ldots, e_r$ be field elements such that e_j is negative in the v_j-th completion but positive in the other real completions. Then A.5 implies that a product $l(e_{i_1}) \ldots l(e_{i_n})$ with $n \geqq 3$ is zero unless $i_1 = \cdots = i_n$. On the other hand the powers $l(e_1)^n, \ldots, l(e_r)^n$ certainly are linearly independent, since they map into linearly independent elements of $\oplus_v k_n F_v$.

Since $F^{\bullet}$ is generated by $e_1, \ldots, e_r$ together with the totally positive elements, it follows immediately that these powers $l(e_1)^n, \ldots, l(e_r)^n$ actually form a basis for $k_n F$, $n \geqq 3$. This completes the proof.

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