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Almost-Primes Represented by Quadratic Polynomials

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1. Introduction and Statement of Results

The only polynomials that are known to represent infinitely many primes are of degree one (Dirichlet's theorem on primes in arithmetic progressions). Some heuristic arguments suggest that any irreducible polynomial in $Z[x]$ with positive leading coefficient and without any fixed prime divisor assumes infinitely many prime values. Schinzel [19] formulated a more general conjecture and Bateman and Horn [1] have given a quantitative form of his conjecture

$$|\{n \leqq x;\ G(n)=p\}| \sim \Gamma_G \frac{x}{\log x} \quad \text{as } x \to \infty,$$

where $\Gamma_G = \frac{1}{g}\prod \left(1-\frac{\rho(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-1}$, $g=\deg G$, and $\rho(p)$ is the number of incongruent solutions of the congruence

$$G(n) \equiv 0 \pmod{p}.$$

The case of quadratic polynomial is the content of the Hardy-Littlewood Conjecture F [6]. In the present state of knowledge the conjecture seems to be extremely difficult. A very nice approximation to the conjecture was offered by Brun's sieve, the powerful tool for studying almost-primes.

Let P_r denote an almost-prime of order r, i.e., a number having at most r prime factors counted with multiplicity. The problem was first investigated by Rademacher [16] who proved that there exist infinitely many n such that $G(n) = P_{4g-1}$, and Ricci [17] reduced r from $4g-1$ to $3g-1$. Great progress has been made since the work of Kuhn [12], who first noticed that if weights of a certain kind are attached to the elements of the sequence to be sifted, the sieve results concerning almost primes become much stronger. By using Selberg's sieve with Kuhn's weights, Kuhn [13] and Wang [21] obtained $r = g + c\log g$. For small values of g a further improvement was obtained by Levin [14].

The problem of finding the optimal weights is rather complicated. Buchstab [2] transfered the problem to linear programming and was led to a very

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complicated set of constant weights. In 1968 Richert [18] discovered his simple weighted sum with logarithmic weights of the type

$$w(a)=1-\lambda\sum_{p|a}\frac{\log^+(y/p)}{\log y}$$

where $\log^+ x=\max(0,\log x)$. Either weight permits one to prove that there exist infinitely many n such that

$$G(n)=P_{g+1}.$$

We should mention that the same result was proved earlier by Kuhn [13] if $g=2$, by Wang [21] if $g=3,4,5$ and by Levin [14] if $g=6,7$ (in his definition of almost-primes of order r the prime factors are not counted with multiplicity).

Our aim is to prove the following result.

Theorem. *Let $G(n)=an^2+bn+c$ be an irreducible polynomial with $a>0$ and $c\equiv 1 \pmod 2$. Then there exist infinitely many integers n such that $G(n)$ has at most two prime factors. Moreover, if x is sufficiently large the following estimate holds*

$$|\{n\leqq x;\ G(n)=P_2\}|>\frac{1}{77}\Gamma_G\frac{x}{\log x}. \tag{1}$$

Actually, in order to avoid some technical complications, we shall restrict our proof to the polynomial $G(n)=n^2+1$, the general case being similar.

The method involves an injection into a new Richert weighted sum [5] of a combination of two ideas due to Hooley [8, 10] with a new form of the error term in the linear sieve [11]. These four innovations are presented in sections 2, 3 and 4.

Acknowledgement. This is due to several people with whom I have discussed the problem since two years ago. I deeply appreciate Professor Friedlander's assistence in studying Hooley's methods [7, 8] (during our sabbaticals at the Scuola Normale Superiore di Pisa in 1976/77). Actually, a slightly weaker estimate of the type (5) was already obtained in our joint paper [4]. I should say that it was the suggestion of Professor Halberstam that the ideas of Chen [3] and Hooley [8] might be used to attack the problem. I am very grateful to Professor Richert for showing me his new unpublished results and for many valuable comments. Finally I would like to express my thanks to Professor Hooley for his critical remarks and some corrections in the first version of the paper. For the financial support I wish to thank the Mittag-Leffler Institut of the Royal Swedish Academy of Sciences.

2. A New Weighted Sum of Richert

Let p_n and $\omega(n)$ denote the least prime factor and the number of prime divisors of the integer n respectively. For given $x>2$ and for all $p<x$, $p|n$ define

$$\omega_p(n)=\begin{cases}1-\dfrac{\log p}{\log x} & \text{if } p=p_n\\[2ex] \dfrac{\log p_n}{\log x} & \text{if } p>p_n \text{ and } p<x^{\frac{1}{2}}\\[2ex] 1-\dfrac{\log p}{\log x} & \text{if } p>p_n \text{ and } p\geqq x^{\frac{1}{2}}.\end{cases}$$

Let $2 \leqq \lambda < 3$ and

$$w(n) = 1 - \frac{1}{3-\lambda} \sum_{p \mid n,\ p < x} \omega_p(n).$$

Lemma 1. *If $n \leqq x^{\lambda}$ and $w(n) > 0$ then n has at most two distinct prime factors.*

Proof. If n had two or more different prime factors less then $x^{\frac{1}{2}}$ its weight would be

$$w(n) \leqq 1 - \frac{1}{3-\lambda}\left\{1 - \frac{\log p_n}{\log x} + \frac{\log p_n}{\log x}\right\} = \frac{2-\lambda}{3-\lambda} \leqq 0.$$

Therefore one can assume that n has at most one prime factor less than $x^{\frac{1}{2}}$. In this case we have

$$w(n) = 1 - \frac{1}{3-\lambda} \sum_{p \mid n} \frac{\log^+(x/p)}{\log x} \leqq 1 - \frac{1}{3-\lambda}\left\{\omega(n) - \frac{\log n}{\log x}\right\} \leqq \frac{3-\omega(n)}{3-\lambda}$$

whence $\omega(n) \leqq 2$.

Let $\mathscr{A}$ be a finite sequence of positive integers $a \leqq x^{\lambda}$. For any $z \leqq x^{\frac{1}{2}}$, by Lemma 1, we have

$$|\{a \in \mathscr{A};\ a = P_2\}| \geqq \sum_{a \in \mathscr{A},\ (a,\ P(z)) = 1}^{*} w(a) = W^*(\mathscr{A}, z), \quad \text{say}$$

where the summation is over all square-free numbers from $\mathscr{A}$ which are coprime with

$$P(z) = \prod_{p < z} p.$$

We shall be concerned with the sequence

$$\mathscr{A} = \{n^2 + 1;\ n \leqq x\}$$

in which case we choose $\lambda = 2$. The non-square-free numbers are easy to handle. We have

$$\sum_{\substack{a \in \mathscr{A},\ (a,\ P(z)) = 1 \\ a - \text{non-square-free}}} 1 \leqq \sum_{z \leqq p \leqq x z^{-\frac{1}{2}}} \sum_{n \leqq x,\ n^2 + 1 \equiv 0 \,(\mathrm{mod}\, p^2)} 1$$
$$\leqq \sum_{z \leqq p \leqq x z^{-\frac{1}{2}}} 2(x p^{-2} + 1) \leqq 4 x z^{-\frac{1}{2}}.$$

Therefore we are led to consider the expression

$$W(\mathscr{A}, z) = \sum_{a \in \mathscr{A},\ (a,\ P(z)) = 1} w(a)$$

with the aim of showing that

$$W(\mathscr{A}, z) > \frac{1}{77} \Gamma_G \frac{x}{\log x} \tag{2}$$

for all sufficiently large x and for some suitable value of z. Letting

$$\mathscr{A}_q=\{a\in\mathscr{A};\ a\equiv 0(\mathrm{mod}\ q)\}$$

the sum $W(\mathscr{A},z)$ may be interpreted in terms of the sifting functions

$$S(\mathscr{A}_q,u)=|\{a\in\mathscr{A}_q;\ (a,P(u))=1\}|.$$

We have

$$W(\mathscr{A},z)=S(\mathscr{A},z)-\sum_{z\leqq p<x}\ \sum_{a\in\mathscr{A}_p,\ (a,\,P(z))=1}\omega_p(a)$$

and the double sum is equal to

$$\sum_{z\leqq p<x^{\frac12}}\left(1-\frac{\log p}{\log x}\right)S(\mathscr{A}_p,p)+\sum_{z\leqq p_1<p<x^{\frac12}}\frac{\log p_1}{\log x}S(\mathscr{A}_{pp_1},p_1)$$
$$+\sum_{x^{\frac12}\leqq p<x}\left(1-\frac{\log p}{\log x}\right)S(\mathscr{A}_p,z).$$

Applying to the middle sum the Buchstab formula

$$\sum_{z\leqq p_1<p}S(\mathscr{A}_{pp_1},p_1)=S(\mathscr{A}_p,z)-S(\mathscr{A}_p,p)$$

we arrive at

$$W(\mathscr{A},z)=S(\mathscr{A},z)+\sum_{z\leqq p<x^{\frac12}}\ \sum_{z\leqq p_1<p}\frac{\log(p_1/p)}{\log x}S(\mathscr{A}_{pp_1},p_1)$$
$$-\sum_{z\leqq p<x^{\frac12}}\left\{\left(1-2\frac{\log p}{\log x}\right)S(\mathscr{A}_p,p)+\frac{\log p}{\log x}S(\mathscr{A}_p,z)\right\}$$
$$-\sum_{x^{\frac12}\leqq p<x}\left(1-\frac{\log p}{\log x}\right)S(\mathscr{A}_p,z).\tag{3}$$

By (3), the proof of (2) is transformed to the linear sieve problem of estimating the sifting function $S(\mathscr{A}_q,u)$. The usual estimates, being not sufficient, will be bettered in two respects, for individual $S(\mathscr{A},z)$ and for $S(\mathscr{A}_q,u)$ on average. We utilize a variant of the Linnik dispersion method (the idea of Hooley used in [10] for the arithmetic progressions). The method when applied directly would yield the improvement for almost all q in the range $x^{14/15}<q<x$ only. Our purpose will be effected by a combination of the Hooley's method with a new very flexible form of the error term in the linear sieve [11].

3. A New Form of the Error Term in the Linear Sieve

Let $\mathscr{B}$ be a finite sequence of integers (not necessary distinct) and suppose that the quantity

$$|\mathscr{B}_d|=|\{b\in\mathscr{B};\ b\equiv 0(\mathrm{mod}\ d)\}|$$

can be approximated by an expression of the form $\frac{\rho(d)}{d}X$ with some parameter $X>0$ independent of d and with some multiplicative function $\rho(d)$ such that $0\leqq\rho(p)<p$ and

$$\prod_{w\leqq p<z}\left(1-\frac{\rho(p)}{p}\right)^{-1}\leqq\frac{\log z}{\log w}\left(1+\frac{K}{\log w}\right)$$

for any $2\leqq w<z$ with some constant $K\geqq 1$. By [11] we have

Lemma 2. *Let $z\geqq 2$, $M\geqq 2$ and $N\geqq 2$. For any $\eta>0$ we have*

$$S(\mathscr{B},z)\leqq V(z)X\{F(s)+E\}+2^{\eta^{-7}}R(\mathscr{B};M,N),$$
$$S(\mathscr{B},z)\geqq V(z)X\{f(s)-E\}-2^{\eta^{-7}}R(\mathscr{B};M,N),$$

where $s=\log MN/\log z$, $E\ll\eta s^2e^K+\eta^{-8}e^{K-s}(\log MN)^{-1/3}$ and

$$V(z)=\prod_{p<z}\left(1-\frac{\rho(p)}{p}\right).$$

The functions $F(s)$ and $f(s)$ are the continuous solutions of the system of differential-difference equations

$$\begin{aligned}
sf(s) &= 0 && \text{if } 0<s\leqq 2,\\
sF(s) &= 2e^C && \text{if } 0<s\leqq 3,\\
(sf(s))' &= F(s-1) && \text{if } s>2,\\
(sF(s))' &= f(s-1) && \text{if } s>3,
\end{aligned}$$

where $C=0.577\ldots$ is the Euler constant. The second error term $R(\mathscr{B};M,N)$ has the form

$$R(\mathscr{B};M,N)=\sum_{m<M,\ n<N,\ mn\mid P(z)}a_mb_nr(\mathscr{B};mn) \tag{4}$$

where $r(\mathscr{B};d)=|\mathscr{B}_d|-\frac{\rho(d)}{d}X$ and the coefficients a_m,b_n are bounded by 1 in absolute value and depend at most on M,N,z and η.

Since $V(z)X$ should be the order of magnitude of $S(\mathscr{B},z)$ the error term $R(\mathscr{B};M,N)$ is required to be $o(V(z)X)$. There are several treatments of the double sum (4) that lead to $R(\mathscr{B};M,N)=o(V(z)X)$ with values of MN larger than those that are possible for the conventional error term

$$R(\mathscr{B};MN)=\sum_{d<MN,\ (d,\ P(z))=1}|r(\mathscr{B},d)|.$$

4. Estimation of the Error Term $R(\mathscr{A};M,N)$

Here we use a method based on the concept of dispersion. For a given sequence of real numbers b_n such that $|b_n|\leqq 1$ and $b_n=0$ unless n is square-free, let us denote

$$B(x;m,N)=\sum_{n<N,\ (n,\ m)=1}b_nr(\mathscr{A};mn).$$

The bulk of this section will be devoted to the proof of the following result.

Proposition 1. *Let $M<x$ and $\varepsilon>0$. We have*

$$\sum_{M<m<2M} B^2(x;m,N) \ll (1+N^{7/2}M^{-5/4}x)\,x^{1+\varepsilon},$$

the implied constant depending on ε.

Corollary. *For any $\varepsilon>0$ we have*

$$\sum_{m<x^{1-4\varepsilon}} \Big| \sum_{n<x^{1/15-\varepsilon},\,(n,m)=1} b_n r(\mathscr{A};mn)\Big| \ll x^{1-\varepsilon}$$

the implied constant depending on ε.

Proof of Corollary. Letting $N=x^{1/15-\varepsilon}$ it is sufficient to proof that for all $M<x^{1-4\varepsilon}$

$$\sum_{M<m<2M} |B(x;m,N)| \ll x^{1-3\varepsilon/2}.$$

If $M<x^{14/15-\varepsilon}$ the result follows from the trivial estimate

$$|B(x;m,N)| \leqq \rho(m) \sum_{n<N} \rho(n) \ll \rho(m)\,N.$$

If $x^{14/15-\varepsilon} \leqq M < x^{1-4\varepsilon}$, by Proposition 1 and the Cauchy-Schwarz inequality we get

$$\sum_{M<m<2M} |B(x;m,N)| \ll \{(Mx)^{1/2}+N^{7/4}M^{-1/8}x\}\,x^{\varepsilon/8} \ll x^{1-3\varepsilon/2}.$$

Remark. Our former treatment of $R(\mathscr{A};M,N)$ (jointly with Friedlander, see [4]) was based on another idea of Hooley [8]. The error terms $r(\mathscr{A};mn)$ were expressed by some exponential sums being estimated through the use of Lemmata 5 and 6. Analysing this method with greater detail one can prove the following estimate

$$R(\mathscr{A};M,N) \ll \{N^{-1/2}+(N^5M^{-1})^{1/8}+(x^4NM^{-5})^{1/12}\}\,MN\,x^{\varepsilon}. \tag{5}$$

We should mention that the method of Motohashi [16] can also give some useful results.

Turning our attention to the proof of Proposition 1 we write

$$B(x;m,N) = \sum_{n<N,\,(n,m)=1} b_n |\mathscr{A}_{mn}| - \frac{\rho(m)}{m}\,x \sum_{n<N,\,(n,m)=1} b_n \frac{\rho(n)}{n} \tag{6}$$

and

$$\sum_{n<N,\,(n,m)=1} b_n |\mathscr{A}_{mn}| = \sum_{\substack{0<v<m \\ v^2+1\equiv 0 \,(\mathrm{mod}\, m)}} \sum_{\substack{n<N \\ (n,m)=1}} b_n \sum_{\substack{k<x,\ k\equiv v\,(\mathrm{mod}\, m) \\ k^2+1\equiv 0\,(\mathrm{mod}\, n)}} 1.$$

Since $(m, n)=1$ one may expect that the sum $\sum_n b_n \sum_k 1$ is approximately

$$\frac{x}{m} \sum_{n<N,\,(n,m)=1} b_n \frac{\rho(n)}{n} = \frac{x}{m} Y(m), \quad \text{say.} \tag{7}$$

We are thus led to consider the expression

$$\begin{aligned}
\mathscr{M}(x; M, N) &= \sum_{M<m<2M} \sum_{\substack{0<v<m \\ v^2+1\equiv 0 \,(\mathrm{mod}\, m)}} \left\{ \sum_{\substack{n<N \\ (n,m)=1}} b_n \sum_{\substack{k<x \\ k\equiv v \,(\mathrm{mod}\, m) \\ k^2+1\equiv 0 \,(\mathrm{mod}\, n)}} 1 - \frac{x}{m} Y(m) \right\}^2 \\
&= \sum_{M<m<2M} \sum_{\substack{0<v<m \\ v^2+1\equiv 0 \,(\mathrm{mod}\, m)}} \left\{ \sum_{\substack{n_1, n_2<N \\ (n_1 n_2, m)=1}} b_{n_1} b_{n_2} \sum_{\substack{k_1, k_2<x \\ k_1\equiv k_2\equiv v \,(\mathrm{mod}\, m) \\ k_1^2+1\equiv 0 \,(\mathrm{mod}\, n_1) \\ k_2^2+1\equiv 0 \,(\mathrm{mod}\, n_2)}} 1 \right. \\
&\quad \left. -2\frac{x}{m} Y(m) \sum_{\substack{n<N \\ (n,m)=1}} b_n \sum_{\substack{k<x,\, k\equiv v \,(\mathrm{mod}\, m) \\ k^2+1\equiv 0 \,(\mathrm{mod}\, n)}} 1 + \left(\frac{x}{m} Y(m)\right)^2 \right\} \\
&= W(x; M, N) - 2x\, V(x; M, N) + x^2\, U(M, N), \quad \text{say.}
\end{aligned}$$

For the estimation of V and W we shall require the following result about the uniform distribution of the solutions of the congruence $\Omega^2+1\equiv 0 \pmod{mq}$.

Lemma 4. *Let q be a square-free number, $d \mid q$, $2 \nmid d$, $(\mu, d)=1$ and $\omega^2+1\equiv 0 \pmod{d}$. Let $M<M_1<2M$ and $0\leqq\alpha<\beta<1$. Denote by $P(M_1, M; q, d, \mu, \omega, \alpha, \beta)$ the number of all pairs of integers m, Ω satisfying simultaneous conditions*

$$M<m<M_1, \quad (m, q)=1, \quad m\equiv\mu \pmod{d},$$
$$\alpha mq\leqq\Omega<\beta mq, \quad \Omega^2+1\equiv 0 \pmod{mq}, \quad \Omega\equiv\omega \pmod{d}.$$

Then for any $\varepsilon>0$ we have

$$P(M_1, M; q, d, \mu, \omega, \alpha, \beta) = \frac{\pi}{4}(\beta-\alpha)(M_1-M)\,\rho\left(\frac{q}{d}\right)\frac{A(q)}{\varphi(d)} + O((qM)^{3/4+\varepsilon}),$$

where $A(q)=\dfrac{1}{(2, q)}(\varphi(q)/q)^2$.

Corollary 1. *Let qq_1 be a square-free number with $\rho(qq_1)\neq 0$. Let $M<M_1<2M$ and $0\leqq\alpha<\beta<1$. Then for any $\varepsilon>0$ we have*

$$\sum_{\substack{M<m<M_1,\,(m, qq_1)=1 \\ \alpha mq\leqq\Omega<\beta mq,\ \Omega^2+1\equiv 0 \,(\mathrm{mod}\, mq)}} 1 = \frac{\pi}{4}(\beta-\alpha)(M_1-M)\,\rho(q)\,A(qq_1) + O((qq_1 M)^{3/4+\varepsilon}).$$

Proof. The sum in question is equal to $\dfrac{1}{\rho(q_1)} P(M_1, M; 1, 1, 1, \alpha, \beta)$.

Corollary 2. *Let q be a square-free number with $\rho(q)\neq 0$. Let $M<M_1<2M$. Then for any $\varepsilon>0$ we have*

$$\sum_{M<m<M_1,\,(m,q)=1}\rho(m)=\frac{\pi}{4}(M_1-M)\,A(q)+O((qM)^{3/4+\varepsilon}).$$

Proof. The sum in question is equal to $\dfrac{1}{\rho(q)}\,P(M_1, M; q, 1, 1, 1, 0, 1)$.

For the proof of Lemma 4 we shall require three lemmata.

Lemma 5. *There exists a one to one correspondence between the solutions $\Omega \pmod D$ of the congruence $\Omega^2+1\equiv 0 \pmod D$ and the pairs of numbers (r, s) such that*

$$D=r^2+s^2, \quad (r,s)=1, \quad |r|<s.$$

The correspondence is given through the formula

$$\Omega=\frac{\bar r}{s}(r^2+s^2)-\frac{r}{s}.$$

The notation $\bar r$, used either in $\dfrac{\bar r}{s}$ or in a congruence $(\bmod s)$ means that $r\bar r\equiv 1 \pmod s$. This result, due of course to Lagrange is to be found in Article 86 of [20]. The second lemma, due to Hooley, (Lemma 3 of [8]) is a consequence of Weil's estimate for Kloosterman sums.

Lemma 6. *If h and s are integers and if $0<r_2-r_1<2s$ then*

$$\sum_{\substack{r_1<r<r_2,\,(r,s)=1\\ r\equiv\lambda \pmod\Delta}} e\left(h\frac{\bar r}{s}\right)\ll s^{\frac12+\varepsilon}(h,s)^{\frac12}$$

the implied constant depending on ε.

We shall also require an approximation of the characteristic function $\psi(t)$ of the interval $[\alpha, \beta) \pmod 1$ by two absolutely convergent Fourier series.

Lemma 7. *Let $2\Delta<\beta-\alpha<1-2\Delta$. There exist two functions $A(t)$ and $B(t)$ such that*

$$|\psi(t)-A(t)|=B(t)$$

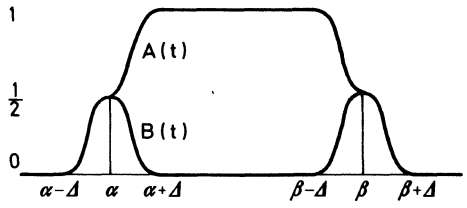
and

$$A(t)=\beta-\alpha+\sum_{h\neq 0}A_h\,e(ht)$$

$$B(t)=\Delta+\sum_{h\neq 0}B_h\,e(ht)$$

with Fourier coefficients A_h, B_h satisfying $|A_h|, |B_h|\leqq\min\left(\dfrac{1}{|h|}, \dfrac{\Delta^{-2}}{|h|^3}\right)=C_h$.

Proof. Take the functions drawn below



Now we are ready to prove Lemma 4. By Lemma 7 we have

$$P(M_1, M; q, d, \mu, \omega, \alpha, \beta) = (\beta-\alpha) \sum_{\substack{M<m<M_1,\ (m,q)=1,\ m\equiv\mu \pmod d \\ 0\leqq\Omega<qm,\ \Omega^2+1\equiv 0 \pmod{mq},\ \Omega\equiv\omega \pmod d}} 1$$
$$+O\{\rho(q)\,\Delta M+\sum_{h\neq 0} C_h \Big| \sum_{\substack{M<m<M_1,\ (m,q)=1,\ m\equiv\mu \pmod d \\ 0\leqq\Omega<qm,\ \Omega^2+1\equiv 0 \pmod{mq},\ \Omega\equiv\omega \pmod d}} e(h\Omega/mq)\Big|\}.$$

The main term is easy to handle. Since $d \mid q$, $(d, mq/d)=1$ and $(\mu, d)=1$ it is equal to

$$\rho(q/d) \sum_{\substack{M<m<M_1,\ (m,q)=1 \\ m\equiv\mu \pmod d}} \rho(m) = \rho(q/d) \sum_{\substack{a\leqq M^{\frac12} \\ (a,q)=1}} \chi_4(a) \sum_{\substack{\frac{M}{a}<b<\frac{M_1}{a},\ (b,q/d)=1 \\ b\equiv\mu\bar a \pmod d}} 1$$
$$+\rho(q/d) \sum_{\substack{b<2M^{\frac12} \\ (b,q)=1}} \sum_{\substack{\max\left(\frac{M}{b}, M^{\frac12}\right)<a<\frac{M_1}{b} \\ a\equiv\mu\bar b \pmod d,\ (a,q/d)=1}} \chi_4(a)$$

$$=\rho(q/d) \sum_{a\leqq M^{\frac12},\ (a,q)=1} \chi_4(a) \left\{\varphi\left(\frac{q}{d}\right)\frac{M_1-M}{aq}+O(d(q))\right\}+O(\rho(q)\,M^{\frac12})$$
$$=\rho\left(\frac{q}{a}\right)\varphi\left(\frac{q}{d}\right)\frac{M_1-M}{q}\,L(1,\chi_4)\prod_{p\mid q}\left(1-\frac{\chi_4(p)}{p}\right)+O(\rho(q)\,M^{\frac12})$$

and this is the required main term. Proceeding to the exponential sum $\sum_{m,\Omega} e(h\Omega/mq)$ we first eliminate the condition $(m, q)=1$ as follows

$$\sum_{m,\Omega} e(h\Omega/mq) = \sum_{l\mid\frac{q}{d}} \mu(l) \sum_{\substack{qM<D<qM_1,\ D\equiv 0 \pmod{lq},\ D\equiv\mu q \pmod{dq} \\ 0\leqq\Omega<D,\ \Omega^2+1\equiv 0 \pmod D,\ \Omega\equiv\omega \pmod d}} e(h\Omega/D).$$

Now we apply Lemma 5. Writing $D=r^2+s^2$, $(r, s)=1$, $|r|<s$, the congruence conditions $D\equiv 0 \pmod{lq}$, $D\equiv\mu q \pmod{dq}$ and $\Omega\equiv\omega \pmod d$ are equivalent to

$$r^2+s^2\equiv 0 \pmod{lq}, \quad r^2+s^2\equiv\mu q \pmod{dq}, \text{ and } r+\omega s\equiv 0 \pmod d.$$

Therefore

$$\sum_{D,\Omega} e(h\Omega/D) \leqq d(q) \sum_{(\frac12 qM)^{\frac12}<s<(2qM)^{\frac12}} \sup_{\substack{r_1, r_2 \\ \lambda, \Lambda}} \left| \sum_{\substack{(r,s)=1 \\ r_1<r<r_2 \\ r\equiv\lambda \pmod\Lambda}} e\left(h\frac{\bar r}{s}-\frac{hr}{s(r^2+s^2)}\right)\right|$$

where the supremum is taken over all integers r_1, r_2, λ, Λ such that $0<r_2-r_1<2s$. By Lemma 6 and partial summation we obtain

$$\sum_{D,\Omega} e(h\Omega/D) \ll \left(1+\frac{h}{qM}\right)(qM)^{1/4+\varepsilon} \sum_{s<(2qM)^{\frac{1}{2}}} (h,s)^{\frac{1}{2}}$$

$$\ll \left(1+\frac{h}{qM}\right)(qM)^{3/4+\varepsilon} d(h),$$

the implied constant depending on ε. Obviously the same holds for the sum $\sum_{m,\Omega} e(h\Omega/qm)$. Since $C_h \leqq \min\left(\frac{1}{|h|}, \Delta^{-2}|h|^{-3}\right)$ we have

$$\sum_{h\neq 0} C_h \left(1+\frac{h}{qM}\right) d(h) \ll \left(1+\frac{1}{\Delta qM}\right)(\log \Delta)^2.$$

On choosing $\Delta = 1/qM$ we complete the proof of Lemma 4.

Estimation of $U(M,N)$

By the definition of $U(M,N)$ given by (7) and (8), we have

$$\begin{aligned} U(M,N) &= \sum_{M<m<2M} \sum_{\substack{0\leqq v<m \\ v^2+1\equiv 0 \,(\mathrm{mod}\, m)}} \frac{1}{m^2} \sum_{\substack{n_1,n_2<N \\ (n_1n_2,m)=1}} b_{n_1} b_{n_2} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} \\ &= \sum_{n_1,n_2<N} b_{n_1} b_{n_2} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} \sum_{\substack{M<m<2M \\ (m,n_1n_2)=1}} \frac{\rho(m)}{m^2} \\ &= \sum_{n_1,n_2<N} b_{n_1} b_{n_2} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} \\ &\quad \cdot \left\{\frac{\pi}{8} A(n_1n_2) M^{-1} + O((n_1n_2M)^{3/4+\varepsilon} M^{-2})\right\} \\ &= \frac{\pi}{8} M^{-1} \sum_{n_1,n_2<N} b_{n_1} b_{n_2} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} A(n_1n_2) + O(M^{-2}(N^2M)^{3/4+\varepsilon}) \end{aligned} \tag{9}$$

the penultimate line having been formed through the use of Corollary 2 of Lemma 4 and partial summation. This is the required estimate for $U(M,N)$.

Estimation of $V(x;M,N)$

Expressing $V(x;M,N)$ as a multiple sum through (7) and (8), we have

$$\begin{aligned} V(x;M,N) = \sum_{\substack{M<m<2M \\ 0\leqq v<m,\, v^2+1\equiv 0 \,(\mathrm{mod}\, m)}} \frac{1}{m} \Bigg(\sum_{\substack{n_1<N \\ (m,n_1)=1}} b_{n_1} \sum_{\substack{k<x \\ k\equiv v \,(\mathrm{mod}\, m) \\ k^2+1\equiv 0 \,(\mathrm{mod}\, n_1)}} 1 \Bigg) \\ \cdot \left(\sum_{\substack{n_2<N \\ (n_2,m)=1}} b_{n_2} \frac{\rho(n_2)}{n_2} \right). \end{aligned}$$

Changing the order of summation, we obtain

$$V(x; M, N) = \sum_{n_1, n_2 < N} b_{n_1} b_{n_2} \frac{\rho(n_2)}{n_2} \sum_{\Theta_{n_1, n_2}} \frac{1}{m}$$
$$= \sum_{n_1, n_2 < N} b_{n_1} b_{n_2} \frac{\rho(n_2)}{n_2} S(n_1, n_2; x, M)$$

where, for given n_1, n_2, the symbol Θ_{n_1, n_2} indicates summation over all pairs of integers m, v and k satisfying simultaneous conditions

$$M < m < 2M, \quad (m, n_1 n_2) = 1,$$
$$0 \leqq v < m, \quad v^2 + 1 \equiv 0 \pmod{m},$$
$$k < x, \quad k \equiv v \pmod{m}, \quad k^2 + 1 \equiv 0 \pmod{n_1}.$$

To evaluate $S(n_1, n_2; x, M)$ it is necessary to reinterpret the conditions implied by Θ_{n_1, n_2}. First the condition $k \equiv v \pmod{m}$ taken with the inequalities satisfied by m, v and k, means that there is a non-negative integer $l < x/M$ such that

$$k = v + ml.$$

Therefore $S(n_1, n_2; x, M)$ can be transformed into a sum over variables of summation m, v and l, where the latter are subject to the conditions

$$M < m < \min\left(2M, \frac{x - v}{l}\right), \quad (m, n_1 n_2) = 1,$$
$$0 \leqq v < m, \quad v^2 + 1 \equiv 0 \pmod{m}, \quad (ml + v)^2 + 1 \equiv 0 \pmod{n_1}. \tag{10}$$

Since $0 \leqq v < m$, if one replaces $\frac{x - v}{l}$ by $\frac{x}{l}$ the total error will be less than

$$\sum_{n_2 < N} \frac{\rho(n_2)}{n_2} \sum_{1 \leqq l < \frac{x}{M}} \sum_{\frac{x - 2M}{l} < m < \frac{x}{l}} m^{-1} \sum_{\substack{0 \leqq v < m \\ v^2 + 1 \equiv 0 \pmod{m}}} \sum_{n_1 \mid (ml + v)^2 + 1} 1 \ll x^{\varepsilon} \tag{11}$$

and this is sufficient for our needs. Let $S^*(n_1, n_2; x, M)$ stand for the modified sum. We shall reinterpret the congruence conditions (10) once more. Letting c to be the solution of

$$c \equiv l \pmod{n_1} \quad \text{and} \quad 0 \leqq c < n_1$$

and defining $\Omega = cm + v$ we obtain

$$cm \leqq \Omega < (c + 1)m \quad \text{and} \quad \Omega^2 + 1 \equiv 0 \pmod{mn_1}.$$

We deduce from the above that

$$S^*(n_1, n_2; x, M) = \sum_{l < \frac{x}{M}} \sum_{\substack{M < m < \min\left(2M, \frac{x}{l}\right), (m, n_1 n_2) = 1 \\ cm \leqq \Omega < (c+1)m,\ \Omega^2 + 1 \equiv 0 \pmod{mn_1}}} m^{-1}.$$

We infer from Corollary 1 of Lemma 4 and partial summation that

$$S^*(n_1, n_2; x, M) = \sum_{1 < x/M} \frac{\pi}{4} \frac{p(n_1)}{n_1} A(n_1 n_2) \cdot \left\{ \log \left(\min \left(2, \frac{x}{lM} \right) \right) + O(M^{-1}(n_1 n_2 M)^{3/4+\varepsilon}) \right\}$$
$$= \frac{\pi}{8} \frac{x}{M} \frac{\rho(n_1)}{n_1} A(n_1 n_2) + O\left(\frac{\rho(n_1)}{n_1} \left(\log x + \frac{x}{M^2} (n_1 n_2 M)^{3/4+\varepsilon} \right) \right) \quad (12)$$

The required estimate for $V(x; M, N)$ is now available. We obtain from (11) and (12) that

$$V(x; M, N) = \frac{\pi}{8} \frac{x}{M} \sum_{n_1, n_2 < N} b_{n_1} b_{n_2} \frac{\rho(n_1)\rho(n_2)}{n_1 n_2} A(n_1 n_2) + O(x^{\varepsilon}) + O(xM^{-2}(N^2 M)^{3/4+\varepsilon}). \quad (13)$$

Estimation of $W(x; M, N)$

By the definition of $W(x; M, N)$ given by (8), after changing the order of summations, we obtain

$$W(x; M, N) = \sum_{n_1, n_2 < N} b_{n_1} b_{n_2} \sum_{\Phi_{n_1, n_2}} 1 = \sum_{n_1, n_2 < N} b_{n_1} b_{n_2} T(n_1, n_2; x, M) \quad (14)$$

where, for given n_1, n_2, the symbol Φ_{n_1, n_2} indicates that the summation is over all sets of integers m, v, k_1 and k_2 satisfying the simultaneous conditions

$$M < m < 2M, \quad (m, n_1 n_2) = 1,$$
$$0 \leqq v < m, \quad v^2 + 1 \equiv 0 \pmod{m},$$
$$k_1 < x, \quad k_2 < x, \quad k_1 \equiv k_2 \equiv v \pmod{m},$$
$$k_1^2 + 1 \equiv 0 \pmod{n_1}, \quad k_2^2 + 1 \equiv 0 \pmod{n_2}.$$

We scrutinize the significance of Φ_{n_1, n_2} much as we did that of Θ_{n_1, n_2}. Writing

$$k_1 = ml_1 + v, \quad k_2 = ml_2 + v,$$

where l_1, $l_2 < x/M$ we see that $T(n_1, n_2; x, M)$ can be transformed into a sum over variables of summation m, v, l_1 and l_2 that are constrained by the conditions

$$M < m < \min \left(2M, \frac{x - v}{l_1}, \frac{x - v}{l_2} \right), \quad (m, n_1 n_2) = 1,$$
$$0 \leqq v < m, \quad v^2 + 1 \equiv 0 \pmod{m}, \quad (ml_1 + v)^2 + 1 \equiv 0 \pmod{n_1},$$
$$(ml_2 + v)^2 + 1 \equiv 0 \pmod{n_2}.$$

Since $0 \leqq v < m$, if one replaces $x - v$ by x the total error will be less than

$$2 \sum_{l_1 \leqq l_2 < \frac{x}{M}} \sum_{\frac{x-2M}{l_2} < m < \frac{x}{l_2}} \sum_{\substack{0 \leqq v < m \\ v^2+1 \equiv 0 \pmod m}} \sum_{\substack{n_1 \mid (ml_1+v)^2+1 \\ n_2 \mid (ml_2+v)^2+1}} 1 \ll x^{1+\varepsilon}$$

and this is just what we need. Let $T^*(n_1, n_2; x, M)$ stand for the modified sum. For computing $T^*(n_1, n_2; x, M)$ we shall further reinterpret the congruence conditions (15), as we did that of (10). Letting c to be the solution of

$$c \equiv l_1 \left(\mathrm{mod}\ \frac{n_1}{(n_1, n_2)}\right), \quad c \equiv l_2 \left(\mathrm{mod}\ \frac{n_2}{(n_1, n_2)}\right),$$
$$c \equiv l_1 \ (\mathrm{mod}\ (n_1, n_2)), \quad 0 \leqq c < [n_1, n_2]$$

and defining $\Omega = cm + v$ we obtain

$$cm \leqq \Omega < (c+1)m \quad \text{and} \quad \Omega^2 + 1 \equiv 0 \ (\mathrm{mod}\ m[n_1, n_2]).$$

If the congruence conditions (15) are soluble then

$$l_1 \equiv l_2 \ (\mathrm{mod}\ (2, n_1, n_2)),$$

so $d = (n_1, n_2)/(n_1, n_2, l_1 - l_2)$ is odd number. From the congruences $(\mathrm{mod}\ n_1)$ and $(\mathrm{mod}\ n_2)$ we derive

$$v \equiv -m\frac{l_1+l_2}{2} \ (\mathrm{mod}\ d) \quad \text{and} \quad \left(m\frac{l_1-l_2}{2}\right)^2 + 1 \equiv 0 \ (\mathrm{mod}\ d).$$

Thus, if μ stands for the class of $m \ (\mathrm{mod}\ d)$ then

$$\left(\mu\frac{l_1-l_2}{2}\right)^2 + 1 \equiv 0 \ (\mathrm{mod}\ d) \quad \text{and} \quad \Omega \equiv \omega \ (\mathrm{mod}\ d)$$

where $\omega = \left(c - \frac{l_1+l_2}{2}\right)\mu = \frac{l_1-l_2}{2}\mu$. We deduce from the above discussion that

$$T^*(n_1, n_2; x, M) = \sum_{\substack{l_1, l_2 < x/M \\ l_1 \equiv l_2 \ (\mathrm{mod}\ (2, n_1, n_2))}} \sum_{\mu} \sum_{\substack{M < m < \min\left(2M, \frac{x}{l_1}, \frac{x}{l_2}\right), \ (m, n_1 n_2) = 1 \\ m \equiv \mu \ (\mathrm{mod}\ d), \ \Omega \equiv \omega \ (\mathrm{mod}\ d) \\ cm \leqq \Omega < (c+1)m, \ \Omega^2 + 1 \equiv 0 \ (\mathrm{mod}\ m[n_1, n_2])}} 1.$$

The inner sum is just $P(M_1, M; q, d, \mu, \omega, \alpha, \beta)$ with $M_1 = \min\left(2M, \frac{x}{l_1}, \frac{x}{l_2}\right)$, $q = [n_1, n_2]$, $\alpha = c/q$ and $\beta = (c+1)/q$, thus by Lemma 4 we obtain

$$T^*(n_1, n_2; x, M)$$

$$= \sum_{\substack{l_1, l_2 < x/M \\ l_1 \equiv l_2 \,(\mathrm{mod}\,(2, n_1, n_2))}} \rho(d) \left\{ \frac{\pi}{4q} \rho\left(\frac{q}{d}\right) \frac{A(q)}{\varphi(d)} \left[\min\left(2M, \frac{x}{l_1}, \frac{x}{l_2}\right) - M \right] + O((qM)^{3/4+\varepsilon}) \right\}$$

$$= \frac{\pi}{2} \frac{\rho([n_1, n_2])}{n_1 n_2} \frac{A(n_1 n_2)}{\varphi(n_1, n_2)} \cdot \sum_{\substack{l_1 < l_2 < x/M \\ l_1 \equiv l_2 \,(\mathrm{mod}\,(2, n_1, n_2))}} \varphi((n_1, n_2, l_1 - l_2)) \left[\min\left(2M, \frac{x}{l_2}\right) - M \right] + O\left(\frac{(n_1, n_2)}{n_1 n_2} \rho(n_1 n_2)\, x\right) + O\left(\left(\frac{x}{M}\right)^2 (n_1 n_2 M)^{3/4+\varepsilon}\right). \tag{16}$$

Writing $\varphi = \psi * 1$ we find $\psi(p) = p - 2$ and hence

$$\sum_{\substack{0 < l_1 < l_2 \\ l_1 \equiv l_2 \,(\mathrm{mod}\,(2, n_1, n_2))}} \varphi((n_1, n_2, l_1 - l_2)) = \sum_{t \mid (n_1, n_2)} \psi(t) \sum_{\substack{0 < l_1 < l_2 \\ l_1 \equiv l_2 \,(\mathrm{mod}\,(2, n_1, n_2)\, t)}} 1$$

$$= \sum_{t \mid (n_1, n_2)} \psi(t) \left\{ \frac{l_2}{(2, n_1, n_2)\, t} + O(1) \right\}$$

$$= \frac{d((n_1, n_2))}{(2, n_1, n_2)} \frac{\varphi((n_1, n_2))}{(n_1, n_2)} l_2 + O(\varphi((n_1, n_2)). \tag{17}$$

We have also

$$\sum_{l_2 < x/M} l_2 \left[\min\left(2M, \frac{x}{l_2}\right) - M \right] = \frac{x^2}{4M} + O(x). \tag{18}$$

Collecting together (16), (17) and (18), by

$$\rho(n_1, n_2) \frac{d((n_1, n_2))}{(2, n_1, n_2)} = \rho(n_1)\, \rho(n_2).$$

we obtain

$$T^*(n_1, n_2; x, M) = \frac{\pi x^2}{8M} \frac{\rho(n_1)\, \rho(n_2)}{n_1 n_2} A(n_1 n_2) + O\left(\frac{\rho(n_1 n_2)}{n_1 n_2} (n_1, n_2)\, x\right) + O\left(\left(\frac{x}{M}\right)^2 (n_1 n_2 M)^{3/4+\varepsilon}\right).$$

Therefore

$$W(x; M, N) = \frac{\pi}{8} \frac{x^2}{M} \sum_{n_1, n_2 < N} b_{n_1} b_{n_2} \frac{\rho(n_1)\, \rho(n_2)}{n_1 n_2} A(n_1 n_2) + O(x^{1+\varepsilon}) + O\left(\left(\frac{x}{M}\right)^2 N^2 (N^2 M)^{3/4+\varepsilon}\right). \tag{19}$$

This is the required estimate for $W(x; M, N)$.

If we introduce (9), (13) and (19) into (8) we find that the leading terms disappear throughout and we are left with

$$\mathscr{M}(x; M, N) \ll (x + N^{7/2} M^{-5/4} x^2) x^{\varepsilon}, \tag{20}$$

the implied constant depending on ε. The proof of Proposition 1 follows from (6), (8), (20), the Cauchy-Schwarz inequality and the inequality $\rho(m) \ll m^{\varepsilon}$.

Concluding this section we remark that assuming Hooley's Hypothesis R about incomplete Kloosterman-Ramanujan's sums (see [9]) the exponent 1/15 in Corollary of Proposition 1 may be increased to 1/9.

5. Estimation of the Sifting Functions

The following result is almost immediate consequence of Lemma 2 and Corollary of Proposition 1.

Proposition 2. *Let* $y = x^{16/15}$, $0 < \gamma < \frac{1}{2}$, $z = x^{\gamma}$, $z \leqq z_q < x^{\frac{1}{2}}$ *and* $0 \leqq c_q \leqq 1$. *Then for any* $\varepsilon > 0$ *we have*

$$\sum_{\substack{q < x^{1-\varepsilon} \\ (q, P(z_q)) = 1}} c_q S(\mathscr{A}_q; z_q)$$

$$< V(z) x \left\{ \sum_{\substack{q < x^{1-\varepsilon} \\ (q, P(z_q)) = 1}} c_q \frac{\rho(q)}{q} F\left(\frac{\log y/q}{\log z_q}\right) \frac{\log z}{\log z_q} + O_{\gamma}(\varepsilon) \right\},$$

provided $x > x_0(\varepsilon, \gamma)$. *A similar result holds for lower bound, the function* $F(s)$ *being replaced by* $f(s)$.

Proof. For any pair of real numbers Q, Z such that $1 \leqq Q < x^{1-\varepsilon}$ and $z \leqq Z < x^{\frac{1}{2}}$ define $H(Q, Z)$ to be the set of that integers q which satisfy simultaneous conditions

$$Q \leqq q < 2Q, \quad Z \leqq z_q < 2Z, \quad (q, P(z_q)) = 1.$$

Now, for each q from $H(Q, Z)$ we apply Lemma 2 to the sequence $\mathscr{B} = \mathscr{A}_q$ with the parameter $X = \dfrac{\rho(q)}{q} x$ as to have $r(\mathscr{B}, d) = r(\mathscr{A}_q, d) = r(\mathscr{A}, qd)$ for all $d \mid P(Z)$. We obtain

$$S(\mathscr{A}_q, z_q) \leqq S(\mathscr{A}_q, Z) < V(Z) \frac{\rho(q)}{q} x \left\{ F\left(\frac{\log MN}{\log Z}\right) + O(\eta) \right\}$$

$$+ 2^{\eta^{-10}} \sum_{m < M,\, m \mid P(Z)} \Big| \sum_{n < N,\, (m, n) = 1} b_n r(\mathscr{A}, qmn) \Big| \tag{21}$$

with arbitrary $\eta > 0$ and $M > 1$, $N > 1$ subject to $MN > c(\gamma, \eta)$. The implied constant in the symbol $O(\eta)$ depends at most on γ. The best choice for M and N is

$$M = x^{1-\varepsilon}/Q \quad \text{and} \quad N = x^{1/15-\varepsilon}.$$

If we multiply (21) throughout by c_q and sum with respect to q from $H(Q,Z)$ we get

$$\sum_{q\in H(Q,\,Z)} c_q S(\mathscr{A}_q, z_q)$$
$$< \left(\sum_{q\in H(Q,\,Z)} c_q \frac{\rho(q)}{q}\right) V(Z)\, x\left\{F\left(\frac{\log(y/Qx^{2\varepsilon})}{\log Z}\right)+O(\eta)\right\}$$
$$+O_\varepsilon(2^{\eta^{-10}}x^{1-\varepsilon}),$$

the latter error term being estimated by means of Corollary of Proposition 1. But we have

$$V(Z)=V(z)\frac{\log z}{\log z_q}\left(1+O\left(\frac{1}{\log z}\right)\right)$$

and

$$F\left(\frac{\log(y/Qx^{2\varepsilon})}{\log Z}\right)=F\left(\frac{\log(y/q)}{\log z_q}\right)+O_\gamma(\varepsilon).$$

Therefore, since the number of disjoint classes $H(Q,Z)$ covering all q can be less than $4(\log x)^2$, the summation over the classes completes the proof.

6. Estimation of $W(\mathscr{A}, z)$

Apart from the short sum

$$\sum_{x^{1-\varepsilon}\le p<x}\left(1-\frac{\log p}{\log x}\right)S(\mathscr{A}_p, z) \tag{22}$$

in (3) the weighted sum $W(\mathscr{A}, z)$ can be estimated by means of Proposition 2. For (22) the crude estimate

$$S(\mathscr{A}_p, z)\ll\frac{x}{p\log(x/p)}$$

is sufficient giving the bound $O(\varepsilon x/\log x)$. If we replace sums by integrals, as we can do by partial summation, we obtain

$$W(\mathscr{A}, z) > V(z)\, x\left\{f\left(\frac{\alpha}{\gamma}\right)+\int_\gamma^{\frac12}\int_\gamma^{u}(u-t)\frac{\gamma}{t}f\left(\frac{\alpha-u-t}{t}\right)\frac{du}{u}\frac{dt}{t}\right.$$
$$-\int_\gamma^{\frac12}\left[(1-2u)\frac{\gamma}{u}F\left(\frac{\alpha-u}{u}\right)+uF\left(\frac{\alpha-u}{\gamma}\right)\right]\frac{du}{u}$$
$$\left.-\int_{\frac12}^{1}(1-u)F\left(\frac{\alpha-u}{\gamma}\right)\frac{du}{u}-\varepsilon\right\}$$
$$=V(z)\,x\{W-\varepsilon\},\quad \text{say}$$

where $\alpha = 16/15$ and ε is any positive number, provided $x > x_0(\varepsilon, \gamma)$. Note that from Mertens prime number theorem we have

$$V(z) \sim \Gamma_G e^{-C} (\log z)^{-1} = \Gamma_G e^{-C} (\gamma \log x)^{-1}.$$

Now our aim is to prepare the integrals for estimation by manual calculations. We succeed partly in reducing the problem to the estimation of few logarithms of rational numbers. The functions $F(s)$ and $f(s)$ are elementary in the intervals $0 < s \leqq 3$ and $0 < s \leqq 4$ only, while our choice of $\gamma = 1/5$ necessitaties our considering the arguments of F and f beyond the elementary ranges.

From the differential-difference equations for $F(s)$ and $f(s)$ it is easy to derive the formulae

$$sF(s) = 2e^C \left\{1 + \int_2^{s-1} \log(u-1) \frac{du}{u}\right\} \quad \text{if } 3 \leqq s \leqq 5,$$

$$sf(s) = 2e^C \left\{\log(s-1) + \int_3^{s-1} \int_2^{t-1} \log(u-1) \frac{du}{u} \frac{dt}{t}\right\} \quad \text{if } 4 \leqq s \leqq 6.$$

Hence one can find that

$$W = 2e^C \gamma \left\{\log(\alpha - \gamma) - \frac{\alpha - 1}{\alpha} \log(\alpha - 1) - \int_2^{\frac{\alpha}{\gamma} - 2} \left[t \log \frac{\alpha(t+1)}{(\alpha-\gamma)(t+2)} + \log\left(1 - \frac{\gamma t}{\alpha - \gamma}\right)(t+1)\right] \frac{\log(t-1)}{t(t+1)} dt\right\}$$

and applying the inequality $\log(1+a) \leqq a$ we obtain

$$W \geqq 2e^C \gamma \left\{\log(\alpha - \gamma) - \frac{\alpha - 1}{\alpha} \log(\alpha - 1) - \frac{\alpha}{\alpha - \gamma} \int_4^{\alpha/\gamma} \left(\frac{1}{t} - \frac{\gamma}{\alpha}\right) \log(t-3)\, dt\right\}$$

$$\geqq 2e^C \gamma \left\{\log(\alpha - \gamma) - \frac{\alpha - 1}{\alpha} \log(\alpha - 1) - \frac{\alpha}{\alpha - \gamma} \int_4^{\alpha/\gamma} \left(\frac{1}{t} - \frac{\gamma}{\alpha}\right)(t-4)\, dt\right\}$$

$$= 2e^C \gamma \left\{\log(\alpha - \gamma) - \frac{\alpha - 1}{\alpha} \log(\alpha - 1) - \frac{\alpha}{\alpha - \gamma} \left(\frac{\alpha}{2\gamma} - 8\frac{\gamma}{\alpha} - 4 \log \frac{\alpha}{4\gamma}\right)\right\}.$$

On substituting $\alpha = 16/15$ and $\gamma = 1/5$ we finally get

$$W > 2e^C \gamma / 154 \quad \text{and} \quad W(\mathcal{A}, z) > \frac{\Gamma_G}{77} \frac{x}{\log x}.$$

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