manuscripta mathematica

Volume 81 1993

Printers: Brühlsche Universitätsdruckerei, Giessen
Printed in Germany – © Springer-Verlag Berlin Heidelberg 1993
Springer-Verlag GmbH & Co. KG, Berlin, Germany

Springer International

Contents

Covered by *Zentralblatt für Mathematik*
and
Current Mathematical Publications

manuscripta math. 81, 1 - 14 (1993)

manuscripta mathematica

A Dirichlet problem for an H-system with variable H

ENRIQUE LAMI DOZO - MARÍA CRISTINA MARIANI

We give conditions on H, a continuous and bounded real function in R^3, to obtain at least two solutions for the problem (Dir) below. H can be far from being constant in the sense of [9]. Our motivation is a better understanding of the Plateau problem for the prescribed mean curvature equation.

1 - Introduction

We consider the Dirichlet problem in the unit disc $B = \{(u,v) \in I\!R^2 ;\ u^2 + v^2 < 1\}$ for a vector function $X : \overline{B} \to I\!R^3$ which satisfies the equation of prescribed mean curvature

$$\text{(Dir)} \quad \begin{cases} \Delta X = 2H(X) X_u \wedge X_v \text{ in } B \\ X = g \text{ on } \partial B \end{cases}$$

where $X_u = \dfrac{\partial X}{\partial u}$, $X_v = \dfrac{\partial X}{\partial v}$, " $\wedge$ " denotes the exterior product in $I\!R^3$ and $H : I\!R^3 \to I\!R$ is a given continuous function. When H is bounded and g is in the Sobolev space $H^1(B, I\!R^3)$ we call $X \in H^1(B, I\!R^3)$ a *weak solution* of (Dir) if for every $\varphi \in C_0^1(B, I\!R^3)$

$$\text{(Sol)} \quad \begin{cases} \int_B (\nabla X \cdot \nabla \varphi + 2H(X) X_u \wedge X_v \cdot \varphi) = 0 \\ X \in T \equiv g + H_0^1(B, I\!R^3) \end{cases}$$

where $H_0^1(B, I\!R^3) = adh_{H^1} C_0^1(B, I\!R^3)$

If $H \equiv H_0 =$ const. in $I\!R$ or if H is near a constant, weak solutions are obtained as critical points in T of the following functional

$$D_H(X) = D(X) + 2V(X)$$

with

$$D(X) = \frac{1}{2}\int_B |\nabla X|^2$$

the Dirichlet integral and

$$V(X) = \frac{1}{3}\int_B Q(X)\cdot X_u \wedge X_v$$

the Hildebrandt volume where the field Q on $I\!R$ satisfies div $Q = 3H$, $Q(0) = 0$ (cf [6]).

For $Q = H_0\xi$ and $g \equiv g_0 =$ const. in $I\!R$, there is only one weak solution $X \equiv g_0$ [10].

For g non constant with $0 < |H_0|\|g\|_\infty < 1$ there are two weak solutions, a *local minimun of D_H in T* [5][7] called a ***stable solution*** and another weak solution which is not a local minimun called an ***unstable solution*** [3][8].

For H variable and H near $H_0 \in I\!R \setminus \{0\}$ for the following distance

$$[H - H_0] = \text{ess sup } (1 + |\xi|)(|H(\xi) - H_0| + |\nabla H(\xi)|) + |Q(\xi) - H_0\xi| + |dQ(\xi) - H_0 \text{ id }|$$

defined in [9], there are also two solutions (stable and unstable) [4].

We will consider prescribed H far from constants and obtain conditions to have at least two solutions.

Finally we recall that (Dir) is motivated for a better understanding of the Plateau's problem of finding a surface with prescribed mean curvature H which is supported by a given curve in $I\!R^3$.

Notations We denote $W^{1,p}(B, I\!R^3)$ the usual Sobolev spaces [1] and

$$H^1(B, I\!R^3) = W^{1,2}(B, I\!R^3)$$

For $X \in H^1(B, I\!R^3)$, $\|X\|_{L^2(\partial B, I\!R^3)} = \left(\int_{\partial B} |TrX|^2\right)^{1/2}$ where $Tr : H^1(B, I\!R^3) \to L^2(\partial B, I\!R^3)$ is the usual trace operator [1] and for $Y \in L^\infty(U, I\!R^n)$, we denote

$$\|Y\|_\infty = \underset{w\in U}{\text{ess sup}} |Y(w)|$$

Concerning D_H (resp V) we denote

$$dD_H(X)(\varphi) = \lim_{t \to 0} \left[\frac{D_H(X + t\varphi) - D_H(X)}{t} \right]$$

wherever this limit exists, (resp. $dV(X)(\varphi)$)

2 - Minima in convex subsets

We find a global minimum of D_H in T when Q lies in a specific convex subset of L^∞ and g is arbitrary in $H^1(B, \mathbb{R}^3)$. When g is harmonic and regular on $\overline{B}$ we find conditions on H ensuring that g is a local minimum in $T \cap W^{1,\infty}(B, \mathbb{R}^3)$.

We give proofs based on technical lemmas from section 4.

Theorem 1 *Let $H : \mathbb{R}^3 \to \mathbb{R}$ be continuous and bounded. If the function $Q \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ associated to H satisfies that $\|Q\|_\infty < 3/2$ and $\frac{\partial Q_i}{\partial \xi_j} \in L^\infty(\mathbb{R}^3)$ for $i \neq j$, then, given $g \in H^1(B, \mathbb{R}^3)$ the functional D_H has a minimum $\underline{X}$ in $T = g + H_0^1(B, \mathbb{R}^3)$ and $\underline{X}$ is a weak solution of (Dir).*

Proof. From

$$|\xi \cdot \eta \wedge \zeta| \leq |\xi| 1/2(|\eta|^2 + |\zeta|^2) \tag{1}$$

for vectors in $\mathbb{R}^3$ we have for $X \in H^1(B, \mathbb{R}^3)$ that

$$|D_H(X)| \leq D(X) + 2/3 \int_B |Q(X) \cdot X_u \wedge X_v| \leq D(X) + 2/3 \|Q\|_\infty D(X) \leq 2D(X)$$

so D_H is finite on T.

From $\|Q(X)\|_\infty < \frac{3}{2}$ for $X \in T$, D_H is weakly lower semicontinuous in $H^1(B, \mathbb{R}^3)$, hence in T, and from $X|_{\partial B} = g, D_H$ is coercive in T [7, Lemma 3.3, p 101]. But T is an affine subspace of $H^1(B, \mathbb{R}^3)$, hence weakly closed, so there is a minimum $\underline{X}$ of D_H in T.

If $\underline{X} \in H^2(B, \mathbb{R}^3)$, then $\underline{X}$ satisfies (Sol) by calculation (integrating by parts) of $D_H(\underline{X})(\varphi)$. For $\underline{X} \in H^1(B, \mathbb{R}^3)$, it follows by a density argument.

Theorem 2 *Let $g \in C^1(\overline{B}, \mathbb{R}^3)$ be harmonic in B. Suppose that $H \in C^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ and g satisfy*

i) $0 < \|H\|_\infty \|g\|_\infty < 3/2$ *and* $Q \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

ii) *For some* $c > \|\nabla g\|_\infty$, *for all* $\xi \in \mathbb{R}^3$

$$|H(\xi)| \leq (\lambda_1/c^2)(|\xi| - \|g\|_\infty)_+$$

where $\lambda_1 > 0$ *is the first eingenvalue of* $-\Delta$ *in* $H_0^1(B)$.

Then g *is a local minimum of* D_H *in* $T \cap W^{1,\infty}(B, \mathbb{R}^3)$.

Proof. From ii) $H(g) = 0$ on B and g is harmonic, so (Dir) holds trivially.
Now, we choose $c_1 > 0$ and $\delta > 0$ such that $\|\nabla g\|_\infty < c_1 < c$, $\delta < \min\{c_1 - \|\nabla g\|_\infty, \frac{3}{2\|H\|_\infty} - \|g\|_\infty\}$ and define

$$M = \{X \in T; \|X - g\|_\infty + \|\nabla(X - g)\|_\infty \leq \delta\}$$

We have that M is a nonempty, convex, closed and bounded subset of $H^1(B, \mathbb{R}^3)$, then M is a weakly compact subset of $H^1(B, \mathbb{R}^3)$, D_H is weakly lower semicontinuous in M because

$$\|Q(X)\|_\infty \leq \|H\|_\infty \|X\|_\infty \leq \|H\|_\infty(\|X - g\|_\infty + \|g\|_\infty) < 3/2$$

for $X \in M$ [7,lemma 3.3, p 101].
Hence, there exists $\overline{X} \in M$ such that

$$D_H(\overline{X}) = \inf_{X \in M} D_H(X)$$

For any $X \in M$,

$$|H(X)| \leq \frac{\lambda_1}{c^2}(|X| - \|g\|_\infty)_+ \leq \frac{\lambda_1}{c^2}(|X| - |g|)_+ \leq \frac{\lambda_1}{c^2}|X - g|$$

a.e. in B and

$$\int_B 2H(X)X_u \wedge X_v \cdot (X - g) \geq -\int_B 2|X_u \wedge X_v|\frac{\lambda_1}{c^2}|X - g|^2 \geq$$
$$-\int_B c_1^2 \frac{\lambda_1}{c^2}|X - g|^2 \geq -\left(\frac{c_1}{c}\right)^2 \|\nabla(X - g)\|_2^2$$

Also $\int_B \nabla g \cdot \nabla(X - g) = -\int_B \Delta g \cdot (X - g) = 0$, because $X - g \in H_0^1(B, \mathbb{R}^3)$.

Finally, from $X_u \wedge X_v$ in L^∞, the first equality in (Sol) holds for X and φ in $H_0^1(B, \mathbb{R}^3)$ since

$$|dV(X)(\varphi)| \leq 3\left(\frac{|B|}{\lambda_1}\right)^{1/2} \|H(X)\|_\infty \|X_u \wedge X_v\|_\infty \|\varphi\|_{H_0^1}$$

for the same λ_1. So, Lemma 1 gives

$$0 \geq dD_H(\overline{X})(\overline{X}-g) \geq \int_B |\nabla(\overline{X}-g)|^2 - \left(\frac{c_1}{c}\right)^2 \|\nabla(\overline{X}-g)\|_2^2 =$$
$$= (1-\left(\frac{c_1}{c}\right)^2)\|\nabla(\overline{X}-g)\|$$

Hence $\overline{X}=g$ and M is a neighborhood of g in $T\cap W^{1,\infty}(B, I\!R^3)$.

A class of functions H satisfying the assumptions of Theorem 1 is given by

$$H(\xi)=\begin{cases} H_0 & \text{if } \xi_i \in (a_i,b_i) \\ 0 & \text{if } \xi_i \notin (a_i-\varepsilon, b_i+\varepsilon)\end{cases}$$

with H_0, ε, a_i, b_i, $i=1,2,3$ positive numbers such that $0 < a_i-\varepsilon$, $a_i < b_i$ and $H_0\left(\sum_{i=1}^{3}(b_i+\varepsilon)^2\right)^{1/2} < 3/2$. Moreover, we suppose that $H\in C^1(I\!R^3)$ and $\|H\|_\infty = H_0$.
Now, from

$$Q(\xi)=\left(\int_0^{\xi_1} H(s,\xi_2,\xi_3)\,ds\,,\int_0^{\xi_2} H(\xi_1,s,\xi_2)\,ds\,,\int_0^{\xi_3} H(\xi_1,\xi_2,s)\,ds\right)$$

we have that

$$|Q(\xi)| \leq \left(\sum_{i=1}^{3}\left(\int_0^{b_i+\varepsilon} H_0\right)^2\right)^{1/2}$$

and then, $\|Q\|_\infty < 3/2$

Finally, $\dfrac{\partial Q_i}{\partial \xi_j} \in L^\infty(I\!R^3)$ for $i\neq j$, $i,j=1,2,3$ because $\dfrac{\partial H}{\partial \xi_i} \in C_0(I\!R^3)$ $i=1,2,3$.

3 - Weak solutions of (Dir) via The Mountain Pass Lemma

We search now the possibility to find other weak solutions of (Dir) in T, taking account the case $H=H_0\in I\!R$ studied in [3][7].

First we give a result which is a variant of The Mountain Pass Lemma [2],[7] and second, sufficient conditions to get from critical points of D_H in convenient closed convex subsets of T, weak solutions of (Dir).

Again, we give proofs based on Lemmas from section 4.

We consider a fixed data $g \in W^{1,\infty}$. For $k > 0$ in $I\!R$, denote

$$M(k) = \{X \in T; \|\nabla(X - g)\|_\infty < k\}$$

$M(k)$ is nonempty and convex, so, $\overline{M(k)}$ is nonempty, convex and closed, and

$$\overline{M(k)} = \{X \in T; \|\nabla(X - g)\|_\infty \leq k\}$$

The slope of D_H in $\overline{M(k)}$ is given, for $X \in \overline{M(k)}$, by

$$\rho(X) = \sup \{dD_H(X)(X - Y); Y \in \overline{M(k)}\}$$

Finally, we define for $\beta \in I\!R$,

$$M_\beta = \{X \in \overline{M(k)}; D_H(X) < \beta\} \text{ and}$$
$$K_\beta = \{X \in \overline{M(k)}; D_H(X) = \beta, \rho(X) = 0\}$$

The following result is a variant of The Mountain Pass Lemma.

Theorem 3 *Let $H \in L^\infty(I\!R^3) \cap C(I\!R^3)$, $Q \in L^\infty(I\!R^3, I\!R^3)$ and suppose that $X_0 \in W^{1,\infty}(B, I\!R^3)$ is a local minimum of D_H in T. If for some $k > 0$ in $I\!R$ there exists $X_1 \in \overline{M(k)}$ such that $D_H(X_1) < D_H(X_0)$, then*

$$\beta = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} D_H(\gamma(t))$$

where $\Gamma = \{\gamma : [0,1] \to \overline{M(k)},\ \gamma$ continuous, $\gamma(0) = X_0,\ \gamma(1) = X_1\}$, is attained by D_H at $X \in \overline{M(k)}$ with $\rho(X) = 0$.

Remark At least one of these X is not a local minimum of D_H in T. We will call it an unstable ρ-critical point.

Corollary 1 *If X_0 and X_1 are in some $\overline{M(k)}$ and both are local minima of D_H in T, with $D_H(X_0) \neq D_H(X_1)$ then there exists an unstable ρ-critical point X in $\overline{M(k)}$.*

Corollary 2 *If there are no unstable critical points in $\overline{M(k)}$ then the set of local minima of D_H in $\overline{M(k)}$ is connected and D_H is constant on it.*

Proof of Theorem 3. It is analogous to the one of Theorem 3.3[7] p. 125. We choose $k > 0$ such that $X_0, X_1 \in \overline{M(k)}$.

For $\bar{\varepsilon} = D_H(X_0) - D_H(X_1) > 0$ and for any neighborhood N of the set K_β in $\overline{M(k)}$, there exists a number $\varepsilon \in (0, \bar{\varepsilon})$ and a deformation Φ with the properties stated in Lemma 4.

If $K_\beta = \phi$, we choose $N = \phi$. By definition of β, there is $\gamma \in \Gamma$ such that $\sup_{t\in[0,1]} D_H(\gamma(t)) < \beta + \varepsilon$

Then $\gamma_1 = \Phi(1,\gamma) \in \Gamma$ because $\gamma_1(0) = \Phi(1, X_0) = X_0$ from $\rho(X_0) = 0$. Also $\gamma_1(1) = \Phi(1, X) = X_1$ because $\bar{\varepsilon} = D_H(X_0) - D_H(X_1) \leq \beta - D_H(X_1)$.
But $\gamma_1([0,1]) \subset M_{\beta-\varepsilon}$ contradicting the definiton of β.

We end as in [7] Theorem 3.3, p. 125.

Proof of Corollary 1 and Corollary 2. Corollary 1 being immediate we proceed with the proof of Corollary 2.

If K_β consists only of local minima of D_H in T, there exists a neighborhood N of K_β in $\overline{M(k)}$ such that N and $\overline{M_\beta} \setminus K_\beta$ are disjoint and then, $M_{\beta-\varepsilon} \cap N = \phi$ for any $\varepsilon > 0$.

By Lemma 4 we can choose Φ corresponding to this N, $\bar{\varepsilon} = 1$ and $\gamma \subset M_{\beta+\varepsilon}$. By property iii) of Φ we obtain that $\gamma_1 = \Phi(1,\gamma) \in \Gamma$, $\gamma_1 \subset M_{\beta-\varepsilon} \cup N$. Hence, $\gamma_1 \subset N$ and X_0, X_1 belong to the same connected component of K_β, and in particular $D_H(X_0) = D_H(X_1) = \beta$.

Now, we give sufficient conditions to get from zero slope a weak solution of (Dir).

Theorem 4 *Let $X \in \overline{M(k)}$ satisfy one of the following conditions:*

i) $X \in M(k)$

ii) $|H(X)(X-g)| \leq 1$ a.e. *in* B *and* $\int_B \nabla X \cdot \nabla g \leq 0$

Then, if $\rho(X) = 0$, X is a weak solution of (Dir)

Proof. From $\|\nabla(X-g)\|_\infty < k$, we obtain that for ε small enough and $\varphi \in C_0^1(B, \mathbb{R}^3)$, $Y = X \pm \varepsilon\varphi \in \overline{M(k)}$.

Then, $dD_H(X)(X - (X \pm \varepsilon\varphi)) = dD_H(X)(\mp\varepsilon\varphi) \leq \rho(X) = 0$ and we deduce $dD_H(X)(\varphi) = 0$

Finally, if $\|H(X)(X-g)\|_\infty \leq 1$ and $\int_B \nabla X \cdot \nabla g \leq 0$, and we suppose that

$dD_H(X)(X-g)<0$, we have that

$$0 > dD_H(X)(X-g) = \int_B \nabla X \cdot \nabla(X-g) + 2H(X)X_u \wedge X_v \cdot (X-g)$$
$$\geq \int_B |\nabla X|^2 - 2|X_u \wedge X_v| - \int_B \nabla X \cdot \nabla g \geq 0.$$

Then, from Lemma 1 X is a weak solution of (Dir).

Theorem 5 *Let $g \in C^1(\overline{B}, I\!R^3)$ be harmonic in B. If M is a nonempty convex subset of T such that*

i) *For some $\delta > 0$, $g + \varphi \in M$ for any $\varphi \in C_0^1(B, I\!R^3)$ with $\|\nabla\varphi\|_\infty \leq \delta$*

ii) $c = \sup_{X \in M} \|\nabla X\|_\infty < +\infty$
and if $H \in C^1(I\!R^3) \cap W^{1,\infty}(I\!R^3)$ satisfies $c^2|H(\xi)| \leq \lambda_1(|\xi| - \|g\|_\infty)_+$ where $\lambda_1 > 0$ is the first eingenvalue of $-\Delta$ in $H_0^1(B)$.

Then any $X \in M$ with slope $\rho(X) = 0$ is a weak solution of (Dir).

Proof. From Lemma 1, X is a weak solution of (Dir) or $dD_H(X)(X-g) < 0$.

The proof that the case $dD_H(X)(X-g) < 0$ is not possible is the same that the one in Theorem 2.

A class of functions H shuch that D_H has a stable and unstable critical point, thanks to Theorem 3 and 4, is as follows:

Let $g(u,v) = X_0(u,v) = (0,a,0), a \in I\!R$, and let $H \in C^1(I\!R^3)$ satisfy

$$H(\xi) = \begin{cases} H_0 & \text{if } \xi_1^2 + \xi_2^2 \leq R^2 \text{ and } \alpha_1 \leq \xi_3 \leq \alpha_2 \\ 0 & \text{if } \xi_1^2 + \xi_2^2 > (R+\varepsilon)^2 \text{ or } \xi_3 \notin (\alpha_1 - \varepsilon, \alpha_2 + \varepsilon) \end{cases}$$

and $|H(\xi)| \leq |H_0|$, where $H_0 \in I\!R$, $\alpha_1, \alpha_2, \varepsilon$, R positive will be fixed later with $\varepsilon < \alpha_1 < \alpha_2$. Then H is zero near $(0,a,0)$ in $I\!R^3$ (the image of g) for any $a \in I\!R$, and $\|Q\|_\infty \leq |H_0|\big((\alpha_2+\varepsilon)^2 + (R+\varepsilon)^2\big)^{\frac{1}{2}}$.

Consider $X_1(u,v) = \big(f(u,v), a, h(u,v)\big)$ with $f, h \in C^\infty(\overline{B})$ and $f = h = 0$ on ∂B. We fix $R^2 \geq \|f\|_\infty^2 + a^2$. Then $X_{1u} \wedge X_{1v} = (0, h_u f_v - h_v f_u, 0)$ and

$$\begin{aligned} 3V(X_1) = &\int_{\alpha_1 \leq h(u,v) \leq \alpha_2} aH_0(h_u f_v - h_v f_u) \\ &+ \int_{\alpha_2 \leq h(u,v) \leq \alpha_2 + \varepsilon} Q_2(X_1)(h_u f_v - h_v f_u) \\ &+ \int_{\alpha_1 - \varepsilon \leq h(u,v) \leq \alpha_1} Q_2(X_1)(h_u f_v - h_v f_u) \end{aligned} \tag{2}$$

because $Q_2(X_1) = \int_0^a H\big(f(u,v), s, h(u,v)\big)\,\mathrm{ds} =$

$$\begin{cases} \int_0^a H_0\,\mathrm{ds} = aH_0 & \text{if } \alpha_1 \le h(u,v) \le \alpha_2 \\ 0 & \text{if } h(u,v) < \alpha_1 - \varepsilon \text{ or } h(u,v) > \alpha_2 + \varepsilon. \end{cases}$$

Besides $X_1 \in \overline{M(k)}$ for $k \ge \left(\|\nabla f\|_\infty^2 + \|\nabla h\|_\infty^2\right)^{\frac{1}{2}}$.

We take α_1, α_2, f, h such that $\int_{\alpha_1 \le h(u,v) \le \alpha_2}(h_u f_v - h_v f_u) \ne 0$, we choose $|H_0| = \left(\|f\|_\infty^2 + \|h\|_\infty^2\right)^{-\frac{1}{2}}$, $a \in I\!R$ verifying $D(X_1) + \frac{2}{3}aH_0 \int_{\alpha_1 \le h(u,v) \le \alpha_2}(h_u f_v - h_v f_u) < 0$, and ε sufficiently close to zero, so $D_H(X_1) < D_H(g) = 0$.

For $\varphi \in H_0^1(B, I\!R^3)$, the formula $D_H(g+\varphi) = D_H(g) + dD_H(g)(\varphi) + \frac{1}{2}d^2D_H(g)(\varphi,\varphi) + o(\|\nabla\varphi\|_2^2)$ holds. To prove it we remark that $D_H(g) = dD_H(g)(\varphi) = 0$ and that $d^2V(g) = 0$, because

$$\begin{aligned} d^2V(X)(\varphi,\psi) = & \int_B (\nabla H(X)\cdot\psi) X_u \wedge X_v \cdot \varphi + \int_B (\nabla H(X)\cdot X_v)\cdot \psi_u \wedge \varphi \\ & + \int_B (\nabla H(X)\cdot X_u)\varphi \wedge \psi_v \cdot X + \int_B H(X)(\psi_u \wedge \varphi_v + \varphi_u \wedge \psi_v)\cdot X \end{aligned}$$

Finally, Lemma A.8 in [**3**] gives

$$\left| \frac{D_H(g+\varphi) - \frac{1}{2}d^2D(g)(\varphi,\varphi)}{\|\nabla\varphi\|_2^2} \right| = \left| \frac{\frac{2}{3}\int_B Q(g+\varphi)\cdot\varphi_u \wedge \varphi_v}{\|\nabla\varphi\|_2^2} \right| \le \frac{2}{3}\bar{c}\|\nabla(Q(g+\varphi))\|_2,$$

and the formula follows from $\|\nabla\big(Q(g+\varphi)\big)\|_2 \to 0$ if $\|\nabla\varphi\|_2 \to 0$, since $\nabla(Q(g)) = 0$.

Hence g is a local minimum of D_H in T, and $D_H(X_1) < 0 = D_H(g)$. If $M_k = \{X \in \overline{M(k)}; \|X - g\|_\infty \le \frac{1}{|H_0|}$ and ρ is the slope of D_H in M_k, from Theorems 3,4 we obtain that there exists $X_2 \in M_k$ with $\rho(X_2) = 0$, and X_2 is a weak solution of (Dir).

To apply Theorems 3 and 5, we take $g(u,v) = (0, a, a^2)$, the same H which is zero in a neighborhood of $(0,a,a^2)$ if $\alpha_1 - \varepsilon > a^2$.

For $c > 0$, we would want ii) in Theorem 2. To obtain it, we take $\alpha_1 = (a^2 + a^4)^{1/2} + \varepsilon_1 = \|g\|_\infty + \varepsilon_1$ with $\varepsilon_1 > 0$ and we choose $H_0 = \frac{\lambda_1\varepsilon_1}{c^2}$.

Denote now $X_1(u,v) = (f(u,v), a, a^2 + h(u,v))$ and fix $c = \|\nabla X_1\|_\infty$. The first term in (2) is

(3)
$$\int_{\alpha_1 < a^2 + h(u,v) < \alpha_2} H_0 a(h_u f_v - h_v f_u) = \frac{\lambda_1}{c^2}\varepsilon_1 a \int_{(a^2+a^4)^{\frac{1}{2}} - a^2 + \varepsilon_1 < h(u,v) < \alpha_2 - a^2} (h_u f_u - h_v f_u).$$

Now, we fix f, h, $\varepsilon_1, \varepsilon_2$ to have $\int_{\frac{1}{2}+\varepsilon_1<h(u,v)<\varepsilon_2}(h_u f_v - h_v f_u) \quad < 0$.

Then, from $\lim_{|a|\to+\infty}\left[(a^2+a^4)^{\frac{1}{2}} - a^2\right] = \frac{1}{2}$, we can take $a > 0$ such that

$$D(X_1) + \frac{2}{3}\frac{\lambda_1 \varepsilon_1 a}{c^2}\int_{(a^2+a^4)^{\frac{1}{2}}-a^2+\varepsilon_1<h(u,v)<\varepsilon_2}(h_u f_v - h_v f_u) \qquad < 0.$$

Finally, we fix $\alpha_2 = \varepsilon_2 + a^2$ in(3), and ε sufficiently close to 0, so from (2), $D_H(X_1) < D_H(g) = 0$ and $\alpha_1 - \varepsilon > a^2$.

As before, g is a local minimum of D_H in T, so Theorem 3 gives X with $\rho(X) = 0$ in $\overline{M(c)}$ and from Theorem 5, X is a weak solution of (Dir).

Remark We can choose $f(r,\alpha) = (1-r^2)r^2\int_0^\alpha \sin^2 t \, dt$ and $h(r,\alpha) = (1-r^2)$ in polar coordinates for both cases. Also $g = (0, a, a^2)$ and the same H are an example for theorem 2.

4 - Technical Lemmas

Lemma 1 *Let $g \in H^1(B, I\!R^3)$. Suppose M a nonempty subset of T such that for some $\delta > 0$, $g + \varphi \in M$ for any $\varphi \in C_0^1(B, I\!R^3)$ with $\|\nabla\varphi\|_\infty \leq \delta$ and such that $dD_H(X)(X-Y)$ exists for X and Y in M. Then, any X with $dD_H(X)(X-g) = 0$ and $dD_H(X)(X-Y) \leq 0$ for all $Y \in M$, is a weak solution of (Dir).*

Proof. Given $\varphi \in C_0^1(B, I\!R^3)$ verifying $\|\nabla\varphi\|_\infty \leq \delta$, let $Y = g + \varphi$. Then, from $0 \geq dD_H(X)(X-Y) = dD_H(X)(X-g) - dD_H(X)(\varphi)$ we conclude $dD_H(X)(\varphi) \geq 0$.

Replacing φ by $-\varphi$, also the opposite inequality holds true, and X is a solution of (Dir).

Lemma 2 *D_H, dD_H and ρ are continuous on $\overline{M(k)}$.*

Proof. Suppose $X_m \to X$ in $H^1(B, I\!R^3)$, X_m, $X \in \overline{M(k)}$

a) D_H is continuous on $\overline{M(k)}$

For each $\delta > 0$, by Egorov's Theorem, there exists $B^\delta \subset B$ with measure $|B^\delta| < \delta$ and a subsecuence, still denoted X_m such that $X_m \to X$ uniformly in $B \setminus B^\delta$. Hence, $H(X_m) \to H(X)$ and $Q(X_m) \to Q(X)$ uniformly in $B \setminus B^\delta$, because $H \in L^\infty(I\!R^3) \cap C(I\!R^3)$ and $Q \in L^\infty(I\!R^3, I\!R^3) \cap C(I\!R^3, I\!R^3)$.

From

$$|V(X_m) - V(X)| \le 2/3 \int_{B\setminus B^\delta} |Q(X_m) - Q(X)||X_{mu} \wedge X_{mv}| +$$
$$2/3 \int_{B^\delta} |Q(X_m) - Q(X)||X_{mu} \wedge X_{mv}| + 2/3\|Q\|_\infty \int_B |X_{mu} \wedge X_{mv} - X_u \wedge X_v|$$

we have that the first term tends to 0 if m grows for a given $\delta > 0$.
Since $|X_{mu} \wedge X_{mv}| \le c^2$ on $\overline{M(k)}$ with $c = k + \|\nabla_g\|_\infty$ and $|Q(X_m) - Q(X)| \le 2\|Q\|_\infty$ a.e. in B, we choose δ to have the second term less than a given ε.
We estimate the last term by

$$\int_B |X_{mu} \wedge X_{mv} - X_u \wedge X_v| \le 2c \int_B |\nabla(X_m - X)| \le 2c|B|^{1/2}\|\nabla(X_m - X)\|_2$$

so it tends also to 0 with $m \to +\infty$.

b) dD_H is continuous.

For this last assertion, we take $\varphi \in H_0^1(B, I\!R^3)$ and estimate

$$\begin{aligned} &|dV(X_m)(\varphi) - dV(X)(\varphi)| \\ (4) \qquad &\le c^2 \int_B |H(X_m) - H(X)||\varphi| + \|H\|_\infty 2c\|\nabla(X_m - X)\|_2\|\varphi\|_2 \\ &\le [c^2\|H(X_n) - H(X)\|_2 + 2c\|H\|_\infty\|\nabla(X_m - X)\|_2]\|\varphi\|_2 \end{aligned}$$

c) ρ is continuous.

Let $(X_n) \subset \overline{M(k)}$ such that $X_n \to X$ in $H^1(B, I\!R^3)$ and fix $\varepsilon > 0$. For $m \in \mathbf{N}$ there exists $Y_m \in \overline{M(k)}$ satisfying $\rho(X_m) \le dD_H(X_m)(X_m - Y_m) + \varepsilon/2$.

We suppose $\rho(X_m) \ge \rho(X)$, then

$$0 \le \rho(X_m) - \rho(X) \le dD_H(X_m)(X_m - Y_m) + \varepsilon/2 - dD_H(X)(X - Y_m) =$$
$$dD_H(X_m)(X_m - X) + dD_H(X_m)(X - Y_m) - dD_H(X)(X - Y_m) + \varepsilon/2$$

From

$$|dD_H(X_m)(X_m - X)| \le \|\nabla X_m\|_2\|\nabla(X_m - X)\|_2 + 2c^2\|H\|_\infty\|X_m - X\|_1$$

and from

$$|(dD_H(X_m) - dD_H(X))(X - Y_m)| \le |(dD(X_m) - dD(X))(X - Y_m)| +$$
$$2/3|(dV(X_m) - dV(X))(X - Y_m)|$$

using (4) with $\varphi = X - Y_m \in H_0^1(B, I\!R^3)$, we deduce that $\lim \rho(X_n) = \rho(X)$

Now, if we denote $\widetilde{\overline{M(k)}} = \{X \in \overline{M(k)}; \rho(X) \neq 0\}$, we call $\widetilde{e}$ a pseudo gradient vector field for D_H on $\overline{M(k)}$ a Lipschitz continuous mapping $\widetilde{e} : \widetilde{\overline{M(k)}} \to H^1(B, I\!R^3)$ such that

i) $\widetilde{e}(X) + X \in \overline{M(k)}$ for $X \in \widetilde{\overline{M(k)}}$

ii) There exists $c_1 > 0$ in $I\!R$ such that

a) $\|\widetilde{e}(X)\|_{H^1} \leq \delta$ with $\delta =$ diameter of $\overline{M(k)}$

b) $dD_H(X)(\widetilde{e}(X)) < -\min\{c_1^{-1}\rho(X)^2, 1\}$

Our slope ρ and pseudo gradient $\widetilde{e}$ being very similar to those in [7], p. 34, so we have that there exists a pseudo gradient vector field $\widetilde{e}$ for D_H on $\overline{M(k)}$.

Now, we will prove that D_H satisfy a Palais-Smale type condition in $\overline{M(k)}$.

Lemma 3 *Any sequence $(X_m) \subset \overline{M(k)}$ such that $\lim_{m\to+\infty} \rho(X_m) = 0$ is relatively compact.*

Proof. $(X_m) \subset \overline{M(k)}$, then $\|X_m - g\|_{H_0^1} \leq k|B|^{1/2}$ and there exists a subsequenc, still denoted (X_m) and $X \in \overline{M(k)}$ such that $X_m - g \to X - g$ weakly in $H_0^1(B, I\!R^3)$.

If $\varphi_m = X_m - X$, we have that

$$dD_H(X_m)(\varphi_m) \leq \rho(X_m) \underset{m\to+\infty}{\longrightarrow} 0.$$

But

$$\begin{aligned} dD_H(X_m)(\varphi_m) &= \int_B \nabla X_m \cdot \nabla\varphi_m + 2\int_B H(X_m) X_{mu} \wedge X_{mv} \cdot \varphi_m \\ &= \|\nabla\varphi_m\|_2^2 + \int_B \nabla X \cdot \nabla\varphi_m + 2\int_B H(X_m) X_{mu} \wedge X_{mv} \cdot \varphi_m. \end{aligned}$$

Now, $\int_B \nabla X \cdot \nabla\varphi_m \to 0$ because $\varphi_m \to 0$ weakly in $H_0^1(B, I\!R^3)$ and

$$|2\int_B H(X_m) X_{mu} \wedge X_{mv} \cdot \varphi_m| \leq c^2 \|H\|_\infty |B|^{1/2} \|\varphi_m\|_2.$$

By the Rellich-Kondrakov Theorem $\varphi_m \to 0$ strongly in $L^2(B, I\!R^3)$, hence

$$\|\nabla\varphi_m\|_2^2 \leq \rho(X_m) - \int_B \nabla X \cdot \nabla\varphi_m - 2\int_B H(X_m) X_{mu} \wedge X_{mv} \cdot \varphi_m \underset{m\to+\infty}{\longrightarrow} 0$$

and $X_m \to X$ in $H^1(B, I\!R^3)$.

Now, recalling $M_\beta = \{X \in \overline{M(k)}; D_H(X) < \beta\}$, $K_\beta = \{X \in \overline{M(k)}; D_H(X) = \beta, \rho(X) = 0\}$; exactly as [7], Lemma 1.9, p. 36, and thanks to the preceding lemmas we obtain the following result

Lemma 4 *Let $\beta \in I\!R$, $\bar{\varepsilon} > 0$, and suppose N is a neighborhood of K_β in $\overline{M(k)}$.*

Then there exists a number $\varepsilon \in (0, \bar{\varepsilon})$ and a continuous one-parameter family Φ : $[0,1] \times \overline{M(k)} \to \overline{M(k)}$ of homeomorphisms $\Phi(t, \cdot)$ of $\overline{M(k)}$ having the properties

i) $\Phi(t, X) = X$ *if* $t = 0$, *or if* $|D_H(X) - \beta| \geq \bar{\varepsilon}$ *or if* $\rho(X) = 0$.

ii) $D_H(\Phi(t, X))$ *is non-increasing in* t.

iii) $\Phi(1, M_{\beta+\varepsilon} \setminus N) \subset M_{\beta-\varepsilon}$ *and* $\Phi(1, M_{\beta+\varepsilon}) \subset M_{\beta-\varepsilon} \cup N$.

Acknowledgements. We thank Michael Struwe and the referee for useful remarks and suggestions.

References

[1] Adams, Robert: Sobolev Spaces, New York, Academic Press, 1975

[2] Ambrosetti, A. - Rabinowitz, P.H.: Dual variational methods in critical point theory and applications, J-Functional Analysis 14 (1973) 349- 381

[3] Brezis, H. - Coron, J.M.: Multiple solutions of H-Systems and Rellich's conjeture, Comm. Pure Appl. Math. 37 (1984), 149-187

[4] Wang, Guofang: The Dirichlet Problem for the equation of prescribed mean curvature, Analyse Nonlinéaire 9 (1992), 643-655

[5] Hildebrandt, S.: On the Plateau problem for surfaces of constant mean curvature, Comm. Pure Appl. Math. 23 (1970), 97-114

[6] Hildebrandt, S.: Randwertprobleme für Flächen mit Vorgeschriebenes mittlerer Krümmung und Anwendungen auf die kapillaritätstheorie Teil I. Fest vorgegebener Rand. Math - Z. 112 (1969), 205-213

[7] Struwe, M.: Plateau's problem and the calculus of variations. (Math. Notes 35, Princeton University Press) Princeton 1989

[8] Struwe, M.: Non-uniquenes in the Plateau problem for surfaces of constant mean curvature, Arch. Rat. Mech. Anal. 93 (1986), 135-157

[9] Struwe, M.: Multiple solutions to the Dirichlet problem for the equation of prescribed mean curvature, Preprint

[10] Wente, H.C.: The differential equation $\Delta X = 2HX_u \wedge X_v$ with vanishing boundary values, Proc. AMS 50 (1975), 59-77

ENRIQUE LAMI DOZO
Inst. Argentino de Matemática, CONICET
Viamonte 1636, 1er. cuerpo, 1er. piso
1055 – Buenos Aires, Argentina
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales, UBA

MARÍA CRISTINA MARIANI
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales, UBA
CONICET

(Received April 24, 1992;
in revised form May 17, 1993)

manuscripta math. 81, 15 - 39 (1993)

manuscripta mathematica

GLOBAL EXISTENCE OF SMALL RADIALLY SYMMETRIC SOLUTIONS TO QUADRATIC NONLINEAR WAVE EQUATIONS IN AN EXTERIOR DOMAIN

NAKAO HAYASHI

We study the initial boundary value problem for the nonlinear wave equation :

$$(*)\qquad \begin{cases} \partial_t^2 u - (\partial_r^2 + \frac{n-1}{r}\partial_r)u = F(\partial_t u, \partial_t^2 u), & t \in \mathbb{R}^+, R < r < \infty, \\ u(0,r) = \epsilon_0 u_0(r), \partial_t u(0,r) = \epsilon_0 u_1(r), & R < r < \infty, \\ u(t,R) = 0, \quad t \in \mathbb{R}^+, \end{cases}$$

where $n = 4, 5$, u_0, u_1 are real valued functions and ϵ_0 is a sufficiently small positive constant. In this paper we shall show small solutions to (*) exist globally in time under the condition that the nonlinear term $F : \mathbb{R}^2 \to \mathbb{R}$ is quadratic with respect to $\partial_t u$ and $\partial_t^2 u$.

§1 Introduction. In this paper we study the nonlinear wave equation

$$(1.1)\qquad \begin{cases} \partial_t^2 u - (\partial_r^2 + \frac{n-1}{r}\partial_r)u = F(\partial_t u, \partial_t^2 u), & t \in \mathbb{R}^+, R < r < \infty, \\ u(0,r) = \epsilon_0 u_0(r), \partial_t u(0,r) = \epsilon_0 u_1(r), & R < r < \infty \end{cases}$$

with the homogeneous Dirichlet boundary condition

$$(1.2)\qquad u(t,R) = 0, \quad t \in \mathbb{R}^+,$$

where $n = 4, 5$, u_0, u_1 are real valued functions, ϵ_0 is a sufficiently small positive constant and the nonlinear term $F : \mathbb{R}^2 \to \mathbb{R}$ is quadratic with respect to $\partial_t u$ and $\partial_t^2 u$.

Remark 1.1. By using our method in this paper mentioned below we can treat the more general nonlinear terms such that $F(\partial_t u, \partial_r\partial_t u, \partial_t^2 u)$. However it seems that our method is not applicable to the case when F depends only on $\partial_r u$ and $\partial_r^2 u$. The codition that the nonlinear term F contains the time derivative of unknown function is needed to prove Lemma 3.1 (3.3) which plays an important role to derive time decay estimates of solutions.

Notation and function spaces. We let $\partial_j = \partial/\partial x_j$, $\partial_r = \partial/\partial r$ and $\partial_u = \partial/\partial u$, where $r = (\sum_j^n x_j^2)^{1/2}$. The following operators are needed to obtain the result stated below :

$$L_r = r\partial_t + t\partial_r, \quad \tilde{Q} = r\partial_r\partial_t + t(\partial_r^2 + \frac{n-1}{r}\partial_r),$$

$$L_0 = r\partial_r + t\partial_t, \quad L_j = x_j\partial_t + t\partial_j.$$

We introduce some function spaces. $\mathcal{S}(B) = \{$ rapidly decreasing infinitely differentialble functions on $B\}$, where $B = \{r; R < r < \infty\}$, $R > 0$. $\mathcal{S}'(B)$ is the dual space of $\mathcal{S}(B)$. $L^P = \{f \in \mathcal{S}'(B); \|f\|_p < \infty\}$, where $\|f\|_p = (\int_R^\infty |f(r)|^p r^{n-1} dr)^{1/p}$ if $1 \le p < \infty$ and $\|f\|_\infty = \sup_{R<r<\infty} |f(r)|$ if $p = \infty$. For simplicity we let $\|f\| = \|f\|_2$. Usual Sobolev space of order m is defined by $H^m = \{f \in L^2; \|f\|_{m,0} = \sum_{j=0}^m \|\partial_r^j f\| < \infty\}$. We let $H_0^1 = \{f \in H^1; f(R) = 0\}$,$(f, g) = \int_R^\infty f \cdot g r^{n-1} dr$, $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$. Different positive constants might be denoted by the same letter C.

Our main result in this paper is

THEOREM 1.1. *We assume that* $n = 4, 5$, $u_0 \in H_0^1 \cap H^6$ *satisfies the compatibility condition of order 5 and* $u_1 \in H_0^1 \cap H^5$ *satisfies the compatibility condition of order 4. Furthermore we assume that* $\|\partial_r u_0\|_{5,0} + \|r\partial_r^2 u_0\|_{4,0} + \|r^2\partial_r^3 u_0\|_{3,0} < \infty$, $\|u_1\|_{5,0} + \|r\partial_r u_1\|_{4,0} + \|r^2\partial_r^2 u_1\|_{3,0} < \infty$ *and* ϵ_0 *is sufficiently small positive constant. Then there exists a unique global solution* u *of (1.1)-(1.2) such that*

$$\partial_r u \in C(\mathbb{R}^+; H^5), \quad \partial_t u \in \bigcap_{k=0}^{4} C^k(\mathbb{R}^+; H_0^1 \cap H^{5-k}).$$

Remark 1.2. The compatibility condition of order k means that

$$\partial_t^j u(0, x) \in H_0^1, \quad (j = 0, 1, \cdots, k).$$

Y.Shibata and Y.Tsutsumi [8] first considered the global existence theorem of the exterior problem for the nonlinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = F(\partial u, \partial^2 u), & (t, x) \in \mathbb{R}^+ \times D, \\ u(0, x) = \epsilon_0 u_0(x), \partial_t u(0, x) = \epsilon_0 u_1(x), & x \in D \end{cases}$$

with the homogeneous Dirichlet boundary condition

$$u(t, x) = 0, \quad (t, x) \in \mathbb{R}^+ \times \partial D,$$

where $\partial u = (\partial_t u, \partial_1 u, \cdots, \partial_n u)$, ϵ_0 is sufficiently small, D is the exterior domain of a compact set in $\mathbb{R}^n$ with smooth boundary ∂D and satisfies a certain assumption on the shape (for details see [8],[9]). They proved that small solutions exist globally in time provided that initial data are sufficiently small in a suitable sense and the nonlinear term F satisfies the following growth conditions : F is quadratic with respect to $(\partial u, \partial^2 u)$ when $n \ge 6$ and F is cubic with respect to $(\partial u, \partial^2 u)$ when $3 \le n \le 5$. They obtained the result by making use of the local energy decay of solutions for the free wave equation in D established in [8] (see also [5],[6]) and the technique used in [7]. In [1] Y.M.Chen obtained the result similar to that of [8] or [9] when the nonlinear term contains unknown function u itself.

Our method in this paper is based on Klainerman's inequality (Lemma 2.5) and an estimate of functions on the boundary (Lemma 2.6).

P.S.Datti [2] announced the result similar to ours. However the details of the proof for estimates of local solutions on the boundary are omitted.

Time decay of soutions is important to the proof of the global existence theorem of small solutions. To obtain the time decay of solutions we use Klainerman's inequality [4] which requires the iterative use of the operators L_0 and L_r. It seems that the operator L_r does not work well for the boundary value problem apparently. However through Lemma 2.6 it becomes clear that the operator L_r is valid for our problem. We note here that Lemma 2.6 depends on the space dimention n and does not hold for $n = 3$.

To prove Theorem 1.1 we introduce the function space Γ_T by

$$\Gamma_T = \{f \in C([0,T]; \mathcal{S}'(B)); |||f|||_{\Gamma_T} = \sup_{t\in[0,T]} \|f(t)\|_\Gamma < \infty\}$$

where

$$\|f(t)\|_\Gamma$$

$$= \sum_{m=0}^{4} (\|\partial_r L_0 \partial_t^m f\| + \|\partial_t L_0 \partial_t^m f\| + \|\partial_r L_r \partial_t^m f\| + \|\partial_t L_r \partial_t^m f\|$$

$$+\|\tilde{Q}\partial_t^m f\| + (1+t)^{-1}(\|r\partial_r\partial_t^m f\| + \|r\partial_t^{m+1} f\|))$$

$$+\sum_{m=0}^{3}((1+t)^{-\epsilon}(\|\tilde{Q}L_0\partial_t^m f\| + \|\partial_r L_0^2 \partial_t^m f\|) + (1+t)^{-1}(\|r\partial_r L_0 \partial_t^m f\| + \|r\partial_t L_0 \partial_t^m f\|))$$

$$+\sum_{j+k=1}^{2}\sum_{m=1}^{3}((1+t)^{-2\epsilon}\|\partial_r L_0^j L_r^k \partial_t^m f\| + \|\Delta L_0 \partial_t^m f\| + (1+t)^{-\epsilon}\|\Delta L_r \partial_t^m f\|)$$

$$+\sum_{m=1}^{5}\|\partial_t^m f\|_{6-m,0} + \|\partial_r f\|_{5,0}$$

and ϵ is a sufficiently small positive constant.

To simplify the argument of the proof of Theorem 1.1 we assume that the following local existence theorem holds.

THEOREM 1.2. *We assume that conditions of Theorem 1.1 are satisfied. Then there exists a finite time interval $[0,T]$ with $T > 2R$ and a unique solution u of (1.1)-(1.2) such that*

$$\partial_r u \in C([0,T]; H^5), \quad \partial_t u \in \bigcap_{k=0}^{5} C^k([0,T]; H_0^1 \cap H^{5-k}),$$

$$|||u|||_{\Gamma_T} \le 1, \quad \sum_{m=1}^{3}\|\partial_t^m u\|_\infty^2 < \frac{1}{2},$$

where T depends on the size of ϵ_0.

For the proof of Theorem 1.2 see, e.g., [10].

For the convenience of the reader we now give a sketch of the proof of Theorem 1.1. From Theorem 1.2 it is clear that Theorem 1.1 follows from a priori estimates of local solutions in the function space Γ_T which will be proved in section 3. Our main tools in section 3 are the first-order differential operators $L_0 = r\partial_r + t\partial_t$, $L_j = x_j\partial_t + t\partial_j$ which have the following commuation properties $[\Box, L_0] = 2\Box$ and $[\Box, L_j] = 0$. We note here that these first-differential operators are used by S.Klainerman first to study a global theory of the Cauchy problem for nonlinear wave equations with quadratic nonlinearities. These operators are useful to obtain the time decay estimates of local solutions. In fact we will prove the time decay estimates of local solutions in section 2 (see Lemma 2.5 and Corollary to Lemma 2.5) by making use of the operators L_0, L_j and Lemmas 2.1-2.4. Most part of section 3 is devoted to prove Lemma 3.3 ((3.8), (3.10)) and Lemma 3.4 (3.31) which are important estimates in the proof. To obtain the estimates of the left hand sides of (3.10) and (3.31) we have to consider the term in our problem

$$-\int_R^\infty \partial_r(t^2(r^{n-2}|\partial_r v|^2 + r^n|\partial_r v|^2)dr,$$

where $v = \partial_t^m u, (m = 0, 1, 2, 3, 4)$. Boundary estimates of functions (Lemma 2.6) is needed to handle the above term. The other terms in the function space Γ_T will be estimated through the systematic use of Lemma 2.7, Lemma 2.8 and the estimates (3.8), (3.10), (3.31).

§2 Preliminaries.

In this section we will give some inequalities and estimates of functions on the boundary which are useful to derive a priori estimates of local solutions to (1.1)-(1.2) proved in section 3.

LEMMA 2.1 (HARDY'S INEQUALITY). *We have for $\alpha > -1$*

$$\int_R^\infty |f|^2 r^\alpha dr \le C \int_R^\infty |\partial_r f|^2 r^{2+\alpha} dr,$$

provided the right hand side is finite.

Proof. It is easy to see that

$$0 \ge \int_R^\infty \partial_r(|f|^2 r^{1+\alpha})dr = (1+\alpha)\int_R^\infty |f|^2 r^\alpha dr + 2\int_R^\infty f \cdot \partial_r f r^{1+\alpha} dr.$$

By this and the Schwarz inequality

$$\int_R^\infty |f|^2 r^\alpha dr \le C(\int_R^\infty |f|^2 r^\alpha dr)^{\frac{1}{2}}(\int_R^\infty |\partial_r f|^2 r^{2+\alpha} dr)^{\frac{1}{2}},$$

from which we have the lemma.

Q.E.D.

LEMMA 2.2. *We have for any f satisfying the condition such that $|f|^2 r^m \to 0$ as $r \to \infty$*

$$-\int_R^\infty \partial_r(|f|^2 r^m)dr \le C\|\partial_r f\|^2$$

provided the right hand side is finite.

Proof. We have by integration by parts

$$-\int_R^\infty \partial_r(|f|^2 r^m)dr = -R^m \int_R^\infty \partial_r(|f|^2)dr$$

$$\le C\int_R^\infty |\partial_r f||f|dr \le CR^{-\frac{n-1}{2}}\|\partial_r f\| \int_R^\infty |f|^2 dr.$$

By Lemma 2.1 the right hand side of the above is estimated by

$$C\|\partial_r f\| \int_R^\infty |\partial_r f|^2 r^2 dr \le CR^{3-n}\|\partial_r f\|^2.$$

Hence we have the lemma.

Q.E.D.

LEMMA 2.3 (SOBOLEV'S INEQUALITY). *Let $n \ge 4$ and $0 < \beta \le 1$. Then we have*

$$\|f\|_4 \le C\|\partial_r f\| \quad \text{for} \quad f \in H^1,$$

$$\|f\|_\infty \le C\|\partial_r f\|^{\frac{\beta}{1+\beta}} \|\partial_r f\|_4^{\frac{1}{1+\beta}} \quad \text{for} \quad f \in H^2,$$

$$\|f\|_\infty \le C\|\partial_r f\|_\infty^{\frac{1}{2}} \|f\|_4^{\frac{1-\beta}{2}} \|f\|^{\frac{\beta}{2}} \quad \text{for} \quad f \in H^2.$$

Proof. Since

$$0 \ge \int_R^\infty \partial_r(|f|^4 r^n)dr = n\int_R^\infty |f|^4 r^{n-1}dr + 4\int_R^\infty f^3 \partial_r f r^n dr,$$

we get by the Schwarz inequality

$$\|f\|_4^4 \le C\|\partial_r f\|(\int_R^\infty |f|^6 r^{n+1}dr)^{\frac{1}{2}}.$$

Therefore,

$$\|f\|_4^2 \le C\|\partial_r f\| \sup_{R<r<\infty} r|f|. \tag{2.1}$$

We have

$$r^2|f|^2 = -\int_r^\infty \partial_r(r^2|f|^2)dr$$
$$= -2\int_r^\infty r|f|^2 dr - 2\int_r^\infty r^2 f\partial_r f dr$$
$$\leq C(\int_R^\infty |\partial_r f|^2 r^3 dr)^{\frac{1}{2}}(\int_R^\infty |f|^2 r dr)^{\frac{1}{2}}.$$

We apply Lemma 2.1 to the above to obtain for $n \geq 4$

$$r|f| \leq C\|\partial_r f\|.$$

This and (2.1) give the first part of the lemma.

We use the Schwarz inequality several times to get

$$|f|^2 = -\int_r^\infty \partial_r(|f|^2)dr = -2\int_r^\infty f\partial_r f dr$$
$$\leq C(\int_R^\infty |f|^{\frac{4}{3}} r^{-1} dr)^{\frac{3}{4}}(\int_R^\infty |\partial_r f|^4 r^3 dr)^{\frac{1}{4}}$$
$$\leq C\|f\|_\infty^{1-\beta}(\int_R^\infty |f|^{\frac{4}{3}\beta} r^{-1} dr)^{\frac{3}{4}}\|\partial_r f\|_4$$
$$\leq C\|f\|_\infty^{1-\beta}(\int_R^\infty |f|^2 r dr)^{\frac{\beta}{2}}\|\partial_r f\|_4$$

from which and Lemma 2.1 we have the second part of the lemma.

By Hölder's inequality

$$|f|^2 = -\int_r^\infty \partial_r(|f|^2)dr \leq C\int_R^\infty |f||\partial_r f|dr$$
$$\leq C\sup_{R<r<\infty}|\partial_r f|(\int_R^\infty |f|^p r^{\alpha p} dr)^{\frac{1}{p}}(\int_R^\infty r^{-\frac{\alpha p}{p-1}} dr)^{1-\frac{1}{p}}.$$

We take $\alpha p = n-1$, $p = 4 - \frac{4\beta}{1+\beta}$ and again use Hölder's inequality to obtain

$$\|f\|_\infty^2 \leq C\|\partial_r f\|_\infty \|f\|_4^{1-\beta}\|f\|^\beta.$$

This implies the last inequality of the lemma.

Q.E.D.

LEMMA 2.4. *We have for $f \in H^1$,*

$$|f| \leq Cr^{-\frac{n-1}{2}}\|f\|^{\frac{1}{2}}\|\partial_r f\|^{\frac{1}{2}}.$$

Proof. Because

$$r^{n-1}|f|^2 = -\int_r^\infty \partial_r(|f|^2 r^{n-1})dr \leq -2\int_r^\infty f\partial_r f r^{n-1} dr$$

we have the lemma by the Schwarz inequality.

Q.E.D.

LEMMA 2.5 (KLAINERMAN'S INEQUALITY). *We have*

$$\|\partial_t f\|_4 \le C(1+t)^{-\frac{3}{4}}(\|\partial_r L_0 f\| + \|\partial_r L_r f\| + \|\partial_t f\|_{1,0}), \tag{2.2}$$

$$\|\partial_r f\|_4 \le C(1+t)^{-\frac{3}{4}}(\|\partial_r L_0 f\| + \|\partial_r L_r f\| + \|\partial_r f\|_{1,0}), \tag{2.3}$$

(2.4)
$$\|\partial_t^2 f\|_\infty \le C(1+t)^{-\frac{3}{2}}(\sum_{1\le j+k\le 2} \|\partial_r L_0^j L_r^k \partial_t f\| + \|\partial_r^2 L_0 \partial_t f\| + \|\partial_r^2 L_r \partial_t f\| + \|\partial_t^2 f\|_{1,0}),$$

$$\|\partial_r \partial_t f\|_\infty \le C(1+t)^{-\frac{3}{2}}(\sum_{1\le j+k\le 2} \|L_0^j L_r^k \partial_r \partial_t f\| + \|\partial_r \partial_t f\|_{1,0}), \tag{2.5}$$

$$\|\partial_t f\|_\infty \le C(1+t)^{-\frac{9}{8}+\frac{3}{8}\beta} \tag{2.6}$$
$$\times(\|\partial_r L_0 f\| + \|\partial_r L_r f\| + \|\partial_t f\|_{1,0} + \sum_{1\le j+k\le 2} \|L_0^j L_r^k \partial_r \partial_t f\| + \|\partial_r \partial_t f\|_{1,0}),$$

$$\|\partial_t L_0 \partial_t f \cdot \partial_t L_0 \partial_t^4 f\| \tag{2.7}$$
$$\le C(1+t)^{-\frac{3}{2}}\{(\|\partial_t L_0 \partial_t f\| + \|\partial_r L_0 \partial_t^2 f\| + \|\partial_r \partial_t^2 f\|)\|\partial_t L_0 \partial_t^4 f\|$$
$$+(\|\partial_r L_0^2 \partial_t^2 f\| + \|\partial_r L_0 \partial_t^2 f\| + \|\partial_r L_0 L_r \partial_t^2 f\| + \|\partial_r L_r \partial_t^2 f\| + \|\partial_r L_0 \partial_t^2 f\|_{1,0}$$
$$+\|\partial_r \partial_t^2 f\|_{1,0})(\|L_0^2 \partial_t^4 f\| + \|L_0 L_r \partial_t^4 f\|) + \|\partial_t L_0 \partial_t f\|_{1,0}\|\partial_t L_0 \partial_t^4 f\|\}.$$

provided the right hand sides are finite, where $0 < \beta \le 1$.

Proof. Form the relation

$$\partial_t = \frac{1}{t^2 - r^2}(tL_0 - rL_r) \tag{2.8}$$

we derive

$$\|\partial_t f\|_4^4 \le \int_R^\infty |\partial_t f|^4 r^{n-1} dr = \int_R^{\frac{t}{2}} |\partial_t f|^4 r^{n-1} dr + \int_{\frac{t}{2}}^\infty |\partial_t f|^4 r^{n-1} dr$$

$$\le C\int_R^{\frac{t}{2}} \frac{1}{|t-r|^4}(|L_r f|^4 + |L_0 f|^4) r^{n-1} dr + C \sup_{\frac{t}{2}<r<\infty} |\partial_t f|^2 \|\partial_t f\|^2.$$

Applying Lemma 2.4 to the above, we obtain

$$\|\partial_t f\|_4^4 \le Ct^{-4}(\|L_r f\|_4^4 + \|L_0 f\|_4^4) + Ct^{-(n-1)}\|\partial_t f\|^3\|\partial_r\partial_t f\| \quad \text{for} \quad t \ge 2R.$$

This and Lemma 2.3 give (2.2). In the same way as in the proof of (2.2) we obtain (2.3) by using the relation

$$\partial_r = \frac{1}{t^2 - r^2}(tL_r - rL_0).$$

We next show (2.4). By (2.8)

$$|\partial_t^2 f| \le \frac{C}{|t-r|}(|L_0\partial_t f| + |L_r\partial_t f|).$$

Hence,

$$\sup_{R<r<\frac{t}{2}} |\partial_t^2 f| \le Ct^{-1}(\|L_0\partial_t f\|_\infty + \|L_r\partial_t f\|_\infty)$$

from which with Lemma 2.3 and (2.3) we obtain

$$\sup_{R<r<\frac{t}{2}} |\partial_t^2 f| \le Ct^{-1}(1+t)^{-\frac{3}{4}(\frac{1}{1+\beta})}$$

$$\times(\|\partial_r L_0\partial_t f\|^{\frac{\beta}{1+\beta}}\{\|\partial_r L_0^2\partial_t f\| + \|\partial_r L_r L_0\partial_t f\| + \|\partial_r L_0\partial_t f\|_{1,0}\}^{\frac{1}{1+\beta}}$$

$$+\|\partial_r L_r\partial_t f\|^{\frac{\beta}{1+\beta}}\{\|\partial_r L_0 L_r\partial_t f\| + \|\partial_r L_r^2\partial_t f\| + \|\partial_r L_r\partial_t f\|_{1,0}\}^{\frac{1}{1+\beta}})$$

and by Lemma 2.4

$$\sup_{\frac{t}{2}<r<\infty} |\partial_t^2 f| \le Ct^{-\frac{n-1}{2}}\|\partial_t^2 f\|^{\frac{1}{2}}\|\partial_r\partial_t^2 f\|^{\frac{1}{2}} \quad \text{for} \quad t \ge 2R.$$

These two inequalities and Lemma 2.4 show (2.4) with $\beta = \frac{1}{2}$.
In the same way as in the proof of [4,p328] we obtain

(2.9) $$|\partial_r\partial_t f| \le C|t-r|^{-\frac{3}{2}} \sum_{0\le j+k\le 2} \|L_0^j L_r^k \partial_r\partial_t f\| \quad \text{for} \quad R \le r \le \frac{t}{2}.$$

From (2.9) and Lemma 2.4 we derive (2.5).
Lemma 2.3 yields

$$\|\partial_t f\|_\infty \le C\|\partial_r\partial_t f\|_\infty^{\frac{1}{2}}\|\partial_t f\|_4^{\frac{1-\beta}{2}}\|\partial_t f\|^{\frac{\beta}{2}}.$$

By this, (2.2) and (2.5) we have (2.6).
We finally prove (2.7). We have

(2.10) $$\|g\partial_t h\|^2 = \int_R^{\frac{t}{2}} |g\partial_t h|^2 r^{n-1}dr + \int_{\frac{t}{2}}^\infty |g\partial_t h|^2 r^{n-1}dr.$$

By Lemma 2.4

$$(2.11)\qquad \int_{\frac{1}{2}}^{\infty} |g\partial_t h|^2 r^{n-1} dr \le Ct^{-(n-1)}\|g\|\|\partial_r g\|\|\partial_t h\|^2.$$

From (2.8), Lemma 2.3 and (2.3) it follows that

$$(2.12)\qquad \int_R^{\frac{t}{2}} |g\partial_t h|^2 r^{n-1} dr \le Ct^{-2}\|g\|_\infty^2(\|L_0 h\|^2 + \|L_r h\|^2)$$

$$\le Ct^{-2}\|\partial_r g\|^{\frac{2\beta}{1+\beta}}\|\partial_r g\|_4^{\frac{2}{1+\beta}}(\|L_0 h\|^2 + \|L_r h\|^2)$$

$$\le Ct^{-2-\frac{3}{2(1+\beta)}}\|\partial_r g\|^{\frac{2\beta}{1+\beta}}(\|\partial_r L_0 g\| + \|\partial_r L_r g\| + \|\partial_r g\|_{1,0})^{\frac{2}{1+\beta}}(\|L_0 h\|^2 + \|L_r h\|^2)$$

for $t \ge 2R$.

We put $g = \partial_t L_0 \partial_t f$,$h = L_0 \partial_t^4 f$ and $\beta = \frac{1}{2}$ in (2.10)-(2.12) and use the Schwarz inequality to obtain

$$\|\partial_t L_0 \partial_t f \cdot \partial_t L_0 \partial_t^4 f\|$$

$$\le Ct^{-\frac{3}{2}}\{(\|\partial_t L_0 \partial_t f\| + \|\partial_r L_0 \partial_t^2 f\| + \|\partial_r \partial_t^2 f\|)\|\partial_t L_0 \partial_t^4 f\|$$

$$+(\|\partial_r L_0^2 \partial_t^2 f\| + \|\partial_r L_0 \partial_t^2 f\| + \|\partial_r L_0 L_r \partial_t^2 f\| + \|\partial_r L_r \partial_t^2 f\|$$

$$+\|\partial_r L_0 \partial_t^2 f\|_{1,0} + \|\partial_r \partial_t^2 f\|_{1,0})(\|L_0^2 \partial_t^4 f\| + \|L_0 L_r \partial_t^4 f\|)\}.$$

From this and Lemma 2.4 we obtain (2.7).

Q.E.D.

COROLLARY TO LEMMA 2.5. *Let $f \in \Gamma_T$. Then we have for $0 < t < T$*

$$\sum_{m=1}^{4} \|\partial_t^m f\|_\infty \le C(1+t)^{-\frac{9}{8}+\frac{3}{8}\beta+2\epsilon}\|f\|_\Gamma,$$

$$\sum_{m=1}^{3} \|\partial_t L_0 \partial_t^m f\|_4 + \sum_{m=1}^{5} \|\partial_t^m f\|_4 \le C(1+t)^{-\frac{3}{4}+2\epsilon}\|f\|_\Gamma,$$

$$\|\partial_t L_0 \partial_t f \cdot \partial_t L_0 \partial_t^4 f\| \le C(1+t)^{-\frac{3}{2}+4\epsilon}\|f\|_\Gamma^2,$$

where ϵ is the same one as that given in the definition of the function space Γ_T.

The following lemma is important in this paper to prove the main result.

LEMMA 2.6. *We have*

$$-t^2\int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr \le CI_\epsilon(f) - Ct^2\int_R^\infty \partial_r(|f|^2)dr$$

provided the right hand side is finite, where $0 < \epsilon < \frac{1}{2}$ and

$$I_\epsilon(f) \equiv \|\tilde{Q}f\|^2 + \|\partial_r L_0 f\|^2 + \|\partial_r f\|_{1,0}^2 + (1+t)^{2\epsilon}(\|\tilde{Q}f\|^2 + \|\partial_r f\|^2)$$

$$+(\|\partial_r L_0 f\| + \|\partial_r f\|)(\|\partial_r f\|^\epsilon + \|r\partial_r f\|^\epsilon)\|\partial_r f\|^{1-2\epsilon}\|r\partial_r f\|^\epsilon.$$

Proof. By a direct calculation

$$-t^2 \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr \tag{2.13}$$

$$= (2(n-2)-\epsilon)t^2 \int_R^\infty |\partial_r f|^2 r^{1+\epsilon}dr - 2t^2 \int_R^\infty \partial_r f \Delta f r^{2+\epsilon}dr$$

$$= (2(n-2)-\epsilon)t^2 \int_R^\infty |\partial_r f|^2 r^{1+\epsilon}dr - 2t \int_R^\infty \partial_r f \cdot \tilde{Q} f r^{2+\epsilon}dr + 2 \int_R^\infty \partial_r f \cdot \partial_r L_0 f r^{3+\epsilon}dr$$

$$+(2+\epsilon) \int_R^\infty |\partial_r f|^2 r^{3+\epsilon}dr - \int_R^\infty \partial_r(|\partial_r f|^2 r^{4+\epsilon})dr \equiv \sum_{j=1}^5 A_j$$

By the Schwarz inequality

$$A_2 \le t^2 \int_R^\infty |\partial_r f|^2 r^{1+2\epsilon}dr + C\|\tilde{Q} f\|^2, \tag{2.14}$$

$$A_3 \le C(\|\partial_r L_0 f\|^2 + \int_R^\infty |\partial_r f|^2 r^{3+2\epsilon}dr) \tag{2.15}$$

$$\le C(\|\partial_r L_0 f\|^2 + \|r\partial_r f\|^{2\epsilon}\|\partial_r f\|^{2-2\epsilon}),$$

$$A_4 \le C\|r\partial_r f\|^\epsilon \|\partial_r f\|^{2-\epsilon}. \tag{2.16}$$

By Lemma 2.2

$$A_5 \le C\|\partial_r^2 f\|^2. \tag{2.17}$$

We consider the term A_1. Integration by parts gives

$$\frac{1}{2(n-2)-\epsilon} A_1 = t^2 \int_R^\infty |\partial_r f|^2 r^{1+\epsilon}dr \tag{2.18}$$

$$= t^2 \int_R^\infty \partial_r(f\partial_r f r^{1+\epsilon})dr - t^2 \int_R^\infty f\Delta f r^{1+\epsilon}dr + (n-2-\epsilon)t^2 \int_R^\infty f\partial_r f r^\epsilon dr$$

$$= t^2 \int_R^\infty \partial_r(f\partial_r f r^{1+\epsilon})dr - t \int_R^\infty f\tilde{Q} f r^{1+\epsilon}dr + t \int_R^\infty f\partial_r\partial_t f r^{2+\epsilon}dr$$

$$+\frac{1}{2}(n-2-\epsilon)t^2 \int_R^\infty \partial_r(|f|^2 r^\epsilon)dr - \frac{1}{2}(n-2-\epsilon)\epsilon \int_R^\infty |tf|^2 r^{-1+\epsilon}dr \equiv \sum_{j=1}^5 B_j.$$

By the Schwarz inequality

$$B_1 \le -\epsilon_1 t^2 \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr - CB_4, \tag{2.19}$$

where ϵ_1 is a sufficiently small positive constant determined later. We use the Schwarz and the Hölder inequalities to obtain

$$B_2 \le C\|\tilde{Q}f\|(\int_R^\infty |tf|^2 r^{-1+2\epsilon} dr)^{\frac{1}{2}}$$

$$\le C\|\tilde{Q}f\|(\int_R^\infty |tf|^2 r dr)^{\frac{\epsilon}{2(2-\epsilon)}}(\int_R^\infty |tf|^2 r^{-1+\epsilon} dr)^{\frac{1-\epsilon}{2-\epsilon}}$$

$$\le C\|\tilde{Q}f\|^{2-\epsilon}(\int_R^\infty |tf|^2 r dr)^{\frac{\epsilon}{2}} - \frac{1}{2}B_5.$$

Applying Lemma 2.1 to the above, we obtain

$$B_2 \le Ct^\epsilon\|\tilde{Q}f\|^{2-\epsilon}\|\partial_r f\|^\epsilon - \frac{1}{2}B_5. \tag{2.20}$$

By integration by parts it is easy to see that

$$B_3 = \int_R^\infty f\cdot\partial_r(L_0 - r\partial_r)fr^{2+\epsilon}dr$$

$$= \int_R^\infty (f\cdot\partial_r L_0 f + (2+\epsilon)f\cdot\partial_r f + |\partial_r f|^2 r)r^{2+\epsilon}dr - \int_R^\infty \partial_r(f\cdot\partial_r f r^{3+\epsilon})dr.$$

We use the Schwarz and the Hölder inequalities to get

$$B_3 \le C(\|\partial_r L_0 f\| + \|\partial_r f\|)\|\frac{1}{r^{1-\epsilon}}f\| + C\|r\partial_r f\|^\epsilon\|\partial_r f\|^{2-\epsilon}$$

$$-\int_R^\infty (\partial_r(\{|f|^2 + |\partial_r f|^2\}r^{3-\epsilon})dr.$$

Using Lemma 2.1 , Lemma 2.2 and the fact that

$$\|r^\epsilon\partial_r f\| \le \|r\partial_r f\|^\epsilon\|\partial_r f\|^{1-\epsilon}$$

which follows from the Schwarz inequality , we arrive at

$$B_3 \le C\{(\|\partial_r L_0 f\| + \|\partial_r f\|)\|r\partial_r f\|^\epsilon\|\partial_r f\|^{1-\epsilon} + \|\partial_r f\|_{1,0}^2\}. \tag{2.21}$$

By (2.19)-(2.21)

$$\frac{1}{2(n-2)-\epsilon}A_1 = t^2\int_R^\infty |\partial_r f|^2 r^{1+\epsilon}dr \tag{2.22}$$

$$\le -\epsilon_1 t^2\int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr + Ct^\epsilon(\|\tilde{Q}f\|^2 + \|\partial_r f\|^2)$$

$$+C\{(\|\partial_r L_0 f\| + \|\partial_r f\|)\|r\partial_r f\|^\epsilon\|\partial_r f\|^{1-\epsilon} + \|\partial_r f\|_{1,0}^2 - Ct^2\int_R^\infty \partial_r(|f|^2 r^\epsilon)dr$$

$$\le -\epsilon_1 t^2\int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr + CI_\epsilon(f) - Ct^2\int_R^\infty \partial_r(|f|^2 r^\epsilon)dr.$$

In the same way as in the proof of (2.22), the first part of the right hand side of (2.14) is bounded from above by

$$-\epsilon_1 t^2 \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+2\epsilon})dr + CI_\epsilon(f) - Ct^2 \int_R^\infty \partial_r(|f|^2 r^\epsilon)dr,$$

from which and (2.14) it follows that

$$A_2 \le -\epsilon_1 t^2 \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+2\epsilon})dr + CI_\epsilon(f) - Ct^2 \int_R^\infty \partial_r(|f|^2 r^\epsilon)dr. \tag{2.23}$$

Because

$$-t^2 \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+2\epsilon})dr = -t^2 R^\epsilon \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr,$$

we have by (2.22), (2.23) and (2.15)-(2.17)

$$-t^2 \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr$$

$$\le -t^2 \epsilon_1(1+R^\epsilon) \int_R^\infty \partial_r(|\partial_r f|^2 r^{2+\epsilon})dr + CI_\epsilon(f) - Ct^2 \int_R^\infty \partial_r(|f|^2)dr.$$

The lemma follows from the above if we take ϵ_1 such that $\epsilon_1(2(n-2)-\epsilon)+R^\epsilon) < \frac{1}{2}$.

Q.E.D.

LEMMA 2.7. *We have*

$$\sum_{j,k=1}^n \|L_j\partial_k f\|^2 \le C(I_\epsilon(f) + \|\partial_t f\|^2 - \int_R^\infty \partial_r(t^2|f|^2 + |\partial_t f|^2)dr)$$

provided the right hand side is finite.

Proof. By [3 , Lemma 3.2] we have

$$\sum_{j,k=1}^n \|L_j\partial_k f\|^2 \le C(\|\tilde{Q}f\|^2 + \|\partial_t f\|^2 - \int_R^\infty \partial_r(|L_r f|^2 r^{n-2})dr)$$

$$\le C(\|\tilde{Q}f\|^2 + \|\partial_t f\|^2 - \int_R^\infty \partial_r(|\partial_t f|^2 + t^2|\partial_r f|^2)dr).$$

Applying Lemma 2.6 to the above, we obtain the lemma.

Q.E.D.

LEMMA 2.8. *We have*

$$\sum_{j,k=1}^n |L_j\partial_k f|^2 = (n-1)t^2|\frac{1}{r}\partial_r f|^2 + |L_r\partial_r f|^2 \tag{2.24}$$

$$(2.25) \qquad |L_r\partial_r L_r f|^2 \le C\{|L_0\partial_r f|^2 + \sum_{j,k,l=1}^{n} |L_j\partial_k L_l f|^2\}$$

Proof. From the relation

$$L_j\partial_k = t(\delta_{jk} - \frac{x_j x_k}{r^2})\frac{1}{r}\partial_r + \frac{x_j x_k}{r^2} L_r\partial_r$$

we have (2.24). By (2.24)

$$(2.26) \qquad \sum_{j,k,l}^{n} |L_j\partial_k L_l f|^2 = \sum_{l=1}^{n}\{(n-1)t^2|\frac{1}{r}\partial_r L_l f|^2 + |L_r\partial_r L_l f|^2\}.$$

By a direct calculation

$$L_r\partial_r L_l = \frac{x_l}{r}(L_r\partial_r L_r + L_0\partial_r)$$

from which and (2.26) we have (2.25).

Q.E.D.

§3 Proof of Theorem 1.1.

In this section we will prove Theorem 1.1 by showing a priori estimates of local solutions to (1.1)-(1.2) in the function space Γ_T. The next lemma is needed to obtain the time decay estimates of local solutions. We note here that the right hand sides of (3.1)-(3.3) do not contain the operator L_r^2 explicitly. In what follows we let $I_\epsilon = I$ for simplicity.

LEMMA 3.1. *Let u be the local solution of (1.1)-(1.2) stated in Theorem 1.2. Then*

$$(3.1) \qquad \sum_{j,k=1}^{n}\sum_{m=0}^{4} \|L_j\partial_k\partial_t^m u\|^2 \le C\sum_{m=0}^{4}(I(\partial_t^m u) + \|\partial_t^{m+1}u\|^2),$$

(3.2)

$$\sum_{j,k=1}^{n}\sum_{m=0}^{3} \|L_j\partial_k L_0\partial_t^m u\|^2 \le C\sum_{m=0}^{3}(I(L_0\partial_t^m u) + \|\partial_t L_0\partial_t^m u\|^2) + C\sum_{m=0}^{4}\|\partial_r\partial_t^m u\|_{1,0}^2,$$

$$(3.3) \qquad \sum_{j,k,l=1}^{n}\sum_{m=1}^{3} \|L_j\partial_k L_l\partial_t^m u\|^2$$

$$\le C\{\sum_{m=0}^{2}(I(L_0\partial_r\partial_t^m u) + I(\partial_r\partial_t^m u) + I(r\partial_t^m F)) + \sum_{m=1}^{3}\|\partial_t L_r\partial_t^m u\|^2$$

$$+\sum_{m=1}^{4} I(\partial_t^m u) + \sum_{m=0}^{3} I(L_0 \partial_t^m u)\}.$$

Proof. From Lemma 2.7, (3.1) follows immediately. Using Lemma 2.7 we have with $v = \partial_t^m u$

$$\sum_{j,k=1}^{n} \|L_j \partial_k L_0 v\|^2 \le C(I(L_0 v) + \|\partial_t L_0 v\|^2 - \int_R^\infty \partial_r(t^2|\partial_r v|^2 + |\partial_r \partial_t v|^2) dr).$$

We apply Lemma 2.2 to the last term of the right hand side of the above to obtain (3.2). We finally prove (3.3). By Lemma 2.7 the left hand side of (3.3) is bounded from above by

$$(3.4)\quad C\sum_{m=1}^{3}\{I(L_l \partial_t^m u) + \|\partial_t L_l \partial_t^m u\|^2 - t^2 \int_R^\infty \partial_r(|L_l \partial_t^m u|^2 + |\partial_t L_l \partial_t^m u|^2) dr\}.$$

We consider the last term of (3.4). Since

$$L_l \partial_t = x_l \partial_t^2 + \partial_l L_0 - \partial_l - x_l \partial_r^2$$

we have by Lemma 2.6

$$-t^2 \int_R^\infty \partial_r(|L_l \partial_t u|^2) dr \le -Ct^2 \int_R^\infty \partial_r(|\partial_r L_0 u|^2 + |\partial_r u|^2) dr$$

$$\le C(I(L_0 u) + I(u) - t^2 \int_R^\infty \partial_r(|\partial_r u|^2) dr).$$

We again use Lemma 2.6 to obtain

$$-t^2 \int_R^\infty \partial_r(|L_l \partial_t u|^2) dr \le C(I(L_0 u) + I(u)).$$

Hence by Lemma 2.6

$$(3.5)\quad -t^2 \int_R^\infty \partial_r(|L_l \partial_t^m u|^2 + |\partial_t L_l \partial_t^m u|^2) dr$$

$$\le -Ct^2 \int_R^\infty \partial_r(|L_l \partial_t^m u|^2 + |L_l \partial_t^{m+1} u|^2 + |\partial_r \partial_t^m u|^2) dr$$

$$\le C(I(L_0 \partial_t^{m-1} u) + I(L_0 \partial_t^m u) + I(\partial_t^m u) + I(\partial_t^{m+1} u)).$$

By (3.4) and (3.5)

$$(3.6)\quad \sum_{j,k,l=1}^{n} \sum_{m=1}^{3} \|L_j \partial_k L_l \partial_t^m u\|^2$$

$$\leq C\{\sum_{l=1}^{n}\sum_{m=1}^{3}(I(L_l\partial_t^m u)+\|\partial_t L_l\partial_t^m u\|^2)+\sum_{m=1}^{4}I(\partial_t^m u)++\sum_{m=0}^{3}I(L_0\partial_t^m u)\}.$$

Since

$$L_l\partial_t^m = \frac{x_l}{r}\{(L_0\partial_r+(n-1)\partial_r)\partial_t^{m-1}u+r\partial_t^{m-1}F\}$$

we have (3.3).

Q.E.D.

By Lemma 3.1 and Lemma 2.9 we have

LEMMA 3.2. *Let u be the solution of (1.1)-(1.2) stated in Theorem 1.2. Then*

$$\sum_{m=0}^{4}\|L_r\partial_r\partial_t^m u\|^2 \leq \quad \text{The R.H.S. of (3.1)},$$

$$\sum_{m=0}^{3}\|L_r\partial_r L_0\partial_t^m u\|^2 \leq \quad \text{The R.H.S. of (3.2)},$$

$$\sum_{m=1}^{3}\|L_r\partial_r L_r\partial_t^m u\|^2 \leq \quad \text{The R.H.S. of (3.3)}.$$

We notice that in the proofs of Lemma 3.3 and Lemma 3.4 it is clear that the iterative use of operators L_0 and ∂_t work well for the boundary value problem.

LEMMA 3.3. *Let u be the solution of (1.1)-(1.2) stated in Theorem 1.2 and let ϵ be the same one as that given in the definition of the function space Γ_T. Then we have for $0 < t < T$*

$$(1+t)^{-2}\sum_{m=0}^{4}(\|r\partial_t^{m+1}u\|^2+\|r\partial_r\partial_t^m u\|^2) \tag{3.7}$$

$$\leq C(\epsilon_0^2+\int_0^t(1+s)^{-\frac{9}{8}+\frac{3}{8}\beta+4\epsilon}\|u(s)\|_\Gamma^3 ds),$$

$$\sum_{m=0}^{4}(\|\partial_r L_0\partial_t^m u\|^2+\|\partial_t L_0\partial_t^m u\|^2-2\int_0^t s\int_R^\infty \partial_r(|\partial_r\partial_s^{m+1}u|^2 r^n)drds) \tag{3.8}$$

$$\leq \text{The R.H.S. of (3.7)},$$

$$(1+t)^{-2}\sum_{m=0}^{3}(\|r\partial_t L_0\partial_t^m u\|^2+\|r\partial_r L_0\partial_t^m u\|^2) \leq \quad \text{The R.H.S. of (3.7)}\ , \tag{3.9}$$

$$(3.10)\quad (1+t)^{-2\epsilon}\sum_{j=1}^{n}\sum_{m=0}^{4}(\|\partial_r L_j\partial_t^m u\|^2+\|\partial_t L_j\partial_t^m u\|^2)\le \quad \textit{The R.H.S. of (3.7)},$$

Proof. Applying the operator L_0 to both sides of (1.1), we obtain with $v=\partial_t^m u$, $G=\partial_t^m F$

$$(3.11)\qquad \partial_t^2 L_0 v-(\partial_r^2+\frac{n-1}{r}\partial_r)L_0 v=2G+L_0 G.$$

Multiplying both sides of (3.11) by $r^{n-1}\partial_t L_0 v$, integrating with respect to r, we have

$$(3.12)\qquad \frac{d}{dt}(\|\partial_t L_0 v\|^2+\|\partial_r L_0 v\|^2)$$

$$-2\int_R^\infty \partial_r(\partial_r L_0 v\cdot\partial_t L_0 v r^{n-1})dr=2(2G+L_0G,\partial_t L_0 v).$$

It is easy to see that

$$(3.13)\qquad \begin{cases}\partial_r L_j &= x_j(\frac{1}{r}\partial_t+\frac{t}{r}\partial_r^2+\partial_t\partial_r),\\ \partial_t L_j &= x_j(\partial_t^2+\frac{1}{r}\partial_r+\frac{t}{r}\partial_t\partial_r),\\ \partial_r L_0 &= t\partial_r\partial_t+\partial_r+r\partial_r^2,\\ \partial_t L_0 &= \partial_t+t\partial_t^2+r\partial_r\partial_t.\end{cases}$$

Hence by (3.13) the second term of the left hand side of (3.12) is equal to

$$-2\int_R^\infty \partial_r(\{-(n-2)\partial_r v+t\partial_r\partial_t v\}\cdot\partial_r\partial_t v r^n)dr$$

$$=(n-2)\frac{d}{dt}\int_R^\infty \partial_r(|\partial_r v|^2 r^n)dr-2t\int_R^\infty \partial_r(|\partial_r\partial_t v|^2 r^n)dr.$$

Therefore we have by (3.12)

$$(3.14)\qquad \frac{d}{dt}\{\|\partial_t L_0 v\|^2+\|\partial_r L_0 v\|^2+(n-2)\int_R^\infty \partial_r(|\partial_r v|^2 r^n)dr\}$$

$$-2t\int_R^\infty \partial_r(|\partial_r\partial_t v|^2 r^n)dr=2(2G+L_0G,\partial_t L_0 v).$$

Integrating (3.14) with respect to t, using Lemma 2.6 we obtain

$$(3.15)\qquad \|\partial_t L_0 v\|^2+\|\partial_r L_0 v\|^2-2\int_0^t s\int_R^\infty \partial_r(|\partial_r\partial_s v|^2 r^n)drds$$

$$\le C(\epsilon_0^2+\|\partial_r^2 v\|^2+|\int_0^t(2G+L_0G,\partial_s L_0 v)ds|).$$

By a direct calculation we see that

$$\partial_t F = \partial_{\partial_t u} F \cdot \partial_t^2 u + \partial_{\partial_t^2 u} F \cdot \partial_t^3 u,$$

$$\partial_t^2 F = a_1 \cdot (\partial_t^2 u)^2 + 2a_2 \cdot (\partial_t^2 u)(\partial_t^3 u) + a_3 \cdot (\partial_t^3 u)^2$$

and

$$L_0 F = \partial_{\partial_t u} F \cdot L_0 \partial_t u + \partial_{\partial_t^2 u} F \cdot L_0 \partial_t^2 u,$$

where a_1,a_2 and a_3 are constants defined by

$$a_1 = \partial_{\partial_t u, \partial_t u} F, \quad a_2 = \partial_{\partial_t u, \partial_t^2 u} F, \quad a_3 = \partial_{\partial_t^2 u, \partial_t^2 u} F.$$

By making use of the above equalities

$$(3.16) \qquad \sum_{m=0}^{4} \left| \int_0^t (L_0 \partial_s^m F, \partial_t L_0 \partial_s^m u) ds \right|$$

$$\leq C \int_0^t (1+s)^{2\epsilon} \Big(\sum_{m=1}^{3} (\|\partial_s^m u\|_\infty (1+s)^{2\epsilon} \|u\|_\Gamma + \|\partial_s^4 u\|_4 \sum_{j=0}^{1} \|\partial_s L_0^j \partial_s^m u\|_4) \|u\|_\Gamma \Big) ds$$

$$+ \left| \int_0^t (\partial_{\partial_s^2 u} F \cdot L_0 \partial_s^6 u, \partial_s L_0 \partial_s^4 u) ds \right|.$$

Because

$$L_0 \partial_t^6 u = \partial_t^2 L_0 \partial_t^4 u - 2\partial_t^6 u$$

the last term of the right hand side of (3.16) is estimated by

$$(3.17) \qquad \left| \int_0^t \frac{d}{ds} (\partial_{\partial_s^2 u} F \cdot \partial_s L_0 \partial_s^4 u, \partial_s L_0 \partial_s^4 u) ds \right| + C \int_0^t \sum_{m=1}^{3} \|\partial_s^m u\|_\infty \|u\|_\Gamma^2 ds.$$

Hence by Corollary to Lemma 2.5, (3.16) and (3.17)

$$(3.18) \quad \sum_{m=0}^{4} \left| \int_0^t (L_0 \partial_s^m F, \partial_s L_0 \partial_s^m u) ds \right| \leq \quad \text{The R.H.S. of (3.7)} + \frac{1}{2} \|\partial_t L_0 \partial_t^4 u\|^2.$$

In the same way as in the proof of (3.18)

$$(3.19) \qquad \sum_{m=0}^{4} \left| \int_0^t (\partial_s^m F, \partial_s L_0 \partial_s^m u) ds \right| \leq \quad \text{The R.H.S. of (3.7)}.$$

We apply (3.18) and (3.19) to the second term of the right hand side of (3.15) to obtain

$$(3.20) \quad \sum_{m=0}^{4} (\|\partial_r L_0 \partial_t^m u\|^2 + \|\partial_t L_0 \partial_t^m u\|^2 - 4 \int_0^t s \int_R^\infty \partial_r (|\partial_r \partial_s^{m+1} u|^2 r^n) dr ds)$$

$$\leq \quad \text{The R.H.S. of (3.7)} + C\sum_{m=0}^{4}\|\partial_r^2\partial_t^m u\|^2.$$

By the ususal energy estimate for $0 \leq m \leq 5$

$$\|\partial_t^{m+1}u\|^2 + \|\partial_r\partial_t^m u\|^2 \leq C(\epsilon_0^2 + |\int_0^t (\partial_s^m F, \partial_s^{m+1}u)ds|). \tag{3.21}$$

In the same way as in the proof of (3.18) we have

$$\sum_{m=0}^{5}|\int_0^t (\partial_s^m F, \partial_s^{m+1}u)ds| \leq \quad \text{The R.H.S. of (3.7)} + \frac{1}{2}\|\partial_t^6 u\|^2. \tag{3.22}$$

By (3.21) and (3.22)

$$\sum_{m=0}^{5}(\|\partial_t^{m+1}u\|^2 + \|\partial_r\partial_t^m u\|^2) \leq \quad \text{The R.H.S. of (3.7)}. \tag{3.23}$$

From (1.1), Lemma 2.3 and (3.23) we obtain

$$\sum_{m=0}^{4}\|\partial_r^2\partial_t^m u\|^2 \leq C\sum_{m=0}^{4}(\|\partial_r\partial_t^m u\|^2 + \|\partial_t^{m+2}u\|^2 + \|\partial_t^m F\|^2)$$

$$\leq \quad \text{The R.H.S. of (3.7)}.$$

We iterate this argument to obtain

$$\sum_{m=1}^{5}\|\partial_t^m u\|_{6-m,0}^2 + \|\partial_r u\|_{5,0}^2 \leq \quad \text{The R.H.S. of (3.7)}. \tag{3.24}$$

By (3.20) and (3.24) we have (3.8).

We next prove (3.7). We have by (1.1)

$$\partial_t^2 rv - (\partial_r^2 + \frac{n-1}{r}\partial_r)rv = rG - 2\partial_r v - \frac{n-1}{r}v. \tag{3.25}$$

Multiplying both sides of (3.25) by $r^{n-1}\partial_t rv$, integrating with respect to r, we find

$$\frac{d}{dt}(\|r\partial_t v\|^2 + \|\partial_r rv\|^2) - 2\int_R^\infty \partial_r(\partial_r rv\cdot\partial_t rv\cdot r^{n-1})dr$$

$$= 2(rG - 2\partial_r v - \frac{n-1}{r}v, r\partial_t v)$$

from which it follows

$$\|r\partial_t v\|^2 + \|\partial_r rv\|^2 + (n-1)\|v\|^2$$

$$\leq C(\epsilon_0^2 + \int_0^t \|\partial_r v\|ds \sup_{t\in[0,T]}\|r\partial_t v\| + |\int_0^t (rG, r\partial_s v)ds|).$$

Hence

$$\sum_{m=0}^{4}(\|r\partial_t^{m+1}u\|^2+\|r\partial_r\partial_t^m u\|^2) \tag{3.26}$$

$$\leq C(\epsilon_0^2+\sum_{m=0}^{4}\{(1+t)\sup_{t\in[0,T]}\|\partial_r\partial_t^m u\|\sup_{t\in[0,T]}\|r\partial_t^{m+1}u\|+|\int_0^t(r\partial_s^m F,r\partial_s^{m+1}u)ds|\}),$$

where we have used the fact that

$$\|r\partial_r v\|^2+\int_R^\infty \partial_r(|v|^2r^n)dr=\|\partial_r rv\|^2+(n-1)\|v\|^2.$$

In the same way as in the proof of (3.18)

$$\sum_{m=0}^{4}|\int_0^t(r\partial_s^m F,r\partial_s^{m+1}u)ds|$$

$$\leq (1+t)^2\sum_{m=0}^{4}|\int_0^t(1+s)^{-2}(r\partial_s^m F,r\partial_s^{m+1}u)ds|+\frac{1}{2}\|r\partial_t^5 u\|^2$$

$$\leq \text{The R.H.S. of (3.7)}+\frac{1}{2}\|r\partial_t^5 u\|^2.$$

Therefore by the Schwarz inequality and (3.24) we have (3.7).

In the same way as in the proof of (3.26)

$$\sum_{m=0}^{3}(\|r\partial_t L_0\partial_t^m u\|^2+\|r\partial_r L_0\partial_t^m u\|^2)$$

$$\leq C(\epsilon_0^2+\sum_{m=0}^{3}\{(1+t)\sup_{t\in[0,T]}\|\partial_r L_0\partial_t^m u\|\sup_{t\in[0,T]}\|r\partial_t L_0\partial_t^m u\|$$

$$+|\int_0^t(rL_0\partial_s^m F+r\partial_s^m F,r\partial_s L_0\partial_s^m u)ds|\}),$$

By this, (3.8) and the same argument as in the proof of (3.18) we obtain (3.9).

We finally prove (3.10). In the same way as in the proof of (3.12) we have

$$\frac{d}{dt}\sum_{j=1}^{n}(\|\partial_t L_j v\|^2+\|\partial_r L_j v\|^2) \tag{3.27}$$

$$-2\sum_{j=1}^{n}\int_R^\infty \partial_r(\partial_r L_j v\cdot\partial_t L_j v r^{n-1})dr=2\sum_{j=1}^{n}(L_j G,\partial_t L_j v).$$

From (3.13) and (3.27) we conclude that

$$\frac{d}{dt}\{\sum_{j=1}^{n}(\|\partial_t L_j v\|^2+\|\partial_r L_j v\|^2)+\int_R^\infty \partial_r(\{(n-1)t^2|\partial_r v|^2-|\partial_r v|^2r^2\}r^{n-2})dr\} \tag{3.28}$$

$$-2t\int_R^\infty \partial_r(|\partial_r\partial_t v|^2 r^n)dr = 2\sum_{j=1}^n (L_jG, \partial_t L_j v).$$

Integrating (3.28) with respect to t, using Lemma 2.6 we obtain

$$\sum_{j=1}^n (\|\partial_t L_j v\|^2 + \|\partial_r L_j v\|^2) - \int_R^\infty \partial_r(|\partial_r v|^2 r^n)dr \tag{3.29}$$

$$\leq C(\epsilon_0^2 + I(v) + |\sum_{j=1}^n \int_0^t (L_jG, \partial_s L_j v)ds|).$$

By (3.7), (3.8) and (3.24) we have

$$(1+t)^{-2\epsilon}\sum_{m=0}^4 I(\partial_t^m u) \leq \quad \text{The R.H.S. of (3.7).} \tag{3.30}$$

The desired estimate (3.10) follows from (3.29),(3.30) and the same argument as in the proof of (3.18).

Q.E.D.

LEMMA 3.4. *Let u be the solution of (1.1)-(1.2) stated in Theorem 1.2 and let ϵ be the same one as that given in the definition of the function space Γ_T. Then for $0 < t < T$*

$$(1+t)^{-2\epsilon}\sum_{m=0}^3 (\|\partial_r L_0^2\partial_t^m u\|^2 + \|\partial_t L_0^2 \partial_t^m u\|^2) \leq \quad \textit{The R.H.S. of (3.7),} \tag{3.31}$$

$$(1+t)^{-2\epsilon}\sum_{m=0}^3 \|\tilde{Q}L_0\partial_t^m u\|^2 \leq \quad \textit{The R.H.S. of (3.7),} \tag{3.32}$$

$$(1+t)^{-2\epsilon}\sum_{m=0}^2 (\|\tilde{Q}L_0\partial_r\partial_t^m u\|^2 + \|\partial_r L_0^2\partial_r\partial_t^m u\|^2 + \|\partial_r L_0\partial_r\partial_t^m u\|^2) \tag{3.33}$$

$$\leq \quad \textit{The R.H.S. of (3.7),}$$

$$(1+t)^{-2}\sum_{m=0}^2 \|r\partial_r L_0\partial_r\partial_t^m u\|^2 \leq \quad \textit{The R.H.S. of (3.7).} \tag{3.34}$$

Proof. Multiplying both sides of (3.11) by L_0, we obtain

$$\partial_t^2 L_0^2 v - (\partial_r^2 + \frac{n-1}{r}\partial_r)L_0^2 v = 4G + 4L_0G + L_0^2G,$$

where $v = \partial_t^m u, G = \partial_t^m F$. In the same way as in the proof of (3.12) we have

$$\frac{d}{dt}(\|\partial_t L_0^2 v\|^2 + \|\partial_r L_0^2 v\|^2) \tag{3.35}$$

$$-2\int_R^\infty \partial_r(\partial_r L_0^2 v \cdot \partial_t L_0^2 v r^{n-1})dr = 2(4G + 4L_0 G + L_0^2 G, \partial_t L_0^2 v).$$

By a direct calculation we see that

$$\partial_r L_0^2 v = ((n-2)^2\partial_r + (5-2n)t\partial_r\partial_t + (t^2+r^2)\partial_r\partial_t^2)v$$

and

$$\partial_t L_0^2 v = ((4-n)r\partial_r\partial_t + 2tr\partial_r\partial_t^2)v$$

on the boundary $r = R$. Hence by (3.35)

$$\frac{d}{dt}\{\|\partial_t L_0^2 v\|^2 + \|\partial_r L_0^2 v\|^2 - 3(2-n)\int_R^\infty \partial_r(t^2|\partial_r\partial_t v|^2 r^n)dr \tag{3.36}$$

$$+(2(n-2)^2 - (n-1)^2(4-n))\int_R^\infty \partial_r(|\partial_r v|^2 r^n)dr$$

$$+(4-n)\int_R^\infty \partial_r((t^2+r^2)|\partial_r\partial_t v|^2 r^n)dr - 4(n-2)^2\int_R^\infty \partial_r(t\partial_r v\cdot\partial_r\partial_t v r^n)dr\}$$

$$+4\int_R^\infty \partial_r(t|\partial_r\partial_t v|^2 r^n)dr - 4\int_R^\infty \partial_r(t(t^2+r^2)|\partial_r\partial_t^2 v|^2 r^n)dr$$

$$= 2(4G + 4L_0 G + L_0^2 G, \partial_t L_0^2 v).$$

Integrating (3.36) with respect to t we obtain

$$\|\partial_t L_0^2 v\|^2 + \|\partial_r L_0^2 v\|^2$$

$$\le C(\epsilon_0^2 - \int_R^\infty \partial_r((1+t^2)(|\partial_r v|^2 + |\partial_r\partial_t v|^2)r^n)dr$$

$$-\int_0^t s\int_R^\infty \partial_r(|\partial_r\partial_s v|^2 r^n)drds + |\int_0^t (4G + 4L_0 G + L_0^2 G, \partial_s L_0^2 v)ds|).$$

Therefore by (3.8), Lemma 2.2, (3.23), Lemma 2.6 and (3.36)

$$\|\partial_t L_0^2 v\|^2 + \|\partial_r L_0^2 v\|^2 \tag{3.37}$$

$$\le \text{The R.H.S. of (3.7)} + C\sum_{m=0}^{4} I(\partial_t^m u)$$

$$+C(\|\partial_r^2 v\|^2 + \|\partial_r^2\partial_t v\|^2 + |\int_0^t (4G + 4L_0 G + L_0^2 G, \partial_s L_0^2 v)ds|)$$

$$\le C(1+t)^{2\epsilon}(\text{The R.H.S. of (3.7)}) + C|\int_0^t (4G + 4L_0 G + L_0^2 G, \partial_s L_0^2 v)ds|.$$

By the same argument as that in the proof of (3.18) we see that the last term of the right hand side of (3.37) is estimated by

$$\text{The R.H.S. of (3.7)} + \frac{1}{2}\|\partial_t L_0^2 \partial_t^3 u\|^2.$$

Thus we have (3.31) by (3.37). By a direct calculation we have

$$\begin{cases} \tilde{Q}L_0 v = \partial_t L_0^2 v - \partial_t L_0 v - r(2G + L_0 G), \\ \tilde{Q}L_0 \partial_r v = \partial_r L_0^2 \partial_t v - \partial_t L_0 \partial_t v - (2r\partial_r G + r\partial_r L_0 G + G), \\ \partial_r L_0 \partial_r v = \partial_t L_0 \partial_t v - \frac{n-1}{r}\partial_r L_0 v + \partial_t^2 v - \partial_r \partial_t v - (2G + L_0 G), \\ r\partial_r L_0 \partial_r v = \partial_r L_0^2 v - 2\partial_r L_0 v + \partial_r v - t\partial_r L_0 \partial_t v. \end{cases}$$

Hence Lemma 3.3 and (3.31) give

$$(1+t)^{-2\epsilon} \sum_{m=0}^{3} \|\tilde{Q}L_0 \partial_t^m u\|^2$$

$$\leq \quad \text{The R.H.S. of (3.7)} + C(1+t)^{-2\epsilon} \sum_{m=0}^{3} (\|r\partial_t^m F\|^2 + \|rL_0 \partial_t^m F\|^2),$$

$$(1+t)^{-2\epsilon} \sum_{m=0}^{2} (\|\tilde{Q}L_0 \partial_r \partial_t^m u\|^2 + \|\partial_r L_0 \partial_r \partial_t^m u\|^2) \leq \quad \text{The R.H.S. of (3.7)}$$

$$+C(1+t)^{-2\epsilon} \sum_{m=0}^{2} (\|r\partial_r \partial_t^m F\|^2 + \|r\partial_r L_0 \partial_t^m F\|^2 + \|\partial_t^m F\| + \|L_0 \partial_t^m F\|)$$

and

$$(1+t)^{-2} \sum_{m=0}^{2} \|r\partial_r L_0 \partial_t^m u\|^2 \leq \quad \text{The R.H.S. of (3.7)}.$$

Applying Lemma 3.3((3.7), (3.9)) and Corollary to Lemma 2.5 to the above inequalities we obtain (3.32)-(3.34).

Q.E.D.

LEMMA 3.6. *Let u be the solution of (1.1)-(1.2) stated in Theorem 1.2 and let ϵ be the same one as that given in the definition of the function space Γ_T. Then we have*

$$(3.38) \quad (1+t)^{-2\epsilon} \sum_{m=0}^{4} (\|\partial_r L_r \partial_t^m u\|^2 + \|\partial_t L_r \partial_t^m u\|^2) \leq \quad \textit{The R.H.S. of (3.7)},$$

$$(3.39) \quad (1+t)^{-4\epsilon} \sum_{m=0}^{3} \|L_r \partial_r L_0 \partial_t^m u\|^2 \leq \quad \textit{The R.H.S. of (3.7)},$$

$$(3.40) \qquad (1+t)^{-4\epsilon}\sum_{m=0}^{3}\|L_r\partial_r L_r\partial_t^m u\|^2 \le \quad \textit{The R.H.S. of (3.7)}.$$

Proof. The first part of the lemma (3.38) follows from Lemma 3.3 ((3.10)) and the relations that

$$\begin{cases} \partial_r L_j = \frac{x_j}{r}\partial_r L_r, \\ \partial_t L_j = \frac{x_j}{r}\partial_t L_r. \end{cases}$$

By Lemma 3.2, (3.8) and (3.24)

$$(3.41) \qquad \sum_{m=0}^{3}\|L_r\partial_r L_0\partial_t^m u\|^2 \le \quad \text{The R.H.S. of (3.7)} + C\sum_{m=0}^{3} I(L_0\partial_t^m u).$$

By the definition of I

$$I(L_0 v) = \|\tilde{Q}L_0 v\|^2 + \|\partial_r L_0^2 v\|^2 + \|\partial_r L_0 v\|_{1,0}^2$$
$$+(1+t)^{2\epsilon}(\|\tilde{Q}L_0 v\|^2 + \|\partial_r L_0 v\|^2)$$
$$+(\|\partial_r L_0^2 v\| + \|\partial_r L_0 v\|)(\|\partial_r L_0 v\|^\epsilon + \|r\partial_r L_0 v\|^\epsilon)\|\partial_r L_0 v\|^{1-2\epsilon}\|r\partial_r L_0 v\|^\epsilon.$$

Hence Lemma 3.4 and Lemma 3.3 yield

$$(3.42) \qquad \sum_{m=0}^{3} I(L_0\partial_t^m u) \le C(1+t)^{4\epsilon}\{\text{The R.H.S. of (3.7)}\} + \sum_{m=0}^{3}\|\partial_r^2 L_0\partial_t^m u\|^2.$$

From (1.1), Lemma 3.3, Lemma 2.3 and (3.24) we see that the last term of the right hand side of (3.42) is estimated from above by

$$(3.43) \quad C\sum_{m=0}^{3}(\|\partial_t L_0\partial_t^{m+1}u\|^2 + \|\partial_t^{m+1}u\|^2 + \|\partial_r L_0\partial_t^m u\|^2 + \|\partial_t^m F\|^2 + \|L_0\partial_t^m F\|^2)$$
$$\le \quad \text{The R.H.S. of (3.7)}.$$

Therefore by (3.41), (3.42) and (3.43) we have (3.39).

We finally prove (3.40). In the same way as in the proof of (3.41) we obtain by Lemma 3.4 ((3.33), (3.34)), (3.42) and Lemma 3.3

$$\text{The R.H.S. of (3.3)} \quad \le \quad C(1+t)^{4\epsilon}\{\text{The R.H.S. of (3.7)}\}.$$

Hence (3.40) follows.

Q.E.D.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We prove

$$(3.44) \qquad \|u(t)\|_\Gamma^2 \le C(\epsilon_0^2 + \int_0^t (1+s)^{-\frac{9}{8}+\frac{3}{8}\beta+4\epsilon}\|u(s)\|_\Gamma^3 ds).$$

By Lemma 3.4 and Lemma 3.5 it is sufficient to prove that

$$(3.45)\qquad \sum_{m=0}^{4}\|\tilde{Q}\partial_t^m u\|^2+\sum_{m=1}^{3}(\|\Delta L_0\partial_t^m u\|^2+(1+t)^{-2\epsilon}\|\Delta L_r\partial_t^m u\|^2)$$

$$\le C(\epsilon_0^2+\int_0^t(1+s)^{-\frac{9}{8}+\frac{3}{8}\beta+4\epsilon}\|u(s)\|_\Gamma^3 ds).$$

In the same way as in the proof of (3.43)

$$(3.46)\qquad \sum_{m=1}^{3}\|\Delta L_0\partial_t^m u\|^2\le \quad \text{The R.H.S. of (3.44).}$$

We have with $v=\partial_t^m u$

$$\tilde{Q}v=L_0\partial_t v-t\partial_t^m F,$$

$$\Delta L_r\partial_t v=\partial_r L_r\partial_t^2 v-\partial_r L_r\partial_t^m F+2\partial_r\partial_t^2 v$$

$$+\frac{n-1}{r^2}\partial_t L_r v-(\partial_r^3+\frac{n-1}{r}\partial_r^2-\frac{n-1}{r^2}\partial_r)v.$$

Hence by (3.46), Lemma 3.3, Lemma 2.3, (3.24) and Corollary to Lemma 2.5 we conclude that

$$(3.47)\qquad \sum_{m=0}^{4}\|\tilde{Q}\partial_t^m u\|^2+\sum_{m=1}^{3}(1+t)^{-2\epsilon}\|\Delta L_r\partial_t^m u\|^2\le \quad \text{The R.H.S. of (3.44).}$$

From (3.46) and (3.47) the desired estimate (3.45) follows. By (3.44) and Gronwall's inequality we have

$$|||u|||_{\Gamma_T}\le C\epsilon_0.$$

This implies Theorem 1.1.

Q.E.D.

Acknowledgements. The author would like to thank the referee for several remarks and Professor Keiichi Kato for his help in Université de Paris-Sud.

References

1. Y.M.Chen, *Initial boundary value problems of nonlinear wave equations in an exterior domain*, J. London Math. Soc. 40 (1989), 519-534
2. P.S.Datti, *Nonlinear wave equations in exterior domains*, Nonlinear Analysis T.M.A. 15 (1990), 321-331.
3. N.Hayashi, *Global existence of small radially symmetric solutins to quadratic nonlinear evolution equations in an exterior domain*, Math. Z. (1993) (to appear).
4. S.Klainerman, *Uniform decay estimates and Lorentz invariant of the classical wave equation*, Comm. Pure Appl. Math. 38 (1985), 321-332.
5. P.Lax, C.Morawetz and R.Philips, *Exponential decay of solutions of the wave equations in the exterior of a star-shaped obstacle*, Comm. Pure Appl. Math. 16 (1963), 477-486

6. R.B.Melrose, *Singularities and energy decay in acoustical scattering*, Duke Math. J. **46** (1979), 43-59
7. Y.Shibata, *On the global existence theorem of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in an exterior domain*, Tsukuba J. Math. **7** (1983), 1-68.
8. Y.Shibata and Y.Tsutsumi, *Global existence theorem of nonlinear wave equation in the exterior domain*, in "Lecture Notes in Num. Appl. Anal. 6," North-Holland, Kinokuniya, 1983, pp. 155-196
9. Y.Shibata and Y.Tsutsumi, *On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain*, Math. Z. **191** (1986), 165-199
10. Y.Shibata and Y.Tsutsumi, *Local existence of solutions for the initial boundary value problem of fully nonlinear wave equations*, Nonlinear Analysis, TMA **11** (1987), 335-365

Department of Mathematics, Faculty of Engineering
Gunma University, Kiryu 376, JAPAN

(Received May 19, 1992;
in revised form February 3, 1993)

manuscripta math. 81, 41 - 56 (1993)

manuscripta mathematica

Fractal Relaxed Dirichlet Problems

Andrea Braides & Lino Notarantonio

Given a *self similar fractal* $K \subset \mathbb{R}^n$ of Hausdorff dimension $\alpha > n-2$, and $c_1 > 0$, we give an easy and explicit construction, using the self similarity properties of K, of a sequence of closed sets $\mathcal{E}_h$ such that for every bounded open set $\Omega \subset \mathbb{R}^n$ and for every $f \in L^2(\Omega)$ the solutions to

$$\begin{cases} -\Delta u_h = f & \text{in } \Omega \setminus \mathcal{E}_h \\ u_h = 0 & \text{on } \partial(\Omega \setminus \mathcal{E}_h) \end{cases}$$

converge to the solution of the relaxed Dirichlet boundary value problem

$$\begin{cases} -\Delta u + u c_1 \mathcal{H}^\alpha{}_{|K} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

($\mathcal{H}^\alpha{}_{|K}$ denotes the restriction of the α-dimensional Hausdorff measure to K). The condition $\alpha > n-2$ is strict.

1. Introduction: "Un fractal venu d'ailleurs"

In their article *Un terme étrange venu d'ailleurs* [4] D. Cioranescu and F. Murat showed that the sequence of the solutions of Dirichlet problems defined on a domain with holes may converge to the solution of a Dirichlet problem with a "strange term". When the holes $\mathcal{E}_h$ are the union of closed sets, with diameters which tend to zero, but at the same time the number of these closed sets increases as $h \to +\infty$, they proved that the solutions u_h of

$$\text{(1.1)} \qquad \begin{cases} -\Delta u_h = f & \text{in } \Omega \setminus \mathcal{E}_h \\ u_h = 0 & \text{on } \partial(\Omega \setminus \mathcal{E}_h) \end{cases}$$

may converge to the solution of a problem of the kind

$$\text{(1.2)} \qquad \begin{cases} -\Delta u + u\mu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

involving a measure $\mu \in W^{-1,\infty}$ which is independent of Ω and $f \in L^2(\Omega)$. In particular they showed by a "homogenization construction" how to obtain, in the limit problem (1.2), μ equal to (a constant times) the Lebesgue measure, or the restriction of the $n-1$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ to an affine hyperplane.

A complete characterization, in a variational framework, of the possible measures which may appear in the problem (1.2) was given by G. Dal Maso and U. Mosco, introducing the so called Relaxed Dirichlet Problems (cf. [6]). A typical relaxed Dirichlet problem has the form (1.2), but now the measure μ involved belongs to the more general class of all positive Borel measures which do not charge (Borel) sets of capacity zero. This class allows to introduce a suitable notion of (weak) solution for a relaxed Dirichlet problem in a variational way. For more details we refer to [6] and to the references therein.

The aim of this paper is to exhibit a class of measures μ, namely measures of the form

$$\mu = c_1 \mathcal{H}^{\alpha}{}_{|K} \tag{1.3}$$

(with K a fractal, and α its Hausdorff dimension), for which a direct construction of sets $\mathcal{E}_h$ is possible and the problems in (1.1) converge to the problem (1.2). We exploit the self similar structure of a class of fractal sets introduced by J.E. Hutchinson in [8] in order to build up the sets $\mathcal{E}_h$ by an iterating construction. It is interesting to remark that in the "trivial" case of the Lebesgue measure (or of the $n-1$-dimensional Hausdorff measure restricted to an hyperplane) the usual homogenization technique can be seen as a particular case of this method. Our construction applies to fractals of dimension larger than $n-2$; this limitation is natural since Hausdorff measures of dimension less than or equal to $n-2$ charge sets of capacity zero. Note that, since we shall see as a by-product (and it is indeed clear *a fortiori* by [6]) that the measure μ in (1.3) does not charge sets of capacity zero, the existence of a sequence of sets $(\mathcal{E}_h)$ for which the solutions of the problems in (1.1) converge to the solution of (1.2) can be deduced using the results of G. Dal Maso and U. Mosco.

The proof of our main result (Theorem 3.1) is essentially based on a derivation and compactness lemma (Lemma 2.4) which is a consequence of the work of G. Buttazzo, G. Dal Maso and U. Mosco (cf. [2], [6]). An application of their technique reduces the proof of the convergence of the problems in (1.1) to the study of the asymptotic behaviour of the capacity of the sets $\mathcal{E}_h$.

The plan of the paper is as follows. In Section 2 we introduce the family of (strictly) self similar fractals, and we recall the fundamental results proven by J.E. Hutchinson on fractals that are constructed as fixed points of iterations of a family of similitudes. Moreover we state the definition and basic properties of the capacity, together with the compactness and derivation Lemma 2.4. The construction of the sets $\mathcal{E}_h$ which "converge" to a fractal K with Hausdorff dimension α, is carried out in Section 3, through an iterative procedure, starting from a "model" set E. The constant c_1 in (1.3) is explicitly given by an espression involving only the measure $\mathcal{H}^\alpha(K)$ and the capacity of E. Beside stating our main result (Theorem 3.1), we provide a pictorial example of the iterating construction. The last section is devoted to the proof of Theorem 3.1: the crucial step is to establish a "superadditivity" result for the capacity of the sets $\mathcal{E}_h$; eventually we make use of Lemma 2.4, and of some elementary properties of the capacity to conclude the proof.

2. Notations. Fractals, Self Similar Sets, Capacity

We shall denote by $\#(E)$ the cardinality of the set E if E is finite, and $+\infty$ otherwise. If μ is a set function defined on the subsets of a set X and $E \subset X$, we shall indicate by $\mu_{|E}(A) = \mu(A \cap E)$ for every $A \subset X$. Moreover if $f : X \to Y$ is a function, then we define $f_{\#}\mu(E) = \mu(f^{-1}(E))$.

We shall denote by $B_r(x)$ the open ball in $\mathbb{R}^n$ with centre x and radius r. In all that follows we shall assume $n \geq 2$. If E is a measurable subset of $\mathbb{R}^n$, and $r > 0$ we shall denote by $\mathcal{H}^r(E)$ its r-dimensional Hausdorff measure, defined as

$$\mathcal{H}^r(E) = \sup_{\varepsilon>0} \inf\{\sum_{i\in I} m(r)(\mathrm{diam}(E_i))^r : \\ : E_i \text{ measurable}\ ,\ E \subset \bigcup_{i\in I} E_i,\ \mathrm{diam}(E_i) \leq \varepsilon\},$$

where

$$m(r) = 2^{-r}\frac{\Gamma(1/2)^r}{\Gamma((r/2)+1)}$$

and Γ is Euler's function. By $\omega_{n-1} = \mathcal{H}^{n-1}(\partial B_1(0))$ we denote the $n-1$-dimensional measure of the unit sphere.

The letter c will denote a strictly positive constant whose value may vary from line to line, depending only on the fixed parameters of the problem. If $F(p)$, $G(p)$ are two quantities depending on a parameter p, then by $F(p) \approx G(p)$ as $p \to \overline{p}$ we mean that the ratio F/G approaches 1 when $p \to \overline{p}$.

Self Similar Sets.

We introduce now the class of (strictly) self similar fractals, as studied by Hutchinson in [8].

We shall denote with $\mathcal{S} = \{S_1, \ldots, S_N\}$ a finite family of similitudes on $\mathbb{R}^n$ with common ratio $\rho < 1$. The *dimension of similitude* of $\mathcal{S}$ will be the number $\alpha = -\log_\rho N$ (*i.e.* $N\rho^\alpha = 1$).

We define the following sets of indices: $\mathrm{C}(N) = \{1, \ldots, N\}^{\mathbb{N}}$, and $\mathrm{C}_p(N) = \{1, \ldots, N\}^p$, for every $p \in \mathbb{N}$. If $(i) = (i_1, i_2, \ldots) \in \mathrm{C}(N)$, then $(i)_p = (i_1, \ldots, i_p) \in \mathrm{C}_p(N)$ will denote the 'projection' of (i) on $\mathrm{C}_p(N)$.

For every $\beta = (\beta_1, \ldots, \beta_p) \in C_p(N)$ we define the similitude $S_\beta = S_{\beta_p} \circ \ldots \circ S_{\beta_1}$, and $s_\beta \in \mathbb{R}^n$ the unique fixed point of S_β. Moreover, for every $(i) \in C(N)$ we set

$$s_{(i)} = \lim_{p \to +\infty} s_{(i)_p}, \tag{2.1}$$

and

$$K_S = \{s_{(i)} : (i) \in C(N)\}. \tag{2.2}$$

The *coordinate map* $\pi : C(N) \to K_S$, $(i) \mapsto s_{(i)}$ is continuous if we consider the product topology on $C(N)$.

We will say that a set $E \subset \mathbb{R}^n$ is *$\mathcal{S}$-invariant* if we have $\bigcup_{i=1}^N S_i(E) = E$.

Theorem 2.1. (Hutchinson [8]) *The set K_S is the unique closed bounded $\mathcal{S}$-invariant set.*

If ν is a regular Borel measure with bounded support and finite mass, we define the measure

$$\mathcal{S}\nu = \sum_{i=1}^N \rho^\alpha S_{i\#}\nu = \sum_{i=1}^N \frac{1}{N} S_{i\#}\nu; \tag{2.3}$$

i.e. $\mathcal{S}\nu(E) = \sum_{i=1}^N \rho^\alpha \nu\Big(S_i^{-1}(E)\Big)$.

On $C(N)$ we shall consider the product measure induced by the probability measure with mass $\frac{1}{N}$ on each $i \in \{1, \ldots, N\}$, and we shall denote it by τ.

Theorem 2.2. (Hutchinson [8]) *The measure $\nu_S = \pi_\#(\tau)$ is the unique regular Borel measure verifying:*

i) ν_S has compact support (the support of ν_S is in fact K_S);

ii) ν_S is a probability measure (i.e. $\nu_S(\mathbb{R}^n) = 1$);

iii) ν_S is $\mathcal{S}$-invariant (i.e. $\mathcal{S}\nu = \nu$).

We will say that a set $J \subset \mathbb{R}^n$ is *self similar* if there exists a family of similitudes $\mathcal{S}$ as above such that

i) J is $\mathcal{S}$-invariant;

ii) J has Hausdorff dimension $k \geq 0$, $\mathcal{H}^k(J) > 0$, and $\mathcal{H}^k(S_i(J) \cap S_j(J)) = 0$ if $i \neq j$.

We will say that the family of similitudes $\mathcal{S}$ satisfies the *open set condition* (see [8]), and that K_S is a *self similar fractal*, if there exists a bounded open set $\mathcal{O} \subset \mathbb{R}^n$ such that $\mathcal{H}^\alpha(K_S \setminus \mathcal{O}) = 0$, $S_i(\mathcal{O}) \subset \mathcal{O}$ for every $i \in \{1, \ldots, N\}$, and $S_i(\mathcal{O}) \cap S_j(\mathcal{O}) = \emptyset$ if $i \neq j$. For a possible choiche of $\mathcal{S}$ and of the set $\mathcal{O}$ satisfying the open set conditionfor the Von Koch curve we refer to [1].

With this definition of self similar fractals, we recover most of the well-known fractals obtained by an iteration construction, such as the Von Koch curve, the Sierpinski gasket etc. (cf. [9], [7], [8] for more details and examples). Let us remark that in this framework are included also "trivial" fractals such as the cube $[0,1]^n$ in $\mathbb{R}^n$ or the k-dimensional cube $[0,1]^k$ in $\mathbb{R}^n$, $k < n$.

The following theorem is the fundamental result for self similar fractals.

Theorem 2.3. (Hutchinson [8]) *If the family of similitudes $\mathcal{S}$ verifies the open set condition, then $K_{\mathcal{S}}$ is a self similar set, the Hausdorff dimension of $K_{\mathcal{S}}$ is α, $0 < \mathcal{H}^\alpha(K_{\mathcal{S}}) < +\infty$, and*

$$\nu_{\mathcal{S}} = \frac{1}{\mathcal{H}^\alpha(K_{\mathcal{S}})}\mathcal{H}^\alpha{}_{|K_{\mathcal{S}}}. \tag{2.4}$$

Capacity.

The main tool for the proof of our results will be a compactness and derivation lemma (Lemma 2.4 below). Before stating it, we shall need to define the notions of *capacity*.

Let F be an open subset of $\mathbb{R}^n$, and E a Borel subset of F; the capacity of E with respect to F is

$$\begin{aligned}\operatorname{cap}(E,F) = \inf\Big\{ \int_F |Du|^2\, dx : u \geq 1 \text{ on an}\\ \text{open neighbourhood of } E, u \in \mathrm{H}^1_0(F)\Big\}.\end{aligned} \tag{2.5}$$

We say that a property $P(x)$ holds for *quasi every* $x \in E$ (or quasi-everywhere in E) if

$$\operatorname{cap}(\{x \in E : P(x) \text{ is not verified}\}, F) = 0.$$

Note that the property of being of capacity zero is independent of the set F.

It can be proven (see [3]) that for every Borel subset E of F, there exists a function $u \in \mathrm{H}^1_0(F)$ such that $u \in \mathrm{H}^1_0(F)$, $u \geq 1$ quasi everywhere on E, and

$$\operatorname{cap}(E,F) = \int_F |Du|^2 dx;$$

this function will be called the *capacitary potential* of E with respect to F.

For example, if we consider two concentric balls $B_r(x) \subset B_R(x)$, $0 < r < R$, then we have

$$\operatorname{cap}(B_r(x), B_R(x)) = (n-2)\omega_{n-1}\frac{(rR)^{n-2}}{R^{n-2} - r^{n-2}} \quad \text{for } n \geq 3$$

and

$$\operatorname{cap}(B_r(x), B_R(x)) = 2\pi \left(\log \frac{R}{r}\right)^{-1} \quad \text{for } n = 2.$$

Notice that from the definition of capacity, we have $\operatorname{cap}(tE, tF) = t^{n-2}\operatorname{cap}(E, F)$ for any real number $t > 0$.

Lemma 2.4. *Let (E_h) be a sequence of closed subsets of $\mathbb{R}^n$ and let ν be a Radon measure such that $\nu(B) = 0$ on sets B of capacity zero. Let us suppose that the following hypotheses are satisfied:*

(i) for every $x \in \mathbb{R}^n$ and for every $T > 0$ we have

$$\liminf_{t\to 0+} \limsup_{k\to+\infty} \frac{\operatorname{cap}(B_t(x) \cap E_k, B_T(x))}{\nu(B_t(x))}$$

$$= \liminf_{t\to 0+} \liminf_{k\to+\infty} \frac{\operatorname{cap}(B_t(x) \cap E_k, B_T(x))}{\nu(B_t(x))} = \varphi(x); \tag{2.6}$$

(ii) $\varphi(x) < +\infty$ for q.e. $x \in \mathbb{R}^n$;

(iii) $\varphi \in L^1(\mathbb{R}^n, \nu)$.

Let us define the measure $\mu = \varphi\nu$. Then, for every bounded open set Ω of $\mathbb{R}^n$, and for every $f \in L^2(\Omega)$, the solutions (u_h) to the Dirichlet problem

$$\begin{cases} -\Delta u_h = f & \text{in } \Omega \setminus E_h \\ u_h \in H^1_0(\Omega \setminus E_h) \end{cases} \tag{2.7}$$

converge in $L^2(\Omega)$, as $h \to +\infty$, to the weak solution u of the Dirichlet problem

$$\begin{cases} -\Delta u + u\mu = f & \text{in } \Omega \\ u \in H^1_0(\Omega) \cap L^1(\Omega, \mu), \end{cases} \tag{2.8}$$

i.e., the unique function $u \in H^1_0(\Omega) \cap L^1(\Omega, \mu)$ such that

$$\int_\Omega DuDv\,dx + \int_\Omega uv\,d\mu = \int_\Omega fv\,dx$$

for every $v \in C^\infty(\Omega)$ with compact support in Ω (cf. [6, Proposition 3.8]).

Proof. The existence of a measure $\tilde{\mu}$ and of a function u which satisfy (2.8) has been proven in the more general framework of the so-called Relaxed Dirichlet Problem (cf. Propositions 4.9 and 4.10 in [6]). In [2, Theorem 5.2] it is proven that $\tilde{\mu} = \mu = \varphi\nu$. □

3. The Main Result

From now on we shall consider as fixed a family of similitudes $\mathcal{S}$ verifying the open set condition, and the corresponding set $K = K_{\mathcal{S}}$, as defined in Section 2.

Let us fix $x_0 \in \mathcal{O}$, and let us define

$$R = \frac{1}{2}\text{dist}(x_0, \partial\mathcal{O}). \tag{3.1}$$

Let $c_0 > 0$; for every $p \in \mathbb{N}$ we shall set

$$R_p = R\rho^p, \tag{3.2}$$

and

$$\rho_p = \begin{cases} c_0(R\rho^p)^{\alpha/(n-2)} = c_0(R_p)^{\alpha/(n-2)} & \text{if } n \geq 3, \\ (R\rho^p)\exp\left(-\frac{1}{c_0}(R\rho^p)^{-\alpha}\right) = (R_p)\exp\left(-\frac{1}{c_0}(R_p)^{-\alpha}\right) & \text{if } n = 2. \end{cases} \tag{3.3}$$

Moreover let us fix a set $E \subset B_1(0)$, with finite capacity with respect to $B_1(0)$, and define for every $\beta \in \mathrm{C}_p(N)$

$$x_\beta = S_\beta(x_0), \qquad B^p_\beta = x_\beta + \rho_p E, \tag{3.4}$$

and

$$\mathcal{E}_p = \bigcup\{\overline{B^p_\beta} : \beta \in \mathrm{C}_p(N)\}. \tag{3.5}$$

Note that we have $|x_\gamma - x_\beta| > 2R_p$ if $\beta, \gamma \in \mathrm{C}_p(N)$, and $\beta \neq \gamma$; moreover, if $\alpha > n-2$, we have for large p

$$\text{dist}(B^p_\gamma, B^p_\beta) > R_p. \tag{3.6}$$

We briefly illustrate the construction in (3.1)–(3.6) with an example.

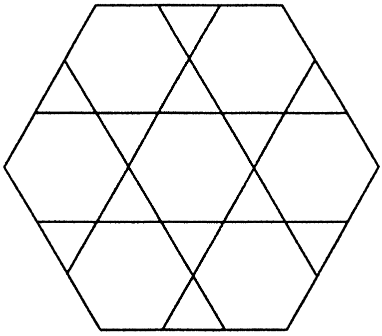


Figure 1

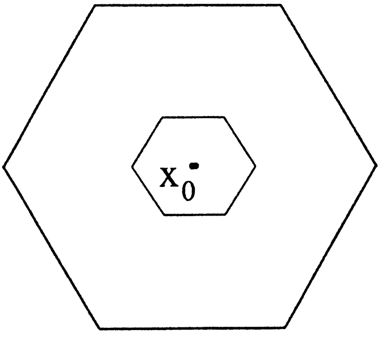


Figure 2

We take as $\mathcal{O}$ the larger hexagon in Fig. 1 and $S_1, \dots, S_7$ the seven similitudes with $\rho = 1/3$ which carry $\mathcal{O}$ into the smaller hexagons. Fig. 2 shows a possible choice of x_0 and R.

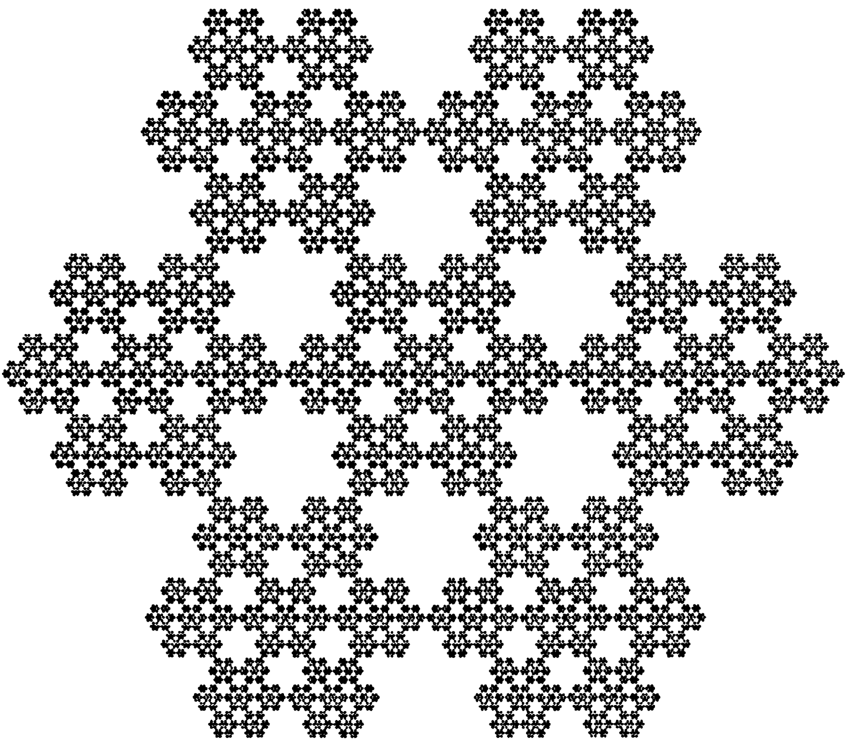


Figure 3

In Fig. 3 it is shown $\mathcal{E}_5$ with $E = B_1(0)$ for a proper choice of c_0.

We are now in a position to state the main result.

Theorem 3.1. *Let K be a self similar fractal of Hausdorff dimension $\alpha > n-2$, R defined by (3.1), $c_0 > 0$, $E \subset B_1(0)$ a set with finite capacity with respect to $B_1(0)$, and let us set*

$$
(3.7) \qquad c_1 = \begin{cases} c_0^{n-2} R^\alpha \frac{1}{\mathcal{H}^\alpha(K)} \mathrm{cap}(E, \mathbb{R}^n) & \text{if } n \geq 3 \\ c_0 R^\alpha \frac{1}{\mathcal{H}^\alpha(K)} \lim_{t \to +\infty} \Big(\log t \ \mathrm{cap}(E, B_t(0)) \Big) & \text{if } n = 2. \end{cases}
$$

Let $\mathcal{E}_p$ be constructed as in (3.1)-(3.6), let Ω be a bounded open subset of $\mathbb{R}^n$, and $f \in L^2(\Omega)$. Then, the weak solutions u_p to the Dirichlet problem

$$
(3.8) \qquad \begin{cases} -\Delta u_p = f & \text{in } \Omega \setminus \mathcal{E}_p \\ u_p \in \mathrm{H}_0^1(\Omega \setminus \mathcal{E}_p) \end{cases}
$$

($p \in \mathbb{N}$) converge in $L^2(\Omega)$, as $p \to +\infty$, to the weak solution u of the Dirichlet problem

$$
\begin{cases} -\Delta u + c_1 u \mathcal{H}^\alpha_{|K\cap\Omega} = f & \text{in } \Omega \\ u \in \mathrm{H}^1_0(\Omega), \end{cases} \tag{3.9}
$$

i.e., the unique function $u \in \mathrm{H}^1_0(\Omega)$ such that

$$
\int_\Omega DuDv\,dx + c_1 \int_{K\cap\Omega} uv\,d\mathcal{H}^\alpha = \int_\Omega fv\,dx \tag{3.10}
$$

for every $v \in \mathrm{H}^1_0(\Omega)$.

Remark 3.2. 1) If we take $E = B_1(0)$, the constant c_1 may be written as

$$
c_1 = \begin{cases} \frac{c_0^{n-2} R^\alpha}{\mathcal{H}^\alpha(K)} \mathrm{cap}(B_1(0), \mathbb{R}^n) & \text{if } n \geq 3 \\ \frac{c_0 R^\alpha}{\mathcal{H}^\alpha(K)} \lim_{t\to+\infty} \Big(\log t \ \mathrm{cap}(B_1(0), B_t(0))\Big) & \text{if } n = 2, \end{cases}
$$

i.e. we have

$$
c_1 = \begin{cases} c_0^{n-2} \frac{(n-2)\omega_{n-1} R^\alpha}{\mathcal{H}^\alpha(K)} & \text{if } n \geq 3 \\ c_0 \frac{2\pi R^\alpha}{\mathcal{H}^\alpha(K)} & \text{if } n = 2. \end{cases}
$$

2) As a particular case of Theorem 3.1, we get the results of D. Cioranescu & F. Murat [4,§2], where they obtain in the limit the Lebesgue measure. In fact, we can consider Ω as a subset of an n-dimensional cube $Q = [-T, T]^n$. The cube Q itself can be seen as a self similar set, by choosing the 2^n similitudes which carry it into 2^n sub-cubes of side length T. In this case, the procedure of iteration coincides with the usual homogenization technique. In the same way, we can obtain as a limit the $(n-1)$-dimensional Hausdorff measure restricted to an $(n-1)$-hyperplane, as in [4].

3) It will be clear from the proof of Theorem 3.1 that the same conclusion holds true when we consider more general sets $\mathcal{E}_p$ obtained as in (3.5) where the $B_p^\beta \subset B_{R_p}$ are not necessarily similar to each other; it suffices that

$$
\mathrm{cap}(B_p^\beta, B_{R_p}) \approx c_1 \rho^{\alpha p} \mathcal{H}^\alpha(K)
$$

uniformly in β as $p \to +\infty$.

4) If $\mathrm{cap}(B_p^\beta, B_{R_p}(x_\beta))\rho^{-\alpha p}$ tends to $+\infty$ as $p \to +\infty$, uniformly in β, then the limit problem (3.9) becomes

$$
\begin{cases} -\Delta u = f & \text{in } \Omega \setminus K \\ u \in \mathrm{H}^1_0(\Omega \setminus K). \end{cases}
$$

5) Theorem 3.1 may be interpreted in term of Γ-*convergence* of the energy functionals related to problems (3.8), (3,9) (cf. the recent book [5] for an introduction to the theory of Γ-convergence). In fact, let $\mathcal{F}_p : \mathrm{H}^1_0(\Omega) \to [0,+\infty]$ be defined as

$$\mathcal{F}_p(u) = \begin{cases} \int_\Omega |Du|^2\,dx & \text{if } u_{|\Omega\setminus\mathcal{E}_p} \in \mathrm{H}^1_0(\Omega\setminus\mathcal{E}_p) \\ +\infty & \text{otherwise;} \end{cases}$$

then the sequence $(\mathcal{F}_p)$ Γ-converges with respect to the $\mathrm{L}^2(\Omega)$-topology, as $p \to +\infty$, to the functional $\mathcal{F} : \mathrm{H}^1_0(\Omega) \to [0,+\infty]$ defined by $\mathcal{F}(u) = \int_{\Omega\setminus K} |Du|^2 dx + \int_{K\cap\Omega} u^2 d\mathcal{H}^\alpha$ (see Proposition 4.10 in [6]).

4. Proof of the main result

We begin by proving two simple results regarding the structure of self similar fractals.

Proposition 4.1. *Let $K = K_S$ be a self similar fractal. Let V be a bounded open subset of $\mathbb{R}^n$ such that $\mathcal{H}^\alpha(\partial V \cap K_S) = 0$. Then*

$$\mathcal{H}^\alpha(V \cap K_S) = \lim_{p\to+\infty} \rho^{p\alpha}\mathcal{H}^\alpha(K_S)\ \#\{\beta \in \mathrm{C}_p(N) : S_\beta(\mathcal{O}) \cap V \neq \emptyset\}, \tag{4.1}$$

and

$$\mathcal{H}^\alpha(V \cap K_S) = \lim_{p\to+\infty} \rho^{p\alpha}\mathcal{H}^\alpha(K_S)\ \#\{\beta \in \mathrm{C}_p(N) : B^p_\beta \cap V \neq \emptyset\}. \tag{4.2}$$

Proof. Let $I = \{\beta \in \mathrm{C}_p(N) : S_\beta(\mathcal{O}) \cap V \neq \emptyset\}$, and let us define

$$V_q = \{x \in \mathbb{R}^n : \mathrm{dist}(x, V) < \rho^q \mathrm{diam}\mathcal{O}\}.$$

We have

$$\mathcal{H}^\alpha(V \cap K_S) = \mathcal{H}^\alpha(\overline{V} \cap K_S) = \lim_{q\to+\infty} \mathcal{H}^\alpha(V_q \cap K_S).$$

If $p \geq q$, then $V_q \supset \bigcup\{S_\beta(\mathcal{O}) : \beta \in I\}$ so that

$$\mathcal{H}^\alpha(V_q \cap K_S) \geq \sum_{\beta\in I} \mathcal{H}^\alpha(K_S \cap S_\beta(\mathcal{O})) = \rho^{p\alpha}\mathcal{H}^\alpha(K_S)\ \#(I).$$

Passing to the limit first as $p \to +\infty$, and then as $q \to +\infty$ we have

$$\lim_{q\to+\infty} \mathcal{H}^\alpha(V_q \cap K_S) \geq \limsup_{p\to+\infty} \rho^{\alpha p}\mathcal{H}^\alpha(K_S)\ \#(I)$$

$$\geq \liminf_{p\to+\infty} \rho^{\alpha p}\mathcal{H}^\alpha(K_S)\ \#(I) \geq \mathcal{H}^\alpha(V \cap K_S),$$

and hence the proof of (4.1) is achieved; in the same way we can prove (4.2). □

Proposition 4.2. *Let $\mathcal{S}$ satisfy the open set condition. For every $R > 0$ there exists $M_R > 0$ such that, for every $p \in \mathbb{N}$, and for every $\beta \in \mathrm{C}_p(N)$, we have*

$$\#\{\gamma \in \mathrm{C}_p(N) : \mathrm{dist}(S_\beta(\mathcal{O}), S_\gamma(\mathcal{O})) < R\rho^p\} \leq M_R. \tag{4.3}$$

Proof. Fixed $R > 0$, let us consider the set

$$D^R_{p,\beta} = \{x \in \mathbb{R}^n : 0 < \mathrm{dist}(x, S_\beta(\mathcal{O})) < \mathrm{diam} S_\beta(\mathcal{O}) + R\rho^p \Big(= \rho^p(\mathrm{diam}\mathcal{O} + R)\Big)\}.$$

We have $|D^R_{p,\beta}| = c_R(\rho^p)^n$ (with c_R a constant depending only on R and $\mathcal{O}$). Let us consider

$$N^R_{p,\beta} = \{\gamma \in \mathrm{C}_p(N) : \mathrm{dist}(S_\beta(\mathcal{O}), S_\gamma(\mathcal{O})) < R\rho^p\}.$$

Since $\mathcal{O}$ is open, it contains a ball of radius R_0, and hence each $S_\gamma(\mathcal{O})$ contains a ball of radius $R_0\rho^p$. We have then

$$|D^R_{p,\beta}| \geq N^R_{p,\beta}(R_0)^n |B_1(0)| \rho^{pn},$$

so that $N^R_{p,\beta}(R_0)^n |B_1(0)| \leq c_R$. We can take then $M_R = c_R(R_0)^{-n}|B_1(0)|^{-1}$. □

We can proceed now in the proof of Theorem 3.1. Let us first notice that, if $\alpha > n-2$, the measure $\mathcal{H}^\alpha{}_{|K}$ belongs to $\mathrm{H}^{-1}_{\mathrm{loc}}(\mathbb{R}^n)$ (and hence it is zero on all sets of capacity zero). In fact, by [10,Th 4.7.5] it is sufficient to remark that for all $x \in K$

$$\int_0^1 \frac{\mathcal{H}^\alpha{}_{|K}(B_r(x))}{r^{n-2}} \frac{dr}{r} \leq c \int_0^1 \frac{dr}{r^{n-2-\alpha+1}} < +\infty.$$

The next step will be to estimate the capacity in (2.6) in order to calculate the limits therein. The estimate from above follows from the (strong) sub-additivity of the capacity, while the estimate from below is proven using the following result.

Lemma 4.3. *Let V be a bounded open subset of $\mathbb{R}^n$. There exists a constant c depending only on n, R and α such that if we define*

$$\delta = \delta(V) = \begin{cases} c(\mathrm{diam}V)^{\alpha-n+2} & \text{if } n \geq 3 \\ c(\mathrm{diam}V)^\alpha |\log(\mathrm{diam}V)| & \text{if } n = 2 \end{cases} \tag{4.4}$$

and we have $\delta < 1$, then for every $T > 2\mathrm{diam}(V)$ and for every $x \in V$,

$$\begin{aligned} &\mathrm{cap}(V \cap \mathcal{E}_p, B_T(x)) \\ &\quad \geq (1-\delta)^2 \#\{\beta \in C_p(N) : B^p_\beta \cap V \neq \emptyset\} \mathrm{cap}(\rho_p E, B_{R_p}(0)). \end{aligned} \tag{4.5}$$

for sufficiently large p.

Proof. Let us define $I = \{\beta \in C_p(N) : B^p_\beta \cap V \neq \emptyset\}$. We claim that, if the capacitary potential u of $V \cap \mathcal{E}_p$ with respect to $B_T(x)$ satisfies $u \leq \delta$ on $\partial B_{R_p}(x_\beta)$ for every $\beta \in I$, then the proof is achieved. In fact, let us assume that $u \leq \delta$ on $\partial B_{R_p}(x_\beta)$ for every $\beta \in I$; let us define $v = \frac{1}{(1-\delta)}(u-\delta)^+$. By the definition of the capacitary potential, it is easy to see that $v \in \mathrm{H}^1_0(B_T(x))$, $v \geq 1$ q.e on $V \cap \mathcal{E}_p$ and $v = 0$ q.e on $\partial B_{R_p}(x_\beta)$, for every $\beta \in I$, hence we have

$$\mathrm{cap}(B^p_\beta, B_{R_p}(x_\beta)) \leq \int_{B_{R_p}(x_\beta)} |Dv|^2 dx$$

and therefore

$$\int_{B_T(x)} |Dv|^2 dx \geq \sum_{\beta \in I} \int_{B_{R_p}(x_\beta)} |Dv|^2 dx \geq \sum_{\beta \in I} \mathrm{cap}(B^p_\beta, B_{R_p}(x_\beta)). \tag{4.6}$$

By definition of v, we have also

$$\begin{aligned}\int_{B_T(x)} |Dv|^2 dx &= \frac{1}{(1-\delta)^2} \int_{B_T(x)} |D(u-\delta)^+|^2 dx \\ &\leq \frac{1}{(1-\delta)^2} \int_{B_T(x)} |Du|^2 dx = \frac{\mathrm{cap}(V \cap \mathcal{E}_p, B_T(x))}{(1-\delta)^2}.\end{aligned} \tag{4.7}$$

We obtain the assertion by (4.6) and (4.7).

Now it remains to prove that $u \leq \delta$ on $\partial B_{R_p}(x_\beta)$ for every $\beta \in I$. We start with the case $n = 2$.

For every $\beta \in I$ consider the function

$$u_\beta(x) = \left(\log \frac{\rho_p}{2T}\right)^{-1} \log \frac{|x - x_\beta|}{2T} \tag{4.8}$$

which is the solution to

$$\begin{cases} -\Delta u_\beta = 0 & \text{in } B_{2T}(x_\beta) \\ u_\beta = 1 & \text{on } \partial B_{\rho_p}(x_\beta) \\ u_\beta = 0 & \text{on } \partial B_{2T}(x_\beta), \end{cases} \tag{4.9}$$

and define

$$z(x) = \sum_{\beta \in I} u_\beta(x); \tag{4.10}$$

observe that z is superharmonic on $\mathbb{R}^2$, as sum of superharmonic functions. Since, for every $\beta \in I$, we have $B_T(x) \subset\subset B_{2T}(x_\beta)$, it follows that $u_\beta(y) \geq 0$ for every $y \in \partial B_T(x)$ and for every $\beta \in I$; so $z \geq 0$ on $\partial B_T(x)$, hence $z \geq u$ in $B_T(x)$, since the capacitary potential is characterized to be smaller than to be smaller than any other positive superharmonic function which is greater than or equal to 1 on $\mathcal{E}_p \cap V$. To achieve the proof it is sufficient, by the maximum principle, to consider a fixed $y \in \partial B_{R_p}(x_\beta)$ and prove that $z(y) \leq \delta$.

By definition of z, we have

$$z(y) = \sum_{\beta \in I} \left(\log \frac{\rho_p}{2T}\right)^{-1} \log \frac{|y - x_\beta|}{2T}.$$

Let $I_q = \{\beta \in I : R\rho^{p-q} \leq |x_\beta - y| < R\rho^{p-q-1}\}$; then

$$z(y) \leq \sum_{q=0}^{\overline{q}} \#(I_q) \left(\log \frac{\rho_p}{2T}\right)^{-1} \log\left(\frac{R\rho^{p-q}}{2T}\right), \tag{4.11}$$

where $\overline{q} = \overline{q}(h,p) = p - \left[\log_\rho\left(\frac{1}{R}\mathrm{diam}V\right)\right]$. Now we have

$$I_q \subset \{\beta \in I : |x_\beta - y| < R\rho^{p-q-1}\},$$

so by Proposition 4.2 we get

$$\#(I_q) \leq N^{q+1}\#\{\gamma \in C_{p-q-1}(N) : \mathrm{dist}(S_{\overline{\beta}}(\mathcal{O}), S_\gamma(\mathcal{O})) < R\rho^{p-q-1}\} \leq MN^{q+1},$$

where $\overline{\beta} \in C_{p-q-1}(N)$ is determined by $x_\beta \in S_{\overline{\beta}}(\mathcal{O})$. Recalling that $N = \rho^{-\alpha}$, we obtain

$$z(y) \leq \frac{MN}{\log\left(\frac{2T}{\rho_p}\right)} \sum_{q=0}^{\overline{q}} \log\left(\frac{2T\rho^{q-p}}{R}\right)\rho^{-q\alpha}. \tag{4.12}$$

We observe that the function

$$x \mapsto \rho^{-x\alpha} \log\left(\frac{2T}{R}\rho^{x-p}\right) = N^x\left(\log\left(\frac{2T}{R}\rho^{-p}\right) + x\log\rho\right)$$

is increasing in $(0, \overline{q}+1)$ for p large enough. Therefore we can estimate the sum at the right hand side by an integration as follows

$$\begin{aligned} \sum_{q=0}^{\overline{q}} \log\left(\frac{2T}{R}\rho^{q-p}\right)\rho^{-q\alpha} &\leq \int_0^{\overline{q}+1} \log\left(\frac{2T}{R}\rho^{x-p}\right)\rho^{-x\alpha}dx \\ &= \left(\log 1/\rho\right)^{-1} \int_1^{\rho^{-\overline{q}-1}} \log\left(\frac{2T}{R}\rho^{-p}y^{-1}\right)y^{\alpha-1}dy \\ &\leq c\rho^{-p\alpha}\left(\mathrm{diam}V\right)^\alpha \left|\log\left(\mathrm{diam}V\right)\right|. \end{aligned} \tag{4.13}$$

From (4.12), we obtain (recalling the definition of ρ_p)

$$z(y) \le c\rho^{-p\alpha}\Big(\mathrm{diam}V\Big)^{\alpha}\Big|\log\Big(\mathrm{diam}V\Big)\Big|\frac{MN}{\log\Big(\frac{T}{\rho_p}\Big)}$$

$$\le c\,\frac{\rho^{-p\alpha}\Big(\mathrm{diam}V\Big)^{\alpha}\Big|\log\Big(\mathrm{diam}V\Big)\Big|}{c_0^{-1}(R\rho^p)^{-\alpha}-\log\Big(\frac{R\rho^p}{T}\Big)} \le c(\mathrm{diam}V)^{\alpha}\Big|\log\Big(\mathrm{diam}V\Big)\Big|. \tag{4.14}$$

We can take then $\delta = c(\mathrm{diam}V)^{\alpha}\Big|\log\Big(\mathrm{diam}V\Big)\Big|$.

In the case $n \ge 3$ we may proceed in the same way as in the case $n = 2$, using in (4.8) the functions

$$u_\beta(x) = \frac{\rho_p^{n-2}}{T^{n-2}-\rho_p^{n-2}}\Big(\Big(\frac{|x-x_\beta|}{T}\Big)^{2-n}-1\Big),$$

which verify (4.9), and defining the function z as in (4.10). We get then, as in (4.11) and (4.12),

$$z(y) \le \sum_{q=0}^{\bar q}\#(I_q)\frac{\rho_p^{n-2}}{T^{n-2}-\rho_p^{n-2}}\Big(\frac{R\rho^{p-q}}{T}\Big)^{2-n}$$

$$\le c\rho_p^{n-2}\sum_{q=0}^{\bar q}\#(I_q)\rho^{(p-q)(2-n)} \le c\,\rho^{p(\alpha-n+2)}\sum_{q=0}^{\bar q}N^q\rho^{(n-2)q}$$

$$= c\,\rho^{p(\alpha-n+2)}\sum_{q=0}^{\bar q}\rho^{(n-2-\alpha)q} \le c\rho^{p(\alpha-n+2)}(\rho^{n-2-\alpha})^{\bar q+1} \le c(\mathrm{diam}V)^{\alpha-n+2},$$

and we achieve the proof of the Lemma. □

We can now conclude the proof of Theorem 3.1. It will suffice to compute the function φ as defined in Lemma 2.4, and show that $\varphi(x) \equiv c_1$ q.e. on K.

We choose $\nu = \mathcal{H}^\alpha_{|K}$, $E_p = \mathcal{E}_p$, and $T > 0$ a fixed real number, and compute then φ for every $x \in K$ using (2.6). Notice that $\mathcal{H}^\alpha(B_t(x)\cap K) > 0$ for every $t > 0$, and we have $\mathcal{H}^\alpha(\partial B_t(x)\cap K) = 0$, except for at most a countable number of t. Let $I = \{\beta \in \mathrm{C}_p(N) : B^p_\beta \cap B_t(x) \ne \emptyset\}$. From (4.2) we obtain

$$\mathcal{H}^\alpha(B_t(x)\cap K) = \lim_{p\to+\infty}\rho^{p\alpha}\mathcal{H}^\alpha(K)\ \#(I). \tag{4.15}$$

Recall that, from the elementary properties of the capacity (cf. [3]), we have

$$\mathrm{cap}(B_t(x)\cap\mathcal{E}_p, B_T(x)) \le \sum_{\beta\in I}\mathrm{cap}(B^p_\beta, B_{R_p}(x_\beta))$$

$$= \#(I)\ \mathrm{cap}(\rho_pE, B_{R_p}(0)),$$

hence by (4.15) we get

$$\liminf_{t\to0}\limsup_{p\to+\infty}\frac{\mathrm{cap}(B_t(x)\cap\mathcal{E}_p, B_T(x))}{\mathcal{H}^\alpha(B_t(x)\cap K)} \le \lim_{p\to+\infty}\frac{\mathrm{cap}(\rho_pE, B_{R_p}(0))}{\rho^{p\alpha}\mathcal{H}^\alpha(K)}.$$

By using Lemma 4.3 and (4.2), we obtain also

$$\liminf_{t\to 0}\liminf_{p\to+\infty}\frac{\operatorname{cap}(B_t(x)\cap\mathcal{E}_p, B_T(x))}{\mathcal{H}^\alpha(B_t(x)\cap K)}$$

$$\geq \liminf_{t\to 0}\liminf_{p\to+\infty}\frac{\operatorname{cap}(\rho_p E, B_{R_p}(0))\,(1-\delta(B_t(x))^2}{\mathcal{H}^\alpha(K)\rho^{p\alpha}}$$

$$= \lim_{p\to+\infty}\frac{\rho^{-\alpha p}}{\mathcal{H}^\alpha(K)}\operatorname{cap}(\rho_p E, B_{R_p}(0)).$$

Hence the hypotheses of Lemma 2.4 are established.

If $n \geq 3$, we have

$$\varphi(x) = \lim_{p\to+\infty}\frac{\rho^{-\alpha p}}{\mathcal{H}^\alpha(K)}\rho_p^{n-2}\operatorname{cap}(E, \rho_p^{-1}B_{R_p}(0))$$

$$= \frac{c_0^{n-2}R^\alpha}{\mathcal{H}^\alpha(K)}\operatorname{cap}(E, \mathbb{R}^n) = c_1;$$

in the case $n = 2$ we have

$$\varphi(x) = \lim_{p\to+\infty}\frac{\rho^{-\alpha p}}{\mathcal{H}^\alpha(K)}\operatorname{cap}(E, \rho_p^{-1}B_{R_p}(0)) = c_1.$$

Finally, we remark that since $\mathcal{H}^\alpha{}_{|K} \in \mathrm{H}^{-1}(\Omega)$, we have that $\mathrm{L}^1(\Omega, \mathcal{H}^\alpha{}_{|K}) \supset \mathrm{H}_0^1(\Omega)$; therefore formula (3.10) is valid for test functions in $\mathrm{H}_0^1(\Omega)$. □

Acknowledgements

We gratefully acknowledge many fruitful discussions with Gianni Dal Maso. This work is part of the research project "Irregular Variational Problems – Discontinuous Structures" of the Italian CNR.

References

[1] A. BRAIDES & P. D'ANCONA: Perimeter on Fractal Sets. *Manuscripta Math.* **72** (1991), 5–25

[2] G. BUTTAZZO, G. DAL MASO & U. MOSCO: A Derivation Theorem for Capacities with respect to a Radon Measure. *J. Funct. Anal.* **71** (1987), 263–278

[3] G. CHOQUET: *Lectures on Analysis, Vol. I.* Mathematics Lecture Notes Series, Benjamin, New York, 1969

[4] D. CIORANESCU & F. MURAT: Un terme étrange venu d'ailleurs, I & II. In H. Brezis & J.L. Lions (eds.)*Nonlinear PDE's and Their Applications. Collège de France Seminar, Vols. II and III.* Research Notes in Mathematics, Pitman, London, Vol. 60 (1982), pp. 98–138, Vol. 70 (1983), pp. 154–178

[5] G. DAL MASO: *An Introduction to Γ-convergence.* Birkhäuser, Boston, book in print

[6] G. DAL MASO & U. MOSCO: Wiener's Criterion and Γ-convergence. *Appl. Math. Optim.* **15** (1987), 15–63

[7] K.J. FALCONER: *The Geometry of Fractals.* Cambridge Univ. Press, Cambridge, 1985

[8] J.E. HUTCHINSON: Fractals and Self Similarity. *Indiana Univ. Math. J.* **30** (1981), 713–747

[9] B.B. MANDELBROT: *Les objects fractals: forme, hazard et dimension.* Flammarion, Paris, 1975

[10] W.P. ZIEMER: *Weakly Differentiable Functions.* Springer, Berlin, 1989.

A. BRAIDES, Dipartimento di Elettronica per l'Automazione, Università di Brescia, via Valotti 9, I-25060 BRESCIA, and SISSA, via Beirut 2-4, I-34014 TRIESTE

L. NOTARANTONIO, SISSA, via Beirut 2-4, I-34014 TRIESTE

This article was processed by the author
using the Springer-Verlag TEX mamath macro package 1990.

(Received July 25, 1992;
in revised form February 1, 1993)

manuscripta math. 81, 57 - 78 (1993)

manuscripta mathematica

MULTIPLICITY RESULTS FOR AN INHOMOGENEOUS NEUMANN PROBLEM WITH CRITICAL EXPONENT

Gabriella Tarantello

In dimension $N \geq 5$, we prove the existence of three solutions for the inhomogeneous Neumann problem with critical Sobolev exponent. One of the solutions is obtained with a changing sign property.

Introduction

In this note we shall discuss some multiplicity results for a class of inhomogeneous Neumann problem involving the critical Sobolev exponent.

We will place particular emphasis on the existence of changing sign solutions which for constant data, will yield non constant solutions. More precisely, let $\lambda > 0$ and let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded domain with smooth boundary $\partial\Omega$. For a given function f we seek solutions for the following problem:

$$(1)_f \begin{cases} -\Delta u + \lambda u = |u|^{2^*-2} u + f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega \end{cases}$$

with n the outward pointing normal on $\partial\Omega$ and $2^* = \frac{2N}{N-2}$ the best exponent in the Sobolev embedding.

Even though our discussion extends to include $f \in H^{-1}$ (the dual of $H^1(\Omega)$), we prefer to simplify the techniqualities and assume $f \in L^{\frac{2N}{N+2}}(\Omega)$.

The homogeneous case, i.e. $f = 0$, has been treated by several authors (cf [A–M], [C–K], [W]). They have established the existence of a positive solution for $(1)_{f=0}$ for all $\lambda > 0$. It must be noticed however that when $f = 0$, problem $(1)_{f=0}$ always admits the constant positive solution $u = \lambda^{\frac{1}{2^*-2}}$. The above mentioned results can guarantee a non–constant positive solution only when λ is large.

The problem of finding non constant solutions for $(1)_{f=0}$ has been examined in [C–T]. There the oddness of the problem has allowed to obtain changing sign solutions for all $\lambda > 0$, provided $N \geq 5$.

Here, we extend these results to include the case where $f \not\equiv 0$. It should be noticed that for $f \not\equiv 0$ problem $(1)_f$ is no longer odd and the techniques used in [C–T] become unsuitable. However, extending an approach introduced in [Ta] for the corresponding Dirichlet problem, we are able to construct "ad hoc" minimization problems which yield the desired solutions.

More precisely, following [Ta] let $c_N = \frac{4}{N-2}\left(\frac{N-2}{N+2}\right)^{\frac{N+2}{4}}$ and define:

$$\mu_f = \inf_{\|u\|_{2^*}=1} \left\{ c_N \left(\|\nabla u\|_2^2 + \lambda \|u\|_2^2\right)^{\frac{N+2}{4}} - \int_\Omega f u \right\}. \tag{1.1}$$

The role of μ_f will become clear from the discussion below. Our main result states the following:

<u>Theorem 1:</u>
Let $N \geq 5$ and $f \not\equiv 0$. If $\mu_f > 0$, then $(1)_f$ admits at least <u>three</u> (weak) solutions one of which necessarily changes sign. Furthermore, if $f \geq 0$ then the other two solutions u_0 and u_1 satisfy: $0 \leq u_0 \leq u_1$. ■

Obviously regular data f will yield classical solutions for $(1)_f$. Also we have, $0 < u_0 < u_1$ in case $f \geq 0$ and $f \not\equiv 0$. Furthermore, putting together the results of [A–M] (see also [W] and [C–K]) and [C–T] we see that the given Theorem continues to hold for $f = 0$; only that, in this case, the "smallest" solution u_0 reduces to the trivial one, i.e., $u_0 = 0$. So, Theorem 1 can be viewed as a bifurcation type result. In fact, the condition $\mu_f > 0$ (to be compared with (*) in [Ta]), is essentially a "smallness" condition on f, since it certainly holds when f satisfies:

$$\|f\|_{\frac{2N}{N+2}} < c_N \left(S_N(\lambda)\right)^{\frac{N+2}{4}}$$

with

$$S_N(\lambda) = \inf_{\|u\|_{2^*}=1} \left\{ \|\nabla u\|_2^2 + \lambda \|u\|_2^2 \right\} \tag{1.2}$$

Incidentally, let us also mention that the minimization problem (1.2) attains its infinum at a positive function in $H^1(\Omega)$ (cf [A–M], [C–K] and [W]).

When $f = \text{constant} > 0$ (not too large), the claimed two positive solutions could correspond to (suitable) constants. While it follows from our construction that this is not the case for λ large, our result asserts that, in any case, problem $(1)_f$ admits nonconstant solutions for all $\lambda > 0$.

We also point out that our result holds in the subcritical case (where one replaces the power 2^* in $(1)_f$ with $p \in (2, 2^*)$) under both Neumann or Dirichlet boundary condition. The proof is simpler in this situation and therefore left to the reader.

Finally, let us mention that our approach can be applied to handle the case $\lambda = 0$ and $\int_\Omega f = 0$. This is done via a dual variational principle as introduced by Clarke [Cl] and discussed in [C–K] in this context.

In this situation one finds a "dual" correspondent for the value μ_f as given by:

$$\mu_f^* = \inf \{ c_N \left(\int_\Omega wKw \right)^{\frac{2-N}{4}} + \int_\Omega wKw;\ w \in E,\ \|w\|_{\frac{2N}{N+2}} = 1 \} \tag{1.3}$$

where

$$E = \{ w \in L^{\frac{2N}{N+2}}(\Omega) : \int_\Omega w = 0 \} \tag{1.4}$$

and $K : E \longrightarrow E$ is the inverse of $-\Delta$ in E, that is:

$$Kf = g \Leftrightarrow -\Delta g = f \text{ in } H^1(\Omega) \text{ and } \int_\Omega g = 0. \tag{1.5}$$

We have:

<u>Theorem 2:</u> Let $N \geq 5$. If $f \not\equiv 0$ satisfies $\int_\Omega f = 0$ and $\mu_f^* > 0$, then the problem:

$$(2)_f \begin{cases} -\Delta u = |u|^{2^*-2} u + f & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{in } \Omega \end{cases}$$

admits at least <u>two</u> (weak) solutions. ∎

Notice that, since $\int_\Omega f = 0$, all solutions of $(2)_f$ must change sign.

Acknowledgements: This work originated from several discussions with M. Comte. It was completed while the author was visiting the Universitá di Roma "La Sapienza" and the S.I.S.S.A. – Trieste. The warm hospitality provided respectively by Prof. F. Pacella and Prof. G. Dell'Antonio are greatly acknowledged.
The author also wishes to thank the referee for some valuable suggestions.

THE EXISTENCE OF THE FIRST TWO SOLUTIONS:

This section will be devoted to prove the following:

Theorem 2.1: Let $N \geq 5$ and $f \not\equiv 0$ satisfy $\mu_f > 0$. Problem $(1)_f$ admits at least two solutions u_0 and u_1. Furthermore, if $f \geq 0$ then $0 \leq u_0 \leq u_1$. ■

Such a result should be compared to the analogous one obtained in [Ta] for the corresponding Dirichlet problem. In fact, the proof is essentially the same and we shall refer to [Ta] for several of the details.

To start, let us observe that (weak) solutions for $(1)_f$ are the critical points for the functional,

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \lambda |u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} - \int_\Omega f u; \quad u \in H^1(\Omega).$$

Denote by $(\cdot, \cdot)$ the scalar product in $H^1(\Omega)$ corresponding to the norm:

$$\| u \|^2 = \| \nabla u \|_2^2 + \lambda \| u \|_2^2, \quad u \in H^1(\Omega).$$

Easy computations show that I is bounded from below in the set,

$$\Lambda = \{ u \in H^1(\Omega) : (I'(u), u) = 0 \}.$$

So, in the search for solution of $(1)_f$, a first candidate would be the minimizer for the following problem:

$$c_0 = \inf_\Lambda I \tag{2.1}$$

On the other hand, to insure that c_0 is indeed critical for I, we require that Λ defines a differentiable manifold (away from zero) without boundary. This is guaranteed if the function:

$$\varphi(t) \equiv I(tu), \quad t \geq 0$$

admits a nonzero critical point for each direction $u \in H^1(\Omega)$, $u \not\equiv 0$

Following [Ta], this corresponds to require that:

$$\langle I'(t_0(u)\,u), u \rangle > 0 \quad \forall\, u \neq 0$$

with

$$t_0(u) = \left[\frac{\|\nabla u\|_2^2 + \lambda \|u\|_2^2}{(2^*-1)\ \|u\|_{2^*}^{2^*}} \right]^{\frac{1}{2^*-2}} \tag{2.2}$$

Equivalently,

$$(2^*-2) \left[\frac{1}{2^*-1} \right]^{\frac{2^*-1}{2^*-2}} \left[\frac{(\|\nabla u\|_2^2 + \lambda \|u\|_2^2)^{2^*-1}}{\|u\|_{2^*}^{2^*}} \right]^{\frac{1}{2^*-2}} - \int_\Omega f\,u > 0, \ \forall\, u \neq 0$$

that is,

$$\|u\|_{2^*} \left[c_N \left[\frac{\|\nabla u\|_2^2 + \lambda \|u\|_2^2}{\|u\|_{2^*}^2} \right]^{\frac{2^*-1}{2^*-2}} - \int_\Omega f \frac{u}{\|u\|_{2^*}} \right] > 0 \quad \forall\, u \neq 0$$

which is exactly the condition $\mu_f > 0$.

A straightforward consequence of this observation is the following,

Lemma 2.1: Assume $\mu_f > 0$.

For every $u \neq 0$ there exists unique $t^-(u) < t^+(u)$ such that,

(i) $0 \leq t^-(u) < t_0(u) < t^+(u)$ ($t_0(u)$ given in (2.2))

(ii) $t^{\pm}(u)u \in \Lambda$

(iii) $I(t^-(u)u) = \min\limits_{t \in [0, t^+(u)]} I(tu)$; $I(t^+(u)u) = \max\limits_{t \geq 0} I(tu)$.

Furthermore, $t^-(u) > 0$ if and only if $\int_\Omega f\,u > 0$. ∎

The proof of Lemma 2.1 follows exactly as in Lemma 2.1 of [Ta].

Next, we derive some other useful consequences from the condition $\mu_f > 0$.

Lemma 2.2: Set,

$$\Lambda_0 = \{ u \in \Lambda : \|\nabla u\|_2^2 + \lambda \|u\|_2^2 - (2^*-1)\,\|u\|_{2^*}^{2^*} = 0 \}.$$

If $\mu_f > 0$ then,

$$\Lambda_0 = \{0\}. \tag{2.3}$$

Furthermore, for every $u \in \Lambda - \{0\}$ there exist $\epsilon > 0$ and a C^1 – map:

$$t : B_\epsilon \longrightarrow \mathbb{R}^+$$

such that,

(i) $t(w)(u+w) \in \Lambda, \quad \forall\ w \in B_\epsilon = \{w \in H^1(\Omega) : \|w\| < \epsilon\}$;

(ii) $t(0) = 1$ and $(t'(0), \varphi) =$

$$= \frac{2\int_\Omega \nabla u \cdot \nabla \varphi + 2\int_\Omega \lambda u \varphi - 2^*\int_\Omega |u|^{2^*-2} u\, \varphi}{\|\nabla u\|_2^2 + \lambda \|u\|_2^2 - (2^*-1)\, \|u\|_{2^*}^{2^*}} \ \forall\ \varphi \in H^1(\Omega).$$

<u>Proof:</u>

To obtain (2.3) let us argue by contradiction and assume that there exist $u \in \Lambda - \{0\}$:

$$\|\nabla u\|_2^2 + \lambda \|u\|_2^2 = (2^* - 1)\, \|u\|_{2^*}^{2^*}.$$

This implies that,

$$\|u\| \geq \gamma \ \text{ for suitable } \gamma > 0 \text{ and } \int_\Omega f u = (2^* - 2)\, \|u\|_{2^*}^{2^*}.$$

But this is impossible since,

$$0 < \gamma\, \mu_f \leq \|u\|_{2^*} \mu_f \leq (2^* - 2) \left[\frac{1}{2^* - 1}\right]^{\frac{2^*-1}{2^*-2}} \left[\frac{\left(\|\nabla u\|_2^2 + \lambda \|u\|_2^2\right)^{2^* - 1}}{\|u\|_{2^*}^{2^*}}\right]^{\frac{1}{2^*-2}} -$$

$$- \int_\Omega f u = (2^* - 2)\, \|u\|_{2^*}^{2^*} - \int_\Omega f u = 0.$$

At this point we obtain the second part of our claim as a straightforward application of the Implicit Function Theorem applied to the function:

$$F(t, w) = t(\|\nabla(u + w)\|_2^2) - t^{2^*-1}\, \|u + w\|_{2^*}^{2^*} - \int_\Omega f(u + w)$$

at the point $(1, 0) \in \mathbb{R} \times H^1(\Omega)$. ■

<u>Remark 2.1:</u>

Notice that necessarily,

$$\|\nabla t^+(u)u\|_2^2 + \lambda \|t^+(u)u\|_2^2 - (2^*-1)\, \|t^+(u)u\|_{2^*}^{2^*} < 0,$$

while for $\int_\Omega f u > 0$ we have:

$$\| \nabla t^-(u)u \|_2^2 + \lambda \| t^-(u)u \|_2^2 - (2^* - 1) \| t^-(u)u \|_{2^*}^{2^*} > 0.$$

Another consequence of Lemma 2.2 is that the manifold Λ is differentiable at every $u \neq 0$. Assertion (2.3) can be strengthened as follows:

Lemma 2.3: Assume $\mu_f > 0$ and let $\{ u_n \} \subset \Lambda$ such that,

$$\lim_{n\to+\infty} \| \nabla u_n \|_2^2 + \lambda \| u_n \|_2^2 - (2^* - 1) \| u_n \|_{2^*}^{2^*} = 0;$$

then, $\liminf_{n\to+\infty} \| u_n \| = 0.$

Proof: Argue by contradiction and assume that $\| u_n \| \geq \gamma > 0, \ \forall n.$

Then,

$$\int_\Omega f u_n = (2^* - 2) \| u_n \|_{2^*}^{2^*} + o(1)$$

and

$$\frac{\|\nabla u_n\|_2^2 + \lambda \|u_n\|_2^2}{(2^* - 1) \ \|u_n\|_{2^*}^{2^*}} = o(1).$$

But this is impossible since, as above, it yields:

$$\gamma \mu_f \leq (2^* - 2) \| u_n \|_{2^*}^{2^*} - \int_\Omega f u_n + o(1) = o(1). \qquad \blacksquare$$

At this point we are ready to establish the following:

Proposition 2.1: If f satisfies $\mu_f > 0$, then the minimization problem:

$$c_0 = \inf_\Lambda I \tag{2.4}$$

attains its infinum at a point u_0 which defines a critical point for I. Furthermore, $u_0 \geq 0$ for $f \geq 0$.

Proof: Let $f \neq 0$, since for $f = 0$ we have $c_0 = 0$ and $u_0 = 0$.

For $u \in \Lambda$ it follows that,

$$I(u) = \left[\frac{1}{2} - \frac{1}{2^*}\right] (\| \nabla u \|_2^2 + \lambda \| u \|_2^2) - (1 - \frac{1}{2^*}) \int_\Omega f u$$

from which we immediately derive that I is bounded below in Λ.

Claim 1: $\quad c_0 < 0 \qquad (2.5)$

Indeed if $v \in H^1(\Omega)$ satisfies $\int_\Omega f v > 0$, then $\frac{d}{dt} I(tv)|_{t=0} = -\int_\Omega fv < 0$ and from Lemma 2.1, there exist $0 < t^-(v) < t^+(v)$ such that $t^-(v) v \in \Lambda$. Thus,

$$c_0 \leq I(t^-(v)v) = \min_{t\in[0,t^+(v)]} I(tv) < 0.$$

Therefore, if $\{u_n\} \subset \Lambda$ is a minimizing sequence for (2.4) then, in view of (2.5) we have,

$$\int_\Omega fu_n \geq \frac{|c_0|}{2} \quad \text{and} \quad \|u_n\|^2 \leq \frac{2(2^*-1)}{2^*-2} \int_\Omega fu_n$$

for n large, which yields,

$$b_1 \leq \|u_n\| \leq b_2 \tag{(2.6)}$$

for suitable $b_1, b_2 > 0$.

This together with the differentiability of Λ at every $u \neq 0$ and Lemma 2.3 allows one to conclude that such a minimizing sequence is in fact a (P.S.) sequence, that is

$$I(u_n) \longrightarrow c_0 \quad \text{and} \quad I'(I_n) \longrightarrow 0.$$

However, we give an alternative way of deriving such a minimizing (P.S.) sequence by means of Ekeland's principle. This approach will turn out useful also in the sequel and it displies clearly the role of Lemma 2.3.

Indeed, Ekeland's principle (cf [A–E]) applies to (2.4) and gives a sequence $\{ u_n \} \subset \Lambda$ satisfying:

(a) $$c_0 \leq I(u_n) \leq c_0 + \frac{1}{n}$$

(b) $$I(u) \geq I(u_n) - \frac{1}{n} \| u_n - u \|, \ \forall u \in \Lambda.$$

Notice that necessarily, $\| \nabla u_n \|_2^2 + \lambda \| u_n \|_2^2 - (2^* - 1) \| u_n \|^{2^*} > 0$.

We show that condition (b) implies $\| I'(u_n) \| \longrightarrow 0$ as $n \longrightarrow +\infty$.

To this purpose, fix n with $I'(u_n) \neq 0$. By Lemma 2.2 and the estimate (2.6), for $\delta > 0$ sufficiently small, we can find $t(\delta) > 0$ such that,

(1) $$u_\delta = t(\delta) \left[u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|} \right] \in \Lambda$$

(2) $t(0) = 1$ and $|t'(0)| \le \dfrac{c}{\|\nabla u_n\|_2^2 + \lambda\|u_n\|_2^2 - (2^*-1)\|u_n\|_{2^*}^{2^*}}$

(c > 0 suitable constant).

On the other hand, since $\| u_n \| \ge b_1$, from Lemma 2.3 also follows that,

$$\liminf_{n \to +\infty} \|\nabla u_n\|_2^2 + \lambda\|u_n\|_2^2 - (2^* - 1)\|u_n\|_{2^*}^{2^*} > 0;$$

which yields

$$|t'(0)| \le a_1 \quad \text{for a suitable } a_1 > 0.$$

Thus,

$$\frac{1}{n} \| u_\delta - u_n \| \ge I(u_n) - I(u_\delta) = \delta\| I'(u_n) \| + o(\| u_n - u_\delta \|)$$

and

$$\| u_n - u_\delta \| \le | 1 - t(\delta) | \, \| u_n \| + \delta \le b_2 \, | 1 - t(\delta) | + \delta.$$

Therefore,

$$\| I'(u_n) \| \le \frac{1}{n}(| t'(0) | + 1) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.$$

So, if we call u_0 the weak limit of (a subsequence of) u_n in $H^1(\Omega)$ we have that, u_0 solves $(1)_f$. Therefore $u_0 \in \Lambda$, and

$$c_0 \le I(u_0) = \frac{1}{N} (\| \nabla u_0 \|_2^2 + \lambda\| u_0 \|_2^2) - \int_\Omega f u_0 \le$$

$$\le \lim_{n \to +\infty} I(u_n) = c_0.$$

Thus, $u_n \longrightarrow u_0$ strongly in $H^1(\Omega)$ and u_0 is the desired minimizer.

Notice that, $t^-(u_0) = 1$ ($t^-(u_0)$ as defined in Lemma 2.1). So for $f \ge 0$, we have,

$$t^-(| u_0 |) \ge 1.$$

Therefore,

$$I(t^-(| u_0 |) \, | u_0 |) \le I(| u_0 |) \le I(u_0).$$

which yields $u_0 \ge 0$. ∎

Remark 2.2: Arguing as in [Ta], one can conclude that u_0 is a local minimum for I.

Set,

$$\Lambda^{+} = \{ u \in \Lambda : \| \nabla u \|_2^2 + \lambda \| u \|_2^2 - (2^* - 1) \| u \|_{2^*}^{2^*} > 0 \} \subset \Lambda.$$

The argument above, shows that, $u_0 \in \Lambda^+$ and,

$$c_0 = \inf_{\Lambda} I = \inf_{\Lambda^+} I.$$

Thus, in the search of our second solution, it is natural to consider the second minimization problem:

$$c_1 = \inf_{\Lambda^-} I \tag{2.7}$$

where

$$\Lambda^{-} = \{ u \in \Lambda : \| \nabla u \|_2^2 + \lambda \| u \|_2^2 - (2^* - 1) \| u \|_{2^*}^{2^*} < 0 \}.$$

We start by describing some nice (topological) properties of Λ^-.
To this purpose set $\Sigma = \{u \in H^1(\Omega) : \| u \| = 1\}$. We have,

<u>Lemma 2.4:</u>

The subset Λ^- is closed in $H^1(\Omega)$. Furthermore, the map: $\Psi : \Sigma \longrightarrow \Lambda^-$ given by,

$$\Psi(u) = t^+(u)u \qquad (t^+(u) \text{ as defined in Lemma 2.1})$$

defines an homeomorphism.

<u>Proof:</u> Note that if $u \in \Lambda^-$ then $\| u \| \geq b > 0$ for a suitable $b > 0$. Thus, in view of Lemma 2.3, every sequence $\{u_n\}$ in Λ^- satisfies

$$\liminf_{n \to +\infty} \| \nabla u_n \|_2^2 + \lambda \| u_n \|_2^2 - (2^* - 1) \| u_n \|_{2^*}^{2^*} < 0$$

which readily gives Λ^- closed.

The continuity of $t^+(u)$ follows immediately from its uniqueness and extremal property. Thus, Ψ is continuous with continuous inverse given by:

$$\Psi^{-1}(u) = \frac{u}{\|u\|}. \qquad \blacksquare$$

We have:

<u>Proposition 2.2:</u> Let $N \geq 5$, then the minimization problem (2.7) attains its infinum at a critical point $u_1 \in \Lambda^-$ of I. In addition, $u_1 \geq 0$ for $f \geq 0$.

<u>Proof:</u> First of all notice that any minimizing sequence $\{ u_n \} \subset \Lambda^-$ for (2.7) satisfies:

$$0 < b_1 \leq \| u_n \| < b_2$$

for suitable b_1 and b_2.

Therefore, exactly as in the proof of Proposition 2.1, via Ekeland's principle (which applies in view of Lemma 2.4) we derive a minimizing sequence $\{ u_n \} \subset \Lambda^-$ satisfying:

$$I(u_n) \longrightarrow c_1$$
$$\| I'(u_n) \| \longrightarrow 0.$$

Since I involves a nonlinearity with critical growth, to be able to carry out the final convergence argument we need some information on the value c_1.

Claim:

$$c_1 < c_0 + \frac{1}{N} \frac{S^{N/2}}{2} \tag{2.8}$$

where S is the best constant in the Sobolev inequality (cf [T]).

To establish (2.8) we follow [Ta] and note that, in view of Lemma 2.4, Λ^- disconnects $H^1(\Omega)$ in exactly two components:

$$U^- = \{ u = 0 \text{ or } u \neq 0 : \| u \| < t^+ (\tfrac{u}{\|u\|}) \},$$
$$U^+ = \{ u : \| u \| > t^+ (\tfrac{u}{\|u\|}) \},$$

and $\Lambda^+ \subset U^-$.

As usual for this type of problem (cf.[B–N]), to obtain (2.8) we use a suitable cut off function u_ϵ of an extremum for the Sobolev inequality as given by the function:

$$U_{\epsilon,y} = \frac{(N(N-2)\epsilon)^{\frac{N-2}{4}}}{(\epsilon + |x-y|^2)^{\frac{N-2}{2}}}$$

with $\epsilon > 0$ fixed sufficiently small and $y \in \partial\Omega$ chosen so that, in a small neighborhood of y, the domain Ω lies on one side of the tangent plane of $\partial\Omega$ at y and the mean curvature of $\partial\Omega$ at y (with respect to the outward normal) is positive. The existence of such a y is guaranteed by the smoothness of $\partial\Omega$.

As well known, $\| \nabla u_\epsilon \|_2^2 = \frac{S^{N/2}}{2} + o(1)$ and $\| u_\epsilon \|_2^2 = o(1)$ as $\epsilon \longrightarrow 0$. Replacing u_ϵ with $-u_\epsilon$ if necessary, we can assume that $\int_\Omega f u_\epsilon \geq 0$ and therefore

$$t^+ \left[\frac{u_0 + R u_\epsilon}{\|u_0 + R u_\epsilon\|} \right] \leq \left[\frac{\|u_0 + R u_\epsilon\|}{\|u_0 + R u_\epsilon\|_p} \right]^{\frac{2^*}{2^*-2}} \longmapsto 1 \text{ as } R \longrightarrow +\infty \text{ and } \epsilon \longrightarrow 0^+.$$

Therefore, for $R_0 > 0$ sufficiently large and $\epsilon_0 > 0$ sufficiently small we have:

$$A_0 = \sup\{t^+\left[\frac{u_0+Ru_\epsilon}{\|u_0+Ru_\epsilon\|}\right]; R \geq R_0,\ 0 < \epsilon < \epsilon_0\} < +\infty.$$

Thus, for $R \geq 2S^{-N/4} A_0 + R_0$ and $\epsilon > 0$ sufficiently small, we derive that,

$$\|\nabla(u_0 + R\,u_\epsilon)\|_2^2 + \lambda\,\|u_0 + R\,u_\epsilon\|_2^2 \geq \|\nabla u_0\|_2^2 + \lambda\|u_0\|_2^2 + \frac{S^{N/2}}{4} R^2 > A_0 \geq$$

$$\geq t^+\left[\frac{u_0+R\ u_\epsilon}{\|u_0+R\ u_\epsilon\|}\right]$$

that is, $u_0 + R\,u_\epsilon \in U^+$. Hence, we find a $t_0 \in (0,1)$ such that,

$$v_\epsilon = u_0 + t_0 R\,u_\epsilon \in \Lambda^-.$$

So, for a suitable constant $C > 0$, this yields:

$$c_1 \leq I(v_\epsilon) \leq I(u_0) + I_0(t_0 R\,u_\epsilon) + C\left[\int_\Omega |u_0|\,u_\epsilon^{2^*-1} + \int_\Omega |u_0|^{2^*-1} u_\epsilon + \int_\Omega |f|\,u_\epsilon\right]$$

where

$$I_0(u) = \frac{1}{2}(\|\nabla u\|_2^2 + \lambda\|u\|_2^2) - \frac{1}{2^*}\|u\|_{2^*}^{2^*}.$$

Direct calculations show that,

$$\int_\Omega |u_0|\,u_\epsilon^{2^*-1} + \int_\Omega |u_0|^{2^*-1} u_\epsilon + \int_\Omega |f|\,u_\epsilon = a\,\epsilon^{\frac{N-2}{4}} + o\left[\epsilon^{\frac{N-2}{4}}\right],\quad a > 0;$$

(cf [B–N]) and $\max_{t\geq 0} I_0(t\,u_\epsilon) = \frac{1}{N}\left[\frac{\|\nabla u_\epsilon\|_2^2 + \lambda\|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2}\right]^{N/2}$.

Therefore,

$$c_1 \leq c_0 + \frac{1}{N}\left[\frac{\|\nabla u_\epsilon\|_2^2 + \lambda\|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2}\right]^{N/2} + 0\left[\epsilon^{\frac{N-2}{4}}\right].$$

On the other hand, our choice of u_ϵ guarantees that, for $N \geq 4$, we have:

$$\left[\frac{\|\nabla u_\epsilon\|_2^2 + \lambda\|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2}\right]^{N/2} \leq \frac{S^{N/2}}{2} - C\,\epsilon^{1/2} + o(\epsilon^{1/2}),\quad C > 0,$$

(see [A–M], [C–K], [W]). Thus, for $N \geq 5$ we conclude:

$$c_1 \le c_0 + \frac{1}{N}\frac{S^{N/2}}{2} - C\epsilon^{1/2} + o(\epsilon^{1/2}) + 0\left[\epsilon^{\frac{N-2}{4}}\right] < c_0 + \frac{1}{N}\frac{S^{N/2}}{2}$$

for $\epsilon > 0$ sufficiently small.

At this point, to show that the sequence $\{u_n\}$ is precompact we use an inequality of Cherrier [Ch] which, for every $\tau > 0$, gives a constant $M_\tau > 0$ such that:

$$\left[\frac{S}{2^{2/N}} - \tau\right] \| u \|^2_{2^*} \le \| \nabla u \|^2_2 + M_\tau \| u \|^2_2, \quad \forall u \in H^1(\Omega).$$

Since u_n is uniformly bounded in $H^1(\Omega)$, after taking a subsequence (which we still call u_n) we find $u_1 \in H^1(\Omega)$ such that $u_n \rightharpoonup u_1$ weakly in $H^1(\Omega)$. In particular, $u_1 \in \Lambda$ and so $I(u_1) \ge c_0$.

Furthermore, if we write $u_n = u_1 + v_n$ with $v_n \rightharpoonup 0$ weakly in $H^1(\Omega)$, we derive:

$$I(u_n) = I(u_1) + \tfrac{1}{2}\| \nabla v_n \|^2_2 - \frac{1}{2^*} \| v_n \|^{2^*}_{2^*} + o(1) \longmapsto c_1 < c_0 + \frac{1}{N}\frac{S^{N/2}}{2}$$

which yields:

$$\lim_{n\to+\infty} \tfrac{1}{2}\| \nabla v_n \|^2_2 - \frac{1}{2^*}\| v_n \|^{2^*}_{2^*} < \frac{1}{N}\frac{S^{N/2}}{2}. \tag{2.9}$$

Moreover,

$$0 = (I'(u_n), u_n) = (I'(u_1), u_1) + \| \nabla v_n \|^2_2 - \| v_n \|^{2^*}_{2^*} + o(1)$$

that is,

$$\| \nabla v_n \|^2_2 - \| v_n \|^{2^*}_{2^*} = o(1). \tag{2.10}$$

Putting together (2.9) and (2.10) we have that,

$$\lim_{n\to+\infty} \| \nabla v_n \|^2_2 := \gamma < \frac{S^{N/2}}{2}. \tag{2.11}$$

Next we show how (2.10) and (2.11) can hold simultaneously only if $\lim_{n\to+\infty} \| \nabla v_n \| = 0$, (i.e., $\gamma = 0$).

Let us argue by contradiction and assume $\gamma > 0$. Take $\tau > 0$ such that:

$$\left[\frac{S}{2^{2/N}} - \tau\right]^{N/2} > \gamma.$$

From (2.10) we have:

$$\| \nabla v_n \|_2^2 = \| v_n \|_{2^*}^{2^*} + o(1) \le \left[\frac{S}{2^{2/N}} - \tau \right]^{-2^*/2} \| \nabla v_n \|_2^{2^*} + o(1),$$

Since $\gamma > 0$, then $\| \nabla v_n \|_2$ is bounded below away from zero. Therefore,

$$\| \nabla v_n \|_2^2 \ge \left[\frac{S}{2^{2/N}} - \tau \right]^{N/2} + o(1)$$

which, in view of our choice of τ, contradicts (2.11).
This gives $u_n \longrightarrow u_1$ strongly in $H^1(\Omega)$ and so u_1 is the desired minimizer.
Finally, for $f \ge 0$ we have:

$$I(t^+(|u_1|)|u_1|) \le I(t^+(|u_1|)\, u_1) \le \max_{t \ge 0} I(t\, u_1) = I(u_1)$$

which yields $u_1 \ge 0$. ■

Obviously $u_0 \ne u_1$. To conclude the proof of Theorem 2.1 set $u_+ = \min\{u_0, u_1\}$; we show that $f \ge 0$, $f \ne 0$ implies the existence of a solution $0 \le u_0^* \le u_+$. To this purpose note that when $f \ge 0$, $(f \ne 0)$ the unique solution u_μ for the problem:

$$-\Delta u + \lambda u = \mu f \text{ in } H^1(\Omega)$$

gives a positive <u>subsolution</u> for $(1)_f$ for all $\mu \in (0,1)$ and $u_\mu \longrightarrow 0$ as $\mu \longrightarrow 0$. On the other hand, $u_+ = \min\{u_0, u_1\}$ defines a <u>supersolution</u> for $(1)_f$. So choosing $\mu > 0$ sufficiently small to guarantee $u_\mu \le u_+$ a.e. in Ω, by the method of sub–super solutions, we obtain a solution u_0^* for $(1)_f$ satisfying:

$$0 < u_\mu \le u_0^* \le u_+.$$

This concludes the proof of Theorem 2.1.

<u>Remark 2.3:</u> Notice that, if u_1 is a minimizer for (2.7) and $f \ne 0$ then

$$\int_\Omega f u_1 > 0.$$

To see this, observe first that if $\int_\Omega f u_1 < 0$ then $I(t^+(-u_1)\,(-u_1)) < I(u_1)$ which is impossible. On the other hand, if $\int_\Omega f u_1 = 0$ and we let $\hat{u}$ be the minimizer for

$$I_0 \text{ on } \Gamma = \{u \neq 0 : \langle I_0'(u), u\rangle = 0\} \text{ with } \int_\Omega f\hat{u} \geq 0,$$

then $u_1 \in \Gamma$ and, $I_0(u_1) = I(u_1) \leq I(t^+(\hat{u})\hat{u}) \leq I_0(t^+(\hat{u})\hat{u}) \leq I_0(\hat{u})$.
Thus, u_1 would have to solve both $(1)_f$ and $(1)_{f=0}$, which is impossible for $f \not\equiv 0$. In conclusion, $\int_\Omega f u_1 > 0$.

EXISTENCE OF CHANGING SIGN SOLUTIONS

In this section we investigate the existence of changing sign solutions for $(1)_f$.

To this purpose, we need to compare between some different minimization problems. To start, define:

$$\Lambda_1^- = \{u = u^+ - u^- \in \Lambda : u^+ \in \Lambda^-\}$$

and

$$\Lambda_2^- = \{u = u^+ - u^- \in \Lambda : -u^- \in \Lambda^-\},$$

where, as usual, we have denoted with $u^+ = \sup\{u, 0\}$ and $u^- = \sup\{-u, 0\}$.
Set

$$\gamma_1 = \inf_{\Lambda_1^-} I \geq c_0 \tag{3.1}$$

and

$$\gamma_2 = \inf_{\Lambda_2^-} I \geq c_0 \tag{3.2}$$

We have:

Proposition 3.1:

If $\gamma_1 < c_1$ then the minimization problem (3.1) attains its infimum at a point which defines a changing sign critical point for I.

Analogously, if $\gamma_2 < c_1$ the same conclusion holds for the minimization problem (3.2).

Proof: As for Λ^-, it is not difficult to show that the condition $\mu_f > 0$ implies that Λ_1^- and Λ_2^- are closed.

We start by discussing (3.1) and observe that (3.2) is handled similarly. As for the previous minimization problems, we shall apply Ekeland's principle to derive a minimizing sequence $\{u_n\} \subset \Lambda_1^-$ with the property that $I(u_n) \longrightarrow \gamma_1$ and

$$I(u) \geq I(u_n) - \frac{1}{n}\| u_n - u \|, \quad \forall u \in \Lambda_1^-.$$

We claim that, in view of the condition $\gamma_1 < c_1$, necessarily $\| u_n^- \| \geq b > 0$ for a suitable b. Indeed, if for a subsequence (which we still call u_n^-) we have $\| u_n^- \| \longrightarrow 0$ then

$$\gamma_1 + o(1) = I(u_n) = I(u_n^+) + I(-u_n^-) \geq c_1 + o(1)$$

which is clearly impossible.

On the other hand, since $u_n^+ \in \Lambda^-$, it follows immediately that $\| u_n^+ \|$ is bounded away from zero. Thus, we derive the existence of suitable constants C_1 and C_2 such that,

$$C_1 \leq \| u_n^{\pm} \| \leq C_2. \tag{3.3}$$

Now, for fixed n with $I'(u_n) \neq 0$, take $\delta > 0$ sufficiently small to find suitable C^1 – function $t_+(\delta) > 0$ and $t_-(\delta)$ such that $t_{\pm}(0) = 1$, and $w_\delta = t_+(\delta)\left[u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|}\right]^+ - t_-(\delta)\left[u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|}\right]^- \in \Lambda_1^-$ (see Lemma 2.2). In view of (3.3) and Lemma 2.2 it follows,

$$| t_{\pm}'(0) | \leq C_3$$

for suitable $C_3 > 0$,

Therefore,

$$\frac{1}{n} \| w_\delta - u_n \| \geq I(u_n) - I(w_\delta) = -(I'(u_n), w_\delta - u_n) + o(\| w_\delta - u_n \|)$$

$$= (1 - t_+(\delta))\left[I'(u_n), \left[u_n - \delta\frac{I'(u_n)}{\|I'(u_n)\|}\right]^+\right] - (1 - t_-(\delta))\left[I'(u_n), \left[u_n - \delta\frac{I'(u_n)}{I'(u_n)\|}\right]^-\right]$$
$$+ \delta \| I'(u_n) \| + o(\| u_n - w_\delta \|).$$

On the other hand, for a suitable $C_4 > 0$ we have:

$$\| w_\delta - u_n \| \leq C_4 (| t_+(\delta) - 1 | + | t_-(\delta) - 1 | + \delta)$$

which yields,

$$\| I'(u_n) \| \leq \frac{C_4}{n}(| t_+'(0) | + | t_-'(0) | + 1) + t_+'(0) (I'(u_n), u_n^+) +$$
$$+ t_-'(0) (I'(u_n), -u_n^-) \leq \frac{1}{n} C_4(1 + 2C_3) \longmapsto 0 \text{ as } n \longrightarrow +\infty.$$

In conclusion the sequence $\{u_n\}$ satisfies

(i) $I(u_n) \longrightarrow \gamma_1 < c_1 < c_0 + \frac{1}{N}\frac{S^{N/2}}{2}$;

(ii) $\| I'(u_n) \| \longrightarrow 0$.

Thus, as we have seen in the proof of Proposition 2.2, conditions (i) and (ii) are sufficient to guarantee a convergent subsequence for $\{u_n\}$ whose (strong) limit will give the desired minimizer. ■

Obviously, Proposition 3.1 would yield the conclusion for Theorem 1 only if the given relations between γ_1, γ_2 and c_1 could be established. While it is not clear whether or not such inequalities should hold, we shall use these values to compare with another minimization problem.

Namely, set

$$\Lambda_*^- = \Lambda_1^- \cap \Lambda_2^- \subset \Lambda^-$$

and define,

$$c_2 = \inf_{\Lambda_*^-} I. \tag{3.4}$$

It is clear that $c_2 \geq c_1$. An upper bound for c_2 is provided by the following:

<u>Lemma 3.1:</u>

For fixed $\epsilon > 0$ and $y \in \partial\Omega$ there exist $s > 0$ and $\mu \in \mathbb{R}$ such that

$$s\, u_1 - \mu\, U_{\epsilon,y} \in \Lambda_*^-.$$

In particular, for $N \geq 5$,

$$c_2 \leq \sup_{s\geq 0,\, t} I(s\, u_1 - t\, U_{\epsilon,y}) < c_1 + \frac{1}{N}\frac{S^{N/2}}{2}$$

for $\epsilon > 0$ sufficiently small and y suitably fixed in $\partial\Omega$.

<u>Proof:</u> We shall show that there exist $s > 0$ and $t \in \mathbb{R}$ such that

$$s(u_1 - tU_{\epsilon,y})^+ \in \Lambda^- \text{ and } -s(u_1 - tU_{\epsilon,y})^- \in \Lambda^-. \tag{3.5}$$

To this purpose let,

$$t_2 = \max_{\bar\Omega} \frac{u_1}{U_{\epsilon,y}} \text{ and } t_1 = \min_{\bar\Omega} \frac{u_1}{U_{\epsilon,y}}.$$

For $t \in (t_1, t_2)$ denote by $s_+(t)$ and $s_-(t)$ the positive values given by Lemma 2.1 according to which we have:

$$s_+(t)\,(u_1 - t\, U_{\epsilon,y})^+ \in \Lambda^-$$

and

$$-s_-(t)(u_1 - t\, U_{\epsilon,y})^- \in \Lambda^-.$$

Note that $s_+(t)$ is a continuous function of t satisfying:

$$\lim_{t\to t_1^+} s_+(t) = t^+(u_1 - t_1 U_{\epsilon,y}) < +\infty \text{ and } \lim_{t\to t_2^-} s_+(t) = +\infty.$$

Similarly, $s_-(t)$ is continuous and,

$$\lim_{t\to t_1^+} s_-(t) = +\infty \text{ and } \lim_{t\to t_2^-} s_-(t) = t^+(t_2 U_{\epsilon,y} - u_1) < +\infty.$$

Therefore, by the continuity of $s_\pm(t)$ we find a value $t_0 \in (t_1, t_2)$ such that

$$s_+(t_0) = s_-(t_0) = s_0 > 0.$$

This gives (3.5) with $t = t_0$ and $s = s_0$. At this point we only need to estimate $I(s\, u_1 - t\, U_{\epsilon,y})$ for $s \geq 0$ and $t \in \mathbb{R}$. To this purpose we fix $y \in \partial\Omega$ as in the proof of proposition 2.2 and let $u_\epsilon = U_{\epsilon,y}$. The structure of I guarantees the existence of $R > 0$ (independent of ϵ) such that $I(s\, u_1 - t\, u_\epsilon) \leq c_1$ for all $s^2 + t^2 \geq R^2$. On the other hand, for $s^2 + t^2 \leq R^2$, we have:

$$I(s\, u_1 - t\, u_\epsilon) \leq I(s\, u_1) + I_0(t\, u_\epsilon) + 0(\epsilon^{\frac{N-2}{4}}) \leq$$

$$\leq \max_{s\geq 0} I(s\, u_1) + \max_{t\in\mathbb{R}} I_0(t\, u_\epsilon) + 0(\epsilon^{\frac{N-2}{4}}) =$$

$$= I(u_1) + \frac{1}{N}\left[\frac{\|\nabla u_\epsilon\|_2^2 + \lambda\|u_\epsilon\|_2^2}{\|u_\epsilon\|_{2^*}^2}\right]^{N/2} + 0(\epsilon^{\frac{N-2}{4}}) \leq$$

$$\leq c_1 + \frac{1}{N}\frac{S^{N/2}}{2} - C\,\epsilon^{1/2} + o(\epsilon^{1/2}) + 0(\epsilon^{\frac{N-2}{4}});\ C > 0$$

where we have used the estimates in [A–M], [C–K] and [W]. Hence, for $N \geq 5$ and $\epsilon > 0$ sufficiently small we readily obtain,

$$c_2 \leq \sup_{\substack{s\geq 0\\ t\in\mathbb{R}}} I(s\, u_1 - t\, u_\epsilon) < c_1 + \frac{1}{N}\frac{S^{N/2}}{2}. \qquad \blacksquare$$

__Proposition 3.2:__ Assume that $\gamma_1 \geq c_1$ and $\gamma_2 \geq c_1$. The minimization problem,

$$c_2 = \inf_{\Lambda_*^-} I$$

attains its infimum at $u_2 \in \Lambda_*^-$ which defines a (changing sign) critical point for I.

Proof: Exactly as in Proposition 3.1, by means of Ekeland's principle, we derive a minimizing sequence $\{u_n\} \subset \Lambda_*^-$ satisfying:

$$I(u_n) \to c_2$$
$$\|I'(u_n)\| \to 0.$$

In particular, we have:

$$0 < a_1 \le \| u_n^\pm \| \le a_2 \tag{3.6}$$

for suitable constant a_1 and a_2. Thus, after taking a subsequence, we obtain

$$u_n^\pm \rightharpoonup u^\pm \in H^1(\Omega) \text{ weakly in } H^1(\Omega).$$

We start by showing that $u^\pm \neq 0$.

Indeed, if by contradiction we assume for instance, that $u^+ = 0$ then we would have:

(i) $\| \nabla u_n^+ \|_2^2 - \| u_n^+ \|_{2^*}^{2^*} = o(1)$

and

(ii) $\lim_{n\to+\infty} \frac{1}{2} \| \nabla u_n^+ \|_2^2 - \frac{1}{2^*} \| u_n^+ \|_{2^*}^{2^*} = \lim_{n\to+\infty} I(u_n^+) \le$

$$\le c_2 - \lim_{n\to+\infty} I(-u_n^-) \le c_2 - c_1 < \frac{1}{N}\frac{S^{N/2}}{2}.$$

But, we have already seen how condition (i) and (ii) can hold simultaneously only if $\lim_{n\to+\infty} \| \nabla u_n^+ \| = 0$ which clearly contradicts (3.6). A similar argument applies to u^-. Thus, $u_2 = u^+ - u^- \neq 0$ is a changing sign solution for $(1)_f$ and in particular, $I(u_2) \ge c_0$.

Set $u_n^+ = u^+ + v_n^+$ and $u_n^- = u^- + v_n^-$ with $v_n^\pm \rightharpoonup 0$ in $H^1(\Omega)$. Note that,

$$\| \nabla v_n^\pm \|_2^2 - \| v_n^\pm \|_{2^*}^{2^*} = o(1) \tag{3.7}$$

In view of (2.8) and Lemma 3.1, we also have:

$$\lim_{n\to+\infty} I(v_n^+) + I(-v_n^-) = \lim_{n\to+\infty} I(u_n) - I(u_2) \leq c_2 - c_0 <$$

$$< \frac{1}{N}\frac{S^{N/2}}{2} + c_1 - c_0 < \frac{1}{N} S^{N/2} \tag{3.8}$$

So, necessarily,

$$\lim_{n\to+\infty} \min\{I(v_n^+), I(-v_n^-)\} < \frac{1}{N}\frac{S^{N/2}}{2}$$

which, in view of (3.7), yields:

$$\| v_n^+ \| \longrightarrow 0 \quad \text{or} \quad \| v_n^- \| \longrightarrow 0$$

that is, $u_2 = u^+ - u^- \in \Lambda_1^-$ *or* $u_2 = u^+ - u^- \in \Lambda_2^-$.

Consequently, since we are in the situation where $\gamma_1, \gamma_2 \geq c_1$, we conclude:

$$I(u_2) \geq c_1.$$

Therefore, if we write $u_n = u_2 + w_n$ with $w_n \rightharpoonup 0$ in $H^1(\Omega)$ we obtain,

$$\| \nabla w_n \|_2^2 - \| w_n \|_{2^*}^{2^*} = o(1)$$

and

$$\lim_{n\to+\infty} \frac{1}{2}\| \nabla w_n \|_2^2 - \frac{1}{2^*}\| w_n \|_{2^*}^{2^*} = \lim_{n\to+\infty} I(u_n) - I(u_2) =$$

$$\leq c_2 - c_1 < \frac{1}{N}\frac{S^{N/2}}{2};$$

This, in the usually way, yields $\| w_n\| \longrightarrow 0$. Thus $u_n \longrightarrow u_2$ strongly in $H^1(\Omega)$ and $u_2 \in \Lambda_*^-$ gives the desired minimizer. ■

<u>The proof of Theorem 1:</u>

For $f \geq 0$ ($f \leq 0$) Theorem 1 is a direct consequence of Theorem 2.1 and Proposition 3.1 and 3.2. If both f and u_0 change sign in Ω and the situation of Proposition 3.1 occurs then necessarily $\gamma_1 > c_0$ and $\gamma_2 > c_0$ and we would be done. So assume that we are in the situation of Proposition 3.2. To conclude it suffices to show that $u_2 \neq u_1$ (since, obviously, $u_2 \neq u_0$).

In fact, argue by contradiction and assume that $u_2 = u_1$. Then,

$c_2 = c_1$, $u_1 \in \Lambda_1^- \cap \Lambda_2^-$ and $\gamma_1 = c_1 = \gamma_2$. On the other hand, $\int_\Omega f\,u_1 > 0$ (see Remark 2.3), thus $\int_\Omega f\,u_1^+ > 0$ or $-\int_\Omega f\,u_1^- > 0$. Assume, for instance, that $\int_\Omega f\,u_1^+ > 0$. From Lemma 2.1 then we obtain a $t^- > 0$ such that

$$t^- u_1^+ \in \Lambda^+ \text{ and } I(u_1^+) > I(t^- u_1^+).$$

This is clearly impossible since, $t^- u_1^+ - u_1^- \in \Lambda_2^-$ and

$$\gamma_2 = I(u_1) = I(u_1^+) + I(-u_1^-) > I(t^- u_1^+) + I(-u_1^-) = I(t^- u_1^+ - u_1^-) \geq \gamma_2.$$

Thus, in all circumstances, a third changing sign solution for $(1)_f$ is guaranteed. ■

<u>Sketch of the proof of Theorem 2</u>

The proof of Theorem 2 follows by considering the (dual) functional,

$$F(w) = \frac{N+2}{2N} \int_\Omega |w|^{\frac{2N}{N+2}} - \frac{1}{2}\int_\Omega wKw + \int_\Omega wKf, \quad w \in E$$

with E and K as defined in (1.4) and (1.5).

As above, the idea is to consider,

$$\Lambda_-^* = \{ w \in E, (F'(w), w) = 0 \text{ and } \frac{N-2}{N+2} \int_\Omega |w|^{\frac{2N}{N+2}} - \int_\Omega w K w > 0 \},$$

and

$$\Lambda_+^* = \{ w \in E, (F'(w), w) = 0 \text{ and } \frac{N-2}{N+2} \int_\Omega |w|^{\frac{2N}{N+2}} - \int_\Omega w K w < 0 \}.$$

One shows that the condition $\overset{*}{\mu}_f > 0$ ($\overset{*}{\mu}_f$ as defined in (1.3)), implies that the corresponding minimization problems:

$$\overset{*}{c}_0 = \inf_{\Lambda_+^*} F \tag{4.1}$$

$$\overset{*}{c}_1 = \inf_{\Lambda_-^*} F \tag{4.2}$$

yield <u>two</u> distinct critical values for F, hence two (distinct) solutions for $(2)_f$.

For the minimization problem (4.1) this follows exactly as for Proposition 2.1 with the obvious modifications.

The minimization problem (4.2) is treated similarly to that in (2.7) and the corresponding compactness argument follows by providing an appropriate upper bound on $\overset{*}{c}_1$. This can be derived using the estimates contained in [C–K]. We leave the details to the interested reader.

REFERENCES

[A–M] Adimurti – G. Mancini: The Neumann problem for elliptic equations with critical non–linearity, A tribute in honour of G. Prodi, *Scuola Norm. Sup. Pisa* (1991) p. 9–25

[A–E] J. P. Aubin – I. Ekeland: Applied Nonlinear Analysis, Wiley Interscience Publications, 1984

[B–N] H. Brezis and L. Nirenberg: Positive solutions of nonlinear elliptic equations involving Critical Sobolev exponent, *Comm. Pure Appl. Math.*, Vol. 36 (1983) p. 437–477

[Ch] P. Cherrier : Meilleures constants dans les inéqualités relatives aux espaces de Sobolev, *Bull. Sc. Math.* 2^{e} serié, **108** (1984), p. 154–206

[Cl] F. Clarke: A classical variational principle for periodic Hamiltonian trajectories, *Proc. A.M.S.*, **76** (1979), p. 186–188

[C–K] M. Comte – M. C. Knaap: Existence of solutions of elliptic equations involving critical Sobolev exponent with Neumann boundary condition in general domains, *Differential and Integral Eq.* **4** n. 6 (1991), p. 1132–1146

[C–T] M. Comte – G. Tarantello: A Neumann problem with critical Sobolev exponent, *Houston J. Math.*, **18** n. 2 (1992), p. 279–294

[T] G. Talenti: Best constant in Sobolev inequality, *Ann. Mat. Pure Appl.*, **110** (1976), p. 353–372

[Ta} G. Tarantello: On nonhomogeneous elliptic equations involving critical Sobolev exponent. *Annales I. H. P. Analyse non linéaire*, **9** n. 3, (1992) p. 281–304

[W] X. J. Wang: Neumann problem of semilinear elliptic equations involving critical Sobolev exponent, *J. Diff. Eq.*, **93** (1991), p. 283–310

Current Address:

Universitá Degli Studi di Roma "Tor Vergata"
Dipartimento di Matematica
Via Fontanile di Carcaricola
00133 Roma Italy

(Received September 5, 1992;
in revised form January 15, 1993)

manuscripta math. 81, 79 - 87 (1993)

manuscripta mathematica

FINITISTIC DIMENSIONS OF SEMIPRIMARY RINGS

Yong Wang

Let R be a semiprimary ring. We show that if the left generalized projective dimension (defined below) of $_R(R/J^2)$ is finite, then the injectively defined left finitistic dimension of R is finite.

Let R be a ring, the projectively (injectively) defined left big finitistic dimension, denoted by $lFPD(R)$ ($lFID(R)$, respectively), is the supremum of the projective (injective) dimensions of the left R-modules with finite projective (injective) dimension. The right versions of the finitistic dimensions $rFPD(R)$ and $rFID(R)$ are defined similarly. It is well-known that $rFPD(R) = lFID(R)$ if R is a right artinian ring. But for a semiprimary ring R, one only has $rFPD(R) \leq lFID(R)$ and the inequality sign may be strict. In general, as pointed in [1], there is no universal inequality relating $rFPD(R)$ and $lFID(R)$. The readers are referred to [1] for these information.

The theory of the finitistic dimension of finite dimensional algebras (artinian rings) has been attracting interests recently, but little attention has been given to that of semiprimary rings, especially, the injectively defined finitistic dimensions. In an early paper of Mochizuki [8], he studied $lFID(R)$ for a semiprimary ring R and gave a sufficient condition for $lFID(R) < \infty$. In [7], an example is given of a semiprimary ring R with $J^4 = 0$ ($J = Rad(R)$ is the Jacobson radical of R) and $rFPD(R) = \infty$, hence $lFID(R) = \infty$. (Note that it has been shown by Green and Zimmermann Huisgen [6] that for a left artinian ring R with $J^3 = 0$, $lfPD(R) < \infty$, where $lfPD(R) = sup\{Pd({}_RM) \mid M$ a finitely generated left R-module of $Pd({}_RM) < \infty\}$). For a semiprimary ring R with $J^2 = 0$, it is fairly easy to see that $lFID(R) < \infty$ since the minimal left projective resolution of R/J has the property that there are only finitely many indecomposable direct summands (up to isomorphism) appearing in its syzygies ([4], Proposition 4.3). In this paper, we are going to extend this result by showing $lFID(R) < \infty$ under the hypothesis that the left generalized projective dimension (defined below) of R/J^2 is finite.

Definition. Let R be a ring. A left R-module ${}_RM$ is said to have a *finite generalized projective dimension*, denoted by $gPd({}_RM) < \infty$, in case there exists an integer $m \geq 0$ such that $Ext_R^{m+1}(M, N) = 0$ for any left R-module N with $Id({}_RN) < \infty$. We define $gPd({}_RM)$ to be the least integer $m \geq 0$ satisfying the above condition. If there is no such a number, then $gPd({}_RM) = \infty$.

Generally, if an artinian ring R has the property $lFPD(R) \neq rFPD(R)$, say $m = lFPD(R) < rFPD(R)$ (such examples can be easily found in finite dimensional monomial algebras), then $m = rFID(R) = lFPD(R) < rFPD(R)$ ([1]). Take any module M with $m < Pd(M_R) < \infty$, we have $gPd(M) < Pd(M) < \infty$. It is not our intention to discuss the *generalized projective dimension* in detail in the present paper, some properties of the generalized projective dimension as well as the generalized injective (flat) dimension, in particular, their relations with the finitistic dimensions, can be found in [9]. However, we want to provide the following proposition to see the ubiquity of this concept. Let us recall that a left R-module ${}_RM$ is said to have *an projective resolution with a strongly redundant image from an integer* $m \geq 1$ in case the mth syzygy Ω_m has a decomposition $\Omega_m = \bigoplus_{i \in I} A_i$ (not necessarily a finite direct sum) such that each A_i is a direct summand of a syzygy Ω_{α_i} for some $\alpha_i > n$ ([3]).

Proposition 1 *Suppose R is a ring and ${}_RM$ has a projective resolution*

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow {}_RM \longrightarrow 0$$

with a strongly redundant image from m. Then $gPd(_RM) \leq m$.

Proof. The proof is similar to the one given in ([5], Lemma 8).

We are going to prove $Ext_R^{m+1}(M,N) = 0$ for any module $_RN$ of finite injective dimension. So let $Id(_RN) = n$. Without loss of generality, we may assume that $m+1 \leq n$. Let Ω_i be the ith syzygy and $\Omega_m = \oplus_i A_i$ such that each A_i appears as a direct summand in some $\Omega_{\alpha_i}, \alpha_i > m$. Then

$$Ext_R^n(M,N) \cong Ext_R^{n-m}(\Omega_m, N) = Ext_R^{n-m}(\oplus_i A_i, N) \cong \prod_i Ext_R^{n-m}(A_i, N),$$

But for A_i,

$$Ext_R^{n-m}(A_i, N) \leq Ext_R^{n-m}(\Omega_{\alpha_i}, N) \cong Ext_R^{\alpha_i+n-m}(M,N) = 0$$

since $\alpha_i + n - m > n$ and $Ext_R^n(M,N) = 0$.

Inductively, suppose

$$Ext_R^n(M,N) = Ext_R^{n-1}(M,N) = \cdots = Ext_R^{n-k}(M,N) = 0$$

for $n-k > m+1$. Consider

$$Ext_R^{n-k-1}(M,N) \cong Ext_R^{n-k-1-m}(\Omega_m, N)$$

$$= Ext_R^{n-k-1-m}(\oplus_i A_i, N) \cong \prod_j Ext_R^{n-k-1-m}(A_i, N),$$

each $Ext_R^{n-k-1-m}(A_i, N) \leq Ext_R^{n-k-1-m}(\Omega_{\alpha_i}, N) \cong Ext_R^{n-k-1-m+\alpha_i}(M,N)$.

If $\alpha_i > m+k+1$, then $n-k-1-m+\alpha_i > n = Id(N)$, it follows that $Ext_R^{n-k-1-m+\alpha_j}(M,N) = 0$. If $\alpha_i \leq m+k+1$, then $n-k \leq \alpha_i + n - k - m - 1 \leq n$. By induction, $Ext_R^{n-k-1-m+\alpha_i}(M,N) = 0$ and $Ext_R^{n-k-1}(M,N) = 0$. □

Remark. It is easily seen that If $gPd(_RM) \leq m$, then $Ext_R^{m+k}(M,N) = 0$ for all integers $k \geq 1$ and left R-module N with $Id(N) < \infty$. The reason is that each cosyzygy of a module N with finite injective dimension still has finite injective dimension.

In the sequel, we suppose that R is a semiprimary ring. Let $V = \{e_1, \ldots, e_n\}$ be a basic set of primitive idempotents. We say that there is an arrow from e_i

to e_j if $e_i(Je_j/J^2e_j) \neq 0$. An e_i is said a sink if there is no arrow starting from e_i. First we have

Lemma 2 *Let R be semiprimary and $\{e_1, \ldots, e_n\}$ be a basic set of primitive idempotents of R. Let M be a left R-module with $Id(M) = m$ and $Ext_R^m(Re_i/Je_i, M) \neq 0$ for some i (note such an i always exists). If there is an arrow from e_i to e_j, then $Ext_R^m(Re_j/J^2e_j, M) \neq 0$. Hence $Ext_R^m(R/J^2, M) \neq 0$.*

Proof. By the hypothesis, Re_i/Je_i is a direct summand of Je_j/J^2e_j, thus

$$Ext_R^m(Je_j/J^2e_j, M) \neq 0.$$

From the exact sequence

$$0 \longrightarrow Je_j/J^2e_j \longrightarrow Re_j/J^2e_j \longrightarrow Re_j/Je_j \longrightarrow 0,$$

we have

$$Ext_R^m(Re_j/J^2e_j, M) \longrightarrow Ext_R^m(Je_j/J^2e_j, M) \longrightarrow Ext_R^{m+1}(Re_j/Je_j, M) = 0$$

exact, hence $Ext_R^m(Re_j/J^2e_j, M) \neq 0$. □

Corollary 3 *Suppose R is semiprimary and R has no sinks (e.g. R is strongly connected in the sense of [2]) and M is a left R-module with $Id(M) = m$. Then $Ext_R^m(R/J^2, M) \neq 0$.* □

Before proceeding our next theorem, we introduce the notation $A \in add(_RB)$ which means $_RA$ is a direct summand of finitely many copies of $_RB$.

Theorem 4 *Suppose R is a semiprimary ring and J/J^2 has s simple modules (up to isomorphism) as its direct summands. If $gPd_R(R/J^2) = m$, then $lFID(R) \leq m + s$.*

Proof. We may assume that $s \geq 1$, for otherwise, R is a semisimple ring and $lFID(R) = 0$.

If the result were false, there would be a left R-module ${}_RN$ with $m+s+1 \leq n = Id(N) < \infty$. we are going to show this leads a contradiction.

Let $\{e_1, \ldots, e_s\}$ be the set such that J/J^2 is precisely decomposed into a direct sum of copies of Re_i/Je_i ($i = 1, 2, \ldots, s$), that is, $J/J^2 \in add(Re_1/Je_1 \oplus \cdots \oplus Re_s/Je_s)$ and $e_iJ/J^2 \neq 0$ for all $i = 1, 2, \ldots, s$; and set $e = e_1 + \cdots + e_s$. By Lemma 2, there must be $Ext^n_R(Re/Je, N) = 0$. For otherwise, there is an i: $1 \leq i \leq s$ such that $Ext^n_R(Re_i/Je_i, N) \neq 0$, but $e_iJ/J^2 \neq 0$, so there exists a j: $1 \leq j \leq s$ such that there is an arrow from e_i to e_j , hence $Ext^n_R(R/J^2, N) \neq 0$. This is a contradiction since $gPd(R/J^2) = m < n$.

Next, if $Ext^{n-1}_R(Je/J^2e, N) \neq 0$, from the exact sequence

$$0 \longrightarrow Je/J^2e \longrightarrow Re/J^2e \longrightarrow Re/Je \longrightarrow 0,$$

we get $Ext^{n-1}_R(Re/J^2e, N) \neq 0$, so $Ext^{n-1}_R(R/J^2, N) \neq 0$, this is contradict to the condition $gPd(R/J^2) = m$. Thus we must have $Ext^{n-1}_R(Je/J^2e, N) = 0$.

Consider $Ext^{n-1}_R(Re/J^2e, N)$. If for each i: $1 \leq i \leq s$, $Ext^{n-1}_R(Re_i/Je_i, N) \neq 0$, (note, then $Je_i \neq 0$), then $Ext^{n-1}_R(Je/J^2e, N) \neq 0$ since $Je/J^2e \in add(Re/Je)$ and $Je/J^2e \neq 0$ (since $Je \neq 0$ and J is nilpotent). This is a contradiction. So we may assume that for $e_1, \ldots, e_{i_1}$ $(i_1 \geq 1)$, $Ext^{n-1}_R(Re_i/Je_i, N) = 0$ $(1 \leq i \leq i_1)$ and for $e_{i_1+1}, \ldots, e_s$, $Ext^{n-1}_R(Re_j/Je_j, N) \neq 0$ $(i_1 + 1 \leq j \leq s)$. Especially, Je/J^2e is a direct sum of copies of $Re_1/Je_1, \ldots, Re_{i_1}/Je_{i_1}$.

Let $e^{(1)} = e_1 + \cdots + e_{i_1}$, then $Ext^{n-1}_R(Re^{(1)}/Je^{(1)}, N) = 0$. We claim that $i_1 < s$. For otherwise, $e^{(1)} = e_1 + \cdots + e_s = e$, then $Ext^{n-1}_R(Re/Je, N) = 0$. Note J/J^2 is a direct sum of copies of $Re_1/Je_1, \ldots, Re_s/Je_s$, so is J^k/J^{k+1} for all $k \geq 1$. Also since $Ext^{n-1}_R(J, N) \cong Ext^n_R(R/J, N) \neq 0$, from the exact sequence $0 \longrightarrow J^2 \longrightarrow J \longrightarrow J/J^2 \longrightarrow 0$, we get

$$0 = Ext^{n-1}_R(J/J^2, N) \to Ext^{n-1}_R(J, N) \to Ext^{n-1}_R(J^2, N) \to Ext^n_R(J/J^2, N) = 0,$$

thus $Ext^{n-1}_R(J^2, N) \cong Ext^{n-1}_R(J, N) \neq 0$. By using induction, we have

$Ext_R^{n-1}(J^l, N) \neq 0$ for all $l \geq 1$. This is a contradiction. So there must be $1 \leq i_1 < s$.

In summary, we obtain

$$Ext_R^n(Re/Je, N) = 0, \qquad Ext_R^n(R/J, N) \neq 0$$

$$Ext_R^{n-1}(Re^{(1)}/Je^{(1)}, N) = 0, \qquad Ext_R^{n-1}(Re/Je, N) \neq 0$$

$$J/J^2 \cong (Re_1/Je_1)^{A_1} \oplus \cdots \oplus (Re_s/Je_s)^{A_s},$$

$$Je/J^2e \cong (Re_1/Je_1)^{B_1} \oplus \cdots \oplus (Re_{i_1}/Je_{i_1})^{B_{i_1}}$$

for some $A_i, B_i \geq 0$, and $1 \leq i_1 < s$.

Generally, suppose the following is true:

(1) $Ext_R^{n-l}(Re^{(l)}/Je^{(l)}, N) = 0$, $Ext_R^{n-l}(Re^{(l-1)}/Je^{(l-1)}, N) \neq 0$ for $l = 0, 1, 2, \ldots, k$, where $e^{(-1)} = 1_R$ is the identity of R, $e^{(0)} = e$, $e^{(l)} = e_1 + \cdots + e_{i_l}$ (for $l \geq 1$), and

$$1 \leq i_k < i_{k-1} < \cdots < i_1 < s \qquad (k < s).$$

(2) $Je^{(j)}/J^2e^{(j)} \cong (Re_1/Je_1)^{A_1^{(j)}} \oplus \cdots \oplus (Re_{i_{j+1}}/Je_{i_{j+1}})^{A_{i_{j+1}}^{(j)}}$ ($j = -1, 0, 1, \cdots, k-1$) for some $A_1^{(j)}, \ldots, A_{i_{j+1}}^{(j)}$. (define $i_0 = s$).

Let us consider $Ext_R^{n-k-1}(Je^{(k)}/J^2e^{(k)}, N)$: If $Ext_R^{n-k-1}(Je^{(k)}/J^2e^{(k)}, N) \neq 0$, from the exact sequence

$$0 \longrightarrow Je^{(k)}/J^2e^{(k)} \longrightarrow Re^{(k)}/J^2e^{(k)} \longrightarrow Re^{(k)}/Je^{(k)} \longrightarrow 0,$$

we have $Ext_R^{n-k-1}(Re^{(k)}/J^2e^{(k)}, N) \neq 0$, a contradiction. (note we are assuming $k < s$). So there must be $Ext_R^{n-k-1}(Je^{(k)}/J^2e^{(k)}, N) = 0$, Since $Je^{(k)}/J^2e^{(k)}$ is a direct summand of $Je^{(k-1)}/J^2e^{(k-1)}$ and the latter is a direct sum of copies of $Re_1/Je_1, \ldots, Re_{i_k}/Je_{i_k}$ by our assumption. Thus

$$Je^{(k)}/J^2e^{(k)} \cong (Re_1/Je_1)^{A_1^{(j)}} \oplus \cdots \oplus (Re_{i_k}/Je_{i_k})^{A_{i_k}^{(j)}}$$

for some $A_i \geq 0$.

Now, if each $Ext_R^{n-k-1}(Re_j/Je_j, N) \neq 0$, $1 \leq j \leq i_k$, (especially, $Je_j \neq 0$, hence $Je^{(k)}/J^2e^{(k)} \neq 0$), we would have $Ext_R^{n-k-1}(Je^{(k)}/J^2e^{(k)}, N) \neq 0$. Contradiction. So we may assume that there is an integer i_{k+1} such that $1 \leq i_{k+1} \leq i_k$ and $Ext_R^{n-k-1}(Re_j/Je_j, N) = 0$ for $j = 1, 2, \ldots, i_{k+1}$, but $Ext_R^{n-k-1}(Re_j/Je_j, N) \neq 0$ for $j = i_{k+1}+1, \ldots, i_k$. Set $e^{(k+1)} = e_1 + \cdots + e_{i_{k+1}}$ $(i_{k+1} \geq 1)$, Thus

$$Ext_R^{n-k-1}(Re^{(k+1)}/Je^{(k+1)}, N) = 0.$$

We want to prove that $i_{k+1} < i_k$. If this were not true, then $e^{(k+1)} = e^{(k)}$ and $Ext_R^{n-k-1}(Re^{(k)}/Je^{(k)}, N) = 0$.

Claim. For any integer $u \geq 1$

$$(\#) \quad J^u e^{(k-1)}/J^{u+1}e^{(k-1)} \cong (Re_1/Je_1)^{(B_1)} \oplus \cdots \oplus (Re_{i_k}/Je_{i_k})^{(B_{i_k})}$$

for some $B_i \geq 0$.

Proof of the Claim. If $u = 1$, (#) is clear by our assumption. Inductively, suppose (#) is true for any number less than u $(u > 1)$. Consider the case u. If

$$e_i\left(\frac{J^u e^{(k-1)}}{J^{u+1}e^{(k-1)}}\right) \neq 0$$

for some number i, then there is a j satisfying $1 \leq j \leq i_{k-1}$ and

$$e_i\left(\frac{J^u e_j}{J^{u+1}e_j}\right) \neq 0.$$

By induction, $J^{u-1}e^{(k-1)}/J^u e^{(k-1)}$ is a direct sum of copies of $Re_1/Je_1, \cdots,$ Re_{i_k}/Je_{i_k}. So we may assume

$$h: \quad Re_1{}^{(A_1)} \oplus \cdots \oplus Re_{i_k}{}^{(A_{i_k})} \longrightarrow J^{u-1}e_j \longrightarrow 0$$

is a projective cover of $J^{u-1}e_j$, where $A_i \geq 0$. Hence $h(e_iJe_1{}^{(A_1)} \oplus \cdots \oplus e_iJe_{i_k}{}^{(A_{i_k})}) = e_iJ^u e_j$ and $h(J^2e_1{}^{(A_1)} \oplus \cdots \oplus J^2e_{i_k}{}^{(A_{i_k})}) = J^{u+1}e_j$. But

$$e_i\left(\frac{J^u e^{(k-1)}}{J^{u+1}e^{(k-1)}}\right) \neq 0,$$

therefore there is an l: $1 \leq l \leq i_k$ such that $e_i(Je_l/J^2e_l) \neq 0$, so $i \in \{1, 2, \ldots, i_k\}$. Thus $J^u e^{(k-1)}/J^{u+1}e^{(k-1)}$ is a direct sum of copies of $Re_1/Je_1, \ldots,$ Re_{i_k}/Je_{i_k}. that is , the claim is true.

Let us return to our proof. From the claim, we especially have

$$Ext_R^{n-k-1}(\frac{J^u e^{(k-1)}}{J^{u+1} e^{(k-1)}}, N) = 0$$

and

$$Ext_R^{n-k}(\frac{J^u e^{(k-1)}}{J^{u+1} e^{(k-1)}}, N) = 0$$

for any $u \geq 1$.

Noticing

$$Ext_R^{n-k-1}(Je^{(k-1)}, N) \cong Ext_R^{n-k}(\frac{Re^{(k-1)}}{Je^{(k-1)}}, N) \neq 0$$

by our assumption and from

$$0 \longrightarrow J^2 e^{(k-1)} \longrightarrow Je^{(k-1)} \longrightarrow \frac{Je^{(k-1)}}{J^2 e^{(k-1)}} \longrightarrow 0,$$

we obtain an exact sequence

$$0 = Ext_R^{n-k-1}(\frac{Je^{(k-1)}}{J^2 e^{(k-1)}}, N) \longrightarrow Ext_R^{n-k-1}(Je^{(k-1)}, N) \longrightarrow$$

$$Ext_R^{n-k-1}(J^2 e^{(k-1)}, N) \longrightarrow Ext_R^{n-k}(\frac{Je^{(k-1)}}{J^2 e^{(k-1)}}, N) = 0.$$

So $Ext_R^{n-k-1}(J^2 e^{(k-1)}, N) \neq 0$ since $Ext_R^{n-k-1}(Je^{(k-1)}, N) \neq 0$. By induction and the above claim, we have $Ext_R^{n-k-1}(J^l e^{(k-1)}, N) \neq 0$ for all $l \geq 1$. This is a contradiction. It follows that $1 \leq i_{k+1} < i_k$ and

$$Je^{(k)}/J^2 e^{(k)} \cong (Re_1/Je_1)^{(A_1)} \oplus \cdots \oplus (Re_{i_{k+1}}/Je_{i_{k+1}})^{(A_{i_{k+1}})}$$

since $Ext_R^{n-k-1}(Je^{(k)}/J^2 e^{(k)}, N) = 0$ and $Ext_R^{n-k-1}(Re_j/Je_j, N) \neq 0$ for $j = i_{k+1}+1, \ldots, i_k$. Also, $Ext_R^{n-k-1}(Re^{(k)}/Je^{(k)}, N) \neq 0$.

Thus, we have proved, by induction, that

$$1 < i_s < i_{s-1} < \cdots < i_2 < i_1 < s.$$

Of course, this is impossible. This completes the proof. □

References

[1] H. Bass. *Finitistic dimensions and a homological generalization of semiprimary rings.* Trans. Amer. Math. Soc. 95 (1960) 466-488

[2] W. Burgess. *The graded Cartan matrix and global dimension of 0-relations algebras.* Proc. Edenburgh Math. Soc. 30 (1987) 351-362

[3] R.R. Colby and K.R. Fuller. *A note on the Nakayama conjectures.* Tsukuba J. Math. Vol.14 No.2 (1990) 343-352

[4] K. Fuller and M. Saorín. *On the finitistic dimension conjecture for artinian rings.* manuscripta math. 74 (1992) 117-132

[5] K. Fuller and Y. Wang. *Redundancy in resolutions and finitistic dimensions of noetherian rings.* to appear

[6] E.L Green and B. Zimmermann Huisgen. *Finitistic dimension of artinian rings with vanishing radical cube.* Math. Z. 206 (1991) 505-526.

[7] E. Kirkman and J. Kuzmanovich. *Algebras with large homological dimension.* Proc. Amer. Math. Soc. 109 (1990) 903-906.

[8] H. Mochizuki. *Finitistic global dimension for rings.* Pac. J. Math. 15 (1965) 249-258

[9] Y. Wang. Ph.D. thesis. Preprint. The University of Iowa. 1992

Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242, USA

(Received October 5, 1992)

manuscripta math. 81, 89 - 94 (1993)

manuscripta mathematica

THE BLOW-UP OF p–HARMONIC MAPS

Martin Fuchs

We show the following theorem of compensated compactness type: If $u_n \rightharpoonup u$ weakly in the space $H^{1,p}(\Omega, \mathbb{R}^k)$ and if also $\lim_{n\to\infty} \partial_\alpha(|\nabla u_n|^{p-2}\partial_\alpha u_n) = 0$ in the sense of distributions then $\partial_\alpha(|\nabla u|^{p-2}\partial_\alpha u) = 0$. This result has applications in the partial regularity theory of p–stationary mappings $\Omega \to S^{k-1}$.

Suppose that M is an m–dimensional Riemannian manifold and let N denote a compact n–manifold embedded in some Euclidean space $\mathbb{R}^k$. For $p \in (1, \infty)$ we introduce the p–energy functional $\varepsilon_p(u) = \int_M |\nabla u|^p \, d\text{vol}$ on the Sobolev space $H^{1,p}(M, N) := \{u \in H^{1,p}(M, \mathbb{R}^k) : u(x) \in N \text{ a.e.}\}$. In recent years much attention has been paid to the partial regularity properties of locally ε_p–minimizing maps $u : M \to N$. Independently and with different methods Hardt-Lin [5], Luckhaus [7] and the author [4] showed $\mathcal{H} - \dim(\operatorname{Sing} u) \le m - [p] - 1$ for the set of interior singular points. The purpose of this note is to give a complete proof of the fact that weak limits of certain blow-up sequences are p–harmonic functions from some domain $\Omega \subset \mathbb{R}^m$ into the space $\mathbb{R}^k$. As demonstrated in [5] this leads to partial regularity of minimizers but the technique can also be applied to weakly p–stationary mappings $M \to S^{k-1}$. More precisely, we will prove the following

THEOREM: Let Ω denote an open bounded set in $\mathbb{R}^m$ and consider a sequence $\{u_n\}$ in $H^{1,p}(\Omega, \mathbb{R}^k)$ such that $u_n \rightharpoonup u$ weakly in this space and

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \, dx \le c_n \cdot \|\varphi\|_{L^\infty(\Omega)}$$

for all $\varphi \in \overset{\circ}{H}{}^{1,p} \cap L^\infty(\Omega, \mathbb{R}^k)$ with $\lim_{n\to\infty} c_n = 0$. Then u is a p–harmonic mapping $\Omega \to \mathbb{R}^k$, i.e. $\partial_\alpha(|\nabla u|^{p-2}\partial_\alpha u) = 0$ in the sense of distributions.

Going through the arguments outlined in Evan's recent paper [2] we obtain as a

COROLLARY: Assume that $u \in H^{1,p}(\Omega, S^{k-1})$ is a weakly p–harmonic map from the domain Ω into the sphere S^{k-1} which in addition satisfies the monotonicity inequality

$$R^{p-m} \int_{B_R(x)} |\nabla u|^p \, dy \ge r^{p-m} \int_{B_r(x)} |\nabla u|^p \, dy$$

for all balls $B_r(x) \subset B_R(x) \subset \Omega$. Then $u \in C^1(\Omega - \Sigma)$ for a relatively closed set $\Sigma \subset \Omega$ such that $\mathcal{H}^{m-p}(\Sigma) = 0$.

In order to get this result one only has to replace Evan's equation (3.13) by the corresponding p-harmonic system which is valid for the limit on account of our Theorem.

REMARK: As mentioned before the Theorem implies partial regularity for local ε_p-minima in the space $H^{1,p}(M, N)$. Since an Euler equation is also true for obstacle problems (i.e. $\partial N \neq \emptyset$) the statement of the Theorem gives an alternative proof of the results obtained in [4].

The proof of the Theorem is organized in several steps, we make crucial use of earlier results due to Frehse [3] and also Landes [6].

LEMMA 1: We have $\nabla u_n(x) \to \nabla u(x)$ a.e. on Ω at least for a subsequence.

Assuming this we fix $\varphi \in C_0^1(\Omega, \mathbb{R}^k)$ and pass to a subsequence with the property

$$W_n := |\nabla u_n|^{p-2} \nabla u_n \rightharpoonup W$$

weakly in $L^q(\Omega)$, $q := \frac{p}{p-1}$, for some function W. This implies

$$\mu_n(B) \to \mu(B), \quad B \subset \Omega,$$

for the measures $\mu_n := \mathcal{L}^m \llcorner W_n$, $\mu := \mathcal{L}^m \llcorner W$ on Ω and by the Theorem of Hahn-Vitali-Saks we find

$$|\mu_n(B)|, |\mu(B)| \leq \varepsilon \tag{1}$$

for any set $B \subset \Omega$ such that $\mathcal{L}^m(B) \leq \delta_\varepsilon$. Here $\varepsilon > 0$ is an arbitrary number in $(0,1)$. By Egoroff's Theorem we find $\Omega' \subset \Omega$ with the property $\nabla u_n \to \nabla u$ uniformly on Ω' and $\mathcal{L}^m(\Omega - \Omega') \leq \delta_\varepsilon$. For $n \geq n_\varepsilon$ we clearly have

$$\sup_{\Omega'} \left| W_n - |\nabla u|^{p-2} \nabla u \right| \leq \varepsilon$$

so that

$$\begin{aligned}
&\left| \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla\varphi \, dx - \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla\varphi \, dx \right| \\
&\quad \leq \int_{\Omega'} |\nabla\varphi| \cdot \left| W_n - |\nabla u|^{p-2} \nabla u \right| dx + \int_{\Omega - \Omega'} |\nabla\varphi| \cdot [\, |W_n| + |\nabla u|^{p-1} \,] \, dx \\
&\quad \leq \mathcal{L}^m(\Omega) \cdot \|\nabla\varphi\|_\infty \cdot \varepsilon + \|\nabla\varphi\|_\infty \cdot \left(\mu_n(\Omega - \Omega') + \int_{\Omega - \Omega'} |\nabla u|^{p-1} \, dx \right).
\end{aligned}$$

W.l.o.g. we may assume $\int_{\Omega - \Omega'} |\nabla u|^{p-1} \, dx \leq \varepsilon$. From (1) we then deduce

$$\lim_{n \to \infty} \int_\Omega W_n \cdot \nabla\varphi \, dx = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla\varphi \, dx$$

so that $\partial_\alpha \left(|\nabla u|^{p-2} \partial_\alpha u \right) = 0$ on Ω. ∎

Lemma 1 is a consequence of

LEMMA 2: Consider an arbitrary subregion $\Omega^* \subset\subset \Omega$. Then we have for all $\alpha > 0$

$$\lim_{n\to\infty} \mathcal{L}^m\Big(\{x \in \Omega^* : |\nabla u_n(x) - \nabla u(x)| \geq \alpha\}\Big) = 0\,. \tag{2}$$

Condition (2) immediately implies $\nabla u_n^*(x) \to \nabla u(x)$ a.e. on Ω^* for a suitable subsequence $\{u_n^*\}$ (compare [1], 19.6) and the claim follows from a diagonal argument.

In order to prove Lemma 2 we proceed similar to [6], §4, and choose a sequence $\{\Omega_\ell\}$ of measurable sets and a suitable subsequence of $\{u_n\}$ with the following properties:

$$\begin{aligned} &\text{i)}\quad u_n \to u \quad \text{strongly in } L^p(\Omega) \text{ and a.e.}\,,\\ &\text{ii)}\quad \Omega_\ell \subset \Omega_{\ell+1}\,,\quad \mathcal{L}^m\Big(\Omega - \bigcup_{\ell=1}^{\infty}\Omega_\ell\Big) = 0\\ &\text{iii)}\quad \lim_{n\to\infty} \|u_n - u\|_{L^\infty(\Omega_\ell)} = 0\,, \text{ especially}\\ &\qquad \|u_n - u\|_{L^\infty(\Omega_\ell)} \leq \frac{1}{\ell} \quad \text{for } n \geq n_\ell\,. \end{aligned} \tag{3}$$

Let Ω^* denote a region as in (2) and fix $\eta \in C_0^1(\Omega,[0,1])$ with $\eta \equiv 1$ on Ω^*. Suppose that we already know

$$\begin{aligned} \alpha_\ell \;:=\; &\limsup_{n\to\infty} \int_{\Omega_\ell} \eta \cdot \Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\Big)\cdot \nabla(u_n - u)dx \\ &\longrightarrow 0 \quad \text{as } \ell \to \infty\,. \end{aligned} \tag{4}$$

Then ellipticity implies

$$\limsup_{n\to\infty} \int_{\Omega_\ell} \eta \cdot |\nabla u_n - \nabla u|^p\,dx \underset{\ell\to\infty}{\longrightarrow} 0\,,$$

hence $\Big([|\nabla u_n - \nabla u| \geq \alpha] := \{x \in \Omega : |\nabla u_n(x) - \nabla u(x)| \geq \alpha\}\Big)$

$$\begin{aligned} &\limsup_{n\to\infty} \mathcal{L}^m\Big(\Omega^* \cap [\,|\nabla u_n - \nabla u| \geq \alpha]\Big)\\ &\quad \leq \limsup_{n\to\infty}\Big\{\int_{\Omega_\ell} \alpha^{-p}\cdot \eta|\nabla u_n - \nabla u|^p dx + \mathcal{L}^m(\Omega^* - \Omega_\ell)\Big\}\\ &\quad \underset{\ell\to\infty}{\longrightarrow} 0 \quad \text{for any } \alpha > 0\,. \end{aligned}$$

In conclusion our result will follow from relation (4). We have (by the weak convergene $u_n \rightharpoonup u$ in $H^{1,p}(\Omega)$)

$$\begin{aligned} 0 \;&\leq\; \limsup_{n\to\infty} \int_{\Omega_\ell} \Big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\Big)\eta \cdot \nabla(u_n - u)dx\\ &=\; \limsup_{n\to\infty} \int_{\Omega_\ell} |\nabla u_n|^{p-2}\nabla u_n\,\eta \cdot \nabla(u_n - u)dx\\ &\underset{(3)}{=}\; \limsup_{n\to\infty} \int_{\Omega_\ell} |\nabla u_n|^{p-2}\nabla u_n\,\eta \cdot \nabla[u_n - u]^L\,dx \;=:\; a_\ell \end{aligned}$$

where $L := \frac{1}{\ell}$ and $v^L := \begin{cases} v & \text{if } |v| \le L \\ \frac{L}{|v|}\cdot v & \text{if } |v| \ge L \end{cases}$ for vector valued functions $v : \Omega \to \mathbb{R}^k$.
We split

$$\begin{aligned} a_\ell &\le \limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\, \eta \cdot \nabla [u_n - u]^L dx \\ &\quad + \limsup_{n\to\infty} \int_{\Omega-\Omega_\ell} |\nabla u_n|^{p-2}\nabla u_n\, \eta \cdot \nabla [u - u_n]^L dx \\ &=: b_\ell + d_\ell , \\ b_\ell &= \limsup_{n\to\infty} \Big\{ \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(\eta \cdot [u_n - u]^L) dx \\ &\qquad - \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\eta \otimes [u_n - u]^L dx \Big\} \\ &\le \limsup_{n\to\infty} \Big(c_n \cdot \|\eta\cdot [u_n - u]^L\|_{L^\infty(\Omega)} \Big) \end{aligned}$$

where we have made use of the assumption in our Theorem and also of the fact that $\lim_{n\to\infty} \|(u_n - u)^L\|_{L^p(\Omega)} = 0$ for fixed ℓ. Since

$$\limsup_{n\to\infty} \Big(c_n \cdot \|\eta\cdot [u_n - u]^L\|_{L^\infty(\Omega)} \Big) = 0$$

it remains to discuss the term d_ℓ. Let $G_n^L := [|u_n - u| \le L] \supset \Omega_\ell$ (on account of (3)). Then

$$\begin{aligned} d_\ell &\le \limsup_{n\to\infty} \int_{G_n^L-\Omega_\ell} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla[u - u_n]^L \eta\, dx \\ &\quad + \limsup_{n\to\infty} \int_{\Omega - G_n^L} \dots\, dx \;=:\; e_\ell + f_\ell , \\ e_\ell &= \limsup_{n\to\infty} \int_{G_n^L-\Omega_\ell} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u - u_n)\eta\, dx \\ &\le \limsup_{n\to\infty} \int_{G_n^L-\Omega_\ell} \eta \cdot \big[\, |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\big] \cdot \nabla(u - u_n) dx \\ &\quad + \limsup_{n\to\infty} \int_{G_n^L-\Omega_\ell} \eta \cdot |\nabla u|^{p-2}\nabla u \cdot \nabla(u - u_n) dx \\ &\le \limsup_{n\to\infty} \int_{G_n^L-\Omega_\ell} \eta |\nabla u|^{p-2}\nabla u \cdot \nabla(u - u_n) dx . \end{aligned}$$

Weak convergence $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega)$ gives

$$\lambda_n(B) := \int_B \eta |\nabla u|^{p-2}\nabla u \cdot \nabla(u - \nabla u_n) dx \underset{n\to\infty}{\longrightarrow} 0$$

for every set $B \subset \Omega$. By the Theorem of Vitali-Hahn-Saks the signed measures λ_n are equi absolutely continuous with respect to $\mathcal{L}^m$. Since $\mathcal{L}^m(G_n^L - \Omega_\ell) \le \mathcal{L}^m(\Omega - \Omega_\ell) \underset{\ell\to\infty}{\longrightarrow} 0$ we find that

$$\lim_{\ell\to\infty} \Big(\limsup_{n\to\infty} \int_{G_n^L-\Omega_\ell} \eta \cdot |\nabla u_n|^{p-2}\nabla u \cdot \nabla(u - u_n) dx \Big) = 0 .$$

Finally we look at

$$f_\ell = \limsup_{n\to\infty} \int_{\Omega - G_n^L} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla[u-u_n]^L \eta \, dx \, .$$

The integrand I_n^ℓ is given by (see [6], §3 Prop. 2)

$$\begin{aligned}
I_n^\ell &= |\nabla u_n|^{p-2} \cdot \eta \cdot \partial_\alpha u_n \cdot L \cdot |u_n - u|^{-1} \\
&\quad \Big[\partial_\alpha(u-u_n) - (u-u_n)\cdot|u-u_n|^{-2}(\partial_\alpha(u-u_n)\cdot(u-u_n)\,)\Big] \\
&= |\nabla u_n|^{p-2} \cdot \eta \cdot L \cdot |u-u_n|^{-3} \\
&\quad \cdot\Big\{-|\nabla u_n|^2 \cdot |u-u_n|^2 + [\partial_\alpha u_n \cdot (u-u_n)]\cdot[\partial_\alpha u_n \cdot (u-u_n)] \\
&\qquad + |u-u_n|^2\, \nabla u_n \cdot \nabla u \\
&\qquad - [\partial_\alpha u_n \cdot (u-u_n)]\cdot[\partial_\alpha u \cdot (u-u_n)]\Big\} \\
&\le |\nabla u_n|^{p-2}\eta \cdot L \cdot |u-u_n|^{-1} \nabla u_n \cdot \nabla u \\
&\quad - |\nabla u_n|^{p-2}\, \eta \cdot L \cdot |u-u_n|^{-3} \cdot [\partial_\alpha u_n \cdot (u-u_n)]\cdot[\partial_\alpha u \cdot (u-u_n)] \;=:\; v_n \, .
\end{aligned}$$

From these estimates we infer

$$\begin{aligned}
f_\ell &\le \limsup_{n\to\infty} \int_{\Omega - G_n^L} |v_n|\, dx \\
&\le \limsup_{n\to\infty} \int_{\Omega - G_n^L} 2\cdot \eta |\nabla u_n|^{p-1} |\nabla u|\, dx
\end{aligned}$$

and since $|\nabla u_n|^{p-1} \rightharpoonup \vartheta$ weakly in $L^{\frac{p}{p-1}}(\Omega)$ for some function ϑ we deduce

$$\int_B 2\cdot \eta |\nabla u_n|^{p-1} |\nabla u|\, dx \underset{n\to\infty}{\longrightarrow} \int_B 2\cdot \eta\vartheta |\nabla u|\, dx$$

for all subsets B of Ω. On the other hand strong convergence in $L^p(\Omega)$ implies

$$\mathcal{L}^m\Big([\,|u_n - u| \ge L]\Big) \underset{n\to\infty}{\longrightarrow} 0$$

so that

$$\lim_{\ell\to\infty}\Big(\limsup_{n\to\infty} \int_{\Omega - G_n^L} 2\cdot\eta\cdot|\nabla u_n|^{p-1}|\nabla u|\, dx\Big) = 0$$

by the Theorem of Hahn-Vitali-Saks. This shows $\limsup_{\ell\to\infty} f_\ell \le 0$ and putting together all our results we arrive at (4). ■

REMARKS: 1) Under the assumptions of the Theorem we have

$$V_n := |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup V := |\nabla u|^{p-2}\,\nabla u$$

weakly in $L^{p/p-1}(\Omega)$ at least for a subsequence.

PROOF: Since $\sup_{n\in\mathbb{N}} \|V_n\|_{L^{p/p-1}(\Omega)} < \infty$ we deduce $V_n \rightharpoonup: \tilde V$ for some function $\tilde V$ in $L^{p/p-1}(\Omega)$. Quoting Lemma 1 and Egoroff's Theorem we find $\Omega_\varepsilon \subset \Omega$ such that

$$\mathcal{L}^m(\Omega - \Omega_\varepsilon) \le \varepsilon\,, \quad V_n \to V \ \text{ uniformly on } \Omega_\varepsilon\,.$$

Clearly

$$\int_{\Omega_\varepsilon}(V_n - \tilde{V}) \cdot \psi \, dx = \int_{\Omega}(V_n - \tilde{V})\chi_{\Omega_\varepsilon} \cdot \psi \, dx \underset{n\to\infty}{\longrightarrow} 0$$

for any $\psi \in L^\infty(\Omega)$ and also

$$\int_{\Omega_\varepsilon}(V_n - V) \cdot \psi \, dx \underset{n\to\infty}{\longrightarrow} 0$$

so that $\int_{\Omega_\varepsilon}(V - \tilde{V}) \cdot \psi \, dx = 0$, hence $V = \tilde{V}$ a.e. on Ω_ε.

Since ε was arbitrary we find $V = \tilde{V}$ a.e. on Ω.

2) An inspection of our arguments shows that the statements of Lemma 1 and Remark 1 remain valid under the assumptions $u_n \rightharpoonup u$ weakly in $H^{1,p}(\Omega)$ and $\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla\varphi \, dx \le c \cdot \|\varphi\|_{L^\infty(\Omega)}$ for all $\varphi \in \overset{\circ}{H}{}^{1,p} \cap L^\infty(\Omega)$ with $c \in (0,\infty)$ fixed.

References

[1] H. Bauer, Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie. Walter de Gruyter · Berlin · New York 1978

[2] L.C. Evans, Partial regularity for stationary harmonic maps into spheres. Arch. Rat. Mech. Anal. 116 (1991) 101–113

[3] J. Frehse, Existence and perturbation theorems for nonlinear elliptc systems. Preprint Nr. 576, SFB 72, Bonn 1983

[4] M. Fuchs, p–harmonic obstacle problems. Annali Mat. Pura Appl. 156 (1990) 127–180

[5] R. Hardt, F.H. Lin, Mappings minimizing the L^p–norm of the gradient Comm. Pure Appl. Math. XL (1987), 555–588

[6] R. Landes, On the existence of weak solutions of perturbed systems with critical growth. J. Reine Ang. Math. 393 (1989), 21–38

[7] S. Luckhaus, Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold. Ind. Univ. Math. J. 37, No. 2(1988) 346–367

Martin Fuchs
Fachbereich Mathematik
Universität des Saarlandes
66041 Saarbrücken

(Received October 15, 1992)

manuscripta math. 81, 95 - 111 (1993)

manuscripta
mathematica

OBSTRUCTIONS TO THE SECTION PROBLEM IN FIBRE BUNDLES

SAMSON SANEBLIDZE

ABSTRACT. For a Serre fibration with a fibre of the $K(\pi, n)$'s product type, obstructions to the section problem in each degree are defined by means of the Hirsch complex of fibration. This allows us to give the homotopy classification of sections (maps) as well as other applications. In particular, for G-bundles, these obstructions are related to the A_∞-module structure on the homology of the fibre and, consequently, some results in the fixed point theory are obtained.

1. INTRODUCTION

A general idea to the section problem for a fibration being realized in the rational homotopy theory is to construct a retraction in a suitable algebraic model of the fibration and then to develop an obstruction theory to the existence of that retraction (cf. [22], [24], [19] for instance). Unfortunately, in ordinary (integral) homotopy theory there are not so far such good models for the fibration, especially for getting an affirmative answer to the existence of a section. There is another approach, based on the Moore-Postnikov decomposition of the fibration and carried out by several authors, see [23], for example, but those computations with higher obstructions became to be difficult. On the other hand, [3] suggests careful examination of the Hirsch model (complex) of a fibration. By definition the Hirsch complex (with integral coefficients) of a Serre fibration $F \to E \xrightarrow{\xi} X$ with $\pi_1(X)$ acting trivially on $H_*(F)$ is a twisted tensor product in the following commutative diagram of complexes (cf. [12], [4], [8], [2])

$$(1)\qquad \begin{array}{ccccc} C_*(X) & \xleftarrow{\xi_*} & C_*(E) & \longleftarrow & C_*(F) \\ \uparrow{=} & & \uparrow{k} & & \uparrow{k_F} \\ C_*(X) & \xleftarrow{j} & (C_*(X)\otimes RH_*(F), \partial_h) & \xleftarrow{i} & (RH_*(F), \partial) \end{array}$$

where k is a chain homotopy equivalence (as well k_F), $(R_{\geq 0}H_q(F), \partial) \to H_q(F)$ is a free group ($\mathbb{Z}$-module) resolution of the integral homology group $H_q(F)$, $\partial_h = \partial^{\otimes} + h\cap_-$, h is a twisting cochain, $h \in C^*(X; Hom(RH_*(F), RH_*(F)))$ (see §2, for details). In fact, originally the Hirsch complex was defined in [12] with coefficients in R, a commutative ring with 1, for $H_*(F; R)$ a free R-module. This has been extended for arbitrary $H_*(F; R)$ in [2].

In particular, it immediately follows from [2] that if ξ has a section, then in (1) there is a chain map

$$i' : C_*(X) \to (C_*(X) \otimes RH_*(F), \partial_h)$$

with $j \circ i' = id$.

---

* Partially supported by the Forschungsprojekt Topologie und nichtkommutative Geometrie der Universität Heidelberg

Briefly speaking what we do here is to show that the existence of the chain map i' above with some additional properties is equivalent to the existence of a section of ξ provided the Hurewicz homomorphism (the H.H.) $\pi_*(F) \longrightarrow H_*(F)$ for the fibre of ξ is split injective, that is F has the homotopy type of a product of Eilenberg-MacLane spaces.

More precisely, suppose we are given a section $s^n : X^n \longrightarrow E$ on n-skeleton of the base X of ξ, and let

$$c(s^n) \in C^{n+1}(X; \pi_n(F))$$

be the obstruction cocycle to an extension of s^n to X^{n+1}. Then we can choose a twisting cochain h whose one component detects the image of the cocycle $c(s^n)$ under the cochain map induced by the H.H. for the fibre (Proposition 2.4; cf. [3]). Moreover, by means of the section s^n and the twisting cochain h we define the subgroup

$$I^{n+1}(s^n, h) \subset H^{n+1}(X; \pi_n(F)),$$

and denoting by $o(s^n)$ the obstruction cohomological class of $c(s^n)$ and by $O^{n+1}(\xi)$ the set of such cohomological classes corresponding to all sections on X^n we have the following

Theorem A. *By hypotheses and notations above in a fibration ξ with a section $s^n : X^n \longrightarrow E$, suppose the H.H. $\pi_i(F) \longrightarrow H_i(F)$ is split injective for $i \leq n$. Then*

$$O^{n+1}(\xi) = o(s^n) + I^{n+1}(s^n, h).$$

It follows from this that the subgroup $I^{n+1}(s^n, h)$ is uniquely determined by the fibration and we denote it by $I^{n+1}(\xi)$. So $O^{n+1}(\xi)$ can be regarded as a single element in the quotient group $H^{n+1}(X; \pi_n(F))/I^{n+1}(\xi)$ and, consequently, we have

Corollary 2.6. *There exists a section of ξ on X^{n+1} if and only if $O^{n+1}(\xi) = 0$.*

In general, $I^{n+1}(\xi)$ has a filtration by the subgroups

$$0 = I_0^{n+1}(\xi) \subset I_1^{n+1}(\xi) \subset \cdots \subset I_{n-1}^{n+1}(\xi) = I^{n+1}(\xi),$$

and we have the following

Theorem 2.7. *Let ξ be a fibration as in Theorem A. Then there is a section, s', of ξ on X^{n+1} with $s'|_{X^m} = s^n|_{X^m}$, $m < n$ if and only if $o(s^n) \in I_{n-m-1}^{n+1}(\xi)$.*

In particular, if $I^{n+1}(\xi) = 0$, i.e., $O^{n+1}(\xi) = o(s^n)$, then $o(s^n)$ becomes an invariant of ξ and this situation occurs in the following theorem (cf. [16], [7]):

Theorem 3.1. *Let ξ with a section $s^n : X^n \longrightarrow E$ satisfy one of the following conditions:*

(i) ξ is a principal fibration, i.e., is induced by a map $X \longrightarrow Y$ from the path fibration $\Omega Y \longrightarrow PY \longrightarrow Y$ (with Y simply connected and the H.H. split injective in degrees $\leq n$ for ΩY);

(ii) The base X of ξ is $\mathbb{Z}$-formal with $H^*(X)$ having the trivial multiplication;

(iii) The base X of ξ is $\mathbf{k}$-formal with $H^*(X; \mathbf{k})$ having the trivial multiplication provided the composition of the H.H. with the canonical homomorphism $H_i(F; \mathbb{Z}) \longrightarrow H_i(F; \mathbf{k})$ is injective and the image of $\pi_i(F)$ is a $\mathbf{k}$-submodule in $H_i(F; \mathbf{k})$ for a field $\mathbf{k}$, $i \leq n$.

Then the obstruction cohomological class $o(s^n)$ is an invariant of ξ, i.e., $O^{n+1}(\xi) = o(s^n)$.

This obstruction theory allows us to give the homotopy classification of sections, too. In particular, the twisting cochain h above restricts to a *spherical twisting cochain*

$$\nu \in C^*(X; Hom(R\pi_*(F), R\pi_*(F)))$$

which defines a new differential, d_ν, on the cochain complex

$$L^k_{(n-1)} = \prod_{j\geq 0} \prod_{q=1}^{n-1} C^{k+j+q}(X; R_j\pi_q(F), d(= d^C + \partial^R)), \qquad k \in \mathbb{Z},$$

by the rule $d_\nu = d + \nu \cup\_$.

The classification has more nice form if ξ is as in the following

Theorem B. *By hypotheses and notations above let ξ with a section $s^n : X^n \to E$ satisfy the following two conditions:*
(i) The H.H. for the fibre of ξ is split injective in degrees $\leq n$;
(ii) ξ has the Hirsch model with

$$\partial_h(C_*(X^n) \otimes R\bar{H}_i(F)) \subset C_*(X) \otimes R\bar{H}_*(F)$$

for some decomposition $H_i(F) = \pi_i(F) \oplus \bar{H}_i(F)$, $i \leq n$, compatible with the H.H..
Then there is a bijection

$$[X^{n-1}, E]_s \approx H^0(L_{(n-1)}, d_\nu)$$

where $[\ ,\ ]_s$ denotes the set of homotopy classes of sections.

In the case of an R-formal base X (e.g. X is a suspension (cf. the proof of Corollary 3.3), $H^*(X; R)$ is polynomial, cf. [1], [9]) computations are more simple, since the cochain complexes $C^*(X; \cdot)$ under consideration can be replaced by the cohomology $H^*(X; \cdot)$ (cf. Theorem 2.10, for example).

On the other hand, because of [14] computations with the Hirsch complex and, therefore, with our obstructions become more convenient for G-bundles; it is shown that an action $G \times F \to F$ induces higher order pairings

$$\mu_n : \otimes^{n-1} H_*(G; R) \otimes H_*(F; R) \to H_*(F; R), \qquad n = 2, 3, \ldots,$$

the A_∞-module structure on $H_*(F; R)$ with μ_2 the canonical induced pairing on the homology (homologies are free R-modules). These pairings define a special twisting cochain of the associated fibre bundle which, in particular, allows us to give in some cases criteria for the existence of a homotopy fixed point of the action in terms of characteristic classes of the fibre bundle (cf. Theorem 3.10).

In §2 we develop our obstruction theory, where in particular main Theorems A and B are proved, and in §3 we give some applications.

Acknowledgement. I would like to thank prof. N. Berikashvili, prof. A. Dold and prof. D. Puppe for useful discussions and suggestions.

2. Obstructions to the section problem in a Serre fibration with fibre of the $K(\pi, n)$'s product type

Throughout this paper we consider a Serre fibration of path connected spaces

$$F \to E \xrightarrow{\xi} X$$

with the base X a polyhedron. We restrict ourself to X of finite type, i.e., it has finitely many simplices in each dimension, although in many cases, this restriction can be avoided.

We will also assume that $\pi_1(X)$ acts trivially on the homology groups $H_*(F)$. For a space (polyhedron) X, $C(X)$ will denote the singular (simplicial) complex with integer coefficients, unless specified otherwise. We use usual sign conventions for graded maps, etc.

We begin by recalling some facts about a twisting cochain of a fibration ξ playing a basic role in the Hirsch complex (1). In particular, the question of estimation of freedom in the choice of a twisting cochain, being a key point in our obstruction theory, leads to the functor D [2].

Fix a free group ($\mathbb{Z}$-module) resolution

$$\rho : (R_{\geq 0}H_q(F), \partial^R) \to H_q(F), \quad \partial^R : R_i H_q(F) \to R_{i-1}H_q(F),$$

of $H_q(F)$. Of course, we can take $i = 0, 1$, however, it is sometimes usefull to consider resolutions of any length, since the functor D does not depend on resolutions used. Consider the cochain complex

$$C^*(X; Hom(RH_*(F), RH_*(F))) = Hom(C_*(X), Hom(RH_*(F), RH_*(F)))$$

which will be denoted by $(\mathcal{H}, \nabla)$. Here $Hom(RH_*(F), RH_*(F), d)$ is the standard bigraded complex: $f^{s,t} : R_j H_q(F) \to R_{j+s}H_{q+t}(F)$ if $f^{s,t} \in Hom^{s,t}(R_*H_*(F), R_*H_*(F))$. Moreover, under the composition of homomorphisms it becomes a (bi)graded differential algebra with 1 and, hence, so does $(\mathcal{H}, \nabla)$ via the $\cup$-product. In fact, $\mathcal{H}$ is tri-graded:

$$\mathcal{H}^{i,j,t} = C^i(X; Hom^{j,t}(RH_*(F), RH_*(F))),$$

and we refer to i as the *base topological degree,* to j as the *fibre resolution degree,* to t as the *fibre weight* and to $n = i - j - t$ as the *total degree* (cf. [20], [18]). Moreover, we refer to $r = i - j$ as the *perturbation* degree, which is mainly exploited by induction arguments below (In particular, when we can take $RH_*(F) = H_*(F)$, then perturbation degree coincides with that of base topological one). Thus, we have

$$\mathcal{H} = \{\mathcal{H}^n\}, \quad \mathcal{H}^n = \prod_{n=r-t} \mathcal{H}^{r,t},$$

$$\nabla : \mathcal{H}^{r,t} \to \mathcal{H}^{r+1,t}.$$

A twisting cochain, h, is an element of $\mathcal{H}$ of total degree 1 and at least of perturbation degree 2 satisfying the condition $\nabla h = -hh$, i.e. h has the form

$$h = h^2 + \cdots + h^r + \cdots, \qquad h^r \in \mathcal{H}^{r,r-1}.$$

With respect to the perturbation degree the condition for a cochain to be twisting reads

$$\nabla(h^2) = 0, \quad \nabla(h^3) = -h^2h^2, \quad \nabla(h^4) = -h^2h^3 - h^3h^2, \ldots .$$

Note that a single superscript to an element of $\mathcal{H}$ will always denote the perturbation degree, while we denote by h_m the component of h in the subcomplex $C^*(X; Hom(RH_m(F), RH_*(F)))$. In particular, we refer to the component h_0 of h as *transgressive* one.

The basic fact about the Hirsch complex is the following theorem [2]

Theorem 2.1. *Let ξ be a Serre fibration as above. Then*

Existence. There are a twisting cochain h and a chain map k such that (1) is satisfied.

Uniqueness. If there are another such $\bar{h}$ and $\bar{k}$, then there is an isomorphism, p, of complexes

$$p : (C_*(X) \otimes RH_*(F), \partial_{\bar{h}}) \to (C_*(X) \otimes RH_*(F), \partial_h)$$

such that

(i) p is defined by an element of $\mathcal{H}$ (denoted by the same symbol) of total degree 0 via $\cap$-product and $p-1$ increases the fibre weight ($H_*(F)$-degree), i.e.,

$$p = 1 + p^1 + \cdots + p^r + \cdots, \quad p^r \in \mathcal{H}^{r,r}.$$

(ii) $k \circ p$ is chain homotopic to $\bar{k}$.

In other words, twisting cochains h and $\bar{h}$ are on the same orbit with respect to an action of the group of those automorphisms, p, which are of the form as in the uniqueness part of the theorem, on the set of all twisting cochains, h, via

$$p * h = php^{-1} + \nabla(p)p^{-1}.$$

The obtained quotient set denoted by $D(X; H_*)$, $H_* = H_*(F)$ (cf. [10]) is the contravariant functor from the category of polyhedrons to the category of pointed sets (for a fixed graded group H_*). So that in this set Theorem 2.1 still assignes to the fibration the element, $d(\xi)$, called the (homological) *predifferential* of ξ [2].

It is of interest to observe that the functor D composed with the suspension functor $X \to SX$ becomes cohomological. More precisely, we have

Theorem 2.2. *There exists a natural equivalence of functors from the category of connected polyhedrons to the category of pointed sets*

$$\phi : D(\cdot\,; H_*) \circ S \xrightarrow{\approx} \prod_{i \geq 2} H^i(\cdot\,; G_i) \circ S,$$

where $G_i = \prod_j Hom(H_j, H_{j+i-1}) \oplus Ext(H_j, H_{j+i})$.

Proof. Consider the standard triangulaton of a suspension, $SY = C_-Y \cup C_+Y$, by adding two verteces to a triangulation of Y. Then the set, $\bar{\mathcal{H}}$, of those cochains in $\mathcal{H} = C^*(SY; Hom(RH_*, RH_*))$ which annihillite on the cone C_+Y is a differential subalgebra of $\mathcal{H}$ with the trivial multiplication. Moreover, the inclusion $\bar{\mathcal{H}} \subset \mathcal{H}$ induces an isomorphism in cohomology. Let $d \in D(SY; H_*)$ and let $h \in d$ be a representative. Since h^2 is a cocycle, there is $\bar{h}^2 \in \bar{\mathcal{H}}$ with $\nabla(\bar{h}^2) = 0$, $\nabla(p^1) = \bar{h}^2 - h^2$, some $p^1 \in \mathcal{H}$. Put $\bar{g} = (1 + p^1) * h$. Clearly, $\bar{g}^2 = \bar{h}^2$ and $\nabla(\bar{g}^3) = \bar{g}^2\bar{g}^2 = 0$. Obviously, one can construct by induction on perturbation degree another representative, $\bar{h}$, of d with $\bar{h} \in \bar{\mathcal{H}}$. But we have $\nabla\bar{h} = \bar{h}\bar{h} = 0$, and then put $\phi(d) = [\bar{h}]_\nabla \in \prod_{i \geq 2} H^i(SY; G_i)$. It is easy to see that ϕ is a bijection. In order to show that ϕ is natural, it is sufficient to remark that for a map $f : Y' \to Y$, an induced homomorphism $(Sf)^* : \mathcal{H}_{SY'} \to \mathcal{H}_{SY}$ sends $\bar{\mathcal{H}}_{SY'}$ into $\bar{\mathcal{H}}_{SY}$. □

Note that for an arbitrary map $g : SY' \to SY$, we have not $H(g) \circ \phi = \phi \circ D(g)$ as shows the Hopf map $S^3 \to S^2$.

Example 2.3. Here we consider those elements (predifferentials) of $D(X; H_*)$ which correspond to certain principal fibrations over suspensions. Let X be a suspension, SY, with $H_*(Y)$ a free group and H_* the tensor algebra over $H_*(Y)$. So by the Bott-Samelson theorem we can regard $H_* = H_*(\Omega SY)$. By Theorem 2.2 we also have the natural bijection with respect to maps being suspensions $\phi : D(SY; H_*) \xrightarrow{\approx} \prod_{i \geq 2} H^i(SY; Hom^{i-1}(H_*, H_*))$. First consider the path fibration $\Omega SY \to PSY \xrightarrow{\pi} SY$. Then a twisting cochain and the predifferential of π are defined as follows. Let $C_*(SY) \to H_*(SY)$ be a coalgebra map dual to the composite map of differential algebras $H^*(SY) \to C^*(SY, C_+Y) \subset C^*(SY)$ inducing an isomorphism in cohomology. We have that the composite

$$\alpha : C_*(SY) \to H_*(SY) \xrightarrow[\approx]{s} H_*(Y) \subset H_*(\Omega SY)$$

defines a twisting cochain, $h \in \prod_{i\geq 2} C^i(SY; Hom^{i-1}(H_*, H_*))$, by $h(\sigma)(a) = a\alpha(\sigma), \sigma \in C_{\geq 2}(SY), a \in H_*$. Therefore, for the predifferential of π we have $\phi(d(\pi)) = u$ with $u(b)(a) = as(b), b \in H_{\geq 2}(SY), a \in H_*(\Omega SY)$. Now if π_{Sf} is a principal fibration over SY' induced from π by a map $Sf, f : Y' \to Y$, then because of functoriality of predifferentials with respect to induced fibrations we simply get $\phi(d(\pi_{Sf})) = H^*(Sf)(\phi(d(\pi)))$.

From now on assume that ξ has a section, s^n, on the n-skeleton of the base, i.e. $s^n : X^n \to E, \ \ \xi \circ s^n(x) = x, \ x \in X^n$. Let

$$c(s^n) \in C^{n+1}(X; \pi_n(F))$$

be the obstruction cocycle. Consider the homomorphism

$$u^* : C^{n+1}(X; \pi_n(F)) \to C^{n+1}(X; H_n(F))$$

induced by the H.H. for the fibre F

$$u : \pi_*(F) \to H_*(F).$$

On the other hand, let

$$(2) \qquad \rho^* : C^*(X; Hom(RH_*(F), RH_*(F))) \to C^*(X; Hom(RH_*(F), H_*(F)))$$

be a homomorphism induced by the resolution map ρ above. Hence, it induces an isomorphism in cohomology. Using the obvious isomorphism $Hom(\mathbb{Z}, G) = G$ for a group G, we can regard $C^*(X; H_*(F))$ as the subcomplex in $C^*(X; Hom(RH_*(F), H_*(F)))$, since $RH_0(F) = \mathbb{Z}$. In particular, it is shown in [3] that a twisting cochain $h \in d(\xi)$ can be chosen in such a way that for the transgressive components h_0^r one has

$$(3) \qquad h_0^r = 0, \quad r \leq n,$$

$$(4) \qquad \rho^*(h_0^{n+1}) = u^*(c(s^n)).$$

Denoting by $u(\pi_*(F)) = \tilde{\pi}_*(F)$, the subgroup of spherical elements of $H_*(F)$, we show below in addition that there is a twisting cochain $h \in d(\xi)$ with coefficients those homomorphisms which preserve a resolution of the subgroup.

Proposition 2.4. *By hypotheses and notations above let ξ have a section $s^n : X^n \to E$. Then there is a twisting cochain $h \in d(\xi)$ satisfying (3), (4) and, moreover, for each simplex $\sigma \in X^n$, $h^r(\sigma) : RH_*(F) \to RH_*(F)$ preserves the subgroup $R\tilde{\pi}_*(F) \subset RH_*(F)$.*

Proof. Recall that an algorithm for construction of a twisting cochain for a Serre fibration ξ is described in [2] (cf. [18]). We have that ξ defines a colocal system of singular chain complexes over the base X: To each simplex $\sigma \in X$ is assigned the complex

$$(C_*(F_\sigma), \gamma_\sigma), \ F_\sigma = \xi^{-1}(\sigma),$$

and to a pair $\tau \subset \sigma$ the induced chain map

$$C_*(F_\tau) \to C_*(F_\sigma).$$

Then $\sigma \to Hom(RH_*(F), C_*(F_\sigma))$ also forms a colocal system over X. Define, $\mathcal{K}$, canonically as the simplicial cochain complex of X with coefficients in the last colocal system:

$$\mathcal{K} = \{\mathcal{K}^{i,j,t}\}, \ \ \mathcal{K}^{i,j,t} = C^i(X; Hom^{j,t}(RH_*(F), C_*(F_\sigma)))$$

(C_* is regarded as bigraded via $C_{0,*} = C_*, C_{j>0,*} = 0$,). Hence, $\mathcal{K}$ becomes a bicomplex via

$$\mathcal{K}^{r,t} = \prod_{r=i-j} \mathcal{K}^{i,j,t},$$

$$\delta : \mathcal{K}^{r,t} \to \mathcal{K}^{r+1,t}, \quad \delta = d^C + \partial^R,$$

$$\gamma : \mathcal{K}^{r,t} \to \mathcal{K}^{r,t-1}, \quad \gamma = \{\gamma_\sigma\}.$$

For convenience we refer to the gradings of $\mathcal{K}$ as in accordance with those of $\mathcal{H}$. Next we have a natural d.g. pairing (defined by the $\cup$-product and by the composition of homomorphisms in coefficients)

$$(\mathcal{K}, \delta + \gamma) \otimes (\mathcal{H}, \nabla) \to (\mathcal{K}, \delta + \gamma),$$

and, since $\gamma(kh) = \gamma(k)h$, an induced d. g. pairing

$$(\mathcal{K}_\gamma, \delta_\gamma) \otimes (\mathcal{H}, \nabla) \to (\mathcal{K}_\gamma, \delta_\gamma)$$

where

$$(\mathcal{K}_\gamma, \delta_\gamma) = (H(\mathcal{K}, \gamma), \delta_\gamma) = C^*(X; Hom(RH_*(F), H_*(F))).$$

Now consider the following equation

$$(\delta + \gamma)(k) = kh \tag{5}$$

with respect to a pair (k, h),

$$k = k^0 + \cdots + k^r + \cdots, \quad k^r \in \mathcal{K}^{r,r},$$

$$h = h^2 + \cdots + h^r + \cdots, \quad h^r \in \mathcal{H}^{r,r-1}.$$

We also have the following initial conditions:

$$\nabla(h) = -hh, \tag{6}$$

$$\gamma(k^0) = 0, \quad [k^0]_\gamma = e \in \mathcal{K}_\gamma^{0,0}, \quad e = \rho^*(1), \quad 1 \in \mathcal{H}, \tag{7}$$

where we assume that for each generator $a \in R_0\tilde{\pi}_i(F)$, $k_i^0(\sigma^0)(a) \in C_i(F_{\sigma^0})$ is a map, $(S^i, x_0) \to (F_{\sigma^0}, k_0^0(\sigma^0))$, representing $\rho(a)$.

In addition, we require for k that its components (transgressive ones),

$$k_0^r \in C^r(X; Hom(RH_0(F), C_r(F_\sigma))) = C^r(X; C_r(F_\sigma)),$$

be determined by a given section, s^n. This means

$$k_0^r(\sigma^r) = s^n|_{\sigma^r}, \quad \sigma^r \in X^n, \tag{8}$$

or simply we will write $k_0^r = s^r$ if there is no confusion. And for h,

$$h(\sigma)(a) \in R\tilde{\pi}_*(F), \quad a \in R\tilde{\pi}_*(F), \quad \sigma \in X^n. \tag{9}$$

Note that a solution of equation (5) with initial conditions (6), (7) is given in [2], while we show below that the existence of the section for ξ specifies solutions in view of (8) and (9). A construction of a desired pair (k, h) goes inductively on its perturbation degree. First consider $\delta(k^0) \in \mathcal{K}^{1,0}$. By hypothesis (7) we can find some $k^1 \in \mathcal{K}^{1,1}$ with $\gamma(k^1) = -\delta(k^0)$, but we choose k^1 as follows: recall that k^1 may have two components

$$k^1 = k^{1,0,1} + k^{0,-1,1},$$

and then for a generator, $a \in R_0\tilde{\pi}_i(F)$, we set $k_i^1(\sigma^1)(a) \in C_{i+1}(F_{\sigma^1})$ to be a homotopy, $G : S^i \times I \to F_{\sigma^1}$, between $k_i^0(\sigma_0^1)(a)$ and $k_i^0(\sigma_1^1)(a)$ (σ_j^1 is the j-vertex of σ^1) such that

$$G|_{x_0 \times I} = k_0^{1,0,1}(\sigma^1)(= s^n|_{\sigma^1});$$

while for a generator, $b \in R_1\tilde{\pi}_i(F)$, we take $k_i^{0,-1,1}(\sigma^0)(b)$ to be a chain factoring through the map, $\alpha_b : C_{\tilde{b}} \to F_{\sigma^0}$, where $C_{\tilde{b}}$ is the mapping cone of the standard map, $\tilde{b} : S^i \to \vee S^i$, corresponding to the representation of the element $\partial^R(b)$ in the basis of $R_0\tilde{\pi}_i(F)$ and then α_b is induced by the composite $S^i \xrightarrow{\tilde{b}} \vee S^i \xrightarrow{k^0(\sigma^0)(\partial^R(b))} F_{\sigma^0}$. Then we get a cochain (here, for simplicity, we can assume $R_{j>1}H_*(F) = 0$),

$$g^2 = [\delta(k^1)]_\gamma, \; g^2 = \{g_i^2\} \in \mathcal{K}_\gamma^{2,1},$$

with the property that for $a \in R_0\tilde{\pi}_i(F)$, $g^2(\sigma)(a) \in \tilde{\pi}_{i+1}(F)$; indeed, the chain $d^C(k_i^{1,0,0})(\sigma^2)(a)$ factors through a map,

$$\alpha : T_{\sigma^2}^{i+1}(a) \to F_{\sigma^2}, \; T_{\sigma^2}^{i+1}(a) = \partial\Delta^2 \times S^i \cup_{\partial\Delta^2 \times x_0} \Delta^2$$

(Δ^n is the standard n-simplex), and there is a map, $\beta_{\sigma_2} : S^{i+1} \to T_{\sigma^2}^{i+1}(a)$, representing the generator of $H_{i+1}(T_{\sigma^2}^{i+1}(a))$. If $b \in R_1\tilde{\pi}_i(F)$, it is easy to see that the class of the γ-cocycle $(\partial^R(k^{1,0,1}) + d^C(k^{0,-1,1}))(\sigma^1)(b)$ is also spherical.

Now since (2) is an epimorphism and a cohomology isomorphism at the same time, there is some ∇-cocycle, $h^2 \in \mathcal{H}^{2,1}$, with $\rho^*(h^2) = g^2$ and $h^2(\sigma)(a) \in R_*\tilde{\pi}_*(F)$ for $a \in R_*\tilde{\pi}_*(F)$, $\sigma \in X^n$.

Next we consider $k^0h^2 - \delta(k^1)$. Clearly, there is some $k^2 \in \mathcal{K}^{2,2}$ with

$$\gamma(k^2) = k^0h^2 - \delta(k^1),$$

but we choose k^2 as follows: for a generator $a \in R_0\tilde{\pi}_i(F)$ we set $k_i^2(\sigma^2)(a)$ to be a chain factoring through a map $Z_1^2 \cup_{S^{i+1}} Z_2^2 \to F_\sigma$ which itself is formed by the maps (chains)

$$\alpha_j : Z_j^2 \to F_{\sigma^2}, \; j = 1, 2,$$

where Z_1^2 is the (reduced) mapping cylinder of the map β_{σ^2} above and Z_2^2 is that of the map $S^{i+1} \to \vee S^{i+1}$ corresponding to the representation of the element $h^2(\sigma^2)(a)$ in the basis of $R_0\tilde{\pi}_{i+1}(F)$, and then α_j are defined obviously. Analogously, $k_i^2(\sigma^1)(b) \in C_{i+2}(F_{\sigma^1})$ is chosen for $b \in R_1\tilde{\pi}_i(F)$.

Suppose by induction that we have constructed a pair $(k^{(n-1)}, h^{(n-1)})$, $k^{(n-1)} = k^0 + \cdots + k^{n-1}$, $h^{(n-1)} = h^2 + \cdots + h^{n-1}$, satisfying equation (5) and initial conditions (7)-(9) in perturbation degrees $\leq n-1$, while condition (6) in degrees $\leq n$. Then again because of (2) we can define $h^n \in \mathcal{H}^{n,n-1}$ with $\nabla(h^n) = -\Sigma_{i+j=n+1}h^ih^j$ and $\rho^*(h^n) = g^n$, g^n is the class of the γ-cocycle, f^n,

$$f^n = \delta(k^{n-1}) - k^{n-2}h^2 - \cdots - k^1h^{n-1},$$

while $k^n \in \mathcal{K}^{n,n}$ by

$$\gamma(k^n) = k^0h^n - f^n;$$

moreover, for a generator, $a \in R\tilde{\pi}_*(F)$, we have that $h_i^n(\sigma^n)(a) \in R\tilde{\pi}_{i+n-1}(F)$, since one can immediately see that the chain $f_i^n(\sigma^n)(a) \in C_{i+n-1}(F_{\sigma^n})$ factors through a map,

$$\alpha : T_{\sigma^n}^{i+n-1}(a) \to F_{\sigma^n},$$

where the generator of $H_{i+n-1}(T_{\sigma^n}^{i+n-1}(a))$ is defined by a map, $\beta_{\sigma^n} : S^{i+n-1} \to T_{\sigma^n}^{i+n-1}(a)$ (in fact, $T_{\sigma^n}^{i+n-1}(a)$ has the homotopy type of a bouquet of spheres with only one component of S^{i+n-1}). Also k^n is chosen for $a \in R\tilde{\pi}_i(F)$, $\sigma \in X^n$, by a chain $k_i^n(\sigma)(a)$ factoring through the map

$$Z_1^n \cup_{S^{i+n-1}} Z_2^n \to F_\sigma,$$

entirely analogously to k_i^2. Thus, we obtain a pair (k, h) as desired. □

From now on assume that the H.H. for the fibre F is split injective in degrees $\leq n$. Then for a pair (s^n, h), s^n is a section and h is a twisting cochain from the previous proposition (satisfying (3) and (4), in particular), we define a subgroup

$$\Gamma^{n+1}(s^n, h) \subset H^{n+1}(X; \pi_n(F))$$

as follows: first consider the following two short exact sequences of complexes

$$0 \to (C_n^*, d) \to (C_{(n)}^*, d_h) \to (C_{(n-1)}^*, d_h) \to 0$$

and

$$0 \to (L_n^*, d) \to (L_{(n)}^*, d_\nu) \to (L_{(n-1)}^*, d_\nu) \to 0$$

in which

$$C_n^k = \prod_{j\geq 0} C^{k+j+n}(X; R_j H_n(F)),$$

d is the total differential of the bicomplex defined by the d^C and the ∂^R,

$$\begin{aligned}
C_{(n)}^k &= \textstyle\prod_{j\geq 0}\prod_{q=1}^n C^{k+j+q}(X; R_j H_q(F)),\\
d_h &= d + h\cup_,\\
L_n^k &= \textstyle\prod_{j\geq 0} C^{k+j+n}(X; R_j \pi_n(F)),\\
L_{(n)}^k &= \textstyle\prod_{j\geq 0}\prod_{q=1}^n C^{k+j+q}(X; R_j \pi_q(F)),\\
d_\nu &= d + \nu\cup_,
\end{aligned}$$

ν is a *spherical twisting cochain*, the restriction of $h^{(n)}$ to $Hom(R\pi_{\leq n}(F), R\pi_{\leq n}(F))$. Here we identify the i-homotopy group, $i \leq n$, with its image under the H.H., so we can regard $L_{(n)}^*$ as the subcomplex of $C_{(n)}^*$.

Consider the boundary operators of the corresponding long exact sequences

$$\delta_n : H^0(C_{(n-1)}, d_h) \to H^1(C_n, d) = H^{n+1}(X; H_n(F))$$

and

$$\delta_n' : H^0(L_{(n-1)}, d_\nu) \to H^1(L_n, d) = H^{n+1}(X; \pi_n(F)).$$

Then we set

$$\Gamma^{n+1}(s^n, h) = \delta_n(N),$$

where N is itself the following subgroup of $H^0(C_{(n-1)}, d_h)$: observing that some fixed splitting $H_i(F) = \pi_i(F) \oplus \bar{H}_i(F)$, $i \leq n$, (compatible with the H.H.) induces a decomposition $C_{(n)}^* = L_{(n)}^* \oplus B_{(n)}^*$ of the graded group $C_{(n)}^*$, we have a sequence of homomorphisms (defined below)

$$\begin{aligned}
\phi_1 &: \{a_{(2)} \in L_{(2)}^0 | d_\nu(a_{(2)}) = 0\} \to B_3^0,\\
\phi_2 &: \{a_{(3)} \in L_{(3)}^0 | d_\nu(a_{(3)}) = 0\} \to B_4^0, \ldots,\\
\phi_i &: \{c_{(i+1)} \in C_{(i+1)}^0 | d_h(c_{(i+1)}) = 0,\ c_{(j+1)} = a_{(j+1)} + b_{(j)},\ a_{j+1} \in L_{j+1}^0,\ b_j \in B_j^0,\\
&\quad b_j = \phi_{j-2}(a_{(j-1)} + b_{(j-2)}),\ 0 \leq j \leq i\} \to B_{i+2}^0, \quad i = 1, \ldots, n-2,
\end{aligned}$$

and, by definition,

$$N = \{[c_{(n-1)}] \in H^0(C_{(n-1)}, d_h) | d_h(c_{(n-1)}) = 0,\ c_{(n-1)} = a_{(n-1)} + b_{(n-1)},\ b_j = \phi_{j-2}(a_{(j-1)} + b_{(j-2)}),\ 3 \le j \le n-1\}.$$

Now we begin the construction of the ϕ_i by induction. Consider the fibration ξ' over $X \times I$ induced from ξ by the projection $X \times I \to X$. To define ϕ_1 first we consider equation (5) for ξ' and show that for a d_ν-cocycle $a_{(2)} \in L^0_{(2)}$ (i.e., $d(a_1) = 0, d(a_2) = \nu^2 a_1$ if $a_{(2)} = a_1 + a_2, a_j \in L^0_j$) there is a solution (k', h') such that $(k', h')|_{X \times 0} = (k, h)$, a solution of (5) for ξ, $k'_0|_{X^3 \times 1} = s'^3$, a section $X^3 \to E$ of ξ, $h'^{(3)}_0(\sigma \times I) = a_{(2)}(\sigma)$. Fix the pair (k, h) on $X \times 0$. In fact, we are interested in transgressive components of an extension pair of (k, h). For convenience, we use here for the space $X \times I$, the cochain complex (algebra) $C^*(X \times I; \cdot)$ corresponding to the standard cellular decomposition of the cylinder. Put, for example, $k'^0_0(X^0 \times 1) = k^0_0(X^0 \times 0)$, $k'^1_0(\sigma^0 \times I) = *$, the constant map at the vertex σ^0, while realize a cochain $k'^1_0(X^1 \times 1)$ as a section $s'^1 : X^1 \to E$ of ξ such that the difference cochain, $d(s^1, s'^1)$, is just $\rho^*(a_1)$. Then we will have

$$g'^2_0(\sigma^1 \times I) = [\delta(k'^1_0)]_\gamma(\sigma^1 \times I) = \rho^*(a_1)(\sigma^1),$$

and we can take $h'^2_0(\sigma \times I) = a_1(\sigma)$. Next define $k'^2_0(\sigma^1 \times I)$ as a homotopy between $\delta(k'^1_0)$ and $k^0_1 h'^2_0(\sigma^1 \times I)$, while $k'^2_0(X \times 1) = s'^2$, a section $s'^2 : X^2 \to E$ extending s'^1 (such section exists, since $0 = d^C(d(s^1, s'^1)) = c(s'^1)$); moreover, we can choose s'^2 with $g'^3_0(\sigma^2 \times I) = [\delta(k'^2_0)]_\gamma(\sigma^2 \times I) = \rho^*(a_2)(\sigma^2)$, and then define $h'^3_0(\sigma \times I) = a_2(\sigma)$. Also we have $c(s'^2) = 0$, so s'^2 extends to a section, $s'^3 : X^3 \to E$, and choose s'^3, i.e., $k'^3_0(X^3 \times 1)$, with $g'^4_0(\sigma^3 \times I) = [\delta(k'^3_0)]_\gamma(\sigma^3 \times I)|_{\pi_3(F)} = 0$. Then (recalling $\rho^*(h'^r) = g'^r$) define ϕ_1 with

$$\phi_1(a_{(2)})(\sigma) = h'^4_0(\sigma \times I), \quad \sigma \in X^n.$$

Suppose we have defined ϕ_j for $j = 1, \ldots, i-1$, and for the d_h-cocycle $c_{(i)} \in C^0_{(i)}$, a pair (k', h') with $k'^{i+1}_0(X^{i+1} \times 1) = s'^{i+1}$, a section $s'^{i+1} : X^{i+1} \to E$, and $h'^{(i+1)}_0(\sigma \times I) = c_{(i)}(\sigma)$. Then for d_h-cocycle $c_{(i+1)}$, we have that

$$c(s'^{i+1}) = \rho^*(h'h'_0)^{i+2} = \rho^*(hc_{(i)})^{i+2} = -\rho^*(d(a_{i+1})) = d^C\rho^*(a_{i+1}).$$

So we can canonically change s'^{i+1} by $\rho^*(a_{i+1})$ to define a section s'^{i+2}, i.e., $k'^{i+2}_0(X^{i+2} \times 1)$, with

$$g'^{i+3}_0(\sigma^{i+2} \times I) = [\delta(k'^{i+2}_0)]_\gamma(\sigma^{i+2} \times I)|_{\pi_{i+2}(F)} = 0.$$

Next define ϕ_i by

$$\phi_i(c_{(i+1)})(\sigma) = h'^{i+3}_0(\sigma \times I), \quad \sigma \in X^n.$$

Thus, the construction of ϕ_i's is finished.

Note that N does not depend on a splitting of $H_*(F)$ above; moreover, if h has the property that for $a \in R\bar{H}_{\le n}(F)$, $h(\sigma)(a) \in R\bar{H}_*(F)$, $\sigma \in X^n$, then it is easy to see that

$$(10) \qquad \Gamma^{n+1}(s^n, h) = \delta'_n(H^0(L_{(n-1)}, d_\nu)).$$

Now let $O^{n+1}(\xi)$ denote as in §1. Then we have the following main theorem:

Theorem 2.5. *By hypotheses and notations above in a fibration ξ with a section $s^n : X^n \to E$ suppose the H.H. $\pi_i(F) \to H_i(F)$ is split injective for $i \le n$. Then*

(i)

$$O^{n+1}(\xi) = o(s^n) + \Gamma^{n+1}(s^n, h);$$

(ii) If in addition the twisting cochain h satisfies $h(\sigma)(a) \in R\bar{H}_{\leq n}(F)$ for $a \in R\bar{H}_{\leq n}(F)$, $\sigma \in X^n$, then

$$O^{n+1}(\xi) = o(s^n) + \delta'_n(H^0(L_{(n-1)}, d_v)).$$

Proof. (i) Let $s'^n : X^n \to E$ be another section of ξ. Consider the fibration ξ' over $X \times I$ (see the construction of the ϕ's above) and equation (5) for it. In the initial conditions we fix the solution (k, h) for ξ on $X \times 0$ and $k'^n_0 = s'^n$ on $X^n \times 1$. Then for an obtained twisting cochain $h' \in d(\xi)$, by putting

$$c_{(n-1)}(\sigma) = h'^{(n)}_0(\sigma \times I),$$

we will have that $d_h(c_{(n-1)}) = 0$ in $C_{(n-1)}$, $[c_{(n-1)}] \in N$, and

$$o(s'^n) = o(s^n) + \delta_n([c_{(n-1)}]).$$

Conversely, if $v \in I^{n+1}(s^n, h)$, then the definition of N shows that there is a section $s''^n : X^n \to E$ such that

$$o(s''^n) = o(s^n) + v.$$

(ii) Follows from (i), because of (10). □

From this it easily follows that the subgroup $I^{n+1}(s^n, h)$ is uniquely determined by the fibration ξ and we denote it by $I^{n+1}(\xi)$. Hence, $O^{n+1}(\xi)$ defines the element in the quotient group $H^{n+1}(X; \pi_n(F))/I^{n+1}(\xi)$, and then there is

Corollary 2.6. *There exists a section of ξ on X^{n+1} if and only if $O^{n+1}(\xi) = 0$.* □

Note that $I^{n+1}(\xi)$ has a filtration by the subgroups

$$0 = I^{n+1}_0(\xi) \subset I^{n+1}_1(\xi) \subset \cdots \subset I^{n+1}_{n-1}(\xi) = I^{n+1}(\xi),$$

as follows. Consider a filtration of the $C^*_{(n-1)}$ by the subcomplexes $(C^k_{m,n-1}, d')$, $m < n$, where $C^k_{m,n-1} = \prod_{j\geq 0}\prod_{q=n-m-1}^{n-1} C^{k+j+q}(X; R_jH_q(F))$ and d' is the restriction of d_h to $C^*_{m,n-1}$. Then we have the induced filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_{n-1} = N$ of the N, and $I^{n+1}_m(\xi) = \delta_n(N_m)$. It is easy to see from the proof of Theorem 2.5 that we have the following

Theorem 2.7. *Let ξ be a fibration as in Theorem 2.5. Then there is a section, s', of ξ on X^{n+1} with $s'|_{X^m} = s^n|_{X^m}$, $m < n$ if and only if $o(s^n) \in I^{n+1}_{n-m-1}(\xi)$.*

The following theorem provides the case when condition (3) is also sufficient for the existence of a section.

Theorem 2.8. *Let ξ be as in the part (ii) of Theorem 2.5 and let the fibre of ξ have the homotopy type of a product of Eilenberg-MacLane spaces. Then*

(i) ξ has a section on X^{n+1} if and only if there is a twisting cochain $h \in d(\xi)$ with zero transgressive components in perturbation degrees $\leq n+1$, i.e., $h^r_0 = 0$, $r \leq n+1$;

(ii) ξ has a section (on X) if and only if $h_0 = 0$.

Proof. (i) Given a section of ξ on X^n, the existence of such twisting cochain follows from Proposition 2.4. Conversely, let $o(s^r), r \geq 1$, be the first non-zero obstruction class (the Euler class) to the extension of s^r on X^{r+1}. Then $o(s^r) = \rho^*(\bar{h}^{r+1}_0)$ and $\bar{h}^i_0 = 0, i \leq r$, for some $\bar{h}$ (cf. (3), (4)); moreover, we can choose $\bar{h}$ with $\bar{h}(\sigma)(a) \in R\bar{H}_{\leq n}(F)$, for $a \in R\bar{H}_{\leq n}(F), \sigma \in X^n$, too. Now consider equation (5) for the fibration ξ' over $X \times I$, and fix the solution (k, h) on $X \times 0$, while the solution $(\bar{k}, \bar{h})$ on $X \times 1$. Then we obtain a twisting cochain $h' \in d(\xi')$ such that for $a_{(n-1)}(\sigma) = h'^{(n)}(\sigma \times I)|_{\pi_*(F)}$, we will have that $d_\nu(a_{(n-1)}) = 0$ in $L_{(n-1)}$, and

$$o(s^n) = \delta'_n([a_{(n-1)}]).$$

Thus by Theorem 2.5 (ii) a section exists on X^{n+1};

(ii) From the proof of the existence of a pair (k, h) for equation (5), one derives that the sections s^n provided by (i) can be chosen with $s^{n+1}|_{X^n} = s^n$. So a global section on X is defined. □

Now we will consider a question about the homotopy classification of sections. We have the following classification theorem:

Theorem 2.9. *Let ξ be as in the part (ii) of Theorem 2.5. Then there is a bijection*

$$[X^{n-1}, E]_s \approx H^0(L_{(n-1)}, d_\nu),$$

where $[\ ,\ ]_s$ denotes the set of homotopy classes of sections.

Proof. First define the map

$$\psi : [X^{n-1}, E]_s \longrightarrow H^0(L_{(n-1)}, d_\nu)$$

as follows: Let $s'^n : X^n \longrightarrow E$ be another section of ξ. Consider the fibration ξ' over $X \times I$ induced from ξ by the projection $X \times I \longrightarrow X$. Then we consider equation (5) for ξ' with the initial conditions where we fix a given solution (k, h) for ξ on $X \times 0$, while $k'^n_0 = s'^n$ on $X \times 1$. Let (k', h') be an obtained solution for ξ', where we require by the choice of k' that

$$k'^{i+1}_0(\sigma^i \times I) = \chi(f_0^{i+1})(\sigma^i \times I)$$

in which f^{i+1} is as in the proof of Proposition 2.4 and χ is some fixed homomorphism,

$$\chi : C^{i+1}(X \times I; ZC_i(F_\sigma)) \longrightarrow C^{i+1}(X \times I; C_{i+1}(F_\sigma)),$$

defined by $\gamma(\chi(c^{i+1}))(\sigma^{i+1}) = c^{i+1} - k^0[c^{i+1}]_\gamma$, ZC denotes the cycles of C. Then define a sequence of cochains

$$\{a_{(n-1)}\}_{s'}, \ a_j \in L_j^0,$$

by $a_j(\sigma) = h'^{j+1}_0(\sigma \times I)|_{R\pi_j(F)}$. It is easy to see that condition (6) for h' implies

$$d_\nu\{a_{(n-1)}\}_{s'^n} = 0.$$

in $L_{(n-1)}$. Moreover, if t'^n is a section homotopic to s'^n, then we consider the fibration ξ'' over $X \times I \times I$ induced from ξ by the projection $X \times I \times I \longrightarrow X$. Consider again equation (5) for ξ'' with the initial conditions where we fix the solution (s'^n, h') for ξ' on $X \times I \times 0$, the solution (t'^n, h') for ξ' on $X \times 0 \times I$, a homotopy between s'^n and t'^n on $X \times I \times 1$, the constant homotopy for s'^n on $X \times 1 \times I$. Let $h'' \in d(\xi'')$ be an obtained twisting cochain. Then for cochains, $\theta_j \in L_j^{-1}$, defined by

$$\theta_j(\sigma) = h''^{j+2}_0(\sigma \times I \times I)|_{R\pi_j(F)}, \ 0 \le j \le n-1,$$

we have that

$$d_\nu(\theta_{(n-1)}) = \{a_{(n-1)}\}_{s'^n} - \{a_{(n-1)}\}_{t'^n}.$$

Thus the assignment $s'^n \longrightarrow \{a_{(n-1)}\}_{s'^n}$ induces the map ψ above.

Conversely, we assign to a d_ν-cocycle, $a_{(n-1)} \in L^0_{(n-1)}$, a section of ξ on X^n as follows. We have that the argument of the proof of Theorem 2.5 defines a section, $s'^n : X^n \longrightarrow E$ (up to homotopy, since we again use the fixed homomorphism χ above), such that if we fix s^n and s'^n respectively on $X \times 0$ and $X \times 1$ in the initial conditions of (5) for ξ', then there is a twisting cochain $h' \in d(\xi')$ with $h'^{(n)}_0(\sigma \times I)|_{R\pi_*(F)} = a_{(n-1)}(\sigma)$.

If $a_{(n-1)}$ were a d_ν-boundary, then by considering the fibration ξ'' over $X \times I \times I$ we would get that s'^n is homotopic to s^n. Therefore, a map,

$$H^0(L_{(n-1)}, d_\nu) \longrightarrow [X^{n-1}, E]_s,$$

is defined which is obviously the converse of ψ. □

Now we observe the case of an R-formal base of the fibration. A space, X, is said to be R-formal with respect to a ring R if there exist a differential graded algebra, A, over R and maps of differential graded algebras

$$C^*(X;R) \longleftarrow A \longrightarrow H^*(X;R)$$

inducing an isomorphism in cohomology (i.e., $C^*(X;R)$ and $H^*(X;R)$ are weak equivalent as algebras, cf. [1], [9]). Using the argument similar to that of the proof of the comparison theorem for the functor D [2] (see also Theorem 4.1 [18]) the algebra maps above transfer a (spherical) twisting cochain, $h \in \mathcal{H}$, into a (spherical) twisting element, h', in $H^*(X; Hom^*(RH_*(F), RH_*(F)))$ such that we have on X a natural isomorphism

$$H^*(C_{(n)}, d_h) \approx H^*(H(C_{(n)}, d^C), \partial^R_{h'}).$$

So that we can replace singular cochain complexes $C^*(X;\cdot)$ by $H^*(X;\cdot)$ above. Details are left to the reader. In particular, we get from the previous theorem the following

Theorem 2.10. *If ξ is as in Theorem 2.9 and the base of ξ is $\mathbb{Z}$-formal, then there is a bijection*

$$[X^{n-1}, E]_s \approx H^0(H^*(L_{(n-1)}, d^C), \partial^R_{\nu'}),$$

where ν' is induced by the spherical twisting cochain ν.

Remark 2.11. *If the composition of the H.H. with the homomorphism $H_i(F;\mathbb{Z}) \longrightarrow H_i(F;\mathrm{k})$ induced by the canonical map $\mathbb{Z} \longrightarrow \mathrm{k}$ is injective and the image of $\pi_i(F)$ is a* k*-submodule in $H_i(F;\mathrm{k})$ for a field* k*, then we can obviously replace $\mathbb{Z}$ by* k*, and respectively, $RH_*(X;\mathbb{Z})$ by $H_*(X;\mathrm{k})$ in all statements above.*

3. Applications

Throughout of this section by $F \longrightarrow E \xrightarrow{\xi} X$ will be denoted a Serre fibration as at the beginning of §2 and, in addition, the H.H. $\pi_i(F) \longrightarrow H_i(F)$ for the fibre will be required to be split injective in degrees $\leq n$, unless specified otherwise.

First we consider the cases when for the fibration ξ with a section $s^n : X^n \longrightarrow E$, the obstruction cohomological class $o(s^n) \in H^{n+1}(X;\pi_n(F))$ is an invariant of ξ, i.e., does not depend on a section (in our terminology this means that $I^{n+1}(\xi) = 0$ or $O^{n+1}(\xi) = o(s^n)$; cf. [16], [7]).

Theorem 3.1. *Let ξ with a section $s^n : X^n \longrightarrow E$ satisfy one of the following conditions:*

(i) ξ is a principal fibration, i.e., is induced by a map $X \longrightarrow Y$ from the path fibration $\Omega Y \longrightarrow PY \longrightarrow Y$ (with Y simply connected and the H.H. split injective in degrees $\leq n$ for ΩY);

(ii) The base X of ξ is $\mathbb{Z}$-formal with $H^*(X)$ having the trivial multiplication;

(iii) The base X of ξ is k*-formal with $H^*(X;\mathrm{k})$ having the trivial multiplication provided the composition of the H.H. with the canonical homomorphism $H_i(F;\mathbb{Z}) \longrightarrow H_i(F;\mathrm{k})$ is injective and the image of $\pi_i(F)$ is a* k*-submodule in $H_i(F;\mathrm{k})$ for a field* k*, $i \leq n$.*

Then the obstruction cohomological class $o(s^n)$ *is an invariant of* ξ, *i.e.*, $O^{n+1}(\xi) = o(s^n)$.

Proof. (i) Since there is a section s^n, the restriction of ξ to X^n is fibre homotopy equivalent to the trivial fibration, so there is a twisting cochain $h \in d(\xi)$ with $h^r = 0, r \leq n$. Hence, by Theorem 2.5 we get $O^{n+1}(\xi) = o(s^n)$;

(ii) Since $C^*(X)$ and $H^*(X)$ are weak equivalent, we can use the comparison theorem [2] for computation of the set $D(X; H(F))$, and since $H^*(X)$ has the trivial multiplication, we then deduce that there is a bijection (cf. Theorem 2.2)

$$D(X; H_*(F)) \approx \prod_{i>0} H^i(X; G_i),$$

where

$$G_1 = \textstyle\prod_j Ext(H_j(F), H_{j+1}(F)),$$
$$G_i = \textstyle\prod_j Hom(H_j(F), H_{j+i-1}(F)) \oplus Ext(H_j(F), H_{j+i}(F)), i > 1.$$

Hence, we have the inclusion $\prod_{r>0} H^{r+1}(X; H_r(F)) \subset D(X; H_*(F))$. In particular, the class of h_0^{n+1} in $H^{n+1}(X; H_n(F))$ does not depend on a twisting cochain from $d(\xi)$. Then (4) completes the proof.

(iii) Is analogous to (ii) in view of Remark 2.11. □

Remark 3.2. *As we can see from the above proof, in fact, to an arbitrary Serre fibration over the base satisfying (ii) (or (iii)) of the theorem, a sequence of elements,* $b_i \in H^i(X; H_{i-1}(F)), i = 2, 3, \ldots,$ *determined by the transgressive component* h_0 *of a twisting cochain* h *of the fibration, is assigned. If the fibration has a section,* s^n, *on the* n*-skeleton, then it follows that* $b_i = 0$, $i \leq n$, *and* $b_{n+1} = u^*(o(s^n))$, *where* $u^* : H^*(X; \pi_*(F)) \longrightarrow H^*(X; H_*(F))$. *If, in addition, the condition of the theorem for the fibre holds (or, more generally, if the homomorphism* u^* *is only injective!), then the triviality of the invariants* b_i *is also sufficient for the existence of a section (cf. Corollary 2.6).*

Corollary 3.3. *Let* ξ *with a section* s^n *have the base* X *being a suspension over a path connected space. Then* $O^{n+1}(\xi) = o(s^n)$.

Proof. In view of Theorem 2.2 the proof is similar to that of Theorem 3.1 (ii). However, the proof of Theorem 2.2 shows that a suspension, SY, is R-formal for any principal ideal domain R. In fact, since there is the trivial subalgebra $C^*(SY, C_+Y; R) \subset C^*(SY; R)$, we have R-module maps

$$C^*(SY, C_+Y; R) \longleftarrow RH^*(SY; R) \longrightarrow H^*(SY; R),$$

where in the middle there is a free R-module resolution of the R-module $H^*(SY; R)$, being algebra maps at the same time (by regarding the complexes with the trivial multiplication) and inducing isomorphisms in cohomology. This means that $C^*(SY; R)$ and $H^*(SY; R)$ are weak equivalent as algebras, so SY is R-formal. □

It is of interest to decide when the elements b_i in Remark 3.2 are natural with respect to induced fibrations. It follows from Theorem 2.2 that this is satisfied for a fibration induced by a suspension map; but not, for example, for the Hopf map $f : S^3 \longrightarrow S^2$, since we have $b_3(\pi) = 0$ but $b_3(\pi_f)$ is the generator of $H^3(S^3; H_2(\Omega S^2)) = \mathbb{Z}$, where π_f denotes the induced fibration from the path fibration π by f (cf. Example 2.3).

Thus from Theorem 2.2 and Remark 3.2 follows

Theorem 3.4. *Let $f : Y' \longrightarrow Y$ be a map, where Y' is a polyhedron of dimension $\leq n+1$ and the homomorphism $u^* : H^i(Y'; \pi_i(\Omega SY)) \longrightarrow H^i(Y'; H_i(\Omega SY))$ is injective for $i \leq n$ (e.g. $H^i(Y')$ and $\pi_i(\Omega SY)$ are free groups for $i \leq n$). Then the suspension map Sf is homotopic to zero if and only if $0 = f^* : H^*(Y; G) \longrightarrow H^*(Y'; G)$ for all finitely generated groups G.*

In particular, since the loop space of any simply connected rational space has the homotopy type of a product of Eilenberg -MacLane spaces, one gets the result of [15]:

Corollary 3.5. *For a map $f : Y' \longrightarrow Y$, the suspension map Sf is rationally homotopic to zero if and only if $0 = f^* : H^*(Y; \mathbb{Q}) \longrightarrow H^*(Y'; \mathbb{Q})$.*

For the homotopy classification of sections, we have considered in [17] the free loop fibration $\Omega X \longrightarrow \Lambda X \longrightarrow X$, since it satisfies the hypotheses of Theorem 2.9 at least X is rational. Here we want to state the following theorem which immediately follows from Theorem 3.1.

Theorem 3.6. *Let ξ be as in Theorem 3.1 with a simply connected fibre. Then there is a bijection*

$$[X^{n-1}, E]_s \approx \prod_{i=1}^{n-1} H^i(X; \pi_i(F)).$$

In particular, for the free loop fibration over a suspension we get

Corollary 3.7. *Let $\Omega SY \longrightarrow \Lambda SY \longrightarrow SY$ be the free loop fibration with Y simply connected and the H.H. split injective in degrees $\leq n$ for ΩSY. Then there is a bijection*

$$[(SY)^n, \Lambda SY]_s \approx \prod_{i=1}^{n} H^i(SY; \pi_{i+1}(SY)).$$

From now on we consider the case of ξ a fibre bundle $F \longrightarrow E \times_G F \xrightarrow{\xi} X$ associated with a principal G-bundle $G \longrightarrow E \xrightarrow{\zeta} X$ by an action $G \times F \longrightarrow F$. Then computations with a twisting cochain of ξ are provided by the method of [14] which yields a special twisting cochain in the Hirsch complex of ξ involving the A_∞-module structure on $H_*(X; \mathrm{k})$ (assuming k is a field). Namely, the action above induces not only the natural pairing

$$\mu_2 : H_*(G; \mathrm{k}) \otimes H_*(F; \mathrm{k}) \longrightarrow H_*(F; \mathrm{k}),$$

but a sequence of higher order pairings

$$\mu_n : \otimes^{n-1} H_*(G; \mathrm{k}) \otimes H_*(F; \mathrm{k}) \longrightarrow H_*(F; \mathrm{k}), \quad n = 2, 3, \ldots,$$

which for $F = G$ convert into A_∞-algebra structure in the sense of Stasheff on $H_*(G; \mathrm{k})$ [21]:

$$m_n : \otimes^n H_*(G; \mathrm{k}) \longrightarrow H_*(G; \mathrm{k}), \quad n = 2, 3, \ldots.$$

To the principal G-bundle ζ a cochain,

$$\varphi \in \prod_{i>0} C^{i+1}(X; H_i(G; \mathrm{k})), \tag{11}$$

satisfying

$$d^C(\varphi) = \Sigma_{n=1}^{\infty} m_n^*(\varphi \cup \cdots \cup \varphi)$$

(where $m_n^* : C^*(X; \otimes^n H_*(G;\mathrm{k})) \to C^*(X; H_*(G;\mathrm{k}))$ is the map induced by m_n), a twisting cochain in the sense of [14], is assigned. It actually defines an ordinary twisting cochain $h \in d(\xi)$ by

$$h(\sigma)(a) = \Sigma_{n>1}\mu_n((\varphi \cup \cdots \cup \varphi)(\sigma) \otimes a), \ \sigma \in X, \ a \in H_*(F;\mathrm{k}).$$

Now we have the following

Theorem 3.8. *Let F be a path connected G-space having the homotopy type of a product of Eilenberg-MacLane spaces, $K(\pi_n, n)$'s, π_n a vector space over a fixed field* k, *G a path connected topological group, and let for an induced A_∞-module structure on $H_*(F;\mathrm{k})$,*

$$\mu_n(\otimes^{n-1}\bar{H}(G;\mathrm{k}) \otimes \bar{H}_{>0}(F)) \subset \bar{H}_*(F), \ n = 2, 3, \ldots,$$

for some decomposition $H_*(F;\mathrm{k}) = \pi_*(F) \oplus \bar{H}_*(F)$ compatible with the H.H.. Then

(i) The fibre bundle $F \to E \times_G F \to X$ associated with the principal G-bundle $G \to E \to X$ has a section if

$$\mu_n(\otimes^{n-1}\bar{H}(G;\mathrm{k}) \otimes 1) = 0, \ 1 \in H_0(F;\mathrm{k}) = \mathrm{k}, \ n = 2, 3, \ldots ;$$

(ii) The fibre bundle $F \to EG \times_G F \to BG$, where $G \to EG \to BG$ is the universal bundle for G, has a section if and only if the condition of (i) holds.

Proof. (i) From the hypotheses of the theorem and the definition of the twisting cochain by the A_∞-module structure above we have that there is a twisting cochain of ξ satisfying the condition of Theorem 2.8, so the statement follows;

(ii) If the fibre bundle has a section, then it is not hard to see that there is some G-map $EG \to F$, so that the naturality of an A_∞-module structure with respect to G-maps finishes the proof. □

Computations with this theorem simplifies in the case $H^*(BG;\mathrm{k}) = \mathrm{k}[w_1, w_2, \ldots]$ is polynomial. In fact, denoting by $\bar{w}$ the dual of the transgressive element of w we will have that the Pontrjagin algebra $H_*(G;\mathrm{k}) = \mathbb{E}[\bar{w}_1, \bar{w}_2, \ldots]$ is an exterior one (cf. [11]) and then using k-formality of BG one can easily show that the twisting cochain (11) for the universal bundle $G \to EG \to BG$ is defined by the following twisting element

$$\Sigma_{n>1} w_n \otimes \bar{w}_n \in H^{*+1}(BG;\mathrm{k}) \hat{\otimes} H_*(G;\mathrm{k}).$$

Consequently, we have

Theorem 3.9. *By the hypotheses and notations above let for a G-space F,*

$$\mu_n(\bar{w}_1 \otimes \cdots \otimes \bar{w}_{n-1} \otimes \bar{H}_{>0}(F)) \subset \bar{H}_*(F), \ n = 2, 3, \ldots .$$

Then the fibre bundle $F \to EG \times_G F \to BG$ has a section if and only if

$$\mu_n(\bar{w}_1 \otimes \cdots \otimes \bar{w}_{n-1} \otimes 1) = 0, \ 1 \in H_0(F;\mathrm{k}) = \mathrm{k}, \ n = 2, 3, \ldots .$$

An action $F \times G \to F$ is said to have a *homotopy fixed point* if there is a G-map $EG \to F$. It is easy to verify that this is equivalent to the existence of a section in the associated fibre bundle. So, Theorem 3.9 implies the following

Theorem 3.10. *Let F be a G-space as in Theorem 3.9. Then the action has a homotopy fixed point if and only if*

$$\mu_n(\bar{w}_1 \otimes \cdots \otimes \bar{w}_{n-1} \otimes 1) = 0, \ 1 \in H_0(F;\mathrm{k}) = \mathrm{k}, \ n = 2, 3, \ldots .$$

Note that any action $G \times F \to F$ induces a diagonal one on $SP^n F$, the nth symmetric power of F, and then the assiciated fibre bundle provides an example of a fibration with the H.H. for the fibre split injective in degrees $< n$, in view of [6] (see also [5]).

Remark 3.11. *Using the technique similar to that of* [13], *one can extend the A_∞-module structure to the integral homologies of G-spaces and, consequently, the statements above to the integral coefficients.*

References

1. D. Anick, *A model of Adams-Hilton type for fibre squares*, Ill. J. Math., **29** (1985), 463–502
2. N. Berikashvili, *On the differentials of spectral sequences (in Russian)*, Proc. Tbilisi Math. Inst., **51** (1976), 1–105
3. ———, *On the obstruction theory in fibre spaces (in Russian)*, Bull. Acad. Sci. Georgian SSR, **125** (1987), 473–475
4. E.H. Brown, *Twisted tensor products,I*, Ann. Math., **69** (1959), 223–246.
5. A. Dold, *Ramified coverings, orbit projections and symmetric powers*, Math. Proc. Camb. Phil. Soc., **99** (1986), 65–72
6. A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische producte*, Ann. Math., **67** (1958), 239–281
7. A. Dold and H. Whitney, *Classification of oriented sphere bundles over 4-complex*, Ann. Math., **69** (1959), 667–677
8. V.K.A.M. Gugenheim, *On the chain complex of a fibration*, Ill. J. Math., **16** (1972), 398–414
9. V.K.A.M. Gugenheim, L.A. Lambe, and J.D. Stasheff, *Algebraic aspects of Chen's twisting cochain*, Ill. J. Math., **34** (1990), 485–502
10. V.K.A.M. Gugenheim and H.J. Munkholm, *On the extended functoriality of Tor and Cotor*, J. of Pure and Appl. alg., **4** (1974), 9–29
11. S. Halperin and J.D. Stasheff, *Differential homological algebra in its own rite*, Proc. Adv. Study Inst. Alg. Top., Aarhus (1970)
12. G. Hirsch, *Sur les groups d'homologies des espaces fibres*, Bull. Soc. Math. de Belg., **6** (1953), 76–96
13. J. Huebschmann and T. Kadeishvili, *Small models for chain algebras*, Math. Z., **207** (1991), 245–280
14. T. Kadeishvili, *The predifferential of a twisted product*, Russian Math. Survey, **41** (1986), 135–147
15. J.-M. Lemaire and F. Sigrist, *Sur les invariants d'homotopie rationnelle lies a la L.S. categorie*, Comment. Math. Helv., **56** (1981), 103–122
16. L. Pontrjagin, *Classification of some skew products*, Dokl. Acad. Nauk. SSSR, **47** (1945), 322–325
17. S. Saneblidze, *Homotopy classification of sections in the free loop fibration*, J. of Pure and Appl. alg., to appear
18. ———, *The set of multiplicative predifferentials and the rational cohomology algebra of fibre spaces*, J. of Pure and Appl. alg., **73** (1991), 277–306
19. ———, *Filtered model of a fibration and rational obstruction theory*, Manuscripta Math., **76** (1992), 111–136
20. M. Schlessinger and J.D. Stasheff, *Deformation theory and rational homotopy type*, preprint
21. J.D. Stasheff, *Homotopy associativity of H-spaces,I,II*, Trans. Amer. Math. Soc., **108** (1963), 275–312
22. D. Tanre, *Homotopie rationnelle: models de Chen, Quillen, Sullivan*, Lect. Notes in Math., **1025** (1982)
23. E. Thomas, *Postnikov invariants and higher order cohomology operations*, Ann.Math., **85** (1967), 184–217
24. J.-C. Thomas, *Rational homotopy of Serre fibrations*, Ann. Inst. Fourier, **31** (1981), 71–90

Mathematical Institute of the
Georgian Academy of Sciences,
Z.Rukhadze Str.1, Tbilisi, 380093,
Republic of Georgia

(Received January 3, 1993;
in revised form June 5, 1993)

manuscripta math. 81, 113 - 127 (1993)

manuscripta mathematica

Some loci in Teichmüller space for genus seven defined by vanishing thetanulls*

Robert D. M. Accola

Let $\theta[\varepsilon](u)$ be a theta function for a Riemann surface W of genus seven. Suppose $\theta[\varepsilon](u)$ vanishes at $u = 0$ for 4 half-integer theta characteristics $[\varepsilon_i]$, $i = 1,2,3,4$ to orders 2, 2, 2, and 3 respectively and $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) = (0)$. Then W is hyperelliptic, elliptic-hyperelliptic or W lies in the closure of a locus in Teichmüller space of Riemann surfaces which admit plane models where the four half-canonical linear series corresponding to the theta-vanishings are clearly evident.

1. Introduction

Let W_p be a compact Riemann surface of genus p. Let $\mathcal{T}_p$, $\mathcal{H}_p$, $(\mathcal{E} - \mathcal{H})_p$ stand for, respectively, Teichmüller space for genus p, the hyperelliptic locus in $\mathcal{T}_p$, and the elliptic-hyperelliptic locus in $\mathcal{T}_p$. Our primary interest is the case $p = 7$. Then $\mathcal{H}_7$ has pure codimension 5 in $\mathcal{T}_7$, and $(\mathcal{E} - \mathcal{H})_7$ has pure codimension 6. In this paper we shall give local defining equations for $\mathcal{H}_7$, $(\mathcal{E} - \mathcal{H})_7$, and a third locus, $\mathcal{N}_7$, in terms of the vanishing properties of the theta function.

The vanishing properties are as follows. Let a canonical homology basis be chosen for W_7, and so corresponding theta functions with half-integer theta characteristics, $\theta[\varepsilon](u)$, are defined. Let $[\varepsilon_i]$, $i = 1,2,3,4$, be theta characteristics so that (i) $\theta[\varepsilon_i](u)$ vanishes at $u = 0$ to orders 2, 2, 2, and 3 for $i = 1,2,3,4$, and (ii) $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) = (0)$.

By Riemann's vanishing theorem ([12, p. 459], and [13]), these vanishing properties are equivalent to the existence on W_7 of 4 complete half-canonical linear series, g_6^1, h_6^1, k_6^1, and ℓ_6^2 whose sum is bicanonical. Such a set of 4 half-canonical linear series will be called a *quartet* and ℓ_6^2 will be called the *leader*. We shall state our theorem in terms of the existence of a quartet.

By the known vanishing properties of theta functions for hyperelliptic and elliptic-hyperelliptic Riemann surfaces of genus 7 ([12, p. 459], [10, Ch. VII], and [3, p. 51]) many

* The author thanks Alan Landman for valuable discussions concerning the material in this paper
A.M.S. Mathematical Subject Classification 1991. Primary 14H45. Secondary 30F10. Key Words: Riemann surface; theta function; half-canonical linear series

quartets exist on such surfaces. The third locus, $\mathcal{N}_7$, will be the closure in $\mathcal{T}_7$ of Riemann surfaces which have a distinctive plane model which will now be described.

Let A_1, A_2, A_3 and A_4 be the 4 points in $\mathbf{P}^2(C)$, $(\pm 1, \pm 1, 1)$. Let P, Q and R be the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, the 3 diagonal points of the quadrangle $A_1A_2A_3A_4$. Let $\mathcal{C}_9$ be a plane curve of degree 9 with singularities only at these 7 points, and each singularity is an ordinary singularity of multiplicity 3, or any other singularity of multiplicity 3 which contributes 3 to the δ-invariant of $\mathcal{C}_9$. $\mathcal{C}_9$ has genus 7 and the dimension of such curves, $\mathcal{C}_9$, in $\mathcal{T}_7$ is 12. The quartet is as follows. The leader, ℓ_6^2, is cut out by cubics passing through all 7 singular points. The other 3 linear series are cut out by lines passing through the 3 diagonal points P, Q, and R. The closure in $\mathcal{T}_7$ of Riemann surfaces admitting such models will be denoted $\mathcal{N}_7$. Models of the above type will be called *general* in contradistinction to *non-general* models now to be described.

Consider a general model as described in the preceding paragraph. Perform an elementary quadratic transformation with fundamental points A_2, A_3, and A_4. The transformed curve, $\mathcal{C}_9'$, is again of degree 9 with an ordinary singularity of multiplicity 3 at A_1' (the transform of A_1) and with (what we shall call) a $(3,6)$-point at P', Q', and R' (the transforms of P, Q, and R.)

Definition. A $(3,6)$-*point* is a singularity of multiplicity 3 where all branches have a common tangent (the $(3,6)$-*tangent*) which intersects the curve at least 6 times at the singularity. (A $(3,6)$-point will contribute at least 6 to the δ-invariant.)

On $\mathcal{C}_9'$ all 3 of the $(3-6)$-tangents will pass through A_1'. The points P', Q', and R' are not collinear. ℓ_6^2 is cut out by cubics through all 4 singularities and tangent at P', Q', and R' to the (3-6)-tangents. g_6^1 will be cut out by conics through P', Q', and R' and tangent to the (3-6)-tangent at P'. Similarly for h_6^1 and k_6^1.

But for a plane curve like $\mathcal{C}_9'$, there is no reason why the 3 points P', Q', and R' cannot be collinear. If they are, the corresponding W_7 will be said to admit a *non-general* model. In this case ℓ_6^2 is cut out as before, but the other linear series are now cut out by lines passing through P', Q', and R'.

We shall also see that in $\mathcal{N}_7$ there are W_7's which are 3-sheeted coverings of tori. Any 3-sheeted covering of a torus will be called *elliptic-trigonal* and the locus of such surfaces in $\mathcal{T}_7$ will be denoted $(\mathcal{E}-\mathcal{T})_7$. By the Riemann-Hurwitz formula, each component of $(\mathcal{E}-\mathcal{T})_7$ has dimension 12.

We now state the theorem.

Theorem. *Suppose W_7, A Riemann surface of genus 7, admits 4 complete half-canonical linear series g_6^1, h_6^1, k_6^1, and ℓ_6^2 whose sum is bicanonical. Then one of the following possibilities holds:*

(i) W_7 is hyperelliptic

(ii) W_7 is elliptic-hyperelliptic

(iii) W_7 is in $\mathcal{N}_7$.

Since W_7's in $\mathcal{H}_7$ or $(\mathcal{E}-\mathcal{H})_7$ admit several half-canonical linear series of dimension 2, we see that if ℓ_6^2 is unique then $W_7 \varepsilon \mathcal{N}_7$.

In $\mathcal{T}_7$ it is known that $\mathcal{H}_7$ and $(\mathcal{E}-\mathcal{H})_7$ are closed and disjoint. Examples show that $(\mathcal{E}-\mathcal{H})_7$ and $\mathcal{N}_7$ are not disjoint. (See the appendix.) We suspect that $\mathcal{H}_7$ and $\mathcal{N}_7$ are disjoint but we have no proof.

The existence of a complete half-canonical ℓ_6^2 is a codimension 3 condition on $\mathcal{T}_7$ ([1], [11], and [14]), and the existence of a complete half-canonical g_6^1 is a codimension one condition. Consequently, the four loci in $\mathcal{T}_7$ corresponding to the existence of a quartet intersect transversally on $(\mathcal{E}-\mathcal{H})_7$ and $\mathcal{N}_7$ since these 2 loci have codimension 6.

This paper is the latest in a series by the author ([2], [3, Part III], [7]) which have considered loci in Teichmüller space for Riemann surfaces of low genus where the hope has been to define, at least locally, each locus (usually $\mathcal{H}_p$ or $(\mathcal{E}-\mathcal{H})_p$) by equations, the number of which is equal to the codimension of the locus. Through genus 6 this has been successful except for $(\mathcal{E}-\mathcal{H})_4$, a case that has defied any characterization in these terms. In genus 6 a third locus, somewhat analogous to $\mathcal{N}_7$, has appeared.

In the present paper, where genus 7 is considered, to obtain the correct count for the codimension of the locus, we have found it necessary to substitute for 3 dimensions in the codimension count the vanishing of $\theta[\varepsilon](u)$ at $u=0$ to order 3, where $[\varepsilon]$ is an odd theta characteristic. Using such a vanishing property, or equivalently, the existence of a complete half-canonical linear series, G_{p-1}^2, on a Riemann surface of genus p, we obtain some other characterizations as follows.

For $p=5$, the existence of a complete half-canonical G_4^2 is equivalent to W_5 being hyperelliptic: (Clifford's theorem).

For $p=6$, the existence of complete half-canonical linear series G_5^2 and G_5^1 is again equivalent to W_6 being hyperelliptic.

For $p=7$, the existence of two complete half-canonical linear series G_6^2 and H_6^2 is equivalent to $W_7 \varepsilon \mathcal{H}_7 \cup (\mathcal{E}-\mathcal{H})_7$. The codimension count is correct for $(\mathcal{E}-\mathcal{H})_7$ [3, p. 74].

For $p=8$, the existence of two complete half-canonical linear series G_7^2 and H_7^2 implies $W_8 \varepsilon \mathcal{H}_8 \cup (\mathcal{E}-\mathcal{H})_8$ or W_8 admits a plane model somewhat analogous to those determining $\mathcal{N}_7$ for $p=7$, and conversely. For W_8's admitting such models and $\mathcal{H}_8$ the codimension count is again correct. [6, p. 254].

These cases exhaust all cases known to the author in the search for the correct count of the codimension of $\mathcal{H}_p$ and $(\mathcal{E}-\mathcal{H})_p$ by the existence of complete half-canonical linear series of dimension one or two.

2. Definitions, notations, and some known results

Let us first recapitulate the subvarieties of $\mathcal{T}_7$ (dimension = 18) which will be of interest.

$\mathcal{H}_7$ - the hyperelliptic locus of pure dimension 13.

$(\mathcal{E}-\mathcal{H})_7$ - the elliptic-hyperelliptic locus of pure dimension 12.

$\mathcal{N}_7$ - The closure in $\mathcal{T}_7$ of surfaces admitting a general model, of pure dimension 12.

$\mathcal{V}_7$ - W_7's admitting a quartet. Components of $\mathcal{V}_7$ will have dimension 12 or more.

We shall also need to consider:

$(\mathcal{E}-\mathcal{T})_7$ - elliptic-trigonal surfaces; that is, W_7's which are 3-sheeted coverings of tori. This locus is also of pure dimension 12.

Our discussion of plane algebraic curves and linear series will follow Walker [15]. In particular, for a plane curve C of degree n and genus p we have the formula

$$p = \frac{(n-1)(n-2)}{2} - \delta(C)$$

where $\delta(C)$ will be referred to as the *δ-invariant*. On a plane curve a singularity of multiplicity n will be referred to as an *n-fold point* on C. An n-fold point on a plane curve will be called "ordinary", with parentheses, if it contributes precisely $n(n-1)/2$ to the δ-invariant.

All linear series will be considered to be complete unless otherwise stated. We shall use the notation g_6^1, h_6^1, k_6^1, and ℓ_6^2 always to refer to the half-canonical linear series in a quartet. Other linear series will be denoted with capital letters, e.g., G_n^r, H_n^r, or small letters g_n^r where $(r,n) \neq (1,6),(2,6)$. For a (complete) linear series G_n^r we have the Riemann-Roch theorem: $r = n-p+i$ where i is the index of specialty of G_n^r. The Brill-Noether formulation of the Riemann-Roch theorem concerns complements with respect to the canonical series $K = G_{2p-2}^{p-1}$ namely

$$G_n^r + G_{2p-2-n}^{p-1-n+r} \equiv K$$

For example, when we show that a Riemann surface, W_7, does not admit a G_3^1 then W_7 also will not admit a G_9^4.

Concerning $(3-6)$-points for a plane curve we will have to admit the possibility that the $(3-6)$-tangent has more than 6 intersections with the curve at the singularity. Thus a $(3-6)$-point will have a singularity in its first neighborhood and perhaps other singularities in further neighborhoods.

An n-point on a Riemann surface will be an integral divisor of degree n. 2-points will be called *pairs* and 3-points will be called *triples*. If D_1 and D_2 are two integral divisors, (D_1, D_2) will denote the greatest common divisor. $(D_1, D_2) = 0$ will mean they are disjoint.

Considering that the 4 linear series in a quartet sum to the bicanonical series we have the following:

$$g_6^1 + h_6^1 \equiv k_6^1 + \ell_6^2$$
$$g_6^1 + k_6^1 \equiv h_6^1 + \ell_6^2$$
$$g_6^1 + \ell_6^2 \equiv h_6^1 + k_6^1$$

where the identity sign in the above equations stands for linear equivalence. We shall also write

$$g_6^1 \equiv D$$

to mean

$$g_6^1 = |D|.$$

We will say a linear series G_n^r imposes t (linear) conditions on a linear series $G_{n'}^{r'}$, if

$$G_{n'}^{r'} = G_n^r + G_{n'-n}^{r'-t}.$$

If G_n^r and $H_{n'}^{r'}$ are linear series we say a divisor D is *common* to G_n^r and $H_{n'}^{r'}$ if there are divisors E, F in G_n^r and $H_{n'}^{r'}$, so that $((E,F),D) = D$.

We also need the fact that if G_m^2 and H_n^2 are simple linear series, then

$$G_m^2 + H_n^2 = G_{m+n}^{6+\varepsilon} \quad (\varepsilon \geq 0)$$

unless $G_m^2 = H_n^2$. [4, p. 364].

We need the fact that if g_{p-1}^r is half-canonical and $m \leq 2r+1$ then a g_m^1 without fixed points imposes at most $[m/2]$ conditions on g_{p-1}^r [5, p. 11]. Thus if ℓ_6^2 is half-canonical on W_7 it must contain any G_4^1 or G_5^1 without fixed points.

If $W_p \to W_q$ is a t-sheeted branched covering of closed Riemann surfaces, the fibers of the covering will be called an *involution* and denoted γ_t. We will say that a (composite) g_n^r on W_p is *compounded of the involution* γ_t if there is a G_m^r on W_q and the lift of G_m^r to W_p, G_{tm}^r, is the fixed-point free part of g_n^r.

We will also need a special case of the inequality of Castelnuovo-Severi. If $(n_1, n_2) = 1$ and W_p covers W_{q_i} in n_i-sheets, $i = 1,2$ then

$$p \leq n_1 q_1 + n_2 q_2 + (n_1 - 1)(n_2 - 1).$$

We use this inequality to show that a W_7 cannot be simultaneously hyperelliptic and elliptic-trigonal.

Finally we give a characterization of an "ordinary"n-fold point on a plane curve C_m of degree m. If W is the Riemann surface for C_m then on W there is a simple g_m^2 giving rise to the plane model C_m. If $\mathcal{D}$ is an n-fold singularity on C_m then on W there is an n-point D corresponding to $\mathcal{D}$ where D imposes one condition on g_m^2 and $g_m^2 = D + g_{m-n}^1$, g_{m-n}^1 being fixed point free. Then $\mathcal{D}$ is "ordinary"if and only if for all E in g_{m-n}^1 the degree of (D,E) is one or zero. (This is a restatement of the fact that there are no singularities in the first neighborhood of $\mathcal{D}$.)

3. Preliminary results

For this section and the next assume we have a Riemann surface W_7 which admits a quartet, but W_7 is not hyperelliptic or elliptic-hyperelliptic.

In this section we show that ℓ_6^2 is simple, and that a divisor common to two of the four linear series in a quartet must be a triple. Also we show that we may assume no two of the four members of a quartet are compounded of the same involution.

Lemma 1. *W_7 is not trigonal.*

Proof. If W_7 admits a G_3^1 then $\ell_6^2 \equiv 2G_3^1$ and $4G_3^1$ is canonical. Moreover, every complete half-canonical G_6^1 is $G_3^1 + D_3$ when D_3 is a fixed divisor of degree 3. Since $2D_3 + 2G_3^1 = 2G_6^1 \equiv 4G_3^1$, we have $2D_3 \equiv 2G_3^1$. Now a simple argument leads to the contradiction $D_3 \equiv G_3^1$. □

Lemma 2. *The leader ℓ_6^2 is simple and without fixed points.*

Proof. If ℓ_6^2 were composite then W_7 would be hyperelliptic, elliptic-hyperelliptic, or trigonal, all possibilities now excluded. If ℓ_6^2, now known to be simple, had a fixed point, then the genus of W_7 would be 6 or less. □

Lemma 3. *The leader ℓ_6^2 is the unique G_6^2 on W_7.*

Proof. If G_6^2 is a second such linear series then the argument of Lemma 2 shows it is simple. Also

$$\ell_6^2 + G_6^2 \equiv G_{12}^s$$

where $s \geq 6$. Thus $s = 6$ and G_{12}^6 is canonical. Since ℓ_6^2 is half-canonical, we have $\ell_6^2 = G_6^2$. □

Because our W_7 is not trigonal, elliptic-hyperelliptic or hyperelliptic, it admits at most 3 G_4^1's, all withouf fixed points. Since ℓ_6^2 is simple we have a plane model of degree 6, C_6, for W_7. $\delta(C_6) = 3$; that is, C_6 has 3 double points suitably counted. The G_4^1's are cut out by lines passing through the double points of C_6, and their sum is canonical. Two or possibly three of the G_4^1's may be equal depending on the nature of the double points of C_6.

Now we consider the possibility that one of our half-canonical G_6^1's has a fixed point.

Lemma 4. *If g_6^1 has a fixed point, then it has 2 fixed points.*

Proof. Suppose $g_6^1 = G_5^1 + x$ where G_5^1 is without fixed points. Since G_5^1 imposes at most 2 linear conditions on our half-canonical ℓ_6^2, we see that $\ell_6^2 = G_5^1 + y$ where $y \neq x$. But $2g_6^1 \equiv 2\ell_6^2$ or $2x \equiv 2y$. This means that W_7 is hyperelliptic, a contradiction. □

Lemma 5. *It is not true that all the linear series g_6^1, h_6^1, and k_6^1 have fixed points.*

Proof. If g_6^1 has a fixed point then $g_6^1 = g_4^1 + P$, $deg\, P = 2$, and $2g_4^1 + 2P$ is canonical. Now $\ell_6^2 \equiv g_4^1 + Q$ where $(P, Q) = 0$, and $2P \equiv 2Q \equiv h_4^1$; so $2\ell_6^2 \equiv 2g_4^1 + h_4^1$. Thus W_7 admits at most two distinct G_4^1's. If $h_6^1 \equiv M_4^1 + R$, then $2M_4^1 + h_4^1$ is canonical and so $M_4^1 = g_4^1$ and $2P \equiv 2Q \equiv 2R \equiv h_4^1$.

Now assume $k_6^1 \equiv g_4^1 + S$, where $2S \equiv h_4^1$. Since $g_6^1 + h_6^1 \equiv k_6^1 + \ell_6^2$ we have $P + R \equiv S + Q$. Similarly $P + S \equiv R + Q$. Consequently, we have 3 distinct G_4^1's, namely $|2P|$, $|P + R|$, $|P + S|$. Contradiction. □

Lemma 6. *None of the linear series g_6^1, h_6^1, and k_6^1 has a fixed point.*

Proof. Suppose $g_6^1 \equiv G_4^1 + P$ and $\ell_6^2 \equiv G_4^1 + Q$ where $2P \equiv 2Q \equiv H_4^1$ and $2G_4^1 + H_4^1$ is canonical. Since

$$g_6^1 + k_6^1 \equiv \ell_6^2 + h_6^1 \tag{1}$$

we have

$$P + G_4^1 + k_6^1 \equiv Q + G_4^1 + h_6^1$$

or

$$(2) \qquad P + k_6^1 \equiv Q + h_6^1 \equiv G_8^s \quad (P, Q) = 0,$$

where $s \geq 2$. (If $s = 1$, h_6^1 and k_6^1 would have a g_4^1 in common, violating Lemma 5.) Thus G_8^s is special, and so $P \subset k_6^1$ and $Q \subset h_6^1$ since these linear series are half-canonical. Interchanging the roles of h_6^1 and k_6^1 (1) shows that $P \subset h_6^1$ and $Q \subset h_6^1$. Thus there are 4-points F_1 and F_2 so that

$$k_6^1 \equiv P + F_1 \equiv Q + F_2$$

and $P + F_1 + Q + F_2$ is canonical.

By (2) above it follows that

$$h_6^1 \equiv P + F_2 \equiv Q + F_1$$

Now suppose h_6^1 is without fixed points. (h_6^1 or k_6^1 must be without fixed points by Lemma 5.) Consider

$$\begin{aligned} G_{10}^s &\equiv G_4^1 + h_6^1 \\ &\equiv G_4^1 + Q + F_1 \\ &\equiv \ell_6^2 + F_1 \end{aligned}$$

If $s = 3$, F_1 imposes one condition on $G_{10}^3 \equiv G_4^1 + h_6^1$. Since we can find a divisor in h_6^1 with no points in F_1, it follows that $F_1 \equiv G_4^1$, a contradiction. If $s = 4$ then G_{10}^4 is special and $G_4^1 \subset h_6^1$, so that again h_6^1 has fixed points. This final contradiction proves the lemma. □

Lemma 7. *No 2 of the 4 linear series in a quartet have a 4-point in common.*

Proof. Suppose $g_6^1 \equiv Q + P$ and $h_6^1 \equiv Q + R$ where $(P, R) = 0$ and $deg\, P = deg\, R = 2$. Then

$$P + k_6^1 \equiv \ell_6^2 + R \equiv G_8^3$$

a special linear series. Consequently, there is a G_4^1 so that $G_4^1 + P + k_6^1 \equiv K$. This implies that $k_6^1 \equiv P + G_4^1$ contradicting Lemma 6.

Now suppose that $g_6^1 \equiv Q + P$, $\ell_6^2 \equiv Q + R$ where $(P, R) = 0$ and $deg\, P = deg\, R = 2$. Then

$$P + Q + R \equiv G_8^3.$$

Again, this implies the existence of a G_4^1 where $G_8^3 + G_4^1 \equiv K$. Thus g_6^1 has fixed points. Contradiction. □

Lemma 8. *If g_6^1 and h_6^1 have a pair P in common then P is part of a common triple.*

Proof. Suppose $g_6^1 \equiv P + G$, $h_6^1 \equiv P + H$ where $(G, H) = 0$ and $deg\, G = deg\, H = 4$. Since $g_6^1 + k_6^1 \equiv \ell_6^2 + h_6^1$ we have

$$G + k_6^1 \equiv \ell_6^2 + H \equiv G_{10}^s \quad (s \geq 3).$$

Since $(G, H) = 0$, G_{10}^s is without fixed points. If $s = 3$ then H imposes one condition on G_{10}^s and so $H \subset k_6^1$, contradicting Lemma 7. If $s = 4$ then G_{10}^s is special and so $H \subset \ell_6^2$, again contradicting Lemma 7. □

Lemma 9. *Suppose there are triples A, B, and C so that $g_6^1 \equiv A + B$ and $k_6^1 \equiv A + C$ where $(B, C) = 0$. Then there exists triples D and E so that*

$$h_6^1 \equiv B + D \equiv C + E$$

and

$$\ell_6^2 \equiv B + E \equiv C + D.$$

where $(D, E) = 0$. Also $(A, D) = (A, E) = 0$.

Proof. Since $g_6^1 + h_6^1 \equiv k_6^1 + \ell_6^2$ we have

$$B + h_6^1 \equiv C + \ell_6^2 \equiv G_9^3$$

and G_9^3 is without fixed points. Therefore, there is a triple D so that $D + G_9^3 \equiv K$; that is,

$$h_6^1 \equiv B + D \quad \text{and} \quad \ell_6^2 \equiv C + D.$$

Now considering $g_6^1 + \ell_6^2 \equiv h_6^1 + k_6^1$ or

$$B + \ell_6^2 \equiv h_6^1 + C \equiv H_9^3$$

we derive the existence of E.

That the various g.c.d.'s are zero follows from Lemma 7. □

Lemma 10. *Suppose there are triples A, B, and C so that $g_6^1 \equiv A + B$ and $\ell_6^2 \equiv A + C$, $(B, C) = 0$. Then there exists a triple D so that*

$$g_6^1 \equiv C + D \quad \textit{and} \quad \ell_6^2 \equiv B + D$$

where $(A, D) = 0$. Also $B \subset h_6^1$ and $C \subset k_6^1$ or $C \subset h_6^1$ and $B \subset k_6^1$.

Proof. $B + A + C \equiv G_9^s \equiv B + \ell_6^2 \equiv g_6^1 + C$. G_9^s is without fixed points. Thus $s = 3$ and there exists D so that $A + B + C + D \equiv K$. The first part of the lemma now follows.

Now consider $g_6^1 + h_6^1 \equiv k_6^1 + \ell_6^2$. Then

$$B + h_6^1 \equiv k_6^1 + C \equiv G_9^s$$

If $s = 3$, G_9^s is special and $B \subset h_6^1$ and $C \subset k_6^1$.

If $s = 2$ then B and C both impose one condition on G_9^s, and so $B \subset k_6^1$ and $C \subset h_6^1$. □

Lemma 11. *If g_6^1 and h_6^1 are compounded of the same involution then W_7 is elliptic-trigonal. If $W_7 \to W_1$ is the 3-sheeted cover and γ_3^1 is the set of fibers of this cover, then g_6^1, h_6^1, k_6^1 and a one-dimensional subseries of ℓ_6^2 are all compounded of γ_3^1.*

Proof. Suppose W_7 covers a Riemann surface W_q in t sheets and g_6^1 and h_6^1 are lifts of linear series $g_{6/t}^1$, $h_{6/t}^1$ on W_q. By Lemma 8 t must equal 3, and since W_q then admits 2 distinct g_2^1's it follows that $q = 1$. Thus the fibers of the 3-sheeted cover $W_7 \to W_1$

form an involution γ_3^1 of which g_6^1 and h_6^1 are compounded. By Lemma 9 it follows that k_6^1 is compounded by γ_3^1 and that ℓ_6^2 contains an infinite number of divisors made up of 2 divisors from γ_3^1. Also, a triple common to any two divisors in the quartet must be in γ_3^1. □

Lemma 12. *If g_6^1 and h_6^1 are compounded of the same involution on W_7, then in $\mathcal{T}_7$ arbitrarily close to W_7 there is a W_7' admitting a quartet where the g_6^1 and h_6^1 on W_7' are not compounded of the same involution.*

Proof. The existence of a quartet (or equivalently, the vanishing of the theta function at certain half-periods to orders 2, 2, 2, and 3) defines in $\mathcal{T}_7$ a variety, which we have called $\mathcal{V}_7$, each component of which has dimension 12 or more. To prove Lemma 12 it suffices to show that W_7's satisfying the hypotheses of Lemma 11 lie in a proper subvariety of $(\mathcal{E}-\mathcal{T})_7$, that is, in varieties of dimension 11 or less. To do this we will show that in every component of $(\mathcal{E}-\mathcal{T})_7$ there are W_7's which do not satisfy the hypotheses of Lemma 11. Since the image of $(\mathcal{E}-\mathcal{T})_7$ in moduli space for genus 7 is irreducible, it suffices to exhibit a Riemann surface in $(\mathcal{E}-\mathcal{T})_7$ which does not admit a quartet.

Assume the contrary; that is, assume that all elliptic-trigonal W_7's satisfy the hypotheses of Lemma 11. Let $\pi : W_7 \to W_1$ be a 3-sheeted cover with 5 branch points of multiplicity 3 and 2 branch points of multiplicity 2. Let B be the branched locus in W_7 of the cover,

$$B = 2x_1 + 2x_2 + \cdots + 2x_5 + x_6 + x_7$$

a canonical divisor. Now g_6^1 on W_7 is lifted from a g_2^1 on W_1, and so $2g_6^1$ is lifted from $2g_2^1(=g_4^3)$; that is $\pi^{-1}(g_4^3) \equiv K_7$, the canonical series on W_7. If $z_j = \pi(x_j)$, $j = 1, 2, \ldots, 7$, then

$$\pi(B) = 2z_1 + 2z_2 + \cdots + 2z_5 + z_6 + z_7$$

and $\pi(B) \equiv 3g_4^3$. It now follows that

$$3K_7 \equiv \pi^{-1}\pi(B) = 6x_1 + 6x_2 + \cdots + 6x_5 + 2x_6 + y_6 + 2x_7 + y_7$$

where $2x_j + y_j$ is the fiber of π containing x_j, $j = 6, 7$. But

$$3K_7 \equiv 3B \equiv 6x_1 + 6x_2 + \cdots + 6x_5 + 3x_6 + 3x_7.$$

Therefore we see that $y_6 + y_7 \equiv x_6 + x_7$; that is, W_7 is hyperelliptic. Since a W_7 cannot be simultaneously hyperelliptic and elliptic-trigonal, we have reached the desired contradiction. □

Since $\mathcal{H}_7$ and $(\mathcal{E}-\mathcal{H})_7$ are closed varieties in $\mathcal{T}_7$ we can assume that the W_7' of Lemma 12 is not in $\mathcal{H}_7$ or $(\mathcal{E}-\mathcal{H})_7$ and admits a quartet where g_6^1 and h_6^1 are not compounded of the same involution. In the next section we show that such a W_7' is in $\mathcal{N}_7$ by showing W_7' admits a general or non-general model as discussed in the introduction.

4. Proof of the Theorem

We assume our Riemann surface W_7 is neither hyperelliptic nor elliptic-hyperelliptic. Moreover W_7 admits a quartet, no two of which are compounded of the same involution.

Lemma 13. *There are six triples P, Q, R, P', Q', R' such that*

$$\begin{aligned} g_6^1 &\equiv Q + R' \equiv Q' + R \\ h_6^1 &\equiv P + R' \equiv P' + R \\ k_6^1 &\equiv P + Q' \equiv P' + Q \\ \ell_6^2 &\equiv P + P' \equiv Q + Q' \equiv R + R' \end{aligned}$$

where $(P,Q) = (P,R) = (Q,R) = 0$ and $2P \equiv 2Q \equiv 2R$ and $(P',Q') = (P',R') = (Q',R') = 0$ and $2P' \equiv 2Q' \equiv 2R'$.

Proof. By [9, p. 282] g_6^1 and ℓ_6^2 have 12 triples in common. Let R be such a triple. Write

$$g_6^1 \equiv R + Q' \quad \text{and} \quad \ell_6^2 \equiv R + R'$$

By Lemma 10 there exists a triple Q so that

$$g_6^1 \equiv R' + Q \quad \text{and} \quad \ell_6^2 \equiv Q + Q'$$

Also we may assume (by relabeling the linear series, if necessary) that $R' \subset h_6^1$ and $Q' \subset k_6^1$. Then there is a triple P so that $k_6^1 \equiv Q' + P$. Applying Lemma 9 to

$$g_6^1 \equiv Q' + R \quad \text{and} \quad k_6^1 \equiv Q' + P$$

we have 2 triples P' and A so that

$$h_6^1 \equiv R + P' \equiv P + A$$

and

$$\ell_6^2 \equiv R + A \equiv P + P'$$

Since $\ell_6^2 \equiv R + R'$ we see that $A = R'$. Finally

$$2\ell_6^2 \equiv Q + Q' + P + P' \equiv 2k_6^1,$$

so

$$k_6^1 \equiv Q' + P \equiv Q + P'.$$

All the assertions about g.c.d.'s follow from Lemma 7. The other assertions follow since all members of the quartet are half-canonical. □

We now consider three cases for the remaining g.c.d.'s for the triples in Lemma 13.

Case i). All 6 divisors of Lemma 13 are mutually disjoint or possibly an equality of the following type holds: $deg(P,Q') = 1$.

Case ii). An inequality of the following type holds: $deg(P,P') \geq 1$.

Case iii). An inequality of the following type holds: $deg(P, Q') \geq 2$. Lemma 15 will show that Case iii) is impossible.

Lemma 14. *If $deg(P, P') \geq 1$ in Lemma 13 then $P = P'$, $Q = Q'$, $R = R'$ and there exist a triple S so that*

$$g_6^1 \equiv Q + R \equiv P + S$$
$$h_6^1 \equiv P + R \equiv Q + S$$
$$k_6^1 \equiv P + Q \equiv R + S$$
$$\ell_6^2 \equiv 2P \equiv 2Q \equiv 2R \equiv 2S$$

Proof. Since $h_6^1 \equiv P + R' \equiv P' + R$ and $deg(P, P') \geq 1$ we see that the divisors $P + R'$ and $P' + R$ are equal. But $(P, R) = 0$; consequently, $P = P'$. It follows that $R = R'$ and $Q = Q'$. If $P + Q + R$ is special the conclusion follows immediately.

So assume $P + Q + R \equiv G_9^2$. Now

$$G_9^2 \equiv P + g_6^1 \equiv Q + h_6^1 \equiv R + k_6^1$$

so that G_9^2 is without fixed points. It also follows that if G_9^2 were composite then g_6^1 and h_6^1 would be compounded of the same involution, contradicting our hypotheses. Consequently, G_9^2 gives rise to a plane model, C_9, of degree 9 in $\mathbf{P}^2$. P, Q, and R correspond to 3-fold singularities of C_9 all lying in a line, L_0.

We assert first that these 3 singularities (call them also P, Q, and R) are "ordinary". Suppose P is not "ordinary". Since $G_9^2 \equiv P + g_6^1$, there is a divisor in g_6^1 containing 2 points of P, say $x_1 + x_2$. Since g_6^1 is half-canonical $g_6^1 + x_1 + x_2 = G_8^s$ where G_8^s is special and $G_8^s \subset G_9^2$. Consequently $s = 2$ and we have contradicted the fact that G_9^2 is without fixed points. The assertion is proved.

The δ-invariant of C_9 is 21 and the 3-fold points P, Q, and R contribute only 9. C_9 must have further singular points not on the line L_0. Such singularities will correspond to n-points common to g_6^1 and h_6^1 and so must be 3-fold singularities.

Let A be a further 3-fold singularity not on L_0. Then A corresponds to a triple (again call it A) common to g_6^1 and h_6^1. If $g_6^1 \equiv A + B$ and $h_6^1 \equiv A + C$, $(B, C) = 0$, then by Lemma 9 we have two triples D, E so that $k_6^1 \equiv B + D \equiv C + E$. Since k_6^1 is cut out by lines through R on C_9, it follows that B, C, D, E all correspond to 3-fold singularities of C_9 not on L_0. A is also a triple in k_6^1. Thus for every singularity, A, not on L_0, there are 3 further singularities on the 3 lines AP, AQ, AR. This leads to too many singularities for C_9. This contradiction concludes the proof. □

Lemma 15. *In Lemma 13, $deg(P, Q') \leq 1$.*

Proof. If $deg(R, R') > 0$ then the result follows by Lemma 14. If $(R, R') = 0$ and $deg(P, Q') \geq 2$ then it follows from Lemma 13 that $P = Q'$ since only triples are common to two members of a quartet. Considering ℓ_6^2 in Lemma 13 we see that $P' = Q$. Then we have

$$g_6^1 \equiv Q + R' \equiv Q' + R$$
$$k_6^1 \equiv 2Q' \equiv 2Q \equiv 2R' \equiv 2R$$

Counting multiplicities, g_6^1 and k_6^1 now have 8 triples in common. Since g_6^1 and k_6^1 are not compounded of the same involution, the maximum numbers of triples common to g_6^1 and k_6^1 is 6. We have reached the desired contradiction. □

Let us summarize the results so far.

Proposition. *At least one of the following two cases holds.*

(1) There are 6 distinct triples P, Q, R, P', Q', R' so that

$$g_6^1 \equiv Q + R' \equiv Q' + R$$
$$h_6^1 \equiv P + R' \equiv P' + R$$
$$k_6^1 \equiv P + Q' \equiv P' + Q$$
$$\ell_6^2 \equiv P + P' \equiv Q + Q' \equiv R + R'$$

For any 2 of the 6 triples, A, B, we have $(A, B) = 0$ unless it is of the type (P', Q) where the degree is at most one. Thus no 2 of the 6 triples have a pair in common. Also

$$2P \equiv 2Q \equiv 2R \quad \textit{and} \quad 2P' \equiv 2Q' \equiv 2R'.$$

(2) There are 4 mutually disjoint triples P, Q, R, S and

$$g_6^1 \equiv Q + R \equiv P + S$$
$$h_6^1 \equiv P + R \equiv Q + S$$
$$k_6^1 \equiv P + Q \equiv R + S$$
$$\ell_6^2 \equiv 2P \equiv 2Q \equiv 2R \equiv 2S$$

Lemma 16. *In case (1) of the Proposition $P + Q' + R$ is not special.*

Proof. Suppose $P + Q' + R$ is special. Then there exists a triple S' so that $P + Q' + R + S'$ is canonical and $g_6^1 \equiv P + S'$. Since $\ell_6^2 \equiv P + P'$, Lemma 10 yields a triple S so that $g_6^1 = P' + S$ and $\ell_6^2 \equiv S + S'$. Now $S + P' + Q + R'$ is special and we have

$$g_6^1 \equiv Q + R' \equiv Q' + R \equiv S + P' \equiv S' + P$$
$$k_6^1 \equiv P + Q' \equiv P' + Q \equiv S + R' \equiv S' + R$$

Also $2k_6^1 \equiv K \equiv P + Q' + S + R'$. Consequently

$$h_6^1 \equiv P + R' \equiv P' + R \equiv S + Q' \equiv S' + Q.$$

By repeating previous arguments we see that no 2 of these 8 triples have a pair in common, and we have reached the desired contradiction. □

Proof of the theorem

Suppose we are in case (1) of the Proposition. Let $\mathcal{C}_9$ be the plane curve determined by $G_9^2 \equiv P + Q' + R \equiv P + g_6^1$. That P is an "ordinary"singularity follows by the argument of the third paragraph of Lemma 14. Now

$$G_9^2 \equiv P + g_6^1 \equiv R + k_6^1 \equiv P + Q + R' \equiv Q + h_6^1$$

and the same argument shows that Q and R are "ordinary"singularities for $\mathcal{C}_9$.

First we show that there are no further 3-fold points on the triangle PQR. Suppose, on the contrary, that Q' is a 3-fold point of $\mathcal{C}_9$. Then $h_6^1 \equiv Q' + A$ where A is a further 3-fold point. If $A = Q'$ and $h_6^1 \equiv 2Q'$ ($\equiv 2P'$) then $2P' \equiv P' + R$ and we arrive at the contradiction $P' = R$. Thus A is not on the triangle PQR. By Lemma 9 there are further 3-fold points, B, C, so that $g_6^1 = A + B$ and $k_6^1 = A + C$. By again applying Lemma 9 we see that the only way to avoid having too many 3-fold points is for $B = P'$ and $C = R'$. Then the 7 3-fold points of $\mathcal{C}_9$ an P, Q, R, P', Q', R' and A. Now ℓ_6^2 is the unique G_6^2 on W_7; therefore, ℓ_6^2 is cut out on $\mathcal{C}_9$ by cubics through the 7 singularities of $\mathcal{C}_9$. The 3 lines PQ, $Q'Q$, and RQ together form such a cubic cutting out $2Q$. Therefore $\ell_6^2 \equiv 2Q \equiv Q+Q'$. Thus we arrive at the contradiction $Q = Q'$. Thus any further singularity of $\mathcal{C}_9$ besides P, Q, R must not lie on the triangle PQR.

Now let A be a 3-fold point of $\mathcal{C}_9$ not on the triangle PQR and let $g_6^1 = A + B$. Then by Lemma 9, B must be a triple in k_6^1 and so must correspond to a singular point of $\mathcal{C}_9$. Thus the singularities of $\mathcal{C}_9$, other than P, Q, and R, lie in pairs on lines through P, Q, and R. Since there can be at most 4 additional singularities (all 3-fold), they must be the vertices of a quadrilateral for which P, Q, and R are the diagonal points. Thus case (1) of the Proposition leads to the general model for a Riemann surface in $\mathcal{N}_7$.

Now suppose we are in case (2) of the Proposition.

First suppose that there is a 5^{th} triple, A, common to g_6^1 and h_6^1, $g_6^1 \equiv A + B$ and $k_6^1 \equiv A + C$ where $(B, C) = 0$. By Lemma 9 there exist triples D and E so that $h_6^1 \equiv B + D \equiv C + E$ and $\ell_6^2 \equiv B + E \equiv C + D$. None of these triples have points in common with P, Q, R, or S. Then g_6^1 and ℓ_6^2 have the triple B in common. We now repeat the argument of Lemma 13 and obtain 6 triples as in the statement of Lemma 13.

If case ii) following Lemma 13 holds for these 6 triples, we obtain by Lemma 14 4 more triples common to g_6^1 and k_6^1 in addition to P, Q, R, and S. This is too many. Consequently we are led to case i) following Lemma 13 which is case (1) of the Proposition, and again we have the general model.

So now we assume, finally, that only the triples P, Q, R, and S are common to the 4 linear series in the quartet as in case (2) of the Proposition. We must analyze how these triples are multiply counted in the linear series of the quartet. To do this we first prove our final lemma.

Lemma 17. *Suppose P, Q, R, and S are the only triples common to the linear series in the quartet as in case (2) of the Proposition. If $Q = Q_1 + Q_2 + Q_3$ then $Q_1 + Q_2$ impose two independent conditions on ℓ_6^2; that is, a divisor in ℓ_6^2 containing Q_1 and Q_2 must be $2Q$.*

Proof. Suppose not. Then $\ell_6^2 = Q_1 + Q_2 + g_4^1 \equiv Q_1 + Q_2 + Q_3 + Q$. Consequently $Q_3 + Q \equiv g_4^1$. But $g_6^1 \equiv Q + R$. Thus g_4^1 and g_6^1 have the triple Q in common. This gives the contradiction $g_4^1 \subset g_6^1$. □

We continue by deriving a plane model, $\mathcal{C}_{12}$, of degree 12. Let $F\,(resp\,K)$ be a meromorphic function on W_7 with polar and zero divisors in $g_6^1\,(resp\,k_6^1)$. Assume these polar and zero divisors together with P, Q, R, and S are all mutually disjoint. Consider the map $\phi : W_7 \to \mathbf{P}^2$ given by

$$\phi(x) = (F(x), K(x), F(x) + K(x)).$$

This gives our plane model, $\mathcal{C}_{12}$, with two singularities, Y and Z, of multiplicity 6. Y and Z must be "ordinary"; otherwise, we would be led to a fifth triple common to g_6^1 and k_6^1. Lines through $Y\,(resp\,Z)$ cut out $g_6^1\,(resp\,k_6^1)$ on $\mathcal{C}_{12}$. $\mathcal{C}_{12}$ also has the singularities P', Q', R', and S' corresponding to the triples P, Q, R, and S on W_7. $\mathcal{C}_{12}$ has no further singularities. P', Q', R', and S' determine a quadrangle, two of whose diagonal points are Y and Z. Since the contribution to $\delta(\mathcal{C}_{12})$ outside Y and Z is 18, we see that one of the 3-fold singularities, say Q', has a singularity in its first neighborhood.

We now transform $\mathcal{C}_{12}$ by an elementary quadratic transformation with fundamental points at Y, Z and Q' to obtain a curve $\mathcal{C}_9$ of degree 9. P'', Q'', R'', and S'' will denote the transforms of P', Q', R', and S'. P'' and R'' are $(3,6)$-points with $(3,6)$-tangents passing through S''. Q'' lies on the line of P'' and R'' and accounts for at least a 2-fold singularity on $\mathcal{C}_9$. Lines through S'' cut out $2P''$ and $2R''$ and so cut out a one-dimensional subseries of ℓ_6^2. But the line through S'' and the singularity in Q'' cuts out a divisor in ℓ_6^2 containing 2 points of Q, and so, by Lemma 17, this line must cut out $2Q$. Consequently, Q'' is also a $(3,6)$-point with the $(3,6)$-tangent passing through S''. We have the non-general model in $\mathcal{N}_7$. q.e.d.

(In the triples suitably counted, common to g_6^1 and k_6^1, P, R, and S are counted once and Q is counted 3 times.)

5. Appendix: A generalization of quartets for some Riemann surfaces of higher genus

The Riemann surfaces occurring in $\mathcal{N}_7$ are part of a sequence of Riemann surfaces admitting 4 half-canonical linear series whose sum is bicanonical.

For $n \geq 1$ let $\mathcal{C}_{3n}$ be a curve of degree $3n$ with ordinary n-fold singularities at the 7 points A_1, A_2, A_3, A_4, P, Q, R as in the general model for $\mathcal{N}_7$. The genus of $\mathcal{C}_{3n}$ is $n^2 - n + 1$.

If $n = 2m + 1$ curves of degree $3m$ with m-fold singularities at the 7 points cut out a $g_{p-1}^{m^2+m}$. Curves of degree $3m - 2$ with an m-fold singularity at one of the diagonal points, say P, and with $(m-1)$-fold singularities at the other six points cut out one of 3 $g_{p-1}^{m^2+m-1}$'s.

If $n = 2m$ we have 4 $g_{p-1}^{m^2-1}$'s by considering curves of degree $3m-3$ with an $(m-2)$-fold singularity at one of the A_i's and $(m-1)$-fold singularities at the other six points.

For $n = 1$ let C_1, C_2, and C_3 be 3 cubic forms corresponding to 3 independent cubics passing through the 7 points (e.g., $C_1 = x(y^2 - z^2)$). Then if $f_n(X, Y, Z)$ is a form of degree n, $f_n(C_1, C_2, C_3) = 0$ will define a curve of degree $3n$ with n-fold singularities at the 7 points. The family of curves arising from the $f_n(C_1, C_2, C_3)$'s will have dimension $n(n+3)/2$. These latter types of curves will be invariant under the elementary quadratic transformation T: $x \to yz$, $y \to zx$, $z \to xy$. Thus T will define an involution on such curves with fixed points at $A_1, \ldots, A_4$. The quotient Riemann surface will have genus $(n-1)(n-2)/2$.

For $n = 2$ we obtain a family of curves of genus 3 and the curves $f_2(C_1, C_2, C_3) = 0$ are hyperelliptic.

For $n = 3$, the case considered in this paper, the $f_3(C_1, C_2, C_3)$'s give a 9-dimensional family of elliptic-hyperelliptic Riemann surfaces in $\mathcal{N}_7$.

References

1. Accola, R. D. M., "Algebraic curves and half-canonical linear series", *Discontinuous Groups and Riemann Surfaces*, Princeton, 1974, 13–22
2. Accola, R. D. M., "Some loci of Teichmüller space for genus five defined by vanishing theta-nulls", *Contributions to Analysis*, Academic Press, 1974, 11-18
3. Accola, R. D. M., "Riemann surfaces, theta functions, and abelian automorphism groups", *Lecture Notes in Mathematics* 483, Springer-Verlag (1975)
4. Accola, R. D. M., "On Castelnuovo's inequality for algebraic curves, Part I", *Transactions of the American Mathematical Society*, **251** (1979), 357-373
5. Accola, R. D. M., "Plane models for Riemann surfaces admitting certain half-canonical linear series, Part I", *Riemann Surfaces and Related Topics* (1980) Princeton University Press, 7-20
6. Accola, R. D. M., "Plane models for Riemann surfaces admitting certain half-canonical linear series, Part II", *Transactions of the American Mathematical Society*, **263** (1981), 243–259
7. Accola, R. D. M., "Some loci in Teichmüller space for genus six defined by vanishing thetanulls", *Differential Geometry and Complex Analysis* (H. E. Rauch Memorial Volume), Springer-Verlag (1985), 49–63
8. Castelnuovo, G.; "Sui multiple di una serie lineare di gruppe di punti appartenente ad una curve algebrica"; *Rendiconti del Circolo Matematico di Palermo*, VII (1893), 89–110 (Memorie scelte 95)
9. Coolidge, J. L., *Algebraic plane curves*, Oxford, 1931 (Dover).
10. Farkas, H. M. and Kra, I.; *Riemann Surfaces*; New York, Heidelberg, Berlin, Springer-Verlag, 1980
11. Harris, J., "Theta characteristics on algebraic curves", *Transactions of the American Mathematical Society*, **271** (1982), 611-638
12. Krazer, A., *Lehrbuch der Thetafunktionen*, B. G. Teubner, Leipsiz, 1903
13. Lewittes, J., "Riemann surfaces and the theta function", *Acta Methematica*, **111** (1964), 37-61
14. Teixidor, Montserrat, "Half-canonical series on algebraic curves", *Transactions of the American Mathematical Society,* 302 (1987), 99–115
15. Walker, R. J.; *Algebraic Curves*, New York, Dover Publications, Inc., 1962

Robert D.M. Accola
Brown University
Mathematics Department
Box 1917
Providence, RI 02912

(Received January 13, 1993)

manuscripta math. 81, 129 - 139 (1993)

manuscripta
mathematica

Bounded degree of weakly algebraic topological Lie algebras

Bienvenido Cuartero, José E. Galé, Angel Rodríguez Palacios and Arkadii M. Slinko

We prove in this paper that a weakly algebraic Lie algebra A which is also a topological Baire algebra over a complete non-discrete valuated field K must be in fact algebraic of bounded degree. Similar results are also proven for p-algebraic restricted Lie p-algebras, and for algebraic non-associative algebras.

0. Introduction

It was noticed a long time ago that under some topological conditions every algebraic algebra A over a field K becomes algebraic over K of bounded degree. Ostrowski [14, 15] showed this for a field A with a complete valuation that extends a complete valuation of K, and Mazur [13] proved the same for a commutative Banach algebra A over $K = \mathbb{R}$ or $\mathbb{C}$. Kaplansky [10] established the corresponding theorem for a complete, metrizable, associative algebra A over an arbitrary field K of type V, giving a proof which required heavy use of the Jacobson's structure theory. Also, Slinko [16] proved an analogous theorem for a Jordan algebra A in just the same formulation but relying upon the Shirshov's combinatorial technique. A much simpler proof that works even for a power-associative algebra A was recently discovered by Cuartero and Galé [8].

The result of Cuartero and Galé cannot be applied to Lie algebras as the notion of algebraic element in a Lie algebra differs from that of the algebras mentioned above. The aim of this note therefore is to extend the quoted results to the Lie case by proving Theorems A and C below. The technique developed also allows us to obtain some previously unknown results for Jordan algebras which will be discussed in section 5.

Finally, we prove that power associativity in the paper [8] can be removed by considering an appropriate definition, due to Albert [1].

1. Formulations of the main results

All the algebras considered in this paper are algebras over a field K.

Definition 1. A subset V of a Lie algebra L is said to be weakly algebraic over K if for any $x \in L$, $y \in V$ there exists a non-zero polynomial $f(t) = f_{x,y}(t) \in K[t]$ depending on x and y such that $xf(\mathrm{ad}\, y) = 0$, where $\mathrm{ad}\, y : x \mapsto [x, y]$ is the adjoint representation of the Lie algebra L. Moreover, if L itself is weakly algebraic and for some positive integer n the inequality $\deg f_{x,y} \leq n$ holds for all $x, y \in L$ and for some $x, y \in L$ the equality $\deg f_{x,y} = n$ takes place, then L is said to be weakly algebraic over K of bounded degree n.

Definition 2. A Lie algebra L is said to be algebraic over K if for any $x \in L$ there exists a non-zero polynomial $f(t) = f_x(t) \in K[t]$ depending on x such that $f(\mathrm{ad}\, x) = 0$. Moreover, if for some positive integer n the inequality $\deg f_x \leq n$ holds for all $x \in L$ and $\deg f_x = n$ for some $x \in L$, then L is said to be algebraic over K of bounded degree n.

Clearly, every algebraic Lie algebra is weakly algebraic but not viceversa.

By a valuated field we shall mean a topological field whose topology is defined by a valuation or an absolute value in the sense of [5, Chapter 6].

Recall that a (Hausdorff) topological space X is of the second category in itself if it cannot be represented as a countable union of closed sets without interior points. Moreover X is a Baire space if the intersection of each countable family of open dense sets is dense or, equivalently, if every non-empty open set is of the second category in itself ([6, Chap. IX §5 No. 3]).

Complete metrizable vector spaces are Baire spaces. They are, of course, the objects of our primary interest inasmuch as Banach spaces are complete and metrizable.

Theorem A. *Let L be a Baire Lie algebra over a complete non-discrete valuated field K. If a non-empty open set V of L is weakly algebraic over K, then L is algebraic over K of bounded degree.*

One important corollary is worth mentioning. To do this, we recall two more definitions.

Definition 1'. A subset V of a Lie algebra L is said to be weakly Engel if for any $x \in L$, $y \in V$ there exists a positive integer n depending on x and y such that $x(\mathrm{ad}\, y)^n = 0$.

Definition 2'. A Lie algebra L is said to be Engel if for any $x \in L$ there exists a positive integer n depending on x such that $(\mathrm{ad}\, x)^n = 0$. Moreover, if for some positive integer n this equality holds for all $x \in L$, then L is said to be Engel of bounded index.

Then Theorem A implies the following.

Corollary B. *Let L be a Baire Lie algebra over a complete non-discrete valuated field K. If a non-empty open set V of L is weakly Engel, then L is Engel of bounded index.*

E. I. Zelmanov proved [17, 12] that, over a field of characteristic 0, every Lie algebra which is Engel of bounded index must be nilpotent. Thus Corollary B can be strengthened in this case.

For restricted Lie algebras we have also another definition of algebraic elements. Recall that a Lie algebra L over a field K of characteristic $p > 0$ is said to be a restricted Lie p-algebra if it has a mapping $x \mapsto x^{[p]}$ satisfying

(i) $(\operatorname{ad} x)^p = \operatorname{ad}(x^{[p]})$ for all $x \in L$;
(ii) $(\alpha x)^{[p]} = \alpha^p \cdot x^{[p]}$ for all $\alpha \in K$, $x \in L$;
(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$, where $is_i(x,y)$ is the coefficient of t^{i-1} in $x\big(\operatorname{ad}(tx+y)\big)^{p-1}$

Note that $s_i(x,y)$ is homogeneous of degree i in x and of degree $p-i$ in y.

Definition 3. Set $x^{[0]} = x$ and $x^{[p^k]} = (x^{[p^{k-1}]})^{[p]}$ for $k > 0$. If for $x \in L$ and some nonzero $\alpha_i \in K$ an equality $\sum_{i=0}^{n} \alpha_i x^{[p^i]} = 0$ holds, the element $x \in L$ is said to be p-algebraic. If n is the minimal positive integer for which such an equality holds then we shall say that the degree of x is equal to p^n. The notions of p-algebraic restricted Lie algebra and p-algebraic restricted Lie algebra of bounded degree now come up in an ordinary manner.

Theorem C. *Let L be a Baire restricted Lie p-algebra over a complete non-discrete valuated field K of prime characteristic $p > 0$. If all elements of some non-empty open set V of L are p-algebraic over K, then L is p-algebraic of bounded degree.*

Although it can be derived from Theorem A our method allows us to prove Theorem C directly.

Our last result shows how the arguments in [8] can be adapted to extend its main result to general nonassociative algebras. Following [1], we shall say that an element x in a (nonassociative) algebra A over a field K is algebraic (over K) if the subalgebra $A(x)$ of A generated by x is finite-dimensional (over K). When this is true for every x in A, we say that A is algebraic. If in fact $\dim_K A(x) \leq m$ for all $x \in A$ and some integer m depending on A only, then the algebraic algebra A is called of bounded degree, and the smallest such a number m is called the (bounded) degree of A. Thus we can prove the following.

Theorem D. *Let A be an (nonassociative) Baire algebra over a complete non-discrete valuated field K. If all elements of some non-empty open set V of A are algebraic over K, then A is algebraic of bounded degree.*

Taking $K = \mathbb{R}$ or $\mathbb{C}$, we get the following consequence.

Corollary E. *Every real or complex algebraic (nonassociative) complete normed algebra is of bounded degree.*

2. Weakly algebraic and algebraic operators

In this section K is an arbitrary field.

Definition 4. A linear operator $\varphi : E \longrightarrow E$ on a vector space E over a field K is said to be weakly algebraic (or locally algebraic, see [11]) if for every $x \in E$ there exists a non-zero polynomial $f(t) \in K[t]$ depending on x such that $xf(\varphi) = 0$. Denote such a polynomial of minimal degree by $f_x(t)$. If for some positive integer n the inequality $\deg f_x(t) \leq n$ holds and n is minimal with this property, we say that φ is weakly algebraic of bounded degree n. A linear operator φ is algebraic if there exists a non-zero polynomial $f(t) \in K[t]$ such that $f(\varphi) = 0$. If $f(t)$ is of minimal degree, then $n = \deg f$ is called the degree of algebraicity of φ.

Proposition 1. *Let $\varphi : E \longrightarrow E$ be a weakly algebraic linear operator of bounded degree on a vector space E over an arbitrary field K. Then φ is algebraic.*

Proof. In a usual way we can consider E as a $K[t]$-module letting

$$x \cdot f(t) = xf(\varphi).$$

Since φ is weakly algebraic, E is a torsion module over a principal ideal domain $K[t]$ and according to [4, Chapter VII, §2, Theorem 1] $E = \sum E_{\pi(t)}$, where

$$E_{\pi(t)} = \{x \in E : x \cdot \pi^n(t) = 0 \text{ for some } n \geq 1\}$$

and $\pi(t)$ runs over the set of all irreducible polynomials of $K[t]$. As φ is weakly algebraic of bounded degree only a finite number of such $E_{\pi(t)}$ is nonzero and in this case φ is algebraic. □

Remark 1. Similar arguments can be found in [2, Chapter 4, §4.8, p. 141]

Remark 2. For an algebraically closed field K see also [11, Lemma 14, p. 41].

3. Weakly linearly dependent sequences of continuous functions

Definition 5. Let K be a field and E, F be two vector spaces over K. We shall say that a sequence of functions $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ is weakly linearly dependent on a subset $E_0 \subseteq E$ if for any $x \in E_0$ there exists a positive integer n depending on x such that the set of vectors $\{f_1(x), \dots, f_n(x)\}$ is linearly dependent. The minimal integer with this property will be called the degree of $\mathcal{F}$ at a point x and it will denoted by $\deg_{\mathcal{F}}(x)$. The sequence $\mathcal{F}$ will be called weakly linearly dependent if it is weakly linearly dependent on E and weakly linearly dependent of bounded degree if there exists an integer N such that $\deg_{\mathcal{F}}(x) \leq N$ for all $x \in E$. In this case we shall call degree of the sequence $\mathcal{F}$ the minimal integer $N(\mathcal{F})$ with the above property.

Definition 6. Let $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ be a weakly linearly dependent sequence of functions of bounded degree N. A point x_0 is said to be regular if the set of vectors $\{f_1(x_0), \dots, f_{N-1}(x_0)\}$ is linearly independent. Let $\mathrm{Reg}(\mathcal{F})$ stands for the set of all regular points of $\mathcal{F}$. For any $x \in \mathrm{Reg}(\mathcal{F})$ we can uniquely define functions $\alpha_1(x), \dots, \alpha_{N-1}(x)$ with values in K such that

$$f_N(x) = \sum_{m=1}^{N-1} \alpha_m(x) f_m(x). \tag{6}$$

The following lemma which is of paramount importance is a nonmetric version of a well-known result of Kaplansky [10].

Lemma 2. *Let K be a complete non-discrete valuated field and let E be a topological vector space over K. Let $(x_{i,n})_{i \in I} (1 \leq n \leq N)$ be a finite set of nets in E converging to linearly independent vectors $x_1, \dots, x_N \in E$. Suppose that for every index $i \in I$ there is a linear combination $y_i = \sum_{n=1}^{N} \lambda_{i,n} x_{i,n}$ such that the net $(y_i)_{i \in I}$ converges to some y. Then for each n the net $(\lambda_{i,n})_{i \in I}$ converges in K.*

When K is discrete, the same result is true if we demand F to be locally balanced, *i.e., to have a base of 0-neighborhoods V such that $K.V = V$.*

Proof. See [7, Lemma 4 and Lemma 6]. □

Lemma 3. *Let E, F be two topological vector spaces over a complete valuated field K, so that F is locally balanced in the discrete case. Let $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ be a sequence of continuous functions. Then for each $N \in \mathbb{N}$ the set*

$$E_N = \{x \in E \mid \deg_{\mathcal{F}}(x) \leq N\}$$

is closed in E.

Proof. Let $(x_i)_{i \in I}$ be a net of $x_i \in E_N$ converging to x. Then $\{f_n(x_i) : 1 \leq n \leq N\}$ is linearly dependent for every $i \in I$. As the functions f_n, $n = 1, \dots, N$, are continuous, the nets $(f_n(x_i))$ converge to $(f_n(x))$, $n = 1, \dots, N$ and so, by Lemma 2, $\{f_n(x) : 1 \leq n \leq N\}$ is also linearly dependent, i. e., $x \in E_N$. □

Note. If moreover $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ is weakly linearly dependent of bounded degree, then the set $\text{Reg}(\mathcal{F})$ is *open*, because $\text{Reg}(\mathcal{F}) = E \setminus E_{N-1}$, $N = N(\mathcal{F})$.

Theorem 1. *Let E, F be two topological vector spaces over a complete valuated field K so that F is locally balanced in the discrete case and let $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ be a weakly linearly dependent sequence of continuous functions of bounded degree N. Then $\alpha_i : E \to K$, $i = 1, \dots, N-1$, are continuous functions on* $\text{Reg}(\mathcal{F})$.

Proof. Let $x \in \text{Reg}(\mathcal{F})$ and $(x_i)_{i \in I}$ be a net in $\text{Reg}(\mathcal{F})$ converging to x. Then

$$f_N(x_i) = \sum_{m=1}^{N-1} \alpha_m(x_i) f_m(x_i). \tag{8}$$

The nets $\big(f_m(x_i)\big)_{i \in I}$ converge to $f_m(x)$, $m = 1, \dots, N-1$, and $\{f_1(x), \dots, f_{N-1}(x)\}$ is a linearly independent set of vectors. Moreover, the net of linear combinations $\left(\sum_{m=1}^{N-1} \alpha_m(x_i) f_m(x_i)\right)_{i \in I}$ converges as

$$\sum_{m=1}^{N-1} \alpha_m(x_i) f_m(x_i) = f_N(x_i) \to f_N(x).$$

By Lemma 2 the nets $\big(\alpha_m(x_i)\big)_{i \in I}$ converge for all $m = 1, \dots, N-1$. Passing to the limit in (8) we deduce that $\alpha_m(x_i) \to \alpha_m(x)$. □

Definition 7. Let E, F be two vector spaces over a field K. A sequence of functions $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ is said to be extremal if for every $n \in \mathbb{N}$ and for every $x, y \in E$ the set

$$\Delta_n(x,y) = \{\lambda \in K \mid \deg_{\mathcal{F}}(x + \lambda y) \leq n\} \tag{9}$$

is finite or the whole of K.

Example. Recall that a mapping $Q : E \to F$ is said to be a homogeneous polynomial of degree n if it is the restriction to the diagonal of some n-linear mapping from E^n to F. It follows that $Q(x + \lambda y)$ is a polynomial function of λ, and using Lemma 2 bis of [8] we can assert now that if for every $n \in \mathbb{N}$ the function f_n is homogeneous of some degree, then the sequence $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ is extremal.

Theorem 2. *Let E be a Baire topological vector space over a complete non-discrete valuated field K, F be an arbitrary topological vector space over K and $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ be a extremal sequence of continuous functions. If $\mathcal{F}$ is* *weakly linearly dependent on a non-empty open subset $V \subseteq E$, then it is weakly linearly dependent of bounded degree.*

Proof. By Lemma 3 the set

$$E_n = \{x \in E \mid \deg_{\mathcal{F}}(x) \leq n\}$$

is closed for each $n \in \mathbb{N}$. Define $V_n = V \cap E_n$. It is clear that $V = \cup V_n$. Since an open set of a Baire space is of the second category, we conclude that some V_n has an interior point $a \in V_n$ and hence there exists a neighborhood of zero $U \subseteq E$ such that $a + U \subseteq V_n \subseteq E_n$. Since K is not discrete for arbitrary $b \in E$ there exist an infinite number of $\lambda \in K$ such that $\lambda(b-a) \in U$ and thus the set $\Delta_n(a, b-a)$ defined by (9) is infinite. As $\mathcal{F}$ is extremal we have $\Delta_n(a, b-a) = K$ and by letting $\lambda = 1$ we obtain $\deg_{\mathcal{F}}(b) \leq n$. Since b is arbitrary the theorem is proved. □

Proof of Theorem C. Set $\mathcal{F} = \{f_n : x \to x^{[p^n]}\}$. A Lie algebra L is p-algebraic if and only if $\mathcal{F}$ is weakly linearly dependent and L is of bounded degree if $\mathcal{F}$ is. Now Theorem C follows from Theorem 2 since f_n is homogeneous of degree p^n. □

Definition 8. Let us say that a sequence of functions $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ is pointwise finite-dimensional if for every $x \in E$ the linear subspace spanned by the set $\{f_n(x)\}_{n \in \mathbb{N}}$ is finite-dimensional. We shall say that $\mathcal{F}$ is pointwise finite-dimensional of bounded degree N if N is the least upper bound for the dimensions of all the subspaces mentioned.

This notion is close to weak linear dependence but different in some situations. An analog of Theorem 2 can also be proved for it.

Theorem 3. *Let E be a Baire topological vector space over a complete non-discrete valuated field K, F be an arbitrary topological vector space over K and $\mathcal{F} = \{f_n : E \to F\}_{n \in \mathbb{N}}$ be a extremal sequence of continuous functions. If $\mathcal{F}$ is pointwise finite-dimensional on a non-empty open subset $V \subseteq E$, then it is pointwise finite-dimensional of bounded degree.*

Proof. As in Lemma 3 we can prove that the set

$$E(i_1, \ldots, i_n) = \{x \in E : f_{i_1}(x), \ldots, f_{i_n}(x) \text{ is linearly dependent}\}$$

is closed for every $n \in \mathbb{N}$ and each choice of $i_1 < \ldots < i_n$, so the sets E_n of vectors x in E such that the linear subspace spanned by $f_j(x)$, $j \in \mathbb{N}$, has dimension $< n$ is also closed, for

$$E_n = \bigcap_{i_1 < \ldots < i_n} E(i_1, \ldots, i_n).$$

Now the proof of Theorem 2 can be repeated to obtain that $E = E_n$ for some n, and this completes the proof. □

4. Continuous operators on Baire spaces and topological representations of Baire algebras

Proposition 2. *Let $\varphi : E \to E$ be a continuous linear transformation of a Baire topological vector space E over a complete non-discrete valuated field K. Then φ is algebraic as soon as it is weakly algebraic.*

Proof. Set $f_n(x) = x\varphi^n$, $x \in E$, $n \in \mathbb{N}$. Then $\mathcal{F} = \{f_n\}$ is a extremal sequence of continuous functions. It is weakly linearly dependent if and only if φ is weakly algebraic and $\mathcal{F}$ is weakly linearly dependent of bounded degree if and only if φ is weakly algebraic of bounded degree. By Theorem 1 φ is weakly algebraic of bounded degree and thus algebraic by Proposition 1. □

For Banach spaces see again [11].

Let K be a topological field and M be a topological vector space over K. Let A be a topological algebra over K of an arbitrary variety of algebras and $(m, a) \mapsto m \cdot a \in A$, $m \in M$, $a \in A$, be a mapping continuous in each variable which is linear in m and is homogeneous of some degree in a. We shall say that M is a topological module over A although actually this is a much more general object as the product may not be jointly continuous and the product may not be linear in a. By letting $\rho_a : m \mapsto m \cdot a$ we obtain a continuous representation $\rho : A \mapsto \mathrm{End}_K(M)$ of A by endomorphisms of M, where the vector space $\mathrm{End}_K(M)$ is endowed with the topology of the pointwise convergence.

Definition 9. We shall say that a representation ρ is weakly algebraic (algebraic) if for every $a \in A$ the operator ρ_a is weakly algebraic (algebraic). A representation ρ is said to be algebraic of bounded degree if all operators ρ_a are algebraic and the degrees of algebraicity of these operators are overall bounded.

Theorem 4. *If ρ is a weakly algebraic representation of a Baire algebra A over a complete non-discrete valuated field K by endomorphisms of a Baire module M, then ρ is algebraic of bounded degree.*

Proof. By Proposition 2, ρ is algebraic. Moreover, the functions $f_n : x \mapsto \rho_x^n$ are continuous. They are also homogeneous and thus the sequence $\mathcal{F} = \{f_n\}$ is extremal. Now Theorem 4 follows from Theorem 2. □

Proof of Theorem A. This is a direct consequence of Theorem 4. Consider $\rho_x = \mathrm{ad}\, x$. □

5. Applications to Jordan algebras

Let J be a Jordan algebra over a field K and M be a Jordan module over J. Define an operator $m \mapsto mU_x = \{xmx\} = 2(mx)x - mx^2$, $m \in M$, $x \in J$. The quadratic representation $U : x \mapsto U_x$ is known to play a very important role in Jordan theory [9]. Theorem 4 allows us to obtain the following.

Corollary 1. *Let J be a Baire Jordan algebra over a complete non-discrete valuated field K and M be a Baire Jordan module over J. Then if the quadratic representation U is algebraic, it is algebraic of bounded degree.* □

There is another interesting quadratic operator in Jordan algebras [9], namely $m \mapsto mN_x = (m, x, x) = (mx)x - mx^2$. Using this operator it is possible to define associator nil and associator algebraic algebras. Of course, for this notions a statement similar to the preceding corollary can be proved.

6. Arbitrary nonassociative algebras

Let $F_K(x)$ be the free nonassociative algebra over K generated by one element x. Elements of $F_K(x)$ are said to be nonassociative polynomials, those obtained without use of summation and multiplication by scalars are said to be nonassociative monomials. Let A be an arbitrary (not necessarily power associative) algebra over K. Given $a \in A$ and $p \in F_K(x)$, respectively, we will denote by $p(a)$ the image of p under the unique homomorphism $F_K(x) \to A$ which maps x to a. We note the intuitively obvious fact that, if A is a topological algebra, then for each p in $F_K(x)$ the mapping $a \in A \to p(a) \in A$ is continuous. This can be formally verified by writing p as a linear combination of elements in the "free monad" generated by $\{x\}$ and then reasoning by induction on the "degree" of such elements (see [9] for details).

Proof of Theorem D. Write down in a sequence $\mathcal{F} = \{f_1(x), \ldots, f_n(x), \ldots\}$ all non-associative monomials of $F_K(x)$. Now consider $f_n(x)$ as a continuous function on A. Since $f_n(x)$ is homogeneous $\mathcal{F}$ is extremal and Theorem 3 can be applied. □

An immediate consequence of Theorem D (and Lemma 2 for the "if" part) is the following.

Corollary 2. *Let A be a Baire algebra over a complete non-discrete valuated field K and B be a dense subalgebra of A. Then A is algebraic if and only if B is algebraic of bounded degree. Moreover, if this is the case, then A is of bounded degree that is equal to the one of B.*

As a consequence, *the completion of a normed algebra A is algebraic if and only if A is algebraic of bounded degree.* It is easy to see that, *if K is an algebraically closed field, then every nonzero algebraic algebra over K with no* *nonzero zero-divisors is isomorphic to K*. Also, by a result of Bott and Milnor ([3]), every finite-dimensional nonzero real algebra with no nonzero zero-divisors is of dimension 1, 2, 4 or 8, hence *algebraic real algebras with no nonzero zero-divisors are of bounded degree less than or equal to 8*. Now we have also the following corollary.

Corollary 3. *Let A be a Baire real algebra containing a dense algebraic subalgebra with no nonzero zero-divisors in itself. Then A is algebraic of bounded degree less than or equal to* 8.

In particular, *the completion of a normed algebraic real algebra with no nonzero zero-divisors is algebraic of bounded degree less or equal* 8.

References

1. Albert, A.A.: Absolute valued algebraic algebras. Bull. Amer. Math. Soc. **55**,763–768 (1949). A note of correction. *Ibid.* **55**, 1191 (1949)
2. Bakhturin, Y.; Mikhalev, A.; Petrogradsky, V. and Zaicev, M.: Infinite-dimensional Lie Superalgebras. Berlin New York: Walter de Gruyter and Co, 1992
3. Bott, R. and Milnor, J.: On the parallelizability of the spheres . Bull. Amer. Math. Soc. **64**, 87–89 (1958)
4. Bourbaki, N.: Algèbre (Chapter 7) Paris: Hermann, 1952
5. Bourbaki, N.: Algèbre commutative (Chapter 5 et 6) Paris: Hermann 1964
6. Bourbaki, N.: Topologie Générale (Chap. 5 à 10) Paris: Hermann 1974
7. Cuartero, B. and Galé, J.E.: Locally PI-algebras over valued fields, in *Aportaciones Matemáticas en Memoria del Profesor V.M. Onieva*, Santander: Universidad de Cantabria 1991
8. Cuartero, B. and Galé, J.E.: Bounded degree of algebraic topological algebras. Comm. in Algebra (to appear)
9. Jacobson, N.: Structure and representations of Jordan algebras. (Colloq. Publ., Vol. **39**) Providence, R.I.: Amer. math. Soc. 1968
10. Kaplansky, I.: Topological methods in valuation theory. Duke Math. J. **14**, 527–541 (1947)
11. Kaplansky, I.: Infinite abelian groups. University of Michigan Press, 1954
12. Kostrikin, A.I.: Around Burnside (Russian), Moscow: Nauka 1986
13. Mazur, S.: Sur les anneaux linéaires. Comptes Rendus de l'Academie des sciences **207**, 1025-1027 (1938)
14. Ostrowski, A.: Über einige Fragen der allgemeinen Körpertheorie. Journal für die reine und angewandte Mathematik **143**, 255–284 (1913)
15. Ostrowski, A.: Über sogenannte perfekte Körper. Journal für die reine und angewandte Mathematik **147**, 191–204 (1917)
16. Slinko, A.M.: Complete metric Jordan algebras. Mat. Zametki **47** no. 5, 100–105 (1990). English translation: Math. Notes **47**, no. 5-6 , 491–494 (1990)
17. Zelmanov, E.I.: On Engel Lie algebras. Soviet Math. Dokl. **35**, 44–47 (1987)

Bienvenido Cuartero and José E. Galé
Departamento de Matemáticas
Universidad de Zaragoza
50009 - Zaragoza
ESPAÑA

Angel Rodríguez Palacios
Departamento de Análisis Matemático.
Facultad de Ciencias.
Universidad de Granada
18071 - Granada
ESPAÑA

Arkadii M. Slinko
Department of Mathematics and Statistics
University of Auckland
Auckland
NEW ZEALAND

This article was processed by the author
using the Springer-Verlag TEX mamath macro package 1990.

(Received April 28, 1993)

manuscripta math. 81, 141 - 147 (1993)

manuscripta mathematica

Homological Characterization of Lean Algebras

ISTVÁN ÁGOSTON[1], VLASTIMIL DLAB[2] AND ERZSÉBET LUKÁCS[1]

Certain classes of lean quasi-hereditary algebras play a central role in the representation theory of semisimple complex Lie algebras and algebraic groups. The concept of a lean semiprimary ring, introduced recently in [1] is given here a homological characterization in terms of the surjectivity of certain induced maps between Ext^1-groups. A stronger condition requiring the surjectivity of the induced maps between Ext^k-groups for all $k \geq 1$, which appears in the recent work of Cline, Parshall and Scott on Kazhdan–Lusztig theory, is shown to hold for a large class of lean quasi-hereditary algebras.

Throughout the paper R will denote a basic semiprimary ring with identity; thus the (Jacobson) radical J of R is nilpotent and R/J is a finite product of division rings. Let us fix a complete ordered set of primitive orthogonal idempotents $(e_1, e_2, \dots, e_n)$ and define for $1 \leq i \leq n$ the idempotent elements $\varepsilon_i = e_i + e_{i+1} + \dots + e_n$; set $\varepsilon_{n+1} = 0$. Thus, we have fixed an order on the set of the corresponding simple (right) R-modules $S(i)$ and their projective covers $P(i) \simeq e_iR$, $1 \leq i \leq n$. The corresponding left R-modules will be denoted by $S^\circ(i)$ and $P^\circ(i)$, respectively.

The *(right) standard modules* $\Delta(i)$ are defined by $\Delta(i) \simeq e_iR/e_iR\varepsilon_{i+1}R$. The submodule $e_iR\varepsilon_{i+1}R$ will be denoted by $V(i)$. Thus we have the exact sequence $0 \to V(i) \to P(i) \to \Delta(i) \to 0$. Similarly, we can define the *left standard modules* $\Delta^\circ(i)$ and the corresponding kernels $V^\circ(i)$.

The module $\Delta(i)$ is *Schurian* if $\mathrm{End}_R(\Delta(i))$ is a division ring. It is easy to see that $\Delta(i)$ is Schurian if and only if $\Delta^\circ(i)$ is Schurian.

1991 Mathematics Subject Classification. Primary 16E99, 16S99. Secondary 17B10

[1] Research partially supported by NSERC of Canada and by Hungarian National Foundation for Scientific Research grant no. 1903

[2] Research partially supported by NSERC of Canada

The ring R is *quasi-hereditary* (see [2]) with respect to the order $(e_1, e_2, \ldots, e_n)$ if $\Delta(i)$ is Schurian for every $1 \le i \le n$ and the regular module R_R has a filtration $R_R = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_\ell \supseteq X_{\ell+1} = 0$ such that every factor X_i/X_{i+1}, $1 \le i \le \ell$ is isomorphic to a standard module $\Delta(j)$ for some $1 \le j \le n$. For basic facts concerning quasi-hereditary algebras we refer the reader to [3] and [4].

Let us now recall the definition of a top embedding ([1]). Let X and Y be arbitrary (right) R-modules. An embedding $f : X \to Y$ is called a *top embedding* if it induces an embedding $\bar{f} : X/\operatorname{rad} X = \operatorname{top} X \to \operatorname{top} Y = Y/\operatorname{rad} Y$. In this case we write $X \overset{t}{\subseteq} Y$. Note that, for a submodule $X \subseteq Y$, the condition $X \overset{t}{\subseteq} Y$ is equivalent to $\operatorname{rad} X = \operatorname{rad} Y \cap X$. A filtration $X = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m \supseteq X_{m+1} = 0$ of a module X is called a *top filtration* of X if $X_i \overset{t}{\subseteq} X$ for every $2 \le i \le m$. If $\mathcal{M}$ is a class of modules, then we will say that X has a *top filtration by* $\mathcal{M}$ if X has a top filtration $X = X_1 \supseteq X_2 \supseteq \ldots \supseteq X_m \supseteq X_{m+1} = 0$ such that the factor modules X_i/X_{i+1} belong to $\mathcal{M}$ for $1 \le i \le m$.

The semiprimary ring R is called *lean* with respect to the order $(e_1, e_2, \ldots, e_n)$ if $e_i J^2 e_j \subseteq e_i J \varepsilon_m J e_j$ for $m = \min\{i, j\}$ and $1 \le i, j \le n$. Theorem 2.1 of [1] asserts that A is lean if and only if $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ and $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ for all $1 \le i \le n$.

Lemma 1. *Let X be an arbitrary R-module and S a semisimple submodule of* $\operatorname{rad} X$. *Denote by Y the factor module X/S. Then the following statements are equivalent:*

(a) $S \overset{t}{\subseteq} \operatorname{rad} X$;

(b) *there exists an extension* $\zeta \in \operatorname{Ext}^1(\operatorname{top} Y, S)$ *such that the following diagram is commutative:*

$$\begin{array}{ccccccccc}
\zeta^\ell : 0 & \longrightarrow & S & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
\zeta : 0 & \longrightarrow & S & \longrightarrow & X' & \longrightarrow & \operatorname{top} Y & \longrightarrow & 0;
\end{array}$$

(c) *there exists a semisimple module T and an extension* $\rho \in \operatorname{Ext}^1(T, S)$ *such that the following diagram is commutative:*

$$\begin{array}{ccccccccc}
\rho^\ell : 0 & \longrightarrow & S & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\
\rho : 0 & \longrightarrow & S & \xrightarrow{\iota} & X'' & \longrightarrow & T & \longrightarrow & 0.
\end{array}$$

Proof. To prove (a) $\Rightarrow$ (b), observe that since S is semisimple, $S \overset{t}{\subseteq} \operatorname{rad} X$ implies that S is a direct summand of $\operatorname{rad} X$. Let C be a direct complement of S in rad X. Then we have the following diagram with the natural maps:

$$\begin{array}{ccccccccc} & & & & C & = & C & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & X/C & \longrightarrow & Y' & \longrightarrow & 0. \end{array}$$

Note that $Y' \simeq X/(C \oplus S) = X/\operatorname{rad} X = \operatorname{top} X \simeq \operatorname{top} Y$.

Since the implication $(b) \Rightarrow (c)$ is trivial, we have to show only that $(c) \Rightarrow (a)$. We need that $XJ^2 \cap S = 0$. Let us assume that $0 \neq S' = XJ^2 \cap S$. Then $0 \neq \iota(S') = \varphi(S') \subseteq \varphi(XJ^2) = \varphi(X)J^2$. But $\varphi(X) \subseteq X''$ and $X''J^2 = 0$, a contradiction. Thus $S \overset{t}{\subseteq} \operatorname{rad} X$. □

Proposition 2. *Let P_R be an indecomposable projective R-module and $V \subseteq \operatorname{rad} P$. Denote by W the factor module P/V. Then the following are equivalent:*

(a) $V \overset{t}{\subseteq} \operatorname{rad} P$;

(b) $\operatorname{Ext}^1(\operatorname{top} W, S) \to \operatorname{Ext}^1(W, S)$ *is an epimorphism for every simple module S.*

Proof. $(a) \Rightarrow (b)$ Consider a non-split exact sequence $0 \to S \to X \to W \to 0$; thus $S \subseteq \operatorname{rad} X$. Using the projectivity of P we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & V & \to & P & \to & W & \to & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \| & & \\ 0 & \to & S & \to & X & \to & W & \to & 0. \end{array}$$

Here ψ is an epimorphism, since $S \subseteq \operatorname{rad} X$. It follows that φ is also an epimorphism. We get that $S \overset{t}{\subseteq} \operatorname{rad} X$ since $V \overset{t}{\subseteq} \operatorname{rad} P$ by assumption. Thus, by Lemma 1, the sequence $0 \to S \to X \to W \to 0$ is a lifting of a sequence $0 \to S \to X' \to \operatorname{top} W \to 0$ along the natural map $W \to \operatorname{top} W$, so it is in the image of $\operatorname{Ext}^1(\operatorname{top} W, S) \to \operatorname{Ext}^1(W, S)$.

$(b) \Rightarrow (a)$ To prove that $V \overset{t}{\subseteq} \operatorname{rad} P$, it is sufficient to show that $V/V' \overset{t}{\subseteq} \operatorname{rad} P/V'$ for an arbitrary maximal submodule V' of the module V. Hence consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & V & \to & P & \to & W & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & V/V' & \to & P/V' & \to & W & \to & 0. \end{array}$$

Since V/V' is simple, (b) implies that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & V/V' & \to & P/V' & \to & W & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & V/V' & \to & Z & \to & \operatorname{top} W & \to & 0\,. \end{array}$$

By Lemma 1, we get that $V/V' \overset{t}{\subseteq} \operatorname{rad} P/V'$. □

Using Proposition 2 and the left dual version of it, we get immediately the following characterization of lean semiprimary rings.

Theorem 3. *Let $(e_1, e_2, \ldots, e_n)$ be a complete set of primitive orthogonal idempotents of the semiprimary ring R and let all standard modules $\Delta(i)$ be Schurian. Then R is lean with respect to the given order of idempotents if and only if the natural maps $\operatorname{Ext}^1(S(i), S(j)) \to \operatorname{Ext}^1(\Delta(i), S(j))$ and $\operatorname{Ext}^1(S^\circ(i), S^\circ(j)) \to \operatorname{Ext}^1(\Delta^\circ(i), S^\circ(j))$ are epimorphisms for all $1 \le i, j \le n$.*

Proof. Proposition 2 implies that the surjectivity of the maps given above is equivalent to the condition that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ and $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ for all $1 \le i \le n$. In turn, by Theorem 2.1 of [1], this is equivalent to the fact that R is lean. □

In what follows, let us restrict our attention to the case when $R = A$ is a finite dimensional K-algebra, where K is a field. For every $1 \le i \le n$, denote by $\nabla(i)$ the K-dual of $\Delta^\circ(i)$, and call the modules $\nabla(i)$ the *(right) costandard modules*. Using this terminology, we get the following characterization of lean quasi-hereditary K-algebras.

Corollary 4. *Let A be a quasi-hereditary K-algebra with respect to the order $(e_1, e_2, \ldots, e_n)$. Then A is lean with respect to the same order if and only if the natural maps $\operatorname{Ext}^1(S(i), S(j)) \to \operatorname{Ext}^1(\Delta(i), S(j))$ and $\operatorname{Ext}^1(S(j), S(i)) \to \operatorname{Ext}^1(S(j), \nabla(i))$ are epimorphisms for $1 \le i, j \le n$.*

In their contributions to the Workshop on Representation Theory held in Ottawa in August 1992, B.J. Parshall and L.L. Scott emphasized the importance of the surjectivity of all natural maps $\operatorname{Ext}^k(S(i), S(j)) \to \operatorname{Ext}^k(\Delta(i), S(j))$, $k \ge 1$, for the Kazhdan–Lusztig theory. In this connection, the following theorem and its corollary seem to be of some interest.

Theorem 5. *Let A be a quasi-hereditary K-algebra with respect to the order $(e_1, e_2, \ldots, e_n)$ such that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$ for $1 \le i \le n$. Suppose that for every $1 \le i \le n$, the module $V(i)$ has a top filtration by $\Delta(j)$'s and $P(j)$'s, $i+1 \le j \le n$. Then the natural maps $\operatorname{Ext}^k(S(i), S(j)) \to \operatorname{Ext}^k(\Delta(i), S(j))$ are surjective for all $1 \le i, j \le n$ and $k \ge 1$.*

For the proof of Theorem 5 we shall need the following simple lemma.

Lemma 6. *Let $0 \to X \xrightarrow{\mu} Y \to Z \to 0$ be a short exact sequence with a top embedding μ. If, for a module S and for some $k \geq 1$, the natural maps $\mathrm{Ext}^k(\mathrm{top}\, X, S) \to \mathrm{Ext}^k(X, S)$ and $\mathrm{Ext}^k(\mathrm{top}\, Z, S) \to \mathrm{Ext}^k(Z, S)$ are surjective, then so is the natural map $\mathrm{Ext}^k(\mathrm{top}\, Y, S) \to \mathrm{Ext}^k(Y, S)$.*

Proof. The bottom sequence of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \to & Y & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{top}\, X & \to & \mathrm{top}\, Y & \to & \mathrm{top}\, Z & \to & 0 \end{array}$$

clearly splits. Thus, applying the functor $\mathrm{Hom}(-, S)$, we can derive easily the following commutative diagram from the long exact sequences:

$$\begin{array}{ccccccccc} \mathrm{Ext}^{k-1}(X,S) & \to & \mathrm{Ext}^k(Z,S) & \to & \mathrm{Ext}^k(Y,S) & \to & \mathrm{Ext}^k(X,S) & \to & \mathrm{Ext}^{k+1}(Z,S) \\ \uparrow & & \uparrow \gamma & & \uparrow \beta & & \uparrow \alpha & & \uparrow \\ 0 & \to & \mathrm{Ext}^k(\mathrm{top}\, Z,S) & \to & \mathrm{Ext}^k(\mathrm{top}\, Y,S) & \to & \mathrm{Ext}^k(\mathrm{top}\, X,S) & \to & 0\,. \end{array}$$

Since α and γ are surjective, we get that β is surjective as well. □

Proof of Theorem 5. We proceed by induction. Proposition 2 implies that the statement holds for $k = 1$. Thus assuming the statement for some $k \geq 1$, we want to show that for every exact sequence

$$(*) \qquad 0 \to S(j) \to X_1 \to \ldots \to X_k \to X_{k+1} \to \Delta(i) \to 0$$

there is a commutative diagram of exact sequences with the natural projection $\Delta(i) \to S(i)$:

$$\begin{array}{ccccccccccccc} 0 & \to & S(j) & \to & Y_1 & \to & \ldots & \to & Y_k & \to & P(i) & \to & \Delta(i) & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & S(j) & \to & Z_1 & \to & \ldots & \to & Z_k & \to & Z_{k+1} & \to & S(i) & \to & 0\,, \end{array}$$

in which the first row is equivalent to $(*)$.

Let us write $(*)$ as the Yoneda composite of the following exact sequences:

$$0 \to S(j) \to X_1 \to \ldots \to X_k \to N \to 0 \ \text{ and } \ 0 \to N \to X_{k+1} \to \Delta(i) \to 0\,.$$

In view of the commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & V(i) & \to & P(i) & \to & \Delta(i) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & N & \to & X_{k+1} & \to & \Delta(i) & \to & 0 \end{array}$$

and

$$\begin{array}{ccccccccccccc} 0 & \to & S(j) & \to & Y_1 & \to & \dots & \to & Y_k & \to & V(i) & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & S(j) & \to & X_1 & \to & \dots & \to & X_k & \to & N & \to & 0 \end{array}$$

the sequence $(*)$ is equivalent to

$$0 \to S(j) \to Y_1 \to \dots \to Y_k \to P(i) \to \Delta(i) \to 0 .$$

Now, by the induction hypothesis and by repeated use of Lemma 6, we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccccccc} 0 & \to & S(j) & \to & Y_1 & \to & \dots & \to & Y_k & \to & V(i) & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \to & S(j) & \to & Z_1 & \to & \dots & \to & Z_k & \to & \operatorname{top} V(i) & \to & 0 . \end{array}$$

Furthermore, in view of Proposition 2, there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & V(i) & \to & P(i) & \to & \Delta(i) & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \operatorname{top} V(i) & \to & Z & \to & \Delta(i) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{top} V(i) & \to & Z_{k+1} & \to & S(i) & \to & 0 . \end{array}$$

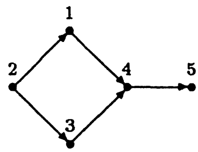
Hence the theorem follows. □

Corollary 7. *Let A be a shallow, medial or replete quasi-hereditary algebra with respect to $(e_1, e_2, \dots, e_n)$. Then the natural maps $\operatorname{Ext}^k(S(i), S(j)) \to \operatorname{Ext}^k(\Delta(i), S(j))$ and $\operatorname{Ext}^k(S^\circ(i), S^\circ(j)) \to \operatorname{Ext}^k(\Delta^\circ(i), S^\circ(j))$ are surjective for all $1 \le i, j \le n$ and $k \ge 1$.*

The definition of shallow, right medial, left medial and replete algebras can be found in [1]. For the convenience of the reader, we wish to recall that these algebras are defined by the fact that $V(i) \overset{t}{\subseteq} \operatorname{rad} P(i)$, $V^\circ(i) \overset{t}{\subseteq} \operatorname{rad} P^\circ(i)$ and, respectively, $V(i)$ and $V^\circ(i)$ have top filtrations by $\Delta(j)$'s and $\Delta^\circ(j)$'s, by $\Delta(j)$'s and $P^\circ(j)$'s, by $P(j)$'s and $\Delta^\circ(j)$'s and, finally, by $P(j)$'s and $P^\circ(j)$'s.

Remark. Let us point out that, in general, lean quasi-hereditary algebras do not satisfy the above surjectivity conditions for higher Ext-groups. Here is a simple example.

Let A be the path algebra of the graph



modulo the relations $\alpha_{14}\alpha_{45} = 0$ and $\alpha_{21}\alpha_{14} = \alpha_{23}\alpha_{34}$ (where α_{ij} denotes the arrow from i to j). Thus the right regular representation of A can be described by the following charts of composition factors:

$$A_A = \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus 5 \; .$$

One can check easily that A is lean. On the other hand $\mathrm{Ext}^2\left(S(2), S(5)\right) = 0$, while $\mathrm{Ext}^2\left(\Delta(2), S(5)\right) \neq 0$.

References

[1] Ágoston, I., Dlab, V., Lukács, E.: Lean quasi-hereditary algebras. *Can. Math. Soc. Conference Proceedings Series* Vol. **13**, 1–14 (1993)

[2] Cline, E., Parshall, B.J., Scott, L.L.: Finite dimensional algebras and highest weight categories. *J. Reine Angew. Math.* **391**, 85–99 (1988)

[3] Dlab, V., Ringel, C.M.: Quasi-hereditary algebras. *Illinois J. of Math.* **35**, 280-291 (1989)

[4] Dlab, V., Ringel, C.M.: The module theoretical approach to quasi-hereditary algebras. *Representations of Algebras and Related Topics*, London Math. Soc. Lecture Note Series **168**, 200–224 (1992)

Mathematical Institute of the Hungarian Academy of Sciences, P.O.Box 127, 1364 Budapest, Hungary
E-mail address: h4134ago@ella.hu

Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada
E-mail address: vdlab@ccs.carleton.ca

Department of Mathematics, Faculty of Transport Engineering, Technical University of Budapest, 1111 Budapest, Hungary
E-mail address: h4091luk@ella.hu

(Received May 6, 1993)

manuscripta math. 81, 149 - 159 (1993)

manuscripta mathematica

Determining Boundary Sets of Bounded Symmetric Domains

José M. Isidro*,** Wilhelm Kaup*

We consider bounded symmetric domains in complex Banach spaces. It is known that each of these domains can be realized as open unit ball D of a uniquely determined complex Banach space E and that every biholomorphic automorphism g of D extends holomorphically to the closure $\overline{D}$ of D in E. We study subsets S of $\overline{D}$ (and in particular of the boundary ∂D) such that every automorphism g is already uniquely determined by its values on S. We also consider subsets S with the analogue topological property: For every sequence (g_n) of automorphisms converging uniformly on S to g the convergence is already uniform on D.

0. Introduction

Consider the open unit disc $\Delta \subset \mathbb{C}$ and denote by G the group of all biholomorphic automorphisms of Δ. Then it is easily verified that every $g \in G$ is uniquely determined within G by its values at any pair of two different points $z_1, z_2 \in \Delta$. It is also classical that every $g \in G$ is linear fractional and extends to a biholomorphic transformation of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Therefore every $g \in G$ also is uniquely determined by its values on any triple of different points z_1, z_2, z_3 in the boundary $\partial\Delta$ of Δ with pairs not sufficing in this situation.

As a consequence of Schwarz Lemma also every $g \in G$ is uniquely determined by $g(a)$ and $g'(a)$ for given $a \in \Delta$. Again, in case $a \in \partial\Delta$ the automorphism g can only be recovered within G from the *three* derivatives $g(a), g'(a)$ and $g''(a)$ in a.

A natural generalization of the open unit disc $\Delta \subset \mathbb{C}$ to higher (and even infinite) dimension are the bounded symmetric domains in their standard realization. These are precisely the open unit balls D of complex Banach spaces for which the group G of all biholomorphic automorphisms acts transitively on D.

In this note we study determining sets $S \subset \overline{D}$ in the closure of the bounded symmetric domain D for the full automorphism group $G = \mathrm{Aut}(D)$, the connected

* Supported by Acción Integrada Hispano-Alemana
** Supported by Xunta de Galicia, project Xuga 20702 B 90

identity component G^0 of G and also for the corresponding infinitesimal group, i.e. the Lie algebra $\mathfrak{g}$ of all complete holomorphic vector fields on D. The case where S contains a point $a \in D$ is conjugate (mod G) to the case $0 \in D$ which reduces the problem to a linear one (every $g \in G$ fixing the origin is well known to be linear). We are therefore mainly interested in the case of determining sets $S \subset \partial D$.

1. Preliminaries

Let D be a bounded domain in a complex Banach space E. By definition, a function $f: D \to E$ is called *holomorphic* if for every $a \in D$ the Fréchet derivative $f'(a) \in \mathcal{L}(E)$ exists as a continuous linear operator on E. A bijection $g: D \to D$ is called *biholomorphic* or an *automorphism of* D if g and g^{-1} are holomorphic. Denote by $G := \mathrm{Aut}(D)$ the group of all biholomorphic automorphisms of D and endow G with the topology of local uniform convergence in D. The domain D is called *symmetric* if to every $a \in D$ there is an automorphism $s = s_a \in G$ with $s^2 = \mathrm{id}$ having a as isolated fixed point. Then it is known that G acts transitively on D and that there is a complex Banach space E (uniquely determined up to an isometric isomorphism) such that D is biholomorphically equivalent to the open unit ball of E (compare [7] for details).

A complex Banach space E is called a *JB*-triple* if the open unit ball $D \subset E$ is symmetric, or equivalently if the automorphism group $G := \mathrm{Aut}(D)$ acts transitively on D. Then there exists a uniquely determined continuous ternary operation (called the *Jordan triple product on* E) $(x, y, z) \mapsto \{xyz\}$ from E^3 to E such that, by writing $x \square y$ for the linear operator $z \mapsto \{xyz\}$ on E, the following axioms are satisfied

(J_1) $\{xyz\}$ is symmetric bilinear in the outer variables x, z and conjugate linear in the inner variable y

(J_2) $[x \square x, y \square y] = \{xxy\} \square y + y \square \{yxx\}$

(J_3) $x \square x$ is hermitian and has spectrum ≥ 0

(J_4) $\|\{xxx\}\| = \|x\|^3$

for all $x, y \in E$ and $[\ ,\ ]$ being the commutator product of linear operators.

On the other hand, every complex Banach space E admitting a continuous mapping $\{,,\}$ with $(J_1) - (J_4)$ is a JB*-triple.

It is known [3] that for every x, y, z in a JB*-triple the following estimate holds

$$\|\{xyz\}\| \leq \|x\| \cdot \|y\| \cdot \|z\|. \tag{1.1}$$

A trivial but usefull consequence is

$$\|(1 + z \square a)^{-1}\| \leq (1 - r\|z\|)^{-1} \leq (1 - r)^{-1} \tag{1.2}$$

for all $z \in \overline{D}$ and $r := \|a\| < 1$.

For instance, every C*-algebra A is a JB*-triple. The triple product then is given by

$$\{xyz\} = (xy^*z + zy^*x)/2. \tag{1.3}$$

Also for every pair H, K of complex Hilbert spaces the space $\mathcal{L}(H,K)$ of all bounded linear operators $H \to K$ endowed with the operator norm and triple product (1.3) is a JB*-triple.

A linear subspace $F \subset E$ is called a *subtriple* if $\{FFF\} \subset F$ holds. Every closed subtriple clearly is a JB*-triple itself. The subtriple F is called an *ideal* in E if $\{EEF\} \cup \{EFE\} \subset F$ holds. For every closed ideal $F \subset E$ also the quotient E/F is a JB*-triple.

For every $\alpha \in E$ the odd powers α^{2n+1} of α are defined as $(\alpha \square \alpha)^n \alpha$. The closed linear span E_α of all odd powers of α in E is a subtriple, called the closed subtriple *generated by* $\alpha \in E$. Then $E_\alpha \square E_\alpha$ is a commutative set of operators in $\mathcal{L}(E)$ by [2] p. 102. Denote by Σ the spectrum of the operator $\alpha \square \alpha | E_\alpha \in \mathcal{L}(E_\alpha)$ and consider $\Omega := \{t > 0 : t^2 \in \Sigma\}$. Then there is a unique (isometric) JB*-triple isomorphism $\tau: E_\alpha \to \mathcal{C}_0(\Omega)$ such that $h := \tau(\alpha)$ is the function $h(\omega) \equiv \omega$ on Ω. For every $f \in \mathcal{C}_0(\Omega)$ we denote the element $\tau^{-1}(f) \in E_\alpha$ by $f(\alpha)$. Corollary (3.5) in [7] implies

$$\|f(\alpha) \square g(\alpha)\| = \|fg\| \tag{1.4}$$

for every $f, g \in \mathcal{C}_0(\Omega)$. As an example, $\tanh: E \to D$ is a real analytic (not holomorphic) mapping with analytic inverse $\tanh^{-1}: D \to E$.

It is known that $G = \mathsf{Aut}(D)$ is a real Banach Lie group in the topology of locally uniform convergence in D. The subgroup $K := \{g \in G : g(0) = 0\}$ is the group of all (surjective) linear isometries of the Banach space E (restricted to D, clearly) and coincides also with the group of all linear triple automorphisms of E (for this and the following compare [7]). It is an algebraic subgroup of $\mathsf{GL}(E)$ in the sense of [5] (over $\mathbb{R}$). The Lie algebra $\mathfrak{g}$ of G can be identified with the space $\mathfrak{aut}(D)$ of all complete holomorphic vector fields on E. By definition, a holomorphic function $f: D \to E$ is called a *complete holomorphic vector field* on D if for every $z \in D$ the differential equation

$$\frac{\partial h_t}{\partial t} = f(h_t) \tag{1.5}$$

has a solution $h_t(z) \in D$ to the initial value $h_0(z) = z$ for all real t. Then $\{h_t : t \in \mathbb{R}\}$ is a one parameter subgroup of G. We conceive vector fields as differential operators and write also $X = f(z)\frac{\partial}{\partial z}$ instead of f and $\exp(tX)$ instead of h_t.

The Lie algebra $\mathfrak{g}$ has a decomposition as direct sum of closed linear subspaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{1.6}$$

where $\mathfrak{k}$ is the Lie subalgebra of all triple derivations of E and $\mathfrak{p}$ is the Lie triple system of all constant-quadratic vector fields $X^\alpha := \alpha - \{z\alpha z\}\frac{\partial}{\partial z}$, $\alpha \in E$. By definition, a linear operator $\delta: E \to E$ is called a triple derivation if

$$\delta\{xyz\} = \{(\delta x)yz\} + \{x(\delta y)z\} + \{xy(\delta z)\}$$

holds for every $x, y, z \in E$. Every derivation of E is automatically continuous [1] and every $\delta = i(\alpha \square \alpha)$ is a derivation by (J_2).

The analogue of the decomposition (1.6) on the group level is

$$G = KP = PK \qquad \text{with} \qquad P := \exp(\mathfrak{p}).$$

P is a closed real-analytic submanifold of G and $\exp: \mathfrak{p} \to P$ is bianalytic. To every $a \in D$ there is a unique $p \in P$ with $p(0) = a$. We denote this automorphism p by g_a in the following. There is a unique vector field $X^\alpha \in \mathfrak{p}$ with $g_a = \exp(X^\alpha)$. The elements a and α generate the same closed subtriple of E and they are related by the formulas $a = \tanh(\alpha)$, $\alpha = \tanh^{-1}(a)$ in the above sense of continuous functional calculus. In particular $g_0 = \mathrm{id}$ and $g_{-a} = g_a^{-1}$ for all $a \in D$.

The Lie group K acts by inner automorphisms on G thereby leaving the subsets P and K invariant. In particular $kg_ak^{-1} = g_{ka}$ holds for every $a \in D$ and $k \in K$. Every $g \in G$ has a unique representation of the form $g = g_ak$ with $k \in K$ and hence $a = g(0)$. Furthermore, the mapping

$$G \to D \times K \qquad \text{defined by} \qquad g \mapsto (a, g_{-a}g), \quad a := g(0)$$

is a K-equivariant isomorphism of real analytic manifolds.

We need several asymptotic estimates on g as g(0) approaches the origin:

1.7 Proposition. *Let $g \in G$ be an arbitrary automorphism and put $a := g(0)$, $r := \|a\|$ and $\lambda := g'(0)$. Then g extends holomorphically to a neighbourhood of $\overline{D}$ and for every $z, w \in \overline{D}$ the following estimates hold*

(i) $\|\lambda\| \le 1$
(ii) $\|g(z) - \lambda(z) - a\| \le r/(1-r)$
(iii) $\|g(z) - g(w)\| \le \|z - w\|/(1-r)^2$.

Proof. (i) is obvious from the integral formula $\lambda(z) = (2\pi i)^{-1} \int_{|t|=1} t^{-2} g(tz)dt$ valid for all $z \in D$. As a consequence of [7] p. 132 one has

$$g(z) = a + \lambda(1 + z \square a)^{-1} z \tag{1.8}$$

for all $z \in D$. This implies with (1.2) that g extends holomorphically to $\overline{D}$. (ii) follows with

$$g(z) - \lambda(z) - a = -\lambda(z \square a)(1 + z \square a)^{-1} z$$

and (iii) follows from

$$g(z) - g(w) = \lambda(1 + z \square a)^{-1}\Big((z - w) + ((z - w) \square a)(1 + w \square a)^{-1} w\Big) . \qquad \square$$

1.9 Proposition. *Let $g, h \in G$ be arbitrary automorphisms and put $a := g(0)$, $b := h(0)$, $\lambda := g'(0)$, $\mu := h'(0)$ and $r := \|a\|$, $s := \|b\|$. Then for all $z \in \overline{D}$*

$$\|g(z) - h(z)\| \le \Big(1 + \frac{1}{(1-r)(1-s)}\Big)\|a - b\| + \frac{1}{1-r}\|\lambda - \mu\| .$$

Proof. Equation (1.8) for h reads $h(z) = b + \mu(1 + z \square b)^{-1} z$. Then $g(z) - h(z) = (a-b) - \mu(1 + z \square a)^{-1}\big(z \square (a-b)\big)(1 + z \square b)^{-1} z + (\lambda - \mu)(1 + z \square a)^{-1} z$ immediately gives the statement. $\square$

As a consequence we get a result already contained in [11]

1.10 Corollary. *On the group $G = \mathsf{Aut}(D)$ the topology of local uniform convergence coincides with the topology of uniform convergence on $\overline{D}$.*

Notice, that (1.9) also contains a quantitave version of the topological Cartan's uniqueness Theorem: *For every $a \in D$ the map $g \mapsto (g(a), g'(a))$ defines a homeomorphism of G to a subspace of $D \times \mathsf{GL}(E)$.* By a result of Vigué this is even true for arbitrary bounded domains in complex Banach spaces. For boundary points $a \in \partial D$ it is known [7] that every $g \in G$ is uniquely determined by the three derivatives $g(a), g'(a), g''(a)$.

1.11 Theorem. *Let $g \in G$ be an arbitrary automorphism and denote by $k := g_{-a}g \in K$ the corresponding isometry of E where $a = g(0)$. Then for every $z \in \overline{D}$ and $r := \|a\|$*

$$\|g(z) - k(z)\| \leq \frac{2r}{1-r}\,.$$

Proof. Since k is an isometry of E it is enough to show the result in the special case $g \in P$ and $k = \mathsf{id}$. Fix an arbitrary $\alpha \in E$ with $\|\alpha\| = 1$ and consider $h_t := \exp(tX^\alpha) \in P$, $a_t := h_t(0) = \tanh(t\alpha) \in D$ and $\lambda_t := h_t'(0) \in \mathcal{L}(E)$ for all $t \in \mathbb{R}$. Since h_t solves (1.5), i.e.

$$\frac{\partial h_t}{\partial t} = \alpha - (h_t \square \alpha) h_t$$

we get by differentiation that λ_t solves

$$\frac{\partial \lambda_t}{\partial t} = -2(a_t \square \alpha)\lambda_t \tag{1.12}$$

to the initial value $\lambda_0 = \mathsf{id}$. This equation has the explicit solution (compare [8])

$$\lambda_t = \exp(-2A(t)) \qquad \text{with} \qquad A(t) := \int_0^t (a_s \square \alpha)\, ds. \tag{1.13}$$

The function tanh is non-decreasing on the real line. Therefore $a_s = \tanh(s\alpha)$ implies $\|a_s \square \alpha\| = \tanh(s)$ by (1.4) and hence

$$\|A(t)\| \leq \int_0^t \tanh(s)\,ds = \ln\cosh(t)$$

for all $t \geq 0$. We derive

$$\begin{aligned} \|\lambda_t - \lambda_0\| &\leq \exp(2\ln\cosh(t)) - 1 = \cosh(t)^2 - 1 \\ &= \frac{\tanh(t)^2}{1-\tanh(t)^2} = \frac{\|a_t\|^2}{1-\|a_t\|^2}\,, \end{aligned} \tag{1.14}$$

giving for $g := g_a$ and $\lambda := g'(0)$ the estimate

$$\begin{aligned} \|g(z) - z\| &\leq \|g(z) - \lambda(z) - a\| + \|a\| + \|\lambda(z) - z\| \\ &\leq \frac{r}{1-r} + r + \frac{r^2}{1-r^2} \leq \frac{2r}{1-r}\,. \end{aligned}$$ □

With $r := \|a\|$ and $\lambda := g_a'(0)$ the estimate (1.14) reads

$$\|\lambda - \mathrm{id}\| \le \frac{r^2}{1-r^2}.$$

On the other hand, $\lambda(a) = a - \{aaa\}$ implies $\|\lambda(a) - a\| = \|\{aaa\}\| = r^2\|a\|$, i.e.

$$r^2 \le \|\lambda - \mathrm{id}\| . \tag{1.15}$$

We conjecture that equality holds in (1.15).

A JB*-triple E is called *abelian* (or *commutative*) if the set of operators $E \square E \subset \mathcal{L}(E)$ is commutative, or equivalently, if E is isomorphic to a subtriple of a commutative C*-algebra. In this case

$$\begin{aligned} g_a(z) &= a + (1 - a\square a)(1 + z\square a)^{-1}z \\ &= (1 + z\square a)^{-1}(z + a) \end{aligned}$$

holds for every $a \in D$.

A JB*-triple E is called *special* or a JC*-triple if it is isomorphic to a subtriple of a C*-algebra (with triple product (1.3)). These spaces were first studied by Harris [4] under the name of J*-algebras. In case E is special the automorphisms $g = g_a$ can be expressed in the C*-algebra product as

$$\begin{aligned} g_a(z) &= a + (1 - aa^*)^{1/2}(1 + za^*)^{-1}z(1 - a^*a)^{1/2} \\ &= (1 - aa^*)^{1/2}(1 + za^*)^{-1}(z + a)(1 - a^*a)^{-1/2} \end{aligned}$$

(compare [4] and also [7] p. 526).

2. Main results

In the following let E be a JB*-triple and D the corresponding bounded symmetric domain, i.e. the open unit ball of E. Denote by $G := \mathrm{Aut}(D)$ the automorphism group, by G^0 the connected identity component of G and by $\mathfrak{g} := \mathfrak{aut}(D)$ the Lie algebra of all complete holomorphic vector fields on D. Then G and G^0 are (real) Banach Lie groups with Lie algebra $\mathfrak{g}$ and G^0 is the subgroup generated by $\exp(\mathfrak{g})$ in G.

For every subset $S \subset E$ denote by $\mathcal{B}(S)$ the Banach space of all bounded continuous functions $f: S \to E$ with the norm $\|f\| := \sup\{\|f(s)\| : s \in S\}$. Every automorphism $g \in G$ extends as a holomorphic function to a neighbourhood of the closure $\overline{D}$, i.e. G may be considered as a subset of $\mathcal{B}(\overline{D})$. It is known [11], compare also (1.10), that the topology on G is the one induced from $\mathcal{B}(\overline{D})$. The same is true for $\mathfrak{g}$ which is a closed $\mathbb{R}$-linear subspace of $\mathcal{B}(\overline{D})$.

2.1 Definition. Let $\mathcal{F}$ be one of the spaces G, G^0 or $\mathfrak{g}$. A subset $S \subset \overline{D}$ is called *determining* (resp., *topologically determining*) *for* $\mathcal{F}$ if the restriction operator $\mathcal{F} \to \mathcal{B}(S)$ is injective (resp., a homeomorphism onto its image). Denote by $[\mathcal{F}]$ (resp., by $[[\mathcal{F}]]$) the set of all subsets of $\overline{D}$ that are determining (resp., topologically determining) for $\mathcal{F}$.

We will use the following Lemma mainly in the group case and $f = \mathrm{id}$

2.2 Lemma. *Let $f \in \mathcal{F}$ be an arbitrary but fixed element and let $S \subset \overline{D}$ be a subset. Then the following are equivalent*

(i) *S is topologically determining for $\mathcal{F}$, i.e. $S \in [[\mathcal{F}]]$*

(ii) *Every sequence (f_n) in $\mathcal{F}$ converging to f uniformly on S also converges to f uniformly on $\overline{D}$.*

Proof. Since $\mathcal{F}$ is metrizable we may use sequences instead of nets. The case $\mathcal{F} = \mathfrak{g}$ being trivial we may assume $\mathcal{F} = G$ or G^0. Suppose (ii) holds and suppose (g_n) is a sequence in $\mathcal{F}$ converging to $g \in \mathcal{F}$ uniformly on S. Then the sequence (f_n) defined by $f_n := fg^{-1}g_n$ for all n converges to f uniformly on S since fg^{-1} is uniformly continuous on $\overline{D}$ by (1.7.iii). But then $\lim f_n = f$ by (ii) and hence $\lim g_n = g$ in $\mathcal{F}$. □

2.3 Corollary. *The automorphism group $G = \mathrm{Aut}(D)$ acts on the power set of $\overline{D}$ in a natural way and clearly leaves all the sets $[[G]], [[G^0]], [[\mathfrak{g}]], [G], [G^0], [\mathfrak{g}]$ invariant.*

Proof. In case $\mathcal{F} = \mathfrak{g}$ use the fact that for every $g \in G$ also the derivative g' is uniformly continuous on $\overline{D}$. □

2.4 Proposition. *For every bounded symmetric domain D in its standard realization and $G := \mathrm{Aut}(D)$, $\mathfrak{g} := \mathfrak{aut}(D)$*

$$\begin{array}{ccccc} [[G]] & \subset & [[G^0]] & & [[\mathfrak{g}]] \\ \cap & & \cap & & \cap \\ [G] & \subset & [G^0] & \subset & [\mathfrak{g}] \end{array}$$

holds. In general none of the inclusions is an equality.

Proof. The inclusions are obvious. The last statement is clear with the following examples (2.6) - (2.8). □

2.5 Remark. We do not know wether the inclusion $[[G^0]] \subset [[\mathfrak{g}]]$ also holds. The proof of the inclusion $[G^0] \subset [\mathfrak{g}]$ is a simple consequence of the fact that $X(a) = 0$ implies $g(a) = a$ for every $a \in D$, every $X \in \mathfrak{g}$ and $g := \exp(X) \in G^0$. Unfortunately the topological analogue is not true: In case $D = \Delta$ and $a \in \partial\Delta$ there exists a sequence (X_n) in $\mathfrak{g}$ with $\lim X_n(a) = 0$ but $\lim g_n(a) \neq a$ for $g_n := \exp(X_n)$.

2.6 Example. Let Ω be a compact topological space and $\varphi: \Omega \to \Omega$ a (surjective) homeomorphism with $\varphi \neq \mathrm{id}$. Then $E := \mathcal{C}(\Omega)$ is a commutative C*-algebra with unit e and $\Phi(f)(\omega) := f(\varphi(\omega))$ defines an automorphism $\Phi \in G$ fixing every element in $S := \{0, e\}$, i.e. $S \notin [G]$. Consider a sequence (g_n) in G^0 with $\lim g_n(0) = 0$ and $\lim g_n(e) = e$. Then there are $a_n, c_n \in E$ with

$$g_n(z) = \frac{c_n z + a_n}{\overline{a}_n c_n z + e}$$

for all n, z. Therefore $\lim a_n = \lim g_n(0) = 0$ and $\lim c_n = \lim g_n(e) = e$ imply $\lim g_n = \mathrm{id}$, i.e. $S \in [[G^0]]$.

2.7 Example. Let H be a complex Hilbert space of dimension > 1 with inner product $(\,|\,)$ and orthonormal basis $\mathcal{E} := \{e_i : i \in I\}$. Then to every $z \in \mathcal{L}(H)$ there is a unique $z' \in \mathcal{L}(H)$ with $(z'e_i|e_j) = (ze_j|e_i)$ for all $i, j \in I$ and

$E := \{z \in \mathcal{L}(H) : z' = z\}$ with triple product (1.3) is a JB*-triple. Denote by S the set of all $z \in \overline{D}$ that are diagonal with respect to $\mathcal{E}$, i.e. such that every e_i is an eigenvector of z. Every $X \in \mathfrak{g}$ is of the form

$$X(z) = i(uz + zu') + (a - za^* z)$$

with suitable $a \in E$ and $u \in \mathcal{L}(H)$ hermitian. This implies easily $S \in [\mathfrak{g}]$. On the other hand, there exists an element $u \in S$ with $u \neq u^2 = \mathrm{id}$. The automorphism $g \in \exp(\mathfrak{g}) \subset G^0$ defined by $z \mapsto uzu$ leaves every point in S invariant without being the identity on D, i.e. $S \notin [G^0]$.

2.8 Example. Consider $E := \mathcal{C}_0(I)$ for $I := \{t \in \mathbb{R} : 0 < t \leq 1\}$ and set $S := \{0, a\}$ with $a \in E$ defined by $a(t) \equiv t$. Consider an automorphism $g \in G$ fixing S pointwise. Then g is a linear isometry of E and hence of the form $g(z)(t) = c(t)z(\varphi(t))$ for all $z \in E$ and $t \in I$ where φ is a homeomorphism of I and c is a continuous complex valued function on I with $|c(t)| \equiv 1$. Evaluation of $g(a) = a$ implies $\varphi(t) \equiv t$, $c(t) \equiv 1$ and hence $g = \mathrm{id}$, i.e. $S \in [G]$. For every $n \in \mathbb{N}$ choose a real valued function $b_n \in E$ with $\|b_n\| = 1$ and $\|b_n a\| < 1/n$. Then $X_n(z) := ib_n z$ defines a divergent sequence (X_n) in $\mathfrak{g}$ with $\lim X_n(a) = 0$, i.e. $S \notin [[\mathfrak{g}]]$. The sequence $(\exp X_n)$ in G^0 shows that also $S \notin [[G^0]]$. It is clear that every element of $[[\mathfrak{g}]]$ must be an infinite set.

In the preceding examples the presence of the origin in S reduces the problem essentially to a linear one. This is true also in a more general situation

2.9 Proposition. *Let $S \subset \overline{D}$ be a subset with $0 \in S$ and let (g_n) be a sequence in G converging uniformly on S to the identity. Then*

(i) *$\lim g_n(c) = c$ for every c in the closed subtriple generated by S in E.*
(ii) *(g_n) also converges uniformly to the identity on the subset $\{SSS\} \subset \overline{D}$.*
(iii) *(g_n) converges to the identity in G if the closed, convex circled hull of S in E has an inner point.*

Proof. By assumption $\lim g_n(0) = 0$ holds. Therefore, by (1.11) there is a sequence of isometries $k_n \in K \subset \mathrm{GL}(E)$ such that $(g_n - k_n)$ converges to 0 uniformly on $\overline{D}$, i.e. also (k_n) converges to the identity uniformly on S. Every k_n is a triple automorphism. Together with (1.1) this implies that (k_n) converges to the identity uniformly on $\{SSS\}$ and this implies (ii). Now (i) and (iii) are also clear. □

Statement (iii) in (2.9) can be improved in the following way: Denote by ccc the operation of forming the closed, convex, circled hull and define for every $S \subset E$ inductively

$$S^1 := \mathrm{ccc}(S)$$
$$S^{n+1} := \mathrm{ccc}(S^n \cup \{SSS^n\} \cup \{SS^nS\})$$
$$S^\infty := \bigcup_n S^n .$$

It is clear that the closed linear span of S^∞ is a JB*-triple. Under the assumption of (2.9) the sequence (g_n) converges uniformly to the identity on every S^n by (ii). Therefore, if S^∞ contains an inner point, every S^n contains an inner point for n large by the Theorem of Baire, i.e. (g_n) converges to $\mathrm{id} \in G$.

2.10 Example. Let E be the JB*-triple of all $n \times n$-matrices over $\mathbb{C}$ (considered as operators on the Hilbert space $\mathbb{C}^n$ with standard inner product). Let $d \in E$ be a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$ satisfying $0 < |d_1| < |d_2| < \ldots < |d_n|$ and let $s = (s_{ij})$ be the shift matrix defined by $s_{ij} = \delta_{i,i+1}$. Then $S := \{d, s\}$ generates E as a JB*-triple. For instance, in case $n = 3$, it is easily checked that S^2 does not span E whereas S^3 has an inner point.

2.11 Proposition. *Suppose $a \in D$ and $c \in \overline{D}$ are different but linearly dependent over $\mathbb{C}$. Suppose furthermore that (g_n) is a sequence in G with $\lim g_n(a) = a$ and $\lim g_n(c) = c$. Then also $\lim g_n(0) = 0$ is true.*

Proof. We may assume that $a = tc$ for some $t \in \mathbb{C}$. Denote by $F \subset E$ the closed subtriple generated by c. The automorphism $h := g_{-a} \in G$ maps $F \cap D$ onto itself and the sequence $(\widetilde{g}_n)$ defined by $\widetilde{g}_n := hg_nh^{-1}$ satisfies $\lim \widetilde{g}_n(0) = 0$ and $\lim \widetilde{g}_n(d) = d$ for $d := h(c)$. We claim that d generates F as a JB*-triple. For this we identify F with $\mathcal{C}_0(\Omega)$, $\Omega \subset \mathbb{R}$ a locally compact subset with $\Omega > 0$ and $\Omega \cup \{0\}$ compact, in such a way that c is the function $c(\omega) = \omega$ for all $\omega \in \Omega$. Then d is the function

$$d = \frac{c-a}{1-\overline{a}c} = (1-t)\frac{c}{1-\overline{t}c\overline{c}}$$

and it is enough to show that $e := c/(1-\overline{t}c\overline{c})$ generates F, or equivalently (apply Weierstrass Approximation Theorem) that the function

$$e\overline{e} = \frac{c\overline{c}}{(1-tc\overline{c})(1-\overline{t}c\overline{c})}$$

does not vanish on Ω (this is obvious) and separates the points of Ω. For this consider the function

$$f(x) := \frac{x}{(1-tx)(1-\overline{t}x)}$$

for all $x \in \mathbb{R}$ with $0 < x \leq \|c\|^2$. Then $|tx| \leq \|tc\overline{c}\| = \|a\overline{c}\| < 1$ implies $f'(x) > 0$ for all $0 < x \leq \|c\|^2$, i.e. f is strictly increasing. Therefore also $e\overline{e}$ separates the points of Ω and our claim is clear. With (2.9.i) we derive $\lim \widetilde{g}_n(-a) = -a = h(0)$ and from this $\lim g_n(0) = 0$. □

So far we always needed for the study of S a point $a \in D$. In section 0 we already pointed out that three different points in the boundary $\mathbb{T} = \partial\Delta$ of the open unit disk $\Delta \subset \mathbb{C}$ are determining for the group $\mathsf{Aut}(\Delta)$. Let us call every subset $\mathbb{T}u \subset E$, $u \neq 0$, a *circle* in E.

2.12 Proposition. *Let $S \subset \partial D$ be a subset containing three different points on a circle and let $g \in G = \mathsf{Aut}(D)$ be an automorphism fixing S pointwise. Then g is linear and in particular $g(0) = 0$ holds.*

Proof. g can be written as

$$g(z) = a + \lambda(1 + z\,\square\,a)^{-1}z$$

for $a := g(0) \in D$ and $\lambda := g'(0) \in \mathsf{GL}(E)$ - compare (1.8).

$$f(z) := (1 + z\,\square\,a)\lambda^{-1}(a - z) + z$$

defines a polynomial map $f : E \to E$ of degree ≤ 2 with

$$f(z) = (1 + z\,\square\,a)\lambda^{-1}(g(z) - z)$$

for all $z \in \overline{D}$. By assumption, f vanishes on three different points on a circle in E, i.e. $0 = f(0) = \lambda^{-1}(a)$ and hence $a = 0$. □

The question arises wether (2.12) is also true in the topological sense. Unfortunately we have only a partial answer

2.13 Proposition. *Suppose E is an abelian JB*-triple and $S \subset \partial D$ contains three different points on a circle. Suppose furthermore that (g_n) is a sequence in G with $\lim g_n(s) = s$ for all $s \in S$. Then also $\lim g_n(0) = 0$ is true.*
Proof. For every $n \in \mathbb{N}$ there is a unique representation

$$g_n(z) = (1 + k_n(z)\square a_n)^{-1}(k_n(z) + a_n)$$

with $a_n = g_n(0)$ and $k_n \in \mathsf{GL}(E)$ an isometry. By assumption there are pairwise different $t_0, t_1, t_2 \in \mathbb{T}$ and $u \in \partial D$ with $t_k u \in S$ for $k = 0, 1, 2$. Put

$$b_n := k_n(u) - u, \qquad c_n := -\{k_n(u) a_n u\}$$

and define $f_n: \mathbb{C} \to E$ by $f_n(t) := a_n + t b_n + t^2 c_n$. Since the Vandermonde matrix $(t_k^j)_{0 \le j,k \le 2}$ is regular we can solve the three equations

$$a_n + t_k b_n + t_k^2 c_n \;=\; f_n(t_k) \qquad k = 0, 1, 2$$

and obtain in particular every a_n as a linear combination of $f_n(t_0), f_n(t_1), f_n(t_2)$ with complex coefficients not depending on n. It is easily checked that

$$f_n(t) = (1 + k_n(tu)\square a_n)(g_n(tu) - tu)$$

holds for all $t \in \mathbb{T}$. This implies $\lim_n f_n(t_k) = 0$ for $k = 0, 1, 2$ and hence $\lim a_n = 0$ as required. □

For every JB*-triple F with open unit ball B and for every compact topological space Ω also $\mathcal{C}(\Omega, F)$ with norm $\|f\| := \sup_\omega \|f(\omega)\|$ is a JB*-triple. Let us assume in the following that $E = \mathcal{C}(\Omega, F)$. Then $D = \mathcal{C}(\Omega, B)$ and $\mathcal{C}(\Omega, \mathsf{Aut}(B))$ may be considered as a subgroup of $G = \mathsf{Aut}(D)$ (simply define $gf \in E$ for $f \in E$ and $g \in \mathcal{C}(\Omega, \mathsf{Aut}(B))$ by $\omega \mapsto g(\omega)f(\omega)$). By [10] $G^0 \subset \mathcal{C}(\Omega, \mathsf{Aut}(B))$ is true. From this the following is easily derived

2.14 Lemma. *Suppose $\widetilde{\Omega} \subset \Omega$ is a dense subset and $S \subset \overline{D}$ has the property that for every $\omega \in \widetilde{\Omega}$ the subset $S(\omega) := \{s(\omega) : s \in S\}$ of $\overline{B}$ is determining for the group $\mathsf{Aut}(B)^0$. Then also S is determining for the group G^0.*

In contrast to the 1-dimensional situation (compare section 0) it is possible in the higher dimensional case that a two point boundary set is determining for G. We use a variation of (2.14) to give an example in every dimension > 1:

2.15 Example. Let $E = \mathcal{C}(T)$ where $T \subset \mathbb{R}$ is a compact set containing 0 as an isolated point with $\inf T = 0$ and $\sup T = 1$. Then $S := \{a, b\} \subset \partial D$ with a the characteristic function of $\{0\} \subset T$ and $b(t) \equiv t$ is determining for G.
Proof. Let $g \in G$ be an automorphism fixing a and b. T can be identified with the space of all maximal triple ideals of E (compare [7]). Therefore there is a homeomorphism φ of T and for every $t \in T$ an automorphism $g_t \in \mathsf{Aut}(\Delta)$ such

that $(gf)(t) = g_t(f \circ \varphi(t))$ for all $t \in T$ and $f \in E$. Evaluating this for $f = a$ gives $\varphi(0) = 0$, $g_0(1) = 1$ and $g_t(0) = 0$ for $t > 0$. Evaluating for $f = b$ implies $g_0(0) = 0$ and $g_t(\varphi(t)) = t$ for all $t > 0$. But this only is possible if $\varphi = \mathrm{id}$ and $g_t = \mathrm{id}$ for all $t \in T$, i.e. g is the identity in G. □

For the special case $T = \{0, 1\}$ in (2.15) we get $E = \mathbb{C}^2$ with norm $\|z\| = \max(|z_1|, |z_2|)$ and $a = (1, 0)$, $b = (0, 1)$.

References

1. Barton, T.J., Friedman, Y.: Bounded derivations of JB*-triples. Quart. J. Math. Oxford **41**, 255-268 (1990)
2. Braun, R., Kaup. W., Upmeier, H.: On the automorphisms of circular and Reinhardt domains in complex Banach spaces. **manuscripta math.** **25**, 97-133 (1978)
3. Friedman, Y., Russo, B.: The Gelfand-Naimark theorem for JB*-triples. Duke Math. J. **53** 139-148 (1986)
4. Harris, L.A.: Bounded symmetric homogeneous domains in infinite dimensional spaces. In: Lecture Notes in Mathematics Vol. 364. Berlin-Heidelberg-New York: Springer 1973
5. Harris, L.A., Kaup, W.: Linear algebraic groups in infinite dimensions. Illinois J. Math. **21**, 666-674 (1977)
6. Kaup, W.: Über die Klassifikation der symmetrischen Hermiteschen Mannigfaltigkeiten unendlicher Dimension I, II. Math. Ann. **257**, 463-483 (1981); **262**, 503-529 (1983)
7. Kaup, W.: A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. Math. Z. **183**, 503-529 (1983)
8. Kaup, W.: Hermitian Jordan triple systems and the automorphisms of bounded symmetric domains. To appear
9. Loos, O.: Bounded symmetric domains and Jordan pairs. Mathematical Lectures. Irvine: University of California at Irvine 1977
10. Vigué, J.P.: Automorphismes analytiques des produits continus de domaines bornés. Ann. Scient. Éc. Norm. Sup. **11** 229-246 (1978)
11. Vigué, J.P., Isidro, J.M.: Sur la topologie du groupe des automorphismes analytiques d'un domaine cerclé borné dans un espace de Banach complexe. Bull. Sci. Math. **106**, 417-426 (1982)

José M. Isidro
Departamento de Análisis Matemático
Facultad de Matemáticas
15706 Santiago de Compostela
Spain

Wilhelm Kaup
Mathematisches Institut der Universität
Auf der Morgenstelle 10
D-72076 Tübingen
Germany

(Received May 10, 1993)

manuscripta math. 81, 161 - 182 (1993)

manuscripta mathematica

ON SAITO–KUROKAWA DESCENT FOR CONGRUENCE SUBGROUPS

M. Manickam, B. Ramakrishnan and T. C. Vasudevan

The conjecture made by H. Saito and N. Kurokawa states the existence of a "lifting" from the space of elliptic modular forms of weight $2k-2$ (for the full modular group) to the subspace of the space of Siegel modular forms of weight k (for the full Siegel modular group) which is compatible with the action of Hecke operators. (The subspace is the so called "Maaß spezialschar" defined by certain identities among Fourier coefficients). This conjecture was proved (in parts) by H. Maaß, A.N. Andrianov and D. Zagier. The purpose of this paper is to prove a generalised version of the conjecture for cusp forms of odd squarefree level.

1. Introduction

The conjecture, made on the basis of numerical calculations of eigenvalues of Hecke operators, formulated independently by H. Saito and by N. Kurokawa [12], asserted the existence of a "lifting" from the space of elliptic modular forms of weight $2k-2$ for the full modular group to a subspace of the Siegel modular forms of weight k for the full Siegel modular group of degree 2 which is compatible with the action of Hecke operators. (This subspace, defined by certain identities among Fourier coefficients, is called the "Maaß spezialschar"). Most of this conjecture was proved by H. Maaß ([13]–[15]), another part by A.N. Andrianov [1] and the remaining part by D. Zagier [21]. The conjectured correspondence is the composition of three isomorphisms

$$\begin{array}{ccc} \text{"Maaß spezialschar"} & \subset & M_k(\Gamma_2) \\ \downarrow\wr & & \\ \text{Jacobi forms of weight } k \text{ and index } 1 & & \\ \downarrow\wr & & \\ \text{Kohnen's "+ space"} & \subset & M_{k-1/2}(\Gamma_0(4)) \\ \downarrow\wr & & \\ M_{2k-2}(\Gamma_1) & & \end{array} \qquad (1)$$

In the above diagram (1), instead of considering the full modular group, if one considers a congruence subgroup of arbitrary level, there is still no clear picture about this conjecture. Here we list some of the attempts made so far (to our knowledge) in this connection. In 1980, M. Eichler submitted a paper in which he proved the above correspondence for arbitrary level N, but the level of the forms in the bottom of the above diagram was left open.(In this connection one can refer [3],p.6). H. Kojima [8]–[10], in his papers obtained a correspondence between Siegel cusp forms (in the Maaß space) and cusp forms of half-integral weight. He proved that the existence of common eigensubspaces of all Hecke operators for these spaces (with normalisation condition on the Fourier coefficients) would imply an isomorphism between these common eigensubspaces.

In this paper we will prove a generalised version of the Saito–Kurokawa conjecture when the level of the congruence subgroup is an odd squarefree natural number and also we restrict ourselves to the case of cusp forms. The idea is to obtain the following diagram

$$\begin{array}{ccc} \text{The space of "newforms" in "Maaß spezialschar"} & \subset & S_k(\Gamma_0^2(M)) \\ \downarrow\wr & & \\ \text{The space of Jacobi (cusp) newforms of weight } k, \text{level } M \text{ and index } 1 & & \\ \downarrow\wr & & \\ \text{Kohnen's "newform" space} & \subset & S^+_{k-1/2}(\Gamma_0(4M)) \\ \downarrow\wr & & \\ S^{new}_{2k-2}(M) & & \end{array} \qquad (2)$$

where M is an odd squarefree natural number.

The first part of the above diagram (2) follows exactly in the same way as in the previous case. The last one is the isomorphism due to W. Kohnen [6]. For $M = 1$, the middle part was proved in [3]. We prove the second isomorphism in the diagram in a different way. Actually, we define the map as done in [3] and obtain the isomorphism by proving that it really maps the corresponding Poincaré series indexed by fundamental discriminants. We also set up the parallel theory of newforms for the spaces of Jacobi cusp forms and the space of "Maaß spezialschar" contained in the space of Siegel cusp forms to obtain the isomorphism.

We remark that N-P. Skoruppa [20] used the methods of [3] to give a correspondence between Jacobi forms of weight k, index m for the full Jacobi group and a subspace of modular forms of half-integral weight. Finally we note that our Theorem 5 along with the Theorem in § 4. of [6] generalise Satz 7 of [11], which states that the dimension of the space of Jacobi forms of weight $k \in \mathbf{N}$, index 1 and level ℓ (ℓ an odd prime) is equal to the dimension of the space of modular forms of weight $2k-2$ and level ℓ, when restricted to the space of cusp forms where the level is a squarefree natural number.

2. Preliminaries

Let $\mathcal{H}$ be the upper half-plane and $\mathcal{H}_2$ be the Siegel upper half-space of degree 2,consisting of 2×2 matrices Z with positive definite imaginary part. We often write $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$.

Let us denote by $\mathbf{C}, \mathbf{R}, \mathbf{Q}, \mathbf{Z}$ and $\mathbf{N}$, the field of complex numbers , the field of real numbers , the field of rational numbers ,the ring of rational integers and the set of natural numbers respectively.

For a natural number M, let $\Gamma_0(M)$ be the congruence subgroup (of level M) of the full modular group $\Gamma_1 = SL_2(\mathbf{Z})$ and let

$$\Gamma_0^2(M) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2 \middle| C \equiv 0 (\bmod\ M) \right\}$$

where $\Gamma_2 = Sp_2(\mathbf{Z})$ is the group of integral symplectic 4×4 matrices and $\Gamma_0(M)^J = \Gamma_0(M) \times \mathbf{Z}^2$ be the subgroup (of level M) of the full Jacobi modular group $\Gamma_1 \times \mathbf{Z}^2$.

For a natural number k, we write $S_k(M)$ for the space of elliptic modular cusp forms of weight k for $\Gamma_0(M)$; $S_k(\Gamma_0^2(M)))$ for the space of Siegel modular cusp forms of weight k, for $\Gamma_0^2(M)$; $J_{k,N}(M)$ (resp. $J_{k,N}^{cusp}(M)$) for the space of holomorphic Jacobi (resp. Jacobi cusp) forms of weight k, index $N \in \mathbf{N}$ for $\Gamma_0(M)^J$; $S_{k-1/2}^+(\Gamma_0(4M))$, for the Kohnen's '+ space' of the space of modular cusp forms of weight $k - 1/2$ for the group $\Gamma_0(4M)$.

Let p be a prime. Let us denote by $T(p), p \nmid M$, $U(p), p \mid M$, the Hecke operators in the space $S_k(M)$; $T^+(p^2)$, $p \nmid M$, $U(p^2)$, $p \mid M$ denote the

Hecke operators on $S^+_{k-1/2}(\Gamma_0(4M))$; and for $n \in \mathbb{N}$, $T_S(n)$ denote the Hecke operator on $S_k(\Gamma_0^2(M))$.

We define the Petersson scalar product $\langle \cdot , \cdot \rangle$ on $S_k(M)$, $S^+_{k-1/2}(\Gamma_0(4M))$, $J^{cusp}_{k,N}(M)$ and $S_k(\Gamma_0^2(M))$ in the usual way. For details we refer ([2]-[6]).

3. Correspondence between Maaß spezialschar and Jacobi forms

Let $\phi(\tau, z) \in J_{k,m}(M)$ and denote by $c(D,r)$ its (D,r)–th Fourier coefficient. We define the linear operator $V_N, N \in \mathbb{N}$, the Hecke operators $T_J(p), p \nmid mM$, $U_J(p), p|M$, as follows.

$$\phi \mid V_N = \sum_{\substack{D\leq 0, r\in \mathbb{Z} \\ D\equiv r^2(4mN)}} \left(\sum_{\substack{d \mid (N,r) \\ D\equiv r^2(4mNd)}} \chi(d)\, d^{k-1} c\left(\frac{DN}{d^2}, \frac{r}{d}\right) \right) e\left(\frac{r^2-D}{4mN}\tau + rz\right) \tag{3}$$

$$\phi \mid T_J(p) = \sum_{\substack{D\leq 0, r\in \mathbb{Z} \\ D\equiv r^2(4m)}} c^*(D,r) e\left(\frac{r^2-D}{4m}\tau + rz\right), \quad \text{for } p \nmid mM \tag{4}$$

where

$$c^*(D,r) = c(p^2D, pr) + p^{k-2}\left(\frac{D}{p}\right) c(D,r) + p^{2k-3} c\left(\frac{D}{p^2}, \frac{r}{p}\right)$$

(here $c(D,r) = 0$ if $D = r^2 - 4nm$ is not an integer) and

$$\phi \mid U_J(p) = \sum_{\substack{D\leq 0, r\in \mathbb{Z} \\ D\equiv r^2(4m)}} c(p^2D, pr)\, e\left(\frac{r^2-D}{4m}\tau + rz\right), \quad \text{for } p \mid M \tag{5}$$

Remark 1.

Note that the operators V_N, $T_J(p)$, $p \nmid mM$, and $U_J(p), p \mid M$ preserve the space of cusp forms and $T_J(p), p \nmid mM$ is hermitian with respect to the Petersson product. Also for $p|M$,

$$V_N\, U_J(p) = U_J(p)\, V_N \quad \text{on } J_{k,1}(M) \tag{6}$$

Let $S_k^*(\Gamma_0^2(M))$ be the subspace (called the Maaß spezialschar) of $S_k(\Gamma_0^2(M))$ consisting of Siegel modular forms

$$F = \sum_{\substack{n,r,m \in \mathbb{Z} \\ r^2 < 4nm}} A(n,r,m)\; e(n\tau + rz + m\tau') = \sum_{m \geq 1} \phi_m(\tau, z)\, e(m\tau')$$

satisfying

$$A(n,r,m) = \sum_{d \mid (n,r,m)} d^{k-1} A\left(\frac{nm}{d^2}, \frac{r}{d}, 1\right) \quad \text{for all} \quad n, r, m$$

which is called the Maaß relation.

Proceeding exactly on the same lines as in ([3],§6) we obtain the connection between Maaß spezialschar and the space of Jacobi forms in the following

Theorem 1. *Let $F \in S_k^*(\Gamma_0^2(M))$ and write the Fourier expansion of F in the form*

$$F(\tau, z, \tau') = \sum_{m=1}^{\infty} \phi_m(\tau, z)\, e(m\tau')$$

Then

$$F(\tau, z, \tau') = \sum_{m=1}^{\infty} (\phi_1 \mid V_m)(\tau, z)\, e(m\tau')$$

where $\phi_1 \in J_{k,1}^{cusp}(M)$. The association $F \mapsto \phi_1$ gives an isomorphism between $S_k^*(\Gamma_0^2(M))$ and $J_{k,1}^{cusp}(M)$.

4. Connection between Jacobi forms and Modular forms of half-integral weight

In this section, we will construct a linear map between $J_{k,1}^{cusp}(M)$ and $S_{k-1/2}^{+}(\Gamma_0(4M))$, where M is an odd squarefree natural number. Since it is known that $J_{k,1}^{cusp}(M) = \{0\}$ if k is odd (cf.[3]), we restrict ourselves to the case when $k > 2$ is even.

For each discriminant $D < 0 (D = r^2 - 4n)$, we denote by $P_{(D,r)}(\tau, z)$, the (D, r)-th Poincaré series in $J_{k,1}^{cusp}(M)$ defined by

$$P_{(D,r)}(\tau, z) = \frac{1}{2} \sum_{\gamma \in \Gamma^J_{1\,\infty} \backslash \Gamma_0(M)^J} e^{n,r} \;|_{k,1}\, \gamma \;\; (\tau, z)$$

where $e^{n,r} = e^{2\pi i(n\tau + rz)}$, "$|_{k,m}$" is the stroke operator as in [3] and

$$\Gamma^J_{1\,\infty} = \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \lambda) \right) \;\middle|\; n, \lambda \in \mathbf{Z} \right\}$$

Then we have the following proposition

Proposition 1. *(cf.[4],[18]).*

i)

$$P_{(D,r)}(\tau, z) \in J^{cusp}_{k,1}(M)$$

ii) For

$$\phi = \sum_{\substack{D'<0, r' \in \mathbf{Z} \\ D' \equiv r'^2 (4)}} c(D', r')\, e\left(\frac{r'^2 - D'}{4} \tau + r' z \right) \in J^{cusp}_{k,1}(M)$$

we have

$$\langle \phi, P_{(D,r)} \rangle = \alpha_k \, |D|^{-k-3/2} \, c(D, r),$$

where

$$\alpha_k = \frac{\Gamma(k - 3/2)}{2\pi^{k-3/2}}$$

iii)

$$P_{(D,r)}(\tau, z) = \sum_{\substack{D'<0, r' \in \mathbf{Z} \\ D' \equiv r'^2 (4)}} g_{(D,r)}(D', r')\, e\left(\frac{r'^2 - D'}{4} \tau + r' z \right)$$

where

$$g_{(D,r)}(D', r') = \delta(D, r, D', r') + i^{-k} \pi \sqrt{2} (D'/D)^{k/2 - 3/4} \times \sum_{c \geq 1} H_{Mc}(D, r, D', r')\, J_{k-3/2}\left(\frac{\pi \sqrt{D'D}}{Mc} \right) \;;$$

with

$$\delta(D, r, D', r') = \begin{cases} 1 & \text{if } D' = D, r' \equiv r(2); \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_c(D,r,D',r') = c^{-3/2} \times \sum_{\substack{\rho(c)^* \\ \lambda(c)}} e_c\left(\rho^{-1}\left(\lambda^2 + r\lambda + \frac{r^2-D}{4}\right) + r'\lambda + \frac{r'^2-D'}{4}\rho\right) \times e_{2c}(-rr')$$

Let $D_0 = r_0^2 - 4n_0$ be a negative fundamental discriminant. Let $\Delta > 0$ be a discriminant divisible by D_0 such that both Δ and Δ/D_0 are squares modulo 4. Let χ_{D_0} be the genus character defined on integral binary quadratic forms [a,b,c] with discriminant Δ.

We put for a,m $\in$ N,

$$S_a(m;\Delta,D_0) = \sum_{\substack{b(2a) \\ b^2 \equiv \Delta(4a)}} \chi_{D_0}\left(\left[a,b,\frac{b^2-\Delta}{4a}\right]\right) e_{2a}(mb). \tag{7}$$

where $\Delta = D_0 D, D = r^2 - 4n$.

Then the following proposition is proved in [4] in connection with correspondences between Jacobi forms and elliptic modular forms.

Proposition 2. *For a,m $\geq$ 1,*

$$S_a(m;\Delta,D_0) = \sum_{d|(a,m)} \left(\frac{D_0}{d}\right)(a/d)^{1/2} H_{a/d}\left(D_0\frac{m^2}{d^2}, r_0\frac{m}{d}, D, r\right) \tag{8}$$

For a negative discriminant D , let us denote by $P_D^+(\tau)$, the D-th Poincaré series in $S^+_{k-1/2}(\Gamma_0(4M))$. Then we have the following

Proposition 3. *(cf.[7],Proposition 3).*

$$P_D^+(\tau) = \sum_{D'\equiv 0,1(4)} g_D(|D'|)\, e(|D'|\tau)$$

where

$$g_D(|D'|) = \frac{2}{3}\delta_{D,D'} + \frac{2\pi\sqrt{2}}{3}(-1)^{[k/2]}(D'/D)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc}(D',D) J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right)$$

with

$$\delta_{D,D'} = \begin{cases} 1 & \text{if } D' = D \\ 0 & \text{otherwise} \end{cases}$$

$$H_c(D',D) = (1-(-1)^{k-1}i)\left(1+\left(\frac{4}{c}\right)\right)\frac{1}{4c}\sum_{\delta(4c)^*}\left(\frac{4c}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k-1/2} \times e_{4c}(|D'|\delta + |D|\delta^{-1})$$

where $\delta^{-1} \in \mathbf{Z}$, *with* $\delta\delta^{-1} \equiv 1\ (4c)$

Proposition 4. *(cf.[7],Proposition 5). For $a, m\geq 1$ and for all negative discriminants D, we have,*

$$S_a(m; D_0D, D_0) = \sum_{d|(a,m)} \left(\frac{D_0}{d}\right)(a/d)^{1/2} H_{a/d}(D, D_0m^2/d^2) \tag{9}$$

Remark 2.

Note that the function $S_{a,D_0,D}(D_0D, m)$ defined in ([7],p.246) coincides with the function $S_a(m; D_0D, D_0)$ defined by (7) in our case.

Putting $m = 1$ in (8) and (9) we obtain

$$H_a(D_0, r_0, D, r) = H_a(D, D_0) \quad \text{for all} \quad a\geq 1 \tag{10}$$

Theorem 2. *The linear map $\mathcal{S}$ defined by*

$$\sum_{\substack{D<0, r\in\mathbf{Z} \\ D\equiv r^2(4)}} c(D,r)\, e\left(\frac{r^2-D}{4}\tau + rz\right) \longmapsto \sum_{\substack{D<0 \\ D\equiv 0,1(4)}} c(|D|)\, e(|D|\tau)$$

maps the (D_0, r_0)*–th Poincaré series in* $J_{k,1}^{cusp}(M)$ *(upto a multiplicative constant) to the* D_0*-th Poincaré series in* $S^+_{k-1/2}(\Gamma_0(4M))$ *where* $D_0 = r_0^2 - 4n_0$ *is a negative fundamental discriminant.*

Remark 3.

Since the Fourier coefficients $c(D,r)$ of a Jacobi form of index 1 depends only on the value of the discriminant D we write $c(|D|)$ instead of $c(D,r)$.

Proof.

Using Proposition 1, the image of $P_{(D_0,r_0)}(\tau,z)$ under the map $\mathcal{S}$ is given by

$$P_{(D_0,r_0)}|\mathcal{S} = \sum_{\substack{D<0 \\ D\equiv 0,1(4)}} g_{(D_0,r_0)}(|D|)e(|D|\tau) \tag{11}$$

Clearly

$$\delta(D_0,r_0,D,r) = \delta(D_0,r_0,D,-r) = \delta_{D_0,D} \tag{12}$$

and

$$H_c(D_0,r_0,D,r) = H_c(D_0,r_0,D,-r) = H_c(D,D_0) \quad \text{(using (7))} \tag{13}$$

Now using (12) and (13) in (11), and using the fact that $i^{-k} = (-1)^{k/2}$, we have

$$P_{(D_0,r_0)} \mid \mathcal{S} = \frac{3}{2}\, P^+_{D_0}$$

This completes the proof.

We shall now extend the result of Theorem 2 for negative discriminants relatively prime to M.

Let $D \equiv r^2 (\text{mod } 4)$ be a negative fundamental discriminant. Then for $(n,M)=1$, we have the following

$$P_{(D,r)}|T_J(n) = n^{2k+3} \sum_{\substack{d|n \\ (d,M)=1}} \left(\frac{D}{d}\right) d^{-k-5} P_{(Dn^2/d^2, rn/d)} \tag{14}$$

and

$$P^+_{|D|}|T^+(n^2) = n^{2k+3} \sum_{\substack{d|n \\ (d,M)=1}} \left(\frac{D}{d}\right) d^{-k-5} P^+_{|D|n^2/d^2} \tag{15}$$

The above two equations along with Theorem 2 and the fact that

$$\mathcal{S}\, T^+(p^2) = T_J(p)\, \mathcal{S} \qquad p \nmid M \tag{16}$$

will give the following

$$P_{(D,r)} \mid \mathcal{S} = \frac{3}{2} P^+_{|D|} \tag{17}$$

where $D \equiv r^2\ (4), D < 0$ and $(D, M) = 1$.

5. Theory of newforms on $J_{k,1}^{cusp}(M)$

Let M be an odd squarefree natural number and let k be even. Define

$$J_{k,1}^{cusp,old}(M) = \sum_{\substack{rd|M \\ 1 \le r < M}} J_{k,1}^{cusp}(r) \mid U_J(d)$$

and put $J_{k,1}^{cusp,new}(M)$ to be the orthogonal complement of $J_{k,1}^{cusp,old}(M)$ in $J_{k,1}^{cusp}(M)$ with respect to the Petersson product.

$\mathcal{W}$ – operator

For $p \mid M$, we define the $\mathcal{W}$-operator on $J_{k,1}^{cusp}(M)$ as follows.

$$\mathcal{W}_p = p^{-2} \sum_{u,\lambda,\mu(p)} \left(\begin{pmatrix} a + cuM/p & u + b/p \\ Mc & p \end{pmatrix}, (\lambda, \mu) \right),$$

where $a, b, c \in \mathbf{Z}$ satisfying $pa - bcM/p = 1$.

We now state the following Propositions and Theorem which can easily be proved.

Proposition 5. *The operator $\mathcal{W}_p$ is independent of the representatives a, b, c and $\mathcal{W}_p$ preserves the space $J_{k,1}^{cusp}(M)$. Also $\mathcal{W}_p$ is hermitian with respect to the Petersson scalar product.*

Proposition 6. *For $p \mid M$,*

i) $U_J(p) + p^{k-2}\,\mathcal{W}_p \;:\; J_{k,1}^{cusp}(M) \longrightarrow J_{k,1}^{cusp}(M/p)$ *and we have*

$$U_J(p) + p^{k-2}\,\mathcal{W}_p = \begin{cases} T_J(p) & \text{on } J_{k,1}^{cusp}(M/p) \\ 0 & \text{on } J_{k,1}^{cusp,new}(M) \end{cases} \tag{18}$$

ii) $\mathcal{W}_p$ *preserves the space* $J_{k,1}^{cusp,new}(M)$

Theorem 3. *The space $J_{k,1}^{cusp,new}(M)$ has a basis of eigenforms with respect to $T_J(p)$, $p \nmid M$, $U_J(p)$, $p \mid M$ and $\mathcal{W}_p$, $p \mid M$.*

If V is the linear span of $P_{(D,r)}$, where $D \equiv r^2 \pmod 4$ is negative with $(D, M) = 1$, we will show that $V = J_{k,1}^{cusp}(M)$ in the following

Proposition 7.

i) V is isomorphic to $S^+_{k-1/2}(\Gamma_0(4M))$ under the linear map $\mathcal{S}$ and is invariant with respect to $U_J(p)$, $p|M$

ii) $V \cap J_{k,1}^{cusp,new}(M) = J_{k,1}^{cusp,new}(M)$

iii) $V \cap J_{k,1}^{cusp,old}(M) = J_{k,1}^{cusp,old}(M)$

Proof.

Using Theorem 1 of [19] for the case M squarefree and χ trivial, we see that the space $S^+_{k-1/2}(\Gamma_0(4M))$ is spanned by all $P^+_{|D|}$, where $D < 0$ is a discriminant with $(D, M) = 1$. Since $\mathcal{S}$ is injective, this proves the first part of *i)*, using (17). The proof of the second part of *i)* is clear using the first part and the fact that $S^+_{k-1/2}(\Gamma_0(4M))$ is invariant under $U(p^2)$ and also using the following fact

$$\mathcal{S}\, U(p^2) = U_J(p)\, \mathcal{S} \qquad p|M \tag{19}$$

To prove *ii)*, let U denote the orthogonal complement of $V \cap J_{k,1}^{cusp,new}(M)$ in $J_{k,1}^{cusp,new}(M)$. Since $U_J(p)$, $p|M$ is hermitian (in $J_{k,1}^{cusp,new}(M)$), we see that U is invariant with respect to $U_J(p)$. Let $\phi \in U$ and let

$$\phi \mid U_J(p) = \lambda_p \phi \qquad p|M \tag{20}$$

Then $\langle \phi , P_{(D,r)} \rangle = 0$ for all $D < 0$, $D \equiv r^2$ (4), $(D, M) = 1$. Denoting the $(D,r)-$th Fourier coefficient of ϕ as $c_\phi(D,r)$, we have

$$c_\phi(D,r) = 0 \quad \text{for all} \quad D < 0, D \equiv r^2 \ (4), (D,M) = 1$$

Also from (20), we obtain that

$$c_\phi(D,r) = 0 \quad \text{for all} \quad D < 0, D \equiv r^2 \ (4), (D,M) > 1$$

This means that $\phi = 0$, proving *ii)*.

Since

$$S_{k-1/2}^{+,old}(\Gamma_0(4M)) = \bigoplus_{\substack{rd|M \\ d<M}} S_{k-1/2}^{+,new}(\Gamma_0(4d)) \mid U(r^2),$$

using induction on the number of prime factors of M and also (19), we see that

$$J_{k,1}^{cusp,old}(M) = \bigoplus_{\substack{rd|M \\ d<M}} J_{k,1}^{cusp,new}(d) \mid U_J(r)$$

and clearly by *ii)*

$$J_{k,1}^{cusp,old}(M) \mid \mathcal{S} = S_{k-1/2}^{+,old}(\Gamma_0(4M))$$

and the pre-image of $S_{k-1/2}^{+,old}(\Gamma_0(4M))$ under $\mathcal{S}$ is a subset of V, proving *iii)*.

Thus we have established the following

Theorem 4. *The spaces* $J_{k,1}^{cusp,new}(M)$ *and* $S_{k-1/2}^{+,new}(\Gamma_0(4M))$ *are Hecke equivariantly isomorphic.*

We summarise the results of this section in the following

Theorem 5.

i)

$$J_{k,1}^{cusp}(M) = \bigoplus_{rd|M} J_{k,1}^{cusp,new}(r) \mid U_J(d)$$

ii) The spaces $J_{k,1}^{cusp,new}(M)$ *and* $S_{k-1/2}^{+,new}(\Gamma_0(4M))$ *are Hecke equivariantly isomorphic. i.e., the isomorphism commutes with the action of Hecke operators.*

iii) The space $J_{k,1}^{cusp,new}(M)$ *has a basis of eigenforms with respect to the Hecke operators* $T_J(p)$, $p \nmid M$ *and* $U_J(p)$, $p \mid M$. *If* $\phi_1 \in J_{k,1}^{cusp,new}(M_1)$ *and* $\phi_2 \in J_{k,1}^{cusp,new}(M_2)$ *be two eigenforms having the same eigenvalues for almost all Hecke operators, then* $\mathbf{C}\,\phi_1 = \mathbf{C}\,\phi_2$, $M_1 = M_2$. *(*M_1 *and* M_2 *are odd squarefree natural numbers). In particular, "strong multiplicity 1" theorem holds in* $J_{k,1}^{cusp,new}(M)$.

iv) If $\phi \in J_{k,1}^{cusp,new}(M)$ *is a newform, then we have*

$$\phi \mid U_J(p) = \pm\, p^{k-2}\, \phi \quad \text{for all} \quad p \mid M.$$

6. Newform Theory in $S_k^*(\Gamma_0^2(M))$

Let $\prod_S^M$ be the Hecke algebra for Siegel cusp forms of degree 2 and level M and let $\prod_J^M$ be the Hecke algebra for Jacobi cusp forms of index 1 and level M. Then it is known that $\prod_S^M$ is generated by $T_S(p)$, $T_S(p^2), p \nmid M$ and $T_S(p)$, $p \mid M$ and $\prod_J^M$ is generated by $T_J(p)$, $p \nmid M$ and $U_J(p)$, $p \mid M$. (cf.[2],[3],[8],[9],[10],[17]).

For $F \in S_k(\Gamma_0^2(M))$ and $p \mid M$ we have

$$F \mid T_S(p) \;=\; \sum_{\substack{n,r,m\in\mathbb{Z}\\ r^2<4nm}} A(np,rp,mp)\, e(n\tau+rz+m\tau') \tag{21}$$

where

$$F \;=\; \sum_{\substack{n,r,m\in\mathbb{Z}\\ r^2<4nm}} A(n,r,m) e(n\tau+rz+m\tau')$$

(Refer [17])

Also if $F\in S_k^*(\Gamma_0^2(M))$, we have

$$F \mid T_S(p) \in S_k^*(\Gamma_0^2(M)) \quad \text{for} \quad p\mid M$$

i.e.,

$$A(n,r,m) \;=\; \sum_{\substack{d\mid(n,r,m)\\ (d,M)=1}} d^{k-1} A\left(\frac{nm}{d^2},\frac{r}{d},1\right)$$

and

$$A(np,rp,mp) \;=\; \sum_{\substack{d\mid(n,r,m)\\ (d,M)=1}} d^{k-1} A\left(\frac{nmp^2}{d^2},\frac{rp}{d},1\right) \tag{22}$$

Remark 4.

We observe that the Hecke operator $T_S(p)$, $p\mid M$ on $S_k^*(\Gamma_0^2(M))$ can be explicitly given by

$$T_S(p) = p^{k-4} \sum_{\substack{v(p^2)\\ \lambda\mu(p)}} \begin{pmatrix} 1/p & 0 & v/p & (\mu-\lambda v)/p \\ \lambda & 1 & \mu & 1 \\ 0 & 0 & p & -p\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let

$$\phi = \sum_{\substack{D<0,r\in\mathbb{Z}\\ D\equiv r^2(4)}} c(D,r)\, e\left(\frac{r^2-D}{4}\tau+rz\right) \;\in\; J_{k,1}^{cusp}(M)$$

and let $\mathcal{V}$ be the inverse map from $J_{k,1}^{cusp}(M)$ to $S_k^*(\Gamma_0^2(M))$ of Theorem 1. Then using (3),(5),(21) and (22), we have for $p|M$,

$$\phi \mid U_J(p) \mid \mathcal{V} = \sum_{\substack{D<0,m,r\in Z \\ D\equiv r^2(4m)}} \left(\sum_{\substack{d|(m,r),(d,M)=1 \\ D\equiv r^2(4md)}} d^{k-1} c\left(\frac{Dmp^2}{d^2}, \frac{rp}{d}\right)\right) \times e\left(\frac{r^2-D}{4m}\tau + rz + m\tau'\right)$$

$$= \phi \mid \mathcal{V} \mid T_S(p)$$

For $p \not| M$, we put

$$T_S'(p) = T_S(p)^2 - T_S(p^2)$$

Let $\phi \in J_{k,1}^{cusp}(M)$ and let $\phi|\mathcal{V} = F \in S_k^*(\Gamma_0^2(M))$. Put $F_1 = F \mid T_S(p)$ and $F_2 = F \mid T_S'(p)$ Let $\phi_1, \phi_2 \in J_{k,1}^{cusp}(M)$ be such that

$$F_1 = \phi_1 \mid \mathcal{V}$$

$$F_2 = \phi_2 \mid \mathcal{V}$$

Then proceeding exactly on the lines of [3],we can prove that

$$\phi_1 = \phi \mid \left(T_J(p) + p^{k-1} + p^{k-2}\right)$$

and

$$\phi_2 = \phi \mid \left((p^{k-1}+p^{k-2})\, T_J(p) + 2p^{2k-3} + p^{2k-4}\right)$$

Thus we have established the following

Theorem 6. *The map $\mathcal{V} : J_{k,1}^{cusp}(M) \to S_k^*(\Gamma_0^2(M))$ is Hecke equivariant in the sense*

$$\phi \mid \mathcal{V} \mid T = \phi \mid \iota(T) \mid \mathcal{V},$$

$\phi \in J_{k,1}^{cusp}(M)$, $T \in \prod_S^M$, *with respect to the homomorphism of the Hecke algebras* $\iota : \prod_S^M \rightarrow \prod_J^M$ *defined on generators by*

$$\begin{aligned}
\iota(T_S(p)) &= T_J(p) + p^{k-1} + p^{k-2} && p \nmid M \\
\iota(T'_S(p)) &= (p^{k-1} + p^{k-2})\, T_J(p) + 2p^{2k-3} + p^{2k-4} && p \nmid M \\
\iota(T_S(p)) &= U_J(p) && p \mid M
\end{aligned}$$

Newform Theory

In $S_k^*(\Gamma_0^2(M))$, we define

$$S_k^{*,old}(\Gamma_0^2(M)) = \sum_{\substack{rd \mid M \\ 1 \leq r < M}} S_k^*(\Gamma_0^2(M)) \mid U_S(d)$$

and put $S_k^{*,new}(\Gamma_0^2(M))$ to be the orthogonal complement of $S_k^{*,old}(\Gamma_0^2(M))$ in $S_k^*(\Gamma_0^2(M))$ with respect to the Petersson product. Since $\mathcal{V} : J_{k,1}^{cusp}(M) \rightarrow S_k^*(\Gamma_0^2(M))$ is an isomorphism (by Theorem 1), using the isomorphism $\mathcal{V}$ and Theorem 4, we have the following

Theorem 7.

i)

$$S_k^*(\Gamma_0^2(M)) = \bigoplus_{rd \mid M} S_k^{*,new}(\Gamma_0^2(r)) \mid U_S(d)$$

ii) $S_k^{*,new}(\Gamma_0^2(M))$ *is Hecke equivariantly isomorphic to* $J_{k,1}^{cusp,new}(M)$.*(i.e., the isomorphism commutes with the action of Hecke operators).*

iii) The space $S_k^{*,new}(\Gamma_0^2(M))$ *has a basis of eigenforms with respect to the Hecke operators* $T_S(p)$, $T'_S(p), p \nmid M$ *and* $T_S(p)$, $p \mid M$. *If two eigenforms* $F_1 \in S_k^{*,new}(\Gamma_0^2(M_1))$ *and* $F_2 \in S_k^{*,new}(\Gamma_0^2(M_2))$ *(*M_1, M_2 *are odd square-free natural numbers) have the same eigenvalues for almost all Hecke operators then* $\mathbf{C}\, F_1 = \mathbf{C}\, F_2$ *and* $M_1 = M_2$. *In particular "strong multiplicity 1" theorem holds in* $S_k^{*,new}(\Gamma_0^2(M))$.

iv) If $F \in S_k^{*,new}(\Gamma_0^2(M))$ is a newform (i.e., a basis element)then for $p \mid M$, we have

$$F \mid T_S(p) = \pm p^{k-2} F.$$

7. Generalised Saito-Kurokawa Conjecture

Let $F \in S_k^*(\Gamma_0^2(M))$ be a Hecke eigenform.Then the Andrianov zeta function $Z_F(s)$ has the Euler product expansion

$$Z_F(s) = \prod_{p \mid M} (1 - \gamma_p\, p^{-s})^{-1}$$
$$\times \prod_{p \nmid M} (1 - \gamma_p p^{-s} + (\gamma_p' - p^{2k-4})p^{-2s} - \gamma_p p^{2k-3-3s} + p^{4k-4-4s})^{-1}$$

where

$$F \mid T_S(p) = \gamma_p F ; \quad F \mid T_S'(p) = \gamma_p' F \quad p \nmid M \text{ and } F \mid T_S(p) = \gamma_p F \, , p \mid M.$$

Let $F \in S_k^{*,new}(\Gamma_0^2(M))$ such that $F = \phi \mid \mathcal{V}$ where $\phi \in J_{k,1}^{cuap,new}(M)$ with

$$\phi \mid T_J(p) = \lambda_p \, \phi \ , \quad p \nmid M$$
$$\phi \mid U_J(p) = \lambda_p \, \phi \ , \quad p \mid M$$

Using Theorem 4 and the isomorphism of Kohnen ([6],Theorem 2), there is a 1-1 correspondence between eigenforms in $S_{2k-2}^{new}(M)$ and $J_{k,1}^{cusp,new}(M)$. Assume that $f \in S_{2k-2}^{new}(M)$ be the normalised Hecke eigenform (with eigenvalue λ_p) corresponding to $\phi \in J_{k,1}^{cusp,new}(M)$.

Consider

$$Z_F(s) = \prod_{p \mid M} (1-\gamma_p\, p^{-s})^{-1} \times \prod_{p \nmid M} (1-\gamma_p p^{-s} + (\gamma'_p - p^{2k-4})p^{-2s} - \gamma_p p^{2k-3-3s} + p^{4k-4-4s})^{-1}$$

Then using Theorem 6,

$\gamma_p = \lambda_p, p \mid M$ and for $p \nmid M$ $\gamma_p = \lambda_p + p^{k-1} + p^{k-2}$ and $\gamma'_p = (p^{k-1} + p^{k-3})\lambda_p + 2p^{2k-3} + p^{2k-4}$.

Therefore, for $p \nmid M$,

$$\begin{aligned}(1-\gamma_p p^{-s} &+ (\gamma'_p - p^{2k-4})p^{-2s} - \gamma_p p^{2k-3-3s} + p^{4k-4-4s}) \\ &= (1-p^{k-1}p^{-s})(1-p^{k-2}p^{-s})(1-\lambda_p p^{-s} + p^{2k-3-2s})\end{aligned}$$

Hence

$$\begin{aligned}Z_F(s) &= \prod_{p \mid M} (1-\lambda_p p^{-s})^{-1} \\ &\times \prod_{p \nmid M} (1-p^{k-1-s})^{-1}(1-p^{k-2-s})^{-1}(1-\lambda_p p^{-s} + p^{2k-3-2s})^{-1} \\ &= \prod_{p \mid M} (1-p^{k-1-s})^{-1}\,(1-p^{k-2-s})^{-1}\, L_f(s)\end{aligned}$$

where

$$L_f(s) = \prod_{p \mid M} (1-\lambda_p p^{-s})^{-1} \prod_{p \nmid M} (1-\lambda_p p^{-s} + p^{2k-3-2s})^{-1}$$

Put

$$Z^*_F(s) = \prod_{p \mid M} (1-p^{k-1-s})^{-1}(1-p^{k-2-s})^{-1} Z_F(s)$$

Then it is clear, from the above equation, that

$$Z_F^*(s) \;=\; \zeta(s-k+1)\,\zeta(s-k+2)\;L_f(s)$$

Thus we have established the following

Theorem 8. **(Generalised Saito–Kurokawa Conjecture).**

The space $S_k^{*,new}(\Gamma_0^2(M))$ *is in 1–1 correspondence with* $S_{2k-2}^{new}(M)$ *where* M *is an odd squarefree natural number. For a Hecke eigenform* $F \in S_k^{*,new}(\Gamma_0^2(M))$ *and a normalised Hecke eigenform* $f \in S_{2k-2}^{new}(M)$*, the correspondence is given by*

$$Z_F^*(s) \;=\; \zeta(s-k+1)\,\zeta(s-k-2)\;L_f(s)$$

Acknowledgements

We thank Prof. V. Kumar Murty for his valuable suggestios. The earlier version of the paper was written when the second author was a Post Doctoral Fellow at The Institute of Mathematical Sciences, Madras. He wishes to thank the Institute for the financial support. The second author also acknowledges the NBHM grant.

REFERENCES

[1] Andrianov, A.N.: Modular Descent and the Saito– Kurokawa Conjecture *Invent. Math.* **53**, 267–280 (1979)

[2] Andrianov, A.N.: Quadratic forms and Hecke operators *Grundlehren. Math.Wiss.* **286** Springer–Verlag, Berlin–Heidelberg–New York 1987

[3] Eichler, M. and Zagier, D.: The Theory of Jacobi forms Progress in Math. **55** Boston–Birkhauser 1985

[4] Gross, B., Kohnen, W. and Zagier, D.: Heegner points and derivatives of L–series II *Math. Ann.* **278**, 497–562 (1987)

[5] Koblitz, N.I.: Introduction to Elliptic curves and Modular forms Springer–Verlag. Berlin/New York 1984

[6] Kohnen, W.: Newforms of half-integral weight *J. Reine Angew Math.* **333**, 32–72 (1981)

[7] Kohnen, W.: On the Fourier coefficients of modular forms of half–integral weight *Math. Ann.* **271**, 237–268 (1985)

[8] Kojima, H.: On construction of Siegel modular forms of degree two *J. Math. Soc. Japan* **34**, 393–412 (1982)

[9] Kojima, H.: Siegel modular cusp forms of degree two *Tohoku Math. J.***33**, 65–75 (1981)

[10] Kojima, H.: Siegel modular forms of degree two *Proc. Japan. Acad.* **58** *Ser.A* , 369–371 (1982)

[11] Kramer, J.: Jacobiformen und thetareihen *Manuscripta Math.* **54** , 279–322 (1986)

[12] Kurokawa, N.: Examples of Eigenvalues of Hecke operators on Siegel cusp forms of Degree Two *Invent. Math.* **49** , 149–165 (1978)

[13] Maaß, H.: Über eine Spezialschar von Modulformen Zweiten Grades *Invent. Math.* **52**, 95–104 (1979)

[14] Maaß, H.: Über eine Spezialschar von Modulformen Zweiten Grades II *Invent. Math.* **53** , 249–253 (1979)

[15] Maaß, H.: Über eine Spezialschar von Modulformen Zweiten Grades III *Invent. Math.* **53** , 255–265 (1979)

[16] Manickam, M., Ramakrishnan, B. and Vasudevan, T.C.: On the theory of newforms of half–integral weight *J. Number Theory* **34**, 210–224 (1990)

[17] Matsuda. I.: Dirichlet series corresponding to Siegel modular forms of degree two, level N *Sci. Papers College Gen. Ed. Univ. Tokyo* **28**, 21–49 (1979) "Correction" *ibid.* **29**, 29–30 (1979)

[18] Ramakrishnan, B.: On Shimura and Shintani Correspondences for modu-

lar forms and Jacobi forms and some results on Hecke operators, Thesis, University of Madras 1989

[19] Serre, J.-P. and Stark, H.M.: Modular forms of weight 1/2. In : Modular Functions of One Variable VI. pp 27–67. Lecture Notes in Math.**627**. Berlin–Heidelberg–New York: Springer (1977)

[20] Skoruppa, N.-P.: Über den Zusammenhang zwischen Jacobiformen und Modulformen halbganzen Gewichts *Bonner Math. Schriften* **159** Bonn (1985)

[21] Zagier, D.: Sur la conjecture de Saito–Kurokawa (d'aprèsH.Maaß), Sém Delange–Pisot–Poitou 1979/80. Progress in Math. **12**, Boston–Basel–Stuttgart , 371–394 (1980)

M.M., T.C.V. :
Department of Mathematics
Ramakrishna Mission Vivekananda
College
Mylapore
Madras 600 004
India

B.R. :
The Mehta Research Institute of
Mathematics and Mathematical Physics
10, Kasturba Gandhi Marg
(Old Kutchery Road)
Allahabad 211 002
India.
e-mail: ramki@mri.ernet.in

(Received May 18, 1993)

manuscripta math. 81, 183 - 202 (1993)

manuscripta mathematica

ON IRRATIONALITY MEASURES OF THE VALUES OF GAUSS HYPERGEOMETRIC FUNCTION

Ari Heimonen, Tapani Matala-aho and Keijo Väänänen

The paper gives irrationality measures for the values of some Gauss hypergeometric functions both in the archimedean and p-adic case. Further, an improvement of general results is obtained in the case of logarithmic function.

Introduction

We shall consider the irrationality measures of the values of Gauss hypergeometric function

$$F(z) = {}_2F_1\left(\begin{matrix} 1\ ,\ b \\ c \end{matrix}\middle| z\right) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} z^n, \tag{1}$$

where $b, c \neq 0, -1, -2, \ldots$ are rational parameters, and $(b)_0 = 1$, $(b)_n = b(b+1)\ldots(b+n-1)$, $n = 1, 2, \ldots$. The irrationality and linear independence measures of the values of F are considered in many works both in the general case and in some interesting special cases, see [1] [2], [4], [6], [7], [8], [9], [10], [11], [13], [14], [15], [16], [17], [18], [19], [20] and [23]. Also the transcendence of the values of F at algebraic points is considered in the important papers [3], [5] and [24].

In the present work we first give using Padé type approximations an irrationality measure for $F(r/s)$ with certain values $r/s \in \mathbb{Q}$, both in the archimedean and p-adic case. In many special values of b and c these general results can be sharpened by the careful consideration of the arithmetic properties of the coefficients of the approximation polynomials. This idea was first realised for the binomial function by Chudnovsky [8], and then in some other cases in [10], [13], [14] and [20]. Here we shall deduce a general criterion to find a common factor for the coefficients of our approximation polynomials and then apply this criterion to the logarithmic function to obtain a generalisation of the nice work of Rukhadze [20].

Results and notations

We shall denote by $\mathbb{Q}_v$ the v-adic completion of $\mathbb{Q}$, where $v \in \{\infty, \text{primes } p\}$, in particular $\mathbb{Q}_\infty = \mathbb{R}$. For an irrational number $\theta \in \mathbb{Q}_v$, we shall call an irrationality measure $m_v(\theta)$ of θ the infimum of m satisfying the following condition: for any $\varepsilon > 0$ there exists an $H_0 = H_0(\varepsilon)$ such that

$$\left|\theta - \frac{P}{Q}\right|_v > H^{-m-\varepsilon}$$

for all rationals P/Q satisfying $H = \max\{|P|, |Q|\} > H_0$. In the following we denote $m_\infty(\theta) = m(\theta)$. All our measures are effective in the sense that H_0 can be effectively determined.

Throughout this paper we shall assume that $c > b > 0$, $b = a/f$, $c = g/h$, where a, f, g, h are natural numbers such that $(a, f) = (g, h) = 1$. Let us denote $B = b - 1 = E/F$, $C = c - b - 1 = G/H$ with $E, G \in \mathbb{Z}$, $F, H \in \mathbb{N}$, $(E, F) = (G, H) = 1$. Further, let $L = \text{l.c.m.}(F, H)$, and use H^* to denote the denominator of h/H (therefore $H^* \mid H$). We shall also need the notations

$$\mu_F = \prod_{p \mid F} p^{\frac{1}{p-1}}, \quad \lambda(h) = \frac{h}{\phi(h)} \sum_{\substack{i=1 \\ (i,h)=1}}^{h} \frac{1}{i}$$

to state the following result.

Theorem 1. *If $r/s \in (-1, 1)$ is a non-zero rational number satisfying*

$$(r, s) = 1, \quad LH^*\mu_L\mu_{H^*}e^{\lambda(h)}\left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1,$$

then

$$m\left(F\left(\frac{r}{s}\right)\right) \leq 1 - \frac{2\ln\left(\sqrt{s} + \sqrt{s-r}\right) + \lambda(h) + \ln(LH^*\mu_L\mu_{H^*})}{2\ln\left|\sqrt{s} - \sqrt{s-r}\right| + \lambda(h) + \ln(LH^*\mu_L\mu_{H^*})}.$$

As a p-adic analogue of this result we state the following sharpening of [17].

Theorem 1p. *Suppose that p is a prime such that $p \nmid fh$. If $r/s > 1$ is a rational number satisfying*

$$|r/s|_p < 1, \quad (r, s) = 1, \quad LH^*\mu_L\mu_{H^*}e^{\lambda(h)}r|r|_p^2 < 1,$$

then

$$m_p\left(F\left(\frac{r}{s}\right)\right) \leq \frac{2\ln|r|_p}{2\ln|r|_p + \ln r + \lambda(h) + \ln(LH^*\mu_L\mu_{H^*})}$$

(in writing $m_p(f(\dots))$ we always think of f as a corresponding p-adic series).

If $b = 1$, $c = 2$, then Theorem 1p implies for the p-adic logarithm the following

Corollary 1p. *If $r/s > 1$ is a rational number satisfying*

$$|r/s|_p < 1, \quad (r,s) = 1, \quad er|r|_p^2 < 1,$$

then

$$m_p\left(\log\left(1 - \frac{r}{s}\right)\right) \leq \frac{2\ln|r|_p}{2\ln|r|_p + \ln r + 1}.$$

In particular, for all $p^l > e$ we have

$$m_p\left(\log\left(1 - p^l\right)\right) \leq \frac{2l\ln p}{l\ln p - 1}.$$

For the real logarithm we obtain, by Theorem 1, the well-known result

$$m\left(\log\left(1 - \frac{r}{s}\right)\right) \leq 1 - \frac{2\ln\left(\sqrt{s} + \sqrt{s-r}\right) + 1}{2\ln\left|\sqrt{s} - \sqrt{s-r}\right| + 1},$$

if $r/s \in [-1, 1)$ is a rational number satisfying

$$e\left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1.$$

To get a sharpening of this result we define, for a rational $\alpha = u/v \in (0,1]$, $u, v \in \mathbb{N}$, $(u,v) = 1$, the subsets I_1 and I_2 of $\{1, \ldots, v-1\}$ such that

$$i \in I_1 \quad \text{iff} \quad [\alpha i] + 1 = \left[\alpha i + \frac{\alpha}{2}\right], \qquad i \in I_2 \quad \text{iff} \quad [\alpha i] = \left[\alpha i + \frac{\alpha}{2}\right].$$

Let then

$$\tau_1(\alpha) = \frac{1}{v}\left(\sum_{i\in I_1}\left(\Psi\left(\frac{1+[\alpha i]}{u}\right) - \Psi\left(\frac{i}{v}\right)\right) + \sum_{i\in I_2}\left(\Psi\left(\frac{2i-[\alpha i]}{2v-u}\right) - \Psi\left(\frac{i}{v}\right)\right)\right),$$

where Ψ is the digamma function (see e.g. [12], pp. 15–20). Further with a given rational $\beta \geq \alpha$ we define

$$A(\alpha,\beta,z) = \min_{0<\rho<|z|+\frac{1}{2}(1-\operatorname{sgn} z)}\left(\frac{(\rho+|z|)(\rho+|z|-\operatorname{sgn} z)^\beta}{\rho^\alpha}\right)$$

for all $z \geq 1$ or $z < 0$, and

$$R(\alpha,\beta,z) = \max_{0\leq t\leq 1}\frac{t(1-t)^\beta}{(1-zt)^\alpha}$$

for all $z \in [-1, 1)$.

Theorem 2. *If*

$$Q(\alpha) = e^{2-\alpha-r_1(\alpha)}|r|^{2-\alpha} A\left(\alpha, 1, \frac{s}{r}\right), \quad R(\alpha) = e^{2-\alpha-r_1(\alpha)}|r|^2 s^{-\alpha} R\left(\alpha, 1, \frac{r}{s}\right),$$

then

$$m\left(\log\left(1-\frac{r}{s}\right)\right) \le \inf_{\alpha}{}^{*}\left\{1 - \frac{\ln Q(\alpha)}{\ln R(\alpha)}\right\},$$

where $\inf_{\alpha}^{*}$ *means that for a given non-zero rational* $r/s \in [-1, 1)$ *the infimum is taken over all rationals* $\alpha \in (0, 1]$ *satisfying* $R(\alpha) < 1$.

As numerical examples we give the following list, where u.b. means the obtained upper bound for $m(\log(1 - r/s))$.

$\frac{r}{s}$	α	u.b.	u.b.($\alpha = 1$)
-1	$\frac{6}{7}$	3.891399 ...	4.6221 ...
$-\frac{2}{3}$	$\frac{18}{19}$	9.7551 ...	11.1449 ...
$-\frac{3}{5}$	$\frac{30}{31}$	53.8149 ...	90.7656 ...
$-\frac{1}{2}$	$\frac{12}{13}$	3.3317 ...	3.5474 ...
$-\frac{1}{3}$	$\frac{16}{17}$	3.1105 ...	3.2240 ...
$-\frac{7}{30}$	$\frac{160}{161}$	619.5803 ...	1798.6314 ...
$-\frac{1}{120}$	$\frac{578}{579}$	2.3854 ...	2.3862 ...
$\frac{1}{15}$	$\frac{68}{69}$	2.6411 ...	2.6535 ...
$\frac{3}{20}$	$\frac{88}{89}$	5.7392 ...	5.7977 ...

In the first row of this list we have Rukhadze's [20] measure for log 2. However we note that in some other cases, e.g. if $r/s = -1/2$, $-1/3$, we are not able to reach the measures announced in [20].

Padé type approximations

We use l, m and n to denote positive integer parameters satisfying $l \le \min\{m, n\}$. In the proof of our theorems 1 and 1p we shall use only the choice $l = m = n$, but in some interesting cases like in Theorem 2 some other choices are better. Therefore we give our next lemmas in the general form.

Let us define the polynomial $A_{l,m,n}(z)$ by

$$\text{(2)} \qquad A_{l,m,n}(z) = \frac{1}{z^B(1-z)^C}\frac{1}{l!}\left(\frac{d}{dz}\right)^l (z^{n+B}(1-z)^{m+C}),$$

$$= (-1)^l z^n (1-z)^{m-l} \sum_{k=0}^{l} \binom{n+B}{k}\binom{m+C}{l-k}\left(\frac{z-1}{z}\right)^k$$

$$= \frac{(-m-C)_l}{l!} z^n (1-z)^{m-l} {}_2F_1\left(\begin{matrix} -n-B\,,\ -l \\ 1+m+C-l \end{matrix}\middle|\, \frac{z-1}{z}\right).$$

Thus the polynomial $A_{l,m,n}$ is of degree $\le n+m-l$ and has a zero of order $\ge n-l$. By defining

$$\begin{aligned} Q_{l,m,n}(z) &= z^{n+m-l} A_{l,m,n}\left(\frac{1}{z}\right) \\ &= \frac{(-m-C)_l}{l!}(z-1)^{m-l}{}_2F_1\left(\begin{matrix}-n-B\ ,\ -l\\ 1+m+C-l\end{matrix}\middle| 1-z\right) \\ &= \frac{(-m-n-B-C)_l}{l!}(z-1)^{m-l}{}_2F_1\left(\begin{matrix}-n-B\ ,\ -l\\ -n-m-B-C\end{matrix}\middle| z\right) \end{aligned}$$

we get a polynomial of degree $\le m$, where the last equality is obtained using the formula 2.10 (1) of [12].

The function $F(z)$ has for all $|z| < 1$ an integral representation

$$F(z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{\omega(t)}{1-zt}\, dt, \quad \omega(t) = t^{b-1}(1-t)^{c-b-1}. \tag{3}$$

Therefore, for all $0 < |z| < 1$,

$$\begin{aligned} Q_{l,m,n}(z)F(z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{Q_{l,m,n}(z)\omega(t)}{1-zt}\, dt \\ &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \left(z^{n+m-l-1} \int_0^1 \frac{A_{l,m,n}(1/z) - A_{l,m,n}(t)}{\frac{1}{z} - t} \omega(t)\, dt \right. \\ &\left. + z^{n+m-l} \int_0^1 \frac{A_{l,m,n}(t)\omega(t)}{1-zt}\, dt \right) = z^{n+m-l-1} B_{l,m,n}(1/z) + R_{l,m,n}(z) \end{aligned}$$

with obvious definitions of the polynomial $B_{l,m,n}$ and the function $R_{l,m,n}$.

We next consider more closely the remainder function $R_{l,m,n}$. If

$$f(z) = z^{n+B}(1-z)^{m+C},$$

then, by partial integration and our assumption $l \le \min\{m,n\}$,

$$\begin{aligned} R_{l,m,n}(z) &= \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \frac{z^{n+m-l}}{l!} \int_0^1 \frac{f^{(l)}(t)}{1-zt}\, dt \\ &= \cdots = \frac{(-1)^l\Gamma(c)}{\Gamma(c-b)\Gamma(b)} z^{n+m} \int_0^1 \frac{f(t)}{(1-zt)^{l+1}}\, dt \\ &= (-1)^l z^{n+m} \frac{\Gamma(c)\Gamma(m+c-b)\Gamma(n+b)}{\Gamma(c-b)\Gamma(b)\Gamma(n+m+c)} {}_2F_1\left(\begin{matrix} l+1\ ,\ n+b\\ n+m+c\end{matrix}\middle| z\right). \\ &= (-1)^l z^{n+m} \frac{(b)_n(c-b)_m}{(c)_{n+m}} {}_2F_1\left(\begin{matrix} l+1\ ,\ n+b\\ n+m+c\end{matrix}\middle| z\right). \end{aligned} \tag{4}$$

Therefore the Taylor expansion of $R_{l,m,n}$ has rational coefficients and vanishes at $z=0$ at least to the order $n+m$.

By the above considerations we now have the approximation formula

$$R_{l,m,n}(z) = Q_{l,m,n}(z)F(z) - P_{l,m,n}(z), \tag{5}$$

where

$$P_{l,m,n}(z) = z^{n+m-l-1} B_{l,m,n}(\frac{1}{z}) \tag{6}$$

is a polynomial of degree $n+m-l-1$. Thus (5) is an identity with rational coefficients, and therefore we can use it also in other metrics if the series converge.

The estimation of the polynomials and the remainder term

Let us suppose that $l = [\alpha n]$, $m = [\beta n]$, where α and β are rationals satisfying $0 < \alpha \leq \min\{1, \beta\}$, and let us denote

$$P_n(z) = P_{l,m,n}(z), \quad Q_n(z) = Q_{l,m,n}(z), \quad R_n(z) = R_{l,m,n}(z).$$

We shall first estimate the remainder term $R_n(z)$. Let $\delta = \delta(v)$ be 1, if $v = \infty$, and 0, if $v = p$. By $c_1, c_2, \ldots$ we shall denote positive constants independent of n. We now obtain the following

Lemma 1. *If $|z|_v < 1$ and in the finite case $v \not| fh$, then we have*

$$|R_n(z)|_v \leq c_1 n^{1-\delta} \left(|z|_v^{1+\beta} R(\alpha, \beta, z)^\delta\right)^n$$

for all $n \geq c_2$. In the archimedean case the bound on the right-hand side of this inequality is an asymptotic for $|R_n(z)|$ $(n \to \infty)$.

Remark 1. In the archimedean case the bound holds at the point $z = -1$, too.

Proof. In the archimedean case the result follows immediately from the integral representation (4) of $R_{l,m,n}(z)$.

To prove the finite case we denote

$$Q_n(z) = \sum_{j=0}^{m} q_j z^j, \qquad F(z) = \sum_{j=0}^{\infty} f_j z^j.$$

By (5) we then have

$$R_n(z) = \sum_{k=m+n}^{\infty} \left(\sum_{j=0}^{m} q_j f_{k-j}\right) z^k = z^{m+n} \sum_{k=0}^{\infty} e_k z^k,$$

where

$$e_k = \sum_{j=0}^{m} q_j f_{k+m+n-j}, \qquad k = 0, 1, \ldots.$$

Because q_j are v-integers (i.e.$|q_j|_v \leq 1$), it follows that

$$|e_k|_v \leq \max_{0 \leq j \leq m} \{|f_{k+m+n-j}|_v\}.$$

Here

$$f_{k+m+n-j} = \frac{h^{k+m+n-j} a(a+f)\ldots(a+(k+m+n-j-1)f)}{f^{k+m+n-j} g(g+h)\ldots(g+(k+m+n-j-1)h)},$$

and therefore $|e_k|_v \le p^{r(k)}$ $(v = p)$, where

$$r(k) \le \max_{0\le j\le m} \sum_{\mu \le \frac{\ln(|g|+(k+m+n)h)}{\ln p}} \left(\left[\frac{k+m+n-j}{p^\mu} \right] + 1 - \left[\frac{k+m+n-j}{p^\mu} \right] \right)$$

$$\le \frac{\ln h(|c| + k + (1+\beta)n)}{\ln p}, \quad k = 0, 1, \ldots .$$

Thus

$$|e_k|_v \le h(|c| + k + (1+\beta)n), \quad k = 0, 1, \ldots,$$

which implies the estimate

$$\left| e_k z^k \right|_v \le h(|c| + k + (1+\beta)n)|z|_v^k \le c_1 n$$

for all $n \ge c_2$. This proves our lemma. □

The function $f(t) = t^{n+B}(1-t)^{m+C}$ is analytic in a complex domain D obtained by cutting the plane from 0 to infinity and from 1 to infinity. We choose these cuts in such a way that they avoid the point z. To estimate the polynomials $P_n(z)$ and $Q_n(z)$ we first consider the polynomial $A_n(z) = A_{l,m,n}(z)$ by using Cauchy's integral formula to get

$$(7) \qquad A_n(z) = \frac{1}{z^B(1-z)^C} \frac{1}{l!} \left(\frac{d}{dz} \right)^l f(z) = \frac{1}{2\pi i} \frac{1}{z^B(1-z)^C} \oint_\Gamma \frac{f(t)}{(t-z)^{l+1}}\, dt,$$

where Γ is a simple closed curve in D.

Lemma 2. *If* $|z| > 1$ *or* $z = -1$, *then*

$$|A_n(z)| \le c_3 \left(A(\alpha, \beta, z)^n + |z|^{-\alpha n} R\left(\alpha, \beta, \frac{1}{z} \right)^n \right),$$

and if $-1 < z < 0$, *then*

$$|A_n(z)| \le c_4 \left(A(\alpha, \beta, z)^n + (1-z)^{-\alpha n} R\left(\frac{\alpha}{\beta}, \frac{1}{\beta}, \frac{1}{1-z} \right)^{\beta n} \right).$$

In the case $\alpha = \beta = 1$ *we have*

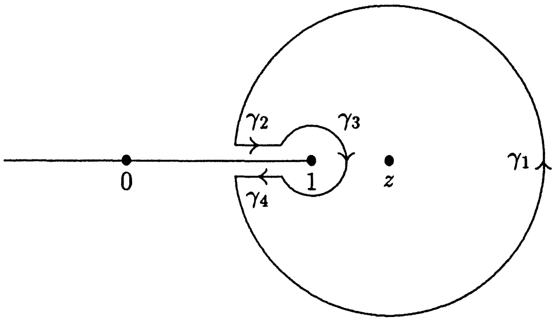
$$|A_n(z)| \le c_5 n^q$$

for all $0 \le z \le 1$, *if* $q = \max\{B, C\} \ge -\frac{1}{2}$.

Proof. We divide our proof into four cases. Let first $z > 1$. Then we cut the plane along the real line from 1 to $-\infty$, and take $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $\gamma_1 : |t - z| = \rho < z$ and $\gamma_3 : |t - 1| = \varepsilon$ with some $\varepsilon > 0$ (see Picture 1). Then

$$\left| \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(t)}{(t-z)^{l+1}}\, dt \right| \le \frac{(\rho+z)^{n+B}(\rho+z-1)^{m+C}}{\rho^l}$$

$$\le \frac{(\rho+z)^B(\rho+z-1)^{C-\{\beta n\}}}{\rho^{-\{\alpha n\}}} \left(\frac{(\rho+z)(\rho+z-1)^\beta}{\rho^\alpha} \right)^n$$



Picture 1.

(this is all we need if $\rho < z - 1$). In the case $z - 1 < \rho < z$ we have (with small ε)

$$\left|\frac{1}{2\pi i}\oint_{\gamma_3}\frac{f(t)}{(t-z)^{l+1}}\,dt\right| \le \left|\frac{1}{2\pi i}\int_{\pi}^{-\pi}\frac{\left(1+\varepsilon e^{i\phi}\right)^{n+B}\left(\varepsilon e^{i\phi}\right)^{m+C}}{(1+\varepsilon e^{i\phi}-z)^{l+1}}\varepsilon\,d\phi\right|$$

$$\le \frac{(1+\varepsilon)^{n+B}\varepsilon^{m+1+C}}{(z-1-\varepsilon)^{l+1}} \to 0, \text{ when } \varepsilon \to 0.$$

Further it follows that

$$\left|\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(t)}{(t-z)^{l+1}}\,dt\right| = \frac{1}{2\pi}\left|\int_{z-\rho}^{1}\frac{f(t)}{(t-z)^{l+1}}\,dt\right|$$

$$\le \frac{1}{2\pi|z|^{l+1}}\int_0^1\frac{f(t)}{(1-\frac{t}{z})^{l+1}}\,dt \le \frac{c_6}{|z|^{\alpha n}}R\left(\alpha,\beta,\frac{1}{z}\right)^n.$$

These estimates give the truth of our lemma in this case.

The cases $z \le -1$ and $-1 < z < 0$ are analogous.

In the case $\alpha = \beta = 1$ our polynomial is connected with the Jacobi polynomial $P_n^{(B,C)}(z)$ by the formula

$$A_n(z) = P_n^{(B,C)}(1-2z), \quad 0 \le z \le 1.$$

Therefore our result follows immediately from Theorem 7.32.1 of [22]. □

Lemma 3. *If* $|z| < 1$ *and* $R(\alpha,\beta,z) \le |z|^{-\alpha}A\left(\alpha,\beta,\frac{1}{z}\right)$, *then*

$$\max\{|Q_n(z)|,|P_n(z)|\} \le c_7\left(|z|^{1+\beta-\alpha}A\left(\alpha,\beta,\frac{1}{z}\right)\right)^n.$$

If $\alpha = \beta = 1$ *and* $z > 1$, *then*

$$\max\{|Q_n(z)|,|P_n(z)|\} \le c_8 n^{q+2}|z|^n.$$

If $\alpha = \beta = 1$, $z < -1$, *and* $\frac{z}{z-1}R\left(1,1,\frac{z}{z-1}\right) < A\left(1,1,\frac{1}{z}\right)$, *then*

$$\max\{|Q_n(z)|,|P_n(z)|\} \le c_9\left(|z|A\left(1,1,\frac{1}{z}\right)\right)^n.$$

Remark 2. Since (5) holds in the archimedean case at $z = -1$, the first part of our lemma is true at $z = -1$.

Proof. Since

$$Q_n(z) = z^{n+m-l} A_n\left(\frac{1}{z}\right),$$

the bounds for $Q_n(z)$ follow from Lemma 2. By (5)

$$Q_n(z)F(z) - P_n(z) = R_n(z)$$

for all $|z| < 1$. If $|P_n(z)| > c_{10}\left(|z|^{1+\beta-\alpha} A(\alpha,\beta,\frac{1}{z})\right)^n$ with a suitable c_{10} we have a contradiction with our hypothesis $R(\alpha,\beta,z) < |z|^{-\alpha} A\left(\alpha,\beta,\frac{1}{z}\right)$. This proves Lemma 3 in the case $|z| < 1$.

Next we assume that $\alpha = \beta = 1$, $z > 1$. From the definition of $P_n(z)$ it follows that

$$P_n(z) = z^{n-1} B_n\left(\frac{1}{z}\right), \quad B_n(u) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 \frac{A_n(u) - A_n(t)}{u-t}\omega(t)\,dt.$$

If $0 < u < 1$, we give the integral in the form $(\frac{u}{2} < \gamma < \frac{1-u}{2})$

$$B_n(u) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\left(\int_0^{u-\gamma} + \int_{u-\gamma}^{u+\gamma} + \int_{u-\gamma}^{1}\right)\frac{A_n(u) - A_n(t)}{u-t}\omega(t)\,dt = I_1 + I_2 + I_3,$$

say. For $|I_1|$ and $|I_3|$ we have the upper bound $2c_{11}n^q/\gamma$ by Lemma 2. Further, by the mean value theorem

$$I_2 = \int_{u-\gamma}^{u+\gamma} A_n'(v)\omega(t)\,dt,$$

where $v = v(t)$ is some point between u and t. Here

$$A_n'(v) = \frac{d}{dv}P_n^{(B,C)}(1-2v) = -2\left(P_n^{(B,C)}\right)'(1-2v),$$

and from Theorem 7.32.4 of [22] we obtain

$$|A_n'(v)| \le c_{12} n^{\max\{2+B,2+C,\frac{1}{2}\}} \le c_{12} n^{q+2}.$$

The case $z < -1$ can be considered in an analogous way. Thus Lemma 3 is true. □

On the properties of the coefficients of P_n and Q_n

Let p be a prime and $r \in \mathbb{Q}$, $r \neq 0$. As usual we define $v_p(r)$ by $r = p^{v_p(r)} R/S$, where $(R,S) = (R,p) = (S,p) = 1$. In the following we shall also need the notation

$$\mu_F(j) = \prod_{p \mid F} p^{v_p(j!)}.$$

Using this notation we see that the coefficients

$$a_j = \binom{n+B}{j}\binom{m+C}{l-j}$$

of the polynomial

$$A_n(z) = \sum_{j=0}^{l}(-1)^m a_j z^{n-j}(z-1)^{m-l+j}$$

satisfy

$$(8) \qquad a_j \in \frac{1}{F^j H^{l-j}\mu_F(j)\mu_H(l-j)}\mathbb{Z} \quad \text{and} \quad a_j \in \frac{1}{L^l \mu_L(l)}\mathbb{Z}, \quad j = 0, 1, \ldots, l,$$

where $L = \text{l.c.m.}(F, H)$. Since

$$Q_n(z) = z^{n+m-l}A_n\left(\frac{1}{z}\right) = \sum_{j=0}^{l}(-1)^m a_j (1-z)^{m-l+j},$$

it follows that

$$(9) \qquad Q_n\left(\frac{r}{s}\right) \in \frac{(r-s)^{m-l}}{s^m L^l \mu_L(l)}\mathbb{Z}.$$

The polynomial $A_n(z)$ can also be given in the form

$$(10) \qquad A_n(z) = \sum_{j=m-l}^{n+m-l} c_j (1-z)^j,$$

where

$$c_j = \sum_{i=\min\{0,j-m\}}^{j-m+l} (-1)^{m-j} a_{j-m+l-i}\binom{n+m-l-j+i}{i}.$$

By (8) we have

$$(11) \qquad c_j \in \frac{1}{F^{j-m+l}H^l \mu_F(j-m+l)\mu_H(l)}\mathbb{Z} \quad \text{and} \quad c_j \in \frac{1}{L^l \mu_L(l)}\mathbb{Z}, \\ j = m-l, \ldots, n+m-l.$$

Next we investigate the polynomial

$$P_n(z) = z^{n+m-l-1}B_n\left(\frac{1}{z}\right) = z^{n+m-l-1}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{A_n\left(\frac{1}{z}\right) - A_n(t)}{\frac{1}{z} - t}\omega(t)\,dt.$$

By (10)

$$\int_0^1 \frac{A_n\left(\frac{1}{z}\right) - A_n(t)}{\frac{1}{z} - t}\omega(t)\,dt = -\sum_{j=m-l}^{n+m-l} c_j \sum_{i=0}^{j-1} \hat{z}^{j-1-i}\frac{\Gamma(b)\Gamma(i+c-b)}{\Gamma(i+c)},$$

where we have used the notation $\hat{z} = 1 - 1/z$. Therefore we immediately obtain

$$P_n(z) = -z^{n+m-l-1} \sum_{j=m-l}^{n+m-l} c_j \sum_{i=0}^{j-1} \hat{z}^{j-1-i} \frac{(C+1)\dots(C+i)}{c(c+1)\dots(c+i-1)}.$$

By the Gauss formula (see [12], p. 104, and [15])

$${}_2F_1\left(\begin{matrix}-i\ ,\ a\\ b\end{matrix}\middle|\, 1\right) = \frac{(b-a)_i}{(b)_i}$$

we get

$$\begin{aligned}\frac{h^i i!}{g(g+h)\dots(g+(i-1)h)} &= \frac{(g/h-(g/h-1))_i}{(g/h)_i}\\ &= 1 + \sum_{j=1}^{i}(-1)^j \binom{i}{j} \frac{g-h}{g+(j-1)h} \in \frac{1}{d_i(g,h)}\mathbb{Z},\end{aligned}$$

where $d_i(g,h) = \text{l.c.m.}\{g, g+h, \dots, g+(i-1)h\}$. Since h and $d_i(g,h)$ have no common prime factors this implies

$$\frac{(C+1)\dots(C+i)}{c(c+1)\dots(c+i-1)} \in \frac{h^i \mu_h(i)}{H^i \mu_H(i) d_i(g,h)}\mathbb{Z}.$$

Combining these facts we are led to the result

$$P_n\left(\frac{r}{s}\right) \in \frac{1}{s^{n+m-l} L^l H^{*\,n+m-l} \mu_L(l) \mu_{H^*}(n+m-l) d_{n+m-l}(g,h)}\mathbb{Z}, \tag{12}$$

where H^* denotes the denominator of h/H.

We now use (9) and (12) to obtain the following

Lemma 4. *If*

$$\Omega_n = s^{n+m-l} L^l H^{*\,n+m-l} \mu_L(l) \mu_{H^*}(n+m-l) d_{n+m-l}(g,h),$$

then

$$\Omega_n Q_n\left(\frac{r}{s}\right),\quad \Omega_n P_n\left(\frac{r}{s}\right) \in \mathbb{Z}.$$

Approximation sequences

The above considerations are performed to find good approximation sequences (q_n, p_n, r_n) for $F(r/s)$, i.e. to find integers q_n, p_n such that

$$q_n F\left(\frac{r}{s}\right) - p_n = r_n,$$

where r_n tends to zero as $n \to \infty$.

In considering the general case we note that

$$\mu_F^{n-p\frac{\ln n}{\ln p}} \le \mu_F(n) \le \mu_F^n$$

and, by [1], Lemma 1,

$$\lim_{n\to\infty} \frac{1}{n} \ln d_n(g,h) = \frac{h}{\phi(h)} \sum_{\substack{i=1\\(i,h)=1}}^{h} \frac{1}{i} = \lambda(h).$$

We now use Lemma 4 to obtain the integers

$$q_n = \Omega_n Q_n\left(\frac{r}{s}\right), \quad p_n = \Omega_n P_n\left(\frac{r}{s}\right).$$

By denoting

$$(13) \qquad \begin{cases} \omega(\alpha,\beta) = L^\alpha {H^*}^{1+\beta-\alpha}\mu_L \mu_{H^*}^{1+\beta-\alpha} e^{(1+\beta-\alpha)\lambda(h)}, \\ \nu(\alpha,\beta) = |r|^{1+\beta-\alpha} A\left(\alpha,\beta,\frac{s}{r}\right), \\ \mu(\alpha,\beta) = |r|^{1+\beta} s^{-\alpha} R\left(\alpha,\beta,\frac{r}{s}\right), \\ Q(\alpha,\beta) = \omega(\alpha,\beta)\nu(\alpha,\beta), \quad R(\alpha,\beta) = \omega(\alpha,\beta)\mu(\alpha,\beta), \end{cases}$$

we get, by Lemmas 1, 3 and 4, the following

Lemma 5. *Let $\varepsilon > 0$ be given.*

(i) If $|r/s| < 1$ and $R(\alpha,\beta) < \min\{1, Q(\alpha,\beta)\}$, then the above q_n, p_n and $r_n = q_n F(r/s) - p_n$ satisfy

$$\max\{|p_n|, |q_n|\} \le Q(\alpha,\beta)^{(1+\varepsilon)n},$$
$$R(\alpha,\beta)^{(1+\varepsilon)n} \le |r_n| \le R(\alpha,\beta)^{(1-\varepsilon)n}$$

for all $n \ge c_{13}$.

(ii) If p is a prime such that $p \nmid fh$ and $|r/s|_p < 1$, then

$$|r_n|_p \le c_1 |r|_p^{(1+\beta-\varepsilon)n}$$

for all $n \ge c_{14}$.

(iii) Suppose that $\alpha = \beta = 1$. If $r/s > 1$ then

$$\max\{|p_n|, |q_n|\} \le (\omega(1,1)|r|)^{(1+\varepsilon)n},$$

and if $r/s < -1$ and $\frac{r}{r-s}R(1,1,r/(r-s)) < A(1,1,s/r)$, then

$$\max\{|p_n|, |q_n|\} \le Q(1,1)^{(1+\varepsilon)n},$$

for all $n \ge c_{15}$.

Some determinants

In the archimedean case we have an asymptotic formula for the remainder term r_n in the lemmas above. On the other hand it seems difficult to obtain such a result in the p-adic case. Therefore we need the nonvanishing of the determinant

$$\Delta_n(z) = \begin{vmatrix} Q_n(z) & P_n(z) \\ Q_{n+1}(z) & P_{n+1}(z) \end{vmatrix}$$

in the p-adic considerations.

Lemma 6. *If $\alpha = \beta = 1$ or $l = m = n$, then we have*

$$\Delta_n(z) = (-1)^n \frac{(b)_n (c-b)_n}{(c)_{2n}} \binom{-n-c}{n+1} z^{2n}.$$

Proof. Clearly $\Delta_n(z)$ is a polynomial in z of $\deg \Delta_n(z) \le 2n$. Since $Q_n(z)F(z) - P_n(z) = R_n(z)$, we have

$$\Delta_n(z) = Q_{n+1}(z)R_n(z) - Q_n(z)R_{n+1}(z).$$

Thus $\operatorname{ord}_{z=0}\Delta_n(z) \ge 2n$ and our lemma follows from (2) and (4). □

Proof of Theorem 1 and 1p

In the archimedean case we may use following well-known result (see e.g. [8], Corollary 3.3). Let $x > 0$ and $y < 0$ be given. Suppose that for each $\varepsilon > 0$ there exists a constant c_{16} and rational integers p_n, q_n satisfying for all $n \ge c_{16}$ the inequalities

$$\frac{1}{n} \ln \max\{|q_n|, |p_n|\} < x + \varepsilon,$$
$$y - \varepsilon < \frac{1}{n} \ln |r_n| < y + \varepsilon,$$

where $r_n = q_n F(r/s) - p_n$. Then the number $F(r/s)$ has an irrationality measure $m(F(r/s))$ not greater than $1 - x/y$.

If $|z| < 1$, then we have

$$R(1,1,z) = (1+\sqrt{1-z})^{-2}, \quad A\left(1,1,\frac{1}{z}\right) = \frac{(1+\sqrt{1-z})^2}{|z|}.$$

Therefore, if $z = r/s$, then (13) implies

$$Q(1,1) = \omega(1,1)\left(\sqrt{s} + \sqrt{s-r}\right)^2,$$
$$R(1,1) = \omega(1,1)\left(\frac{r}{\sqrt{s}+\sqrt{s-r}}\right)^2 = \omega(1,1)\left(\sqrt{s} - \sqrt{s-r}\right)^2.$$

The assumption $LH^* \mu_L \mu_{H^*} e^{\lambda(h)} \left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1$ means that $R(1,1) < 1$. Thus the use of Lemma 5 gives us an upper bound

$$1 - \frac{\ln Q(1,1)}{\ln R(1,1)}$$

for the irrationality measure of $F(r/s)$. This proves our Theorem 1.

To give our p-adic results we prove the following simple lemma.

Lemma 7. *Let $\theta \in \mathbb{Q}_p$ be such that there exists a sequence (q_n, p_n) of integers satisfying for all $n \geq c_{17}$*

$$\max\{|q_n|, |p_n|\} \leq Q(p)^n, \quad p_n q_{n+1} - q_n p_{n+1} \neq 0, \quad |r_n|_p \leq c_1 R(p)^n,$$

where $r_n = q_n\theta - p_n$. If $Q(p)R(p) < 1$, then θ has an irrationality measure

$$m_p(\theta) \leq \frac{\ln R(p)}{\ln R(p) + \ln Q(p)}.$$

Proof. We shall find a lower bound for $|L|_p = |Q\theta - P|_p$, where (Q, P) is a non-trivial pair of integers with $H = \max\{|Q|, |P|\}$. Since $Q(p)R(p) < 1$, the inequality

$$\frac{1}{2c_1 H} \leq (Q(p)R(p))^n \tag{14}$$

has only a finite number of solutions $n \in \mathbb{N}$. Let $\overline{n}$ denote the greatest of these. We choose H large enough, say $H \geq H_0$, to satisfy $\overline{n} \geq c_{17}$. From the assumption $p_n q_{n+1} - q_{n+1} p_n \neq 0$ it follows that there exists a natural number N either $= \overline{n}+1$ or $= \overline{n}+2$ such that

$$\Delta = \begin{vmatrix} q_N & -p_N \\ Q & -P \end{vmatrix} = \begin{vmatrix} q_N & r_N \\ Q & L \end{vmatrix}$$

is a non-zero integer. Hence

$$1 \leq |\Delta||\Delta|_p \leq 2HQ(p)^N \left|q_N L - Q r_N\right|_p.$$

By our choice of N we have

$$2HQ(p)^N \left|Q r_N\right|_p \leq 2c_1 H (Q(p)R(p))^N < 1,$$

and therefore, by (14),

$$|L|_p \geq |q_N L|_p \geq \frac{1}{2HQ(p)^N} \geq c_{18} H^{-1+\ln Q(p)/\ln(Q(p)R(p))}.$$

This proves our lemma. □

By (ii) and (iii) of Lemma 5 we may use Lemma 7, where

$$Q(p) = (\omega(1,1)r)^{1+\varepsilon} = \left(LH^* \mu_L \mu_{H^*} e^{\lambda(h)} r\right)^{1+\varepsilon}, \quad R(p) = |r|_p^{2-\varepsilon}.$$

Since $\varepsilon > 0$ may be chosen arbitrarily small, our Theorem 1p follows immediately.

The assumption $r/s > 1$ is of course not necessary. To consider other cases we only have to use part (i) or the second part of (iii) of Lemma 5.

A common factor of the coefficients of P_n and Q_n

It turns out that in many cases the coefficients of the polynomials P_n and Q_n have a big common factor which must be eliminated to get sharp irrationality measures. This kind of idea appears already in Siegel's [21] paper, and it was used in an ingenious way by Chudnovsky [8] to consider certain binomial series, see also [11]. Later this idea combined with Padé-type approximations is used e.g. in [13], [14] and [20]. We shall now introduce a general criterion (see Lemma 10 below) to find a common factor of the coefficients of P_n and Q_n, and then this criterion will be applied to the consideration of the logarithms. Using (2) and the definition of $Q_n(z)$ we see that each common factor of

$$\binom{n+B}{i}\binom{m+C}{l-i}, \qquad i=0,1,\ldots,l, \tag{15}$$

is also a common factor for all the coefficients of Q_n and P_n. Therefore we shall find out which primes $p > c_{19}\sqrt{n}$ divide the numbers (15). It was Chudnovsky's [8] observation that only these big primes are really important here. To find a criterion for primes dividing the numbers (15) we first give two lemmas.

To state our lemmas we use for a rational number r the notations $p\,|\,r$ or $r \equiv 0 \pmod p$, if $v_p(r) \geq 1$. Further, if $v_p(r) \geq 0$, then there exists a unique $\overline{r} \in \{0,1,\ldots,p-1\}$ satisfying $\overline{r} \equiv r \pmod p$.

Lemma 8. *Let $r = R/S \in \mathbb{Q}$, $(R,S)=1$, $S>0$, $i \in \mathbb{N}$, and let p be a prime satisfying $p \not| S$, $p^2 > \max\{i, \max_{1\leq j\leq i}\{|R+(j-1)S|\}\}$. Let $i = Ap + \overline{i}$. Then*

$$v_p((r)_i) = A+1 \quad \textit{if and only if} \quad \overline{-r} < \overline{i}.$$

Further

$$p \,\Big|\, \binom{r}{i}$$

if and only if $\overline{r} < \overline{i}$.

Proof. First we suppose that $0 \leq \overline{i} \leq \overline{-r}$. Then

$$\begin{aligned} v_p((r)_i)) &= v_p(R(R+S)\ldots(R+(i-1)S)) \\ &= v_p(R(R+S)\ldots(R+(Ap-1)S)) \\ &+ v_p\left((R+ApS)\ldots(R+(Ap+\overline{i}-1)S)\right) = A+0 = A \end{aligned}$$

because $R + \overline{(-r)}S \equiv 0 \pmod p$. On the other hand we have

$$v_p\left((R+ApS)\ldots(R+(Ap+\overline{i}-1)S)\right) = 1,$$

if $\overline{-r} < \overline{i}$. Thus we have $v_p((r)_i)) = A+1$ in this case. This proves the first part of our lemma.

To prove the second part we note that

$$\binom{r}{i} = (-1)^i \frac{(-r)_i}{i!}.$$

Then $v_p((-r)_i)) = A+1$ if and only if $\overline{r} < \overline{i}$ by the above consideration. Moreover $v_p(i!) = A$, which completes the proof. □

Lemma 9. *Let $r_1 = R_1/S_1$, $r_2 = R_2/S_2$ denote rationals satisfying $(R_1, S_1) = (R_2, S_2) = 1$, $S_1 > 0$ and $S_2 > 0$, and let p be a prime satisfying $v_p(r_1) \geq 0$, $v_p(r_2) \geq 0$ and*

$$p^2 > \max\left\{l, \max_{1\leq j\leq l}\{|R_1 + (j-1)S_1|, |R_2 + (j-1)S_2|\}\right\}.$$

If

$$\overline{r}_1 + \overline{r}_2 + 1 \leq \overline{l}, \tag{16}$$

then

$$p \left| \binom{r_1}{i}\binom{r_2}{l-i}\right., \quad i = 0, 1, \ldots, l.$$

Proof. Let us suppose that $\overline{l} \leq \overline{i}$. Then we have, by our assumption (16),

$$\overline{r_1} + 1 \leq \overline{r}_1 + \overline{r}_2 + 1 \leq \overline{l} \leq \overline{i}.$$

By Lemma 8 it follows that $p \left| \binom{r_1}{i} \right.$.

We now consider the case $\overline{l} \geq \overline{i} + 1$. If $\overline{r}_1 + 1 \leq \overline{i}$, then $p \left| \binom{r_1}{i} \right.$, again by Lemma 8. If $\overline{r}_1 \geq \overline{i}$, then (16) implies

$$\overline{r}_2 + 1 \leq \overline{l} - \overline{r}_1 \leq \overline{l} - \overline{i} = \overline{l - i}.$$

We use once again Lemma 8 and obtain $p \left| \binom{r_2}{l-i} \right.$. This proves Lemma 9. □

Lemma 9 gives immediately the following.

Lemma 10 (Divisibility criterion for the coefficients of Q_n). *Let $P(n, \alpha, \beta)$ denote the set of all primes satisfying $p \nmid FH$,*

$$p^2 > \max\left\{l, \max_{1\leq j\leq l}\{|E + (n-j)F|, |G + (n-j)H|\}\right\},$$

$$\overline{n+B} + \overline{m+C} + 1 \leq \overline{l}.$$

Then

$$\left(\prod_{p\in P(n,\alpha,\beta)} p\right) \left| \binom{n+B}{i}\binom{m+C}{l-i}\right., \quad i = 0, 1, \ldots, l.$$

We now apply this criterion to the logarithmic function. In this case $B = C = 0$, and we further choose $\beta = 1$, i.e. $m = n$. To use Lemma 10 we have to characterize the primes $p \geq c_{20}\sqrt{n}$ satisfying $2\overline{n} + 1 \leq \overline{l}$. By denoting $\overline{n} = n - Np, \overline{l} = l - Lp$ this condition becomes

$$0 \leq 2(n - Np) \leq l - Lp - 1 \leq p - 2 \tag{17}$$

or

$$\max\left\{\frac{2n - l + 1}{2N - L}, \frac{l+1}{L+1}\right\} \leq p \leq \frac{n}{N}. \tag{18}$$

Conversely, if p is in this interval for some L and $N(\geq L)$, then p satisfies (17).

We now consider carefully the inequalities (18) assuming $n \geq c_{21}$. If

$$\frac{2n-l+1}{2N-L} \leq \frac{l+1}{L+1} < \frac{n}{N}$$

or

(i) $$\alpha N - 1 < L \leq \alpha N - 1 + \frac{\alpha}{2},$$

then all primes in the interval

$$\left(\frac{l+1}{L+1}, \frac{n}{N}\right)$$

satisfy (17). Further, if

$$\frac{l+1}{L+1} < \frac{2n-l+1}{2N-L} < \frac{n}{N}$$

or

(ii) $$\alpha N - 1 + \frac{\alpha}{2} < L < \alpha N,$$

then all the primes in

$$\left(\frac{2n-l+1}{2N-L}, \frac{n}{N}\right)$$

also satisfy (17).

We assume that $\alpha = u/v$, $u, v \in \mathbb{N}$, $(u,v) = 1$, and set $N = vK + i$, where $i \in \{0, 1, \ldots, v-1\}$. Then (i) is of the form

$$uK + \alpha i - 1 < L \leq uK + \alpha i + \frac{\alpha}{2} - 1.$$

Therefore, if

(19) $$[\alpha i + \alpha/2] = [\alpha i] + 1,$$

then $L = uK + [\alpha i]$ satisfies (i) and all the primes p in the interval

$$\left(\frac{[\alpha n]+1}{uK+[\alpha i]+1}, \frac{n}{vK+i}\right)$$

satisfy our condition (17).

By the prime number theorem it follows (see [8]) that the product of all the primes in above intervals is asymptotically equal to $e^{n\Sigma_1}$, where Σ_1 is equal to

$$\sum_{i\in(19)} \sum_{K=0}^{\infty} \left(\frac{1}{vK+i} - \frac{u/v}{uK+[\alpha i]+1}\right) = \frac{1}{v} \sum_{i\in(19)} \sum_{K=0}^{\infty} \left(\frac{1}{K+\frac{i}{v}} - \frac{1}{K+\frac{1+[\alpha i]}{u}}\right),$$

and here $i \in (19)$ means that i satisfies (19). By the well-known properties of the digamma function Ψ (see [12], 1.7, (3)) we obtain

$$\Sigma_1 = \frac{1}{v} \sum_{i \in (19)} \left(\Psi\left(\frac{1+[\alpha i]}{u}\right) - \Psi\left(\frac{i}{v}\right)\right).$$

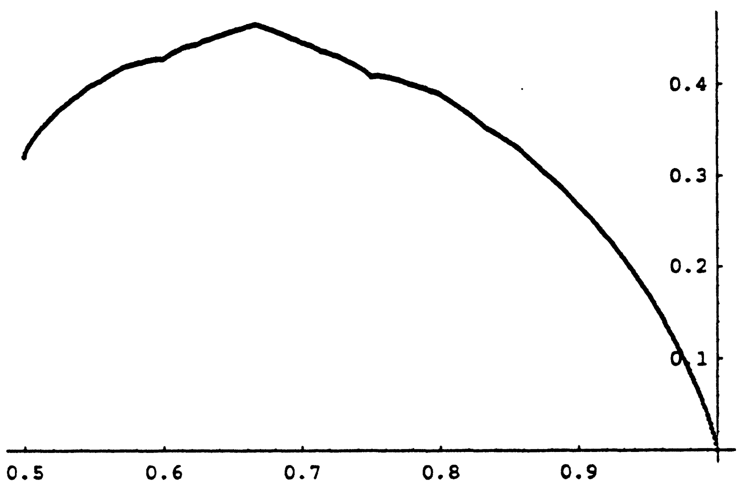
In the same way, if

$$\alpha i \neq [\alpha i] = [\alpha i + \alpha/2], \tag{20}$$

then $L = uK + [\alpha i]$ satisfies (ii) and this case gives an asymptotic $e^{n\Sigma_2}$, where

$$\Sigma_2 = \frac{1}{v} \sum_{i \in (20)} \left(\Psi\left(\frac{2i-[\alpha i]}{2v-u}\right) - \Psi\left(\frac{i}{v}\right)\right).$$

Combining the above considerations we obtain an asymptotic $e^{n\tau_1(\alpha)}$, where $\tau_1(\alpha) = \Sigma_1 + \Sigma_2$. The values of $\tau_1(\alpha)$ are given in the following graph (the interval of the subsequent arguments in the graph is of length 1/1000):



Picture 2.

Proof of Theorem 2

Let us assume that $B = C = 0$, $\beta = 1$. From the above considerations it follows that for a given rational $\alpha \in (0, 1]$ there exists a common factor D_n of the coefficients of P_n and Q_n asymptotically equal to $e^{n\tau_1(\alpha)}$. Thus the use of (13) and Lemma 5 immediately gives us the following result concerning the integers

$$q_n = \frac{\Omega_n Q_n(r/s)}{D_n}, \quad p_n = \frac{\Omega_n P_n(r/s)}{D_n},$$

and the remainder term

$$r_n = q_n\, {}_2F_1\left(\begin{matrix} 1, 1 \\ 2 \end{matrix} \middle| \frac{r}{s} \right) - p_n.$$

Lemma 11. *Let $\varepsilon > 0$ be given, and let*

$$\omega_1 = \omega_1(\alpha) = e^{2-\alpha-\tau_1(\alpha)}, \quad Q(\alpha) = \omega_1\nu(\alpha,1), \quad R(\alpha) = \omega_1\mu(\alpha,1).$$

If $|r/s| < 1$ and $R(\alpha) < 1$, then we have

$$\max\{|p_n|, |q_n|\} \le Q(\alpha)^{(1+\varepsilon)n},$$
$$R(\alpha)^{(1+\varepsilon)n} \le |r_n| \le R(\alpha)^{(1-\varepsilon)n}$$

for all $n \ge c_{22}$.

By using this lemma we now get the truth of Theorem 2 analogously to the proof Theorem 1.

We note that Lemma 10 may be used to obtain improvements of Theorem 1 in some other special cases, too. These will be considered in another work.

Remark 3. All the numerical computations including Picture 2 are made using MATHEMATICA programs.

References

1. Alladi K. & Robinson M.L., *Legendre polynomials and irrationality*, J. Reine Angew. Math. **318**, 137–155 (1980)
2. Baker A., *Rational approximation to $\sqrt[3]{2}$ and other algebraic numbers*, Q. J. Math. Oxf. II Ser. **15**, 375–383 (1964)
3. Beukers F., *Algebraic values of G-functions*, J. Reine Angew. Math. **434**, 45–65 (1993)
4. Beukers F. & Matala-aho T. & Väänänen K., *Remarks on the arithmetic properties of the values of hypergeometric functions*, Acta Arith. **XLII**, 281–289 (1983)
5. Beukers F. & Wolfart J., *Algebraic values of hypergeometric functions*, in: New advances in transcendence theory, Cambridge 1988
6. Bundschuh P., *Zur Approximation gewisser p-adischer algebraischer Zahlen durch rationalen Zahlen*, J. Reine Angew. Math. **265**, 154–159 (1974)
7. Chudnovsky G.V., *Number theoretic applications of polynomials with rational coefficients defined by extremality conditions*, in: Arithmetic and geometry, Birkhäuser, Boston 1983, 61–105
8. Chudnovsky G.V., *On the method of Thue-Siegel*, Ann. of Math., II. Ser. **117**, 325–382 (1983)
9. Danilov L. V., *Rational approximations of some functions at rational points*, (Russian), Mat. Zametki **24, no. 4**, 449–458 (1978)
10. Dubitskas A. K., *Approximation on $\pi/\sqrt{3}$ by rational fractions*, (Russian), Vestn. Mosk. Univ., Ser. I **no. 6**, 73–76 (1987)
11. Easton D., *Effective Irrationality Measures for Certain Algebraic Numbers*, Math. Comput. **46, no 174**, 613–622 (1986)
12. Erdélyi A. (ed.), *Higher transcendental functions*, vol. 1, McGraw-Hill Book Company, Inc., New York, 1953

13. Hata M., *Legendre type polynomials and irrationality measures*, J. Reine Angew. Math. **407**, 99–125 (1990)

14. Hata M., *Irrationality measures of the values of hypergeometric functions*, Acta Arith. **LX**, 335–347 (1992)

15. Huttner M., *Problème de Riemann et irrationalité d'un quotient de deux fonctions hypergéométriques de Gauss*, C.R. Acad. Sci., Paris Sér. I **302**, 603–606 (1986)

16. Huttner M., *Irrationalité de certaines integrales hypergéométriques*, J. Number Theory **26**, 166–178 (1987)

17. Kuukasjärvi M. & Matala-aho T. & Väänänen K., *On irrationality measures of certain P-adic numbers*, Acta Univ. Oulu., Ser A., Sci. Rerum Nat. **207**, 1–16 (1989)

18. Nikisin E.M., *Arithmetic properties of the Markov function for the Jacobi weight*, Anal. Math. **8**, 39–46 (1982)

19. Rhin G., *Approximants de Padé et mesures effectives d'irrationalité*, Prog. Math. **71**, 155–164 (1987)

20. Rukhadze E. A., *Lower estimate for rational approximations of* ln 2, (Russian), Vestn. Mosk. Univ., Ser. I **no. 6**, 25–29 (1987)

21. Siegel C. L., *Über einige Anwendungen diophantischer Approximationen*, Abh. d. Preuss. Akad. d. Wiss., Phys.-mat. **Kl. 1 No. 1** (1929)

22. Szegö G., *Orthogonal Polynomials*, American Mathematical Society, New York, 1939

23. Vasilenko O.N., *On the irrationality of the values of the Gaussian hypergeometric function*, (Russian), Vestn. Mosk. Univ., Ser. I **no 3**, 15–18 (1985)

24. Wolfart J., *Werte hypergeometrischer Funktionen*, Invent. Math. **92**, 187–216 (1988)

Department of Mathematics
University of Oulu
90570 Oulu
Finland

(Received May 24, 1993)

manuscripta math. 81, 203 - 222 (1993)

manuscripta mathematica

WILLMORE SUBMANIFOLDS OF THE MÖBIUS SPACE AND A BERNSTEIN-TYPE THEOREM

Marco Rigoli and Isabel M. C. Salavessa

We study Willmore immersed submanifolds $f : M^m \to S^n$ into the n-Möbius space, with $m \geq 2$, as critical points of a conformally invariant functional $\mathcal{W}$. We compute the Euler-Lagrange equation and relate this functional with another one applied to the conformal Gauss map of immersions into S^n. We solve a Bernestein-type problem for compact Willmore hypersurfaces of S^n, namely, if $\exists a \in I\!R^{n+2}$ such that $< \gamma_f, a > \neq 0$ on M, where γ_f is the hyperbolic conformal Gauss map and $<\ ,\ >$ is the Lorentz inner product of $I\!R^{n+2}$, and if f satisfies an additional condition, then $f(M)$ is an $(n-1)$-sphere.

1 Introduction

Conformal Geometry is concerned with the properties of figures and objects of S^n, invariant under the action of the Möbius group, that is, invariant under an arbitrary conformal transformation of the sphere S^n, equipped with its usual Riemannian structure of constant positive sectional curvature. The geometry of the Möbius space S^n and of the induced conformal structure of an immersed submanifold is described by, for example, Schiemangk and Sulanke [6], Bryant [2] and Rigoli [4], which authors use Cartan's method of moving frames by considering S^n as a homogeneous space G/H where G is the Möbius group. Conformal invariants of the Riemannian geometry of submanifolds of $I\!R^n$ can be interpreted as invariants of conformal geometry, thinking of S^n as $I\!R^n$ with a point at infinity through stereographic projection. For example, the Willmore integrand for immersed surfaces $F : M^2 \to I\!R^3$ into the 3-dimensional Euclidean space, which is invariant under conformal transformations of $I\!R^3$, plus the "point at infinity", can be interpreted as the Riemannian version of a conformally invariant 2-form Ω_F on M, endowed with the induced conformal structure by the Möbius space S^3. In this way, Bryant [2] studied the Willmore functional and the associated variational problem, deriving its Euler–Lagrange equation. The critical points are called Willmore immersed surfaces. This procedure allowed Rigoli [4] to generalize in a natural manner the concept of Willmore immersed submanifolds $f : M^m \to S^n$ of the Möbius space S^n as critical points of the variational problem associated with a functional $\mathcal{W}(f)$. However, in that paper the Euler–Lagrange equation is only derived for the case $m = 2$ and n arbitrary. In section 3, we will solve the Euler-Lagrange equation for any dimension $m \leq n$. This equation is, in local frames, a fourth-order partial differential equation, which is quite simple for $m = 2$, less simple for $m = 4$, and rather complicated for $m \geq 6$. If $m = 3$ or 5, we can only derive the equation outside the umbilic points. This variational problem is related to the one of a different conformally invariant functional, applied to the conformal Gauss map $\gamma_f : M^m \to Q_{n-m}(I\!R^{n+2})$ for an immersion $f : M^m \to S^n$. This relation was first pointed out in [2], in the $m = 2$,

$n = 3$ case, and in [4], for $m = 2$, $n \le 3$. Also, we will solve a Bernstein-type problem for Willmore hypersurfaces of S^n in section 5, which generalizes the one solved in [4] for surfaces of S^3. The present work is largely based on part of the Ph.D. thesis of Salavessa [5].

2 The Geometry of Submanifolds of S^n

Let $\mathcal{L}^+$ denote the connected component of the light cone of $\mathbb{R}^{n+2}$ with the Lorentz inner product $<\ ,\ >$ of signature $-+\ldots+$,

$$\mathcal{L}^+ = \left\{x = (x^0, x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+2} : -(x^0)^2 + (x^1)^2 + \ldots + (x^{n+1})^2 = 0,\ x^0 > 0\right\}.$$

Henceforth, we adopt on the index ranges $0 \le a, b, \ldots \le n+1$, $1 \le A, B, \ldots \le n$, $1 \le i, j, \ldots \le m$, $m+1 \le \alpha, \beta, \ldots \le n$, and we use the index-summation convention on repeated indices.

Let us fix a right-handed basis $\{\eta_0, \eta_A, \eta_{n+1}\}$ of $\mathbb{R}^{n+2}$, with $\eta_0, \eta_{n+2} \in \mathcal{L}^+$ such that $<\ ,\ >$ is represented in this basis by the matrix

$$S = \begin{bmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then, the Lorentz group of dimension $\frac{(n+1)(n+2)}{2}$, $O(1, n+1)$, can be identified (though not canonically) with the group $\{P \in M_{(n+2)^2} : {}^tPSP = S\}$, and the Möbius group with its identity component

$$G = G(n) = \{P \in O(1, n+1) : \det P = 1,\ P(\mathcal{L}^+) \subset \mathcal{L}^+\}\ .$$

The submersion

$$\begin{array}{rccc} T: & \mathcal{L}^+ & \longrightarrow & S^n \subset \mathbb{R}^{n+1} \\ & \begin{bmatrix} c \\ v \\ s \end{bmatrix} & \longrightarrow & \left(\frac{s-c}{s+c}, \frac{\sqrt{2}v}{s+c}\right) \end{array}$$

identifies $\mathcal{L}^+/_\sim$, where $\sim$ is the relation of equivalence on $\mathbb{R}^{n+2}\backslash\{0\}$ given by: $x \sim y$ iff $\exists \lambda \neq 0 : x = \lambda y$, with the unit sphere S^n of $\mathbb{R}^{n+1}$. By embedding S^n into $\mathbb{R}P^{n+1}$ in this way, we call it the *Möbius space* , and G acts transitively on S^n by

$$\begin{array}{ccc} G \times S^n & \longrightarrow & S^n \\ (P, [x]_\sim) & \longrightarrow & [P(x)]_\sim \end{array},$$

which represents the action of the group of orientation-preserving conformal transformations of the n-sphere. Let $\mathbf{x_0} = [\eta_0]_\sim$ and $\mathbf{x_\infty} = [\eta_\infty]_\sim$ be the *origin* and *Möbius point,* respectively, of S^n. The isotropic subgroup of G at $\mathbf{x_0}$ is

$$G_0 = \left\{ \begin{bmatrix} r^{-1} & {}^tXA & \frac{1}{2}r\,{}^tXX \\ 0 & A & rX \\ 0 & 0 & r \end{bmatrix} : \begin{array}{l} A \in SO(n) \\ X \in \mathbb{R}^n \\ r \in \mathbb{R}^+ \end{array} \right\}. \tag{2.1}$$

Then, S^n is identified with the homogeneous space G/G_0 and the canonic projection of G onto the quotient space G/G_0 is given by

$$\begin{array}{rcl} \Pi : G & \longrightarrow & G/G_0 \equiv S^n \\ P & \longrightarrow & [P(\eta_0)]_\sim \end{array}.$$

Let Φ denote the Maurer–Cartan form of G. Denoting by $\mathcal{O}(n)$ the Lie algebra of $O(n)$, Φ takes values on the Lie algebra $\mathcal{G}$ of G,

$$\mathcal{G} = \left\{ \begin{bmatrix} a & {}^t\xi & 0 \\ v & D & \xi \\ 0 & {}^tv & -a \end{bmatrix} : \begin{array}{c} a \in \mathbb{R} \\ v, \xi \in \mathbb{R}^n \\ D \in \mathcal{O}(n) \end{array} \right\} .$$

The matrix $\left[\Phi^a_b\right]$ of left-invariant 1-forms constituted by the components of Φ satisfies

$$\Phi^0_0 = -\Phi^{n+1}_{n+1}, \quad \Phi^A_0 = \Phi^{n+1}_A, \quad \Phi^0_A = \Phi^A_{n+1}, \quad \Phi^A_B = -\Phi^B_A, \quad \Phi^{n+1}_0 = \Phi^0_{n+1} = 0 . \tag{2.2}$$

The Maurer-Cartan equation $d\Phi = -\Phi \wedge \Phi$ gives the structure equations of G

$$\begin{cases} d\Phi^0_0 &= -\Phi^0_A \wedge \Phi^A_0 \\ d\Phi^A_0 &= -\Phi^A_0 \wedge \Phi^0_0 - \Phi^A_B \wedge \Phi^B_0 \\ d\Phi^0_A &= -\Phi^0_0 \wedge \Phi^0_A - \Phi^0_B \wedge \Phi^B_A \\ d\Phi^A_B &= -\Phi^A_0 \wedge \Phi^0_B - \Phi^A_C \wedge \Phi^C_B - \Phi^0_A \wedge \Phi^B_0 \, . \end{cases} \tag{2.3}$$

If $s : S^n \to G$, defined on an open set of S^n, is a local section of $\Pi : G \to S^n$, denoting by $\phi = s^*\Phi$ the associated $\mathcal{G}$-valued 1-form on S^n, one easily sees that $(\phi^A_0)_{1\leq A\leq n}$ determine a conformal structure and an orientation on S^n. More generally, let $f : M^m \to S^n$ be a smooth immersion of an oriented m-manifold M with $m \geq 2$. Then, a conformal structure is assigned to M, induced by f from the conformal structure of S^n as described in [4] and which we summarize here for simplicity. A *zeroth-order frame along* f is a map $e : M \to G$ defined on an open set of M such that $\Pi \circ e = f$. To each of such frames, we associate the $\mathcal{G}$-valued 1-form on M

$$\phi = e^*\Phi = e^{-1}de$$

with components $\phi^a_b = e^*\Phi^a_b$. If $\tilde{e} : M \to G$ is another zeroth-order frame along f, then $\tilde{e} = eK$, with $K : M \to G_0$ smooth. Then

$$\tilde{\phi} = \tilde{e}^*\Phi = K^{-1}\phi K + K^{-1}dK \, . \tag{2.4}$$

From this equation we get the transformation of $\tilde{\phi}^a_b$. In particular (see (2.1) for notations), $\tilde{\phi}^B_0 = r^{-1}A^C_B\phi^C_0 \ \forall B$. A *first order frame* $e : M \to G$ is a frame satisfying

$$\phi^\alpha_0 = 0 \qquad \forall\alpha \, ,$$

and $\phi^1_0, \ldots, \phi^m_0$ is a direct basis of T^*M at each point of the domain of e.

If e and $\tilde{e}$ are first-order frames, then $\tilde{e} = eK$, with $K : M \to G_1$, where $G_1 \subset G_0$ is the isotropic subgroup of first-order frames, i.e.,

$$G_1 = \left\{ \begin{bmatrix} r^{-1} & {}^tXA & {}^tYB & \frac{1}{2}r({}^tXX + {}^tYY) \\ 0 & A & 0 & rX \\ 0 & 0 & B & rY \\ 0 & 0 & 0 & r \end{bmatrix} : \begin{array}{l} A \in SO(m) \\ B \in SO(n-m) \\ X \in \mathbb{R}^m, \ Y \in \mathbb{R}^{n-m} \\ r \in \mathbb{R}^+ \end{array} \right\} . \tag{2.5}$$

For such frames one has $\tilde{\phi}^i_0 = r^{-1}A^j_i\phi^j_0$. Thus, $\sum_i \phi^i_0 \otimes \phi^i_0$ defines the same conformal structure on M, for any first-order frame e. From the structure equations (2.3) we have $0 = d\phi^\alpha_0 = -\phi^\alpha_i \wedge \phi^i_0$. Then, by Cartan's lemma,

$$\phi^\alpha_i = h^\alpha_{ij}\phi^j_0 \qquad \text{with } \ h^\alpha_{ij} = h^\alpha_{ji} \, . \tag{2.6}$$

A *Darboux G-frame* $e : M \to G$ along f is a first-order frame such that

$$h^\alpha_{ii} = 0 \qquad \forall\alpha\ . \tag{2.7}$$

We can construct a Darboux frame around each point of M (cf. [5] and compare with section 6). If e and $\tilde{e}$ are Darboux frames, then $\tilde{e} = eK$ with $K : M \to G_D$, where $G_D \subset G_1$ is the isotropic subgroup of Darboux frames, i.e.

$$G_D = \left\{ \begin{bmatrix} r^{-1} & {}^tXA & 0 & \frac{1}{2}r{}^tXX \\ 0 & A & 0 & rX \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & r \end{bmatrix} : \begin{array}{l} A \in SO(m) \\ B \in SO(n-m) \\ X \in I\!R^m \\ r \in I\!R^+ \end{array} \right\} . \tag{2.8}$$

Differentiating (2.6) and using the structure equations and Cartan's lemma, we get, for any Darboux frame,

$$dh^\alpha_{ij} = h^\alpha_{ik}\phi^k_j + h^\alpha_{kj}\phi^k_i - h^\beta_{ij}\phi^\alpha_\beta - h^\alpha_{ij}\phi^0_0 - \delta_{ij}\phi^0_\alpha + h^\alpha_{ijk}\phi^k_0\ , \tag{2.9}$$

where $h^\alpha_{ijk} = h^\alpha_{jik} = h^\alpha_{ikj}$. Defining

$$p^\alpha_k = \frac{1}{m}h^\alpha_{iik}\ , \tag{2.10}$$

we have

$$\phi^0_\alpha = p^\alpha_k\phi^k_0\ . \tag{2.11}$$

By differentiation of the latter equation, we obtain

$$dp^\alpha_j = p^\alpha_k\phi^k_j + h^\alpha_{kj}\phi^0_k - p^\beta_j\phi^\alpha_\beta - 2p^\alpha_j\phi^0_0 + p^\alpha_{jk}\phi^k_0\ , \tag{2.12}$$

where $p^\alpha_{jk} = p^\alpha_{kj}$. From equation (2.9), and using the symmetry properties of the indices,

$$h^\alpha_{ij}dh^\alpha_{ij} = -\sum_{ij\alpha}(h^\alpha_{ij})^2\phi^0_0 + h^\alpha_{ij}h^\alpha_{ijk}\phi^k_0\ .$$

Using the vanishing of $d(h^\alpha_{ij}dh^\alpha_{ij})$, the structure equations, and Cartan's lemma, we obtain

$$d(h^\alpha_{ij}h^\alpha_{ijk}) = -3h^\alpha_{ij}h^\alpha_{ijk}\phi^0_0 - \sum_{ij\alpha}(h^\alpha_{ij})^2\phi^0_k + h^\alpha_{ij}h^\alpha_{ijp}\phi^p_k + H_{kr}\phi^r_0\ , \tag{2.13}$$

where H_{kr} are smooth functions with the symmetry property $H_{kr} = H_{rk}$.

A symmetric tensor $\mathcal{N} \in C^\infty(\bigodot^2 T^*M)$ is defined, locally in a Darboux frame, by $\mathcal{N} = \mathcal{N}_{jk}\phi^j_0 \otimes \phi^k_0$, where (cf. [4])

$$\mathcal{N}_{jk} = h^\alpha_{ik}h^\alpha_{ji}\ .$$

A point $p \in M$ is said to be *umbilic* if Trace$\mathcal{N}(p) = \mathcal{N}_{jj}(p) = h^\alpha_{ij}(p)h^\alpha_{ij}(p) = 0$, or, equivalently, if $\mathcal{N}(p) = 0$. The immersion f is said to be *Möbius-flat* (or *totally umbilical*) if all points of M are umbilic. By a result of [6] (see also [4]), if M is connected and $m \geq 2$, then $\mathcal{N} \equiv 0$ iff there exists a $S^m \subset S^n$ such that $f(M) \subset S^m$. Moreover, in this case, if M is compact, then f is a diffeomorphism of M onto S^m.

If e and $\tilde{e}$ are two Darboux G-frames along f, say $\tilde{e} = eK$ with $K : M \to G_D$, then ϕ, h^α_{ij}, h^α_{ijk}, p^α_k, p^α_{ij}, and H_{ij} transform by the following laws

$$\begin{aligned}
\tilde{\phi}^0_0 &= \phi^0_0 - X_i\phi^i_0 - d\log r\,; \qquad \tilde{\phi}^i_0 = r^{-1}A^j_i\phi^j_0\\
\tilde{\phi}^0_i &= rA^j_i(dX_j + X_j\phi^0_0 - X_jX_k\phi^k_0 + \phi^0_j - X_k\phi^k_j + \tfrac{1}{2}X_kX_k\phi^j_0)\\
\tilde{\phi}^0_\alpha &= rB^\beta_\alpha(\phi^0_\beta - X_i\phi^i_\beta)\,; \qquad \tilde{\phi}^i_j = A^k_i(X_j\phi^k_0 - X_k\phi^j_0 + A^t_j\phi^k_t + dA^k_j)\\
\tilde{\phi}^i_\alpha &= A^j_iB^\beta_\alpha\phi^j_\beta\,; \qquad \tilde{\phi}^\alpha_\beta = B^\gamma_\alpha(B^\rho_\beta\phi^\gamma_\rho + dB^\gamma_\beta)\\
\tilde{h}^\alpha_{ij} &= rB^\beta_\alpha A^k_iA^l_jh^\beta_{kl}\,; \qquad \tilde{p}^\alpha_i = r^2B^\beta_\alpha A^k_i(p^\beta_k + h^\beta_{kj}X_j)\\
\tilde{h}^\alpha_{ijk} &= r^2B^\beta_\alpha\big(A^u_iA^v_jA^w_kh^\beta_{uvw} - A^u_iA^v_jA^w_kX_vh^\beta_{uw} - A^u_iA^v_jA^w_kX_uh^\beta_{vw}\\
&\qquad - A^u_iA^v_jA^w_kX_wh^\beta_{uv} + \delta_{ij}A^v_kX_uh^\beta_{uv} + \delta_{ik}A^v_jX_uh^\beta_{uv} + \delta_{jk}A^v_iX_uh^\beta_{uv}\big)\\
\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ijk} &= r^3A^w_k(h^\beta_{uv}h^\beta_{uvw} - X_wh^\beta_{uv}h^\beta_{uv})\\
\tilde{p}^\alpha_{ij} &= r^3B^\beta_\alpha\big(A^k_iA^t_jp^\beta_{kt} + A^k_iA^t_jX_ph^\beta_{pkt} - A^k_iA^t_jX_tX_ph^\beta_{pk} - A^k_iA^t_jX_kX_ph^\beta_{pt}\\
&\qquad -\tfrac{1}{2}A^k_iA^t_jX_pX_ph^\beta_{kt} - 2A^k_iA^t_jX_tp^\beta_k - 2A^k_iA^t_jX_kp^\beta_t + \delta_{ij}X_pX_th^\beta_{pt} + \delta_{ij}X_pp^\beta_p\big)\\
\tilde{p}^\alpha_{ii} &= r^3B^\beta_\alpha\big(p^\beta_{tt} + (m-2)(2X_pp^\beta_p + X_pX_th^\beta_{pt})\big)\\
\tilde{H}_{kt} &= r^4\big\{A^r_kA^p_tH_{rp} - 3A^r_kA^p_th^\beta_{uv}h^\beta_{uvp}X_r - 3A^r_kA^p_th^\beta_{uv}h^\beta_{uvr}X_p\\
&\qquad + \delta_{tk}h^\beta_{uv}h^\beta_{uvp}X_p + \big(\sum_{i,j,\alpha}(h^\alpha_{ij})^2\big)(3A^u_kA^p_tX_uX_p - \tfrac{1}{2}\delta_{kt}X_pX_p)\big\}\\
\tilde{h}^\alpha_{kt}\tilde{H}_{kt} &= r^5B^\gamma_\alpha\big(h^\gamma_{kt}H_{kt} - 6h^\gamma_{kt}h^\beta_{uv}h^\beta_{uvk}X_t + 3\big(\sum_{i,j,\beta}(h^\beta_{ij})^2\big)h^\gamma_{kt}X_kX_t\big)\,.
\end{aligned}$$

Denoting by $\phi^{1\ldots\hat{i}\ldots m}$ the $(m-1)$-form $\phi^1_0 \wedge \ldots \wedge \phi^{i-1}_0 \wedge \phi^{i+1}_0 \wedge \ldots \wedge \phi^m_0$, then

$$\tilde{\phi}^{1\ldots\hat{k}\ldots m} = (-1)^{i+k}r^{1-m}A^i_k\phi^{1\ldots\hat{i}\ldots m} \qquad (k \text{ is fixed}).$$

Finally,

$$d\tilde{V} = \tilde{\phi}^1_0 \wedge \ldots \wedge \tilde{\phi}^m_0 = r^{-m}\phi^1_0 \wedge \ldots \wedge \phi^m_0 = r^{-m}dV\,,$$

and explicitly we have

$$\tilde{e} = [\tilde{e}_0, \tilde{e}_i, \tilde{e}_\alpha, \tilde{e}_{n+1}] = [r^{-1}e_0\,, A^j_i(X_je_0 + e_j)\,, B^\beta_\alpha e_\beta\,, r(\tfrac{1}{2}{}^tXXe_0 + X_je_j + e_{n+1})]\,.$$

3 Willmore Submanifolds

If $f : M^2 \to \mathbb{R}^3$ is an immersed closed surface into the Euclidean 3-space, f is said to be a *Willmore surface* if it is a critical point of the Willmore functional

$$\mathcal{W}(f) = \int_M (H^2 - K)dA\,, \tag{3.1}$$

where H is the scalar mean curvature and K the Gaussian curvature. The interest of this functional was aroused by Willmore [8] when he posed the problem of finding $\inf_f \mathcal{W}(f)$. The Euler-Lagrange equation for the associated variational problem is given by

$$\triangle H + 2H(H^2 - K) = 0\,, \tag{3.2}$$

which is invariant under conformal transformations of $\mathbb{R}^3$ plus the point at infinity, just as the integrand $(H^2 - K)dA$ ([1], [7], [3]). Observe that, if e_1, e_2 is an orthonormal basis of T_pM, $p \in M$, for the induced metric g and ν is a normal to $df_p(T_pM)$, then, denoting by $h_{ij} = \left\langle \nabla df(e_i, e_j), \nu \right\rangle$ the components of the second fundamental form of f, we have $H = \frac{1}{2}(h_{11} + h_{22})$ and, by the Gauss equation, $K = h_{11}h_{22} - h_{12}^2$. So,

$$H^2 - K = \frac{1}{4}(h_{11} - h_{22})^2 + h_{12}^2 = \frac{1}{2}\|\nabla df - Hg\|^2 .$$

In particular, $\mathcal{W}(f) \geq 0$, with equality iff f is totally umbilical, i.e. $f(M)$ is either a part of a plane or a sphere. Since $(H^2 - K)dA$ is a conformally invariant 2-form, we may translate it in the conformal setting. That is, considering $\mathbb{R}^3$ embedded into S^3, from a Darboux frame $E: M \to \mathbb{E}(3)$ ($\mathbb{E}(n)$ is the group of the Euclidean motions of $\mathbb{R}^n$) along $f: M^2 \to \mathbb{R}^3$ we construct a Darboux G-frame $\tilde{e}: M \to G$ along $f: M^2 \to S^3$ (see section 6), and $(H^2 - K)dA$ is in this frame locally written as

$$(H^2 - K)dA = \tfrac{1}{2}(\mathrm{Trace}\, \tilde{\mathcal{N}})\tilde{\phi}_0^1 \wedge \tilde{\phi}_0^2 .$$

Now let $f : M^m \to S^n$ be an immersion of an oriented m-dimensional manifold into the Möbius space. Then, one can define on M a global m-form (cf. [4])

$$\Omega_f = \frac{1}{m}(\mathrm{Trace}\, \mathcal{N})^{m/2} dV . \tag{3.3}$$

On a domain of a Darboux G-frame $e : M \to G$ along f, Ω_f takes the expression

$$\Omega_f = \frac{1}{m}\Big(\sum_{ij\alpha} (h_{ij}^\alpha)^2 \Big)^{\frac{m}{2}} \phi_0^1 \wedge \ldots \wedge \phi_0^m .$$

If e is defined from a Darboux frame $E : M \to \mathbb{E}(n)$ along $f : M^m \to \mathbb{R}^n$, then one can see that Ω_f has the following Riemannian expressions (cf. [4])

$$\Omega_f = \tfrac{1}{m}(m(m-1)\|H\|^2 - R)^{\frac{m}{2}} dV = (\|\nabla df\|^2 - m\|H\|^2)^{\frac{m}{2}} dV = \|\nabla df - Hg\|^m dV,$$

where $R =$ Trace Ricci is the scalar curvature and H the mean curvature.

If m=2 and $n \geq 3$, the Euler–Lagrange equation of the variational problem associated with the functional

$$\mathcal{W}_D(f) = \int_D \Omega_f \tag{3.4}$$

acting on immersions $f : D \to S^n$, where D is a compact domain of M, is given by (see [2] for $n = 3$ and [4] for $n \geq 3$)

$$(p_{11}^\alpha + p_{22}^\alpha)dV = 0 , \qquad \alpha = 3, \ldots, n \tag{3.5}$$

which is conformally invariant as we can see from the transformation laws at the end of section 2. We can see from the relations in section 6 that, for an immersion into $\mathbb{R}^n$ the Euler-Lagrange equation (3.5), in the Riemannian Geometry of M, takes the form

$$\Delta H - 2\|H\|^2 H + \tilde{A}(H) = 0 ,$$

which for the case $n = 3$ is equivalent to (3.2). M is said to be a *immersed Willmore submanifold of* S^n ([4]) if f is a critical point of the functional $\mathcal{W}_D$ defined as in (3.4), for smooth variations $(f_t)_{t \in (-\epsilon, \epsilon)}$ of f keeping ∂D fixed. We further observe that if $f(M) \subset S^m \subset S^n$, then $\mathcal{W}_D(f) = 0$, that is, f is a trivial Willmore submanifold.

In the following, we calculate the Euler-Lagrange equation for any $2 \leq m \leq n$. Let $v : D\times(-\epsilon',\epsilon') \to S^n$ be a smooth variation of f through immersions $f_t = v(\cdot,t)$, with compact support $C' \subset D\backslash\partial D$, i.e. $f_t(p) = f(p)\ \forall t \in (-\epsilon',\epsilon'),\ p\in D\backslash C'$. Now we are going to compute $\frac{\partial}{\partial t}\mathcal{W}_D(f_t)$ at $t = 0$. To that end we take smooth maps $e : U\times(-\epsilon,\epsilon) \to G$, where U is a neighbourhood of a given point $p_0 \in D$ and $0 < \epsilon \leq \epsilon'$, satisfying the properties

$$\begin{cases} \text{(i)} & e(p,t) = e(p,0) \qquad \forall p\in U\backslash C,\ t\in(-\epsilon,\epsilon) \\ \text{(ii)} & \forall t\in(-\epsilon,\epsilon),\ e_t = e(\cdot,t) : M \to G \text{ is a Darboux} \\ & G\text{-frame along } f_t \text{ defined on } U\ , \end{cases} \tag{3.6}$$

where C is a compact set such that $C' \subset C \subset D\backslash\partial D$. It is easy to construct these maps e, using a method similar to when constructing Darboux G-frames (for details see [5], compare also with [2]). For such a map we define the $\mathcal{G}$-valued 1-form on $U\times(-\epsilon,\epsilon)$

$$\phi = e^*\Phi = e^{-1}de\ ,$$

with components ϕ^a_b satisfying the relations (2.2) and the structure equations (2.3). For each $t\in(-\epsilon,\epsilon)$, let $\phi(t)$ denote the $\mathcal{G}$-valued 1-form on U,

$$\phi(t) = e_t^*\Phi\ ,$$

with components $\phi^a_b(t)$. Then, at a point $(p,t)\in U\times(-\epsilon,\epsilon)$

$$\phi_{(p,t)} = \phi(t)_p + \overline{\lambda}(p,t)dt \tag{3.7}$$

with the meaning $\phi_{(p,t)}(u,h) = \phi(t)_p(u) + \overline{\lambda}(p,t)h,\ \forall u \in T_pM,\ h \in \mathbb{R}$, where $\overline{\lambda} : U\times(-\epsilon,\epsilon) \to \mathcal{G}$ with components $\overline{\lambda}^a_b$ is given by $\overline{\lambda}(p,t) = \Phi_{e_t(p)}\left(\frac{\partial}{\partial t}e(p,t)\right)$. From (3.6)(i), we have $\overline{\lambda}^a_b(p,t) = 0$ and $\phi^a_b{}_{(p,t)} = \phi^a_b{}_{(p,0)},\ \forall t \in (-\epsilon,\epsilon),\ p\in U\backslash C$. Setting $\lambda^A_0 = \overline{\lambda}^A_0$ for $1 \leq A \leq n$, and since e_t is a Darboux G-frame, then $\phi^\alpha_0{}_{(p,t)} = \lambda^\alpha_0(p,t)dt$. Since $\phi^\alpha_i(t)_p = h^\alpha_{ij}(p,t)\phi^j_0(t)_p$ with $h^\alpha_{ii} = 0,\ h^\alpha_{ij} = h^\alpha_{ji}$, and $h^\alpha_{ij}(p,t) = h^\alpha_{ij}(p,0)\ \ \forall p\in U\backslash C,\ t\in(-\epsilon,\epsilon)$, we have

$$\phi^\alpha_i{}_{(p,t)} = h^\alpha_{ij}(p,t)\phi^j_0(t)_p + \overline{\lambda}^\alpha_i(p,t)dt = h^\alpha_{ij}(p,t)\phi^j_0{}_{(p,t)} + \lambda^\alpha_i(p,t)dt\,,$$

where $\lambda^\alpha_i = -h^\alpha_{ij}\overline{\lambda}^j_0 + \overline{\lambda}^\alpha_i : U\times(-\epsilon,\epsilon) \to \mathbb{R}$ is a smooth map satisfying $\lambda^\alpha_i(p,t) = 0\ \ \forall p\in U\backslash C,\ t\in(-\epsilon,\epsilon)$. Differentiating some of the above equations and using the structure equations and Cartan's lemma,

$$d\lambda^\alpha_0 = \lambda^\alpha_0\phi^0_0 + \lambda^\alpha_i\phi^i_0 - \lambda^\beta_0\phi^\alpha_\beta + \mu^\alpha dt \tag{3.8}$$

with $\mu^\alpha(p,t) = 0\ \ \forall p\in U\backslash C,\ t\in(-\epsilon,\epsilon)$. Analogously, and using the linear independence of $(\phi^1_0,\ldots,\phi^m_0,dt)$, we obtain

$$\begin{aligned} dh^\alpha_{ij} - h^\alpha_{ik}\phi^k_j - h^\alpha_{jk}\phi^k_i + h^\beta_{ij}\phi^\alpha_\beta + h^\alpha_{ij}\phi^0_0 + \delta_{ij}\phi^\alpha_0 &= h^\alpha_{ijk}\phi^k_0 + \lambda^\alpha_{ij}dt \\ d\lambda^\alpha_i - \lambda^\alpha_0\phi^0_i - \lambda^\alpha_j\phi^j_i + \lambda^\beta_i\phi^\alpha_\beta + \lambda^\beta_0 h^\alpha_{ij}h^\beta_{jk}\phi^k_0 &= \lambda^\alpha_{ik}\phi^k_0 + \mu^\alpha_i dt \end{aligned}\ , \tag{3.9}$$

where $h^\alpha_{ijk},\ \lambda^\alpha_{ij} : U\times(-\epsilon,\epsilon) \to \mathbb{R}$ are smooth functions such that $h^\alpha_{ijk} = h^\alpha_{jik} = h^\alpha_{ikj}$, $\lambda^\alpha_{ij} = \lambda^\alpha_{ji}$, and $h^\alpha_{ijk}(p,t) = h^\alpha_{ijk}(p,0),\ \ \lambda^\alpha_{ij}(p,t) = 0\ \ \forall p\in U\backslash C,\ t\in(-\epsilon,\epsilon)$. Hence,

$$h^\alpha_{ij}dh^\alpha_{ij} = -h^\alpha_{ij}h^\alpha_{ij}\phi^0_0 + h^\alpha_{ij}h^\alpha_{ijk}\phi^k_0 + h^\alpha_{ij}\lambda^\alpha_{ij}dt\ . \tag{3.10}$$

If $\tilde{e} : \tilde{U} \times (-\epsilon, \epsilon) \to G$ is another map satisfying (3.6), then $\tilde{e}_t = e_t K_t$, where $K_t : U \cap \tilde{U} \to G_D$. Set $K : U \cap \tilde{U} \times (-\epsilon, \epsilon) \to G_D$, $K(p,t) = K_t(p)$. So, $\tilde{e} = eK$ and $\phi = \tilde{e}^{-1} d\tilde{e} = K^{-1}\phi K + K^{-1}dK$. Hence (see (2.8) for notations),

$$\tilde{\phi}_0^j = r^{-1} A_j^i \phi_0^i \ ; \qquad \tilde{\phi}_0^\beta = \tilde{\lambda}_0^\beta dt = r^{-1} B_\beta^\alpha \phi_0^\alpha = r^{-1} B_\beta^\alpha \lambda_0^\alpha dt \,,$$

which implies the transformations

$$\tilde{\lambda}_0^j = r^{-1} A_j^i \lambda_0^i \ ; \qquad \tilde{\lambda}_0^\beta = r^{-1} B_\beta^\alpha \lambda_0^\alpha \ . \tag{3.11}$$

Furthermore, we obtain

$$\begin{aligned} \tilde{\phi}_i^\alpha &= \tilde{h}_{ij}^\alpha \tilde{\phi}_0^j + \tilde{\lambda}_i^\alpha dt = B_\alpha^\beta \phi_0^\beta X_j A_i^j + B_\alpha^\beta \phi_j^\beta A_i^j \\ &= B_\alpha^\beta A_i^j h_{jk}^\beta \phi_0^k + (\lambda_j^\beta B_\alpha^\beta A_i^j + \lambda_0^\beta B_\alpha^\beta A_i^j X_j) dt \ , \end{aligned} \tag{3.12}$$

which gives us the transforms $\tilde{h}_{ij}^\alpha$ and $\tilde{\lambda}_i^\alpha$. As a final remark on maps with property (3.6), we observe that, given a point $p_0 \in M$, one can always find a variation $(f_t)_{t\in(-\epsilon,\epsilon)}$ of f with compact support C' contained in a domain $D \backslash \partial D$, such that x_0 lies in the interior of C', and a map e satisfying (3.6) with arbitrary $\lambda_0^\alpha(\cdot, 0)$ as long as $\mathrm{supp}\lambda_0^\alpha \subset C'$ (cf. [5], see also [2]). Now we have

Theorem 1 *Let $f : M^m \to S^n$ be an immersion of an oriented m-manifold into the Möbius space. Then we have:*
for $m = 2$, f is a Willmore immersed surface iff (cf. [2], [4])

$$p_{jj}^\alpha = 0 \quad \forall \alpha = 3, \ldots, n$$

for $m = 4$, f is a Willmore immersed 4-submanifold iff

$$(\mathrm{Trace}\, \mathcal{N})\,(3p_{jj}^\alpha + h_{kj}^\alpha h_{ji}^\beta h_{ik}^\beta) + 2h_{ij}^\alpha H_{ij} + 12 p_i^\alpha H_{st}^\gamma h_{sti}^\gamma = 0 \quad \forall \alpha = 5, \ldots, n$$

for $m = 3$ or $m = 5$ with the assumption that f has no umbilic points, or for $m \geq 5$ without any non-degeneracy condition, then f is a Willmore immersed m-submanifold, iff

$$\begin{aligned} &(\mathrm{Trace}\, \mathcal{N})^{\frac{m-2}{2}} \left((m-1)p_{jj}^\alpha + h_{kj}^\alpha h_{ji}^\beta h_{ik}^\beta\right) + \\ +\ &(m-2)(\mathrm{Trace}\, \mathcal{N})^{\frac{m-4}{2}} \left(h_{ij}^\alpha H_{ij} + 2(m-1) p_i^\alpha h_{st}^\gamma h_{sti}^\gamma\right) + \\ +\ &(m-2)(m-4)(\mathrm{Trace}\, \mathcal{N})^{\frac{m-6}{2}} h_{ij}^\alpha h_{st}^\gamma h_{stj}^\gamma h_{pv}^\nu h_{pvi}^\nu = 0 \qquad \forall \alpha = m+1, \ldots, n \,, \end{aligned} \tag{3.13}$$

where the quantities h_{ij}^α, h_{ijk}^α, p_i^α, p_{ij}^α, H_{ij}, and $\mathrm{Trace}\, \mathcal{N} = \mathcal{N}_{jj} = h_{st}^\beta h_{st}^\beta$, are as defined in equations (2.6), (2.9), (2.10), (2.12) and (2.13), relative to a Darboux G-frame $e : M \to G$ of $\Pi : G \to S^n$ along f. The above equations are conformally invariant, that is, they do not depend on the choice of Darboux frames.

Proof. Let $v : \overline{D} \times (-\epsilon', \epsilon') \to S^n$ be a smooth variation of f through immersions with compact support $C' \subset D \backslash \partial D$. Let $e : U \times (-\epsilon, \epsilon) \to G$ be a map satisfying (3.6) with U a neighbourhood of $p_0 \in D$. Let Ω be the m-form on $U \times (-\epsilon, \epsilon)$ given by

$$\Omega_{(p,t)} = \frac{1}{m}\Big(\sum_{ij\alpha} \big(h_{ij}^\alpha(p,t)\big)^2\Big)^{\frac{m}{2}} \phi_{0\,(p,t)}^1 \wedge \ldots \wedge \phi_{0\,(p,t)}^m \ .$$

Since $\phi^a_{b\,(p,t)}\big|_{T_pM} = \phi^a_b(t)$, the restriction of $\Omega_{(p,t)}$ to the tangent space T_pM is just $\Omega_{f_t(p)}$ and so it defines a global m-form on all D. Note that Ω does not depend on e, as one can check by the transformation laws of these maps e satisfying (3.6) described as above, which gives us $\tilde{h}^\alpha_{ij} = rB^\beta_\alpha A^k_i A^l_j h^\beta_{kl}$, $\tilde{\phi}^j_0 = r^{-1}A^i_j\phi^i_0$, and so $\tilde{\Omega}_{(p,t)} = \Omega_{(p,t)}$. Therefore, Ω is a globally well-defined m-form on $D\times(-\epsilon,\epsilon)$. We also observe that, by compactness of D, we may take $\epsilon = \epsilon'$ independent of the point p_0. By elementary calculations, we obtain

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \int_D \frac{\partial}{\partial t}\Big|_{t=0}(\Omega_{f_t}) = \int_D \frac{\partial}{\partial t}\Big|_{t=0}\big(\Omega_{(p,t)}\big|_{TM}\big) = \int_D \big(L_{\frac{\partial}{\partial t}}\Omega\big)\Big|_{\substack{t=0\\TM}}.$$

Since $L = \rfloor \circ d + d\circ\rfloor$, we have

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \int_D \big(\frac{\partial}{\partial t}\rfloor d\Omega\big)\Big|_{\substack{t=0\\TM}} + d\big(\frac{\partial}{\partial t}\rfloor\Omega\big)\Big|_{\substack{t=0\\TM}}.$$

Henceforth we shall use the notations $\phi^{1\ldots m} = \phi^1_0\wedge\ldots\wedge\phi^m_0$ and $\phi^{1\ldots\hat{i}\ldots m} = \phi^1_0\wedge\ldots\wedge\phi^{i-1}_0\wedge\phi^{i+1}_0\wedge\ldots\wedge\phi^m_0$. Using the structure equations, we have

$$d\phi^{1\ldots m} = m\phi^0_0\wedge\phi^{1\ldots m} \tag{3.14}$$
$$d\phi^{1\ldots\hat{i}\ldots m} = (m-1)\phi^0_0\wedge\phi^{1\ldots\hat{i}\ldots m} + (-1)^{k+1}\phi^k_i\wedge\phi^{1\ldots\hat{k}\ldots m}. \tag{3.15}$$

Hence, from (3.14) and (3.10) we obtain

$$d\Omega = \Big(\sum_{ij\beta}(h^\beta_{ij})^2\Big)^{\frac{m-2}{2}} h^\alpha_{st}\lambda^\alpha_{st}\,dt\wedge\phi^{1\ldots m}.$$

Thus,

$$\frac{\partial}{\partial t}\rfloor d\Omega = \Big(\sum_{ij\beta}(h^\beta_{ij})^2\Big)^{\frac{m-2}{2}} h^\alpha_{st}\lambda^\alpha_{st}\phi^{1\ldots m}(t),$$

where $\phi^{1\ldots m}(t) = \phi^1_0(t)\wedge\ldots\wedge\phi^m_0(t)$. Now, we have

$$\frac{\partial}{\partial t}\rfloor\phi^{1\ldots m} = \frac{\partial}{\partial t}\rfloor\Big((\phi^1_0(t)+\lambda^1_0 dt)\wedge\ldots\wedge(\phi^1_0(t)+\lambda^1_0 dt)\Big) = (-1)^{k-1}\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}(t).$$

Hence,

$$\big(\frac{\partial}{\partial t}\rfloor d\Omega\big)\Big|_{\substack{t=0\\TM}} = \Big(\sum_{ij\beta}(h^\beta_{ij})^2\Big)^{\frac{m-2}{2}} h^\alpha_{st}\lambda^\alpha_{st}\phi^{1\ldots m}$$

and

$$d\big(\frac{\partial}{\partial t}\rfloor\Omega\big)\Big|_{\substack{t=0\\TM}} = d\Big(\big(\frac{\partial}{\partial t}\rfloor\Omega\big)\Big|_{\substack{t=0\\TM}}\Big) = d\Big(\frac{1}{m}\Big(\sum_{ij\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}}(-1)^{k-1}\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}\Big),$$

where from now on $\phi^{1\ldots\hat{k}\ldots m} = \phi^{1\ldots\hat{k}\ldots m}(0)$ and $\phi^{1\ldots m} = \phi^{1\ldots m}(0)$ denote forms on U only, with ϕ^i_0, h^α_{st} relative to the frame $e_0\colon U\to G$ of Π along $f = f_0$, and where λ^k_0 and λ^α_{st} are considered as functions of the variable $p\in U$ only, fixing $t=0$.

Thus, we have obtained

$$\begin{aligned}\big(\frac{\partial}{\partial t}\rfloor d\Omega\big)\Big|_{\substack{t=0\\TM}} + d\big(\frac{\partial}{\partial t}\rfloor\Omega\big)\Big|_{\substack{t=0\\TM}} &= \Big(\sum_{ij\beta}(h^\beta_{ij})^2\Big)^{\frac{m-2}{2}} h^\alpha_{st}\lambda^\alpha_{st}\phi^{1\ldots m} + \\ &\quad + d\Big(\frac{1}{m}\Big(\sum_{ij\alpha}(h^\alpha_{ij})^2\Big)^{\frac{m}{2}}(-1)^{k-1}\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}\Big),\end{aligned}$$

where λ_0^α, λ_i^α and λ_{ij}^α have support in $C \cap U \subset D \backslash \partial D$ and satisfy (see (3.8), (3.9))

$$d\lambda_0^\alpha = \lambda_0^\alpha \phi_0^0 + \lambda_i^\alpha \phi_0^i - \lambda_0^\beta \phi_\beta^\alpha \tag{3.16}$$
$$d\lambda_i^\alpha = \lambda_0^\alpha \phi_i^0 + \lambda_j^\alpha \phi_i^j - \lambda_i^\beta \phi_\beta^\alpha - \lambda_0^\beta h_{ij}^\alpha h_{jk}^\beta \phi_0^k + \lambda_{ik}^\alpha \phi_0^k \tag{3.17}$$

with ϕ_b^a, h_{ij}^α relative to the Darboux G-frame e_0 along f. For the sake of notational simplicity, we define

$$\|h\| = \sqrt{\sum_{ij\alpha} (h_{ij}^\alpha)^2} = \sqrt{\mathrm{Trace}\,\mathcal{N}}\ .$$

From (2.9) we have

$$d\|h\|^r = -r\|h\|^r \phi_0^0 + r\|h\|^{r-2} h_{st}^\gamma h_{stk}^\gamma \phi_0^k\ . \tag{3.18}$$

Now we evaluate the expression $\|h\|^{m-2} h_{st}^\alpha \lambda_{st}^\alpha \phi^{1\ldots m}$. Using (3.17), we get for i, j fixed

$$\begin{aligned}\lambda_{ij}^\alpha \phi^{1\ldots m} &= (-1)^{j-1} \lambda_{ij}^\alpha \phi_0^j \wedge \phi^{1\ldots\hat{j}\ldots m} = (-1)^{j-1} (\lambda_{ik}^\alpha \phi_0^k) \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &= (-1)^{j-1} (d\lambda_i^\alpha - \lambda_0^\alpha \phi_i^0 - \lambda_k^\alpha \phi_i^k + \lambda_i^\beta \phi_\beta^\alpha + \lambda_0^\beta h_{ik}^\alpha h_{kl}^\beta \phi_0^l) \wedge \phi^{1\ldots\hat{j}\ldots m}\ .\end{aligned}$$

Hence

$$\begin{aligned}&\|h\|^{m-2} h_{ij}^\alpha \lambda_{ij}^\alpha \phi^{1\ldots m} = \\ &= (-1)^{j-1} \|h\|^{m-2} h_{ij}^\alpha d\lambda_{ij}^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2} h_{ij}^\alpha \lambda_k^\alpha \phi_i^k \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^{j-1} \|h\|^{m-2} h_{ij}^\alpha \lambda_i^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2} h_{ij}^\alpha \lambda_0^\alpha \phi_i^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +\|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m} \\ &= d\big((-1)^{j-1} \|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m}\big) + (-1)^j h_{ij}^\alpha \lambda_i^\alpha d\big(\|h\|^{m-2}\big) \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^j \|h\|^{m-2} \lambda_i^\alpha dh_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha d\phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^j \|h\|^{m-2} \lambda_k^\alpha h_{ij}^\alpha \phi_i^k \wedge \phi^{1\ldots\hat{j}\ldots m} + (-1)^{j-1} \|h\|^{m-2} \lambda_i^\beta h_{ij}^\alpha \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^j \|h\|^{m-2} h_{ij}^\alpha \lambda_0^\alpha \phi_i^0 \wedge \phi^{1\ldots\hat{j}\ldots m} + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m}.\end{aligned}$$

Using (3.18), (2.9), (2.11), (3.15) and (2.12), and assuming $m \neq 3$ unless $\|h\| \neq 0$, we get

$$\begin{aligned}&\|h\|^{m-2} h_{ij}^\alpha \lambda_{ij}^\alpha \phi^{1\ldots m} = \\ &= d\big((-1)^{j-1} \|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha \phi^{1\ldots\hat{j}\ldots m}\big) + (-1)^{j-1}(m-2) \|h\|^{m-2} h_{ij}^\alpha \lambda_i^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad -(m-2) \|h\|^{m-4} h_{ij}^\alpha \lambda_i^\alpha h_{st}^\gamma h_{stj}^\gamma \phi^{1\ldots m} + (-1)^j \|h\|^{m-2} \lambda_i^\alpha h_{kj}^\alpha \phi_i^k \wedge \phi^{1\ldots\hat{j}\ldots m} + \\ &\quad +(-1)^j \|h\|^{m-2} \lambda_i^\alpha h_{ik}^\alpha \phi_j^k \wedge \phi^{1\ldots\hat{j}\ldots m} + (-1)^{j-1} \|h\|^{m-2} \lambda_i^\alpha h_{ij}^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^{j-1} \|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} + \|h\|^{m-2} \lambda_j^\alpha p_j^\alpha \phi^{1\ldots m} \\ &\quad -\|h\|^{m-2} \lambda_i^\alpha h_{ijj}^\alpha \phi^{1\ldots m} + (-1)^j (m-1) \|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^k \|h\|^{m-2} \lambda_i^\alpha h_{ij}^\alpha \phi_j^k \wedge \phi^{1\ldots\hat{k}\ldots m} + (-1)^j \|h\|^{m-2} h_{ij}^\alpha \lambda_k^\alpha \phi_i^k \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^{j-1} \|h\|^{m-2} h_{ij}^\alpha \lambda_i^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2} \lambda_0^\alpha dp_j^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^{j-1} \|h\|^{m-2} \lambda_0^\alpha p_i^\alpha \phi_j^i \wedge \phi^{1\ldots\hat{j}\ldots m} + (-1)^j \|h\|^{m-2} \lambda_0^\alpha p_j^\beta \phi_\beta^\alpha \wedge \phi^{1\ldots\hat{j}\ldots m} \\ &\quad +(-1)^j 2 \|h\|^{m-2} \lambda_0^\alpha p_j^\alpha \phi_0^0 \wedge \phi^{1\ldots\hat{j}\ldots m} + \|h\|^{m-2} \lambda_0^\alpha p_{jj}^\alpha \phi^{1\ldots m} + \|h\|^{m-2} h_{ij}^\alpha h_{ik}^\alpha \lambda_0^\beta h_{kj}^\beta \phi^{1\ldots m}\ .\end{aligned}$$

In this expression, we get several simple cancellations by permuting indices when necessary and using the symmetry properties of the coefficients and forms involved.

By applying (2.10), we obtain

$$\begin{aligned}
&\|h\|^{m-2}h^\alpha_{ij}\lambda^\alpha_{ij}\phi^{1\dots m} = \\
&= d\big((-1)^{j-1}\|h\|^{m-2}\lambda^\alpha_i h^\alpha_{ij}\phi^{1\dots\hat{j}\dots m}\big) - (m-2)\|h\|^{m-4}h^\alpha_{ij}\lambda^\alpha_i h^\gamma_{st}h^\gamma_{stj}\phi^{1\dots m} \\
&\quad +(1-m)\|h\|^{m-2}\lambda^\alpha_j p^\alpha_j\phi^{1\dots m} + d\big((-1)^j\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_j\phi^{1\dots\hat{j}\dots m}\big) \\
&\quad +(-1)^{j-1}\|h\|^{m-2}p^\alpha_j d\lambda^\alpha_0\wedge\phi^{1\dots\hat{j}\dots m} + (-1)^{j-1}p^\alpha_j\lambda^\alpha_0 d(\|h\|^{m-2})\wedge\phi^{1\dots\hat{j}\dots m} \\
&\quad +(-1)^{j-1}p^\alpha_j\lambda^\alpha_0\|h\|^{m-2}d(\phi^{1\dots\hat{j}\dots m}) + (-1)^{j-1}\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_i\phi^i_j\wedge\phi^{1\dots\hat{j}\dots m} \\
&\quad +(-1)^j\|h\|^{m-2}\lambda^\alpha_0 p^\beta_j\phi^\alpha_\beta\wedge\phi^{1\dots\hat{j}\dots m} + (-1)^j 2\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_j\phi^0_0\wedge\phi^{1\dots\hat{j}\dots m} \\
&\quad +\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_{jj}\phi^{1\dots m} + \|h\|^{m-2}h^\alpha_{ij}h^\alpha_{ik}\lambda^\beta_0 h^\beta_{kj}\phi^{1\dots m}\ .
\end{aligned}$$

Using (3.16), (3.18), and (3.15), we get, after some obvious cancelations and rearrangements,

$$\begin{aligned}
\|h\|^{m-2}h^\alpha_{ij}\lambda^\alpha_{ij}\phi^{1\dots m} &= d\big((-1)^{j-1}\|h\|^{m-2}\lambda^\alpha_i h^\alpha_{ij}\phi^{1\dots\hat{j}\dots m} + (-1)^j\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_j\phi^{1\dots\hat{j}\dots m}\big) \\
&\quad -(m-2)\|h\|^{m-4}h^\alpha_{ij}\lambda^\alpha_i h^\gamma_{st}h^\gamma_{stj}\phi^{1\dots m} + (2-m)\|h\|^{m-2}\lambda^\alpha_j p^\alpha_j\phi^{1\dots m} \\
&\quad +(m-2)\|h\|^{m-4}p^\alpha_j\lambda^\alpha_0 h^\gamma_{st}h^\gamma_{stj}\phi^{1\dots m} + \|h\|^{m-2}\lambda^\alpha_0 p^\alpha_{jj}\phi^{1\dots m} \\
&\quad +\|h\|^{m-2}h^\alpha_{ij}h^\alpha_{ik}\lambda^\beta_0 h^\beta_{kj}\phi^{1\dots m}\ . \qquad (3.19)
\end{aligned}$$

This expression will also serve for later use. Fixing i, substituting the factor $\lambda^\alpha_i\phi^{1\dots m}$ in the second term of the right-hand side as

$$\lambda^\alpha_i\phi^{1\dots m} = (-1)^{i-1}\lambda^\alpha_i\phi^i_0\wedge\phi^{1\dots\hat{i}\dots m} = (-1)^{i-1}\lambda^\alpha_k\phi^k_0\wedge\phi^{1\dots\hat{i}\dots m}\ , \qquad (3.20)$$

and using (3.16), we derive

$$\begin{aligned}
\|h\|^{m-2}h^\alpha_{ij}\lambda^\alpha_{ij}\phi^{1\dots m} &= d\big((-1)^{j-1}\|h\|^{m-2}\lambda^\alpha_i h^\alpha_{ij}\phi^{1\dots\hat{j}\dots m} + (-1)^j\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_j\phi^{1\dots\hat{j}\dots m}\big) \\
&\quad +(-1)^i(m-2)\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}(d\lambda^\alpha_0 - \lambda^\alpha_0\phi^0_0 + \lambda^\beta_0\phi^\alpha_\beta)\wedge\phi^{1\dots\hat{i}\dots m} \\
&\quad +(2-m)\|h\|^{m-2}\lambda^\alpha_j p^\alpha_j\phi^{1\dots m} + (m-2)\|h\|^{m-4}p^\alpha_j\lambda^\alpha_0 h^\gamma_{st}h^\gamma_{stj}\phi^{1\dots m} \\
&\quad +\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_{jj}\phi^{1\dots m} + \|h\|^{m-2}h^\alpha_{ij}h^\alpha_{ik}\lambda^\beta_0 h^\beta_{kj}\phi^{1\dots m} \\
&= d\big((-1)^{j-1}\|h\|^{m-2}(\lambda^\alpha_i h^\alpha_{ij} - \lambda^\alpha_0 p^\alpha_j)\phi^{1\dots\hat{j}\dots m}\big) \\
&\quad +d\big((-1)^i(m-2)\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}\lambda^\alpha_0\wedge\phi^{1\dots\hat{i}\dots m}\big) \\
&\quad +(-1)^{i-1}(m-2)\|h\|^{m-4}h^\alpha_{ij}\lambda^\alpha_0 d(h^\gamma_{st}h^\gamma_{stj})\wedge\phi^{1\dots\hat{i}\dots m} \\
&\quad +(-1)^i(m-2)\|h\|^{m-4}\lambda^\alpha_0 h^\gamma_{st}h^\gamma_{stj}dh^\alpha_{ij}\wedge\phi^{1\dots\hat{i}\dots m} \\
&\quad +(-1)^{i-1}(m-2)\lambda^\alpha_0 h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}d(\|h\|^{m-4})\wedge\phi^{1\dots\hat{i}\dots m} \\
&\quad +(-1)^{i-1}(m-2)\|h\|^{m-4}\lambda^\alpha_0 h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}d\phi^{1\dots\hat{i}\dots m} \\
&\quad +(-1)^{i-1}(m-2)\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}\lambda^\alpha_0\phi^0_0\wedge\phi^{1\dots\hat{i}\dots m} \\
&\quad +(-1)^i(m-2)\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}\lambda^\beta_0\phi^\alpha_\beta\wedge\phi^{1\dots\hat{i}\dots m} \\
&\quad +(2-m)\|h\|^{m-2}\lambda^\alpha_j p^\alpha_j\phi^{1\dots m} + (m-2)\|h\|^{m-4}p^\alpha_j\lambda^\alpha_0 h^\gamma_{st}h^\gamma_{stj}\phi^{1\dots m} \\
&\quad +\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_{jj}\phi^{1\dots m} + \|h\|^{m-2}h^\alpha_{ij}h^\alpha_{ik}\lambda^\beta_0 h^\beta_{kj}\phi^{1\dots m}\ .
\end{aligned}$$

From (2.13), (2.9), (3.18) and (2.10), and assuming $m \neq 5$ (unless $\|h\| \neq 0$ everywhere), we obtain, after performing some simplifications,

$$\begin{aligned}
\|h\|^{m-2}h^\alpha_{ij}\lambda^\alpha_{ij}\phi^{1\dots m} &= d\big((-1)^{j-1}\|h\|^{m-2}(\lambda^\alpha_i h^\alpha_{ij} - \lambda^\alpha_0 p^\alpha_j)\phi^{1\dots\hat{j}\dots m}\big) \\
&\quad +d\big((-1)^i(m-2)\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}\lambda^\alpha_0\phi^{1\dots\hat{i}\dots m}\big) +
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^i(m-2)\|h\|^{m-2}h^\alpha_{ij}\lambda^\alpha_0\phi^0_j\wedge\phi^{1\ldots\hat{i}\ldots m}\\
&+(m-2)\|h\|^{m-4}h^\alpha_{ij}\lambda^\alpha_0 H_{ij}\phi^{1\ldots m}\\
&+m(m-2)\|h\|^{m-4}\lambda^\alpha_0 h^\gamma_{st}h^\gamma_{stj}p^\alpha_j\phi^{1\ldots m}\\
&+(m-2)(m-4)\|h\|^{m-6}\lambda^\alpha_0 h^\gamma_{st}h^\gamma_{stj}h^\alpha_{ij}h^\nu_{pv}h^\nu_{pvi}\phi^{1\ldots m}\\
&+(2-m)\|h\|^{m-2}\lambda^\alpha_j p^\alpha_j\phi^{1\ldots m}+\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_{jj}\phi^{1\ldots m}\\
&+\|h\|^{m-2}h^\alpha_{ij}h^\alpha_{ik}\lambda^\beta_0 h^\beta_{kj}\phi^{1\ldots m}\ .
\end{aligned}
$$

Now we separately compute the term $(2-m)\|h\|^{m-2}p^\alpha_i\lambda^\alpha_i\phi^{1\ldots m}$. Using (3.20), (3.16), (2.12), (3.18) and (3.15), we have

$$
\begin{aligned}
\|h\|^{m-2}p^\alpha_i\lambda^\alpha_i\phi^{1\ldots m} &= d\big((-1)^{i-1}\|h\|^{m-2}p^\alpha_i\lambda^\alpha_0\phi^{1\ldots\hat{i}\ldots m}\big)\\
&\quad+(-1)^i\|h\|^{m-2}\lambda^\alpha_0 h^\alpha_{ki}\phi^0_k\wedge\phi^{1\ldots\hat{i}\ldots m}-\|h\|^{m-2}\lambda^\alpha_0 p^\alpha_{ii}\phi^{1\ldots m}\\
&\quad-(m-2)\|h\|^{m-4}\lambda^\alpha_0 p^\alpha_i h^\gamma_{st}h^\gamma_{sti}\phi^{1\ldots m}\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|h\|^{m-2}h^\alpha_{ij}\lambda^\alpha_{ij}\phi^{1\ldots m} &= d\big((-1)^{j-1}\|h\|^{m-2}(\lambda^\alpha_i h^\alpha_{ij}-\lambda^\alpha_0 p^\alpha_j)\phi^{1\ldots\hat{j}\ldots m}\big)+\\
&+d\big((-1)^i(m-2)\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}\lambda^\alpha_0\phi^{1\ldots\hat{i}\ldots m}\big)+d\big((-1)^i(m-2)\|h\|^{m-2}p^\alpha_i\lambda^\alpha_0\phi^{1\ldots\hat{i}\ldots m}\big)\\
&+\lambda^\alpha_0\big((m-1)\|h\|^{m-2}p^\alpha_{ii}+(m-2)\|h\|^{m-4}h^\alpha_{ij}H_{ij}+2(m-1)(m-2)\|h\|^{m-4}p^\alpha_i h^\gamma_{st}h^\gamma_{sti}+\\
&\quad+(m-2)(m-4)\|h\|^{m-6}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}h^\nu_{pv}h^\nu_{pvi}+\|h\|^{m-2}h^\alpha_{kj}h^\beta_{ji}h^\beta_{ik}\big)\phi^{1\ldots m}\ .
\end{aligned}
$$

Thus, on U,

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial t}\rfloor d\Omega\Big)\Big|_{\substack{t=0\\TM}}+d\Big(\frac{\partial}{\partial t}\rfloor\Omega\Big)\Big|_{\substack{t=0\\TM}}=\\
&d\Big(\frac{1}{m}(-1)^{k-1}\|h\|^m\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}\Big)+d\big((-1)^{j-1}\|h\|^{m-2}(\lambda^\alpha_i h^\alpha_{ij}-\lambda^\alpha_0 p^\alpha_j)\phi^{1\ldots\hat{j}\ldots m}\big)\\
&+d\big((-1)^i(m-2)\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}\lambda^\alpha_0\phi^{1\ldots\hat{i}\ldots m}\big)+d\big((-1)^i(m-2)\|h\|^{m-2}p^\alpha_i\lambda^\alpha_0\phi^{1\ldots\hat{i}\ldots m}\big)\\
&+\lambda^\alpha_0\big((m-1)\|h\|^{m-2}p^\alpha_{ii}+(m-2)\|h\|^{m-4}h^\alpha_{ij}H_{ij}+2(m-1)(m-2)\|h\|^{m-4}p^\alpha_i h^\gamma_{st}h^\gamma_{sti}\\
&\quad+(m-2)(m-4)\|h\|^{m-6}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}h^\nu_{pv}h^\nu_{pvi}+\|h\|^{m-2}h^\alpha_{kj}h^\beta_{ji}h^\beta_{ik}\big)\phi^{1\ldots m}\ .
\end{aligned}
$$

Using now the transformation laws (3.11) and (3.12) for the λ^i_0, λ^α_0, λ^α_i under a change of map $e: M\times(-\epsilon,\epsilon)\to G$ under the conditions (3.6), and the transformation laws for Darboux G-frames along f given in section 2, we can easily verify that the local forms

$$
\begin{aligned}
\Upsilon_1 &= \lambda^\alpha_0\big((m-1)\|h\|^{m-2}p^\alpha_{ii}+(m-2)\|h\|^{m-4}h^\alpha_{ij}H_{ij}+2(m-1)(m-2)\|h\|^{m-4}p^\alpha_i h^\gamma_{st}h^\gamma_{sti}\\
&\quad+(m-2)(m-4)\|h\|^{m-6}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}h^\nu_{pv}h^\nu_{pvi}\big)\phi^{1\ldots m}\ ,\\
\Upsilon_2 &= \lambda^\alpha_0\|h\|^{m-2}h^\alpha_{kj}h^\beta_{ji}h^\beta_{ik}\phi^{1\ldots m}\ ,\qquad \Upsilon_3=(-1)^{k-1}\|h\|^m\lambda^k_0\phi^{1\ldots\hat{k}\ldots m}\ ,\qquad(3.21)\\
\Upsilon_4 &= (-1)^{i-1}\|h\|^{m-2}(\lambda^\alpha_j h^\alpha_{ij}-\lambda^\alpha_0 p^\alpha_i)\phi^{1\ldots\hat{i}\ldots m}\ ,\\
\Upsilon_5 &= (-1)^i(m-2)\big(\lambda^\alpha_0\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}+\lambda^\alpha_0\|h\|^{m-2}p^\alpha_i\big)\phi^{1\ldots\hat{i}\ldots m}\ ,\\
\Upsilon_6 &= \lambda^\alpha_0\big(\|h\|^{m-2}p^\alpha_{ii}+(m-2)\|h\|^{m-4}p^\alpha_i h^\gamma_{st}h^\gamma_{sti}\big)\phi^{1\ldots m}+\\
&\quad-(m-2)\lambda^\alpha_i\big(\|h\|^{m-2}p^\alpha_i+\|h\|^{m-4}h^\alpha_{ij}h^\gamma_{st}h^\gamma_{stj}\big)\phi^{1\ldots m}
\end{aligned}
$$

are well-defined global forms on all D.

Hence,

$$\left(\frac{\partial}{\partial t}\rfloor d\Omega\right)\Big|_{\substack{t=0\\TM}} + d\left(\frac{\partial}{\partial t}\rfloor\Omega\right)\Big|_{\substack{t=0\\TM}} = d\zeta + \theta$$

with ζ and θ globally well-defined $(m-1)$- and m-forms on M, respectively. Moreover, ζ has compact support in $C \subset D\backslash\partial D$, just as λ_0^α, λ_0^i and λ_i^α. Therefore, integrating over D and applying Stokes theorem, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \int_D \lambda_0^\alpha\big((m-1)\|h\|^{m-2}p_{ii}^\alpha + (m-2)\|h\|^{m-4}h_{ij}^\alpha H_{ij} \\ + 2(m-1)(m-2)\|h\|^{m-4}p_i^\alpha h_{st}^\gamma h_{sti}^\gamma \\ + (m-2)(m-4)\|h\|^{m-6}h_{ij}^\alpha h_{st}^\gamma h_{stj}^\gamma h_{pv}^\nu h_{pvi}^\nu \\ + \|h\|^{m-2}h_{kj}^\alpha h_{ji}^\beta h_{ik}^\beta\big)\phi^{1\dots m}\ .\end{aligned}$$

Since λ_0^α may be any function with compact support $C \subset D\backslash\partial D$, we conclude that f is a critical point of $\mathcal{W}_D$ iff (3.13) holds. The Euler-Lagrange equation as in the theorem for the case $m = 4$ follows immediately, and the case $m = 2$ is obtained observing that $h_{kj}^\alpha h_{ij}^\beta h_{ik}^\beta = 0$ holds as a consequence of $h_{ii}^\alpha = 0$. □

Remark. The condition

$$h_{kj}^\alpha h_{ij}^\beta h_{ik}^\beta = 0 \qquad \forall\alpha = m+1,\dots,n \tag{3.22}$$

is conformally invariant, i.e. it does not depend on the Darboux G-frame $e : M \to G$ along f. So it defines a condition for the immersion f. This condition allways holds for $m = 2$. For arbitrary m, it is equivalent to $mp_{ii}^\alpha = h_{iikk}^\alpha = 0 \quad \forall\alpha = m+1,\dots,n$, where h_{ijkl}^α is defined in the obvious way (cf. [5]). In the Riemannian geometry of a submanifold of $\mathbb{R}^n$, $f : M^m \to \mathbb{R}^n$, (3.22) means (see section 6)

$$\tilde{B} - 2\tilde{A}(H) + (2m\|H\|^2 - \|\nabla df\|^2)H = 0\,,$$

where $\tilde{B}$ and $\tilde{A}$ are defined in that section.

4 The Conformal Gauss Map

In Riemannian geometry, there exist well-known relations between the mean curvature of immersed submanifolds of the Euclidean space and the respective Gauss maps. Something similar can be done for immersed m-submanifolds $f : M^m \to S^n$ of the Möbius space. In [2] Bryant defined a (hyperbolic) conformal Gauss map for immersions $f : M^2 \to S^3$ as a map $\gamma_f : M^2 \to Q$, with Q the hyperboloid of $\mathbb{R}^5$

$$Q = \{x \in \mathbb{R}^5 : \langle x, x\rangle = 1\}\,,$$

given by $\gamma_f(p) = e_3(p)$, where $e : M \to G$ is an arbitrary Darboux G-frame along f defined on a neighbourhood of the point p. From the transformation laws in section 2 we see that γ_f is well-defined. The above definition is extended to the case of an immersion $f : M^m \to S^n$ for any $m \le n$, as follows [4]. Let $G_{n-m}(\mathbb{R}^{n+2})$ denote the Grassmannian manifold of the $n-m$ planes of $\mathbb{R}^{n+2}$. Fix $\mathbf{0} = \mathrm{span}\{\eta_{m+1},\dots,\eta_n\}$ as the origin of $G_{n-m}(\mathbb{R}^{n+2})$. Then, G acts on the left on $G_{n-m}(\mathbb{R}^{n+2})$ by matrix multiplication. The conformal Grassmannian is the open orbit of the origin, $Q_{n-m}(\mathbb{R}^{n+2}) = G(\mathbf{0})$, which is a submanifold of $G_{n-m}(\mathbb{R}^{n+2})$. G acts transitively on $Q_{n-m}(\mathbb{R}^{n+2})$ and the isotropic subgroup of G at $\mathbf{0}$, $H_{\mathbf{0}}$, is isomorphic to $G(m) \times SO(n-m)$. Thus, $Q_{n-m}(\mathbb{R}^{n+2})$ can be identifyed with the homogeneous space G/H_0 with canonic projection $\hat{\Pi} : G \rightarrow Q_{n-m}(\mathbb{R}^{n+2})$, $\hat{\Pi}(P) = \text{span}\{P(\eta_{m+1}), \ldots, P(\eta_n)\}$. The conformal Grassmannian has dimension $(n-m)(m+2)$ and carries a pseudo-metric with signature $(-\ldots-,+\ldots+)$, $n-m$ minus signs and $(m+1)(n-m)$ plus signs, given by

$$d\ell^2 = -\zeta^*\Phi^0_\alpha \otimes \zeta^*\Phi^\alpha_0 - \zeta^*\Phi^\alpha_0 \otimes \zeta^*\Phi^0_\alpha + \zeta^*\Phi^i_\alpha \otimes \zeta^*\Phi^i_\alpha,$$

where $\zeta : Q_{n-m}(\mathbb{R}^{n+2}) \rightarrow G$ is a local section of the principal bundle $\hat{\Pi} : G \rightarrow G/H_0$.

The *conformal Gauss map* γ_f of an immersion $f : M^m \rightarrow S^n$ is then given by

$$\begin{array}{rccl} \gamma_f : & M^m & \longrightarrow & Q_{n-m}(\mathbb{R}^{n+2}) \\ & p & \longrightarrow & \text{span}\{e_{m+1}(p), \ldots, e_n(p)\}, \end{array}$$

where $e = [e_0, e_i, e_\alpha, e_{n+1}] : M \rightarrow G$ is a Darboux G-frame along f. When $m = n-1$, $Q_1(\mathbb{R}^{n+2})$ can be identified with the projectivisation of the 1-fold hyperboloid $Q = \{x \in \mathbb{R}^{n+2} : \langle x, x \rangle = 1\}$ supplied with the Lorentz inner product induced by the one of $\mathbb{R}^{n+2}$, still to be denoted by $d\ell^2$. In this case, it is more practical to use the hyperbolic conformal Gauss map, still to be denoted as γ_f, given by

$$\begin{array}{rccl} \gamma_f : & M^{n-1} & \longrightarrow & Q \\ & p & \longrightarrow & e_n(p), \end{array}$$

which generalises the conformal Gauss map for immersed surfaces in S^3 used by Bryant. In the general case, the equality (cf. [4])

$$\gamma_f^* d\ell^2 = \mathcal{N} \tag{4.1}$$

holds, with $\mathcal{N}$ defined in section 2, leading to the following proposition:

Proposition ([2][4]) *Let $f : M^m \rightarrow S^n$ be an immersion of an m-manifold M endowed with the induced conformal structure. Then, for $p \in M$, $d\gamma_f(p)$ is not injective iff $\mathcal{N}(p)$ is a degenerate symmetric bilinear map. Let $c(\gamma_f)$ be the set of points p in these conditions. In the case $m = 2$, $c(\gamma_f)$ is the set of umbilic points of f. In the general case, outside $c(\gamma_f)$ γ_f induces a positive-definite metric on M that belongs to the conformal class of $M \backslash c(\gamma_f)$, iff $\mathcal{N}$ does so. This is allways the case when $m = 2$.*

Another variational problem mentioned in [4] is the one associated with the functional

$$\eta_D(\rho) = \frac{1}{m} \int_D |\text{Trace}(\rho^* d\ell^2)|^{\frac{m}{2}} dV, \tag{4.2}$$

applied to maps $\rho : D \rightarrow Q_{n-m}(\mathbb{R}^{n+2})$, and where Trace and dV are taken relative to any metric belonging to the conformal class of M. Moreover, for $m = 2$, $\eta_D(\rho)$ is the energy functional. From (4.1), one has $\mathcal{W}(f) = \eta(\gamma_f)$. In [4], the Euler-Lagrange equation is calculated in the case $2 = m \leq n$, for the functional $\eta(\rho)$ with $\rho = \gamma_f$. Here we are going to discuss the case where $f : M \rightarrow S^n$ is an immersion of a hypersurface, i.e. $m = n-1$. For convenience, we consider the functional (4.2) to act on maps $\rho : D \rightarrow Q$ satisfying (only for m odd) $\text{Trace}(\rho^* d\ell^2) \geq 0$, where now $d\ell^2$ denotes the induced Lorentz inner product of Q. One can easily derive the Euler-Lagrange equation of this functional, obtaining (for $m \neq 3$)

$$\text{Trace}\nabla\Big(\big(\text{Trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} d\rho\Big) = \tag{4.3}$$

$$\text{Trace}\left\{\frac{m-2}{2}\big(\text{Trace}(\rho^* d\ell^2)\big)^{\frac{m-4}{2}} d\big(\text{Trace}(\rho^* d\ell^2)\big) \otimes d\rho + \big(\text{Trace}(\rho^* d\ell^2)\big)^{\frac{m-2}{2}} \nabla d\rho\right\} = 0,$$

where M is considered with one of the metrics out of its conformal class, and both M and Q with the respective Levi-Civita connections. Let us suppose now that $\rho = \gamma_f : M \to Q$ is the hyperbolic conformal Gauss map of an immersion $f : M^m \to S^n$. Let $p_0 \in M$ and $e : M \to G$ be a Darboux G-frame defined near p_0. Then, $\gamma_f(p) = e_n(p)$ near p_0. From $\phi = e^{-1}de$ and (2.2), (2.11), (2.6), we have

$$d\gamma_f = de_n = p^n_j \phi^j_0 e_0 - h^n_{ij}\phi^j_0 e_i \,.$$

Now we evaluate the Euler-Lagrange equation (4.3) for $\rho = \gamma_f$. To that end, we compute $\mathrm{Trace}\nabla d\gamma_f$, whereby considering M to be supplied with the Levi-Civita connection ∇ corresponding to the Riemannian metric $g = \phi^i_0 \otimes \phi^i_0$, and Q with the induced Lorentz metric $d\ell^2$. One can immediately conclude from the structure equations that this connection on M is defined by the connection forms

$$v^i_k = \phi^i_k + \mu_k \phi^i_0 - \mu_i \phi^k_0 \,,$$

where $\phi^0_0 = \mu_k \phi^k_0$. Let E_i denote the linear frame of M dual to the co-frame ϕ^i_0. Then, $\nabla E_i = v^k_i E_k$. The Levi-Civita connection on Q satisfies $(\nabla^Q_u X)_{(e_n)} = d(X)_{e_n}(u) - < d(X)_{e_n}(u), e_n > e_n$, where $X \in C^\infty(TQ)$, $u \in T_{e_n}Q$, and, on the right-hand side, X is considered as a map from Q to $\mathbb{R}^{n+2}$. Let $\nabla^{Q'}$ denote the pullback connection on $\gamma_f^{-1}TQ$. We have $\nabla d\gamma_f(E_i, E_i) = \nabla^{Q'}_{E_i}(d\gamma_f(E_i)) - d\gamma_f(\nabla_{E_i}E_i)$. From (2.9), (2.12) and the equations

$$\left\{ \begin{array}{rcl} de_n(E_i) &=& p^n_i e_0 - h^n_{ki} e_k \\ de_0(E_i) &=& \mu_i e_0 + e_i \\ de_k(E_i) &=& \phi^0_k(E_i)e_0 + \phi^j_k(E_i)e_j + h^n_{kj}e_n + \delta_{ki}e_{n+1} \,, \end{array} \right. \tag{4.4}$$

we get

$$d(d\gamma_f(E_i))(E_i) = (-m+2)p^n_i e_i - \mu_i p^n_i e_0 + p^n_k \phi^k_i e_0 + p^n_{ii} e_0 - h^n_{kj}\phi^j_i(E_i)e_k + \mu_i h^n_{ki} e_k - h^n_{ki} h^n_{ki} e_n$$

and so

$$\nabla^{Q'}_{E_i}(d\gamma_f(E_i)) = (-m+2)p^n_i e_i - \mu_i p^n_i e_0 + p^n_k \phi^k_i(E_i) e_0 + p^n_{ii} e_0 - h^n_{kj}\phi^j_i(E_i)e_k + \mu_i h^n_{ki} e_k \,.$$

Hence, we obtain

$$\nabla d\gamma_f(E_i, E_i) = -(m-2)(p^n_k + \mu_i h^n_{ik})e_k + (p^n_{ii} + (m-2)\mu_i p^n_i)e_0 \,.$$

Therefore, for $\rho = \gamma_f$, (4.3) becomes

$$\begin{array}{rcl} (4.3) &=& \frac{m-2}{2}(\mathrm{Trace}\mathcal{N})^{\frac{m-4}{2}} d(\mathrm{Trace}\mathcal{N}) \otimes de_n(E_i, E_i) + (\mathrm{Trace}\mathcal{N})^{m-2}\nabla d\gamma_f(E_i, E_i) \\ &=& (m-2)(\mathrm{Trace}\mathcal{N})^{\frac{m-4}{2}} h^n_{st} h^n_{sti} p^n_i e_0 - (m-2)(\mathrm{Trace}\mathcal{N})^{\frac{m-4}{2}} h^n_{st} h^n_{sti} e_i + \\ && -(m-2)(\mathrm{Trace}\mathcal{N})^{\frac{m-2}{2}} p^n_k e_k + (\mathrm{Trace}\mathcal{N})^{\frac{m-2}{2}} p^n_{ii} e_0 \,. \end{array}$$

Consequently, since e_0, e_k are linearly independent, we conclude

Proposition 4.1 *Let $f : M \to S^n$ be an immersed hypersurface of the Möbius space. Then, the hyperbolic conformal Gauss map γ_f is a critical point of η iff*

$$\left\{ \begin{array}{ll} (m-2)(\mathrm{Trace}\mathcal{N})^{\frac{m-4}{2}} h^n_{st} h^n_{sti} p^n_i + (\mathrm{Trace}\mathcal{N})^{\frac{m-2}{2}} p^n_{ii} = 0 & \text{and} \\ (m-2)\left((\mathrm{Trace}\mathcal{N})^{\frac{m-4}{2}} h^n_{st} h^n_{sti} h^n_{ik} + (\mathrm{Trace}\mathcal{N})^{\frac{m-2}{2}} p^n_k\right) = 0 & \forall k = 1, \ldots, m \,. \end{array} \right. \tag{4.5}$$

The vanishing of the latter system is independent of the choice of the Darboux G-frame. Observe that, if $m = 2$ and $n = 3$, this system reduces to the equation $p^3_{ii} = 0$, which is the Euller-Lagrange equation of $\mathcal{W}$. Consequently, in this case γ_f is a critical point of η iff f is a critical point of $\mathcal{W}$. Now we analyse the case $m = n-1$ arbitrary. Let $f : M^{n-1} \to S^n$ be an immersion, such that γ_f is a critical point of the functional η. Then, following the computations in the proof of Theorem 1, we obtain for a variation f_t of f the equation (3.19), yielding

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \int_D \Big\{ d\Big(\frac{\|h\|^m}{m}(-1)^{k-1}\lambda^k_0 \phi^{1\ldots\hat{k}\ldots m}\Big) + d\Big((-1)^{j-1}\|h\|^{m-2}(\lambda^n_i h^n_{ij} - \lambda^n_0 p^n_j)\phi^{1\ldots\hat{j}\ldots m}\Big)$$
$$-\lambda^n_i\Big((m-2)\|h\|^{m-4} h^n_{ij} h^n_{st} h^n_{stj} + (m-2)\|h\|^{m-2} p^n_i\Big)\phi^{1\ldots m}$$
$$+\lambda^n_0\Big((m-2)\|h\|^{m-4} p^n_j h^n_{st} h^n_{stj} + \|h\|^{m-2} p^n_{jj} + \|h\|^{m-2} h^n_{ij} h^n_{ik} h^n_{kj}\Big)\phi^{1\ldots m}\Big\} .$$

Taking into account that the expressions given in (3.21) define tensors, the λ^α_i, λ^A_0 have compact support, and (4.5) holds, we obtain by using Stokes' theorem

$$\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\Big|_{t=0} = \int_D \|h\|^{m-2}\lambda^n_0 h^n_{ij} h^n_{ik} h^n_{kj}\phi^{1\ldots m} .$$

Hence, we conclude that f is a critical point of $\mathcal{W}$ iff (3.22) holds. Naturally, one can wonder if the converse is also true: if f is a Willmore submanifold satisfying (3.22), is then γ_f a critical point of η? This does not seem to be the case, because, in the above expression for $\frac{\partial}{\partial t}\mathcal{W}_D(f_t)\big|_{t=0}$, the λ^α_0 can be chosen arbitrarily, but not necessarilly the λ^α_i. Thus, we conclude

Theorem 2 *Let $f : M^m \to S^n$ be an immersion of an oriented hypersurface into the Möbius space $(m = n-1)$. Then,*
for $m = 2$, f is a Willmore immersed surface iff γ_f is a critical point of η [4]*;*
for $m > 3$, if γ_f is a critical point of η, then f is a Willmore hypersurface iff condition (3.22) *holds.*

We observe that it is possible to obtain exactly Theorem 2 for any $m < n$, by deriving (4.5) in such a more general context. Theorem 2 also shows that condition (3.22) looks quite natural. Moreover, it may have far-reaching geometrical consequences, as we will see in the next section on a conformal Bernstein-type theorem.

5 A Conformal Bernstein-type Theorem

In this section, we will formulate a Bernstein-type theorem for immersed Willmore hypersurfaces of the Möbius space, which generalises the special case of immersed surfaces in S^3 treated in [4].

Let $f : M^2 \to \mathbb{R}^3$ be an oriented Willmore surface immersed into the Euclidean 3-space. Let $\nu_f : M \to S^2$ be the spherical Gauss map. Let $\sigma_f : M \to \mathbb{R}^3$ be the map defined by

$$\sigma_f(p) = \nu_f(p) + Hf(p) .$$

Then, the following theorem can be formulated [4]:

Theorem ([4]) *Let $f : M^2 \to \mathbb{R}^3$ be a complete, oriented, immersed Willmore* *surface. If there exists an $a \in \mathbb{R}^3$ with $v = <\sigma_f, a>_{\mathbb{R}^3} \neq 0$ on M, then $f(M)$ is either a sphere or a plane.*

This theorem is the analogue of the weak form of the parametric Bernstein theorem, which states that a complete, oriented, minimal immersed surface $f : M^2 \to \mathbb{R}^3$, with spherical Gauss map lying in a hemisphere of S^3, is a plane. Furthermore, it was reformulated in the conformal geometry of surfaces of S^3 by the same author. Embedding $\mathbb{R}^3$ into S^3 through its cannonic immersion i (section 6), we may consider $f : M^2 \to S^3$ as an immersed Willmore surface into the Möbius 3-space. Let $E = [E_0, E_1, E_2, E_3, E_4] : M \to \mathbb{E}(3)$ be a Darboux frame along $f : M \to \mathbb{R}^3$. Let $\tilde{e} : M \to G$ be the Darboux G-frame constructed from E as described in section 6. Thus, $\tilde{e}_3 = E_3 + HE_0$ can be identifyed as σ_f. That is, σ_f corresponds to the hyperbolic conformal Gauss map γ_f of $f : M \to S^3$. The following theorem is the conformal version of the previous one:

Theorem ([4]) *Let $f : M \to S^3$ be a compact, connected, oriented Willmore surface with hyperbolic conformal Gauss map γ_f. If there exists an $a \in \mathbb{R}^5$ such that $<\gamma_f, a> \neq 0$ on M, then $f(M)$ is a 2-Möbius space.*

Now we derive a generalization of this theorem. Let $f : M^{n-1} \to S^n$ be an immersion of a hypersurface into the Möbius space, and let $\gamma_f : M \to Q$ be the hyperbolic conformal Gauss map of f. Observe that, if M is the Möbius space S^{n-1} and f is the inclusion map, then f is a trivial Willmore hypersurface and $\gamma_f = \eta_n$. In particular, $<\gamma_f, \eta_n> \neq 0$ on all M. The following theorem shows that this property, with an additional condition, characterises the hyperspheres of S^n.

Theorem 3 *Suppose $n \neq 4$ and $n \neq 6$. Let $f : M^{n-1} \to S^n$ be a compact, oriented, connected Willmore hypersurface immersed into S^n with hyperbolic conformal Gauss map γ_f. If there exists an $a \in \mathbb{R}^{n+2}$, such that $<\gamma_f, a> \neq 0$ on all M, and f satisfies condition* (3.22), *then $f(M)$ is an $(n-1)$-Möbius space.*

Proof. Set $m = n - 1$. Obviously, we may assume $<\gamma_f, a>$ positive on all M. Let $e : M \to G$ be a Darboux G-frame along f and let $\|h\| = \sqrt{\mathrm{Trace}\mathcal{N}} = \sqrt{h_{ij}h_{ij}}$. Consider the local $(m-1)$-form on M given by

$$\begin{aligned}\omega &= (-1)^{i-1}\|h\|^{m-2}\big((m-1)p_i^n < e_0, a > -h_{ik}^n < e_k, a > \big)\phi^{1\ldots\hat{i}\ldots m} \\ &\quad +(-1)^{i-1}(m-2) < e_0, a > \|h\|^{m-4}h_{ik}^n h_{st}^n h_{stk}^n \phi^{1\ldots\hat{i}\ldots m} .\end{aligned}$$

One can straightforwardly verify that ω is a well-defined global $(m-1)$-form on M. Using Eqs. (2.12), (2.13), (4.4), (3.18), (3.15) and (2.10), we have

$$\begin{aligned}d\omega &= < e_0, a > \big\{(m-1)\|h\|^{m-2}p_{ii}^n + 2(m-1)(m-2)\|h\|^{m-4}p_i^n h_{st}^n h_{sti}^n + \\ &\quad +(m-2)(m-4)\|h\|^{m-6}h_{ik}^n h_{st}^n h_{stk}^n h_{uv}^n h_{uvi}^n + (m-2)\|h\|^{m-4}h_{ik}^n H_{ik}\big\}\phi^{1\ldots m} \\ &\quad -\|h\|^m < \gamma_f, a > \phi^{1\ldots m} .\end{aligned}$$

If f is a Willmore hypersurface, then, using the Euler-Lagrange equation derived in Theorem 1, we obtain

$$d\omega = -(< e_0, a > \|h\|^{m-2}h_{ij}^n h_{jk}^n h_{ki}^n + \|h\|^m < \gamma_f, a >)\phi^{1\ldots m} ,$$

which is a global m-form on M. Now, since f satisfies, by assumption, condition (3.22), application of Stokes' theorem yields

$$0 = \int_M d\omega = -\int_M \|h\|^m < \gamma_f, a > \phi^{1\ldots m} .$$

As $<\gamma_f, a>$ is positive on all M, necessarily $\sqrt{\sum_{ij}(h^n_{ij})^2} = 0$. Therefore, $f(M)$ is an $(n-1)$-sphere. □

By slightly modifying the above proof, we get the following result:

Theorem 4 *Suppose $n \neq 4$. Let $f : M^{n-1} \to S^n$ be a compact, oriented, connected immersed hypersurface into S^n with hyperbolic conformal Gauss map $\gamma_f : M \to Q$. If γ_f is a critical point of the functional η given in (4.2)and if there exists an $a \in \mathbb{R}^{n+2}$, such that $< \gamma_f, a > \neq 0$ on all M, then $f(M)$ is an $(n-1)$-sphere.*

Proof. Set $m = n-1$, and let $e : M \to G$ as in the proof of Theorem 3. Consider the local $(m-1)$-form on M given by

$$\omega = (-1)^{i-1}\|h\|^{m-2}(p^n_i < e_0, a > -h^{n}_{ik} < e_k, a >)\phi^{1\ldots\hat{i}\ldots m} .$$

We can easily verify that ω is a well-defined global $(m-1)$-form on M, and that

$$\begin{aligned} d\omega =& < e_0, a > \left((m-2)\|h\|^{m-4}p^n_i h^n_{st}h^n_{sti} + \|h\|^{m-2}p^n_{ii}\right)\phi^{1\ldots m} \\ &+ < e_i, a > (2-m)\left(\|h\|^{m-2}p^n_i + \|h\|^{m-4}h^n_{ik}h^n_{st}h^n_{stk}\right)\phi^{1\ldots m} - \|h\|^m < \gamma_f, a > \phi^{1\ldots m}. \end{aligned}$$

Since γ_f is a critical point of η, (4.5) holds, i.e

$$d\omega = -\|h\|^m < \gamma_f, a > \phi^{1\ldots m} .$$

Now the conclusion follows as in the proof of Theorem 3. □

6 Embedding $\mathbb{R}^n$ into S^n

We can embed the Euclidean space $\mathbb{R}^n$ into the Möbius space S^n through the map

$$\begin{array}{cccc} i: & \mathbb{R}^n & \hookrightarrow & S^n \\ & v & \longrightarrow & \begin{bmatrix} 1 \\ v \\ \frac{1}{2}\|v\|^2 \end{bmatrix}_{\sim} \end{array} .$$

Under the identification of the Euclidean n-sphere with S^n through the map T of section 2, the present map i corresponds to the stereographic projection from $\mathbf{x_0}$. We can identify $\mathbb{R}^n$ with the homogeneous space

$$\mathbb{R}^n \cong S^n \backslash \{\mathbf{x}_\infty\} \cong \mathbb{E}(n)/\mathbb{E}_0(n),$$

where

$$\mathbb{E}(n) = \left\{ [Z, A] = \begin{bmatrix} 1 & 0 & 0 \\ Z & A & 0 \\ \frac{1}{2}{}^tZZ & {}^tZA & 1 \end{bmatrix} : \begin{array}{c} A \in SO(n) \\ Z \in \mathbb{R}^n \end{array} \right\}$$

is a Lie subgroup of G and is isomorphic to the Lie group $\mathbb{R}^n \times SO(n)$ of rigid motions of $\mathbb{R}^n$, and where the isotropic subgroup at the origin $\mathbb{E}_0(n)$ is identified with $SO(n)$. $\mathbb{R}^n \times SO(n)$ acts on $v \in \mathbb{R}^n$ as $\mathbb{E}(n)$ acts on $i(v)$. The canonic projection $\bar{\Pi} : \mathbb{E}(n) \to \mathbb{R}^n$ is given by $\bar{\Pi}([Z, A]) = Z$. The Maurer-Cartan form of $\mathbb{E}(n)$ is given by $\Psi = j^*\Phi$, where $j : \mathbb{E}(n) \to G$ is the inclusion map, and its components satisfy $\Psi^A_0 = \Psi^{n+1}_A$, $\Psi^A_B = -\Psi^B_A$, $\Psi^a_b = 0$ otherwise. A Darboux $\mathbb{E}(n)$-frame $E : M \to \mathbb{E}(n)$ along an immersion $F : M \to \mathbb{R}^n$ of an m-manifold is a map, defined on an open set of M, satisfying $\bar{\Pi} \circ E = F$ and $\psi_0^\alpha = 0, \forall m+1 \leq \alpha \leq n$, where $\psi = E^*\Psi$. Then, ψ satisfies the structure equations,

$$d\psi_0^i = -\psi_j^i \wedge \psi_0^j$$
$$d\psi_j^i = -\psi_k^i \wedge \psi_j^k + \bar{\Omega}_j^i, \quad \text{with } \bar{\Omega}_j^i = -\psi_\alpha^i \wedge \psi_j^\alpha .$$

Differentiating $\psi_0^\alpha = 0$ and Cartan's lemma yield $\psi_i^\alpha = \bar{h}_{ij}^\alpha \psi_0^j$ with $\bar{h}_{ij}^\alpha = \bar{h}_{ji}^\alpha$. The functions $\bar{h}_{ij}^\alpha$ are the coefficients of the second fundamental form of the immersion F with respect to the frame E. If we assume E to be of the form $E = \rho \circ F$, where $\rho : I\!R^n \to I\!E(n)$ is a local section of $\bar{\Pi}$, then $\psi = E^*\Psi = F^*(\rho^*\Psi)$. It is easy to see that $dt^2 = \sum_{A=1,\ldots,n}(\rho^*\Psi_0^A)^2$ defines the usual Euclidean metric on $I\!R^n$, by taking for example the particular case where ρ is given by $\rho(w) = [w, I]$, with $w \in I\!R^n$. Then, $d\ell^2 = \sum_{i=1,\ldots,m}(\psi_0^i)^2 = F^*dt^2$ is the metric of M induced by f from the metric dt^2 of $I\!R^n$. If we take $U_1, \ldots, U_n$ the orthonormal frame of $(I\!R^n, dt^2)$ dual to $\rho^*\Psi_0^1, \ldots, \rho^*\Psi_0^n$, then $U_i = dF(X_i)$, with $X_1, \ldots, X_m$ the local orthonormal frame of $(M, d\ell^2)$ dual to $\psi_0^1, \ldots, \psi_0^m$, $U_{m+1} \circ F, \ldots, U_n \circ F$ a local orthonormal frame of the normal bundle V, and $< \nabla dF(X_i, X_j), U_\alpha >= \bar{h}_{ij}^\alpha = \psi_i^\alpha(X_j)$. We also remark that two Darboux frames E, $\tilde{E} : M \to I\!E(n)$ must satisfy $\tilde{E} = EK$, where $K : M \to I\!E_0(n)$ takes values in the subgroup $SO(m) \times SO(n-m)$.

Now we set $f = i \circ F : M \to S^n$. We will construct a Darboux G-frame $\tilde{e} : M \to G$ along f from a Darboux $I\!E(n)$-frame $E : M \to I\!E(n)$. This construction allows us to compare the Riemannian Geometry of F with the Conformal Geometry of f. Set $e = j \circ E : M \to G$ and $\phi = e^*\Phi$. Then, $\Pi \circ e = (i \circ \bar{\Pi} \circ j^{-1}) \circ (j \circ E) = i \circ \bar{\Pi} \circ E = i \circ F = f$, i.e e is a zeroth-order G-frame along f. Moreover, $\phi = E^*(j^*\Phi) = E^*\Psi = \psi$, and so e is a first-order G-frame along f with $h_{ij}^\alpha = \bar{h}_{ij}^\alpha$. Thererfore, $\tilde{e} = eK$, where

$$K = \begin{bmatrix} 1 & 0 & {}^tY & \frac{1}{2}{}^tYY \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_{n-m} & Y \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and where $Y^\beta = \frac{1}{m}h_{kk}^\beta = H^\beta$ is the β-component of the mean curvature H of F, gives a Darboux G-frame along f. Summarising, we have

$$[\tilde{e}_0, \tilde{e}_i, \tilde{e}_\alpha, \tilde{e}_{n+1}] = [E_0, E_i, E_\alpha + H^\alpha E_0, \tfrac{1}{2}H^\alpha H^\alpha E_0 + H^\alpha E_\alpha + E_{n+1}]$$
$$\tilde{\phi}_0^0 = 0; \quad \tilde{\phi}_0^i = \psi_0^i; \quad \tilde{\phi}_i^0 = H^\alpha(\tfrac{1}{2}H^\alpha\delta_{ij} - h_{ij}^\alpha)\psi_0^j = -H^\alpha(\psi_i^\alpha + \tfrac{1}{2}H^\alpha\psi_0^i)$$
$$\tilde{\phi}_\alpha^0 = dH^\alpha - H^\beta\psi_\alpha^\beta; \quad \tilde{\phi}_0^\alpha = \psi_0^\alpha = 0; \quad \tilde{\phi}_j^i = \psi_j^i$$
$$\tilde{\phi}_i^\alpha = (h_{ij}^\alpha - H^\alpha\delta_{ij})\psi_0^j; \quad \tilde{\phi}_\beta^\alpha = \psi_\beta^\alpha$$

If E is of the type $E = \rho \circ F$, where $\rho = I\!R^n \to I\!E(n)$ is a local section of $\bar{\Pi} : I\!E(n) \to I\!R^n$, then, denoting by ∇ and $\nabla^\perp$ the Levi-Civita connections of $(M, d\ell^2)$ and V respectively, and by $\nabla dF \in \odot^2 T^*M \otimes V$ the second fundamental form of F, we have $\tilde{h}_{ij}^\alpha = -\delta_{ij}H^\alpha + h_{ij}^\alpha$, and so $p \in M$ is an umbilic point iff $(\nabla dF)_p^\alpha = H_p^\alpha d\ell_p^2$, iff $\tilde{h}_{ij}^\alpha(p) = 0$. Also, we conclude that

$$\psi_j^i(X_k) = < \nabla_{X_k}X_j, X_i >_{dt^2}; \qquad \psi_j^\alpha(X_k) = h_{kj}^\alpha = < \nabla dF(X_k, X_j), U_\alpha >_{dt^2}$$
$$< H, U_\alpha >_{dt^2} = \frac{1}{m}h_{kk}^\alpha; \qquad \psi_\alpha^\beta(X_k) = < \nabla_{X_k}^\perp U_\alpha, U_\beta >_{dt^2}$$

$$\tilde{\phi}_k^0(X_i) = - < H, \nabla dF(X_i, X_k) >_{dt^2} + \tfrac{1}{2}\delta_{ik}\|H\|^2$$
$$\tilde{h}_{ij}^{\alpha} = < -\delta_{ij}H + \nabla dF(X_i, X_j), U_\alpha >_{dt^2}$$
$$\mathrm{Trace}\,\tilde{\mathcal{N}} = \tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\alpha} = \|\nabla dF\|^2 - m\|H\|^2 = \|\nabla dF - Hg\|^2$$
$$\tilde{h}_{kj}^{\alpha}\tilde{h}_{ji}^{\beta}\tilde{h}_{ik}^{\beta} = < \tilde{B} - 2\tilde{A}(H) + 2m\|H\|^2 H - \|\nabla df\|^2 H, U_\alpha >_{dt^2},$$

where

$$\tilde{A}(H) = < H, \nabla dF(X_i, X_r) >_{dt^2} \nabla dF(X_i, X_r)$$
$$\tilde{B} = < \nabla dF(X_i, X_j), \nabla dF(X_i, X_k) >_{dt^2} \nabla dF(X_k, X_j)\ .$$

If $m = 2$, $\tilde{\mathcal{N}} = \|H\|^2 d\ell^2 - \mathrm{Ricci} = (\|H\|^2 - K)d\ell^2 = \frac{1}{2}\mathrm{Trace}\,\tilde{\mathcal{N}}(\psi_0^1 \otimes \psi_0^1 + \psi_0^2 \otimes_0^2)$. Applying (2.9), (2.11), (2.12) and (2.13), we get, respectively,

$$\tilde{h}_{ijk}^{\alpha} = < \nabla_{X_k}\nabla dF(X_i, X_j), U_\alpha >_{dt^2}; \qquad \tilde{p}_k^{\alpha} = < \nabla_{X_k}^{\perp} H, U_\alpha >_{dt^2};$$
$$\tilde{p}_{ik}^{\alpha} = \Big\langle \nabla^{\perp 2} H(X_k, X_i) - < \nabla dF(X_k, X_i), H >_{dt^2} H + \tfrac{1}{2}\delta_{ik}\|H\|^2 H +$$
$$+ < \nabla dF(X_k, X_r), H >_{dt^2} \nabla dF(X_i, X_r) - \tfrac{1}{2}\|H\|^2 \nabla dF(X_i, X_k)\ ,\ U_\alpha \Big\rangle_{dt^2},$$

and, in particular,

$$\tilde{p}_{ii}^{\alpha} = < \Delta H - m\|H\|^2 H + \tilde{A}(H), U_\alpha >_{dt^2}\ .$$

Similarly, we may identify the n-hyperbolic space H^n with an open submanifold of the Möbius space. Also, we may compare the Riemannian Geometry of S^n with its Conformal Geometry.

References

[1] W. Blaschke, *Vorlesungen über Differentialgeometrie III*, Springer Berlin (1929)

[2] R.L. Bryant, *A Duality Theorem for Willmore Surfaces*, J. Diff. Geom. **20**, 23-53 (1984)

[3] B.Y. Chen, *On a Variational Problem on Hypersurfaces*, J. London Math. Soc. **6** (2), 321-325 (1973)

[4] M. Rigoli, *The Conformal Gauss Map of Submanifolds of the Möbius Space*, Ann. Global Anal. Geom. **5**, (1987)

[5] I.M.C. Salavessa, *Graphs with Parallel Mean Curvature and a Variational Problem in Conformal Geometry*, Ph.D. Thesis, University of Warwick (1988)

[6] Ch. Schiemangk and R. Sulanke, *Submanifolds of the Möbius space*, Math. Nachr. **96**, 165-183 (1980)

[7] J.H. White, *A Global Invariant of Conformal Mappings in Space*, Proc. Am. Math. Soc. **38**, 162-164 (1973).

[8] T.J. Willmore, *Minimal Conformal Immersions*, Lect. Notes in Math. **392**, (1974), Géométrie Différentielle, Coll. Santiago de Compostela 1972

Dipartimento di Matematica "Federigo Enriques "
Università di Milano
Via C. Saldini, 50
20133 MILANO, Italy

Centro de Física da Matéria Condensada
Av. Prof. Gama Pinto, 2
1699 LISBOA Codex, Portugal

(Received May 27, 1993)

manuscripta math. 81, 223 - 224 (1993)

manuscripta mathematica

ERRATUM TO
ON CONTINUOUS DYNAMICS OF AUTOMORPHISMS OF $\mathbf{C}^2$

Chiara de Fabritiis

In the proof of Theorem 1.4 of [1] a couple of coefficients was missing by a misprint. This sort of misprint does not change the final results of the paper. To recover the missing coefficients substitute the definition of E_1 with

$$E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cx + f(y) \\ c^{-1}y + \beta \end{pmatrix}, \;\middle|\; c \in \mathbf{C}^*,\ \beta \in \mathbf{C},\ \ f \in \mathrm{Hol}(\mathbf{C}, \mathbf{C}) \right\}.$$

(the coefficient c was missing). Then the intersection $A_1 \cap E_1$ is given by

$$A_1 \cap E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} cx + \alpha y + a \\ c^{-1}y + \beta \end{pmatrix}, \;\middle|\; c \in \mathbf{C}^*,\ \alpha, \beta, a \in \mathbf{C} \right\}.$$

In order to prove that $L_1 \cap L_3 = \{0\}$, in the proof of of Theorem 1.4 of [1] we must take

$$g_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + f_1(y) \\ y \end{pmatrix},\ g_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx + \alpha y \\ \beta x + dy \end{pmatrix} \text{ and } g_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + f_3(y) \\ y \end{pmatrix},$$

where f_1 and f_3 are non-linear elements of $\mathcal{H}_0(\mathbf{C})$, $cd - \alpha\beta = 1$ and $\beta \neq 0$ (the coefficients c and d were missing). Then the correct expression for $g_3 \circ g_2 \circ g_1$ is given by

$$g_3 \circ g_2 \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} = g_3 \begin{pmatrix} c(x + f_1(y)) + \alpha y \\ \beta x + \beta f_1(y) + dy \end{pmatrix} = \begin{pmatrix} c(x + f_1(y)) + \alpha y + f_3(\beta x + \beta f_1(y) + dy) \\ \beta x + \beta f_1(y) + dy \end{pmatrix}.$$

If $\langle x, y \rangle \cap \langle g_3 \circ g_2 \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_3 \circ g_2 \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2 \rangle \neq \{0\}$ there exist $\gamma, \delta \in \mathbf{C}$ such that $|\gamma| + |\delta| > 0$ and

$$\gamma[c(x + f_1(y)) + \alpha y + f_3(\beta x + \beta f_1(y) + dy)] + \delta[\beta x + \beta f_1(y) + dy]$$

is linear in x and y. Then $(c\gamma + \delta\beta) f_1(y) + \gamma f_3(\beta x + dy + \beta f_1(y))$ is linear in x and y, and therefore taking the derivative with respect of x we find that $\gamma\beta f_3'(\beta x +$

$\beta f_1(y)+dy)$ is constant. As f_3 is non-linear and $\beta x+\beta f_1(y)+dy$ is non-constant, then $\gamma\beta = 0$, and since $\beta \neq 0$, $\gamma = 0$. Thus $\delta\beta f_1(y)$ is linear in x and y against the fact that $\delta\beta \neq 0$ and f_1 is non-linear.

Proposition 2.6 of [1] must be replaced by the new

Proposition 2.6. *All the one-parameter groups in E_1 are expressed (up to conjugation) by*

$$\Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}e^{ta}x+f_t(y)\\e^{-ta}y\end{pmatrix}\qquad \Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+f_t(y)\\y+ts\end{pmatrix},$$

where $a \in \mathbf{C}$ and in the first case f_t satisfies $f_{t+\tau}(y) = e^{\tau a}f_t(y) + f_\tau(e^{ta}y)$, while in the second it satisfies f_t satisfies $f_{t+\tau}(y) = f_t(y + \tau s) + f_\tau(y)$.

Proof. The fact that $\Phi_t\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\alpha_t x+f_t(y)\\\alpha_t^{-1}y+\beta_t\end{pmatrix}$ satisfies the composition rule is equivalent to the fact that $\alpha_t^{-1}y+\beta_t$ is a one parameter group of affine transformations of $\mathbf{C}$, hence it can be conjugated to obtain $y \mapsto e^{ta}y$ or $y \mapsto y+t$, the relation on f follows immediately. □

With these replacements all other statements and proofs remain as they are.

[1] C. de Fabritiis, On continuous dynamics of automorphisms of $\mathbf{C}^2$, Manuscripta Mathematica, 77, 337-359 (1992)

Chiara de Fabritiis
Scuola Internazionale Superiore di Studi Avanzati
Via Beirut 2/4
34014, Trieste—Italy
E.mail FABRITII@SISSA.bitnet

(Received January 25, 1993;
in revised form October 17, 1993)

manuscripta math. 81, 225 - 242 (1993)

manuscripta mathematica

MODULI SPACES FOR FUNDAMENTAL GROUPS AND LINK INVARIANTS DERIVED FROM THE LOWER CENTRAL SERIES

Martin MARKL and Ştefan PAPADIMA

We consider an algebraic parametrization for the set of (Mal'cev completed) fundamental groups of the spaces with fixed first two Betti numbers, having in mind applications in low-dimensional topology and especially in link theory. The factor set of (restricted) isomorphism types of these groups acquires the structure of a 'moduli space', giving rise to invariants which, in the case of links, detect the isotopy type. We indicate two methods of computation for these invariants. We also prove a rigidity result for the associated graded Lie algebra of the fundamental group. A lot of examples are given.

Introduction

The methods of rational homotopy theory offer the possibility of describing moduli spaces for homotopy types of spaces with prescribed invariants (such as ranks of homology/homotopy groups, cohomology algebra/homotopy Lie algebra) in an algebraic reasonably manageable form. We shall focus here on the variation of the fundamental group with fixed first two Betti numbers, with an eye for link theory (where the Betti numbers are entirely determined by the number of knots, as is well known).

Our enterprise was prompted by the observation that natural geometric equivalence relations, such as link isotopy, translate nicely on the algebraic side, thus making possible *the definition of invariants* landing in *convenient* moduli spaces. Here by convenient we mean moduli spaces endowed with a simple decision algorithm (inductive and linear at each step) allowing us to answer the basic question: are the invariants of two given objects the same, up to some (arbitrarily prescribed) degree of approximation?

Our notion of equality up to k-th order approximation is given by the existence of a (special kind of) isomorphism between Mal'cev completions of the nilpotent quotients of the corresponding fundamental groups by k-fold commutators. The relevant algebraic moduli space turns out to be of the form M/U, where M is a rational representation of the unipotent group U. A bijection between topological and algebraic 'moduli spaces' is established in our main result (Theorem 1.2). The algebraic decision algorithm is presented in 1.8. For a comparison between our algorithm and the general isomorphism problem for nilpotent groups, see Example 1.7.

The bridge between topology and algebra is supplied by Chen's [8] formal power series connection approach to π_1 (1.3 and 1.5), which allows us to inductively construct the Lie power series expansion of the algebraic invariants $\partial \in M$ by a differential form method.

The initial terms of the series expansion of ∂ give rise to well-defined numerical invariants defined on M/U (by unipotency). In the second section we use the duality between differentials of differential graded Lie algebra models and Massey products [29, Ch. V] to compute initial terms with the aid of first (possibly) nonvanishing Massey products (Theorem 2.1). Combining this with the easy part of the Porter-Turaev theorem [25], we obtain for links an explicit formula for the initial terms, involving Milnor's $\overline{\mu}$-invariants (Corollary 2.2).

In the next section we indicate how the computation of initial terms may be used to obtain useful information on related traditional invariants associated to the lower central series. The main result (Theorem 3.1) says that when the initial terms (which are homogeneous elements of a certain free graded Lie algebra) satisfy a certain independence condition called 'inertia' (see [19, 2, 16]) the graded Lie algebra associated to the lower central series is rigid, i.e. it does not depend on the Taylor rest of our Lie power series expansion of the ∂-invariants. Related results can be found in [19, 14, 1, 22].

The ∂-invariants may be used, via 1.8, to distinguish isotopy types of links. For this it is necessary in general to go beyond initial terms (Example 4.5). Replacing forms by submanifolds Hain [14] has developed a 'stable intersection calculus' for the computation of initial terms. In the final section we show by several examples how his method may be combined, in certain situations, with disjoint support arguments, to obtain explicit computations for higher terms, too (see e.g. Proposition 4.1).

The above results were presented to the 11th Winter School 'Geometry and Physics', January 1991, Srní, Czechoslovakia, in a series of lectures by both authors together with B. Berceanu (cf. the announcement [21]). Further results along similar lines (also including 3-manifolds) are to be found in [3, 4].

0. Conventions

All spaces will be supposed to be connected with finite Betti numbers. All algebraic objects, if not stated otherwise, will be defined over the field $\mathbb{Q}$ of rationals. For a vector space V, let $\#V$ denote the dual, $\#V = \mathrm{Hom}(V, \mathbb{Q})$. For a graded vector space W_*, $\uparrow W_*$ (resp. $\downarrow W_*$) will denote the suspension (resp. desuspension), i.e. the graded vector space with $(\uparrow W)_q = W_{q-1}$ (resp. $(\downarrow W)_q = W_{q+1}$) for all q. We will systematically use the Koszul sign convention, that is we introduce the sign $(-1)^{pq}$ whenever we commute two objects of degrees p and q. The symbol $cl(z)$ will denote the (co)homology class of the (co)cycle z.

1. Moduli spaces for marked fundamental groups

Let $V_* = \oplus_{i\geq 1} V_i$ be a finite type graded vector space. We shall consider various categories of *homologically marked* objects associated to V_*. For example an *h_*-marked* space S is a space together with a degree zero linear map $\mu : V_* \to H_*(S;\mathbf{Q})$ (*marking*). A morphism $f : (S_1, \mu_1) \to (S_2, \mu_2)$ of h_*-marked spaces is a continuous map $f : S_1 \to S_2$ such that $f_*\mu_1 = \mu_2$, where f_* is the induced map on $\mathbf{Q}$-homology. Having fundamental groups in mind, we shall be mostly concerned with *$h_{1,2}$-markings* and continuous maps which are compatible with them. The definitions are a particular case of the above ones, namely the case when $V_i = 0$ for any $i > 2$.

Fix then $V_1 =: X$ and $V_2 =: Y$ and consider the set $\mathscr{S}$ of *$h_{1,2}$-marked weak equivalence classes* of spaces, that is the set of $h_{1,2}$-marked spaces S whose markings give *isomorphisms* $X \cong H_1(S;\mathbf{Q})$ and $Y \cong H_2(S;\mathbf{Q})$, modulo the equivalence relation generated by *elementary $h_{1,2}$-marked weak equivalence*, i.e. the existence of an $h_{1,2}$-marked map $f : S \to S'$ such that $H_i(f;\mathbf{Q})$ is an isomorphism for any i.

Similarly we may consider the category of *$h_{1,2}$-marked Lie algebras*, whose objects are Lie algebras E together with degree zero linear marking maps $\mu : V_1 \oplus V_2 \to H_1(E;\mathbf{Q}) \oplus H_2(E;\mathbf{Q})$ and whose morphisms $f : (E,\mu) \to (E',\mu')$ are the Lie algebra maps $f : E \to E'$ with the property that $f_*\mu = \mu'$. Denote by $\mathscr{E}$ the set of isomorphisms classes of objects of this category. One may also consider the category of *towers of $h_{1,2}$-marked Lie algebras*, i.e. the category of inverse systems of $h_{1,2}$-marked Lie algebras indexed by the positive integers. Denote by $\mathscr{T}$ the associated set of isomorphism classes of objects. For any $k \geq 1$ there is an obvious projection functor pr_k, which associates to a tower of marked Lie algebras $(E_n)_{n\geq 1}$ the marked Lie algebra E_k, which induces a set map $\mathrm{pr}_k : \mathscr{T} \to \mathscr{E}$.

One may associate to an isomorphically $h_{1,2}$-marked space S the tower of $h_{1,2}$-marked nilpotent Mal'cev groups $((\pi_1 S/\Gamma_k \pi_1 S)\otimes \mathbf{Q})_{k\geq 2}$. Here Γ_k = the k-th term of the lover central series and $(-)\otimes\mathbf{Q}$ = the Mal'cev completion, as in [26, Appendix A]. The homological marking of the tower is induced from S via $S \to K(\pi_1 S, 1)$. The relevant definitions pertaining to (towers of) $h_{1,2}$-marked groups are exactly as above – just replace 'Lie algebra' by 'group'. One may easily verify that the isomorphism type of this associated tower of $h_{1,2}$-marked groups remains unchanged under elementary $h_{1,2}$-marked weak equivalence [28]. Given the categorial equivalence between nilpotent Mal'cev groups and nilpotent Lie algebras ([26]) one may view the above tower as a tower of $h_{1,2}$-marked Lie algebras (note that the h_1-marking of each Lie algebra of the tower will be an isomorphisms). By taking the isomorphism class of this marked Lie tower we have thus constructed a well-defined map $P : \mathscr{S} \to \mathscr{T}$, together with maps $P_k : \mathscr{S} \to \mathscr{E}$, $P_k = \mathrm{pr}_k \circ P$, for any $k \geq 1$.

Define *the set $\mathscr{P}$ of marked Mal'cev completions of fundamental groups of spaces with fixed first two Betti numbers* ($b_1 = \dim X$, $b_2 = \dim Y$) by $\mathscr{P} = P(\mathscr{S})$, and likewise the nilpotent versions $\mathscr{P}_k = P_k(\mathscr{S})$, for any k.

1.1. *Example.* The above approach provides a very natural framework for handling the rational invariants coming from the lower central series of link groups. This is due to the existence of natural homology bases for the link complement. Our specific choice here will be the following: given a link $L \subset \mathbf{S}^3$, whose components are *numbered* (say, by $0,\ldots,m$) and *oriented*, the duals of oriented Seifert surfaces $\sigma_0,\ldots,\sigma_m$ which span L ($\partial\sigma_i = L_i$) will provide a basis for $H^1(\mathbf{S}^3 \setminus L)$, while the duals of oriented arcs $\alpha_1, \ldots, \alpha_m$ with α_k joining the 0th and kth component, will give a basis for $H^2(\mathbf{S}^3 \setminus L)$, as in [14]. There is thus an induced h_*-marking, and the point is that (*order and orientation preserving*) isotopy obviously does not affect the $h_{1,2}$-marked weak type of the link complement.

We shall now define the algebraic counterparts of $\mathscr{P}$ and $\mathscr{P}_k$. To this end, consider the free bigraded Lie algebra $\mathsf{L}_*^* \overset{\text{def.}}{=} \mathsf{L}(X \oplus Y)$; the lower degrees come from setting $|X| = 0$ and $|Y| = 1$ and the upper degree = bracket length. There is an associated free complete graded Lie algebra $\widehat{\mathsf{L}}_*$ of formal Lie series, obtained from (lower) degreewise completion with respect to the filtration $F_k\mathsf{L} = \mathsf{L}^{\geq k}$. Set $\widehat{M} = \mathrm{Der}_{-1}^{+}\widehat{\mathsf{L}}$, the complete vector space of continuous derivations ∂ of $\widehat{\mathsf{L}}$, which are homogeneous of lower degree -1 and have the property $\partial F_k\widehat{\mathsf{L}} \subset F_{k+1}\widehat{\mathsf{L}}$, $\forall k$, filtered by $F_s\widehat{M} = \{\partial \mid \partial F_k\widehat{\mathsf{L}} \subset F_{k+s}\widehat{\mathsf{L}}, \forall k\}$.

Taking the restriction to the free generators, one may think of $\partial \in \widehat{M}$ in the more familiar form of a series $\partial = \partial_2 + \partial_3 + \cdots$, $\partial_i \in \mathrm{Hom}(Y, \mathsf{L}^i X)$, also noticing that $\partial \in F_s\widehat{M} \iff \partial_{\leq s} = 0$. Consider next the group U of continuous degree zero Lie algebra automorphisms u of $\widehat{\mathsf{L}}_*$ with the property that $\mathrm{gr}^1 u = \mathrm{id}$ (with respect to the filtration $F_k\widehat{\mathsf{L}}$). It is a prounipotent group which acts by conjugation on the filtered vector space $\widehat{M}$. Setting $\mathscr{M} = \widehat{M} \bmod U$ and $\mathscr{M}_k = \widehat{M}/F_k\widehat{M} \bmod U$, we thus obtain a canonical projection between *algebraic moduli spaces*, $\mathrm{pr}_k : \mathscr{M} \to \mathscr{M}_k$, for any $k \geq 2$.

The bridge between topology and algebra will be provided by the map $T : \mathscr{M} \to \mathscr{T}$, to be constructed as follows. Given $\partial \in \widehat{M}$, a tower of $h_{1,2}$-marked nilpotent Lie algebras $(E_k) \overset{\text{def.}}{=} T(\partial)$ may be described by: $E_k = \mathsf{L}X/\mathscr{J}_k + \mathsf{L}^{\geq k}X$, where $\mathscr{J}_k =$ ideal generated by $\partial_{<k}Y$, $E_{k+1} \to E_k$ is the canonical projection, the (isomorphic) h_1-marking is induced by the inclusion $X \subset \mathsf{L}X$ via the identification $H_1E_k = E_k/[E_k, E_k]$, and the h_2-marking is given by $\partial_{\leq k} : Y \to \mathscr{J}_k + \mathsf{L}^{\geq k}X/[\mathsf{L}X, \mathscr{J}_k] + \mathsf{L}^{\geq k+1}X$, via Hopf's formula for H_2E_k, see e.g. [18]. If $u\partial = \partial' u$ for some $u \in U$, it is easy to see that the restriction $u : \widehat{\mathsf{L}}X \xrightarrow{\sim} \widehat{\mathsf{L}}X$ induces a tower isomorphism $T(\partial) \xrightarrow{\sim} T(\partial')$, which is also compatible with the markings, due to the fact that $\mathrm{gr}^1 u = \mathrm{id}$, whence T is well-defined. It is equally easy to notice that $\partial' \equiv \partial \bmod F_k\widehat{M}$ implies that $\partial'_{\leq k} = \partial_{\leq k}$, therefore T also induces $T_k : \mathscr{M}_k \to \mathscr{E}$, such that $\mathrm{pr}_k \circ T = T_k \circ \mathrm{pr}_k$, for any k. Our main result in this section is

Theorem 1.2. *For any k, there is an induced bijection*

$$T_k : \mathscr{M}_k \xrightarrow{\sim} \mathscr{P}_k.$$

The proof will rely upon a variation on the theme of the construction of the commutative differential graded algebra of cochains on a given dgl (differential graded Lie algebra) (see e.g. [29, I.1]) involving complete free Lie algebras which are not necessarily 1-connected. Let then $W_* = \oplus_{i \geq 0} W_i$ be a finite type graded vector space. Set $\mathsf{L}_*^* =$ free (bi) graded Lie algebra on W and $\widehat{\mathsf{L}}_* =$ free complete graded Lie algebra on W (where completion is taken with respect to upper degree = bracket length). By a *complete minimal dgl* we shall mean a dgl of the form $(\widehat{\mathsf{L}}, \partial)$, where ∂ is a continuous degree -1 graded Lie derivation with $\partial^2 = 0$, and $\partial F_k\widehat{\mathsf{L}} \subset F_{k+1}\widehat{\mathsf{L}}$, $\forall k$ (minimality). Define a bigraded vector space Z_*^* by $Z_i^j = \#\widehat{\mathsf{L}}_{j-1}^{i+1}$, $\forall i \geq 0, j \geq 1$, and consider the free (bi)graded algebra $\wedge Z^*_*$. Noting that $\dim Z^j_i < \infty$, for any i, j, vector space duality may be safely used to construct two (upper) degree 1 graded algebra derivations of $\wedge Z^*$, l and q. On the free generators: $l = \#\partial : Z \to Z$ and $q = \#[-,-] : Z \to \wedge^2 Z$. Setting $d = l + q$, one verifies that $d^2 = 0$ as in [29], hence we have the connected commutative differential graded algebra (dga) of cochains on $(\widehat{\mathbb{L}}, \partial)$, $\mathscr{C}^*(\widehat{\mathbb{L}}, \partial) \stackrel{\text{def.}}{=} (\wedge Z^*, d)$. The construction is functorial with respect to continuous dgl maps. One has that $Z^*_0 \subset \mathrm{Ker} d$ by minimality. Finally it is not hard to see that the canonical map $Z^*_0 \to \overline{H}^*(\wedge Z, d)$ is an isomorphism (compare [29, II.8(9)]).

Let $V_* \to \overline{H}_*(S; \mathbb{Q})$ be an isomorphic h_*-marking, and endow S with the induced isomorphic $h_{1,2}$-marking. Let $(\widehat{\mathbb{L}}(\downarrow V_*), \partial)$ be a complete minimal dgl, giving rise by restriction to $\widehat{\mathbb{L}}(X \oplus Y)$ to $\partial_| \in \widehat{M}$. Consider also Sullivan's deRham dga of S, $A^*_{PL}(S)$, together with the isomorphism $H^*(A_{PL}(S)) \xrightarrow{\sim} H^*(S; \mathbb{Q})$ given by integration, see [28, §7]. Use it and the h_*-marking to identify $\overline{H}^*(A_{PL}(S))$ with $\#V^*$, and also recall the canonical identification of $\overline{H}^*\mathscr{C}^*(\widehat{\mathbb{L}}, \partial)$ with $Z^*_0 = \#V^*$.

The following basic result is Chen's approach to π_1 [8], which relates dga objects (such as A^*_{PL}) and dgl objects (such as $(\widehat{\mathbb{L}}(\downarrow V_*), \partial)$) to the topology of S. We include a proof, for the sake of completeness.

Proposition 1.3. **(Chen)** *Let S and ∂ be as above. If there exists a dga map $\rho : \mathscr{C}^*(\widehat{\mathbb{L}}, \partial) \to A^*_{PL}(S)$ such that $H^*\rho = \mathrm{id}$, via the above identifications, then $P(S)$ and $T(\partial_|)$ are isomorphic as towers of $h_{1,2}$-marked Lie algebras.*

Proof. Set $K = \mathrm{Ker}(l_{|Z^1})$. The commutation condition $lq + ql = 0$ (see [29, I.1]) implies that $(\wedge K, q)$ is a subdga of $(\wedge Z^*, d)$. It is immediately seen that this subdga inclusion induces an isomorphism on H^1 and is monic on H^2. We also know that q is homogeneous of lower degree -1, by construction. The filtration $\mathcal{F}_k K \stackrel{\text{def.}}{=} K \cap Z_{\le k}$ has thus the property that $q\mathcal{F}_k \subset \wedge^2 \mathcal{F}_{k-1}$, for any k, hence $(\wedge K, q)$ represents the 1-minimal model of S and the general theory of [28, 7, 26] may be used to construct the Lie algebra form (E_k) of the tower $((\pi_1 S/\Gamma_k \pi_1 S) \otimes \mathbb{Q})$ as follows. Define *the canonical filtration* by $F_{-1}K = 0$ and inductively $F_k K = q^{-1}(\wedge^2 F_{k-1})$. Then the (classical) cochains on $E_k = (\wedge F_{k-2}, q_|)$. It will thus suffice to show that $F_k = \mathcal{F}_k$, for any k. Indeed one has $\mathscr{C}^*(\mathbb{L}X/\mathcal{J}_k + \mathbb{L}^{\ge k}X) = (\wedge \mathcal{F}_{k-2}, q)$, by construction, where $\mathcal{J}_k$ = ideal generated by $\partial_{<k}(Y)$, as in the definition of $T(\partial_|)$. We may then apply $\mathscr{C}^{*-1}$ and finally use $H^*\rho = \mathrm{id}$ to also infer that the $h_{1,2}$-markings of $P_k(S)$ and $T_k(\partial_|)$ coincide, in order to finish the proof. The claim on filtrations follows in turn inductively, starting trivially with $k = -1$. Assuming $F_{k-1} = \mathcal{F}_{k-1}$, we know by definition that $qz \in \wedge^2 \mathcal{F}_{k-1}$, for any $z \in F_k K$. On the other hand everything takes place in $(\wedge Z^1_*, q) = \mathscr{C}^*(\mathbb{L}^* X)$, where one knows [22, 1.6] that $F_k Z^1 = Z^1_{\le k}$, which implies by the definition of the canonical filtration, that $z \in Z^1_{\le k}$, hence $F_k \subset \mathcal{F}_k$. Finally the other inclusion is an easy inductive consequence of the definition of F_k, combined with the already mentioned property $q\mathcal{F}_k \subset \wedge^2 \mathcal{F}_{k-1}$, for any k. ■

Corollary. *$T(\mathscr{M}) \subset \mathscr{P}$ and $T_k(\mathscr{M}_k) \subset \mathscr{P}_k$, for any k.*

Proof. Plainly it is enough to verify only the first inclusion. Given $\partial \in \widehat{M}$, consider the complete minimal dgl $(\widehat{\mathbb{L}}(X \oplus Y), \partial)$ ($\partial^2 = 0$ is automatically satisfied, for trivial degree reasons). The geometric realization functor ([6, 28]) applied to the dga $\mathscr{C}^*(\widehat{\mathbb{L}}, \partial)$ provides a space S together with a dga map $\rho : \mathscr{C}^*(\widehat{\mathbb{L}}, \partial) \to A^*_{PL}(S)$

inducing a homology isomorphism. Use $H^*\rho$ to produce an h_*-marking of S and then apply the previous result to infer that $T(\partial) = P(S) \in \mathscr{P}$, as desired. ∎

Lemma 1.4. *T_k is injective, for any k.*

Proof. We are given a canonically $h_{1,2}$-marked Lie algebra $E_k = \mathsf{L}X/\mathcal{J}_k + \mathsf{L}^{\geq k}X$, $\mathcal{J}_k = \text{ideal}(\partial_{<k}Y)$, associated to $\partial \in \widehat{M}$, a similar E'_k associated to ∂', together with an $h_{1,2}$-marked Lie isomorphism $f : E_k \xrightarrow{\sim} E'_k$. It lifts to an automorphism ϕ of $\mathsf{L}X$ which sends $\mathcal{J}_k$ into $\mathcal{J}'_k + \mathsf{L}^{\geq k}X$. Since f is h_1-marked, we know that $\phi x \equiv x \bmod \mathsf{L}^{\geq 2}X$, for any $x \in X$. Writing down explicitly that f is also h_2-marked we find, for any basis element $y_i \in Y$, a Lie polynomial $\psi(y_i) \in \mathsf{L}(X \oplus Y)$, which is a sum of monomials of the form $\text{ad}_{z_1} \cdots \text{ad}_{z_r}(y)$, $z_1, \ldots, z_r \in \mathsf{L}X$, $r \geq 1$, $y \in Y$, with the property that $\phi\partial_{<k}(y_i) \equiv \partial'_{<k}(y_i) + \overline{\psi}(y_i) \bmod \mathsf{L}^{\geq k+1}X$, where $\overline{\psi}(y_i) \in \mathsf{L}X$ is obtained by making in $\psi(y_i)$ the substitution $y \mapsto \partial'_{<k}y$, for any $y \in Y$. Define $u \in U$ on the free generators by $ux = \phi x$, for any $x \in X$, and $uy_i = y_i + \psi y_i$, for any i. To see that $\partial' \equiv u\partial u^{-1} \bmod F_k\widehat{M}$ and thus finish the proof it is plainly enough to check that $u\partial \equiv \partial' u \bmod \mathsf{L}^{\geq k+1}X$, on each free generator y_i, which readily follows from the construction of u. ∎

We shall conclude the proof of our theorem by showing that $T : \mathscr{M} \to \mathscr{P}$ is surjective. Along the way we shall also describe a practical procedure for inductively constructing a representative of $T_k^{-1}P_k(S)$, for a given $h_{1,2}$-marked S, to be used in the sequel. Extend first the given $h_{1,2}$-marking to an isomorphic marking $V_* \xrightarrow{\sim} \overline{H}_*(S; \mathbb{Q})$. We shall next construct a complete minimal dgl $(\widehat{\mathsf{L}}(\downarrow V_*), \partial)$ and a dga map ρ such that $H^*\rho = \text{id}$ (as in Proposition 1.3) to get the claimed surjectivity of $T : \mathscr{M} \to \mathscr{P}$. This may be accomplished by the use of Chen's formal power series connection method [8]. We have however chosen to follow the approach and the sign conventions of [29, Ch. IV]. This simply means (for the expert) to translate Chen's method into the language of bifiltered models of Félix [10], without 1-connectivity assumptions. The outcome is that everything still goes on smoothly and that the bifiltered model approach is ideally suited for the computation of Massey products, see [29, Ch. V], a fact to be exploited in the next section. We shall thus review Chen's method, à la Tanré [29]. No explicit mention of the bifiltered models will be needed.

Let then (A^*, d) be a dga with h^*-marking $\overline{H}^*A \xrightarrow{\sim} \#V_*$. Put $W_* = \downarrow V_*$, and pick an homogeneous basis of W_*, say (x_α). By an *order $s-1$ ($s \geq 2$) connection on* (A, d) we shall mean a pair (Ω, ∂), where ∂ is a continuous degree -1 Lie derivation of $\widehat{\mathsf{L}}_* \stackrel{\text{def.}}{=} \widehat{\mathsf{L}}_*W$, written on the free generators as a formal series $\partial = \partial_2 + \partial_3 \cdots$, $\partial_i : W_* \to \mathsf{L}^i_{*-1}W$, and likewise $\Omega \in A\widehat{\otimes}\mathsf{L}^*_*$, $\Omega = \Omega^1 + \Omega^2 + \cdots$, each $\Omega^i = \Omega^{i0} + \Omega^{i1} + \cdots$, with $\Omega^{ij} \in A^{j+1} \otimes \mathsf{L}^i_j W$, satisfying the conditions

$$
\begin{aligned}
(1)_s \qquad & \partial_i = 0 \text{ and } \Omega^i = 0, \text{ for any } i \geq s,\\
(2) \qquad & \Omega^1 = \sum \omega_\alpha \otimes x_\alpha \text{ with } d\omega_\alpha = 0, \text{ any } \alpha, \text{ and}\\
& \qquad (cl(\omega_\alpha)) \text{ in duality with } (\uparrow x_\alpha),\\
(3)_s \qquad & d\Omega \equiv \partial\Omega + \frac{1}{2}[\Omega, \Omega] \bmod A\widehat{\otimes}\mathsf{L}^{\geq s}.
\end{aligned}
$$

If we drop conditions $(1)_s$ and replace $(3)_s$ by the honest equality

$$(3) \qquad d\Omega = \partial\Omega + \frac{1}{2}[\Omega,\Omega],$$

we obtain the definition of a *formal connection on* (A,d). As in [8],[29], a formal connection gives rise to a complete minimal dgl $(\widehat{\mathsf{L}}_*W,\partial)$, i.e. $\partial^2=0$ follows from (2) and (3). The connection form Ω may be in turn used to produce a graded algebra map $\rho : \mathscr{C}^*\widehat{\mathsf{L}}W \to A^*$. By a standard duality trick the integrability condition (3) implies that ρ also commutes with the differentials, while the marking condition (2) says that $H^*\rho = \mathrm{id}$, as in [29, IV.2].

We thus see that the existence of a connection on the h^*-marked deRham algebra $A^*_{PL}(S)$ gives the surjectivity of $T:\mathscr{M}\to\mathscr{P}$ and closes the proof of Theorem 1.2. In what follows we shall use T_k to identify $\mathscr{M}_k$ and $\mathscr{P}_k$ and denote the composition $T_k^{-1}P_k : \mathscr{S}\to\mathscr{M}_k$ simply by P_k.

The existence proof goes by induction. Picking $\Omega^{\leq 1}=\Omega^1$, as in (2), and $\partial_{\leq 1}=0$ we get a first order connection on (A,d). Assume we have a connection of order $s-1$, given by $\Omega^{<s}=\Omega^1+\cdots+\Omega^{s-1}$ and $\partial_{<s}=\partial_2+\cdots\partial_{s-1}$. We may extend these data to an order s connection $\Omega^{\leq s}=\Omega^{<s}+\Omega^s$, $\partial_{\leq s}=\partial_{<s}+\partial_s$ by defining

$$(4) \qquad \mathcal{O}_s = A\widehat{\otimes}\mathsf{L}^s - \text{component of } d\Omega^{<s} - \partial_{<s}\Omega^{<s} - \frac{1}{2}[\Omega^{<s},\Omega^{<s}],$$

noting that $d\mathcal{O}_s=0$ by the induction assumptions and finally solving for Ω^s and ∂_s in the integrability equation

$$(5) \qquad \mathcal{O}_s = \partial_s\Omega^1 - d\Omega^s$$

as in [8]. We eventually arrive at a genuine connection (Ω,∂).

Corollary 1.5. *If $(\Omega^{\leq k},\partial_{\leq k})$ is a connection of order k on the deRham algebra of S, then $\partial_{\leq k}|_Y$ represents $P_k(S)$ in $\mathscr{M}_k$.*

Proof. Extend to a connection (Ω,∂), recall our previous discussion and use 1.3 to get $P(S)=T(\partial|_Y)$, whence $P_k(S)=T_k(\partial|_Y \bmod F_k\widehat{M})=T_k(\partial_{\leq k}|_Y)$. ∎

1.6. *Remark.* Assume $A^{>3}=0$ and $H^3A=0$ (e.g. $A^*=\Omega^*_{dR}(\mathbf{S}^3\setminus L)$). As it was noticed in [14] (remember however our different sign conventions!) one may use the above inductive method to construct a connection with simpler connection form Ω, namely $\Omega=\omega+\eta$, $\omega\in A^1\widehat{\otimes}\mathsf{L}_0(X\oplus Y)$ and $\eta\in A^2\widehat{\otimes}\mathsf{L}_1(X\oplus Y)$.

1.7. *Example.* Mal'cev completion translates the isomorphism problem for nilpotent finitely generated groups into the same problem for finite dimensional nilpotent Lie algebras. We want to stress the fact that our point here is related in fact to the extra structure provided by $h_{1,2}$-markings. Given $r,r'\in\mathsf{L}^kX$, $k\geq 2$, consider the nilpotent finite dimensional Lie algebra $E=\mathsf{L}X/\mathcal{J}+\mathsf{L}^{\geq k+1}X$, $\mathcal{J}=\mathrm{ideal}(r)$, and similarly E' endowed with canonical $h_{1,2}$-markings, as explained in the construction of $T_{k+1}:\mathscr{M}\to\mathscr{E}$. It is easy to see that E and E' are isomorphic if and only if r and r' are conjugate under the natural action of $GL(X)\times GL_1$ on L^kX (by linear changes of coordinates and multiplication by nonzero scalars), while E and

E' are isomorphic as Lie algebras with $h_{1,2}$-markings if and only if $r = r'$. Thus the unrestricted isomorphism problem is in general very complicated, while the restricted (= marked) second problem is trivial. In the subsequent paragraph we shall go further and explain the usefulness of working with the moduli spaces $\mathcal{M}_k$ from the point of view of the general *decision problem*: decide whether $P_k(S)$ and $P_k(S')$ represent the same invariant, viewed in $\mathcal{P}_k$.

1.8. *A recipe for decision.* We are going to describe a simple algorithm, inductive and linear at each stage, for the decision problem in $\mathcal{M}_k = \widehat{M}/F_k\widehat{M}$ mod U. Here the unipotence plays the key rôle.

We shall write the points of $\widehat{M}$ in the form $p = p_1+p_2+\cdots$, $p_i \in \mathrm{Der}^i_{-1}\mathsf{L}(X\oplus Y) = \mathrm{Hom}(Y, \mathsf{L}^{i-1}X)$. There is also the whole bigraded Lie algebra $\mathrm{Der}^*_*\mathsf{L}(X\oplus Y)$, with bracket given by the graded (with respect to lower degrees) commutator of derivations, and a graded subalgebra $D^* = \mathrm{Der}^{\geq 1}_0$, with standard filtration given by the upper degrees, giving rise by completion to a complete Lie algebra $\widehat{D}$. The adjoint representation of Der^*_* gives $\widehat{M}$ the structure of a filtered $\widehat{D}$-module. There is also a filtered induced $\exp \mathrm{ad}(\widehat{D})$-action on $\widehat{M}$, given by the usual formula

$$\exp \mathrm{ad}\theta(p) = \sum_{i\geq 0} \frac{1}{i!}(\mathrm{ad}\theta)^i(p), \text{ for } p \in \widehat{M},\ \theta \in \widehat{D}. \tag{6}$$

By unipotence, logarithms are also available, as in classical Lie theory, and consequently $\widehat{M}$ as a filtered U-module is the same thing as $\widehat{M}$ as a filtered $\exp \mathrm{ad}(\widehat{D})$-module.

Given $p, p' \in \widehat{M}$, the decision question 'is $p' \equiv p$ in $\widehat{M}/F_k$ mod $\exp \mathrm{ad}(\widehat{D})$?' is trivial for $k = 2$ (being equivalent with 'is $p'_1 = p_1$?', like in the previous example). Suppose then that we know that $u_k p' \equiv p$ mod F_k, for some $u_k \in \exp \mathrm{ad}(\widehat{D})$. Define then $\Delta \in \mathrm{Der}^k_{-1}$ by $p - u_k p' \equiv \Delta$ mod F_{k+1}. Consider next the solutions of the linear equation

$$\sum_{i+j<k} [\theta_i, p_j] = 0, \text{ where } \theta_i \in \mathrm{Der}^i_0 \text{ and } i \geq 1. \tag{7}$$

Then we may state: $p' \equiv p$ in $\widehat{M}/F_{k+1}$ mod $\exp \mathrm{ad}(\widehat{D})$ if and only if

$$\Delta = \sum_{i+j=k} [\theta_i, p_j], \text{ where } \theta = \theta_1 + \cdots + \theta_{k-1} \text{ satisfies (7).} \tag{8}$$

Moreover, if this happens then $u_{k+1}p' \equiv p$ mod F_{k+1} with $u_{k+1} = \exp \mathrm{ad}(\theta) \cdot u_k$.

A two-lines proof may be found in [4] (it is implicit in [27]). The above algorithm is also applicable to many other interesting unipotent moduli spaces arising in rational homotopy theory, see [27],[4]. Related algorithms are present in [17, 13].

1.9. *Remarks.* We may also handle in this way links L which are not numbered or oriented. Pick first an arbitrary numbering and orientation. This will produce a well-defined h_*-marking $\mu : V_* \xrightarrow{\sim} \overline{H}_*(\mathbf{S}^3 \setminus L; \mathbf{Q})$, as explained in Example 1.1. Next it easily follows from 1.1 that there is a representation $r : \Sigma_{m+1} \ltimes \mathbf{Z}_2^{m+1} \to GL(X) \times GL(Y)$, where $m+1$ is the number of components, with the property that for any other choice of numbering and orientation the associated marking will be of the form μg, for some $g \in \mathrm{image}(r)$.

Here the semidirect product structure comes from the usual permutation representation of Σ_{m+1} on $\mathbf{Z}_2^{m+1}$, and r is constructed as follows. Under the identification $\mathbf{Z}_2 = \{\pm 1\}$, $\Sigma_{m+1} \ltimes \mathbf{Z}_2^{m+1}$ acts in the standard manner on the basis $(x_0, \ldots, x_m)$ of X, by permutations and changes of sign. Finally the action on $Y = \mathrm{Span}(y_1, \ldots, y_m)$ factors through a representation of Σ_{m+1}. To describe it, consider the quotient vector space $Y' = \wedge^2 X$ modulo the subspace generated by $\{x_i \wedge x_j + x_j \wedge x_k - x_i \wedge x_k;\ 0 \le i, j, k \le m\}$, on which Σ_{m+1} naturally acts, and note that (y_i = class of $x_0 \wedge x_i$; $1 \le i \le m$) will represent a basis of Y', corresponding to the basis of $H^2(\mathbf{S}^3 \setminus L; \mathbf{Q})$, $(\alpha_1, \ldots, \alpha_m)$, described in Example 1.1.

We recall that U is a normal subgroup of $\mathcal{A} \overset{\text{def.}}{=}$ continuous graded Lie algebra automorphisms of $\widehat{\mathbb{L}}_*(X \oplus Y)$, which also acts by conjugation on the filtered module $\widehat{M}$. The larger group $\mathcal{A}$ also contains $GL(X) \times GL(Y)$, which is embedded in a standard manner. With these observations, we may spell out the following claim: $P_k(\mathbf{S}^3 \setminus L$ with marking $\mu g) = g^{-1} P_k(\mathbf{S}^3 \setminus L$ with marking $\mu)$ (equality in $\mathscr{M}_k$). As a consequence $P_k : \mathscr{S} \to \mathscr{M}_k$ induces for any k well-defined isotopy invariants of links (no numbering and no orientation) with values in the quotient moduli spaces $\mathscr{M}_k$ mod $\Sigma_{m+1} \ltimes \mathbf{Z}_2^{m+1}$, where the decision algorithm 1.8 still works well, since $\Sigma_{m+1} \ltimes \mathbf{Z}_2^{m+1}$ is finite. As far as our claim is concerned, it readily follows from the existence of a connection, giving rise to a dga map $\rho : \mathscr{C}^*(\widehat{\mathbb{L}}_*(X \oplus Y), \partial) \to A^*(\mathbf{S}^3 \setminus L)$ which is marked with respect to μ. Setting $\partial' = g^{-1} \partial g$, we get an isomorphism $g^{-1} : (\widehat{\mathbb{L}}_*, \partial) \xrightarrow{\sim} (\widehat{\mathbb{L}}_*, \partial')$ and a dga map $\rho \mathscr{C}^*(g^{-1}) : \mathscr{C}^*(\widehat{\mathbb{L}}_*(X \oplus Y), \partial') \to A^*_{PL}(\mathbf{S}^3 \setminus L)$ which is marked with respect to μg. We may now invoke 1.3 and 1.2.

1.10. *Remark.* With a little more care it is possible to show that $T : \mathscr{M} \xrightarrow{\sim} \mathscr{P}$ is also bijective. We have however decided not to include this result, since our primary interest is in the finitely-decidable moduli spaces $\mathscr{M}_k$ (see 1.8).

2. Computation of initial terms: Massey products, $\overline{\mu}$-invariants and deRham theory

We have constructed an invariant for $h_{1,2}$-marked spaces, $P = \varprojlim P_s : \mathscr{S} \to \varprojlim \mathscr{M}_s$ (Corollary 1.5). Here $\mathscr{M}_{s+1} \to \mathscr{M}_s$ is induced by the canonical projection $\widehat{M}/F_{s+1} \to \widehat{M}/F_s$. Given a point $p = (p_s) \in \varprojlim \mathscr{M}_s$, we may inductively assume that $p_s = 0$ in $\mathscr{M}_s$, for any $s \le k-1$, for some $k \ge 2$. It follows that there exists a well-defined numerical invariant $\partial_k : Y \to \mathbb{L}^k X$ with the property that if $\partial^s = \partial_2^s + \partial_3^s + \cdots$ is any representative of p_s, then $\partial^s_{<k} = 0$ and $\partial^s_k = \partial_k$ for any $s \ge k$ (this is an easy consequence of the fact that $\mathrm{gr}^1 u = \mathrm{id}$, for any $u \in U$). We shall say that ∂_k is *the initial term of degree* k of p. If $\partial_k = 0$ then plainly $p_k = 0$ in $\mathscr{M}_k$, so we may iterate.

The following theorem gives a recipe for the computation of the initial terms of $P(S)$ via Massey products in $H^*(S; \mathbf{Q})$. Picking bases $(x_1, \ldots, x_{b_1})$ for X with dual basis $(v_1, \ldots, v_{b_1})$ for $H^1(S; \mathbf{Q})$ and $(y_1, \ldots, y_{b_2})$ for Y, our explicit formula reads

Theorem 2.1. *Let $\partial_k : Y \to \mathbb{L}^k X$ be the initial term of degree k of $P(S)$. Then all Massey products in $H^1(S; \mathbf{Q})$ of order $< k$ are trivial, the Massey products of order k are strictly defined and*

$$\partial_k(y_j) = (-1)^{k+1} \sum_I \langle \langle v_{i_1}, \ldots, v_{i_k} \rangle | y_j \rangle\, x_{i_1} \cdots x_{i_k},\quad 1 \le j \le b_2,$$

as elements of $\bigotimes^k X$*. Here the summation is taken over all* k*-tuples* $I = (i_1, \dots, i_k)$.

As a corollary we get the following very explicit method of computation for ∂_k of a link complement in terms of $\overline{\mu}$-invariants of the link, an alternative for Hain's method for initial terms, based on his 'stable intersection calculus' (see [14, Lemma 9]). So, let L be an $m+1$-component link, numbered and oriented as in Example 1.1, $X = \operatorname{span}(x_0, \dots, x_m)$ and $Y = \operatorname{span}(y_1, \dots, y_m)$.

Corollary 2.2. *Assume that all Milnor* $\overline{\mu}$*-invariants of* L *of order* $< k$ *are trivial. Then* $P_{k-1}(S^3 \setminus L) = 0$ *in* $\mathcal{M}_{k-1}$ *and*

$$\partial_k y_j = -\sum_{I,i} \overline{\mu}(i i_1 \cdots i_{k-2} j) x_i x_{i_1} \cdots x_{i_{k-2}} x_j - \overline{\mu}(j i_1 \cdots i_{k-2} i) x_j x_{i_1} \cdots x_{i_{k-2}} x_i,$$

for any $j = 1, \dots, m$*, where the summation is taken over all* i*,* $0 \le i \le m$*, and all* $(k-2)$*-tuples* $(i_1, \dots, i_{k-2})$ *with* $0 \le i_s \le m$*,* $1 \le s \le k-2$.

Proof of the Theorem. Extend the given marking to an h_*-marking $V_* \xrightarrow{\sim} \overline{H}_*(S; \mathbb{Q})$, use a formal connection (Ω, ∂) on the dga $A^*_{PL}(S)$ to get a dga map $\rho : \mathscr{C}^*(\widehat{\mathbb{L}}{\downarrow}V, \partial) \to A^*_{PL}(S)$ with $H^*\rho = \mathrm{id}$, as explained in the previous section, and invoke Corollary 1.5 to infer that $\partial = \partial_k +$ higher terms.

On the other hand the Massey product structure of $H^*(S; \mathbb{Q})$ is the same as that of $\mathscr{C}^*(\widehat{\mathbb{L}}{\downarrow}V, \partial) = (\wedge Z^*, l+q)$. This is an easy consequence of general naturality properties (see [23]): ρ is a multiplicative weak equivalence, while the integration map $\int : A^*_{PL}(S) \to C^*_{\mathrm{sing}}(S; \mathbb{Q})$ is a strong homotopy multiplicative weak equivalence (see [6]).

Recall next the following convenient description of $(\wedge Z^1_*, q)$ from [29, I.4(2)]. Consider the dual of the inclusion map $\mathbb{L}^s X \hookrightarrow \bigotimes^s X$, $\pi_s : \bigotimes^s Z^1_0 \twoheadrightarrow Z^1_{s-1}$ and set $\pi_s(u_1 \otimes \cdots \otimes u_s) = u_{1\dots s}$. Then

$$qu_{1\dots s} = -\sum_{t=1}^{s-1} u_{1\dots t} \wedge u_{t+1\dots s}, \text{ for any } u_1, \dots, u_s \in Z^1_0 = H^1(S; \mathbb{Q}). \tag{1}$$

Note also that our vanishing hypothesis on ∂ translates by duality to $l|_{Z^1_{\le k-2}} = 0$ and that the map $l\pi_k$ is the dual of $\partial_k : Y \to \bigotimes^k X$.

By definition (see [29, V.4]) the equalities (1) provide defining systems for Massey products of the form $\langle cl(u_1), \dots, cl(u_s)\rangle$ in $(\wedge Z^*, l+q)$, for any $s \le k$, which vanish for $s < k$. The (uniquely defined) k-fold Massey product $\langle cl(u_1), \dots, cl(u_k)\rangle$ is also seen to be equal to $-cl(lu_{1\dots k})$.

The proof ends by duality and a straightforward sign correction, due to the fact that the definition of Massey products given in [29] differs slightly from the classical one of [23],[25]. ∎

Proof of the Corollary: use induction and express the Massey products in the theorem in terms of $\overline{\mu}$-invariants, using [25]. ∎

2.3. *Example.* The case $k = 2$ of the corollary gives ∂_2 of an arbitrary link L in terms of the linking numbers $l_{ij} = \mathrm{lk}(L_i, L_j) = \overline{\mu}(ij)$, namely

$$\partial_2 y_j = \sum_{i=0}^{m} l_{ji}[x_j, x_i], \text{ for } j = 1, \cdots, m. \tag{2}$$

In general, it is equally easy to see that ∂_2 of an arbitrary space $= -\#(\cup : H^1 \wedge H^1 \to H^2)$, where $\cup$ = cup-product. Also it is not hard to see that for a *formal* space S (see [28, 17]) one has $P(S) = T(\partial_2)$ in $\mathscr{T}$. Here it is interesting to recall that the complements of all algebraic links are formal [15].

2.4. *Example.* For a 2-component link L with vanishing linking number l_{01} it is known (see [9, Appendix B]) that all the order 3 $\overline{\mu}$-invariants also vanish. Next (see again [9]) all possibly nontrivial invariants of order 4 are

$$\overline{\mu}(0011) = \overline{\mu}(1001) = \overline{\mu}(0110) = \overline{\mu}(1100) \text{ and } \overline{\mu}(0101) = \overline{\mu}(1010) = -2\overline{\mu}(0011).$$

Consequently $\partial_4 y_1 = \overline{\mu}(0011)[x_0, [x_1, [x_0, x_1]]]$, with $\overline{\mu}(0011) = 1$ for the Whitehead link, see e.g. [11]. For a further discussion in this direction see also [21].

3. A rigidity result

Consider a finite order formal connection (Ω, ∂), $\partial : Y \to \mathsf{L}X$, on the deRham algebra of a given marked space S, as in Section 1. Set $\partial y_j = r_j +$ higher terms, where $r_j \in \mathsf{L}^{m_j+1}X$, for $j = 1, \ldots, b_2$. The knowledge of the 'initial terms' r_j (possibly of different degrees!) may also provide useful information on the graded Lie algebra associated to the lower central series, denoted by $\mathrm{gr}^*\pi_1 S$ (where the Lie bracket is induced by the group commutator). We shall thus see that if the sequence $(r_1, \ldots, r_{b_2})$ is *inert* in $\mathsf{L}X$, in the sense of [16] (see also [2]) then it determines the graded Lie algebra $\mathrm{gr}^*\pi_1 S$. Related results may be found in [22, Theorem B'], [1, Theorem 1.1] or [19, Theorem 1].

Theorem 3.1. *If the sequence $(r_1, \ldots, r_{b_2})$ is inert in $\mathsf{L}X$, then $(\mathrm{gr}^*\pi_1 S) \otimes \mathbb{Q} \cong \mathsf{L}^*X/\mathrm{ideal}(r_1, \ldots, r_{b_2})$, as graded Lie algebras.*

Proof. Extend Ω and ∂ to get a formal connection, as explained in Section 1, then use 1.3 to identify $P(S)$ and $T(\partial|_Y)$, as towers of Lie algebras. As is well-known (see [26]), the Campbell-Hausdorff equivalence between nilpotent Mal'cev groups and Lie algebras preserves the lower central series, whence $(\mathrm{gr}^*\pi_1 S) \otimes \mathbb{Q} \cong \mathrm{gr}^*(\widehat{\mathsf{L}}X/\mathcal{J})$, where $\mathcal{J}$ = ideal generated by ∂Y, and the filtration of $\widehat{\mathsf{L}}X/\mathcal{J}$ is induced by the standard bracket length filtration of $\widehat{\mathsf{L}}X$. On the other hand the identity map of X gives rise to a graded Lie algebra surjection $\mathsf{L}^*X \twoheadrightarrow \mathrm{gr}^*(\widehat{\mathsf{L}}X/\mathcal{J})$, which obviously factors through $\mathcal{I}$ = ideal generated by $r_1, \ldots, r_{b_2}$. For a fixed degree $* = n$ this gives rise to a canonical surjection $\mathsf{L}^nX/\mathsf{L}^nX \cap \mathcal{I} \twoheadrightarrow \mathsf{L}^nX/\mathsf{L}^nX \cap (\widehat{\mathsf{L}}^{\geq n+1}X + \mathcal{J}) = \mathrm{gr}^n(\widehat{\mathsf{L}}X/\mathcal{J})$, where $\widehat{\mathsf{L}}^{\geq n+1} = F_{n+1}\widehat{\mathsf{L}}$. We shall use the inertia assumption to prove that $\mathsf{L}^nX \cap (\widehat{\mathsf{L}}^{\geq n+1}X + \mathcal{J}) \subset \mathcal{I}$, in order to finish our proof.

Extend the map $x_i \mapsto 0$, $y_j \mapsto r_j$, to a (lower) degree -1 derivation of $\widehat{\mathsf{L}}(X \oplus Y)$, to be denoted by $\partial_{\mathcal{I}}$. Setting $\deg(x_i) = 1$ and $\deg(y_j) = m_j + 1 - n$, $\partial_{\mathcal{I}}$ will also be homogeneous of degree n with respect to these upper degrees. By construction if $c \in \widehat{\mathbb{L}}(X \oplus Y)$, $c = c_k +$ higher terms, then $\partial c = \partial_{\mathcal{I}} c_k +$ higher terms, with respect to the new upper degrees. If $\lambda_n \in \mathbb{L}^n X \cap (\widehat{\mathbb{L}}^{\geq n+1} X + \mathcal{J})$, then there exists $a \in \widehat{\mathbb{L}}_1(X \oplus Y)$ such that $\partial a = \lambda_n +$ higher terms. It is then enough to show that a may be chosen to be of the form $a = a_0 +$ higher terms, because this would imply that $\lambda_n = \partial_{\mathcal{I}} a_0 \in \mathcal{I}$ as desired.

Assume then inductively that $a = a_k +$ higher terms, with $k < 0$. Looking at the initial terms we see that $\partial_{\mathcal{I}} a_k = 0$. Now the inertia comes into play. Recall from [16, 2] that the inertia is equivalent with the vanishing of $H_{\geq 1}(\mathbb{L}(X \oplus Y), \partial_{\mathcal{I}})$. This implies the existence of $b \in \mathbb{L}_2^{k-n}(X \oplus Y)$ with the property that $\partial_{\mathcal{I}} b = a_k$. Setting $c = a - \partial b$ we plainly have $\partial a = \partial c$ and $c = c_{k+1} +$ higher terms, so the induction may well go on. ■

3.2. *Examples.* For $S =$ a link complement, Corollary 1.5 and 2.3(2) give the explicit form of the degree 2 initial terms $r_1, \ldots, r_m$. As it was proved in [1] (see also [22]), this sequence is inert if and only if the linking diagram of L is connected.

The fact that an one-element sequence (r) is inert if and only if $r \neq 0$ [16] together with Corollary 2.2 give, for a 2-component link L, that either there is some k with the property that all $\overline{\mu}$-invariants of order $< k$ are trivial but not all $\overline{\mu}$-invariants of order k, in which case $\mathrm{gr}^*\pi_1(\mathrm{S}^3 \setminus L) \otimes \mathbb{Q} \cong \mathbb{L}^*(x_0, x_1)/\mathrm{ideal}(r)$, with

$$r = \sum_{0 \leq i_1, \cdots, i_{k-2} \leq 1} \overline{\mu}(1I0) x_1 x_I x_0 - \overline{\mu}(0I1) x_0 x_I x_1$$

or all $\overline{\mu}$-invariants vanish and consequently (see [24]) the nilpotent completion of the link group is isomorphic to that of a free group, in particular $\mathrm{gr}^*\pi_1(\mathrm{S}^3 \setminus L) \otimes \mathbb{Q} \cong \mathbb{L}^*(x_0, x_1)$. More examples may be found in [14, 21].

4. Computations beyond initial terms

As we have seen in Section 2, the initial terms in the Lie power series expansion of the ∂-part of a formal connection on the deRham algebra of a marked space S are well-defined numerical invariants, which may be used to answer the marked isomorphism problem for the rationalized nilpotent quotients of $\pi_1 S$. The same thing happens with Milnor's [24] first nonvanishing $\overline{\mu}$-invariants (the precise relationship between ∂ and $\overline{\mu}$ was described in Corollary 2.2).

Like the higher $\overline{\mu}$-invariants the higher terms of ∂ are not well-defined. The indeterminacy of $\overline{\mu}$-invariants is measured by taking certain residue classes of μ-integers (see [24]), while the indeterminacy of ∂ is described by the action of the group $\exp \mathrm{ad}(\widehat{D})$ in a manner which is very well suited for answering the marked isomorphism problem (see 1.8). Thus the higher terms are very important too (see Example 4.5).

As we saw in Example 2.3 the degree 2 initial terms of a link complement are given by linking numbers, therefore their computation may be achieved by intersection theory. It is well-understood that in general the initial terms may be computed by some sort of intersection calculus. From the Massey product point of view of

Theorem 2.1 this may be found in [12],[25],[11]. Via the formal connection approach this was done in [14, Lemma 9]. Our examples of computation begin by a result pointing out that Hain's 'stable intersection calculus', which in general gives only the initial terms [14, p. 58], may be combined in certain cases with disjoint support arguments to produce information on higher terms too.

The method of [14] systematically exploits Sullivan's idea of doing infinitesimal computations in which the differential forms are replaced by submanifolds, as explained in [12, pp. 154–157]. As in [14, p. 61], we are going to work with *ϵ-Thom classes* of submanifolds, that is with representatives of Thom classes which are supported in a sufficiently small tubular neighborhood. For standard manipulations with Thom classes (like the correspondence between intersection and wedge product, boundary and exterior differentiation) see [5, 12, 14].

Let then $L = L_0 \sqcup \cdots \sqcup L_m$ be a link in $\mathbf{S}^3$ (numbered, oriented , as usual). Set $M = \mathbf{S}^3 \setminus L$. The collection of submanifolds we are going to associate to L will be understood to be in general position.

Following [14], start with a *Seifert system* $\mathcal{S} = \{\sigma_0, \ldots, \sigma_m\}$ and an *arc system* $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$, like in Example 1.1. We shall moreover require these systems to be *(geometrically) compatible*, that is $\mathrm{int}(\alpha_p) \cap \sigma_i = \emptyset$, for any $1 \le p \le m$, $0 \le i \le m$. It is always easy to construct compatible systems.

Following again [14, p. 51], we next look for a collection $\{\sigma_{ij}\}_{0 \le i < j \le m}$ of geometric 2-chains (= integral combinations of embedded compact connected oriented surfaces) which satisfy

$$(\star)_{ij} \qquad \partial\sigma_{ij} \equiv \sigma_i \cap \sigma_j + l_{ij}(\alpha_i - \alpha_j) \text{ modulo } L$$

(here ∂ = boundary, $l_{ij} = \mathrm{lk}(L_i, L_j)$ and $\alpha_0 = \emptyset$). Again it is easy to solve the above geometric equations (compare [14, p. 60]).

Setting $\tilde{S} = S \cap M$, for any subset $S \subset \mathbf{S}^3$, and $cl(N)^*$ = the cohomology class on M which is dual to the proper submanifold N, we may state the following very explicit formula for ∂_3, which involves only intersectional computations.

Proposition 4.1. *Assuming that $\sigma_i \cap \sigma_j \cap \sigma_k = \emptyset$, for any distinct i, j, k, one has, for any $1 \le p \le m$*

$$\partial_3 y_p = \sum_{\substack{0 \le i \le m \\ 0 \le j < k \le m}} \langle cl(\mathrm{int}(\tilde{\sigma}_{jk}) \cap \tilde{\sigma}_i)^*, y_p \rangle [x_i, [x_j, x_k]].$$

Proof. By corollary 1.5 and the formula 1(5), we need a second order connection on $A^* = \Omega^*_{dR} M$. Set $\partial = \partial_2$, as in 2(2), and

$$\Omega = \omega^1 + \omega^2 + \eta,\ \omega^1 = \sum_{i=0}^{m} \omega_i \otimes x_i,\ \omega^2 = \sum_{0 \le i < j \le m} \omega_{ij} \otimes [x_i, x_j],\ \eta = \sum_{p=1}^{m} \eta_p \otimes y_p,$$

where ω_i = ϵ-Thom class of σ_i, ω_{ij} = ϵ-Thom class of σ_{ij} and η_p = ϵ-Thom class of α_p.

By construction $\Omega^1 = \omega^1 + \eta$ satisfies the marking condition 1(2). Geometric compatibility translates to *algebraic compatibility*, that is $\eta_p \wedge \omega_i = 0$, for any p

and i, and this readily gives that

$$d\Omega - \partial\Omega - \frac{1}{2}[\Omega,\Omega] \equiv \sum_{0\le i<j\le m} (d\omega_{ij} - \omega_i \wedge \omega_j - l_{ij}(\eta_i - \eta_j)) \otimes [x_i, x_j] \bmod A\widehat{\otimes}\mathsf{L}^{\geq 3}.$$

On the other hand, $(\star)_{ij}$ algebraically translates to

$$(\star\star)_{ij} \qquad d\omega_{ij} = \omega_i \wedge \omega_j + l_{ij}(\eta_i - \eta_j)$$

which means that the integrability condition $1(3)_3$ is also satisfied. Setting $K_3 =$ $A\widehat{\otimes}\mathsf{L}^3X$-component of $\mathcal{O}_3$ (see 1(4)) it is straightforward to see that

$$K_3 = \sum_{\substack{i \\ j<k}} \omega_{jk} \wedge \omega_i \otimes [x_i, [x_j, x_k]].$$

Formula 1(5) tells us that

$$cl(K_3) = \sum_p cl(\eta_p) \otimes \partial_3 y_p, \text{ in } H^2M \otimes \mathsf{L}^3X.$$

A simple comparison shows that

$$\partial_3 y_p = \sum_{\substack{i \\ j<k}} \langle cl(\omega_{jk} \wedge \omega_i), y_p\rangle [x_i, [x_j, x_k]].$$

Finally our assumptions on supports will be invoked to infer that

$$cl(\omega_{jk} \wedge \omega_i) = cl(\mathrm{int}(\tilde\sigma_{jk}) \cap \tilde\sigma_i)^*, \text{ for any } i, \text{ and any } j < k.$$

If i is distinct from j and k, the boundary points of $\tilde\sigma_{jk}$ will cause no difficulties, since $\tilde\sigma_i$ will be disjoint from both coming from $\tilde\alpha_j \cup \tilde\alpha_k$ (compatibility) and those coming form $\tilde\sigma_j \cap \tilde\sigma_k$ (by hypothesis). If $i = j$ or k, the potential contribution concentrated on the 'unstable' intersection points of $\tilde\sigma_i$ and $\tilde\sigma_{jk}$ contained in the boundary components sitting inside $\tilde\sigma_j \cap \tilde\sigma_k$ may be safely ignored, due to the technical lemma below, and our proof is complete ■

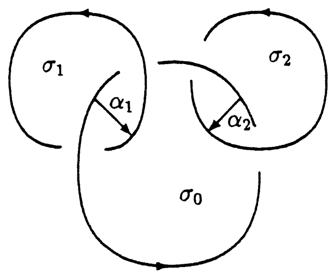
Lemma 4.2. *Let σ_1, σ_2 be a transverse pair of proper oriented 2-dimensional submanifolds (without boundary) of M. Let β be a union of connected components of $\sigma_1 \cap \sigma_2$ and let α be a proper oriented 1-dimensional submanifold (without boundary) of M. Assume $\alpha \cap \sigma_i = \emptyset$, $i = 1, 2$. Let σ be another proper oriented 2-dimensional submanifold ($\sigma \pitchfork \sigma_i$, $i = 1, 2$) such that $\partial\sigma = \beta \sqcup \alpha$. Pick ϵ-Thom classes ω_i, η and ω for σ_i, α and σ, with the property that $d\omega = \omega_1 \wedge \omega_2 + \eta$. We then have equalities in H^2M*

$$cl(\omega_i \wedge \omega) = cl(\sigma_i \cap \mathrm{int}\sigma)^*, \textit{ for } i = 1, 2.$$

Proof of the lemma. It is almost easier than its statement. One may in fact derive a sharper conclusion, namely that $\omega_i \wedge \omega =$ some ϵ-Thom class of $\sigma_i \cap \mathrm{int}(\sigma) + d\xi$, where ξ is supported in an ϵ-tube T of β. One has only to first notice that the class of $\omega_i \wedge \omega$ in $H^2(T, \partial T)$ is independent of the various choices and next that it represents zero, by making canonical choices near β and performing an explicit easy computation. ■

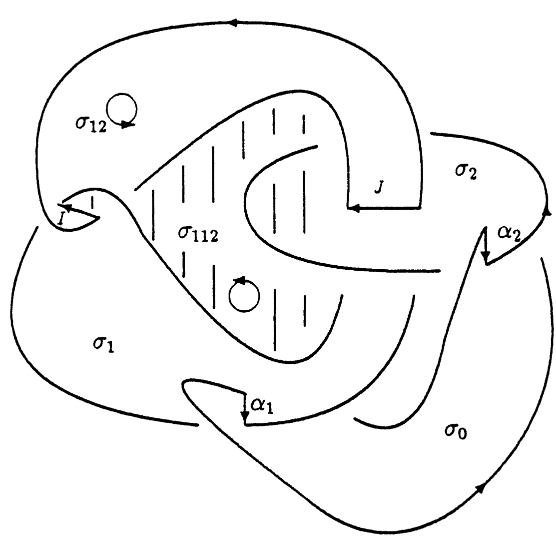
In practice there are several other helpful tricks, related to support arguments.

4.3. *Example.* It is sometimes useful to allow the arcs in $\mathcal{A}$ to lie on the surfaces in $\mathcal{S}$. Let us look at the Mickey Mouse link L



Here $\alpha_p = \sigma_0 \cap \sigma_p$, $p = 1, 2$. Obviously $\partial_2 y_p = [x_p, x_0]$, $p = 1, 2$. Set $\partial = \partial_2$ and $\Omega = \sum_{i=0}^{2} \omega_i \otimes x_i + \sum_{p=1}^{2} \omega_0 \wedge \omega_p \otimes y_p$, where ω_i = ϵ-Thom class of σ_i. Thus 1(2) holds by construction. On the other hand $\omega_1 \wedge \omega_2 = 0$ (disjoint supports) which readily implies that $d\Omega = \partial\Omega + \frac{1}{2}[\Omega, \Omega]$, that is the whole marked Mal'cev tower of $\pi_1(\mathbf{S}^3 \setminus L)$ is determined by ∂, see Section 1. Since $\partial = \partial_2$, it is easily seen that $\mathbf{S}^3 \setminus L$ must be formal, though L cannot be algebraic, due to the vanishing of l_{12}, compare to Example 2.3.

4.4. *Example.* It is also useful to exploit naturality properties coming from sublinks. Consider the following link L, which is a simple modification of the Whitehead link.



Consider the Whitehead sublink $L' = L_1 \sqcup L_2$. First we are going to construct a connection of order 3 on the deRham algebra $A' = \Omega^*_{dR}(\mathbf{S}^3 \setminus L')$, (Ω', ∂'). Set $\partial' = 0$. The Seifert system $\mathcal{S}' = \{\sigma_1, \sigma_2\}$ of L' has σ_1 = immersed disk with normal crossings having I as double points (allowing singular surfaces of this type is harmless, see [14]). Denote by $\{x'_1, x'_2\}$ the corresponding distinguished basis of H_1, and by y' the marking of H_2 corresponding to an arc joining L_1 to L_2 (as in

Example 1.1, modulo reindexing in a convenient way for future computations for L). Pick ω'_i = ϵ-Thom class in A' of σ_i, $0 \leq i \leq 2$. Set

$$\Omega' = \omega'_1 \otimes x'_1 + \omega'_2 \otimes x'_2 + \eta' \otimes y' + \omega'_{12} \otimes [x'_1, x'_2] + \omega'_{112} \otimes [x'_1, [x'_1, x'_2]] + \omega'_{212} \otimes [x'_2, [x'_1, x'_2]].$$

Here $\eta' = \omega'_0 \wedge (\omega'_2 - \omega'_1)$ and thus 1(2) is seen to hold. The higher terms are constructed as follows. First σ_{12} = a copy of the upper half of σ_1 with opposite orientation. It has the property that $\partial\sigma_{12} \equiv \sigma_1 \cap \sigma_2 \bmod L'$, therefore $d\omega'_{12} = \omega'_1 \wedge \omega'_2$, where ω'_{12} = ϵ-Thom class of σ_{12}. Next solve the geometric equations $\partial\sigma_{p12} \equiv \sigma_p \cap \mathrm{int}\sigma_{12} \bmod L'$, for $p = 1, 2$. We may take $\sigma_{212} = \emptyset$ and σ_{112} as indicated in the figure. Finally we may apply Lemma 4.2 to $\sigma_{1,2}$ and $M = \mathbf{S}^3 \setminus L'$, with $\beta = J$, $\alpha = \emptyset$ and $\sigma = \sigma_{12}$ to get that $\omega'_p \wedge \omega'_{12} = d\omega'_{p12}$, with ω'_{p12} = (ϵ-Thom class of σ_{p12}) + ξ_{p12}, where ξ_{p12} is supported near J, for $p = 1, 2$. The algebraic compatibility equations $\eta' \wedge \omega'_1 = \eta' \wedge \omega'_2 = \eta' \wedge \omega'_{12} = 0$ (coming from disjoint supports) and the differential equations satisfied by ω'_{12}, ω'_{112} and ω'_{212} give the integrability condition $1(3)_4$. Taking homology classes of $A'\hat{\otimes}\mathsf{L}^4(x'_1, x'_2)$-components in 1(5) we find out that

$$cl(\omega'_1 \wedge \omega'_{112}) \otimes [x'_1, [x'_1, [x'_1, x'_2]]] + cl(\omega'_2 \wedge \omega'_{212}) \otimes [x'_2, [x'_2, [x'_1, x'_2]]] +$$
$$+ cl(\omega'_1 \wedge \omega'_{212} + \omega'_2 \wedge \omega'_{112}) \otimes [x'_1, [x'_2, [x'_1, x'_2]]] = -cl(\eta') \otimes \partial'_4 y'.$$

On the other hand the initial term $\partial'_4 y'$ equals $[x'_1, [x'_2, [x'_1, x'_2]]]$, see Example 2.4, whence we get $cl(\omega'_p \wedge \omega'_{p12}) = 0$ for $p = 1, 2$ and $cl(\omega'_1 \wedge \omega'_{212} + \omega'_2 \wedge \omega'_{112}) = -cl(\eta')$ (equalities in $H^2(\mathbf{S}^3 \setminus L')$).

By naturality the above initial-term computations for L' may be used for non-initial computations for L as follows. As immediately seen, ∂_2 of L is the same as ∂_2 of the previous Mickey Mouse link. Set $\partial_{\leq 3} = \partial_2$ and

$$\Omega^{\leq 3} = \sum_{i=0}^{2} \omega_i \otimes x_i + \sum_{p=1}^{2} \eta_p \otimes y_p + \omega_{12} \otimes [x_1, x_2] + \sum_{p=1}^{2} \omega_{p12} \otimes [x_p, [x_1, x_2]].$$

We claim that this defines a connection of order 3 on $A = \Omega^*_{dR}(\mathbf{S}^3 \setminus L)$, as soon as we follow the intersectional pattern of L' and take ω_i = image of ω'_i in A (and similarly for ω_{12} and ω_{p12}) and $\eta_p = \omega_0 \wedge \omega_p$. As before, $1(3)_4$ follows from disjointedness of supports plus the naturally induced differential equations in A. Take now homology classes of $A\hat{\otimes}\mathsf{L}^4(x_0, x_1, x_2)$-components in 1(5) to get $-\sum_{p=1}^{2} cl(\eta_p) \otimes \partial_4 y_p = \sum_{p=1}^{2} cl(\omega_0 \wedge \omega_{p12}) \otimes [x_0, [x_p, [x_1, x_2]]] + \sum_{p=1}^{2} cl(\omega_p \wedge \omega_{p12}) \otimes [x_p, [x_p, [x_1, x_2]]] + cl(\omega_1 \wedge \omega_{212} + \omega_2 \wedge \omega_{112}) \otimes [x_1, [x_2, [x_1, x_2]]]$. The first sum of the right hand side vanishes (by disjoint supports) and the rest equals $(cl(\eta_1) - cl(\eta_2)) \otimes [x_1, [x_2, [x_1, x_2]]]$, by naturality and the information provided by L'. Finally we get

$$\partial_4 y_1 = -[x_1, [x_2, [x_1, x_2]]] \text{ and } \partial_4 y_2 = [x_1, [x_2, [x_1, x_2]]].$$

We end by illustrating the usefulness of the decision algorithm 1.8.

4.5. *Example.* We are going to assemble the previous two examples. As we have just seen the P_4-invariants of the above two link complements have representatives in $\mathscr{M}_4$ which are equal in $\mathscr{M}_3$. Consequently their link groups modulo 3-fold commutators have ($h_{1,2}$-marked) isomorphic Mal'cev completions (by Theorem 1.2). To distinguish them we have to resort to higher terms and work in $\mathscr{M}_4$, following the recipe 1.8. We may take $u_3 = 1$ and $\Delta = \partial_4$ of the modified Whitehead link. Noting that in our case $p_2 = 0$ and p_1 = common ∂_2 of Examples 4.3 and 4.4, it follows from 1(7–8) that their equality in $\mathcal{M}_4$ is equivalent with $\partial_4 = \theta\partial_2 - \partial_2\theta$, for some $\theta \in \mathrm{Der}_0^2\mathsf{L}(X \oplus Y)$. This in turn implies the equalities

$$\begin{aligned} -[x_1,[x_2,[x_1,x_2]]] &\equiv [x_1,\theta x_0] \text{ modulo } x_0 \\ [x_1,[x_2,[x_1,x_2]]] &\equiv [x_2,\theta x_0] \text{ modulo } x_0. \end{aligned}$$

Setting $x_0 = 0$ (which sends θx_0 to some $r \in \mathsf{L}^3(x_1,x_2)$) gives $[x_1+x_2,r] = 0$, hence $r = 0$, therefore $[x_1,[x_2,[x_1,x_2]]] = 0$, a contradiction.

It is actually easy to see now that the two link invariants are still distinct in $\mathcal{M}_4$ mod $\Sigma_3 \ltimes \mathbf{Z}_2^3$, hence the two links are not isotopic (without any restrictions on numbering or orientation on the isotopy, see 1.9)

Acknowledgement. The referee's remarks helped us with improving and simplifying the exposition, especially in Section 1 (e.g. Corollary of Proposition 1.3, Remark 1.9).

References

1. Anick, D.: Inert sets and the Lie algebra associated to a group. Journ. of Algebra, **111(1)**, 154–165 (1987)
2. Anick, D.: Non-commutative graded algebras and their Hilbert series. Journal of Algebra, **78(1)**, 120–140 (1982)
3. Berceanu, B. and Papadima, Şt.: Cohomologically generic 2-complexes and 3-dimensional Poincaré complexes, submitted
4. Berceanu, B. and Papadima, Şt.: Moduli spaces for generic low-dimensional complexes, to appear in Journ. Pure Appl. Algebra
5. Bott, R. and Tu, L.W.: Differential Forms in Algebraic Topology. Springer-Verlag, 1982
6. Bousfield, A.K. and Gugenheim, V.K.A.M.: On PL deRham Theory and Rational Homotopy Type, volume 179 of Memoirs of the AMS, 1986
7. Cenkl, B. and Porter, R.: Malcev's completion of a group and differential forms. Journ. of Differential Geometry, **15**, 531–542 (1980)
8. Chen, K.-T.: Extension of C^∞ function algebra by integrals and Malcev completion of π_1. Adv. in Mathem., **23**, 181–210 (1977)
9. Cochran, T.D.: Derivatives of Links: Milnor's Concordance Invariants and Massey's Products, volume 427 of Memoirs of the AMS, 1990
10. Félix, Y.: Modèles bifiltrés: une plaque tournante en homotopie rationnelle. Canad. J. Math., **33(6)**, 1448–1458 (1981)
11. Fenn, R.: Techniques of Geometric Topology. Cambridge Univ. Press, 1983
12. Griffiths, P.A and Morgan, J.W.: Rational Homotopy Theory and Differential Forms. Progress in Math. 16. Birkhäuser, 1981
13. Habegger, N. and Lin, X.-S.: The classification of links up to link-homotopy. Journal of AMS, **3(2)**, 389–419 (1990)
14. Hain, R.M.: Iterated integrals, intersection theory and link groups. Topology, **24(1)**, 45–66 (1985)

15. Hain, R.M. and Durfee, A.: Mixed Hodge structures on the homotopy of links. Math. Ann., **280**, 69–83 (1988)
16. Halperin, S. and Lemaire, J.-M.: Suites inertes dans les algèbres de Lie graduées. Mathematica Scandinavica, **61**, 39–67 (1987)
17. Halperin, S. and Stasheff, J.: Obstructions to homotopy equivalences. Advances in Math., **32**, 233–279 (1979)
18. Hilton, P.J. and Stammbach, U.: A Course in Homological Algebra. Graduate texts in Mathematics 4. Springer-Verlag, 1971
19. Labute, J.P.: The determination of the Lie algebra associated to the lower central series of a group. Trans. Amer. Math. Society, **288(1)**, 51–57 (1985)
20. Lehmann, D.: Théorie Homotopique des Forms Différentielles, volume 54 of Astérisque, 1977
21. Markl, M.: Formal computations in low dimensional topology: Links and group presentations. Rend. Circ. Mat. Palermo 1992, to appear
22. Markl, M. and Papadima, Şt.: Homotopy Lie algebras and fundamental groups via deformation theory. Ann. Inst. Fourier, **42(4)**, 905–935 (1992)
23. May, J.P.: Matrix Massey products. Journal of Algebra, **12**, 533–568 (1969)
24. Milnor, J.: Isotopy of links. In Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz, pages 280–306. Princeton Univ. Press, 1957
25. Porter, R.: Milnor's $\overline{\mu}$-invariants and Massey products. Trans. Amer. Math. Soc., **257**, 39–71 (1980)
26. Quillen, D.: Rational homotopy theory. Annals of Mathematics, **90**, 205–295 (1969)
27. Schlessinger, M. and Stasheff, J.D.: Deformation theory and rational homotopy type, preprint
28. Sullivan, D.: Infinitesimal computations in topology. Publ. IHES, **47**, 269–331 (1977)
29. Tanré, D.: Homotopie Rationnelle: Modèles de Chen, Quillen, Sullivan. Lect. Notes in Math. 1025. Springer-Verlag, 1983

Mathematical Institute of the Academy
Žitná 25
115 67 Praha 1
Czechoslovakia
e-mail: markl@csearn.bitnet

Institute of Mathematics of the Academy
P.O. Box 1-764
RO 70700 Bucharest
Romania

(Received January 20, 1992;
in revised form June 15, 1993)

manuscripta math. 81, 243 - 262 (1993)

manuscripta mathematica

RELATIVE COHERENCE AND PREENVELOPES

Nanqing Ding and Jianlong Chen

We define coherence relative to an arbitrary torsion theory and characterize it in terms of preenvelopes of modules. In particular, some known results are obtained as corollaries.

1. Introduction

Let R be a ring. Chase [5] characterized right coherent rings as the rings R for which any direct product of R is a flat left R—module. His characterization of coherence has led to a number of similar characterizations of coherence with respect to particular torsion theories (see, e. g., [13, 14, 16] and their references). On the other hand, right coherent rings are characterized by the condition that every left R—module has a flat preenvelope [8], or equivalently, every finitely presented left R—module has a flat preenvelope [1]. In this paper, we introduce the concepts of flat modules and coherent rings with respect to an arbitrary torsion theory (called τ—flat modules and τ—coherent rings respectively) in such a way that, when one considers the trivial torsion theory in which every module is torsion—free (torsion), τ—flat modules and τ—coherent rings are flat modules and coherent rings (finitely projective modules and Π—coherent rings) respectively, and generalize the above results to the τ—coherent rings. It is shown that R is right τ—coherent if and only if every left R—module has a τ—flat preenvelope (see Theorem 3.10); R is right τ—coherent and injective left R—modules are τ—flat if and only if every left R—module has a τ—flat preenvelope which is a monomorphism (see Theorem 4.1); and R is right τ—coherent and submodules of τ—flat left R—modules are τ—flat if and only if every left R—module has a τ—flat preenvelope which is an epimorphism (see Theorem 5.1). In particular, some known results appearing in [1, 2, 4, 6, 8, 9] are obtained as corollaries. In the meantime, right FC rings, right semihereditary rings, strongly right coherent rings, right coherent and left perfect rings and quasi—Frobenius rings are characterized in terms of preenvelopes of modules.

2. Preliminaries

In this section we shall recall some known notions and facts which we need in the later sections.

1) Finite (local) projectivity. An R—module M is called finitely (resp. locally) projective [2, 3] if, for any finitely generated submodule M_0 of M, there exist a finitely generated free module F and homomorphisms $f: M_0 \to F$ (resp. $f: M \to F$) and $g: F \to M$ such that $g(f(x))=x$ for all $x \in M_0$. The concept of finitely projective modules was called f—projective modules by Jones [14], while the concept of locally projective modules was called universally torsionless modules by Garfinkel [11]. Clearly, every locally projective module is finitely projective. In general, projective $\Rightarrow$ locally projective $\Rightarrow$ finitely projective $\Rightarrow$ flat, but no two of these concepts are equivalent (see [2] and [14]).

2) Preenvelopes. A projective preenvelope of an R—module M is a homomorphism $f: M \to P$ with P a projective module such that for any $g: M \to Q$ with Q projective there exists $h: P \to Q$ with $g = hf$. If, furthermore, the endomorphisms h of P satisfying $hf = f$ are automorphisms, then f is called a projective envelope of M. By analogy, locally, finitely projective, flat and other types of (pre)envelopes can be defined [8].

3) Coherent rings. A ring R is said to be right coherent if every finitely generated right ideal of R is finitely presented , or equivalently, any direct product of R is a flat left R—module [5]. R is called right Π—coherent [4, 10] if every finitely generated torsionless right R—module is finitely present-

ed, or equivalently, any direct product of R is a finitely projective left R—module [14]. R is called a left * —ring (star ring) [4, 10] provided that every finitely generated left R—module has finitely generated dual. R is called strongly right coherent [20] if any direct product of R is a locally projective left R—module.

4) Notations and conventions. Throughout this paper, R will denote an associative ring with identity and all modules will be unitary. R—Mod will denote the category of left R—modules. $\tau = (T, F)$ will mean a torsion theory for R—Mod. Let M be an R—module, and I a set, M^I will be the direct product of copies of M indexed by I, $M^* = Hom_R(M, R)$ will denote the dual module of M, and the injective envelope of M will be denoted by E(M). For R—modules M, N, Hom(M, N) will denote $Hom_R(M, N)$, and similarly $M\otimes N$ will mean $M\otimes_R N$.

Let P and M be left R—modules. There is a natural homomorphism

$$\sigma = \sigma_{P,M}: P^*\otimes M \rightarrow Hom(P, M)$$

defined via $\sigma(f\otimes m)(p) = f(p)m$ for $f \in P^*$, $m \in M$, $p \in P$. If P is finitely generated projective, then σ is an isomorphism [6].

For each right R—module M and each family $\{L_i\}_{i\in I}$ of left R—modules, there is a natural homomorphism

$$\varphi: M\otimes\prod_{i\in I} L_i \rightarrow \prod_{i\in I}(M\otimes L_i)$$

given by $\varphi(x\otimes(z_i)) = (x\otimes z_i)$. In particular, there is a natural homomorphism $\varphi: M\otimes R^I \rightarrow M^I$ given by $\varphi(x\otimes(a_i)) = (xa_i)$ [19].

Our reference for background in ring theory will be Stenström [19].

3. Relative coherence and preenvelopes

Definition 3. 1. Let R be a ring and $\tau = (T, F)$ a torsion theory for R—Mod. A left R—module M is said to be τ — *finitely generated* if

M/M' $\in$ T for some finitely generated submodule M' of M. M is said to be τ—*finitely presented* if there exists an exact sequence $0 \to K \to F \to M \to 0$ with F finitely generated free and K τ—finitely generated. M is called τ—*flat* if every homomorphism from a τ—finitely presented left R—module into M factors through a finitely generated free module, i. e. , for any τ—finitely presented left R—module P and any homomorphism f: $P \to M$, there exist a finitely generated free module F and homomorphisms g: $P \to F$ and h: $F \to M$ such that f = hg. A submodule M of a left R—module N is called τ—*pure in N* if the natural map Hom(P, N) $\to$ Hom(P, N/M) is epic for all τ—finitely presented left R—modules P.

Remark 1. (1. 1) Every module in T is τ—finitely generated. If M is finitely generated (resp. finitely presented), then M is τ—finitely generated (resp. τ—finitely presented). Each τ—finitely presented module is finitely generated. Every τ—finitely presented τ—flat module is projective. Every τ—pure submodule is pure.

(1. 2) If T = $\{0\}$, then M is τ—finitely generated (τ—finitely presented) if and only if M is finitely generated (finitely presented). If T = R—Mod, then M is τ—finitely presented if and only if M is finitely generated.

(1. 3) It is clear that every finitely projective module is τ—flat, and every τ—flat module is flat. If T = $\{0\}$ (resp. R—Mod), then M is τ—flat if and only if M is flat (resp. finitely projective).

(1. 4) Suppose that $0 \to E \to F \to G \to 0$ is exact. It is easy to see that E is τ—pure in F if G is τ—flat . Conversely if F is τ—flat and E is τ—pure in F then G is τ—flat.

Definition 3. 2. We say that R is right τ—*coherent* if any direct product of R is a τ—flat left R—module.

Remark 2. (2. 1) The τ—coherence defined here is different from those introduced by Jones and others (see, e. g. , [13, 16] and their references).

(2. 2) Let T=$\{0\}$ (resp. R—Mod), then τ—coherence coincides with the usual definition of coherence (resp. Π—coherence).

(2. 3) It is clear that τ—coherent rings with respect to every torsion theory are coherent, but a coherent ring need not be τ—coherent. For example,

there are coherent rings that are not Π—coherent [4].

Lemma 3. 3. *The following are equivalent for a left R—module M.*

1) *M is τ—flat.*

2) *For every τ—finitely presented left R—module P, $\sigma_{P,M}$ is an isomorphism.*

3) *For every τ—finitely presented left R—module P, $\sigma_{P,M}$ is an epimorphism.*

Proof. 1) ⇒ 2). Let P be a τ—finitely presented left R—module and f $\in$ Hom(P, M). Since M is τ—flat, f factors through a finitely generated free module R^n, i. e. , there exist $g: P \to R^n$ and $h: R^n \to M$ such that $f = hg$. If $p_i: R^n \to R$ be the ith projection and $\lambda_i: R \to R^n$ the ith injection, put $g_i = p_i g$ and $m_i = h(\lambda_i(1))$, then $g_i \in P^*$ and $m_i \in M$, $i = 1, 2, \cdots, n$. It is easy to check $f = \sigma_{P,M}(\sum_{i=1}^{n} g_i \otimes m_i)$. This shows $\sigma_{P,M}$ is an epimorphism. On the other hand, since P is finitely generated, there exists a finitely generated free module F such that $F \to P \to 0$ is exact. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0 & \to & P^* \otimes M & \to & F^* \otimes M \\ & & \downarrow \sigma_{P,M} & & \downarrow \sigma_{F,M} \\ 0 & \to & \mathrm{Hom}(P, M) & \to & \mathrm{Hom}(F, M). \end{array}$$

Since $\sigma_{F,M}$ is an isomorphism, $\sigma_{P,M}$ is a monomorphism. Consequently, 2) follows.

2) ⇒ 3) is trivial.

3) ⇒ 1) . Let P be any τ—finitely presented left R—module and $f \in \mathrm{Hom}(P, M)$. By 3), $f = \sigma_{P,M}(\sum_{i=1}^{n} g_i \otimes m_i)$ for some $g_i \in P^*$ and $m_i \in M$, $i = 1, 2, \cdots, n$. Put $F = R^n$, and define $g: P \to F$ by $g(x) = (g_1(x), \cdots, g_n(x))$ for $x \in P$, $h: F \to M$ by $h(r_1, \cdots, r_n) = \sum_{i=1}^{n} r_i m_i$ for $r_i \in R$, $i = 1, 2, \cdots, n$. Then $f = hg$, and so 1) holds. □

Remark 3. The Lemma 3. 3 parallels Lazard's theorem on flat modules [6, p. 240].

To illustrate τ—flat modules, we give the following three examples.

Example 1. Let $\tau=(T,F)$ with $T=\{X \mid X^{*}=0\}$ and M a left R—module. The M is τ—flat if and only if M is flat. In fact, the "only if" part is clear. Conversely, suppose that M is flat. Let N be any τ—finitely presented left R—module and $f \in Hom(N, M)$. There is an exact sequence

$$0 \to K \to P \to N \to 0$$

in which P is a finitely generated free module and K contains a finitely generated submodule L such that $(K/L)^{*}=0$. Thus we get an exact sequence

$$0 \to K/L \xrightarrow{j} P/L \xrightarrow{\pi} N \to 0.$$

Since P/L is finitely presented and M is flat, there exist a finitely generated free module F and homomorphisms $h: P/L \to F$ and $k: F \to M$ such that $f\pi = kh$. But $(K/L)^{*}=0$, and hence $Hom(K/L, F)=0$. Now $hj \in Hom(K/L,F)$, it follows that $K/L \subseteq Ker\ h$. Thus there exists $t: N \to F$ such that $t\pi = h$. Therefore we have $f\pi=(kt)\pi$, whence $f=kt$ (for π is epic), which proves that M is τ—flat.

In general, a flat module need not be τ—flat, as shown by the following example.

Example 2. Let F be any field, $R=\prod_{I} F$ (an infinite product of F) and $K=\bigoplus_{I} F$. If $\tau=(T,F)$ is generated by K (in the sense of [19, p. 139], then R/K is flat but not τ—flat. In fact, R/K is flat since R is von Neumann regular. Now K is τ—finitely generated (note that $K \in T$), so that R/K is τ—finitely presented. If R/K is τ—flat, then it must be projective, whence K is a direct summand of R. But it is easily seen that K is not a direct summand of R. Thus R/K is not τ—flat.

Example 3. Let R be a left self—injective ring and M an injective left R—module. If $\tau=(T,F)$ is cogenerated by M, then M is τ—flat if and only if M is flat and $K^{*} \otimes M=0$ for every torsion submodule K of a finitely presented left R—module.

First we claim that if M is flat and there is an exact sequence of left R—modules

$$0 \to K \to N \to P \to 0$$

with $K \in T$, N finitely presented and P τ—finitely presented, then $\sigma_{P,M}$ is an isomorphism if and only if $K^* \otimes M = 0$. In fact, by the hypotheses, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & P^* \otimes M & \to & N^* \otimes M & \to & K^* \otimes M & \to & 0 \\ & & \downarrow \sigma_{P,M} & & \downarrow \sigma_{N,M} & & \downarrow \sigma_{K,M} & & \\ 0 & \to & \mathrm{Hom}(P,M) & \to & \mathrm{Hom}(N,M) & \to & \mathrm{Hom}(K,M) & \to & 0. \end{array}$$

Since M is flat and N is finitely presented, $\sigma_{N,M}$ is an isomorphism by Lemma 3. 3. Thus $\sigma_{P,M}$ is an isomorphism if and only if $\sigma_{K,M}$ is an isomorphism by Five Lemma ([18, Lemma 3. 32]). Now K is torsion, and so $\mathrm{Hom}(K,M)=0$. Therefore $\sigma_{P,M}$ is an isomorphism if and only if $K^* \otimes M = 0$.

If M is τ—flat, then M is obviously flat. Let K be a torsion submodule of a finitely presented left R—module N, then

$$0 \to K \to N \to P \to 0$$

is exact, where $P=N/K$. In view of [13, Corollary 2. 6], P is τ—finitely presented (for K is τ—finitely generated). Thus $K^* \otimes M = 0$ follows from the first part of the proof and Lemma 3. 3.

Conversely, for any τ—finitely presented left R—module P, there exists an exact sequence

$$0 \to L \to F \to P \to 0$$

where F is a finitely generated free module and L contains a finitely generated submodule L_0 such that $L/L_0 \in T$. Put $K=L/L_0$ and $N=F/L_0$. Then $K \in T$, N is finitely presented and the sequence

$$0 \to K \to N \to P \to 0$$

is exact. By hypothesis, M is flat and $K^* \otimes M = 0$. Hence $\sigma_{P,M}$ is an isomorphism by the first part of the proof. So M is τ—flat by Lemma 3. 3.

The next proposition shows that right τ—coherent rings are precisely those rings for which τ—flat left modules are preserved by direct products.

Proposition 3. 4. *The following are equivalent.*

1) *R is right τ—coherent.*

2) *Any direct product of τ—flat left R—modules is τ—flat.*

3) *P^* is finitely presented (finitely generated) for every τ—finitely presented left R—module P.*

Proof. 1) ⇒ 3). Let P be a τ—finitely presented left R—module. For every index set I, we have the following commutative diagram:

$$\begin{array}{ccc} P^* \otimes R^I & \xrightarrow{\sigma} & \mathrm{Hom}(P, R^I) \\ \downarrow \varphi & & \downarrow \theta \\ (P^*)^I & \xrightarrow{1} & (P^*)^I \end{array},$$

where θ is the canonical isomorphism. Since R^I is τ—flat by 1), σ is an isomorphism by Lemma 3. 3. Thus φ is an isomorphism, and hence P^* is finitely presented by [19, Lemma 13. 2].

3) ⇒ 2). Let $\{M_i\}_{i\in I}$ be a family of τ—flat left R—modules. The following diagram is commutative for any τ—finitely presented left R—module P:

$$\begin{array}{ccc} P^* \otimes \prod\limits_{i\in I} M_i & \xrightarrow{\sigma} & \mathrm{Hom}(P, \prod\limits_{i\in I} M_i) \\ \downarrow \varphi & & \downarrow \theta \\ \prod\limits_{i\in I} (P^* \otimes M_i) & \xrightarrow{t} & \prod\limits_{i\in I} \mathrm{Hom}(P, M_i) \end{array},$$

where θ is the canonical isomorphism and $t = \prod\limits_{i\in I} \sigma_{P,M_i}$ is an isomorphism for each M_i is τ—flat. Since P^* is finitely generated by 3), φ is an epimorphism by [19, Lemma 13.1]. Then σ is an epimorphism, whence $\prod_{i\in I} M_i$ is τ—flat by Lemma 3.3, i.e., 2) follows.

2) ⇒ 1) is trivial. □

Let T = {0} in Proposition 3.4, we have

Corollary 3.5. ([6, *Proposition* 1]). *For any ring R, the following are equivalent.*

1) *R is right coherent.*

2) *P^* is finitely generated for every finitely presented left R—module P.*

3) *P^* is finitely presented for every finitely presented left R—module P.*

If T = R—Mod, then we get

Corollary 3.6. *The following are equivalent for any ring R.*

1) *R is right Π—coherent.*

2) *Any direct product of finitely projective left R—modules is finitely projective.*

3) *R is a left ∗—ring.*

Remark 4. The equivalence of 1) and 3) in the Corollary 3.6 above is due to Camillo [4].

The following lemma shows that the τ—flatness of a module is inherited by pure submodules.

Lemma 3.7. *Every pure submodule of a τ—flat left R—module is τ—flat.*

Proof. Let M be a τ—flat left R—module and N a pure submodule of M, choose a free module F such that $F \xrightarrow{\beta} M \to 0$ is exact. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & K & \xrightarrow{\lambda} & F & \xrightarrow{\pi\beta} & M/N & \to & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow 1 & & \\ 0 & \to & N & \xrightarrow{j} & M & \xrightarrow{\pi} & M/N & \to & 0, \end{array}$$

where K = Ker $\pi\beta$, λ and j are inclusion maps. Diagram—chasing shows that there exists a homomorphism $\alpha: K \to N$ making the left square commute. For any τ — finitely presented left R — module P and any $f \in \mathrm{Hom}(P,N)$, $jf \in \mathrm{Hom}(P,M)$. Since M is τ—flat, there exist a finitely generated free module F_1 and homomorphisms $g: P \to F_1$ and $h: F_1 \to M$ such that $jf = hg$. By the projectivity of F_1, there is a homomorphism $l: F_1 \to F$ such that $h = \beta l$. Thus we have

$$\pi\beta lg(P) = \pi hg(P) = \pi jf(P) = 0,$$

which implies that $lg(P) \subseteq K$. Since P is finitely generated, $lg(P)$ is a finitely generated submodule of K. By hypothesis, N is pure in M, whence M/N is flat (for M is flat). From [18, Theorem 3. 57] we get a homomorphism $k: F \to K$ such that $klg(p) = lg(p)$ for all $p \in P$. Put $h_1 = \alpha kl$, then, for any $p \in P$, we have

$$\begin{aligned} h_1 g(p) &= jh_1 g(p) = j\alpha klg(p) = \beta\lambda klg(p) \\ &= \beta klg(p) = \beta lg(p) = hg(p) = jf(p) = f(p), \end{aligned}$$

i. e. , $f = h_1 g$, which shows that N is τ—flat. □

Let $T = R$—Mod in Lemma 3. 7, we have

Corollary 3. 8. [2, *Proposition* 14]. *Every pure submodule of a finitely projective module is finitely projective.*

Since every τ—pure submodule is pure, we get

Corollary 3. 9. *Every τ—pure submodule of a τ—flat left R—module is τ—flat.*

Theorem 3. 10. *The following are equivalent.*

1) *R is right τ—coherent.*

2) *Every τ—finitely presented left R—module has a τ—flat preenvelope.*

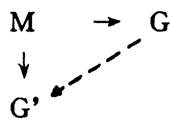
3) *Every left R—module has a τ—flat preenvelope.*

4) *Every τ—finitely presented left R—module has a finitely projective preenvelope.*

5) *Every τ—finitely presented left R—module has a (locally) projective preenvelope.*

Proof. 1) ⇒ 3). The proof is obtained by adapting Enochs' arguments [8, Proposition 5. 1]. If R is any ring and $\aleph_\alpha$ is an infinite cardinal, there is an infinite cardinal $\aleph_\beta$ such that if S is a submodule of a τ—flat module F

with Card(S) $\leqq \aleph_\alpha$, there is a pure, hence τ—flat by Lemma 3. 7, submodule G of F with S $\subseteqq$ G and Card(G) $\leqq \aleph_\beta$. This observation means that if M is any left R—module with Card(M) $\leqq \aleph_\alpha$, any homomorphism M $\to$ F with F τ—flat can be "cut down" to a homomorphism M $\to$ G, G $\subseteqq$ F, Card(G) $\leqq \aleph_\beta$, G τ—flat, which agrees with the original. Setting two such homomorphisms M $\to$ G, M $\to$ G' equivalent if



can be completed by an isomorphism, and letting X be the set of representatives of such M $\to$ G, M $\to$ $\prod$G will be a τ—flat preenvelope if R is right τ—coherent, since $\prod$G is τ—flat by Proposition 3. 4.

3) $\Rightarrow$ 2) is trivial.

2) $\Rightarrow$ 5). Let M be a τ—finitely presented left R—module, then M has a τ—flat preenvelope ψ: M $\to$ F. Hence ψ factors through a finitely generated free module F_1, i. e. , there exist g: M $\to$ F_1 and h: F_1 $\to$ F such that ψ = hg. It is easily seen that g: M $\to$ F_1 is a (locally) projective preenvelope of M, i. e. , 5) holds.

5) $\Rightarrow$ 4) follows since every (locally) projective preenvelope of a τ—finitely presented left R—module M is a finitely projective preenvelope of M.

4) $\Rightarrow$ 1). We shall show that, for each index set I, R^I is a τ—flat left R—module. Let M be any τ—finitely presented left R—module. By 4), M has a finitely projective preenvelope g: M $\to$ P, then g factors through a finitely generated free module F (for M is finitely generated), i. e. , there are f: M $\to$ F and h: F $\to$ P such that g = hf. Thus f: M $\to$ F is a projective preenvelope of M, and hence the sequence

$$\mathrm{Hom}(F, Q) \to \mathrm{Hom}(M, Q) \to 0$$

is exact for every projective left R—module Q. In particular, we have an exact sequence

$$F^* \to M^* \to 0.$$

Hence, for each index set I,

$$(F^*)^I \to (M^*)^I \to 0$$

is exact. Now since $(F^*)^I \cong \mathrm{Hom}(F, R^I)$ and $(M^*)^I \cong \mathrm{Hom}(M, R^I)$, we have that every homomorphism from M to R^I factors through a finitely generated free module F, which shows that R^I is a τ—flat left R—module. So 1) follows. □

Specializing Theorem 3. 10 to the case T={0}, we have

Corollary 3. 11. *The following are equivalent for a ring R.*

1) *R is right coherent.*

2) *Every (finitely presented) left R—module has a flat preenvelope.*

3) *Every finitely presented left R—module has a (finitely or locally) projective preenvelope.*

Remark 5. The equivalence of 1) and 2) in the Corollary 3. 11 was shown in [1] and [8].

Let T = R—Mod in Theorem 3. 10, we get

Corollary 3. 12. *The following are equivalent for a ring R.*

1) *R is right Π—coherent.*

2) *Every (finitely generated) left R—module has a finitely projective preenvelope.*

3) *Every finitely generated left R—module has a (locally) projective preenvelope.*

When does every left R — module have a (locally) projective preenvelope ? we obtain

Proposition 3. 13. *The following are equivalent for a ring R.*

1) *R is strongly right coherent.*

2) *Every left R—module has a locally projective preenvelope.*

Proof. 1) ⇒ 2). By [3, Proposition 7], every pure submodule of a locally projective module is locally projective. By [20, Theorem 4. 2], R is strongly right coherent if and only if any direct product of locally projective left R—modules is locally projective. Thus, using the proof of 1) ⇒ 3) in Theorem 3.10, we get 1) ⇒ 2).

2) ⇒ 1). We shall show that, for any index set I, R^I is a locally projective left R—module. By 2), R^I has a locally projective preenvelope $R^I \to P$, hence

$$P^* \to (R^I)^* \to 0$$

is exact. This yields the exactness of

$$(P^*)^I \to ((R^I)^*)^I \to 0.$$

But $(P^*)^I \cong \mathrm{Hom}(P, R^I)$ and $((R^I)^*)^I \cong \mathrm{Hom}(R^I, R^I)$, whence

$$\mathrm{Hom}(P, R^I) \to \mathrm{Hom}(R^I, R^I) \to 0$$

is exact. This shows that R^I is locally projective, and so 1) follows. □

Proposition 3.14. *The following are equivalent for a ring R.*

1) *R is right coherent and left perfect.*

2) *Every left R—module has a projective preenvelope.*

Proof. 1) ⇒ 2) by Corollary 3.11. 2) ⇒ 1) by [5, Theorem 3.3] and the proof of 2) ⇒ 1) in Proposition 3.13. □

Remark 6. By definitions, we have the following chain of implications:

R is right coherent and left perfect ⇒ R is strongly right coherent ⇒ R is right Π—coherent ⇒ R is right coherent.

No two of these properties are equivalent in general. Let R be a right Noetherian but not right Artinian ring, then R is strongly right coherent but not left perfect, hence the converse of the first implication is false. So is the converse of the third implication (see [4, p. 76]). A counterexample to the second is immediate: Let k be a semiprime commutative Noetherian ring and $R = k[X_1, X_2, \cdots, X_n, \cdots]$, then R is Π—coherent by [4, Theorem 6] but not strongly coherent by [11, Example 5.2].

Remark 7. It is interesting to compare the results of Corollary 3.11 and 3.12 and Proposition 3.13 and 3.14. We have that R is right coherent (resp. right Π—coherent, strongly right coherent, right coherent and left perfect) if and only if every left R—module has a flat (resp. finitely projective, locally projective, projective) preenvelope.

4. Preenvelopes which are monomorphisms

In Section 3, we prove that R is right τ—coherent if and only if every left R—module has a τ—flat preenvelope. In general, a τ—flat preenvelope need not be a monomorphism. In this section, we investigate when every left R—module has a τ—flat preenvelope which is a monomorphism.

Theorem 4.1. *The following are equivalent.*

1) *R is right τ—coherent and injective left R—modules are τ—flat.*

2) *Every τ—finitely presented left R—module has a τ—flat preenvelope which is a monomorphism.*

3) *Every left R—module has a τ—flat preenvelope which is a monomorphism.*

4) *Every τ—finitely presented left R—module has a finitely projective preenvelope which is a monomorphism.*

5) *Every τ—finitely presented left R—module has a (locally) projective preenvelope which is a monomorphism.*

Proof. 1) ⇒ 3). By Theorem 3.10, every left R—module M has a τ—flat preenvelope $f: M \to F$. Since E(M) is τ—flat by 1), f must be a monomorphism, i.e., 3) holds.

3) ⇒ 2) is trivial.

2) ⇒ 5) ⇒ 4) is similar to that of Theorem 3.10.

4) ⇒ 1). R is right τ—coherent by Theorem 3.10. Let E be an injective left R—module. For any τ—finitely presented left R—module M and $f \in \mathrm{Hom}(M, E)$, there exists a finitely generated free module F such that $i: M \to F$ is a monomorphism by 4) since M is finitely generated. By the injectivity of E, there is $g: F \to E$ with $f = gi$. This shows that f factors through F, and hence E is τ—flat by definition. □

Recall that a ring R is called left IF if every injective left R—module is flat [6]. R is called right FC if R is right self—FP—injective and right coherent [7]. R is right FC if and only if R is right coherent and left IF [12, Theorem 3.10]. Let T = {0} in Theorem 4.1, we have

Corollary 4.2. *The following are equivalent for a ring R.*

1) *R is right FC.*

2) *Every (finitely presented) left R—module has a flat preenvelope which is a monomorphism.*

3) *Every finitely presented left R—module has a (finitely or locally) projective preenvelope which is a monomorphism.*

We recall that a ring R is left FGF if every finitely generated left R—module embeds in a free left R—module [10]. R is left FGF if and only if every injective left R—module is finitely projective [14, Theorem 2.10]. Take T = R—Mod in Theorem 4.1, we have

Corollary 4.3. *For any ring R, the following are equivalent.*

1) *R is right Π—coherent and left FGF.*

2) *Every (finitely generated) left R—module has a finitely projective preenvelope which is a monomorphism.*

3) *Every finitely generated left R—module has a (locally) projective preenvelope which is a monomorphism.*

We conclude this section with the following easy results for completeness.

Proposition 4.4. *The following are equivalent for a ring R.*

1) *R is strongly right coherent and injective left R—modules are locally projective.*

2) *Every left R—module has a locally projective preenvelope which is a monomorphism.*

Proof. 1) ⇒ 2). Every left R—module M has a locally projective preenvelope f: M → P by Proposition 3.13. Since E(M) is locally projective, f is a monomorphism.

2) ⇒ 1) follows from Proposition 3.13. □

Proposition 4.5. *The following are equivalent for a ring R.*

1) *R is a quasi—Frobenius ring.*

2) *Every left R—module has a projective preenvelope which is a monomorphism.*

Proof. 1) ⇒ 2) by Proposition 3.14. 2) ⇒ 1) follows since every injective left R—module is projective. □

5. Preenvelopes which are epimorphisms

In this section we consider when every left R—module has a τ—flat preenvelope which is an epimorphism.

Theorem 5.1. *The following are equivalent.*

1) *R is right τ—coherent and submodules of τ—flat left R—modules are τ—flat.*

2) *Every τ—finitely presented left R—module has a τ—flat preenvelope which is an epimorphism.*

3) *Every left R—module has a τ—flat preenvelope which is an epimorphism.*

4) *Every τ—finitely presented left R—module has a finitely projective preenvelope which is an epimorphism.*

5) *Every τ—finitely presented left R—module has a (locally) projective preenvelope which is an epimorphism.*

Proof. 1) ⇒ 3). Since R is right τ—coherent, every left R—module M has a τ—flat preenvelope $f: M \to F$ by Theorem 3.10. Let $F_1 = \mathrm{Im}\, f$, then F_1 is τ—flat by 1). Hence $M \xrightarrow{f} F_1$ is a τ—flat preenvelope which is an epimorphism.

3) ⇒ 2) is trivial.

2) ⇒ 5). Let M be a τ—finitely presented left R—module, then M has a τ—flat preenvelope $\psi: M \to F$ which is an epimorphism. Since F is τ—flat, ψ factors through a finitely generated free module P, i.e., there exist $f: M \to P$ and $g: P \to F$ such that $\psi = gf$. But P is τ—flat, hence there exists $h: F \to P$ with $f = h\psi$. Thus $\psi = (gh)\psi$, and so $gh = 1_F$ since ψ is an epimorphism. Therefore F is finitely generated projective, and 5) follows.

5) ⇒ 4) is easy.

4) ⇒ 1). R is right τ—coherent by Theorem 3.10. Let N be a τ—flat

left R—module and N_1 a submodule of N. For any τ—finitely presented left R—module M and any $f \in Hom(M, N_1)$, let $j: N_1 \to N$ be the inclusion map, then jf factors through a finitely generated free module F, i. e. , there are $g: M \to F$ and $h: F \to N$ such that $jf = hg$. By 4), M has a finitely projective preenvelope $\psi: M \to P$ which is an epimorphism, then there is $t: P \to F$ with $g = t\psi$. Thus $jf = (ht)\psi$, whence $Ker\ \psi \subseteq Ker\ f$. Define $s: P \to N_1$ via $s(\psi(x)) = f(x)$ for $x \in M$. It is clear that s is well—defined and $f = s\psi$, i. e. , f factors through P. Since M is finitely generated, P is a finitely generated finitely projective module, and so P is projective. This shows that f factors through a finitely generated free module. Hence N_1 is τ—flat, and 1) follows. □

It is well known that R is right semihereditary if and only if R is right coherent and submodules of flat left R—modules are flat. Let $T = \{0\}$ in Theorem 5. 1, we have

Corollary 5. 2. *The following are equivalent for a ring R.*

1) *R is right semihereditary.*

2) *Every (finitely presented) left R—module has a flat preenvelope which is an epimorphism.*

3) *Every finitely presented left R—module has a (finitely or locally) projective preenvelope which is an epimorphism.*

Remark 8. We note that it was shown in [9, Theorem 3. 1] that R is right semihereditary if and only if every finitely presented left R—module has a projective preenvelope which is an epimorphism.

It is easy to see that R is left semihereditary if and only if submodules of finitely projective left R—modules are finitely projective (cf. , [14, p. 106]). Let T = R—Mod in Theorem 5. 1, we get

Corollary 5. 3. *The following are equivalent for a ring R.*

1) *R is right Π—coherent and left semihereditary.*

2) *Every (finitely generated) left R—module has a finitely projective preenvelope which is an epimorphism.*

3) *Every finitely generated left R—module has a (locally) projective preenvelope which is an epimorphism.*

Finally, we state the following results for the sake of completeness.

Proposition 5. 4. *The following are equivalent for a ring R.*

1) *R is strongly right coherent and submodules of locally projective left R—modules are locally projective.*

2) *Every left R—module has a locally projective preenvelope which is an epimorphism.*

Proof. 1) ⇒ 2). By Proposition 3. 13, every left R—module M has a locally projective preenvelope f: M → P. Thus M → f(M) is a locally projective preenvelope which is an epimorphism.

2) ⇒ 1). R is strongly right coherent by Proposition 3. 13. Let M be a locally projective left R—module and N a submodule of M. By 2), N has a locally projective preenvelope f: N → P which is an epimorphism. Let i: N → M be the inclusion map, then there exists g: P → M such that i = gf, which implies that f is a monomorphism. Thus N ≅ P is locally projective, whence 1) follows. □

Proposition 5. 5. *The following are equivalent for a ring R.*

1) *R is right semihereditary and left perfect.*

2) *Every left R—module has a projective preenvelope which is an epimorphism.*

Proof. 1) ⇒ 2) by Corollary 5. 2.

2) ⇒ 1). On the one hand R is right coherent and left perfect by Proposition 3. 14; on the other hand, R is right semihereditary by Corollary 5. 2. Consequently, R is right semihereditary and left perfect. □

6. Existence of flat envelopes

It is clear that preenvelopes considered in Section 5 are envelopes. In this section we turn to the existence of flat envelopes. For a ring R we will denote by wD(R) the weak global dimension of R. It is shown that if R is a ring with wD(R) ⩽ 2 then R is right coherent if and only if every left R—module has a flat envelope. This result removes the unnecessary hypothesis that R is commutative from [17, Theorem 2. 11].

Lemma 6. 1. *If R is right coherent with wD(R) ⩽ 2, then every finitely presented left R—module has a flat envelope.*

Proof. Let M be any finitely presented left R—module, then M^* is finitely generated projective by [15, Corollary 9]. Thus M has a flat envelope by [1, Proposition 1]. □

Theorem 6. 2. *Let R be a ring with wD(R) ⩽ 2, then the following are equivalent.*

1) *R is a right coherent ring.*

2) *Every left R—module has a flat envelope.*

Proof. 2)⇒1) by Corollary 3. 11. 1)⇒2) can be proved in the similar manner as in the proof of [17, Theorem 2. 11] using the Lemma 6. 1 above and [17, Proposition 2. 10]. □

Corollary 6. 3. *Let R be a ring with wD(R) ⩽ 2. Then the following are equivalent.*

1) *R is right coherent and left perfect.*

2) *Every left R—module has a projective envelope.*

Proof. 1) ⇒ 2) by Theorem 6. 2. 2) ⇒ 1) follows from Proposition 3. 14. □

Acknowledgment. We wish to thank the referee for the helpful suggestions which have been incorporated herein.

REFERENCES

[1] Asensio Mayor, J., Martinez Hernandez, J.: Flat envelopes in commutative rings. Israel J. Math. 62, 123—128(1988)

[2] Azumaya, G.: Finite splitness and finite projectivity. J. Algebra 106, 114—134(1987)

[3] Azumaya, G.: Some characterizations of regular modules. Publ. Mat. 34, 241—248(1990)

[4] Camillo, V.: Coherence for polynomial rings. J. Algebra 132, 72—76(1990)

[5] Chase, S. U.: Direct products of modules. Trans. Amer. Math. Soc. 97, 457—473(1960)

[6] Colby, R. R.: Rings which have flat injective modules. J. Algebra 35, 239—252(1975)

[7] Damiano, R. F.: Coflat rings and modules. Pacific J. Math. 81, 349—369(1979)

[8] Enochs, E. E.: Injective and flat covers, envelopes and resolvents. Israel J. Math. 39, 189—209(1981)

[9] Enochs, E. E. , Jenda, O. : Resolvents and dimensions of modules and rings. Arch. Math. 56, 528—532(1991)
[10] Faith, C. : Embedding torsionless modules in projectives. Publ. Mat. 34, 379—387(1990)
[11] Garfinkel, G. S. : Universally torsionless and trace modules. Trans. Amer. Math. Soc. 215, 119—144(1976)
[12] Jain, S. : Flat and FP—injectivity. Proc. Amer. Math. Soc. 41, 437—442(1973)
[13] Jones, M. F. : Coherence relative to a hereditary torsion theory. Comm. Algebra 10, 719—739(1982)
[14] Jones, M. F. : Flatness and f—projectivity of torsion free modules and injective modules. Lecture Notes in Math. 951, 94—116(1982)
[15] Jones, M. F. , Teply, M. L. : Coherent rings of finite weak global dimension. Comm. Algebra 10, 493—503(1982)
[16] Jones, M. F. , Teply, M. L. , Harui, H. : Relative coherence and TTF classes. Comm. Algebra 10, 2019—2029(1982)
[17] Martinez Hernandez, J. :Relatively flat envelopes. Comm. Algebra 14, 867—884 (1986)
[18] Rotman, J. J. : An Introduction to Homological Algebra. New York San Francisco London: Academic Press 1979
[19] Stenström, B. : Rings of Quotients. Berlin — Heidelberg — New York: Springer 1975
[20] Zimmermann—Huisgen, B. : Pure submodules of direct products of free modules. Math. Ann. 224, 233—245(1976)

Nanqing Ding

Department of Mathematics, Nanjing University

Nanjing, 210008, P. R. China

Jianlong Chen

Department of Mathematics and Mechanics, Southeast University

Nanjing, 210018, P. R. China

(Received December 8, 1992;
in revised form June 1, 1993)

manuscripta math. 81, 263 - 282 (1993)

manuscripta mathematica

A homological criterion for reducibility of analytic spaces, with application to characterizing the theta divisor of a product of two general principally polarized abelian varieties

Roy Smith and Robert Varley

A closed subset of pure codimension one in an analytic space, consisting entirely of local normal crossings double points, is called an ordinary rank two double locus. We give a topologically computable upper bound on the number of connected components of an ordinary rank two double locus in a given space. This leads to criteria for global reducibility of spaces. The first is that a simply connected space with a non empty ordinary rank two double locus is always reducible. A finer criterion implies that a principally polarized abelian variety A is isomorphic to a product of two positive dimensional principally polarized abelian varieties, each with smooth theta divisor, if and only if the theta divisor of A contains a non empty ordinary rank two double locus. Analogous reducibility results apply to certain complete intersection varieties, and to divisors on such varieties.

Introduction:

One of the prettiest results in elementary plane curve theory is the theorem that a reduced projective plane curve C of degree d with more than $(1/2)(d-1)(d-2) = p_a(C)$ singular points, must be reducible. One can see this from a topological point of view by choosing vanishing cycles on a generic smoothing C_t, one cycle near each singular point of C, hence mutually disjoint. If C is irreducible these cycles are independent, and the bound $\text{genus}(C_t) = p_a(C)$ on the maximal dimension of a totally isotropic subspace of $H_1(C_t)$ implies that the number of singular points of C is also bounded by $p_a(C)$.

In this paper we shall partially generalize this result to higher dimensional singularities to give a bound on the number of components the "ordinary rank two double locus" of an irreducible analytic space X can have, in terms of homological invariants of X. (The complete statement is in the theorem given at the beginning of section 1 below.) An ordinary rank two double locus is a pure codimension one closed subset of X, every point of which has an analytic neighborhood isomorphic to a neighborhood of a singular point on a transverse union of two smooth hypersurfaces. On a curve, such a locus is just a set of ordinary double points. In higher dimensions it represents the simplest codimension one singularities an analytic space can possess.

Applications of the reducibility criterion include the following: 1) If M is a smooth complete intersection of dimension ≥ 3 in $\mathbb{P}^m$, then an effective divisor X on M which has a non empty ordinary rank two double locus as its only singular points is reducible.

2) Recall that if a principally polarized abelian variety (A,Θ) is a product of two others $(A,\Theta) = (A_1,\Theta_1)\times(A_2,\Theta_2)$, and if both Θ_1,Θ_2 are smooth, then Θ (the theta divisor of A) has an ordinary rank two double locus as its singular locus. We prove conversely that for any p.p.a.v. (A,Θ) with dimension of A at least three, if the singular locus of Θ contains a non empty ordinary rank two double locus, then Θ is the transverse union of two smooth components, hence (A,Θ) is a product of two p.p.a.v.'s, each with smooth theta divisor. Thus a product of two general p.p.a.v.'s is characterized by the local geometry of the singularities of its theta divisor. The principal open question of this type (which we do not resolve), is whether in fact Θ is already reducible (and hence (A,Θ) a product of lower dimensional p.p.a.v.'s) whenever sing(Θ) contains an irreducible component of codimension one in Θ, without any further assumption on the nature of those singularities. This stronger result (the "$\mathfrak{N}_{g-2}$ conjecture") is at present apparently known only for p.p.a.v.'s of dimension $g \leq 5$, ([2], Th.4.10 p.170, Prop.6.4 p.177) The result in the present paper is to our knowledge the first global one on reducibility of theta divisors to be proved in all dimensions. (We gave a corresponding local result in ([12], p.254).) Since our general result does not apply to p.p.a.v's of dimension two, we give here also a short independent proof of the stronger theorem that a theta divisor on an abelian surface is reducible if and only if it is singular. In a separate note, we will give a class of examples, suggested by M. Nori, which show that irreducible spaces with ordinary rank two double loci abound, when the homological conditions of our reducibility criterion do not hold. The logical organization of the paper is as follows: in section 1 we state the main theorem and a number of corollaries and give self contained proofs for many of the corollaries; in section 2 we give the proof of the main theorem. We remark that the arguments given here generalize to prove the following statement: If X is a space satisfying a "partial Poincare Duality" condition, then a closed subset Σ separates X globally if and only if Σ separates X locally, in a certain precise sense, (see Remark 7 following Corollary 7).

Conventions and Terminology:

Although we may append remarks about more general situations, in all theorems and corollaries below we assume we are working with varieties and spaces defined over the complex numbers $\mathbb{C}$; in particular the unmodified word "dimension" refers to the complex dimension. We use the word "unibranch" to mean locally analytically irreducible. Thus we say a variety X is "unibranch at p", or "p is unibranch" (on X) if the analytic germ of X at p is irreducible, and "X is unibranch" if X is unibranch at every point. Similarly a point p is r-branched on X if there are exactly r local analytic branches of X at p, or equivalently if there are exactly r points over p on the normalization of X. Thus "smooth" implies "normal" implies "unibranch". The use of the word "variety" does not imply any irreducibility. The question of whether a space is "reducible" always refers to whether the underlying reduced space is reducible.

Acknowledgments:

We are grateful to O. Debarre, W. Fulton, N. Habegger, C. McCrory, and M. Nori, for helpful discussions, and to the National Science Foundation for partial support through grants #DMS - 88 03487 and #DMS - 92 08282, and for travel and lodging allowances during the special year 1992-93 in algebraic geometry at MSRI in Berkeley. We thank also the organizers of the special year and the staff of MSRI for a particularly stimulating and supportive environment in which to work during the preparation of this paper. We appreciate particularly the referee's help in correcting the statement of Corollary 4, simplifying the proof of Corollary 7, and clarifying the exposition.

Section 1:

Fact: *If X is a pure dimensional reduced compact complex space, if $\Sigma = sing(X)$ denotes its singular locus, if $\tilde{X}$ = normalization of X, and if $n = \dim_{\mathbb{R}}(X)$ = the dimension of X as a real analytic space, then the following statements are equivalent:*

(i) X is reducible,

(ii) $(X-\Sigma)$ is disconnected,

(iii) $\tilde{X}$ is disconnected,

(iv) $H_n(X,\mathbb{Z}/2)$ has rank ≥ 2,

(v) $H^0(\tilde{X},\mathbb{Z}/2)$ has rank ≥ 2.

Properties (i), (ii), (iii), and (v) are equivalent even if X is not pure dimensional; and if Σ denotes only a union of connected components of the singular locus of X, and $\tilde{X}$ = the partial normalization of X along Σ, then properties (ii), (iii), (iv), (v) each imply (i).

References for the Fact: ([14], Thm. 1B, p. 251, and Thm. Q, p.148).

Definition: *If X is a reduced compact complex space, we say that $\Sigma \subset X$ is an ordinary rank two double locus if Σ is closed in X and for each point p of Σ there is an integer m and an analytic neighborhood of p in X which is isomorphic to an analytic neighborhood of 0 in the set $\{z_1 z_2 = 0\} \subset \mathbb{C}^{m+1}$, and so that Σ corresponds to the subset $\{z_1 = z_2 = 0\}$. Such a locus Σ is both a smooth closed subspace, and a union of connected components, of $sing(X)$.*

The following theorem is the principal result of this paper. After it we shall state and prove a number of corollaries. Since most of them are derived from Corollaries 1 and 5, which do not require the full force of the theorem, we choose to give short direct proofs of those two corollaries, saving the proof of the full theorem until section 2 of the paper. In this way we try to give each result only as much proof as it requires.

Theorem: *Let X be a pure (positive) dimensional reduced compact complex space, and $n = \dim_{\mathbb{R}}(X) = 2\dim_{\mathbb{C}}(X)$. Assume $\Sigma \subset X$ is a (possibly empty) ordinary rank two double locus, $[X] \in H_n(X, \mathbb{Z}/2)$ is the fundamental homology class of X, and $\cdot \cap [X]$ denotes the cap product map $\cdot \cap [X]: H^1(X, \mathbb{Z}/2) \to H_{n-1}(X, \mathbb{Z}/2)$. Then the inequality #(irred. comps. of (X)) $\geq h^0(X, \mathbb{Z}/2) + h^0(\Sigma, \mathbb{Z}/2) - rk(\ker \cdot \cap [X])$, holds. (Hence, if the number of components of Σ is greater than $h^1(X, \mathbb{Z}/2)$ then X is reducible.) Moreover, if (X - Σ) is unibranch (i.e. if the local analytic germ of X, at each point not in Σ, is irreducible), then the previous inequality is actually an equality.*

Corollary 1: *If X is any reduced compact analytic space with $h^1(X, \mathbb{Z}/2) = 0$, and X has a non empty ordinary rank two double locus, then X is reducible. (For example, a simply connected irreducible space cannot have a non empty ordinary rank two double locus.)*

Proof:: A proof of this corollary (and later of Corollary 5), without assuming the theorem, will be based on a generalization of a classical construction in the theory of Prym varieties ([15] p.118; [2], Prop.5.2i p.174; [5], p.60). If $\Sigma \subset X$ is a non empty ordinary rank two double locus in any reduced complex space, we construct an etale double cover of X, the "Wirtinger cover", as follows. (If Σ is empty, the construction yields the trivial double cover.) Let $\nu: \tilde{X}_\Sigma \to X$ be the partial normalization of X along Σ only, and let $\tilde{X}_\Sigma(1)$ and $\tilde{X}_\Sigma(2)$ denote two disjoint isomorphic copies of $\tilde{X}_\Sigma$. We shall glue $\tilde{X}_\Sigma(1)$ to $\tilde{X}_\Sigma(2)$ with a twist along Σ. I.e. in each copy $\tilde{X}_\Sigma(j)$ of $\tilde{X}_\Sigma$, the preimage $\tilde{\Sigma}(j) = \nu_j^{-1}(\Sigma)$, of Σ under the (partial) normalization map $\nu_j: \tilde{X}_\Sigma(j) \to X$, is an etale double cover $\tilde{\Sigma}(j) = \nu_j^{-1}(\Sigma) \to \Sigma$, (possibly trivial), of Σ, and hence $\tilde{\Sigma}(j)$ carries a non trivial, fixed point free involution $\sigma_j: \tilde{\Sigma}(j) \to \tilde{\Sigma}(j)$. We use this involution for the gluing: i.e. in the disjoint union ($\tilde{X}_\Sigma(1) \amalg \tilde{X}_\Sigma(2)$), identify $\tilde{\Sigma}(1)$ with $\tilde{\Sigma}(2)$ via $\sigma = (\sigma_2 \circ$ "id"): ($\tilde{\Sigma}(1) \to \tilde{\Sigma}(2) \to \tilde{\Sigma}(2)$), where "id": $\tilde{\Sigma}(1) \to \tilde{\Sigma}(2)$ is the restriction of the isomorphism of $\tilde{X}_\Sigma(1)$ and $\tilde{X}_\Sigma(2)$, to form an identification variety

$$W_\Sigma = (\tilde{X}_\Sigma(1) \amalg \tilde{X}_\Sigma(2)) / \{ p \sim \sigma(p): \text{for all p in } \tilde{\Sigma}(1)\}.$$

Since the two partial normalization maps $\nu_j: \tilde{X}_\Sigma(j) \to X$ agree on the identified points, i.e. $\nu_1(p) = \nu_2(\sigma(p))$ for for all p in $\tilde{\Sigma}(1)$, they induce a map $W_\Sigma \to X$ which is an etale double cover of X. This is the Wirtinger double cover of X associated to Σ.

Assume now X is connected; (otherwise X is reducible). If X is irreducible, or even if some connected component of Σ intersects only one irreducible component of X, then W_Σ is connected, hence X cannot be simply connected. In fact from the theory of covering spaces a connected W_Σ corresponds to an index two subgroup of $\pi_1(X)$, hence to an index two subgroup of $H_1(X, \mathbb{Z}/2)$ which is the kernel of a unique non zero homomorphism $H_1(X, \mathbb{Z}/2) \to \mathbb{Z}/2$, and thus corresponds to a unique non zero element $[\omega_\Sigma]$ of $H^1(X, \mathbb{Z}/2)$. QED for Corollary 1.

Remark 1: This purely topological argument shows that if $\tilde{X}$ is any topological space to which the classification theory of covering spaces applies, if $\tilde{\Sigma} \subset \tilde{X}$ is a closed subset whose removal does not disconnect $\tilde{X}$, and $\sigma : \tilde{\Sigma} \to \tilde{\Sigma}$ is a fixed point free involution, and if we define X as the quotient space of $\tilde{X}$ obtained by identifying p with $\sigma(p)$, for p in $\tilde{\Sigma}$, then $H^1(X,\mathbb{Z}/2) \neq 0$. Thus the crucial point in order for the Wirtinger construction to apply, is that along an ordinary rank two double locus the singular points of X are all two-branched singularities, i.e. the normalization map is exactly two to one over every singular point. In particular Corollary 1 can be formulated also for real analytic spaces.
Remark 2: As observed in the proof of Corollary 1, W_Σ is connected whenever some component of Σ meets only one irreducible component of X; thus if $h^1(X,\mathbb{Z}/2) = 0$, so that W_Σ is disconnected, then at every point p of the ordinary rank two double locus the two branches of X are distinct irreducible components of X, and thus p is non singular on each irreducible component of X which contains it.

In view of Corollary 1, we recall some examples of varieties X with $h^1(X,\mathbb{Z}/2) = 0$. We shall abbreviate "Z is a local complete intersection" by "Z is l.c.i." Recall that any hypersurface and any smooth subvariety in $\mathbb{P}^m$ is l.c.i.
Corollary 2: *If $X = (\cap_j Y_j) \subset \mathbb{P}^m$, where for each j, $Y_j \subset \mathbb{P}^m$ is a closed l.c.i. projective subvariety of pure codimension d_j with $2d_j \leq (m-1)$, and if $(m - \Sigma_j d_j) \geq 2$, and X has a non empty ordinary rank two double locus, then X is reducible. (For example, an irreducible Segre surface, the intersection of two quadrics in $\mathbb{P}^4$, cannot have a non empty ordinary rank two double locus.)*
Proof: A variety X satisfying these hypotheses is (connected and) simply connected, by the "Lefschetz" theorems given in ([7], Cor 9.7 p.81, Rmk. 9.9 p.82). Hence $h^1(X,\mathbb{Z}/2) = 0$ and Corollary 1 applies. QED for Corollary 2.

The Lefschetz theorems cited above also give conditions when the property $h^1(X,\mathbb{Z}/2) = 0$ is inherited by subvarieties of varieties other than $\mathbb{P}^m$, as follows.
Corollary 3: *Let M be a reduced compact l.c.i. variety of pure complex dimension = n ≥ 3, with $h^1(M, \mathbb{Z}/2) = 0$, and let D be an ample divisor on M. Let $Y_1, \ldots, Y_{n-2}$ be effective divisors on M, not necessarily all distinct, with $Y_j \in |r_j D|$, $r_j \geq 1$, and put $X = (\cap_j Y_j) \subset M$. If X has a non empty ordinary rank two double locus Σ, then X is reducible; (for example an irreducible ample divisor X, in a smooth simply connected projective threefold, cannot have a non empty ordinary rank two double curve).*
Proof: It follows from the Lefschetz theorem in ([7], Rmk. 9.9 p.82), that under these conditions $\pi_2(M,X) = \pi_1(M,X) = 0$, and hence M and X have the same π_1, hence the same $H_1(\mathbb{Z})$, and thus the same $h^1(\mathbb{Z}/2)$. (In the notation of Corollary 3 if r is a large enough multiple of $\prod r_j$, then we can embed M in projective space via |rD|, and for each j

= 1,...., n-2, the divisor $(r/r_j)Y_j$ belongs to |rD|; hence as a set Y_j is then cut out on M by a hyperplane. Thus X is cut out on M by n-2 hyperplanes, not necessarily distinct, and ([7], Rmk. 9.9 p. 82) applies.) Then $h^1(X, \mathbb{Z}/2) = 0$, and Corollary 1 applies to show X is reducible. QED for Corollary 3.

The Lefschetz theorems also provide specific examples of varieties M, and divisors Y_j with the properties described in Corollary 3, as follows:

Corollary 4: *Let $M = (\cap_j Z_j) \subset \mathbb{P}^m$, where for each j, $Z_j \subset \mathbb{P}^m$ is a closed l.c.i. subvariety of pure codimension d_j with $2d_j \leq (m-2)$, and $(m - \sum_j d_j) \geq 3$, and assume M is smooth. Then M is irreducible of dimension = $n \geq 3$. Let $Y_1, \ldots, Y_{n-2}$ be (not necessarily distinct) effective divisors on M, put $X = (\cap_j Y_j) \subset M$, and assume X has a non empty ordinary rank two double locus Σ. Then X is reducible. (For example an irreducible divisor on a smooth quadric hypersurface in $\mathbb{P}^4$, cannot have a non empty ordinary rank two double curve). Furthermore, if X is a divisor in M and $(X - \Sigma)$ is unibranch then X is a transverse union of two irreducible components, and if in fact $(X - \Sigma)$ is smooth then X has global normal crossings.*

Proof: We know from the argument in Corollary 2 that $h^1(M,\mathbb{Z}/2) = 0$, so in order to apply Corollary 3 it remains to show that all the divisors Y_j on M are multiples of the same ample divisor D. Since all Y_j are effective Cartier divisors, and M is projective, it suffices to show that $\mathrm{Pic}(M) \cong \mathbb{Z}$. By ([7], p.82) again, and the relative Hurewicz isomorphisms ([10], Th.7.5.4), both the relative homotopy and relative homology groups of the pair $(\mathbb{P}^n, M)$ vanish up through dimension 3, hence the inclusion $M \subset \mathbb{P}^m$ induces isomorphisms of absolute homology and (by universal coefficients) cohomology groups up through dimension two. Thus $H^1(M,\mathbb{Z}) \cong H^1(\mathbb{P}^n,\mathbb{Z}) = 0$, and $H^2(M,\mathbb{Z}) \cong H^2(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$. Hence $H^1(M,\mathbb{C}) = 0$ by universal coefficients, and then $H^1(M,\mathcal{O}) = 0$ by the Hodge - Dolbeault decomposition $H^1(M,\mathbb{C}) \cong H^1(M,\mathcal{O}) \oplus H^0(M,\Omega^1)$. Hence the following part of the l.e.s. of the exponential map: $H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \to H^2(M,\mathbb{Z})$ becomes $0 \to H^1(M,\mathcal{O}^*) \to \mathbb{Z}$, so that $\mathrm{Pic}(M) \cong H^1(M,\mathcal{O}^*) \subset \mathbb{Z}$. Since M is projective $0 \neq \mathrm{Pic}(M) \cong \mathbb{Z}$, and Corollary 3 applies to prove X is reducible. If X is a divisor on M, then every irreducible component of X is an ample Cartier divisor, and $\dim(M) \geq 3$, so any three components of X would meet at a point of X which is neither an ordinary rank two double point nor unibranch; thus if $(X - \Sigma)$ is unibranch there are at most two irreducible components of X. Since these two components meet at non unibranch points of X, their intersection locus is contained in Σ. A component of Σ which meets only one irreducible component Y of X would contradict Corollary 3 applied to Y (or Remark 2), so Σ equals the intersection of the two components of X, which is therefore transverse. If $(X - \Sigma)$ is smooth thus X has global normal crossings. QED for Corollary 4.

Remark 3: In Corollary 4, if M is not smooth but only normal and l.c.i., we can still conclude reducibility of X, by using Grothendieck's result that then $Pic^0(M)$ is compact ([8], p.236-11), so that $Pic^0(M) = H^1(M,\mathcal{O}) = 0$, hence $Pic(M) \subset \mathbb{Z}$, and again applying Corollary 3. Alternatively, if the number of the divisors Y_j is only $(m - \Sigma_j d_j - 2)$, for example if M is the "proper" intersection of the Z_j, i.e. $n = (m - \Sigma_j d_j)$, we can still conclude reducibility of X from Corollary 2, with no other assumptions on M.

The next result is a useful refinement of the reducibility criterion in Corollary 1.

Corollary 5: *If X is any pure dimensional reduced compact complex analytic space with* $\dim_{\mathbb{R}}(X) = 2\dim_{\mathbb{C}}(X) = n$, *if* $\Sigma \subset X$ *is a non empty ordinary rank two double locus, and if the "partial Poincare Duality" condition* $\ker(\cdot\cap[X]: H^1(X,\mathbb{Z}/2) \to H_{n-1}(X,\mathbb{Z}/2)) = 0$ *holds, then X is reducible.*

Proof: It suffices to show that the class $[\omega_\Sigma]$ in $H^1(X,\mathbb{Z}/2)$, of the Wirtinger cover constructed in the proof of Corollary 1, belongs to the kernel of the cap product map $\cdot\cap[X]: H^1(X,\mathbb{Z}/2) \to H_{n-1}(X,\mathbb{Z}/2)$. For this, look at the commutative compatibility diagram for cap product with respect to the homomorphism ν_* induced by the partial normalization $\nu: \tilde{X}_\Sigma \to X$ of X along Σ; (for brevity of notation we will write just $\tilde{X}$ for $\tilde{X}_\Sigma$). The commutativity of the following diagram is implied by ([6], 12.6, p.239), and the fact that $\nu_*([\tilde{X}]) = [X]$ since ν is birational.

$$\begin{array}{ccc} \cdot\cap[X]: H^1(X) & \to & H_{n-1}(X) \\ \downarrow \nu^* & & \uparrow \nu_* \\ \cdot\cap[\tilde{X}]: H^1(\tilde{X}) & \to & H_{n-1}(\tilde{X}) \end{array}$$

Thus $\cdot\cap[X]([\omega_\Sigma]) = 0$ would follow if $\nu^*([\omega_\Sigma]) = 0$. Since $\nu^*([\omega_\Sigma])$ is the element classifying the etale double cover $(\tilde{W}_\Sigma \to \tilde{X})$ which is the pull back (via the partial normalization map ν) of the Wirtinger cover $W_\Sigma \to X$, it suffices to show that the Wirtinger double cover becomes trivial after normalization of X along Σ. In fact $(\tilde{W}_\Sigma \to \tilde{X})$ is the trivial cover $(\tilde{X}(1) \amalg \tilde{X}(2) \to \tilde{X})$. To check that, we will prove triviality of the double cover defined by the left hand vertical arrow in the following fiber product diagram, by constructing a section of it.

$$\begin{array}{ccc} \tilde{W}_\Sigma & \to & W_\Sigma \\ \downarrow & & \downarrow \\ \tilde{X} & \to & X \end{array}$$

By the universal property of fiber product it suffices to construct a map $\tilde{X} \to W_\Sigma$ such that the composition $\tilde{X} \to W_\Sigma \to X$ is the normalization map $\nu: \tilde{X} \to X$. Since by definition $W_\Sigma = (\tilde{X}(1) \amalg \tilde{X}(2))/\{p \sim \sigma(p)\}$, we may map $\tilde{X}$ isomorphically to either $\tilde{X}(1)$ or $\tilde{X}(2)$, and then to W_Σ. QED for Corollary 5.

Remark 4: As before, the argument for Corollary 5 is more general than the statement (see Remark 1 following Corollary 1); in particular Corollary 5 can be formulated for real analytic spaces.

Remark 5: Remark 2 following Corollary 1 is valid now also whenever $\ker(\cdot \cap [X]: H^1(X,\mathbb{Z}/2) \to H_{n-1}(X,\mathbb{Z}/2)) = 0$.

Remark 6: It is precisely the fact that the double cover $W_\Sigma \to X$ becomes trivial under normalization that was used by Wirtinger ([15] p.118; [3], p.417), to give a double cover of a nodal curve C such that the associated Prym variety is the Jacobian variety of the normalization of C. Since this double cover is a specialization of an etale cover of a smooth curve, the conclusion is that the Jacobian locus is in the closure of the locus of ordinary Prym varieties.

One can use the Lefschetz theorems again to verify the partial Poincare Duality condition in Corollary 5 when X is an effective ample divisor, or more generally an intersection of effective multiples of some ample divisor, in a manifold M, by a calculation with the cohomology class of X in M, as follows.

Corollary 6: *Let there be given a smooth irreducible complex projective manifold M, with $\dim_{\mathbb{C}}(M) = m \geq 3$, an ample divisor D on M, and effective divisors $Y_1 \in |r_1 D|$, ..., $Y_{m-2} \in |r_{m-2}D|$, $r_j \geq 1$, not necessarily all Y_j distinct, and assume:*

(i) $X = (Y_1 \cap .... \cap Y_{m-2})$ has pure (complex) codimension $k \leq (m-2)$ in M,

(e.g. we could assume $X \subset M$ is any effective ample divisor on M, or X is a complete intersection of some projective embedding of M with $k \leq m-2$ hypersurfaces);

(ii) the cup product homomorphism $\cdot \cup \{X\}_M: H^1(M,\mathbb{Z}/2) \to H^{1+2k}(M,\mathbb{Z}/2)$ is injective, where $\{X\}_M$ = the Poincare dual in $H^{2k}(M,\mathbb{Z}/2)$ of the homology class $[X]_M \in H_{2m-2k}(M,\mathbb{Z}/2)$;

(iii) $\Sigma \subset X$ is a non empty ordinary rank two double locus.

Then X is reducible. (For example, on a smooth threefold M, if cup product with the chern class of an ample line bundle $\mathcal{O}(D)$ is injective on $H^1(M,\mathbb{Z}/2)$, then no irreducible divisor in the linear series $|D|$ can have a non empty ordinary rank two double locus.)

Proof:

Lemma: *The hypotheses (i) and (ii) of Corollary 6 imply the partial Poincare duality condition $\ker(\cdot \cap [X]: H^1(X,\mathbb{Z}/2) \to H_{n-1}(X,\mathbb{Z}/2)) = 0$, where $n = 2(m-k) = \dim_{\mathbb{R}}(X)$.*

Proof of Lemma: Briefly, since M is smooth, cup and cap product on M are equivalent via Poincare Duality, and the Lefschetz theorem implies that since cap product is then injective on M, the restriction to X remains injective. In more detail, there are three steps:

Step 1) The square below commutes, where the vertical maps are induced by inclusion $i: X \subset M$, and the left hand vertical arrow $i^*: H^1(M,\mathbb{Z}/2) \to H^1(X,\mathbb{Z}/2)$ is an isomorphism.

$$\begin{array}{ccc} \cdot\cap[X]: H^1(X,\mathbb{Z}/2) & \to & H_{n-1}(X,\mathbb{Z}/2) \\ \cong\uparrow i^* & & \downarrow i_* \\ \cdot\cap[X]_M: H^1(M,\mathbb{Z}/2) & \to & H_{n-1}(M,\mathbb{Z}/2) \end{array}$$

Proof of Step 1: Commutativity: Let α belong to $H^1(M,\mathbb{Z}/2)$; then $i_*(i^*(\alpha) \cap [X]) = \alpha \cap i_*[X] = \alpha \cap [X]_M$. (For this "projection formula", use naturality of cap product ([6], 12.6, p.239).) The fact that $i^*: H^1(M) \to H^1(X)$ is an isomorphism follows again by the Lefschetz theorem in ([7], p.82). QED for Step 1.

Consequently, the partial PD condition for X will follow if we prove injectivity of the cap product map $\cdot\cap[X]_M: H^1(M) \to H_{n-1}(M)$, on the space M.

Step 2) The following square also commutes, (recall $\dim_{\mathbb{R}}(M) = n + 2k$).

$$\begin{array}{ccc} \cdot\cap[X]_M: H^1(M,\mathbb{Z}/2) & \to & H_{n-1}(M,\mathbb{Z}/2) \\ \downarrow \cdot\cup\{X\}_M & & \downarrow = \\ \cdot\cap[M]: H^{1+2k}(M,\mathbb{Z}/2) & \to & H_{n-1}(M,\mathbb{Z}/2) \\ & \cong & \end{array}$$

Proof of Step 2: Recalling that "homology is a module over cohomology", this is associativity of the module operation. I.e. cohomology is a ring via cup product, and operates on homology by cap product. Hence the associativity says that $(\alpha \cup \{X\}_M) \cap [M] = \alpha \cap (\{X\}_M \cap [M]) = \alpha \cap [X]_M$. Reference: ([6], 12.7 "associativity", p.238-239). The bottom map $\cdot\cap[M]$ is the Poincare duality isomorphism. QED for Step 2.

Step 3) The map $\cdot\cap[X]: H^1(X) \to H_{n-1}(X)$ is injective.

Proof of Step 3: By Step 2, the injectivity of $\cdot\cup\{X\}_M$ implies that of $\cdot\cap[X]_M$, which implies by Step 1 that of $\cdot\cap[X]: H^1(X) \to H_{n-1}(X)$. QED for Step 3 and Lemma.

Now Corollary 6 follows from Corollary 5. QED for Corollary 6.

Corollary 7: *(Characterization of a product of two general p.p.a.v.'s by the local geometry of the singularities of the theta divisor.) Assume (A,Θ) is a principally polarized abelian variety with theta divisor $\Theta \subset A$, and $\dim_{\mathbb{C}}(A) = g \geq 3$. Then $(A,\Theta) \cong (A_1,\Theta_1)\times(A_2,\Theta_2)$ is isomorphic to a product of two indecomposable (positive dimensional) p.p.a.v.'s, each with smooth irreducible theta divisor, if and only if Θ contains a non empty ordinary rank two double locus $\Sigma \subset \Theta$. More generally, $(A,\Theta) \cong (A_1,\Theta_1)\times(A_2,\Theta_2)$ with both Θ_1, Θ_2 unibranch if and only if for some irreducible component Σ of sing(Θ), there is a cover of Σ by analytic open sets U_j of Θ, such that each U_j has exactly two irreducible components and whose intersection is $\Sigma \cap U_j$.*

Proof: A theta divisor is ample by definition (or by Lefschetz's embedding theorem, classically) and a zero dimensional theta divisor is one point; hence Θ is connected, and irreducible whenever unibranch. If $(A,\Theta) \cong (A_1,\Theta_1)\times(A_2,\Theta_2)$ is a (non trivial) product then by definition of the product polarization $\Theta = (A_1\times\Theta_2) \cup (\Theta_1\times A_2)$; hence Θ has exactly two irreducible components whenever Θ_1 and Θ_2 are unibranch. If Θ_1 and Θ_2 are also smooth then the two components $(A_1\times\Theta_2)$ and $(\Theta_1\times A_2)$ of Θ are smooth and meet transversely along the smooth intersection $(\Theta_1\times\Theta_2)$; hence this intersection is an ordinary rank two double locus $\Sigma = \text{sing}(\Theta)\subset\Theta$. If Θ_1 and Θ_2 are only unibranch, the two components $(A_1\times\Theta_2)$ and $(\Theta_1\times A_2)$ of Θ are likewise unibranch; hence their intersection $\Sigma = (\Theta_1\times\Theta_2)$ is still an irreducible component of $\text{sing}(\Theta)$ of codimension one in Θ, consists entirely of two-branched points of Θ (since each point of Σ is on one branch of each component), and is everywhere locally the intersection of the two branches. This shows that product p.p.a.v.'s of the given types have the singularities claimed.

To prove conversely that such singularities characterize these products we will use an argument like that in Corollary 6 with $X = \Theta$. To verify the condition that the cup product $\cdot\cup\{\Theta\}_A:H^1(A,\mathbb{Z}/2) \to H^3(A,\mathbb{Z}/2)$ is injective, we calculate using a standard basis of the exterior algebra of $H^1(A,\mathbb{Z}/2)$, and a standard representative of the class $\{\Theta\}_A$ in $H^2(A,\mathbb{Z}/2)$, denoted simply by $\{\Theta\}$. We claim $\{\Theta\} = \sum_j \alpha_j\wedge\beta_j$ where $\{\alpha_j,\beta_j\}$ in $H^1(A,\mathbb{Z}/2)$ is Kronecker dual to a symplectic basis of $H_1(A,\mathbb{Z}/2)$. I.e. the symplectic form $<,>$ on $H_1(A,\mathbb{Z}/2)$, is given by $<a,b>[A] = a\times b\times[\Theta]$, where '$\times$' denotes Pontrjagin product, $[A]$ is the fundamental class of A in $H_{2g}(A,\mathbb{Z}/2)$, and where 'symplectic basis' means that the matrix of values of the pairing on this basis is the standard $\mathbb{Z}/2$ symplectic matrix, with zeros in the upper left and lower right hand blocks, and the identity matrix in the lower left and upper right hand blocks. Then one may calculate the class of theta as $\{\Theta\} = \sum_j \alpha_j\wedge\beta_j$ in $H^2(A,\mathbb{Z}/2)$.

Now we compute. If α is an arbitrary non zero element of $H^1(A,\mathbb{Z}/2)$, choose a symplectic basis $\{\alpha_j,\beta_j\}$ of $H^1(A,\mathbb{Z}/2)$ with $\alpha = \alpha_1$, and assume that the following cup product is zero: $\alpha\wedge\{\Theta\} = \alpha_1\wedge(\sum_j \alpha_j\wedge\beta_j) = \sum_j \alpha_1\wedge\alpha_j\wedge\beta_j = \sum_{1<j} \alpha_1\wedge\alpha_j\wedge\beta_j = 0$. We remark that the four sets of elements: $\{\alpha_i\wedge\alpha_j\wedge\beta_k\}_{i<j}$, $\{\alpha_i\wedge\alpha_j\wedge\alpha_k\}_{i<j<k}$, $\{\beta_i\wedge\beta_j\wedge\beta_k\}_{i<j<k}$, $\{\alpha_i\wedge\beta_j\wedge\beta_k\}_{j<k}$, together give a basis for the space $H^3(A,\mathbb{Z}/2)$, and in particular the elements $\{\alpha_i\wedge\alpha_j\wedge\beta_k\}_{i<j}$ in the first set are independent. Then the relation $\sum_{1<j} \alpha_1\wedge\alpha_j\wedge\beta_j = 0$ above gives a contradiction when the complex dimension of A is ≥ 2, (i.e. when the sum is non empty). Thus the map $\cdot\cup\{\Theta\}:H^1(A,\mathbb{Z}/2) \to H^3(A,\mathbb{Z}/2)$ is injective when $g \geq 2$.

When $g \geq 3$, and $\Sigma \subset \Theta$ is a non empty ordinary rank two double locus, Corollary 6 implies Θ is reducible, and by ([4], Lemma 3.20 p.29; [1], pf. of Lemma 11, p.221) (A,Θ) is a product. If $\Sigma \subset \text{sing}(\Theta)$ is only an irreducible subset in Θ, consisting of

two-branched points of Θ, and Σ is locally the intersection of the branches, then the Wirtinger construction described in the proof of Corollary 1 can be adapted to this setting to construct a double cover of Θ by gluing two copies of the partial normalization of Θ, i.e. by gluing two copies of the space obtained by separating the local analytic branches of Θ along Σ. The conclusion is that Θ has a 2 - sheeted analytic covering space that trivializes after normalizing Θ, hence (pf. Cor.5) is represented by a cohomology class in the kernel of the cap product $\cdot \cap [\Theta]$: $H^1(\Theta,\mathbb{Z}/2) \to H_{2g-3}(\Theta,\mathbb{Z}/2)$; this class must then be zero by the Lemma in the proof of Corollary 6 and the cup product calculation above. Thus the covering space of Θ was already trivial, at each point of Σ the two local analytic branches of Θ arise from distinct irreducible components of Θ, Σ lies in the intersection of exactly two of the irreducible components of Θ, and (A, Θ) is again isomorphic to a product $(A_1, \Theta_1)\times \ldots\ldots \times(A_r, \Theta_r)$ where each Θ_j is irreducible, by [4], p.29. By definition of the product polarization, Θ has the irreducible components
$(\Theta_1\times A_2\times\ldots\times A_r) \cup (A_1\times\Theta_2\times\ldots\times A_r) \cup \ldots\ldots\ldots\ldots \cup (A_1\times\ldots\times A_{r-1}\times\Theta_r)$.

We claim that r = 2. Suppose the two components containing Σ are the first two, i.e. $\Sigma \subset (\Theta_1\times A_2\times\ldots\times A_r) \cap (A_1\times\Theta_2\times\ldots\times A_r) = (\Theta_1\times\Theta_2\times A_3\times\ldots\times A_r)$, an irreducible set of codimension one in Θ. Since Σ also has codimension one in Θ, this inclusion is an equality. Then Σ contains the ($\geq$) r-branched points $(\Theta_1\times\Theta_2\times\ldots\times\Theta_r)$ belonging to all r components; since all the points of Σ are two-branched, r = 2. Hence $(A, \Theta) \cong (A_1, \Theta_1)\times(A_2, \Theta_2)$ is a product of just two indecomposable factors, $\Theta = (\Theta_1\times A_2) \cup (A_1\times\Theta_2)$ is a union of just two irreducible components, and $\Sigma = (\Theta_1\times A_2) \cap (A_1\times\Theta_2) = (\Theta_1\times\Theta_2)$. If Σ is actually an ordinary rank two double locus, then it is smooth, and hence both Θ_1 and Θ_2 are smooth. It remains to show that, if we assume only that Σ consists entirely of 2 - branched points of Θ, then both Θ_1 and Θ_2 are unibranch. If Θ_1, for example, were not unibranch, say at p, then for any q on Θ_2, the pair (p,q) would lie on at least two branches of the component $(\Theta_1\times A_2)$ of Θ, and also on at least one branch of the component $(A_1\times\Theta_2)$ of Θ. On the other hand (p,q) lies on $(\Theta_1\times\Theta_2) = \Sigma$, which consists entirely of 2-branched points of Θ, a contradiction. QED for Corollary 7.

Remark 7: For comparison with Corollary 7, if the "$\mathfrak{N}_{g-2}$ conjecture" (cf. Introd.) were known, one would have a slightly stronger statement, that $(A,\Theta) \cong (A_1,\Theta_1)\times(A_2,\Theta_2)$ with both Θ_1,Θ_2 irreducible, if and only if some irreducible component Σ of sing(Θ) consists entirely of "two - branched" points, i.e. Σ has a cover by analytic open sets U_j of Θ, such that each U_j has exactly two irreducible components. With only our present methods, we can give the following characterization of the theta divisor on an arbitrary product : Θ is reducible, and hence (A,Θ) is a non trivial product, if and only if there is a non empty closed subset $\Sigma \subset \Theta$ whose removal allows a compatible local bipartite separation of Θ; i.e. there is a cover of Σ by analytic open subsets U_j of Θ, such that for each j, $(U_j - \Sigma)$ can be written as a union of two disjoint, non empty (not necessarily connected) sets, and such that for each point p of Σ the induced decomposition of the set of local analytic components of Θ at p into two disjoint non empty subsets is independent of j. Equivalently there exists a "partial normalization" $\nu: \Theta' \to \Theta$ of Θ, between Θ and its full normalization, $\tilde{\Theta} \to \Theta' \to \Theta$, so that the induced map $\nu^{-1}(\Sigma) \to \Sigma$, is a precisely 2 sheeted topological covering map of Σ (which can be taken to be etale), and such that ν is an isomorphism over the complement of Σ. To prove this characterization one checks that under these conditions a Wirtinger construction yields an unramified double cover of Θ which trivializes after normalization, so that Θ is disconnected by the removal of Σ, hence is reducible. If Θ is known to be reducible, the decomposition of the local components of Θ along $\Sigma = (\Theta_1 \times A_2) \cap (A_1 \times \Theta_2) = (\Theta_1 \times \Theta_2)$ induced by any global decomposition of Θ into two sets of irreducible components $\Theta = (\Theta_1 \times A_2) \cup (A_1 \times \Theta_2)$ satisfies the conditions. Similarly, our methods imply that on any analytic space X satisfying the partial PD condition $\ker(\cdot \cap [X]: H^1(X,\mathbb{Z}/2) \to H_{n-1}(X,\mathbb{Z}/2)) = 0$, X admits a compatible local bipartite separation along a closed subset Σ if and only if Σ separates X globally.

Remark 8: The calculation in the proof of Corollary 7 shows that all odd positive multiples of Θ give injective cup product maps (mod two), and hence the same reducibility result holds for divisors D in any linear series $|(2n+1)\Theta|$, where $n \geq 0$. The situation is quite different for divisors which are even multiples of Θ. Debarre pointed out for instance that there are irreducible divisors in $|2\Theta|$ having a non empty ordinary rank two double locus. When dim(A) = 2 the inverse image of a general plane section through a node of the Kummer surface, and when dim(A) ≥ 2 the construction of Nori (which we describe in a separate note), give examples.

Remark 9: Corollary 7 shows that a "generic" decomposable p.p.a.v. (i.e. the product of two p.p.a.v.'s each with smooth theta divisor), of dimension $g \geq 3$, is characterized by the local nature of the singular locus of its theta divisor. The case of p.p.a.v.'s of dimension two can be settled by a short direct argument as follows: given (A,Θ), with dim(A) = 2, and Θ singular, one knows Θ is a reduced ([1], p.207) and connected curve, (since Θ is ample, Lefschetz ([7], p.82) gives connectedness), and the adjunction formula shows that $p_a(\Theta) = 2$; hence $0 \leq g(\tilde{\Theta}) < 2$, where $\tilde{\Theta}$ is the normalization of Θ. If $\tilde{\Theta}$ is disconnected, Θ is reducible and we are done. If $\tilde{\Theta}$ is connected, since the map $\tilde{\Theta} \to A$ is non constant, Abel's theorem implies $g(\tilde{\Theta}) > 0$, hence $g(\tilde{\Theta}) = 1$. Then $\tilde{\Theta} \to A$ is a holomorphic map of abelian varieties, hence a translate of a homomorphism, and thus the image Θ is smooth. Therefore on any principally polarized abelian surface, Θ is either smooth or reducible. QED for Remark 9.

Remark 10: Kollar has recently shown [oral communication], using the Kawamata -

Viehweg Vanishing theorem ([9], p.46; [13], p.1), that the theta divisor on an arbitrary p.p.a.v. has only "semi log canonical" singularities. He deduces that the locus of points of multiplicity d on Θ has codimension in Θ at least (d-1), and that the general singularity of codimension one in Θ is a local normal crossings double point. In particular Θ cannot have points of multiplicity > g = dim(A), the codimension in Θ of the set of unibranch singularities is at least two, and Θ is unibranch $\Leftrightarrow$ nonsingular in codimension one $\Leftrightarrow$ normal. It remains to determine whether general codimension one singularities can degenerate to unibranch singularities to understand further the question of reducibility of Θ. Kollar pointed out that his result suggests one may generalize the "$\mathfrak{N}_{g-2}$ conjecture" (cf. Introd.) by asking whether Θ must have $\geq$ d components if, for some d with $2 \leq d \leq g$, the set of d- fold points has codimension (d-1) in Θ.

Remark 11: Let M be any smooth simply connected projective threefold with Pic(M) $\cong \mathbb{Z}$, and Y any irreducible smooth surface. Then our results imply that an immersion of Y in M by a map whose fibers contain at most two points is actually an embedding. I.e. such an immersed surface X is an ample divisor, hence is simply connected by Lefschetz. Consequently the immersion has degree one, and the singular locus Σ of X will be everywhere exactly 2 branched. Then the Wirtinger construction, in the topological category, shows X is not simply connected unless Σ is empty. This is a partial generalization of the theorem of Fulton and Hansen ([7], Th.5.1 p.45) which implies Y cannot be immersed in $\mathbb{P}^3$ without being embedded. Using the Lefschetz theorems, as in the proof of Corollary 2, we also obtain that a smooth connected variety of dimension at least two cannot be immersed as a complete intersection in $\mathbb{P}^n$ by a map whose fibers contain at most two points, without being embedded. (This is slightly more general than Corollary 2, since the branches of an immersed variety can be tangent, rather than transverse, at a multiple point.) We do not know whether in these statements the restriction to at most double point fibers is needed. Does there exist, e.g., a simply connected smooth projective threefold M with Pic(M) $\cong \mathbb{Z}$, and an irreducible (ample) divisor $X \subset M$ such that X has only local normal crossings (necessarily including triple points)? Also, can a smooth irreducible surface ever be immersed, without being embedded, as a complete intersection say in $\mathbb{P}^4$? For instance can a smooth irreducible surface be immersed and not embedded in a smooth quadric threefold?

The remaining corollaries do not follow from Corollary 5, i.e. from the existence and properties of the Wirtinger cover given above, but require the full theorem stated in the beginning of this section and proved below. First we specialize that theorem to the case of curves, just to make clear what aspect of the influence of curve singularities on reducibility has been generalized to higher dimensions.

Corollary 8: *If C is a connected reduced curve with k nodes (and possibly other* *singularities), and if* $r = rank(ker(\cdot\cap[C]:H^1(C,\mathbb{Z}/2)\to H_1(C,\mathbb{Z}/2)))$, *then C has* $\geq 1+(k-r)$ *irreducible components. Hence if C is also irreducible then* $k \leq r$.

Remark 12: The role of nodes in Corollary 8 makes clear that in our theorem we have not generalized directly the reducibility argument for curves given at the beginning of the Introduction. That argument proceeded from the fact that any set of k singular points on a reduced irreducible smoothable curve has associated to it at least k independent "vanishing (real) 1- cycles" on a nearby smoothing of the curve. This statement does not seem to generalize directly to higher dimensions; more precisely, even when the singular locus of a smoothable reduced irreducible variety has pure (complex) codimension one, it is not always true that each set of k connected components of the singular locus has associated to it a set of k independent vanishing cycles of real codimension one on a nearby smoothing. For instance, on "Whitney's umbrella" W : $\{x^2w = y^2z\}$ in $\mathbb{P}^3$, the (reduced) singular locus is isomorphic to $\mathbb{P}^1$, hence connected and of pure codimension one in W. However, since a nearby smoothing of W is a non singular cubic surface in $\mathbb{P}^3$, it has no non zero homology cycles at all in real codimension one, in particular no non trivial "vanishing (real) 3-cycles". This phenomenon is "caused" by the presence of pinch points on the singular locus, which explains the restriction on the singularities in our theorem to the case of ordinary double loci.

The second difference from the earlier argument for curves is the total absence of mention of vanishing cycles, their role being replaced by the kernel of the cap product. Unlike the restriction to ordinary double loci, this modification in the argument is not essential (we also have a vanishing cycle version of the argument), but is an alternative which is computable and available in the case of ordinary rank two double loci. The relationship between cap product and vanishing cycles is as follows: when a smooth connected curve acquires some singularities, but remains irreducible, one may choose a symplectic basis of 1 - cycles so that each vanishing cycle at a node is paired with a "transverse cycle" which does not vanish but passes through the node of the singular curve. Dual to these transverse cycles are the elements of H^1 which are supported on them, and hence which become zero upon normalizing the singular curve, since normalization cuts apart the transverse cycles. Finally, as in the proof of Corollary 5, the kernel of the cap product contains the kernel of the map induced by normalization, hence contains the 1 - cocycles supported on the cycles transverse to the vanishing cycles at the nodes. For more complicated singularities there need not be any non-zero transverse cycles on the singular variety, and one will presumably need to analyze the vanishing cycles to understand such singularities by topological methods. Moreover, for any reduced connected curve C, $\text{rank}(\ker(\cdot\cap[C])) \leq p_a(C)$, so that Corollary 8 implies $\#\{\text{nodes}\} \leq p_a(C)$ for irreducible C. [Consider the normalization map $\nu:\tilde{C} \to C$, and the induced maps on the H^1 groups associated to the exponential sequences of $\tilde{C}$, C. The kernels of these maps form a short exact sequence exhibiting a subspace of $H^1(C,\mathcal{O})$ as the universal covering group of a product of $\mathbb{C}$'s and $\mathbb{C}^*$'s (the kernel of the map of generalized jacobians $\nu^*:J(C) \to J(\tilde{C})$), with kernel a lattice of rank = rk(ker($\cdot \cap [C]$)) (by the Lemma in the proof of Prop. 2 (iii) below), which thus equals the # of $\mathbb{C}^*$'s in J(C). Thus rk(ker($\cdot \cap [C]$)) $\leq \dim(H^1(C,\mathcal{O})) = p_a(C)$. For any reduced C, with singular set Σ, we can pursue this analysis a bit to get: the # of $\mathbb{C}^*$'s in J(C) = rk(ker($\cdot \cap [C]$)) = rk(ker($\nu^*:H^1(C,\mathbb{Z}/2) \to H^1(\tilde{C},\mathbb{Z}/2)$)) = $h^1(C) - h^1(\tilde{C}) = h^1(C) - h^1(\tilde{C}) - h^2(C) + h^2(\tilde{C})$ $= \chi(\tilde{C}) - \chi(C) + h^0(C) - h^0(\tilde{C}) = \#(\nu^{-1}(\Sigma)) - \#(\Sigma) + h^0(C) - h^0(\tilde{C})$, where $\#(\nu^{-1}(\Sigma))$ is the total number of local branches at all singular points. To summarize, # of $\mathbb{C}^*$'s in J(C) = rk(ker($\cdot \cap [C]$)) = $\#(\nu^{-1}(\Sigma)) - \#(\Sigma) + h^0(C) - h^0(\tilde{C})$.] Also, the formula $p_a(C) = \delta + p_a(\tilde{C})$, from the sequence of structure sheaves induced by ν for irreducible C, implies #{singular points} $\leq \delta \leq p_a(C)$, and it may be that these coherent sheaf methods should be used also to study higher dimensional singularities.

Corollary 9: *If M is a smooth connected complex projective variety, dim(M) = m ≥ 3, X ⊂ M is an irreducible pure dimensional intersection of some projective embedding of M with k ≤ m-2 hypersurfaces, and rank(ker(* $\cdot \cup \{X\}_M$*)) = r (with the notation of Corollary 6), then an ordinary rank two double locus on X has at most r components. In particular if X is an irreducible divisor in the linear series* $|2\Theta|$ *on a p.p.a.v. of dimension g ≥ 3, then an ordinary rank two double locus on X has at most 2g components.*

Proof: The diagrams in the proof of Cor. 6 imply rk(ker ($\cdot \cap [X]$)) ≤ rk(ker($\cdot \cup \{X\}_M$)), and then the Theorem implies that $h^0(\Sigma,\mathbb{Z}/2) \leq$ rk(ker ($\cdot \cap [X]$)) ≤ r. QED Corollary 9.

Remark 13: We do not know if this bound is sharp, even in case the class in $H^2(M,\mathbb{Z}/2)$ Poincare dual to [X] is zero, as it is for X in $|2\Theta|$. For g = 2, the inverse image of a plane section through k ≤ 3 nodes of a Kummer surface, seems to give an irreducible element of $|2\Theta|$ with k nodes, but a plane section through four nodes (and its inverse image), if reduced, would be reducible.

Section 2:

Now we begin the proof of the main theorem (stated near the beginning of Section 1). Since all the (co)homology spaces we use have coefficients in $\mathbb{Z}/2$, we frequently omit the coefficients from the notation. We begin the proof of the Theorem with a proposition which computes the number of irreducible components of a space X in terms of a homological invariant involving a partial normalization of X. A second proposition will then replace this invariant with one computable entirely on X, in terms of the cap product.

Proposition 1: *Assume X is a pure dimensional reduced compact complex space, n = $\dim_{\mathbb{R}}(X)$, and $\Sigma \subset X$ is a (possibly empty) ordinary rank two double locus, and let $\nu: \tilde{X} \to X$ denote the partial normalization of X along Σ. Then the relations*
#(irred. comps. of X) $\geq h^0(\tilde{X}) = h^0(X) + h^0(\Sigma) - rk(\ker(\nu^*: H^1(X) \to H^1(\tilde{X})))$, hold. Moreover, if $(X - \Sigma)$ is unibranch, then equality holds throughout the previous formula.
Proof: Since each connected component of a normal space is irreducible, the number of irreducible components of X equals the number of connected components of the full normalization of X, which is at least as great as $h^0(\tilde{X})$ = the number of connected components of $\tilde{X}$, so the first inequality holds. To compute $h^0(\tilde{X})$, consider $\Sigma \subset X$, and $\nu: \tilde{X} \to X$ the partial normalization map. If $\tilde{\Sigma} = \nu^{-1}(\Sigma)$, then the restriction $\nu: \tilde{\Sigma} \to \Sigma$ is an etale double cover. We want to look at the map of Mayer-Vietoris sequences induced by ν and its restriction, and at the associated spectral sequence. Let U = an open neighborhood of Σ in X, V = X-Σ, B = U$\cap$V. Define $\tilde{U} = \nu^{-1}(U)$, $\tilde{V} = \tilde{X} - \tilde{\Sigma}$, $\tilde{B} = \tilde{U} \cap \tilde{V}$, and note that $\tilde{V} \cong V$, and $\tilde{B} \cong B$, since ν is an isomorphism away from $\tilde{\Sigma}, \Sigma$.

Fact: *The neighborhood U can be chosen so that there exists a strong deformation retract of $\tilde{U}$ onto $\tilde{\Sigma}$, which descends via ν to a strong deformation retract of U onto Σ.*
Reference for the Fact: Use ([11], Lemma 0.13 p.354) for the retraction of $\tilde{U}$ onto $\tilde{\Sigma}$, since $\tilde{\Sigma}$ is a submanifold of the manifold $\tilde{X}$, and note that it descends.

Now look at the map of cohomology (M-V) sequences (with $\mathbb{Z}/2$ coefficients) induced by ν, using the observations that $\tilde{V} \cong V$, and $\tilde{B} \cong B$ to eliminate some of the tildas in the upper row, and using the deformation retracts to replace U and $\tilde{U}$ with Σ and $\tilde{\Sigma}$:

$$\begin{array}{ccccccccccccc}
0 \to & H^0(\tilde{X}) & \to & H^0(\tilde{\Sigma}) \oplus H^0(X-\Sigma) & \to & H^0(B) & \to & H^1(\tilde{X}) & \to & H^1(\tilde{\Sigma}) \oplus H^1(X-\Sigma) & \to & H^1(B) & \to \cdots \\
& \uparrow \nu_1^* & & \uparrow \nu_2^* & & \uparrow \nu_3^* & & \uparrow \nu_4^* & & \uparrow \nu_5^* & & \uparrow \nu_6^* & \\
0 \to & H^0(X) & \to & H^0(\Sigma) \oplus H^0(X-\Sigma) & \to & H^0(B) & \to & H^1(X) & \to & H^1(\Sigma) \oplus H^1(X-\Sigma) & \to & H^1(B) & \to \cdots
\end{array}$$

See ([10], Cor. 9, p.239, excisive couple, p.188, Th.3 p.188: Given a cover of a space by two open sets U,V, then {U,V} is an excisive couple.) Let the cokernel of the vertical map ν_j^* above be denoted by C_j, and the kernel by K_j.

Assertion: *There is an exact sequence $(*)$: $0 \to C_1 \to C_2 \to K_4 \to K_5 \to 0$.*

Proof of Assertion: Consider the two spectral sequences associated to the double complex formed by the two rows of (M-V) sequences above. For one spectral sequence the horizontal homology of the double complex gives the E_1 terms, and since those rows are exact, these E_1 terms are zero. Consequently this spectral sequence abuts to zero.

Now consider the other spectral sequence. For this sequence the vertical homology, i.e. the spaces C_j and K_j are the E_1 terms, and the differentials d_1 are the induced horizontal maps $\alpha_j: C_j \to C_{j+1}$, and $\beta_j: K_j \to K_{j+1}$, from the (M-V) sequences. The E_2 terms are thus $D_j = (\ker(\alpha_j) / \mathrm{im}(\alpha_{j-1}))$, and $F_j = (\ker(\beta_j) / \mathrm{im}(\beta_{j-1}))$. Since there are only two rows $E_3 = E_\infty$, and since the abutments are zero $E_\infty = 0$. Therefore the differentials $d_2^j : D_j = (\ker(\alpha_j) / \mathrm{im}(\alpha_{j-1})) \to (\ker(\beta_{j+2}) / \mathrm{im}(\beta_{j+1})) = F_{j+2}$, are all isomorphisms. The sequence $(*)$: $0 \to C_1 \to C_2 \to K_4 \to K_5 \to 0$ will be obtained by the concatenation of two exact sequences, $(**)$: $0 \to C_1 \to C_2 \to D_2 \to 0$, and $(***)$: $0 \to F_4 \to K_4 \to K_5 \to 0$, so it suffices to establish these two sequences and then use the fact that $d_2^2 : D_2 \to F_4$ is an isomorphism. For $(**)$, α_1 is injective because $\ker(\alpha_1) = D_1 \cong F_3 = 0$, ($F_3$ is a subquotient of $K_3 = 0$). Similarly $C_3 = \mathrm{coker}(\nu_3{}^*) = \mathrm{coker}(\mathrm{id}) = 0$. Thus we have $D_2 = \ker(C_2 \to C_3)/\mathrm{im}(C_1 \to C_2) = C_2/\mathrm{im}(C_1)$. For the sequence $(***)$, since $K_6 = 0$, we conclude $K_5 / \mathrm{im}(K_4) \cong F_5 \cong D_3 = 0$, ($D_3$ is a subquotient of C_3); hence the sequence is exact at K_5. Since $K_3 = 0$, the kernel of $K_4 \to K_5$ is F_4. QED Assertion.

To deduce Proposition 1, we take the Euler characteristic of the exact sequence $(*)$, $\chi(0 \to C_1 \to C_2 \to K_4 \to K_5 \to 0) = \mathrm{rk}(C_1) - \mathrm{rk}(C_2) + \mathrm{rk}(K_4) - \mathrm{rk}(K_5) = 0$. Now since $C_1 = \mathrm{coker}\,(H^0(X) \to H^0(\tilde{X}))$, $\mathrm{rk}\,(H^0) = \#$ conn. comps., and the map $H^0(X) \to H^0(\tilde{X})$ is injective, (being dual to the surjection on H_0), $\mathrm{rk}(C_1) = \#$(conn. comps. of $(\tilde{X})$) - #(conn. comps. of X). Similarly, $\mathrm{rk}(C_2) = \#$ comps. of $\tilde{\Sigma}$ - # comps. of $\Sigma = \tau =$ # comps. of Σ over which the etale double cover $\tilde{\Sigma} \to \Sigma$ is trivial. By definition, $K_4 = \ker(\nu^* : H^1(X) \to H^1(\tilde{X}))$. Finally, $K_5 \cong \ker\,(H^1(\Sigma) \to H^1(\tilde{\Sigma}))$, and in $\mathbb{Z}/2\mathbb{Z}$ coefficients $\mathrm{rk}(K_5)$ is then the number σ of components over which the double cover $\tilde{\Sigma} \to \Sigma$ is non trivial. The alternating sum is thus $0 = \mathrm{rk}(C_1) - \mathrm{rk}(C_2) + \mathrm{rk}(K_4) - \mathrm{rk}(K_5) = h^0(\tilde{X}) - h^0(X) - \tau + \mathrm{rk}\,(\ker\,(\nu^* : H^1(X) \to H^1(\tilde{X}))) - \sigma$; and $(\sigma + \tau) = h^0(\Sigma)$.
Thus, $h^0(\tilde{X}) = h^0(X) + h^0(\Sigma) - \mathrm{rk}(\ker\,(\nu^* : H^1(X) \to H^1(\tilde{X})))$. QED for Proposition 1.

Proposition 2: *Assume X is a pure (positive) dimensional reduced compact complex space, $n = \dim_{\mathbb{R}}(X)$, $\Sigma \subset X$ is a (possibly empty) ordinary rank two double locus, let $\nu : \tilde{X} \to X$ denote the (partial) normalization of X along Σ, and let $\cdot \cap [X]$ denote the cap product map $\cdot \cap [X] : H^1(X, \mathbb{Z}/2) \to H_{n-1}(X, \mathbb{Z}/2)$.*
Then (i): $\mathrm{rk}\,(\ker \cdot \cap [X]) \geq \mathrm{rk}\,(\ker(\nu^* : H^1(X, \mathbb{Z}/2) \to H^1(\tilde{X}, \mathbb{Z}/2)))$.
Moreover, if $(X - \Sigma)$ is unibranch, then
(ii) #(irred. comps. of X) $= h^0(\tilde{X})$, and
(iii) $\mathrm{rk}\,(\ker \cdot \cap [X]) = \mathrm{rk}\,(\ker(\nu^* : H^1(X, \mathbb{Z}/2) \to H^1(\tilde{X}, \mathbb{Z}/2)))$.
Proof: Statement (i) follows from the commutativity of the diagram in the proof of Corollary 5 in section 1:

$$\begin{array}{ccc} \cdot \cap [X] : H^1(X) & \to & H_{n-1}(X) \\ \downarrow \nu^* & & \uparrow \nu_* \\ \cdot \cap [\tilde{X}] : H^1(\tilde{X}) & \to & H_{n-1}(\tilde{X}) \end{array}$$

QED. for (i).

For statement (ii), note that if $(X - \Sigma)$ is unibranch, then the partial normalization $\tilde{X}$ of X along Σ is everywhere unibranch, hence everywhere locally irreducible. Therefore the connected components of $\tilde{X}$ are all irreducible. Thus #(irred. comps. of X) = #(irred. comps. of $\tilde{X}$) = $h^0(\tilde{X})$. QED. for (ii).

For (iii) it suffices to prove the following:

Lemma: *Assume X is a pure (positive) dimensional reduced compact complex space, n = $\dim_{\mathbb{R}}(X)$, $\Sigma \subset X$ is a union of connected components of sing(X), $(X - \Sigma)$ is unibranch, and $\nu: \tilde{X} \to X$ is the partial normalization of X along Σ. Then the two maps $\cdot \cap [X] : H^1(X, \mathbb{Z}/2) \to H_{n-1}(X, \mathbb{Z}/2)$ and $\nu^*: H^1(X, \mathbb{Z}/2) \to H_{n-1}(X, \mathbb{Z}/2)$, have the same kernels.*

Proof: First we prove the following Claim (where $f: Z \to W$ will represent either $\nu: \tilde{X} \to X$ or a desingularization $\mu: Y \to \tilde{X}$, of $\tilde{X}$).

Claim: *Assume Z, W are pure (positive) dimensional compact complex analytic varieties, n = $\dim_{\mathbb{R}}(Z) = \dim_{\mathbb{R}}(W)$, $A \subset Z$, $B \subset W$ are compact complex subvarieties both of real dimension at most n - 2, $f: Z \to W$ is a surjective morphism, $f^{-1}(B) = A$, and the restriction of f to the open complements $f:(Z-A) \to (W-B)$, is an isomorphism. Then $f_*: H_{n-1}(Z) \to H_{n-1}(W)$ is injective.*

Proof of Claim: Look at the map of homology sequences of the pairs (Z,A) and (W,B) induced by f:

$$\begin{array}{ccccccc} \cdots \to & H_{n-1}(B) & \to & H_{n-1}(W) & \to & H_{n-1}(W,B) & \to \cdots \\ & \uparrow f_* & & \uparrow f_* & & \uparrow f_* & \\ \cdots \to & H_{n-1}(A) & \to & H_{n-1}(Z) & \to & H_{n-1}(Z,A) & \to \cdots \end{array}$$

The hypotheses imply that the map $Z/A \to W/B$ induced by f on the identification spaces in which the subvarieties A, B are each collapsed to a point, is a homeomorphism, and since the pairs are triangulable, $H_{n-1}(Z,A) \cong H_{n-1}(Z/A)$, and $H_{n-1}(W,B) \cong H_{n-1}(W/B)$, (by [10], 4.8.9, since $n-1 \geq 1$). Consequently $f_*: H_{n-1}(Z,A) \to H_{n-1}(W,B)$, the right vertical arrow in the diagram, is an isomorphism. Furthermore the group $H_{n-1}(A) = 0$, since $n-1 = \dim_{\mathbb{R}}(Z) - 1 > \dim_{\mathbb{R}}(A)$. Consequently the map $H_{n-1}(Z) \to H_{n-1}(Z,A)$ in the diagram above is injective. Thus the composition $H_{n-1}(Z) \to H_{n-1}(Z,A) \to H_{n-1}(W,B)$ is injective, and by the commutativity of the diagram also the map $f_*: H_{n-1}(Z) \to H_{n-1}(W)$ is injective as desired. QED. for the Claim.

Proof of the Lemma: Now, assuming the hypotheses of the Lemma, look again at the commutative compatibility diagram used in the proof of statement (i), for ν^*, ν_*, and cap product. Applying the Claim to $\nu: \tilde{X} \to X$, the Lemma will follow from the fact that the bottom horizontal arrow in that diagram, $\cdot \cap [\tilde{X}]: H^1(\tilde{X}) \to H_{n-1}(\tilde{X})$, is injective when $\tilde{X}$ is unibranch. To prove that, let $\mu: Y \to \tilde{X}$ be a desingularization of $\tilde{X}$ and consider the following diagram analogous to the one in the proof of statement (i):

$$\begin{array}{ccc} \cdot\cap[\tilde{X}]: H^1(\tilde{X}) & \to & H_{n-1}(\tilde{X}) \\ \downarrow \mu^* & & \uparrow \mu_* \\ \cdot\cap[Y]: H^1(Y) & \to & H_{n-1}(Y) \end{array}$$

To prove the map across the top is injective, we will prove the other three maps are injective. Since $\tilde{X}$ is unibranch each of its connected components is irreducible; hence each connected component of Y is the resolution of exactly one connected component of $\tilde{X}$, and we may assume Y and $\tilde{X}$ are both connected if we wish. The map across the bottom is injective by Poincare duality for the smooth space Y, and the right vertical map is injective by the Claim proved above, applied to the resolution map μ. To prove the injectivity of μ^* on H^1, it suffices to prove surjectivity of μ_* on H_1, which will in turn be implied by the surjectivity of μ_* on π_1. We argue this next.

First represent a loop in $\tilde{X}$ by a finite simplicial loop, and then replace any edge lying wholly in the singular locus by the two other sides of a small triangle not wholly in the singular locus, to obtain a simplicial representative γ which intersects the singular locus only a finite number of times. Then, at a vertex p lying in the singular locus, lift each edge of γ terminating at that vertex into Y. By considering separately each branch of γ passing through p, we may assume there are exactly two edges of γ terminating at p. The inverse image of each edge is a compact real semi analytic set in Y, hence Y can be finitely triangulated with the inverse image of the closed edge as a subcomplex. Then the inverse image of each half open edge (omitting p) is a union of edges in Y terminating in a vertex in the exceptional locus. The two edges in $\tilde{X}$ terminating at p lift to two simplicial arcs terminating at two possibly different vertices in the exceptional locus in Y. Thus we get lifts of both edges terminating at p, but the lifts do not necessarily meet at the same point of Y. The two terminal points do however lie in the inverse image of p, which is connected by the "connectedness theorem" (a birational proper map to a unibranch variety has connected fibers). Thus one can join up the two points over p by a simplicial arc in $\mu^{-1}(p)$ to obtain a lift of γ whose image back in $\tilde{X}$ is at least homotopic to γ. I.e. the image in $\tilde{X}$ of the lift differs from the original γ only by having an additional degenerate segment at p. This proves that μ_* is surjective on π_1.

QED for the Lemma, and hence for the Proposition.

Proof of the Theorem: The Theorem follows directly from the statements of Propositions 1 and 2. QED for Theorem.

Bibliography

1. Andreotti, A. and Mayer, A.: On period relations for abelian integrals on algebraic curves. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **21**, 189-238 (1967)
2. Beauville, A.: Prym varieties and the Schottky problem. Invent. Math. **41**, 149-196 (1977)
3. Clemens, C.H.: Applications of the theory of Prym varieties. Proc. Int. Cong. of Mathematicians, Vancouver 1974, Vol. 1. Canadian Math. Congress, 415-421 (1975)
4. Clemens, C.H. and Griffiths, P.: The intermediate Jacobian of the cubic threefold. Ann. Math., II. Ser. **95**, 281-356 (1972)
5. Donagi, R. and Smith, R.: The structure of the Prym map. Acta Math. **146**, 25-102 (1981)
6. Dold, A.: Lectures on Algebraic Topology. (Die Grundlehren der Mathematischen Wissenschaften, Bd. 200) Berlin-Heidelberg-New York: Springer-Verlag 1972.
7. Fulton, W. and Lazarsfeld, R.: Connectivity and its applications in algebraic geometry. (Lect. Notes Math. **862**) Berlin-Heidelberg-New York: Springer-Verlag, 26-92 (1981)
8. Grothendieck, A.: Technique de descente et theoremes d'existence en geometrie algebrique VI, les schemas de Picard: proprietes generales. Semin. Bourbaki, t. 14, 1961/62, no.236
9. Kawamata, Y.: A generalization of Kodaira - Ramanujam's vanishing theorem. Math. Ann. **261**, 43-46 (1982)
10. Spanier, E.: Algebraic Topology. Berlin-Heidelberg-New York: Springer-Verlag 1966
11. Smith, R. and Varley, R.: Components of the locus of singular theta divisors of genus five. Algebraic Geometry, Sitges 1983. (Lect. Notes Math. **1124**) Berlin-Heidelberg-New York: Springer-Verlag, 338-416 (1985)
12. Smith, R. and Varley, R.: Singularity theory applied to Θ-divisors. Algebraic Geometry, Proceedings, Chicago 1989. (Lect. Notes Math. **1479**) Berlin-Heidelberg-New York: Springer-Verlag, 238-257 (1991)
13. Viehweg, E.: Vanishing theorems. J. Reine Angew. Math., Bd. **335**, 1-8 (1982)
14. Whitney, H.: Complex Analytic Varieties. Reading, Mass.: Addison-Wesley 1972
15. Wirtinger, W.: Untersuchungen uber Thetafunktionen. Berlin: Teubner 1895

Roy Smith (roy@joe.math.uga.edu)
Robert Varley (robert@joe.math.uga.edu)
Dept. of Math.
Boyd Grad. Studies
University of Georgia
Athens, Georgia, 30602
USA

(Received December 17, 1992;
in revised form June 20, 1993)

manuscripta math. 81, 283 - 297 (1993)

manuscripta mathematica

α-Compactness of Reduced Products and Filter Quotients

A. Laradji

1. Introduction

Throughout this paper, R is a ring and $\{M_i\}_{i\in I}$ is a non-empty family of left R modules. By a filter on I, is meant a non-empty subset F of $\mathcal{P}(I)$ such that whenever $X, Y \in F$ and $X \subseteq Z \subseteq I$, we have $X \bigcap Y \in F$ and $Z \in F$. If for each $m \in \prod M_i$, we denote by $z(m)$ the set $\{i \in I : m(i) = 0\}$, we can then define the filter sum of the M_i to be the module $\{m \in \prod M_i : z(m) \in F\}$, denoted by $\sum_F M_i$. It is easy to see that if G is another filter on I with $F \subseteq G$, then $\sum_F M_i$ is a submodule of $\sum_G M_i$. Clearly, filter quotients $\sum_G M_i / \sum_F M_i$ are generalizations of reduced direct products $\prod M_i / \sum_F M_i$. The study of the algebraic compactness of these quotients has largely been motivated by the classical result of Balcerzyk [1] asserting that the abelian group $Z^N / Z^{(N)}$ is algebraically compact. For example, it was proved by Hulanicki in [9] that $\Pi M_i / \oplus M_i$ is algebraically compact when $I = N$ and the M_i are abelian groups, while in [8], Gerstner proved that this quotient is not algebraically compact when I is uncountable and each

$M_i = Z$. In a generalization of Hulanicki's result, and using a more direct argument, Fuchs [6] proved that $\sum_{F_{\aleph_0}} M_i / \sum_F M_i$ is algebraically compact, where, for an infinite cardinal α, F_α denotes the filter obtained from F by adding α intersections. The special case of Fuchs' theorem and its proof, when $F_{\aleph_0} = \mathcal{P}(I)$, were further put into the more general setting of universal algebra by Mycielski in his seminal paper [14], in which he introduced the important notion of equational compactness, and which, for modules, coincides with algebraic compactness.

Dugas and Göbel [3] and Franzen [5] established several results relating the $\aleph_o$-compactness of quotients of the form $\sum_G M_i / \sum_F M_i$ to certain conditions on the filters F and G (see also Eda [4] in the context of quasi–sheaves of abelian groups). However, although $\aleph_0$-compactness and algebraic compactness coincide for modules over countable rings, and in particular for abelian groups (see [7] and also [15] and [16] for a universal algebra approach), this is not the case for arbitrary modules or algebras. In [10] and [11] for example, it is proved that the algebraic compactness of certain $\aleph_0$–compact reduced products of modules over some rings R forces R to have a specific structure. A study of the α-compactness (α an infinite cardinal) of quotients of the form $\sum_G M_i / \sum_F M_i$ is therefore worthwhile, and in fact, an interesting problem in this direction is to determine when the reduced product $\prod M_i / \sum_F M_i$ and, more generally, a filter quotient is α-compact given that it is β-compact for all $\beta < \alpha$. As mentioned above, when $\alpha = \aleph_0$ and $F_{\aleph_0} = \mathcal{P}(I)$, the problem has a positive answer, i.e. the quotient is $\aleph_0$-(equationally) compact (Fuchs [6]). However, for higher cardinals α (with the obvious replacement of $F_{\aleph_0}$ by F_α), the situation is quite different (see [12]):

(a) The quotient $\prod M_i / \sum_F M_i$ is α-compact if it is β-compact for all $\beta < \alpha$ and if the M_i are isomorphic.

(b) The quotient is α-compact if α is not weakly inaccessible and if $\{i \in I : M_i \text{ is } \beta\text{-compact }\} \in F$ for all $\beta < \alpha$.

(c) The quotient can be β-compact for all $\beta < \alpha$, without being α-compact.

The aim of this paper is, on the one hand, to extend Fuchs' theorem to arbitrary modules, and on the other hand, to determine some conditions for which the problem described above has an affirmative answer. In so doing, several generalizations to uncountable cardinals of results of Franzen [5], de Marco [13], Mycielski [14] will be obtained. Although we restrict our attention to modules, a number of results given here can be extended to general algebras. A theory of ordinals is assumed where each cardinal is an initial ordinal and each ordinal $x = \{y : y < x, y \text{ an ordinal}\}$. Also, for each cardinal α, α^+ will denote the infinite successor cardinal of α, and for any set X, $|X|$ is the cardinality of X.

2. Preliminary Definitions and Results

In this section we introduce the basic definitions and notations as well as some preliminary results needed for the later sections.

Definitions. Let F be a filter on the non-empty set I and let α be an infinite cardinal.

1. F is called *α-complete*, or simply an *α-filter*, if for each cardinal $\beta < \alpha$, F is closed under β-intersections (so that a filter is an $\aleph_0$-filter). It is clear that F is an α-filter if and only if $F = F_\beta$ for all $\beta < \alpha$ and that F_α is an α^+-filter on I containing F (recall that for an infinite cardinal γ, F_γ denotes the

set $\{X \subseteq I : \bigcap_{j<\gamma} A_j \subseteq X$, for some $A_j \in F\})$. Further, it is easy to prove that $F_\alpha = F_{cf(\alpha)}$.

2. Let G be a filter on I such that $F \subseteq G$. We shall say that F is *α-pure* in G, if for every descending chain $\{A_j\}_{j<\alpha}$ of members of G, there exist $X \in G$ and a descending family $\{B_j\}_{j<\alpha}$ of members of F such that for each $j < \alpha$ we have

$$X \cap \left(\bigcap_{t<j} B_t\right) \subseteq A_j.$$

Definition 2 generalizes the notion of purity of filters introduced by Dugas and Gobel in [3] for abelian groups, and used by Franzen in [5] in the case of modules. However, although every filter is $\aleph_0$-pure in itself, we can prove using Fodor's theorem, that if α is an uncountable regular cardinal then the Fréchet filter $\{X \subseteq \alpha : |\alpha \backslash X| < \alpha\}$ is an α-filter which is not α-pure in itself. We have however the following:

Proposition 1. *Let F be a filter on a non-empty set I and let α be an infinite cardinal, then F is α-pure in F_α.*

Proof. Let $\{A_j\}_{j<\alpha}$ be a descending chain in F_α. For each $j < \alpha$, there exists a descending family $\{B_{jt}\}_{t<\alpha}$ in F such that $A_j \supseteq \bigcap_{t<\alpha} B_{jt}$. If we let $X = \bigcap_{j<\alpha} \bigcap_{t<\alpha} B_{jt}$, $B_j = I \quad (j < \alpha)$, then clearly $X \cap \left(\bigcap_{t<j} B_t\right) \subseteq A_j$ for each j $< \alpha$.

Definition. A submodule N of an R-module M is *α-pure* in M for some infinite cardinal α if every system of less than α-equations over N which is solvable in M is solvable in N. In particular, $\aleph_0$-pure means pure in the usual sense.

If G is an α-filter on I, it can easily be proved that $\sum_G M_i$ is α-pure in $\prod M_i$, so that for any filter F on I with $F \subseteq G$ we also have $\sum_G M_i / \sum_F M_i$ α-pure in $\prod M_i / \sum_F M_i$. The following proposition gives a sufficient condition for $\sum_G M_i / \sum_F M_i$ to be α^+-pure in $\prod M_i / \sum_F M_i$.

Proposition 2. *Let F and G be α- filters on I and suppose that F is α-pure in G, then $\sum_G M_i / \sum_F M_i$ is α^+-pure in $\prod M_i / \sum_F M_i$.*

Proof. Let the system of equation $\sum_{k \in K} r_{jk} x_k = \overline{a}_j \quad (j < \alpha)$, where $\overline{a}_j \in \sum_G M_i / \sum_F M_i$ be solvable in $\prod M_i / \sum_F M_i$ by $\overline{m_k} \quad (k \in K)$, say. Put $A_j = \bigcap_{t \leq j} z\left(\sum_{k \in K} r_{tk} m_k - a_t\right)$, then $\{A_j\}$ is a descending family of members of F. Clearly, for each $u < \alpha$, $\bigcap_{j<u} A_j \subseteq \bigcap_{j<u} z\left(\sum r_{jk} m_k - a_j\right)$. Also, if $B_u = \bigcap_{j \leq u} z(a_j)$, then $\{B_u\}_{u<\alpha}$ is a descending chain of members of G, and so there exist $X \in G$ and a descending chain $\{C_u\}_{u<\alpha}$ in F such that for each $j < \alpha, X \cap \left(\bigcap_{t<j} C_t\right) \subseteq B_j$. Now, put $D_j = \left(\bigcap_{t<j} C_t\right) \cap A_j$ and define $\mu_k \quad (k \in K)$ in $\prod M_i$ by

$$\mu_k(i) = \begin{cases} m_k(i) & \text{if } i \in X^c \\ 0 & \text{otherwise} . \end{cases}$$

Clearly $X \subseteq z(\mu_k)$, i.e. $\mu_k \in \sum_G M_i \ (k \in K)$. We claim that $D_j \subseteq z\left(\sum_{k \in K} r_{jk} \mu_k - a_j\right)$ for each $j < \alpha$, from which it will follow that $\{\overline{\mu}_k\}_{k \in K}$ is a solution in $\sum_G M_i / \sum_F M_i$ of our system of equations, since $D_j \in F$. Let $i \in D_j$, then either $i \in X$ or $i \in X^c$. If $i \in X$ then $i \in z(\mu_k)$ and $i \in \bigcap_{t<j} C_t$, so that $i \in B_j$. Hence $\mu_k(i) = 0$ and $a_j(i) = 0$, i.e. $i \in z\left(\sum r_{jk} \mu_k - a_j\right)$. Now suppose $i \in X^c$.

In this case, either $i \in X^c \cap \left(\bigcap_{t<u} D_t \backslash D_u\right)$ for some $u > j$, so that $i \in \bigcap_{t<u} A_t \subseteq \bigcap_{t<u} z\left(\sum r_{tk}m_k - a_t\right) \subseteq z\left(\sum r_{jk}m_k - a_j\right)$ and hence $i \in z\left(\sum r_{jk}\mu_k - a_j\right)$, or $i \in \bigcap_{t<\alpha} D_t$, and so $i \in \bigcap_{t<\alpha} A_t$ which also implies that $i \in z\left(\sum r_{jk}\mu_k - a_j\right)$. This completes the proof.

Let F be an α-filter on I and let $T \in F$. Denote by $F(T)$ the set $\{X \cap T : X \in F\}$. It is easy to see that $F(T)$ is an α-filter on T. The following result will be referred to later, and is easy to prove.

Proposition 3. *Let F and G be filters on I with $F \subseteq G$ and let $T \in F$. Then there is an isomorphism $\sum_G M_i / \sum_F M_i \cong \sum_{G(T)} M_i / \sum_{F(T)} M_i$.*

3. Generalizations of Fuchs' Theorem

Recall that a module (or a general algebra) M is α-compact, where α is an infinite cardinal, if every finitely solvable system of α equations over M has a solution in M. If M is α-compact for all α, M is then said to be algebraically (or equationally) compact.

For the next proposition, we need the following easy result.

Lemma. *Let $0 \to A \to B \to C \to 0$ be a pure-exact sequence of R-modules and let β be a cardinal.*

1. *If A and C are β-compact then so too is B.*
2. *If A is α-pure in B for some $\alpha > \beta$ and B is β-compact, then so too is A.*

Proposition 4. *Let F be an α-filter on I and let G be a filter on I with $F \subseteq G$. Then for each cardinal $\beta < \alpha$, $\sum_G M_i$ is β-compact if and only if both $\sum_F M_i$ and $\sum_G M_i / \sum_F M_i$ are β-compact.*

Proof. Suppose first that $\sum_G M_i$ is β-compact. Since $\sum_F M_i$ is β-pure in $\sum_G M_i$, it follows that $\sum_F M_i$ is β-compact by the lemma. To prove that $\sum_G M_i / \sum_F M_i$ is β-compact, consider the system

$$\sum_{k \in K} r_{jk} x_k = \overline{a}_j \qquad (\overline{a}_j \in \textstyle\sum_G M_i / \sum_F M_i, \quad j < \beta) \tag{1}$$

and assume it is finitely solvable. For each finite subset L of β denote by $\overline{m(k, L)}$ a solution of the subsystem of (1) obtained when restricting j to L. Let $\mathcal{L} = \{L \subseteq \beta : L \text{ is finite }\}$ and let $S = \bigcap_{L \in \mathcal{L}} \bigcap_{j \in L} z\ (\sum r_{jk} m(k, L) - a_j)$. Since $|\mathcal{L}| = \beta < \alpha$, it follows that $S \in F$. Define $a'_j \quad (j < \beta)$ in $\prod M_i$ by

$$a'_j(i) = \begin{cases} a_j(i) & \text{if } i \in S \\ 0 & \text{otherwise}, \end{cases}$$

and for each $L \in \mathcal{L}$, define $\mu(k, L) \quad (k \in K)$ in $\prod M_i$ by

$$\mu(k, L)(i) = \begin{cases} m(k, L)(i) & \text{if } i \in S \\ 0 & \text{otherwise}. \end{cases}$$

Since $z(m(k, L)) \subseteq z(\mu(k, L))$ and $z(a_j) \subseteq z(a'_j)$, it follows that $\mu(k, L) \quad (k \in K)$ and $a'_j \quad (j < \beta)$ are in $\sum_G M_i$. Clearly, $\sum r_{jk} \mu(k, L) = a'_j \quad (j \in L)$, and so the system $\sum r_{jk} x_k = a'_j \quad (j < \beta)$ is finitely solvable in $\sum_G M_i$, and therefore has a solution in $\sum_G M_i$, $\mu_k \quad (k \in K)$ say. Since $S \in F$, $\sum r_{jk} \overline{\mu}_k = \overline{a'_j} = \overline{a_j} \quad (j < \beta)$, i.e., $\{\overline{\mu_k}\}_{k \in K}$ solves (1). This proves that $\sum_G M_i / \sum_F M_i$ is β-compact. The converse follows from the lemma.

Corollary 1. *Let F be an α-filter on I and let $\beta < \alpha$. If for each $i \in I$, M_i is β-compact, then $\sum_F M_i$ and $\prod M_i / \sum_F M_i$ are β-compact.*

Proof. Use Proposition 4 with $G = \mathcal{P}(I)$.

Corollary 2. *Let F be an α-filter on I and let G be a β^+-filter on*

I with $F \subseteq G$ and $\beta < \alpha$. If $\{i \in I : M_i$ is β-compact$\} \in F$, then $\sum_G M_i / \sum_F M_i$ is β-compact.

Proof. Put $T = \{i \in I : M_i \text{ is } \beta\text{-compact}\}$. Then $G(T)$ is a β^+-filter on T and so, by Corollary 1, $\sum_{G(T)} M_i$ is β-compact. By Proposition 4, $\sum_{G(T)} M_i / \sum_{F(T)} M_i$ is also β-compact. Now use the isomorphism $\sum_G M_i / \sum_F M_i \cong \sum_{G(T)} M_i / \sum_{F(T)} M_i$ (see Proposition 3).

The next result is a generalization of Fuchs' Theorem in [6].

Proposition 5. *Let F be an α-filter on I and for each $j < \alpha$, put $T_j = \{i \in I : M_i$ is $|j|$-compact$\}$. Suppose that there is a descending chain $\{S_j\}_{j<\alpha}$ of members of F such that $\bigcap_{t<j} S_t \subseteq T_j$ $(j < \alpha)$. Then $\sum_{F_\alpha} M_i / \sum_F M_i$ is α-compact.*

Proof. Observe first that $T_j \in F$ $(j < \alpha)$. Next, consider the system $\{R_j\}_{j<\alpha}$ over the filter quotient $Q = \sum_{F_\alpha} M_i / \sum_F M_i$, where R_j is the equation $\sum_{k\in K} r_{jk} x_k = \overline{a}_j$ $(\overline{a}_j \in Q)$, and suppose it is finitely solvable. By Corollary 2, Q is β-compact for each $\beta < \alpha$, and so the system is β-solvable in Q for each $\beta < \alpha$. Therefore, for each $u < \alpha$, there is a solution $\{\overline{m(k,u)}\}_{k\in K}$, say, in Q of the subsystem $\{R_j\}_{j\le u}$. Clearly, if we denote by R_{ji} the equation $\sum_{k\in K} r_{jk} x_k = a_j(i)$, then, for each $u < \alpha$, the set $A_u = \{i \in I : R_{ji}$ $(j \le u)$ is solvable in $M_i\}$ contains the set $\bigcap_{j\le u} z\left(\sum r_{jk} m(k,u) - a_j\right)$ which is in F. Hence, $\{A_u\}_{u<\alpha}$ is a descending chain in F. Also, since $\bigcap_{u<\alpha} z(a_u) \in (F_\alpha)_\alpha = F_\alpha$, there exists a descending family $\{E_u\}_{u<\alpha}$ in F such that $\bigcap_{u<\alpha} E_u \subseteq \bigcap_{u<\alpha} z(a_u)$. Fix $u < \alpha$, and let $i \in \bigcap_{j<\alpha} W_j$, where $W_j = A_j \cap E_j \cap S_j$. We claim that the system $\{R_{ji}\}_{j<u}$ is finitely solvable in M_i. For, let $j_1 < j_2 < \cdots < j_n < u$, say, then $i \in W_{j_n} \subseteq A_{j_n}$, and so the system $\{R_{ji}\}_{j\le j_n}$ is solvable in M_i. *A fortiori* so too is the system $\{R_{j_1}, R_{j_2}, \ldots, R_{j_n}\}$, as required. Now,

$i \in \bigcap_{j<u} W_j \subseteq \bigcap_{j<u} S_j \subseteq T_u$ and so M_i is $|u|$-compact, which implies that the system R_{ji} $(j < u)$ is solvable in M_i by $\{m'(k,u)(i)\}_{k\in K}$, say. Define μ_k $(k \in K)$ in $\prod M_i$ by

$$\mu_k(i) = \begin{cases} m'(k,u)(i) & \text{if } i \in \bigcap_{j<u} W_j \backslash W_u, \text{ for some } u < \alpha \\ 0 & \text{otherwise .} \end{cases}$$

Clearly, $W_u \in F$ for each $u < \alpha$, and $\bigcap_{u<\alpha} W_u \subseteq z(\mu_k)$, so that $\mu_k \in \sum_{F_\alpha} M_i$ for all $k \in K$. Next, let $i \in W_j$. Then, either there exists $u < \alpha$ (with $u > j$), such that $i \in \bigcap_{t<u} W_t \backslash W_u$ and so $\mu_k(i) = m'(k,u)(i)$, i.e. $\sum r_{jk}\mu_k(i) = \sum r_{jk}m'(k,u)(i) = a_j(i)$, or $i \in \bigcap_{t<\alpha} W_t$ and so $\mu_k(i) = 0$ which implies $\sum r_{jk}\mu_k(i) = 0 = a_j(i)$, since $i \in \bigcap_{t<\alpha} E_t$. Therefore, $W_j \subseteq z\left(\sum r_{jk}\mu_k - a_j\right)$ for all $j < \alpha$, and this proves that $\{\overline{\mu}_k\}_{k\in K}$ is a solution of $\{R_j\}_{j<\alpha}$ in Q.

As a consequence of Proposition 5, we next obtain the following generalizations of Franzen [5, Theorem (1.1)]

Proposition 6. *Let F and G be α-filters on I such that F is α-pure in G and $G \subseteq F_\alpha$ and suppose that there is a descending chain $\{S_j\}_{j<\alpha}$ in F such that $\bigcap_{t<j} S_j \subseteq T_j$, where $T_j = \{i \in I : M_i$ is $|j|$-compact$\}$, for each $j < \alpha$. Then $\sum_G M_i / \sum_F M_i$ is α-compact.*

Proof. By Proposition 2, $\sum_G M_i / \sum_F M_i$ is α^+-pure in $\prod M_i / \sum_F M_i$, so it is α^+-pure in $\sum_{F_\alpha} M_i / \sum_F M_i$ which is α-compact by Proposition 5. Therefore, $\sum_G M_i / \sum_F M_i$ is α-compact.

Proposition 7. *Let F and G be α-filters on I such that F is α-pure in G. If $\{i \in I : M_i$ is α-compact$\} \in F$, then $\sum_G M_i / \sum_F M_i$ is α-compact.*

Proof. Put $A = \sum_{F_\alpha} M_i / \sum_F M_i$, $B = \sum_{G_\alpha} M_i / \sum_F M_i$, $C =$

$\sum_{G_\alpha} M_i / \sum_{F_\alpha} M_i$, and consider the exact sequence

$$0 \to A \to B \to C \to 0.$$

By Proposition 5, A is α-compact, and by Corollary 2, C is α-compact. Clearly, A is pure in B and so by the previous lemma, B is α-compact. Now, $\sum_G M_i / \sum_F M_i$ is α^+-pure in $\sum_{G_\alpha} M_i / \sum_F M_i$ (using Proposition 2), and therefore, $\sum_G M_i / \sum_F M_i$ is α-compact.

Corollary 3. *Let F and G be α-filters on I such that F is α-pure in G and $G \subseteq F_\alpha$.*

(a) If $\{i \in I : M_i$ is β-compact for each $\beta < \alpha\} \in F$, then $\sum_G M_i / \sum_F M_i$ is α-compact.

(b) If α is not weakly inaccessible and for each $\beta < \alpha$, $\{i \in I : M_i$ is β-compact$\} \in F$, then $\sum_G M_i / \sum_F M_i$ is α-compact.

Proof. (a) Put $T = \{i \in I : M_i$ is β-compact for each $\beta < \alpha\}$, and for each ordinal $u < \alpha$, let $T_u = \{i \in I : M_i$ is $|u|$-compact$\}$. Clearly, $T \subseteq T_u$, and putting $S_u = T \quad (u < \alpha)$, we see that $\bigcap_{j<u} S_j \subseteq T_u$. Now, use Proposition 6.

(b) We distinguish two cases. (i) α is singular. Then $F = F_{cf(\alpha)} = F_\alpha$, so that $F = G$ and there is nothing to prove. (ii) α is regular. In this case $\alpha = \gamma^+$ for some cardinal γ, since α is not inaccessible, and $T = \{i \in I : M_i$ is γ-compact$\} \in F$. By (a), $\sum_G M_i / \sum_F M_i$ is α-compact.

Corollary 4. *Let F be an α-filter on I such that $F_\alpha = \mathcal{P}(I)$.*

(a) If $\{i \in I : M_i$ is α-compact$\} \in F$, then $\prod M_i / \sum_F M_i$ is α-compact.

(b) *If α is not weakly inaccessible, and if for each $\beta < \alpha$, $\{i \in I : M_i$ is β-compact$\} \in F$, then $\prod M_i / \sum_F M_i$ is α-compact.* ([12, Theorem 2]).

Proof. (a) is a particular case of Proposition 7. (b) follows from Corollary 1(b), since $\sum_{F_\alpha} M_i = \prod M_i$.

Now, for each infinite cardinal α, let us denote by $I(\alpha)$ the filter $\{X \subseteq I : |I \backslash X| < \alpha\}$. Then $I(\alpha^+) = \{X \subseteq I : |I \backslash X| \leq \alpha\}$, and it is easy to show that $I(\alpha)$ is a $cf(\alpha)$-filter on I with $I(\alpha)_{cf(\alpha)} = I(\alpha^+)$. By Proposition 1, $I(\alpha)$ is $cf(\alpha)$-pure in $I(\alpha^+)$ and so, by Corollary 1, we obtain

Proposition 8. *Let α be an infinite cardinal and suppose that the family $\{M_i\}_{i \in I}$ of R-modules is such that $\{i \in I : M_i$ is β-compact for all $\beta < cf(\alpha)\} \in I(\alpha)$. Then $\sum_{I(\alpha^+)} M_i / \sum_{I(\alpha)} M_i$ is $cf(\alpha)$-compact.*

As a corollary, we deduce the following well–known theorem of Balcerzyk [2] for modules.

Corollary 5. *Let α be a cardinal with $cf(\alpha) = \aleph_0$, then $\sum_{I(\alpha^+)} M_i / \sum_{I(\alpha)} M_i$ is $\aleph_0$-compact.*

4. α-Compactness of Filter Quotients

As mentioned in the introduction, the reduced product $\prod M_i / \sum_F M_i$, where F is an α-filter on I with $F_\alpha = \mathcal{P}(I)$, is α-compact if it is β-compact for all cardinals $\beta < \alpha$ and if all the M_i's are isomorphic. In this section, we prove that this is so for more general families $\{M_i\}_{i \in I}$ of R-modules and more general filter quotients. Let β be a cardinal and let $i \in I$. For notational convenience, let us denote by X_i the set $\{j \in I : M_i$ is isomorphic to a β^+-pure submodule of $M_j\}$. Our main result in this section is the following:

Proposition 9. *Let F and G be filters on I, let β be a cardinal such that $F \subseteq G \not\subseteq F_\beta$ and let $T = \{i \in I : X_i \in F_\beta\}$. If $\sum_G M_i / \sum_F M_i$ is β-compact, then $T \subseteq \{i \in I : M_i$ is β-compact$\}$.*

Proof. Without loss of generality, we may assume that for each $j \in X_i$, M_i is β^+–pure in M_j. Fix $t \in T$ and let $\sum_{k\in K} r_{jk}x_k = a_j$ $(j < \beta,\ a_j \in M_t)$ be a finitely solvable system over M_t. Since $G \not\subseteq F_\beta$, there exists $V \in G \backslash F_\beta$. Define b_j in $\prod M_i$ by

$$b_j(i) = \begin{cases} a_j & \text{if } i \in X_t \backslash V \\ 0 & \text{otherwise .} \end{cases}$$

Clearly, $V \subseteq z(b_j)$, so that $b_j \in \sum_G M_i$. Furthermore, the system $\sum_{k\in K} r_{jk}x_k = \overline{b}_j$ $(j < \beta)$ is easily seen to be finitely solvable in $\sum_G M_i / \sum_F M_i$, and therefore it has a solution $\overline{m}_k$ $(k \in K)$ in $\sum_G M_i / \sum_F M_i$. Hence, the set $U = \bigcap_{j<\beta} z\left(\sum r_{jk}m_k - b_j\right) \in F_\beta$. Since $X_t \in F_\beta$, it follows that $X_t \cap U \not\subseteq V$ and so we can choose an element h in $(X_t \cap U)\backslash V$. Clearly, $\sum r_{jk}m_k(h) = b_j(h) = a_j \in M_t$. Finally, M_t is β^+-pure in M_h (since $h \in X_t$), and so there exist $\mu_k \in M_t$ $(k \in K)$ with $\sum_{k\in K} r_{jk}\mu_k = a_j$, which implies that M_t is β-compact.

Remark. If we assume further that $T \in F$ in Proposition 9, then we obtain $\{i \in I : M_i$ is β-compact$\} \in F$. This means that when the assumptions of Proposition 9 are satisfied, a converse to [14, Theorem 2] is deduced in the following sense:

Proposition 10. *Let F be a filter on I and let α and β be infinite cardinals such that $\beta < \alpha$ and $F_\beta \neq \mathcal{P}(I)$. Suppose also that the R-modules $\{M_i\}_{i\in I}$ are such that $\{i \in I : X_i \in F_\beta\} \in F$. If*

$\prod M_i / \sum_F M_i$ *is* α*-compact, then* $\{i \in I : M_i$ *is* β*-compact*$\} \in F$.

The next result provides some conditions for which the problem described in the introduction has a positive answer.

Proposition 11. *Let* F *and* G *be* α*-filters on* I *with* $F \subseteq G \subseteq F_\alpha$ *and* F α*-pure in* G*, and suppose that the* R*-modules* $\{M_i\}_{i\in I}$ *satisfy:* $\{i \in I : X_i \in F\} \in F$. *If* $\sum_G M_i / \sum_F M_i$ *is* β*-compact for all* $\beta < \alpha$*, then it is* α*-compact.*

Proof. We may assume that $F \subset G$. Since $F = F_\beta$ for all $\beta < \alpha$, it can easily be shown, using Proposition 9, that $\{i \in I : M_i$ is β-compact for all $\beta < \alpha\} \in F$. By Corollary 3(a), $\sum_G M_i / \sum_F M_i$ is α-compact.

Corollary 6. *Let* F *be an* α*-filter on* I *such that* $F_\alpha = \mathcal{P}(I)$*, and suppose that the* R*-modules* $\{M_i\}_{i\in I}$ *satisfy:*

$$\{i \in I : \{j \in I : M_i \text{ is } \alpha\text{-pure in } M_j\} \in F\} \in F.$$

If $\prod M_i / \sum_F M_i$ *is* β*-compact for all* $\beta < \alpha$*, then it is* α*-compact.*

Remarks.

1. It is easy to show that the above result is true in the more general case when the $\{M_i\}_{i\in I}$ are similar relational structures. This corollary is consequently an extension of [12, Theorem 3] to more general families of relational structures, and also provides a weaker version of [14, Theorem 1].

2. An interesting special case of Corollary 6 is when α is a regular cardinal, $|I| = \alpha$, and the M_i form an ascending chain such that $\bigcup_{j<i} M_j$ is α-pure in M_i for each $i \in I$. It is then clear that $\prod M_i / \sum_{I(\alpha)} M_i$ is α-compact if and only if it is β-compact for all $\beta < \alpha$.

The following is a generalization of results of de Marco [13], and of Franzen [5] to higher cardinals.

Corollary 7. *Let F and G be filters on I, and let β be a cardinal such that $F \subseteq G \not\subseteq F_\beta$. If the R-module M is such that $\sum_G M/\sum_F M$ is β-compact, then M is β-compact.*

Proof. Use Proposition 9 with $M_i = M$ for each $i \in I$.

Remark. Putting $F = I(\aleph_0)$, $G = \mathcal{P}(I)$, we obtain that if M is an R–module such that $M^I/M^{(I)}$ is β–compact for some $\beta < |I|$, then M is β-compact.

If we combine Corollaries 2 and 7, we obtain

Corollary 8. *Let F be α–filter on I and G be a β^+–filter on I, where $\beta < \alpha$, and suppose that $F \subset G$. Then, an R–module M is β–compact if and only if $\sum_G M/\sum_F M$ is β–compact.*

Acknowledgement. The author gratefully acknowledges the support of King Fahd University of Petroleum and Minerals.

REFERENCES

1. Balcerzyk, S. On factor groups of some subgroups of the complete direct sum of infinite cyclic groups. Bull. Acad. Polon. Sci. **7**, 141–142 (1959)

2. Balcerzyk, S. On groups of functions defined on Boolean algebras, Fund. Math. **50**, 347–367 (1962)

3. Dugas, M. and Göbel, R. Algebraisch kompakte Faktorgruppen, J. Reine angew. Math. **307/308**, 341–352 (1979)

4. Eda, K. A generalized direct product of abelian groups, J. Algebra, **92**, 33–43 (1985)

5. Franzen, B. Algebraic compactness of filter quotients, Lecture Notes in Math. **874**, Springer, 228–241 (1981)

6. Fuchs, L. Note on factor groups in complete direct sums, Bull. Acad. Polon. Sci. **11**, 39–40 (1963)

7. Fuchs, L. Algebraically compact modules over noetherian rings, Ind. J. Math. **9**, 357–374 (1967)

8. Gerstner, O. Algebraische Kompaktheit bei Faktorgruppen von Gruppen ganzzahliger Abbildungen, Manuscripta Math. **11**, 104–109 (1974)

9. Hulanicki, A. The structure of the factor groups of the unrestricted sum by the restricted sum of abelian groups, Bull. Acad. Polon. Sci. **10**, 77–80 (1962)

10. Jensen, C. and Lenzing, H. Algebraic compactness of reduced products and applications to pure global dimension, Comm. Algebra **11**, 305–325 (1983)

11. Jensen, C. and Zimmermann–Huisgen, B. Algebraic compactness of ultrapowers and representation type, Pac. J. Math. **139**, 251–265 (1989)

12. Laradji, A. and Pawlikowski, J. On a problem of saturation of certain reduced direct products, Colloq. Math. **62**, 189–191 (1991)

13. de Marco, G. On the algebraic compactness of some quotients of product groups, Rend. Sem. Math. Univ. Padova **53**, 329–333 (1975)

14. Mycielski, J. Some compactifications of general algebras, Colloq. Math. **13**, 1–9 (1964)

15. Mycielski, J. and Ryll–Nardzewski, C. Equationally compact algebras II, Fund. Math. **61**, 271–281 (1968)

16. Wentzel, H. Equational compactness, Appendix 6 in: Grätzer, G. Universal Algebra, 2nd Ed., Springer, 417–447 (1979)

Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia

(Received February 11, 1993;
in revised form July 16, 1993)

manuscripta math. 81, 299 - 310 (1993)

manuscripta mathematica

Primitive Line Bundles on Abelian Threefolds*

Ch. Birkenhake, H. Lange, S. Ramanan

Introduction

Let L be an ample line bundle on an abelian variety X. According to Lefschetz's Theorem L^n is very ample for every $n \geq 3$. The second power L^2 is globally generated and by a theorem of Ohbuchi one knows exactly when L^2 is very ample (see [CAV]). It remains to consider the analogous questions for primitive L, i.e. for those L which are not of the form M^n for some $n \geq 2$ and an ample line bundle M on X. Recall that L is said to be primitive if it is of type $(1, d_2, \ldots, d_g)$.

On an abelian surface X the following is known. Let L be an ample line bundle of type $(1, d)$ on X defining an irreducible polarization. Then L is globally generated if and only if $d \geq 3$ and is very ample if and only if $d \geq 5$ and there is no elliptic curve E on X with $(L \cdot E) = 2$ (see [CAV] Theorem 10.4.1).

For abelian varieties of dimension $n \geq 3$ not much is known. Recently Ein and Lazarsfeld proved a theorem on global generation of adjoint line bundles on arbitrary smooth projective threefolds (see [E-L]). In the case of an abelian threefold their result can be stated as follows:

Theorem (Ein-Lazarsfeld). *Let L be an ample line bundle of type $(1, d_2, d_3)$ on X with $d_2 \cdot d_3 \geq 5$. Suppose there is no curve $C \subset X$ with $(L \cdot C) \leq 29$ and there is no surface $S \subset X$ with $(L^2 \cdot S) \leq 16$. Then L is globally generated.*

If $d_2 \cdot d_3$ is bigger, then one can choose the upper bounds for $(L \cdot C)$ and $(L^2 \cdot S)$ to be smaller (see [E-L]). Nevertheless the existence of such curves and surfaces is difficult to check. For instance in Section 4 we give examples of primitive line bundles L on abelian varieties X of arbitrary dimension ≥ 3 defining irreducible polarizations of type $(1, \ldots, 1, 2d)$ with arbitrarily high d which are not globally generated. The above theorem implies that there exists either a curve C or a surface S in X with small intersection number with L.

An immediate consequence of the above theorem is that for the general polarized abelian threefold (X, L) of type $(1, d_2, d_3)$ with $d_2 \cdot d_3 \geq 5$ the line bundle L is

* Supported by DFG-contract La 318/4 and EC-contract SC1-0398-C(A)

globally generated. This can be also easily seen via degeneration. So the problem is to find out whether a given primitive line bundle is globally generated or not, or equivalently to determine the corresponding subsets of the moduli space.

Since according to Proposition 4.3 there is no numerical criterion for global generation for primitive line bundles, any sufficient criterion requires an extra condition apart from the type. We express such an extra condition in terms of an isogeny $(X, L) \to (Y, P)$ with (Y, P) a principally polarized abelian variety.

To state our results note that a principally polarized abelian threefold (Y, P) is of one of the following three types (see [CAV] Section 10.8):

a) $(J(C), \mathcal{O}(\Theta))$, the Jacobian of a smooth curve C of genus 3,
b) $(J(C) \times E, p_1^*\mathcal{O}_{J(C)}(0) \otimes p_2^*\mathcal{O}_E(0))$, with C a smooth curve of genus 2 and an elliptic curve E,
c) $(E_1 \times E_2 \times E_3, p_1^*\mathcal{O}_{E_1}(0) \otimes p_2^*\mathcal{O}_{E_2}(0) \otimes p_3^*\mathcal{O}_{E_3}(0))$ with elliptic curves E_1, E_2, E_3.

In each of these cases we have a criterion for L to be globally generated. So let L be a primitive ample line bundle on the abelian threefold X. We may assume that L is of type $(1, 1, d)$ (see Remark 1.5). Furthermore according to the Decomposition Theorem [CAV] 4.3.1 we may assume that L defines an irreducible polarization.

Theorem 1. *Let $\pi: (X, L) \to (Y, P)$ be an isogeny onto a principally polarized abelian variety.*

a) Suppose $Y = J(C)$ is the Jacobian of a smooth curve C of genus 3. If $d \geq 5$ and C is not hyperelliptic, then L is globally generated.
b) Suppose $Y = J(C) \times E$ is the product of the Jacobian of a smooth curve of genus 2 and an elliptic curve E. Let $l \in \mathrm{Pic}^0(Y)$ be a line bundle defining the isogeny $\pi: X \to Y$. If the restriction $l|J(C)$ is of order $d \geq 5$, then L is globally generated.
c) Suppose $Y = E_1 \times E_2 \times E_3$ is the product of three elliptic curves. Let $l \in \mathrm{Pic}^0(Y)$ be a line bundle defining the isogeny $\pi: X \to Y$. Suppose the restriction $l|E_i$ is of order ≥ 4 for $i = 1, 2, 3$, then L is globally generated.

In part c) of the Theorem the assumption implies that $d \geq 4$. Moreover part c) easily generalizes to a result on abelian varieties of arbitrary dimension (see Theorem 4.1).

The proofs of the different parts of the theorem use different methods: For a) we apply curve theory, in particular a result on the Clifford index (see Section 2). The proof of b) proceeds via restriction using the analogous results on abelian surfaces (see Section 3). Finally in the proof of c) we profit by an explicit basis of theta functions for L (see Section 4).

The second part of the paper deals with the question whether a primitive ample line bundle on an abelian threefold is very ample. The easiest way to give an example of very ample primitive line bundles on an abelian surface is the method of Comessatti: If X is the Jacobian of a curve admitting two principal polarizations P and P' such that $L = P \otimes P'$ is of type $(1, d)$, then it is easy to see that L is very ample if $d \geq 5$.

For abelian threefolds this method yields: Let X be the Jacobian of a smooth curve of genus 3 and P and P' two principal polarizations on X with $(P \cdot P'^2) \geq 10$. Assume moreover that X is a simple abelian variety. Then $L = P \otimes P'$ is very ample. The proof applies Sakai's version of Reiders' Theorem to all translates of a theta divisor for P. This can be used for the proof of the following

Theorem 2. *Let d be an integer ≥ 13, $\neq 14$. There is a 3-dimensional family of abelian threefolds (X, L) such that the complete linear system $|L|$ gives an embedding $X \hookrightarrow \mathbb{P}_{d-1}$.*

In the proof these families are explicitly constructed. If d is squarefree, then L is necessarily of type $(1, 1, d)$. If d is not squarefree, the type depends not only on the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(X)$ but on the subring $\mathrm{End}(X)$. It is shown in [B] that the type $(1, 1, d)$ always can be realized in this way. Hence we get as a consequence, since very ampleness is an open condition for line bundles on abelian varietes:

Corollary. *For a general polarized abelian threefold (X, L) ot type $(1, 1, d)$, $d \geq 13$, $\neq 14$, the line bundle L is very ample.*

The last named author would like to thank the University of Erlangen-Nürnberg for its hospitality, which enabled him to engage in this cooperation.

1 Preliminaries

Let L be an ample line bundle on an abelian variety of dimension g over the field of complex numbers. The polarization induced by L is by definition the first Chern class $c_1(L)$ or equivalently the algebraic equivalence class of L. In the notation we often do not distinguish between L and its polarization and always write (X, L) for the polarized abelian variety. The polarization L induces an isogeny

$$\phi_L : X \to \widehat{X}\ , \quad x \mapsto t_x^* L \otimes L^{-1}$$

onto the dual abelian variety $\widehat{X} := \mathrm{Pic}^0(X)$. The kernel $K(L)$ of ϕ_L is of the form

$$K(L) \simeq \left(\bigoplus_{i=1}^{g} \mathbb{Z}/d_i\mathbb{Z} \right)^2$$

with positive integers $d_1, \ldots, d_g$ and $d_i | d_{i+1}$ for $i = 1, \ldots, g-1$. The vector $(d_1, \ldots, d_g)$ is called the *type* of the polarization L.

A polarized abelian variety (X, L) of type $(d_1, \ldots, d_g)$ admits an isogeny onto a principally polarized abelian variety $\pi : (X, L) \to (Y, P)$ such that $\pi^* P \simeq L$: The kernel of π and thus also the kernel of the dual isogeny $\widehat{\pi} : \widehat{Y} \to \widehat{X}$ is isomorphic to $\bigoplus_{i=1}^{g} \mathbb{Z}/d_i\mathbb{Z}$. The isogeny $\phi_M : Y \to \widehat{Y}$ is an isomorphism, P being a principal polarization. Hence the isogeny π determines the subgroup $Z := \phi_P^{-1}(\ker \widehat{\pi}) \simeq \bigoplus_{i=1}^{g} \mathbb{Z}/d_i\mathbb{Z}$ of Y.

Conversely any finite subgroup Z of a principally polarized abelian variety (Y, P) determines an isogeny $\pi: X \to Y$, namely π is the dual of the isogeny $Y \simeq \widehat{Y} \to \widehat{X} := Y/Z$. We call $\pi: X \to Y$ the isogeny associated to Z. When Z is a cyclic subgroup one can easily compute the type of the line bundle $L = \pi^* P$:

(1.1) Lemma. *Let Z be a cyclic subgroup of order d of a principally polarized abelian variety (Y, P) and $\pi: X \to Y$ the associated isogeny. Then $L = \pi^* P$ is of type $(1, \ldots, 1, d)$.*

Proof. The isogeny ϕ_L fits into the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_L} & \widehat{X} \\ {\scriptstyle\pi}\downarrow & & \uparrow{\scriptstyle\widehat{\pi}} \\ Y & \xrightarrow[\phi_P]{\sim} & \widehat{Y} \end{array}$$

According to the diagram $K(L) = \ker \phi_L = \pi^{-1}(Z)$ and hence $K(L)$ is of order d^2. On the other hand $K(L)$ contains a point of order d. Since $K(L)$ is a direct sum of an abelian group with itself, this implies $K(L) \simeq (\mathbb{Z}/d\mathbb{Z})^2$. □

The following lemma is an essential tool for the proof of Theorem 1.

(1.2) Lemma. *Let $\pi: (X, L) \to (Y, P)$ be an isogeny onto a principally polarized abelian variety (Y, P) associated to a finite subgroup $Z \subset Y$.*

a) There is a canonical decomposition

$$H^0(L) \simeq \bigoplus_{z \in Z} H^0(t_z^* P) \tag{1}$$

induced by the embeddings $\pi^*: H^0(t_z^* P) \to H^0(L)$.

b) (1) is the decomposition of $H^0(L)$ into eigenspaces with respect to the action of the Galois group $\ker \pi$.

Proof. a) First note that

$$\pi_* \mathcal{O}_X = \bigoplus_{z \in Z} \phi_P(z) = \bigoplus_{z \in Z} t_z^* P \otimes P^{-1} .$$

This is well known in the cyclic case and follows by an easy induction for an arbitrary finite subgroup $Z \subset Y$. Using the projection formula this implies

$$\begin{aligned} H^0(L) = \pi^* H^0(\pi_* \pi^* P) \simeq H^0(\pi_* \pi^* P) &= H^0(\pi_* \mathcal{O}_X \otimes P) \\ = \bigoplus_{z \in Z} H^0(\phi_P(z) \otimes P) &= \bigoplus_{z \in Z} H^0(t_z^* P). \end{aligned}$$

b) Recall that the theta group $\mathcal{G}(L)$ of L is an extension of $K(L)$ by $\mathbb{C}^*$. The commutator map of this extension induces a nondegenerate alternating form $e^L: K(L) \times K(L) \to \mathbb{C}^*$, with respect to which $\ker \pi$ is an isotropic subgroup. Hence there is a section $\ker \pi \hookrightarrow \mathcal{G}(L)$. It is easy to compute (use [CAV] Proposition 6.4.2) that the canonical action of $\mathcal{G}(L)$ on $H^0(L)$ restricts to the action of $\ker \pi$ on $H^0(t_z^* P)$ given by $(g, \vartheta) \mapsto e^L(\widetilde{z}, g)\vartheta$ where $\widetilde{z} \in \pi^{-1}(z)$. So the assertion follows from the fact that any character of $\ker \pi$ is of the form $e^L(\tilde{z}, \cdot)$, since e^L is nondegenerate. □

In the rest of this section we compile some easy results on global generation of ample line bundles on abelian varieties. In accordance with common usage we often write "base point free" instead of "globally generated". First note that the notion of global generation for an ample line bundle depends only on the polarization, since any two algebraically equivalent ample line bundles differ only by a translation.

(1.3) Lemma. *The base locus of an ample line bundle L on an abelian variety X contains a divisor if and only if the polarized abelian variety (X, L) contains a principally polarized factor (Y, P), i.e. $(X, L) \simeq (Z, M) \times (Y, P)$ as polarized abelian varieties.*

This is an immediate consequence of the Decomposition Theorem [CAV] 4.3.1. □

Now let L be an ample line bundle of type $(1, d_2, \ldots, d_g)$ on X and write $d = d_2 \cdot \ldots \cdot d_g$. In the sequel we denote by $\mathrm{bs}(L)$ the base locus of the complete linear system $|L|$. It is a closed subscheme of X.

(1.4) Lemma. *a) If $d < g$, then* $\mathrm{bs}(L)$ *contains at least a curve.*
b) If $d = g$, then L admits at least $d \cdot g!$ base points.
c) If $d > g!$ and L admits base points, then $\dim \mathrm{bs}(L) \geq 1$.

For the proof note that by the Riemann-Roch Theorem $h^0(L) = d$. So the assertions a) and b) follow from elementary intersection theory. As for c), suppose the base locus consists only of $b > 0$ points. The action of the group $K(L)$ on $|L|$ clearly induces a fixed-point free action on $\mathrm{bs}(L)$. This implies $\#K(L) = d^2 \leq b$. On the other hand $b \leq (L^g) = d \cdot g!$ □

Part c) of Lemma 1.4 generalizes the fact that line bundles of type $(1, d)$ on abelian surfaces defining an irreducible polarization are globally generated for $d \geq 3$.

(1.5) Remark. Finally suppose that L is an ample line bundle of type $(1, d_2, d_3)$ on an abelian threefold X. Our aim is to give sufficient criteria for L to be base point free.

First we point out that, in a sense, it suffices to consider line bundles of type $(1, 1, d)$. To see this note that there is a cyclic isogeny $\pi_1 : (X, L)) \to (X_1, L_1)$ of degree d_2 such that L_1 is of type $(1, 1, d_3)$. In the next sections we prove a criterion for L_1 to be base point free for $d_3 \geq 4$. Via π_1^* this gives analogous results for L if $d_3 \geq 4$. So it remains to consider the types $(1, 2, 2)$ and $(1, 3, 3)$:

According to [N-R] any ample line bundle of type $(1, 2, 2)$ (indeed of type $(1, 2, \ldots, 2)$ in general) admits base points. On the other hand for a general abelian threefold an ample line bundle of type $(1, 3, 3)$ is base point free. To see this note that a polarized product $(A, L_1) \times (E, L_2)$ with (A, L_1) an abelian surface of type $(1, 3)$ and (E, L_2) an elliptic curve of type (3) is base point free.

2 Proof of Theorem 1 a)

Let L be an ample line bundle of type $(1,1,d)$ on an abelian threefold X and $\pi: (X,L) \to (J, \mathcal{O}_J(\Theta))$ an isogeny onto the Jacobian J of a smooth curve C of genus 3. We have to show that if C is not hyperelliptic and $d \geq 5$, then L is base point free. For this we need the following

(2.1) Lemma. *Suppose C is nonhyperelliptic. For any $x \in X$ there is a translate $\widetilde{C} \subset J(C)$ of C with $x \in D := \pi^{-1}(\widetilde{C})$ such that the restriction map $H^0(X,L) \to H^0(D, L|D)$ is bijective.*

Proof. Recall that the group $J^\nu := \mathrm{Pic}^\nu(C)$ of line bundles of degree ν on C is a principal homogeneous space for $J(C)$. We may identify $J^2 = J(C)$, so $\pi(x)$ will considered as a point in J^2. Recall that $\Theta = \{l \in J^2 \mid h^0(l) \geq 1\}$ is a copy of the theta divisor in J^2. Any translate of C in J^2 is of the form $C_\alpha = \{\alpha \otimes \mathcal{O}_C(y) \mid y \in C\}$ for some $\alpha \in J^1$.

*S*tep I: There exists a translate C_α with $\pi(x) \in C_\alpha \not\subset \Theta$.Suppose the contrary, i.e. for every $\alpha \in J^1$ with $\pi(x) \in C_\alpha$ we have $C_\alpha \subset \Theta$. This can be expressed in terms of line bundles on C as follows: Note that $\alpha = \pi(x) \otimes \mathcal{O}(-p)$ for some $p \in C$. For these α we have $C_\alpha \subset \Theta$ if and only if $\pi(x) \otimes \mathcal{O}(-p) \otimes \mathcal{O}(q) \in \Theta$ for some $q \in C$. Hence our assumption translates to $\pi(x) \otimes \mathcal{O}(q-p) \in \Theta$ for all $p, q \in C$ and thus $\pi(x) \otimes \mathcal{O}(q) = \omega_C \otimes \mathcal{O}(-z)$ for some $z \in C$, since $h^0(C, \pi(x) \otimes \mathcal{O}(q)) = 2$. Hence for every $q \in C$ there exists a $z \in C$ such that

$$\omega_C \otimes \pi(x)^{-1} = \mathcal{O}(q+z) \ .$$

This means that $h^0(\omega_C \otimes \pi(x)^{-1}) \geq 2$ and thus C is hyperelliptic, a contradiction.

*S*tep II: Since the set of translates C_α of C in J^2 with $\pi(x) \in C_\alpha \not\subset \Theta$ is open in the set of all translates of C passing through $\pi(x)$, there exists a translate $\widetilde{C}$ with $\pi(x) \in \widetilde{C}$ and $\widetilde{C} \not\subset \bigcup_{z \in Z} t_z^*\Theta$ where Z is the cyclic subgroup of J associated to the isogeny $\pi: X \to J$. Let $D = \pi^{-1}(\widetilde{C})$ and assume that the restriction

$$r: H^0(L) \to H^0(D, L|D)$$

is not injective. Since r is Galois equivariant and $H^0(L) = \bigoplus_{z \in Z} H^0(J, \mathcal{O}(t_z^*\Theta))$ is the Galois decomposition according to Lemma 1.2, there is a $z \in Z$ such that $H^0(J, \mathcal{O}(t_z^*\Theta)) \subset \ker r$. But this means that $H^0(J, \mathcal{O}(t_z^*\Theta))$ is zero when restricted to $\widetilde{C}$, i.e. $\widetilde{C} \subset t_z^*\Theta$, a contradiction.

Finally, r is injective if and only if r is bijective, since $h^0(L) = h^0(D, L|D) = d$. □

Proof of Theorem 1 a). Suppose $x \in \mathrm{bs}(L)$. Since the Galois group $\ker \pi$ acts on L this implies $\pi^{-1}\pi(x) \subset \mathrm{bs}(L)$. If D and $\widetilde{C}$ are curves on X and J as in Lemma 2.1, then

$$h^0(D, L|D \otimes \mathcal{O}_D(-\pi^{-1}\pi(x))) = h^0(D, L|D) = d \ .$$

So for the Clifford index of $l := L|D \otimes \mathcal{O}_D(-\pi^{-1}\pi(x))$ we have

$$\mathrm{Cliff}(l) = \deg l - 2(h^0(l) - 1) = 2d - 2(d-1) = 2 \ .$$

This implies for the Clifford index of the curve D

$$\mathrm{Cliff}(D) \leq \mathrm{Cliff}(l) = 2\ .$$

According to [M] any curve of Clifford index ≤ 2 is hyperelliptic, trigonal, tetragonal, plane quartic, plane sextic or bielliptic. But D is neither a plane quintic nor a plane sextic, since it is of genus $2d+1 \geq 11$. Suppose $f: D \to F$ is a morphism of degree n onto a smooth curve F of genus g' with $(n, g') = (2,0)$, $(2,1)$, $(3,0)$, $(4,0)$. If τ denotes an automorphism of D generating the Galois group of $D \to \widetilde{C}$ and $f\tau \neq f$, then Castelnuovo's inequality (see [ACGH] Exercise C-1, p. 366) implies $11 \leq 2d+1 = g(D) \leq (n-1)^2 + 2ng(F) \leq 9$, a contradiction. Hence $f\tau = f$ and thus the morphism f factorizes via $\pi: D \to C$ which is absurd. □

3 Proof of Theorem 1 b)

Let L be an ample line bundle of type $(1,1,d)$ on an abelian threefold X and $\pi: (X, L) \to (J \times E, P)$ a cyclic isogeny of degree d onto the product of a Jacobian J of a smooth curve C of genus 2 and an elliptic curve E with canonical polarization $P = p_1^* \mathcal{O}_J(C) \otimes p_2^* \mathcal{O}_E(0)$. Suppose the isogeny π is associated to the subgroup $Z \subset J \times E$ generated by a point $z = (z_1, z_2) \in J \times E$. We have to show that L is base point free if z_1 is of order $d \geq 5$. For this we need some preparations.

Define A and F by the following cartesian diagrams

$$\begin{array}{ccccc} A & \hookrightarrow & X & \hookleftarrow & F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ J & \hookrightarrow & J \times E & \hookleftarrow & E \end{array}$$

Since z_1 is of order d, $\pi: A \to J$ is a cyclic isogeny of degree d of abelian surfaces. If z_2 is of order d', then $d' | d$ and F is the disjoint union of $\frac{d}{d'}$ copies of an elliptic curve. Note that $\ker \pi$ is a cyclic subgroup of X of order d. Since moreover $A \cap F = \ker \pi$ this implies $(A \cdot F) = d$.

Any $x \in X$ can be written in the form $x = x_1 + x_2$ with $x_1 \in A$ and $x_2 \in F$. Any two such decompositions of x differ at most by an element of $\ker \pi$. We omit the easy proofs of the following two lemmas.

(3.1) Lemma. *For any $x = x_1 + x_2 \in X$ with $x_1 \in A$ and $x_2 \in F$ we have*

$$\mathcal{O}_X(t_x^* A) = \mathcal{O}_X(A) \otimes \pi^* \phi_P \pi(x_2).$$

(3.2) Lemma. *Let $x = x_1 + x_2 \in X$ be as above. Then $h^i(X, L(-t_x^* A)) = 0$ for $i = 2,3$ and*

$$h^i(X, L(-t_x^* A)) = \begin{cases} 0 & \text{if} \quad \pi(x_2) \notin \langle z_2 \rangle \\ 1 & \quad \pi(x_2) \in \langle z_2 \rangle \end{cases} \quad \text{for } i = 0,1.$$

Moreover we need the following result.

(3.3) Lemma. *Consider* $V = \bigoplus_{\nu=1}^{d-1} H^0(J, t_{\nu z}^* \mathcal{O}_J(C))$ *as a subvector space of* $H^0(A, L)$ *according to Lemma 1.2 a). If* $d \geq 5$*, then the linear system* $|V|$ *is base point free.*

Proof. It suffices to show that the intersection of the divisors $t_z^* C, \dots, t_{(d-1)z}^* C$ in J is empty.

First we claim that any two translates of the curve C in J intersect in exactly two points (counted with multiplicities). To see this we identify J with $\mathrm{Pic}^1(C)$. Then C embeds canonically into J via $p \mapsto \mathcal{O}_C(p)$. Any translate of C in J is of the form $t_x^* C = \{\mathcal{O}_C(p) \otimes x^{-1} \mid p \in C\}$ with $x \in \mathrm{Pic}^0(C)$. If $\mathcal{O}_C(p) \in C \cap t_x^* C$, then there exists a $q \in C$ such that $\mathcal{O}_C(p) = \mathcal{O}_C(q) \otimes x^{-1}$, i.e. $\mathcal{O}_C(p - q) = x^{-1}$. But the anti-diagonal map $\delta: C \times C \to \mathrm{Pic}^0(C)$, $(p, q) \mapsto \mathcal{O}_C(p - q)$ is surjective, and of degree 2 outside the diagonal $\Delta \subset C \times C$. So for $x \neq 0$ the equation $\mathcal{O}(p) = \mathcal{O}(q) \otimes x^{-1}$ admits exactly 2 solutions counted with multiplicities. This proves the claim.

Suppose now $t_z^* C \cap t_{2z}^* C = \{x, y\}$. Then $t_{2z}^* C \cap t_{3z}^* C = \{x - z, y - z\}$ and $t_{3z}^* C \cap t_{4z}^* C = \{x - 2z, y - 2z\}$. But obviously these three sets do not intersect, since $z \neq 0 \neq 2z$. □

Proof of Theorem 1 b). It suffices to show that the restriction $|L||t_x^* A$ of the linear system $|L|$ to $t_x A$ is base point free for every $x = x_1 + x_2 \in X$. Consider the exact sequence

$$0 \to H^0(L(-t_x^* A)) \to H^0(L) \xrightarrow{r} H^0(L|t_x^* A) \to H^1(L(-t_x^* A)) \to 0. \qquad (3)$$

If $\pi(x_2) \notin \langle z_2 \rangle$, then by Lemma 3.2 $|L||t_x^* A$ is a complete linear system on $t_x^* A$, which is base point free, since the restricion of L to the abelian surface $t_x^* A$ is of type $(1, d)$.

Suppose now $\pi(x_2) \in \langle z_2 \rangle$. Without loss of generality we may assume $x = 0$. Since (3) is a sequence of Galois modules, the image of the restriction map $r: H^0(L) \to H^0(L|t_x^* A)$ is the span of the eigenspaces corresponding to the non-trivial characters of $\ker \pi$. So the assertion follows from Lemma 3.3. □

4 Proof of Theorem 1 c)

In this section we give a proof of Theorem 1 c). Since it works in arbitrary dimension, we assume more generally that X is an abelian variety of dimension g and L a line bundle on X defining an irreducible polarization of type $(1, \dots, 1, d)$. Moreover suppose X admits an isogeny $\pi: (X, L) \to (E_1 \times \cdots E_g, P)$ onto a product of elliptic curves with canonical principal polarization. Let $z = (z_1, \dots, z_g) \in E_1 \times \cdots \times E_g$ be a generator of the subgroup $Z \subset E_1 \times \cdots \times E_g$ associated to the isogeny π. Theorem 1 c) is a special case of the following

(4.1) Theorem. *If* z_i *is of order* $\geq g + 1$ *in* E_i *for* $i = 1, \dots, g$*, then* L *is base point free.*

Proof. We may assume that

$$D = \sum_{i=1}^{g} E_1 \times \cdots \times E_{i-1} \times \{0\} \times E_{i+1} \times \cdots \times E_g$$

is the unique divisor in the linear system $|P|$. According to the decomposition of Lemma 1.2, $H^0(L) = \bigoplus_{\nu=0}^{d-1} H^0(t^*_{\nu z} P)$, the line bundle L is base point free if and only if the divisors $t^*_{\nu z} D$, $\nu = 0, \ldots, d-1$, have no common point of intersection. Now $t^*_{\nu z} D = \sum_{i=1}^{g} E_1 \times \cdots \times \{\nu z_i\} \times E_{i+1} \times \cdots \times E_g$ implies

$$\bigcap_{\nu=1}^{g} t^*_{\nu z} D = \{(\sigma(1) z_1, \ldots, \sigma(g) z_g) \mid \sigma \in \mathfrak{S}_g\} \, .$$

Since by assumption $\sigma(\nu) z_\nu \neq 0$ for all $\nu = 1, \ldots, g$, no point $(\sigma(1) x_1, \ldots, \sigma(g) x_g)$ lies in D, so L is base point free. □

The method of the proof of the Theorem works also for line bundles of other types. We omit a general statement, since it is technically more complicated. On the other hand one can use the method to construct line bundles with base points.

(4.2) Example. Let $E_1, \ldots, E_g$ be elliptic curves and $\pi: X \to E_1 \times \cdots \times E_g$ be the cyclic isogeny associated to the subgroup $Z \subset E_1 \times \cdots \times E_g$ generated by a $2d$-division point $z = (z_1, \ldots, z_g)$. Suppose $z_1, \ldots, z_{g-1}$ are of order 2 and z_g is of order $2d$. The divisor

$$D = \sum_{\nu=1}^{g} E_1 \times \cdots \times E_{\nu-1} \times \{0\} \times E_{\nu+1} \times \cdots \times E_g$$

defines the canonical principal polarization on $E_1 \times \cdots \times E_g$. According to Lemma 1.1 the line bundle $L = \pi^* \mathcal{O}(D)$ is of type $(1, \ldots, 1, 2d)$.

We claim that the base locus $\mathrm{bs}(L)$ of L is of codimension 2. For this it suffices to show that $\pi^{-1}((z_1, 0) \times E_3 \times \cdots \times E_g) \subset \mathrm{bs}(L)$, since obviously $\mathrm{bs}(L)$ does not contain a divisor. But $(z_1, 0) \times E_3 \times \cdots \times E_g$ is contained in any divisor $t^*_{\nu z} D$, $\nu = 0, \ldots, 2d-1$. Since according to the decomposition (1) the divisors $\pi^* t^*_{\nu z} D$, $\nu = 0, \ldots, 2d-1$, are a basis for the linear system $|L|$, this implies the assertion. □

As a consequence we obtain

(4.3) Proposition. *There is no numerical criterion for global generation of ample line bundles on abelian varieties of dimension $g \geq 3$.*

5 Very Ample Line Bundles on Abelian Varieties

In this section we generalize the method of Comessatti (see [L1]) to prove Theorem 2 of the introduction. We need some preliminaries.

(5.1) Lemma. *Let C be a smooth curve of genus 3, $J = J(C)$ its Jacobian and Θ a theta divisor on J. For any two points $x, y \in J$ and any tangent vector $t \in T_xJ$*
a) there is a point $z \in J$ such that $t_z^*\Theta$ contains both x and y,
b) there is a point $z \in J$ such that $t_z^*\Theta$ contains x and t is tangential to $t_z^*\Theta$ at x.

Proof. a) Since the map $(s, z) \mapsto s - z$ of $\Theta \times t_x^*\Theta$ into J is surjective, there are $s \in \Theta$ and $z \in t_x^*\Theta$ such that $y = s - z$. In particular $y \in \Theta - z = t_z^*\Theta$. On the other hand $z \in t_x^*\Theta$ is equivalent to $x \in t_z^*\Theta$, which implies the assertion.

b) $x \in t_z^*\Theta$ if and only if $z \in t_x^*\Theta$. Hence we have to show that $\bigcup_{z \in t_x^*\Theta} T_x(t_z^*\Theta) = T_xJ$. Note that $\bigcup_{z \in t_x^*\Theta} T_x(t_z^*\Theta) = \bigcup_{z \in t_x^*\Theta} T_{x+z}\Theta = \bigcup_{y \in \Theta} T_y\Theta$. If we consider every tangent vector as tangent vector in $T_0J = H^0(\omega_C)$ via translation, then it suffices to show that $\bigcup_{y \in \Theta} T_y\Theta = T_0J = H^0(\omega_C)$.

But the Gauss map $G: \Theta_{\text{smooth}} \to P(H^0(\omega_C)^*)$, $y \mapsto T_y\Theta$ is dominant (see [CAV], Proposition 4.4.2). If C is nonhyperelliptic then Θ is smooth and thus G is surjective. This implies the assertion in this case. If C is hyperelliptic the same argument works, one has only to blow up the unique singular point. □

(5.2) Lemma. *Suppose (X, L) is a polarized abelian variety of dimension g and C an irreducible reduced curve on X generating X. Then $(L \cdot C) \geq g$.*

Proof. Let $\nu: \widetilde{C} \to C \subset X$ be the normalization and $J = J(\widetilde{C})$ the Jacobian of $\widetilde{C}$. According to the universal property of the Jacobian ν extends to a homomorphism $\psi: J \to X$. Since C generates X the homomorphism ψ is surjective.

Denote $n = (L \cdot C)$ and let $D \subset |L|$. Define a rational map $h: X \dashrightarrow S^n\widetilde{C}$ by $x \mapsto \nu^* t_x^* D$ and let $\alpha: S^n\widetilde{C} \to J$ be the map $\alpha(p_1 + \cdots + p_n) := \mathcal{O}_{\widetilde{C}}(p_1 + \cdots + p_n - \nu^*D)$. Then the following diagram is commutative

$$\begin{array}{ccc} X & \overset{h}{\dashrightarrow} & S^n\widetilde{C} \\ {\scriptstyle \phi_L}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \widehat{X} & \xrightarrow{\nu^*} & J \end{array} .$$

The homomorphism ν^* has finite kernel, since $-\nu^*$ is the dual of ψ according to [CAV] Corollary 11.4.2, and ψ is surjective. Hence

$$g = \dim X = \dim \operatorname{im} \nu^*\phi_L = \dim \overline{\operatorname{im} \alpha h} \leq \dim \overline{\operatorname{im} h} \leq n. \quad \square$$

(5.3) Proposition. *Let (J, Θ) be the Jacobian of a smooth curve C of genus 3. Assume that J is a simple abelian variety. Suppose Θ' is another principal polarization on J with $(\Theta^2 \cdot \Theta') = d_1$ and $(\Theta \cdot \Theta'^2) = d_2 \geq 10$. Then $L = \mathcal{O}_J(\Theta + \Theta')$ is very ample and embeds J into $\mathbb{P}_{\frac{1}{2}(d_1+d_2)+1}$.*

Proof. According to Lemma 5.4 it suffices to show that the restriction of the linear system $|L|$ to any translate $t_x^*\Theta$ is very ample. Since $L(-t_x^*\Theta)$ is algebraically equivalent to $\mathcal{O}_J(\Theta')$, we have $H^1(L(-t_x^*\Theta)) = 0$. Hence the restriction map $H^0(L) \to H^0(L|t_x^*\Theta)$ is surjective and it suffices to show that the line bundle $L|t_x^*\Theta$ is very ample on the surface $t_x^*\Theta$. But $L|t_x^*\Theta = K_{t_x^*\Theta} \otimes M$ with a line bundle M algebraically equivalent to $\mathcal{O}_{t_x^*\Theta}(\Theta')$. We have on the one hand for the intersection numbers

$$(M^2)_{t_x^*\Theta} = (\Theta'^2 \cdot \Theta)_J = d_2 \geq 10.$$

On the other hand any curve C on $t_x^*\Theta \subset J$ generates J, since J is simple. So we have by Lemma 5.5

$$(M \cdot C)_{t_x^*\Theta} = (\Theta' \cdot C)_J \geq 3.$$

Hence Reider's Theorem yields that $L|t_x^*\Theta$ is very ample: Note that if the curve C is nonhyperelliptic, then Θ is smooth and Reider's original theorem applies (see [R]). In the hyperelliptic case, Θ has a rational double point, so we may apply Sakai's version of Reider's Theorem (see [S]). □

It remains to construct Jacobian threefolds with suitable extra principal polarizations as in the Proposition. For this let K denote a totally real number field of degree 3 over $\mathbb{Q}$. Let $\mathfrak{o}$ denote the maximal order of K and $\mathfrak{d}_{K|\mathbb{Q}}$ the different of K over $\mathbb{Q}$. Consider the free $\mathbb{Z}$-module $\mathcal{M} = \mathfrak{o} \oplus \mathfrak{d}_{K|\mathbb{Q}}^{-1}$ of rank 6. According to [CAV] Proposition 10.2.1 the moduli space $\mathcal{A}(\mathcal{M})$ of abelian threefolds X with endomorphism structure in K, i.e. an embedding $K \subseteq \mathrm{End}_{\mathbb{Q}}(X)$, associated to the module $\mathcal{M}$ is of dimension 3. According to [L2] Lemma 2.1 every X in an open subset $U \subseteq \mathcal{A}(\mathcal{M})$ (with respect to the euclidean topology) admits a principal polarization Θ. On the other hand according to a theorem of Shimura (see [CAV] Theorem 10.9.1) for a general $X \in \mathcal{A}(\mathcal{M})$ we have $K = \mathrm{End}_{\mathbb{Q}}(X)$. In particular any such X is a simple abelian variety. Hence a general element X of U is simple and as such the Jacobian of a smooth curve of genus 3 (see [CAV] Corollary 11.8.2).

Fix a Jacobian threefold (J, Θ) with $\mathrm{End}_{\mathbb{Q}}(J) = K$ a totally real cubic number field as above. Since the Rosati involution on $\mathrm{End}_{\mathbb{Q}}(J)$ defined by Θ is the identity, the assignment $\Theta' \mapsto \varphi_{\Theta'} = \phi_\Theta^{-1}\phi_{\Theta'}$ gives a bijection between the set of principal polarizations on J and the set of totally positive automorphisms of J (see [CAV] Theorem 5.2.5). Thus any totally positive unit $\eta \in \mathfrak{o}^* \subset K$ corresponds to a principal polarization Θ_η of J. Moreover, if η is of order 3 over $\mathbb{Q}$ with minimal polynomial $\varphi_{\Theta_\eta} = x^3 - a_1x^2 + a_2x - 1$ (η being totally positive means that the roots of the minimal polynomial are all positive), then

$$(\Theta^2 \cdot \Theta_\eta) = 2a_1 \quad \text{and} \quad (\Theta \cdot \Theta_\eta^2) = 2a_2$$

(see [CAV] Proposition 5.2.3). Hence in order to complete the proof of Theorem 5.1 it suffices to show the following

(5.4) Lemma. *For any integer $d \geq 13$, $\neq 14$ there is an irreducible polynomial $f(x) = x^3 - a_1x^2 + a_2x - 1$ over $\mathbb{Q}$ with only positive roots such that $d = a_1 + a_2 + 2$ and $a_2 \geq 5$.*

Proof. Suppose f is not irreducible, then 1 or -1 is a root of f. But $f(-1) = 0$ if and only if $a_1 + a_2 = -2$, which is impossible since a_1 and a_2 are positive, and $f(1) = 0$ if and only if $a_1 = a_2$. So for $d = 2m + 1 \geq 13$ choose $a_1 = m$ and $a_2 = m - 1$ and for $d = 2m \geq 16$ choose $a_1 = m$, and $a_2 = m - 2$. An easy computation shows that the discriminant of f is positive in these cases. So f has only positive roots. □

(5.5) Remark. For special positive integers there is an easy method to show that for a general polarized abelian threefold (X, L) of type $(1, 1, d)$ the line bundle L is very ample.

Let d_1 and d_2 be coprime integers with $d_1 \geq 3$ and $d_2 \geq 5$. Then the general abelian variety of type $(1, 1, d_1 \cdot d_2)$ is very ample.

Proof. Suppose (E, L_1) is a polarized elliptic curve of type (d_1) and (A, L_2) a general polarized abelian surface of type $(1, d_2)$. According to [CAV] the line bundle L_2 is very ample on A. Hence the product $p_1^* L_1 \otimes p_2^* L_2$ is very ample on $E \times A$. Moreover $p_1^* L_1 \otimes p_2^* L_2$ is necessarily of type $(1, 1, d_1 \cdot d_2)$, since $(d_1, d_2) = 1$. □

References

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P.A., Harris, J.: Geometry of algebraic curves I, Springer Grundlehren 267 (1985)

[B] Birkenhake, Ch.: Tensor Products of Ample Line Bundles on Abelian Varieties, to appear

[E-L] Ein, L., Lazarsfeld, R.: Global Generation of Pluricanonical and Adjoint Linear Series on Smooth Projective Threefolds, Preprint (1992)

[L1] Lange, H.: Jacobian Surfaces in $\mathbb{P}_4$, J. reine angew. Math. 372 (1989), 71–86

[L2] Lange, H.: Abelian Varieties with several principal polarizations, Duke Math. Journal, 55 (1987), 617–628

[CAV] Lange, H., Birkenhake, Ch.: Complex Abelian Varieties, Springer Grundlehren 302 (1992)

[M] Martens, G.: Funktionen von vorgegebener Ordnung auf komplexen Kurven, J. reine angew. Math. 320 (1980), 68–85

[N-R] Nagaraj, D.S, Ramanan, S.: Abelian varieties with polarisation of type $(1, 2, \ldots, 2)$, to appear

[R] Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces, Ann. Math. 127 (1988), 309–316

[S] Sakai, F.: Reider-Serano's method on normal surfaces, in: Algebraic Geometry: Proceedings, L'Aquila 1988, Lect. Notes in Math. 1417 (1990), 301–319

Ch. Birkenhake, H. Lange
Mathematisches Institut
Bismarckstraße $1\frac{1}{2}$
D-8520 Erlangen, Germany

S. Ramanan
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005, India

(Received February 22, 1993;
in revised form June 11, 1993)

manuscripta math. 81, 311 - 328 (1993)

manuscripta mathematica

Generalized Castelnuovo Varieties

Vincenzo Di Gennaro

0. Introduction

In this paper we will show that if $V \subset \mathbb{P}^r_{\mathbb{C}}$ is an irreducible nondegenerate projective variety of dimension n and degree d, not contained on any irreducible subvariety of $\mathbb{P}^r_{\mathbb{C}}$ of dimension n+1 and degree < s, and if d is large with respect to s, then the genus of V (arithmetic if n=1; geometric if n ≥ 2) is bounded above by a number $G = \frac{d^{n+1}}{(n+1)!s^n} + O(d^n)$ (see Section 1) which depends only on n,r,d and s. In Section 4 we will see that this upper bound G is sharp (at least in some cases): *generalized Castelnuovo varieties* are varieties whose genus is G (see Section 3 and 4).

In the case of nondegenerate curves (i.e. n=1, s=r-1) this result is well known since the end of last century (see [Hp], [Ca]) and at the beginning of the 1980's it has been proved in [GP] for n=1, r=3, s ≥ 2, d » s and in [EH] for n=1, r ≥ 3, r-1 ≤ s ≤ 2r-3, d » s; the general result has been only recently proved for n=1, r ≥ 3, s ≥ r-1 and d » s ([CCD]). The case n ≥ 1 has been analyzed in 1981 ([H]) for s=r-n (in this case the varieties which achieve the bound G are called "Castelnuovo varieties") and in 1990 ([NV]) for codimension two arithmetically Cohen-Macaulay subvarieties V of $\mathbb{P}^r$ (in this case, when d » s, our bound is more precise than the bound obtained in [NV] (see Remark 1.9)).

We begin in Section 1 by establishing the bound. In Section 2 we make some remarks on the arithmetic genus of a projective variety which generalize an old result of Jongmans (see Remark 2.3 and [J] p.27). In Section 3 we give the definition of generalized Castelnuovo varieties and show some of their geometric properties. Finally in Section 4 we present some examples and in the particular cases d ≡ 0 (mod s), s ≡ 0 (mod r-n-1) and d » s » 0 we study the Hilbert scheme of generalized Castelnuovo varieties.

Our approach in proving such results is to use "Castelnuovo theory" which is roughly the study of varieties in $\mathbb{P}^r$ by means of the Hilbert function of their generic hyperplane section. In particular we will use the methods developed in [H], [CCD] and in Section 2. Finally, we refer to this paper for some questions that we left open (e.g. see Problem 3.8).

I would like to thank C. Ciliberto and L. Chiantini for many helpful discussions and suggestions and for their encouragement. Special thanks go to C. Ciliberto for introducing me Castelnuovo theory.

1. A bound on the geometric genus of projective varieties

We begin with some preliminary notation.

Let n, r and s be integers such that $1 \leq n \leq r-2$ and $s \geq r-n$.

Define a, b by dividing: $s-2=a(r-n-1)+b$, $0 \leq b < r-n-1$ and put

$$d_0(n,r,s) := d_0 := \max\left\{(2r-n-2)(s-1)+a(a-1)(r-n-1)+2ab-2,\ \frac{2s}{r-n-1}\prod_{i=1}^{r-n-1} \sqrt[r-n-i]{(r-n)!s}\right\}.$$

Observe that :

$$d_0 < \max\left\{2s^2+rs,\ \frac{2s}{r-n-1}[(r-n)!s]^{1+\log(r-n-1)}\right\}$$

Let $d > d_0(n,r,s)$ be an integer and $h := h(r-n+1,d,s)$ the numerical function introduced in [CCD] for $r' := r-n+1$, d and s. That is, put:

$$d-1=ms+\varepsilon \qquad 0 \leq \varepsilon < s$$
$$s-1=(r'-2)w+v \qquad 0 \leq v < r'-2$$

and define:

$$h(r',d,s)(i) := h(i) := \sum_{j=0}^{i} \Delta h(j),$$

where:

$$\Delta h(j) := \begin{cases} 0 & j<0 \\ (r'-2)j+1 & 0 \leq j \leq w \\ s & w < j \leq m \\ s+k-(r'-2)(j-m) & m < j \leq m+\delta \\ s+k-(r'-2)(j-m)-1 & m+\delta < j \leq w+m+e \\ 0 & j > w+m+e \end{cases}$$

and k,δ,e are defined by:

$$\begin{cases} \text{if } \varepsilon < w(r'-1-v) & \text{then } e=0\ \ \varepsilon=kw+\delta \quad 0 \leq \delta < w \\ \text{if } \varepsilon \geq w(r'-1-v) & \text{then } e=1\ \ \varepsilon+r'-2-v=k(w+1)+\delta \quad 0 \leq \delta < w+1. \end{cases}$$

Note that $h(n) \leq d$ for all n and $h(n) = d$ for $n \geq w+m+e$.

Finally put:

$$G(n,r,d,s) := G_n := \sum_{i=1}^{\infty} \binom{i-1}{n-1}(d-h(i)).$$

An easy calculation shows that $G_n = \dfrac{d^{n+1}}{(n+1)!s^n} + O(d^n)$.

The purpose of this section is to show the following:

Theorem 1.1 *Let* V *be an irreducible nondegenerate projective variety of dimension* n *and degree* d *in the projective space* $\mathbb{P}^r$ *over the complex field. Assume* V *not contained on any irreducible subvariety of* $\mathbb{P}^r$ *of dimension* n+1 *and degree* < s. *Assume* $1 \le n \le r-2$, $s \ge r-n$ *and* $d > d_0(n,r,s)$.
Let $V = V_n \supset V_{n-1} \supset \dots \supset V_1 \supset V_0 = \Gamma$ *be generic linear sections of* V *with* $\dim V_j = j$. *Let* $p_g(V_j)$ *be the geometric genus of* V_j *(i.e.* $p_g(V_j) := \dim_{\mathbb{C}} H^0(V_j^{\sim}, \Omega^j_{V_j^{\sim}})$ *where* $V_j^{\sim}$ *is a nonsingular projective variety birational to* V_j (see [H] p.41)*). Then, for each* j=1,2, ... ,n, *we have* $p_g(V_j) \le G(j,r-n+j,d,s)$. *In particular:*

$$p_g(V) \le G(n,r,d,s).$$

In order to prove Theorem 1.1 we need some preliminary results.

Proposition 1.2 *With the notation of* Theorem 1.1, *let* h_Γ *be the Hilbert function of* Γ *(= intersection of* V *with the generic* (r-n)*-dimensional subspace of* $\mathbb{P}^r$*). Then for each* j = 1,2, ... ,n *we have:*

$$p_g(V_j) \le \sum_{i=1}^{\infty} \binom{i-1}{j-1} (d-h_\Gamma(i)).$$

Proof: Following ([H], p.42) we can work on a resolution of V (i.e. a smooth projective variety $V^{\sim}$ mapping holomorphically and birationally to V). So that we may assume that $V_1, \dots, V_n$ are nonsingular.

Put $C := V_1$. Since the points of Γ impose $h_\Gamma(l)$ conditions on the linear system of hypersurfaces of degree l in $\mathbb{P}^r$ then, a fortiori, they impose at least $h_\Gamma(l)$ conditions on the complete linear system $|\mathcal{O}_C(l)|$ on C. Hence, for all l we have : $h^0(C,\mathcal{O}_C(l)) - h^0(C,\mathcal{O}_C(l-1)) \ge h_\Gamma(l)$. It follows that:
$0 \le h^0(C,\Omega^1_C(-l+1)) - h^0(C,\Omega^1_C(-l)) = d - (h^0(C,\mathcal{O}_C(l)) - h^0(C,\mathcal{O}_C(l-1))) \le d - h_\Gamma(l)$.

Let $M \gg 0$ be such that $h_\Gamma(M+1) = d$. Since for all $l \gg 0$ $h^0(C,\Omega^1_C(-l)) = 0$ we have: $h^0(C,\Omega^1_C(-M)) = 0$, $h^0(C,\Omega^1_C(-M+1)) \le d - h_\Gamma(M)$, $h^0(C,\Omega^1_C(-M+2)) \le 2d - (h_\Gamma(M) + h_\Gamma(M-1))$, ... , $h^0(C,\Omega^1_C(-M+l)) \le ld - \sum_{i=0}^{l-1} h_\Gamma(M-i)$. (1)

If l = M we have: $p_g(V_1) = h^0(C,\Omega^1_C) \le Md - \sum_{i=0}^{M-1} h_\Gamma(M-i) = \sum_{i=1}^{\infty}(d-h_\Gamma(i))$ so that we are done for V_1.

Put $S=V_2$, and consider the Poincaré residue sequence tensored with $\mathcal{O}_S(-l)$:

$$0 \to \Omega_S^2(-l) \to \Omega_S^2(-l+1) \to \Omega_C^1(-l) \to 0$$

We have: $0 \le h^0(S, \Omega_S^2(-l+1)) - h^0(S, \Omega_S^2(-l)) \le h^0(C,\Omega_C^1(-l))$. Since for $l \gg 0$ $h^0(S, \Omega_S^2(-l)) = 0$ and $h^0(C,\Omega_C^1(-M)) = 0$ then: $h^0(S, \Omega_S^2(-M+1)) = 0$, $h^0(S, \Omega_S^2(-M+2)) \le h^0(C,\Omega_C^1(-M+1))$, $h^0(S, \Omega_S^2(-M+3)) \le h^0(C,\Omega_C^1(-M+1)) + h^0(C,\Omega_C^1(-M+2))$, ... , $h^0(S, \Omega_S^2(-M+l)) \le \sum_{i=1}^{l-1} h^0(C,\Omega_C^1(-M+i))$. (2)

By (1) and (2) it follows that: $p_g(V_2) = h^0(S, \Omega_S^2) \le \sum_{l=1}^{M-1}(ld - \sum_{i=0}^{l-1} h_\Gamma(M-i)) = \sum_{i=1}^{\infty}(i-1)(d- h_\Gamma(i))$, so that we are done for V_2.

Put $T = V_3$. From the exact sequence: $0 \to \Omega_T^3(-l) \to \Omega_T^3(-l+1) \to \Omega_S^2(-l) \to 0$ we deduce: $0 \le h^0(T,\Omega_T^3(-l+1)) - h^0(T,\Omega_T^3(-l)) \le h^0(S,\Omega_S^2(-l))$. Hence as before we get

$$p_g(V_3) = h^0(T, \Omega_T^3) \le \sum_{l=2}^{M-1} h^0(S, \Omega_S^2(-M+l)) \le \sum_{l=2}^{M-1} (\sum_{i=1}^{l-1} h^0(C,\Omega_C^1(-M+i))) \le$$

$$\le \sum_{l=2}^{M-1} (\sum_{i=1}^{l-1} (id - \sum_{i'=0}^{i-1} h_\Gamma(M-i'))) = \sum_{i=1}^{\infty} \binom{i-1}{2} (d - h_\Gamma(i))$$ so that we are done for V_3.

Continuing in this fashion we obtain the bound for each $j=1,2, \dots ,n$.●

Theorem 1.3 (Lifting) *Let* $V \subset \mathbb{P}^r$ *be a nondegenerate irreducible projective variety of dimension* n *and degree* d. *Let* B *be a smooth irreducible scheme and* $f:B \to \mathrm{Grass}(h,r)$ *a dominant smooth morphism,* $h+n \ge r$. *For any* $t \in B$, *let* L_t *be the* h*-plane corresponding to the point* $f(t) \in \mathrm{Grass}(h,r)$. *Let* $\mathcal{W}$ *in* $B \times \mathbb{P}^r$ *be a family of projective varieties, flat over* B. *For* $t \in B$ *let* W_t *be the fibre of* $\mathcal{W}$ *over* t. *Suppose that the generic fibre* W_t *of* $\mathcal{W}$ *is irreducible, of dimension* $h+n+1-r$ *and degree* σ, *and that for* $t \in B$ *one has* $L_t \supseteq W_t \supseteq V_t := V \cap L_t$. *If* $d > (r+h-3)\sigma + \alpha(\alpha-1)(r-n-1) + 2\alpha\beta - 2$, *where* $\sigma-1 = \alpha(r-n-1) + \beta$, $0 \le \beta < r-n-1$, *then the image* W *of* $\mathcal{W}$ *in* $\mathbb{P}^r$ *is a variety of dimension* $n+1$ *and degree* σ *containing* V *and such that* $W_t = W \cap L_t$ *for* $t \in B$.

<u>Proof</u>: See ([CC1] p.2, Theorem 0.2).●

Remark 1.4 It may be interesting to point out that if $d > 2^n(n+2)!(\sigma+1)^{n+1} + \sigma+1$ then one can easily prove Theorem 1.3 without the differential geometric techniques employed in [CC1] but only relying on Castelnuovo's theory.
We briefly sketch our different proof which is a slight modification of the proof of ([CCD] Proposition 3.1).

Put $s = \sigma+1$ and $d-1=ms+\varepsilon$, $0 \leq \varepsilon < s$. All hypersurfaces of $\mathbb{P}^r$ of degree m passing through V must contain W. Let $\mathcal{T}$ be the intersection of these hypersurfaces and T an irreducible component of $\mathcal{T}$ containing W. In order to prove our assertion it is sufficient to show that $\dim T \leq n+1$. To see this, put $k = \dim T$ and observe that:
$md^n \geq h^0\mathcal{O}_V(m) \geq h_V(m) = h_T(m) \geq \binom{m+k-1}{k}(r-k) + \binom{m+k}{k}$ ([EH] , Proposition 3.23, p.117). Since $d > 2^n(n+2)!s^{n+1} + s$ then $k \leq n+1$.●

Corollary 1.5 *Let* $V \subset \mathbb{P}^r$ *be a nondegenerate irreducible projective variety of dimension* n *and degree* d. *If the curve intersection of* V *with the generic* $(r-n+1)$*-dimensional subspace* H *of* $\mathbb{P}^r$ *is contained in some irreducible surface of* H *of degree* $< s$ *and if* $d > d_0(n,r,s)$ *then* V *is contained in some irreducible subvariety of* $\mathbb{P}^r$ *of dimension* $n+1$ *and degree* $< s$.●

Theorem 1.6 *Let* $C \subset \mathbb{P}^r$ *be an irreducible nondegenerate curve of arithmetic genus* $p_a(C)$ *and degree* $d > d_0(1,r,s)$. *Let* Γ *be a generic hyperplane section of* C *and* h_Γ *its Hilbert function. Assume* C *not contained on any surface of degree* $< s$. *Then:*
(a) $h_\Gamma(i) \geq h(i)$ *for all* $i \geq 0$
(b) $p_a(C) \leq G(1,r,d,s)$.
Proof: See the main theorem in [CCD] and its proof.●

Now we can prove Theorem 1.1.
Let $V_1 = V \cap H$ be the intersection of V with the generic $(r-n+1)$-dimensional subspace H of $\mathbb{P}^r$. By Corollary 1.5, V_1 is not contained on any surface of H of degree $< s$. By Theorem 1.6 it follows that $h_\Gamma(i) \geq h(i)$ for all $i \geq 0$. The conclusion follows by Proposition 1.2.●

Remark 1.7 If we put in the Theorem 1.1 n=1 (and assume that the curve V is smooth) then we obtain the main theorem of [CCD] . On the other hand if we put s=r-n we get the bound of [H] p.44.●

Theorem1.8 *Let V be as in the* Theorem 1.1. *Put* $x = \max\{2, [\frac{n}{s}]+1\}$ *and assume* $d > d_0(n,r,s)$ *if* $n \le 2$ *and* $d > \max\{d_0(n,r,s), (xs)^{r-n}\}$ *if* $n > 2$.

If $p_g(V) = G(n,r,d,s)$ *then* $h_\Gamma(i) = h(i)$ *for all* $i \ge 0$.

Proof: Put $R = r-n$ and $\Gamma = V \cap L$ where L ($= \mathbb{P}^R$) is a generic R-dimensional subspace of $\mathbb{P}^r$. By Proposition 1.2 and Theorem 1.6 we have:

$$\Sigma_{i=1}^{\infty}\binom{i-1}{n-1}(h_\Gamma(i)-h(i))= 0 \qquad \text{so that:}$$

$$(3) \qquad h_\Gamma(i) = h(i) \text{ for all } i \ge n.$$

Hence the claim is obvious for n=1,2. Assume $n \ge 3$.

With the previous notation, let $\mathcal{W}_L$ be the intersection of all hypersurfaces of L of degree xs containing Γ (possibly $\mathcal{W}_L = L$ if there is no such hypersurface). Since $d > (xs)^R$, by Bézout's Theorem $\mathcal{W}_L$ must contain a positive-dimensional component W_L containing one or more of the points of Γ ; and since the points of Γ are in uniform position ([EH]) then W_L must contain all of Γ.

If $\dim W_L > 1$ then we get $h_\Gamma(xs) = h_{W_L}(xs) \ge$ ([EH], Proposition 3.23) $\ge \binom{xs+1}{2}(R-2) + \binom{xs+2}{2} \ge \frac{1}{2}(xs+2)(xs+1) > xs^2 + 1 \ge h(xs)$. This is in contrast with (3) because $xs \ge n$. Hence W_L is necessarily a curve and, by Theorem 1.3,

$$(4) \qquad s \le \deg W_L$$

Since $h_\Gamma(xs) = h_{W_L}(xs)$ and $n \le xs$ then $h_\Gamma(i) = h_{W_L}(i)$ for all $i = 0, \dots ,n$. Hence by (3) we have: $h(n) = h_\Gamma(n) = h_{W_L}(n) \ge$ ([CCD], Lemma 1.4) $\ge h_{W_L}(n-1) + \min\{\deg W_L, n(R-1)+1\} = h_\Gamma(n-1) + \min\{\deg W_L, n(R-1)+1\} \ge$ (Theorem 1.6) $\ge h(n-1) + \min\{\deg W_L, n(R-1)+1\}$.

If $n(R-1)+1 \le s$ then $\min\{\deg W_L, n(R-1)+1\} = n(R-1)+1$ by (4) and $n \le w$ (see definition of the function h). In such case we have $h(n-1) + \min\{\deg W_L, n(R-1)+1\} = h(n-1) + n(R-1)+1 = h(n)$. It follows that $h(n) \ge h_\Gamma(n-1) + n(R-1)+1 \ge h(n-1) + n(R-1)+1 = h(n)$ and hence $h_\Gamma(n-1) = h(n-1)$.

If $n(R-1)+1 > s$ then $\min\{\deg W_L, n(R-1)+1\} \ge s$ (by (4)) and $n > w$. In such case we have $h(n) \ge h_\Gamma(n-1) + s \ge h(n-1) + s = h(n)$ and $h_\Gamma(n-1) = h(n-1)$. Hence *in any case* we get $h_\Gamma(n-1) = h(n-1)$.

Now we can restart with $h_\Gamma(n-1) = h(n-1)$ and repeating the previous argument we get $h_\Gamma(n-2) = h(n-2), \dots, h_\Gamma(2) = h(2), h_\Gamma(1) = h(1), h_\Gamma(0) = h(0)$.●

Remark 1.9 We notice that, asymptotically, the bound $G_n = \frac{d^{n+1}}{(n+1)!s^n} + O(d^n)$ is more precise than the bound obtained in ([NV] Corollary 8 p.170) for codimension two arithmetically Cohen-Macaulay subvarieties V of $\mathbb{P}^r$. In fact, if we apply the formula of [NV] we get the bound $G'_n = \frac{d^{n+1}}{(n+1)!f(s)^n} + O(d^n)$ where $f(s) = \frac{s^2+s-2}{2s-2}$. Hence, asymptotically, $G_n < G'_n$ (when $s \geq 3$). In particular, for $d \gg s$, the bound of [NV] is not sharp.●

2. A bound on the arithmetic genus of projective varieties

Using the numbers G_j previously defined, in this Section we establish a bound involving the arithmetic genus. More precisely we have the following:

Theorem 2.1 *Let* $V \subset \mathbb{P}^r$ *be a nondegenerate irreducible projective variety of dimension* n *and degree* d. *Assume* V *not contained on any irreducible subvariety of* $\mathbb{P}^r$ *of dimension* n+1 *and degree* < s. *Assume* $1 \leq n \leq r-2$, $s \geq r-n$ *and* $d > d_0(n,r,s)$. *Let* $V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0 = \Gamma$ *be generic linear sections of* V *with* $\dim V_j = j$. *For each* j=1,2, ... ,n *let* $p_a(V_j)$ *be the arithmetic genus of* V_j *and* $G_j := G(j,r-n+j,d,s)$. *Put:* p_V = the Hilbert polynomial of V, h_V = the Hilbert function of V, $\rho(V) = \min\{t \in \mathbb{N} : p_V(i) \geq h_V(i) \text{ for all } i \geq t\}$ *and fix an integer* t *such that* $t \geq \max\{\rho(V), m+w+e\}$ *(see notation of Section1). Then:*

$$(5) \qquad \sum_{i=0}^{n-1}(-1)^{n+i+1}\binom{t+i-1}{i}(G_{n-i} - p_a(V_{n-i})) \geq 0 \ . \ ●$$

We make some preliminary remark.

Remark 2.2 In [EH] p.82 the authors claim that if $V \subset \mathbb{P}^r$ is an irreducible projective variety of dimension n and degree d then:

$$(6) \qquad h^p(V,\mathcal{O}_V(i)) = 0 \ \text{ for all } p > 0 \text{ and } i > \frac{d-1}{r-n}.$$

This would imply that $\rho(V) \leq \frac{d-1}{r-n} + 1$. Unfortunately property (6) is false (see [N]). However, in the case of nonsingular surfaces one has $\rho(V) \leq d+1-r$ (see [L]). Hence in formula (5), in the case n=2, we can put t = d+1-r. In the general case $\dim V \geq 3$ see again [N] for bounds on $\rho(V)$.●

Remark 2.3 By Theorem 2.1 we have: for n=1, $p_a(V) \leq G_1$ (see [CCD]); for n=2, $p_a(V) \geq G_2 - t(G_1 - p_a(V_1))$ (see [J], p.27, for s=r-2); for n=3, $p_a(V) \leq G_3 - t(G_2 - p_a(V_2)) + \binom{t+1}{2}(G_1 - p_a(V_1))$; etc....●

Lemma 2.4 *Let* $V \subset \mathbb{P}^r$ *be an irreducible nondegenerate projective variety of dimension* $n \geq 1$. *Let* Γ *be the intersection of* V *with a generic* $(r-n)$-*dimensional subspace of* $\mathbb{P}^r$. *Let* h_V *(resp.* h_Γ *) be the Hilbert function of* V *(resp. of* Γ*). Then we have :*

$$h_V(t) \geq \sum_{i=0}^{t} \binom{t+n-1-i}{n-1} h_\Gamma(i) \quad \textit{for all } t \geq 0$$

Proof: Let $V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0 = \Gamma$ be generic linear sections of V with $\dim V_j = j$. By Castelnuovo theory we have ([EH]):

$h_{V_1}(t) \geq h_{V_1}(t-1)+h_\Gamma(t) \geq h_{V_1}(t-2)+h_\Gamma(t-1)+h_\Gamma(t) \geq \ldots \geq \sum_{i=0}^{t} h_\Gamma(i)$. Hence:

$$h_{V_2}(t) \geq \sum_{i=0}^{t} h_{V_1}(i) \geq \sum_{i=0}^{t}\left(\sum_{j=0}^{i} h_\Gamma(j)\right) = \sum_{i=0}^{t}(t+1-i)h_\Gamma(i) = \sum_{i=0}^{t}\binom{t+2-1-i}{2-1}h_\Gamma(i).$$

Again:

$$h_{V_3}(t) \geq \sum_{i=0}^{t}\left(\sum_{j=0}^{i}\left(\sum_{l=0}^{j} h_\Gamma(l)\right)\right) = \sum_{i=0}^{t}\binom{t+2-i}{2}h_\Gamma(i) = \sum_{i=0}^{t}\binom{t+3-1-i}{3-1}h_\Gamma(i).$$

Continuing in this fashion we get the inequality for $V_n = V$. ●

Corollary 2.5 *With the same assumptions as in* Theorem 2.1, *let* h *be the numerical function defined in Section 1. Then for all* $t \geq 0$ *we have:*

$$h_V(t) \geq \sum_{i=0}^{t}\binom{t+n-1-i}{n-1}h_\Gamma(i) \geq \sum_{i=0}^{t}\binom{t+n-1-i}{n-1}h(i).$$

Proof: It follows by Lemma 2.4, Corollary 1.5 and Theorem 1.6. ●

Remark 2.6 Observe that Corollary 2.5 generalizes Proposition 3.23 i) and ii) in [EH] p.117. ●

Lemma 2.7 *Let* $V \subset \mathbb{P}^r$ *be an irreducible nondegenerate projective variety of dimension* $n \geq 1$. *Let* $V = V_n \supset V_{n-1} \supset \ldots \supset V_1 \supset V_0 = \Gamma$ *be generic linear sections of V with* $\dim V_j = j$. *Then:*

$$(-1)^{n+1}p_V(t) = \sum_{i=1}^{n-1}(-1)^{i+1}\binom{t+i-1}{i}p_a(V_{n-i}) + (-1)^{n-1}\binom{t+n-1}{n}d + (-1)^{n+1}\binom{t+n-1}{n-1} - p_a(V).$$

Proof: Use induction on n, the well known formula $\Delta_z\binom{z}{h} = \binom{z-1}{h-1}$ and the fact that $\Delta_t p_{V_n}(t) = p_{V_{n-1}}(t)$. ●

The proof of the following lemma is purely computational and therefore we omitted:

Lemma 2.8 *In* $\mathbb{Q}[t]$ *we have:*

(7) $(-1)^n\binom{t+n-1-i}{n-1} = \sum_{j=0}^{n-1}(-1)^{j+1}\binom{t+j-1}{j}\binom{i-1}{n-j-1}$ for all $i \geq 1$ and $n \geq 2$.

(8) $\sum_{i=0}^{n}(-1)^i\binom{t+i-1}{i}\binom{t}{n-i} = 0$ for all $n \geq 2$. ●

Corollary 2.9 *With the same assumptions as in* Lemma 2.7, *let* $\varphi: \mathbb{Z}\to\mathbb{Z}$ *be a numerical function such that* $\varphi(0) = 1$. *Then:*

$$(-1)^n[\sum_{i=0}^{t}\binom{t+n-1-i}{n-1}\varphi(i) - p_V(t)] = \sum_{j=0}^{n-1}(-1)^j\binom{t+j-1}{j}[\sum_{i=1}^{t}\binom{i-1}{n-j-1}(d-\varphi(i)) - p_a(V_{n-j})]$$

Proof: By (7) of Lemma 2.8 we can cancel the terms $\varphi(i)$ for $i \geq 1$. Moreover, by (8) of Lemma 2.8 we have: $(-1)^{n-1}\binom{t+n-1}{n}d = \sum_{j=0}^{n-1}(-1)^j\binom{t+j-1}{j}[\sum_{i=1}^{t}\binom{i-1}{n-j-1}]\, d$.

Hence the claim follows by Lemma 2.7.●

We now are able to prove Theorem 2.1.

By the assumptions and by Corollary 2.5 we get:

$$p_V(t) \geq h_V(t) \geq \sum_{i=0}^{t}\binom{t+n-1-i}{n-1}h_\Gamma(i) \geq \sum_{i=0}^{t}\binom{t+n-1-i}{n-1}h(i)$$

So that: $p_V(t) - \sum_{i=0}^{t}\binom{t+n-1-i}{n-1}h(i) \geq 0$. Hence, if we put $h = \varphi$, Theorem 2.1 follows by Corollary 2.9.●

We recall the following:

Definition 2.10 Let $V \subset \mathbb{P}^r$ be a projective variety of dimension $n \geq 1$.
We say that V is *arithmetically Cohen-Macaulay* (shortly a.C.M.) if all the restriction maps $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(V, \mathcal{O}_V(k))$ $(k \in \mathbb{Z})$ are surjective and $H^j(V, \mathcal{O}_V(k))=0$ for all $k \in \mathbb{Z}$ and $1 \leq j \leq n-1$ (see [S]).
Observe that if $\dim V \geq 2$ then V is a.C.M. if and only if its generic hyperplane section is.●

Corollary 2.11 *With the assumptions of* Theorem 2.1, *fix an integer* $t_0 \geq \max\{\rho(V), m+w+e, d-r+n\}$. *Assume that:*

$$\sum_{i=0}^{n-1}(-1)^{n+i+1}\binom{t_0+i-1}{i}(G_{n-i} - p_a(V_{n-i})) = 0.$$

Then:

1) $h_\Gamma(i) = h(i)$ *for all* $i \geq 0$.

2) $V_1, V_2, \ldots, V_{n-1}, V_n = V$ *are arithmetically Cohen-Macaulay*.

3) $p_a(V_j) = G_j$ *for all* $j=1,2, \ldots, n$.

Proof: By the proof of Theorem 2.1 we have:

$$p_V(t_o) = \sum_{i=0}^{t_o}\binom{t_o+n-1-i}{n-1}h_\Gamma(i) = \sum_{i=0}^{t_o}\binom{t_o+n-1-i}{n-1}h(i).$$

Since $h_\Gamma(i) \geq h(i)$, 1) follows. Moreover, by the proof of Lemma 2.4 we have:

(9) $h_{V_1}(i) - h_{V_1}(i-1) = h_\Gamma(i)$ for all $i=0,1, ..., t_0$.

Since $t_0 \geq d-r+n$ then (9) holds for $i \geq t_0$ too (see [GLP]). It follows that V_1 and $V_2, ..., V_n= V$ are a.C.M. (see [EH] p.84). Since V_1 is a.C.M. then $p_a(V_1) = ([C1]) = \sum_{i=1}^{\infty}(d-h_\Gamma(i)) = G_1$. Since V_2 is a.C.M., by Corollary 2.9 applied to V_2 for $t \gg 0$ and by 1), we get $p_a(V_2) = G_2 - t(G_1 - p_a(V_1)) = G_2$. Continuing in this fashion we get 3) for all $j=1,2, ..., n$. ●

3. Generalized Castelnuovo Varieties

Let $\mathcal{V}(n,r,d,s)$ be the set of all projective varieties V such that:

(a) V is an irreducible nondegenerate subvariety of $\mathbb{P}^r$ of dimension n and degree d, not contained on any irreducible subvariety of $\mathbb{P}^r$ of dimension n+1 and degree < s.

(b) $d \geq r-n+1$, $s \geq r-n$ and $1 \leq n \leq r-2$.

(c) if $n = 1$ then $p_a(V) = G(1,r,d,s)$; if $n \geq 2$ then $p_g(V) = G(n,r,d,s)$.

In other words, when $d \gg s$, $\mathcal{V}(n,r,d,s)$ is the set of those irreducible nondegenerate projective varieties $V \subset \mathbb{P}^r$ whose genus (arithmetic if n=1; geometric if $n \geq 2$) is maximal according to their dimension n, to their degree d and to the property not to be contained on any subvariety of $\mathbb{P}^r$ of dimension n+1 and degree < s (see Theorem 1.1 and Theorem 1.6).

The elements of $\mathcal{V}(n,r,d,r-n)$ are the so called <u>Castelnuovo varieties</u> (see [C2] for curves and [H] for $n \geq 2$). <u>Castelnuovo curves</u> (i.e. the elements of $\mathcal{V}(1,r,d,r-1)$) are well known (see for example [Ca] , [Hp] , [GP] , [GH] , [ACGH] , [EH] , [C2]). When $d \gg s$ the study of $\mathcal{V}(1,r,d,s)$ has been introduced in [CCD] (see also [GP] , [EH] , [CC2]).

In this Section we will concentrate our attention on the general case $n \geq 1$ rather than on curves. Many properties have been established for Castelnuovo varieties (i.e. s=r-n) when $d \geq n(r-n)+2$, $n \geq 2$ ([H]) ; among the most important we recall that such varieties are arithmetically Cohen-Macaulay and their hyperplane sections are again Castelnuovo varieties. As far as we know, the case $n \geq 2$ and $s > r-n$ is completely new.

It seems natural to define "generalized Castelnuovo variety" as variety $V \in \mathcal{V}(n,r,d,s)$ when $s > r-n$. But since we do not know whether a variety $V \in \mathcal{V}(n,r,d,s)$ is a.C.M. (with the exception of the case of surfaces (i.e. n=2; see Theorem 3.7 and Problem 3.8 below)) we propose the following definition:

Definition 3.1 Let $V \subset \mathbb{P}^r$ be an irreducible nondegenerate projective variety of dimension n and degree d. We say that *V is a Castelnuovo variety of type (n,r,d,s)* (or *"V is a generalized Castelnuovo variety"*) if $V \in \mathcal{V}(n,r,d,s)$ and is arithmetically Cohen-Macaulay.●

Put: $\mathcal{V}_{reg}(n,r,d,s)$:= the set of $V \in \mathcal{V}(n,r,d,s)$ which are nonsingular; $\mathcal{V}^0(n,r,d,s)$:= the set of the Castelnuovo varieties of type (n,r,d,s) and $\mathcal{V}^0_{reg}(n,r,d,s)$:=the set of $V \in \mathcal{V}^0(n,r,d,s)$ which are nonsingular.

Remark 3.2 Since $\mathcal{V}^0(n,r,d,r-n) = \mathcal{V}(n,r,d,r-n)$ for $n \geq 1$ and $d \geq n(r-n) + 2$ (see [EH] for n=1 and [H] for $n \geq 2$) then, in the case s=r-n, we find again the definition of Castelnuovo variety. Observe also that if $d > d_0(n,r,s)$ then $\mathcal{V}^0(1,r,d,s) = \mathcal{V}(1,r,d,s)$ ([CCD])●

We now show some of the geometric properties of generalized Castelnuovo varieties. The first does not depend on the property a.C.M.. With the notation of Theorem 1.8, put:

$$d_1(n,r,s) := \max\{d_0(n,r,s), (xs)^{r-n})\}$$

Theorem3.3 *Let* $V \in \mathcal{V}(n,r,d,s)$ *and assume* $d > d_1(n,r,s)$. *Then* V *is contained on a unique irreducible variety* $W \subset \mathbb{P}^r$ *of dimension* n+1*and degree* s *whose generic* (r-n)-*linear section is a Castelnuovo curve of degree s in* $\mathbb{P}^{r-n}$ *(i.e. an element of* $\mathcal{V}(1,r-n,s,r-n-1)$*). Moreover, if* $s \geq 2(r-n)+1$,*then the intersection of all quadrics through* V *is an irreducible subvariety* T *of* $\mathbb{P}^r$ *of dimension* n+2 *and minimal degree* r-n-1, *containing* W.

Proof: Let L be the generic (r-n)-plane of $\mathbb{P}^r$ and put $\Gamma = V \cap L$. By Theorem 1.8 we have $h_\Gamma(i) = h(i)$ for all $i \geq 0$. Using the same argument in the proof of Theorem 1.8, one can see that Γ is contained on some irreducible curve $W_L \subset L$ of degree exactly s. Since $h(s) = h_\Gamma(s) = h_{W_L}(s)$ then the arithmetic genus of W_L can be computed by $h(s) = s^2 - p_a(W_L) +1$ ([GLP]): one easily sees that the genus of W_L is maximal in L ([EH]). By Theorem 1.3 these curves W_L can be lifted in $\mathbb{P}^r$ to a (n+1)-fold W, irreducible of degree s, which has W_L as generic (r-n)-linear section. Moreover, the intersection of all quadrics containing W is a variety T' of dimension n+2 and degree r-n-1 by ([H] p.45 and proof). It is well known that $h_{T'}(2) = \binom{r+2}{2} - \binom{r-n-1}{2}$. On the other hand

$h_V(2) \geq$ (by direct computation (see Section1 and Lemma 2.4)) $\geq$

$\binom{n+1}{n-1} + \binom{n}{n-1}(r-n+1) + 3(r-n) = h_{T'}(2)$. Since $V \subset T'$, then $h_{T'}(2) = h_V(2)$ so that $T = T'$.●

Theorem 3.4 *Let V be a Castelnuovo variety of type* (n,r,d,s) *and assume* $d > d_1(n,r,s)$. *Let* $V = V_n \supset V_{n-1} \supset \dots \supset V_1$ *be generic linear sections of* V *with* $\dim V_j = j$. *Then:*

$$p_a(V_j) = G(j,r-n+j,d,s) \text{ for each } j=1,2, \dots ,n.$$

In particular, all $V \in \mathcal{V}^0(n,r,d,s)$ *have the same* Hilbert polynomial (=: $p(n,r,d,s)(t)$ =: $p(t)$). *Hence* $\mathcal{V}^0(n,r,d,s)$ *is contained in the* Hilbert scheme $\mathbb{H}ilb_r^{p(t)}$.

<u>Proof:</u> By Theorem 1.8 we have $h_\Gamma(i) = h(i)$ for all $i \geq 0$. Since $V = V_n, V_{n-1}, \dots, V_1$ are a.C.M., by Lemma 2.4 we have $h_V(t) = \sum_{i=0}^{t}\binom{t+n-1-i}{n-1}h(i)$ for all $t \geq 0$. By Corollary 2.9 (put $h = \varphi$) it follows that $\sum_{i=0}^{n-1}(-1)^{n+i+1}\binom{t+i-1}{i}(G_{n-i} - p_a(V_{n-i})) = 0$ for all $t \gg 0$. By Corollary 2.11 we get $p_a(V_j) = G(j,r-n+j,d,s)$ for each $j=1,2, \dots ,n$. Moreover, since the coefficients of the Hilbert polynomial of V depend only on n, r, d and on the arithmetic genus $G(j,r-n+j,d,s)$ of generic linear sections of V (see Lemma 2.7), then all $V \in \mathcal{V}^0(n,r,d,s)$ have the same Hilbert polynomial.●

Theorem 3. 5 *Let* $V \subset \mathbb{P}^r$ *be a smooth variety of dimension* $n > 1$ *and degree* $d > d_1(n,r,s)$. V *is a Castelnuovo variety of type* (n,r,d,s) *if and only if its generic hyperplane section* H *is a Castelnuovo variety of type* (n-1,r-1,d,s).

<u>Proof:</u> If $V \in \mathcal{V}^0_{reg}(n,r,d,s)$ then, by Theorem 3.4, $p_a(H) = G(n-1, r-1, d, s)$. Since H is smooth and a.C.M. then $p_a(H) = p_g(H)$. By Theorem 1.3 it follows that $H \in \mathcal{V}^0_{reg}(n-1,r-1,d,s)$. If $H \in \mathcal{V}^0_{reg}(n-1,r-1,d,s)$ then by Theorem 1.8 we have $h_\Gamma(i) = h(i)$ for all $i \geq 0$. Since V is a.C.M., by using the same argument in the proof of Theorem 3.4, we get $p_a(V) = G_n$. Since V is smooth we have also $p_g(V) = p_a(V) = G_n$.●

Remark 3.6 The proof of Theorem 3.5 should be compared with the proof of [H] p.67.●

In the particular case of smooth surfaces we have:

Theorem 3.7 *Let* $S \subset \mathbb{P}^r$ *be an irreducible nondegenerate smooth surface of degree* d, *not contained on any irreducible subvariety of* $\mathbb{P}^r$ *of dimension* 3 *and degree* < s. *Assume* $r \geq 4$, $s \geq r-2$ and $d > \max\{d_0(1,r,s), d_0(2,r,s)\}$. *Put:* $p_a(S)$ = *arithmetic genus of* S, $p_g(S)$ = *geometric genus of S*, H = *generic hyperplane section of* S *in* $\mathbb{P}^r$, $p_a(H)$ = *arithmetic genus of H, and* $G_j = G(j,r-2+j,d,s)$ *for* j=1,2. *Then we have:*

1) $G_2 - (d-r+1)(G_1 - p_a(H)) \leq p_a(S) \leq p_g(S) \leq G_2$ (see [J] p.27 for s = r-2)

2) $q(S) := p_g(S) - p_a(S) \leq (d-r+1)(G_1 - p_a(H))$

3) $q(S) \leq \frac{d^3}{2s} + O(d^2)$

4) $S \in \mathcal{V}_{reg}(2,r,d,s)$ *if and only if* $S \in \mathcal{V}^0_{reg}(2,r,d,s)$

Proof: 1), 2) and 3) follow by Theorem 2.1 and Remark 2.2. We now prove 4) by assuming $S \in \mathcal{V}_{reg}(2,r,d,s)$. Look at the proof of Theorem 1.1 : since $p_g(S) = G_2$ then, working back to the generic hyperplane section H, we get for M » 0 (put H = C):

$$h^0\mathcal{O}_H(M) - h^0\mathcal{O}_H(M-1) = h_\Gamma(M) = h(M)$$

$$h^0\mathcal{O}_H(M-1) - h^0\mathcal{O}_H(M-2) = h_\Gamma(M-1) = h(M-1)$$

$$\dots\dots\dots\dots\dots$$

$$h^0\mathcal{O}_H(2) - h^0\mathcal{O}_H(1) = h_\Gamma(2) = h(2) \qquad (10)$$

We want to show that H is a.C.M.. First we show that H is linearly normal. Suppose not and let ξ be the linear system cut out on H by the hyperplanes of $\mathbb{P}^{r-1}$ ($H = S \cap \mathbb{P}^{r-1}$). Let η be a linear system on H such that: $\xi \subset \eta \subseteq |\mathcal{O}_H(1)|$ and $\dim \eta = r$ ($= 1 + \dim \xi$).

η determines in $\mathbb{P}^r$ a curve $H^\sim$ of degree d such that H is a birational projection of $H^\sim$ in $\mathbb{P}^{r-1}$. As well as the curve H, $H^\sim$ is not contained on any surface of $\mathbb{P}^r$ of degree < s. Hence, if we denote by $\Gamma^\sim$ the generic hyperplane section of $H^\sim$ then: $h^0\mathcal{O}_H(2) - h^0\mathcal{O}_H(1) = h^0\mathcal{O}_{H^\sim}(2) - h^0\mathcal{O}_{H^\sim}(1) \geq h_{\Gamma^\sim}(2) \geq$ (by Theorem 1.6 when $s \geq 2r-3$) $\geq 3(r-1) > 3(r-2) = h(2)$ which is in contrast with (10). It follows that H is necessarily linearly normal (a similar argument works also when $r-2 \leq s < 2r-3$). Hence we have $h^0\mathcal{O}_H(1) = h_H(1)$. By (10) we get: $h(2) = h^0\mathcal{O}_H(2) - h_H(1) \geq h_H(2) - h_H(1) \geq h_\Gamma(2) = h(2)$ which implies that $h^0\mathcal{O}_H(2) = h_H(2)$. Reiterating this procedure we get $h^0\mathcal{O}_H(l) = h_H(l)$ for all $l \geq 0$ i.e. H is a.C.M.. Hence also S is (see Definition 2.10).●

Problem 3.8 It would be interesting to know whether a (smooth) variety $V \in \mathcal{V}(n,r,d,s)$ is arithmetically Cohen-Macaulay. For $d \gg s$ this is true when V is a curve ([CCD]) or when V is a surface (Theorem 3.7). •

4. Examples

Examples of (generalized or not) Castelnuovo curves can be found in the quoted references. We now give examples of smooth generalized Castelnuovo varieties of dimension $n \geq 1$. This shows that the upper bound introduced in Section 1 is sharp (at least in the cases which we are going to consider).

Example 4.1 By direct computation one can sees that:
in codimension 2, a complete intersection in $\mathbb{P}^r$ of type $V = F_s \cap F_p$, $\forall s\ \forall d = ps \gg s$, is a (nonsingular) Castelnuovo variety of type $(r-2,r,d,s) = (r-2,r,ps,s)$;
in codimension 3, a complete intersection in $\mathbb{P}^r$ of type $V = F_2 \cap F_\sigma \cap F_p$, $\forall s = 2\sigma \geq 4\ \forall d = ps \gg s$, is a (nonsingular) Castelnuovo variety of type $(r-3,r,ps,2\sigma)$;
in codimension 4, a complete intersection in $\mathbb{P}^r$ of type $V = F_2 \cap F_2 \cap F_2 \cap F_p$, $\forall d = 8p \gg 0$, is a (nonsingular) Castelnuovo variety of type $(r-4,r,8p,8)$.•

Next example is a generalization of ([CCD] Example 6.5) and provides nonsingular and not complete intersection generalized Castelnuovo varieties.

Example 4.2 Let $T \subset \mathbb{P}^r$ be a nonsingular rational normal scroll of dimension n+2 (i.e. a (n+2)-dimensional smooth scroll of (n+1)-planes of degree r-n-1 in $\mathbb{P}^r$; this is the case whenever $r \geq 2n+3$; in such case the generic rational normal scroll is nonsingular ([H])). Let H be the hyperplane class of T and F the class representing a (n+1)-plane of T. Let $s \geq (n+1)(r-n-1)+2$ and assume $s-1 = w(r-n-1)+v$ with $0 < v < r-n-1$. Consider the linear system $|(w+1)H - (r-n-2-v)F|$ on T. If $s \gg r$ then a generic $W \in |(w+1)H - (r-n-2-v)F|$ is irreducible, nonsingular and a Castelnuovo variety of type (n+1,r,s,r-n-1) ([H]).

Let $d-1 = ms + \varepsilon$, assume $(0 \leq)\ w(r-n-1-v) \leq \varepsilon < s$ and put (see Section 1) $\varepsilon+r-n-1-v = k(w+1) + \delta$ $\ 0 \leq \delta < w+1$. Let h and f be the images of H and F in the Picard group of W. We may choose W such that some divisor f contains a component of degree δ: call D the class of this component. If $d \gg s$ then the generic $V \in |(m+1)h - (r-n-1-k)f + D|$ is irreducible and nonsingular. We want to show that $V \in \mathcal{V}^0_{reg}(n,r,d,s)$.

For this purpose we intersect T with a generic (r-n+1)-plane L of $\mathbb{P}^r$ and put: $T' = T \cap L$, $W' = W \cap L$, $V' = V \cap L$, $D' = D \cap L$, $h' = h \cap L$ and $f' = f \cap L$. The canonical class $K_{W'}$ of the Castelnuovo surface W' is $(w-2)h' + (v-1)f'$ and $V' \in |(m+1)h' - (r-n-1-k)f' + D'|$. The genus g of the smooth curve V' is computed by the adjunction formula $2g-2 = V'(V'+K_{W'})$ and one easily sees that $g = G(1,r-n+1,d,s)$. When $d \gg s$, by Bézout V' is not contained on surfaces of degree < s so $V' \in \mathcal{V}^0_{reg}(1,r-n+1,d,s)$ (see Remark 3.2). By Theorem 3.5 it follows that $V \in \mathcal{V}^0_{reg}(n,r,d,s)$.●

To conclude we will make some remarks on the Hilbert scheme of generalized Castelnuovo varieties.

Since all $V \in \mathcal{V}^0(n,r,d,s)$ have the same Hilbert polynomial $p(t) :=: p(n,r,d,s,t)$ (Theorem 3.4) then V is contained in $\mathbb{H}ilb_r^{p(t)}$. Put:

$$I_{n,r,d,s} := \text{the closure of } \mathcal{V}^0(n,r,d,s) \text{ in } \mathbb{H}ilb_r^{p(t)}.$$

In the classical case of curves, i.e. $n = 1$ and $s = r - 1$, there are many known results on $I_{1,r,d,r-1}$ (see [EH], [C2]). Here we will consider the general case $n \geq 1$ and $s \geq r - n$ but, just to work out an example, in the particular range $d \equiv 0$ (mod s) and $s \equiv 0$(mod r-n-1). More precisely we prove the following:

Theorem 4.3 *Let* n,r,d,s *be integers such that:* $1 \leq n \leq r-2$, $r \geq 2n+3$, $r \neq n+5$, $d \gg s \gg 0$, $s=(w+1)(r-n-1) =(w+1+\alpha)(r-n-2) +\beta$ $0 \leq \beta < r-n-2$, $d=(m+1)s = ps$. *We have:*

(a) $\mathcal{V}^0(n,r,d,s)$ *is nonempty locally closed set and* $I_{n,r,d,s}$ *is an irreducible component of* $\mathbb{H}ilb_r^{p(t)}$. *In particular* $\mathbb{H}ilb_r^{p(t)}$ *is nonempty.*

(b) *The generic* $V \in I_{n,r,d,s}$ *is nonsingular in* $\mathbb{P}^r$.

(c) *The generic* $V \in I_{n,r,d,s}$ *represents a smooth point of* $\mathbb{H}ilb_r^{p(t)}$.

(d) *In particular there exists a nonempty open set in* $\mathbb{H}ilb_r^{p(t)}$ *reduced and irreducible (and contained in* $\mathcal{V}^0(n,r,d,s)$*).*

(e) $\dim I_{n,r,d,s} =$

$$[(r-n-1)(r+n+3)-3]+[\binom{n+w+2}{n+1}+(r-n-1)\binom{n+w+2}{n+2}-1]+[s\binom{n+p-w-\alpha-1}{n+1}+\sum_{i=0}^{w+\alpha+1}\binom{n+p-i}{n}((r-n-1)i+1)-1].$$

<u>Proof:</u> We will proceed in several steps.

Since $r \geq 2n+3$ and $r \neq n+5$ then the varieties in $\mathbb{P}^r$ of dimension n+2 and minimal degree r-n-1 are only the rational normal scrolls. Let $\mathcal{T}$ be the space of these rational normal scrolls. It is well known that $\mathcal{T}$ is irreducible,

the generic $T \in \mathcal{T}$ is nonsingular($r \geq 2n+3$) and $\dim\mathcal{T}= (r-n-1)(r+n+3)-3$ (see [H], [EH]). Let $\mathcal{E}$ be the set:
$\{(W,T)$: $W \subset T$, $T \in \mathcal{T}$ and W is a nondegenerate irreducible variety of dimension n+1 and degree s whose generic (r-n)-linear section is an irreducible curve of degree s and maximal genus in $\mathbb{P}^{r-n}\}$.
The generic fibre $\varphi^{-1}(T)$ of the projection φ: $\mathcal{E} \to \mathcal{T}$ is an open set of the linear system $|(w+1)H|$ (H= hyperplane section of T) (see [H]). Since $\mathcal{T}$ is irreducible then $\mathcal{E}$ is irreducible and $\dim\mathcal{E}=\dim\mathcal{T} + h^0(T, \mathcal{O}_T(w+1)) - 1$.
We compute $h^0(T, \mathcal{O}_T(w+1))$ by using Lemma 2.4 and the fact that the generic (r-n-1)-linear section $T \cap L$ of T is a rational normal curve of degree r-n-1 in $L = \mathbb{P}^{r-n-1}$.
Hence we have: $h^0(T,\mathcal{O}_T(w+1)) = h_T(w+1) =$

$$\sum_{i=0}^{w+1}\binom{n+w+2-i}{n+1}h_\Gamma(i) = \binom{n+w+2}{n+1}+(r-n-1)\binom{n+w+2}{n+2}.$$

Summarizing: $\mathcal{E}$ is irreducible and $\dim\mathcal{E}$ =

$$[(r-n-1)(r+n+3)-3]+[\binom{n+w+2}{n+1}+(r-n-1)\binom{n+w+2}{n+2}-1].$$

Let $\mathfrak{I}$ be the set:

$$\{ (V,W,T): V \subset W \subset T, V\in \mathcal{V}^0(n,r,d,s) \text{ and } (W,T) \in \mathcal{E}\}.$$

Lemma 4.4 *If* $(V',W',T') \in \mathfrak{I}$ *then* $V' \in |(m+1)h'|$ (h'=hyperplane section of W').

Proof: Let $V \subset W$ be as in the Example 4.2 (in our range). We have the exact sequence $0 \to \mathcal{O}_W(l-p) \to \mathcal{O}_W(l) \to \mathcal{O}_V(l)\to 0$; since V' (resp. W') has the same Hilbert function of V (resp. of W) then $h_{V'}(l)= h_V(l) = h_W(l) - h_W(l-p) = h_{W'}(l) - h_{W'}(l-p)$. It follows that there exists an irreducible hypersurface F_p of degree p containing V' and not W'. It is obvious that $V' = W' \cap F_p$ and so $V' \in |(m+1)h'|$.●

By Theorem 3.3, Lemma 4.4 and by the projections $\mathfrak{I} \to I_{n,r,d,s}$ and $\mathfrak{I} \to \mathcal{E}$ we have that $\mathcal{V}^0(n,r,d,s)$ is locally closed and $\mathfrak{I}$ is irreducible. In particular $I_{n,r,d,s}$,which is dominated by $\mathfrak{I}$, is irreducible. Moreover $\dim I_{n,r,d,s} = \dim\mathcal{E} + h^0\mathcal{O}_W(p) -1$. As before by Lemma 2.4 we get (e); (b) follows because the generic $T \in\mathcal{T}$ is smooth.

Now, in order to conclude the proof of Theorem 4.3, it suffices giving for $d \gg s \gg 0$ an example $V \subset W \subset T$ with V,W,T smooth and $h^0\mathcal{N}_{V,\mathbb{P}^r} \leq \dim I_{n,r,d,s}$ ($\mathcal{N}$ = normal sheaf, see [S]). For this purpose look at the Example 4.3 and consider the exact sequence:

$$(11) \qquad 0 \to \mathcal{N}_{V,W} \to \mathcal{N}_{V,\mathbb{P}^r}\to \mathcal{N}_{W,\mathbb{P}^r}|_V\to 0$$

Since $V \in |ph|$ (h =hyperplane section of W) then we have: $\mathcal{N}_{V,W} =\mathcal{O}_V(p)$. By (11) we get

(12) $$h^0\mathcal{N}_{V,\mathbb{P}^r} \le h^0\mathcal{N}_{W,\mathbb{P}^r}|_V + (h^0\mathcal{O}_W(p) - 1)$$

(observe that one has $0 \to \mathcal{O}_W \to \mathcal{O}_W(p) \to \mathcal{O}_V(p) \to 0$ and $h^1\mathcal{O}_W = 0$, [H]).

Next consider:

(13) $$0 \to \mathcal{N}_{W,\mathbb{P}^r}(-V) \to \mathcal{N}_{W,\mathbb{P}^r} \to \mathcal{N}_{W,\mathbb{P}^r}|_V \to 0; \qquad \mathcal{N}_{W,\mathbb{P}^r}(-V) = \mathcal{N}_{W,\mathbb{P}^r}(-p).$$

Since W is smooth then we have $h^1\mathcal{N}_{W,\mathbb{P}^r}(-p) = 0$ for $d \gg 0$ ([Ht] p.243). Hence by (13) and (12) we get:

(14) $h^0\mathcal{N}_{V,\mathbb{P}^r} \le h^0\mathcal{N}_{W,\mathbb{P}^r} - h^0\mathcal{N}_{W,\mathbb{P}^r}(-p) + (h^0\mathcal{O}_W(p) - 1) = h^0\mathcal{N}_{W,\mathbb{P}^r} + (h^0\mathcal{O}_W(p) - 1)$.

As before consider: $0 \to \mathcal{N}_{W,T} \to \mathcal{N}_{W,\mathbb{P}^r} \to \mathcal{N}_{T,\mathbb{P}^r}|_W \to 0$. Since $\mathcal{N}_{W,T} = \mathcal{O}_W(w+1)$ and $h^1\mathcal{O}_W(w+1) = 0$ then we have by (14): $h^0\mathcal{N}_{V,\mathbb{P}^r} \le h^0\mathcal{N}_{T,\mathbb{P}^r}|_W + (h^0\mathcal{O}_T(w+1) - 1) + (h^0\mathcal{O}_W(p) - 1)$. Since for $s \gg 0$ $h^0\mathcal{N}_{T,\mathbb{P}^r}(-W) = h^1\mathcal{N}_{T,\mathbb{P}^r}(-W) = h^1\mathcal{N}_{T,\mathbb{P}^r}(-w-1) = 0$ then by the exact sequence $0 \to \mathcal{N}_{T,\mathbb{P}^r}(-W) \to \mathcal{N}_{T,\mathbb{P}^r} \to \mathcal{N}_{T,\mathbb{P}^r}|_W \to 0$ we obtain :

$h^0\mathcal{N}_{V,\mathbb{P}^r} \le (h^0\mathcal{O}_T(w+1) - 1) + (h^0\mathcal{O}_W(p) - 1) + h^0\mathcal{N}_{T,\mathbb{P}^r} = \dim I_{n,r,d,s}$ because nonsingular rational normal scrolls are smooth points for their Hilbert scheme $\mathcal{T}$ (see Lemma 4.5 below).●

Lemma 4.5 *If* $T \subset \mathbb{P}^r$ *is a nonsingular rational normal scroll then* T *is unobstructed in* $\mathbb{P}^r$ (see [P] for dimT=2).

<u>Proof:</u> It is sufficient to show that $h^1\mathcal{N}_{T,\mathbb{P}^r} = 0$ ([S], Corollary 8.6). Using the exact Euler sequences $0 \to \mathcal{O}_T \to \mathcal{O}_T(1)^{r+1} \to \mathcal{T}_{\mathbb{P}^r}|_T \to 0$, $0 \to \mathcal{T}_T \to \mathcal{T}_{\mathbb{P}^r}|_T \to \mathcal{N}_{T,\mathbb{P}^r} \to 0$ ($\mathcal{T}$ = tangent bundle) and $h^1\mathcal{O}_T(1) = h^2\mathcal{O}_T = 0$, we need only show $H^2(T, \mathcal{T}_T)=0$.

Put h = dimT. Since T is a nonsingular rational normal scroll then there exists a vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ of rank h such that $T = \mathbb{P}(\mathcal{E})$. Let $\pi: \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ be the stucture morphism; π gives an exact sequence: $0 \to \mathcal{T}_{T,\mathbb{P}^1} \to \mathcal{T}_T \to \pi^*\mathcal{T}_{\mathbb{P}^1} \to 0$ where $\mathcal{T}_{T,\mathbb{P}^1} = \mathcal{H}om(\Omega_{T,\mathbb{P}^1}, \mathcal{O}_T)$ and $\Omega_{T,\mathbb{P}^1}$ is the sheaf of relative differentials. We only prove $H^2(T, \pi^*\mathcal{T}_{\mathbb{P}^1}) = H^2(T, \mathcal{T}_{T,\mathbb{P}^1}) = 0$.

Using the projection formula we have: $R^i\pi_*(\pi^*\mathcal{T}_{\mathbb{P}^1}) = R^i\pi_*(\mathcal{O}_T) \otimes \mathcal{T}_{\mathbb{P}^1} = 0$ for all $i > 0$ (it is known that $R^i\pi_*(\mathcal{O}_T(j)) = 0$ for all $i > 0$ and $j > -h$ (see [Ht], Ex. 8.4, p.253, for $0 < i < h$ and use Theorem 12.11, p.288, for $i \ge h$)). Hence, by the Leray spectral sequence, we get $H^2(T, \pi^*\mathcal{T}_{\mathbb{P}^1}) = H^2(\mathbb{P}^1, \pi_*\pi^*\mathcal{T}_{\mathbb{P}^1}) = 0$.

Finally, in order to prove that $H^2(T, \mathcal{T}_{T,\mathbb{P}^1}) = 0$, we consider the exact sequence:

$$0 \to \mathcal{O}_T \to (\pi^*(\mathcal{E}^\vee))(1) \to \mathcal{T}_{T,\mathbb{P}^1} \to 0.$$

As before, using the projection formula, we see that $R^i\pi_*((\pi^*(\mathcal{E}^\vee))(1)) = 0$ for all $i > 0$. Hence we get $H^2(T, (\pi^*(\mathcal{E}^\vee))(1)) = H^2(\mathbb{P}^1, \pi_*((\pi^*(\mathcal{E}^\vee))(1))) = 0$. Since $H^3(T, \mathcal{O}_T) = 0$ it follows that $H^2(T, \mathcal{T}_{T,\mathbb{P}^1}) = 0$.●

We conclude here the presentation of the examples but we have in mind to give more information in a forthcoming paper.● ●

References

[ACGH] Arbarello E., Cornalba M., Griffiths P.A., Harris J., *Geometry of algebraic curves* Volume I, Springer-Verlag, A series of Comprehensive Studies in Mathematics, 1985, 267

[Ca] Castelnuovo G., *Ricerche di geometria sulle curve algebriche*, Zanichelli (1937) in Memorie Scelte

[CC1] Chiantini L., Ciliberto C., *A few remarks on the lifting problem*, preprint

[CC2] Chiantini L., Ciliberto C., *Genus of curves in $\mathbb{P}^4$*, preprint

[CCD] Chiantini L., Ciliberto C., Di Gennaro V., *The genus of projective curves*, to appear in the Duke Math. J.

[C1] Ciliberto C., *Hilbert functions on finite sets of points and the genus of a curve in a projective space*, Springer Lect. Notes 1266 (1987), 24-73

[C2] Ciliberto C., *On the Hilbert scheme of curves of maximal genus in a projective space*, Math. Z. 194(1987)no.3,351-363

[EH] Eisenbud D., Harris J., *Curves in projective space*, Presses de l'Université de Montreal(1982)

[GH] Griffiths P.A., Harris J., *Principles of Algebraic Geometry*, John Wiley 1978

[GP] Gruson L., Peskine C., *Genre des courbes dans l'espace projectif*, Springer Lect. Notes 687(1978), 31-59

[GLP] Gruson L., Lazarsfeld R., Peskine C., *On a theorem of Castelnuovo and the equations defining space curves*, Inv. Math. 72(1983),491-506

[Hp] Halphen G., *Mémoire sur la classification des courbes gauches algébriques*, Oeuvres complètes t. III

[H] Harris J., *A bound on the geometric genus of projective varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 8 (1981), 35-68

[Ht] Hartshorne R., *Algebraic Geometry*, GTM, Springer-Verlag, 1983

[J] Jongmans F., *Contribution à la théorie des variétés algébriques*, Thèse d'Agrégation de l'Einsegnement supérieur, Bruxelles, M. Hayez, Imprimeur de l'Académie Royale de Belgique 1947

[L] Lazarsfeld R., *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. 55 (1987) 423-429

[N] Nagel U., *On Castelnuovo's regularity and Hilbert functions* , Compositio Mathematica, 76, 265-279, 1990

[NV] Nagel U., Vogel W., *Bounds for Castelnuovo's regularity and the genus of projective varieties*, in Topics in Algebra, Banach Center Publications, Volume 26, PWN-Polish Scientific Publishers, Warsaw 1989 (Part 2, 1990) 163-183

[P] Pinkham H.C., *Deformations of Cones with Negative Grading*, J. of Algebra 30, 92-102 (1974)

[S] Sernesi E., *Topics on families of projective schemes*, Queen's Pap. Pure Appl. Math. 73(1986)

VINCENZO DI GENNARO - Università degli

Studi di Salerno - Dipartimento di Ingegneria

dell'Informazione e Matematica Applicata -

Sede distaccata - Via S. Allende - 84081 -

Baronissi (SALERNO) - ITALY

(Received February 24, 1993;
in revised form June 10, 1993)

manuscripta math. 81, 329 - 337 (1993)

manuscripta mathematica

Transcendency of Local Conjugacies in Complex Dynamics and Transcendency of Their Values*

Paul-Georg Becker[†]

and

Walter Bergweiler

Let p and q be polynomials of the same degree. A classical result of Böttcher says that there exists a function f conformal in a neighborhood of infinity such that $f(p(z)) = q(f(z))$. We show that f is transcendental and takes transcendental values at algebraic points unless p and q are linearly conjugate to monomials or Chebychev polynomials. As an application, we show that the conformal map from the exterior of the Mandelbrot set onto the exterior of the unit disk takes transcendental values at algebraic points. A second application is the solution of a transcendency problem posed by Golomb.

I. Introduction and main results

Let p and q be polynomials of degree $d \geq 2$. Denote by a and b the leading coefficients of p and q, that is, $p(z) = az^d + \ldots$ and $q(z) = bz^d + \ldots$, and let λ be a value satisfying $\lambda^{d-1} = a/b$. A classical theorem of Böttcher (see e.g. [Bd, Theorem 6.10.1] or [S1, § 3.3]) says that there exists a unique function f defined and analytic in a neighborhood of ∞ such that $f(z) \sim \lambda z$ as $z \to \infty$ and

$$f(p(z)) = q(f(z)) \tag{1}$$

for all large z, that is, p and q are locally conjugate in a neighborhood of ∞. Such a conjugating function f is called a *Böttcher function* with respect to p and q. It is clear from this definition that there exist precisely $d-1$ different Böttcher functions. We remark that the theorem is usually stated only in the case $p(z) = z^d$ or $q(z) = z^d$, but the version formulated above follows easily from this special case.

Under suitable hypotheses, it was shown in [Be] that if f is a transcendental solution of (1), then f takes on transcendental values at algebraic points. Therefore,

*1991 Mathematics Subject Classification: Primary 11 J 91. Secondary: 30 D 05, 58 F 08

[†]Supported by a Habilitationsstipendium of the Deutsche Forschungsgemeinschaft

it is of interest to know which Böttcher functions are transcendental and which are algebraic. We start with some examples of algebraic Böttcher functions.

1) f is linear, that is, $f(z) = \alpha z + \beta$, where $\alpha, \beta \in \mathbf{C}, \alpha \neq 0$. Then, for any polynomial p, there exists a unique polynomial q satisfying (1). In this case, we say that p and q are *linearly conjugate.*

2) $p(z) = M_d(z)$ and $q(z) = T_d(z)$, where $M_d(z) = z^d$ and where $T_d(z) = 2^{d-1}z^d + \ldots$ denotes the d-th Chebychev polynomial, that is, $\cos dz = T_d(\cos z)$. Here we find the Böttcher functions

$$f(z) = \frac{1}{2}(\rho z + \frac{1}{\rho z}), \tag{2}$$

where ρ runs through E_{d-1}, the set of $(d-1)$-th roots of unity.

3) $p(z) = M_d(z)$ and $q(z) = -T_d(z)$. We have the Böttcher functions (2), but now with those ρ satisfying $\rho^{d-1} = -1$.

4) Since inverse functions of algebraic functions and compositions of algebraic functions are algebraic, it follows from Examples 2 and 3 that the Böttcher functions with respect to $T_d(z)$ and $-T_d(z)$ are also algebraic. In fact, we have

$$f(z) = \frac{1}{2}\left((\rho + \frac{1}{\rho})z + (\rho - \frac{1}{\rho})\sqrt{z^2-1}\right), \tag{3}$$

where $\rho^{d-1} = -1$. Of course, if d is even, then $f(z) = -z$ is among these functions.

5) Similarly as in Example 4, the Böttcher functions for $p(z) = q(z) = T_d(z)$ are algebraic. Here the identity is always a Böttcher function, and if d is odd, so is $f(z) = -z$. The other Böttcher functions are nonlinear and given by (3), where $\rho \in E_{d-1} \setminus \{1, -1\}$.
It is easily seen that these are also the Böttcher functions for $p(z) = q(z) = -T_d(z)$.

Our first result is that the above examples provide a list of all algebraic Böttcher functions, except for permutation of p and q and combination with linear conjugations.

Theorem 1. *Let p and q be polynomials of degree $d \geq 2$ and let f be an algebraic Böttcher function with respect to p and q. Then f is linear or both p and q are linearly conjugate to M_d, T_d, or $-T_d$, where $M_d(z) = z^d$ and where T_d is the d-th Chebychev polynomial.*

It is clear from Theorem 1 and the above examples that for fixed p and q there cannot be transcendental as well as nonlinear algebraic Böttcher functions.

Examples 4 and 5 show that one may have linear as well as nonlinear algebraic Böttcher functions. One can check that $p(z) = q(z) = z^d + 1$ allows only one linear Böttcher function, namely $f(z) = z$, while all others are transcendental by our theorem.

Now we are able to prove the following transcendency result for the values of Böttcher functions.

Theorem 2. *Let p and q be polynomials of degree $d \geq 2$ having algebraic coefficients. Suppose that at least one of them is not linearly conjugate to M_d, T_d, or $-T_d$. Let f be a nonlinear Böttcher function with respect to p and q and suppose that f is defined and analytic in a punctured neighborhood G of ∞ such that $p(G) \subset G$ and $p^m|_G \to \infty$ as $m \to \infty$, where p^m denotes the m-th iterate of p. Then $f(\alpha)$ is transcendental for any algebraic $\alpha \in G$.*

We remark that $G = \{z \mid |z| > R\}$ satisfies the hypotheses of Theorem 1 for sufficiently large R.

The proof of Theorem 1 is based on the following general result concerning the algebraic solutions of the functional equation

$$f(p(z)) = Q(z, f(z)), \tag{4}$$

where $Q(z, y)$ is a rational function.

We should remark that $a \in \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is called a singularity of f if and only if a is an algebraic branch point of f. (Thus poles are not considered as singularities.)

Theorem 3. *Let p be a polynomial of degree $d \geq 2$ and let Q be a rational function in the variables z and y. Suppose that f is an algebraic function satisfying the functional equation (4). If f is not a rational function, then one of the following conditions is satisfied.*

(i) f has exactly one finite singularity σ and another singularity at ∞. Furthermore, there is an integer $t \geq 2$ such that $f(\varphi(z^t))$ with $\varphi(z) = z + \sigma$ is a rational function.

(ii) f has exactly two finite singularities σ_1 and σ_2. Furthermore, p is linearly conjugate to T_d or $-T_d$ and the conjugation map is given by $\varphi(z) = \frac{1}{2}((\sigma_1 - \sigma_2)z + (\sigma_1 + \sigma_2))$. There is an integer $t \geq 1$ such that $f(\varphi(\frac{1}{2}(z^t + z^{-t})))$ is a rational function.

Remarks. 1) The following examples show that the situations (i) and (ii) mentioned in Theorem 3 may in fact both occur. Let $d \in \mathbf{Z}$, $d \geq 2$ and $r \in \mathbf{Q} \setminus \mathbf{Z}$. Then $f_1(z) = z^r \notin \mathbf{C}(z)$ is an algebraic function having its only finite singularity at $z = 0$. Clearly, f_1 satisfies the linear functional equation $f_1(z^d) = f_1(z)^d$. Another functional equation satisfied by f_1 is $f_1(z^d) = z^n f_1(z)$, which is of the form (4) if $n = (d-1)r \in \mathbf{Z}$.

To get an example for the situation (ii) one can take the nonlinear Böttcher functions from the Examples 3, 4, and 5. Here one may take $t = 1$. Another type of examples is constructed as follows. Let d and r be as above and take $f_2(z) = (z + \sqrt{z^2 - 1})^r$. Then we have $f_2(T_d(z)) = f_2(z)^d$. Furthermore, $f_2(\frac{1}{2}(z^t + z^{-t}))$ is a rational function for any multiple t of the denominator of r.

2) In [MFP] Mendès–France and van der Poorten discussed some examples of non–rational algebraic functions satisfying functional equations of the form (4) with $Q(z,y) \in \mathbf{C}(z)[y]$ of degree 1 in y. Theorem 3 explains why the functions of their Examples 3.1 – 3.5 have only two singularities and why their transformations $\varphi(x)$, which play the role of our $p(z)$, are always linearly conjugate to a Chebychev polynomial.

3) The question whether the solutions of functional equations of a similar type are algebraic or transcendental has been considered earlier by Mahler [M], Ostrowski [O], Loxton and van der Poorten [LP], Kubota[K], Gramain, Mignotte and Waldschmidt [GMW], and Nishioka [N]. Their analysis, however, was restricted to linear functional equations or to suitable n–dimensional generalizations of the special transformation $p(z) = z^d$.

II. Applications

The Mandelbrot set M is defined to be the set of all $c \in \mathbf{C}$ such that the Julia set of $p_c(z) = z^2 + c$ is connected. Equivalently,

$$M = \{c \mid p_c^n(0) \not\to \infty \text{ as } n \to \infty\}.$$

Douady and Hubbard ([DH1], [DH2, § 8.1], see also [Bd, § 9.10], [S1, § 6.2]) have shown that M is connected. In fact, they constructed a conformal map

$$\Phi(z) = z + c_0 + \frac{c_1}{z} + \dots$$

from the complement of M onto $\{z \mid |z| > 1\}$. This map was given by $\Phi(c) = \varphi_c(c)$, where φ_c is the unique Böttcher function with respect to $z^2 + c$ and z^2. From Theorem 2 we obtain

Corollary 1. *$\Phi(\alpha)$ is transcendental for all algebraic $\alpha \in \mathbf{C} \setminus M$.*

Of course, the analogous result holds for the inverse function Ψ of Φ, that is, $\Psi(\alpha)$ is transcendental for algebraic α, $|\alpha| > 1$. It is a well-known open problem whether the boundary of M is locally connected or, equivalently, whether $\Psi(z)$ has a continuous extension to $|z| = 1$ (see [DH1], [DH2]). Douady and Hubbard have shown that $c_\theta = \lim_{r\to 1} \Psi(re^{2\pi i\theta})$ exists if θ is rational. It is of interest to note that c_θ is algebraic if $\theta = r/s$, where r is an even and s is an odd integer. This does, however, by no means exclude the possibility of a continuous extension of $\Psi(z)$ to $|z| = 1$. An example of a function analytic for $|z| < 1$ and continuous for $|z| \le 1$ which takes on transcendental values at all algebraic points α with $|\alpha| < 1$ but fails to have this property for all roots of unity is given in [Be].

Our second application concerns a question of Golomb [G]. For $r = 0, 1, 2, \dots$ he defines the sequence $\{\beta_n^{(r)}\}_{n\in\mathbf{N}}$ by $\beta_1^{(r)} = 1 + \frac{r}{2}$ and

$$\beta_{n+1}^{(r)} = (\beta_n^{(r)})^2 + c$$

for $n \in \mathbf{N}$, where $c = \frac{r}{2}(1 - \frac{r}{2})$. Since the successive terms of the sequence are approximately obtained by successive squaring, Golomb asks whether there is a positive real number $\Theta(r)$ with the property $\Theta(r)^{2^n} \sim \beta_{n+1}^{(r)}$. He shows that this is true for

$$\Theta(r) = \beta_1^{(r)} \prod_{i=1}^{\infty} (1 + \frac{c}{(\beta_i^{(r)})^2})^{2^{-i}}.$$

Obviously, $\Theta(0) = 1$ and $\Theta(2) = 2$. Golomb poses the question for which values of $r \neq 0, 2$ the number $\Theta(r)$ is transcendental. It was shown in [AS, p. 435] and [FG, p. 456] that $\Theta(4) = (3 + \sqrt{5})/2$. A final answer to Golomb's question is given by

Corollary 2. *$\Theta(r)$ is transcendental for $r \neq 0, 2, 4$.*

We remark that $\Theta(r)$ may also be defined for any complex r with $\beta_n^{(r)} \to \infty$ for $n \to \infty$. Again we find that $\Theta(r)$ is transcendental for such algebraic $r \neq 2, 4$.

III. Proofs

Proof of Theorem 1. Let p, q, and f be as in the hypotheses of the theorem. Because $f(z) \sim \lambda z$, $\lambda \neq 0$, as $z \to \infty$, there exists a branch of f^{-1} analytic in a neighborhood of ∞ such that $f^{-1}(z) \sim z/\lambda$ as $z \to \infty$. From (1) we deduce that

$$f^{-1}(q(z)) = p(f^{-1}(z)). \tag{5}$$

Of course, by Weierstraß's principle on the permanence of functional equations, (1) and (5) remain valid under analytic continuation of f and f^{-1}.

Suppose now that f is nonlinear. Then f or f^{-1} has a singularity. We see from (1) and (5) that replacing f by f^{-1} corresponds to interchanging the roles of p and q. Therefore we may assume without loss of generality that f has a singularity.

Since ∞ is not a singularity of f, it is clear that f satisfies condition (ii) of Theorem 3. Hence p is linearly conjugate to T_d or $-T_d$. If f^{-1} has also singularities, this argument may be repeated with q instead of p thus showing that q is also linearly conjugate to T_d or $-T_d$. Hence we are left with the case that f^{-1} does not have singularities, that is, f^{-1} is rational. Clearly, f^{-1} is nonlinear and this implies that, besides the simple pole at ∞, f^{-1} must have at least one finite pole. From (5) we deduce that its set of poles is completely invariant with respect to q. Thus the poles of f^{-1} are exceptional in the sense of [Bd, Definition 4.1.1]. Now [Bd, Theorem 4.1.2] implies that q is linearly conjugate to M_d. ∎

Proof of Theorem 2. Suppose that p, q, f, G, and α satisfy the assumptions of the theorem. Let $\beta \in \mathbf{C}$ be a fixed point of p. Since f has a simple pole at ∞, we introduce $\tilde{f}(z) = f(z)/(z - \beta)$. Define $U = G \cup \{\infty\}$ and $Tz = p(z)$. We show that $\tilde{f}$, U, and T satisfy the hypotheses of the main theorem in [Be].

Clearly, T is meromorphic in the neighborhood U of $\omega = \infty$, ω is a fixed point of T of order d and $T(U) \subset U$. Since $p^m|_U \to \infty$ as $m \to \infty$, we have $\beta \notin U$. Thus $\tilde{f}$ is holomorphic in U.

The functional equation (1) leads to a recursion formula for the Laurent coefficients of $f(z)$ at $z = \infty$. Hence the algebraicity of the coefficients of p and q

guarantees that f has algebraic Laurent coefficients, too. Thus the power series expansion of $\tilde{f}$ at ω has only algebraic coefficients. Theorem 1 allows to conclude the transcendency of $f(z)$ over $\mathbf{C}(z)$ from the assumption that not both, p and q, are linearly conjugate to M_d, T_d, or $-T_d$.

Let
$$\tilde{P}(z,u,w) = (p(z)-\beta)w - q((z-\beta)u).$$
$\tilde{P}$ is a polynomial with algebraic coefficients and it is easily seen that
$$\tilde{P}(z,\tilde{f}(z),\tilde{f}(Tz)) = 0$$
for $z \in U$.

Now, α is an algebraic number with $T^m\alpha \to \infty$ for $m \to \infty$. Since p is a polynomial, we have $T^m\alpha \neq \infty$ for $m = 0,1,\ldots$. Furthermore, $T^m\alpha \neq \beta$ implies that $\tilde{P}(T^m\alpha, \tilde{f}(T^m\alpha), w)$ is not identically zero.

Using the terminology of [Be] we have $h(T) = h_w(\tilde{P}) = d$, $d(T) = d_w(\tilde{P}) = 1$, and $\operatorname{ord}_\omega T = d$. Thus condition (2) of the main theorem in [Be] is also satisfied. We conclude that $\tilde{f}(\alpha)$ and hence $f(\alpha)$ is a transcendental number. ∎

Proof of Theorem 3. Let f, p, and Q be as required in the statement of the theorem. Since f is algebraic, but not rational, it has only finitely many, say $n \geq 1$, singularities in $\hat{\mathbf{C}}$. This already implies $n \geq 2$. Otherwise we would have an algebraic function with exactly one singularity, say a, and f would be single-valued in $\hat{\mathbf{C}} \setminus \{a\}$ by the monodromy theorem, a contradiction.

Now, suppose that $p(\sigma) \in \mathbf{C}$ is a singularity of f. Suppose also that $p'(\sigma) \neq 0$. Then σ is a singularity of $f \circ p$ and hence, by the functional equation (1), σ is a singularity of f.

Let $\sigma_1,\ldots,\sigma_n$ be the finite singularities of f. Then there are nd inverse images of these singularities under p, counted according to multiplicity. These inverse images must be singularities of f or zeros of p'. Let $\omega_1,\ldots,\omega_r$ be the zeros of p' and denote by $\nu_1,\ldots,\nu_r$ their respective multiplicities. Then $r \leq d-1$ and $\sum_{j=1}^{r} \nu_j = d-1$. The above observations yield
$$dn \leq n + \sum_{j=1}^{r} (\nu_j + 1) = n + d - 1 + r.$$
Hence
$$n \leq 1 + \frac{r}{d-1} \leq 2. \tag{6}$$
Thus f has either one or two finite singularities. We treat these two case separately.

(A) f has exactly one finite singularity, say σ. Then $\tilde{f}(z) = f(z+\sigma)$ has its only finite singularity at $z = 0$. Since $\tilde{f}$ must have a second singularity in $\hat{\mathbf{C}}$, it has the singularities 0 and ∞. But, as $\tilde{f}$ is algebraic, there exist integers t_0,$t_\infty \geq 2$ such that $\tilde{f}(z^{t_a})$ is regular at $z = a$ for $a = 0,\infty$. Let $t = t_0 t_\infty$. Clearly, $\tilde{f}(z^t)$ has no singularities in $\hat{\mathbf{C}}$, hence it must be a rational function.

(B) f has exactly two finite singularities, say σ_1 and σ_2. From (6) we conclude $r = d-1$, that is, all zeros of p' are simple. Furthermore, we have

$$p(\omega_1),\ldots,p(\omega_r),\ p(\sigma_1),\ p(\sigma_2) \in \{\sigma_1,\sigma_2\}.$$

Let $\varphi(z) = \frac{1}{2}((\sigma_1-\sigma_2)z + (\sigma_1+\sigma_2))$, $\tilde{f}(z) = f(\varphi(z))$, $\tilde{p}(z) = \varphi^{-1}(p(\varphi(z)))$, and $\tilde{Q}(z,y) = Q(\varphi(z),y)$. It is easily seen that

$$\tilde{f}(\tilde{p}(z)) = \tilde{Q}(z,\tilde{f}(z))$$

and that $\tilde{f}$ has the two finite singularities ± 1. Since all zeros of $\tilde{p}'$ are simple, ± 1 are algebraic branch points of $\tilde{f}$ of order 2.

Following an argument of Steinmetz (see [S1, p. 143] or [S2]), we consider the polynomials $d^2(\tilde{p}(z)^2-1)$ and $(z^2-1)\tilde{p}'(z)^2$. These polynomials have the same leading coefficients and the same zeros, and hence are equal. Differentiation of the identity $d^2(\tilde{p}(z)^2-1) = (z^2-1)\tilde{p}'(z)^2$ yields

$$(z^2-1)\tilde{p}''(z) + z\tilde{p}'(z) - d^2\tilde{p}(z) = 0,$$

i.e. the differential equation for the Chebychev polynomials. The polynomial solutions of this differential equation are given by $\tilde{p}(z) = cT_d(z)$, where $c \in \mathbf{C}$. Since $T_d(1) = 1$ and $\tilde{p}(1) = \pm 1$, we have $\tilde{p} = T_d$ or $\tilde{p} = -T_d$.

From the fact that $(z+z^{-1})/2$ has the fixed points ± 1, each of order 2, and that ± 1 are also algebraic branch points of $\tilde{f}$ of order 2, we conclude that they are regular points of $g(z) = \tilde{f}((z+z^{-1})/2)$. Furthermore, g satisfies the functional equation

$$g(z^d) = Q((z+z^{-1})/2, g(z))$$

and has either no singularity or exactly one finite singularity at $z=0$. Hence g is either rational or, by the results established in part (A), there exists an integer $t \geq 2$ such that $g(z^t) = f(\varphi((z^t+z^{-t})/2))$ is a rational function. ∎

Proof of Corollary 1. The polynomial $p_{-2} = z^2-2$ is linearly conjugate to $T_2(z) = 2z^2-1$, and also to $-T_2(z)$. For $c \in \mathbf{C}\setminus\{0,-2\}$, p_c is not linearly conjugate to M_2, T_2, or $-T_2$. Suppose now that $\alpha \in \mathbf{C}\setminus M$ is algebraic. Because $\{0,-2\} \subset M$ we conclude that p_α is not linearly conjugate to M_2, T_2, or $-T_2$. We define

$$A(\infty) = \{z \in \mathbf{C} \mid \lim_{n\to\infty} p_\alpha^n(z) = \infty\}$$

and

$$G = \{z \in A(\infty) \mid g(z) > g(0)\},$$

where g denotes the Green's function of $A(\infty)$. It follows from the analysis in [Bd, § 9.10] that the Böttcher function φ_α with respect to p_α and M_2 is defined and analytic in G. Moreover, $\alpha \in G$, $p_\alpha(G) \subset G$, and $p_\alpha^n|_G \to \infty$ as $n \to \infty$. Thus, by Theorem 2, $\varphi_\alpha(z)$ is transcendental for each algebraic $z \in G$. In particular, $\Phi(\alpha) = \varphi_\alpha(\alpha)$ is transcendental. ∎

Proof of Corollary 2. Let z be a complex number with sufficiently large absolute value. Let $p(z) = z^2 + c$ with $c = \frac{r}{2}(1 - \frac{r}{2})$. We define

$$f(z) = z \prod_{i=0}^{\infty} (1 + \frac{c}{p^i(z)^2})^{2^{-i-1}},$$

where the branch of the 2^{i+1}-th root is chosen in such a way that $(1 + cz^{-1})^{2^{-i-1}}$ takes on the value 1 at $z = \infty$.

It is easily seen that $f(z)$ satisfies the functional equation

$$f(p(z)) = f(z)^2 \tag{7}$$

and has a simple pole at ∞. Thus f has to be the unique Böttcher function with respect to p and M_2.

Now, we are going to apply Theorem 2. Since $p(z) = z^2 + c$ and M_2 are linearly conjugate only for $c = 0$, $f(z)$ is nonlinear in all other cases. Furthermore, let G be a suitable neighborhood of infinity with $p(G) \subset G$ and $p^n|_G \to \infty$ for $n \to \infty$. (Since ∞ is an attractive fixed point of p, such a neighborhood exists.) Clearly, $p^m(1 + \frac{r}{2}) \in G$ if m is sufficiently large. Thus, by (7), the assertion of the corollary is true, if we can show the the transcendency of $f(p^m(1 + \frac{r}{2}))$ in the case that $p(z)$ is not linearly conjugate to M_2 or T_2. But that is now immediate from Theorem 2. ∎

Remark. Our corollaries concern only the special case $q(z) = z^d$. It is also possible to give proofs of Corollaries 1 and 2 based on the results established in [GMW, § 4], where the arithmetic properties of the solutions of the functional equation

$$f(z^d) = af(z)^d + bz^h$$

are studied. Here a and b are complex numbers, h is an integer, and f is a power series at $z = 0$. It turns out that such a proof would require the study of suitable modifications of the inverses of the Böttcher functions used in the proofs given above.

Acknowledgement. We are grateful to Professor S. W. Golomb for drawing our attention to the papers [AS] and [FG].

References

[AS] *A. V. Aho, N. J. A. Sloane*, Some doubly exponential sequences, Fibonacci Quart. **11**, (1973), 429–437

[Bd] *A. Beardon*, Iteration of Rational Functions, Springer, New-York 1991

[Be] *P.-G. Becker*, Transcendence of the values of functions satisfying generalized Mahler type functional equations (to appear in: J. reine angew. Math.)

[DH1] *A. Douady, J. Hubbard*, Itération des polynômes quadratiques complexes, C. R. Acad. Sci. Paris **294** (1982), 123–126

[DH2] *A. Douady, J. Hubbard*, Étude dynamique des polynômes complexes I & II, Publ. Math. Orsay 84-02 (1984) & 85-04 (1985)

[FG] *J. N. Franklin, S. W. Golomb*, A function–theoretic approach to the study of nonlinear recurring sequences, Pac. J. Math. **56**, (1975), 455–468

[G] *S. W. Golomb*, On certain nonlinear recurring sequences, Amer. Math. Monthly **70** (1963), 403–405

[GMW] *F. Gramain, M. Mignotte, M. Waldschmidt*, Valeurs algébriques de fonctions analytiques, Acta Arith. **47** (1986), 97–121

[K] *K. K. Kubota*, On the algebraic independence of holomorphic solutions of certain functional equations and their values, Math. Ann. **227** (1977), 9–50

[LP] *J. H. Loxton, A. J. van der Poorten*, A class of hypertranscendental functions, Aequationes Math. **16** (1977), 93–106

[M] *K. Mahler*, Über das Verschwinden von Potenzreihen mehrerer Veränderlicher in speziellen Punktfolgen, Math. Ann. **103** (1930), 573–587

[MFP] *M. Mendès-France, A. J. van der Poorten*, From geometry to Euler identities, Theor. Comput. Sci. **65** (1989), 213–220

[N] *K. Nishioka*, Algebraic function solutions of a certain class of functional equations, Arch. Math. **44** (1985), 330–335

[O] *A. Ostrowski*, Über einige Verallgemeinerungen des Eulerschen Produktes $\prod_{v=0}^{\infty}(1+x^{2^v}) = 1/(1-x)$, in: *Ostrowski, Collected Mathematical Papers*, Birkhäuser, Basel 1984

[S1] *N. Steinmetz*, Rational Iteration, Walter de Gruyter, Berlin 1993

[S2] *N. Steinmetz*, Dendrites and Jordan arcs (preprint 1992)

Paul-Georg Becker
Mathematisches Institut der Universität zu Köln
D-50923 Köln
Germany

Walter Bergweiler
Lehrstuhl II für Mathematik der RWTH Aachen
D-52056 Aachen
Germany

(Received March 3, 1993;
in revised form June 16, 1993)

manuscripta math. 81, 339 - 359 (1993)

manuscripta mathematica

A DEFORMATION LEMMA ON A C^1 MANIFOLD

Alexis BONNET

The classical construction of deformations by mean of pseudo-gradient vector fields requires the $C^{1,1}$ regularity. Here, we are concerned with a deformation lemma for a C^1 function on a manifold defined by a C^1 functional. We will assume some coupled Palais-Smale conditions between the two functions. The deformation is constructed with the help of integral lines of pseudo-gradient vector fields on a foliation of the manifold. Three different constructions are used for a sub-manifold of codimension 1 in finite dimension, then in infinite dimension and lastly a sub-manifold of any finite codimension in an infinite dimensional Banach space.

1. Introduction

In this article we are concerned with a deformation lemma for a C^1 functional on a manifold which is itself defined by a C^1 functional. We will assume some coupled Palais-Smale conditions between the two functionals. The goal of this paper is to establish deformation lemmas in this situation.

The classical deformation lemma was introduced by Palais to prove the results of Ljusternik-Schnirelman theory on a C^2 manifold S for a C^1 functional f satisfying the Palais-Smale condition ([5], [6]).

The deformation lemma is constructed with the help of integral lines of a pseudo-gradient vector field of f on S ([7]). In order to apply the results of the theory of differential equations, the vector field is required to be locally Lipschitz continuous. Thus, it seems necessary to assume that S is at least $C^{1,1}$ (a mapping is of class $C^{1,1}$ if it is differentiable with its derivative locally Lipschitz continuous).

However, in many applications to partial differential equations, the $C^{1,1}$ assumption imposes some further technical restrictions which seem unnatural for the problem. An example of this situation can be found in a work by H. Berestycki, T.Gallouet and O. Kavian (see [1] and [2]).

Let E be a real Banach space. Our purpose here is to prove a deformation lemma (Theorems 2.5 and 2.6) on a concrete C^1 manifold S, globally defined by:

$$S = \{x \in E \mid g(x) = 1\}$$

where g is a C^1 functional and 1 is not a critical value of g. Let f be a C^1 functional defined on a neighborhood of S and let a be a regular value of $f/_S$.

Denote by $\| f'/_{S_x} \| = \inf_{\lambda \in \mathbb{R}} \| f'(x) - \lambda g'(x) \|$ the norm of the derivative at x of the restriction of f to the level surface of g.

In infinite dimension (§3), we assume a coupled Palais-Smale condition on f and g which is roughly speaking:

For any sequence $(x_n) \in E$ such that $g(x_n) \longrightarrow 1$ and $f(x_n) \longrightarrow a$ and either $\| f'/_{S_{x_n}} \| \longrightarrow 0$ or $\| g'(x_n) \| \longrightarrow 0$ as $n \longrightarrow +\infty$ there exists a convergent subsequence.

In §5 we will prove an extension of the results to a submanifold of codimension k, $1 \leq k < +\infty$:

$$S = \{x \in E \mid g_1(x) = 1, \ldots, g_k(x) = 1\}.$$

The proof, here, involves the same kind of coupled Palais-Smale conditions as above between the functions g_i, $i = 1, \ldots, k$ and f.

In finite dimension (§4) the Palais-Smale condition on f and the compactness of bounded closed sets allow us to prove the deformation lemma on $S = \{x \in E \mid g(x) = 1\}$, without any Palais-Smale condition on g.

2. Definitions and statement of the main results

Let E be a real Banach space, S the C^1 manifold defined by:

$$S = \{x \in E \mid g(x) = 1\}$$

where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value for g.

We model a family of manifolds on S. We will note F_α (resp. F^*_α) the family of sets: $S_\beta = \{x \in E \mid g(x) = 1 + \beta\}$ where $\beta \in [0, \alpha]$ (resp. $\beta \in [-\alpha, 0]$).

Let f be a C^1 real functional defined on a neighborhood of S. $f/_S$ is a C^1 functional on the C^1 manifold S and $(f/_S)'(x) = f'(x)/_{TS(x)}$ where $TS(x)$ is the tangent space of S at x.

Definition 2.1. *We say that $f/_S$ satisfies the Palais-Smale condition at the level a (henceforth denoted by $(PS)_a$) if any sequence $(x_m) \subset S$ for which $f/_S(x_m) \to a$ and $f'/_S(x_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.*

For $f/_S$ satisfying $(PS)_a$ we have the following classical result:

Lemma 2.2. *If $f/_S$ satisfies $(PS)_a$ where a is a regular value of $f/_S$ then there are constants δ, ε_1 such that*

$$\forall x \in S, | f(x) - a | \leq \varepsilon_1 \Rightarrow \| f'/_S(x) \| > \delta \tag{1}$$

Definition 2.3. *Let a be a regular value of $f/_S$. A family F_α is said to be admissible for f at $a \in \mathbb{R}$ if f is defined on S_β for all $\beta \in [0, \alpha]$ and if there exist constants $\mu, \delta, \varepsilon_1 > 0$ such that:*

$$\forall \beta \in [0, \alpha], \forall x \in S_\beta, | f(x) - a | \leq \varepsilon_1 \Rightarrow \left(\| f'/_{S_\beta}(x) \| > \delta \text{ and } \| g'(x) \| > \mu\right). \tag{2}$$

Remark 2.4. The existence of δ, ε_1 for S is a consequence of $(PS)_a$ as we saw in Lemma 2.2 above, here we suppose that (1) is satisfied on S_β for $\beta \in [0, \alpha]$.

In fact (2) is a consequence of the following assumption:

all sequences $(x_n) \in S_{\varepsilon_n}, \varepsilon_n > 0$ such that $\varepsilon_n \to 0$ and ($\| f'/_{S_{\varepsilon_n}}(x_n) \| \to 0$ or $\| g'(x_n) \| \to 0$) and $| f(x_n) - a | \to 0$ as $n \to \infty$ possesses a convergent subsequence.

This assumption contains the Palais-Smale condition on f which is fondamental in this context. As g has only the regularity C^1, the assumption that 1 is a regular value of g is not sufficient. There is a lack of compactness. Therefore, Palais-Smale like conditions are introduced on g.

Of course, analogous definitions and remarks can be stated for the family F^*_α.

In the next section we will prove the following main theorem.

Theorem 2.5. *Let $S = \{g(x) = 1\}$ be a C^1 submanifold of a Banach E, f a C^1 functional on a neighborhood of S, a a non-critical value of $f/_S$ and α such that F_α or F^*_α is admissible for f at the value a. Then there exists $\hat{\varepsilon}$ such that for all $\varepsilon < \hat{\varepsilon}$ there exists an homeomorphism η of S onto S such that:*

a) $\eta(x) = x$ if $f(x) \notin [a - \hat{\varepsilon}, a + \hat{\varepsilon}]$,

b) $f(\eta(x)) \leq f(x)$ for all $x \in S$,

c) $f(\eta(x)) \leq a - \varepsilon$ for all x such that $f(x) \leq a + \varepsilon$,

d) if S is symetric ($S = -S$) and if f is even then η is odd.

In finite dimension we have a stronger form of Theorem 2.5:

Theorem 2.6. *Let $S = \{g(x) = 1\}$ be a C^1 submanifold of a Banach E, $\dim(E) < +\infty$, $f \in C^1(U, \mathbb{R})$ where U is a neighborhood of S. If $f/_S$ satisfies the Palais-Smale condition at a which is a regular value of $f/_S$, then there exists $\hat{\varepsilon} > 0$ such that for all $0 < \varepsilon < \hat{\varepsilon}$ there exists η homeomorphism of S onto S such that:*

a) $\eta(x) = x$ if $f(x) \notin [a - \hat{\varepsilon}, a + \hat{\varepsilon}]$,

b) $f(\eta(x)) \leq f(x)$ for all $x \in S$,

c) $f(\eta(x)) \leq a - \varepsilon$ for all x such that $f(x) \leq a + \varepsilon$,

d) if S is symetric ($S = -S$) and if f is even then η is odd.

3. Proof of Theorem 2.5

In the following we suppose that F_α is admissible for f. For the case F^*_α the proof is exactly the same.

The proof is based on the construction of two homeomorphisms, one of S onto $S_{\frac{\alpha}{2}}$ and one of $S_{\frac{\alpha}{2}}$ onto S. Each one allows us to diminish the value taken by f.

The two homeomorphisms will be constructed as the solutions of suitably modified gradient flows for f and g.

3.1. Construction of locally Lipschitz continuous vector fields

Let $\mu, \varepsilon_1, \delta$ be given by assumption (2).

Let $A_\varepsilon = \{x \in E, -\varepsilon < f - a < \varepsilon\}$.

Step 1.

Lemma 3.1. *For each $\nu > 0$ there exist two locally Lipschitz continuous vector fields V_1 and W_1 which have the following properties:*

$$\begin{aligned} &\forall \beta \in [0,\alpha], \forall x \in S_\beta \bigcap A_{\varepsilon_1}, \\ &\langle f'(x), V_1(x)\rangle \in (-(\delta+\nu), -\delta) \text{ and } \langle g'(x), V_1(x)\rangle \in (\tfrac{1}{2}, 1), \\ &\langle f'(x), W_1(x)\rangle \in (\delta, \delta+\nu) \text{ and } \langle g'(x), W_1(x)\rangle \in (\tfrac{1}{2}, 1), \\ &\exists M > 0, \| V_1 \| < M \text{ and } \| W_1 \| < M. \end{aligned} \tag{3}$$

Proof. We will note $\Sigma_\alpha = \{x \in S_\beta, \beta \in [0,\alpha]\}$.

For all x in $\Sigma \bigcap A_{\varepsilon_1}$ we can find some vectors v and w in E such that properties (3) are satisfied. This is a consequence of the fact that $\| g' \| > \mu$ and that $\| f'/_{S_\beta}(x) \| > \delta$ as F_α is admissible. Indeed, $\| f'/_{S_\beta}(x) \| = \inf_{\lambda \in \mathbb{R}} \| f' - \lambda g' \| > \delta$ implies that f' and g' are linearly independent in the dual space E^* of E. M is given as a function of μ and δ. The continuity of f' and g' then shows that v and w satisfy (3) in an open neighborhood N_x of x. Since $\{N_x, x \in \Sigma_\alpha \bigcap A_{\varepsilon_1}\}$ is an open covering of $\Sigma_\alpha \bigcap A_{\varepsilon_1}$ it possesses a locally compact refinment which will be noted by $\{M_j\}$. Let $\rho_j(x)$ denote the distance from x to the complementary of M_j. Then $\rho_j(x)$ is Lipschitz continuous and $\rho_j(x) = 0$ if $x \notin M_j$. Set:

$$\kappa_j(x) = \frac{\rho_j(x)}{\sum_k \rho_k(x)}.$$

The denominator of κ_j is only a finite sum since each $x \in E$ belongs to only finitely many sets M_k. Each of the sets M_k lies in some N_{x_j} , let $v_j = v(x_j)$.

$$\begin{aligned} \text{Set } V_1(x) &= \sum_j v_j \kappa_j(x) \\ \text{and } W_1(x) &= \sum_j w_j \kappa_j(x). \end{aligned} \tag{4}$$

Since $0 \le \kappa_j(x) \le 1$ and $\sum_j \kappa_j(x) = 1$, for each $x \in \Sigma_\alpha \bigcap A_{\varepsilon_1}$, $V_1(x)$ and $W_1(x)$ are convex combinations of vectors satisfying (3) at x. Moreover V_1 and W_1 are locally Lipschitz continuous. The proof of Lemma 3.1 is completed.

At this stage of the proof we are able to construct (see §3.3 Proposition 3.5), using the flow of V_1, a local homeomorphism going from S to $S_{\frac{\alpha}{2}}$ and an other one from $S_{\frac{\alpha}{2}}$ to S using W_1. Their composition would be a local homeomorphism of S but we are looking for an homeomorphism whose restriction to $S \backslash A_{\varepsilon_1}$ is the identity which is not the case of the previous one. For this purpose, we will now introduce two new vector fields V and W constructed after V_1 and W_1. In fact we want to obtain an homeomorphism whose restriction to $S \backslash A_{\frac{\varepsilon_1+\varepsilon}{2}}$ is the identity. Indeed as V_1 and W_1 are defined only on A_{ε_1}, we choose $S \backslash A_{\frac{\varepsilon_1+\varepsilon}{2}}$ instead of $S \backslash A_{\varepsilon_1}$ such that the orbits involved in the construction of the homeomorphism remain in A_{ε_1}.

Step 2.

The vector fields V and W, we are introducing here, are chosen such that the flows constructed as in (8) and (14) (see section 3.2) look like in fig 1.

Let $\tau = \frac{\varepsilon_1+\varepsilon}{2} - \mid f(x) - a \mid, \tau_1 = \frac{\varepsilon_1-\varepsilon}{4}$. Let $\beta(x) = g(x) - 1$, we have $x \in S_{\beta(x)}$. Let h be the real function defined by $h(x) = x$ if $x > 0$ and $h(x) = 0$ if $x \leq 0$.

Define γ on $\Sigma_\alpha \bigcap A_{\varepsilon_1}$ as follows:

$$\text{(5)}\quad \begin{aligned} &\text{if } \frac{\varepsilon_1+\varepsilon}{2} < \mid f(x) - a \mid, \text{ then } \gamma = 0;\\ &\text{if } \varepsilon + \frac{\varepsilon_1-\varepsilon}{4} \leq \mid f(x) - a \mid \leq \frac{\varepsilon_1+\varepsilon}{2}, \text{ then } \gamma = h\big(1 - \frac{\tau_1^2\beta}{\alpha\tau^2}\big);\\ &\text{if } \varepsilon < \mid f(x) - a \mid < \varepsilon + \frac{\varepsilon_1-\varepsilon}{4}, \text{ then } \gamma = \big(1 - \frac{\tau_1^2\beta}{\alpha\tau^2}\big)\frac{(2\tau_1 - \tau)}{\tau_1} + \frac{\tau - \tau_1}{\tau_1};\\ &\text{if } \mid f(x) - a \mid \leq \varepsilon, \text{ then } \gamma = 1. \end{aligned}$$

γ is Lipschitz continuous on $(\Sigma_\alpha \bigcap A_{\varepsilon_1}) \backslash (S \bigcap A^c_{\frac{\varepsilon_1+\varepsilon}{2}})$.

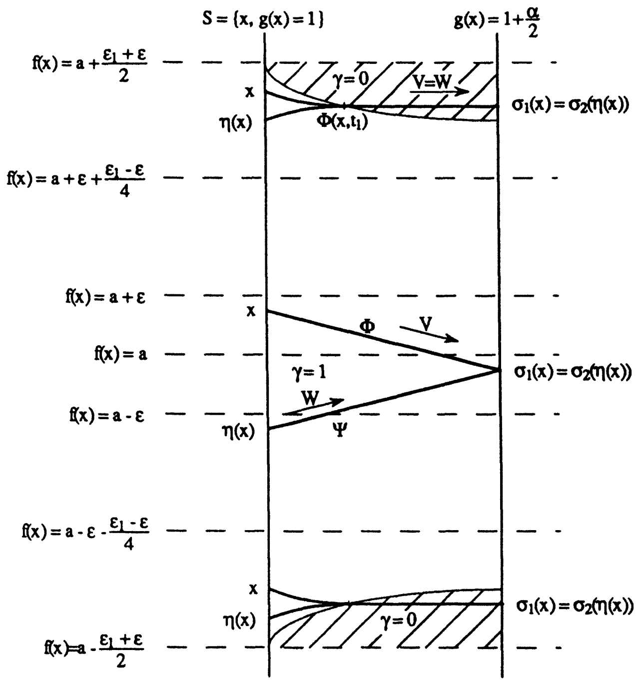


Fig. 1

We then define:

$$\text{(6)}\qquad V = \frac{(1+\gamma)}{2}V_1 + \frac{(1-\gamma)}{2}W_1,$$

$$\text{(7)}\qquad W = \frac{(1-\gamma)}{2}V_1 + \frac{(1+\gamma)}{2}W_1.$$

V and W are locally Lipschitz continuous on $(\Sigma_\alpha \bigcap A_{\varepsilon_1}) \backslash (S \bigcap A^c_{\frac{\varepsilon_1+\varepsilon}{2}})$. V and W are not Lipschitz continuous on a neighborhood of some point x in S where $\mid f(x) - a \mid = \frac{\varepsilon_1+\varepsilon}{2}$, that's why we choose $(\Sigma_\alpha \bigcap A_{\varepsilon_1})\backslash(S \bigcap A^c_{\frac{\varepsilon_1+\varepsilon}{2}})$ in the statement above. In the hatched area of Fig. 1, we have $\gamma = 0$ and $V = W$. The flows Φ and Ψ that will be constructed on V and W in Section 3.2. are identical in this area.

Remark 3.2. (i) In $A_\varepsilon, V = V_1$ and $W = W_1$.
(ii) When $\gamma = 0$, then $V = W = \frac{V_1+W_1}{2}$.
(iii) We still have $\langle g'(x), V(x)\rangle \in (\frac{1}{2}, 1)$, $\langle g'(x), W(x)\rangle \in (\frac{1}{2}, 1)$, $\| V_1 \| < M$ and $\| W_1 \| < M$.

Remark 3.4. If S is symetric and f is even, we replace $g(x)$ by $\frac{g(x)+g(-x)}{2}$, V by $\frac{V(x)-V(-x)}{2}$ and W by $\frac{W(x)-W(-x)}{2}$. Then S_β is symetric and V and W are even. The flows constructed on V and W are even as well.

3.2. Construction of the solutions of modified gradient flows for f and g

Step 1. Consider the Cauchy problem:

$$\frac{d\Phi}{dt} = V(\Phi), \Phi(x, 0) = x. \tag{8}$$

Our aim here is to associate to each point of $S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ a point of $S_{\frac{\alpha}{2}}$ on the same orbit solution of problem (8). For this purpose we shall prove that the orbits solutions of (8) go from x to $S_{\frac{\alpha}{2}}$ and therefore stay in $\Sigma_\alpha \bigcap A_{\varepsilon_1}$ where V is defined.

The basic existence uniqueness theorem for ordinary differential equations implies that for each $x \in S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$, (8) has a unique solution defined for t in a maximal interval $[0, t^+(x))$. We claim that we can choose $\nu > 0$ in Lemma 3.1 such that there exists some $\theta \in [0, t^+(x))$ satisfying:

$$g(\Phi(x, \theta)) = 1 + \frac{\alpha}{2}. \tag{9}$$

$\Phi(x, \theta)$ is the point on $S_{\frac{\alpha}{2}}$ we are looking for.

As $\langle g'(x), V\rangle > \frac{1}{2}$, we just need to prove that $t^+(x) \geq \alpha$.

Let $(t_n)_n$ be an increasing sequence such that $t_n \to t^+(x)$. Integrating (8) shows:

$$\| \Phi(x, t_n) - \Phi(x, t_{n+1}) \| \leq M \mid t_{n+1} - t_n \mid$$

since $\| V \| < M$. But then $\Phi(x, t_n)$ is a Cauchy sequence and hence converges to some $\bar{x}$ as $t_n \to t^+(x)$. Let us now derive some a priori bounds for $\beta(\bar{x})$ and $f(\bar{x})$ assuming $t^+(x) < \alpha$.

$$t^+(x) < \alpha \text{ leads to } \beta(\bar{x}) < \alpha \quad \text{since} \quad \langle g'(x), V\rangle < 1.$$

Using (3) and (6), (7) we get:

$$\langle f', V\rangle \in (\frac{(1+\gamma)}{2}(-\delta - \nu) + \frac{(1-\gamma)}{2}\delta \, , \, -\frac{(1+\gamma)}{2}\delta + \frac{(1-\gamma)}{2}(\delta + \nu)),$$

$$\text{hence} \quad \langle f', V\rangle \in (-\gamma\delta - \frac{(1+\gamma)}{2}\nu \, , \, -\gamma\delta + \frac{(1-\gamma)}{2}\nu).$$

$\bar{x}$ must be on the boundary of $\Sigma_\alpha \bigcap A_{\varepsilon_1}$. As $t^+ < \alpha$, $\mid f(\bar{x}) - a \mid = \varepsilon_1$ and then $\mid f(\Phi(x, t)) \mid \geq \frac{\varepsilon_1+\varepsilon}{2}$ on a maximal interval $[t_1, t^+(x)]$. On this interval $\gamma = 0$ and therefore $\langle f', V\rangle \in (-\frac{\nu}{2}, \frac{\nu}{2})$. An integration shows:

$$\mid f(\bar{x}) - f(\Phi(x, t_1)) \mid \leq \nu(t^+ - t_1) \leq \nu\alpha. \tag{10}$$

We recall that $f(\bar{x}) = \varepsilon_1$ and $f(\Phi(x,t_1)) = \frac{\varepsilon_1+\varepsilon}{2}$. Then for $\nu < \frac{\varepsilon_1-\varepsilon}{2\alpha}$, (9) gives $| f(\bar{x}) - f(\Phi(x,t_1)) | < \frac{\varepsilon_1-\varepsilon}{2}$, a contradiction.

Hence for $t^+(x) < \alpha$ and $\nu < \frac{\varepsilon_1-\varepsilon}{2\alpha}$, $\bar{x}$ is interior to $\Sigma_\alpha \bigcap A_{\varepsilon_1}$. Integrating (8) with $\bar{x}$ as initial data then furnishes a continuation of $\Phi(x,t)$ to values $t > t^+(x)$ contradicting the maximality of $t^+(x)$. This completes the proof of (9).

By the implicit function theorem, $\theta(x)$ defined by (9) is a C^1 function and we can define a C^1 function σ_1,

$$\sigma_1 : S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}} \longrightarrow S_{\frac{\alpha}{2}}, x \longmapsto \sigma_1(x) = \Phi(x,\theta(x)). \tag{11}$$

Step 2. We then derive some properties for σ_1:

Proposition 3.5. *(i) For all x in $S \bigcap A_\varepsilon$, we have $f(\sigma_1(x)) \le \sup(a+\varepsilon-\delta\frac{\alpha}{2}, a-\varepsilon)$,*
(ii) σ_1 is an homeomorphism of $S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ onto $\sigma_1(S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$.

Proof. V is constructed such that:

$$\forall x \in \Sigma_\alpha \bigcap A_\varepsilon, V = V_1, \text{ and then } \langle f'(x), V(x)\rangle < -\delta \tag{12}$$

$$\text{and } \langle g'(x), V(x)\rangle \in (\frac{1}{2}, 1). \tag{13}$$

By a straightforward integration, (13) implies that $\frac{\alpha}{2} < \theta(x) < \alpha$. Moreover, we have for all $x \in S \bigcap A_\varepsilon$:

$$f\big(\sigma_1(x)\big) < f(x) - \int_0^{\theta(x)} \langle f\big(\Phi(x,t)\big), V\big(\Phi(x,t)\big)\rangle dt,$$

where $\theta(x)$ is defined by (9).

If $f(x) < a+\varepsilon$ and $f\big(\sigma_1(x)\big) > a-\varepsilon$ then $\Phi(x,t) \in \Sigma_\alpha \bigcap A_\varepsilon$ for all $t \in [0,\theta(x)]$ and with (12), $f\big(\sigma_1(x)\big) < f(x) - \frac{\delta\alpha}{2}$; (i) follows immediatly.

Let us now prove (ii). Inequality (10) and the fact that $\theta(x) < \alpha$ for all x in $S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ leads to:

$$\sigma_1\big(S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}\big) \quad \subset \quad S_{\frac{\alpha}{2}} \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}+\nu\alpha}.$$

Taking $\nu < \frac{\varepsilon_1-\varepsilon}{4\alpha}$, we can prove then, as for the Cauchy problem (8), the following lemma:

Lemma 3.6. *For each $y \in S_{\frac{\alpha}{2}} \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}+\nu\alpha}$, the differential equation*

$$\frac{d\Phi}{dt} = V(\Phi), \Phi(y,0) = y$$

is integrable on a maximum interval $\big(t^-(y), t^+(y)\big)$ with $\lim_{t\to t^-(x)} \Phi(y,t) \in S \bigcap A_{\varepsilon_1}$. Let $\bar{\sigma}_1(y)$ be this limit.

As V is locally Lipschitz continuous, there is only one orbit of Φ passing through some z in $\big(\Sigma_\alpha \bigcap A_{\varepsilon_1}\big)\backslash\big(S \bigcap A^c_{\frac{\varepsilon_1+\varepsilon}{2}}\big)$. Then $\bar{\sigma}_1/_{\sigma_1\left(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}\right)}$ is a continuous map and it is the inverse of σ_1. This completes the proof of (ii).

Step 3. As we did with the fields V, we can consider the Cauchy problem:

$$\frac{d\Psi}{dt} = W(\Psi), \Psi(x,0) = x \tag{14}$$

We construct, for $\nu < \frac{\varepsilon_1-\varepsilon}{4\alpha}$ as before with V, a continuous mapping σ_2 of $S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ onto $S_{\frac{\alpha}{2}}$ with the following properties.

Proposition 3.7. *(i) σ_2 is an homeomorphism of $S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ onto $\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$,*

(ii) $\forall y\in\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}), a-\varepsilon\le f(y)\le a+\varepsilon\Rightarrow f(\sigma_2^{-1}(y))\le\sup(f(y)-\delta\frac{\alpha}{2},a-\varepsilon)$.

The proof is exactly the same as the one of Proposition 3.5.

3.3. Construction of an homeomorphism of S onto S with the properties of Theorem 2.5.

Our aim here is to prove that $\sigma_2^{-1}\circ\sigma_1$ is an homeomorphism of $S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ onto $S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ (step 1) then that $\sigma_2^{-1}\circ\sigma_1$ can be extended by Id on $S\backslash(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$ to give an homeomorphism of S onto S (step 2). We will then show in §3.4 that this homeomorphism satisfies the properties required by Theorem 2.5.

Step 1. We will now prove the following proposition:

Proposition 3.8.

$$\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})=\sigma_1(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}).$$

Proof. Let $y\in\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$. As for V and σ_1, (10) gives $y\in S_{\frac{a}{2}}\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}+\nu\alpha}$ and then $\bar\sigma_1(y)$ exists. Let t^- be defined as in Lemma 3.6. We will then use the following result.

Lemma 3.9. *Let $y\in\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$. If $\bar\sigma_1(y)\notin(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$ and $\nu<\frac{\varepsilon_1-\varepsilon}{4\alpha}$ then for all $t\in(t^-,0)$ we have $\gamma(\Phi(y,t))=0$.*

Proof. If not, let (t^-,t_0) be a maximum interval on which $\gamma(\Phi(y,t))=0$. We will choose $\nu<\delta$ in the following, this assumption simplifies the calculus. We will examine the cases $t_0>t^-$ and $t_0=t^-$ successively. We recall that $\gamma=0$ if $|f(x)-a|\le\frac{\varepsilon_1+\varepsilon}{2}$ or $1-\frac{\tau_1^2\beta}{\alpha\tau^2}\le 0$ where $\tau=\frac{\varepsilon_1+\varepsilon}{2}-|f(x)-a|$, $\tau_1=\frac{\varepsilon_1-\varepsilon}{4}$ and $\beta(x)=g(x)-1$ as in the setting of γ (assertion (5)).

a) If $t_0>t^-$, let us derive some inequalities on $g(\Phi(y,t_0))$ and $f(\Phi(x,t_0))$:

By Remark 3.2 we have $g(\Phi(x,t))\in(1+\frac{1}{2}(t-t^-),1+(t-t^-))$ and then,

$$\beta(\Phi(x,t))\in(\frac{1}{2}(t-t^-),(t-t^-)), \tag{15}$$

$$\gamma=0\quad\text{on}\quad(t^-,t_0)\quad\text{then}\quad\langle f'(\Phi(y,t)),V\rangle\in(-\frac{\nu}{2},\frac{\nu}{2}), \tag{16}$$

$$\sigma_1(y)\notin A_{\frac{\varepsilon_1+\varepsilon}{2}}\quad\text{then}\quad|f(\bar\sigma_1(y))-a|>\frac{\varepsilon_1+\varepsilon}{2}. \tag{17}$$

Assertions (16) and (17) imply by integration between t^- and t_0 that:

$$|f(\Phi(y,t_0))-a|>\frac{\varepsilon_1+\varepsilon}{2}-\frac{\nu}{2}(t_0-t^-), \tag{18}$$

thus, either $|f(\Phi(y,t_0))-a|\ge\frac{\varepsilon_1+\varepsilon}{2}$ and there exists $t_1>t_0$ such that $\gamma(\Phi(y,t))=0$ on (t_0,t_1) contradicting the maximality of t_0, or $|f(\Phi(y,t_0))-a|<\frac{\varepsilon_1+\varepsilon}{2}$ and

$$\tau(\Phi(y,t_0))<\frac{\nu}{2}(t_0-t^-). \tag{19}$$

Using (15) and (19) one can get easily:

$$1-\frac{\tau_1^2\beta}{\alpha\tau^2}<1-\frac{\tau_1^2\frac{1}{2}(t_0-t^-)}{\alpha\left(\frac{\nu}{2}(t_0-t^-)\right)^2}$$
$$<1-\frac{2\tau_1^2\beta}{\alpha\nu^2(t_0-t^-)}.$$

By assumption, $g(y)=\frac{\alpha}{2}$ and $t_0<0$. Estimate (15) leads then to $t^->-\frac{\alpha}{2}$ and $t_0-t^-<\frac{\alpha}{2}$. We deduce that:

$$1-\frac{\tau_1^2\beta}{\alpha\tau^2}<1-\frac{8\tau_1^2}{\alpha^2\nu^2}.$$

For $\nu\alpha<\frac{\varepsilon_1-\varepsilon}{4}=\tau_1$, which has been already assumed in §3.3, we get $1-\frac{\tau_1^2\beta}{\alpha\tau^2}<-7$ and $\gamma=0$ in a neighborhood of $\Phi(y,t_0)$, contradicting the maximality of t_0.

b) If $t_0=t^-$ then $|\ f(\bar\sigma_1(y))\ |=\frac{\varepsilon_1+\varepsilon}{2}$. As $\langle f',V\rangle\in(-(\delta+\nu),\frac{\nu}{2})$, $|\ \langle f',V\rangle\ |<2\delta$ (ν is chosen less than δ). Then $\forall t>t^-$, $|\ f(\Phi(y,t))-a\ |>\frac{\varepsilon_1+\varepsilon}{2}-2\delta(t-t^-)$ and either $|\ f(\Phi(y,t))-a\ |\geq\frac{\varepsilon_1+\varepsilon}{2}$ or

$$\tau<2\delta(t-t^-). \tag{20}$$

Assertions (15) and (20) give for $|\ f(\Phi(y,t))-a\ |<\frac{\varepsilon_1+\varepsilon}{2}$ that:

$$1-\frac{\tau_1^2\beta}{\alpha\tau^2}<1-\frac{\tau_1^2}{8\alpha\delta^2(t-t^-)},$$

which is negative for $(t-t^-)<\frac{\tau_1^2}{8\alpha\delta^2}$, then $\gamma=0$ on $(t^-,t_1]$ where $t_1=t^-+\frac{\tau_1^2}{8\alpha\delta^2}>t^-$ contradicting the maximality of $t_0=t^-$. This completes the proof of Lemma 3.9.

We are now able to prove that $\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})\subset\sigma_1(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$.

Indeed if $\bar\sigma_1(y)\notin(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$, then $\gamma=0$ on the orbit of Φ passing through y, so $V=W$ and the orbit of Φ is also the orbit of Ψ and $\bar\sigma_1(y)=\sigma_2^{-1}(y)\in(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$, a contradiction. Thus $\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})\subset\sigma_1(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$.

We can prove the inclusion $\sigma_1(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})\subset\sigma_2(S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}})$ in the same way. This completes the proof of Proposition 3.8.

Let us now construct σ as

$$\sigma=\sigma_2^{-1}\circ\sigma_1.$$

σ is obviously an homeomorphism of $S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ onto $S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$.

Step 2. Let η be defined by

$$\begin{aligned}\eta&=\sigma \quad \text{on} \quad S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}\\ \eta&=Id \quad \text{on} \quad S\bigcap(A_{\frac{\varepsilon_1+\varepsilon}{2}})^c.\end{aligned} \tag{21}$$

For a subset G of E, G^c denotes the complementary of G in E. The function $\eta:S\mapsto S$ is one to one and onto. η and η^{-1} are continuous at each point of $S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ and of the interior in S of $S\bigcap(A_{\frac{\varepsilon_1+\varepsilon}{2}})^c$. It remains to see the case of points on the boundary of $S\bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$; this is done in the following lemma.

Lemma 3.10. *η is continuous on each point of $G = \{x \in S, \mid f(x) - a \mid = \frac{\varepsilon_1+\varepsilon}{2}\}$.*

Proof. Let $x \in \left(S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}\right)$ we will first estimate $\mid \eta(x) - x \mid = \mid \sigma(x) - x \mid$ by a function of τ. As in the proof of Lemma 3.9 we have at $\Phi(x,t)$:

$$1 - \frac{\tau_1^2\beta}{\alpha\tau^2} < 1 - \frac{\tau_1^2 t}{2\alpha(\tau(x) + 2\delta t)^2},$$

negative for $t_0 = \frac{1}{2\delta}\tau(x)$ if $\tau < \frac{\tau_1^2}{16\alpha\delta}$.

Then $\gamma(\Phi(x,t)) = 0$ for a maximum interval $[t_1, t_2] \ni t_0, t_2 > t_1$. There is a $\nu_0 > 0$ independent of bounded τ such that for $\nu < \nu_0$, t_2 satisfies $t_2 \geq t_1 + \theta$, where $\Phi(x, t_1 + \theta) = \sigma_1(x)$. (This is derived by a direct calculus as in part (a) of the proof of Lemma 3.9). Therefore, for all t in $[-\theta, 0]$, we have $\gamma(\Phi(\sigma_1(x), t)) = 0$ whence $V(\Phi(\sigma_1(x), t)) = W(\Phi(\sigma_1(x), t))$ and thus $\Psi(\sigma_1(x), t) = \Phi(\sigma_1(x), t)$. In particular, $\Phi(x, t_1) = \Phi(\sigma_1(x), -\theta) = \Psi(\sigma_1(x), -\theta)$.

By definition of σ, there exists t_3 such that $\sigma(x) = \Psi(\sigma_1(x), t_3)$. As $\langle g', V\rangle \in (\frac{1}{2}, 1)$ and $\langle g', W\rangle \in (\frac{1}{2}, 1)$, we have $\beta(\Phi(x, t_1)) < t_1 \leq t_0$ and thus, $\mid t_3 + \theta \mid \leq 2t_0$ by Remark 3.2 concerning inequality (4). For $\tau(x) < \frac{\tau_1^2}{16\alpha\delta}$, one can now derive the following main inequality:

$$\begin{aligned} \| x - \sigma(x) \| &\leq \| x - \Phi(x, t_0) \| + \| \Psi(\sigma_1(y), -\theta) - \sigma(x) \| \\ &\leq 3Mt_0 = \frac{6M}{\delta}\tau(x). \end{aligned} \tag{22}$$

Let $x_0 \in G$ and let $B(x_0, r)$ be a neighborhood of $\eta(x_0) = x_0$ in S. As $\tau(x) = 0$ and as τ is continuous there exists $\mathcal{W}$ neighborhood of x_0 in S such that:

$$\forall x \in \mathcal{W} \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}, \tau(x) < \inf\left(\frac{2\tau_1}{\alpha}, \frac{\tau_1^2}{16\alpha\delta}, \frac{r}{2}\frac{\delta}{6M}\right),$$

then using (22) we get:

$$\forall x \in \mathcal{W} \bigcap B(x_0, \frac{r}{2}) \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}, \ \| \eta(x) - \eta(x_0) \| \leq \| \sigma(x) - x \| + \| x - x_0 \| < r$$

$$\text{and} \quad \forall x \in \mathcal{W} \bigcap B(x_0, \frac{r}{2}) \bigcap A^c_{\frac{\varepsilon_1+\varepsilon}{2}}, \| \eta(x) - \eta(x_0) \| = \| x - x_0 \| \leq \frac{r}{2} < r.$$

In conclusion, for all $B(x_0, r)$ neighborhood of x_0 in S, for all x in $\mathcal{W} \bigcap B(x_0, \frac{r}{2})$ neighborhood of x_0 in S, $\eta(x)$ is in $B(x_0, r)$ and thus, η is continuous at x_0; this completes the proof of Lemma 3.10. The continuity of η^{-1} can be proved in the same way and thus η is an homeomorphism of S onto S.

3.4. Properties of η

We have seen in §3.3 that η is an homeomorphism of S onto S. Let us now show that η satisfies the properties required in Theorem 2.5.

First, property a) is obviously satisfied.

For property b) let us prove:

Lemma 3.11. *ν can be chosen small enough such that $f(\eta(x)) \leq f(x)$*

Proof. With the notation of the proof of Lemma 3.10 we are going to prove that

$$\begin{aligned} & f(\Phi(x,t_1)) \leq f(x) \\ \text{and} \quad & f(\Psi(\sigma_1(x),-\theta)) \geq f(\sigma(x)), \end{aligned} \tag{23}$$

and, as $\Phi(x,t_1) = \Psi(\sigma_1(x),-\theta)$ we will conclude $f(\sigma(x)) \leq f(x)$.

To prove the first inequality of (23) we remark that :

$$\frac{d}{dt} f(\Phi(x,t)) = \langle f'(x), V(\Phi(x,t))\rangle. \tag{24}$$

The proof starts with the estimate $\frac{d}{dt}f(\Phi(x,t)) < -\frac{\delta}{4}$ derived on an interval $[0, C_1\tau^2]$. Then, it is shown that the subset of $(0,t_1)$, on which $\frac{d}{dt}f(\Phi(x,t)) > 0$, is of measure less than $C_2\tau^2$. The constants C_1 and C_2 are independent of τ and ν. Lastly, as $\langle f', V\rangle < \frac{\nu}{2}$, we have:

$$f(\Phi(x,t_1)) \leq f(x) - \frac{C_1\delta}{4}\tau^2 + \frac{\nu}{2}C_2\tau^2. \tag{25}$$

and for $\nu < \frac{C_1\delta}{2C_2}$ we get $f(\Phi(x,t_1)) \leq f(x)$.

Step 1. Determination of C_1.

We first remark that, by definition of γ (formula (5)) either $\gamma \geq \frac{1}{2}$ or $\gamma \geq 1 - \frac{\tau_1^2\beta}{\alpha\tau^2}$. If $\gamma > \frac{1}{2}$ then $\frac{d}{dt}f(\Phi(x,t)) = \langle f'(\Phi(x,t), V(\Phi(x,t))\rangle < -\frac{\delta}{4}$.

V and ν are chosen such that $\mid \langle f'(x), V\rangle \mid \leq \delta + \nu < 2\delta$, this leads to:

$$\mid \tau(x) - \tau(\Phi(x,t)) \mid \leq 2\delta t. \tag{26}$$

We know that $\beta \leq t$ and then at $\Phi(x,t)$:

$$\frac{\tau_1^2\beta}{\alpha\tau^2} \leq \frac{\tau_1^2 t}{\alpha(\tau(x) - 2\delta t)^2}. \tag{27}$$

For $t < \frac{\tau}{4\delta}$, we have $\frac{\tau(x)}{2} \leq \tau(\Phi(x,t)) \leq \frac{3}{2}\tau(x)$ and at $\Phi(x,t)$,

$$\frac{\tau_1^2\beta}{\alpha\tau^2} \leq \frac{4\tau_1^2 t}{\alpha\tau^2(x)}. \tag{28}$$

which is less than $\frac{1}{2}$ whenever $t < \frac{\alpha}{8\tau_1^2}\tau^2(x)$.

As τ is bounded we can choose $C_1 < \frac{\alpha}{8\tau_1^2}$ such that $t < C_1\tau^2(x)$ implies $t < \frac{\tau}{4\delta}$, therefore (28) gives for $t < C_1\tau^2(x)$ that $\gamma \geq \frac{1}{2}$ or $\gamma \geq 1 - \frac{\tau_1^2\beta}{\alpha\tau^2} \geq \frac{1}{2}$ and then $\frac{d}{dt}f(\Phi(x,t)) = \langle f'(\Phi(x,t), V(\Phi(x,t))\rangle < -\frac{\delta}{4}$.

Step 2. Determination of C_2.

Now suppose that $\frac{d}{dt}f(\Phi(x,t))$ is positive, then $\gamma(\Phi(x,t)) < \frac{1}{2}$ and $\gamma(\Phi(x,t)) = h(1 - \frac{\tau_1^2\beta}{\alpha\tau^2})$. We will prove that $\gamma(\Phi(x, t + C_2\tau^2)) = 0$ where the constant C_2 will be made precise further. For this purpose we will estimate the derivative of $\frac{\beta}{\tau^2}$. At $\Phi(x,t)$, we have:

$$\frac{d}{dt}\left(\frac{\beta}{\tau^2}\right) = \frac{1}{\tau^2}\frac{d}{dt}\beta - \frac{\beta}{\tau^3}\frac{d}{dt}\tau. \tag{29}$$

But $\gamma = 0$ if $\frac{\tau_1^2\beta}{\alpha\tau^2} \geq 1$, therefore while $\gamma(\Phi(x,t)) \neq 0$ we have $\frac{\beta}{\tau^2} < \frac{\alpha}{\tau_1^2}$. Moreover $| \frac{d}{dt}\tau(\Phi(x,t)) | \leq \frac{d}{dt}f(\Phi(x,t)) < \frac{\nu}{2}$ and thus $| \frac{\beta}{\tau^3}\frac{d}{dt}\tau | < \frac{\nu}{2}\frac{\alpha}{\tau_1^2}\frac{1}{\tau}$.

For ν small enough, so that $\frac{\alpha}{\tau_1^2} < \frac{1}{4\tau^2}$ (which is satisfied for $\nu < \frac{\varepsilon_1-\varepsilon}{8\alpha}$), inequality $\frac{d}{dt}\beta > \frac{1}{2}$ and identity (29) give at $\Phi(x,t)$:

$$\frac{d}{dt}\left(\frac{\beta}{\tau^2}\right) > \frac{1}{4\tau^2}. \tag{30}$$

From (30) we will now derive a lower bound of $\frac{d}{dt}\left(\frac{\beta}{\tau^2}\right)$ depending only on $\tau(x)$ and not on $\tau(\Phi(x,t))$ as above.

We have seen that $\gamma(\Phi(x,t)) = 0$ for $\tau(x) < \frac{\tau_1^2}{16\alpha\delta}$ and $t \leq \frac{1}{2\delta}\tau(x)$, thus, for

$$\tau(x) < \frac{\tau_1^2}{16\alpha\delta},$$

we have

$$t_1 < \frac{1}{2\delta}\tau(x)$$

by minimality of t_1. By (26) we get $\tau(\Phi(x,t)) \leq 2\tau(x)$. Then at $\Phi(x,t)$:

$$\frac{d}{dt}\left(\frac{\beta}{\tau^2}\right) > \frac{1}{4\tau^2} > \frac{1}{16\tau^2(x)}.$$

Therefore, when $\langle f'(\Phi(x,\tilde{t}), V(\Phi(x,\tilde{t}))\rangle > 0$ for $\tilde{t} < t_1$ we have

$$\frac{d}{dt}\gamma(\Phi(x,\tilde{t})) < -\frac{1}{16\tau^2(x)} \tag{31}$$

and γ decrease to 0 on an interval $(\tilde{t}, t_1)$ of length less then $16\tau^2(x)$ as $\gamma \leq 1$.

We have proved that for $\tau(x) < \frac{\tau_1^2}{16\alpha\delta}$, $\frac{d}{dt}\gamma(\Phi(x,t))$ is positive on a set of measure less then $16\tau(x)^2$. We know that $\sigma_1(x) = \Phi(x, \theta(x))$ with $\theta(x) < \alpha$, then for $\tau(x) \geq \frac{\tau_1^2}{16\alpha\delta}$, there exists $C_2 > 16$ such that $\alpha < C_2\tau^2$ and then $\frac{d}{dt}f(\Phi(x,t))$ is negative on a set of measure less than $\theta < \alpha < C_2\tau^2$. This completes the proof of the first inequality of (23). The second one can be treated in the same way. The proof of Lemma 3.11 is completed and assertion b) of Theorem 2.5 is satisfied by η.

We know by Proposition 3.5 and 3.7 that:

$$\text{if}\quad a-\varepsilon \leq f(x) \leq a+\varepsilon \quad\text{then}\quad f(\eta(x)) \leq \sup(f(x) - \frac{\delta\alpha}{2}, a-\varepsilon),$$

thus, for $\varepsilon < \frac{\delta\alpha}{2}$, $f(\eta(x)) < a - \varepsilon$. Then with Lemma 3.9 assertion c) of Theorem 2.5 is satisfied and we set $\hat{\varepsilon} = \inf(\varepsilon_1, \frac{\delta\alpha}{2})$.

The last point of Theorem 2.5 follows from Remark 3.4. If S is symetric and if f is even, the flows Φ and Ψ are even and g can be chosen odd and thus, the constructions of σ_1 and σ_2 make them odd. In conclusion, σ and η are odd. This completes the proof of Theorem 2.5.

4. The case of the finite dimension

In this case we will use the compactness of bounded closed sets, to obtain the inequalities assumed in Definition 2.3. Thus, we will construct some deformations on bounded sets extended by composition to the whole manifold S.

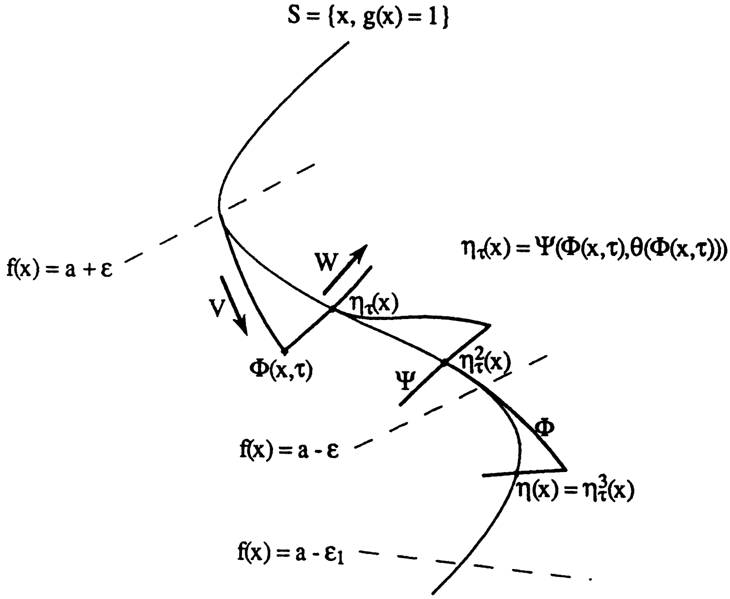


Fig. 2

Sketch of the proof:

As in §3 we will construct the local deformation by composition of two homeomorphisms, solutions of suitably modified gradient flows for f and g. Here the first flow (noted Φ) will remain close to S and will allow us to reduce the value taken by f. For $\tau > 0$ fixed we associate to x in S the point $\Phi(x, \tau)$. The second flow (noted Ψ) will allow us to come back to S changing just slightly the value of f (see Fig. 2). We will prove then that the continuous function η_τ constructed on a bounded subset of S by associating to $\Phi(x, t)$ the point $\eta_\tau(x)$ of S belonging to the same orbit of the flow defined by Ψ, is an homeomorphism whenever τ is small enough. η_τ or a finite power of η_τ is then a deformation. The last step of the proof is to extend these local deformation to the manifold S by composition of deformations on chosen bounded sets of S.

4.1. Construction of locally Lipschitz continuous vector fields

Let ε_1, δ be given by Lemma 2.2.

Let Ω be a bounded set, by compactness there exist $\mu > 0$ such that $| g'(x) |> \mu$ for all x in $S\bigcap\Omega$ and there exists $\alpha > 0$ such that:

$$\forall\beta\in[-\alpha,\alpha],\forall x\in S_\beta\bigcap A_{\varepsilon_1}\bigcap\Omega, \mid g'(x)\mid>\mu \quad \text{and} \quad \mid f'/_{S_\beta}(x)\mid>\delta \tag{32}$$

Lemma 4.1 *For all $\varepsilon < \varepsilon_1$, let $\eta < \inf\left(\frac{\delta}{2},\frac{\varepsilon_1-\varepsilon}{3}\right)$, there exist two locally Lipschitz continuous vector fields V_1 and W_1 and a real $M > 1$ satisfying:*

$$\begin{aligned}&\forall\beta\in[-\alpha,\alpha],\forall x\in S_\beta\bigcap A_{\varepsilon_1}\bigcap\Omega,\\ &\langle f'(x),V_1(x)\rangle < -\delta \text{ and } \mid\langle g'(x),V_1(x)\rangle\mid<\frac{1}{M},\\ &\mid\langle f'(x),W_1(x)\rangle\mid<\eta \text{ and } \langle g'(x),W_1(x)\rangle>1,\\ &\parallel V_1\parallel<1 \text{ and } \parallel W_1\parallel<M.\end{aligned} \tag{33}$$

Proof. The proof is almost the same as in Lemma 3.1: we just need to construct w and find M before dealing with v. M is taken bigger than 1 for the inequality $\mid\langle g'(x),V_1(x)\rangle\mid<\frac{1}{M}$ to be satisfied. The fact that $\mid V_1\mid<$ is less than 1 is a consequence of $\parallel f'/_{S_{\beta(x)}}\parallel>\delta$.

Then let $\Omega_1,\Omega_2,\Omega_3$ be defined as:

$$\begin{aligned}\Omega_1 &=\{x\in\Omega, d(x,\partial\Omega)>1\},\\ \Omega_2 &=\{x\in\Omega, d(x,\partial\Omega)>2\},\\ \Omega_3 &=\{x\in\Omega, d(x,\partial\Omega)>3\}.\end{aligned} \tag{34}$$

In the following, Ω will be choosen such that Ω_3 is not empty.

Let ρ be a C^∞ function, with $0\le\rho\le1$ such that:

$$\begin{aligned}\rho=0 \quad &\text{on} \quad \Omega_1^c\bigcup A^c_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}\\ \rho=1 \quad &\text{on} \quad \Omega_2\bigcap A_{\varepsilon+\frac{\varepsilon_1-\varepsilon}{3}}\end{aligned} \tag{35}$$

Then, Let V and W be defined by

$$\begin{aligned}V(x) &=\rho(x)V_1(x) \quad \text{on} \quad \Omega,\\ W(x) &=W_1(x) \quad \text{on} \quad \Omega_1^c\bigcap A_{\varepsilon_1}.\end{aligned} \tag{36}$$

Actually, for $x\notin A_{\varepsilon_1}$, V_1 is not defined but $\rho(x)=0$ and we can take $V(x)=0$. V and W are then, locally Lipschitz continuous. The sets $\Omega_1,\Omega_2,\Omega_3$ are chosen such that the deformation acts on Ω_3; the set $\Omega\backslash\Omega_1$ on which $V=0$ allows us to extend continuously the local deformation by Id outside Ω.

4.2. Construction of the solution of modified gradient flows for f and g

We now construct the flows associated to V and W in Ω:

$$\begin{aligned}\frac{d\Phi}{dt} &=V(\Phi), \Phi(x,0)=x,\\ \frac{d\Psi}{dt} &=W(\Psi), \Psi(x,0)=x.\end{aligned} \tag{37}$$

As $\mid V \mid\leq 1$ and $V = 0$ in $\Omega - \Omega_1$, the flow $\Phi(x,t)$ is defined for all x in Ω for $t \in (-\infty, +\infty)$ and we have $\Phi(x,t) \in \Omega$. By the definition of ρ, $\Phi(x,t) = x$ for x in $\Omega_1^c \bigcup A^c_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}$.

Let $x \in S \bigcap \Omega$, we then derive the following inequalities on Φ:

$$
\begin{aligned}
f(\Phi(x,t)) - f(x) &= \int_0^t \langle f'(\Phi(x,t)), V(\Phi(x,t))\rangle dt \\
&= \int_0^t \langle f'(\Phi(x,t)), V_1(\Phi(x,t))\rangle \rho(\Phi(x,t)) dt \\
&\leq -\delta \int_0^t \rho(\Phi(x,t)) dt.
\end{aligned}
\tag{38}
$$

Indeed, either $\rho(y) = 0$ or $y \in A_{\varepsilon_1} \bigcap \Omega_1$ but then $\langle f'(y), V_1(y)\rangle < -\delta$. Remarking that $\mid V \mid\leq\mid V_1 \mid\leq 1$ and $\mid \langle g'(y), V(y)\rangle \mid\leq \frac{1}{M}$ we get:

$$
\begin{aligned}
&\mid \Phi(x,t) - x \mid \leq t, \\
&\mid g(\Phi(x,t)) - 1 \mid \leq \frac{1}{M} \int_0^t \rho(\Phi(x,t)) dt
\end{aligned}
\tag{39}
$$

For $0 < \tau \leq 1$ fixed, let $\bar{x} = \Phi(x,\tau)$ we will prove the lemma:

Lemma 4.2. *There exists a unique θ such that $\Psi(\bar{x},\theta) \in S$ and then $\mid \theta \mid< \frac{\tau}{M}$.*

Proof. $\Psi(\bar{x},\theta) \in S$ is equivallent to $g(\Psi(\bar{x},\theta)) = 1$.

We have seen that $\bar{x} \in \Omega$. Then, $\Psi(\bar{x},t)$ is defined on a maximum interval (t_0,t_1). As $\mid W \mid< M$, if $\mid t_i \mid< +\infty$ ($i = 0$ or 1) then $\Psi(\bar{x},t)$ has a limit at t_i noted $\Psi(\bar{x},t_i)$ such that $\Psi(\bar{x},t_i) \in \partial(\Omega \bigcap A_{\varepsilon_1})$, if not, integrating (37) gives a contradiction to the maximality of t_i.

If $x \in \Omega_1^c \bigcup A_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}$ then $\bar{x} = x$ and $\Psi(\bar{x},0) = \bar{x} = x \in S$, $\theta = 0$ is a solution to our problem.

We assume now that $x \in \Omega_1 \bigcap A_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}$. As $\Phi(x,t) = Id$ on $\Omega_1^c \bigcup A_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}$, $\Phi(x,t)$ remains in $\Omega_1 \bigcap A_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}$ for all t.

As $\mid \langle f'(x), W(x)\rangle \mid< \inf\left(\frac{\delta}{2}, \frac{\varepsilon_1-\varepsilon}{3}\right)$ and $\mid W \mid\leq M$,

$$
\mid t \mid\leq \frac{1}{M} < 1 \text{ implies } \mid f(\Psi(\bar{x},t)) - f(\bar{x}) \mid< \eta < \frac{\varepsilon_1 - \varepsilon}{3} \text{ and } \mid \Psi(\bar{x},t) - \bar{x} \mid< t \leq 1.
\tag{40}
$$

From these inequalities and the hypothesis $\bar{x} \in \Omega_1 \bigcap A_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}$, we derive that if $\mid t \mid\leq \frac{1}{M}$ then $\Psi(\bar{x},t) \in \Omega \bigcap A_{\varepsilon_1}$, thus $|t_i| > \frac{1}{M}$. We recall that W is chosen such that $\langle g'(x), W(x)\rangle > 1$. Inequality (39) leads to $\mid g(\bar{x}) - 1 \mid< \frac{\tau}{M} < \frac{1}{M}$ and then one of the $g(\Psi(\bar{x},t_i)) - 1$ is necessarily of the opposite sign of $g(\bar{x}) - 1$. This gives immediatly the existence of θ. Moreover:

$$
\mid \theta \mid< \frac{\tau}{M},
\tag{41}
$$

and as θ is unique, $g\big(\Psi(x,t)\big)$ is strictly increasing with respect to t. The proof of Lemma 4.2 is then completed.

Remark 4.3. As $\| V \|< 1$ and $\| W \|< M$ we have $\| x - \Phi(x,\tau) \|\leq \tau$ and $\| \Phi(x,\tau) - \Psi\big(\Phi(x,\tau), \theta(\Phi(x,\tau))\big) \|\leq \tau$.

By the implicit function theorem, θ is of class C^1 on a neighborhood of $\bar{x}$, this is still true for $x \in A_{\varepsilon_1} \bigcap A^c_{\varepsilon+\frac{\varepsilon_1-\varepsilon}{3}} \bigcap S$ for which $x = \bar{x}$. This remark allows us to define a C^1 function η_τ (its construction is shown on Fig. 2 above):

$$\begin{aligned}\eta_\tau(x) &= x \text{ in } \Omega_1^c \bigcup A^c_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}},\\ \eta_\tau(x) &= \Psi\big(\Phi(x,\tau),\theta(\Phi(x,\tau))\big) \text{ in } \Omega_1 \bigcap A_{\varepsilon+2\frac{\varepsilon_1-\varepsilon}{3}}.\end{aligned} \tag{42}$$

By Remark 4.3 we have $\| \eta_\tau(x) - x \| \leq 2\tau$.

4.3. Properties of the function $\eta_\tau(x)$

Here we will prove that η_τ or a finite power of η_τ is a deformation on Ω_3, that it satisfies the properties a),b),c) and d) of Theorem 2.6, and then that for τ small enough η_τ is a diffeomorphism.

By (39) and as $\langle g', W\rangle > 1$ we have:

$$\begin{aligned}| \theta(\bar{x}) | &\leq \int_0^t \rho(\Phi(x,t)),\\ f(\eta_\tau(x)) - f(\bar{x}) &\leq \frac{\delta}{2} | \theta(\bar{x}) | \leq \frac{\delta}{2} \int_0^\tau \rho(\Phi(x,t))dt\\ \text{and } f(\bar{x}) - f(x) &\leq -\delta \int_0^\tau \rho(\Phi(x,t))dt.\end{aligned} \tag{43}$$

Thus,

$$f(\eta_\tau(x)) - f(x) \leq -\frac{\delta}{2} \int_0^\tau \rho(\Phi(x,t))dt. \tag{44}$$

Inequality (44) leads in particular to $f(\eta_\tau(x)) \leq f(x)$. Let $x \in \Omega_3 \bigcap S \bigcap A_\varepsilon$, we know that $\| V \| \leq 1$ then, for $\tau < 1$ and $\forall t \in [0,\tau]$, we have $\Phi(x,t) \in \Omega_2$. Moreover, if $x \in \Omega_3 \bigcap S \bigcap A_\varepsilon$ and if $f(\bar{x}) \geq a - (\varepsilon + \frac{\varepsilon_1-\varepsilon}{3})$ then $\forall t \in (0,\tau)$, we have $\Phi(x,t) \in \Omega_2 \bigcap A_{\varepsilon+\frac{\varepsilon_1-\varepsilon}{3}}$ and $\rho(\Phi(x,t)) = 1$. This yields:

$$f(\eta_\tau(x)) \leq -\frac{\delta}{2}\tau \tag{45}$$

If $x \in \Omega_3 \bigcap S \bigcap A_\varepsilon$ and if $f(\bar{x}) \leq a - (\varepsilon + \frac{\varepsilon_1-\varepsilon}{3})$ then as $| f(\eta_\tau(x)) - f(\bar{x}) | \leq \frac{\varepsilon_1-\varepsilon}{3}$ we get:

$$f(\eta_\tau(x)) \leq a - \varepsilon.$$

In conclusion, we have proved the lemma

Lemma 4.4. *For all x in S, $f(\eta_\tau(x)) \leq f(x)$.*
$\forall x \in S \bigcap A_\varepsilon$, if $f(x) \leq a + \varepsilon$ then $f(\eta_\tau(x)) \leq \sup(a - \varepsilon, f(x) - \frac{\delta\tau}{2})$.
$\eta_\tau(x) = x$ if $f(x) \notin (a - \varepsilon_1, a + \varepsilon_1)$.

Then, taking $\eta^* = (\eta_\tau)^m$ with $\frac{m\delta\tau}{2} > 2\varepsilon$, η^* satisfies the properties a) and b) of Theorem 2.6 and c) is satisfied for $x \in \Omega_3$.

The case S symetric and f even is solved as in §3 (Remark 3.4) with a symetric set Ω.

The problem is almost solved in Ω_3, but η_τ is not necessary an homeomorphism. For this point we will prove:

Proposition 4.5. *For τ small enough, η_τ is a diffeomorphism.*

Indeed, note Φ_τ the function $\Phi(.,\tau)$, Φ_τ is a diffeomorphism from S onto $\Phi_\tau(S)$ but the function

$$h_\tau : \Phi_\tau(S) \longrightarrow S, y \longmapsto \Psi\big(y, \theta(y)\big) \tag{46}$$

is not necessary a diffeomorphism.

We need to prove that η_τ is a diffeomorphism for τ small enough. For this purpose, we will build the inverse of h_τ for small τ's.

Let $y \in S$, to find t such that $\Psi(y,t) \in \Phi_\tau(S)$ is equivalent to find t such that

$$g\big(\Phi_{-\tau}(\Psi(y,t))\big)= 1 \tag{47}$$

For $t = 0$ using (33) we get $\mid g\big(\Phi_{-\tau}(y)\big) - 1 \mid< \frac{\tau}{M}$. We would like to prove that the derivative with respect to t of the left side of equality (47) is bigger than 1 on an interval $(-\frac{\tau}{M}, \frac{\tau}{M})$. This derivative is

$$D_{y,\tau}(t) = \langle g'\Big(\Phi - \tau(\Psi(y,t))\Big) \circ \Phi'_{-\tau}\Big(\Psi(y,t)\Big), W\big(\Psi(y,t)\big)\rangle \tag{48}$$

If $\tau = t = 0$ then $D_{y,\tau}(t) = \langle g'(y), W(y)\rangle > 1$; then by continuity of g', Φ, Ψ, Φ', W and as $S \bigcap \Omega$ is relatively compact, there exists $\alpha > 0$ such that, for all $y \in S \bigcap \Omega$ and for all $\tau, \mid t \mid< \alpha$

$$D_{y,\tau}(t) > 1 \tag{49}$$

As $M > 1$, in particular (49) is true for $t \in (-\frac{\tau}{M}, \frac{\tau}{M})$. Thus, as $\mid g\big(\Phi_{-\tau}(\Psi(y,0))\big) - 1 \mid< \frac{\tau}{M}$, there exists a unique t in $(-\frac{\tau}{M}, \frac{\tau}{M})$ such that:

$$g\big(\Phi_{-\tau}(\Psi(y,t))\big)= 1.$$

Moreover by the implicit function theorem, $t(y)$ is C^1.

Now, the unicity of θ and t in $(-\frac{\tau}{M}, \frac{\tau}{M})$ yields easily:

$$\begin{aligned} \theta\big(\Psi(y, t(y))\big) &= -\, t(y) \\ \text{and} \quad t\big(\Psi(z, \theta(z))\big) &= -\, \theta(z) \quad \text{for } z \in \Phi_\tau(S). \end{aligned} \tag{50}$$

In conclusion, $\Psi\big(., t(.)\big)$ is the C^1 inverse of h_τ, h_τ is a diffeomorphism and η_τ is diffeomorphism as well.

This complete the proof of Proposition 4.5.

Take $\tau < \alpha$ and $\eta^* = (\eta_\tau)^m$ on $S \bigcap \Omega$, $\eta^* = \mathrm{Id}$ on $S \bigcap \Omega^c$, with $\frac{m\delta\tau}{2} > 2\varepsilon$. The function η^* satisfies the property a), b), d) of Theorem 2.6. Property c) is satisfied on Ω_3.

4.4. Construction of a deformation on S

To conclude let

$$\begin{aligned} \Omega^n &= \{x \in E, 4n - 7 <\| x \|< 4n + 4\}, \\ \text{by definition} \quad \Omega_3^n &= \{x \in E, 4n - 4 <\| x \|< 4n + 1\}. \end{aligned} \tag{51}$$

Ω_3^n is not empty for $n \geq 0$, the sets Ω^n are bounded and

$$\bigcup_{n=0}^{+\infty} \Omega_3^n = E. \tag{51}$$

Then let η_n be the deformation constructed on Ω^n as in §4.3, define

$$\eta = \prod_{n=+\infty}^{0} \eta_n. \tag{52}$$

That means $\eta = \ldots \circ \eta_n \circ \ldots \circ \eta_3 \circ \eta_2 \circ \eta_1$.

For every bounded set $B \subset S$, the intersection $B \bigcap \Omega^n$ is not empty for only a finite number of n and then $\eta_n/_B = \mathrm{Id}/_B$ except for a finite number of n. Thus $\eta(x)$ is well defined and it is a diffeomorphism from S onto S. Property a) of Theorem 2.6 is obviously satisfied with $\hat{\varepsilon} = \varepsilon_1$, property b) is true for each η_n and thus, by composition, for η.

We will now prove by induction that property c) is satisfied for η.

By Remark 4.3 we have proved that $\| \eta_\tau(x) - x \| \leq 2\tau$, thus $\| \eta_n(x) - x \| \leq 2m\tau$ where m is such that $\frac{m\delta\tau}{2} > 2\varepsilon$, that is to say $m > \frac{4\varepsilon}{\delta\tau}$. We would like to have $\| \eta_n(x) - x \| \leq 1$ for the following. Let $\hat{\varepsilon} = \inf(\varepsilon_1, \frac{\delta}{16})$. Proposition 4.5 is true for τ small enough, thus, for each η_n, we can choose $\tau < \frac{1}{4}$ and define $m \in \mathbb{N}$ by:

$$\frac{4\varepsilon}{\delta\tau} < m \leq \frac{4\varepsilon}{\delta\tau} + 1, \tag{53}$$

then $2m\tau \leq \frac{8\varepsilon}{\delta} + 2\tau < \frac{8\hat{\varepsilon}}{\delta} + \frac{1}{2} \leq 1$ and

$$\| \eta_n(x) - x \| \leq 2m\tau < 1. \tag{54}$$

Therefore, it is possible to choose η_n, deformation on Ω^n, such that $\| \eta_n(x) - x \| < 1$.

On Ω_3^0, property c) is satisfied for η_0 and then also for $\eta = \left(\prod_{n=+\infty}^{1} \eta_n\right) \circ \eta_0$ thanks to property b).

Assume that c) is satisfied by η on Ω_3^k for all $k = 0 \,..\, (n-1)$. Let $x \in S \bigcap \Omega_3^n$ then x can be in Ω^k only for $k = n$, $n-1$ or $n+1$. If $x \in \Omega_3^{n-1}$ then c) is satisfied. If not then $x \in \Omega_4^n$ or $x \in (\Omega_3^n - \Omega^{n-1})$. In the former case, as $\| \eta_{n-1}(x) - x \| < 1$ then $\prod_{k=n-1}^{1} \eta_n(x) \in \Omega_3^n$ and c) is satisfied for x, in the later case $\eta_{n-1}(x) = x$ and $\prod_{k=n}^{1} \eta_n(x) = \eta_n(x)$ and c) is again satisfied for x. Property c) is then satisfied for Ω_3^n and by induction for every n and then on S. This completes the proof of Theorem 2.6.

5. Extension of the results to any finite codimension

In this section we will extend the results of §3 to some submanifold S of finite codimension k bigger than 1.

Let S be defined by k C^1 functionals:

$$S = \{x \in E \mid g_1(x) = 1, \ldots, g_k(x) = 1\} \tag{55}$$

where $g_i \in C^1(E, \mathbb{R})$, 1 is a regular value for g_i and the family of k submanifolds $H_i = \{g_i = 1\}$, $i = 1, \ldots, k$ is transverse.

Remark 5.1. If g_1 is C^1 and g_i for $i = 2, \ldots, k$ is of class $C^{1,1}$ than we can apply §3 on the $C^{1,1}$ submanifold

$$\tilde{E} = \{x \in E \mid g_2(x) = 1, \ldots, g_k(x) = 1\}. \tag{56}$$

Indeed, as $\tilde{E}$ is $C^{1,1}$ we are able to construct some $C^{1,1}$ vector fields tangent to $\tilde{E}$ which are integrable and which allow us to stay inside $\tilde{E}$.

In the general case, g_i is of class C^1 for $i = 1, \ldots, k$, we will again introduce some coupled Palais-Smale conditions on g_i and f.

With the same notations as above in §1, 2, 3, let f be a C^1 functional defined on a neighborhood of S and let a be a non-critical value of $f/_S$. Denote by $\| f'/_{S_x} \| = \inf_{\lambda_1,\ldots,\lambda_k \in \mathbb{R}} \| f'(x) - \Sigma_{i=1}^k \lambda_i g_i'(x) \|$ the norm of the derivative of f on the level surface of $(g_1, \ldots, g_k)$, and by $tg(x) = \inf_{\lambda_1,\ldots,\lambda_k \in S^{k-1}} \| \Sigma_{i=1}^k \lambda_i g_i'(x) \|$. In some sense, tg measures the transversality of $H_1, \ldots, H_k$ and the slope of the g_i.

Definition 5.2. *$f, g_1, \ldots, g_k$ and a are admissible for our problem if for any sequence $(x_n) \in E$ such that $g_1(x_n) \to 1, \ldots, g_k(x_n) \to 1$ and $f(x_n) \to a$ and either $\| f'/_{S_{x_n}} \| \to 0$ or $tg(x) \to 0$ as $n \to +\infty$ there exists a convergent subsequence.*

Roughly speaking, the condition on $tg(x)$ allows us to keep the $\| g_i' \|$ big enough and the $\frac{g_i'}{\|g_i'\|}$ far enough from each other which will be useful in the construction of vector fields adapted to each functional (see condition (58)).

The main theorem of this section is:

Theorem 5.3. *Let S (codim $S = k$) and f defined as above, let a be a non-critical value of $f/_S$ if $f, g_1, \ldots, g_k$ and a satisfy the condition of Definition 5.2, then there exists $\hat{\varepsilon}$ such that for all $\varepsilon < \hat{\varepsilon}$ there exists an homeomorphism η of S onto S such that:*

a) $\eta(x) = x$ if $f(x) \notin [a - \hat{\varepsilon}, a + \hat{\varepsilon}]$,

b) $f(\eta(x)) \leq f(x)$ for all $x \in S$,

c) $f(\eta(x)) \leq a - \varepsilon$ for all x such that $f(x) \leq a + \varepsilon$,

d) if S is symetric ($S = -S$) and if f is even then η is odd.

Proof.

Sketch of the proof: as in §3 we will construct η as the composition $\sigma_2 \circ \sigma_1$ of two homeomorphisms σ_1, σ_2 from S to an other submanifold:

$$S_{\frac{\alpha}{2}} = \{x \in E \mid g_1(x) = 1 + \frac{\alpha}{2}, \ldots, g_k(x) = 1 + \frac{\alpha}{2}\} \tag{57}$$

As in section 2 and 3 the choice of α and $\hat{\varepsilon}$ is a concequence of the coupled Palais-Smale conditions of Definition 5.2. More precisely:

Lemma 5.4. *There exist $\delta > 0, \mu > 0, \alpha > 0, \varepsilon_1 > 0$ such that for $-\alpha \leq g_1(x) \leq \alpha$ $| f(x) - a | \leq \varepsilon_1$ we have $\| f'/_{S_x} \| \geq \delta$ and $tg(x) \geq \mu$.*

As in §3 we start with the construction of adapted vector fields. The existence of $\delta, \mu, \alpha, \varepsilon_1$ give us easily the lemma:

Lemma 5.5. *For each $\nu > 0, \zeta > 0$ there exist $2k$ locally Lipschitz continuous vector fields V_1^i and W_1^i which have the following properties:*

$$
(58)\qquad
\begin{aligned}
&\forall x \in E \text{ such that } | f(x) - a | \leq \varepsilon_1 \text{ and } | g_i(x) - 1 | \leq \alpha, \text{ for } i = 1, \ldots, k,\\
&\langle f'(x), V_1^i(x)\rangle \in (-(\delta+\nu), -\delta)\ ,\ \langle g_i'(x), V_1^i(x)\rangle \in (\frac{1}{2}, 1),\\
&\qquad\qquad \text{and for } j \neq i \quad \langle g_j'(x), V_1^i(x)\rangle \in (0, \zeta),\\
&\langle f'(x), W_1^i(x)\rangle \in (\delta, \delta+\nu)\ ,\ \langle g_i'(x), W_1^i(x)\rangle \in (\frac{1}{2}, 1),\\
&\qquad\qquad \text{and for } j \neq i \quad \langle g_j'(x), W_1^i(x)\rangle \in (0, \zeta),\\
&\exists M > 0, \| V_1^i \| < M \text{ and } \| W_1^i \| < M.
\end{aligned}
$$

M is a function of μ and δ. The value of ν and ζ will be made precise in the following.

With the help of V_1^i, W_1^i we will now build two other vector fields as in §3.

Let $\beta_i(x) = g_i(x) - 1$ and $\beta(x) = \frac{1}{n}\Sigma_{i=1}^k \beta_i(x)$. Then let $\gamma(x)$ be defined as in equations (5) of section 3 and V^i, W^i defined by mean of γ as in the equations (6), (7) of §3.

Then as, in §3.2, we construct gradient flows on V^i and W^i. The construction of σ_1, σ_2 will be done in several steps. With the flow Φ^i:

$$
(59)\qquad \frac{d\Phi^i}{dt} = V^1(\Phi^i),\ \Phi^i(x,0) = x
$$

let us define the C^1 submanifold $F_1 = \{\Phi^1(x,t), x \in S, t \in \mathbb{R}\}$ and by induction $F_2 = \{\Phi^2(x,t), x \in F_1, t \in \mathbb{R}\}, \ldots, F_k = \{\Phi^k(x,t), x \in F_{k-1}, t \in \mathbb{R}\}$.

Remark 5.6. $\Phi^i(x,t)$ is not defined for every $t \in \mathbb{R}$, we have to stay in the domain of V_1^i and W_1^i defined by Lemma 5.5. The submanifolds F^i are included in this domain.

For ζ, α small enough, by Lemma 5.4, F_1 is of codimension $k-1$, …, and F_k is of codimension 0.

Therefore, for $\varepsilon_2 < \varepsilon_1$ given and α, ζ, ν small enough we have $S_{\frac{\varepsilon}{2}} \bigcap A_{\varepsilon_2} \subset F_k$. Then as F_{k-1} is of codimension 1, to each point of $S_{\frac{\varepsilon}{2}} \bigcap A_{\varepsilon_2}$ we can associate the only point of F_{k-1} on the same orbit of the flow Φ^k, this correspondance is an homeomorphism by choice of V^i. By induction we build an homeomorphism denoted σ_1 from $S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ to a subset of $S_{\frac{\varepsilon}{2}}$. In the same way, with the help of $G_1 = \{\Psi^1, x \in S, t \in \mathbb{R}\}, G_2 = \{\Psi^2, x \in G_1, t \in \mathbb{R}\}, \ldots, G_k = \{\Psi^k, x \in G_{k-1}, t \in \mathbb{R}\}$, we can build an homeomorhism σ_2 from $S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}$ to a subset of $S_{\frac{\varepsilon}{2}}$.

Moreover there is a path from $x \in S$ to $\sigma_1(x) \in S$:

$$
x \to x_1 = \Phi^1(x,t_1) \in F_1, x_1 \to x_2 = \Phi^2(x,t_2) \in F_2, \ldots x_{k-1} \to \sigma_1(x) = \Phi^k(x,t_k) \in S_{\frac{\varepsilon}{2}}.
$$

A similar path exists from x to $\sigma_2(x)$. Using these paths we can prove as in Proposition 3.8 that:

$$
(60)\qquad \sigma_2(S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}) = \sigma_1(S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}).
$$

Then let η defined by

$$
(61)\qquad
\begin{aligned}
\eta &= \sigma \quad \text{on} \quad S \bigcap A_{\frac{\varepsilon_1+\varepsilon}{2}}\\
\eta &= Id \quad \text{on} \quad S \bigcap (A_{\frac{\varepsilon_1+\varepsilon}{2}})^c.
\end{aligned}
$$

The same kind of estimates of Lemma 3.10 can be developped here for η proving its bicontinuity.

Lastly, the property a),b),c),d) of Theorem 5.3. are satisfied by the same argument as in §3, which completes the proof of the theorem.

REFERENCES

[1] H. Berestycki, T. Gallouet and O. Kavian, *Equations de champs scalaires euclidiens non linéaires dans le plan,* Note de C.R. Acad. Sc. Paris, Série I, 297 (1983) p. 307-310

[2] H. Berestycki, T. Gallouet and O. Kavian, *Semilinear elliptic problems in* $\mathbb{R}^2$. , Rapports du Laboratoire d'analyses numériques, Université Paris 6

[3] S.N. Chow and J.K. Hale, *Methods of Bifurcation Theory,* Springer-Verlag, New York, 1982

[4] L. Ljusternik and L. Schnirelmann, *Méthodes Topologiques dans les Problèmes Variationels,* Hermann, Paris, 1934

[5] R.S. Palais, *Ljusternik-Schnirelmann theory on Banach Manifolds,* Topology 5 (1966), 115-132

[6] R.S. Palais, *Critical point theory and the minimax principle,* Proc. Sym. Pure Math., Vol 15, Amer. Math. Soc., Providence, R.I., 1970, pp. 185-212

[7] P.H Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations,* CBMS 65, Amer. Math. Soc., Providence, R.I., 1986

[8] A. Szulkin, *Critical point theory of Ljusternik-Schnirelmann type and applications to partial differential equations,* Séminaires de Mathématiques Supérieures, Les Presses de l'Université de Montréal, to appear

[9] A. Szulkin, *Ljusternik-Schnirelmann Theory on* C^1 *manifold, Ann. Inst. H. Poincaré, Anal. Non Linéaires, 5 (1988), no 2, 171-197*

Alexis BONNET
Ecole Nationale des Ponts et Chaussées
CERMICS
Central 2, La Courtine, 93167 Noisy-Le-Grand Cedex
and
Ecole Normale Supérieure
DMI
45, rue d'Ulm, 75230 Paris Cedex 05

(Received March 23, 1993;
in revised form June 24, 1993)

manuscripta math. 81, 361 - 378 (1993)

manuscripta mathematica

Weierstrass Weight of Gorenstein Singularities with One or Two Branches

Arnaldo Garcia[1] and R.F. Lax[2]

We obtain formulas for the Weierstrass weight of singularities with either one or two branches on a Gorenstein curve in characteristic zero. These formulas generalize results of C. Widland in the cases of simple cusps and ordinary nodes. The formulas arise from a study of the semigroup of values of such a singularity and the relation between this semigroup and a basis for the dualizing differentials on the curve adapted to the singular point.

0. Let X denote an integral, projective Gorenstein curve over an algebraically closed field k. Weierstrass points have been defined in the case of a line bundle on X when k is of characteristic 0 in [20,12,21], and in the case of a linear system on X in arbitrary characteristic in [5]. In order to arrive at formulas for the weights of unibranch and two-branch singularities that are not overly complicated, we will assume throughout that the characteristic of k is zero. One can modify these formulas, taking into account the order sequences of the linear systems involved, and obtain analogous results in positive characteristic. In particular, the formulas for weight that we will derive give a lower bound on the weight of a singularity of the given type on a classical canonical curve in positive characteristic.

In the first section, we briefly review the definition of Weierstrass points of linear systems on Gorenstein curves in characteristic 0. In the second section, we compute the Weierstrass weight of a unibranch singularity in terms of its semigroup of values. This generalizes a result of C. Widland [20,13] in the case of a simple cusp.

In the final section, we compute the Weierstrass weight of a singularity with precisely two branches. This generalizes a result of Widland [20] in the case of an ordinary node. Our result depends heavily on the structure of the semigroup of values of the singularity. These semigroups have been studied by the first author [4] and by F. Delgado [1, 2], among others. In the course of the argument used to derive our weight formula, we construct a basis for the dualizing differentials that is analogous to a (Hermitian) basis of regular differentials adapted to a point in the smooth case. We also compute the number of smooth Weierstrass points on

[1] Partially supported by a CNPq grant
[2] Partially supported by NSF grant INT-9101337

a Gorenstein curve with only unibranch or two-branch singularities that are not "overweight."

1. We recall the definition of Weierstrass points of a linear system on a Gorenstein curve in characteristic zero. Let K denote the field of rational functions on X. Let $\pi : \tilde{X} \to X$ denote the normalization of X. Let $\omega = \omega_X$ denote the sheaf of dualizing differentials on X. We recall (cf. [18]) that if $P \in X$, then ω_P consists of all rational differentials τ on X such that

$$\sum_{Q \to P} \operatorname{Res}_Q(f\tau) = 0 \quad \text{for all } f \in \mathcal{O}_P,$$

where the sum is over all points on $\tilde{X}$ lying over P, and where $\mathcal{O}_P$ denotes the local ring of the structure sheaf of X at P. Since X is Gorenstein, ω is an invertible sheaf.

Let $\mathcal{L}$ be an invertible sheaf on X. Assume that $\mathcal{L}$ has nontrivial global sections and let $V \subseteq H^0(X, \mathcal{L})$ be a subspace of dimension $s > 0$. Choose a basis $\phi_1, \phi_2, \ldots, \phi_s$ of V. Suppose $\{U_\alpha\}$ is an open covering of X such that $\mathcal{L}(U_\alpha)$ and $\omega(U_\alpha)$ are free $\mathcal{O}_X(U_\alpha)$-modules generated by ψ_α and τ_α, respectively. Write $\phi_j|_{U_\alpha} = f_j\psi_\alpha$ for some $f_j \in \mathcal{O}_X(U_\alpha)$ and $j = 1, 2, \ldots, s$. Then $f_1, f_2, \ldots, f_s$ are linearly independent rational functions over k.

Let $\mathcal{C} = \operatorname{Ann}(\pi_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) = \operatorname{Ann}(\omega_X/\pi_*\omega_{\tilde{X}})$ denote the conductor sheaf. The dualizing differential τ_α is of the form

$$\tau_\alpha = \frac{dt}{h_\alpha},$$

where t is some separating element for K over k and where $h_\alpha \in \mathcal{C}(U_\alpha)$. Put

$$\rho_\alpha = \det\left(h_\alpha^i d^i f_j/dt^i \right) \psi_\alpha^s \tau_\alpha^N,$$

where $i = 0, 1, \ldots, s-1$, $j = 1, 2, \ldots, s$, and $N = (s-1)s/2$. We note that although $d^i f_j/dt^i$ may not be in $\mathcal{O}_X(U_\alpha)$, it is in $\pi_*\mathcal{O}_{\tilde{X}}(U_\alpha)$, and the product $h_\alpha^i d^i f_j/dt^i$ is thus in $\mathcal{O}_X(U_\alpha)$. Proceeding in this way on each U_α, we obtain local sections ρ_α and it is not hard to show, using properties of determinants as in [19, Proposition 1.4], that the ρ_α patch to define a section $\rho \in H^0(X, \mathcal{L}^{\otimes s} \otimes \omega^{\otimes N})$, which we refer to as a wronskian. It follows from the linear independence of $f_1, f_2, \ldots, f_s$ that ρ is not identically zero. A similar definition for the wronskian may be given in positive characteristic (see [5]) using the iterative (Hasse-Schmidt) derivative with respect to t and taking into account the order sequence of the linear system V.

If $P \in X$ and if ψ generates $\mathcal{L}_P$ and τ generates ω_P, then we may write $\rho = f\psi^s\tau^N$ for some nonzero $f \in \mathcal{O}_P$. We define $\operatorname{ord}_P\rho$ to be $\operatorname{ord}_P f = \dim \mathcal{O}_P/(f)$. This order of vanishing is independent of the choices of the basis of V and the generators for $\mathcal{L}_P$ and ω_P.

(1.1) Definitions. Put $W_V(P) = \operatorname{ord}_P\rho$ and call this number the V-Weierstrass weight of P. The point P is called a V-Weierstrass point if $W_V(P) > 0$. If $V = H^0(X, \mathcal{L})$, then we write $W_\mathcal{L}(P)$ for $W_V(P)$ and a V-Weierstrass point is called an $\mathcal{L}$-Weierstrass point. The Weierstrass points of X are the ω-Weierstrass points. We will write $W_X(P)$, or simply $W(P)$ if it is clear to what curve we are referring, instead of $W_\omega(P)$.

This definition may be viewed as a generalization of the definitions of Matzat [15], Laksov [9], and Stöhr-Voloch [19] to the Gorenstein case. Recently Laksov and Thorup [10,11] have given a more general definition of Weierstrass points of "Wronski systems," and our definition may be viewed as a concrete realization in our setting of their rather abstract definition.

Let g denote the arithmetic genus of X.

(1.2) Proposition. *The number of V-Weierstrass points, counting multiplicities, is $s \deg \mathcal{L} + (g-1)(s-1)s$.*

Proof. This number is the degree of $\mathcal{L}^{\otimes s} \otimes \omega^{\otimes N}$. ∎

Let $\tilde{\mathcal{O}}_P$ denote the normalization of $\mathcal{O}_P$. Put $\delta_P = \dim_k \tilde{\mathcal{O}}_P/\mathcal{O}_P$. One often calls δ_P the degree of the singularity at P. Suppose that $\tau \in H^0(X,\omega)$ generates ω_P. Then, locally at P, we may write τ in the form $\tau = dt/h$, where t is a rational function such that $\mathrm{ord}_Q t = 1$ for all Q on $\tilde{X}$ lying over P, and where h is some generator (in $\tilde{\mathcal{O}}_P$) of the conductor ideal of $\mathcal{O}_P$ in $\tilde{\mathcal{O}}_P$. Since $\mathcal{O}_P$ is a Gorenstein ring, we have $\mathrm{ord}_P h = 2\delta_P$.

(1.3) Proposition. $W_X(P) = \delta_P(g-1)g + \mathrm{ord}_P \det(d^i f_j/dt^i)$, *where* $i = 0,1,\ldots,g-1$ *and* $j = 1,2,\ldots,g$.

Proof. Let $f_1\tau, f_2\tau, \ldots, f_g\tau$ be local representations at P for a basis for $H^0(X,\omega)$. Then we have

$$\begin{aligned} W_X(P) &= \mathrm{ord}_P \det(h^i d^i f_j/dt^i) \\ &= \mathrm{ord}_P h^{(g-1)g/2} + \mathrm{ord}_P \det(d^i f_j/dt^i) \\ &= \delta_P(g-1)g + \mathrm{ord}_P \det(d^i f_j/dt^i). \end{aligned}$$ ∎

In particular, if $N > 0$ and if P is a singular point, then we have $W_X(P) > 0$, since $\delta_P > 0$.

2. In this section, we will consider unibranch singularities.

(2.1) Definition. By a *numerical semigroup* we mean a subsemigroup of the natural numbers (under addition) with finite complement. The elements in the complement set are called the *gaps* of the semigroup. If S is a numerical semigroup with gaps $l_1, l_2, \ldots, l_\delta$, then we define the *weight of S*, denoted $wt(S)$, by

$$wt(S) = \sum_{j=1}^{\delta} (l_j - j).$$

We define the weight of $\mathbf{N}$ to be 0. The element $c = l_\delta + 1$ is called the *conductor* of the semigroup.

(2.2) Remark. We always have $l_\delta \le 2\delta - 1$ (and the equality occurs if and only if S is symmetric). To see this, note that if c is the conductor of S and if $x + y = c - 1 = l_\delta$, then either x or y must be a gap. Hence among the nonnegative integers less than c, there are at least as many gaps as there are nongaps.

(2.3) Lemma. *If S is a numerical semigroup with δ gaps and if $0 = n_0, n_1, \ldots, n_{\delta-1}$ are the elements of S less than 2δ, then*

$$\sum_{i=0}^{\delta-1}(n_i - i) = (\delta - 1)\delta - wt(S).$$

Proof. Let $l_1, l_2, \ldots, l_\delta$ denote the gaps of S. We have

$$\sum_{j=1}^{\delta} l_j + \sum_{i=0}^{\delta-1} n_i = \sum_{k=0}^{2\delta-1} k = \delta(2\delta - 1).$$

Hence

$$\begin{aligned} wt(S) &= \sum_{j=1}^{\delta} l_j - \delta(\delta+1)/2 \\ &= (\delta - 1)\delta - \sum_{i=0}^{\delta-1}(n_i - i). \ \blacksquare \end{aligned}$$

The next theorem treats the weight of a unibranch singularity on a Gorenstein curve. This theorem generalizes a result of C. Widland [20,13] in the case of a simple cusp. We note that E. Kunz [8] showed that a unibranch singularity is Gorenstein if and only if the semigroup of values associated to the singularity is a symmetric numerical semigroup. We recall that if P is a singular point of X, then by the partial normalization of X at P one means the curve obtained from X by desingularizing only the singularity P.

(2.4) Theorem. *Suppose X is a Gorenstein curve of arithmetic genus g. Suppose $P \in X$ is a unibranch singularity. Let S denote the semigroup of values at P. Let Y denote the partial normalization of X at P and let Q denote the point of Y that lies over P. Then*

$$W_X(P) = \delta_P(g-1)(g+1) - wt(S) + W_Y(Q).$$

Proof. Put $\delta = \delta_P$. Let $\sigma_1, \sigma_2, \ldots, \sigma_{g-\delta}$ be a basis of $H^0(Y, \omega_Y)$. Locally at Q, write $\sigma_i = f_i\, dt$, for $i = 1, \ldots, g - \delta$, where t is a local coordinate on Y centered at Q. Put $r_i = \mathrm{ord}_Q f_i$. We will assume that the basis of differentials has been chosen so that $r_1 < r_2 < \cdots < r_{g-\delta}$.

Let $\sigma \in H^0(X, \omega)$. Locally at P, write $\sigma = dt/h$, where t is a local coordinate centered at Q. Let $\mathrm{ord}_P h = m + 1$. If $m \in S$, then there would be a function $f \in \mathcal{O}_P$ such that $\mathrm{ord}_P f = m$; but then the residue of $f\sigma$ at Q would not be zero. Therefore, m must be a gap of S. It follows that there exist linearly independent dualizing differentials $\tau_1, \tau_2, \ldots, \tau_\delta$ such that if we write $\tau_j = dt/h_j$, then $\mathrm{ord}_P h_j = l_j + 1$, where the $l_j, j = 1, \ldots, \delta$, are the gaps of S. Note that $l_\delta + 1 = c$, the conductor of S. For a generator of ω_P, we may take $\tau = \tau_\delta = dt/h_\delta$. Put $h = h_\delta$.

The differentials

$$\tau_1, \tau_2, \ldots, \tau_\delta, \sigma_1, \sigma_2, \ldots, \sigma_{g-\delta}$$

then form a basis for $H^0(X,\omega_X)$. As in the proof of Proposition (1.3), we have

$$W_X(P) = \delta(g-1)g + \mathrm{ord}_Q W_t(1, h/h_{\delta-1}, h/h_{\delta-2}, \dots, h/h_1, hf_1, hf_2, \dots, hf_{g-\delta}),$$

where W_t denotes the ordinary wronskian (obtained by differentiating with respect to t) of the given functions and where we have used the fact that the order of a function in $\mathcal{O}_P$ is the same at P as it is at Q. Notice that each of the functions $1, h/h_{\delta-1}, \dots, h/h_1$, $hf_1, \dots, hf_{g-\delta}$ has a different order at Q. Indeed, we have $\mathrm{ord}_Q h/h_{\delta-i} = c - (l_{\delta-i}+1) = n_i$, where $n_0, n_1, \dots, n_{\delta-1}$ are the elements in S that are less than 2δ, and $\mathrm{ord}_Q hf_j = c + r_j = 2\delta + r_j$. Hence the order of the wronskian of these functions at Q may be easily computed as in [3, p.82]. We have

$$\begin{aligned}\mathrm{ord}_Q W_t(1, h/h_{\delta-1}, \dots, hf_{g-\delta}) &= \sum_{i=0}^{\delta-1}(n_i - i) + \sum_{j=1}^{g-\delta}(2\delta + r_j - (\delta + j - 1)) \\ &= \sum_{i=0}^{\delta-1}(n_i - i) + \delta(g-\delta) + \sum_{j=1}^{g-\delta} r_j - j + 1 \\ &= \sum_{i=0}^{\delta-1}(n_i - i) + \delta(g-\delta) + W_Y(Q).\end{aligned}$$

The Theorem now follows from Lemma (2.3). ∎

In [5], we gave the following definition.

(2.5) Definition. Suppose X is a rational curve and P is a unibranch singularity on X with semigroup of values S. Suppose the conductor of S is c. Let $0, n_1, n_2, \dots, n_r$ denote the nonnegative integers less than c that are in S. We will call P a monomial unibranch singularity if the local ring $\mathcal{O}_P$ at P is of the form

$$\mathcal{O}_P = k + kt^{n_1} + kt^{n_2} + \cdots + kt^{n_r} + t^c\tilde{\mathcal{O}}_P,$$

where $\tilde{\mathcal{O}}_P$ denotes the normalization of $\mathcal{O}_P$ and where t is a uniformizing parameter of $\tilde{\mathcal{O}}_P$ that generates the function field of X.

The following result about a rational curve with a single unibranch singularity follows from Theorem (2.3) in [5].

(2.6) Theorem. *Let S be a symmetric numerical semigroup. Suppose X is a rational curve of arithmetic genus g with a unique singular point P such that P is a monomial unibranch singularity with semigroup of values S. Then we have:*

$$\begin{aligned} W(P) &= g^3 - g - wt(S) \\ W(P_\infty) &= wt(S), \end{aligned}$$

where P_∞ represents the pole of the function t appearing in Definition (2.5). Moreover, there are no other Weierstrass points.

Using Theorems (2.4) and (2.6), one may compute the weight of each singularity on a rational curve with precisely two monomial unibranch singularities.

(2.7) Theorem. *Suppose X is a rational Gorenstein curve of arithmetic genus g with two monomial unibranch singularities P_1 and P_2 as its only singularities. For $i = 1, 2$ put $\delta_i = \delta_{P_i}$, let S_i denote the semigroup of values at P_i, and let t_i denote the uniformizing parameter at P_i that generates the function field of X used to define the local ring at P_i as in Definition (2.5). Let Q_1 and Q_2 be the points on $\mathbf{P}^1$ lying over P_1 and P_2, respectively. Then*

1) If Q_1 and Q_2 are the poles of the functions t_2 and t_1, respectively (i.e., if t_1t_2 is a nonzero constant), then

$$W_X(P_1) = \delta_1(g-1)(g+1) - wt(S_1) + wt(S_2)$$
$$W_X(P_2) = \delta_2(g-1)(g+1) - wt(S_2) + wt(S_1),$$

and there are no smooth Weierstrass points on X.
2) If Q_1 is the pole of the function t_2, but Q_2 is not the pole of the function t_1, then

$$W_X(P_1) = \delta_1(g-1)(g+1) - wt(S_1) + wt(S_2)$$
$$W_X(P_2) = \delta_2(g-1)(g+1) - wt(S_2),$$

and there are $wt(S_1)$ smooth Weierstrass points on X, counting multiplicities.
3) If Q_1 and Q_2 are not the poles of the functions t_2 and t_1, respectively, then

$$W_X(P_1) = \delta_1(g-1)(g+1) - wt(S_1)$$
$$W_X(P_2) = \delta_2(g-1)(g+1) - wt(S_2),$$

and there are $wt(S_1) + wt(S_2)$ smooth Weierstrass points on X, counting multiplicities.

Proof. By Theorem (2.4) we have

$$W_X(P_1) = \delta_1(g-1)(g+1) - wt(S_1) + W_{Y_1}(Q_1),$$

where Y_1 is the partial normalization of X at P_1 and Q_1 is the point on Y_1 that lies over P_1. Now, Y_1 is a rational curve with the unique monomial unibranch singularity P_2. Hence we see from Theorem (2.6) that $W_{Y_1}(Q_1) = wt(S_2)$ if Q_1 is the pole of the function t_2 and is 0 otherwise. A similar argument holds with regard to $W_X(P_2)$. The assertions about the number of smooth Weierstrass points on X follow by adding the weights of P_1 and P_2 and subtracting from $g^3 - g$, which is the total of all the weights. Note that $g = \delta_1 + \delta_2$. ∎

(2.8) Example. Suppose that X is the rational curve obtained from $\mathbf{P}^1$ by creating two monomial unibranch singularities P_0 and P_1 at 0 and 1, each with semigroup of values generated by 3 and 4. Then X has arithmetic genus 6 and the Weierstrass weight of each singularity is 103. The total Weierstrass weight is 210, and it may be seen, by computing the wronskian on the smooth locus of X, that there are four distinct smooth Weierstrass points (each of weight one). We note that the point at infinity is not a Weierstrass point on X, but it is a Weierstrass point on Y_1, the partial normalization of X at P_1. The existence of a function with a zero of order 3 at P_0 shows that 3 is a nongap at infinity on the curve Y_1, but this function is not regular at P_1 on X and 3 is not a nongap at infinity on X.

The situation in part (3) of Theorem (2.7) may be generalized as follows.

(2.9) Theorem. *Suppose that X is a Gorenstein curve of arithmetic genus g and geometric genus $\tilde{g}$ with unibranch singularities $P_1, P_2, \ldots, P_n$ as its only singularities. For $i = 1, 2, \ldots, n$, let S_i denote the semigroup of values at P_i, and assume that the point Q_i that lies over P_i is not a Weierstrass point of the partial normalization Y_i of X at P_i. Then the number of smooth Weierstrass points on X, counting multiplicities, is $\tilde{g}(g-1)(g+1) + \sum_{i=1}^{n} wt(S_i)$.*

Proof. Since by hypothesis $W_{Y_i}(Q_i) = 0$, it follows from Theorem (2.4) that the number of smooth Weierstrass points on X, counting multiplicities, is

$$g^3 - g - \sum_{i=1}^{n} W_X(P_i) = g^3 - g - \sum_{i=1}^{n} (\delta_i(g-1)(g+1) - wt(S_i))$$
$$= \tilde{g}(g-1)(g+1) + \sum_{i=1}^{n} wt(S_i),$$

since $g = \tilde{g} + \sum_{i=1}^{n} \delta_i$. ∎

3. We now consider singularities with precisely two branches. Let X be an integral, projective Gorenstein curve over an algebraically closed field of characteristic zero. Suppose $P \in X$ is a singularity with precisely two branches. Let $\pi : Y \to X$ denote the partial normalization of X at P and let $Q_1, Q_2 \in Y$ denote the points lying over P. Let ν_1 and ν_2 denote the discrete valuations associated to Q_1 and Q_2, respectively. The value semigroup S at P is given by

$$S = \{(\nu_1(f), \nu_2(f)) \in \mathrm{N} \times \mathrm{N} : f \in \mathcal{O}_P, f \neq 0\}.$$

Such semigroups have been studied by, among others, the first author [4] in the case of plane curves, and by F. Delgado in the cases of plane curves [1] and Gorenstein curves [2] with an arbitrary number of branches. Let $\xi = (\xi_1, \xi_2)$ denote the conductor of S; i.e., ξ is the minimum element in S, with respect to the product order on $\mathrm{N} \times \mathrm{N}$, such that $\xi + \mathrm{N} \times \mathrm{N} \subseteq S$.

We think of the semigroup S as being a set of points in the plane. Let $S_i = \pi_i(S)$, for $i = 1, 2$, denote the projections of S onto the coordinate axes. For $i = 1, 2$, let δ_i denote the number of gaps of S_i and let c_i denote the conductor of S_i. We note that S_1 and S_2 need not be symmetric semigroups (see Example (3.10) below).

Since $\mathcal{O}_P$ is a Gorenstein ring, the semigroup S has certain symmetry properties, which we now recall.

(3.1) Definitions. For $x \in \mathrm{N}$, the vertical fiber at x, denoted $\mathrm{VF}(x)$, is defined by

$$\mathrm{VF}(x) = \{(x, y') \in \mathrm{N} \times \mathrm{N} : (x, y') \in S\}.$$

For $y \in \mathrm{N}$, the horizontal fiber at y, denoted $\mathrm{HF}(y)$, is defined similarly. A point (x_1, y_1) is said to be above (resp. to the right of) another point (x_2, y_2) if $x_1 = x_2$ and $y_1 > y_2$ (resp. if $y_1 = y_2$ and $x_1 > x_2$). Put

$$\Delta((x, y)) = \{(x', y') \in S : (x', y') \text{ is either above or to the right of } (x, y)\}.$$

A point $(x, y) \in S$ is called a *maximal point* (or simply a *maximal*) if $\Delta((x, y)) = \emptyset$.

(3.2) Lemma.

1) Suppose $n \in S_1$ *and* $n < \xi_1$. *Then*

$$(n, \xi_2) \in S \Leftrightarrow \xi_1 - 1 - n \notin S_1 \Leftrightarrow VF(n) \textit{ is infinite.}$$

2) Suppose $n \in S_2$ *and* $n < \xi_2$. *Then*

$$(\xi_1, n) \in S \Leftrightarrow \xi_2 - 1 - n \notin S_2 \Leftrightarrow HF(n) \textit{ is infinite.}$$

Proof. [2, Lemma (1.8) and Theorem (2.3)] ∎

Put $\mu = (\xi_1 - 1, \xi_2 - 1)$. From [2, Corollary (2.7)], we have that μ is a maximal point in S. This point plays a role in S analogous to the number $c - 1$ in a symmetric numerical semigroup. More precisely, one has the following result.

(3.3) Proposition. *(Symmetry properties of* S*). The semigroup* S *has the following symmetry properties.*

1) For any $(x, y) \in \mathbf{Z} \times \mathbf{Z}$,

$$(x, y) \in S \Leftrightarrow \Delta(\mu - (x, y)) = \emptyset.$$

2) For any $(x, y) \in \mathbf{N} \times \mathbf{N}$,

$$(x, y) \textit{ is a maximal of } S \Leftrightarrow \mu - (x, y) \textit{ is a maximal of } S.$$

Proof. Delgado [2, Theorem (2.8)] establishes the first property above and notes ([2, Remark (2.9)]) that the second property also holds. ∎

(3.4) Lemma.

1) Suppose $n < \xi_1$. *Then* $VF(n)$ *is infinite if and only if* $\xi_1 - 1 - n$ *is a gap of* S_1.
2) Suppose $n < \xi_2$. *Then* $HF(n)$ *is infinite if and only if* $\xi_2 - 1 - n$ *is a gap of* S_2.

Proof. Suppose VF(n) is infinite, with $n < \xi_1$. Then there exists a point $(n, y) \in S$ with $y > \xi_2$. By adding the function corresponding to this point with the function corresponding to (ξ_1, ξ_2), we see that $(n, \xi_2) \in S$. Therefore, by Lemma (3.2), we have that $\xi_1 - 1 - n \notin S_1$. Conversely, if $n < \xi_1$ and $\xi_1 - 1 - n \notin S_1$, then we claim that $n \in S_1$. For consider the point $\alpha = (n, \xi_2 - c_2)$. Then $\mu - \alpha = (\xi_1 - 1 - n, c_2 - 1)$. But $\xi_1 - 1 - n \notin S_1$ and $c_2 - 1 \notin S_2$, so $\Delta(\mu - \alpha) = \emptyset$. It follows from Proposition (3.3) that $\alpha \in S$ and so $n \in S_1$. Hence, if $n < \xi_1$ and $\xi_1 - 1 - n \notin S_1$, we can conclude from Lemma (3.2) that VF(n) is infinite.

The proof of (2) is similar. ∎

(3.5) Proposition. *The symmetry properties in Proposition (3.3) are equivalent.*

Proof. (1) $\Rightarrow$ (2): By the symmetrical form of statement (2) in Proposition (3.3), it suffices to show that if (x, y) is a maximal of S, then $\mu - (x, y)$ is also a maximal of S. Now, if (x, y) is a maximal of S, then $(x, y) \in S$ and $\Delta((x, y)) = \emptyset$. But then, by applying (1) of Proposition (3.3) in both directions, we see that $\Delta(\mu - (x, y)) = \emptyset$ and $\mu - (x, y) \in S$. Therefore, $\mu - (x, y)$ is a maximal of S.

(2) $\Rightarrow$ (1): Assume $(x, y) \in S$. Suppose there exists a point in S above $\mu - (x, y)$. (A similar argument applies if there exists a point in S to the right of $\mu - (x, y)$.) But then $(x, y) + (\mu_1 - x, \mu_2 - y + z)$, for some $z > 0$, is in S, contradicting the fact that μ is a maximal point.

Conversely, suppose that $\Delta(\mu - (x, y)) = \emptyset$. Since $\mathrm{VF}(\mu_1 - x)$ and $\mathrm{HF}(\mu_2 - y)$ are then finite, it follows, from Lemma (3.4), that $x \in S_1$ and $y \in S_2$. We have then two possibilities: either $\mathrm{VF}(\mu_1 - x)$ is empty or nonempty. If $\mathrm{VF}(\mu_1 - x)$ is empty, it follows from Lemma (3.4) that $\mathrm{VF}(x)$ is infinite. If $\mathrm{VF}(\mu_1 - x)$ is nonempty, then the maximal point of this fiber, call it $(\mu_1 - x, z)$, satisfies $z \leq \mu_2 - y$. Hence, applying (2) of Proposition (3.3), we have a maximal point of the form $(x, \mu_2 - z)$ with $\mu_2 - z \geq y$. So, in any case, one has a point in the semigroup S of the form (x, y') with $y' \geq y$. Similarly, one has a point in S of the form (x', y) with $x' \geq x$. If $y' = y$ or $x' = x$, we are finished. We can then assume that $y' > y$ and $x' > x$. Then the sum of the functions in the local ring $\mathcal{O}_P$ corresponding to (x', y) and (x, y') is a function f satisfying $\nu_1(f) = x$ and $\nu_2(f) = y$, showing that $(x, y) \in S$. ∎

Put I equal to the number of maximal points in S. From [4], we have that if X is a plane curve, then I is also equal to the intersection number of the two branches and the conductor of S is $(I + 2\delta_1, I + 2\delta_2)$. We now show that these results also hold for any Gorenstein curve.

(3.6) Proposition. *The coordinates of the conductor of S are $\xi_1 = I + 2\delta_1, \xi_2 = I + 2\delta_2$.*

Proof. Consider the vertical fibers $\mathrm{VF}(x)$ for $0 \leq x < \xi_1$. We will count how many of these fibers are infinite, empty, or finite and nonempty. From Lemma (3.4), we see that the number of these vertical fibers that are infinite is δ_1. The number of empty vertical fibers is also equal to δ_1. The number of nonempty finite fibers is equal to I, the number of maximal points. Therefore, $\xi_1 = I + 2\delta_1$. A similar argument using horizontal fibers shows that $\xi_2 = I + 2\delta_2$. ∎

(3.7) Corollary. *$\delta_P = I + \delta_1 + \delta_2$ and I is the intersection number of the two branches at P.*

Proof. From Proposition (3.6) and the fact that $\mathcal{O}_P$ is Gorenstein, we have $2\delta_P = 2I + 2\delta_1 + 2\delta_2$. Therefore, $\delta_P = I + \delta_1 + \delta_2$. It then follows from [7, Proposition 4] that I is the intersection number of the two branches at P. ∎

(3.8) Corollary. *Suppose the maximal points of S are*

$$(a_0, b_0), (a_1, b_1), \ldots, (a_{I-1}, b_{I-1}).$$

Then we have

$$\sum_{i=0}^{I-1} a_i = I(I-1)/2 + \delta_1 I$$

$$\sum_{i=0}^{I-1} b_i = I(I-1)/2 + \delta_2 I.$$

Proof. By Propositions (3.3) and (3.6), a_i is the first coordinate of a maximal point if and only if $I+2\delta_1-1-a_i$ is also the first coordinate of a maximal. Hence we have

$$\sum_{i=0}^{I-1} a_i = I(I+2\delta_1-1) - \sum_{i=0}^{I-1} a_i,$$

and the first equality in the statement of the Corollary follows. A similar argument applies to the second coordinates of the maximal points. ∎

Consider the rectangle (with one vertex deleted)

$$R = \{(x,y) \in \mathrm{N}\times\mathrm{N} : x \le \xi_1 \text{ and } y < \xi_2 \text{ or } x < \xi_1 \text{ and } y \le \xi_2\}.$$

It follows from Lemma (3.4) that the points in S that are on the top edge of R are of the form $(\xi_1 - 1 - l, \xi_2)$, where l is a gap of S_1, and the points in S that are on the right edge of R are of the form $(\xi_1, \xi_2 - 1 - l')$, where l' is a gap of S_2. If S_1 and S_2 are symmetric, then one can write these points in a nicer form.

(3.9) Corollary.

1) If S_1 is symmetric and if $m_0, m_1, \dots m_{\delta_1 - 1}$ are the elements in S_1 that are less than c_1, then the points that are in S and on the upper edge of the rectangle R are the points $(I+m_j, \xi_2), j = 0,1,\dots,\delta_1 - 1$.

2) If S_2 is symmetric and if $n_0, n_1, \dots, n_{\delta_2 - 1}$ are the elements in S_2 that are less than c_2, then the points that are in S and on the right edge of R are the points $(\xi_1, I+n_k), k = 0,1,\dots,\delta_2 - 1$.

Proof. Suppose S_1 is symmetric. Then $c_1 = 2\delta_1$ and $c_1 - 1 - n \in S_1$ if and only if $n \notin S_1$. Therefore, from Lemma (3.4) and Proposition (3.6),

$$n \notin S_1 \Leftrightarrow \mathrm{VF}(I + c_1 - 1 - n) \text{ is infinite.}$$

Thus, the infinite vertical fibers are of the form $\mathrm{VF}(I+m)$, where $m \in S_1$. The analogous statement for S_2 is proved similarly. ∎

We thank Professor K.-O. Stöhr for the following example of a two-branch Gorenstein singularity having one branch that is not Gorenstein.

(3.10) Example. Let H be a numerical semigroup with g gaps $l_1, l_2, \dots, l_g$ such that $l_g = 2g-2$. Clearly, $(g-1) \notin H$. Take $S_1 = (g-1)\mathrm{N} + H$. We claim that

$$S_1 = H \cup \{g-1, 2g-2\};$$

i.e., that S_1 has $g-2$ gaps. In fact, if α belongs to S_1 but not to $H \cup \{g-1, 2g-2\}$, then we can write $\alpha = (g-1) + h$, for some $h \in H$. Since $\alpha \neq g-1$ and α is a gap of H we have, from [16, Prop. 1.2], that

$$2g - 2 - \alpha = g - 1 - h = h_1 \in H.$$

Hence $g - 1 = h + h_1 \in H$, a contradiction.

Take $\mathcal{O}_P \subseteq \tilde{\mathcal{O}}_P = k[[t]] \times k[[u]]$ to be the local ring given by $\mathcal{O}_P =$

$$\left\{ \left(\sum_{i=0}^{\infty} a_i t^i, \sum_{i=0}^{\infty} b_i u^i \right) : a_0 = b_0, a_{g-1} = b_1, a_{2g-2} = b_2, a_i = 0 \text{ for all } i \notin S_1 \right\}.$$

Since S_1 has $(g-2)$ gaps, we have

$$\delta_P = \dim_k \tilde{\mathcal{O}}_P / \mathcal{O}_P = (g-2) + 3 = g + 1.$$

Clearly, the conductor ideal C of $\mathcal{O}_P$ in $\tilde{\mathcal{O}}_P$ is

$$C = t^{2g-1}\, k[[t]] \times u^3\, k[[u]]$$

and $\dim \tilde{\mathcal{O}}_P / C = 2g-1+3 = 2g+2$. This shows that $\mathcal{O}_P$ is Gorenstein. However, the semigroup of the first branch, namely S_1, is not symmetric if $3 \notin H$. In fact, if S_1 is symmetric, then its conductor c_1 satisfies $c_1 = 2(g-2) = 2g-4$. Hence $2g-5$ is a gap of S_1 and of H. Again by [16, Prop. 1.2], we have $2g-2-(2g-5) = 3 \in H$.

We now want to describe how to find a basis of dualizing differentials on X that have certain orders at P. This process is analogous to finding a "Hermitian" basis (or basis "adapted to a point") of regular differentials at a point on a smooth curve. We want to show that we can choose linearly independent dualizing differentials on X whose orders at P are related to the maximal points of S and the points of S that lie on the upper edge and right edge of the rectangle R. These differentials will be those dualizing differentials in a "Hermitian" basis at P that are not regular on Y either at Q_1 or at Q_2 (or at both points if the differential corresponds to a maximal point of S).

We will use the following Riemann-Roch Theorem for zero-dimensional subschemes on a Gorenstein curve, which was proved in [21] (also cf. [6]). If J is a proper ideal of $\mathcal{O}_P$, we let $\mathcal{I}(J)$ denote the sheaf of $\mathcal{O}_X$-ideals defined by $\mathcal{I}(J)_P = J$ and $\mathcal{I}(J)_Q = \mathcal{O}_Q$ for all $Q \neq P$. Put

$$\begin{aligned} h(J) &= \dim_k \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}(J), \mathcal{O}_X) \\ \iota(J) &= \dim_k H^0(X, \mathcal{I}(J) \otimes \omega) \\ d(J) &= \dim_k \mathcal{O}_P / J. \end{aligned}$$

The elements of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}(J), \mathcal{O}_X)$ may be identified with rational funtions f on X such that $fJ \subseteq \mathcal{O}_P$ and $f \in \mathcal{O}_Q$ for all $Q \neq P$.

(3.11) Theorem. $h(J) - \iota(J) = d(J) + 1 - g.$

Let C denote the conductor of $\mathcal{O}_P$ in its normalization $\tilde{\mathcal{O}}_P$.

(3.12) Lemma. $h(C) = 1$.

Proof. This follows from the fact that $\mathrm{Hom}_{\mathcal{O}_P}(C, \mathcal{O}_P) = \tilde{\mathcal{O}}_P$ (cf. the proof of Proposition (2.2) of [14]). ∎

(3.13) Proposition. *Suppose $\tau \in H^0(X, \omega)$ generates ω_P. Suppose that*

$$\mathcal{O}_P = J_0 \supset J_1 \supset \cdots \supset J_n \supseteq C$$

is a strictly decreasing chain of $\mathcal{O}_P$-ideals such that $d(J_i) = i$ for $i = 0, 1, \ldots, n$. Then there exist n linearly independent dualizing differentials $\tau_1, \tau_2, \ldots, \tau_n$ in $H^0(X, \omega)$ such that, locally at P, we have $\tau_i = f_i \tau$ with $f_i \in J_{i-1} \setminus J_i$ for $i = 1, 2, \ldots, n$.

Proof. Since $J_i \supseteq C$ and $h(C) = 1$, it follows that $h(J_i) = 1$ for all i. Therefore, from Theorem (3.11), we have $\iota(J_i) = \iota(J_{i-1}) - 1$ for $i = 1, 2, \ldots, n$. Thus, there exists $\tau_i \in H^0(X, \mathcal{I}(J_{i-1}) \otimes \omega) \setminus H^0(X, \mathcal{I}(J_i) \otimes \omega)$ for $i = 1, 2, \ldots, n$. Then, locally at P, we have $\tau_i = f_i \tau$ for some $f_i \in J_{i-1} \setminus J_i$. ∎

(3.14) Definition. For $(x, y) \in \mathbf{N} \times \mathbf{N}$, put

$$J(x, y) = \{f \in \mathcal{O}_P : \nu_1(f) \geq x \text{ and } \nu_2(f) \geq y\}.$$

(3.15) Definition. Suppose $\sigma \in H^0(X, \omega)$ and τ generates ω_P. Locally at P, write $\sigma = f\tau$, with $f \in \mathcal{O}_P$. Then put $\nu_1(\sigma) = \nu_1(f)$ and $\nu_2(\sigma) = \nu_2(f)$.

(3.16) Theorem. *There exist δ_P linearly independent dualizing differentials*

$$\tau_0, \tau_1, \ldots, \tau_{\delta_P - 1} \in H^0(X, \omega)$$

such that

1) For each maximal point $(a, b) \in S$, there is a $\tau_i, 0 \leq i \leq I - 1$, such that $\nu_1(\tau_i) = a$, and $\nu_2(\tau_i) = b$.

2) For each point in S of the form (r, ξ_2), with $r < \xi_1$, there is a $\tau_j, I \leq j \leq I + \delta_1 - 1$, such that $\nu_1(\tau_j) = r$, and $\nu_2(\tau_j) \geq \xi_2$.

3) For each point in S of the form (ξ_1, s), with $s < \xi_2$, there is a $\tau_k, I + \delta_1 \leq k \leq \delta - 1$, such that $\nu_1(\tau_k) \geq \xi_1$ and $\nu_2(\tau_k) = s$.

Proof. Let $0 = x_1, x_2, \ldots, x_{I+\delta_1}$ be the nonnegative integers such that $x_k < \xi_1$ and $VF(x_k) \neq \emptyset$. Let $(\xi_1, s_0), (\xi_1, s_1), \ldots, (\xi_1, s_{\delta_2 - 1})$ denote the points in S on the right edge of the rectangle R. (These points correspond to infinite horizontal fibers that lie below the line $y = \xi_2$.) Let C denote the conductor of $\mathcal{O}_P$ in its normalization. Consider the following chain of ideals in $\mathcal{O}_P$:

$$\begin{aligned} &J(0,0) \supset J(x_2, 0) \supset \cdots \supset J(x_{I+\delta_1}, 0) \supset \\ &J(\xi_1, s_0) \supset J(\xi_1, s_1) \supset \cdots \supset J(\xi_1, s_{\delta_2 - 1}) \supset C. \end{aligned} \qquad (*)$$

This is a proper chain of ideals as in Proposition (3.13). Hence there exist δ_P linearly independent dualizing differentials $\sigma_0, \sigma_1, \ldots, \sigma_{\delta_P - 1}$ as in Proposition (3.13).

The last δ_2 of these differentials, call them $\tau_{I+\delta_1}, \ldots, \tau_{\delta_P - 1}$, satisfy condition (3) in the statement of the Theorem.

In a similar manner, we may find δ_1 differentials, call them $\tau_I, \ldots, \tau_{I+\delta_1-1}$, satisfying condition (2) in the statement of the Theorem.

Suppose the maximal points of S are

$$(a_0, b_0) < (a_1, b_1) < \cdots < (a_{I-1}, b_{I-1}),$$

ordered lexicographically. One of the differentials, call it σ, that we found using the chain $(*)$ above satisfies

$$\nu_1(\sigma) = a_{I-1}, \nu_2(\sigma) \leq b_{I-1}.$$

If $\nu_2(\sigma) \neq b_{I-1}$, then $\nu_2(\sigma) = s_k$ for some $k, 0 \leq k \leq \delta_2 - 1$. In that case, a suitable linear combination of σ and the differential $\tau_{I+\delta_1+k}$ will yield a differential $\bar{\sigma}$ such that

$$\nu_1(\bar{\sigma}) = a_{I-1} \text{ and } \nu_2(\bar{\sigma}) > \nu_2(\sigma).$$

If $\nu_2(\bar{\sigma}) = b_{I-1}$, then $\bar{\sigma}$ is one of the differentials we need to satisfy condition (1) in the statement of the Theorem and we will put $\tau_{I-1} = \bar{\sigma}$. If not, then $\nu_2(\bar{\sigma}) = s_{k'}$ for some k' with $k < k' \leq \delta_2 - 1$. Then, by adding a suitable multiple of $\tau_{I+\delta_1+k'}$, we obtain a differential with a greater order on the second branch (while leaving the order on the first branch unchanged). In this way, we obtain a differential, call it τ_{I-1}, such that $\nu_1(\tau_{I-1}) = a_{I-1}$ and $\nu_2(\tau_{I-1}) = b_{I-1}$.

We continue by (descending) induction, assuming that we have found the differentials $\tau_{I-t+1}, \ldots, \tau_{I-1}$ corresponding to the maximal points (a_{I-t+1}, b_{I-t+1}), $\ldots, (a_{I-1}, b_{I-1})$. Consider the maximal point (a_{I-t}, b_{I-t}). One of the differentials we found above using chain $(*)$, call it ρ, satisfies $\nu_1(\rho) = a_{I-t}$. If $\nu_2(\rho) \neq b_{I-t}$, then we add to ρ a suitable multiple of either $\tau_{I+\delta_1+k}$ if $\nu_2(\rho) = s_k$ for some k, or τ_{I-u} if $\nu_2(\rho) = b_{I-u}$ for some $u, 1 \leq u \leq t-1$. Continuing in this way, we can increase the order of the differential on the second branch, without changing the order on the first branch, until we obtain a differential τ_{I-t} such that $\nu_1(\tau_{I-t}) = a_{I-t}$ and $\nu_2(\tau_{I-t}) = b_{I-t}$. By this inductive process, we obtain differentials $\tau_0, \ldots, \tau_{I-1}$ satisfying condition (1) in the statement of the Theorem.

The differentials $\tau_0, \tau_1, \ldots, \tau_{\delta-1}$ are easily seen to be linearly independent by considering their orders on the two branches at P. ∎

A basis of g linearly independent dualizing differentials on X may be obtained by taking the union of a basis of $g - \delta_P$ dualizing differentials on Y and the δ_P differentials in Theorem (3.16). We will divide such a basis into four subsets and will use the following notation. Let

$$\tau_0, \tau_1, \ldots, \tau_{I-1}$$

denote the differentials corresponding, as in (1) of Theorem (3.16), to the maximal points of S. Let

$$\zeta_0, \zeta_1, \ldots, \zeta_{\delta_1 - 1}$$

denote the differentials corresponding, as in (2) of Theorem (3.16), to certain points in S with first coordinate $\xi_1 - 1 - l$, where l is a gap of S_1. Note that on Y each of the ζ_j's is regular at Q_2 and has a pole at Q_1. Let

$$\eta_0, \eta_1, \ldots, \eta_{\delta_2 - 1}$$

denote the differentials corresponding, as in (3) of Theorem (3.16), to certain points in S with second coordinate $\xi_2 - 1 - l'$, where l' is a gap of S_2. On Y, each of the η_k's is regular at Q_1 and has a pole at Q_2. Finally, let

$$\sigma_0, \sigma_1, \ldots, \sigma_{g-\delta_P-1}$$

be a basis of the dualizing differentials on Y.

To state the main result of this section, we must also introduce two linear systems on Y. Let

$$V_1 \subseteq H^0(Y, \omega_Y(-c_2Q_2))$$

be the linear system generated by

$$\eta_0, , \ldots, \eta_{\delta_2-1}, \sigma_0, \ldots, \sigma_{g-\delta_P-1}.$$

Then V_1 has dimension $g - \delta_P + \delta_2$ and $\dim_k H^0(Y, \omega_Y(-c_2Q_2)) = g - \delta_P + c_2 - 1$, assuming $c_2 > 0$. If $c_2 = 0$, then $V_1 = H^0(Y, \omega_Y)$, while if $c_2 > 0$, then the codimension of V_1 in $H^0(Y, \omega_Y(-c_2Q_2))$ is $c_2 - 1 - \delta_2$. Hence $V_1 = H^0(Y, \omega_Y(-c_2Q_2))$ if and only if the semigroup $S_2 = \{n \in \mathrm{N} : n = 0 \text{ or } n \geq c_2\}$ (e.g., if P is a simple cusp on the second branch).

Let

$$V_2 \subseteq H^0(Y, \omega_Y(-c_1Q_1))$$

be the linear system generated by

$$\zeta_0, \ldots, \zeta_{\delta_1-1}, \sigma_0, \ldots, \sigma_{g-\delta_P-1}.$$

Similar remarks to those made just above also hold concerning $H^0(Y, \omega_Y(-c_1Q_1))$ and V_2.

(3.17) Theorem. *Suppose X is a Gorenstein curve of arithmetic genus g. Suppose P is a singularity with precisely two branches. Let Q_1 and Q_2 be the two points on the partial normalization Y of X at P that correspond to the branches at P. Let V_1 and V_2 denote the linear systems on Y defined above. Then we have*

$$W_X(P) = \delta_P(g-1)(g+1) - I(g-1) - wt(S_1) - wt(S_2) + W_{V_1}(Q_1) + W_{V_2}(Q_2).$$

Proof. Locally at P, write

$$\begin{aligned} \tau_i &= F_i\tau, & i &= 0, 1, \ldots, I-1 \\ \zeta_j &= G_j\tau, & j &= 0, 1, \ldots, \delta_1 - 1 \\ \eta_k &= H_k\tau, & k &= 0, 1, \ldots, \delta_2 - 1 \\ \sigma_l &= M_l\tau, & l &= 0, 1, \ldots, g - \delta_P - 1. \end{aligned}$$

Put

$$(\hat{F},\hat{G},\hat{H},\hat{M}) = (F_0,\ldots,F_{I-1},G_0,\ldots,G_{\delta_1-1},H_0,\ldots,H_{\delta_2-1},M_0,\ldots,M_{g-\delta_P-1}).$$

Then, as follows from Proposition (1.3), we have

$$W_X(P) = \delta_P(g-1)g + \operatorname{ord}_{Q_1} W_t(\hat{F},\hat{G},\hat{H},\hat{M}) + \operatorname{ord}_{Q_2} W_t(\hat{F},\hat{G},\hat{H},\hat{M}),$$

where t is a local coordinate at Q_1 and Q_2 and W_t denotes the ordinary Wronskian (obtained by differentiating with respect to t). Notice that each of the functions $F_0,\ldots,F_{I-1}$, $G_0,\ldots,G_{\delta_1-1}$ has a different order at Q_1. By forming linear combinations of the H_k's and M_l's, if necessary, we may assume that each of the functions $H_0,\ldots,H_{\delta_2-1}$, $M_0,\ldots,M_{g-\delta_P-1}$ also has a different order at Q_1 and that, of these functions, H_0 has the lowest order at Q_1, with that order being $I+2\delta_1$. Then we have

$$\begin{aligned}
\operatorname{ord}_{Q_1} W_t(\hat{F},\hat{G},\hat{H},\hat{M}) &= \sum_{i=0}^{I-1} \operatorname{ord}_{Q_1} F_i + \sum_{j=0}^{\delta_1-1} \operatorname{ord}_{Q_1} G_j \\
&\quad + \sum_{k=0}^{\delta_2-1} \operatorname{ord}_{Q_1} H_k + \sum_{l=0}^{g-\delta_P-1} \operatorname{ord}_{Q_1} M_l - \sum_{n=0}^{g-1} n \\
&= \sum_{i=0}^{I-1} (\operatorname{ord}_{Q_1} F_i - i) + \sum_{j=0}^{\delta_1-1} (\operatorname{ord}_{Q_1} G_j - (I+j)) \\
&\quad + \sum_{k=0}^{\delta_2-1} (\operatorname{ord}_{Q_1} H_k - (I+\delta_1+k)) \\
&\quad + \sum_{l=0}^{g-\delta_P-1} (\operatorname{ord}_{Q_1} M_l - (\delta_P+l)) \\
&= \sum_{i=0}^{I-1} (a_i - i) + \sum_{j=0}^{\delta_1-1} (I+2\delta_1-1-l_{j+1}-(I+j)) \\
&\quad + \sum_{k=0}^{\delta_2-1} (\operatorname{ord}_{Q_1} H_k - (I+\delta_1+k)) \\
&\quad + \sum_{l=0}^{g-\delta_P-1} (\operatorname{ord}_{Q_1} M_l - (\delta_P+l)) \\
&= \delta_1 I + (\delta_1-1)\delta_1 - wt(S_1) \\
&\quad + \sum_{k=0}^{\delta_2-1} (\operatorname{ord}_{Q_1} H_k - (I+\delta_1+k)) \\
&\quad + \sum_{l=0}^{g-\delta_P-1} (\operatorname{ord}_{Q_1} M_l - (\delta_P+l)),
\end{aligned}$$

where $l_1, l_2, \dots, l_{\delta_1}$ are the gaps of S_1 and where, in the last equality, we have used Corollary (3.8) and the fact that

$$\sum_{j=0}^{\delta_1-1}(I+2\delta_1-1-l_{j+1}-(I+j))=(\delta_1-1)\delta_1-\sum_{j=1}^{\delta_1}(l_j-j).$$

Now, to compute $W_{V_1}(Q_1)$, we must express $\eta_0, \dots, \eta_{\delta_2-1}, \sigma_0, \dots, \sigma_{g-\delta_P-1}$ in terms of a generator of $\omega_Y(-c_2Q_2)$ at Q_1. Let η be a generator of $\omega_Y(-c_2Q_2)$ at Q_1. Then η has order 0 at Q_1. Note that the rational function $H = \eta/\tau$ has a zero of order $I + 2\delta_1$ at Q_1. We then have

$$\begin{aligned} W_{V_1}(Q_1) =&\mathrm{ord}_{Q_1} W_t(H_0/H, \dots, H_{\delta_2-1}/H, M_0/H, \dots, M_{g-\delta_P-1}/H) \\ =&\sum_{k=0}^{\delta_2-1}((\mathrm{ord}_{Q_1} H_k-(I+2\delta_1))-k) \\ &+\sum_{l=0}^{g-\delta_P-1}((\mathrm{ord}_{Q_1} M_l-(I+2\delta_1))-(\delta_2+l)) \\ =&\sum_{k=0}^{\delta_2-1}(\mathrm{ord}_{Q_1} H_k-(I+\delta_1+k))-\delta_2\delta_1 \\ &+\sum_{l=0}^{g-\delta_P-1}(\mathrm{ord}_{Q_1} M_l-(\delta_P+l))-(g-\delta_P)\delta_1. \end{aligned}$$

Thus,

$$\begin{aligned} &\mathrm{ord}_{Q_1} W_t(\hat{F}, \hat{G}, \hat{H}, \hat{M}) \\ &\quad=\delta_1 I+(\delta_1-1)\delta_1-wt(S_1)+\delta_2\delta_1+(g-\delta_P)\delta_1+W_{V_1}(Q_1) \\ &\quad=\delta_1(I+\delta_1-1+\delta_2+g-\delta_P)-wt(S_1)+W_{V_1}(Q_1) \\ &\quad=\delta_1(g-1)-wt(S_1)+W_{V_1}(Q_1), \end{aligned}$$

since $\delta_P = I + \delta_1 + \delta_2$.

Similarly, we have

$$\mathrm{ord}_{Q_2} W_t(\hat{F}, \hat{G}, \hat{H}, \hat{M}) = \delta_2(g-1) - wt(S_2) + W_{V_2}(Q_2).$$

The Theorem now follows by adding these two orders and using the fact that $\delta_1 + \delta_2 = \delta_P - I$. ∎

In the case of an ordinary node, we have $I = 1, \delta_1 = \delta_2 = 0$ and Theorem (3.17) reduces to the following result of Widland [20].

(3.18) Corollary. *If P is an ordinary node, then*

$$W_X(P) = (g-1)g + W_Y(Q_1) + W_Y(Q_2).$$

We will call a singularity P *overweight* if its Weierstrass weight is greater than the "expected" number. A unibranch singularity P is not overweight if the point lying over P on the partial normalization at P is not a Weierstrass point (see Theorem (2.4)). A singularity P with two branches is not overweight if, with the notation of Theorem (3.17), $W_{V_1}(Q_1) = W_{V_2}(Q_2) = 0$. The following result, a generalization of Theorem (2.9) to the case of a Gorenstein curve with only unibranch or two-branch singularities, follows from Theorems (2.4) and (3.17).

(3.19) Theorem. *Suppose X is a Gorenstein curve of arithmetic genus g and geometric genus $\tilde{g}$ with only unibranch and two-branch singularities that are not overweight. Let $U_1, U_2, \ldots, U_n$ denote the unibranch singular points and let $T_1, T_2, \ldots, T_m$ denote the two-branch singular points on X. Then the number of smooth Weierstrass points on X, counting multiplicities, is*

$$\tilde{g}(g-1)(g+1) + \sum_{i=1}^{n} wt(S_i) + \sum_{j=1}^{m} [(g-1)I_j + wt(S_{1j}) + wt(S_{2j})],$$

where S_i is the semigroup of values at U_i, I_j is the intersection multiplicity of the two branches at the point T_j, and S_{1j} and S_{2j} are the semigroups of values of the two branches at T_j.

References

1. Delgado, F.: The semigroup of values of a curve singularity with several branches, Manuscripta Math. **59**, 347–374 (1987)
2. Delgado,F.: Gorenstein curves and symmetry of the semigroup of values, Manuscripta Math. **61**, 285–296 (1988)
3. Farkas, H., Kra, I.: Riemann surfaces, New York: Springer 1980
4. Garcia, A.: Semigroups associated to singular points of plane curves, J. Reine Angew. Math. **336**, 165–184 (1982)
5. Garcia, A, Lax, R.F.: Weierstrass points on Gorenstein curves in arbitrary characteristic, preprint.
6. Hartshorne, R.: Generalized divisors on Gorenstein curves and a theorem of Noether, J. Math. Kyoto Univ. **26**, 375–386 (1986)
7. Hironaka, H.: On the arithmetic genera and the effective genera of algebraic curves, Memoirs of the College of Science, Univ. Kyoto, Series A, **30**, 178–195 (1957)
8. Kunz, E.: The value-semigroup of a one-dimensional Gorenstein ring, Proc. Amer. Math. Soc. **25**, 748–751 (1970)
9. Laksov, D.: Weierstrass points on curves, Astérisque **87–88**, 221–247 (1981)
10. Laksov, D., Thorup, A.: The Brill-Segre formula for families of curves. In: Enumerative algebraic geometry (Copenhagen, 1989), 131–148, Contemp. Math. **123**. Providence, RI: Amer. Math. Soc. 1991
11. Laksov, D. , Thorup, A.: On gap sequences for families of curves, preprint
12. Lax, R.F.: On the distribution of Weierstrass points on singular curves, Israel J. Math. **57**, 107–115 (1987)
13. Lax, R. F., Widland, C.: Weierstrass points on rational cuspidal curves, Boll. U.M.I. **2**-A, 65–71 (1988)
14. Lax, R.F., Widland, C.: Gap sequences at a singularity, Pac. J. Math. **150**, 111–122 (1991)
15. Matzat, B.H.: Ein Vortrag über Weierstrasspunkte, Karlsruhe, 1975
16. Oliveira, G.: Weierstrass semigroups and the canonical ideal of non-trigonal curves, Manuscripta Math. **71**, 431–450 (1991)
17. Schmidt, F.K.: Die Wronskiche Determinante in beliebigen differenzierbaren Funktionenkörpern, Math. Z. **45**, 62–74 (1939)

18. Serre, J.-P.: Groupes Algébriques et Corps des Classes. Paris: Hermann 1959
19. Stöhr, K.-O., Voloch, J.F.: Weierstrass points and curves over finite fields, Proc. London Math. Soc. (3) **52**, 1–19 (1986)
20. Widland, C.: On Weierstrass points of Gorenstein curves, Ph.D. dissertation, Louisiana State University 1984
21. Widland, C., Lax, R.F.: Weierstrass points on Gorenstein curves, Pac. J. Math. **142**, 197–208 (1990)

Arnaldo Garcia
I.M.P.A.
Estrada Dona Castorina 110
22.460 Rio de Janeiro
Brasil
garcia@impa.br

Robert F. Lax
Department of Mathematics
LSU
Baton Rouge, LA 70803
U.S.A.
lax@marais.math.lsu.edu

(Received March 30, 1993;
in revised form June 14, 1993)

manuscripta math. 81, 379 - 386 (1993)

manuscripta mathematica

Iwasawa invariants of imaginary quadratic fields

Shu-Leung Tang

Let K be an imaginary quadratic field and p an odd prime which splits in K. We study the Iwasawa invariants for $\mathbb{Z}_p$-extensions of K. This is motivated in part by a recent result of Sands. The main result is the following. Assume p does not divide the class number of K. Let K_∞ be a $\mathbb{Z}_p$-extension of K. Suppose K_∞ is not totally ramified at the primes above p. Then the μ-invariant for K_∞/K vanishes. We also show that if $\mu = 0$ for all $\mathbb{Z}_p$-extensions of K, then the λ-invariant is bounded as K_∞ runs through all such extensions.

1 Introduction

Motivated by the analogous situation in function fields, Iwasawa initiated the study of $\mathbb{Z}_p$-extensions of number fields and made the conjecture that $\mu = 0$ for $\mathbb{Z}_p$-extensions of number fields. Later, however, he [10] constructed examples of non-cyclotomic $\mathbb{Z}_p$-extensions for which μ can be arbitrarily large. But for cyclotomic $\mathbb{Z}_p$-extensions, the question is still open. For abelian extensions of $\mathbb{Q}$, this has been proved by Ferrero-Washington [2] .

In this paper, we shall study the Iwasawa invariants of $\mathbb{Z}_p$-extensions of imaginary quadratic fields. Let K be an imaginary quadratic field and p an odd prime which splits in K: $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$, $\mathfrak{p} \neq \mathfrak{p}^*$. Consider the following question:

(A) Does $\mu = 0$ for any $\mathbb{Z}_p$-extension K_∞ of K?

This is partly inspired by a recent result of Sands [12], who showed that (A) is true provided p does not divide the class number h_K of K and the λ-invariant for the cyclotomic $\mathbb{Z}_p$-extension K_∞^c of K is at most two, in which case he also showed that the λ-invariant is bounded by two as K_∞ varies over all $\mathbb{Z}_p$-extensions of K. We shall actually consider a more general question. Let K_∞/K be a $\mathbb{Z}_p$-extension, $\mathcal{M}_\infty$ the maximal abelian $\mathfrak{p}$-ramified p-extension of K_∞ and $\mathfrak{X} = \mathrm{Gal}(\mathcal{M}_\infty/K_\infty)$. Then $\mathfrak{X}$ is known to be a finitely generated torsion module over the completed group ring $\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$.

(B) Does $\mu(\mathfrak{X}) = 0$?

Here $\mu(\mathfrak{X})$ denotes the Iwasawa μ-invariant of $\mathfrak{X}$. The validity of (B) for all $\mathbb{Z}_p$-extensions of K would have some bearing on the anti-cyclotomic main conjecture for K and p (cf. [8, Thm. 2]). We remark that Gillard [3] has shown that if F is an abelian extension of K and F_∞ is the unique $\mathbb{Z}_p$-extension of F unramified outside the primes above $\mathfrak{p}$ with $p \geq 5$, then $\mu(\mathfrak{X}_F) = 0$ where $\mathfrak{X}_F$

is the Galois group of the maximal abelian $\mathfrak{p}$-ramified p-extension of F_∞ over F_∞. A similar result was obtained independently by Schneps [13]. We shall give a partial answer to (A) and (B) as follows.

Theorem 1 *Assume $p \nmid h_K$. If K_∞/K is not totally ramified at the primes above p, then $\mu = 0$. Moreover, if K_∞/K is not totally ramified at $\mathfrak{p}^*$, then $\mu(\mathfrak{X}) = 0$.*

Here is an outline of the paper. In Section 2, we introduce the notation used and give some basic facts needed later. In Section 3, we give some necessary and sufficient conditions for $\mu = 0$. In Section 4, we give a proof of Theorem 1. We also show that for the anti-cyclotomic $\mathbb{Z}_p$-extension K_∞^a of K, $\mu = 0$ implies $\mu(\mathfrak{X}) = 0$. Parallel to Sands' result quoted above, we show that if $\mu = 0$ for all $\mathbb{Z}_p$-extensions of K, then the λ-invariant is bounded as K_∞ varies over all $\mathbb{Z}_p$-extensions of K (cf. Proposition 5.1).

2 Notation and Basic Facts

Let K be an imaginary quadratic field and let p be an odd prime. Unless stated otherwise, we assume that p splits in K: $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^*$, $\mathfrak{p} \neq \mathfrak{p}^*$, and does not divide the class number h_K of K. Let K_∞ be a $\mathbb{Z}_p$-extension of K, $\Gamma = \mathrm{Gal}(K_\infty/K)$, $\Gamma_n = \Gamma/\Gamma^{p^n}$ and $\Lambda = \Lambda_{K_\infty} = \varprojlim \mathbb{Z}_p[\Gamma_n]$ the Iwasawa algebra, the inverse limit being over all n with respect to the natural maps. Let M_∞ (resp. $\mathcal{M}_\infty$, resp. $\mathcal{M}_\infty^*$, resp. L_∞) be the maximal abelian p-ramified (resp. $\mathfrak{p}$-ramified, resp. $\mathfrak{p}^*$-ramified, resp. unramified) p-extension of K_∞ and let $X = X_{K_\infty} = \mathrm{Gal}(M_\infty/K_\infty)$, $\mathfrak{X} = \mathfrak{X}_{K_\infty} = \mathrm{Gal}(\mathcal{M}_\infty/K_\infty)$ and $Y = Y_{K_\infty} = \mathrm{Gal}(L_\infty/K_\infty)$. It is known that X, $\mathfrak{X}$ and Y are finitely generated Λ-modules. Moreover, $\mathfrak{X}$ and hence Y are torsion over Λ (cf. [11, II 2.2, Prop. 20]).

Fix a topological generator γ of Γ. Let K_n be the fixed field of Γ^{p^n} and A_n the p-primary part of the ideal class group of K_n. For each $n \geq 0$, let U_n be the principal semi-local units of K_n at p, E_n the global units of K_n, $\overline{E}_n$ the closure of $E_n \cap U_n$ in the natural topology of U_n under the diagonal embedding and M_n the maximal abelian p-ramified p-extension of K_n. For a prime $\mathfrak{l}$ of a number field F, we write $[\mathfrak{l}]$ for the projection of the ideal class represented by $\mathfrak{l}$ to the p-primary part of the ideal class group of F.

By class field theory, we have for each n, an exact sequence

$$0 \to U_n/\overline{E}_n \to \mathrm{Gal}(M_n/K_n) \to A_n \to 0. \qquad (1)$$

By our assumption on h_K, $A_0 = 1$. Since K is imaginary quadratic and p splits in K, $U_0 \cong \mathbb{Z}_p^2$ and $\overline{E}_0 = 1$. Thus for $n = 0$, (1) gives $\mathrm{Gal}(M_0/K) \cong \mathbb{Z}_p^2$. In other words, the maximal abelian p-ramified p-extension of K is precisely the $\mathbb{Z}_p^2$-extension of K. Note also that the inertia groups $I_\mathfrak{p}$ and $I_{\mathfrak{p}^*}$ at $\mathfrak{p}$ and $\mathfrak{p}^*$ in $\mathrm{Gal}(M_0/K)$ respectively are both isomorphic to $\mathbb{Z}_p$. Thus if K_∞/K is ramified at $\mathfrak{p}$ and $\mathfrak{p}^*$, then M_0/K_∞ is unramified. Since $\mathfrak{p}$ does not split completely in the fixed field of $I_\mathfrak{p}$, it follows in this case that the decomposition group of a prime of L_∞ above p in Y is non-trivial, hence isomorphic to $\mathbb{Z}_p$.

The following give some information about the structure of X, $\mathfrak{X}$ and Y over Λ.

Lemma 2.1 $X \cong \Lambda$.

Proof. By [7, Thm. 1], X is of rank one over Λ. Since M_0 is the maximal abelian p-ramified p-extension of K, $X/(\gamma - 1)X \cong \mathrm{Gal}(M_0/K_\infty)$. So X is cyclic over Λ by Nakayama's lemma. Hence $X \cong \Lambda$. □

Corollary 2.1 *$\mathfrak{X}$ and Y are cyclic over Λ.*

Proof. As $\mathfrak{X}$ and Y are quotients of X, the corollary follows from Lemma 2.1. □

Corollary 2.2 $\mathrm{Gal}(M_n/K_n) \cong \mathbb{Z}_p^{1+p^n}$ *for all n.*

Proof. This follows from Lemma 2.1 and the fact $\mathrm{Gal}(M_n/K_\infty) \cong X/\omega_n X$ where $\omega_n = \gamma^{p^n} - 1 \in \Lambda$. □

3 Some conditions for $\mu = 0$

In this section, we give some necessary and sufficient conditions for $\mu = 0$. To study the μ-invariant of K_∞, it is convenient to divide into three cases:

1. One of $\mathfrak{p}$ and $\mathfrak{p}^*$ does not ramify in K_∞/K;
2. Both $\mathfrak{p}$ and $\mathfrak{p}^*$ ramify totally in K_∞/K;
3. Both $\mathfrak{p}$ and $\mathfrak{p}^*$ ramify in K_∞/K but one of them does not ramify totally.

In Case 1, it follows from [9] that $A_n = 1$ for all $n \geq 0$. Thus $\mu = 0$ in this case. In fact, if K_∞ is the unique $\mathbb{Z}_p$-extension of K unramified outside $\mathfrak{p}$, then $\mathcal{M}_\infty = K_\infty$ since $A_0 = 1$, so $\mathfrak{X} = 0$. Henceforth, we assume that both $\mathfrak{p}$ and $\mathfrak{p}^*$ ramify in K_∞/K.

We now give a sufficient condition for $\mu = 0$. Let $n \geq 0$ be the smallest integer such that K_∞/K_n is totally ramified at the primes above p. As Y is cyclic over Λ by Corollary 2.1, we can write $Y = \Lambda y$ for some $y \in Y$. Let x be the Frobenius substitution of a prime $\mathfrak{P}$ of L_∞ above p. Then $x = hy$ for some $h \in \Lambda$. Since the primes above p are totally ramified in K_∞/K_n, the ideal classes generated by them are invariant under Γ^{p^n}. So we have $(\gamma^{p^n} - 1)hy = (\gamma^{p^n} - 1)x = 0$ for n sufficiently large. If we can show that $x \notin Y^p$ or equivalently $[\mathfrak{p}_m] \notin A_m^p$ for some $m > n$ where $\mathfrak{p}_m$ is the prime of K_m below $\mathfrak{P}$, then clearly h is not divisible by p in Λ, so $\mu = 0$. Thus we have proved

Proposition 3.1 *If $[\mathfrak{p}_n] \notin A_n^p$ for some $n \geq 1$ and some prime $\mathfrak{p}_n$, then $\mu = 0$. In particular, if A_1 is elementary, then $\mu = 0$.*

The condition in Proposition 3.1 is, however, not necessary, as the following example shows. Let $K = \mathbb{Q}(\sqrt{-164})$ and K_∞ the cyclotomic $\mathbb{Z}_3$-extension of K. By the tables in [4], $A_0 = 1$ and $|A_1| = 27$. From the proof of Proposition 3.3

below, 3-rank $A_1 \le 2$. By [1, Cor. to Prop. 5], 3-rank A_1 cannot be one. Hence $A_1 \cong \mathbb{Z}/3^2\mathbb{Z} \bigoplus \mathbb{Z}/3\mathbb{Z}$. By genus theory, $A_1^{\Gamma_1}$ is order of 3, generated by the classes of $\mathfrak{p}_1$and $\mathfrak{p}_1^*$ where for each $n \ge 1$, $\mathfrak{p}_n$ and $\mathfrak{p}_n^*$ are the primes of K_n above $\mathfrak{p}$ and $\mathfrak{p}^*$. The kernel of the map $\pi : A_1 \longrightarrow A_1/A_1^3$ is of order 3 and is a $\mathbb{Z}_3[\Gamma_1]$-submodule of A_1. It follows that $A_1^{\Gamma_1}$ is the kernel of π and so $[\mathfrak{p}_1]$, $[\mathfrak{p}_1^*] \in A_1^3$. Since K_∞ is the cyclotomic $\mathbb{Z}_3$-extension of a CM-field, the map $A_1 \longrightarrow A_n$ induced by the inclusion of ideals is injective (cf. [14, Prop. 13.26]). So $[\mathfrak{p}_n]$, $[\mathfrak{p}_n^*] \in A_n^3$ for all $n \ge 1$. However, $\mu = 0$ by the theorem of Ferrero-Washington.

In view of Proposition 3.1, it is nevertheless natural to study the structure of $A_n^{\Gamma_n}$. Suppose K_∞/K is totally ramified at the primes above p. By genus theory, $A_n^{\Gamma_n}$ has order $p^n \cdot |A_0|$, generated by the classes of A_0, $\mathfrak{p}_n$ and $\mathfrak{p}_n^*$. For the next proposition, we do not assume that $A_0 = 1$.

Proposition 3.2 *Suppose K_∞/K is totally ramified at the primes above p. Let n_0 be the largest integer such that $\mathfrak{p}_n$ and $\mathfrak{p}_n^*$ split completely in $\tilde{K}_n/K_n$ where $\tilde{K}_n$ is the compositum of the n-th layers of all $\mathbb{Z}_p$-extensions of K. Then the classes $[\mathfrak{p}_n]$ and $[\mathfrak{p}_n^*]$ have the same order in A_n for all $n \ge n_0$.*

Proof. We write $|x|$ for the order of the group $< x >$ generated by an element x in a group. Since $|A_{n+1}^{\Gamma_{n+1}}| = p\,|A_n^{\Gamma_n}|$ and $\tilde{K}_n/K$ is unramified, it follows that $|[\mathfrak{p}_{n+1}]| = p\,|[\mathfrak{p}_n]|$ and $|[\mathfrak{p}_{n+1}^*]| = p\,|[\mathfrak{p}_n^*]|$ for all $n \ge n_0$. Suppose $|[\mathfrak{p}_n^*]| < |[\mathfrak{p}_n]|$ for some $n \ge n_0$ and hence for all $n \ge n_0$. As in [5], we have, for n sufficiently large, $A_n^{\Gamma_n} \cong < [\mathfrak{p}_n] > \bigoplus H_n$ where H_n is the subgroup of $A_n^{\Gamma_n}$ which capitulates in K_∞. Then $[\mathfrak{p}_n^*] = [\mathfrak{p}_n]^{a_n} \cdot x_n$ with $p \mid a_n$ and $x_n \in H_n$. Let ϕ and ϕ^* be the Frobenius elements of $\mathfrak{p}_\infty$ and $\mathfrak{p}_\infty^*$ in $\mathrm{Gal}(L_\infty/K_\infty)$. Then $\phi^* = \varprojlim [\mathfrak{p}_n^*] = \varprojlim ([\mathfrak{p}_n]^{a_n} \cdot x_n) = \phi^a \cdot x$ where $a = \varprojlim a_n \in p\mathbb{Z}_p$ and $x = \varprojlim x_n$ belongs to the $\mathbb{Z}_p$-torsion submodule of Y. If $\overline{\phi}$, $\overline{\phi}^*$ and $\overline{x}$ denote the images of ϕ, ϕ^* and x in $\mathrm{Gal}(M_0/K_\infty)$, then $\overline{\phi}^* = \overline{\phi}^a \cdot \overline{x} = \overline{\phi}^a$ since $\mathrm{Gal}(M_0/K_\infty)$ is $\mathbb{Z}_p$-free. But this is impossible since $\mathrm{Gal}(M_0/K_\infty) = \varprojlim \mathrm{Gal}(\tilde{K}_n/K_n)$ and $< \overline{\phi}_n > = < \overline{\phi}_n^* >$ for all n where $\overline{\phi}_n$ and $\overline{\phi}_n^*$ are the projections of $\overline{\phi}$ and $\overline{\phi}^*$ in $\mathrm{Gal}(\tilde{K}_n/K_n)$. Hence we have $|[\mathfrak{p}_n]| = |[\mathfrak{p}_n^*]|$ for all $n \ge n_0$. This proves the proposition. □

Remark. Proposition 3.2 is known to be true for the cyclotomic $\mathbb{Z}_p$-extension of K.

On the other hand, we have the following criterion for $\mu > 0$, which is a quantitative version of the well-known fact that $\mu = 0 \Leftrightarrow p$-rank A_n is bounded as $n \longrightarrow \infty$.

Proposition 3.3 *Let e be the smallest integer such that K_∞/K_e is totally ramified at the primes above p. Then $\mu > 0$ if and only if p-rank $A_n = p^n - p^e$ for all $n \ge e$.*

Proof. The ($\Leftarrow$) part is clear. We prove the ($\Rightarrow$) part. Let $\nu_{n,e} = \gamma^{p^n} - 1/\gamma^{p^e} - 1$ for $n \ge e$. By [14, Lemma 13.15], there exists a Λ-submodule Z of finite index in Y such that $A_n \cong Y/\nu_{n,e}Z$ for all $n \ge e$. Now by genus theory, $A_e = 1$.

So we have $Y = Z$. Let $\mathrm{Ann}(Y)$ be the annihilator ideal of Y in Λ and $f \in \Lambda$ a characteristic power series of Y. Then we have $Y \cong \Lambda/\mathrm{Ann}(Y)$. Let $g_1, \ldots, g_r \in \Lambda$ be a minimal set of generators for $\mathrm{Ann}(Y)$ and g the *gcd* of the g_i's. We claim that g is a characteristic power series of Y. If $r = 1$, this is clear. Assume $r \geq 2$. Consider the exact sequence

$$0 \to g\Lambda/\mathrm{Ann}(Y) \to \Lambda/\mathrm{Ann}(Y) \to \Lambda/g\Lambda \to 0. \tag{2}$$

The first term of (2) is isomorphic to $\Lambda/(g'_1, \ldots, g'_r)$ where $g'_i = g_i/g$ for $i = 1, \ldots, r$ and is finite. This proves the claim. So for $n \geq e$, $A_n \cong \Lambda/(\mathrm{Ann}(Y), \nu_{n,e})$ and the latter surjects onto $\Lambda/(f, \nu_{n,e})$. Thus if $\mu > 0$, then p-rank $A_n \geq p^n - p^e$. Suppose $Y/(\nu_{n,e}, p)Y = Y/(\nu_{n,e}(\gamma - 1), p)Y$ for some $n \geq e$. By the above claim, we can find $h \in \mathrm{Ann}(Y)$ such that $h = p^\mu h'$ with $h' \in \Lambda$ not divisible by p. Then $(\nu_{n,e}, p)Y = (\nu_{n,e}(\gamma - 1), p)Y$, so

$$\nu_{n,e}Y + pY = \nu_{n,e}(\gamma - 1)Y + pY.$$

Hence

$$p^{\mu-1}h'\nu_{n,e}Y = p^{\mu-1}h'\nu_{n,e}(\gamma - 1)Y,$$

so $p^{\mu-1}h'\nu_{n,e}Y = 0$ by Nakayama's lemma. Thus $p^{\mu-1}h'\nu_{n,e} \in \mathrm{Ann}(Y)$. But this contradicts the definition of μ. This shows that p-rank $A_n < p$-rank $Y/\nu_{n,e}(\gamma - 1)Y$. This, combined with the above, proves the proposition. □

4 Proof of Theorem 1

We shall now prove Theorem 1. Note that if K_∞/K is not totally ramified at $\mathfrak{p}$, then the conjugate K_∞^τ of K_∞ over $\mathbb{Q}$ is not totally ramified at $\mathfrak{p}^*$ and vice versa. To prove Theorem 1, it suffices therefore to show that $\mu(\mathfrak{X}) = 0$ for K_∞/K not totally ramified at $\mathfrak{p}^*$. Let $n \geq 1$ be the smallest integer such that K_∞/K_n is totally ramified at the primes above $\mathfrak{p}^*$. Let $\mathcal{M}_{\infty,n}$ be the maximal elementary subextension of $\mathcal{M}_\infty/K_\infty$ which is abelian over K_n and let $\mathcal{M}_n$ denote the fixed field of the subgroup $\mathcal{I}$ of $\mathrm{Gal}(\mathcal{M}_{\infty,n}/K_n)$ generated by the inertia groups $I_{\mathfrak{p}_n^*}$ at the primes $\mathfrak{p}_n^*$ of K_n above $\mathfrak{p}^*$. Then $\mathcal{I}$ has p-rank at most p^n since $I_{\mathfrak{p}_n^*} \cong \mathbb{Z}_p$ for any $\mathfrak{p}_n^*$ and there are at most p^n primes of K_n above $\mathfrak{p}^*$. Assume that $\mu(\mathfrak{X}) > 0$. Then p-rank $\mathrm{Gal}(\mathcal{M}_{\infty,n}/K_\infty) = p^n$ by Corollary 2.2. Thus $\mathcal{M}_n$ is a non-trivial abelian $\mathfrak{p}$-ramified p-extension of K_n. As $\mathcal{M}_n$ is Galois over K, $\mathcal{M}_n/K_n$ contains a non-trivial subextension abelian over K. But this is impossible by the assumption $A_0 = 1$. So $\mu(\mathfrak{X}) = 0$. This proves Theorem 1.

Let K_∞^a be the anti-cyclotomic $\mathbb{Z}_p$-extension of K, i.e., the $\mathbb{Z}_p$-extension of K Galois over $\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q})$ acts by -1 on $\mathrm{Gal}(K_\infty^a/K)$. If we assume $\mu(Y) = 0$ for K_∞^a, then we have the following result similar to the second part of Theorem 1.

Proposition 4.1 *Let $K_\infty = K_\infty^a$. If $\mu(Y) = 0$, then $\mu(\mathfrak{X}) = 0$.*

Proof. Let $\tilde{\mathcal{M}}_\infty = \mathcal{M}_\infty \mathcal{M}^*_\infty$, $\tilde{\mathfrak{X}}^* = \mathrm{Gal}(\tilde{\mathcal{M}}_\infty/K_\infty)$, $N = \mathrm{Gal}(\tilde{\mathcal{M}}_\infty/\mathcal{M}^*_\infty)$ and $N^* = \mathrm{Gal}(\tilde{\mathcal{M}}_\infty/\mathcal{M}_\infty)$. Then $N \cong N^*$ under the conjugation by a lift of the generator τ of $\mathrm{Gal}(K/\mathbb{Q})$ to $\mathrm{Gal}(\tilde{\mathcal{M}}_\infty/\mathbb{Q})$. Note that if $f(\gamma - 1)$ is a generator of char(N), then $f(\gamma^{-1} - 1)$ is a generator of char(N^*). Since $\mathrm{Gal}(\mathcal{M}^*_\infty/L_\infty) = N \oplus N^*$, we have the exact sequence

$$0 \to N \oplus N^* \to \tilde{\mathfrak{X}} \to Y \to 0. \tag{3}$$

We deduce from (3) the following exact sequence

$$Y^{\Gamma^{p^n}} \to (N \oplus N^*)_{\Gamma^{p^n}} \to \tilde{\mathfrak{X}}_{\Gamma^{p^n}} \to Y_{\Gamma^{p^n}} \to 0. \tag{4}$$

Now if E is the elementary Λ-module pseudo-isomorphic to Y, then

$$|(p-\text{rank } Y^{\Gamma^{p^n}}) - (p - \text{rank } E^{\Gamma^{p^n}})|$$

is bounded for all n. Since the latter is bounded for all n, so is the former. It is easy to see that if M is a finitely generated torsion Λ-module and $\mu(M) > 0$, then p-rank $M_{\Gamma^{p^n}} \geq p^n + O(1)$ for n large. If $\mu(N) > 0$, then $\mu(N^*) > 0$ and so from (4) for n large,

$$p\text{-rank } \tilde{\mathfrak{X}}_{\Gamma^{p^n}} \geq p\text{-rank } (N \oplus N^*)_{\Gamma^{p^n}} + O(1) \geq 2p^n + O(1). \tag{5}$$

But (5) contradicts Corollary 2.2 for n sufficiently large. Thus $\mu(N) = 0$. Since $\mu(Y) = 0$ by assumption, $\mu(\mathfrak{X}) = 0$. This proves the proposition. □

5 Boundedness of λ

So far, we have only been concerned with the μ-invariant of $\mathbb{Z}_p$-extensions. Suppose $\mu(Y) = 0$ for all $\mathbb{Z}_p$-extensions of K. Then we have the following interesting consequence concerning the λ-invariant. Here we do not assume $A_0 = 1$.

Proposition 5.1 *If $\mu = 0$ for all $\mathbb{Z}_p$-extensions of K, then the λ-invariant is bounded as K_∞ varies over all $\mathbb{Z}_p$-extensions.*

First, we recall some notation and results from [6]. Let $\mathcal{E}$ be the set of all $\mathbb{Z}_p$-extensions of K. For $K_\infty \in \mathcal{E}$ and $n \in \mathbb{N}$, let $\mathcal{E}(K_\infty, n) = \{K'_\infty \in \mathcal{E} : [K_\infty \bigcap K'_\infty : K] \geq p^n\}$. As in [6, §3], taking the $\mathcal{E}(K_\infty, n)$ as a basis of neighborhoods of K_∞ defines a topology on $\mathcal{E}$ which then becomes a compact topological space. Let $\tilde{\Lambda}$ be the completed group ring of $\mathrm{Gal}(M_0/K)$ over $\mathbb{Z}_p$. For $K_\infty \in \mathcal{E}$, there is a surjective homomorphism $\mathrm{Gal}(M_0/K) \to \mathrm{Gal}(K_\infty/K)$ which induces a surjective ring homomorphism $\tilde{\Lambda} \to \Lambda_{K_\infty}$. The kernel of this map is plainly the ideal $(\gamma_{K_\infty} - 1)\tilde{\Lambda}$ of $\tilde{\Lambda}$ with γ_{K_∞} a topological generator of $\mathrm{Gal}(M_0/K_\infty)$. Let $\tilde{Y}$ be the Galois group of the maximal unramified abelian p-extension of M_0 over M_0. Then by [6, Thm. 1], $\tilde{Y}$ is a finitely generated torsion $\tilde{\Lambda}$-module. We can view $\tilde{Y}_{K_\infty} \stackrel{\text{def}}{=} \tilde{Y}/(\gamma_{K_\infty} - 1)\tilde{Y}$ as a finitely generated module over $\Lambda_{K_\infty} \cong \tilde{\Lambda}/(\gamma_{K_\infty} - 1)\tilde{\Lambda}$.

Lemma 5.1 *If $\mu = 0$ for all $K_\infty \in \mathcal{E}$, then $\tilde{Y}_{K_\infty}$ is a torsion Λ_{K_∞}-module and $\mu(\tilde{Y}_{K_\infty}) = 0$ for all $K_\infty \in \mathcal{E}$.*

Proof. Consider first the case where K_∞/K is ramified at both $\mathfrak{p}$ and $\mathfrak{p}^*$. Then as was explained before, M_0/K is unramified. So $\tilde{Y}_{K_\infty}$ is a Λ_{K_∞}-submodule of Y_{K_∞} and hence is torsion over Λ_{K_∞}. Since $\mu(Y_{K_\infty}) = 0$ by assumption, $\mu(\tilde{Y}_{K_\infty}) = 0$. Next suppose K_∞/K is unramified at $\mathfrak{p}$, say. Let L_{K_∞} be the fixed field of $(\gamma_{K_\infty} - 1)\tilde{Y}$. Since $\mathfrak{p}$ is finitely decomposed in K_∞/K, the set Σ of primes of K_∞ above $\mathfrak{p}$ is finite. Let $\mathfrak{p}_\infty \in \Sigma$ and let $I_{\mathfrak{p}_\infty}$ be the inertia subgroup of $G = \mathrm{Gal}(L_{K_\infty}/K_\infty)$ for $\mathfrak{p}_\infty$. Note that M_0/K_∞ is ramified at $\mathfrak{p}_\infty$ and so $I_{\mathfrak{p}_\infty}$ is isomorphic to $\mathbb{Z}_p$ under the restriction map. Moreover, the quotient of G by the subgroup generated by $I_{\mathfrak{p}_\infty}$, $\mathfrak{p}_\infty \in \Sigma$, is Y_{K_∞}. Since $\mu(Y_{K_\infty}) = 0$ by assumption, G is a finitely generated $\mathbb{Z}_p$-module and hence is Λ_{K_∞}-torsion. It follows that $\mu(\tilde{Y}_{K_\infty}) = 0$ in this case too. The case where K_∞/K is unramified at $\mathfrak{p}^*$ is similar. This proves the lemma. □

In the notation of [6], Lemma 5.1 shows that $\mathcal{E}(\tilde{Y}) = \mathcal{E}$ where $\mathcal{E}(\tilde{Y})$ is the set of all K_∞ for which $\tilde{Y}_{K_\infty}$ is Λ_{K_∞}-torsion. To finish the proof of Proposition 5.1, we need the following theorem.

Theorem 2 ([6, Thm. 3]) *If $\mu(\tilde{Y}_{K_\infty}) = 0$ for some $K_\infty \in \mathcal{E}(\tilde{Y})$, then there exists an open $U \subset \mathcal{E}(\tilde{Y})$ containing K_∞ such that $\mu(Y_{K'_\infty}) = 0$ and $\lambda(Y_{K'_\infty})$ is bounded for all $K'_\infty \in U$.*

Proof of Proposition 5.1. By Lemma 5.1 , $\mu(\tilde{Y}_{K_\infty}) = 0$ for all $K_\infty \in \mathcal{E}$. So our proposition follows from Theorem 2 and the compactness of $\mathcal{E}$. □

References

[1] Cornell, G. and Rosen, M.: Group-theoretic constraints on the structure of the class group. J. Number Theory **13**, 1-11 (1980)

[2] Ferrero, B. and Washington, L.: The Iwasawa invariant μ_p vanishes for abelian number fields. Ann. of Math. **109**, 377-395 (1979)

[3] Gillard, R.: Fonctions L p-adiques des corps quadratiques imaginaries et leurs extensions abéliennes. J. Reine. Angew. Math. **358**, 76-91 (1986)

[4] Gold, R.: Examples of Iwasawa invariants, II. Acta. Arith. **26**, 233-240 (1975)

[5] Grandet, M. and Jaulent, F.: Sur la capitulation dans une $\mathbb{Z}_l$-extension. J. Reine. Angew. Math. **362**, 213-217 (1986)

[6] Greenberg, R.: The Iwasawa invariants of Γ-extensions of a fixed number field. Am. J. Math. **95**, 204-214 (1973)

[7] Greenberg, R.: On the structure of certain Galois groups. Invent. Math. **47**, 85-99 (1978)

[8] Hida, H. and Tilouine, J.: Katz p-adic L-functions, congruence modules and deformations of Galois representations. In: Coates, J., Taylor, M. (eds.) L-functions and Arithmetic. (Lond. Math. Soc. Lect. Note Ser., vol. 153). Cambridge, London: Cambridge University Press 1991

[9] Iwasawa, K.: A note on class numbers of algebraic number fields. Abh. Math. Sem. U. Hamburg **20**, 257-258 (1956)

[10] Iwasawa, K.: On the μ-invariants of $\mathbb{Z}_l$ -extensions. In: Number Theory, Algebraic Geometry and Commutative Algebra (in honor of Y. Akizuki). Tokyo: Kinokuniya 1973

[11] Perrin-Riou, B.: Arithmétiques des elliptiques et théorie d'Iwasawa. Bull. Soc. Math. France Suppl., Mémorie **17** (1984)

[12] Sands, J.: On small Iwasawa invariants and imaginary quadratic fields. Proc. Amer. Math. Soc. **112**, 671-684 (1991)

[13] Schneps, L.: On the μ-invariant of p-adic L-functions attached to elliptic curves with complex multiplication. J. Number Theory **25**, 20-33 (1987)

[14] Washington, L.: Introduction to cyclotomic fields. (Graduate Texts in Mathematics, vol. 83). Berlin Heidelberg New York: Springer 1982

Department of Mathematics
The Ohio State University
Columbus, OH 43210-1174
U.S.A.

(Received June 8, 1993)

manuscripta math. 81, 387 - 412 (1993)

manuscripta mathematica

An Elliptic Boundary Value Problem Occurring in Magnetohydrodynamics

M. Faierman[†], R. Mennicken, M. Möller[†]

We derive results concerning the spectral properties of an elliptic boundary value problem arising in the mathematical theory of magnetohydrodynamics which are of basic importance for the further development of this theory.

1. Introduction

The object of this paper is to derive some results concerning the spectral properties of an elliptic boundary value problem arising in magnetohydrodynamic theory [9, Chapter 9] and which are required for a further development of this theory. Accordingly, let $I_{\alpha,\beta}$ denote the region in $\mathbb{R}^2$ defined by the inequalities $\alpha_1 < x_1 < \beta_1$, $-\infty < x_2 < \infty$, where $\alpha_1, \beta_1 \in \mathbb{R}$, let $\Omega = \{x = (x_1, x_2) \in I_{\alpha,\beta} | 0 < x_2 < 2\pi\}$, and let $\Gamma = \partial\Omega$, where ∂ denotes boundary. Let $\alpha_2 = 0$, $\beta_2 = 2\pi$, and for $r = 1, 2$ (resp. $r = 3, 4$), let Γ_r denote the face of Ω: $x_r = \alpha_r$, $\alpha_{3-r} < x_{3-r} < \beta_{3-r}$ (resp. $x_{r-2} = \beta_{r-2}$, $\alpha_{5-r} < x_{5-r} < \beta_{5-r}$). On Ω we introduce the elliptic operator L defined by

$$Lu = -\sum_{i=1}^{2} D_i\Big(\sum_{j=1}^{2} a_{ij}(x)D_j u + a_i(x)u\Big) + \sum_{i=1}^{2} \overline{a_i}(x)D_i u + a_0(x)u,$$

where $D_i = \partial/\partial x_i$, $(a_{ij}(x))$ is a positive definite Hermitian matrix for each $x \in \overline{\Omega}$ (the closure of Ω) whose entries are the restrictions to Ω of Lipschitz continuous functions on $\overline{I}_{\alpha,\beta}$ which are periodic in x_2 of period 2π, the $a_i(x)$, $1 \le i \le 2$, are the restrictions to Ω of complex-valued, Lipschitz continuous functions on $\overline{I}_{\alpha,\beta}$ which are periodic in x_2 of period 2π, $\overline{a}_i(x)$ denotes the complex conjugate of $a_i(x)$, and $a_0(x)$ is the restriction to Ω of a real-valued function in $L^\infty(\overline{I}_{\alpha,\beta})$ which is periodic in x_2 of period 2π. Then we shall be concerned here with the eigenvalue problem

(1.1) $$Lu = \lambda u \text{ in } \Omega,$$

(1.2) $$u = 0 \text{ on } \Gamma_r \text{ for } r = 1 \text{ and } r = 3,$$
$$u(x_1, 0) = u(x_1, 2\pi) \text{ for } x_1 \in \Omega',$$

[†]The work of this author was supported in part by the FRD of South Africa

where Ω' denotes the interval (α_1, β_1).

We shall treat the problem (1.1–2) from a variational point of view, and hence after introducing in §2 the function spaces about which we shall be concerned, we introduce in §3 the sesquilinear form B associated with (1.1–2). By arguing with the form B we are able to derive important information concerning the eigenvalues and eigenfunctions of (1.1–2). In particular, we show that the eigenvalues form a denumberably infinite subset of $\mathbb{R}$ having no finite points of accumulation. Moreover, we obtain regularity results for the eigenfunctions right up to the boundary of Ω. In §4 we establish some results concerning the asymptotics of the eigenvalues of our problem. In this section we also prove the existence of the Green's function for (1.1–2), and with the aid of the Green's function we are able to derive some further relevant results. We remark that the arguments used in the construction of the Green's function are by no means standard due to the appearance of singularities arising from the periodicity of the functions concerned. Finally, there is an appendix, Appendix A, in which we establish some results that are required in §4.

2. Preliminaries

For $\mathbb{Z} \ni m \geq 0$ and E an open set in $\mathbb{R}^2$, let $H^m(E)$ denote the usual Sobolev space of order m related to $L^2(E)$ and let $(\ ,\ )_{m,E}$ and $\|\ \|_{m,E}$ denote the inner product and norm, respectively, in $H^m(E)$. Also let $C^\infty_{0,\pi}$ denote the class of functions $u \in C^\infty(\overline{I}_{\alpha,\beta})$ which are periodic in x_2 of period 2π and which satisfy the condition $u(x) = 0$ for $x \in \partial I_{\alpha,\beta}$. Lastly, let $H^m_{0,\pi}(\Omega)$ denote the closure of $C^\infty_{0,\pi}|\Omega$ in $H^m(\Omega)$.

Of particular importance to us is the space $H^1_{0,\pi}(\Omega)$, and in order to deal with this space we introduce the following notation. Recalling from above the definition of Ω', suppose that $u \in H^1(\Omega)$ and let tr denote trace. Then for brevity we shall henceforth write $\operatorname{tr} u|_{\Gamma_2} = \operatorname{tr} u|_{\Gamma_4}$ if $(\operatorname{tr} u)(x_1, 0) = \operatorname{tr} u(x_1, 2\pi)$ in $L^2(\Omega')$. On the other hand, if u is just a function which is defined and continuous on $\Gamma_2 \cup \Gamma_4$, then we shall write $u|_{\Gamma_2} = u|_{\Gamma_4}$ if $u(x_1, 0) = u(x_1, 2\pi)$ for $x_1 \in \Omega'$.

Proposition 2.1. *It is the case that $H^1_{0,\pi}(\Omega)$ is the set of all $u \in H^1(\Omega)$ such that* $\operatorname{tr} u = 0$ *on Γ_r for $r = 1$ and $r = 3$, while* $\operatorname{tr} u|_{\Gamma_2} = \operatorname{tr} u|_{\Gamma_4}$.

Proof. Suppose firstly that $u \in H^1_{0,\pi}(\Omega)$. Then we may appeal to Nikodym's theorem [13, p. 73] and argue as in the proof of Lemma 3.5 of [13, p. 178] to deduce that $\operatorname{tr} u = 0$ on Γ_r for $r = 1, 3$. Next let $\Omega_1 = \{x \in I_{\alpha,\beta} | 0 < x_2 < 4\pi\}$ and let $H^1_{0,\pi}(\Omega_1)$ denote the closure of $C^\infty_{0,\pi}|\Omega_1$ in $H^1(\Omega_1)$. Then it is clear that $H^1_{0,\pi}(\Omega) = H^1_{0,\pi}(\Omega_1)|\Omega$, and hence we can again argue as in [13, pp. 178–180] to arrive at the conclusion that $\operatorname{tr} u|_{\Gamma_2} = \operatorname{tr} u|_{\Gamma_4}$.

Suppose next that $u \in H^1(\Omega)$ and $\operatorname{tr} u = 0$ on Γ_r for $r = 1, 3$, while $\operatorname{tr} u|_{\Gamma_2} = \operatorname{tr} u|_{\Gamma_4}$. If u_0 denotes the extension of u to $I_{\alpha,\beta}$ by periodicity, then we may argue in a manner somewhat similar to that in [13, pp. 184–185] to show that $u_0 \in H^1(\Omega^\dagger)$ for every bounded open set $\Omega^\dagger \subset I_{\alpha,\beta}$ and that $\operatorname{tr} u_0 = 0$ on the faces $x_1 = \alpha_1$ and $x_1 = \beta_1$ of $I_{\alpha,\beta}$. Now let $\phi_1(x_1) \in C_0^\infty(\mathbb{R})$ such that $0 \leq \phi_1(x_1) \leq 1$, $\phi_1(x_1) = 1$ for $(5\alpha_1 - \beta_1)/4 \leq x_1 \leq (5\beta_1 - \alpha_1)/4$, and $\operatorname{supp} \phi_1 \subset ((3\alpha_1 - \beta_1)/2, (3\beta_1 - \alpha_1)/2)$, where supp denotes support, and let us introduce the extension u_1 of u_0 to $\mathbb{R}^2$ as follows:

$$u_1(x) = \begin{cases} u_0(x) \text{ for } x \in I_{\alpha,\beta}\,, \\ -\phi_1(x_1)u_0(T_1x) \text{ for } (3\alpha_1 - \beta_1)/2 < x_1 < \alpha_1\,, \\ -\phi_1(x_1)u_0(S_1x) \text{ for } \beta_1 < x_1 < (3\beta_1 - \alpha_1)/2, \\ 0 \text{ for } x_1 \leq (3\alpha_1 - \beta_1)/2 \text{ and for } x_1 \geq (3\beta_1 - \alpha_1)/2, \end{cases}$$

where T_1 and S_1 denote the transformations in $\mathbb{R}^2$ given by:

$$T_1 : \; x = (x_1, x_2) \mapsto (2\alpha_1 - x_1, x_2),$$
$$S_1 : \; x = (x_1, x_2) \mapsto (2\beta_1 - x_1, x_2).$$

Then again we can argue as in [13, pp. 184–185] to show that $u_1 \in H^1_{\text{loc}}(\mathbb{R}^2)$ and that u_1 is periodic in x_2 of period 2π.

For $t \in \mathbb{R}$ let

$$j^{\dagger}(t) = \begin{cases} c\exp\{-(1-t^2)^{-1}\} \text{ for } |t| < 1 \\ 0 \text{ for } |t| \geq 1, \end{cases}$$

where the constant c is chosen so that $\int_{\mathbb{R}} j^{\dagger}(t)\,dt = 1$, and for $\varepsilon > 0$ let $j^{\dagger}_{\varepsilon}(t) = \varepsilon^{-1} j^{\dagger}(\varepsilon^{-1}t)$. For $p \in \mathbb{N}$ satisfying $1/p < \min_{1 \leq r \leq 2}\{(\beta_r - \alpha_r)/4\}$, let $j_{1/p}(x) = \prod_{r=1}^{2} j^{\dagger}_{1/p}(x_r)$ and $f_p(x) = (j_{1/p} * u_1)(x)$ for $x \in \mathbb{R}^2$, where $*$ denotes convolution. Then $f_p(x) \in C^{\infty}(\mathbb{R}^2)$, and moreover, it is not difficult to verify that $f_p \in C^{\infty}_{0,\pi}$. Since $f_p|\Omega \to u$ in $H^1(\Omega)$ as $p \to \infty$, the proof of the proposition is complete. □

Proposition 2.2. *If* $u \in H^1_{0,\pi}(\Omega) \cap H^2(\Omega)$, *then* $\operatorname{tr} D_1 u|_{\Gamma_2} = \operatorname{tr} D_1 u|_{\Gamma_4}$ *and* $\operatorname{tr} D_2 u = 0$ *on* Γ_r *for* $r = 1, 3$.

Proof. To prove the first assertion, let v denote the trace of u on Γ_2, where we refer to the statements preceding Proposition 2.1 for terminology. Then for $\phi \in C^{\infty}_0(\Omega')$ we have

$$\int_{\Omega'} v(x_1)(D_1\phi)(x_1)\,dx_1 = \lim_{k\to\infty} \int_{\Omega'} u(x_1, t_k)(D_1\phi)(x_1)\,dx_1$$
$$= -\lim_{k\to\infty} \int_{\Omega'} (D_1 u)(x_1, t_k)\phi(x_1)\,dx_1 = -\int_{\Omega'} (\operatorname{tr} D_1 u)(x_1)\phi(x_1)\,dx_1,$$

in view of Nikodym's theorem and [13, p. 178], where $\{t_k\}$ denotes a null sequence of positive numbers and $\operatorname{tr} D_1 u$ denotes the trace of $D_1 u$ on Γ_2. Since an analogous result holds for the trace of u on Γ_4, the first assertion of the proposition follows immediately from Proposition 2.1. The proof of the second assertion is similar. □

Proposition 2.3. *It is the case that*

$$H^2_{0,\pi}(\Omega) = \{u | u \in H^1_{0,\pi}(\Omega) \cap H^2(\Omega),\ \operatorname{tr} D_2 u|_{\Gamma_2} = \operatorname{tr} D_2 u|_{\Gamma_4}\}.$$

Proof. Suppose firstly that $u \in H^2_{0,\pi}(\Omega)$. Then clearly $u \in H^1_{0,\pi}(\Omega) \cap H^2(\Omega)$, while arguments similar to those used in the first part of the proof of Proposition 2.1 show that $\operatorname{tr} D_2 u|_{\Gamma_2} = \operatorname{tr} D_2 u|_{\Gamma_4}$.

Conversely, suppose that $u \in H^1_{0,\pi}(\Omega) \cap H^2(\Omega)$ and $\operatorname{tr} D_2 u|_{\Gamma_2} = \operatorname{tr} D_2 u|_{\Gamma_4}$. Then turning to the second part of the proof of Proposition 2.1 and beginning with the u under consideration here, let us define the functions $\{u_j\}_0^1$ precisely as we did before. We can again argue as in [13, pp. 184–185] to show that $u_0 \in H^2(\Omega^\dagger)$ for every bounded open subset $\Omega^\dagger \subset I_{\alpha,\beta}$ and that $u_1 \in H^2_{\text{loc}}(\mathbb{R}^2)$. Thus if we define the f_p as before, then $f_p \in C^\infty_{0,\pi}$ and $f_p|\Omega \to u$ in $H^2(\Omega)$ as $p \to \infty$. Since this is precisely the result we wanted, the proof of the proposition is complete. □

Finally for later use we shall require one further result. To this end let C^∞_π denote the class of functions $u \in C^\infty(\overline{I}_{\alpha,\beta})$ which are periodic in x_2 of period 2π, and for $\mathbb{Z} \ni m \geq 0$, let $H^m_\pi(\Omega)$ denote the closure of $C^\infty_\pi|\Omega$ in $H^m(\Omega)$. Also we shall henceforth put $\mathcal{H} = L^2(\Omega)$ and let $(\ ,\ )$ and $\|\ \|$ denote the inner product and norm, respectively, in $\mathcal{H}$, while if γ denotes the multi-index (γ_1, γ_2), then we let $|\gamma| = \sum_{j=1}^{2} \gamma_j$ and $D^\gamma = D_1^{\gamma_1} D_2^{\gamma_2}$.

Proposition 2.4. *It is the case that* $H^0_\pi(\Omega) = H^0_{0,\pi}(\Omega) = \mathcal{H}$, *while for* $m > 0$,

$$H^m_\pi(\Omega) = \{u | u \in H^m(\Omega),\ \operatorname{tr} D^\gamma u|_{\Gamma_2} = \operatorname{tr} D^\gamma u|_{\Gamma_4} \text{ for } 0 \leq |\gamma| < m\}.$$

Proof. It is clear that $C^\infty_0(\Omega) \subset H^0_{0,\pi}(\Omega) \subset H^0_\pi(\Omega) \subset \mathcal{H}$, and hence the first statement follows since $C^\infty_0(\Omega)$ is dense in $\mathcal{H}$.

When $m > 0$, then the proof of the proposition is similar to that of Proposition 2.1. Indeed, if $u \in H^m_\pi(\Omega)$, then we can argue as in the proof just cited to show that $\operatorname{tr} D^\gamma u|_{\Gamma_2} = \operatorname{tr} D^\gamma u|_{\Gamma_4}$ for $0 \leq |\gamma| < m$. Conversely, if $u \in H^m(\Omega)$, $\operatorname{tr} D^\gamma u|_{\Gamma_2} = \operatorname{tr} D^\gamma u|_{\Gamma_4}$ for $0 \leq |\gamma| < m$, and u_0 denotes the extension of u to $I_{\alpha,\beta}$ by periodicity, then we can show as before that $u_0 \in H^m(\Omega^\dagger)$ for every bounded open set $\Omega^\dagger \subset I_{\alpha,\beta}$. However, with $\phi_1(x_1)$ as above, we now define the extension u_1 of u_0 to $\mathbb{R}^2$ as follows:

$$u_1(x) = \begin{cases} u_0(x) \text{ for } x \in I_{\alpha,\beta}, \\ \phi_1(x_1) \sum_{k=1}^{m} c_k u_0(T_{1k}x) \text{ for } (3\alpha_1 - \beta_1)/2 < x_1 < \alpha_1, \\ \phi_1(x_1) \sum_{k=1}^{m} c_k u_0(S_{1k}x) \text{ for } \beta_1 < x_1 < (3\beta_1 - \alpha_1)/2, \\ 0 \text{ for } x_1 \leq (3\alpha_1 - \beta_1)/2 \text{ and for } x_1 \geq (3\beta_1 - \alpha_1)/2, \end{cases}$$

where the c_k are chosen so that $\sum_{k=1}^{m} c_k(-1/k)^j = 1$ for $j = 0, \ldots, m-1$, and T_{1k} and S_{1k} denote the transformations in $\mathbb{R}^2$ given by:

$$T_{1k}:\ x = (x_1, x_2) \mapsto (k^{-1}[(k+1)\alpha_1 - x_1], x_2),$$
$$S_{1k}:\ x = (x_1, x_2) \mapsto (k^{-1}[(k+1)\beta_1 - x_1], x_2).$$

Then as before we can show that $u_1 \in H^m_{\text{loc}}(\mathbb{R}^2)$ and is periodic in x_2 of period 2π. Hence if we let $f_p = j_{1/p} * u_1$, then it follows immediately that $f_p \in C^\infty_\pi$ and $f_p|\Omega \to u$ in $H^m(\Omega)$ as $p \to \infty$. This completes the proof of the proposition. □

3. Some main results

We are now going to use the foregoing results to derive some of our main results. To this end let $V = H^1_{0,\pi}(\Omega)$ and on V let us introduce the sesquilinear form

(3.1)

$$B(u,v) = \sum_{i,j=1}^{2}(a_{ij}D_ju, D_iv) + \sum_{i=1}^{2}[(a_iu, D_iv) + (\overline{a_i}D_iu, v)] + (a_0u, v), \quad u, v \in V.$$

It is clear that B is densely defined in $\mathcal{H}$ and is symmetric, while it is also a simple matter to verify that B is continuous and coercive on V [2, p. 141]. Thus B is bounded from below and we henceforth let κ denote the lower bound of B [8, p. 310]. Then referring to [8, p. 315] for terminology, we have

Proposition 3.1. *It is the case that B is a closed form.*

Proof. It is clear that V, when equipped with the inner product $(\ , \ )_{1,\Omega}$, is a Hilbert space. Hence in view of [8, Theorem 1.11, p. 314], we see that the proposition will be proved if we can show that the inner products $(\ , \ )_{1,\Omega}$ and $\langle \ , \ \rangle = B(\ , \ ) - (\kappa - 1)(\ , \ )$ induce equivalent norms on V. Since the coefficient functions in B are bounded, there is a constant $C > 0$ such that $\langle u, u\rangle \leq C\|u\|^2_{1,\Omega}$ for all $u \in V$. On the other hand, it follows from the coerciveness of B that there exist constants $c_0 > 0$ and $c \geq 0$ such that $\langle u, u\rangle \geq c_0\|u\|_{1,\Omega} - (\kappa + c - 1)\|u\|^2$ for $u \in V$. By the definition of κ, $\langle u, u\rangle \geq \|u\|^2$, and hence $(1 + |\kappa + c - 1|)\langle u, u\rangle \geq c_0\|u\|^2_{1,\Omega}$. □

We henceforth let A denote the selfadjoint operator in $\mathcal{H}$ associated with B [8, Theorem 2.6, p. 323]. Then in order to characterize the domain of A, $D(A)$, let us introduce the following variational problem: given $f \in \mathcal{H}$, find an element $u \in V$ such that

$$B(u, v) = (f, v) \text{ for every } v \in V. \tag{3.2}$$

Proposition 3.2. *Let u be a solution of (3.2) corresponding to f. Then $u \in H^2(\Omega)$ and $Lu = f$ in $\mathcal{H}$.*

In order to prove the proposition we require some preliminary results. To this end, let us introduce the following notation. For $x^0 \in \Omega$, choose R so that $0 < R < \text{dist}\{x^0, \Gamma\}$. If $x^0 \in \Gamma_r$ for $r = 1$ or $r = 3$, then choose R so that $0 < R < \text{dist}\{x^0, \Gamma \setminus \Gamma_r\}$. If $x^0 \in \Gamma_2 \cup \Gamma_4$, then choose R so that $0 < R < \min\{\pi, \text{dist}\{x^0, \Gamma \setminus (\Gamma_2 \cup \Gamma_4)\}\}$ and let $x^1 = x^0 + 2\pi e_2$ (resp. $x^0 - 2\pi e_2$) if $x^0 \in \Gamma_2$ (resp. $x^0 \in \Gamma_4$), where for $1 \leq j \leq 2$, e_j denotes the unit vector in $\mathbb{R}^2$ parallel to and pointing in the direction of the positive j-th axis. Lastly, if $x^0 \in \Gamma \setminus \bigcup_{r=1}^{4} \Gamma_r$, then choose R so that $0 < 2R < \text{dist}\{x^0, \Gamma \setminus \bigcup_{j=1,2} \overline{\Gamma'_j}\}$, where $\Gamma'_j = \Gamma_j$ if $x^0_j = \alpha_j$ and $\Gamma'_j = \Gamma_{j+2}$ if $x^0_j = \beta_j$, and let $x^1 = x^0 + 2\pi e_2$ (resp. $x^0 - 2\pi e_2$) if $x^0_2 = 0$ (resp. $x^0_2 = 2\pi$).

Let R' be any number satisfying $0 < R' < R$, let $R'' = (R + R')/2$, $R''' = (R + R'')/2$, and for $x \in \mathbb{R}^2$ and $r > 0$, let $S_r(x)$ denote the open ball in $\mathbb{R}^2$ with centre x and radius r and put $\Sigma_r(x) = S_r(x) \cap \Omega$, $\overline{\Sigma}_r(x) = S_r(x) \cap \overline{\Omega}$. If $x^0 \in \Omega$, then let $G = S_R(x^0)$, $G' = \overline{G}' = S_{R'}(x^0)$, $G'' = S_{R''}(x^0)$, $G''' = S_{R'''}(x^0)$. If $x^0 \in \Gamma_r$ for $r = 1$ or $r = 3$, then let $G = \Sigma_R(x^0)$, $G' = \Sigma_{R'}(x^0)$, $\overline{G}' = \overline{\Sigma}_{R'}(x^0)$, $G'' = \Sigma_{R''}(x^0)$, and $G''' = \Sigma_{R'''}(x^0)$. If $x^0 \in \Gamma_2 \cup \Gamma_4$ or if $x^0 \in \Gamma \setminus \bigcup_{r=1}^{4} \Gamma_r$, then let $G = \Sigma_R(x^0) \cup \Sigma_R(x^1)$, $G' = \Sigma_{R'}(x^0) \cup \Sigma_{R'}(x^1)$, $\overline{G}' = \overline{\Sigma}_{R'}(x^0) \cup \overline{\Sigma}_{R'}(x^1)$, $G'' = \Sigma_{R''}(x^0) \cup \Sigma_{R''}(x^1)$, and $G''' = \Sigma_{R'''}(x^0) \cup \Sigma_{R'''}(x^1)$. Finally, let $\mathcal{F}$ denote the class of functions in $C^\infty(\mathbb{R}^2)$ which are periodic in x_2 of period 2π and with the property that for every $\psi \in \mathcal{F}$, $\operatorname{supp}\psi \cap \{x \in \mathbb{R}^n | x_2^0 - \pi \le x_2 < x_2^0 + \pi\} \subset S_R(x^0)$.

In the following lemma we introduce the form B_G which is obtained from B by replacing in (3.1) the inner product $(\ ,\ )$ by the inner product $(\ ,\ )_{0,G}$.

Lemma 3.1. *Suppose that $u \in H^1(G)$ and that $\psi u \in V$ for every $\psi \in \mathcal{F}$, where ψu is defined to be zero in $\Omega \setminus G$. Suppose also that $f \in L^2(G)$ and that $B_G(u, \psi\phi) = (f, \psi\phi)_{0,G}$ for every $\psi \in \mathcal{F}$ and $\phi \in C_{0,\pi}^\infty$. Then $D_k u \in H^1(G')$ and there is a constant c depending only upon L, x^0, R, and R' such that*

$$\|D_k u\|_{1,G'} \le c[\|f\|_{0,G} + \|u\|_{1,G}], \tag{3.3}$$

where: (1) $1 \le k \le 2$ *if* $x^0 \in \Omega$, (2) $k = 2$ *if* $x^0 \in \Gamma_1$ *or* $x^0 \in \Gamma_3$, (3) $k = 1$ *if* $x^0 \in \Gamma_2 \cup \Gamma_4$, *and* (4) $k = 2$ *if* $x^0 \in \Gamma \setminus \bigcup_{r=1}^{4} \Gamma_r$.

Proof. We will prove the lemma by the difference quotient method. Accordingly, let ζ be an element of $\mathcal{F}$ satisfying $0 \le \zeta(x) \le 1$, $\zeta(x) = 1$ for $x \in S_{R'}(x^0)$, $\zeta(x) = 0$ for $x \notin S_{R''}(x^0)$ and $x_2^0 - \pi \le x_2 < x_2^0 + \pi$. Then by hypothesis there exist a $\psi \in \mathcal{F}$ and $v \in V$ such that $\psi u = v|G$, $u|G''' = v|G'''$, and

$$B(v, \zeta\phi) = (f, \zeta\phi)_{0,G} \text{ for every } \phi \in C_{0,\pi}^\infty. \tag{3.4}$$

Now let us firstly fix our attention upon the case $x^0 \in \Omega$, and for $0 < |h| < (R - R'')/4$ and $1 \le k \le 2$, let $v_h(x) = (\delta_h \zeta v)(x)$, where $(\delta_h g)(x) = (g(x + he_k) - g(x))/h$. Then observing that $v_h \in V$, we can argue with the expression $B(v_h, \phi)$ precisely as in the proof of Lemma 9.2 of [2, p. 107] and appeal to (3.4) to deduce that

$$|B(v_h, v_h)| \le C\|v_h\|_{1,G}[\|f\|_{0,G} + \|v\|_{1,G'''}], \tag{3.5}$$

where here and below C denotes a generic constant which may vary from inequality to inequality, but can depend only upon those quantities as asserted for the constant c in the statement of the lemma. Then since B is coercive over V, we see that $\|v_h\|_{1,G} \le C[\|f\|_{0,G} + \|v\|_{1,G'''}]$, and hence we conclude that $D_k\zeta v \in H^1(G)$, $D_k v \in H^1(G')$, and

$$\|D_k v\|_{1,G'} \le C[\|f\|_{0,G} + \|v\|_{1,G'''}]. \tag{3.6}$$

If $x^0 \in \Gamma_r$ for $r = 1$ or $r = 3$, then we can define v_h as above, except now in the definition of δ_h we choose $k = 2$ and argue with the expression $B(v_h, \phi)$ as above to deduce that $D_2 v \in H^1(G')$ and that (3.6) holds with $k = 2$. The case

$x^0 \in \Gamma_2 \cup \Gamma_4$ can be treated analogously; here, in the definition of δ_h, we choose $k = 1$.

Turning finally to the situation where $x^0 \in \Gamma \setminus \bigcup_{r=1}^{4} \Gamma_r$, let us henceforth agree to take $k = 2$ in the definition of δ_h above. Then it is clear that $\zeta v \in V$, while this will also be the case for v_h provided that we define $v(x)$ for $x_2 < 0$ and $x_2 > 2\pi$ by periodicity (see the proof of Proposition 2.1). Hence defining $v(x)$ in this way for $x_2 < 0$ and $x_2 > 2\pi$, we can argue with the expression $B(v_h, \phi)$ in a manner somewhat similar to that in [2, pp. 109–110] and appeal to (3.4) to arrive at the inequality (3.5). In light of this result we can now argue as before to show that $D_2 v \in H^1(G')$ and that (3.6), with $k = 2$, is again valid.

In light of the foregoing results and the definition of v, the proof of the lemma is complete. □

Proof of Proposition 3.2. It is clear that $u|G$ and $f|G$ satisfy all the hypotheses of Lemma 3.1. Hence if $x^0 \in \Omega$, then we conclude from Lemma 3.1 that $u \in H^2(G')$ and that

$$\|u\|_{2,G'} \le C[\|f\|_{0,G} + \|u\|_{1,G}], \tag{3.7}$$

where the constant C depends only upon those quantities as asserted for the constant c of Lemma 3.1. It follows immediately that $u \in H^2_{\text{loc}}(\Omega)$, and hence if in (3.1) we take $v \in C_0^\infty(\Omega)$ and transfer all the differentiation from v onto u, then we conclude from (3.2) that $Lu = f$ in $\mathcal{H}$.

If $x^0 \in \Gamma_r$ for $1 \le r \le 4$, then we can appeal to Lemma 3.1 as well as to the fact that $Lu = f$ in $\mathcal{H}$, and argue as in [2, pp. 111–112] to deduce that $u \in H^2(G')$ and that (3.7) remains valid.

It follows from these results and Lemma 3.1 that each of the distributional derivatives $D_k D_2 u (= D_2 D_k u)$, $1 \le k \le 2$, is in $\mathcal{H}$, and hence if $x^0 \in \Gamma \setminus \bigcup_{r=1}^4 \Gamma_r$, then the same arguments as applied to the case $x^0 \in \Gamma_r$ show that $u \in H^2(G')$ and that (3.7) remains valid.

In light of these results all the assertions of the proposition now follow. □

Proposition 3.3. *It is the case that* $D(A) = H^2_{0,\pi}(\Omega)$ *and* $Au = Lu$ *in* $\mathcal{H}$ *for* $u \in D(A)$. *Moreover,*

$$\|u\|_{2,\Omega} \le c[\|Au\| + \|u\|] \text{ for } u \in D(A),$$

where the constant c *depends only upon* Ω *and* L.

Proof. Suppose that $u \in D(A)$. Then it follows from [8, Theorem 2.1, p. 322] that $B(u, v) = (Au, v)$ for every $v \in V$, and hence we conclude from Proposition 3.2 that $u \in H^2(\Omega)$ and $Au = Lu$ in $\mathcal{H}$. Now let $\phi \in C_0^\infty(\Omega')$, and in (3.2) let us take $v(x) = \phi(x_1)\cos x_2$, $f = Au$. Then, since arguments similar to those used in the proof of Proposition 2.2 show that $\operatorname{tr} D_1 u|_{\Gamma_2} = \operatorname{tr} D_1|_{\Gamma_4}$, we can, by appealing to Nikodym's theorem, transfer all the differentiation from v onto u in the expression on the left side of (3.2) and make use of the fact that $Lu = f$ in $\mathcal{H}$ to deduce that $\int_{\Omega'} a_{22}(x_1, 0)\chi(x_1)\phi(x_1)\, dx_1 = 0$, where $\chi(x_1) = (\operatorname{tr} D_2 u)(x_1, 2\pi) - (tr D_2 u)(x_1, 0)$. Thus it follows that $\operatorname{tr} D_2 u|_{\Gamma_2} = \operatorname{tr} D_2|_{\Gamma_4}$, and hence, in light of Proposition 2.3, we conclude that $u \in H^2_{0,\pi}(\Omega)$.

Conversely, if $u \in H^2_{0,\pi}(\Omega)$ and $v \in C^\infty_{0,\pi}$, then we may appeal to Nikodym's theorem and Propositions 2.2–3 to deduce that $B(u,v) = (Lu, v)$. Bearing in mind that $C^\infty_{0,\pi}$ is a core of B (see the proof of Proposition 3.1), we conclude from [8, Theorem 2.1, p. 322] that $u \in D(A)$ and $Au = Lu$ in $\mathcal{H}$. Thus all the assertions of the proposition, except the final one, have now been proved. However, the final assertion is an immediate consequence of the closed graph theorem if we bear in mind that $H^2_{0,\pi}(\Omega)$ is a closed subspace of $H^2(\Omega)$. □

In the sequel, when we refer to the spectrum, resolvent set, eigenvalues, or eigenfunctions of the problem (1.1–2), then this is to be understood as the spectrum, resolvent set, eigenvalues, or eigenvectors, respectively, of the operator A.

Theorem 3.1. *The spectrum of the problem* (1.1–2) *consists solely of eigenvalues of finite multiplicity which form a denumberably infinite subset of* $\mathbb{R}$ *having no finite points of accumulation. Moreover, these eigenvalues, counted according to multiplicity, may be arranged into a sequence* $\{\lambda_j\}_1^\infty$, *where* $\kappa = \lambda_1 \leq \lambda_2 \leq \ldots$, *and* $\lambda_j \to \infty$ *as* $j \to \infty$. *Finally, the corresponding eigenfunctions, suitably normalized, form an orthonormal basis for* $\mathcal{H}$.

Proof. If $\lambda < \kappa$, then it follows from the closed graph theorem that the mapping $(A - \lambda I)^{-1} : \mathcal{H} \to H^2(\Omega)$ is bounded, and so we conclude from Rellich's theorem [2, p. 30] that A has compact resolvent. Hence all the assertions of our theorem now follow from the above results and the spectral theorem. □

Theorem 3.2. *If* $u \in D(A)$, *then we may modify* u *on a set of measure zero and extend* u *to all of* $\overline{\Omega}$ *by continuity so that* $u \in C^{0,\mu}(\overline{\Omega})$ *and satisfies the boundary conditions* (1.2) *in the classical sense, where* μ *is any number satisfying* $0 < \mu < 1$. *Thus the eigenfunctions of the problem* (1.1–2) *are Hölder continuous in* $\overline{\Omega}$ *and satisfy the boundary conditions* (1.2) *in the classical sense.*

Proof. The assertions of the theorem are an immediate consequence of the Sobolev imbedding theorem [1, Theorem 5.4, p. 97], [2, Theorem 3.9, p. 32], and Proposition 2.1. □

In order to investigate the regularity of the eigenfunctions of the problem (1.1–2) we require the following results; and here we employ the same terminology as that of Lemma 3.1 and put $\Gamma^\dagger = \Gamma \setminus \bigcup_{r=1}^{4} \Gamma_r$. Moreover, we will adopt the convention that if a function defined on a proper subset of Ω is extended to all of Ω, then this extension is achieved by defining the function to be zero in the complementary set.

Lemma 3.2. *Suppose that* $x^0 \in \bigcup_{r=1}^{4} \Gamma_r$, *that* $u \in H^1(G)$, *and that* $\psi u \in V$ *for every* $\psi \in \mathcal{F}$. *Suppose also that* $u \in H^2(G')$ *and that* k *is the integer defined in Lemma* 3.1. *Then* $\psi D_k u \in V$ *for every* $\psi \in \mathcal{F}$ *for which* $\operatorname{supp}\psi \cap S_R(x^0) \subset S_{R'}(x^0)$.

Proof. If $\psi \in \mathcal{F}$ and $\operatorname{supp}\psi \cap S_R(x^0) \subset S_{R'}(x^0)$, then it is clear that $(D_k\psi)u \in V$ and $\psi u \in V \cap H^2(\Omega)$. Hence the lemma is an immediate consequence of Propositions 2.1–2. □

Definition 3.1. Recalling our assumptions concerning the coefficients of L given in § 1, we will henceforth say that the coefficients of L satisfy the condition (m),

$m \in \mathbb{N}$, if each a_{ij} and each a_i, for $1 \le i \le 2$, is actually the restriction to Ω of a function of class $C^{m,1}$ in $\overline{I}_{\alpha,\beta}$ which is periodic in x_2 of period 2π, while a_0 is actually the restriction to Ω of a function of class $C^{m-1,1}$ in $\overline{I}_{\alpha,\beta}$ which is periodic in x_2 of period 2π.

Lemma 3.3. *Suppose that m is a non-negative integer, that the coefficients of L satisfy the condition (m) if $m > 0$, and that either $x^0 \in \Omega$ or $x^0 \in \Gamma_r$ for $r = 1$ or $r = 3$. Suppose also that $u \in H^1(G)$ and that $\psi u \in V$ for every $\psi \in \mathcal{F}$. Finally, suppose that $f \in H^m(G)$ and that $B_G(u, \psi\phi) = (f, \psi\phi)_{0,G}$ for every $\psi \in \mathcal{F}$ and $\phi \in C^\infty_{0,\pi}$. Then $u \in H^{m+2}(G')$ and there is a constant c depending only upon m and those quantities as asserted for the constant c of Lemma 3.1, such that*

$$\|u\|_{m+2,G'} \le c[\|f\|_{m,G} + \|u\|_{1,G}]. \tag{3.8}$$

Proof. If k is the integer of Lemma 3.1, then our lemma is proved by arguing with the expression $B_G(u, D_k(\psi\phi))$ as in the proof of Theorem 9.7 of [2, p. 125] and by appealing to Lemma 3.2. □

Lemma 3.4. *Suppose that m is a non-negative integer, that the coefficients of L satisfy the condition (m) if $m > 0$, and that either $x^0 \in \Gamma_2 \cup \Gamma_4$ or $x^0 \in \Gamma^\dagger$. Suppose also that $u \in H^1(G)$ and that $\psi u \in V$ for every $\psi \in \mathcal{F}$. Finally, suppose that $f \in H^m_\pi(\Omega)|G$ and that $B_G(u, \phi\psi) = (f, \phi\psi)_{0,G}$ for every $\psi \in \mathcal{F}$ and $\phi \in C^\infty_{0,\pi}$. Then $u \in H^{m+2}_\pi(\Omega)|G'$ and (3.8) is valid.*

Proof. Let us put $R'_0 = (R' + R'')/2$, $G'_0 = \Sigma_{R'_0}(x^0) \cup \Sigma_{R'_0}(x^1)$ and firstly fix our attention upon the case $m = 0$. Then it follows from Lemma 3.1, with G' there replaced by G'_0, that $D_k u \in H^1(G'_0)$ and that (3.3) holds. Moreover, it follows from Lemma 3.3 that $u \in H^2_{\text{loc}}(G)$, and hence we can argue as in the proof of Proposition 3.2 to deduce that $u \in H^2(G'_0)$, that (3.8), with G' replaced by G'_0, is valid, and that $Lu = f$ in $L^2(G'_0)$. Now if $\psi \in \mathcal{F}$, $\operatorname{supp}\psi \cap S_R(x^0) \subset S_{R'_0}(x^0)$ and $\psi(x) = 1$ for $x \in S_{R'}(x^0)$, then $\psi u \in V \cap H^2(\Omega)$, and hence it follows from Proposition 2.2 that $\operatorname{tr} D_1(\psi u)|_{\Gamma_2} = \operatorname{tr} D_1(\psi u)|_{\Gamma_4}$ and that for $x^0 \in \Gamma^\dagger$, the trace of $D_2(\psi u)$ is zero on Γ_1 if $x^0_1 = \alpha_1$ and is zero on Γ_3 if $x^0_1 = \beta_1$. Since it follows from integration by parts, Proposition 2.1, and our above results that

$$\begin{aligned}(f, \overline{\psi}\phi)_{0,G} &= (Lu, \overline{\psi}\phi)_{0,G'_0}\\ &= B_G(u, \overline{\psi}\phi) + \int_{\Omega'} (a_{22}\overline{\phi} \operatorname{tr} D_2(\psi u))(x_1, 0)\, dx_1 - \int_{\Omega'} (a_{22}\overline{\phi} \operatorname{tr} D_2(\psi u))(x_1, 2\pi)\, dx_1\end{aligned}$$

for $\phi \in C^\infty_{0,\pi}$, we conclude that $\operatorname{tr} D_2(\psi u)|_{\Gamma_2} = \operatorname{tr} D_2(\psi u)|_{\Gamma_4}$. Thus as a consequence of these results and Propositions 2.1 and 2.4 we see that the lemma is true for $m = 0$.

Suppose next that $m > 0$ and that the lemma is true with m replaced by $m - 1$ and G' by G''. Then by the inductive process we also know that $Lu = f$ in $L^2(G'')$, that the trace of $D_2 u$ is zero on Γ'_1 if $x^0_1 = \alpha_1$ and is zero on Γ'_3 if $x^0_1 = \beta_1$, where $\Gamma'_j = \Gamma_j \cap (S_{R''}(x^0) \cup S_{R''}(x^1))$, and we may suppose, moreover, that if $v \in H^1(G'')$, $g \in H^{m-1}_\pi(\Omega)|G''$, $\psi v \in V$ for every $\psi \in \mathcal{F}_1 = \{\psi \in \mathcal{F} | \operatorname{supp}\psi \cap S_R(x^0) \subset S_{R''}(x^0)\}$, and $B_{G''}(v, \psi\phi) = (g, \psi\phi)_{0,G''}$ for every $\psi \in \mathcal{F}_1$ and $\phi \in C^\infty_{0,\pi}$, then $v \in H^{m+1}_\pi(\Omega)|G'_0$ and

$$\|v\|_{m+1,G'_0} \le c[\|g\|_{m-1,G''} + \|v\|_{1,G''}], \tag{3.9}$$

where the form $B_{G''}$ is obtained from the form B by replacing in (3.1) the inner product $(\, , \,)$ by the innner product $(\, , \,)_{0,G''}$, and where the constant c depends only upon those quantities as asserted for the constant c of (3.8). Hence if k denotes the integer of Lemma 3.1, then we may appeal to the foregoing results and argue with the expression $B_{G''}(u, D_k(\psi\phi))$ for $\psi \in \mathcal{F}_1$ and $\phi \in C^\infty_{0,\pi}$ precisely as in the proof of Theorem 9.7 of [2, p. 215] to deduce that $B_{G''}(D_k u, \psi\phi) = (f_k, \psi\phi)_{0,G''}$, where $f_k \in H^{m-1}_\pi(\Omega)|G''$. Since it follows from the foregoing results and Proposition 2.1 that $\psi D_k u \in V$ for every $\psi \in \mathcal{F}_1$, we conclude that $D_k u \in H^{m+1}_\pi(\Omega)|G'_0$ and that (3.9) holds with v replaced by $D_k u$ and g by f_k. We can also argue as in [2, pp. 127–128] to deduce that $u \in H^{m+2}(G'_0)$ and that (3.8) is valid. Thus in order to complete the proof of the lemma it remains only to show that $u \in H^{m+2}_\pi(\Omega)|G'$. To this end let $\psi \in \mathcal{F}_1$ such that $\operatorname{supp}\psi \cap S_{R''}(x^0) \subset S_{R'_0}(x^0)$ and $\psi(x) = 1$ for $x \in S_{R'}(x^0)$. Then it follows from above that $\psi u \in H^{m+2}(\Omega)$ and that $\operatorname{tr} D^\gamma(\psi u)|_{\Gamma_2} = \operatorname{tr} D^\gamma(\psi u)|_{\Gamma_4}$ for $|\gamma| < m+2$ and $\gamma \neq \gamma^\dagger$, where $\gamma^\dagger$ denotes the multi-index $(0, m+1)$ if $x^0 \in \Gamma_2 \cup \Gamma_4$ and the multi-index $(m+1, 0)$ otherwise. Hence we can now appeal to the fact that $Lu = f$ in $L^2(G'_0)$ and argue with Nikodym's theorem to deduce that $\operatorname{tr} D^{\gamma^\dagger}(\psi u)|_{\Gamma_2} = \operatorname{tr} D^{\gamma^\dagger}(\psi u)|_{\Gamma_4}$, which, in light of the above results and Proposition 2.4, completes the proof of the lemma. □

Proposition 3.4. *Suppose that $m \in \mathbb{N}$ and the coefficients of L satisfy the condition (m). Suppose also that $u \in D(A)$, $Au = f$, and $f \in H^m_\pi(\Omega)$. Then $u \in H^{m+2}_\pi(\Omega)$ and*

$$\|u\|_{m+2,\Omega} \leq c[\|f\|_{m,\Omega} + \|u\|],$$

where the constant c depends only upon m, Ω, and L. Moreover, we may modify u on a set of measure zero and extend u to all of $\overline{\Omega}$ by continuity so that $u \in C^{m,\mu}(\overline{\Omega})$, where the exponent μ is defined in Theorem 3.2.

Proof. Observing from the proof of Proposition 3.3 that u is a solution of (3.2) corresponding to f, we see that $\overline{\Omega}$ can be covered by means of a finite number of the sets $\overline{G}'$, where with G' denoting the sets of Lemmas 3.3–4, the $\overline{G}'$ are defined in the statements preceding Lemma 3.1. Hence all the assertions of the proposition are now an immediate consequence of Lemmas 3.3–4, the Interpolation Theorem [2, p. 24], and the Sobolev imbedding theorem. □

As a consequence of Propositions 3.3–4 we finally have

Theorem 3.3. *Suppose that $m \in \mathbb{N}$ and that the coefficients of L satisfy the condition (m). Then the eigenfunctions of the problem (1.1–2) are in $C^{m,\mu}(\overline{\Omega})$.*

4. Further results.

In this section we are going to establish some further results which will complement those of §3. Accordingly, let us write $L(x, D)$ for L (see the notation preceeding Proposition 2.4), let $L_0(x, D)$ denote the principal part of $L(x, D)$, and for $\xi \in \mathbb{R}^2$ let $\omega(x) = |\{\xi \in \mathbb{R}^2| - 1 < L_0(x, \xi) < 0\}|$, where here $|\ |$ denotes the 2-dimensional Lebesgue measure, and put $c_2 = (2\pi)^{-2} \int_\Omega \omega(x)\, dx$. Hence if for $t > \kappa$ we let $N(t)$ denote the number of eigenvalues, counted according to multiplicity, of the problem (1.1–2) which lie in the interval $[\kappa, t]$, then

Theorem 4.1. *For any ε satisfying $0 < \varepsilon < 1/8$ we have*

$$N(t) = c_2 t + O(t^{\frac{7}{8}+\varepsilon}) \text{ as } t \to \infty.$$

Proof. Without loss of generality we can suppose that $\kappa \geq 0$ since this can always be achieved , if necessary, by a shift in the spectral parameter λ. Then for λ not on the non-negative real axis, we know from [3] that $R_\lambda = (A - \lambda I)^{-1}$ is an integral operator with kernel $K_\lambda^\dagger(x,y)$ such that for each $x \in \Omega$, $\|K_\lambda^\dagger(x,\cdot)\| \leq C_1\|R_\lambda\|_{2,\Omega}^{1/2}\|R_\lambda\|_{0,\Omega}^{1/2}$, and that the mapping $K_\lambda^\dagger(x,\cdot) : \Omega \to \mathcal{H}$ is continuous, where the constant c_1 does not depend upon x nor λ and for $0 \leq j \leq 2$, $\|R_\lambda\|_{j,\Omega}$ denotes the supremum of $\|R_\lambda f\|_{j,\Omega}$ over the set $\{f \in \mathcal{H} | \|f\| = 1\}$. Moreover, we note from [2, Theorem 13.5, p. 211] that R_λ is generated by a Hilbert-Schmidt kernel $K_\lambda^\#(x,y)$, and it is not difficult to verify that $K_\lambda^\#(x,\cdot) - K_\lambda^\dagger(x,\cdot) = 0$ and $K_\lambda^\#(\cdot,x) - \overline{K_{\bar\lambda}^\dagger(x,\cdot)} = 0$ in $\mathcal{H}$ for almost every $x \in \mathcal{H}$. Also, we can appeal to Proposition 3.3 and the Interpolation Theorem [2, p.24] to show that $\|R_\lambda\|_{j,\Omega} \leq C_2|\lambda|^{j/2}/d(\lambda)$ for $|\lambda| \geq 1$ and $0 \leq j \leq 2$, where the constant C_2 does not depend upon j nor λ and $d(\lambda) = |\operatorname{Im}\lambda|$ if $\operatorname{Re}\lambda > 0$ and $d(\lambda) = |\lambda|$ otherwise. Hence it follows that if μ is not on the non-negative real axis and λ denotes the square root of μ having non-negative real part, then $(A^2 - \mu I)^{-1} = R_\lambda R_{-\lambda}$ is a trace class operator (see [2,Theorem 12.21, p. 205]) as well as an integral operator with kernel $K_\mu(x,y) = \int_\Omega K_\lambda^\dagger(x,t)\overline{K_{-\bar\lambda}^\dagger(y,t)}\,dt$ which is continuous and bounded in $\Omega\times\Omega$, and in this set $|K_\mu(x,y)| \leq C_3|\mu|^{1/2}/d(\mu)$ for $|\mu| \geq 1$, where the constant C_3 does not depend upon x, y, nor μ.

Next for $x \in \Omega$, let

$$g(x) = (2\pi)^{-2}\int_{\mathbb{R}^2}(L_0^2(x,\xi)+1)^{-1}\,d\xi$$

and $\delta(x) = \min\{1, \operatorname{dist}\{x,\Gamma\}\}$. Then by appealing to the foregoing result and by arguing precisely as in the proof of Theorem 3 of [4] we readily deduce that if a, b, and δ are any positive numbers satisfying $a + b < 1$, $100\delta < \min\{\beta_1 - \alpha_1, 2\pi\}$, and if $\sigma = 1 - a/4$, then

$$K_\mu(x,x) - (-\mu)^{-1/2}g(x) = O(|\mu|^\tau) \tag{4.1}$$

for $d(\mu) \geq |\mu|^\sigma$, $|\mu| \geq \delta^{-4/b}$, $x \in \Omega_\delta = \{x \in \Omega | \delta(x) > \delta\}$, where that branch of $(-\mu)^{-1/2}$ is taken which is positive for $\mu < 0$, $\tau = \frac{1}{2}(a-1) - b/4$, and the constant implied in the O symbol does not depend upon μ nor δ. On the other hand, since it is easy to show that

$$\int_{\Omega-\Omega_\delta} K_\mu(x,x)\,dx = (-\mu)^{-1/2}\int_{\Omega-\Omega_\delta} g(x)\,dx + O(|\mu|^\tau)$$

for $d(\mu) \geq |\mu|^\sigma$ and $|\mu| = \delta^{-4/b}$, where the constant implied in the O symbol does not depend upon μ nor δ, and since δ is arbitrary, we conclude from (4.1) that

$$\int_\Omega K_\mu(x,x)\,dx = (-\mu)^{-1/2}\int_\Omega g(x)\,dx + O(|\mu|^\tau) \text{ as } |\mu| \to \infty \tag{4.2}$$

for $d(\mu) = |\mu|^\sigma$. Finally, since we know from [4] that $\int_\Omega K_\mu(x,x)\,dx = \int_0^\infty (t^2 - \mu)^{-1}\,dN(t)$, the assertion of the theorem follows immediately from from the Tauberian theorem of Malliavin (cf. [4]) if we put $a + b = 1 - \varepsilon'$, $0 < \varepsilon' < 1$, and bear in mind that $\int_\Omega g(x)\,dx = \pi c_2/2$. □

For the remainder of this section it will be supposed that $\kappa > 0$, in which case $0 \in \rho(A)$; and for reasons explained in the proof of Theorem 4.1 it is clear that this supposition involves no loss of generality. Then we know from the proof just cited that A^{-1} is an integral operator with a Hilbert-Schmidt kernel; and we are now going to show that if the coefficients of L are sufficiently smooth, then this kernel is precisely the Green's function of the problem (1.1–2). Note that by a Green's function of the problem (1.1–2) we mean a function $G(x,y)$ of the two variables $x, y \in \overline{\Omega}$ such that G is measurable on $\overline{\Omega} \times \overline{\Omega}$, $\int_{\overline{\Omega}\times\overline{\Omega}} |G(x,y)|^2\, dx\, dy < \infty$, and there holds the identity

$$u(y) = \int_{\Omega} (Lu)(x)\overline{G(x,y)}\, dx \tag{4.3}$$

for $y \in \Omega$ and $u \in C^{2,\mu}_{0,\pi}(\overline{\Omega}) = C^{2,\mu}(\overline{\Omega}) \cap H^2_{0,\pi}(\Omega) \cap H^4_{\pi}(\Omega)$, where $0 < \mu < 1$. We shall see below that such a $G(x,y)$ exists and is unique.

We will suppose for the remainder of this section that the coefficients of L satisfy the condition (2) (see Definition 3.1) and for $x \in \overline{\Omega}$ let $(b_{ij}(x))$ denote the inverse matrix of $(\operatorname{Re} a_{ij}(x))$ and let $\delta(x) = \det(\operatorname{Re} a_{ij}(x))$. For $k \in \mathbb{Z}$ and $x, y \in \overline{\Omega}$, let

$$\begin{aligned} \rho_k(x,y) &= \sum_{i,j=1}^{2} b_{ij}(y)(x_i - y_i(k))(x_j - y_j(k)), \\ \rho_k^{\alpha}(x,y) &= \rho_k(x,y) + 4a_{11}(y)^{-1}(x_1 - \alpha_1)(y_1 - \alpha_1), \\ \rho_k^{\beta}(x,y) &= \rho_k(x,y) + 4a_{11}(y)^{-1}(\beta_1 - x_1)(\beta_1 - y_1), \end{aligned}$$

where $y(k) = y + 2\pi k e_2$, and put

$$H_1(x,y) = \left(2\pi\delta(y)^{1/2}\right)^{-1} \sum_{k=-1}^{1} \log[\rho_k(x,y)/\rho_k^{\alpha}(x,y)\rho_k^{\beta}(x,y)]^{-1/2}.$$

Lastly, let $X = \{(x,y) \in \overline{\Omega} \times \overline{\Omega} |$ either $x = y$ or $x_1 = y_1$ and $|x_2 - y_2| = 2\pi\}$ and for $0 < \varepsilon < (\beta_1 - \alpha_1)/2$ let $\Omega(\varepsilon) = \{x \in \overline{\Omega} | \alpha_1 + \varepsilon \le x_1 \le \beta_1 - \varepsilon\}$. Then writing D_x for D (see the notation preceeding Proposition 2.4) in order to indicate that the differentiation is with respect to the x_j, we have next

Theorem 4.2. *It is the case that the Green's function, $G(x,y)$, of the problem (1.1–2) exists, is unique, and has the following properties: (1) $\chi(x,y) = G(x,y) - H_1(x,y)$ is continuous on $\overline{\Omega} \times \overline{\Omega}$, $\partial\chi(x,y)/\partial x_j$ is continuous in $(\overline{\Omega} \times \overline{\Omega}) \setminus X$ and in this set $\partial\chi(x,y)/\partial x_j = O(\sum_{k=-1}^{1} |\log|x - y(k)||)$, while $\partial^2\chi(x,y)/\partial x_i \partial x_j$ is continuous in $(\Omega(\varepsilon) \times \overline{\Omega}) \setminus X$ and in this set $\partial^2\chi(x,y)/\partial x_i\partial x_j = O(\sum_{k=-1}^{1} |\log|x - y(k)||/|x - y(k)|)$; (2) for each $y \in \Omega$, $L(x, D_x)G(x,y) = 0$ for $x \in \Omega \setminus \{y\}$, $G(x,y) = 0$ for $x_1 = \alpha_1$ or $x_1 = \beta_1$, and $D_x^{\gamma}G(\cdot,y)|_{\Gamma_2} = D_x^{\gamma}G(\cdot,y)|_{\Gamma_4}$ for $|\gamma| \le 1$; (3) $G(y,x) = \overline{G(x,y)}$ for $x, y \in \Omega$, $x \ne y$.*

Proof. We shall prove the existence of the Green's function with the aid of a parametrix (see [7]). Accordingly, for $y \in \overline{\Omega}$, let $L_0^y(D_x)$ denote $L_0(x,D)$ when the coefficients of L_0 are frozen at their values at the point y, and where we have written D_x for D for reasons explained above (in the sequel we will also write D_y for D when the differentiation is with respect to the y_j). Then arguing with the equation $L_0^y u = f$ in a manner somewhat similar to that in [14, pp. 4–5], we are led to take as parametrix the function $H(x,y)$ defined by

$$\begin{aligned} \pi\delta(y)^{1/2}H(x,y) &= \delta_1(y)(\beta_1 - x_1)(y_1 - \alpha_1)/2(\beta_1 - \alpha_1) \\ &+ \sum_{k=1}^{\infty} \frac{\sinh\{k\delta_1(y)(\beta_1 - x_1)\}\sinh\{k\delta_1(y)(y_1 - \alpha_1)\}}{k\sinh\{k\delta_1(y)(\beta_1 - \alpha_1)\}} \times \\ &\times \cos k\{\delta_2(y)(x_1 - y_1) - (x_2 - y_2)\} \\ &\text{for } y_1 \le x_1, \end{aligned} \tag{4.4}$$

while for $y_1 > x_1$, $\pi\delta(y)^{1/2}H(x,y)$ is given by the right side of (4.4) with $\beta_1 - x_1$ replaced by $\beta_1 - y_1$ and $y_1 - \alpha_1$ replaced by $x_1 - \alpha_1$, and where $\delta_1(y) = \delta(y)^{1/2}/a_{11}(y)$ and $\delta_2(y) = \operatorname{Re} a_{12}(y)/a_{11}(y)$ (observe from the definition that we may consider $H(x,y)$ as a function of the variables x and y for $x, y \in \overline{I}_{\alpha,\beta}$, with a similar remark also holding for the function $H_1(x,y)$ introduced above). Letting $\Omega_\pi = \{x \in \mathbb{R}^2 | \alpha_1 < x_1 < \beta_1, -\pi/2 < x_2 < 5\pi/2\}$, it is not difficult to verify that for $x, y \in \overline{\Omega}_\pi$, $H(x,y) = H_1(x,y) + H_2(x,y)$, where $H_2(x,y) \in C^{2,1}(\overline{\Omega}_\pi \times \overline{\Omega}_\pi)$ and $D_x^{\gamma^*} D_y^{\gamma^\dagger} H_2(x,y) \in C^0(\overline{\Omega}_\pi \times \overline{\Omega}_\pi)$ for $|\gamma^*| \ge 0$ and $|\gamma^\dagger| \le 2$. Moreover, for $(x,y) \in (\overline{\Omega} \times \overline{\Omega}) \backslash X$ we also have $L_0^y(D_x)H(x,y) = 0$ and $H(x,y) = 0$ for x_1 or y_1 equal to α_1 (resp. β_1), while for $y \in \Omega$, $D_x^\gamma H(\cdot,y)|_{\Gamma_2} = D_x^\gamma H(\cdot,y)|_{\Gamma_4}$ for $|\gamma| \ge 0$. Similarly, if $K(x,y) = (L(x,D) - L_0^y(D_x))H(x,y)$, then we can show that $K(x,y)$, $\partial K(x,y)/\partial x_j$, and $\partial K(x,y)/\partial y_j$ are continuous in $(\overline{\Omega} \times \overline{\Omega}) \setminus X$ with $K(x,y) = O(\sum_{k=-1}^{1} |x - y(k)|^{-1})$, $\partial K(x,y)/\partial x_j$ and $\partial K(x,y)/\partial y_j$ are $O(\sum_{k=-1}^{1} |x - y(k)|^{-2})$, while, for $|\gamma| \le 1$, $D_x^\gamma K(\cdot,y)|_{\Gamma_2} = D_x^\gamma K(\cdot,y)|_{\Gamma_4}$ for $y \in \Omega$ and $D_y^\gamma K(x,\cdot)|_{\Gamma_2} = D_y^\gamma K(x,\cdot)|_{\Gamma_4}$ for $x \in \Omega$. Lastly, we also have $K(x,y) = 0$ for $(x,y) \in (\overline{\Omega} \times \overline{\Omega}) \setminus X$ and $y_1 = \alpha_1$ or $y_1 = \beta_1$.

We will try to find a $G(x,y)$ of the form

$$G(x,y) = H(x,y) + \int_\Omega H(x,t)F(t,y)\,dt + \sum_{i=1}^{l} \phi_i(x)\overline{\psi_i(y)} \tag{4.5}$$

for suitable functions F, ϕ_i, and ψ_i and suitable integer l. Here we require $F(x,y)$ to be continuous in $(\overline{\Omega} \times \overline{\Omega}) \setminus X$ such that in this set $F(x,y) = O(\sum_{k=-1}^{1} |x - y(k)|^{-1})$, $F(x,y) = 0$ for $y_1 = \alpha_1$ or $y_1 = \beta_1$, while $F(\cdot,y)|_{\Gamma_2} = F(\cdot,y)|_{\Gamma_4}$ for $y \in \Omega$ and $F(x,\cdot)|_{\Gamma_2} = F(x,\cdot)|_{\Gamma_4}$ for $x \in \Omega$. We also require that $\phi_i \in C_{0,\pi}^{2,\mu}(\overline{\Omega})$ and $\psi_i \in C_{0,\pi}^{0,\mu}(\overline{\Omega})$ for $i = 1,\ldots,l$, where letting $C_\pi^0(\overline{\Omega}) = \{u \in C^0(\overline{\Omega}) | u|_{\Gamma_2} = u|_{\Gamma_4}\}$ and $C_\pi^{0,\mu}(\overline{\Omega}) = C_\pi^0(\overline{\Omega}) \cap C^{0,\mu}(\overline{\Omega})$, $C_{0,\pi}^{0,\mu}(\overline{\Omega}) = \{u \in C_\pi^{0,\mu}(\overline{\Omega}) | u(x) = 0 \text{ for } x_1 = \alpha_1 \text{ and } x_1 = \beta_1\}$. These functions are to be determined in such a manner so that (4.3) holds identically in u. Hence if we substitute (4.5) into (4.3), argue as in [7], and observe from Stoke's formula [12, p. 19] that

$$\int_\Omega Lu(x)\overline{H(x,y)}\,dx = \int_\Omega u(x)\overline{K(x,y)}\,dx + u(y) \text{ for } y \in \Omega,$$

then we conclude that in order that the expression (4.5) be a Green's function for the problem (1.1–2), it is necessary and sufficient that

$$(4.6)\quad F(x,y)+\int_\Omega K(x,t)F(t,y)\,dt=-K(x,y)-\sum_{i=1}^{l}L\phi_i(x)\overline{\psi}_i(y)\text{ for }x,y\in\Omega.$$

If we let $K_1(x,y)=K(x,y)$ and $K_r(x,y)=\int_\Omega K(x,t)K_{r-1}(t,y)\,dt$ for $r>1$, then we know from the proof of Lemma A.1 of Appendix A that $K_r(x,y)$ is continuous in $\overline{\Omega}\times\overline{\Omega}$ for $r\geq 3$, and hence we have all the Fredholm theory at our disposal in order to deal with the integral equation (4.6). In particular if the homogeneous equation

$$(4.7)\qquad v(x)+\int_\Omega K(x,y)v(y)\,dy=0$$

has only the trivial solution, the integral equation (4.6), with $\phi_i=\psi_i\equiv 0$ for $i=1,\ldots,l$, has a unique solution $F(x,y)$. Moreover, observing that for some prime $p>3$ the integral equation: $v(x)+\int_\Omega K_p(x,y)v(y)\,dy=0$ and the integral equation (4.7) have the same solutions [10, p. 84], we conclude from Corollary A.1 that $F(x,y)$ has the properties asserted above.

If (4.7) has a non-trivial solution, then the adjoint equation

$$(4.8)\qquad w(x)+\int_\Omega K^*(x,y)w(y)\,dy=0$$

has a finite number of linearly independent solutions $\{w_j(x)\}_1^l$, where $K^*(x,y)=\overline{K(y,x)}$. Putting $K_1^*(x,y)=K^*(x,y)$, $K_r^*(x,y)=\int_\Omega K^*(x,t)K_{r-1}^*(t,y)\,dt$ for $r>1$, let us observe that for some prime $p>3$ the integral equation: $w(x)+\int_\Omega K_p^*(x,y)w(y)\,dy=0$ and the integral equation (4.8) have the same solutions, and since $K_p^*(x,y)=(K_p(x,y))^*$, we also know from the proof of Lemma A.1 that K_p^* is continuous in $\overline{\Omega}\times\overline{\Omega}$. Thus it follows that the w_j are continuous in $\overline{\Omega}$, and hence we conclude from Remark A.2 and arguments similar to those used in the last paragraph of the proof of Lemma A.2 that $w_j\in C_{0,\pi}^{0,\mu}(\overline{\Omega})$ for $j=1,\ldots,l$. Now turning to the solutions of (4.6), we know that the condition for the existence of a solution of this equation is that

$$(4.9)\qquad w_j(y)-\sum_{i=1}^{l}(w_j,L\phi_i)\psi_i(y)=0\text{ for }j=1,\ldots,l,$$

and hence if we can find vectors $\{\phi_i\}_1^l$ in $C_{0,\pi}^{2,\mu}(\overline{\Omega})$ such that

$$(4.10)\qquad (w_j,L\phi_i)=\delta_{ij}\,,$$

where δ_{ij} is the Kronecker delta, then (4.9) is satisfied with $\psi_j=w_j$ for $j=1,\ldots,l$. Thus the existence of a Green's function for the problem (1.1–2) is reduced to the problem of finding vectors $\{\phi_j\}_1^l$ of the kind described satisfying (4.10). However, if we suppose that it is not possible to find such vectors, then we can argue as in [7] to show that there exists a non-trivial linear combination of the w_j, say w, such that $(w, L\phi) = 0$ for every $\phi \in C^{2,\mu}_{0,\pi}(\overline{\Omega})$, and hence in light of Propositions 3.3–4 and Theorems 3.1, 3.3 we arrive at a contradiction. We therefore conclude that the Green's function for the problem (1.1–2) exists, while in light of Corollary A.1 we also know that F and the ϕ_i and ψ_i have the properties asserted above.

Assertion (1) of the theorem as well as those parts of assertion (2) concerning the boundary values of G are an immediate consequence of (4.5) and Lemma A.2. In order to prove the remaining assertions we can now argue with Stoke's formula to deduce that $G(y,x) = \overline{G(x,y)}$ and $L(x, D_x)G(x,y) = 0$ for $x, y \in \Omega$, $x \neq y$. Finally, since we shall show in the proof of Theorem 4.4 below that $G(x,y)$ is precisely the kernel of the integral operator associated with A^{-1}, we conclude from our definition of a Green's function and Theorem 3.1 that it must be unique. This completes the proof of the theorem. □

Referring to [12, p. 2], Theorem 4.2, and the proofs of Lemmas A.1–2 for terminology, we have next

Theorem 4.3. *If $f \in C^{0,\mu}(\Omega) \cap L^1(\Omega)$ and $u(x) = \int_\Omega G(x,y)f(y)\,dy$, then $u \in C^{2,\mu}(\Omega)$, and for $x \in \Omega$ we have:*

$$(4.11) \qquad \partial u(x)/\partial x_j = \int_\Omega \partial G(x,y)/\partial x_j \, f(y)\,dy,$$

$$(4.12) \qquad \partial^2 u(x)/\partial x_k \partial x_j = -\frac{1}{2} b_{kj}(x) f(x) + \int_\Omega^* \partial^2 G(x,y)/\partial x_k \partial x_j f(y)\,dy,$$

$$(4.13) \qquad L(x, D_x)u(x) = f(x),$$

where the asterisk over the integral denotes the limit as $\varepsilon \to 0$ of the integral over $\Omega \setminus I_\varepsilon(x)$. Moreover, if $f \in C^{0,\mu}_\pi(\overline{\Omega})$, then $u \in C^{1,\mu}(\overline{\Omega})$ and $u = 0$ for $x_1 = \alpha_1$ or $x_1 = \beta_1$, $D^\gamma_x u|_{\Gamma_2} = D^\gamma_x u|_{\Gamma_4}$ for $|\gamma| \leq 1$, while $\partial^2 u/\partial x_2^2 \in C^{0,\mu}(\overline{\Omega})$ and $\partial^2/\partial x_k \partial x_j \in C^{0,\mu}(\Omega(\varepsilon))$ if $k + j < 4$.

Proof. If we recall from the proof of Lemma A.2 that $H(x,y) - R_0(x,y)$ is of class $C^{2,1}$ in $\Omega^1_\tau \times \overline{\Omega}$, then the first assertion of the lemma as well as (4.11–13) follow from [12, pp. 29–31] if we bear in mind (4.5) and Lemma A.2.

Suppose next that $f \in C^{0,\mu}_\pi(\overline{\Omega})$. Then it follows from what we have just shown that $u \in C^{2,\mu}(\Omega^1_\tau)$, and hence let us suppose that $x \in \Omega^2_\tau$. Then we see from (4.5) and the proof of Lemma A.2 that

$$(4.14) \qquad u(x) = \int_{\Omega^\dagger_\pi} R_0(x,y) f^\dagger(y)\,dy + g_2(x) \text{ for } x \in \Omega^2_\tau,$$

where $f^\dagger(y) = f(y) + \int_\Omega F(y,t)f(t)\,dt$, $g_2(x) \in C^{2,\mu}(\Omega^2_\tau)$, and $f^\dagger(y)$ is defined by periodicity in that portion of $\Omega^\dagger_\pi$ for which $y_2 < 0$ (see Lemma A.3). By arguing with the integral on the right side of (4.14) as in [12, pp. 29–30], we readily deduce that $u \in C^{2,\mu}(\Omega^2_\tau)$.

Turning to the case $x \in \Omega^3_\tau$, we see from (4.5) and the proof of Lemma A.2 that

$$(4.15) \qquad u(x) = \int_\Omega P_0(x,y) f^\dagger(y)\,dy + g_3(x),$$

where $g_3(x) \in C^{2,\mu}(\Omega_\tau^3)$. If $h(x)$ denotes the integral on the right side of (4.15), then a standard argument shows that $h \in C^{1,\mu}(\Omega_\tau^3)$, and hence so does u. Moreover, since it follows from the definition of P_0 that

$$\partial P_0(x,y)/\partial x_2 + \partial P_0(y,x)/\partial y_2 = O(1)$$
$$\partial^2 P_0(x,y)/\partial x_2^2 - \partial^2 P_0(y,x)/\partial y_2^2 = O(|x-y|^{-1})$$

for $x, y \in \overline{\Omega}$, $x \neq y$ (however, at least one of these bounds fails to hold for the other partial derivatives concerned), we can now argue as in [12, p. 30] to deduce that

$$\partial^2 h(x)/\partial x_2^2 = \int_\Omega [\frac{\partial^2 P_0(x,y)}{\partial x_2^2} f^\dagger(y) - \frac{\partial^2 P_0(y,x)}{\partial y_2^2} f^\dagger(x)]\, dy$$
$$+ f^\dagger(x)[\int_{\alpha_1}^{\beta_1} \frac{\partial P_0(y,x)}{\partial y_2}\Big|_{y_2=2\pi} dy_1 - \int_{\alpha_1}^{\beta_1} \frac{\partial P_0(y,x)}{\partial y_2}\Big|_{y_2=0} dy_1]$$

for $x \in \Omega_\tau^3$. It follows from this last equation that $\partial^2 h/\partial x_2^2 \in C^{0,\mu}(\Omega_\tau^3)$, and hence so does $\partial^2 u/\partial x_2^2$.

If $x \in \Omega_\tau^4$, then it follows from (4.5) and the proof of Lemma A.2 that

$$u(x) = \int_{\Omega_\pi^\dagger} P_0(x,y) f^\dagger(y)\, dy + g_4(x), \tag{4.16}$$

where $g_4 \in C^{2,\mu}(\Omega_\tau^4)$ and $f^\dagger(y)$ is defined by periodicity for $y_2 < 0$. We can now argue with (4.16) as we argued with (4.15) above to deduce that $u \in C^{1,\mu}(\Omega_\tau^4)$ and $\partial^2 u/\partial x_2^2 \in C^{0,\mu}(\Omega_\tau^4)$.

By continuing the foregoing arguments to the remaining portion of $\overline{\Omega}$, we can complete the verification of all the assertions of the theorem except those concerning the boundary values of u. Since these latter assertions can easily be proved by appealing to the properties of G cited in Theorem 4.2 and by using arguments similar to those used in the last paragraph of the proof of Lemma A.2, the proof of the theorem is complete. □

Recalling the definition of $H^2_{0,\pi}(\Omega)$ given in §2 and bearing in mind Proposition 3.3, we have next

Theorem 4.4. *If $f \in \mathcal{H}$ and $u(x) = \int_\Omega G(x,y) f(y)\, dy$, then $u \in H^2_{0,\pi}(\Omega)$ and $\|u\|_{2,\Omega} \le c\|f\|$, where the constant c does not depend upon f. Moreover,* (4.11–13) *are valid almost everywhere in Ω.*

Proof. From the properties of $G(x,y)$ and $H(x,y)$ cited in Theorem 4.2 and in the proof of Lemma A.2, respectively, it is easy to see that $\int_\Omega |D_x^\gamma G(x,y)| dy$ and $\int_\Omega |D_x^\gamma G(x,y)| dx$ are bounded in $\overline{\Omega}$ for $|\gamma| \le 1$. It follows immediately that $u \in H^1(\Omega)$ and that the distributional derivative of u with respect to x_j is given by the expression on the right side of (4.11). Thus by Nikodym's theorem we see that (4.11) holds almost everywhere in Ω.

Next let us observe from the properties of $G(x,y)$ and $H(x,y)$ that $G(x,y)$ is square-integrable on $\Omega \times \Omega$. Now suppose that $f \in C_0^\infty(\Omega)$. Then it follows from Theorem 4.3 that $u \in V \cap H^2_{\text{loc}}(\Omega)$ and that (4.11–13) hold for $x \in \Omega$, where V is defined in §3, while a simple argument also shows that $B(u,v) = (f,v)$ for every

$v \in C^{\infty}_{0,\pi}$. Since $C^{\infty}_{0,\pi}|\Omega$ is a core of B (see the proof of Proposition 3.1), we conclude from [8, Theorem 2.1, p. 322] that $u \in D(A)$. Hence we can now argue with a countable subset of $C^{\infty}_0(\Omega)$ which is dense in $\mathcal{H}$ to readily deduce that $G(x,y)$ is precisely the Hilbert-Schmidt kernel associated with the operator A^{-1}. Moreover, since $A^{-1} : \mathcal{H} \to H^2(\Omega)$ is bounded [2, Lemma 13.4, p. 210], it follows from (4.12) that if $M : C^{\infty}_0(\Omega) \to \mathcal{H}$ denotes the operator $(Mf)(x) = \int^*_\Omega \frac{\partial^2 G(x,y)}{\partial x_k \partial x_j} f(y)\,dy$, then M extends by continuity to a bounded linear operator from $\mathcal{H}$ into itself. Thus in view of these results and Nikodym's theorem, the proof of the theorem is complete. □

Remark 4.1. For the case where Ω is smooth and G is a Levi function [12, p. 18], Theorem 4.4, with $H^2_{0,\pi}(\Omega)$ replaced by $H^2(\Omega)$, is also given in [11], [12, Assertion 13, III, p. 31], and has been obtained by appealing to some results of Calderon and Zygmund (see [5], [15]).

Finally, we have

Theorem 4.5. *If $f \in H^1_\pi(\Omega)$ and $u(x) = \int_\Omega G(x,y)\partial f(y)/\partial y_j\,dy$, then $u \in V$ and*

$$\|u\|_{1,\Omega} \le \begin{cases} c\|f\| & \text{if } j = 2, \\ c_\varepsilon \|f_\varepsilon\| & \text{if } j = 1, \end{cases}$$

where ε is an arbitrary positive number, $f_\varepsilon(y) = (y_1 - \alpha_1)^{-\varepsilon}(\beta_1 - y_1)^{-\varepsilon} f(y)$, and the constants c, c_ε do not depend upon f, but c_ε depends upon ε.

Proof. That $u \in V$ is an immediate consequence of Theorem 4.4; and it also follows from this latter theorem that $\|u\|_{1,\Omega} \le C\|f\|_{1,\Omega}$ independently of j, where the constant C does not depend upon f. Thus it remains only to verify the bounds asserted in the present theorem. To this end let us firstly suppose that $f \in C^{\infty}_{\pi}|\Omega$. Then it follows from Theorem 4.3 that $u \in C^{1,\mu}(\overline{\Omega})$. Furthermore, bearing in mind Theorem 4.2, we readily deduce from an integration by parts that

$$\begin{aligned} u(x) &= -\int_\Omega \partial G(x,y)/\partial y_j f(y)\,dy \\ &= -\int_\Omega [\partial G(x,y)/\partial y_j + \partial G(x,y)/\partial x_j] f(y)\,dy + \int_\Omega \partial G(x,y)/\partial x_j f(y)\,dy \\ &= -u_1(x) + u_2(x) \quad \text{for } x \in \Omega. \end{aligned}$$

Since we already know from Theorem 4.4 that $u_2 \in H^1(\Omega)$ and $\|u_2\|_{1,\Omega} \le c\|f\|$ (here and below c denotes a generic constant which may vary from inequality to inequality, but does not depend upon f), we need only fix our attention henceforth upon u_1. Then it is clear from the definition of G that $u_1(x) = v_1(x) + v_2(x)$ for $x \in \Omega$, where

$$\begin{aligned} v_1(x) &= \int_\Omega \partial H(y,x)/\partial y_j f(y)dy + \int_\Omega \overline{F(t,x)}\Big(\int_\Omega \partial H(y,t)/\partial y_j f(y)\,dy\Big)dt \\ &\quad + \sum_{i=1}^{l} \psi_i(x) \int_\Omega \partial\overline{\phi_i(y)}/\partial y_j\, f(y)\,dy, \end{aligned} \tag{4.17}$$

$$v_2(x) = \int_\Omega \partial H(y,x)/\partial x_j f(y)dy + \int_\Omega \partial H(x,t)/\partial x_j \Big(\int_\Omega F(t,y)f(y)\,dy\Big)\,dt \tag{4.18}$$
$$+ \sum_{i=1}^{l} \partial\phi_i(x)\partial x_j \int_\Omega \overline{\psi_i(y)} f(y)\,dy,$$

while it also follows from Lemma A.3 and Remark A.2 that $v_j \in C_\pi^{0,\mu}(\overline{\Omega})$ for $j = 1, 2$.

In order to deal with the second term on the right side of (4.17), let us observe from Remark A.1 and the statements made in the last paragraph of the proof of Lemma A.1 that $F(t,x) = g(t,x) - \int_\Omega R(t,s)g(s,x)\,ds$, where $g(x,y)$ denotes the expression on the right side of (A.6), $R(t,s)$ is continuous in $\overline{\Omega}\times\overline{\Omega}$ and in this set $R(t,s) = 0$ if $s_1 = \alpha_1$ or $s_1 = \beta_1$, $R(t',s) - R(t,s) = O(|t'-t|^\mu)$ for $t' \in \overline{\Omega}$, $R(t,s') - R(t,s) = O(|s'-s|^\mu)$ for $s' \in \overline{\Omega}$, while $R(\cdot,s)|_{\Gamma_2} = R(\cdot,s)|_{\Gamma_4}$ for $s \in \overline{\Omega}$ and $R(t,\cdot)|_{\Gamma_2} = R(t,\cdot)|_{\Gamma_4}$ for $t \in \overline{\Omega}$. Hence if we put

$$h_1(t) = \int_\Omega \partial H(y,t)/\partial y_j\, f(y)\,dy,\ h^\#(t) = \overline{h_1(t)} + \int_\Omega R(y,t)\overline{h_1(y)}\,dy, \text{ and}$$
$$h^\dagger(t) = -\Big[h^\#(t) + \sum_{r=1}^{p-1}(-1)^r \int_\Omega K_r(s,t)h^\#(s)\,ds\Big],$$

then it follows from (4.17–18) and the above observation that $v_1(x) + v_2(x) = \sum_{j=1}^5 h_j(x)$ for $x \in \Omega$, where

$$h_2(x) = \int_\Omega \partial H(x,y)/\partial x_j\, f(y)\,dy,\ \overline{h_3(x)} = \int_\Omega K(t,x)h^\dagger(t)\,dt,$$
$$h_4(x) = \int_\Omega \partial H(x,t)/\partial x_j \Big(\int_\Omega F(t,y)f(y)\,dy\Big)\,dt,$$
$$h_5(x) = \sum_{i=1}^{l}[c_i(f)\psi_i(x) + d_i(f)\partial\phi_i(x)/\partial x_j],$$

and where $\sum_{i=1}^{l}[|c_i(f)| + |d_i(f)|] \le c\|f\|$. Thus, since

$$\int_\Omega |Q(x,y)|dy \text{ and } \int_\Omega |Q(x,y)|dx \text{ are bounded in } \Omega \tag{4.19}$$

for either $Q(x,y) = \partial H(x,y)/\partial x_j$ or $Q(x,y) = F(x,y)$ or $Q(x,y) = K_r(x,y)$ $(r \ge 1)$, we conclude that $\|h_j\| \le c\|f\|$ for $j = 1, \dots, 5$. Hence in light of Remark A.2, Nikodym's theorem, and the fact that $\psi_i(x) = -\int_\Omega \overline{K(y,x)}\psi_i(y)\,dy$ for $i = 1, \dots, l$, it is clear that in order to complete the proof of the theorem it remains only to show that the partial derivatives of the h_j, $j = 1, \dots, 4$, and the ψ_i, $i = 1, \dots, l$, exist almost everywhere in Ω and are in $\mathcal{H}$, and that for these values of j, to obtain estimates for the $\|\partial h_j/\partial x_k\|$.

Referring again to the proofs of Lemmas A.1–2 for terminology, let us suppose firstly that $x \in \Omega_\tau^1$. Then bearing in mind the results given in the statements following (4.4), we may argue in a manner similar to that in [12, pp. 29–30] to deduce that $\partial h_1(x)/\partial x_k$ exists and

$$\partial h_1(x)/\partial x_k$$
$$= -\lim_{\varepsilon\to 0}\int_{\Omega\setminus S_\varepsilon(x)} \partial^2 R_0(y,x)/\partial y_k\partial y_j\, f(y)\,dy + f(x)z_1(x) + \int_\Omega Z_1(y,x)f(y)\,dy,$$

where $z_1(x)$ is continuous in Ω_τ^1 and (4.19) holds for $Q(x,y) = Z_1(y,x)$ provided that in (4.19) we restrict x to the set Ω_τ^1 instead of to Ω. In light of [2, Lemma 11.1, p. 152] and a result of Calderon and Zygmund [5, Theorem 2], it follows immediately that $\|\partial h_1/\partial x_k\|_{0,\Omega_\tau^1} \le c\|f\|$. Similarly we can show that analogous results hold for h_j if $2 \le j \le 4$ and also that for $i = 1,\dots,l$, $\partial\psi_i(x)/\partial x_k$ exists everywhere in Ω_τ^1 and $\|\partial\psi_i/\partial x_k\|_{0,\Omega_\tau^1} < \infty$.

Suppose next that $x \in \mathring{\Omega}_\tau^2$, where $\mathring{\Omega}_\tau^j$ denotes the interior of Ω_τ^j. Then for this case we have

$$\begin{aligned} h_1(x) = &\int_{\Omega_\pi^\dagger} \partial R_0(y,x)/\partial y_j\, f(y)\,dy + \int_{\Omega^*} \partial R_0(y,x)/\partial y_j\, f(y)\,dy \\ &+ \int_{\Omega^\#} \partial R_1(y,x)/\partial y_j\, f(y)\,dy + \int_\Omega Z_2(y,x)f(y)\,dy, \end{aligned} \tag{4.20}$$

where $f(y)$ is defined by periodicity in that portion of $\Omega_\pi^\dagger$ for which $y_2 < 0$ and $Z_2 \in C^1(\overline{\Omega} \times \Omega_\tau^2)$. Since the first integral on the right side of (4.20) can be dealt with by arguing as in the previous paragraph, it follows immediately that $\partial h_1(x)/\partial x_k$ exists everywhere in $\mathring{\Omega}_\tau^2$ and $\|\partial h_1/\partial x_k\|_{0,\mathring{\Omega}_\tau^2} \le c\|f\|$. Similarly we can show that analogous results hold for h_j if $2 \le j \le 4$ and also that for $i = 1,\dots,l$, $\partial\psi_i/\partial x_k$ exists everywhere in $\mathring{\Omega}_\tau^2$ and $\|\partial\psi_i/\partial x_k\|_{0,\mathring{\Omega}_\tau^2} < \infty$.

Turning our attention to the case $x \in \mathring{\Omega}_\tau^3$, we then have $\partial h_{1,2}(x)/\partial x_k$ exists and

$$\begin{aligned} \partial h_{1,2}(x)/\partial x_k = &\left(\pi\delta(x)^{1/2}a_{11}(x)\right)^{-1}\delta_{j1}\int_\Omega \partial[(x_1+y_1-2\alpha_1)/\rho_0^\alpha(y,x)]/\partial x_k\, f(y)\,dy \\ &+ \int_\Omega Z_3(y,x)f(y)\,dy, \end{aligned} \tag{4.21}$$

where $h_{1,2}(x) = h_1(x) + h_2(x)$, δ_{rs} denotes the Kronecker delta, and (4.19) holds for $Q(x,y) = Z_3(y,x)$ provided that in (4.19) we restrict x to the set $\mathring{\Omega}_\tau^3$ instead of to Ω. Hence if $j = 2$, then it follows immediately that $\|\partial h_{1,2}/\partial x_k\|_{0,\mathring{\Omega}_\tau^3} \le c\|f\|$. On the other hand, if $j = 1$, then it is easy to show that the modulus of the integrand of the first integral on the right side of (4.21) is bounded by

$$c_1|f(y)|/(x_1-\alpha_1)^\lambda(y_1-\alpha_1)^\lambda|y_1-x_1|^\nu|y_2-x_2|^\nu,$$

where $0 < \lambda < 1/2 < \nu < 1$, $\lambda + \nu = 1$, and the constant c_1 does not depend upon f, x, nor y. Hence it follows directly from this fact and (4.21) that when $j = 1$,

$$\|h_{12}/\partial x_k\|_{0,\mathring{\Omega}_\tau^3} \le c_2\|f\| + c_3\left[\int_\Omega \left||\log(y_1-\alpha_1)|^{1/2} f(y)/(y_1-\alpha_1)^\lambda\right|^2 dy\right]^{1/2},$$

where the constants c_j do not depend upon f, but c_3 depends upon λ. It is also not difficult to verify that $\partial h_3(x)/\partial x_k$ exists and

$$\begin{aligned}\overline{\partial h_3(x)}/\partial x_k = \lim_{\varepsilon\to 0}\int_{\Omega\setminus S_\varepsilon(x)} \partial T_0(t,x)/\partial x_k\, h^\dagger(t)\,dt + h^\dagger(x)z_3(x)\\ + \int_\Omega Z_{3,1}(t,x)(t_1-\alpha_1)^{-\nu}h^\dagger(t)\,dt + \int_\Omega Z_{3,2}(t,x)h^\dagger(t)\,dt,\end{aligned}\tag{4.22}$$

where $z_3(x)$ is continuous in Ω_τ^3, $0<\nu<1/4$, and (4.19) holds for $Q(x,y) = Z_{3,j}(y,x)$, $j=1,2$, provided that in (4.19) we restrict x to the set $\mathring{\Omega}_\tau^3$ instead of to Ω. Since we may appeal to the properties of K and H cited in the proofs of Lemmas A.1–2 to deduce that $\|(t_1-\alpha_1)^{-\nu}h^\dagger\| \le c\|f\|$, we can now use the Calderon-Zygmund theorem cited above in dealing with the first integral on the right side of (4.22) to arrive at the conclusion that $\|\partial h_3/\partial x_k\|_{0,\mathring{\Omega}_\tau^3} \le c\|f\|$. Similarly we can show that analogous results hold for h_4 and that for $i=1,\ldots,l$, $\partial\psi_i(x)/\partial x_k$ exists everywhere in $\mathring{\Omega}_\tau^3$ and $\|\partial\psi_i(x)/\partial x_k\|_{0,\mathring{\Omega}_\tau^3} < \infty$.

If $x\in\mathring{\Omega}_\tau^4$ and if we define $f(t)$, $h^\dagger(t)$, $f^\dagger(t) = \int_\Omega F(t,y)f(y)\,dy$, and $\psi_i(t)$ $(1\le i\le l)$ by periodicity in that portion of $\Omega_\pi^\dagger$ for which $t_2<0$, then $h_1(x) - \int_{\Omega_\pi^\dagger}\partial P_0(y,x)/\partial y_j\, f(y)\,dy$, $h_2(x) - \int_{\Omega_\pi^\dagger}\partial P_0(x,y)/\partial x_j\, f(y)\,dy$, $\overline{h_3(x)} - \int_{\Omega_\pi^\dagger} Q_0(y,x)\, h^\dagger(y)\,dy$, $h_4(x) - \int_{\Omega_\pi^\dagger}\partial P_0(x,y)/\partial x_j\, f^\dagger(y)\,dy$, and $\overline{\psi_i(x)} + \int_{\Omega_\pi^\dagger} Q_0(y,x)\overline{\psi_i(y)}\,dy$ are all of the form

$$\int_{\Omega^*} Z_{4,1}(x,y)g(y)\,dy + \int_{\Omega^\#} Z_{4,2}(x,y)g(y)\,dy + \int_\Omega Z_{4,3}(x,y)g(y)\,dy,$$

where $g(y)$ is either $f(y)$ or $h^\dagger(y)$ or $f^\dagger(y)$ or $\overline{\psi_i(y)}$, depending upon which of the h_j we are dealing with or whether we are dealing with ψ_i, and $Z_{4,1}$, $Z_{4,2}$, and $Z_{4,3}$ are of class C^1 in the sets $\Omega_\tau^4\times\overline{\Omega^*}$, $\Omega_\tau^4\times\overline{\Omega^\#}$, and $\Omega_\tau^4\times\overline{\Omega}$, respectively. Hence we can deal with the case $x\in\mathring{\Omega}_\tau^4$ precisely as we dealt with the case $x\in\mathring{\Omega}_\tau^3$ to arrive at analogous results.

By continuing the foregoing arguments to the remaining portion of Ω, we arrive at the proof of the theorem for the case $f\in C_\pi^\infty|\Omega$. The proof for the case $f\in H_\pi^1(\Omega)$ now follows from a standard approximation procedure if we bear in mind Sobolev's imbedding theorem. □

Appendix A.

In the proof of Theorem 4.2 we made certain assertions concerning $F(x,y)$. We now prove these assertions as well as some other results which are required in § 4.

Lemma A.1. *If $\chi_1(x,y) = F(x,y)+K(x,y)$, then χ_1 is continuous in $(\overline{\Omega}\times\overline{\Omega})\setminus X$ and in this set*

$$\chi_1(x,y) = O\Big(\sum_{k=-1}^{1} |\log|x-y(k)||\Big),\tag{A.1}$$

$\chi_1(x,y)=0$ for $y_1=\alpha_1$ or $y_1=\beta_1$, while $\chi_1(\cdot,y)|_{\Gamma_2} = \chi_1(\cdot,y)|_{\Gamma_4}$ for $y\in\Omega$ and $\chi_1(x,\cdot)|_{\Gamma_2} = \chi_1(x,\cdot)|_{\Gamma_4}$ for $x\in\Omega$. Moreover,

$$\chi_1(x',y)-\chi_1(x,y) = O\Big(|x'-x|^\mu \sum_{k=-1}^{1} |x-y(k)|^{-1}\Big),\tag{A.2}$$

for $(x,y) \in (\overline{\Omega} \times \overline{\Omega}) \setminus X$, $x' \in \overline{\Omega}$, *and* $9|x'-x| \le \min_{-1\le k\le 1} |x-y(k)|$.

Proof. Let us show that the lemma is true when $\chi_1(x,y)$ is replaced by $K_2(x,y)$. To this end let $R_k(x,y) = g(y)\log \rho_k(x,y)^{-1/2}$, $R_k^\alpha(x,y) = g(y)\log \rho_k^\alpha(x,y)^{-1/2}$, $R_k^\beta(x,y) = g(y)\log \rho_k^\beta(x,y)$, $T_k(x,y) = (L(x,D) - L_0^y(D_x))R_k(x,y)$, $T_k^\alpha(x,y) = (L(x,D) - L_0^y(D_x))R_k^\alpha(x,y)$, and $T_k^\beta(x,y) = (L(x,D) - L_0^y(D_x))R_k^\beta(x,y)$, for $-1 \le k \le 1$, where $g(y) = 1/2\pi\delta(y)^{1/2}$. Then for $(x,y) \in (\overline{\Omega}_\pi \times \overline{\Omega}_\pi) \setminus X_\pi$, where $X_\pi = \{(x,y) \in \overline{\Omega}_\pi \times \overline{\Omega}_\pi |$ either $x = y$ or $x_1 = y_1$ and $|x_2 - y_2| = 2\pi\}$, we have

$$K(x,y) = \sum_{k=-1}^{1} [T_k(x,y) - T_k^\alpha(x,y) - T_k^\beta(x,y)] + (L(x,D) - L_0^y(D_x))H_2(x,y),$$

while we also note that $T_k^\alpha(x,y) = O(|x-y(k)|^{-1})$ and $T_k^\beta(x,y) = O(|x-y(k)|^{-1})$ for $-1 \le k \le 1$. Now let us observe that if τ is a positive number satisfying $100\tau < \min\{\beta_1 - \alpha_1, 2\pi\}$ and

$$\begin{aligned}
\Omega_\tau^1 &= \{x \in \Omega | \alpha_1 + \tau \le x_1 \le \beta_1 - \tau,\ \tau \le x_2 \le 2\pi - \tau\},\\
\Omega_\tau^2 &= \{x \in \overline{\Omega} | \alpha_1 + \tau \le x_1 \le \beta_1 - \tau,\ 0 \le x_2 \le \tau\},\\
\Omega_\tau^3 &= \{x \in \overline{\Omega} | \alpha_1 \le x_1 \le \alpha_1 + \tau,\ \tau \le x_2 \le 2\pi - \tau\},\\
\Omega_\tau^4 &= \{x \in \overline{\Omega} | \alpha_1 \le x_1 \le \alpha_1 + \tau,\ 0 \le x_2 \le \tau\},
\end{aligned}$$

then: (1) $K(x,y) - T_0(x,y)$ is of class $C^{1,1}$ in $\Omega_\tau^1 \times \overline{\Omega}$, (2) $K(x,y) - T_0(x,y) - T_{-1}(x,y)$ is of class $C^{1,1}$ in $\Omega_\tau^2 \times \overline{\Omega}$, (3) $K(x,y) - T_0(x,y) + T_0^\alpha(x,y)$ is of class $C^{1,1}$ in $\Omega_\tau^3 \times \overline{\Omega}$, (4) $K(x,y) - T_0(x,y) + T_0^\alpha(x,y) - T_{-1}(x,y) + T_{-1}^\alpha(x,y)$ is of class $C^{1,1}$ in $\Omega_\tau^4 \times \overline{\Omega}$, with analogous results holding in the remaining portion of $\overline{\Omega} \times \overline{\Omega}$.

Suppose firstly that $x \in \Omega_\tau^1$. Then

$$K_2(x,y) = \int_\Omega T_0(x,t)K(t,y)\,dt + g_1(x,y), \tag{A.3}$$

where $g_1(x,y)$ is continuous in $\Omega_\tau^1 \times \overline{\Omega}$ and in this set $g_1(x',y) - g_1(x,y) = O(|x'-x|)$ for $x' \in \Omega_\tau^1$. If $f_1(x,y)$ denotes the integral on the right side of (A.3), then we can show by means of standard arguments (cf. [6, p. 136], [12, pp. 29–30]) that $D_x^\gamma f_1(x,y)$ is continuous in $(\Omega_\tau^1 \times \overline{\Omega}) \setminus X$ for $|\gamma| \le 1$ and that in this set (A.1) holds when χ_1 is replaced by f_1 and $\partial f_1(x,y)/\partial x_j = O(\sum_{k=-1}^{1} |\log|x-y(k)||/|x-y(k)|)$. It follows immediately that (A.2) holds with χ_1 replaced by f_1 for $(x,y) \in (\Omega_\tau^1 \times \overline{\Omega}) \setminus X$, $x' \in \Omega_\tau^1$, and $2|x'-x| \le \min_{-1\le k\le 1} |x-y(k)|$. In light of (A.3) we conclude that the assertions just made concerning the continuity and Hölder continuity of f_1 also hold for K_2.

Suppose next that $x \in \Omega_\tau^2$. Then for this case we have

$$K_2(x,y) = \int_{\Omega_\pi^\dagger} T_0(x,t)K(t,y)\,dt + g_2(x,y), \tag{A.4}$$

where $\Omega_\pi^\dagger = \{x \in \Omega_\pi | |x_2| < \pi/2\}$, $g_2(x,y)$ is continuous in $\Omega_\tau^2 \times \overline{\Omega}$, and in this set $g_2(x',y) - g_2(x,y) = O(|x'-x|)$ for $x' \in \Omega_\tau^2$. By arguing with the integral on the right side of (A.4) as in the previous paragraph, it is not difficult to deduce that $K_2(x,y)$ is continuous in $(\Omega_\tau^2 \times \overline{\Omega}) \setminus X$ and that in this set (A.1) holds when χ_1 is replaced by K_2, while (A.2) also holds with χ_1 replaced by K_2 for $(x,y) \in (\Omega_\tau^2 \times \overline{\Omega}) \setminus X$, $x' \in \Omega_\tau^2$, and $2|x'-x| \le \min_{-1\le k\le 1} |x-y(k)|$.

Turning our attention to the case $x \in \Omega_\tau^3$, we then have

$$K_2(x,y) = \int_\Omega Q_0(x,t)K(t,y)\,dt + g_3(x,y), \tag{A.5}$$

where $Q_k(x,y) = T_k(x,y) - T_k^\alpha(x,y)$ for $-1 \le k \le 1$ and $g_3(x,y)$ is continuous in $\Omega_\tau^3 \times \overline{\Omega}$ and in this set $g_3(x',y) - g_3(x,y) = O(|x'-x|)$ for $x' \in \Omega_\tau^3$. Observe that $Q_0(x,y)$ is continuously differentiable in $(\Omega_\tau^3 \times \overline{\Omega}) \setminus X$ and that in this set $Q_0(x,y) = O(|x-y|^{-1})$, while $Q_0(x,y) = 0$ if $x \neq y$ and $y_1 = \alpha_1$. Now let $\lambda_1(y)$ denote the smallest eigenvalue of the matrix $(b_{ij}(y))$, put $\lambda_1 = \inf_{\overline{\Omega}} \lambda_1(y)$, and for $\varepsilon > 0$ let $I_\varepsilon(x) = \{y \in \mathbb{R}^2 | \rho_0(x,y) < \varepsilon^2\}$. Then for $0 < d < \tau$ and $\varepsilon < \lambda_1^{1/2} d/2$, let us consider the integral $f_\varepsilon(x,y) = \int_{\Omega\setminus I_\varepsilon(x)} Q_0(x,t)K(t,y)\,dt$ under the supposition that $|x-y| > d$. It is easily verified that $f_\varepsilon(x,y)$ is continuous in $\Omega_\tau^3 \times \overline{\Omega}$, $|x-y|>d$, and that in this set $f_\varepsilon(x,y) \to f(x,y) = \int_\Omega Q_0(x,t)K(t,y)\,dt$ as $\varepsilon \to 0$ independently of x and y. Since d is arbitrary, we have thus shown that $f(x,y)$ is continuous in $(\Omega_\tau^3 \times \overline{\Omega}) \setminus X$ and clearly (A.1) holds when $\chi_1(x,y)$ is replaced by $f(x,y)$. However, we cannot use the same type of argument to establish the differentiability of $f(x,y)$ since $\frac{\partial Q_0(x,t)}{\partial x_j} - \frac{\partial Q_0(t,x)}{\partial t_j} = O(|x-t|^{-2})$ (see [12, p. 30]), and hence the method used in the previous cases to establish Hölder continuity is no longer valid. On the other hand we can apply a standard method for dealing with Hölder continuity (cf. [6, p. 136], [11]) to show that (A.2) holds with χ_1 replaced by f for $(x,y) \in (\Omega_\tau^3 \times \overline{\Omega}) \setminus X$, $x' \in \Omega_\tau^3$, and $8|x'-x| \le \min_{-1\le k\le 1} |x-y(k)|$. In light of (A.5) we conclude that the assertions just made for f also hold for K_2.

If $x \in \Omega_\tau^4$, then

$$K_2(x,y) = \int_{\Omega_\tau^\dagger} Q_0(x,t)K(t,y)\,dt + g_4(x,y),$$

where $g_4(x,y)$ is continuous in $\Omega_\tau^4 \times \overline{\Omega}$ and in this set $g_4(x',y) - g_4(x,y) = O(|x'-x|)$ for $x' \in \Omega_\tau^4$. Hence we can argue as we did in the previous paragraph to show that $K_2(x,y)$ is continuous in $(\Omega_\tau^4 \times \overline{\Omega}) \setminus X$ and in this set (A.1) holds when χ_1 is replaced by K_2, and also that (A.2) holds with χ_1 replaced by K_2 for $(x,y) \in (\Omega_\tau^4 \times \overline{\Omega}) \setminus X$, $x' \in \Omega_\tau^4$, and $8|x'-x| \le \min_{-1\le k\le 1} |x-y(k)|$.

By continuing the foregoing arguments to the remaining portion of $\overline{\Omega}$, we finally deduce that all of the assertions of the lemma, except those concerning the boundary values of χ_1, are valid when χ_1 is replaced by K_2, while the assertions concerning the boundary values can be proved for K_2 by applying the known results for K to a suitable subregion of Ω and the employing a limiting procedure.

Similarly we can show that $K_3(x,y)$ is continuous in $\overline{\Omega} \times \overline{\Omega}$ and $K_3(x',y) - K_3(x,y) = O(|x'-x|^\mu)$ for $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, $x' \in \overline{\Omega}$, and $9|x'-x| \le \min_{-1\le k\le 1} |x - y(k)|$. Furthermore, for $r \ge 4$, $K_r(x,y)$ is continuous in $\overline{\Omega} \times \overline{\Omega}$ and in this set $K_r(x',y) - K_r(x,y) = O(|x'-x|^\mu)$ for $x' \in \overline{\Omega}$, while if $g_0(x,y) = \sum_{i=1}^{l} L\phi_i(x)\overline{\psi_i(y)}$ and $g_r(x,y) = \int_\Omega K_r(x,t)g_0(t,y)\,dt$ for $r \ge 1$, then for $r \ge 0$, $g_r(x,y)$ is continuous in $\overline{\Omega} \times \overline{\Omega}$ and in this set $g_r(x',y) - g_r(x,y) = O(|x'-x|^\mu)$ for $x' \in \overline{\Omega}$. It is also not difficult to verify that the assertions of the lemma concerning the boundary values of χ_1 apply in full force to the K_r and g_r.

Finally, observing that

$$(A.6)\quad F(x,y)+\int_{\Omega}K_p(x,t)F(t,y)\,dt=f(x,y)+\sum_{r=1}^{p-1}(-1)^r\int_{\Omega}K_r(x,t)f(t,y)\,dt,$$

where $f(x,y)=-K(x,y)-g_0(x,y)$ and p is the prime introduced in the proof of Theorem 4.2, the assertions of the lemma follow immediately from the foregoing results, the results given in the proof of Theorem 4.2, and from equations (6.1.12), (6.3.3) of [6, pp. 129 and 133]. □

As an immediate consequence of the above lemma we have

Corollary A.1. *$F(x,y)$ is continuous in $(\overline{\Omega}\times\overline{\Omega})\setminus X$ and in this set $F(x,y)=O(\sum_{k=-1}^{1}|x-y(k)|^{-1})$, $F(x,y)=0$ for $y_1=\alpha_1$ or $y_1=\beta_1$, while $F(\cdot,y)|_{\Gamma_2}=F(\cdot,y)|_{\Gamma_4}$ for $y\in\Omega$ and $F(x,\cdot)|_{\Gamma_2}=F(x,\cdot)|_{\Gamma_4}$ for $x\in\Omega$. Moreover,*

$$F(x',y)-F(x,y)=O\big(|x'-x|^{\mu}\sum_{k=-1}^{1}|x-y(k)|^{-1-\mu}\big)$$

for $(x,y)\in(\overline{\Omega}\times\overline{\Omega})\setminus X$, $x'\in\overline{\Omega}$, and $9|x'-x|\le\min_{-1\le k\le 1}|x-y(k)|$.

Remark A.1. By using arguments similar to those used in the proof of the above lemma, it is also not difficult to verify that for $(x,y)\in(\overline{\Omega}\times\overline{\Omega})\setminus X$, $y'\in\overline{\Omega}$, and $9|y'-y|\le\min_{-1\le k\le 1}|x-y(k)|$, $K_2(x,y')-K_2(x,y)=O\big(|y'-y|^{\mu}\sum_{k=-1}^{1}|x-y(k)|^{-1}\big)$ and $K_3(x,y')-K_3(x,y)=O(|y'-y|^{\mu})$, while for $(x,y)\in\overline{\Omega}\times\overline{\Omega}$ and $r\ge 4$, $K_r(x,y')-K_r(x,y)=O(|y'-y|^{\mu})$ for $y'\in\overline{\Omega}$.

Lemma A.2. *If $\chi_2(x,y)=\int_{\Omega}H(x,t)F(t,y)\,dt$, then $\chi_2(x,y)$ is continuous in $\overline{\Omega}\times\overline{\Omega}$, $\partial\chi_2(x,y)/\partial x_j$ is continuous in $(\overline{\Omega}\times\overline{\Omega})\setminus X$ and in this set*

$$\partial\chi_2(x,y)/\partial x_j=\int_{\Omega}\partial H(x,t)/\partial x_j\,F(t,y)\,dt=O\big(\sum_{k=-1}^{1}|\log|x-y(k)||\big),$$

while $\partial^2\chi_2(x,y)/\partial x_k\partial x_j$ is continuous in $(\Omega(\varepsilon)\times\overline{\Omega})\setminus X$ and in this set

$$\partial^2\chi_2(x,y)/\partial x_k\partial x_j=O\big(\sum_{k=-1}^{1}|\log|x-y(k)||/|x-y(k)|\big).$$

Moreover, $\chi_2(x,y)=0$ for $x_1=\alpha_1$ or $x_1=\beta_1$, and for $y\in\Omega$, $D_x^{\gamma}\chi_2(\cdot,y)|_{\Gamma_2}=D_x^{\gamma}\chi_2(\cdot,y)|_{\Gamma_4}$ for $|\gamma|\le 1$.

Proof. Returning again to the proof of Lemma A.1, we observe that

$$H(x,y)=\sum_{k=-1}^{1}[R_k(x,y)-R_k^{\alpha}(x,y)-R_k^{\beta}(x,y)]+H_2(x,y)$$

and that: (1) $H(x,y)-R_0(x,y)$ is of class $C^{2,1}$ in $\Omega^1_\tau \times \overline{\Omega}$, (2) $H(x,y)-R_0(x,y)-R_{-1}(x,y)$ is of class $C^{2,1}$ in $\Omega^2_\tau \times \overline{\Omega}$, (3) $H(x,y)-R_0(x,y)+R_0^\alpha(x,y)$ is of class $C^{2,1}$ in $\Omega^3_\tau \times \overline{\Omega}$, (4) $H(x,y)-R_0(x,y)+R_0^\alpha(x,y)-R_{-1}(x,y)+R_{-1}^\alpha(x,y)$ is of class $C^{2,1}$ in $\Omega^4_\tau \times \overline{\Omega}$, with analogous results holding in the remaining portion of $\overline{\Omega}\times\overline{\Omega}$.

Suppose firstly that $x \in \Omega^1_\tau$. Then

$$\chi_2(x,y) = \int_\Omega R_0(x,t)F(t,y)\,dt + \int_\Omega g_1(x,t)F(t,y)\,dt, \tag{A.7}$$

where $D_x^\gamma \int_\Omega g_1(x,t)F(t,y)\,dt = \int_\Omega D_x^\gamma g_1(x,t)F(t,y)\,dt$ is continuous in $\Omega^1_\tau \times \overline{\Omega}$ for $|\gamma| \le 2$. If $f_1(x,y)$ denotes the first integral on the right side of (A.7), then we can argue as in [12, pp. 29–30] (see also [6, p. 136]) to show that all of the assertions of the lemma, except the final ones, are true when $\chi_2(x,y)$ is replaced by $f_1(x,y)$ provided that we restrict x to the set Ω^1_τ. Hence it follows from (A.7) that the same result also holds for χ_2.

Suppose next that $x \in \Omega^2_\tau$. Then for this case we have

$$\begin{aligned}\chi_2(x,y) &= \int_{\Omega^\dagger_\pi} R_0(x,t)F(t,y)\,dt + \int_{\Omega^*} R_0(x,t)F(t,y)\,dt \\ &\quad + \int_{\Omega^\#} R_{-1}(x,t)F(t,y)\,dt + \int_\Omega g_2(x,t)F(t,y)\,dt,\end{aligned} \tag{A.8}$$

where $\Omega^\dagger_\pi$ is defined in the proof of Lemma A.1, $F(t,y)$ is defined by periodicity in that portion of $\Omega^\dagger_\pi$ for which $t_2 < 0$, $\Omega^* = \{x \in \Omega | x_2 \ge \pi/2\}$, $\Omega^\# = \{x \in \Omega | x_2 \le 3\pi/2\}$, and $D_x^\gamma \int_\Omega g_2(x,t)F(t,y)\,dt = \int_\Omega D_x^\gamma g_2(x,t)F(t,y)\,dt$ is continuous in $\Omega^2_\tau \times \overline{\Omega}$ for $|\gamma| \le 2$. Since we can argue with the first integral on the right side of (A.8) as we argued with $f_1(x,y)$ in the previous paragraph, it follows from (A.8) that all of the assertions of the lemma, except the final ones, are certainly true provided that we restrict x to the set Ω^2_τ.

Turning our attention to the case $x \in \Omega^3_\tau$, we then have

$$\chi_2(x,y) = \int_\Omega P_0(x,t)F(t,y)\,dt + \int_\Omega g_3(x,t)F(t,y)\,dt \tag{A.9}$$

where $P_k(x,y) = R_k(x,y) - R_k^\alpha(x,y)$ for $-1 \le k \le 1$ and $D_x^\gamma \int_\Omega g_3(x,t)F(t,y)\,dt = \int_\Omega D_x^\gamma g_3(x,t)F(t,y)\,dt$ is continuous in $\Omega^3_\tau \times \overline{\Omega}$ for $|\gamma| \le 2$. If $f(x,y)$ denotes the first integral on the right side of (A.9), then we can argue as in the [6, p. 136] to show that $f(x,y)$ is continuous in $\Omega^3_\tau \times \overline{\Omega}$. To deal with the derivatives of $f(x,y)$, let d, ε, and $I_\varepsilon(x)$ be defined as in that part of the proof of Lemma A.1 concerning Ω^3_τ, and let us consider the integral $f_\varepsilon(x,y) = \int_{\Omega \setminus I_\varepsilon(x)} P_0(x,t)F(t,y)\,dt$ under the supposition that $|x-y| > d$. Then it is readily verified that

$$\partial f_\varepsilon(x,y)/\partial x_j = \int_{\Omega\setminus I_\varepsilon(x)} \partial P_0(x,t)/\partial x_j\, F(t,y)\,dt - \int_{\partial I_\varepsilon(x)\cap\Omega} P_0(x,t)F(t,y)X_j\,d\sigma_t,$$

where X_j denotes the j-th component of the unit exterior normal to $\partial I_\varepsilon(x)$ and σ_t denotes arc length. It is easily shown that $f_\varepsilon(x,y)$ and $\partial f_\varepsilon(x,y)/\partial x_j$ are continuous in $\Omega^3_\tau \times \overline{\Omega}$, $|x-y| > d$, and that in this set $f_\varepsilon(x,y) \to f(x,y)$,

$\partial f_\varepsilon(x,y)/\partial x_j \to \int_\Omega \partial P_0(x,t)/\partial x_j\, F(t,y)\,dt$ as $\varepsilon \to 0$ independently of x and y. Since d is arbitrary, it follows that in $(\Omega_\tau^3 \times \overline{\Omega})\setminus X$, $\partial f(x,y)/\partial x_j$ is continuous and is equal to $\int_\Omega \partial P_0(x,t)/\partial x_j\, F(t,y)\,dt$, while a standard argument shows that this latter integral is $O\big(\sum_{k=-1}^{1} |\log|x-y(k)||\big)$. We conclude from these arguments that the initial assertions of the lemma concerning χ_2 and $\partial\chi_2/\partial x_j$ are certainly true provided that we restrict x to the set Ω_τ^3.

If $x \in \Omega_\tau^4$, then

$$\chi_2(x,y) = \int_{\Omega_\pi^\dagger} P_0(x,t)F(t,y)\,dt + \int_{\Omega^*} P_0(x,t)F(t,y)\,dt + \int_{\Omega^\#} P_{-1}(x,t)F(t,y)\,dt + \int_\Omega g_4(x,t)F(t,y)\,dt, \tag{A.10}$$

where $D_x^\gamma \int_\Omega g_4(x,t)F(t,y)\,dt = \int_\Omega D_x^\gamma g_4(x,t)F(t,y)\,dt$ is continuous in $\Omega_\tau^4 \times \overline{\Omega}$ for $|\gamma| \le 2$, and we refer to the discussion above for Ω_τ^2 for the remaining terminology. Since we can argue with the first integral on the right side of (A.10) as we argued with $f(x,y)$ in the previous paragraph, it follows immediately from (A.10) that the initial assertions of the lemma concerning χ_2 and $\partial\chi_2/\partial x_j$ are certainly true provided that we restrict x to the set Ω_τ^4.

By continuing the foregoing arguments to the remaining portion of $\overline{\Omega}$, we finally deduce the validity of all of the assertions of the lemma except the final ones. However, these final assertions can be proved by applying the known results for H and F to a suitable subregion of Ω and then employing a limiting procedure. □

Referring to Theorem 4.2 for terminology, we have next

Lemma A.3. *If $f \in L^\infty(\Omega)$ and $\chi_3(x) = \int_\Omega F(x,y)f(y)\,dy$, then $\chi_3 \in C_\pi^{0,\mu}(\overline{\Omega})$.*

Proof. Turning again to Lemma A.1 and bearing in mind Corollary A.1, a standard argument (cf. Assertion 12, II of [12,p. 25]) shows that $\chi_3 \in C^{0,\mu}(\Omega_\tau^1)$. If $x \in \Omega_\tau^2$, then

$$\chi_3(x) = \int_{\Omega_\pi^\dagger} F(x,y)f(y)\,dy + \int_{\Omega^*\cap\Omega^\#} F(x,y)f(y)\,dy, \tag{A.11}$$

where $F(x,y)$ and $f(y)$ are defined by periodicity in that portion of $\Omega_\pi^\dagger$ for which $y_2 < 0$ and all the remaining terms are defined in Lemmas A.1–2. The first integral on the right side of (A.11) can be shown to be of class $C^{0,\mu}(\Omega_\tau^2)$ by arguing as for the case $x \in \Omega_\tau^1$, while the second integral is clearly of class $C^{0,\mu}(\Omega_\tau^2)$, and hence so is χ_3. By arguing in a similar fashion in the remaining portion of $\overline{\Omega}$, one finally deduces that $\chi_3 \in C^{0,\mu}(\overline{\Omega})$. That $\chi_3 \in C_\pi^{0,\mu}(\overline{\Omega})$ can be shown by using arguments similar to those used in the last paragraph of the proof of Lemma A.2. □

Remark A.2. It is not difficult to verify that the conclusions of Lemma A.3 remain valid if we replace $\chi_3(x)$ there by any one of the functions $\int_\Omega F(y,x)f(y)\,dy$, $\int_\Omega K(x,y)f(y)\,dy$, $\int_\Omega K(y,x)f(y)\,dy$, $\int_\Omega \partial H(x,y)/\partial x_j\, f(y)\,dy$, and $\int_\Omega \partial H(y,x)/\partial y_j\, f(y)\,dy$.

REFERENCES

[1] Adams, R. A.: Sobolev Spaces. New York: Academic Press 1975

[2] Agmon, S.: Lectures on Elliptic Boundary Value Problems. Princeton, N. J.: Van Nostrand 1965

[3] Agmon, S.: On kernels, eigenvalues, and eigenfunctions of operators related to elliptic problems. Comm. Pure Apll. Math. **18**, 627–663 (1965)

[4] Beals, R.: Asymptotic behavior of the Green's function and spectral functions of an elliptic operator. J. Funct. Anal. **5**, 484–503 (1970)

[5] Calderon, A. P., Zygmund, A.: On singular integrals. Amer. J. Math. **78**, 289–309 (1956)

[6] Duff, G. F. D. : Partial Differential Equations. Totonto: University of Toronto Press 1956

[7] John, F.: General properties of solutions of linear elliptic partial differential equations. in: Proc. Symposium on Spectral Theory and Differential Problems, Stillwater, Oklahoma, 1951, 113–175

[8] Kato, T.: Perturbation Theory for Linear Operators. 2nd ed., New York: Springer 1976

[9] Lifschitz, A. E.: Magnetohydrodynamics and Spectral Theory. Boston: Kluwer 1989

[10] Mikhlin, S. G.: Linear Integral Equations. Delhi: Hindustan Publishing Corp. 1960

[11] Miranda, C.: Sulla proprietà di regolarità di certe transformazioni integrali. Mem. Acc. Lincei **7**, 303–336 (1956)

[12] Miranda, C.: Partial Differential Equations of Elliptic Type. 2nd ed., Nwe York: Springer 1970

[13] Mizohata, S.: The Theory of Partial Differential Equations. Cambridge: University Press 1973

[14] Titchmarsh, E. C.: Eigenfunction Expansions. Part II, Oxford: Clarendon Press 1958

[15] Zygmund, A.: On singular integrals. Rend. di Mat. **16**, 468–505 (1957)

University of the Witwatersrand, Department of Mathematics, Johannesburg, WITS 2050, South Africa

Universität Regensburg, NWF I – Mathematik, 8400 Regensburg, Germany

University of the Witwatersrand, Department of Mathematics, Johannesburg, WITS 2050, South Africa

(Received June 8, 1993)

manuscripta math. 81, 413 - 435 (1993)

manuscripta mathematica

Etage initial d'une $\mathbb{Z}_\ell$-extension

Hervé Thomas

For a number field F that contains ζ_ℓ a ℓ^{th} root of unity (ℓ is a prime number), we determine the x such that $F(\sqrt[\ell]{x})$ can be embedded in a $\mathbb{Z}_\ell$-extension. We approach the corresponding Kummer radical with the notion of being locally everywhere embedded in a $\mathbb{Z}_\ell$-extension. An idelic description of Galois group is appropriate especially as we utilize the ℓ-adic group of idele of [15]. The illustration concerns ℓ=3 and biquadratic field $\mathbb{Q}(\zeta_3, \sqrt{d})$. We detail the step of the calculus and fournish numerical tables.

Introduction : Si F est un corps de nombres contenant ζ_ℓ, nous lui associons le radical kummerien $\mathfrak{Z}_F$ construit avec les étages n°1 des $\mathbb{Z}_\ell$-extensions de F. Sa $\mathbb{F}_\ell$-dimension est, sous la conjecture de Leopoldt (cf [4],[12], [16]), le nombre de places complexes plus un. Sa connaissance numérique repose principalement sur celle des ℓ-unités, des classes de F, sur les théories du corps de classes et de Kummer, sur la solution de problèmes locaux. Notre méthode diffère sensiblement de celle de G.Gras, [10], car nous utilisons les énoncés de la théorie du corps de classes « à la Chevalley» (spécialisés aux ℓ-extensions dans [15]) et, de plus, comme meilleure approche possible de $\mathfrak{Z}_F$ par des radicaux définis par des conditions purement locales (ce qui n'est nullement le cas de $\mathfrak{Z}_F$), nous regardons un certain $\mathfrak{H}_F$ associé à une ℓ-extension introduite dans [3]. Pour nous, l'appartenance de $x \cdot (F^\times)^\ell$ à $\mathfrak{H}_F$ signifie « $F(\sqrt[\ell]{x})$ localement partout $\mathbb{Z}_\ell$-plongeable » .

Nous regardons ici en détail les étages initiaux des $\mathbb{Z}_3$-extensions des corps biquadratiques $F = \mathbb{Q}(\zeta_3, \sqrt{d})$, ceci pour des valeurs de d entre 67 et 29.10^9. Kramer et Candiotti [18] calculèrent $\mathfrak{Z}_F$ pour $\mathbb{Q}(\zeta_3, \sqrt{d})$ (ℓ=3), d dans $\mathbb{Z}$, sans facteur carré et $|d| < 200$, à l'exception de 5 valeurs de d qui amènent d'autres calculs effectués par D.Hemard [13]. Un cas est aussi traité dans [10] pour $\mathbb{Q}(\sqrt{-3}, \sqrt{-586})$. Notons que pour cette famille de corps, les radicaux kummeriens et groupes de Galois étudiés se « partagent» en composantes isotypiques en raison de leur structure de $\mathbb{Z}_3[\Delta]$-module (où $\Delta = Gal(F/\mathbb{Q})$), ce qui permet, par exemple, d'écrire une « Spiegelungsrelation » pour certains groupes de classes d'idéaux et de traduire les liens étroits entre $\mathbb{Q}(\sqrt{d})$ et $\mathbb{Q}(\sqrt{-3d})$. Nous utilisons aussi une belle notion liée à la précédente : le miroir de Leopoldt. Ces calculs ont été rendus possibles grâce au système informatique PARI (originaire de Bordeaux), très efficace pour aller au cœur de problèmes numériques en théorie des nombres.

Notations de base :

ℓ : un nombre premier impair, $\mathbb{Z}_\ell \simeq \varprojlim_n (\mathbb{Z}/\ell^n\mathbb{Z})$ l'anneau des entiers ℓ-adiques.

ζ_n : une racine $n^{ième}$ de l'unité, locale ou globale suivant le contexte. ζ_n engendre le groupe $< \zeta_n >$.

F : un corps de nombres ([F:$\mathbb{Q}$] $< \infty$) contenant $< \zeta_\ell >$, $F = \mathbb{Q}(\zeta_3, \sqrt{d})$ à partir du §3.

$\mathbb{Z}_F$: l'anneau des entiers du corps F.

$\mathfrak{P}$: une place ultramétrique de F, divisant p, premier de $\mathbb{Z}$.
N : la norme depuis F jusqu'à $\mathbb{Q}$, $N\mathfrak{P} = N_{F/\mathbb{Q}}\mathfrak{P}$.
S : l'ensemble des places sauvages , celles qui divisent ℓ.
E_F (resp. E_F^S): les unités (resp. les ℓ-unités) de F.
Cl_F : le groupe des classes d'ideaux de F (au sens ordinaire , pas au sens restreint).
$Cl_F(S)$ son sous-groupe « sauvage» , engendré par les classes des places divisant ℓ.
Cl_F^S :le groupe des ℓ-classes d'ideaux. $Cl_F^S \stackrel{\text{def}}{=} Cl_F/Cl_F(S)$.
$e_\mathfrak{P}$ l'indice d'inertie de $\mathfrak{P}$ et $f_\mathfrak{P}$ le degré résiduel ($p^{f_\mathfrak{P}} = (\mathbb{Z}_F : \mathfrak{P}) = N\mathfrak{P}$).
$F_\mathfrak{P}$: le corps local associé à la place $\mathfrak{P}$, $v_\mathfrak{P}$ la valuation $\mathfrak{P}$-adique normalisée.
$F_\mathfrak{P}^\times$ le groupe multiplicatif des éléments non nuls de $F_\mathfrak{P}$.
$F^{ab(\ell)}$ (resp. $F_\mathfrak{P}^{ab(\ell)}$) la ℓ-extension abélienne maximale de F (resp. de $F_\mathfrak{P}$). $\mu_\mathfrak{P}$: le sous-groupe des racines de l'unité de $F_\mathfrak{P}^\times$, avec $\mu_\mathfrak{P} = \mu_\mathfrak{P}^0 \oplus \mu_\mathfrak{P}^1$, qui a pour cardinal $m_\mathfrak{P} = (N\mathfrak{P} - 1)\cdot p^{r_\mathfrak{P}}$ où

$\mu_\mathfrak{P}^0$ a pour cardinal $N\mathfrak{P} - 1$ (et rassemble les racines modérées de l'unité),
$\mu_\mathfrak{P}^1$ a pour cardinal $p^{r_\mathfrak{P}}$ (et rassemble les racines sauvages de l'unité).

$l(\mathfrak{P}) = \ell^{v_\ell((N\mathfrak{P}-1)\cdot p^{r_\mathfrak{P}})}$ désigne la partie ℓ-primaire du cardinal de $\mu_\mathfrak{P}$.
$U_\mathfrak{P}^i$: le groupe des unités $\mathfrak{P}$-adiques congrues à 1 modulo $\mathfrak{P}^i$. C'est un $\mathbb{Z}_\ell$-module libre, de dimension égale à $e_\mathfrak{P}.f_\mathfrak{P}$, pour i > $\left[\frac{e_\mathfrak{P}}{p-1}\right]$, [.] = partie entière.
$\pi_\mathfrak{P}$ une uniformisante de l'anneau des entiers de $F_\mathfrak{P}$ ($v_\mathfrak{P}(\pi_\mathfrak{P}) = 1$).
$F_\mathfrak{P}^\times \simeq \mu_\mathfrak{P}^0 \oplus U_\mathfrak{P}^1 \oplus \pi_\mathfrak{P}^\mathbb{Z}$ ($\pi_\mathfrak{P}$ ayant été choisie) et $U_\mathfrak{P}^1 \simeq \mu_\mathfrak{P}^1 \oplus (U_\mathfrak{P}^1)^{libre}$.
Symbole de Hilbert associé à une place finie $\mathfrak{P}$ puis symbole modéré avec sa formule explicite :

$$\left(\frac{\cdot,\cdot}{\mathfrak{P}}\right)\begin{cases} F_\mathfrak{P}^\times \times F_\mathfrak{P}^\times \longrightarrow \mu_\mathfrak{P} = \mu_\mathfrak{P}^0 \oplus \mu_\mathfrak{P}^1 & \text{de cardinal } m_\mathfrak{P} = (N\mathfrak{P}-1)\cdot p^{r_\mathfrak{P}} \\ (x,y) \longmapsto \left(\frac{x,y}{\mathfrak{P}}\right) = \sqrt[m_\mathfrak{P}]{x}^{\,\omega_\mathfrak{P}(y)-1} & \omega_\mathfrak{P} \text{ application d'Artin locale (§1)} \end{cases}$$

$$(\cdot,\cdot)_\mathfrak{P} = \left(\frac{\cdot,\cdot}{\mathfrak{P}}\right)^{p^{r_\mathfrak{P}}}\begin{cases} F_\mathfrak{P}^\times \times F_\mathfrak{P}^\times \longrightarrow \mu_\mathfrak{P}^0 \simeq (\mathbb{Z}_F/\mathfrak{P})^\times & \text{de cardinal } N\mathfrak{P}-1 \\ (x,y) \longmapsto \left((-1)^{v_\mathfrak{P}(x)\cdot v_\mathfrak{P}(y)}\cdot\frac{x^{v_\mathfrak{P}(y)}}{y^{v_\mathfrak{P}(x)}}\right) \bmod \mathfrak{P}. \end{cases}$$

Nous avons donc pris comme symbole de Hilbert celui de profondeur maximale (plus généralement ([21]), $\mu_\mathfrak{P} \simeq (K_2F_\mathfrak{P})^{continu}$) et pour $m|m_\mathfrak{P}$ le symbole de Hilbert de profondeur m sera : $\left(\frac{\cdot,\cdot}{\mathfrak{P}}\right)_m \stackrel{\text{def}}{=} \left(\frac{\cdot,\cdot}{\mathfrak{P}}\right)^{m_\mathfrak{P}/m}$. De même, $(\cdot,\cdot)_{\mathfrak{P},m} \stackrel{\text{déf}}{=} (\cdot,\cdot)_\mathfrak{P}^{\frac{N\mathfrak{P}-1}{m}}$. (voir §4 pour la loi de réciprocité).

$\mathbb{C}_\ell$ le complété d'une clôture algébrique de $\mathbb{Q}_\ell$ ($\mathbb{C}_\ell$ est d'ailleurs algébriquement clos).
Pour un groupe G (avec une loi $\times$) et un entier n de $\mathbb{N}$, ${}_nG \stackrel{\text{def}}{=} \{g \in G, g^n = 1\}$,
${}^nG \stackrel{\text{def}}{=} G/G^n$, $G(p) \stackrel{\text{def}}{=}$ p-Sylow de G (p un nombre premier), $|G|$ étant le cardinal de G.

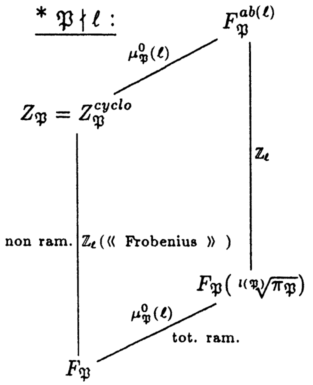
§1 - Brefs rappels de la théorie du corps de classes pour les ℓ-extensions.

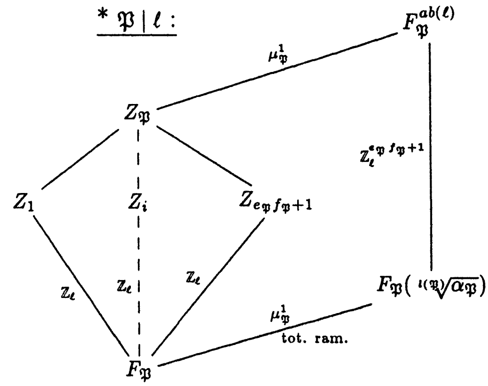
Le langage des ℓ-idèles généralisés de J.F.Jaulent provient de celui des idèles au sens classique (de Chevalley) et de certains résultats de [2]. Les objets créés permettent un emploi facilité de la théorie du corps de classes pour les ℓ-extensions[1]. Nous rappelons juste certaines définitions et résultats; pour plus de détails, consulter [15], [16], [2]. Pour chaque $\mathfrak{P}$, nous choisissons une uniformisante $\pi_\mathfrak{P}$.

$$Gal(F_\mathfrak{P}^{ab(\ell)}/F_\mathfrak{P}) \stackrel{\omega_\mathfrak{P}}{\simeq} \varprojlim_n (F_\mathfrak{P}^\times/(F_\mathfrak{P}^\times)^{\ell^n}) = \begin{cases} \mu_\mathfrak{P}^0(\ell)\cdot\pi_\mathfrak{P}^{\mathbb{Z}_\ell} & \text{pour } \mathfrak{P} \text{ étranger à } \ell \\ \mu_\mathfrak{P}^1\cdot(U_\mathfrak{P}^1)^{libre}\cdot\pi_\mathfrak{P}^{\mathbb{Z}_\ell} & \text{pour } \mathfrak{P} \text{ divisant } \ell\ . \end{cases}$$

[1] si on prend un sous-corps d'une ℓ -extension (algébrique) de F ,de degré fini sur F , alors ce degré est une puissance de ℓ

En formant cette limite projective, nous avons délesté $F_{\mathfrak{P}}^{\times}$ de son sous-groupe ℓ-divisible maximal et nous obtenons un $\mathbb{Z}_\ell$-module de type fini (donc compact) que nous notons $\widehat{F_{\mathfrak{P}}^{\times}}$. Visualisons ceci sur les schémas suivants où apparaissent toutes les ℓ-extensions abéliennes du corps local considéré, $Z_{\mathfrak{P}}$ représentant le composé des $\mathbb{Z}_\ell$-extensions de $F_{\mathfrak{P}}$. $l(\mathfrak{P}) = \ell^{v_\ell(|\mu_{\mathfrak{P}}|)}$ désigne la ℓ-partie du cardinal de $\mu_{\mathfrak{P}}$ (rappel de notation).





Lorsque $\zeta_\ell \in F_{\mathfrak{P}}^{\times}$, $Z_{\mathfrak{P}}^{cyclo} = F(\bigcup_{n\geq 1} \zeta_{\ell^n})$. Notons aussi que la connaissance de $\alpha_{\mathfrak{P}}$ présuppose celle de l'action de $\omega_{\mathfrak{P}}(\mu_{\mathfrak{P}}^1)$. Lorsque $< \zeta_\ell > \subset F^{\times}$, soulignons le critère :

$$F(\sqrt[\ell]{x}) \text{ localement } \mathbb{Z}_\ell\text{-plongeable pour la place } \mathfrak{P} \iff (\sqrt[\ell]{x})^{\omega_{\mathfrak{P}}(\zeta_{l(\mathfrak{P})})-1} = 1.$$

Pour les ℓ-extensions globales, qui sont moins bien « rangées » , parlons du ℓ-groupe des idèles généralisés, $\mathcal{J}_F$, construit avec les $\widehat{F_{\mathfrak{P}}^{\times}}$ en lieu et place des $F_{\mathfrak{P}}^{\times}$ qui définissent $J_F = \prod_{\mathfrak{P}}^{rest.} F_{\mathfrak{P}}^{\times}$.

$$\mathcal{J}_F \overset{\text{def}}{=} \prod_{\mathfrak{P}}^{\text{restreint}} \widehat{F_{\mathfrak{P}}^{\times}} = \left(\prod_{\mathfrak{P}\nmid\ell}^{\text{restreint}} \mu_{\mathfrak{P}}^0(\ell) \cdot \pi_{\mathfrak{P}}^{\mathbb{Z}_\ell} \right) \cdot \left(\prod_{\mathfrak{P}|\ell} \mu_{\mathfrak{P}}^1 \cdot (U_{\mathfrak{P}}^1)^{\text{libre}} \cdot \pi_{\mathfrak{P}}^{\mathbb{Z}_\ell} \right).$$

Le ℓ-adifié du groupe des idèles principaux $\mathcal{R}_F \overset{\text{def}}{=} \mathbb{Z}_\ell \otimes_{\mathbb{Z}} F^{\times}$ s'envoie injectivement dans $\mathcal{J}_F$ via « l'injection diagonale » ; son image est fermée et nous l'identifions à $\mathcal{R}_F$. Le principal intêret de cette construction est que nous avons un homéomorphisme défini par l'application de réciprocité globale entre les $\mathbb{Z}_\ell$-modules compacts $\mathcal{J}_F/\mathcal{R}_F$ et $Gal(F^{ab(\ell)}/F)$.

$$\omega \left\{ \begin{array}{ccc} \mathcal{J}_F/\mathcal{R}_F & \longrightarrow & Gal(F^{ab(\ell)}/F) \\ (i_{\mathfrak{P}})_{\mathfrak{P}}\mathcal{R}_F & \longmapsto & \prod_{\mathfrak{P}} \omega_{\mathfrak{P}}(i_{\mathfrak{P}}) \end{array} \right.$$

L'image $\omega_{\mathfrak{P}}\left(\widehat{F_{\mathfrak{P}}^{\times}}\right)$ s'identifie au groupe de décomposition attaché à la place $\mathfrak{P}$, $\omega_{\mathfrak{P}}(U_{\mathfrak{P}})$ à son sous-groupe d'inertie. Avec cet homéomorphisme, nous pouvons écrire des groupes de Galois infinis sans nous préoccuper des normes universelles car celles-ci ont disparu de par leur structure en groupe divisible.

§2 - Les ℓ-extensions abéliennes ℓ-ramifiées étudiées

Nous supposons dorénavant $< \zeta_\ell > \subset F^{\times}$. Notons que F n'a pas de place réelle et que nous pouvons aussi parler de radicaux kummeriens. Le radical associé à une extension K/F d'exposant ℓ est le sous-groupe de $F^\times/(F^\times)^\ell$ constitué des classes $x(F^\times)^\ell/(F^\times)^\ell$ avec $F(\sqrt[\ell]{x}) \subset K$ (nous écrirons $\bar{x} \in radical(K/F)$)[2]. Nous nous plaçons dans le contexte de la ℓ-ramification c'est à dire ici des ℓ-extensions qui ne sont pas ramifiées aux places étrangères à ℓ. Notons M la ℓ-extension abélienne (la pro-ℓ-extension abélienne) qui contient toutes les autres : la ℓ-ramifiée abélienne maximale. Son groupe de Galois est de type fini :

$$Gal(M/F) \simeq \text{pro-}\ell \text{ partie} \left(\frac{J_F}{\prod_{\mathfrak{P}\nmid\ell} U_{\mathfrak{P}} \cdot F^\times} \right) \simeq \frac{\mathcal{J}_F}{\prod_{\mathfrak{P}\nmid\ell} \mu^0_{\mathfrak{P}}(\ell) \cdot \mathcal{R}_F} \simeq \mathbb{Z}_\ell^{1+c_F+\delta_F} \oplus Torsion(fini).$$

δ_F mesure le défaut de la conjecture de Leopoldt. Le composé des $\mathbb{Z}_\ell$-extensions de F est fixé par la torsion de $Gal(M/F)$ et nous le notons Z. Nous allons réduire un peu la liberté que l'on laisse pour M aux places sauvages en introduisant après F.Bertrandias et J.-J. Payan l'extension H qui se loge entre Z et M définie comme ℓ-extension abélienne maximale qui rassemble toutes les ℓ-extensions cycliques de F qui sont localement partout $\mathbb{Z}_\ell$-plongeables. Dans [3], on parle plutôt de composé des ℓ-extensions cycliques de F globalement plongeable pour tous n dans une extension cyclique de degré ℓ^n sur F mais ces deux notions coïncident d'après [2] chap.10. La première façon de voir laisse moins d'incertitude pour savoir si ce sont des conditions globales ou locales qui caractérisent le radical initial associé. Il est aisé de voir $Z \subset H$ (condition d'emboîtement pour les corps cycliques de degré ℓ^n sur F) et $H \subset M$ repose sur l'argument servant à dire qu'une $\mathbb{Z}_\ell$-extension est ℓ-ramifiée. Remarquons aussi que les extensions cycliques de degré ℓ^n sur F qui interviennent ne sont pas ℓ-ramifiées dès que n est assez grand pour peu que l'extension cyclique de H considérée ne soit pas dans Z. A ces trois extensions, nous associons les trois sous-extensions Z', H', M' maximales et d'exposant ℓ c'est à dire les composés des « premiers étages » (voir le diagramme de fin de ce §2) et trois radicaux kummeriens $\mathfrak{Z}_F$, $\mathfrak{H}_F$, $\mathfrak{M}_F$. Caractérisons ces radicaux.

(rad. modéré) $\bar{x} \in \mathfrak{M}_F \Leftrightarrow (v_{\mathfrak{P}}(x) \equiv 0[\ell], \mathfrak{P} \nmid \ell) \Leftrightarrow ((x, \zeta_{l(\mathfrak{P})})_{\mathfrak{P},\ell} = 1, \mathfrak{P} \nmid \ell) \Leftrightarrow \left(\left(\frac{x,\zeta_{l(\mathfrak{P})}}{\mathfrak{P}}\right)_\ell = 1, \mathfrak{P} \nmid \ell\right)$.

L'appartenance à $\mathfrak{M}_F$ est une condition uniquement basée sur les valuations, la recherche numérique de ce radical se scindant en une partie concernant les ℓ-unités et l'autre le ℓ-groupe des ℓ-classes d'idéaux :

$$1 \longrightarrow {}^\ell E_F^S \longrightarrow \mathfrak{M}_F \xrightarrow{\bar{x} \mapsto cl\left(\sqrt[\ell]{\prod_{\mathfrak{P}\nmid\ell} \mathfrak{P}^{v_{\mathfrak{P}}(x)}}\right)} {}_\ell Cl_F^S \longrightarrow 1$$

(rad. hilbertien) $\bar{x} \in \mathfrak{H}_F \Leftrightarrow ((\sqrt[\ell]{x})^{\omega_{\mathfrak{P}}(\zeta_{l(\mathfrak{P})})-1} = 1 \text{ pour tous } \mathfrak{P}) \Leftrightarrow \left(\left(\frac{x,\zeta_{l(\mathfrak{P})}}{\mathfrak{P}}\right)_\ell = 1, \text{ tous } \mathfrak{P}\right)$.

Les mêmes conditions aux places modérées perdurent. Importantes, sont bien sûr les formules explicites (cf [2] chapitre 12 au signe près) :

$$\text{pour } \mathfrak{P}|\ell\,,\ \left(\frac{x,\zeta_\ell}{\mathfrak{P}}\right)_\ell = \zeta_\ell^{-\frac{1}{\ell}\cdot Trace_{F_{\mathfrak{P}}/\mathbb{Q}_\ell}(log_{\mathfrak{P}}(x))} = \zeta_\ell^{-\frac{1}{\ell}\cdot log_\ell\left(N_{F_{\mathfrak{P}}/\mathbb{Q}_\ell}(x)\right)}$$

$log_{\mathfrak{P}}$ désignant l'unique logarithme $\mathfrak{P}$-adique prolongeant log_ℓ qui est nul sur la torsion et sur $\ell^{\mathbb{Z}}$. D'autres radicaux non précédemment évoqués s'inscrivent aussi dans le schéma de ℓ-ramification abélienne :

$$\mathfrak{D}_F = radical(D'_F/F) = \{\bar{x} \in \mathfrak{M}_F, (x_{\mathfrak{P}})_{\mathfrak{P}} = (u_{\mathfrak{P}} \cdot \pi_{\mathfrak{P}}^{\ell i_{\mathfrak{P}}})_{\mathfrak{P}\nmid\ell} \cdot (t_{\mathfrak{P}}^\ell)_{\mathfrak{P}|\ell}\}$$
$$\mathfrak{C}_F = \text{radical}(C'_F/F) = \{\bar{x} \in \mathfrak{M}_F, (x_{\mathfrak{P}})_{\mathfrak{P}} = (u_{\mathfrak{P}} \cdot \pi_{\mathfrak{P}}^{\ell i_{\mathfrak{P}}})_{\mathfrak{P}\nmid\ell}((u_{\mathfrak{P}}^{nr})^{a_{\mathfrak{P}}} \cdot u_{\mathfrak{P}}^\ell \cdot \pi_{\mathfrak{P}}^{\ell i_{\mathfrak{P}}})_{\mathfrak{P}|\ell}\}$$
$$\mathfrak{H}_F^{cy} = \text{radical}(H'^{cy}_F/F) = \{\bar{x} \in \mathfrak{M}_F, (x_{\mathfrak{P}})_{\mathfrak{P}} = (u_{\mathfrak{P}} \cdot \pi_{\mathfrak{P}}^{\ell i_{\mathfrak{P}}})_{\mathfrak{P}\nmid\ell} \cdot (\zeta_{\ell^{r_{\mathfrak{P}}}}^{i_{\mathfrak{P}}} \cdot t_{\mathfrak{P}}^\ell)_{\mathfrak{P}|\ell}\}$$

[2] **K/F d'exposant ℓ : abélienne à groupe de Galois d'exposant ℓ**

Ici, $(x_{\mathfrak{P}})_{\mathfrak{P}}$ désigne l'image de $x \in F^\times$ dans $\prod_{\mathfrak{P}} F_{\mathfrak{P}}^\times$. $u_{\mathfrak{P}}$ est une unité $\mathfrak{P}$-adique, $t_{\mathfrak{P}}$ un élément quelconque de $F_{\mathfrak{P}}^\times$, $u_{\mathfrak{P}}^{nr}$ une unité principale bien choisie (cf §6 – 3 pour les exemples avec $F = \mathbb{Q}(\zeta_3, \sqrt{d})$), $(i_{\mathfrak{P}})_{\mathfrak{P}}$ et $(a_{\mathfrak{P}})_{\mathfrak{P}|\ell}$ deux familles d'entiers relatifs.

D'_F est d'exposant ℓ sur F, ℓ-ramifiée ℓ-décomposée[3] et maximale. $Gal(D'_F/F) \simeq {}^\ell Cl_F^S$.

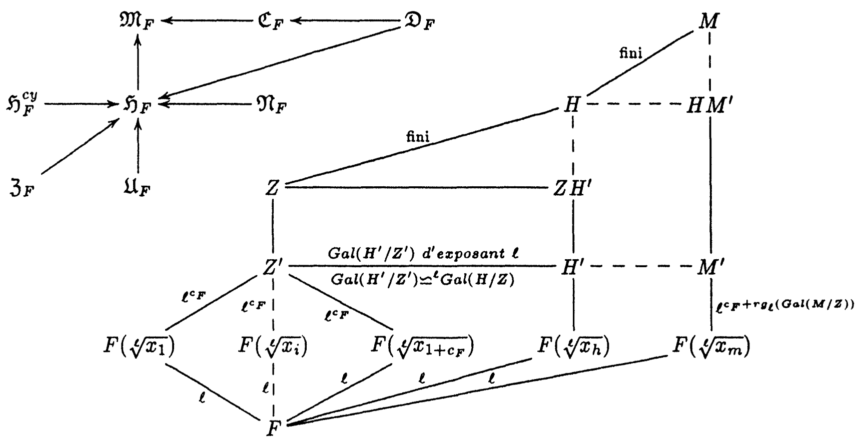
C'_F assemble les extensions d'exposant ℓ contenues dans le ℓ-corps de classes de Hilbert, $Gal(C'_F/F) \simeq {}^\ell Cl_F$.

H'^{cy}_F est le composé des extensions d'exposant ℓ localement partout $\mathbb{Z}_\ell$-plongeable dans la $\mathbb{Z}_\ell$-extension cyclotomique.

Les intersections exactes de ces derniers radicaux avec $\mathfrak{Z}_F$ sont a priori mystérieuses; un éclaicissement n'apparait qu'après quelques étapes dans les calculs (cf §6 – 3). Tous les radicaux écrits sauf $\mathfrak{Z}_F$, et à un degré moindre $\mathfrak{N}_F$ (le radical construit avec les unités logarithmiques [17], accessible si nous connaissons toutes les écritures locales des ℓ-unités), ne demandent à un x global relevant un $\overline{x}$ de $F^\times/(F^\times)^\ell$ que des propriétés locales pour l'appartenance à l'un d'entre eux (des « auto-propriétés locales »). Il « suffit » de regarder la famille des écritures locales d'un élément global x (et pour les places modérées, ce sont toujours les mêmes conditions aux valuations). Le contraste avec $\mathfrak{Z}_F$ provient de l'insuffisance de toute recherche de ce dernier basée sur des propriétés locales. Il est l'orthogonal de $Gal(H'/Z')$ dans la dualité kummérienne:

$$\begin{cases} \mathfrak{H}_F & \times & Gal(H'/Z') & \longrightarrow & <\zeta_\ell> \\ (\overline{x} & , & \sigma\ ) & \longmapsto & (\sqrt[\ell]{x})^{\sigma-1} \end{cases}$$

L'approche de $\mathfrak{Z}_F$ par des radicaux « locaux » a atteint son apogée avec $\mathfrak{H}_F$: nous ne pouvons pas imposer d'autres conditions aux places sauvages sans entrer dans des cas particuliers (cf tables). Visualisons certains objets cherchés dans le cas où $Gal(Z/F) \simeq \mathbb{Z}_\ell^{1+c_F}$, c_F représentant le nombre de places complexes de F (ici $c_F = \frac{[F:\mathbb{Q}]}{2}$):



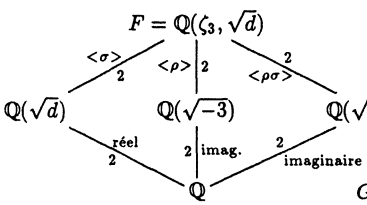
Ici, $\mathfrak{U}_F$ est le radical de Tate, $\mathfrak{U}_F \overset{\text{def}}{=} \{\overline{x} \in \frac{F^\times}{(F^\times)^\ell}\ ,\ \{x, \zeta_\ell\} = 1\ dans K_2K\ \}$. Sa $\mathbb{F}_\ell$-dimension est $1 + c_F$ (voir [27], [28]). Posons $Cl_F(3) = \bigoplus_{0 \leqslant i \leqslant n} < cl(\ {}^{3^{l_i}}\!\sqrt{(\alpha_i)}) >$ avec $\alpha_i \in F^\times$ et $n = \mathrm{rg}_3(Cl_F)$. Alors la structure de Gal(H/Z) dépend des écritures locales aux places sauvages des unités de F et des α_i (cf écriture de Gal(H/Z) au §4 et §4 -2-étape 4). H peut contenir strictement ZH', mais $[H : ZH'] > 1$ seulement si $[H' : Z'] > 1$. Voir aussi §4 -2-étape 4 et les tables numériques pour $F = \mathbb{Q}(\sqrt{d}, \sqrt{-3})$ où de nombreux cas de figure se rencontre pour les degrés $[H' : Z']$ et $[H : ZH']$. Gal(M/H) est de

[3] c'est à dire totalement décomposée aux places divisant ℓ

nature très différente (de même $\mathfrak{M}_F - \mathfrak{H}_F$ et $\mathfrak{H}_F - \mathfrak{Z}_F$) et s'exprime avec les groupes de racines de l'unité locales $(\mu_{\mathfrak{P}}(\ell))_{\mathfrak{P}|\ell}$ et globales $\mu_F(\ell)$ (cf §4 étape 2). $[M : H] > 1$ et $HM' = H$ peuvent arriver simultanément lorsque $[H : Z] > 1$.

Les flèches d'inclusion du diagramme des radicaux sont valables pour tous les corps contenant $< \zeta_\ell >$ mais des intersections plus précises peuvent être explicitées pour des familles de corps avec une ℓ-arithmétique simple. Dans cette optique, les corps ℓ-réguliers [12] vérifient les deux conditions exigeantes suivantes : (i) il n'y a qu'une place sauvage (au dessus de ℓ), (ii) $Cl_F^S(\ell) = 1$, ce qui minimise $dim_{\mathbb{F}_\ell}\mathfrak{M}_F$ à $1 + c_F$ et entraîne $\mathfrak{M}_F = \mathfrak{H}_F = \mathfrak{Z}_F = \mathfrak{U}_F = \mathfrak{N}_F$, $\mathfrak{D}_F = 1$.

§3 - Généralités et notations sur les corps biquadratiques $F = \mathbb{Q}(\zeta_3, \sqrt{d})$, d dans $\mathbb{N}$, $d > 1$ et sans facteur carré



$\Delta \overset{\text{déf}}{=} Gal(F/\mathbb{Q}) \simeq < \rho >< \sigma > \simeq C_2 \times C_2$

$k \overset{\text{déf}}{=} \mathbb{Q}(\sqrt{d})$, $k^* \overset{\text{déf}}{=} \mathbb{Q}(\sqrt{-3d})$

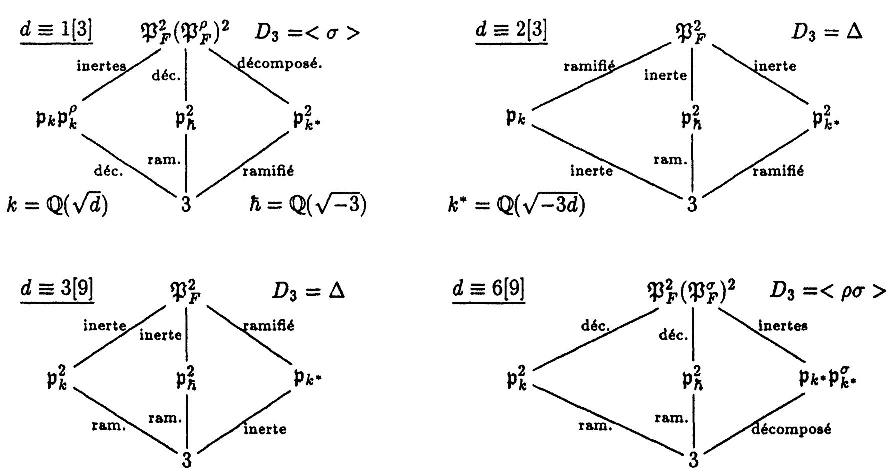
$Gal(F/k) \simeq < \sigma >$, $Gal(F/\mathbb{Q}(\sqrt{-3})) \simeq < \rho >$

$\mathbb{Z}$-bases d'entiers et discriminant (à l'aide de [29]). ϵ_d

Lemme 1. *Si $d = 3$, $E_F =< \zeta_{12} > \cdot \left(\frac{2+\sqrt{3}+i}{2}\right)^{\mathbb{Z}}$. Sinon, nous avons : $E_F =< \zeta_6 > \cdot \epsilon^{\mathbb{Z}}$ avec $\epsilon = \epsilon_d$ ou $\epsilon^2 = \pm\epsilon_d$, ϵ_d étant l'unité fondamentale ($\epsilon_d > 1$) de $k = \mathbb{Q}(\sqrt{d})$. $F(\sqrt[3]{\epsilon}) = F(\sqrt[3]{\epsilon_d})$.*

Lemme 2. $\mathcal{E}_F^S = \mathbb{Z}_3 \otimes_Z E_F^S \simeq \begin{cases} \zeta_3^{\mathbb{F}_3} \cdot \epsilon_d^{\mathbb{Z}_3} \cdot 3^{\mathbb{Z}_3} & \text{pour } d \equiv 2[3] \text{ ou } d \equiv 3[9] \\ \zeta_3^{\mathbb{F}_3} \cdot \epsilon_d^{\mathbb{Z}_3} \cdot 3^{\mathbb{Z}_3} \cdot \pi_k^{\mathbb{Z}_3} & \text{pour } d \equiv 1[3] \\ \zeta_3^{\mathbb{F}_3} \cdot \epsilon_d^{\mathbb{Z}_3} \cdot 3^{\mathbb{Z}_3} \cdot \pi_{k^*}^{\mathbb{Z}_3} & \text{pour } d \equiv 6[9] \end{cases}$ *où π_k (resp. π_{k^*}) est une 3-unité convenable, choisie parmi les générateurs de E_k^S (resp. $E_{k^*}^S$). Nous avons aussi $Cl_F^S(3) = Cl_F(3)$ pour $d \equiv 2[3]$ ou $d \equiv 3[9]$.*

Preuve : cela résulte des différents cas de ramification pour 3. (D_3:groupe de décomposition).



Proposition 1. *Les radicaux $\mathfrak{M}_F, \mathfrak{H}_F, \mathfrak{Z}_F, \mathfrak{N}_F, \mathfrak{U}_F$ sont des $\mathbb{F}_3[\Delta]$-modules .*

$$\begin{cases} \mathbb{F}_3[\Delta] \times \mathcal{R}adical & \longrightarrow \mathcal{R}adical \\ \left(\sum_{i=1}^{i=4} \alpha_i\delta_i \,,\, \overline{x}\right) & \mapsto \overline{x^{\sum a_i\delta_i}} \qquad \text{où } x \text{ relève } \overline{x} \in \frac{F^\times}{(F^\times)^3} \text{ et } a_i \in \mathbb{Z} \text{ relève } \alpha_i \in \mathbb{F}_3. \end{cases}$$

De même pour $Gal(H/F) \simeq \frac{\mathcal{J}_F}{\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}$, $Gal(Z/F) \simeq$ partie libre $(Gal(H/F)) \simeq \frac{\mathcal{J}_F}{\sqrt{\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}}$.

Preuve : cela provient du fait que M, H, Z, sont galoisiennes sur $\mathbb{Q}$, de l'action $\{x,y\}^\delta = \{x^\delta, y^\delta\}$ dans K_2F ($\delta \in \Delta$). Δ agit sur le $\mathbb{Z}_3$-module $\mathcal{J}_F = \overset{rest.}{\widehat{\prod_{\mathfrak{P}}}} F_{\mathfrak{P}}^\times$ par $((i_{\mathfrak{P}})_{\mathfrak{P}})^\delta = (i_{\mathfrak{P}}^\delta)_{\mathfrak{P}^\delta}$. Les dénominateurs des groupes de Galois sont stables par Δ.

Ici $\Delta \simeq C_2 \times C_2$ est abélien, le groupe des caractères sur Δ, noté Δ^* est donc aussi un $C_2 \times C_2$, formé avec les quatre caractères absolument irréductibles considérés comme allant dans $\mathbb{C}_3^\times$. Nous formons les quatre idempotents primitifs de l'algèbre $\mathbb{C}_3[\Delta]$.

Δ^*	caractère	1	ρ	σ	$\rho\sigma$	idempotent associé
1	unité	1	1	1	1	$e_1 = \frac{1}{4}(1+\rho+\sigma+\rho\sigma) = \frac{1}{4}N_{F/\mathbb{Q}}$
φ	réel	1	1	-1	-1	$e_\varphi = \frac{1}{4}(1+\rho-\sigma-\rho\sigma) = \frac{1}{4}(1-\sigma)N_{F/k}$
ω	cyclotomique	1	-1	1	-1	$e_\omega = \frac{1}{4}(1-\rho+\sigma-\rho\sigma) = \frac{1}{4}(1-\rho)N_{F/\kappa}$
φ^*	imaginaire	1	-1	-1	1	$e_{\varphi^*} = \frac{1}{4}(1-\rho-\sigma+\rho\sigma) = \frac{1}{4}(1-\rho)N_{F/k^*}$

Ces idempotents orthogonaux deux à deux sont ici dans $\mathbb{Z}_3[\Delta]$ (4 et 3 sont étrangers et les composantes cycliques de Δ sont des C_2) ce qui nous permet d'écrire l'algèbre $\mathbb{Z}_3[\Delta]$ comme composé direct des facteurs locaux correspondants aux idempotents primitifs : $\mathbb{Z}_3[\Delta] = \bigoplus_{\chi\in\Delta^*} \mathbb{Z}_3[\Delta]\cdot e_\chi$. Tout $\mathbb{Z}_3[\Delta]$-module A (ici en notation multiplicative) est somme directe de composantes isotypiques : $A = \bigoplus_{\chi\in\Delta^*} A^{e_\chi}$. Dans les corps biquadratiques $F = \mathbb{Q}(\zeta_3, \sqrt{d})$, ce découpage de modules galoisiens se révèle particulièrement bien adapté en raison du grand nombre de renseignements que l'on peut tirer des trois sous-corps quadratiques. Par exemple, nous retrouvons le résultat classique $Cl_F(3) \simeq Cl_k(3) \oplus Cl_{k^*}(3)$ en identifiant $Cl_k(3)$ à $Cl_F(3)^{e_\varphi}$ et $Cl_{k^*}(3)$ à $Cl_F(3)^{e_{\varphi^*}}$. Nous avons le même découpage pour un $\mathbb{F}_3[\Delta]$-module. Considérons une 3-extension K de $F = \mathbb{Q}(\zeta_3, \sqrt{d})$ à groupe de Galois d'exposant 3 – qui n'est pas forcément dans le schéma de 3-ramification précédent –, il lui correspond un radical kummerien noté $\mathcal{R}ad(K/F)$, isomorphe à $\mathrm{Hom}(Gal(K/F), <\zeta_3>)$. Mais maintenant si $\mathcal{R}ad(K/F)$ et $Gal(K/F)$ sont tous deux des $\mathbb{F}_3[\Delta]$-modules, l'action du second sur le premier « passe-t-elle » aux composantes isotypiques? La réponse (le miroir) donnée par Leopoldt peut s'énoncer ainsi ici :

$$\left(\sqrt[3]{x^\delta}\right)^{g-1} = \left(\sqrt[3]{x}\right)^{g^{\omega(\delta)\delta^{-1}}-1} \qquad \text{pour } \delta\in\Delta\ ,\ g\in Gal(K/F)$$

$$\left(\sqrt[3]{x^{e_\chi}}\right)^{g-1} = \left(\sqrt[3]{x}\right)^{g^{e_{\omega\chi^{-1}}}-1} \overset{ici}{=} \left(\sqrt[3]{x}\right)^{g^{e_{\omega\chi}}-1} \qquad x \text{ relevant } \overline{x} \in \mathcal{R}ad(K/F) \text{ et } \sqrt[3]{x^{e_\chi}} \text{ signifie}$$

que nous considérons une racine cubique d'un représentant de $\overline{x}^{e_\chi}$. Une composante isotypique de $Gal(K/F)$ associée à un caractère irréductible χ imprime une rotation (dans le plan complexe) à la composante isotypique de $\mathcal{R}ad(K/F)$ qui est associée au caractère $\omega\chi^{-1}$ et laisse fixes les autres. Les caractères associés à ces deux $\mathbb{F}_3[\Delta]$-modules se déduisent l'un de l'autre par l'involution du miroir, cf [20].

Proposition 2. *Le caractère associé à $\mathfrak{Z}_F$ pour $F = \mathbb{Q}(\zeta_3, \sqrt{d})$ par $\mathfrak{Z}_F \mapsto \sum_{\chi\in\Delta^*} (dim_{\mathbb{F}_3} \mathfrak{Z}_F^{e_\chi})\cdot\chi$ est $1+\omega+\varphi$. Nous pouvons commencer une $\mathbb{F}_3$- base de $\mathfrak{Z}_F$ par les classes de 3 et de $j =<\zeta_3>$ modulo $F^{\times 3}$, et la compléter par la classe d'un élément convenable de $k = \mathbb{Q}(\sqrt{d})$ que l'on recherche à l'aide de la composante réelle de $\mathfrak{M}_F$.*

Preuve : considérons la suite exacte suivante qui peut servir à prouver que $Gal(M/F)$ est un $\mathbb{Z}_3$-module de type fini.

$$1 \longrightarrow \frac{\prod_{\mathfrak{P}|3} U_{\mathfrak{P}}}{s_3(\mathcal{E}_F)} \longrightarrow Gal(M/F) \simeq \frac{\mathcal{J}_F}{\prod_{\mathfrak{P}\nmid 3} \mu_{\mathfrak{P}}{}^0(3)\cdot \mathcal{R}_F} \xrightarrow{\overline{(i_{\mathfrak{P}})_{\mathfrak{P}}} \mapsto cl\left(\prod_{\mathfrak{P}} \mathfrak{P}^{v_{\mathfrak{P}}(i_{\mathfrak{P}})}\right)} Cl_F \longrightarrow 1$$

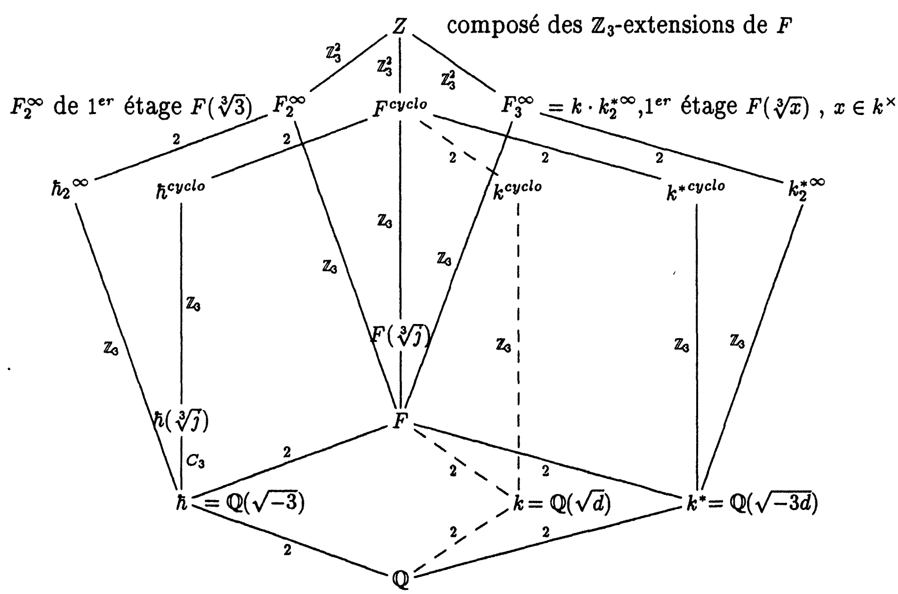
Par s_3, nous entendons la surjection de semi-localisation induite par l'injection diagonale notée i_F.

$s_3 : \mathcal{R}_F = \mathbb{Z}_3 \otimes_{\mathbb{Z}} F^\times \xrightarrow{\prod_\lambda a_\lambda \otimes x_\lambda \mapsto \prod_\lambda (i_F x_\lambda)^{a_\lambda}} \prod_{\mathfrak{P}|3} \widehat{F_{\mathfrak{P}}^\times}$, où $\{a_\lambda\}_\lambda$ est une famille d'entiers 3-adiques. Cette conjecture est vraie dans le cas F abélienne sur $\mathbb{Q}$, voir [4] pour les arguments. Ici, le nombre de Dirichlet est 1 ce qui assure l'injectivité de s_3 restreint à $\mathcal{E}_F = \mathbb{Z}_3 \otimes_{\mathbb{Z}} E_F$ (d'une manière générale, une formulation de la conjecture de Leopoldt est : s_ℓ restreint à $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} E_F$ est injective). $\dim_{\mathbb{F}_3} \mathfrak{Z}_F \overset{\text{ici}}{=} 3$ est « d'autant plus vraie » . Nous avons donc les isomorphismes :

$$Gal(Z/F) \simeq \frac{\prod_{\mathfrak{P}|3} (U_{\mathfrak{P}}^1)^{libre}}{\sqrt{s_3^{rest}(\mathcal{E}_F^{libre})}} \simeq \frac{\mathcal{J}_F}{\sqrt{\prod_{\mathfrak{P}\nmid 3} \mu_{\mathfrak{P}}^0(3)\cdot \mathcal{R}_F}} \simeq \frac{\mathcal{J}_F}{\sqrt{\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot \mathcal{R}_F}}$$

Or le caractère du $\mathbb{Z}_3[\Delta]$-module $\prod_{\mathfrak{P}|3} (U_{\mathfrak{P}}^1)^{libre}$ est le caractère dit régulier $1+\varphi+\omega+\varphi^*$. Maintenant le lemme 1 nous dit que le caractère du dénominateur de $Gal(Z/F)$ est φ, d'où celui de $Gal(Z/F)$: $1+\omega+\varphi^*$. Celui de $\mathfrak{Z}_F$, qui est son reflet via l'involution du miroir, devient ainsi $1+\omega+\varphi$.

Dans le dessin ci-dessous qui illustre le lemme 1, K^{cyclo} représente la $\mathbb{Z}_3$-extension cyclotomique d'un corps K.



§4 - Pratique de la théorie du corps de classes

1 - Le groupe de Galois Gal(H'/Z')

$Gal(H/Z)$ est le noyau de la flèche naturelle de restriction de $Gal(H/F)$ vers $Gal(Z/F)$, c'est ici la torsion de $Gal(H/F)$.

$$Gal(H/Z) \simeq \mathrm{Ker}\left(\frac{\mathcal{J}_F}{\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot \mathcal{R}_F} \xrightarrow{\text{restrition}} \frac{\mathcal{J}_F}{\sqrt{\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot \mathcal{R}_F}} \right) \simeq \frac{\sqrt{\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot \mathcal{R}_F}}{\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot \mathcal{R}_F}$$

Quant à $Gal(H'/Z')$, il s'identifie au quotient 3-primaire ${}^3Gal(H/Z)$ (cf schéma §2). L'extension H a son groupe de Galois mieux adapté à une description idélique que M car le « flou » aux places sauvages a disparu. Nous avons $rg_3(Gal(M/Z)) = dim_{\mathbb{F}_3}\mathfrak{M}_F - dim_{\mathbb{F}_3}\mathfrak{Z}_F$.

Proposition 3. $Gal(H/Z) \simeq Gal(H_k/Z_k) \oplus Gal(H_{k^*}/Z_{k^*}) \simeq \frac{\sqrt{\prod_{\mathfrak{p}} \mu_{\mathfrak{p}}(3)\cdot \mathcal{R}_k}}{\prod_{\mathfrak{p}} \mu_{\mathfrak{p}}(3)\cdot \mathcal{R}_k} \oplus \frac{\sqrt{\prod_{\mathfrak{p}} \mu_{\mathfrak{p}}(3)\cdot \mathcal{R}_{k^*}}}{\prod_{\mathfrak{p}} \mu_{\mathfrak{p}}(3)\cdot \mathcal{R}_{k^*}}$

$rg_3(Gal(H/Z)) = dim_{\mathbb{F}_3}(\mathfrak{H}_F^{cy})^{e_\varphi} + dim_{\mathbb{F}_3}(\mathfrak{H}_F^{cy})^{e_{\varphi^*}}$ *Voir* §2 *pour* $\mathfrak{H}_F^{cy}$.

H_{k^*} désignant la 3-extension hilbertienne maximale de k^*, Z_{k^*} le composé des $\mathbb{Z}_3$-extensions de k^* (idem pour k).

Preuve : on sait que $Gal(H/Z) \simeq \bigoplus_{\chi\in\Delta^*} Gal(H/Z)^{e_\chi}$. Nous énoncons ici la trivialité des composantes unités et cyclotomiques et l'identification possible de $Gal(H_k/Z_k)$ et $Gal(H_{k^*}/Z_{k^*})$ aux composantes réelles et imaginaires. En effet, si nous considérons un sous-corps $\mathfrak{f}$ de F ,le groupe des idèles fixes par $Gal(F/\mathfrak{f})$ s'identifie à $\mathcal{J}_{\mathfrak{f}}$; d'où les assimilations $\mathcal{J}_F^{e_1} \simeq \mathcal{J}_{\mathbb{Q}}$, $\mathcal{J}_F^{e_\varphi} \simeq \mathcal{J}_k$, $\mathcal{J}_F^{e_\omega} \simeq \mathcal{J}_{\hbar}$, $\mathcal{J}_F^{e_{\varphi^*}} \simeq \mathcal{J}_{k^*}$. Mais $\hbar = \mathbb{Q}(\sqrt{-3})$ est principal avec 0 pour nombre de Dirichlet et ainsi $Gal(M_\hbar, Z_\hbar) = 1$ (idem pour $\mathbb{Q}$).Maintenant, les idèles $(i_{\mathfrak{P}})_{\mathfrak{P}}$ intéressants pour construire $Gal(H/Z)$ vérifient $(i_{\mathfrak{P}})_{\mathfrak{P}}^{3^t} = (\zeta_{\mathfrak{P}}\cdot \mathfrak{x}_{\mathfrak{P}})_{\mathfrak{P}}$ où $(\zeta_{\mathfrak{P}})_{\mathfrak{P}}$ est une famille de racines de l'unité d'ordre 3-primaire et $(\mathfrak{x}_{\mathfrak{P}})_{\mathfrak{P}}$ est l'idèle principal correspondant à un élément global noté $\mathfrak{x}$. La classe de ce dernier modulo $F^{\times 3}$ appartient donc à $\mathfrak{H}_F^{cy}$ ce qui donne l'équation au rang. Remarquons que les idèles principaux recherchés correspondent à des éléments globaux avec des valuations multiples de 3 ce qui va orienter nos recherches vers le 3-Sylow du groupe des classes de F (cf §4 – 2).

Signalons que le cardinal de la torsion de $Gal(M_k/k)$ a un aspect analytique p-adique donné par J.Coates [7]. Pour k totalement réel avec R_p le régulateur p-adique (au signe près) non nul (c'est la prime conjecture de Leopoldt : k a une seule $\mathbb{Z}_p$-extension, la cyclotomique), nous avons :

$$|Gal(M_k/k)^{torsion}| = [M_k : Z_k] \underset{p\text{-adique près}}{\overset{\text{à une unité}}{\approx}} \frac{p^{n_0+1} h_k R_p}{\mathrm{disc}_{k/\mathbb{Q}}{}^{1/2}} \prod_{\mathfrak{p}|p} \frac{1}{N\mathfrak{p}}$$

avec $p^{n_0} = [\mathbb{Q}^{\mathbb{Z}_p\text{-ext.}} \cap k : \mathbb{Q}]$, $h_k = |Cl_k|$. Pour k quelconque, le produit $R_p \cdot \mathrm{disc}_{k/\mathbb{Q}}{}^{-1/2}$ peut être défini dans $\mathbb{Q}_p$ sans ambiguïté ([1]). Cela a des conséquences sur l'écriture locale des unités aux places sauvages, nous verrons lesquelles pour les corps $\mathbb{Q}(\sqrt{d}) \subset \mathbb{Q}(\zeta_3, \sqrt{d})$ de manière plus générale lors du §4-2 -étape 2. Nous pouvons ainsi calculer l'ordre de $Gal(M/Z)$ dans le cadre des corps $\mathbb{Q}(\zeta_3, \sqrt{d})$.

L'inconnue étant ici la composante réelle de $\mathfrak{Z}_F$, la proposition 2 et la Spiegelungsrelation, nous permettent de dire que seule la composante imaginaire de $Gal(H/Z)$ nous sera utile ici.

$$\mathfrak{Z}_F^{e_\varphi} \text{ est l'orthogonal de } Gal(H'/Z')^{e_{\varphi^*}} \text{ dans } \begin{cases} \mathfrak{H}_F^{e_\varphi} \times Gal(H'/Z')^{e_{\varphi^*}} \longrightarrow \ <\zeta_3> \\ \overline{x}^{e_\varphi},\ \overline{(i_{\mathfrak{P}})_{\mathfrak{P}}}^{e_{\varphi^*}} \mapsto \sqrt[3]{x^{4e_\varphi}}^{\,\omega((i_{\mathfrak{P}})_{\mathfrak{P}}{}^{e_{\varphi^*}})-1} \end{cases}$$

$\omega((i_{\mathfrak{P}})_{\mathfrak{P}}{}^{e_{\varphi^*}})$ veut désigner ici la restriction à $F(\sqrt[3]{x})$ de l'image par l'application d'Artin globale de l'idèle en question. Nous avons ainsi les relations aux dimensions :

$$dim_{\mathbb{F}_3}(\mathfrak{H}_F^{e_\varphi}) = dim_{\mathbb{F}_3}((\mathfrak{H}_F^{cy})^{e_{\varphi^*}}) + 1\,,\quad dim_{\mathbb{F}_3}(\mathfrak{H}_F^{e_{\varphi^*}}) = dim_{\mathbb{F}_3}((\mathfrak{H}_F^{cy})^{e_\varphi}),$$

puis : $\mathfrak{H}_F^{e_1} \simeq <\overline{3}>$, $\mathfrak{H}_F^{e_\omega} \simeq <\overline{j}>$ la barre signifiant modulo les cubes.

2 - Quelques étapes dans les calculs

L'ordre évoqué ci-dessous est révélateur de l'algorithme utilisé. Pour écrire le programme, nous nous sommes servi d'une manière omniprésente du système PARI et de la possibilité de créer des variables dites « GEN » , en particulier celles de types 15 et 16 : « formes quadratiques » . Je remercie M[rs] Batut, Cohen, Olivier et les gens du groupe de travail « algorithmique arithmétique » de Bordeaux.

Etape 1 : explicitons $\mathfrak{M}_F$. Rappelons la suite exacte $1 \longrightarrow {}^3E_F^S \longrightarrow \mathfrak{M}_F \longrightarrow {}_3Cl_F^S \longrightarrow 1$ que l'on peut « descendre » dans les corps quadratiques $\mathbb{Q}(\sqrt{d})$ et $\mathbb{Q}(\sqrt{-3d})$ car les objets s'y prêtent bien (§4). Pour un corps $\mathbb{Q}(\sqrt{a})$ nous devons connaître numériquement les 3-unités et les idéaux (une $\mathbb{Z}$-base) $\mathfrak{a}_1,\dots,\mathfrak{a}_n$ dont les classes engendrent ${}_3Cl^S_{\mathbb{Q}(\sqrt{a})}$. A la loi de groupe sur les classes d'ideaux au sens restreint de $\mathbb{Q}(\sqrt{a})$ correspond la composition gaussienne de formes quadratiques (cf [8]). Sauf dans les corps quadratiques imaginaires, il n'y a pas qu'un unique idéal réduit (de petite norme) dans une classe et dans le cas quadratique réel les formes quadratiques réduites équivalentes sont rassemblées en cycles. Par forme quadratique interposée, les idéaux dont les classes engendrent ${}_3Cl_F^S$ nous sont connus. Puis nous recherchons des générateurs de tels idéaux élevés à une plus petite puissance principale.

Pour les corps réels $\mathbb{Q}(\sqrt{d})$ de discriminant D, nous utilisons pour ceci une belle notion issue des pas de réduction d'une forme quadratique : la fonction distance de Shanks ([8] chap.5). Dans les corps quadratiques, nous disposons d'algorithmes de réduction explicites (cf [8]). A $\mathfrak{a}$ un idéal entier principal de $\mathbb{Z}$-base $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$, de générateur $\tau = \frac{a_1+a_2\sqrt{d}}{2}$, nous associons $f = ax^2 + bxy + cy^2$ la forme quadratique $\frac{Norme(x\omega_1+y\omega_2)}{N\mathfrak{a}}$. Puis, après un nombre *fini* de pas successifs de réduction, le cycle principal correspondant est atteint (au plus $2 + \lceil log_2(|c|/\sqrt{D})\rceil$ cf [8] prop 5.6.6), puis la forme unité $(\pm 1,.,.)$ notée $\mathbf{1}$:

$f = (a,b,c)$ une forme « principale »

$f \xrightarrow[\text{step}]{\text{reduction}} \partial(f) \xrightarrow[\text{nbr. fini d'étapes}]{\text{réduction totale}}$ cycle principal (fini)

les formes quad. réduites équivalentes à $\mathbf{1}$ sont toutes dans le cycle

$\downarrow^{\partial}$ $\quad \rightarrow^{\partial}$ $\mathbf{1}$

$$\text{distance}(f,\partial(f)) \stackrel{\text{déf}}{=} \frac{1}{2} ln \left|\frac{b+\sqrt{D}}{b-\sqrt{D}}\right|$$

et nous sommes amenés à résoudre $\begin{cases} \frac{1}{2} ln \left|\frac{\tau}{\tau^{\rho}}\right| = \text{distance}(f,\mathbf{1}) \\ \tau \cdot \tau^{\rho} = N\mathfrak{a} = a \end{cases}$

La distance entre f et $\mathbf{1}$ est définie à un multiple du régulateur près suivant le nombre de tour effectués dans le cycle principal ce qui nous donne un générateur à une unité près. Ces calculs nécessitent une précision gigantesque dans les opérations avec les réels manipulés.

Pour $\mathbb{Q}(\sqrt{a})$ imaginaire, nous lions la recherche des éléments de norme $N = \prod_{p|N} p^{v_p(N)}$ à celle d'une $\mathbb{Z}$-base de $N\mathbb{Z}_{\mathbb{Q}(\sqrt{a})} = \prod_{p|N} (\mathfrak{p}^{\text{id ou } \sigma})^{v_p(N)}$ avec par convention $\mathfrak{p} = \mathbb{Z} \oplus \frac{b+\sqrt{D}}{2}\mathbb{Z}$, b positif . On a supposé que si $\mathfrak{p}|(N)$ alors $\mathfrak{p}^{\sigma} \nmid (N)$ et inversement avec $<\sigma> \simeq Gal(\mathbb{Q}(\sqrt{a})/\mathbb{Q})$. En particulier, seuls des premiers décomposés interviennent. A partir des $\mathbb{Z}$-bases plus accessibles des idéaux premiers et en utilisant la forme normale d'Hermite de produits d'idéaux, nous disposons de $2^{|\{p|N\}|-1}$ candidats $\mathbb{Z}$-bases pour $N\mathbb{Z}_{\mathbb{Q}(\sqrt{a})}$. Suit soit l'algorithme de Cornacchia (cf [8] mais ici le contexte est plus général) soit l'algorithme de Gauss qui cherche un vecteur minimal d'un réseau engendré par deux vecteurs dans un espace euclidien (cf...[8]).

Etape 2 : de $\mathfrak{M}_F$ à $\mathfrak{H}_F$.

Proposition 4. *Pour* $F = \mathbb{Q}(\zeta_3, \sqrt{d})$ *, les extensions* M *et* H *coïncident lorsque* $d \equiv 2[3]$ *ou* $d \equiv 3[9]$ *. D'où* $\mathfrak{M}_F = \mathfrak{H}_F$ *dans ces cas là .*

Si $d \equiv 1[3]$, $[M:H] = 3$, $dim_{\mathbb{F}_3}\mathfrak{M}_F^{e_\varphi} - 1 \leq dim_{\mathbb{F}_3}\mathfrak{H}_F^{e_\varphi} \leq dim_{\mathbb{F}_3}\mathfrak{M}_F^{e_\varphi}$, $\mathfrak{H}_F^{e_\varphi^*} = \mathfrak{M}_F^{e_\varphi^*}$

Si $d \equiv 6[9]$, $[M:H] = 3$, $dim_{\mathbb{F}_3}\mathfrak{M}_F^{e_\varphi^*} - 1 \leq dim_{\mathbb{F}_3}\mathfrak{H}_F^{e_\varphi^*} \leq dim_{\mathbb{F}_3}\mathfrak{M}_F^{e_\varphi^*}$, $\mathfrak{H}_F^{e_\varphi} = \mathfrak{M}_F^{e_\varphi}$

Preuve : $Gal(M/H)$ est le noyau de la flèche de restriction de $Gal(M/Z)$ vers $Gal(H/Z)$.

$$Gal(M/H) \simeq \text{Ker}\left(\frac{\mathcal{J}_F}{\prod\limits_{\mathfrak{P}\nmid 3}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F} \xrightarrow{\text{restrition}} \frac{\mathcal{J}_F}{\prod\limits_{\mathfrak{P}}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}\right) \simeq \frac{\prod\limits_{\mathfrak{P}}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}{\prod\limits_{\mathfrak{P}\nmid 3}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}$$

$$\simeq \frac{\prod\limits_{\mathfrak{P}|3}\mu_{\mathfrak{P}}(3)}{\prod\limits_{\mathfrak{P}|3}\mu_{\mathfrak{P}}(3)\cap\mathcal{R}_F} \underset{\text{Leopoldt}}{\overset{\text{sous conjecture}}{\simeq}} \frac{\prod\limits_{\mathfrak{P}|3}\mu_{\mathfrak{P}}(3)}{s_3(\mu_F(3))} \quad \text{(ici, conjecture vraie.)}$$

En effet, supposer l'injectivité de s_3 (cf proposition 2) restreinte aux unités est équivalent à annoncer l'intersection ci-dessus. Rappelons qu'ici, la conjecture de Leopoldt est forcément vraie, le nombre de Dirichlet étant 1. Le groupe $s_3(\mu_F(3))$ est l'image solénoïdale du 3-Sylow du groupe des racines de l'unité. Les calculs d'indices et de dimensions découlent de la ramification en 3 et du fait que $|\mu_{\mathfrak{P}}^1|$ est ici toujours 3 et jamais 9 ou plus. Un peu moins précis est la seule utilisation de la loi de réciprocité d'ordre 3 : $\left(\frac{x,j=\zeta_3}{\mathfrak{P}}\right)_3 = 1$ qui prend ici la forme $\prod\limits_{\mathfrak{P}\nmid 3}(j^{-v_{\mathfrak{P}}(x)} \bmod \mathfrak{P})^{\frac{N\mathfrak{P}-1}{3}} \cdot \prod\limits_{\mathfrak{P}|3}\left(\frac{x,j}{\mathfrak{P}}\right)_3 = 1$, ceci pour x représentant une classe $\bar{x} \in \mathfrak{M}_F$. La partie modérée du produit est 1 et lorsque $d \equiv 2[3]$ ou $d \equiv 3[9]$, nous pouvons aussi conclure $\mathfrak{M}_F = \mathfrak{H}_F$ car il n'y a qu'une seule place sauvage avec $|\mu_{\mathfrak{P}}^1| = 3$.

Lemme 3. *Considérons une classe* $\bar{x}$ *de* $\mathfrak{H}_F$ *puis l'image* $(x_{\mathfrak{P}})_{\mathfrak{P}}$ *d'un* x *de* $F^\times$ *représentant cette dernière via l'injection diagonale :* $F^\times \hookrightarrow J_F = \overset{rest.}{\prod\limits_{\mathfrak{P}}} F_{\mathfrak{P}}^\times$. *Alors aux places sauvages, nous avons :*

Si $d \equiv 1[3], d \equiv 6[9]$, $(x_{\mathfrak{P}})_{\mathfrak{P}|3} \in \left(<\zeta_6> (1+3)^{3\mathbb{Z}_3}(1+3\sqrt{-3})^{\mathbb{Z}_3}\sqrt{-3}^{\mathbb{Z}}\text{ , } idem\right)$

Si $d \equiv 2[3], d \equiv 3[9]$, $x_{\mathfrak{P}|3} \in <\zeta_{24}> (1+\sqrt{3})^{a+3\mathbb{Z}_3}(1+3)^{-a+3\mathbb{Z}_3}(1+3i)^{\mathbb{Z}_3}(1+3\sqrt{-3})^{\mathbb{Z}_3}\sqrt{3}^{\mathbb{Z}}$

$a \in \{0,1,2\}$. *Aux places modérées ,* $x_{\mathfrak{P}} \in \mu_{\mathfrak{P}} \cdot U_{\mathfrak{P}}^1 \cdot \pi_{\mathfrak{P}}^{3\mathbb{Z}}$ *i.e.* $x_{\mathfrak{P}} = \zeta_{m_{\mathfrak{P}}}^{a_{\mathfrak{P}}} \cdot \text{cube}$, $a_{\mathfrak{P}} \in \mathbb{Z}$.

Preuve : pour $\mathfrak{P}$ sauvages, cela découle de la forme d'une base multiplicative des éléments non nuls de $\mathbb{Q}(\zeta_3, \sqrt{d})$ (cf étape 3) et de la formule explicite donnant $\left(\frac{x,j}{\mathfrak{P}}\right)_3$ (cf §2) . Pour $\mathfrak{P} \nmid 3$ rappelons que $v_{\mathfrak{P}}(x) \equiv 0[3]$. L'écriture locale de l'unité fondamentale est compatible avec le fait évoqué au §4-1 : $\mathcal{R}_3 \cdot \text{disc}_{F/\mathbb{Q}}^{-1/2} \in \mathbb{Q}_3$.

Avec ceci, nous pouvons calculer $\mathfrak{H}_F$ ainsi que les unités logarithmiques et le cardinal de $\widetilde{C\ell}_F$, le groupe des classes de diviseurs logarithmiques (cf [17]).

Etape 3 : le 3-Sylow de $Cl_{\mathbb{Q}(\sqrt{-3d})}$ nous fournit des idèles principaux intéressants. Nous trouvons les idéaux générateurs de $Cl_{k^*}(3) = \bigoplus\limits_{1\leqslant m\leqslant rg_3(Cl_{k^*})} < cl(\mathfrak{a}_m) > \simeq C_{3^{i_1}} \times \ldots \times C_{3^{i_n}}$ puis nous cherchons les idèles principaux $(\mathfrak{X}_m)_{m=1,\ldots,n}$ de $\mathcal{J}_{k^*}$ correspondants aux générateurs des $\mathfrak{a}_m^{3^{i_m}}$ (méthode de l'étape 1). Ces recherches se justifient car les idèles de $\sqrt{\prod\limits_{\mathfrak{p}}\mu_{\mathfrak{p}}(3)\cdot\mathcal{R}_{k^*}}$ sont en liaison avec des idèles principaux dont les valuations sont multiples de 3 et $((\mathfrak{X}_m)_{\mathfrak{p}})_{\mathfrak{p}} = (\zeta_{3(\mathfrak{p})} \cdot \pi_{\mathfrak{p}}^{3\cdot\frac{v_{\mathfrak{p}}(\mathfrak{X}_m)}{3}})_{\mathfrak{p}\nmid 3} \cdot ((\mathfrak{X}_m)_{\mathfrak{p}})_{\mathfrak{p}|3}$. Peu importe la famille de racine de l'unité $(\zeta_{3(\mathfrak{p})})_{\mathfrak{p}\nmid 3}$ (car $\prod\limits_{\mathfrak{P}}\mu_{\mathfrak{P}}(3)$ laisse Z fixe); les composantes sauvages sont éléments de $\prod\limits_{\mathfrak{p}|3} k_{\mathfrak{p}}^{*\times}$ qui est ici :

$d \equiv 1[3]$: $\mathbb{Q}_3(\sqrt{-3})^\times = < -1 > < \frac{-1+\sqrt{-3}}{2} > (1+3)^{\mathbb{Z}_3}(1+3\sqrt{-3})^{\mathbb{Z}_3}\sqrt{-3}^{\mathbb{Z}}$

$d \equiv 2[3]$: $\mathbb{Q}_3(\sqrt{3})^\times =< -1 > (1+\sqrt{3})^{\mathbb{Z}_3}(1+3)^{\mathbb{Z}_3}\sqrt{3}^{\mathbb{Z}}$
$d \equiv 3[9]$: $\mathbb{Q}_3(i)^\times =< \frac{-\sqrt{-2}+i\sqrt{-2}}{2} > (1+3)^{\mathbb{Z}_3}(1+3i)^{\mathbb{Z}_3}3^{\mathbb{Z}}$ où $\frac{-\sqrt{-2}+i\sqrt{-2}}{2} = \zeta_8$
$d \equiv 6[9]$: $\mathbb{Q}_3^\times \times \mathbb{Q}_3^\times =< -1 > (1+3)^{\mathbb{Z}_3}3^{\mathbb{Z}} \times < -1 > (1+3)^{\mathbb{Z}_3}3^{\mathbb{Z}}$

Les exposants inconnus des écritures aux places sauvages des $(\mathfrak{X}_l)_{l=1,\dots,n}$ sont cherchés jusqu'à la « profondeur » $81 = 3^4$. Donnons ici les plus petits discriminants connus (par l'auteur) en fonction du 3-rang de Cl_K (source : articles de Diaz y Diaz, Quer, Llorente, Shanks).

3-rang	disc($\mathbb{Q}(\sqrt{a})$),$a > 0$	disc($\mathbb{Q}(\sqrt{-a})$),$a > 0$	nombre
1	229	−23	∞,∞
2	32009	−3299	∞,∞
3	39 345 017	−3 321 607	∞,∞
4	314 582 172 161	−653 329 427	∞,∞
5	1 225 104 664 623 522 549	−35 102 371 403 731	?,?
6	?	−408 368 221 541 174 183	?,?

Etape 4 : construction des idèles adéquats. Nous remarquons $rg_3(Gal(H'/Z')^{e_{\varphi^*}}) = rg_3(Cl_{k^*})$
$-(0ou1)$ d'après la proposition 3. Plus le 3-rang de Cl_{k^*} est important et plus les composantes cycliques de $Cl_{k^*}(3)$ sont « profondes » plus c'est délicat. Les idèles principaux de l'étape 3 sont notre « matière première » car ils sont du type $(\zeta_\mathfrak{p})_\mathfrak{p}.(\pi_\mathfrak{p}^{3^{x_\mathfrak{p}}a_\mathfrak{p}})_{\mathfrak{p}\nmid 3}.(u_\mathfrak{p}^{3^{y_\mathfrak{p}}a_\mathfrak{p}}.\pi_\mathfrak{p}^{3^{z_\mathfrak{p}}a_\mathfrak{p}})_{\mathfrak{p}|3}$ où $(\zeta_\mathfrak{p})_\mathfrak{p}$ est une famille de racine de l'unité, $u_\mathfrak{p} \in (U_\mathfrak{p}^1)^{libre}$, $a_\mathfrak{p}, x_\mathfrak{p}, y_\mathfrak{p}, z_\mathfrak{p} \in \mathbb{Z}$. Une étude de toutes les puissances $(3^{x_\mathfrak{p}}, 3^{y_\mathfrak{p}}, 3^{z_\mathfrak{p}})$ pour chaque idèle principal regardé nous permet, après des produits ad hoc, de former des idèles principaux de la forme $(\zeta_\mathfrak{p})_\mathfrak{p}\cdot(\mathfrak{i}_\mathfrak{p}^{3^l})_\mathfrak{p}$ avec la classe de $(\mathfrak{i}_\mathfrak{p})_\mathfrak{p}$ modulo $\prod_\mathfrak{p}\mu_\mathfrak{p}(3)\cdot\mathcal{R}_{k^*}$ qui engendre une composante cyclique de profondeur 3^l de $Gal(H/Z)^{e_{\varphi^*}} \simeq Gal(H_{k^*}/Z_{k^*}) \simeq \sqrt{\prod_\mathfrak{p}\mu_\mathfrak{p}(3)\cdot\mathcal{R}_{k^*}}/\prod_\mathfrak{p}\mu_\mathfrak{p}(3)\cdot\mathcal{R}_{k^*}$.
Entre les corps $k = \mathbb{Q}(\sqrt{d})$ et $k^* = \mathbb{Q}(\sqrt{-3d})$ existent des liens étroits pour que l'existence du groupe $Gal(H'/Z')^{e_{\varphi^*}}$ avec le bon rang : $dim_{\mathbb{F}_3}\mathfrak{H}_F^{e_\varphi} - 1$ soit effective.

Maintenant soit un idèle $(\pi_\mathfrak{p}^{a_\mathfrak{p}})_{\mathfrak{p}\nmid 3}\cdot(\mathfrak{i}_\mathfrak{p})_{\mathfrak{p}|3}$ dont la classe modulo $\prod_\mathfrak{p}\mu_\mathfrak{p}\mathcal{R}_{k^*}$ engendre une composante cyclique de $Gal(H_{k^*}/Z_{k^*})$. Nous le hissons par $\{ext_{F/k^*}((\pi_\mathfrak{p}^{a_\mathfrak{p}})_{\mathfrak{p}\nmid 3}\cdot(\mathfrak{i}_\mathfrak{p})_{\mathfrak{p}|3})\}^{e_{\varphi^*}}$ avec ext_{F/k^*} l'homomorphisme d'extension des idèles de $\mathcal{J}_{k^*}$ vers $\mathcal{J}_F$. Pour ceci, lorsqu'une place $\mathfrak{p}$ de k^* se décompose dans F, nous utilisons l'action de Δ pour les places sauvages et la connaissance d'une $\mathbb{Z}$-base des idéaux $\mathfrak{P}$ de F au dessus de p avec $\mathfrak{P}|\mathfrak{p}|p$ pour $\mathfrak{p}$ modérée.

Posons ici $Cl_F(3) \simeq Cl_k(3) \oplus Cl_{k^*}(3) = \bigoplus_{0\leqslant i\leqslant r} < cl\ \sqrt[3^{l_i}]{(\alpha_i)} > \oplus \bigoplus_{0\leqslant i\leqslant r^*} < cl\ \sqrt[3^{l_i}]{(\alpha_i^*)} >$ avec $r = rg_3(Cl_k), \alpha_i \in k^\times, r^* = rg_3(Cl_k), \alpha_i^* \in k^{*\times}$ (rappelons le critère de Scholz (cf [26]): $r^* \leqslant r \leqslant r^*+1$). Si nous supposons vraies les conjectures basées sur un principe heuristique de [9], en particulier à chaque structure de $Cl_F(3)$ correspond une densité > 0, il paraît légitime de croire que les écritures locales aux places sauvages qui se rencontre pour l'unité fondamentale de k, les α_i, les α_i^*, permettent toutes les structures pour $Gal(H/Z)$; d'où tous les couples $([H':Z'],[H:ZH']) \in 3^\mathbb{N}\times 3^\mathbb{N}$ (cf dessin §2) se rencontreraient avec la restriction : $[H:ZH'] > 1$ seulement si $[H':Z'] > 1$.

Etape 5 : calculs de symboles et théorie du corps de classes effective.
Nous traduisons maintenant la dualité entre $\mathfrak{H}_F^{e_\varphi}$ et $Gal(H'/Z')^{e_{\varphi^*}}$ (§4 -1).

$$\begin{cases} \mathfrak{H}_F \times Gal(H'/Z') \longrightarrow <\zeta_3> \quad (\text{rappel : } Gal(H/Z) \simeq \sqrt{\prod_\mathfrak{P}\mu_\mathfrak{P}(3)\cdot\mathcal{R}_F}/\prod_\mathfrak{P}\mu_\mathfrak{P}(3)\cdot\mathcal{R}_F) \\ \overline{x}\,,\ \overline{(\mathfrak{i}_\mathfrak{P})_\mathfrak{P}} \mapsto \sqrt[3]{x}^{\,\omega((\mathfrak{i}_\mathfrak{P})_\mathfrak{P})-1} = \prod_\mathfrak{P}\sqrt[3]{x}^{\,\omega_\mathfrak{P}(\mathfrak{i}_\mathfrak{P})-1} = \prod_{\mathfrak{P}\nmid 3}(x,\mathfrak{i}_\mathfrak{P})_{\mathfrak{P},3}\cdot\prod_{\mathfrak{P}|3}\left(\frac{x,\mathfrak{i}_\mathfrak{P}}{\mathfrak{P}}\right)_3 \end{cases}$$

L'application d'Artin globale est produit des applications locales via les identifications du groupe de décomposition $D_{\mathfrak{P}}$ d'une place $\mathfrak{P}$ à $\widehat{F_{\mathfrak{P}}^{\times}} \simeq Gal(F_{\mathfrak{P}}^{ab(\ell)}/F_{\mathfrak{P}})$. $\omega((\mathfrak{i}_{\mathfrak{P}})_{\mathfrak{P}})$ restreint à l'extension $F(\sqrt[3]{x})$ ramifiée en un nombre fini de places a une action triviale sauf pour un nombre fini de composantes. Seule la partie en $\pi_{\mathfrak{P}}$ de la composante en $\mathfrak{P} \nmid 3$ d'un idèle a une action non triviale a priori sur x, $(x, \pi_{\mathfrak{P}})_{\mathfrak{P},3}$ traduit l'action du Frobenius de $\mathfrak{P}$ (cf §4-3 pour les calculs). Une précaution : les représentants de $\mathfrak{H}_F^{e_\varphi}$ sont choisis « étrangers aux idèles » dont les classes engendrent $Gal(H'/Z')^{e_{\varphi^*}}$ (cf §6 – 1 : théorème de Chebotarev effectif). En multipliant l'idèle $(\mathfrak{i}_{\mathfrak{P}})_{\mathfrak{P}}$ par un idèle principal $(\mathfrak{x}_{\mathfrak{P}})_{\mathfrak{P}}$ tel que $(\mathfrak{x}_{\mathfrak{P}})_{\mathfrak{P}|3} \in (\mathfrak{i}_{\mathfrak{P}})_{\mathfrak{P}|3}^{-1} \cdot \mathcal{J}_F^3$, nous neutraliserions les composantes sauvages mais au prix de l'apparition de nouveaux symboles modérés. Ceci dit, il est préférable comme nous disposons déjà des écritures locales des éléments, de calculer les symboles de Hilbert d'ordre 3 aux places sauvages (cf §4-3).

Etape 6 : le radical $\mathfrak{Z}_F$. Soient $(x_c)_{c=1,\ldots,n+1}$ des éléments de k dont les classes $\overline{x_c}^{e_\varphi}$ engendrent $\mathfrak{H}_F^{e_\varphi}$. Puis notons $Gal(H'/Z')^{e_{\varphi^*}} =< \sigma_1 > \ldots < \sigma_n >$ avec n le 3-rang de $Gal(H'/Z')^{e_{\varphi^*}}$. Grâce aux calculs de l'étape 5, on forme une matrice R appartenant à $\mathcal{M}_{n,n+1}\left(\frac{\mathbb{Z}}{3\mathbb{Z}}\right)$ dont les coefficients sont égaux à $r_{l,c} = (\sqrt[3]{x_c^{4e_\varphi}})^{\sigma_l - 1}$.

$$R = \left(\quad (\sqrt[3]{x_c^{4e_\varphi}})^{\sigma_l - 1} \quad \right) = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n+1} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,1} & r_{n,2} & \cdots & r_{n,n+1} \end{pmatrix} = \begin{pmatrix} \vec{l_1} \\ \vec{l_2} \\ \vdots \\ \vec{l_n} \end{pmatrix}$$

Les $\vec{l_a} = (l_{a,1}, \ldots, l_{a,n+1})$ sont dans l'ensemble des vecteurs à $n+1$ coefficients dans $\mathbb{Z}/3\mathbb{Z}$ muni du « produit scalaire » $\vec{l_a} \cdot \vec{l_b} = \sum_{t=1}^{t=n+1} l_{a,t} \cdot l_{b,t}$ dans $\mathbb{Z}/3\mathbb{Z}$. On peut extraire un mineur d'ordre n de la matrice R de déterminant non nul dans $\mathbb{Z}/3\mathbb{Z}$ si et seulement si il existe un unique vecteur à $n+1$ coordonnées dans $\mathbb{Z}/3\mathbb{Z}$, $\vec{e} = (e_1, \ldots, e_{n+1})$ avec $\vec{l_a} \cdot \vec{e} = 0$ dans $\mathbb{Z}/3\mathbb{Z}$ pour a allant de 1 à n.

Dans ces conditions on peut prendre comme premier étage de trois $\mathbb{Z}_3$-extensions linéairement disjointes de $F = \mathbb{Q}(\zeta_3, \sqrt{d})$, les extensions $F(\sqrt[3]{3})$, $F(\sqrt[3]{j})$, $F(\sqrt[3]{x_1^{e_1} \ldots x_{n+1}^{e_{n+1}}})$.

3 - Calculs des symboles de Hilbert

Pour $x \in F^\times$, $\pi_{\mathfrak{P}}$ une uniformisante associée à une place $\mathfrak{P}$ ultramétrique de F étrangère à 3 et à x, nous avons avec les notations du §3 :

$$\left(\frac{x, \pi_{\mathfrak{P}}}{\mathfrak{P}}\right)_3 = (x, \pi_{\mathfrak{P}})_{\mathfrak{P},3} = \left((\sqrt[3]{x})^{\left(\frac{F(\sqrt[3]{x})/F}{\mathfrak{P}}\right)-1}\right)_3 = \left(\frac{x}{\mathfrak{P}}\right)_3 \equiv \left(x^{\frac{N\mathfrak{P}-1}{3}} \bmod \mathfrak{P}\right) \in < \zeta_3 >$$

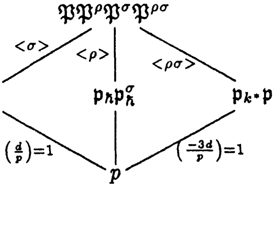
La congruence définissant l'action du symbole d'Artin ou de Frobenius va permettre les calculs.

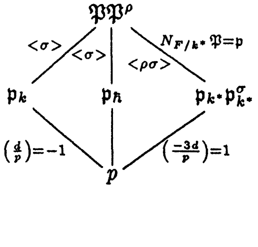
Lemme 6. *Prenons $x \in k^\times \subset F^\times$ et une place $\mathfrak{p}_{k^*}$ de k^* étrangère à x et à 3. Nous avons alors* $\prod_{\mathfrak{P}|\mathfrak{p}_{k^*}} \left(\frac{x}{\mathfrak{P}}\right)_3 = 1$ *sauf si $\mathfrak{p}_{k^*}$ correspond à un p décomposé dans k^*.*

Preuve : on le vérifie aisément compte tenu de :

$$x^{\frac{N\mathfrak{P}-1}{3}} \equiv \left(\frac{x}{\mathfrak{P}}\right)_3 \bmod \mathfrak{P} \iff x^{\frac{N\mathfrak{P}-1}{3}} \equiv \left(\frac{x}{\mathfrak{P}}\right)_3^2 \bmod \mathfrak{P}^\sigma \quad \text{où } < \sigma >= Gal(F/k)$$

Seuls les deux cas suivants amènent des calculs pour nous :





Pour le premier cas, trouvons une $\mathbb{Z}$-base d'un des idéaux $\mathfrak{P}$ au dessus de p à l'aide d'une $\mathbb{Z}$-base de l'anneau des entiers (§4), ce qui nécessite des calculs non triviaux que l'on n'abordera pas ici. Puis on teste les congruences recherchées. Le second cas nécessite des calculs de normes pour descendre au préalable dans k^* puis des congruences.

Pour les symboles sauvages, considérons le schéma de corps:

$$\mathbb{Q}_\ell \xrightarrow{\ell-1} \mathbb{Q}_\ell(\zeta_\ell) \xrightarrow{\ell^{n-1}} \mathbb{Q}_\ell(\zeta_{\ell^n}) \xrightarrow{K \text{ quelconque}} K$$

Notons $\lambda = 1 - \zeta_\ell$ une uniformisante de $\mathbb{Q}_\ell(\zeta_\ell)$, π une uniformisante de K avec $(\pi) = \mathfrak{P}$ et β une unité au hasard dans K. Puis soient f et g deux polynômes dont les coefficients sont des unités de K avec $\zeta_{\ell^n} = g(\pi)$ et $\beta = f(\pi)$. $log_{\mathfrak{P}}$ est le logarithme $\mathfrak{P}$-adique nul sur la torsion et sur ℓ. Nous avons d'après Shankar Sen [23] :

$$\left(\frac{\alpha,\pi}{\mathfrak{P}}\right)_{\ell^n} = \zeta_{\ell^n}^{\frac{1}{\ell^n}\cdot \mathrm{Trace}_{K/\mathbb{Q}_\ell}\left(\frac{\zeta_{\ell^n}}{g'(\pi)\pi}\cdot log_{\mathfrak{P}}(\alpha)\right)} \quad \text{pour } \alpha \equiv 1 \pmod{\lambda^2\pi}$$

$$\left(\frac{\alpha,\beta}{\mathfrak{P}}\right)_{\ell^n} = \zeta_{\ell^n}^{\frac{1}{\ell^n}\cdot \mathrm{Trace}_{K/\mathbb{Q}_\ell}\left(\frac{\zeta_{\ell^n}}{g'(\pi)}\cdot\frac{f'(\pi)}{f(\pi)}\cdot log_{\mathfrak{P}}(\alpha)\right)} \quad \text{pour } \alpha \equiv 1 \pmod{\lambda^2}$$

Dans [24], d'autres formules sont données et les restrictions sur les congruences que doit vérifier α sont enlevées.

§6 - De nombreux exemples

1 - Loi de réciprocité et validité des résultats

Le premier test de validité est le fait que la matrice construite au §4-3 soit de rang maximal. Et si nous changions la base du groupe de Galois $Gal(H'/Z')$, retrouverions-nous le même radical $\mathfrak{Z}_F$?

Pour modifier la base de $Gal(H'/Z')^{e_{\varphi^*}}$, cherchons des idéaux premiers $(\mathfrak{p}_i)_{i=1,\ldots,n}$, de normes minimales avec $Cl_{k^*}(3) \simeq < cl(\mathfrak{a}_1) > \ldots < cl(\mathfrak{a}_n) > = < cl(\mathfrak{p}_1) > \ldots < cl(\mathfrak{p}_n) >$ (cf §4). Une exploration lointaine va-t-elle s'avérer nécessaire c'est à dire peut-on trouver un représentant de chaque classe d'idéaux sous la forme d'un idéal premier de norme raisonnable ? En utilisant les résultats sur la version effective du théorème de densité de Chebotarev que l'on trouve dans [19] et [25], nous pouvons dire que sous (GRH), toute classe d'idéaux de $Cl_{k^*}(3)$ est représentable par un idéal premier de norme $\leqslant 2\cdot|Cl_{k^*}(3)|^2\cdot ln^2(d_{k^*})$ (nous pourions même prendre vraisemblablement une meilleure constante que 2). Nous vérifions à chaque fois la condition nécessaire à (GRH) et nous retrouvons toujours un $\mathfrak{Z}_F$ identique. Même pour les valeurs de d importantes (§6−2), si nous avons deux bases de $Gal(H'/Z')^{e_{\varphi^*}}$, la loi de réciprocité (cf ci-dessous) avec les idèles principaux est toujours vérifiée dans la pratique (changer de base revient à modifier les idèles par des éléments de $\prod_{\mathfrak{P}}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F$ cf §4).

Nous rapelons ici « des » lois de réciprocité en raison du rôle prépondérant qu'elles jouent lors de la vérification des calculs (cf tables). Pour une place réelle $\mathfrak{R}$ de F le symbole continu est d'ordre 2, on peut le voir comme un symbole de Hilbert ou modéré. Les places complexes sont devenues « neutres » pour la théorie du corps de classes et pour $\mathbb{C}$, tout symbole continu est trivial. Pour a et $< \zeta_m >$ dans $F^\times$, si nous considérons l'extension kummerienne $F(\sqrt[m]{a})$, la loi de réciprocité d'Artin sur les idèles principaux permet d'écrire la formule du produit (qui est d'ailleurs la seule relation entre les symboles de Hilbert ([6])) :

$$\prod_{\mathfrak{P} \text{ place de } F}\left(\frac{a,b}{\mathfrak{P}}\right)_m = 1 = \prod_{\mathfrak{P}\nmid(m,\infty)}(a,b)_{\mathfrak{P},m}\cdot\prod_{\mathfrak{P}\mid(m,\infty)}\left(\frac{a,b}{\mathfrak{P}}\right)_m .$$

Pour énoncer la loi de réciprocité d'ordre m, lorsque a, b et m sont étrangers au sens des valuations pour les places finies, on pose, avec les notations « habituelles » :

$$\left(\frac{a}{b}\right)_m \overset{\text{def}}{=} \prod_{\mathfrak{P}|(b\mathbb{Z}_F)} \left(\frac{a}{\mathfrak{P}}\right)_m^{v_{\mathfrak{P}}(b)} \overset{\text{def}}{=} \prod_{\mathfrak{P}|(b\mathbb{Z}_F)} \left((\sqrt[m]{a})^{\left(\frac{F(\sqrt[m]{a})/F}{\mathfrak{P}}\right)-1}\right)_m^{v_{\mathfrak{P}}(b)} \overset{\text{ici}}{=} \prod_{\mathfrak{P}|(b\mathbb{Z}_F)} (a,b)_{\mathfrak{P},m} \quad .$$

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)^{-1} = \prod_{\mathfrak{P}|(m,\infty)} \left(\frac{a,b}{\mathfrak{P}}\right)_m^{-1} = \left(\prod_{\mathfrak{P}\nmid(m,\infty)} \left(\frac{a,b}{\mathfrak{P}}\right)_m = \prod_{\mathfrak{P}\nmid(m,\infty)} (a,b)_{\mathfrak{P},m}\right).$$

2 - Comparaison de $\mathfrak{Z}_F$ avec les autres radicaux

Nous indiquons les cas les plus simples au sens des conditions à vérifier où des conclusions peuvent être tirées avant la fin des calculs (cf questions soulevées lors du §2). Ici, nous considérons seulement le cas $dim_{\mathbb{F}_3}\mathfrak{H}_F^{e_\varphi} = 2$. Supposons de plus $Cl_{k^*}(3) =< cl(\mathfrak{a}) >=< cl(\sqrt[3^l]{(\alpha)}) >\simeq C_{3^l}$ avec $l \geq 2$. ϵ_d est l'unité fondamentale de $\mathbb{Q}(\sqrt{d})$ et $\mathfrak{p}_3^*$ un idéal de k^* au dessus de 3.

Supposons $log_{\mathfrak{p}_3^*}(\alpha)$ dans $3\mathfrak{p}_3^*$ mais pas dans $3^l\mathfrak{p}_3^*$. (cf §2 pour le logarithme $\mathfrak{p}_3^*$ -adique)

Si $d \equiv 1[3]$, $3\mathbb{Z}_{k^*} = (\mathfrak{p}_3^*)^2$ et

1) Si nous écrivons $3\mathbb{Z}_k = \mathfrak{p}_3\overline{\mathfrak{p}_3}$, l'ordre de $cl(\mathfrak{p}_3)$ dans Cl_k n'est pas divisible par 3.

2) Si $3 \nmid |Cl_k/ < \mathfrak{p}_3 > |$, $\mathfrak{Z}_F =< \overline{3} >< \overline{j} >< \overline{\epsilon_d} >$.

Si $d \equiv 2[3]$ ou $d \equiv 3[9]$, alors, $\mathfrak{Z}_F^{e_\varphi} = \mathfrak{C}_F^{e_\varphi} = \mathfrak{D}_F^{e_\varphi} = (\mathfrak{H}_F^{cy})^{e_\varphi}$ (cf §2).

Si $d \equiv 6[9]$, avec α premier à 3, alors $\mathfrak{Z}_F^{e_\varphi} = \mathfrak{C}_F^{e_\varphi}$.

Ici, avec les notations du §2: $\overline{x} \in \mathfrak{C}_F \Longleftrightarrow (x_{\mathfrak{P}})_{\mathfrak{P}} \in \left\{\left(u_{\mathfrak{P}} \cdot \pi_{\mathfrak{P}}^{3i_{\mathfrak{P}}}\right)_{\mathfrak{P}\nmid 3} ((1+3\sqrt{-3})^{\mathbb{Z}_3} \cdot t_{\mathfrak{P}}^3)_{\mathfrak{P}|3}\right\}$.

La condition sur le logarithme correspond à des jeux de « profondeur » entre les places sauvages et modérées pour les écritures locales des éléments qui nous fournissent les idèles souhaités (cf étape 4). Notons C le 3-corps de classes de Hilbert de F.

Proposition 5. *Soit* $F = \mathbb{Q}(\sqrt{d}, \zeta_3)$, $k^* = \mathbb{Q}(\sqrt{-3d})$. *Supposons* $Cl_{k^*}(3) =< cl(\mathfrak{p}^*) >=< cl(\sqrt[3^l]{(\alpha)}) >\simeq C_{3^l}$. $\mathfrak{p}^*$ *est un idéal premier de* k^* *étranger à 3. et supposons* $log_{\mathfrak{p}_3^*}(\alpha) \in 3\mathfrak{p}_3^*$.
Alors $\mathfrak{Z}_F^{e_\varphi} = \mathfrak{C}_F^{e_\varphi} \Longleftrightarrow log_{\mathfrak{p}_3^*}(\alpha) \notin 3^l\mathfrak{p}_3^*$. *Notons que lorsque* $dim_{\mathbb{F}_3}\mathfrak{H}_F^{e_\varphi} = rg_3(Cl_{k^*})$, $\mathfrak{H}_F^{e_\varphi} = \mathfrak{C}_F^{e_\varphi}$.

Preuve : l'écriture de l'idèle principal α dans J_{k^*} donne $(..., 1, \pi_{\mathfrak{p}^*}^{3^l}, (u_{\mathfrak{p}_3^*}^{3^k})_{\mathfrak{p}_3^*|3}, 1, ...)$ où $\pi_{\mathfrak{p}^*}$ est une uniformisante associée à $\mathfrak{p}^*$, $u_{\mathfrak{p}_3^*}$ une unité $\mathfrak{p}_3^*$-adique et $0 < k < l$. Le fait $log_{\mathfrak{p}_3^*}(\alpha) \notin 3^l\mathfrak{p}_3^*$ assure que l'idèle qui « engendre » $Gal(H'/Z')^{e_\varphi^*}$ a ses composantes modérées toutes triviales, et dans ces conditions les calculs explicites de la dualité ne se font qu'aux seules places sauvages d'où le sens $\Longleftarrow$. Maintenant, soit $log_{\mathfrak{p}_3^*}(\alpha) \in 3^l\mathfrak{p}_3^*$. Si nous avions $\mathfrak{Z}_F^{e_\varphi} = \mathfrak{C}_F^{e_\varphi}$, l'idèle $\text{ext}_{L/k^*}(..., 1, \pi_{\mathfrak{p}^*}, 1, ...)$ fixerait $\mathfrak{H}_F^{e_\varphi}$. C'est à dire suivant les deux cas possibles d'écriture de $\mathfrak{p}^*\mathbb{Z}_L$, $\left(\frac{C/F}{\mathfrak{P}\mathfrak{P}^{\rho c}}\right)$ ou $\left(\frac{C/F}{\mathfrak{P}}\right)$ fixerait $\mathfrak{C}_F^{e_\varphi}$ ($\mathfrak{P}$ divise $\mathfrak{p}^*$). Mais $Gal(C/F)^{e_\varphi^*} \simeq Cl_{k^*}(3)$ via l'application d'Artin (ou de Frobenius) sur les idéaux, d'où la contradiction. Si $Cl_{k^*}(3)$ n'est plus cyclique mais admet une composante cyclique du type de la proposition 5, alors $\mathfrak{Z}_F^{e_\varphi} \subset \mathfrak{C}_F^{e_\varphi}$. Précisons un peu la position de C dans le schéma de la 3-ramification abélienne de F:

$$Gal(H/C) \simeq \text{Ker}\left(\frac{\mathcal{J}_F}{\prod_{\mathfrak{P}}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F} \xrightarrow{\text{restrition}} \frac{\mathcal{J}_F}{\prod_{\mathfrak{P}|3}U_{\mathfrak{P}}^1 \cdot \prod_{\mathfrak{P}\nmid 3}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}\right) \simeq \frac{\prod_{\mathfrak{P}|3}U_{\mathfrak{P}}^1 \cdot \prod_{\mathfrak{P}\nmid 3}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}{\prod_{\mathfrak{P}}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}$$

$$\simeq \mathbb{Z}_3^3 \oplus \sqrt{\prod_{\mathfrak{P}|3}\left(U_{\mathfrak{P}}^1\right)^{\text{libre}} \cap \prod_{\mathfrak{P}}\mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F}$$

La partie de torsion dépendant seulement des unités, cela permet ici d'écrire :

Proposition 6. *$Gal(H/C)$ est libre* $\Longleftrightarrow (\epsilon_d)_{\mathfrak{p}_3} \notin \mu^1_{\mathfrak{p}_3}\cdot(U^1_{\mathfrak{p}_3})^3$ *où* $\mathfrak{p}_3 \mid 3\mathbb{Z}_k, \epsilon_d$ *l'unité fond. de k.*
Si cela est, $H = Z.C$ et de plus, « $C \subset Z \Longleftrightarrow dim_{\mathbb{F}_3} = 3$ » .

Pour $d \equiv 1[3]$ et $67 \leq d < 10000$, il y a 208 cas où $\mathfrak{H}_F^{e_\nu}$ est engendré par les classes de ϵ_d et d'une 3-unité (d=67,103,106,139...). Parmi eux, 45 voient l'égalité $\mathfrak{Z}_F^{e_\nu} = \mathfrak{N}_F^{e_\nu}$ (d=295,397,745,...), où $\mathfrak{N}_F$ est le radical des unités logarithmiques (cf [17]). Pour 45 cas $\mathfrak{N}_F^{e_\nu} =<\overline{\epsilon_d}>$ (d=238,610, 727,...), et 41 fois on a $\mathfrak{Z}_F^{e_\nu} =<\overline{\epsilon_d}>$ dont 38 avec $Cl_{k^*}(3)$ cyclique (d=607,787,1669,...).

Pour $d \equiv 2[3]$ et $326 \leq d < 10000$, 210 cas amènent des calculs (d=326,359,443,...). Parmi eux 196 se font avec $Cl_{k^*}(3)$ cyclique; pour les autres, $\mathfrak{H}_F^{e_\nu} = \mathfrak{C}_F^{e_\nu} = \mathfrak{D}_F^{e_\nu}$. Pour 37 cas sur 196, $\mathfrak{Z}_F^{e_\nu} = \mathfrak{C}_F^{e_\nu} = \mathfrak{D}_F^{e_\nu}$ (d=659,1091,1373,...), Z étant donc linéairement disjointe du 3-corps de classes de Hilbert pour les cas restants. 49 fois nous avons $\mathfrak{C}_F^{e_\nu} =<\overline{\epsilon_d}>$ (d=326,359,506,...) et 48 fois $\mathfrak{Z}_F^{e_\nu} =<\overline{\epsilon_d}>$ (d=473,785,899,...).

Pour $d \equiv 3[9]$ et $786 \leq d < 30000$, 213 cas nécessitent des calculs pour connaître $\mathfrak{Z}_F^{e_\nu}$ (d=786,
894,993,...), dont 12 avec $dim_{\mathbb{F}_3}\mathfrak{C}_F^{e_\nu} = 2$ ($\Longleftrightarrow dim_{\mathbb{F}_3}\mathfrak{D}_F^{e_\nu} = 2 \Longleftrightarrow rg_3(Cl_{k^*}) = 2$). 38 fois sur 201, arrive l'égalité $\mathfrak{Z}_F^{e_\nu} = \mathfrak{C}_F^{e_\nu} = \mathfrak{D}_F^{e_\nu}$ (d=1101,3054,3261,...). 46 fois $\mathfrak{C}_F^{e_\nu} =<\overline{\epsilon_d}>$ (d=993,1866,2055,...) et 47 fois $\mathfrak{Z}_F^{e_\nu} =<\overline{\epsilon_d}>$ (d=3594,3846,4215,...). Signalons que nous sommes pleinement en accord avec G.Gras pour le cas d=1758 ($\mathbb{Q}(\sqrt{-3},\sqrt{-586})$) que l'on trouve dans [10].

Pour $d \equiv 6[9]$ et $321 \leq d < 20000$, $dim_{\mathbb{F}_3}\mathfrak{H}_F^{e_\nu} = 2$ arrive 139 fois (d=321,906,1086,...), 29 cas voient l'égalité $dim_{\mathbb{F}_3}\mathfrak{D}_F^{e_\nu} = 1$ ($\Longleftrightarrow rg_3(Cl^S_{k^*}) = 1$) (d=2922,4227,5073,...) et parmi eux 6 voient $\mathfrak{H}_F^{e_\nu} = \mathfrak{C}_F^{e_\nu}$. Pour les 133 cas restants l'égalité $\mathfrak{Z}_F^{e_\nu} = \mathfrak{C}_F^{e_\nu}$ se rencontre 33 fois (d=1086,1257,
3579,...), $\mathfrak{Z}_F^{e_\nu} = \mathfrak{D}_F^{e_\nu}$ 8 fois (d=4227,5073,5901,...), $\mathfrak{C}_F^{e_\nu} =<\overline{\epsilon_d}>$ 30 fois (d=321,3957, 4011,...) et $\mathfrak{Z}_F^{e_\nu} =<\overline{\epsilon_d}>$ 26 fois (d=1599,3138, 3606,...).

3 - Explications du contenu des tables

Dans [22], [11], nous trouvons des corps quadratiques imaginaires avec $1 \leqslant rg_3(Cl_{k^*}) \leqslant 2$. Puis dans la thèse de 3ième cycle de F.Diaz y Diaz, de nombreuses valeurs de d avec $rg_3(Cl_{k^*}) = 3, 4$ et d'autres avec $rg_3(Cl_k) = 3$ sont données. Nous avons traité tous les discriminants rencontrés dans ces tables avec pour celles de F.Diaz y Diaz (que je remercie beaucoup) la limite $\min\{disc(k), |disc(k^*)|\} \leqslant 10^{10}$ ce qui nous fait des valeurs de d entre 67 et 29 719 097 841.

La colonne 1 contient la valeur absolue du discriminant soit de k^* soit de k suivant l'indication. La congruence indiquée est celle de d qui détermine le corps biquadratique (cf §4).

La colonne 2 renseigne sur la structure du 3-Sylow du groupe des classes d'idéaux de F. Celui-ci est séparé en les deux parties indiquées.

La colonne 3 fournit un système de représentants dans $k^\times$ de chaque classe modulo $F^{\times^3}$ dans $\mathfrak{M}_F^{e_\nu}$; L'unité fondamentale de k sera notée ϵ ($\epsilon > 1$) ou ϵ_d. Ces représentants sont trouvés par des conditions normiques (cf étape 1). Lorsque nous savons qu'un idéal premier de k engendre une composante cyclique de ${}_3Cl^S_k$, nous travaillons avec la forme quadratique correspondant à ce couple d'idéaux décomposés dans k. Elevée à sa plus petite puissance principale, puis la distance de Shanks initialisée à zéro, elle nous livre un élément $\frac{a \pm b\sqrt{d}}{2}$ de norme $\pm N$ avec $a > 0$ et $b > 0$, déterminé par l'algorithme décrit à l'étape 1. Nous le notons x_n avec $n^3 = N$, lorsque N est un cube. Sinon, nous avons affaire à une 3-unité de norme $\pm 3^l$ notée x_{3^l} (trouvée grâce au même algorithme).

Pour le groupe de Galois de la colonne 4, nous donnons des idèles de $\mathcal{J}_F$ dont les classes modulo $\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3)\cdot\mathcal{R}_F$ constituent un système générateurs de $Gal(H/Z)^{e_{\nu^*}}$.

Regardons modulo $\prod_{\mathfrak{P}} \mu_{\mathfrak{P}}(3) \cdot \mathcal{R}_F \cdot \mathcal{J}_F^3$, nous avons une base de $Gal(H'/Z')^{e_{\varphi^*}}$ qui est d'exposant 3. Lorsque nous écrivons un idèle, seules les composantes non triviales apparaissent et nous commencons par les places modérées (étrangères à 3) pour finir par les places sauvages (divisant 3). Sur la droite de la case où est écrit un idèle figure entre parenthèse l'ordre de sa classe dans $Gal(H/Z)$. Pour les places modérées, uniquement celles divisant des p décomposés dans k^* fournissent des composantes non triviales ici et celles-ci consistent en des puissances d'uniformisantes (§4). Les deux cas de la factorisation de $p\mathbb{Z}_F$ concernant de tels p sont évoqués au §4 – 3. Lorsque $p\mathbb{Z}_F = \mathfrak{P}\mathfrak{P}^\rho\mathfrak{P}^\sigma\mathfrak{P}^{\rho\sigma}$ (notations : cf §4), $\mathfrak{P}$ est par convention celui qui admet une $\mathbb{Z}$-base construite à partir de [29] et avec des « décalages » tous positifs (si $n > 0$, $\sqrt{-n} = e^{i\frac{\pi}{2}}\sqrt{n}$). Cela signifie que si par exemple nous avons $d \equiv 5[12]$, nous prenons $\mathbb{Z}_F = \mathbb{Z} \oplus \mathbb{Z}\left(\frac{1+\sqrt{d}}{2}\right) \oplus \mathbb{Z}\left(\frac{1+\sqrt{-3}}{2}\right) \oplus \left(\frac{1+\sqrt{d}+\sqrt{-3}+\sqrt{-3d}}{4}\right)$ et pour les p totalement décomposés comme ci-dessus (de densité 1/4): $\mathfrak{P} = p\mathbb{Z} \oplus \mathbb{Z}\left(\frac{a_2+\sqrt{d}}{2}\right) \oplus \mathbb{Z}\left(\frac{a_3+\sqrt{-3}}{2}\right) \oplus \mathbb{Z}\left(\frac{a_4+\sqrt{d}+\sqrt{-3}+\sqrt{-3d}}{4}\right)$ avec $a_2 > 0, a_3 > 0, a_4 > 0$. Une uniformisante de $F_{\mathfrak{P}}$ sera notée $[p]$ pour p au dessous de $\mathfrak{P}$, $[p]_\rho$ représente une uniformisante de $F_{\mathfrak{P}^\rho}$ (idem pour les couples $([p]_\sigma, F_{\mathfrak{P}^\sigma}), ([p]_{\rho\sigma}, F_{\mathfrak{P}^{\sigma\rho}})$). Notons que dans ce cas là, l'idèle écrit contient soit le couple $([p], [p]_{\rho\sigma})$ soit le couple $([p]_\rho, [p]_\sigma)$ car cela provient d'un idèle remonté de $\mathcal{J}_{k^*}$. L'autre cas de factorisation est $p\mathbb{Z}_F = \mathfrak{P}\mathfrak{P}^\rho = \mathfrak{P}\mathfrak{P}^\sigma$ et nous avons $p\mathbb{Z}_{k^*} = \mathfrak{p}^*\mathfrak{p}^{*\sigma}$. $\mathfrak{p}^*$ est inerte relativement à F/k^* et par convention $\mathfrak{p}^* = p\mathbb{Z} \oplus \mathbb{Z}(a_1 + \sqrt{\frac{-3d}{(3,d)^2}})$ avec $a_1 > 0$ lorsque $\frac{-3d}{(3,d)^2} \equiv 2, 3[4]$. Si $\frac{-3d}{(3,d)^2} \equiv 1[4]$, une convention analogue vaut. Nous noterons $[p]$ une uniformisante de $F_{\mathfrak{P}}$ pour $\mathfrak{P}$ au dessus de $\mathfrak{p}^*$ et $\overline{[p]}$ est elle associée à $F_{\mathfrak{P}^\sigma}$. Il y a une ou deux places qui divisent 3 (cf §4 – 2) et elles sont ordonnées d'une manière similaire s'il y en a deux.

La colonne 5 indique le cardinal de $Gal(M/Z)$, la torsion de $Gal(M/F)$, noté ici $|\mathcal{T}|$.

Les résultats de la dualité kummerienne constituant l'étape 6 du §4-2 sont donnés en colonne 6 en respectant les ordres des colonnes 3 et 4. Les coefficients des matrices de cette colonne s'entendent modulo 3.

Enfin, si nous considérons l'extension de F construite en lui adjoignant une racine cubique de l'élément de la colonne 7, nous avons un étage initial d'une $\mathbb{Z}_3$-extension de F. Cette dernière étant disjointe du composé des deux $\mathbb{Z}_3$-extensions de F dont les premiers étages peuvent être pris de la forme $F(\sqrt[3]{3})$ et $F(\sqrt[3]{\zeta_3})$.

$\mathrm{disc}(\mathbb{Q}(\sqrt{-3d}))$	$Cl_k(3)\oplus Cl_{k^*}(3)$	$\mathfrak{H}_F^{e_\varphi}$	$Gal(H/Z)^{e_{\varphi^*}}$		$\lvert T\rvert$	dualité	$\mathfrak{Z}_F^{e_\varphi}$
12 755 172	$C_3\cdot C_3$ $\oplus$	ϵ, x_{3^3}, x_7	$([15013]_\rho,[15013]_\sigma,1,(1+3\sqrt{-3})^2)$ $([10141],[10141]_{\rho\sigma},1,1)$	(9) (3)	$3^4.3^3$	$\begin{pmatrix}2&10&2\\4&2&2\end{pmatrix}$	$\epsilon.x_{3^3}$
$d\equiv 1[3]$	$C_{27}\cdot C_3\cdot C_3$	$\epsilon, x_{3^3}, x_{11}$	$([409],[409]_{\rho\sigma},1,1)$ $([7],[7]_{\rho\sigma},[199],[199]_{\rho\sigma},1,1)$	(9) (3)		$\begin{pmatrix}4&2&2\\2&10&8\end{pmatrix}$	$\epsilon.x_{3^3}$
12 897 383	$C_3\cdot C_3$ $\oplus$	ϵ, x_5, x_{11}	$([36343]_\rho^2,[36343]_\sigma^2,[165601]_\rho,[165601]_\sigma)$ $([11251],[11251]_{\rho\sigma},(1+3)^2,(1+3))$	(9) (3)	$3^3.3^4$	$\begin{pmatrix}4&2&10\\0&2&0\end{pmatrix}$	$\epsilon^2.x_{11}$
$d\equiv 6[9]$	$C_9\cdot C_3\cdot C_3$	ϵ, x_5, x_{11}	$([19],[19]_{\rho\sigma},[165601]_\rho,[165601]_\sigma,4.3^2,4^2)$ $([37]_\rho,[37]_\sigma,(1+3),(1+3)^2)$	(9) (3)		$\begin{pmatrix}7&2&7\\0&4&0\end{pmatrix}$	$\epsilon^2.x_{11}$
13 974 943	$C_3\cdot C_3$ $\oplus$	$\epsilon, x_{41}, x_{103}$	$([5413]_\sigma,[5413]_\sigma,1)$ $([31333]_\rho,[31333]_\sigma,(1+3i))$	(3) (3)	$3^3.3^2$	$\begin{pmatrix}4&0&2\\2&0&2\end{pmatrix}$	x_{41}
$d\equiv 3[9]$	$C_9\cdot C_3\cdot C_3$	$\epsilon, x_{41}, x_{103}^+$	$(\overline{[2]}^4,[47],1)$ $(\overline{[2]}^{19},[23]^2,(1+3i)^2)$	(3) (3)		$\begin{pmatrix}2&9&4\\2&42&23\end{pmatrix}$	x_{41}
14 227 223	$C_9\cdot C_3$ $\oplus$	ϵ, x_7, x_{59}	$([26557]_\rho,[26557]_\sigma,[38629],[38629]_{\rho\sigma},4^2,4^2)$ $([97729],[97729]_{\rho\sigma},(1+3),1)$	(3) (9)	$3^3.3^4$	$\begin{pmatrix}6&4&0\\4&0&4\end{pmatrix}$	$\epsilon^2.x_{59}$
$d\equiv 6[9]$	$C_9\cdot C_3\cdot C_3$	ϵ, x_7, x_{59}	$(\overline{[2]}^2,[79]_\rho,[79]_\sigma,,(1+3)^2.3^8,(1+3))$ $(\overline{[2]}^4,\overline{[23]},(1+3)^2,(1+3).3^2)$	(3) (9)		$\begin{pmatrix}6&14&12\\7&6&7\end{pmatrix}$	$\epsilon^2.x_{59}$
14 935 391	$C_3\cdot C_3\cdot C_3$ $\oplus$	$\epsilon, x_{11}, x_{13}, x_{419}$	$([16057]_\rho,[16057]_\sigma,1,1)$ $([[29683],[29683]_{\rho\sigma},(1+3),(1+3)^2)$ $((1+3)^2,(1+3))$	(3) (3) (9)	$3^5.3^5$	$\begin{pmatrix}0&0&2&2\\2&2&4&4\\0&0&8&0\end{pmatrix}$	$\epsilon^2.x_{11}$
$d\equiv 6[9]$	$C_{81}\cdot C_3\cdot C_3$	$\epsilon, x_{11}, x_{13}, x_{17}$	$(\overline{[2]}^7,3^2,(1+3))$ $(\overline{[2]}^2,[5],(1+3),3^4)$ $((1+3),(1+3)^2)$	(3) (3) (9)		$\begin{pmatrix}0&15&4&22\\2&8&10&8\\0&0&10&8\end{pmatrix}$	$\epsilon^2.x_{11}$
15 476 323	$C_9\cdot C_3$ $\oplus$	$\epsilon, x_{157}, x_{223}$	$([41983],[41983]_{\rho\sigma},(1+3i)^2)$ $([24373],[24373]_{\rho\sigma},1)$	(3) (3)	$3^7.3^2$	$\begin{pmatrix}4&2&4\\4&0&2\end{pmatrix}$	$\epsilon.x_{157}^2.x_{223}$
$d\equiv 3[3]$	$C_3\cdot C_3\cdot C_3$	$\epsilon, x_{157}, x_{223}$	$([1181],(1+3i)^2)$ $([181]_\rho,[181]_\sigma,(1+3i))$	(3) (3)		$\begin{pmatrix}2&1&2\\2&0&4\end{pmatrix}$	$\epsilon.x_{157}^2.x_{223}$
16 042 291	$C_3\cdot C_3$ $\oplus$	ϵ, x_7, x_{11}	$([15271]_\rho,[15271]_\sigma,[72949]_\rho^2,[72949]_\sigma^2,1)$ $([65827]_\rho,[65827]_\sigma,[72949]_\rho,[72949]_\sigma,(1+3i))$	(3) (3)	$3^3.3^3$	$\begin{pmatrix}4&10&0\\2&4&4\end{pmatrix}$	$\epsilon^2.x_7.x_{11}$
$d\equiv 3[9]$	$C_9\cdot C_3\cdot C_3$	ϵ, x_{11}, x_{17}	$([7]_\rho^2,[7]_\sigma^2,\overline{[197]}^2,[263],1)$ $([7]_\rho,[7]_\sigma,\overline{[197]},[571],[571]_{\rho\sigma},(1+3i))$	(3) (3)		$\begin{pmatrix}2&6&12\\1&5&9\end{pmatrix}$	x_{17}
16 564 388	$C_3\cdot C_3$ $\oplus$	$\epsilon, x_{59}, x_{109}$	$([4201]^2,[4201]_{\rho\sigma}^2,[55621]_\rho,[55621]_\sigma,1,(1+3)^2)$ $([82609]_\rho,[82609]_\sigma,(1+3)^2,(1+3)^2)$	(3) (9)	$3^3.3^4$	$\begin{pmatrix}4&12&8\\4&2&2\end{pmatrix}$	$\epsilon.x_{109}$
$d\equiv 6[9]$	$C_9\cdot C_3\cdot C_3$	$\epsilon, x_{59}, x_{109}$	$(\overline{[17]}^2,[29]^2,\overline{[71]},3^2,(1+3)^2)$ $([151],[151]_{\rho\sigma},(1+3)^2.3^2,(1+3)^2)$	(3) (9)		$\begin{pmatrix}7&3&5\\5&1&4\end{pmatrix}$	$\epsilon.x_{109}$
16 572 804	$C_3\cdot C_3$ $\oplus$	ϵ, x_7, x_{13}	$([3109]_\rho,[3109]_\sigma,[7549],[7549]_{\rho\sigma},(1+\sqrt{3}))$ $([7549]^2,[7549]_{\rho\sigma}^2,[21937],[21937]_{\rho\sigma},(1+\sqrt{3}))$	(3) (3)	$3^4.3^2$	$\begin{pmatrix}4&4&4\\0&4&8\end{pmatrix}$	$\epsilon.x_7.x_{13}$
$d\equiv 2[3]$	$C_3\cdot C_3\cdot C_3$	ϵ, x_{23}, x_{53}	$(\overline{[5]}^2,[43]^2,[43]_{\rho\sigma}^2,[223]_\rho^2,[223]_\sigma^2,(1+\sqrt{3})^2)$ $(\overline{[5]}^4,[7]^2,[7]_{\rho\sigma}^2,[13]^2,[13]_{\rho\sigma}^2,[43]^4,[43]_{\rho\sigma}^4,(1+\sqrt{3}))$	(3) (3)		$\begin{pmatrix}20&12&4\\36&12&8\end{pmatrix}$	x_{23}

$\Delta(\mathbb{Q}(\sqrt{-3d}))$	$Cl_k(3) \oplus Cl_{k^*}(3)$	$\mathfrak{H}_F^{e_\varphi}$	$Gal(H/Z)^{e_{\varphi^*}}$		$\|T\|$	dualité	$\mathfrak{Z}_F^{e_\varphi}$
19 233080 $d \equiv 6[9]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{719}, x_{1129}$	$([7681], [7681]_{\rho\sigma}, [135241]_\rho, [135241]_\sigma, (1+3)^2, (1+3))$ $([281], (1+3), 1)$	(9) (3)	$3^2.3^4$	$\begin{pmatrix} 2 & 2 & 4 \\ 1 & 0 & 0 \end{pmatrix}$	$x_{719}.x_{1129}$
27 840799 $d \equiv 3[9]$	$C_3 \cdot C_3 \oplus$ $C_{27} \cdot C_3 \cdot C_3$	$\epsilon, x_{89}, x_{137}$	$((1+3i)^2)$ $([10831]_\rho, [10831]_\sigma, (1+3i))$	(9) (3)	$3^3.3^3$	$\begin{pmatrix} 8 & 0 & 8 \\ 2 & 0 & 0 \end{pmatrix}$	x_{37}
33 889487 $d \equiv 6[9]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{151}, x_{199}$	$([17569], [17569]_{\rho\sigma}, [61819], [61819]_{\rho\sigma}, (1+3)^2, (1+3)^2)$ $([37171], [31171]_{\rho\sigma}, (1+3)^2, (1+3))$	(9) (3)	$3^2.3^4$	$\begin{pmatrix} 2 & 6 & 4 \\ 0 & 2 & 0 \end{pmatrix}$	$\epsilon.x_{199}$
35 180884 $d \equiv 3[9]$	$C_3 \cdot C_3 \oplus$ $C_{81} \cdot C_3 \cdot C_3$	$\epsilon, x_{179}, x_{313}$	$(\overline{[5]}^2, \overline{[59]}^2, [2549]^2, (1+3i))$ $([2549], 1)$	(3) (3)	$3^2.3^2$	$\begin{pmatrix} 8 & 12 & 8 \\ 2 & 2 & 1 \end{pmatrix}$	$\epsilon.x_{179}.x_{313}^2$
41 889531 $d \equiv 1[3]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{3^6}, x_{19}$	$([7]_\rho^4, [7]_\sigma^4, \overline{[11]}^2, [31]^4, [31]_{\rho\sigma}^4, [41]^2, 1, 1)$ $([5]^4, \overline{[11]}^4, [41]^4, 1, (1+3\sqrt{-3})^2)$	(3) (9)	$3^3.3^3$	$\begin{pmatrix} 22 & 24 & 24 \\ 20 & 16 & 24 \end{pmatrix}$	x_{19}
43 987623 $d \equiv 2[3]$	$C_3 \cdot C_3 \oplus$ $C_{81} \cdot C_3 \cdot C_3$	ϵ, x_{11}, x_{41}	$(\overline{[2]}^{22}, [337]_\rho^2, [337]_\sigma^2, (1+\sqrt{3})^2)$ $(\overline{[2]}, \overline{[239]}, (1+\sqrt{3}))$	(9) (3)	$3^3.3^3$	$\begin{pmatrix} 30 & 4 & 52 \\ 2 & 0 & 3 \end{pmatrix}$	$x_{11}^2.x_{41}$
46 677864 $d \equiv 1[3]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{3^3}, x_{17}$	$([5]^4, [1033]_\rho^2, [1033]_\sigma^2, 1, 1)$ $(\overline{[11]}^2, [31]^2, [31]_{\rho\sigma}^2, 1, 1)$	(9) (3)	$3^2.3^3$	$\begin{pmatrix} 4 & 8 & 12 \\ 0 & 4 & 10 \end{pmatrix}$	$\epsilon.x_{3^3}.x_{17}^2$
49 723123 $d \equiv 3[9]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	ϵ, x_5, x_{17}	$([233], [337]_\rho, [337]_\sigma, (1+3i))$ $([1123]_\rho, [1123]_\sigma, (1+3i)^2)$	(3) (9)	$3^3.3^3$	$\begin{pmatrix} 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$	$\epsilon.x_5^2.x_{17}$
55 247159 $d \equiv 6[9]$	$C_3 \cdot C_3 \oplus$ $C_{27} \cdot C_3 \cdot C_3$	$\epsilon, x_{157}, x_{173}$	$(\overline{[2]}^5, [5]^2, [83], (1+3)^2, (1+3)^2.3^{14})$ $([5], [83]^2, [359], (1+3), 3^4)$	(3) (27)	$3^2.3^5$	$\begin{pmatrix} 14 & 12 & 22 \\ 4 & 3 & 6 \end{pmatrix}$	x_{157}
60 912499 $d \equiv 3[9]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{139}, x_{179}$	$([47]^2, \overline{[101]}, 1)$ $(\overline{[5]}, \overline{[11]}, [61], [61]_{\rho\sigma}, (1+3i))$	(9) (3)	$3^2.3^3$	$\begin{pmatrix} 0 & 1 & 6 \\ 4 & 1 & 5 \end{pmatrix}$	$\epsilon.x_{179}$
72 673979 $d \equiv 6[9]$	$C_3 \cdot C_3 \oplus$ $C_{81} \cdot C_3 \cdot C_3$	$\epsilon, x_{59}, x_{101}$	$([263], [219031]_\rho, [219031]_\sigma, (1+3)^2, 1)$ $([263], (1+3)^2, (1+3)^2)$	(3) (3)	$3^2.3^4$	$\begin{pmatrix} 4 & 4 & 3 \\ 2 & 0 & 1 \end{pmatrix}$	$\epsilon.x_{59}^2.x_{101}$
74 251236 $d \equiv 1[3]$	$C_9 \cdot C_3 \oplus$ $C_{27} \cdot C_3 \cdot C_3$	ϵ, x_{3^9}, x_7	$(\overline{[29]}, [89], [2617], [2617]_{\rho\sigma}, 1, (1+3\sqrt{-3}))$ $([19]_\rho, [19]_\sigma, [127]_\rho, [127]_\sigma, 1, (1+3\sqrt{-3})^2)$	(27) (3)	$3^3.3^4$	$\begin{pmatrix} 3 & 14 & 2 \\ 2 & 22 & 4 \end{pmatrix}$	$\epsilon.x_{3^9}.x_7^2$
76 933327 $d \equiv 3[9]$	$C_3 \cdot C_3 \oplus$ $C_{27} \cdot C_3 \cdot C_3$	ϵ, x_5, x_{11}	$([13], [13]_{\rho\sigma}, [71], \overline{[137]}, 1)$ $(\overline{[2]}, \overline{[137]}, (1+3i)^2)$	(27) (3)	$3^2.3^4$	$\begin{pmatrix} 2 & 4 & 3 \\ 4 & 4 & 2 \end{pmatrix}$	$\epsilon.x_5.x_{11}^2$
87 227535 $d \equiv 1[3]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{3^6}, x_{41}$	$([61]_\rho^2, [61]_\sigma^2, \overline{[1553]}, 1, (1+3\sqrt{-3}))$ $(\overline{[2]}^2, [7], [7]_{\rho\sigma}, \overline{[131]}, \overline{[1553]}, 1, (1+3\sqrt{-3}))$	(3) (9)	$3^2.3^3$	$\begin{pmatrix} 0 & 6 & 2 \\ 5 & 17 & 8 \end{pmatrix}$	$\epsilon^2.x_{3^6}$
90 847527 $d \equiv 2[3]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	ϵ, x_5, x_{11}	$(\overline{[2]}^2, [43]_\rho, [43]_\sigma, [103], [103]_{\rho\sigma}, [433]_\rho^2, [433]_\sigma^2, (1+\sqrt{3})^2)$ $(\overline{[2]}^2, [19]^2, [19]_{\rho\sigma}^2, 1)$	(3) (9)	$3^3.3^3$	$\begin{pmatrix} 12 & 8 & 8 \\ 12 & 2 & 4 \end{pmatrix}$	ϵ
96 012987 $d \equiv 2[3]$	$C_3 \cdot C_3 \oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{11}, x_{167}$	$([17]^2, [337]_\rho, [337]_\sigma, 1)$ $([3733], [3733]_{\rho\sigma}, 1)$	(3) (9)	$3^3.3^3$	$\begin{pmatrix} 6 & 4 & 4 \\ 2 & 2 & 4 \end{pmatrix}$	$\epsilon.x_{11}.x_{167}^2$
97 685956 $d \equiv 3[9]$	$C_3 \cdot C_3 \oplus$ $C_{81} \cdot C_3 \cdot C_3$	ϵ, x_{11}, x_{47}	$([461]^2, (1+3i))$ $(\overline{[5]}, \overline{[857]}, 1)$	(9) (3)	$3^2.3^3$	$\begin{pmatrix} 2 & 4 & 2 \\ 3 & 4 & 4 \end{pmatrix}$	$\epsilon.x_{11}^2.x_{47}$

$\Delta(\mathbb{Q}(\sqrt{d}))$	$Cl_k(3) \oplus Cl_{k^*}(3)$	$\mathfrak{H}_F^{e_\varphi}$	$Gal(H/Z)^{e_{\varphi^*}}$		$\|T\|$	dualité	$\mathfrak{Z}_F^{e_\varphi}$
39 345017 $d \equiv 2[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{13}, x_{17}, x_{19}$	$([691], [691]_{\rho\sigma}, (1+\sqrt{3})^2)$ $([643], [643]_{\rho\sigma}, (1+\sqrt{3})^2)$ $([2707]_\rho, [2707]_\sigma, (1++\sqrt{3})^2)$	(3) (3) (9)	$3^3.3^4$	$\begin{pmatrix} 2 & 2 & 4 & 6 \\ 6 & 2 & 8 & 4 \\ 3 & 5 & 6 & 1 \end{pmatrix}$	$\epsilon^2.x_{13}.x_{19}$ $\not\subset \mathfrak{C}_F$
86 814697 $d \equiv 1[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{3^3}.x_{19}^2, x_{3^3}.x_{61}$	$([307]_\rho^2, [307]_\sigma^2, 1, 1)$ $([19891], [19891]_{\rho\sigma}, 1, 1)$	(3) (3)	$3^4.3^2$	$\begin{pmatrix} 0 & 4 & 8 \\ 0 & 0 & 4 \end{pmatrix}$	ϵ $\subset \mathfrak{D}_F$
88 215377 $d \equiv 2[3]$	$C_9 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{19}, x_{31}, x_{101}$	$(\overline{[17]}, \overline{[59]}, (1+\sqrt{3})^2)$ $(\overline{[317]}^2, (1+\sqrt{3})^2)$ $(\overline{[5]}, [173], (1+\sqrt{3}))$	(3) (3) (3)	$3^5.3^3$	$\begin{pmatrix} 1 & 7 & 4 & 3 \\ 4 & 4 & 2 & 4 \\ 1 & 6 & 1 & 4 \end{pmatrix}$	$x_{19}.x_{31}^2.x_{101}$ $\not\subset \mathfrak{C}_F$
292 118845 $d \equiv 1[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{81} \cdot C_3 \cdot C_3$	$\epsilon, x_{3^6}, x_{59}$	$([37]_\rho, [37]_\sigma, [97], [97]_{\rho\sigma}, [127]_\rho^2, [127]_\sigma^2, 1, 1)$ $(\overline{[2]}^2, [11]^2, [19]_\rho^2, [19]_\sigma^2, 1, 1)$	(81) (3)	$3^4.3^5$	$\begin{pmatrix} 10 & 12 & 14 \\ 6 & 12 & 8 \end{pmatrix}$	x_{3^6} $\subset \mathfrak{D}_F$
436 612965 $d \equiv 6[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{81} \cdot C_3 \cdot C_3$	$\epsilon, x_{61}, x_{71}, x_{389}$	$(\overline{[2]}^4, [17]^2, (1+3), 3^4)$ $(\overline{[2]}^2, [83], 1, 3^2)$ $((1+3)^4, (1+3)^2)$	(3) (3) (3)	$3^4.3^4$	$\begin{pmatrix} 10 & 18 & 12 & 1 \\ 4 & 5 & 4 & 8 \\ 8 & 10 & 8 & 8 \end{pmatrix}$	$x_{61}.x_{71}$ $\subset \mathfrak{D}_F$
500 251001 $d \equiv 2[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{27} \cdot C_3 \cdot C_3$	$\epsilon, x_{31}, x_{139}, x_{191}$	$(\overline{[11]}, \overline{[653]}, (1+\sqrt{3})^2)$ $(\overline{[11]}^2, [37]_\rho, [37]_\sigma, (1+\sqrt{3})^2)$ $((1+\sqrt{3})^2)$	(3) (3) (9)	$3^3.3^4$	$\begin{pmatrix} 8 & 4 & 7 & 4 \\ 10 & 4 & 12 & 8 \\ 4 & 4 & 4 & 4 \end{pmatrix}$	$x_{31}.x_{191}^2$ $\subset \mathfrak{C}_F$
623 754661 $d \equiv 1[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{27} \cdot C_3 \cdot C_3$	$\epsilon.x_{3^3}^2, \epsilon.x_5^2, \epsilon.x_{13}$	$(\overline{[2]}^2, \overline{[809]}^2, [11587]_\rho, [11587]_\sigma, 1, (1+3\sqrt{-3}))$ $([11587]_\rho^2, [11587]_\sigma^2, 1, (1+3\sqrt{-3}))$	(3) (3)	$3^3.3^2$	$\begin{pmatrix} 14 & 6 & 6 \\ 18 & 4 & 4 \end{pmatrix}$	$x_5.x_{13}$ $\subset \mathfrak{C}_F$
638 191733 $d \equiv 2[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{11}, x_{13}, x_{41}$	$(\overline{[2]}, [7]_\rho, [7]_\sigma, \overline{[281]}, (1+\sqrt{3})^2)$ $(\overline{[2]}^2, [5], [101], (1+\sqrt{3}))$ $(\overline{[2]}, [7]_\rho, [7]_\sigma, [19]_\rho, [19]_\sigma, \overline{[29]}, (1+\sqrt{3}))$	(3) (3) (9)	$3^4.3^4$	$\begin{pmatrix} 2 & 8 & 6 & 12 \\ 5 & 8 & 4 & 8 \\ 1 & 6 & 12 & 11 \end{pmatrix}$	$\epsilon.x_{11}^2.x_{13}.x_{41}$ $\not\subset \mathfrak{C}_F$
641 515649 $d \equiv 2[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_{71}, x_{109}$	$([10513]_\rho, [10513]_\sigma, (1+\sqrt{3})^2)$ $(\overline{[17]}, \overline{[233]}, (1+\sqrt{3})^2)$ $([3001]_\rho, [3001]_\sigma, (1+\sqrt{3})^2)$	(3) (3) (3)	$3^3.3^3$	$\begin{pmatrix} 4 & 6 & 4 & 6 \\ 5 & 4 & 4 & 8 \\ 6 & 4 & 6 & 8 \end{pmatrix}$	$x_5.x_{109}$ $\not\subset \mathfrak{C}_F$
652 955917 $d \equiv 1[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$x_{3^3}, \epsilon.x_{11}, \epsilon.x_{13}$	$(\overline{[2]}, \overline{[5]}, [17], \overline{[29]}, 1, (1+3\sqrt{-3})^2)$ $(\overline{[2]}^7, [5]^2, \overline{[29]}^2, [227], 1, (1+3\sqrt{-3}))$	(3) (3)	$3^3.3^2$	$\begin{pmatrix} 10 & 3 & 5 \\ 15 & 20 & 16 \end{pmatrix}$	$\epsilon^2.x_{3^3}.x_{11}.x_{13}$ $\subset \mathfrak{D}_F$
725 031941 $d \equiv 2[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_7, x_{13}$	$(\overline{[2]}^4, [11]^2, [19]^2, [19]_{\rho\sigma}^2, (1+\sqrt{3})^2)$ $(\overline{[17]}, [19]_\rho, [19]_\sigma, (1+\sqrt{3})^2)$ $((1+\sqrt{3})^2)$	(3) (3) (3)	$3^3.3^3$	$\begin{pmatrix} 24 & 8 & 16 & 18 \\ 11 & 10 & 11 & 10 \\ 4 & 2 & 4 & 4 \end{pmatrix}$	$x_5.x_7$ $\subset \mathfrak{C}_F$

$\Delta(\mathbb{Q}(\sqrt{d}))$	$Cl_k(3) \oplus Cl_{k^*}(3)$	$\mathfrak{H}_F^{e_\varphi}$	$Gal(H/Z)^{e_{\varphi^*}}$		$\lvert T\rvert$	dualité	$\mathfrak{Z}_F^{e_\varphi}$
779 966689 $d \equiv 1[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$x_{3^9}, \epsilon.x_5^2, \epsilon.x_{17}$	$([1831], [1831]_{\rho\sigma}, [2371]^2, [2371]^2_{\rho\sigma}, 1, (1+3\sqrt{-3})^2)$ $([937]^2_\rho, [937]^2_\sigma, [1831], [1831]_{\rho\sigma}, 1, (1+3\sqrt{-3})^2)$	(3) (3)	$3^3.3^2$	$\begin{pmatrix} 6 & 10 & 2 \\ 14 & 6 & 6 \end{pmatrix}$	$\epsilon^2.x_5^2.x_{17}$ $\subset \mathfrak{D}_F$
845 338289 $d \equiv 2[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{11}, x_{463}, x_{487}$	$([599]^2, 1)$ $(\overline{[23]}^2, [199]^2, [199]^2_{\rho\sigma}, (1+\sqrt{3}))$ $([13]^2, [13]^2_{\rho\sigma}, [53]^2, (1+\sqrt{3})^2)$	(3) (3) (3)	$3^4.3^3$	$\begin{pmatrix} 0 & 2 & 4 & 2 \\ 4 & 7 & 4 & 4 \\ 10 & 10 & 8 & 16 \end{pmatrix}$	$x_{11}^2.x_{487}$ $\not\subset \mathfrak{C}_F$
1107 069709 $d \equiv 1[3]$	$C_9 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{3^9}.x_{11}, x_{3^9}.x_{59}$	$(\overline{[2]}^5, \overline{[23]}, [283]^2_\rho, [283]^2_\sigma, [463], [463]_{\rho\sigma}, 1, (1+3\sqrt{-3})^2)$ $(\overline{[2]}^8, \overline{[17]}^2, [179]^2, [283]^4_\rho, [283]^4_\sigma, 1, (1+3\sqrt{-3})^2)$	(3) (3)	$3^5.3^2$	$\begin{pmatrix} 9 & 26 & 30 \\ 10 & 32 & 40 \end{pmatrix}$	$\epsilon^2.x_{3^9}.x_{59}$ $\subset \mathfrak{D}_F$
2109 568332 $d \equiv 3[9]$	$C_9 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{827}, x_{1009}, x_{1019}$	$(\overline{[5]}, [457], [457]_{\rho\sigma}, 1)$ $(\overline{[5]}^2, \overline{[89]}^2, 1)$ $([13]^2, [13]^2_{\rho\sigma}, [37], [37]_{\rho\sigma}, 1)$	(3) (3) (3)	$3^6.3^3$	$\begin{pmatrix} 2 & 6 & 3 & 5 \\ 4 & 6 & 4 & 6 \\ 8 & 8 & 12 & 8 \end{pmatrix}$	$\epsilon^2.x_{1009}.x_{1019}$ $\not\subset \mathfrak{C}_F$
3893 450664 $d \equiv 3[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{27} \cdot C_3 \cdot C_3$	$\epsilon, x_{113}, x_{257}, x_{367}$	$([7001], 1)$ $([97]_\rho, [97]_\sigma, [103], [103]_{\rho\sigma}, (1+3i))$ $((1+3i))$	(3) (3) (3)	$3^3.3^3$	$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 1 & 2 & 8 & 7 \\ 1 & 2 & 0 & 1 \end{pmatrix}$	$x_{113}.x_{367}$ $\subset \mathfrak{C}_F$
4626 343356 $d \equiv 6[9]$	$C_9 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{37}, x_{47}, x_{109}$	$([67]^2, [67]^2_{\rho\sigma}, (1+3).3^8, 1)$ $([19]^2_\rho, [19]^2_\sigma, 3^{16}, 1)$ $((1+3)^2, (1+3))$	(3) (3) (3)	$3^5.3^4$	$\begin{pmatrix} 8 & 16 & 20 & 8 \\ 12 & 8 & 16 & 16 \\ 10 & 8 & 8 & 10 \end{pmatrix}$	$x_{37}.x_{109}$ $\subset \mathfrak{C}_F,\ \not\subset \mathfrak{D}_F$
6804 188844 $d \equiv 6[9]$	$C_9 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{311}, x_{1753}, x_{2017}$	$([8933]^2, (1+3), 1)$ $([181], [181]_{\rho\sigma}, (1+3)^2.3^4, (1+3))$ $([223]^2_\rho, [223]^2_\sigma, (1+3).3^4, (1+3))$	(3) (3) (3)	$3^5.3^4$	$\begin{pmatrix} 8 & 4 & 8 & 10 \\ 12 & 2 & 12 & 10 \\ 22 & 4 & 16 & 22 \end{pmatrix}$	$\epsilon^2.x_{1753}$ $\not\subset \mathfrak{C}_F$
7005 297324 $d \equiv 3[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3$	$\epsilon, x_{11}, x_{73}, x_{83}$	$([1877], (1+3i))$ $([709]^2_\rho, [709]^2_\sigma, (1+3i))$ $([41]^4, 1)$	(3) (3) (9)	$3^3.3^4$	$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 5 & 4 & 6 & 8 \\ 0 & 8 & 8 & 0 \end{pmatrix}$	$\epsilon^2.x_{11}^2.x_{73}$ $\not\subset \mathfrak{C}_F$
7644 861816 $d \equiv 6[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{27} \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_{19}, x_{23}$	$([7]^2_\rho, [7]^2_\sigma, [569]^2, (1+3)^2.3^4, (1+3)^2)$ $([499]_\rho, [499]_\sigma, 3^2, (1+3)^2)$ $((1+3)^2, (1+3))$	(3) (3) (3)	$3^4.3^4$	$\begin{pmatrix} 12 & 18 & 10 & 16 \\ 6 & 4 & 6 & 8 \\ 8 & 8 & 0 & 8 \end{pmatrix}$	$\epsilon.x_5.x_{19}^2.x_{23}$ $\subset \mathfrak{C}_F,\ \not\subset \mathfrak{D}_F$
8839 684221 $d \equiv 6[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_{17}, x_{47}$	$(\overline{[2]}, [613], [613]_{\rho\sigma}, 3^2, 1)$ $(\overline{[2]}^2, [7], [7]_{\rho\sigma}, 3^8, 1)$ $(\overline{[2]}^2, [107], [107]_{\rho\sigma}, (1+3)^2.3^2, 1)$	(3) (3) (3)	$3^4.3^4$	$\begin{pmatrix} 16 & 3 & 10 & 2 \\ 14 & 8 & 12 & 4 \\ 12 & 3 & 6 & 2 \end{pmatrix}$	$\epsilon^2.x_5.x_{17}$ $\not\subset \mathfrak{C}_F$
9894 469848 $d \equiv 6[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{27} \cdot C_3 \cdot C_3$	$\epsilon, x_{47}, x_{71}, x_{197}$	$([7]^2_\rho, [7]^2_\sigma, [13]^2, [13]^2_{\rho\sigma}, (1+3).3^8, (1+3)^2)$ $(\overline{[11]}, (1+3).3^{10}, 1)$ $((1+3), (1+3)^2)$	(3) (3) (3)	$3^4.3^4$	$\begin{pmatrix} 12 & 8 & 16 & 28 \\ 9 & 5 & 1 & 14 \\ 8 & 0 & 0 & 8 \end{pmatrix}$	$x_{47}.x_{71}$ $\subset \mathfrak{C}_F,\ \not\subset \mathfrak{D}_F$

$\lvert\Delta(\mathbb{Q}(\sqrt{-3d}))\rvert$	$Cl_k(3) \oplus Cl_{k^*}(3)$	$\mathfrak{H}_F^{e_\varphi}$	$Gal(H/Z)^{e_{\varphi^*}}$		$\lvert T\rvert$	dualité	$\mathfrak{Z}_F^{e_\varphi}$
653 329427	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_{11}, x_{23}$	$([2061]_\rho, [2061]_\sigma, (1+3).3^4, (1+3))$ $([\overline{17}], [97]_\rho, [97]_\sigma, (1+3)^2, (1+3)^2)$ $([2729], (1+3)^2, (1+3))$	(3) (3) (3)	$3^4.3^4$	$\begin{pmatrix} 2 & 6 & 4 & 8 \\ 5 & 4 & 4 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	$\epsilon.x_{11}$ $\not\subset \mathfrak{D}_F$
$d \equiv 6[9]$		$\epsilon, x_5, x_{11}, x_{23}$	$([106591], [106591]_{\rho\sigma}, (1+3)^2, 1)$ $([144961]_\rho, [144961]_\sigma, [241117]_\rho, [241117]_\sigma, 1, (1+3)^2)$ $([62743], [62743]_{\rho\sigma}, (1+3)^2, 1)$	(3) (3) (3)		$\begin{pmatrix} 2 & 4 & 4 & 4 \\ 2 & 2 & 4 & 8 \\ 0 & 2 & 0 & 2 \end{pmatrix}$	$\epsilon.x_{11}$
2520 963512	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{17}, x_{41}, x_{71}$	$([7673], (1+3), (1+3))$ $([467]^2, [1063]^2, [1063]^2_{\rho\sigma}, (1+3).3, (1+3)^2)$ $([53]^2, [73]^2_\rho, [73]^2_\sigma, [1063]^4, [1063]^4_{\rho\sigma}, (1+3).3, (1+3)^2)$	(3) (3) (3)	$3^4.3^4$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 16 & 10 & 8 \\ 8 & 30 & 18 & 6 \end{pmatrix}$	$x_{17}.x_{71}$ $\not\subset \mathfrak{D}_F$
$d \equiv 6[9]$		$\epsilon, x_{17}, x_{41}, x_{71}$	$([82183]_\rho, [82183]_\sigma, (1+3)^2, 1)$ $([271927]_\rho, [271927]_\sigma, [517609]^2, [517609]^2_{\rho\sigma}, (1+3)^2, (1+3))$ $([237313], [237313]_{\rho\sigma}, [517609]^2, [517609]^2_{\rho\sigma}, (1+3)^2, (1+3))$	(3) (3) (3)		$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 4 & 8 & 8 & 4 \\ 2 & 6 & 12 & 0 \end{pmatrix}$	$x_{17}.x_{71}$
3146 813128 $d \equiv 3[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{81} \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_{23}, x_{29}$	$([787]^4_\rho, [787]^4_\sigma, [6329], (1+3i))$ $([13], [13]_{\rho\sigma}, [43]_\rho, [43]_\sigma, [\overline{47}], [6329], (1+3i))$ $([6329], 1)$	(3) (3) (9)	$3^8.3^4$	$\begin{pmatrix} 1 & 2 & 2 & 10 \\ 11 & 7 & 5 & 8 \\ 1 & 2 & 2 & 2 \end{pmatrix}$	$\epsilon.x_5$
5252 241199 $d \equiv 3[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{139}, x_{263}, x_{281}$	$(\overline{[2]}^5, \overline{[11]}^2, [47]^2, 1)$ $(\overline{[2]}^4, \overline{[11]}^2, [22369], [22369]_{\rho\sigma}, (1+3i))$ $(\overline{[2]}^{11}, \overline{[11]}^4, (1+3i))$	(3) (3) (3)	$3^4.3^3$	$\begin{pmatrix} 9 & 12 & 13 & 12 \\ 8 & 8 & 8 & 10 \\ 19 & 22 & 19 & 22 \end{pmatrix}$	$\epsilon^2.x_{139}$
5288 116947 $d \equiv 2[3]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_{27} \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_{11}, x_{13}$	$([761]^4, [1259]^2, (1+\sqrt{3})^2)$ $([1259]^4, 1)$ $([\overline{17}], [\overline{143}], (1+\sqrt{3}))$	(3) (9) (3)	$3^4.3^4$	$\begin{pmatrix} 2 & 8 & 4 & 4 \\ 4 & 8 & 0 & 0 \\ 3 & 2 & 2 & 1 \end{pmatrix}$	$\epsilon.x_5.x_{11}.x_{13}$
7311 232679 $d \equiv 6[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{97}, x_{151}, x_{233}$	$([\overline{2}], [15467], 1, 1)$ $([\overline{2}], [13]_\rho, [13]_\sigma, [\overline{17}], [653]^2,$ $(1+3)^2.3^4, (1+3)^2.3^2)$ $([\overline{2}], [5]^4, [653], (1+3)^2.3^2, 1)$	(3) (3) (3)	$3^5.3^4$	$\begin{pmatrix} 3 & 3 & 2 & 2 \\ 10 & 9 & 12 & 8 \\ 3 & 4 & 6 & 2 \end{pmatrix}$	$\epsilon.x_{97}.x_{151}^2.x_{233}$ $\subset \mathfrak{D}_F$
8308 370723 $d \equiv 6[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_9 \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_5, x_{17}, x_{41}$	$([\overline{53}], [61], [61]_{\rho\sigma}, (1+3), 1)$ $([61], [61]_{\rho\sigma}, [\overline{701}], (1+3)^2.3^8, (1+3)^2)$ $([21649]_\rho, [21649]_\sigma, (1+3)^2, 1)$	(3) (3) (9)	$3^4.3^5$	$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 10 & 8 & 4 & 11 \\ 0 & 4 & 0 & 2 \end{pmatrix}$	$\epsilon^2.x_5.x_{41}$ $\not\subset \mathfrak{D}_F$
9775 810067 $d \equiv 6[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{13}, x_{43}, x_{59}$	$([571], [571]_{\rho\sigma}, [1889], (1+3).3^2, (1+3))$ $([1409], (1+3), (1+3)^2)$ $([151]_\rho, [151]_\sigma, [571]^2, [571]^2_{\rho\sigma}, (1+3).3^8, (1+3))$	(3) (3) (3)	$3^8.3^4$	$\begin{pmatrix} 3 & 4 & 4 & 2 \\ 1 & 0 & 1 & 0 \\ 6 & 10 & 12 & 4 \end{pmatrix}$	$\epsilon^2.x_{13}.x_{43}.x_{59}^2$ $\not\subset \mathfrak{D}_F$
9906 365947 $d \equiv 3[9]$	$C_3 \cdot C_3 \cdot C_3$ $\oplus$ $C_3 \cdot C_3 \cdot C_3 \cdot C_3$	$\epsilon, x_{11}, x_{13}, x_{17}$	$([\overline{29}], [67]_\rho, [67]_\sigma, [7253]^2, (1+3i)^2)$ $([2579], [7253], (1+3i))$ $([2039], [7253]^2, (1+3i))$	(3) (3) (3)	$3^4.3^3$	$\begin{pmatrix} 7 & 4 & 2 & 6 \\ 4 & 1 & 3 & 3 \\ 4 & 1 & 4 & 5 \end{pmatrix}$	$\epsilon^2.x_{11}$

REFERENCES

1. Y.AMICE ET J.FRESNEL, Fonctions zêta p-adiques des corps de nombres abeliens réels, *Acta Arithmetica* **XX**(1972), 353-384
2. E.ARTIN ET J.TATE, "Class field theory", Benjamin, New York - Amsterdam, 1967
3. F.BERTRANDIAS ET J.-J. PAYAN,Γ-extensions et invariants cyclotomiques, *Ann. Sci. E.N.S.* **5** (1972), 517-543
4. A.BRUMER, On the units of algebraic number field, *Mathematika* **14** (1967), 121-124
5. J.E.CARROLL,On determining the quadratic subfields of $\mathbb{Z}_2$-extensions of complex quadratic fields, *Compositio Mathematica* **30 Fasc. 3**(1975), 259-271
6. S.U.CHASE ET W.C.WATERHOUSE, Moore's theorem on uniqueness of reciprocity laws, *Inv. Math.* **16**(1972), 267-270
7. J.COATES, "p-adic L-functions and Iwasawa theory", Durham symposium in algebraic number field
(A.Frölich éd.) Academic Press, New York, London, 1977
8. H.COHEN, "A course in computational in algebraic number theory", Grad. Texts in Math. **138** Springer Verlag 1993
9. H.COHEN AND H.W.LENSTRA JR, Heuristics on class groups of number fields, Lect. Notes in Math. **1068** 1984, 33-62
10. G.GRAS, Plongements kummeriens dans les $\mathbb{Z}_p$-extensions, *Compositio Mathematica* **55**(1985), 383-396
11. G.GRAS, Extensions abeliennes non ramifiées de degré premier d'un corps quadratique. *Bull. Soc. Math. France,***100** (1972), 177-193
12. G.GRAS ET J.-F.JAULENT, Sur les corps de nombres réguliers. *Math. Zeitschrift* **202**(1989), 343-365
13. D.HEMARD, Modules galoisiens de torsion et plongement dans les $\mathbb{Z}_p$-extensions, *Journal of Number Theory* **30** (1988), 357-374
14. G.J.JANUSZ, "Algebraic number fields", Academic Press, New York and London, 1973
15. J.-F.JAULENT, "L'arithmétique des ℓ-extensions" (thèse), Pub. Math. Fac. Sci. Besancon, Théorie des Nombres (1986)
16. J.-F.JAULENT, Sur les conjectures de Leopoldt et de Gross, *Astérisque* **147-148** (1987), 107-120
17. J.-F.JAULENT, Noyau universel et valeurs absolues, *Astérisque* **198-199-200** (1991), 187-208
18. K.KRAMER ET A.CANDIOTTI, On K_2 and $\mathbb{Z}_\ell$- extensions of numbers fields, *Amer. J. Math.* **100** (1978), 177-196.
19. J.C. LAGARIAS, H.L. MONTGOMERY, A.M.ODLYZKO, A bound for the least prime ideal in the Chebotarev density theorem, *Inv. Math.* **54** (1979), 271-296.
20. H.W. LEOPOLDT, Zur Struktur der l-Klassengruppe galoissher Zahlkörper, *J. reine angew. Math.* **199** (1958), 165-174
21. J.MILNOR, "Introduction to algebraic K-theory", Annals of mathematics studies 72, (1971) Princeton
22. B.ORIAT, Tables de groupes de classes de corps quadratiques, Pub. Math. Fac. Sci. Besancon, Théorie des Nombres (1986/87)
23. S.SEN, On explicit reciprocity laws, *J. reine angew. Math.* **313** (1980), 1-26
24. S.SEN, On explicit reciprocity laws. II, *J. reine angew. Math.* **323** (1981), 68-87
25. J.-P. SERRE, Quelques applications du théorème de densité de Chebotarev, *Pub. Math. I.H.E.S.* **54** (1981)
26. A.SCHOLZ, Über die Beziehung der Klassenzahlen quadratischer Körper Zueinander, *J. reine angew. Math.* **166** (1932), 201-204
27. J.TATE, Relation between K_2 and Galois cohomology. *Inv. Math.* **36** (1976), 257-274
28. J.TATE, On the torsion in K_2 of fields, International Symposium, Kyoto 1976, 243-261, S.Iyanaga (Ed.)
29. K.S. WILLIAMS, Integers of biquadratic fields, *Canad. Math. Bull.* **13(4)**(1970), 519-528

Hervé Thomas
Centre de recherche en mathématiques de Bordeaux
351, cours de la Libération
F-33405 Talence cedex

(Reçu le 11 juin 1993;
Version révisée reçu le 28 juin 1993)

manuscripta math. 81, 437 - 443 (1993)

manuscripta mathematica

An affirmative answer to a question of Ciliberto

Masahiro OHNO

We give an example of a nondegenerate n-dimensional smooth projective variety X in $\mathbf{P}^{2n+1}$ with the canonical bundle ample a variety X whose tangent variety Tan$X$ has dimension less than $2n$ over an algebraically closed field of any characteristic when $n \geq 9$. This variety X is not ruled by lines and the embedded tangent space at a general point of X intersects X at some other points, so that this yields an affirmative answer to a question of Ciliberto.

1 Introduction

Ciro Ciliberto asked the following question in [1, Problem 15].

> Let X be a smooth nondegenerate closed variety in $\mathbf{P}^r$ over an algebraically closed field k, $\dim X = n$, with $r = 2n+1$, X not ruled by lines. Is it possible that the tangent n-space at a general point of X intersects X at some other points ?

The purpose of this article is to answer the question above in the affirmative in the case that $n \geq 9$. To be precise, we give an example of a nondegenerate n-dimensional smooth projective variety X in $\mathbf{P}^{2n+1}$ with the canonical bundle ample a variety X whose tangent variety Tan$X$ in $\mathbf{P}^{2n+1}$ has dimension less than $2n$ over an algebraically closed field k of any characteristic when $n \geq 9$ (Theorem 3.5). Then we show that this variety is tangentially degenerate and not ruled by lines (Corollary 3.6). Here we say that a projective variety X in $\mathbf{P}^r$ is tangentially degenerate if the embedded tangent space at a general point of X intersects X at some other points (Definition 2.1). Note that, on the contrary, in the case $n = 1$ and $char.k = 0$, Kaji [5, Th.3.1] shows that it is impossible, independent of the question of Ciliberto.

Acknowledgement. The author would like to express his hearty thanks to Professor H.Kaji, who taught him the question of Ciliberto and the corollary [2, Cor.4] of Fulton and Hansen and gave him kind advice and encouragement. The author would also like to express his gratitude to Professors T. Fujita and M. Kobayashi, who pointed out the subcases 1.2 and 2.2 of the proof of Theorem 3.4 and gave him kind advice and encouragement.

Notation and conventions

We work over an algebraically closed field k of any characteristic unless otherwise stated. We follow the notation and terminology of [4]. A closed immersion $X \hookrightarrow \mathbf{P}^r$ is said to be nondegenerate if X is not contained in any hyperplane of $\mathbf{P}^r$. For a smooth projective variety X, we denote by ω_X the canonical bundle of X and by $\kappa(X)$ the Kodaira dimension of X. We use the word *point* to mean a closed point and the word *line* to mean a smooth rational curve of degree 1. For a smooth projective variety X in $\mathbf{P}^r$, we call the secant variety of X the closure of the union of all secant lines and denote it by Sec$X$, and we call the tangent variety of X the union of all embedded tangent spaces and denote it by Tan$X$. For a smooth variety X in $\mathbf{P}^r$ and a point x of X, T_xX denotes the embedded tangent space of X at x and the correspondence $\{(x,y) \in X \times \mathbf{P}^r | y \in T_xX\}$ is called the projectivized tangent bundle of X in $\mathbf{P}^r$ and denoted by $P(X, \mathbf{P}^r)$ or simply $P(X)$ if no confusion occurs. We denote by $G(r,1)$ the Grassmannian of lines in $\mathbf{P}^r$. For a variety V in a hyperplane H of $\mathbf{P}^{r+1}$ and a point p of $\mathbf{P}^{r+1} - H$, $C_p(V)$ denotes the projective cone over V with vertex p.

2 Tangential degeneration

We begin with the definition of tangential degeneration.

Definition 2.1 A smooth projective variety X in $\mathbf{P}^r$ is said to be tangentially degenerate if there exists a dense open subset U of X such that for every point x of U the embedded tangent space T_xX at x intersects X at some other points. (This definition of tangential degeneration differs from that of [5, Def.1.11] when $\dim X \geq 2$.)

Remark 2.2 A smooth projective variety X in $\mathbf{P}^r$ with $r < 2\dim X$ is always tangentially degenerate. Thus this notion is meaningful only when $r \geq 2\dim X$.

Recall that an inequality $\dim \mathrm{Tan} X \leq 2\dim X$ always holds for a smooth variety X in $\mathbf{P}^r$.

Proposition 2.3 *Let X be a smooth variety in $\mathbf{P}^r$. If an inequality* $\dim \mathrm{Tan} X \leq 2\dim X - 1$ *holds, then X is tangentially degenerate.*

Proof. Let Δ be the graph of the closed immersion $X \hookrightarrow \mathbf{P}^r$, and let $\varphi : P(X) \to \mathrm{Tan} X$ be the projection induced from $X \times \mathbf{P}^r \to \mathbf{P}^r$. Since $\dim \mathrm{Tan} X \leq 2\dim X - 1 = \dim P(X) - 1$ and φ is surjective, every irreducible component of a fibre of φ is positive dimensional by [4, Chap.2, Ex.3.22(b)]. Thus for every point x of X the closure $\overline{\varphi^{-1}(x) - \{(x,x)\}}$ of $\varphi^{-1}(x) - \{(x,x)\}$ is $\varphi^{-1}(x)$. Therefore $\overline{\varphi^{-1}(X) - \Delta} = \varphi^{-1}(X)$. Let $\pi : P(X) \to X$ be the projection induced from $X \times \mathbf{P}^r \to X$. Then $\overline{\pi(\varphi^{-1}(X) - \Delta)} = \pi(\overline{\varphi^{-1}(X) - \Delta}) = \pi(\varphi^{-1}(X)) = X$. Thus there exists an open dense subset U in $\pi(\varphi^{-1}(X) - \Delta)$. Hence X is tangentially degenerate. ∎

We will give an example of a nondegenerate n-dimensional smooth projective variety in $\mathbf{P}^{2n+1}$ whose tangent variety has dimension less than $2n - 2$.

Example 2.4 Let $m \geq 3$ be an integer, and embed $G(m,1)$ in $\mathbf{P}^r$ with $r = m(m+1)/2 - 1$ by the plücker embedding. Then it is well known that $\dim \mathrm{Sec} G(m,1) \leq 2\dim G(m,1) - 3$. (see, for example, [3, Example 3] in the case $m = 5$, and the similar in the case that $m \geq 3$.) Thus $\dim \mathrm{Tan} G(m,1) = \dim \mathrm{Sec} G(m,1)$ by [2, Cor.4]. Assume, furthermore, that $m \geq 7$. Then $r \geq 2\dim G(m,1) + 1$. Thus one can embed $G(m,1)$ in $\mathbf{P}^{2\dim G(m,1)+1}$ by the projection from some linear subvariety of $\mathbf{P}^r$. Since the plücker embedding is nondegenerate, this embedding is also nondegenerate. Hence this is the example we want to get.

3 An affirmative answer to the question of Ciliberto

First recall the definition of being ruled by lines.

Definition 3.1 Let X be a projective variety in $\mathbf{P}^r$. X is said to be ruled by lines if there exist a normal variety W and a birational morphism $\mathbf{P}^1 \times W \rightarrow X$ such that for every point w of W the image of the morphism $\mathbf{P}^1 \times w \rightarrow X \hookrightarrow \mathbf{P}^r$ is a line.

Remark 3.2 Recall that a projective variety X is said to be ruled if there exists a complete normal variety W such that X is birational to $\mathbf{P}^1 \times W$. Thus being ruled by lines implies ruledness.

Let us introduce the notion of being uniruled by lines.

Definition 3.3 Let X be a projective variety in $\mathbf{P}^r$. X is said to be uniruled by lines if there exists an open dense subset U of X such that for every point x of U there exists a line through x on X.

Remark 3.4 The following are equivalent for a projective variety X in $\mathbf{P}^r$.

1. X is uniruled by lines.

2. There exist a normal variety W and a generically finite dominant morphism $\mathbf{P}^1 \times W \rightarrow X$ such that for every point w of W the image of the morphism $\mathbf{P}^1 \times w \rightarrow X \hookrightarrow \mathbf{P}^r$ is a line.

The following theorem is a key to answer the question of Ciliberto in the affirmative.

Theorem 3.5 *There exists a nondegenerate n-dimensional smooth projective variety X in $\mathbf{P}^{2n+1}$ with the canonical bundle ample such that* $\dim \mathrm{Tan} X \leq 2n-1$ *if* $n \geq 9$.

Proof. We divide the proof into two cases.

Case 1 ($n \geq 11$). As we have seen in the example (2.4), there exists a nondegenerate m-dimensional smooth projective variety V in $\mathbf{P}^{2m+1}$ such that $\dim \mathrm{Sec} V \leq 2m-3$ if m is even and $m \geq 12$.

Subcase 1.1 (n even). Embed $\mathbf{P}^{2m+1}$ in $\mathbf{P}^{2m+2}$ as a hyperplane, and let $p \in \mathbf{P}^{2m+2} - \mathbf{P}^{2m+1}$. Then $C_p(\mathrm{Sec} V) = \mathrm{Sec} C_p(V)$. Let Y be the intersection of $C_p(V)$ and a general hypersurface in $\mathbf{P}^{2m+2}$ of degree $d \geq 2$. Then $\dim\mathrm{Sec}Y \leq \dim\mathrm{Sec}C_p(V) \leq 2m-2$. Let X be the image of Y under the projection from the point outside of Sec$Y$. Then X is a nondegenerate m-dimensional smooth projective variety in $\mathbf{P}^{2m+1}$. Since $\dim\mathrm{Sec}X = \dim\mathrm{Sec}Y \leq 2m-2$, $\dim\mathrm{Tan}X \leq 2m-2$ by [2, Cor.4]. Furthermore, since $\omega_X \cong \omega_V \otimes \mathcal{O}_X(d-1)$, taking d large enough, we get the canonical bundle ω_X ample.

Subcase 1.2 (n odd). Since $\dim\mathrm{Sec}V \leq 2m-3$, we can embed V in $\mathbf{P}^{2m-1}$. Let X be the intersection of V and a general hypersurface in $\mathbf{P}^{2m-1}$ of degree $d \geq 2$. Then X is a nondegenerate $(m-1)$-dimensional smooth projective variety in $\mathbf{P}^{2m-1}$. Since $\dim\mathrm{Sec}X \leq \dim\mathrm{Sec}V \leq 2m-3$, $\dim\mathrm{Tan}X \leq 2m-3$ by [2, Cor.4]. Furthermore, since $\omega_X \cong \omega_V \otimes \mathcal{O}_X(d)$, taking d large enough, we get the canonical bundle ω_X ample.

Case 2 ($9 \leq n \leq 10$). Let $W = G(6,1)$, and embed W in $\mathbf{P}^{20}$ by the plücker embedding. Note that W is a nondegenerate 10 dimensional smooth projective variety in $\mathbf{P}^{20}$ such that $\dim\mathrm{Sec}W \leq 17$.

Subcase 2.1 ($n = 10$). Embed $\mathbf{P}^{20}$ in $\mathbf{P}^{21}$ as a hyperplane, and let $p \in \mathbf{P}^{21} - \mathbf{P}^{20}$. Then $C_p(\mathrm{Sec}W) = \mathrm{Sec}C_p(W)$. Let X be the intersection of $C_p(W)$ and a general hypersurface in $\mathbf{P}^{21}$ of degree $d \geq 2$. Then X is a nondegenerate 10-dimensional smooth projective variety in $\mathbf{P}^{21}$. Since $\dim\mathrm{Sec}X \leq \dim\mathrm{Sec}C_p(W) \leq 18$, $\dim\mathrm{Tan}X \leq 18$ by [2, Cor.4]. Furthermore, since $\omega_X \cong \omega_V \otimes \mathcal{O}_X(d-1)$, taking d large enough, we get the canonical bundle ω_X ample.

Subcase 2.2 ($n = 9$). Since $\dim\mathrm{Sec}W \leq 17$, we can embed W in $\mathbf{P}^{19}$. Let X be the intersection of W and a general hypersurface in $\mathbf{P}^{19}$ of degree $d \geq 2$. Then X is a nondegenerate 9-dimensional smooth projective variety in $\mathbf{P}^{19}$. Since $\dim\mathrm{Sec}X \leq \dim\mathrm{Sec}W \leq 17$, $\dim\mathrm{Tan}X \leq 17$ by [2, Cor.4]. Furthermore, since $\omega_X \cong \omega_V \otimes \mathcal{O}_X(d)$, taking d large enough, we get the canonical bundle ω_X ample. ∎

The following corollary answers the question of Ciliberto affirmatively.

Corollary 3.6 *There exists a nondegenerate n-dimensional nonruled tangentially degenerate smooth projective variety X in $\mathbf{P}^{2n+1}$ if $n \geq 9$.*

Proof. Note that a smooth projective variety Y with the canonical bundle ample has the Kodaira dimension $\kappa(Y) = \dim Y$. On the other hand, a ruled smooth projective variety Z has the Kodaira dimension $\kappa(Z) = -\infty$. Thus the corollary follows from Proposion (2.3) and Theorem (3.5) immediately. ∎

If $char.k = 0$, we can obtain a result concerning the non-uniruled case.

Corollary 3.7 *If $char.k = 0$ and $n \geq 9$, then there exists a nondegenerate n-dimensional non-uniruled tangentially degenerate smooth projective variety X in $\mathbf{P}^{2n+1}$. Needless to say, such a variety X is not uniruled by lines.*

Proof. See [6] for the definition of uniruledness. Note that a smooth projective variety with the canonical bundle ample is not uniruled if $char.k = 0$ by [6, Th.1]. Thus the corollary follows from Proposition (2.3) and Theorem (3.5) immediately. ∎

References

[1] Ballico, E. and Ciliberto, C.: Open problems, Algebraic curves and projective geometry Proceedings Trento 1988. Springer Lecture Notes in Math. 1389, 277-285 (1988)

[2] Fulton, W. and Hansen, J.: A connectedness theorem for projective varieties, with application to intersections and singularities of mappings. Annals of Math. 110, 159-166 (1979)

[3] Fujita, T. and Roberts, J.: Varieties with small secant varieties: the extremal case. Am. J. Math. 103, 953-976 (1981)

[4] Hartshorne, R.: Algebraic geometry. (Graduate Texts in Mathematics, 52.) Berlin Heidelberg New York: Springer 1983

[5] Kaji, H.: On tangentially degenerate curves. J. London Math. Soc. (2) 33, 430-440 (1986)

[6] Miyaoka, Y. and Mori, S.: A numerical criterion for uniruledness. Annals of Math. 124, 65-69 (1986)

Masahiro OHNO
Department of Mathematics
School of Science and Engineering
Waseda University
3-4-1, Okubo Shinjuku-ku
Tokyo, 169 JAPAN
e-mail: ohno@math.waseda.ac.jp

(Received July 7, 1993)

manuscripta math. 81, 445 - 446 (1993)

manuscripta mathematica

Errata to: Local Mappings on Spaces of Differentiable Functions [1]

Giovanni Alberti and Giuseppe Buttazzo

Some errors occurred in page 82 of the above paper [1].

l.10: "$|\alpha| \leq k$" *becomes* "$|\alpha| = k$",

l.15: "$\alpha \in I(k)$" *becomes* "α with $|\alpha| \leq k$",

l.19: "... a multi–index in $I(k)$" *becomes* "... a multi–index with norm $|\alpha| \leq k$",

l.25: "$\alpha \in I(k)$" *becomes* "α with $|\alpha| \leq k$",

l.27: "... for all $\alpha \in I(k)$." *becomes* "... for all α with $|\alpha| \leq k$." .

After these corrections, lines 10–30 in page 82 should become:

Let $I(k)$ be the set of all multi-indeces α with $|\alpha| = k$; if u is a function from Ω into $\mathbf{R}^m$ we denote by $D^k u$ the k-th derivative of u, i.e. the function of Ω into $(\mathbf{R}^m)^{I(k)}$ defined by $\left(D^k u(x)\right)_\alpha = D^\alpha u(x)$ for all $\alpha \in I(k)$ and $x \in \Omega$.

In the paper the following function spaces are used.

$C^k(\Omega, \mathbf{R}^m)$ the space of all functions of Ω into $\mathbf{R}^m$ such that $D_\alpha u$ is a continuous function for every α with $|\alpha| \leq k$; $C^k(\Omega, \mathbf{R}^m)$ is usually endowed with the structure of Fréchet space induced by the seminorms $\phi_{\alpha,K}$ given by

$$\phi_{\alpha,K}(u) = \sup_{x \in K} \left|D_\alpha u(x)\right| \quad \text{for all } u,$$

where α is a multi-index with norm $|\alpha| \leq k$ and K is a compact subset of Ω.

$C_c^k(\Omega, \mathbf{R}^m)$ the space of all functions $u \in C^k(\Omega, \mathbf{R}^m)$ with compact support in Ω.

$C_0^k(\Omega, \mathbf{R}^m)$ the space of all functions u such that $D_\alpha u$ is a continuous function which vanish at infinity, i.e. such that for every $\varepsilon > 0$ there exists a compact set K which satisfies $|D_\alpha u(x)| < \varepsilon$ for all $x \in \Omega \setminus K$ for every multi-index α with $|\alpha| \leq k$.

$C^k(\overline{\Omega}, \mathbf{R}^m)$ the space of all u such that $D_\alpha u$ admits a continuous extension to the set $\overline{\Omega}$ for all α with $|\alpha| \leq k$. Both $C_0^k(\Omega, \mathbf{R}^m)$ and $C^k(\overline{\Omega}, \mathbf{R}^m)$ are closed subspace of the Sobolev space $W^{k,\infty}(\Omega, \mathbf{R}^m)$ and are usually endowed with the norm

$$\|u\|_{W^{k,\infty}} = \sum_{\alpha \in I(k)} \|D_\alpha u\|_\infty.$$

References

[1] Alberti, G., Buttazzo, G.: Local mappings on spaces of differentiable functions. Manuscripta Mathematica **79**, 81–97 (1993)

Giovanni Alberti, Giuseppe Buttazzo
Dipartimento di Matematica,
via Buonarroti 2,
56127 Pisa, Italy

(Received October 18, 1993)