$$\therefore\ (1 + p_1^2)\frac{d^2w}{dt^2} + c - \frac{gp_1}{c^2h^2} + \frac{3cp_1 + q_1}{cp_1}\ w = 0.$$

And from this equation, by the ordinary method, we get for the angle between the apsides, the expression

$$\pi\sqrt{\frac{cp_1(1 + p_1^2)}{3cp_1 + q_1}},$$

$$\text{or } \pi\sqrt{\frac{p_1(1 + p_1^2)}{3p_1 + q_1 a}},\ \text{ if } c = \frac{1}{a}.$$

J. F. H.

III.—RESEARCHES ON THE THEORY OF ANALYTICAL TRANSFORMATIONS, WITH A SPECIAL APPLICATION TO THE REDUCTION OF THE GENERAL EQUATION OF THE SECOND ORDER.

By G. Boole, Waddington, near Lincoln.

Let P be a function of x and y: then it is clear that, whatever value we give to those variables, (and they are in this instance supposed to be without limitation,) P will assume some corresponding value, real or imaginary. Let us now suppose that x and y bear such relations to two other variable quantities, x' and y', that for every pair of values the former may be supposed to assume, the latter receive corresponding values. This is equivalent to supposing

$$x = f(x', y'), \quad y = f'(x', y') \text{ .... } (1),$$

and does not in any way limit the generality which we suppose x and y to possess.

If we substitute for x and y the values supposed to be given in (1), we shall have the general equation

$$P = P';$$

and this will be true for all supposable values of x and y.

Suppose, now, $P = 0$ to be the equation of a curve, P being, as before, a function of x and y. This equation we may consider under two distinct points of view: first, as expressing a relation between x and y for each point of the curve, which is the ordinary, and I believe hitherto the only method of considering the subject; or, secondly, as expressing a particular state or condition of the function P. Geometrically speaking, this latter view is tantamount to considering any plane curve $\phi(x, y) = 0$, as formed by the intersection of the surface $\phi(x, y) = z$ with the plane $z = 0$, that is, with the plane x, y.

Let it now be required to transform the equation $P = 0$ into the equation $P' = 0$ by the substitution of the values $x = f(x', y')$, $y = f'(x', y')$.

The order of proceeding it is here important to observe. We should first substitute in the function P the *general* values of x and y, and afterwards introduce the particular condition $P = 0$. The transformation of P into P′ is therefore independent of any relation among the variables supposed to be expressed in the condition $P = 0$. We may, therefore, by the reasoning of the preceding section, make $P = P'$, whether the values attributed to x and y satisfy the primitive equation to the curve or not. The same remark may be made respecting the various orders of differentials; we shall therefore have the following system of equations universally true:

$$P = P',$$

$$\frac{dP}{dx} = \frac{dP'}{dx} = \frac{dP'}{dx'}\frac{dx'}{dx} + \frac{dP'}{dy'}\frac{dy'}{dx},$$

$$\frac{dP}{dy} = \frac{dP'}{dy} = \frac{dP'}{dx'}\frac{dx'}{dy} + \frac{dP'}{dy'}\frac{dy'}{dy},$$

&c. &c.

In applying these principles to the transformation of any particular equation, we are at liberty, after performing the requisite differentiations, to replace the primitive condition $P = 0$ by any other which the nature of the problem may render it advisable to introduce. Interpreted into geometrical language, the above implies, that if any curve line in the plane xy be considered as formed by the intersection of the surface whose equation is

$$\phi(x, y) = z$$

with that plane; and if the co-ordinates x, y be transferred into another system x', y', not only will the line of intersection continue the same as before, but the intersecting surface also, throughout its whole extent.

In making use of the differential equations of the first or higher orders,

$$\frac{dP}{dx} = \frac{dP'}{dx}, \qquad \frac{dP}{dy} = \frac{dP'}{dy},$$

$$\frac{d^2P}{dx^2} = \frac{d^2P'}{dx^2}, \qquad \frac{d^2P}{dx\,dy} = \frac{d^2P'}{dx\,dy},$$

&c. &c.

we must be careful to introduce new conditions only after performing the differentiations, or in such a way as to produce the same result. If, for example, we have in P′ a term y'^2, and suppose $y' = \phi(x, y) = 0$, we may in the first differentiation neglect such a term, because y' will be retained as a coefficient; but we cannot do this in taking the partial differentials of the second order which do not contain y'. Although the sum of the partial differentials of any order is zero, yet those partial differentials themselves will be susceptible of real values, which it will be necessary to take account of.

As a first application of the preceding theory, let it be required to transform the general equation of the second order for two variables,

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D = 0,$$

into a new equation with rectangular co-ordinates x', y', and of the form

$$A_1x'^2 + A_1'y'^2 + D_1 = 0.$$

By the reasoning of the preceding sections, the first members of these two equations must be equal for all values of x and y. As we are at liberty to introduce a new condition, and as one object to be determined is the inclination of the axes x and x', let us assume $y' = 0$; then will $x'^2 = (x - a)^2 + (y - b)^2$, making a and b the co-ordinates of the new centre.

The equation $P = P'$ gives

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D = A_1x'^2 + A_1'y'^2 + D_1 \ \ldots \ (1).$$

The equation $\frac{dP}{dx} = \frac{dP'}{dx}$ gives, on substituting for x' its value, and making $y' = 0$ in the results,

$$Ax + By + C = A_1(x \quad a) \ \ldots\ldots\ldots \ (2).$$

The equation $\frac{dP}{dy} = \frac{dP'}{dy}$ gives, moreover,

$$A'y + Bx + C' = A_1(y - b) \ \ldots\ldots \ \ldots \ (3).$$

From the equation (2) we have, on differentiating,

$$\frac{dy}{dx} = \frac{A_1 - A}{B} = \tan x\,x';$$

and from (3), by a like process,

$$\frac{dy}{dx} = \frac{B}{A_1 - A'} = \tan x\,x';$$

equating these two expressions, we have

$$\frac{A_1 - A}{B} = \frac{B}{A_1 - A'},$$

which becomes, on reduction,

$$A_1^2 - (A + A')\,A_1 + AA' - B^2 = 0.$$

This equation virtually includes the system given in the former investigations, and its two roots determine A_1 and A_1'.

Had we, indeed, in lieu of the condition last named, made $x' = 0$, which would have given the equation

$$y'^2 = (x - a)^2 + (y - b)^2,$$

we should, as is evident from the symmetrical form of the equations, have obtained

$$\frac{dy}{dx} = \frac{A_1' - A}{B} = \tan xy',$$

$$\frac{dy}{dx} = \frac{B}{A_1' - A'} = \tan xy';$$

whence, by reduction,

$$A_1'^2 - (A + A')A_1' + AA' - B^2 = 0.$$

This equation will give the same values for A_1' as the one last obtained for A_1, and shews that the solution of either is sufficient to determine the values required.

In the equations (1), (2), and (3), assume $x = a$ and $y = b$, which we are allowed to do, since the values of a and b express the position of a point in the axis of x', for which alone these equations are true; then, observing that x' becomes equal to 0, we have

$$Aa^2 + A'b^2 + 2Bab + 2Ca + 2C'b + D = D_1 \quad \ldots\ldots(4),$$

$$Aa + Bb + C = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$A'b + Ba + C' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

From (5) and (6), by elimination,

$$a = \frac{BC' - A'C}{AA' - B^2}, \quad b = \frac{BC - AC'}{AA' - B^2}.$$

Multiplying (5) by a, and (6) by b, and subtracting half the sum from (4), we have

$$D_1 = Ca + C'b + D.$$

The process of differentiation employed in the solution of this problem, I propose to call *Differentiating along the New Axes.*

Ex. 2. To reduce the general equation of the second order for three variables,

$$Ax^2 + A'y^2 + A''z^2 + 2Bxy + 2B'xz + 2B''yz + 2Cx + 2C'y + 2C''z + D = 0,$$

to the more simple form

$$A_1x'^2 + A_1'y'^2 + A_1''z'^2 + D_1 = 0.$$

Equating these expressions, and differentiating along the axis x', by aid of the condition

$$x'^2 = (x - a)^2 + (y - b)^2 + (z - c)^2,$$

a, b, and c being co-ordinates of the new centre, we obtain

$$Ax + By + B'z + C = A_1(x - a) \quad \ldots\ldots (1),$$

$$A'y + Bx + B''z + C' = A_1(y - b) \quad \ldots\ldots (2),$$

$$A''z + B'x + B''y + C'' = A_1(z - c) \quad \ldots\ldots (3).$$

For the new centre at which $x = a$, $y = b$, $z = c$, the above equations become

$$Aa + Bb + B'c + C = 0 \quad \ldots\ldots (4),$$
$$A'b + Ba + B''c + C' = 0 \quad \ldots\ldots (5),$$
$$A''c + B'a + B''b + C'' = 0 \quad \ldots\ldots (6),$$

whence a, b, and c are determined; and by proceeding as in the last example,

$$D_1 = Ca + C'b + C''c + D.$$

If we subtract the equations (4), (5), (6) respectively from the corresponding ones (1), (2), and (3), we have, after transposing to one side,

$$(A - A_1)(x - a) + B(y - b) + B'(z - c) = 0 \ldots\ldots (7),$$
$$(A' - A_1)(y - b) + B(x - a) + B''(z - c) = 0 \ldots\ldots (8),$$
$$(A'' - A_1)(z - c) + B'(x - a) + B''(y - b) = 0 \ldots\ldots (9).$$

Whence, on eliminating $x - a$, $y - b$, $z - c$, we obtain

$$A_1^3 - (A + A' + A'')A_1^2 + (AA' + AA'' + A'A'' - B^2 - B'^2 - B''^2)A_1$$
$$- AA'A'' - 8BB'B'' + AB''^2 + A'B'^2 + A''B^2 = 0 \ldots\ldots (10),$$

a cubic, whose roots determine A_1, A_1', A_1''.

Finally, if from (7) and (8) we eliminate $z - c$, and from (8) and (9) $y - b$, and compare the results, we shall obtain as the symmetrical equations of the axis of x',

$$\{(A - A_1)B'' - BB'\}(x - a) = \{(A' - A_1)B' - BB''\}(y - b)$$
$$= \{(A'' - A_1)B - B'B''\}(z - c).$$

In the last obtained system, it is only necessary to change A_1 into A_1' and A_1'' to exhibit the symmetrical equations of the axes of y' and z'.

In the preceding applications of theory, it has only been necessary to differentiate the even powers, x'^2, y'^2, of the new co-ordinates; an operation which is immediately effected by the aid of the characteristic equation

$$x'^2 + y'^2 + z'^2 = (x - a)^2 + (y - b)^2 + (z - c)^2.$$

When odd powers occur, it is most convenient to employ the first of the linear formulæ of transformation,

$$x' = \cos xx' . (x - a) + \cos yx' . (y - b) + \cos zx' . (z - c);$$

whence

$$\frac{dx'}{dx} = \cos xx', \quad \frac{dx'}{dy} = \cos yx', \quad \frac{dx'}{dz} = \cos zx'.$$

Ex. To reduce the general equation of the second order for three variables to the form

$$A_1 x'^2 + A_1' y'^2 + A_1'' z'^2 + 2C_1 x' = 0.$$

The solution of this problem will differ from that of the last only in consequence of the constant terms, $C_1 \cos xx'$, $C_1 \cos yx'$, $C_1 \cos zx'$, respectively added to the second members of (1), (2), (3). Hence the cubic equation determining A_1, A_1', A_1'', and the symmetrical equations of the axes x', y', z', will be the same as before. It will at once be seen, that the constants a, b, c, and C_1, will be determined by the system,

$$Aa + Bb + B'c + C = C_1 \cos xx',$$
$$A'b + Ba + B''c + C' = C_1 \cos yx',$$
$$A''c + B'a + B''b + C'' = C_1 \cos zx',$$
$$Aa^2 + A'b^2 + A''c^2 + 2Bab + 2B'ac + 2B''bc + 2Ca + 2C'b + 2C''c + D = 0.$$

The quantities $\cos xx'$, $\cos yx'$, $\cos zx'$, are known, being determined by the coefficients of the equation of the axis x'.

The last of the above equations reduces to a simple one by the process adopted in the last example.

From the examination of (10) in the preceding example, it is apparent, that when the equation of the second order designates a paraboloid, its coefficients must satisfy the condition

$$AA'A'' + 8BB'B'' - AB''^2 - A'B'^2 - A''B^2 = 0;$$

and that the determination of A_1, A_1', A_1'', one of which, in this instance, becomes 0, will be effected by a quadratic.

In the preceding investigations we have supposed the form of the reduced equation known. In the following example I shall give an illustration of a more general method of solution, by which all the possible forms of the equation are determined, together with the general laws of the coefficients.

Assume

$$Ax^2 + A'y^2 + 2Bxy + 2Cx + 2C'y + D$$
$$= A_1x'^2 + A_1'y'^2 + 2B_1x'y' + 2C_1x' + 2C_1'y' + D_1 \ldots (A).$$

Differentiating along the axis of x' with respect to x and y, we have

$$Ax + By + C = A_1(x-a) + B_1x' \cos xy' + C_1 \cos xx' + C_1' \cos xy',$$
$$A'y + Bx + C' = A_1(y-b) + B_1x' \cos yy' + C_1 \cos yx' + C_1' \cos yy'.$$

Now $x' \cos xy' = -(y-b)$ and $x' \cos yy' = x - a$;

whence the above equations become

$$Ax + By + C = A_1(x-a) - B_1(y-b) + C_1 \cos xx' + C_1' \cos xy' \ldots (1),$$
$$A'y + Bx + C' = A_1(y-b) + B_1(x-a) + C_1 \cos yx' + C_1' \cos yy' \ldots (2).$$

Differentiating (1) and (2), we get

$$\frac{dy}{dx} = \frac{A_1 - A}{B_1 + B} = \frac{B_1 - B}{A' - A_1} = \tan xx' \ldots\ldots (3);$$

whence, on reduction,

$$A_1^2 - (A + A')A_1 + AA' - B^2 + B_1^2 = 0 \ldots\ldots (4).$$

Either of the expressions for $\frac{dy}{dx}$, given in (3), determines the value of tan xx'. The equation (4) determines A_1 and A_1', and is evidently equivalent to the remarkable system

$$A_1 + A_1' = A + A' \text{ .... } (5),$$

$$A_1A_1' - B_1^2 = AA' - B^2 \text{ .... } (6).$$

In (A), and in (1) and (2), making $x = a$, $y = b$, and observing that under these suppositions x' vanishes, we have

$$Aa^2 + A'b^2 + 2Bab + 2Ca + 2C'b + D = D_1 \text{......} (7),$$

$$Aa + Bb + C = C_1 \cos xx' + C_1' \cos xy' \text{.........} (8),$$

$$A'b + Ba + C' = C_1 \cos yx' + C_1' \cos yy' \text{.........} (9),$$

of which the first (7) is reducible, as in former examples, to a simple equation.

From the inspection of (3) it appears that the terms A_1, A_1', B_1, and tan xx', are connected by two necessary relations, and that we are therefore at liberty to impose two others. The quantities a, b, C_1, C_1', D_1, are connected by three equations, (7), (8), (9); here, therefore, we may impose likewise two new relations. If we assume

$$B_1 = 0, \quad C_1 = 0, \quad C_1' = 0,$$

we obtain the equations previously given for the discussion of the central system, (Ex. 1). If, on the contrary, we make

$$A_1 = 0, \quad C_1' = 0, \quad D_1 = 0,$$

we obtain a solution true for the case of the parabola.

As yet no use has been made of the higher system of equations

$$\frac{d^2P}{dx^2} = \frac{d^2P'}{dx^2}, \quad \frac{d^2P}{dx\,dy} = \frac{d^2P'}{dx\,dy}, \quad \frac{d^2P}{dy^2} = \frac{d^2P'}{dy^2}, \text{ \&c.}$$

From these a very interesting class of solutions may be obtained. The resulting equations will generally involve a quadratic surd, and will afford a remarkable illustration of the varied and multiform combinations under which the same class of mathematical truths may be presented.

I shall here subjoin a few additional remarks and illustrations with reference to the preceding investigations. The following problem has not, so far as I am aware, been directly solved before.

Given the equations of the projections of a line of the second order on two rectangular co-ordinate planes, to determine the equations of the principal axes; together with the corresponding primitive equation of the curve.

I shall at present merely consider the case in which the equations are of the form

$$ax^2 + a'y^2 + 2bxy + c = 0 \text{ ...... } (1),$$

$$a_1x^2 + a_1'z^2 + 2b_1xz + c_1 = 0 \text{ ...... } (2).$$

The relation among the quantities x, y, z, is evidently linear, since the curve is supposed to be coincident with a plane. Assume, therefore,

$$z = px + qy,$$

and substituting in (2), we have

$$(a_1 + a_1'p^2 + 2b_1p)\,x^2 + (2a_1'p + 2b_1)\,qxy + a_1'q^2y^2 + c_1 = 0;$$

hence, by comparison with (1),

$$a = a_1 + a_1'p^2 + 2b_1p, \quad b = (a_1'p + b_1)\,q, \quad a' = a_1'q^2, \quad c = c_1,$$

the solution of which gives

$$p = b\,\frac{\sqrt{a_1'} - b_1\sqrt{a'}}{a_1'\,.\sqrt{a'}}, \qquad q = \sqrt{\frac{a'}{a_1'}},$$

together with the necessary equations of condition,

$$\frac{aa' - b^2}{a'} = \frac{a_1a_1' - b_1^{\,2}}{a_1'}, \quad c = c_1.$$

The fundamental equation

$$x^2 + y^2 + z^2 = x'^2 + y'^2$$

becomes, on substituting as before for z,

$$(1 + p^2)\,x^2 + 2pq\,xy + (1 + q^2)\,y^2 = x'^2 + y'^2 \;\ldots\ldots\; (3).$$

We are now prepared to apply the principle of transformation, which it is the object of this paper to develope.

Assume, therefore,

$$ax^2 + a'y^2 + 2bxy + c = \mathrm{A}x'^2 + \mathrm{A}'y'^2 + \mathrm{C} \ldots\ldots\ldots (4).$$

Differentiating along the axis of x' with respect to x and y, we obtain

$$\left.\begin{aligned} ax + by &= \mathrm{A}\,(1 + p^2)\,x + \mathrm{A}pqy \\ bx + a'y &= \mathrm{A}pqx + \mathrm{A}\,(1 + q^2)\,y \end{aligned}\right\}$$

or

$$\{a - \mathrm{A}\,(1 + p^2)\}\,x + (b - \mathrm{A}pq)\,y = 0 \ldots\ldots (5),$$

$$(b - \mathrm{A}pq)\,x + \{a' - \mathrm{A}\,(1 + q^2)\}\,y = 0 \ldots\ldots (6),$$

whence, eliminating x and y,

$$\{a - \mathrm{A}\,(1 + p^2)\}\,\{a' - \mathrm{A}\,(1 + q^2)\} - (b - \mathrm{A}pq)^2 = 0 \;\ldots\ldots\; (7),$$

an equation whose roots determine A and A′.

Of equations (5) and (6), either is sufficient, when combined with the linear equation

$$z = px + qy,$$

to determine the position of the axis x'. Thus, from (6) we have

$$y = \frac{\mathrm{A}pq - b}{a' - \mathrm{A}\,(1 + q^2)}\cdot x \ldots\ldots (8);$$

and this expression, substituted in the value of z, gives

$$z = \frac{(a' - \mathrm{A})\,p - bq}{a' - \mathrm{A}\,(1 + q^2)}\cdot x.$$

whence, by comparison with (8),

$$\frac{x}{a' - \mathrm{A}\,(1 + q^2)} = \frac{y}{\mathrm{A}pq - b} = \frac{z}{(a' - \mathrm{A})\,p - bq},$$

the symmetrical equation of the axis x'. On changing A into A′, we obtain the equation of the axis y'.

Finally, on making x' and y' respectively 0 in (3) and (4), we have evidently

$$c = \mathrm{C}.$$

The problem is therefore completely resolved.

Since the general equation of the second order may be represented under the symmetrical form

$$a\,(x - \alpha)^2 + 2b\,(x - \alpha)\,(x - \beta) + a'\,(y - \beta)^2 + c = 0,$$

it is evident that the solution for the more general form, both of the original and the reduced equations, may be easily derived from the above.

It is unnecessary to show that the method which has been here employed in the transformation of functions of two and three variables, is equally applicable to corresponding functions of any number of variables whatever. The transformation of equations of the third and higher orders, is likewise, on the same principle, made to depend on the solution of a *minimum* number of final equations. Neither of these cases being of any importance, I have not thought proper to extend the investigation beyond its present limits. In the case of equations of the third and higher orders, it may, however, be observed, that there would seem to exist more than one system of axes, and more than one relation among the coefficients, of the new and old equations, according to which the transformation may be effected, so as to result in a proposed form, or to satisfy given conditions.

With one or two remarks on the application of the preceding principles to the transformations required in the problem of rotation, and in the undulatory theory of light, I shall bring this communication to a close.

Let U be a function of x, y, z; then, by Maclaurin's theorem,

$$\mathrm{U} = (\mathrm{U}) + \left(\frac{d\mathrm{U}}{dx}\right) x + \left(\frac{d\mathrm{U}}{dy}\right) y + \left(\frac{d\mathrm{U}}{dz}\right) z$$

$$+ \frac{1}{1.2}\left\{\left(\frac{d^2\mathrm{U}}{dx^2}\right) x^2 + \left(\frac{d^2\mathrm{U}}{dy^2}\right) y^2 + \left(\frac{d^2\mathrm{U}}{dz^2}\right) z^2 + 2\left(\frac{d^2\mathrm{U}}{dx\,dy}\right) xy\right.$$

$$\left. + 2\left(\frac{d^2\mathrm{U}}{dx\,dz}\right) xz + 2\left(\frac{d^2\mathrm{U}}{dy\,dz}\right) yz\right\}$$

$$+ \text{\&c. \&c.}$$

Imagine U to be transformed into a function of x', y', z', the equations of transformation being homogeneous and of the first degree, then

$$U = (U) + \left(\frac{dU}{dx'}\right) x' + \left(\frac{dU}{dy'}\right) y' + \left(\frac{dU}{dz'}\right) z'$$

$$+ \frac{1}{1.2}\left\{\left(\frac{d^2U}{dx'^2}\right) x'^2 + \left(\frac{d^2U}{dy'^2}\right) y'^2 + \left(\frac{d^2U}{dz'^2}\right) z'^2 + 2\left(\frac{d^2U}{dx'\,dy'}\right) x'y' + \&c.\right\}$$

Now, from the nature of the relation between x, y, z and x', y', z', the above values of U cannot be equal, unless each aggregate of homogeneous terms in the one be equivalent to the corresponding aggregate in the other. Those of the second order give, on comparison,

$$\left(\frac{d^2U}{dx^2}\right) x^2 + \left(\frac{d^2U}{dy^2}\right) y^2 + \left(\frac{d^2U}{dz^2}\right) z^2$$

$$+ 2\left(\frac{d^2U}{dx\,dy}\right) xy + 2\left(\frac{d^2U}{dx\,dy}\right) xz + 2\left(\frac{d^2U}{dy\,dz}\right) yz$$

$$= \left(\frac{d^2U}{dx'^2}\right) x'^2 + \left(\frac{d^2U}{dy'^2}\right) y'^2 + \left(\frac{d^2U}{dz'^2}\right) z'^2$$

$$+ 2\left(\frac{d^2U}{dx'\,dy'}\right) x'y' + 2\left(\frac{d^2U}{dx'\,dz'}\right) x'z' + 2\left(\frac{d^2U}{dy'\,dz'}\right) y'z'.$$

In the transformation of an equation of the second degree, it is therefore necessary that the coefficients should express the values assumed by a certain system of partial differential coefficients, when the variables vanish; and conversely, when such special values of the required system of partial differential coefficients present themselves, they may be transformed by considering them as coefficients of the equation of the second degree. The former of these cases is the one met with in the problem of rotation, the latter in the undulatory theory of light.

IV.—ON THE FAILURE OF FORMULÆ IN THE INVERSE PROCESSES OF THE DIFFERENTIAL CALCULUS.

If we apply the rule for integrating any power of x to the particular case when the index of the power is -1, we obtain a result having 0 in the denominator, and which is therefore nugatory. This is only one instance of several in which a certain relation of the subject to an inverse operation makes the general formulæ fail; and as these cases give rise to some difficulty, we shall here consider two of the most important of them. The instance to which we have alluded is so well known, that we need do no more than mention it; and for the more general case of failure when the index is of any value, the reader is referred to Art. VI. Vol. I. p. 109. The method of arriving at the true value in these cases of failure, is the same as that which we shall pursue in those we